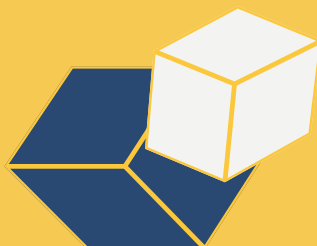


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Contents

In Memoriam Professor Antonio Maturo

Editorial

Giovannina Albano, Chiara Andrà, Domenico Brunetto, and Maria Antonietta Lepellere 1

Addressing learning loss through a mathematical web-app: insights from user behaviour

Giovannina Albano, Chiara Andrà, Domenico Brunetto, and Maria Antonietta Lepellere 10

Learning probability at grade 2: exploring the role of drawings for children's understanding

Chiara Andrà, Domenico Brunetto, and Maria Polo 34

Parental stereotypes and early gender gaps in mathematical attitudes. Evidence from Italian primary schools

Teodora Erika Uberti 49

Children as mediators of home-school continuity in early mathematics: the case of Betta-the-Bee

Raffaele Casi and Carlotta Soldano 64

Mortino and the Blue Pearl Heart: a structured-narrative journey through pupils attitude in mathematics

Cristina Coppola and Angela Sassone 78

From classroom to outdoors and back: orchestrating the field of experience of sun and shadows in mathematics education

Daniele De Giorgi 97

Mathematical modelling in outdoor learning: exploring analog strategies and cooperative processes in a primary school context

Giorgia Distefano and Eugenia Taranto 121

Weaving mathematics across contexts: a theoretical–methodological framework for researching classroom–museum continuity

Raffaele Casi 139

Exploring empathy in mathematics feedback: a comparative study of human and AI-generated responses in informal learning contexts

Gennaro Cordasco, Umberto Dello Iacono, Anna Esposito, Antonio Vitale, and Carl Vogel 154

- Formative feedback and artificial intelligence: new perspectives for pre-service teacher education
Michele Giuliano Fiorentino, Antonella Montone, Rosalba Romito, and Candida Tosques 167
- Reducing students learning gap in multicultural classrooms through mathematics teacher training: an Italian case study
Giuseppe Bianco, Benedetto Di Paola, and Giovanni Giuseppe Nicosia 180



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In Memoriam Professor Antonio Maturo

The Italian Journal of Pure and Applied Mathematics (IJPAM) unfortunately can no longer benefit from the precious collaboration of Professor Antonio Maturo, who passed away this year.

He was a very close friend of Professor Piergiulio Corsini, the founder of the Journal, and contributed greatly to IJPAM, not only as an Editorial Board member but also by organizing conferences at the Università degli Studi 'Gabriele D'Annunzio' Chieti-Pescara with the aim of giving more visibility to the Journal.

The entire Staff and the Editorial Board of IJPAM express their deep sorrow for this sad loss. In particular, Maria Antonietta Lepellere, Piergiulio Corsini and his family, Professor Maturo's son Fabrizio Maturo, Irina Cristea, Violeta Fotea, and Domenico Chillemi regret the loss of Professor Antonio Maturo, who was a great man of science. All those who knew him will always remember his scientific value and his human qualities.

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Continuity in mathematics learning beyond the summer: classroom, outdoor environments, and the design of resilient mathematical experiences

1. Introduction: continuity in mathematics learning beyond classroom boundaries

Continuity is a familiar concern in mathematics education. It is often discussed in relation to curriculum progression, transitions between school levels, or the coherence of mathematical contents over time. The centrality of this construct is discussed in what follows. Students do need opportunities to revisit, deepen, and reorganise mathematical ideas across grades and institutional transitions, and in any occasion of possible leap within their learning trajectory. Yet the contributions collected in this special issue invite us to broaden the meaning of continuity. Continuity in mathematics learning is not only a matter of what comes before and after in a curriculum. It also concerns the possibility for mathematical experiences, practices, meanings, and forms of participation to remain available when the conditions of schooling change.

The contributions collected address this matter through a range of situated cases: a summer web app for children and families, a kindergarten game re-enacted at home, outdoor explorations of shadows and math trails, museum-based work on spirals and mathematical machines, online and AI-mediated feedback and teacher education initiatives focused on intercultural and formative mediation. Across these cases, continuity is not treated as the mere persistence of content, but as the possibility of reactivating mathematical work through different mediations, settings, and forms of participation.

The title of this special issue, *Continuity and learning beyond the summer. Mathematics between classroom and outdoor environments: research perspectives in dialogue*, captures this double movement. On the one hand, the summer break foregrounds the temporal dimension of continuity: what happens to mathematical activity when the ordinary rhythm of schooling is interrupted? On the other hand, the reference to classroom and outdoor environments foregrounds the spatial and cultural dimension of continuity: how does mathematical activity change when it moves across places, artefacts, social relations, institutional aims, and forms of mediation?

The issue originates as a follow-up to the conference held at the University of Salerno on 19 February 2026, within the PRIN2022 project *Coming to terms not only with the pandemic. Mathematics learning loss in primary school: underlying factors and interventions* (Prot. 2022TWCJAS, CUP D53D23013070006). However, it is not limited to the results of that project, nor to the papers presented at the conference. Its aim is broader: to offer a composite view of research, design experiences, and teacher education initiatives concerned with the continuity of mathematics learning across classroom, outdoor, home, digital, museum-based,

intercultural, or other non-strictly formal educational settings.

The special issue is therefore framed by a shift of perspective: from summer learning loss understood as decline to continuity understood as the design of sustained opportunities for mathematical activity during the summer; from the classroom as the only legitimate site of mathematical work to a network of settings that includes homes, schoolyards, museums, digital environments and teacher education workshops; from isolated activities to trajectories in which mathematical meanings are prepared, transformed, documented, and taken up again.

2. From learning loss to learning continuity

The PRIN2022 framework places the special issue in a post-pandemic research landscape, but the problem addressed in this Special Issue, four years later, is broader than pandemic-related interruption. The pandemic made visible forms of discontinuity that were already present in educational systems: unequal access to learning opportunities, fragile connections between school and family contexts, and difficulties in sustaining mathematical engagement when ordinary classroom routines are interrupted. The summer break provides a recurrent interruption in school time through which researchers can examine how mathematical engagement changes when regular classroom routines, teacher mediation, and shared school rhythms are temporarily suspended.

In mathematics, this issue is particularly delicate because mathematical learning depends not only on remembering procedures, but also on maintaining opportunities for reasoning, representing, arguing, modelling, and problem solving over time. The contributions connected to the MaTEs project address summer continuity from different angles: the design and use of a web app for children and families, the analysis of young pupils' probabilistic representations, and the study of parental stereotypes in early mathematical attitudes. In these studies, summer learning loss is treated not only as the possible decline of procedural fluency, but as the fragility of sustained engagement with non-routine problems, narrative modelling, drawings, justification, and adult-mediated mathematical discussion during periods in which school routines are reduced or absent.

The contribution by Albano et al. addresses this issue directly through the MaTEs web app, designed to support children and families during the summer period. The study does not treat the app simply as a device for assigning additional exercises. Rather, it frames it as a hybrid environment in which digital prompts, offline mathematical work, adult mediation, narrative contexts, documentation of children's productions, and affective-metacognitive reflection are integrated. The app is explicitly not conceived as an automated learning environment, but as a mediating device between the child, the adult, and the mathematical activity.

The empirical analysis of log data from 315 users does not claim to measure learning outcomes directly. It examines usage trajectories during the summer, identifying different patterns in terms of completion, continuity, and temporal distribution. This is important because it prevents a simplistic relation between access to a resource and educational continuity. The findings show that use was often non-uniform and sometimes concentrated toward the end of the summer, while the largest group displayed a more regular and sustained pattern. The result gives empirical substance to one of the issue's central claims: continuity cannot be inferred from the provision of a tool alone. It has to be examined through temporal patterns of use, the distribution of activity across the summer, and the forms of adult-mediated engagement that the resource makes possible.

At the same time, the issue invites caution toward a narrow deficit-oriented understanding of learning loss. Bianco et al. enrich the notion of loss by relocating it within multicultural classrooms. Their argument is that, in some cases, mathematical knowledge is not lost but becomes institutionally unrecognisable when students' algorithms, languages, symbolic systems, or epistemological habits do not coincide with those expected by the dominant school culture.

In this perspective, what is often interpreted as missing knowledge may instead be knowledge that has been set aside because it is not readily interpretable within the host classroom.

This reframing is crucial for the special issue. It prevents learning loss from becoming a purely compensatory category. The issue is not only how to recover what may have been lost, but how to recognise what has remained invisible, how to sustain what risks being interrupted, and how to design mathematical experiences that can remain meaningful across changes in time, place, language, and social expectation. The response proposed by Bianco, Di Paola and Nicosia is not remediation in the narrow sense, but teacher education aimed at preparing teachers to act as cultural brokers: professionals able to recognise, interpret, and value mathematical knowledge even when it is produced in distant or unfamiliar contexts.

In this sense, the movement from learning loss to learning continuity is not merely terminological. This does not mean dismissing the empirical problem of learning loss, nor the need to document changes in students' mathematical competence across interruptions. Rather, it means placing such documentation within a broader design question: which forms of mathematical activity remain available to students when school time is suspended, and which forms require intentional mediation in order not to disappear?

2.1. Mathematics between classroom, outdoor environments, and other educational settings

Outdoor mathematics education occupies a central position in this special issue, but the contributions make clear that outdoor learning should not be understood as the simple relocation of classroom tasks to an open-air setting. In the papers focused on outdoor or place-based settings, the educational relevance of the environment does not derive simply from being outside the classroom. It emerges when the environment becomes part of the mathematical task: shadows cast by gnomons in the schoolyard, measurements and observations required by a math trail, spirals encountered through museum objects and bodily movement.

De Giorgi's contribution develops this point through the field of experience of sun and shadows. The paper connects Outdoor Education, fields of experience, instrumental orchestration, semiotic mediation, and teacher professional development. The analysed episode in a fourth-grade classroom and schoolyard shows that the outdoor setting is not an accessory to learning. It becomes part of the mathematical work itself. Students explore shadows through gnomons, drafting triangles, bodily movement, and observations of objects in the courtyard; the teacher orchestrates these explorations by drawing attention to relationships such as direction, parallelism, shadow length, and the relation between light source and object.

The provisional notion of *Outfield Education* is useful here because it avoids a romanticised view of outdoor work. It describes an education that takes place in, about, and through the field of experience, where the field itself functions as a learning environment connecting formal inquiry, outdoor exploration, digital simulations, and students' lived worlds. At the same time, De Giorgi's analysis makes visible the fragility of such work. Teachers' reflections point to the need for further teacher education, more confidence in managing outdoor mathematical activities, and stronger curricular integration, possibly through multi-year pathways.

Taranto and Distefano similarly show that outdoor mathematics depends on design, mediation, and reflection. Their study of a non-digital math trail in a fifth-grade class demonstrates that some core features of digital math trails can be preserved when smartphones are unavailable or restricted. Treasure-hunt activities support spatial orientation; paper envelopes with progressive hints scaffold problem solving; cooperative roles organise participation. Yet the analog redesign does not fully replace the affordances of digital tools. The absence of immediate digital feedback makes validation and assessment more demanding, shifting greater responsibility to classroom discussion and to the teacher's reconstruction of students' written and oral processes.

The study also warns against assuming that cooperation automatically supports all phases of mathematical modelling. While group work can foster engagement and shared reasoning, interpretation and validation still require explicit scaffolding and opportunities for collective discussion.

Casi's theoretical-methodological paper extends this reflection to classroom–museum continuity. It does not treat museums as outdoor settings in a classificatory sense. Rather, it uses an outdoor lens analytically, to make visible place, materiality, movement, mediation, and contextual difference in classroom–museum trajectories. The point is not that the museum is outside school in a simple spatial sense, but that it configures mathematical work through objects, rhythms, institutional purposes, and forms of participation that differ from classroom practice.

The paper argues that continuity cannot be reduced to the recurrence of a mathematical topic across classroom and museum. In the worked example, students move between classroom explorations of circumferences and a museum workshop on spirals, where distance from a centre, rotation, variation of distance, bodily movement, architectural details, and mathematical machines such as the spiralograph and the helicograph become part of the same broader trajectory. What travels across contexts is not only a topic, but a set of ways of seeing, moving, describing, conjecturing, and using artefacts mathematically.

Together, these contributions suggest that the classroom/outdoor relation should not be conceived as a binary opposition. The classroom remains essential as a place for preparation, discussion, institutionalisation, comparison, and reflection. Outdoor environments, museums, schoolyards, and local territories offer forms of experience that cannot be fully reproduced in the classroom. Continuity emerges when these settings are not treated as isolated episodes, but as parts of trajectories in which mathematical meanings can be prepared, transformed, documented, and reactivated.

2.2. Mediation across home, family beliefs, and cultural recognition

Across the issue, the relation between school and what lies beyond it appears in several different forms. In Soldano and Casi, it is a home-school relationship mediated by children who re-enact a mathematical game with parents. In Uberti, it is a family belief system that shapes children's early mathematical attitudes and gendered self-perceptions. In Bianco et al. it is a cultural relation between students' mathematical backgrounds and the dominant norms of the classroom. These are not the same phenomenon, but they converge on one point: continuity depends on whether mathematical activity is recognised as meaningful by those who participate in it.

Soldano and Casi's study on *Betta-the-Bee* offers a particularly clear example. A mathematically rich card game is introduced at kindergarten and then taken home by children during the Christmas break. The game itself is structured around the formulation of yes/no questions, the recognition of variables, the interpretation of answers, and the elimination of possibilities. Its mathematical specificity therefore lies not only in the visible features of the deck, but in the rule-governed reasoning enacted during play.

The findings show that children do not simply transport a material object. They mediate a school-shaped mathematical practice. They introduce the game, explain the rules, correct adult misunderstandings, and keep the interaction aligned with what counts as proper play. The continuity observed in the study depends on more than the portability of the game: what children carry home is a rule-governed practice already socialised at school.

This has consequences for parental recognition of mathematics. After the home experience, parents no longer refer mainly to visible numerical content. They more often identify logical thinking, grouping, spatial organisation, and the role of questioning. The window of what becomes noticeable as mathematics appears to widen. Continuity here is therefore not only

the reuse of the same game across settings, but the transformation of what parents are able to recognise as mathematical within that activity.

Uberti's analysis of parental stereotypes and early gender gaps in mathematical attitudes adds another layer. Drawing on data from the MATES project, the study shows that gender differences in mathematical attitudes are already observable among 8-year-old children: boys display slightly higher self-efficacy and stronger liking for mathematics, while girls report higher overall school enjoyment. No significant differences emerge in the use dimension. The parental dimension is particularly important: stereotypes such as the belief that males are more naturally suited to mathematics are associated with lower liking for mathematics and self-efficacy among girls, while reinforcing more positive attitudes among boys.

This contribution shows that continuity is shaped not only by tasks and resources, but also by the symbolic environment in which children learn to see themselves as mathematical subjects. Family beliefs do not simply accompany learning; they contribute to the formation of expectations, self-efficacy, and early perceptions of who can legitimately see themselves as mathematically competent.

Bianco et al. broaden the problem further by shifting the focus from family beliefs to cultural recognition in multicultural classrooms. Survey data from 560 in-service teachers reveal that only a small proportion had received training specifically focused on mathematics teaching in plurilingual and multicultural contexts. Teachers identify language barriers as the most significant difficulty, but also mention cultural differences, didactic issues, institutional constraints, and lack of family involvement.

The authors' response is grounded in teacher education and co-design. The shared planning scheme developed within the "Cultures Count" course aims to help teachers attend to class composition, students' interests and habits, prior knowledge, linguistic and cultural differences, and the relation between institutional objectives and students' lived mathematical experiences. The teacher is positioned not merely as a transmitter of content, but as a cultural mediator and broker of meanings. Continuity here means reducing the distance between classroom and home cultures, between dominant and subaltern mathematical practices, and between institutional expectations and students' previous knowledge.

Taken together, these papers suggest that continuity cannot be designed only at the level of tasks or materials. It also depends on mechanisms of recognition: whether parents come to see questioning and elimination as mathematical in a game; whether girls and boys receive different implicit messages about who is "naturally" suited to mathematics; whether teachers can recognise culturally different algorithms and mathematical practices as resources rather than deficits.

2.3. Storytelling, representation, affect, and feedback

A further cross-cutting theme concerns the affective, narrative, representational, and feedback-mediated conditions of continuity. Students do not sustain mathematical engagement only because tasks are cognitively meaningful. They also need reasons to participate, emotional conditions that make persistence possible, representations that allow them to organise thinking, and feedback that supports revision without reducing mathematics to correctness alone.

Coppola and Sassone's contribution on *Mortino and the Blue Pearl Heart* explores this issue through a structured narrative and gamified pathway in a second-grade classroom. The study does not present storytelling and gamification as devices that simply remove anxiety. Rather, it shows how a sustained narrative trajectory can provide a shared context in which pupils externalise uncertainty, persist in demanding tasks, and begin to experience mathematical problems as meaningful challenges within a storyworld. The design is layered: the narrative forms the outer shell, while gamification constitutes an intermediate level, and mathematics remains the core of the experience. Identification with Mortino supports children's sense of

responsibility and participation, while the mathematical challenges allow the story to progress.

This contribution is important for the special issue because it frames continuity not only as temporal persistence, but also as narrative and affective coherence. Mathematical activity is embedded in a layered design in which the narrative provides continuity over time, gamification sustains participation, and mathematical tasks remain the core through which the story progresses.

The paper by Andrà et al. on probability and drawings foregrounds representation as another condition for continuity, especially within the MaTEs concern for designing meaningful mathematical tasks for young children beyond routine school exercises. By asking Grade 2 pupils to draw and justify their answers in a probabilistic task, the study shows how drawings may function as semiotic resources through which children coordinate intuition, narrative elements, and mathematical structure. The authors deliberately move beyond a view of probabilistic thinking centred only on misconceptions and instead attend to the intertwined nature of intuitions, procedures, and representations. Their analysis shows that even at this young age, children can identify and employ mathematical features of a probabilistic task in order to answer it correctly, while many still struggle to provide written justifications for correct choices.

This result is relevant beyond probability. It suggests that continuity between intuition, representation, and formal reasoning cannot be assumed. It must be supported through semiotic resources that allow children to move from experience to mathematical articulation. Drawings function as intermediate semiotic resources: they allow pupils to display aspects of the probabilistic situation that may not yet be articulated in written justification.

Two contributions address feedback and AI, extending the issue's concern with continuity to digital and feedback-mediated environments. Vitale et al. examine informal online help-seeking in formal logic, where mathematical support is asynchronous, voluntary, and affectively fragile. Their study compares human feedback with LLM-generated feedback, evaluating empathy through emotion recognition, perspective-taking, and emotional transmission. The findings indicate that LLM-generated feedback, when supported by structured prompts, is rated as more empathetic than human feedback, especially in emotion recognition and emotional transmission, while perspective-taking remains the most complex dimension.

Fiorentino et al. examine a different setting: pre-service teacher education. Their study interprets an Artificial Agent developed through ChatGPT 5.2 as a possible dynamic semiotic mediator within formative feedback processes. The activity involved 300 pre-service teachers working in pairs on an open-ended mathematical problem, followed by interaction with the Artificial Agent and collective discussion. The qualitative analysis of 150 protocols shows a tendency to move from local, incomplete, or poorly argued responses toward broader, more structured, and more generalised responses. At the same time, the authors identify critical issues: misleading feedback, overly verbose outputs, and the risk of mechanical use. These become formative only when discussed and mediated by the teacher.

The two AI-related contributions extend the issue's concern with continuity in two distinct directions. Vitale et al. examine continuity as affective and cognitive support in informal digital help-seeking, where feedback may sustain or discourage learners' willingness to continue working on mathematical problems. Fiorentino et al. examine continuity as revision within a designed formative sequence, where AI-generated feedback becomes productive only through prompt design, peer interaction, and collective teacher-mediated discussion. In both cases, the central issue is not whether AI can replace human mediation. The issue is how feedback, human or artificial, can become part of a designed trajectory of revision, reflection, and mathematical meaning-making.

3. The rationale and scope of the special issue

The contributions collected in this issue do not form a homogeneous set in terms of methods, levels, or theoretical frameworks. They include empirical case studies, quantitative analyses, theoretical-methodological proposals, teacher education studies and digital learning studies. This plurality is not a weakness. It is precisely what allows the special issue to address continuity in its complexity.

Across the papers, continuity appears in several forms. It is temporal, when mathematical engagement is sustained, interrupted, postponed, or intensified during the summer break. It is spatial, when mathematical work moves between classroom, schoolyard, museum, home and digital environments. It is cultural, when learning depends on whether students' prior knowledge, family beliefs, language, gendered expectations, and cultural practices are recognised. It is affective, when students' willingness to engage depends on confidence, anxiety, narrative identification, feedback, and perceived competence. It is methodological, when researchers need tools to document learning across less standardised settings. It is professional, when teachers must learn to design, orchestrate, observe, and interpret mathematical activity beyond ordinary classroom routines.

The issue therefore does not propose outdoor education, summer activities, digital tools, storytelling, AI feedback, or family involvement as simple solutions. Rather, it treats them as contexts in which the problem of continuity becomes visible and can be studied. Each contribution shows, in its own way, that mathematical meanings do not travel automatically. They need mediation. They need tasks, artefacts, representations, discussions, routines, feedback, documentation, and institutional support.

This is the central rationale of the special issue: to examine how mathematical activity can be sustained across interruptions and differences without being reduced either to school repetition or to informal spontaneity. The question is not how to export classroom mathematics unchanged into other spaces. It is how to design trajectories in which mathematical experiences can be transformed while remaining recognisably mathematical.

4. Implications for research, teaching, and teacher education

For research in mathematics education, the special issue suggests the need to study trajectories rather than isolated events. Outdoor lessons, museum visits, summer apps, family games, AI feedback, narrative pathways, and teacher education workshops acquire educational meaning when they are connected to what prepares them, what happens within them, and what follows from them. Methodologically, this requires heterogeneous data: observations, artefacts, digital traces, interviews, drawings, written productions, discussions, focus groups, and forms of longitudinal documentation.

For teaching, the issue highlights the importance of designing mathematical activities that are both situated and mathematically explicit. The outdoor environment, the story, the web app, the family game, the museum object, or the AI feedback system should not merely decorate mathematical content. They should organise attention, support reasoning, and create opportunities for representation, discussion, modelling, argumentation, and validation.

At the same time, the contributions warn against romanticising less formal contexts. Outdoor activities need scaffolding; cooperation needs mathematical guidance; family involvement needs intelligibility; AI feedback needs critical mediation; narrative engagement needs didactical structure; intercultural inclusion needs teacher awareness and institutional support. Less formal or non-classroom settings are not automatically more inclusive, more meaningful, or more effective. They become educationally significant when they are designed and interpreted with mathematical and pedagogical care.

Teachers need to be prepared to recognise mathematics in non-standard forms, to orchestrate

learning across environments, to work with families and cultural differences, to use feedback formatively, and to connect situated experiences with disciplinary knowledge. The studies on Outfield Education, intercultural teacher training, and AI-mediated formative feedback show that professional development must address not only new tools and new settings, but new forms of professional judgement.

Teachers are not simply implementers of activities. They are designers and mediators of continuity. They decide what can be prepared in the classroom, what can be explored outdoors, how an experience can be documented, how students' productions can be discussed, how family participation can be made meaningful, how digital tools can support rather than replace interaction, and how culturally diverse mathematical knowledge can be recognised rather than dismissed.

At the institutional level, the issue points toward the need for stronger alliances among schools, universities, families, museums, local communities, and digital learning environments. Summer learning cannot be addressed only through homework. Outdoor mathematics cannot depend only on individual teacher enthusiasm. Family participation cannot be assumed without support. Teacher education cannot remain confined to general principles without concrete design work. Continuity requires infrastructures: shared planning time, research-practice collaboration, accessible resources, documentation tools, and institutional recognition of mathematical learning beyond the classroom.

4.1. Concluding remarks: towards resilient and situated mathematics learning

The special issue closes without offering a single model of mathematics learning beyond the classroom. Instead, it proposes a research agenda. Future work should investigate how summer interventions can sustain regular and meaningful engagement rather than last-minute completion; how schools, families, universities, museums, and local communities can collaborate without transferring responsibility unevenly to families; how outdoor and museum-based learning can be documented and evaluated without reducing their richness to standard classroom measures; how teachers can be prepared to design mathematical activities that cross spatial, cultural, and temporal boundaries; and how curricula can become more permeable to informal, semi-formal and digital forms of mathematical experience.

The challenge is not simply to compensate for interruption. It is to imagine mathematics education as a more resilient and situated field of experience. The summer break, the home, the schoolyard, the museum and the digital forum should not be treated as peripheral spaces in which school mathematics is merely applied or repeated. In the studies collected are settings in which mathematical activity is reorganised through different mediations: adult support, children's expertise, artefacts, bodily movement, feedback, narrative identification, cultural recognition, and teacher orchestration.

4.2. Acknowledgments

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Addressing learning loss through a mathematical web-app: insights from user behaviour

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Abstract: This study stems from the MaTEs project which aims to address summer learning loss in mathematics through a web-based app designed for pupils and their families. This paper presents some results focusing on the design and use of a web-based app aimed at supporting continuity of engagement with mathematical activities during the summer period. The study conceptualizes learning as a situated and mediated process extending beyond school contexts. The web app offers narrative-based problem-solving activities supported by adult mediation, integrating cognitive, metacognitive, and affective dimensions. Using log data from 315 users, the study adopts a quantitative approach based on Time Series K means clustering to identify patterns of behaviour over the summer period. Four distinct profiles emerge in terms of completion, continuity, and temporal distribution of activity. Findings highlight a general tendency toward non-uniform use, with activity concentrated at the end of the summer, but the largest group corresponding to a regular and sustained pattern of use. Notably, no significant differences are found across profiles in metacognitive reflections, suggesting that discontinuity of use cannot be simply explained by different levels of appreciation or perceived experience. The study sheds light on heterogeneous attitudes towards the summer homework and raises critical questions about how continuity, rather than intensity alone, supports meaningful mathematical learning beyond the classroom.

2020 Mathematics Subject Classification: Primary 97C70; Secondary 97D50, 97C20.

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Keywords: summer learning loss; informal learning; storytelling; mathematical modelling; affect and engagement; time series K-Means cluster.

1. Introduction

This paper presents the results of the MaTEs project², which investigates summer learning loss in mathematics, conceptualized as a mechanism similar to that observed during the COVID-19 pandemic, in terms of reduced school engagement and the increased role of families.

Continuity is a key element in ensuring effective learning: it enables students to consolidate knowledge and skills over time, progressively building stable cognitive structures. When this continuity is disrupted, disconnection arises between previous and subsequent learning experiences, with the risk that part of the acquired competencies may be forgotten or not properly consolidated. Thus, more generally, learning loss can be defined as the decline in skills and knowledge that occurs during periods when instructional activity is reduced or absent [34, 23, 32].

School discontinuity is closely related to learning loss. In particular, during extended breaks, students may experience a decrease in abilities, especially in areas such as mathematics and reading comprehension, which require consistent practice. This effect is often more pronounced among students from socio-economically disadvantaged backgrounds, where opportunities for informal learning during school interruptions are more limited [31].

This study, as part of a wider project aimed at addressing such a problem, focuses on educational intervention designed to counteract math learning loss, based on a web app for families and children. As the first investigation, this work does not assess learning achievements directly, but aims to investigate to what extent the use of the designed web-app can foster continuity supporting student engagement in informal learning activities during summer.

The conceptual background presented in this paper primarily informs the design of the MaTEs web-app, clarifying why the activities were structured around storytelling, modelling, adult mediation, and affective-metacognitive reflection. The empirical analysis reported here does not evaluate learning outcomes directly, but examines how the web-app was used over time, considering usage patterns as traces of engagement with the designed mathematical activities.

2. Conceptual background

The project is grounded in three interconnected strands of research: (i) learning loss and informal learning, (ii) storytelling and mathematical modelling, and (iii) affect and engagement in mathematics education. These perspectives are brought together to conceptualize learning as a situated, mediated, and multidimensional process that extends beyond formal classroom settings.

We use the term learning loss not to frame pupils or families from a deficit perspective, but to refer to the risk that discontinuities in school-based mathematical practices may reduce opportunities for sustained engagement, especially when access to informal learning resources and adult mediation is unevenly distributed.

2.1. Learning loss and informal learning

The phenomenon of *learning loss* has been widely documented in educational research, particularly in relation to prolonged interruptions of formal schooling, such as summer breaks or the COVID-19 pandemic. Studies consistently show that the absence of structured learning opportunities leads to a decline in students' academic achievement, especially in mathematics [4, 22]. Moreover, learning loss is not evenly distributed, as it tends to exacerbate existing

²MaTEs is the acronym of the Italian expression *Matematica per Tutti in Estate*, which can be translated as "Mathematics for Everyone in Summer".

inequalities due to differences in access to resources and support outside school [31].

Recent reviews have also pointed out that the notion of summer learning loss is not always theoretically stable and is described in the literature through different labels, such as summer slide, summer setback, or summer learning gap. For this reason, it is important to treat summer learning loss not as an automatic and uniform outcome of school interruption, but as a phenomenon connected to differences in opportunities, contexts, and forms of support available to students during the summer period [32]. From this perspective, the issue is not simply whether students “lose” knowledge, but how opportunities to participate in meaningful mathematical practices are maintained, transformed, or reduced across school and out-of-school contexts.

In this context, *informal learning environments*, such as home-based activities, play a crucial role. However, their effectiveness depends on the nature of the learning experiences they offer. Research suggests that simply extending school-like tasks into the home is insufficient; rather, informal learning should be structured in ways that promote active engagement, autonomy, and meaning-making [19]. This calls for the design of learning environments that can bridge formal and informal contexts, supporting continuity in students’ learning trajectories while taking into account the specific characteristics of out-of-school settings. In such contexts, digital tools may therefore act not as autonomous learning environments, but as mediating devices that organize access to tasks, support adult guidance, and structure opportunities for children’s participation in meaningful activities.

In this perspective, families – and particularly parents – may play a crucial role in shaping learning opportunities during school interruptions. Research on parental engagement highlights that home routines, homework support, and parents’ mediation of learning activities significantly influence children’s academic continuity and engagement [25]. In the context of summer learning loss, differences in family support and access to learning opportunities contribute to the unequal distribution of achievement trajectories across students [2].

During summer breaks, learning opportunities become increasingly dependent on family practices, parental mediation, and access to informal educational resources. Research shows that families differ significantly in the extent to which they maintain continuity with school-related activities through structured routines, shared learning practices, and autonomy - supportive forms of engagement [29, 27]. These differences are not neutral, as they influence both the quantity and the quality of students’ engagement with learning activities, contributing to unequal educational opportunities outside school [8, 24]. In this perspective, summer learning differences should not be understood only as the consequence of reduced formal instruction, but also as the result of broader social and relational conditions shaping children’s access to meaningful learning experiences [2, 41].

2.2. Storytelling, mathematical modelling and representation

Research in mathematics education frequently highlights a widespread disaffection toward the subject, often associated with negative attitudes rooted in the perception of mathematics as a set of rules to be memorized and mechanically applied. This procedural view limits students’ engagement and reduces their ability to construct meaning from mathematical ideas. In response, storytelling has been proposed as a pedagogical approach capable of supporting meaning-making in mathematics learning. This perspective can be further enriched by the idea of mathematics curriculum as story by Dietiker [17], who argues that mathematical content can be interpreted as a narrative sequence in which mathematical objects, actions, settings, and plots unfold over time. From this point of view, storytelling in mathematics is not limited to adding an external narrative context to a task, but concerns the way mathematical ideas are progressively introduced, transformed, and connected within a meaningful sequence.

Storytelling is effective in education because narratives provide powerful structures for or-

ganizing information and interpreting human experience [42, 43]. Stories typically involve characters who face conflicts or challenges and attempt to resolve them. Unlike expository or scientific texts, stories engage both emotions and imagination. Narratives may orient the listener's feelings toward what is being told, making emotional involvement a central component of understanding. Consequently, storytelling fosters both cognitive and emotional engagement with mathematical content [44].

In mathematics education, stories can serve multiple functions: they may contextualize activities, introduce tasks, explain abstract concepts, or structure questions. In primary school, mathematical modelling is often introduced through word problems intended as simplified narratives. However, these problems are frequently stripped of narrative richness, encouraging superficial strategies such as keyword identification rather than genuine comprehension [43]. This reinforces the perception of mathematics as a meaningless procedure. To address this issue, Zan [42] emphasizes embedding mathematical questions within meaningful narrative contexts, where problems arise naturally from the story rather than being externally imposed, making tasks more authentic and closer to real-life situations.

The theoretical foundation for this approach lies in Bruner's distinction between narrative and paradigmatic thinking [11]. Narrative thinking focuses on human intentions, actions, and emotions, while paradigmatic thinking emphasizes logical structure and formal proof. Although distinct, these modes are complementary rather than hierarchical. In mathematical story-problems, mathematical reasoning should not be understood as exclusively paradigmatic, since it may also unfold through narrative structures as pupils interpret actions, goals, constraints, and consequences. Thus, narrative is not merely an external context, but can shape how mathematical relationships are noticed, connected, and justified [17].

Mathematical modelling involves meaning making through the connection between reality and mathematics. It is defined as the process of translating real-world situations into mathematical representations, working within mathematics, and interpreting and validating results in context [9]. The modelling cycle includes iterative steps such as understanding, simplifying/structuring, mathematizing, working mathematically, interpreting, validating, and presenting, though in practice it is rarely linear [10].

Bruner's distinction should not be directly mapped onto the distinction between reality and mathematics in modelling: the former concerns modes of sense-making, whereas the latter concerns the domains involved in the modelling process. In story-based modelling tasks, however, the two are connected, as narrative thinking supports pupils' understanding of the situation, while modelling guides the transition toward mathematical representations and their interpretation in context.

A key step is understanding the situation, which is essential for successful modelling [13]. However, students may struggle to move between narrative understanding and formal mathematical reasoning, either remaining at a descriptive level or focusing only on manipulation without interpretation [42]. Therefore, graphical representation plays a crucial role in understanding the situation and supporting mathematical modelling and meaning making. Drawing is increasingly understood not only as a representational tool but also as a form of mathematical thinking [37]. Throughout the modelling cycle, drawings evolve from situational representations that help students understand and structure a problem, to more abstract mathematical drawings that support reasoning and problem solving [36, 6]. They also aid in interpreting and validating results by connecting mathematical outcomes to real-world contexts. Research shows that drawing skills and accuracy are positively related to modelling performance, highlighting their importance as a bridge between reality and mathematics [36].

Overall, storytelling and modelling are complementary approaches for fostering meaningful learning in mathematics education. Storytelling provides an emotionally engaging entry point into mathematical ideas, while modelling offers a structured bridge between real-world con-

texts and formal mathematics. Their integration supports mathematical activity through the interaction of narrative sense-making, representation, modelling, imagination, experience, and logical reasoning.

2.3. Affect and engagement

Given the focus of the present study, we do not address the affective domain in its full complexity. Rather, we focus on three dimensions that are directly operationalized in the MaTEs web-app: task enjoyment, perceived performance, and perceived difficulty.

Research in mathematics education increasingly recognizes that learning and teaching mathematics involve not only cognitive processes but also a broad affective dimension, including emotions, motivation, attitudes, beliefs, values, and volition. These elements are deeply intertwined with thinking processes and significantly influence how students engage with mathematics. As Radford [35] argues, mathematics cannot be separated from affect, since thinking itself is always accompanied by emotional experience. From this perspective, affect is not an external component of learning but an intrinsic dimension of mathematical activity.

Given the complexity of the affective domain, research often focuses on specific constructs. In the MaTEs project, particular attention is given to attitudes toward mathematics. According to Deci and Ryan [15], positive attitudes are associated with greater autonomy, engagement, and improved learning outcomes. Di Martino and Zan [16] define attitudes as composed of three interconnected dimensions: interest in mathematics, perceived competence, and emotional disposition. Interest refers to enjoyment in mathematical activity.

Perceived competence is similarly central: drawing on Merleau-Ponty's notion of the "I can," Di Martino and Zan [16] describe it as a felt sense of capability rather than a purely cognitive judgment. Although this phenomenological perspective differs from Bandura's [5] psychological construct of self-efficacy, both approaches highlight the importance of how learners perceive their own capacity to act. Confidence, therefore, plays a crucial role in mathematical engagement. Emotional disposition arises from the interaction between cognitive appraisal and physiological responses [12, 33], highlighting the close relationship between cognition and emotion. In this study, task enjoyment is considered as an indicator of pupils' emotional disposition toward the activity, while perceived performance is related to perceived competence and self-efficacy.

During problem solving, students encounter difficulties influenced by both internal factors (knowledge, skills, beliefs) and external factors (task structure and wording). These factors influence perceived difficulty, which is not an objective property of a task but a subjective experience [40]. Often related to metacognition, perceived difficulty can be understood as a monitoring process of ongoing cognitive activity [21], influencing self-regulation, emotions, and strategy use [18]. Perceived difficulty is particularly relevant here because it captures pupils' subjective evaluation of the task, rather than an objective property of the activity. As such, it can influence persistence, self-regulation, and willingness to continue engaging with mathematical tasks.

Alongside attitudes and metacognition, values also play a central role in mathematics learning. Mathematics is culturally and socially situated rather than value-neutral [7, 38]. Different frameworks distinguish epistemological, social, cultural, and personal values [28], as well as mathematical, educational, and general educational values [7]. Values are deeply held beliefs about what is important, guiding actions and decisions [14] and they differ from beliefs, which concern what is considered true [39].

In line with this perspective, Albano et al. [1], focusing on Grade-2 pupils' interaction with one story-problem, showed that students who reported more positive attitudes toward the activity also tended to provide more elaborated answers, confirming the role of affective dimensions in mathematical problem-solving processes.

2.4. Integrating the perspectives

Taken together, these three strands suggest that addressing learning loss requires more than providing additional practice. It calls for the design of learning environments that are meaningful, engaging, and supportive of both cognitive and affective processes. In this perspective, informal learning contexts can become valuable spaces for mathematical activity if they are carefully structured to include: (i) meaningful and contextualized tasks, (ii) opportunities for explanation and reflection, and (iii) forms of mediation that support engagement without reducing autonomy.

Accordingly, learning is understood here not merely as the accumulation or retention of measurable knowledge, but as sustained participation in meaningful mathematical activity, involving reasoning, representation, argumentation, affective engagement, and adult mediation. This perspective shifts the focus from a deficit view of summer interruption to the conditions that support continuity of mathematical practices beyond the classroom.

The integration of storytelling and modelling offers a promising approach in this direction, as it combines narrative engagement with mathematical reasoning. At the same time, attention to affect and engagement ensures that students are not only able, but also willing, to participate in the learning process. This integrated framework provides the theoretical foundation for the design and analysis of learning environments, such as the MaTEs project, which aim to support meaningful mathematical activity beyond the classroom.

3. The design of the MaTEs web-app

3.1. The rational underpinning the design

The MaTEs web-app is a digital tool designed to support mathematics activities during the summer period, such as informal context, with the aim of keeping students' skills active through meaningful and contextualized problems [44, 42]. The web-app's activities are not intended as review exercises, but as tools to keep mathematical thinking active over the summer, fostering meaningful and lasting learning. In relation to the phenomenon of learning loss, this type of activity seems to be particularly relevant, as the loss of learning mainly affects reasoning, problem solving, and argumentative skills, rather than basic knowledge. The activities proposed by the web-app are therefore designed with attention to the three fundamental dimensions of learning: cognitive, metacognitive, and affective [35, 16]. The three dimensions are operationalized through the design of the tasks. The cognitive dimension is addressed by non-routine story-problems involving understanding, representation, modelling, and argumentation; the metacognitive dimension by prompts asking pupils to explain strategies and reflect on their work; and the affective dimension by narrative contexts, symbolic feedback, and self-reported evaluations of enjoyment, perceived performance, and perceived difficulty. These principles are further detailed in Sections 3.2 and 3.3.

Alongside problem solving, the web-app promotes explicit moments of reflection in which students are invited to explain the strategies they adopted, review their work, and develop awareness of their own learning process. Therefore, the MaTEs web-app is built to support pupils' engagement with non-routine tasks, encourage argumentation, and make visible the interplay between mathematical reasoning, affective dimensions, and social values [1, 3]. Thus, the design of the MaTEs web-app is not merely a technical organization of tasks, but an operational translation of the conceptual framework: informal learning is addressed through adult-mediated home activities; storytelling supports sense-making and engagement; modelling and representations structure pupils' mathematical work; and affective and metacognitive prompts make pupils' subjective experience and reflection visible.

It should be noted that the web-app is not intended as an automated learning environment, but rather as a mediating device between the child, the adult, and the mathematical activity

according to the informal learning approach (see Section 2.1). The use of the web-app is not the responsibility of the child, but it is an experience mediated by a reference adult (a parent or another adult). In fact, the web-app is a tool in the hands of the adult, used both to present the problem-story and the individual activities (through text, audio, and images) and to collect the students' work. Moreover, as a distinctive feature of the web-app, the adult can and is required to upload the child's work, typically in the form of images (photos of drawings) and audio recordings, which document the thinking process and not just the final result. The role of the adult is crucial in this process, as it supports the child in interpreting the task, verbalizing their reasoning, and reflecting at the end, making it possible to use the web-app in a way that is not merely procedural, but deeply formative.

The MaTEs web-app can therefore be understood as a hybrid environment, in which digital and offline dimensions are integrated. Its functioning is based on some pivotal elements: centrality of narration, active role of the child, mediation by the adult, the valorisation of the process as well as the result and the inclusion of metacognitive reflection moments. Thus, the web-app is not a substitute for the teacher, but a device that makes it possible, even in the summer context, to offer a guided learning experience in which the child can continue to engage meaningfully with mathematics.

3.2. The structure of the web-app

The structure of the web-app reflects the design principles described above by organizing each activity as a guided but flexible sequence, in which digital prompts, adult mediation, offline mathematical work, and documentation are progressively integrated.

The web-app is organized into stories, each structured as a sequence of 6 or 7 tasks. Through the interface, the user can select a story and progressively access the related tasks. As shown in Figure 1, each task is presented with a clear structure: an initial screen with a title, image, and text, accompanied by the option to start audio playback that reads the instructions. Each task consists of a short narrative situation, a request (e.g., draw, solve a problem, explain own reasoning), suggestions for the reference adult, guiding questions to stimulate the child's reflection and a final phase of feedback and metacognition.

This structure creates a continuous alternation between digital moments (presentation, uploading, feedback) and concrete moments (paper-based work, dialogue, reflection). The web-app introduces and guides the activities, but meaning making mainly occurs in offline work. From the child's perspective, the interaction is simple and structured: they listen to or read a story, work on a task, explain what they did, and receive symbolic feedback (the sunflower grows). At the same time, the adult is involved as a facilitator, providing guidance that supports the child without directing them toward a single answer. The web-app offers a clear structure that allows the adult to know what to propose, how to accompany and support the child, and how to collect and document the work.

The interface is simple but structured: the presence of buttons (audio, photo, microphone), status messages (task completed), and visual feedback makes the experience accessible and guided, while still leaving room for the child's autonomy in the problem-solving process. The content is presented to the child through different modes: the adult may read the text, play the audio, or invite the child to read independently. This flexibility allows the activity to be adapted to the child's characteristics and the context. The task is not carried out directly within the app: after reading or listening, the device is set aside, and the work continues offline.

The central phase of the activity takes place outside the web-app. The child develops the response using paper and pencil, drawings, or oral explanations. The adult accompanies the process without replacing the child, supporting him/her in the comprehension of the text, the exploration of different strategies and the verbalization of his/her reasoning. The suggestions provided in the web-app are specifically addressed to the adult and serve to guide this medi-



Figure 1. Screenshots of the MaTEs web app interface showing (from left to right): login page, activity selection dashboard, structure of a story with sequential tasks, and an example of a task including instructions, suggestions, upload, and feedback features.

ation. Another distinctive element is the presence of a symbolic feedback system (the growth of the sunflower), which accompanies the pathway and contributes to building a narrative dimension also at a motivational level.

3.3. The story-problems: features and contents

The story-problems constitute the main design component through which the conceptual framework is enacted. Their features are directly connected to the three strands discussed above: informal learning through adult mediation, storytelling and modelling through narrative mathematical tasks, and affective engagement through meaningful contexts, openness, argumentation, and reflection.

The story-problems in the web-app share several common design features, based on the conceptual background. They allow for freedom of exploration: students can follow different paths and arrive at different solutions, moving beyond the idea of a single correct answer. This openness is closely connected to the request for argumentation: they explicitly require argumentation, shifting the focus from the final result to the reasoning process and the justification of choices made. In this process, a key element is the use of representations, particularly drawing (see Section 2.2), which is introduced from the earliest stages of the activity as a way to understand and reinterpret the story. The activities are also designed to encourage the emergence of multiple possible mathematical models, challenging the idea of mathematics as unique and absolute. Moreover, the story-problems integrate the mathematical dimension with social and ethical aspects, as the characters' decisions and proposed solutions involve values such as fairness, justice, and cooperation.

Each story is structured as a pedagogical sequence involving alternating phases of web-app mediation, adult support, and pupils' offline mathematical activity. The web-app mainly supports the presentation, guidance, and documentation of the activity, while the core mathematical work is carried out by the pupil, typically mediated by the adult. The sequence unfolds through the following phases; in parenthesis, we indicate who is directly involved:

- (i) *Narrative introduction*: (pupil supported by the adult within the app) presentation of

characters and context through text and audio, allowing the child to build an initial understanding of the situation. This phase contributes most to the understanding of the real situation in the modelling cycle.

- (ii) *Initial representation (drawing)*: (pupil supported by the adult) the child is invited to draw the story, activating a personal representation of the situation. This phase contributes most to the structuring and presenting steps in the modelling cycle.
- (iii) *Exploration*: (pupil supported by the adult) initial questions guide observation, comparison, and hypothesis formulation. This phase supports mostly the transition from situational understanding to early mathematization steps.
- (iv) *Structuring and modelling*: (pupil) activities require identifying relationships between quantities and developing solution strategies, corresponding mainly to the mathematizing steps.
- (v) *Openness to multiple solutions*: (pupil) some tasks allow for more than one possible answer, promoting flexibility and comparison of strategies, contributing mainly to the mathematizing and working mathematically steps.
- (vi) *Argumentation*: (pupil supported by the adult with the app) the child is asked to explain and justify their reasoning, often through audio recordings. This phase contributes most to the interpreting, and working mathematically steps.
- (vii) *Re-elaboration*: (pupil) the introduction of new information or constraints requires revising and refining strategies, contributing mainly to interpreting, validating, and presenting steps.
- (viii) *Documentation and metacognitive reflection*: (pupil supported by the adult within the app) each activity ends with the upload of the produced work (drawing and audio) and a reflection on enjoyment, perceived success, and difficulty. This phase contributes most to the interpreting presenting steps.

This structure operationalizes the mathematical modelling process [9]. Although there is not a rigid one-to-one correspondence between phases and modelling steps, the activities are designed to support iterative and intertwined transitions from reality to mathematics and back again through the steps of the modelling cycle (see section 2.2).

This design choice shows how narrative mathematical tasks can open up multiple modelling paths and make explicit the values underlying different mathematical solutions, such as equity, fairness, and cooperation [3].

It is worthwhile to highlight that, taken together, the app's stories constitute a didactic tool aimed at developing deep mathematical competencies, particularly problem solving, argumentation, and modelling, through meaningful and contextualized activities. In relation to the phenomenon of learning loss, these activities are especially relevant, since learning loss in mathematics mainly concerns these competencies rather than basic knowledge.

In the following we briefly describe the content of the six stories presented in the app:

Story 1: Grandma Adele's Candies The story revolves around the preparation of a party, where children must manage bags of candies of different flavours. It introduces situations involving choice and the distribution of objects (candies), engaging the four operations with natural numbers up to 100.

Story 2: Naima and the Parking Lot The story is based on an everyday situation: a morning of errands in the city. Here, children work on spatial and organizational aspects related to the arrangement of objects in space.

Story 3: The Farm of Animals The story takes place on a farm and presents a classic problem reinterpreted in a narrative form: the heads and legs problem. The narrative introduces pre-algebraic and arithmetic elements.

Story 4: The Kingdom of Regiomonte This story introduces a fantasy narrative setting in which a prince must manage bridges between islands in order to recover pieces of the crown. It addresses geometric and topological content through playful, experiential contexts (paths and mazes) that are familiar to students from early childhood education.

Story 5: Chocolate Bars The story starts from a familiar situation (a birthday party) and progressively introduces more complex mathematical concepts. It introduces problems related to measurement and equivalence (e.g., comparing a 4×4 chocolate bar with a 2×8 one), gradually leading to the concept of fractions as a ratio between natural numbers.

Story 6: Beremiz, the Man Who Knew How to Count This story, adapted from a narrative tale, introduces a scenario of sharing and reward. It presents a situation involving the division of resources that allows for multiple mathematical solutions.

3.4. Implementation and usage of the app

Interaction with the web-app unfolds in several phases, alternating online and offline work. The adult accesses the platform, identifies the story to propose, and selects the task, then acts as a director, deciding timing and modes of presentation. Thus, the use of the web-app is explicitly designed as an experience mediated by the reference adult, who plays a central role in all phases of the activity.

By selecting a story, one accesses a screen that presents a recurring structure: a reference image representing the situation, the task instructions, an introductory text, a button to listen to the story audio and the list of tasks that make up the pathway, a section dedicated to suggestions for the reference adult. Each task appears as a selectable element; once completed, it changes status (e.g., from orange to green), making progress visible. Once the task work is completed, the adult helps the child upload the product (drawing) and an audio comment to the app.

The web-app requires both: photo and audio. Only after both are provided the “done” button becomes active. This constraint is designed to encourage verbalization and argumentation of the work carried out.

As the final step, the children are guided along a metacognitive reflection, where they are invited to express their own point of view about the task enjoyment, the perceived performance and the perceived difficulty, by answering three questions: *Did you like it? Do you think you did well? Did it seem difficult?* using a three-point scale (not at all / somewhat / very). In addition, an audio comment is required, guided by specific prompts (e.g., about the drawing or the strategy used).

Because of the structure of the web-app, the design of the activities, and the intended mode of interaction, the term user does not refer exclusively to either the adult or the pupil, but to the adult-pupil pair. Adults are users in that they interact with the web-app from a technical perspective and facilitate pupils’ access to the mathematical activities. Pupils, in turn, are users in that they engage with the mathematical content and actively construct meaning. At the same time, pupils also use the web-app to record their responses (e.g., through audio), while adults may or may not be directly involved in the mathematical work.

4. Research Question

The web-app, designed according to the conceptual background and the aim of the project, can provide fine-grained traces of students' activity, allowing the identification of different behaviour patterns over time.

These patterns are not interpreted as direct evidence of learning outcomes, nor as equating web-app use with learning. Rather, these patterns may reflect distinct trajectories of usage, which in turn can offer insights into the continuity of learning beyond the classroom. Thus, while the conceptual framework justifies the design of the learning environment, the present analysis focuses on its use, investigating whether log data reveal different ways in which users engaged with the proposed mathematical activities over the summer. Against this background, the present study addresses the following research question:

To what extent do different behaviour patterns of the web app during summer reflect distinct trajectories of usage, and what do they reveal about continuity of engagement with mathematical activity?

5. Methodology

5.1. Participants and data collection

The participants are pupils who used the web app during the summer 2024 between the end of grade 2 and the beginning of grade 3 (September 2024). They come from 44 school institutes in two different metropolitan cities (Milan and Naples). The web app recorded 616 families, but 315 of whom completed at least one task.

The activities were released progressively over the summer period according to a predefined schedule (see Figure 2), with sequential access requiring completion of prior activities. In particular, the six stories were made available at regular intervals between early July and early September 2024. A longer interval was introduced between Story 3 and Story 4, with the latter released on August 18, reflecting an intentional pause during the mid-August holiday period. The platform remained accessible until September 23, marking the end of the observation period, one week after the start of the school year.

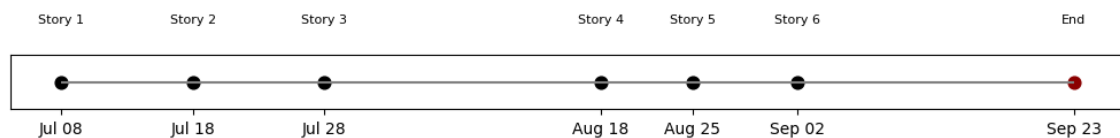


Figure 2. Timeline of the release of the six activities during the summer period. Each point represents the release date of a story, while the final marker indicates the end of the observation period.

We collect data from user logs. More specifically, for each user identified by an ID number, we obtain the following information: the story accessed, the task accessed, whether the user completed the task.

The dataset consists of anonymized log data extracted from the MaTEs web-app, where each record corresponds to a user-task interaction. For each entry, the dataset includes: a user identifier, timestamps of submission and last update, the story and task identifiers (with corresponding titles), and information on task completion: the images submitted (i.e., photos of drawings made by the child), and the audio recordings (see Section 3.3). Additional variables report the users' metacognitive reflection on the activity, collected through three evaluation items (see Section 3.4).

5.2. Methods of analysis

This study adopts a quantitative approach based on the cluster analysis of log data to identify patterns of user behaviour and statistical tests to compare different profiles of interaction with the web-app .

The preprocessing of the log data was carried out starting from the original Excel file containing users' interactions with the web-app . The dataset was first filtered by retaining only completed tasks and removing duplicate records at the user-task level, keeping the earliest completion timestamp. Timestamps were then discretized into six predefined time windows (DT, i.e., discrete time windows between two release dates), corresponding to the release schedule of the activities (see Figure 2). Based on this discretization, a user-by-time matrix was constructed, where each entry represents the number of tasks completed by a given user within each time window DT. From this representation, cumulative trajectories were computed by applying a cumulative sum across time windows, thus capturing the progressive evolution of users' activity.

In addition, a set of derived temporal variables was constructed to summarize different aspects of user behaviour. These included the total number of completed tasks (i.e., completion CT), the number of active time windows (AC), the index of the first and last active window (FW, LW), the maximum number of consecutive active windows (MC) and the proportion of activity concentrated in the final time windows (LT). These variables were selected to capture complementary dimensions of engagement that are particularly relevant in the context of informal learning during extended school breaks. Specifically, CT reflects the overall intensity of engagement, while AC provides an indication of its temporal distribution across the observation period. FW and LW allow identifying the timing of activation and disengagement, thus distinguishing early starters, late participants, or sustained users. MC captures continuity of engagement by measuring the extent to which activity is maintained across consecutive time windows. Finally, LT reflects the degree of temporal concentration of activity in the final phase, allowing the identification of "last-minute" usage patterns that resemble homework completion behaviours. Taken together, these variables provide a multidimensional and interpretable representation of users' temporal trajectories, enabling a more fine-grained comparison of engagement patterns across clusters.

To explore usage patterns of the web app in a data-driven way, a time series clustering analysis based on Time Series K-Means methodological approach [26] was conducted. For each user, a time series was constructed representing the cumulative number of completed tasks within each interval. This representation allows capturing users' trajectories of progression over time, focusing on the evolution of engagement rather than on isolated events.

Clustering was performed using the python TimeSeriesKMeans algorithm with Euclidean distance applied to the cumulative series, without prior standardization, in order to preserve information about absolute activity levels in addition to trajectory shapes. The number of clusters was determined using the Elbow method by analysing the trend of inertia as a function of k . In particular, the relative decrease in inertia between successive values of k was examined to identify a point of inflection beyond which increasing the number of clusters yields only marginal improvements in within-cluster variability. The final choice of k was therefore informed by both the behaviour of inertia and the interpretability of the resulting patterns.

To assess the statistical significance of the differences among clusters, a non-parametric approach was adopted. Given the non-normal distribution of the derived variables and the unequal group sizes, the Kruskal–Wallis test [30] was applied to compare the distributions of each variable across clusters. This test evaluates whether at least one group differs from the others in terms of median values. For each variable, the null hypothesis of equal distributions across clusters was tested independently. When the Kruskal–Wallis test indicated significant

differences, post-hoc pairwise comparisons were conducted using Dunn’s test with Bonferroni correction [20] to control for multiple comparisons. This procedure allowed identifying which specific pairs of clusters differed significantly for each variable of interest, providing a detailed characterization of the statistical separation between clusters. Violin plots were used to represent the variability of each cluster with respect to both the time windows and the chosen derived variables.

To investigate whether clusters differ in terms of metacognitive variables (see Section 3.4) data were extracted from the original dataset and aggregated at the user level. Since metacognitive variables were collected at the task level, average scores were computed for each user across all completed tasks, obtaining a single value per metacognitive variable and per user. More specifically, after each task, pupils answered three metacognitive feedback questions (“Did you like it?”, “Do you think you did well?”, “Did it seem difficult?”) using a three-point response scale. Numerical scores were assigned to these responses (1 = low evaluation, 2 = intermediate evaluation, 3 = high evaluation), and the reported scores therefore represent the average level of pupils’ self-reported feedback across the completed tasks. These variables, coded as a discrete scale from 1 to 3, were then merged with the cluster assignments obtained from the time series clustering procedure.

Given the ordinal nature of the data and the non-normal distribution of the scores, differences across clusters were assessed using the Kruskal-Wallis test. When appropriate, Dunn’s post-hoc test with Bonferroni correction was applied to perform pairwise comparisons between clusters. In addition bar plots were used to visually inspect the distributions of feedback scores across clusters.

6. Data Analysis

As first preliminary analysis, we focused on the temporal distribution of completed tasks (see Figure 3). It shows a highly uneven pattern over time, with a moderate level of activity in the initial phase, followed by a substantial increase and a pronounced peak toward the end of the observation period. The moving average highlights a clear upward trend culminating in a concentration of activity in the final weeks, followed by a sharp decline. This overall pattern suggests that use is not uniformly distributed but rather characterized by periods of low activity interspersed with phases of intense usage.

This aggregate representation raises important questions. Although it clearly shows a global tendency toward late concentration of activity, it does not allow us to distinguish whether this pattern is shared by all users or driven by specific subgroups. In other words, the observed peak may result from heterogeneous behaviors - such as consistently active users, late starters, or users concentrating their activity at the end - that are not visible at the aggregated level. This limitation highlights the need for a more fine-grained analysis capable of identifying distinct usage trajectories, motivating the adoption of a clustering approach to profile users based on their temporal patterns of engagement.

This section unfolds as follows: first, we determine the number k of possible clusters corresponding to a specific profile using the Elbow method (Subsection 6.1), then the cluster trajectory and variability are reported (Subsection 6.2), concluding with statistical tests to characterize clusters (Subsections 6.3 and 6.4).

6.1. Determining the Number of Clusters

The inertia curve in Figure 4 shows a sharp decrease for lower values of k , followed by a more gradual reduction, with a visible change in slope around $k = 4$. The analysis of the inertia values is further supported by the examination of the relative reduction between consecutive values of k . The decrease in inertia remains high up to $k = 4$ (26.09% from $k = 2$ to $k = 3$ and 26.36% from $k = 3$ to $k = 4$), while a marked drop is observed from $k = 4$ to $k = 5$ (14.11%).

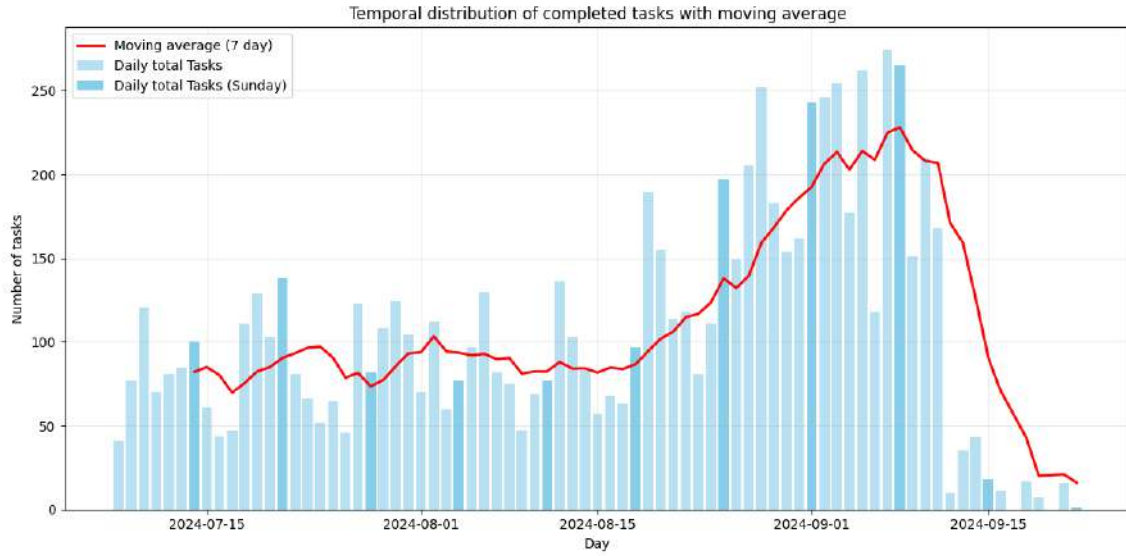


Figure 3. Daily number of completed tasks over time (bars) with a 7-day moving average (red line). The distribution highlights a non-uniform pattern of activity, with a marked increase and concentration toward the final phase of the observation period.

For higher values of k , the relative improvements progressively decrease, stabilizing around lower values. This pattern indicates that most of the structural variability in the data is captured by $k = 4$, while additional clusters account for increasingly marginal refinements. On this basis, the range $k = 4 - 6$ was selected for further investigation.

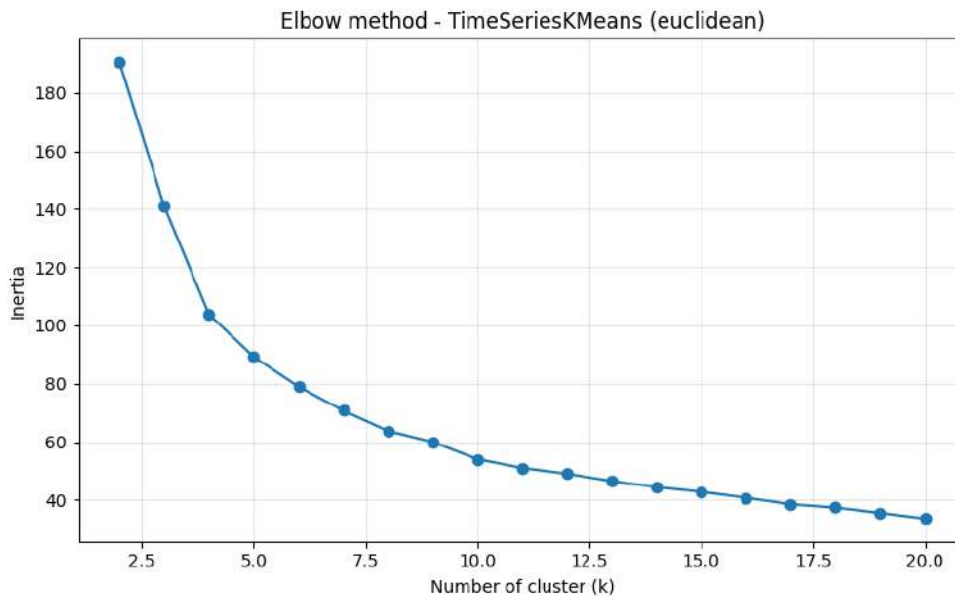


Figure 4. Inertia as a function of the number of clusters (k).

To that end, we employed cluster analysis with the three different values of k . While all configurations showed statistically significant differences across clusters with respect to the behavioural variables, as confirmed by the Kruskal-Wallis tests, the post-hoc comparisons revealed important differences in terms of separability. In particular, increasing the number of clusters led to a reduction in statistical distinctiveness, with several cluster pairs becoming indistinguishable across multiple variables for $k \geq 5$. This indicates that higher values of k introduce partially overlapping or redundant groups.

This interpretation is further supported by the proportion of non-significant pairwise comparisons observed in the Dunn post-hoc tests (Table 1), that is, comparisons between all possible pairs of clusters for each derived variable. The proportion increases from 22.2% for $k = 4$ to 30.0% for $k = 5$ and 34.4% for $k = 6$. This trend indicates a progressive reduction in cluster separability as k increases.

k	Non-significant pairs	Total pairs	Percentage
4	8	36	22.2%
5	18	60	30.0%
6	31	90	34.4%

Table 1. Percentage of non-significant pairwise comparisons (Dunn test) across clustering solutions, showing reduced separability as k increases.

Moreover, increasing k does not provide additional explanatory power with respect to metacognitive variables, which remain statistically homogeneous across clusters for all configurations. Taken together, these elements support the selection of $k = 4$ as the most appropriate solution, as it achieves a balance between statistical separability, interpretability, and parsimony.

6.2. Cluster Trajectories and Variability

The centroids obtained from the Time Series K-Means clustering with $k = 4$ show four distinct trajectories in terms of cumulative task completion over the considered time windows (DT1–DT6). Each centroid represents the average profile of the users assigned to the corresponding cluster. The size of the clusters are reported in Table 2.

Cluster	0	1	2	3
n. users	41	86	58	130
% of users	13.0%	27.3%	18.4%	41.3%

Table 2. Distribution of users across the four clusters.

All clusters exhibit a monotonic increase, as expected from the cumulative nature of the data, but differ in both slope and curvature (see Figure 5). Cluster 0 presents a different pattern, with low values in the initial intervals and a sharp increase in the final time windows, resulting in a pronounced change in slope between DT4 and DT6 time window. Cluster 1 is characterized by the lowest values throughout the entire time span, with a gradual and limited increase. Cluster 2 also shows a strong increase, particularly in the earlier intervals, followed by a more moderate growth in later stages. Cluster 3 displays the highest values across all time windows, with a relatively steep and a consistent growth.

Overall, the centroids highlight differences in magnitude, rate of increase, and temporal distribution of cumulative activity, indicating that the clustering procedure successfully identifies distinct trajectories within the dataset.

Figure 6 reports the violin plots which represent the distribution of cumulative task completion across time windows (DT1–DT6) for each cluster, together with the corresponding centroid trajectory. For all clusters, the distributions shift progressively toward higher values across time, reflecting the cumulative nature of the variable.

Cluster 0 exhibits a markedly different pattern, with very low values and narrow distributions in the initial windows, followed by a substantial increase in both central tendency and dispersion in the final windows (DT5–DT6). Cluster 1 is characterized by low cumulative values across all time windows, with distributions concentrated near the lower range and moderate spread. Cluster 2 also presents relatively high values, with wider distributions especially

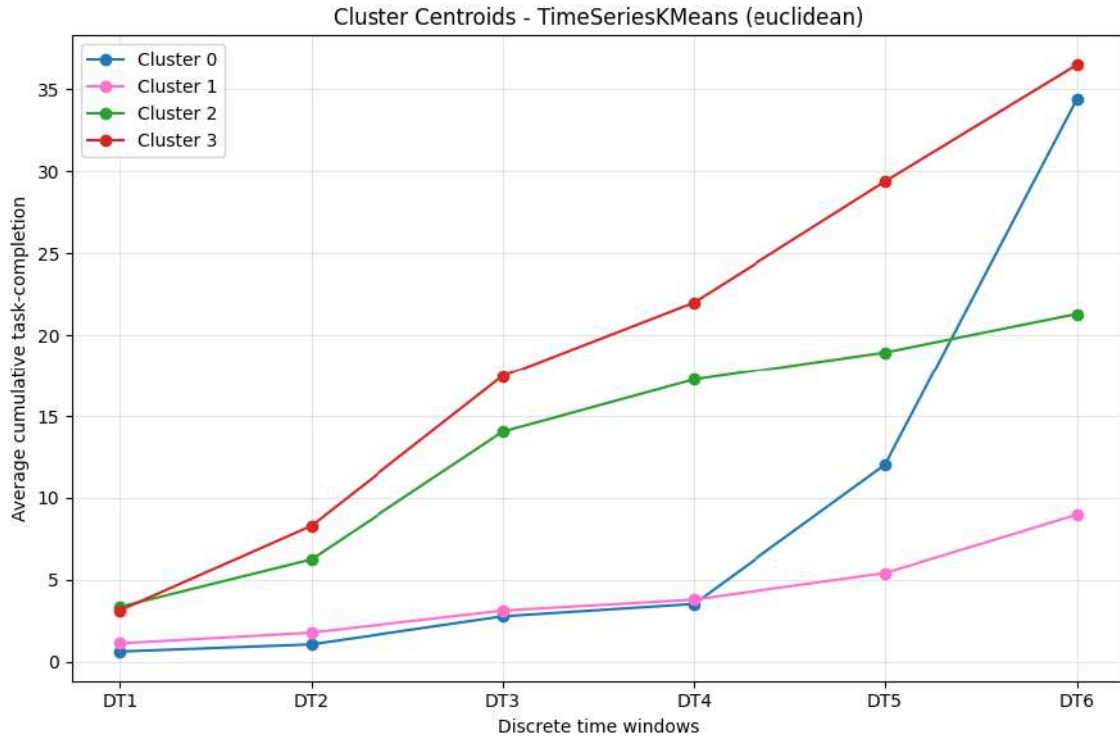


Figure 5. Centroids of the four clusters obtained with Time Series K-Means (Euclidean distance) based on cumulative task completion. The vertical axis represents the average of the cumulative number of completed tasks of each cluster, while the horizontal axis represents the discrete time windows (DT1–DT6). Each line represents the average trajectory of a cluster, obtained by averaging their cumulative task-completion time series.

in the intermediate windows (DT2–DT4), suggesting greater variability in the trajectories. Cluster 3 shows consistently higher central tendencies across all time windows, with relatively concentrated distributions around the centroid, indicating lower dispersion. Across clusters, differences emerge not only in the magnitude of cumulative values but also in the spread of the distributions, with some clusters displaying more compact shapes and others showing broader variability, particularly in later time windows.

6.3. Statistical Validation of Cluster Differences

The statistical validation of the clusters was carried out using the Kruskal-Wallis test on a set of derived variables capturing different dimensions of users' engagement (overall completion, temporal distribution, continuity, and concentration of activity). The results show that all variables (CT, AC, FW, LW, MC, LT) defined in Section 5.2 significantly differ across clusters with very low p -value ($p \ll 0.001$) indicating that the identified clusters correspond to statistically distinct patterns of usage.

The distribution of temporal variables across clusters is shown in Figure 7, using violin plots to represent both central tendency and variability within each group. Each subplot corresponds to a different variable, allowing comparison of how clusters differ across multiple dimensions of activity. The distributions reveal clear differences across clusters for several variables. In particular, completion (CT) shows strong separation, with clusters characterized by distinct ranges of cumulative activity. Similarly, AW (active_windows) and MC (max_consecutive) highlight differences in the continuity and spread of activity over time, with some clusters exhibiting more concentrated and others more distributed patterns. The variables FW (first_active) and LW (last_active) capture differences in the timing of engagement, with clusters showing distinct

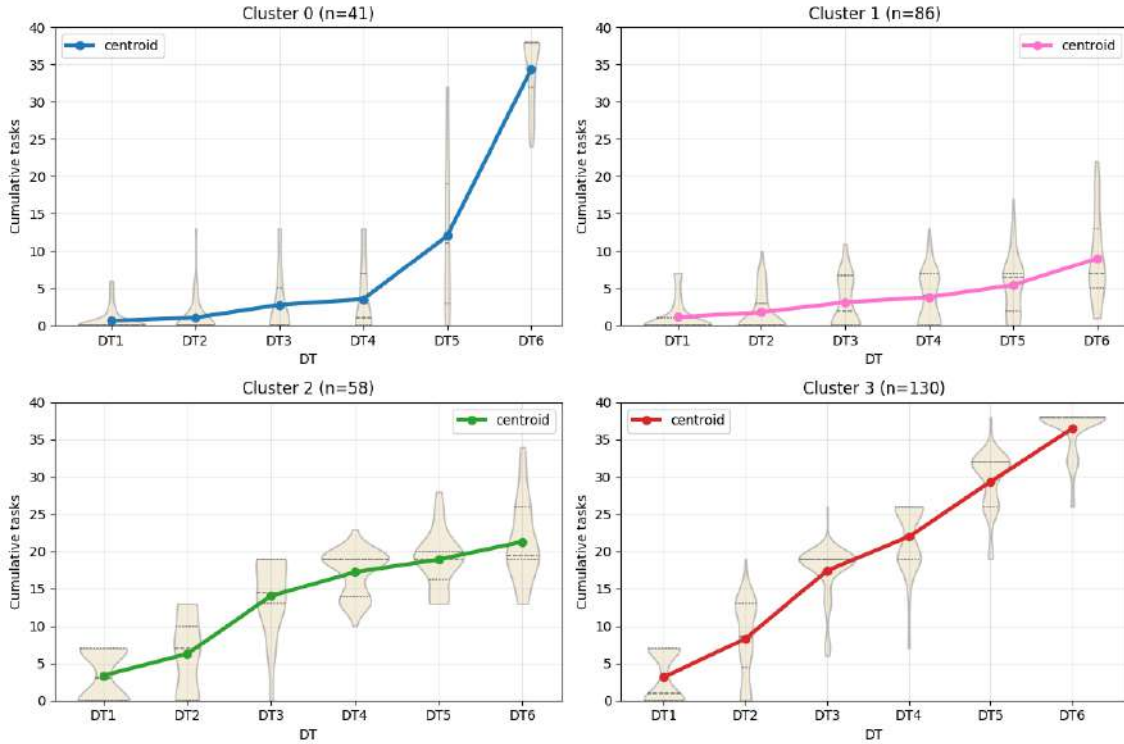


Figure 6. Distribution of cumulative tasks across time windows for each cluster ($k = 4$). Violin plots represent the variability of users’ trajectories within each cluster, while the overlaid line indicates the cluster centroid.

positions along the temporal axis. Finally, *LT* (*share_last2*) exhibits a highly skewed distribution in some clusters, indicating strong concentration of activity in the final time windows for specific groups. Overall, the violin plots confirm that clusters differ not only in terms of overall magnitude of activity but also in timing and continuity, with varying degrees of dispersion across clusters depending on the variable considered.

Post-hoc comparisons using Dunn’s test with Bonferroni correction confirm the structure of these differences. Most pairwise comparisons are statistically significant, especially for *CT* and *LT*, suggesting a strong differentiation between clusters in terms of overall engagement and temporal concentration of activity. At the same time, some non-significant comparisons emerge for specific variables (e.g., *CT* between clusters 0 and 3, or *FW* between clusters 0 and 1), indicating partial overlaps between certain groups. Table 3 summarizes the pairwise differences between clusters, reporting the variables for which statistically significant differences are observed. Overall, these results support the robustness of the clustering solution, while also suggesting the presence of both clearly distinct profiles and more nuanced intermediate behaviours.

	C0	C1	C2
C1	CT, AC, LW, MC, LT	–	–
C2	CT, FW, LW, LT	CT, AC, FW, MC, LT	–
C3	FW, MC, LT	CT, AC, FW, LW, MC	CT, AC, LW, MC, LT

Table 3. Pairwise comparison of clusters (C0, C1, C2, C3) based on Dunn post-hoc tests (Bonferroni corrected). Each cell reports the variables for which the two clusters differ significantly ($p \ll .001$). Abbreviations: completion (CT), active windows (AC), first active window (FW), last active window (LW), maximum consecutive activity (MC).

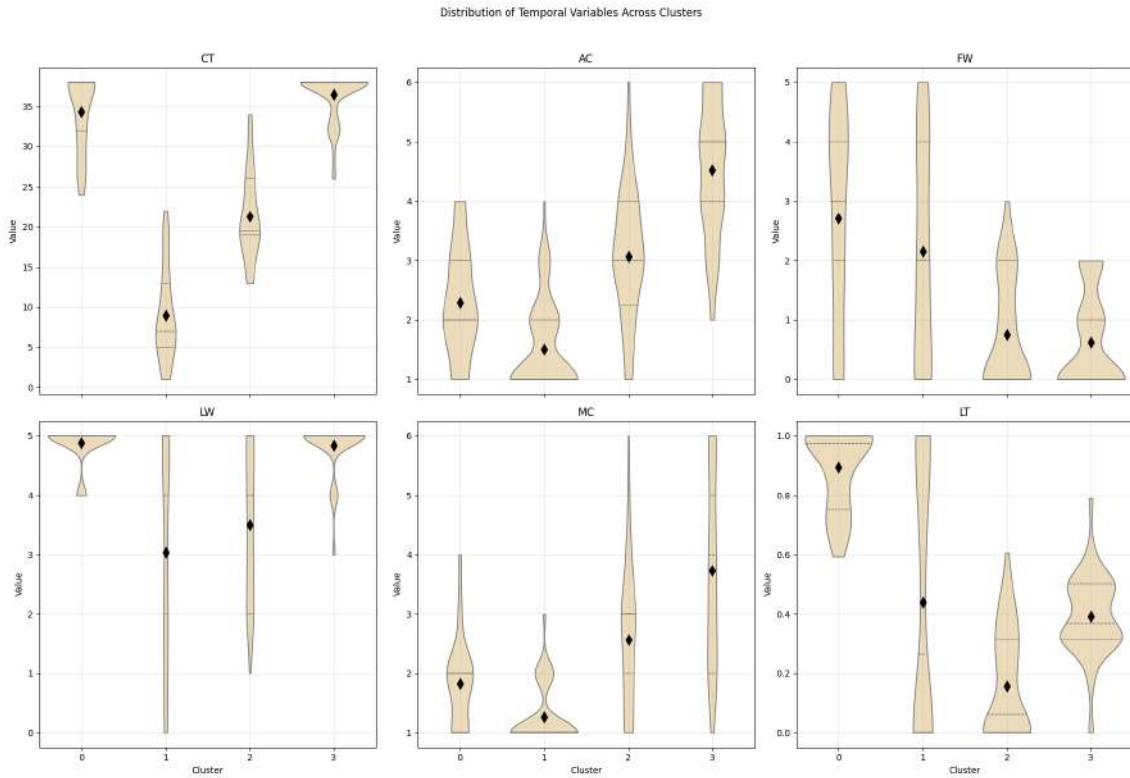


Figure 7. Distribution of temporal variables across clusters. Violin plots show the variability within each cluster for completion (CT), active windows (AC), first active window (FW), last active window (LW), maximum consecutive activity (MC), and share of activity in the final time windows (LT). Differences across clusters are observed in magnitude, timing, and continuity of activity.

6.4. Metacognition Distribution Across Clusters

Figure 8 shows the normalized distribution of metacognitive reflection scores across clusters for the three variables. For each cluster, the proportions of scores (1, 2, and 3) are displayed, allowing comparison independent of cluster size.

Across all three metacognitive variables, the distributions are highly similar between clusters. In each case, score 3 represents the largest proportion of responses, followed by score 2, while score 1 appears only marginally. The shapes of the distributions are largely overlapping across clusters, with only minor variations in the relative proportions of scores.

For task enjoyment, all clusters show a predominance of score 3, with moderate presence of score 2 and very limited occurrence of score 1. For perceived performance, the concentration on score 3 is even stronger, particularly for Cluster 2 and Cluster 3, while score 2 remains secondary and score 1 negligible.

For perceived difficulty, the pattern remains consistent, with score 3 dominating across all clusters and only slight differences in the proportion of score 2.

However, the statistical analysis of metacognitive variables did not reveal significant differences across clusters. The Kruskal-Wallis test showed non-significant results for all three feedback variables (p -value $> .05$), indicating that the distributions of user metacognitive reflections are comparable across clusters. Consistently, Dunn's post-hoc tests did not identify any significant pairwise differences between clusters. These findings suggest that, despite the clear differentiation in temporal usage patterns identified through clustering, users' metacognitive reflections of the activities remain relatively homogeneous. In other words, distinct engagement trajectories do not correspond to significantly different perceived experiences, highlighting a decoupling between patterns of use and subjective experience.

Overall, the normalized distributions confirm a high degree of similarity across clusters and a general skewness toward higher feedback values.

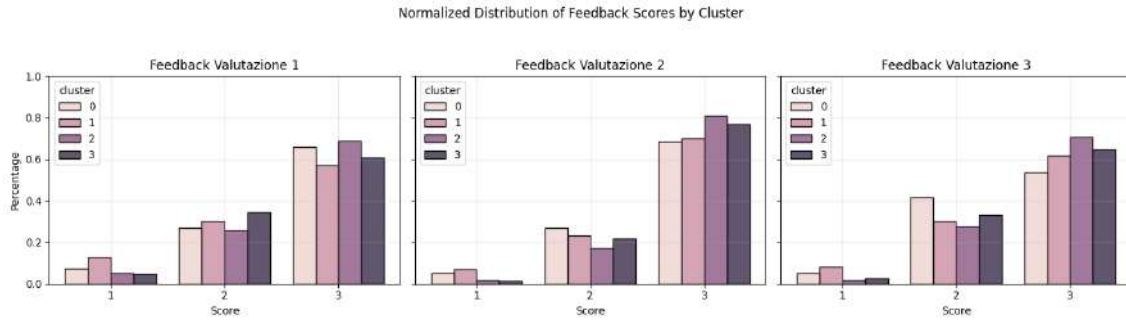


Figure 8. Normalized distribution of feedback scores across clusters. Bars represent the proportion of responses within each cluster, allowing comparison independent of cluster size.

7. Discussion and conclusion

In the previous sections we report the clustering analysis which allows us to identify four distinct usage profiles characterized by different levels of engagement, temporal distribution, and continuity of activity. This section aims at addressing the research question: “To what extent do different behaviour patterns of the web app during summer reflect distinct trajectories of usage, and what do they reveal about continuity of engagement with mathematical activity?”. Accordingly, the profiles are interpreted not as direct indicators of learning outcomes, but as traces of how adult-pupil pairs organized participation in mathematical activities over the summer period. To that end, we identify profiles characterized by specific behaviour patterns and interpret them from the didactical perspective, as reported in Table 4.

Profiles [Cluster] (numerosity)	Description	Didactical interpretation
Late sprinter [0] (41)	High completion achieved through strong concentration of activity in the final time windows	Usage concentrated at the end of the summer break (homework-like pattern)
Low engagement [1] (86)	Low completion, limited activity across time windows, low continuity	Limited engagement with the activities
Early fader [2] (58)	Good initial engagement followed by a reduction of activity in later phases	Early activation not sustained over time
Ideal [3] (130)	High completion, activity distributed across time windows, high continuity	Regular and sustained use

Table 4. Characterization of the four clusters identified through time series clustering ($k = 4$), based on completion, temporal distribution, and continuity of activity.

The profiles arisen from the data analysis provide some insights and sometimes allow to substantiate some common behaviour. It is worth noting that the largest cluster is the Ideal profile, including 130 users (about 40% of the sample). At the same time, about 60% of users fall into other profiles, indicating that regular and sustained engagement was the most represented single trajectory, but not the dominant behaviour overall.

Looking at Late Sprinter, we can note that it is characterized by low initial activity followed by a sharp increase in the final phase. This profile is consistent with the well-known tendency to concentrate on school-related tasks at the end of the summer period (e.g., “homework completed at the last minute”). This allows us to argue this behaviour is not anecdotal but clearly identifiable as a distinct trajectory.

Given that the activities in the web-app are designed to foster reasoning, problem solving, and reflection rather than procedural rehearsal, comparing Late Sprinter profile with the Ideal one shows that both exhibit a high level of task completion; however, the former works intensively toward the end, while the latter works continuously. It is plausible that effectiveness of the MaTEs activities depends on continuity rather than intensity alone. However, this hypothesis cannot be directly tested within the current dataset and should be addressed in future studies.

The profile Early Fader shows initial participation followed by a progressive decline. This suggests that initial activation does not necessarily translate into sustained engagement over time, although the data do not allow identifying the causes of this decline, requiring further investigations.

The profile Low Engagement characterized by very low and sporadic activity indicates that a portion of users engages only marginally with the app. In the context of the MaTEs project, the web app is designed as a mediated environment involving both children and adults. Therefore, low engagement may reflect not only individual factors but also contextual and relational conditions of use. However, unfortunately the current data do not allow disentangling these aspects and should be considered in future research.

A particularly relevant result concerns the relationship between behavioural engagement and users’ metacognitive reflection. Despite the clear differentiation in usage patterns, no significant differences emerge across clusters in terms of metacognitive variables. Recalling that the user is the pair adult-pupil, this suggests that discontinuity or late concentration of use cannot be straightforwardly attributed to lower appreciation of the app or to a more negative perceived experience. In other words, different trajectories of participation correspond to similar perceived experiences.

Overall, this study shows that interaction with a digital learning environment during the summer is a complex and heterogeneous phenomenon. The identified profiles do not simply describe different levels of use but reflect distinct ways in which adult-pupil pairs access, organize and sustain mathematical activity over time. Rather than providing direct evidence about the reduction of learning loss or improvement in achievement, the findings highlight the role that the web app may play in fostering continuity of engagement with mathematical activities during the summer period. In particular, the web app appears to support the construction of routines of participation and shared adult-pupil activities around mathematics, even in informal contexts outside school. From this perspective, the relevance of the intervention lies not only in the proposed tasks themselves, but also in the possibility of sustaining continuity of mathematical practices beyond the classroom. The collected data leads us towards further and deepening investigations as mentioned above, raising further questions such as: To what extent does late and concentrated engagement support meaningful mathematical activity, compared to distributed engagement over time? What factors influence the sustainability of engagement in informal learning environments over extended periods? Addressing such questions required complementing this quantitative analysis with a qualitative analysis of the pupils’ protocols and adult mediation practices, in order to investigate whether discontinuity is related to family routines, contextual constraints, or other conditions of use.

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Learning probability at grade 2: exploring the role of drawings for children's understanding

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Abstract: One of the challenges of Mathematics Education in primary school is to introduce probabilistic thinking when the understanding and use of ratios is not yet well developed. Seminal studies in Mathematics Education about probabilistic thinking focus their attention on misconceptions. Misconceptions refer to intuitive thinking and are generally understood as flaws in logical thinking. Notwithstanding the importance of this form of thinking, our study draws on an alternative approach to probabilistic thinking, which shifts the attention from the mere realm of intuitions to the intertwined nature of intuitions, procedures and representations. More specifically, we resort to storytelling and drawing to involve children in probabilistic activities. Within the MaTEs Italian project, a task on probability is assigned to grade-2 students, who are invited to make a drawing and to write their answers. The drawings have been coded according to the degree of narrative and mathematical elements they represent, and the written justifications provide insights on children's reasoning. The analysis of children's drawings reveal that also children at this young age are able to identify and employ the mathematical features of a probabilistic task in order to answer it correctly, but the majority fails to provide a written justification for their (correct) choices.

2020 Mathematics Subject Classification: Primary 97C70; Secondary 97K50, 97C20.

Keywords: probabilistic thinking; primary school; drawings; storytelling; mathematical modelling; meaning-making.

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1. Introduction and literature review

Since the turn of the millennium, there has been a noticeable and accelerating global trend toward incorporating probability education into early school curricula, often before students have mastered the concept of ratios, which is traditionally considered fundamental to the classical and frequentist approach to probability (Borovcnik & Kapadia, 2014). This shift is primarily driven by the escalating importance of probabilistic reasoning across a multitude of fields, including economics, business, the natural and social sciences, and political decision-making. This curricular evolution has, in turn, arisen a vigorous academic debate and research agenda focused on the fundamental nature of probability and the most effective pedagogical strategies for its learning and teaching [5].

In response to these new challenges, numerous theoretical models have been developed to delineate the process of probabilistic thinking (for a comprehensive review, see [11]). These models typically analyze probabilistic reasoning by identifying, categorizing, and assessing the presence or absence of specific (partial) conceptions in students. A significant portion of this research has focused on explaining learners' difficulties by attributing them to the existence of deep-seated misconceptions or to missing or incomplete conceptual frameworks. It is crucial to note that the findings from these pioneering studies are often context-bound, focusing on discrete probabilistic concepts. For instance, Bar-Hillel and Wagenaar [3] meticulously investigated people's intuitive notions of randomness using both judgment tasks – such as evaluating the likelihood of a specific series of outcomes from a fair die – and production tasks, where participants were asked to generate a seemingly “random” sequence. This work revealed common intuitive biases, which Falk et al. [9] further characterized as a tendency toward “equal total frequencies” (the belief that all outcomes must appear equally often in a short run) and an “excess of alternation” (the tendency to switch between outcomes more frequently than is statistically likely, reflecting an over-compensation for perceived patterns).

Within this complex framework of probability learning, the prospect of engaging very young children with probability concepts, particularly in informal settings such as a summer break program, initially presents itself as a significant pedagogical obstacle – a seemingly “mission impossible.” However, alternative theoretical approaches offer a path forward. Andrà and Stanja [2] advocate for the adoption of a cultural semiotic perspective on probabilistic thinking. This perspective fundamentally reframes probability not merely as a set of mathematical rules, but as a rich, historical, and dynamic “cultural product.” This approach is deeply informed by the work of the philosopher and mathematician Brian Rotman, who proposes a foundational characterization of mathematics itself “as a practice, as an ongoing cultural endeavor” [13, p. 3]. By embracing this framework, Andrà and Stanja [2] emphasize the crucial mediating role played by artifacts and signs in the communication and construction of probabilistic understanding. Consequently, their research allows us to hypothesise that the deliberate use of visual representations, specifically children's own drawings, can serve as powerful cognitive tools that actively support children's problem-solving processes when they are confronted with probabilistic tasks.

This background establishes the context for the core inquiry of the present research, which seeks to contribute to the existing literature regarding modeling and drawings.

The research questions guiding this investigation are:

- (i) Which kinds of drawings emerge when children are engaged in tasks that necessitate probabilistic thinking?
- (ii) How do these various drawings function to support and mediate the meaning-making process for the children involved?

To address these research questions, our study employs a combined theoretical approach. It relies on a theoretical framework concerning the analysis and function of children's drawings,

and simultaneously anchors its understanding of the mathematical content – probability – in Rotman’s conception of mathematics as a cultural practice. These theoretical underpinnings, essential for the analysis and interpretation of the empirical data, will be further elaborated upon in the subsequent section of this paper.

2. Theoretical framework

Our study is substantially grounded in the conceptualizations of Rotman, whose work provides a crucial lens for examining the core subject matter. Our theoretical framework deliberately incorporates and builds upon Zan’s understanding of narrative thinking. This reliance on Zan’s perspective is particularly vital as it directly connects to the methodology of the study, specifically concerning the interpretation and utilization of drawings as a medium for expressing and analyzing thought processes. By integrating these two major theoretical streams, the research aims to establish a foundation for a nuanced and comprehensive analysis.

2.1. Rotman’s frame: ideas, symbols and procedures

Rotman [13] posits that the endeavor of ‘doing mathematics’ is fundamentally underpinned by the coordination of three distinct, yet interwoven, theoretical discourses: Idea, Symbol, and Procedure [13, p. 1676]. Mathematical thought, according to this perspective, necessitates the fluent coordination among these three components. In fact, Idea is the domain of intuition and human thought; Symbol is the domain of signs, communication and objectification; and Procedure is the domain of action, transformation and operation.

More specifically, the domain of Idea is synonymous with human thought, rooted in the individual’s conceptualization and articulation, often initially expressed through natural language narratives [13]. It constitutes the core intuitive understanding of mathematical concepts. Within the specific context of probability, Andrà and Stanja [2] interpret Idea as the realm of intuitive approaches to uncertain situations and fundamental concepts such as variability. This domain is where an individual’s pre-mathematical, often spontaneous, understanding takes shape. Furthermore, Andrà and Santi (2011) characterize the intuition of mathematical concepts as an experience where access to the object’s distinctive features is self-evident, coercive, and global. This immediacy and self-evidence are linked by Andrà and Santi (referencing Radford, 2008) spatial-temporal, sensorimotor and perceptive activity that semiotic means of objectification accomplish, support, foster” (Andrà & santi, 2011, p. 115). Consequently, the domain of Idea concerning probability is inherently situated within space and time, serving as the experiential ground from which intuitions emerge and are subsequently accounted for through narrative expression.

The journey from personal intuition to shared mathematical practice – the development of personal and collective knowledge – is intrinsically linked to the productive engagement with artifacts as instruments for knowledge-building. This transition leads directly to the domain of Symbol. Rotman [13] defines this as the domain of signs, encompassing all forms of communication and semiotic practices, ranging from specific notational devices (like variables or operators) to entire linguistic systems. The profound significance of symbols in the process of mathematics learning is underscored by Duval [8], who asserts that *noesis* without *semiosis* is impossible; that is, any process of meaning-making inherently requires a system of signs to materialize. In the realm of probability, Andrà and Stanja [2] define the Symbol domain as the comprehensive set of signs and symbols conventionally employed, including Venn diagrams, algebraic formulas, tables, histograms, and other visual or symbolic representations. Crucially, a sign is understood as an artifact that has been culturally imbued with a meaning, serving as a representation for a mathematical idea. For an artifact to transcend its mere physical existence and function as a mathematical sign [14], it must establish a meaningful relationship with a corresponding mathematical Idea. In elementary probability, this often involves the

critical translation from the physical domain of artifacts (situated in space and time, such as dice or spinners) to the abstract domain of numbers and symbols. The use of a simple artifact like a spinner can evoke two pivotal cultural conceptualizations of probability: from one side, the classical/Laplacian probability (i.e., the notion of probability as a ratio between favorable outcomes and the total number of possible outcomes); from the other side, the frequentist probability, namely the understanding that probability is assigned based on the relative frequency of an event over a series of repetitions, an understanding particularly salient with common, repeatable artifacts (e.g., dice, cards, coins, spinners), as highlighted by the condition that “probability can be assigned only to an event that can be repeated” [10, p. 2].

Procedure encompasses the domain of purposeful actions, processes, and operations performed on and with artifacts and signs [13]. This resonates strongly with the perspective of Duval [8], who identifies mathematical learning as the progressive capacity to execute correct, meaningful actions on mathematical signs. Mathematical actions are fundamentally transformations, which can manifest in two key ways: (i) treatment – a transformation that occurs *within* the same semiotic register, for example algebraically manipulating one formula into an equivalent but distinct form; (ii) conversion – a transformation that occurs *between* two different semiotic registers, for instance converting an algebraic formula into its graphical representation, or vice versa. While Procedures can, and ideally should, be mathematically explained or justified by the underlying Ideas, Rotman notes that they may sometimes be performed purely algorithmically, without explicit reference to or reflection upon those foundational Ideas.

2.2. Drawings and the power of narrative in Mathematics Education

The integration of storytelling methodology in the mathematics classroom, often referred to as storytelling in mathematics, provides a powerful means for students to construct meaning and develop a deeper connection with the subject matter. Stories, fundamentally, serve as a robust framework for organizing, transmitting, and interpreting information, allowing individuals to create personal and contextual meaning in their lives. A critical feature that distinguishes stories from other forms of narration, such as historical accounts or scientific reports, is their intrinsic capacity to direct and engage our feelings toward the content being shared. In essence, stories evoke an emotional response. This emotional engagement, coupled with the stimulation of students’ imagination concerning the curriculum content, is precisely where the significant educational value of stories lies [17]. In an attempt to connect these theoretical considerations with Rotman’s [13] model, we can say that storytelling in mathematics is related to the realm of Idea, i.e. the intuitive grounding of mathematical concepts.

The pedagogical application of storytelling in mathematics utilizes different types of narratives, each designed to convey a mathematical concept in a specific manner. Among them, the so-called question-posing stories are narratives that are structured to generate a mathematical problem or inquiry. In primary education, this category is often approximated by “word problems” found in textbooks, which are intended to emulate a story-based questioning [17] efficacy of the “question-posing story,” particularly as manifested in the traditional word problem, has often been severely diminished. All too frequently, these “problems” are stripped of the rich, well-organized narrative details that are essential for engaging and motivating students and for shaping the emotional response of the reader or listener. Consequently, they lose their power to foster genuine engagement. While there is widespread agreement that contextualizing mathematical problems within concrete, familiar, and realistic situations benefits children both motivationally and cognitively, the prevalent use of concise, artificial, and often falsely relatable texts proves counterproductive. Rather than fostering comprehension and engagement, these poor-quality texts inadvertently encourage the use of “cognitive shortcuts.” Students learn to employ a selective reading strategy, focusing only on the search for numerical data and mathematical “key words” [16]. This pattern of interaction reinforces the damaging miscon-

ception that mathematics is merely a sterile collection of procedures to be applied without true meaning or conceptual understanding. In Rotman's [13] words, we could say that this kind of approach fosters the operational capability of Procedure, yet risks disconnecting it from both Idea and Sign.

To counteract these negative effects, the narrative context must be rich and profoundly meaningful for the students, namely, in Rotman's [13] words, it emerges a need to connect Idea and Procedure. As suggested by Zan [15], the mathematical questions must arise "in" the story itself, organically growing out of the plot and the protagonists' motivations, rather than being artificially imposed "about the story." The problem must be of genuine interest to the characters within the tale. This requirement highlights a significant contrast with traditional classroom problems, which are often "externalized" – meaning the person who poses the problem is different from the person who is expected to solve it, a dynamic unlike real-life problem-solving scenarios.

Furthermore, comprehending a narrative engages what Bruner [7] narrative thinking, which he contrasted with paradigmatic or logical-scientific thinking. Narrative Thinking is focused on interpreting human actions, people, their intentions, and feelings, and its output is a plausible and reasonable story. Logical-Scientific Thinking is dedicated to categorizing reality, recognizing the order of things, and its output is demonstrative arguments and logical proofs.

While Bruner argued that these two modes of thought are fundamentally irreducible, the distinction should not be interpreted as narrative thinking being an impediment to learning or teaching mathematics. On the contrary, the strong presence of narrative thinking in everyday life and its complementarity with logical thinking means that it can actively support the development of logical thought processes [16].

This synergistic relationship is particularly effective in what Rosetta Zan refers to as "story problems" – those mathematical problems that successfully incorporate the rich, meaningful, and protagonist-driven characteristics described above. In these contexts, the two types of thinking – narrative and logical-scientific – can effectively support each other, thereby deepening students' comprehension of abstract mathematical concepts. The story provides the human, contextual, and emotional foundation, while the mathematics provides the logical framework for resolution.

A crucial distinction should be made between the different types of information embedded within a word problem. A problem contains information necessary for the construction of a mental representation (the 'story' data) and information necessary for the execution of the mathematical solution (the 'solution' data). As emphasized by Zan [15], "the point is that the data a child needs to represent the problem are not necessarily the data he or she needs to use in the solution" (p. 6). This highlights a potential cognitive gap: students must first successfully construct the narrative model, and then transition to identifying and utilizing the quantitative data relevant for the mathematical operation. A successful problem-solving process requires two critical alignments: (i) alignment of solution-relevant information and (ii) alignment of representation-relevant information. The former can be understood as the quantitative, operative data that should be clearly and logically connected to the established narrative structure. If the data points feel arbitrary or divorced from the story, the student's ability to apply them correctly is diminished. With respect to the latter, the elements used to build the narrative representation must be logically consistent and coherent, allowing the student to form a stable and understandable mental model of the situation.

The integration of drawings when the students deal with word problems is a powerful yet underexplored area of research, particularly concerning how different visual representations support distinct cognitive and communicative skills [1]. These drawings serve a crucial mediating role, bridging the gap between stringent mathematical rigor and the narrative sensemaking essential for complex problem-solving. Drawings can offer access to Rotman's [13] domain of

Sign and their representational power in mathematics. Visual tools act as a vital link between real-world contexts and abstract mathematical structures, connecting Idea with Procedure [13].

Despite a relatively limited body of research, existing studies have demonstrated a positive correlation between the strategic use of drawings and enhanced mathematical modeling performance [12, 4]. Specifically, the work of Rellensmann et al. [12] highlights that students' strategic knowledge of drawing is positively associated with their modeling performance. This relationship is further nuanced and mediated by the *type* and *accuracy* of the drawings generated. Their findings suggest a complex relationship: while the accuracy of *situational drawings* is only indirectly related to performance, there is a clear and robust correlation between the accuracy of *mathematical drawings* and overall modeling success. Conversely, Bassi and Brunetto [4] suggest that this strict demarcation between drawing types and their impact may become less distinct when students are engaged in more sophisticated and open-ended modeling activities, implying that the boundaries between situational and mathematical drawing functions can be creatively overcome in complex tasks.

Building upon existing literature [12] and the insights derived from the findings of the research project of which this study is part, drawings have been identified, each characterized by a unique function and level of abstraction. The first one is situational drawing. This type of drawing visually depicts the objects and context described in the problem exactly as they would appear in a real-world setting. It is an immediate, direct representation of the problem's narrative environment. It involves a low degree of abstraction, as the primary goal is fidelity to the physical reality of the problem elements. In Rotman's [13] terms, it is as if the student is accessing the realm of Idea through the narrative of the problem, its characters and its elements.

The second category is mathematical drawing. It is highly selective, including only the essential elements of the problem that are directly relevant to constructing the mathematical solution. The objects within the drawing are simplified and idealized to represent their core mathematical properties (e.g., a car might become a point mass, a building a simple rectangle). This type is significantly more abstract. It fundamentally functions as an externalization of the mathematical model itself, translating the real-world scenario into a structured, solvable mathematical framework. In Rotman's [13] terms, we could say that the realm of Idea becomes, in this case, a blending of both narrative and mathematical elements and this can offer access to Procedure.

The third category is narrative drawing, which is defined as a specific sub-type of situational drawing. The narrative drawing incorporates elements that aid the student in constructing meaning from the narrative problem context. While it remains rooted at the situational level, its choice of visual emphasis and inclusion suggests an active engagement with the problem's underlying logic. As conceptualized by Zan [15], the designation "narrative" underscores the fact that these drawings, while situational, demonstrate the activation of logical thinking. This logical processing is often supported and facilitated by narrative thinking, which helps guide the meaning construction process and the initial comprehension of the problem's demands.

The fourth category is dynamic mathematical drawing, which is configured as a specific sub-type of mathematical drawing, the dynamic mathematical drawing includes conventional or intuitive graphical signs (e.g., arrows, vectors, trajectories) that are capable of highlighting a movement, change, or state-transition of the represented objects. This movement or evolution within the drawing is directly associated with the progression or change over time inherent in the problem-solving process itself, making the drawing a visualization not just of the model, but of the *process* of reaching a solution. In other words, dynamic mathematical drawings allow for a connection between Procedure and Idea, reducing the risk of pure algorithmic performance.

3. Methodology

The study involved a cohort of 363 Italian pupils who had recently completed Grade 2 of primary school. Data collection was carried out during the 2024 summer break, the period spanning between the completion of Grade 2 and the commencement of Grade 3. This activity was integrated into the required summer homework assigned by their respective teachers, fostering participation and relevance to the curriculum.

A stratified sampling method was employed across various Italian schools as part of an Italian National project funded by the Ministry of Education. Following the sampling, school deans were formally requested to invite their Grade 2 teachers to participate in the project. 95% of the invited teachers consented to participate.

Given that data collection occurred during the summer when schools were closed and direct teacher-pupil interaction was suspended, a web app was developed specifically for the project. This application served the crucial dual purpose of presenting the mathematical tasks to the children and facilitating the collection of their work. Critically, the children's elaborations were made using traditional paper and pencil, which were then uploaded via the app, bridging the gap between physical execution and digital collection.

Consensus for collecting anonymous children data has been obtained from their parents via a form, distributed by the schools.

The core of the study involved six distinct mathematical modelling activities. Each activity was designed to commence with an engaging story, providing a context for the subsequent tasks. As the activity progressed and different tasks were assigned, new fragments of the story were revealed, maintaining engagement and building a narrative structure.

The very first task across all activities consistently required the pupils to "draw the story you heard." This initial task aimed to capture the children's interpretation and visual representation of the narrative context before engaging with the mathematical problem-solving.

This particular paper zeroes in on the first two tasks of the first activity. Thematic focus for the first activity is the topic of candies, suggesting an accessible and relatable context for the young participants. Activity 1, like the others, starts with an introductory story to set the scene for the subsequent tasks:

Every time she visits her grandchildren, Elisa and Matteo, Grandma Adele brings candies, which the children are supposed to put in the kitchen drawer and then to eat candy every now and then.

The story goes on, setting the first task:

Today, Grandma arrived with three bags – one green, one white, and one red – and she said to the children: "The green one contains only cherry candies. The white one contains four mint candies and three orange candies, while the red one contains three mint candies and four orange candies."

Then, the first task consists in drawing the story already heard.

The second task is as follows:

Matteo likes orange candies most. Which bag is best for him to choose from? Why?

We undertook an analysis of the children's drawings produced during the execution of the first two tasks. This analysis is framed by and interpreted through the theoretical lenses described in the preceding sections of this paper. In particular, we firstly classify the drawings made in the first task into one of the four aforementioned categories. We, thus, focused on the elements that the children report in their drawings: if a character and/or an object that

is a strong narrative element but is not, or partly, related to the mathematical solution of the problem, is drawn, we started with labelling the drawing as situational or narrative, and then we look for mathematical elements. If none is present, the drawing is labeled as situational, otherwise it is labeled as narrative. If mathematical elements are present and narrative ones are absent, the drawing is labeled as mathematical. After this first classification, a detailed analysis of the content of the drawing is carried out. This classification helps us find an answer to the first research question about the kinds of drawings that emerge in a task involving probabilistic thinking. Then, we look at how the students deal with the second task and we attempt an answer to the second research question about how the different drawings can support the meaning making process.

4. Results

This section details the findings from the analysis of student work across the two aforementioned tasks designed to examine their understanding and representation of probabilistic scenarios. We first examine the drawings produced during Task 1, classifying them into four categories: situational, narrative, mathematical, and dynamical mathematical. Subsequently, we investigate how the same students approached Task 2, which required them to select the most advantageous bag for Matteo, a boy who likes orange-flavored candies.

The initial analysis focuses on the drawings created by the students to interpret the problem description. A significant portion of the drawings fell into the situational and narrative categories, indicating an early focus on the contextual and story elements of the problem rather than the underlying mathematical structure. Figure 1 provides compelling examples of situational drawings. The drawing reported in Figure 1 (left) reproduces the story's setting, depicting "grandma Adele" holding three distinct bags: one white, one red, and one green. The drawing is a faithful visual report of the textual narrative, devoid of any attempt to represent the quantitative details (the number of candies) or the probabilistic nature of the situation. Crucially, the student's approach to Task 2 confirms this lack of mathematical engagement; they submitted the *exact same drawing* without any modification or accompanying mathematical reasoning. This response is interpreted as a failure to solve the task, as it demonstrates an inability to translate the situational context into a mathematical decision-making framework.



Figure 1. Two examples of situational drawings.

The drawing reported in Figure 1 (right) is categorized as situational, as it includes the core objects – the three bags (green, white, and red) – but attempts a rudimentary representation of the contents. The student has drawn candies of various colors (pink, green, and orange) scattered around the bags, intending to suggest the distribution of flavors. Specifically, the visual arrangement suggests an approximation: two pink candies with the green bag, three

green and three orange candies with the white bag, and another three green and three orange candies with the red bag. It is noteworthy that the *number* of candies drawn does not precisely match the numbers specified in the original story prompt. This suggests that the student is prioritizing the *qualitative* representation of the situation (i.e., *what* flavors are in *which* bag) over the *quantitative* accuracy necessary for a probability calculation.

The transition from drawing the scenario (Task 1) to solving the probability question (Task 2) reveals different levels of cognitive shift. For students whose drawings were purely narrative, the transition often failed, as seen in the Figure 1 (left) example. The student who made the drawing reported in Figure 1 (right) offered the following written justification for their choice in Task 2:

The orange is orange, hence the candies are in the orange bag.

This response is highly problematic and indicative of a profound conceptual misunderstanding. The student appears to be making a non-mathematical, purely linguistic association (“orange-flavored” implies “orange bag”). This interpretation is flawed on two key counts: on one side, it confuses the color of the *flavor* (orange-flavored candies) with the color of the *container* (the bag), and on the other side the premise is incorrect, as the story explicitly states there is no orange bag among the three options (only white, red, and green).

Therefore, we conclude that this response, rooted in a color association rather than a probabilistic comparison of the bag contents, is incorrect and highlights a critical barrier in moving from a concrete visual representation to abstract mathematical reasoning.

To recall, a narrative drawing incorporates elements from the problem’s context or story, thereby aiding the student in constructing meaning. Crucially, while this type of drawing remains anchored at the *situational level* – depicting the objects and characters of the story – its choice of visual emphasis and the inclusion of specific, non-mathematical details signals an active, albeit often implicit, engagement with the problem’s underlying mathematical nature. Figure 2 showcases the work of a single student as they tackled Tasks 1 and 2. The drawing corresponding to Task 1 (Figure 2 left) is highly descriptive and meticulous in its representation of the problem. It depicts three bags: in the green bag, the student has included a symbol of a cherry. Notably, closer inspection reveals evidence of revision: the student had initially drawn two pink candies inside this bag but subsequently erased them. This erasure suggests a process of cognitive checking and correction, pointing towards a generalisation. The white bag contains four green (assumed mint-flavored) and three orange (assumed orange-flavored). Similarly, the red bag contains seven candies, but the ratio is reversed: three green candies and four orange candies.

A significant feature distinguishing this narrative drawing from purely *situational* drawings (like the one discussed in Figure 1, right) is the containment and accurate enumeration of the objects. The candies are explicitly drawn *inside* the bags and are counted correctly according to the problem statement. This careful and accurate representation of the quantities and their locations demonstrates a strong effort to map the text onto a concrete visual model, a necessary precursor to formal mathematical calculation. The drawing for Task 2 (Figure 2 right) shifts focus from the static containers to an active, human element. It features a recognizable character – Matteo – portrayed with a seemingly happy or determined expression. The key action depicted is Matteo taking the red bag. Our interpretation of this visual choice suggests a sophisticated, if unstated, inference on the student’s part. In the context of a probability problem that likely asked about the chance of getting an orange-flavoured candy, the red bag is the container that offers the highest probability of success, as it holds the greatest number of orange candies (four orange versus three mint). Therefore, by drawing Matteo choosing the red bag, it seems that the student is visually encoding a mathematically-informed decision.

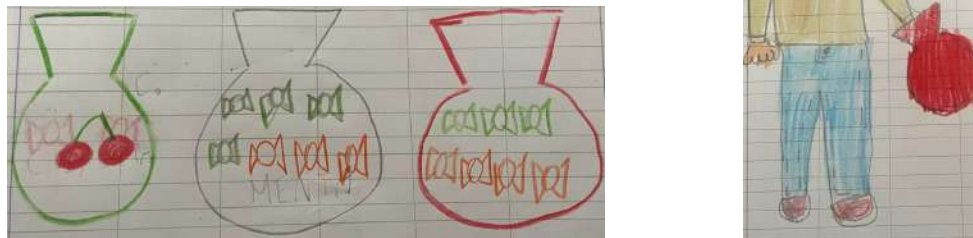


Figure 2. An example of a narrative drawing. To the left, the answer to Task 1. To the right, the answer to Task 2.

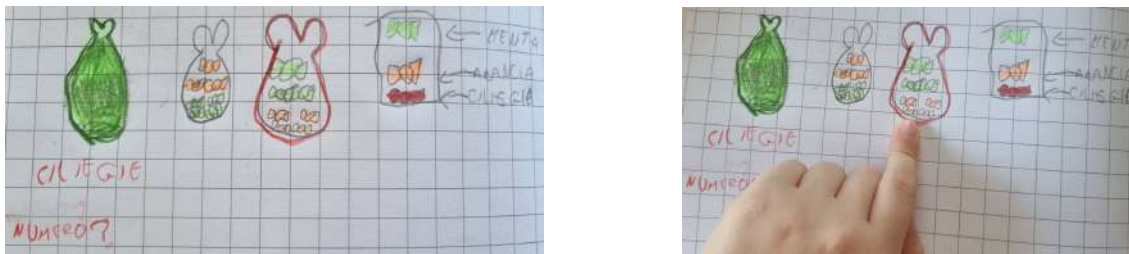


Figure 3. Another example of a narrative drawing. To the left, the answer to Task 1. To the right, the answer to Task 2.

Another instance of a narrative drawing, rich in situational context yet containing crucial elements for the mathematical problem’s resolution, is presented in Figure 3. This figure illustrates the same student tackling both Task 1 and Task 2. Focusing firstly on the representation for Task 1, shown in Figure 3 (left), the student employs distinct visual markers to represent the components of the problem. They draw a prominent green bag. Attached to this green bag is a handwritten question: “Ciliegie, numero?” which translates to “Cherries, number?”. This annotation is significant because it highlights the student’s awareness that the initial problem statement is incomplete or ambiguous regarding the quantity of cherry-flavoured candies. Adjacent to this clearly green bag, the student has drawn two other bags: one in white and one in red. The contents depicted within these bags bear a strong resemblance to the visual representation provided by the same student in Figure 2, suggesting a consistent mental model for the problem’s setup. To clarify the connection between the colours and the candy flavours, the student has included a legend or key: green is assigned to ‘mint’ flavour, orange is associated with ‘orange’ flavour, red is designated for ‘cherry’ flavour. This visual and textual organization confirms the student’s method of identifying and compartmentalizing the different variables involved in the problem – the colours (representing flavours) and the associated quantities.

Moving to the student’s engagement with Task 2, depicted in Figure 3 (right), we observe the student’s own finger pointing directly towards the red bag. Given the structure of the problem (as previously discussed in the context of Figure 2), the red bag is, in fact, the correct item to identify for the solution to Task 2. However, while the *selection* is implicitly correct based on the underlying mathematical nature of the task, the drawing itself and the moment captured lack an explicit, articulated justification for this choice. This common pattern, where

a student correctly identifies the answer visually but fails to provide the reasoning, suggests that the drawing serves as an intuitive or situational model rather than a fully developed mathematical proof. The visual element confirms the choice but leaves the mathematical justification unstated, mirroring the observational findings from the analysis of Figure 2. In Rotman's (2003) terms, we could say that the Idea is fully developed, but the Signs lack for language tools to express the correct intuition.

We conclude our analysis of students' drawings by examining three exemplary solutions to Task 1, presented in Figure 4, which distinctly prioritize mathematical and numerical elements over narrative ones. These examples showcase a significant reduction in storytelling to focus on the problem's mathematical elements. The student's drawing on the left of Figure 4 features three distinctly colored bags – green, white, and red – each clearly labeled with its contents. The narrative is limited but crucial for identifying the *type* of candy. A speech bubble indicates the content of the green bag as “caramelle ciliegia” (cherry candy). The presence of only cherry candies in this bag is clear for the student. The white bag is labeled with the quantities and types of candies: “caramelle: 3 arancia, 4 menta” (candies: 3 orange, 4 mint). Similarly, the red bag is labeled “caramelle: 4 arancia, 3 menta” (candies: 4 orange, 3 mint). We can see the emergence, in this drawing, of a transition from a purely visual narrative to a coded representation, where written text acts as the primary carrier of essential numerical information.

The drawing in the middle of Figure 4 maintains the visual presence of the three bags, arranged on what appears to be a table. The numerical and descriptive information is separated from the bags and linked to them using arrows, suggesting a more formalized symbolic mapping. Specifically, the descriptor “solo ciliegia” (“only cherry”) is connected to the green bag with an arrow, reinforcing the qualitative nature of its contents. The writing “4 menta + 3 arancia” (4 mint + 3 orange) is connected to the white bag and we can see the emergence of a formal mathematical sign, namely the '+' sign, suggesting a summation or a collection of items, further emphasizing the numerical composition. The text “3 menta + 4 arancia” (3 mint + 4 orange) is connected to the red bag. In this drawing, the embedded narrative is reduced in favor of external, structured annotation, moving closer to a diagrammatic representation of the problem's data.

In the drawing to the right of Figure 4, the contents of the bags are depicted directly using color-coded dots, standing for the candies. An arbitrary number of red dots (representing cherry candies) are drawn inside the green bag. The lack of an explicit count underscores that the composition, rather than the total number, is the key mathematical feature (i.e., zero orange candies). The numerical composition of the white bag is visually concrete: 3 orange dots and 4 green dots are drawn inside it. Similarly, 4 orange dots and 3 green dots are drawn inside the red bag.

The differences in representational style across Task 1 does not correspond to a deeper conceptual understanding, as revealed by the students' answers to the subsequent Task 2 (Figure 5). The students who produced the drawings on the left and in the middle of Figure 4, despite their clear representation of the numerical data, provided solutions to Task 2 that were similar to those observed in Figures 2 and 3. This indicates they selected the red bag (the correct solution) but failed to provide a robust, mathematically sound justification for their choice. We can further notice that, for the drawing presented in Figure 5 (right), a significant linguistic and interpretative shift becomes evident. The accompanying written statement, “Ho scelto questo” (“I chose this”), is crucial, as it employs the first singular person (“I”). This use of the first-person pronoun marks a clear departure from the narrative perspective observed in Figure 2. In those instances, the characters and the actions described were framed in the third person, referring to Matteo. The student's statement “Ho scelto questo” signifies a fundamental change in the relationship between the student and the action being documented. A first-person interpretation and ownership of the choice or action performed emerges. In

our view, this implies that the student is not simply documenting a scenario but is actively asserting their participation and decision-making process within the task, potentially offering insight into their metacognitive awareness or their personal connection to the mathematical or thematic content of the drawing.

The student who created the highly minimalist drawing on the right offered a tentative justification, writing “4 caramelle” (“4 candies”). While concise, this suggests an attempt to articulate the reason *why* the red bag was the optimal choice – namely, because it contained the maximum number of orange candies (4) compared to the other non-cherry bag (3). This answer hints at a developing concept of maximizing favorable outcomes, the core principle behind maximizing the chance of getting an orange candy. This correlation suggests that focusing on the purely numerical structure in the visual representation might facilitate a more analytical approach to the subsequent justification phase.

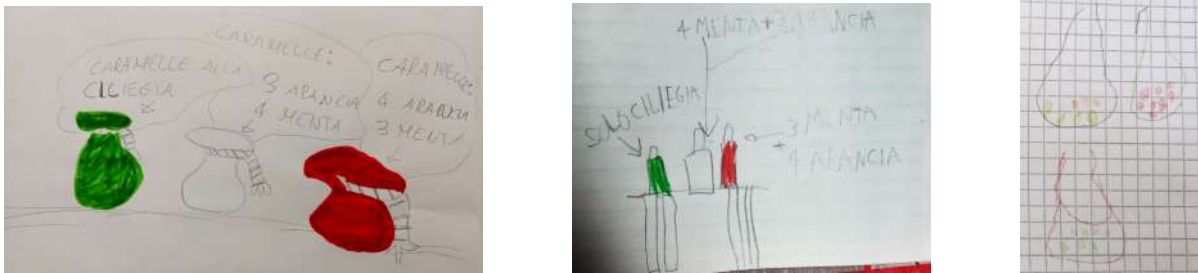


Figure 4. Three examples of mathematical drawings while students are dealing with Task 1.



Figure 5. The same three students' mathematical drawings while dealing with Task 2.

5. Discussion

In this study, we showcased examples of situational, narrative and mathematical drawings taken from a large sample of 363 children who dealt with a probabilistic task about candies. We recall that the participants in our study are grade-2 students who have not been exposed to probability content at school before this activity, which is carried out during the summer break. Some drawings belong to the narrative category, as they are strongly related to the elements of the story (grandma Adele, the boy Matteo, the presence of candies of various colors), but they also represent the candies inside each bag, mirroring the quantities mentioned in the story and hence accessing the mathematical symbols and procedures necessary to solve the task mathematically. The students, in fact, correctly identify the red bag as the one that maximises the probability of getting an orange-flavoured candy, but their seemingly weak ability

to represent, in their own words, their reasoning limits their ability to justify their (yet correct) choice. We employ Rotman's [13] framework to interpret this phenomenon: the intuitive realm of Idea is well supported by the story problem and the children's own drawings, but in the realm of Sign the children seem to lack the language tools to account for their choices. We also acknowledge the potential of distinguishing between situational and narrative drawing. The analysis of the examples in Figure 1 underscores the challenge students face in disentangling the narrative elements from the mathematical structure of probability problems. The tendency to remain within a situational framework often prevents them from identifying and comparing the relevant ratios of desired outcomes to total outcomes.

The observations about the drawings in Figure 2 highlights a fundamental characteristic of narrative drawings: they are not mathematical drawings. They do not use formal mathematical symbols, diagrams, or structures to convey the solution. Instead, the student embeds their reasoning and potential solution within the narrative itself. The visual story – Matteo's choice – becomes a proxy for the mathematical conclusion (the highest probability is associated with the red bag). The student does not feel the need to provide a written explanation or a numerical calculation because the answer is encoded within the chosen action of the drawn character. This feature underscores the distinction of narrative drawings as visual tools that illuminate a student's intuitive understanding and qualitative reasoning before or alongside the development of formal mathematical skills. We could say that the students stay in the realm of Idea [13].

In the mathematical drawings, a significant departure from the narrative elements emerges and more abstract representations are present. A number of the students who made this kind of drawing was able to (partly) justify their answer, as if the lightening of the cognitive load given by not feeling obliged to represent the narrative elements of the story too faithfully allowed them to concentrate on the mathematical elements, choosing symbols that allow them a more direct access to the mathematical ideas and therefore also to the words necessary to justify their choices. For these drawings, an entanglement of idea, Sign and Procedure [13] emerges.

The study adopts a descriptive and qualitative approach, seeking to identify the processes through which young children construct an understanding of probability by employing drawings and narrative thought. This methodology moves beyond the conventional focus on merely highlighting misconceptions. The analysis of drawings presents inherent complexity, as multiple interpretations by different researchers are plausible, and a single drawing may lend itself to various classifications. This potential ambiguity can be mitigated, firstly, by involving multiple researchers, and, more significantly, by redirecting the focus from simple classification towards an examination of how drawings function as pivotal components in the meaning-making process. Consequently, the primary objective of this research is not the creation of classifications, but rather the demonstration of the potential inherent in a methodology that emphasizes representations beyond verbal expression.

6. Conclusions

The essence of this research lies in employing descriptive and interpretive methodologies for drawing analysis, rather than mere correction, premised on the notion that children frequently lack the requisite semiotic tools to substantiate their choices. The pedagogical implications for mathematics teachers underscore the necessity of encouraging students to create drawings when addressing mathematical problems, as opposed to concentrating solely on verbal expression, the written text, and the procedural steps. Indeed, Rotman's theoretical framework posits that the integration of all these components collectively constitutes mathematical thinking in its entirety.

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Parental stereotypes and early gender gaps in mathematical attitudes. Evidence from Italian primary schools

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Abstract: This article investigates gender differences in mathematical attitudes among 8-year-old students at the beginning of Grade 3 in two Italian metropolitan areas, Milano and Napoli. Drawing on data from the MATES project—a PRIN-funded study involving 3,312 students across 180 classes—we examine differences in students' liking for mathematics, confidence in mathematics, and perceived everyday use of mathematics. Although both boys and girls report broadly positive attitudes at this early age, gender differences are already observable: girls report higher overall school enjoyment, while boys display slightly higher self-efficacy and stronger liking for mathematics. No significant differences emerge in the use dimension. A key contribution lies in the analysis of parental influences (based on a subsample of 1,452 students): parental attitudes toward mathematics are positively associated with children's attitudes, while the endorsement of gender stereotypes—such as the belief that males are more naturally suited to mathematics—plays a critical and asymmetric role. These stereotypes are associated with lower liking for mathematics and self-efficacy among girls, while reinforcing more positive attitudes among boys. The findings highlight the role of parental beliefs in children's early attitude formation and suggest that gender stereotypes held by parents may contribute to gender gaps well before differences in performance become pronounced.

2020 Mathematics Subject Classification: Primary 97C70; Secondary 97D50, 97C20.

Keywords: mathematics education; gender differences; primary school; mathematical attitudes; self-efficacy; parental stereotypes.

1. Introduction

Mathematics education in the early years of schooling plays a foundational role in shaping children's long-term academic trajectories. Despite extensive research on gender disparities in STEM fields, the specific mechanisms through which attitudinal and motivational differences emerge remain a subject of active scholarly debate [13, 21], and studies focusing on very young

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students, such as primary school children, remain relatively rare [16, 6]. Recent reviews confirm that research on mathematical attitudes has been predominantly conducted with adolescents and adults, and that the emergence of gender differences during the early primary years represents an understudied area [30, 12]. Nevertheless, available evidence suggests that attitudinal and motivational gender differences can already be detected as early as second grade, before meaningful performance gaps appear [11], underscoring the importance of investigating this critical developmental window.

A recent European Commission report highlights a critical paradox: while girls frequently match or exceed boys' academic performance in early schooling, they consistently report lower self-efficacy, which serves as a primary predictor of future STEM persistence [20]. These disparities are tied to a "leaky pipeline" and entrenched societal stereotypes that begin to erode girls' STEM identities as early as primary education.

Italy presents a particularly instructive case. National standardised assessment data from IN-VALSI consistently document gender gaps in mathematical performance, yet these patterns have received comparatively little scrutiny in the quantitative education research literature, especially at the primary school level [9].

The MATES project (MAtematica per Tutti in Estate), funded under the Italian PRIN programme and conducted across the metropolitan areas of Milano and Napoli, offers a rare opportunity to investigate these dynamics.

With 3,312 students enrolled in Grade 3 across 180 classes, MATES combines standardised cognitive assessments, attitudinal questionnaires, sociometric network data, and parental surveys to analyse summer learning loss in mathematics. This article focuses specifically on attitudinal and motivational dimensions related to mathematics, asking whether 8-year-old girls and boys differ in terms of self-reported school enjoyment and perceptions of mathematics. We argue that gender differences in mathematical attitudes are already detectable at age 8, before the more pronounced divergences typically documented at the secondary school level, and that parents' attitudes and stereotypes play a crucial role in shaping them [14, 29, 16, 38, 39, 37, 11, 22, 9, 30].

The remainder of the article is structured as follows. Section 2 reviews the relevant literature on gender and mathematical attitudes. Section 3 describes MATES project design, sample characteristics, measures, and empirical strategy. Section 4 presents the main empirical findings, while Section 5 discusses the results. Section 6 concludes.

2. Theoretical Framework and Literature Review

The relationship between gender and mathematical achievement has been extensively studied since at least the 1970s; early meta-analyses, beginning in the 1990s, documented a male advantage in specific domains such as spatial reasoning and problem-solving, although these differences have narrowed substantially across cohorts [26, 27]. More recent scholars have shifted attention from performance gaps to the attitudinal and motivational mechanisms that may precede and sustain them [17, 39].

From an economic perspective, early mathematical skills and attitudes represent key components of human capital formation [23, 10]. Cognitive and non-cognitive skills develop dynamically and interact over time, implying that early differences in motivation, confidence, and preferences may generate persistent inequalities in educational trajectories and labour market outcomes. In this framework, gender differences in mathematical attitudes can be interpreted as early inputs into later choices, particularly in STEM-related fields [3]. Attitudinal gaps—operating through reduced expectations of success and lower self-efficacy—function as early

negative inputs into the human capital accumulation process, potentially generating persistent inequalities well before performance differences become measurable.

Within the psychological literature, Eccles and colleagues' Expectancy-Value Theory (EVT) provides a central framework for understanding these processes [17, 18]. EVT posits that students' academic engagement is shaped by two main factors: their expectations of success and the subjective value they attach to a domain. Both are influenced by family and school environments, including parental beliefs, teacher expectations, and peer relations [40]. The economic logic maps naturally onto EVT: attitudinal gaps can be modelled as distortions in the formation of expectations and valuations that, once established early, generate persistent misallocation of effort and human capital investment.

Self-efficacy—defined as the belief in one's ability to succeed in a specific task [5]—is a key construct within this framework. Within the EVT framework, self-efficacy can be understood as the psychological mechanism underlying the expectancy component: a child's belief in her own mathematical ability directly shapes her expectation of success, which in turn determines engagement and persistence in the domain. In mathematics, a robust body of evidence shows that boys tend to report higher self-efficacy than girls [34, 31]. These differences emerge early: girls as young as six already report lower self-efficacy in mathematics [22]. Experimental evidence from economics further supports the importance of confidence in shaping educational decisions, showing that gender differences in beliefs about ability influence willingness to compete and to enter mathematically intensive fields [32, 35].

A related dimension concerns affective responses to mathematics, particularly enjoyment and anxiety. Mathematics anxiety—a feeling of tension that interferes with numerical tasks [36]—is consistently higher among girls across countries and age groups [24]. At the same time, boys tend to report slightly higher levels of mathematics enjoyment, although the magnitude of these differences varies [33].

A third dimension concerns the perceived utility of mathematics—the extent to which students view mathematics as useful in everyday life. This “use” dimension has attracted increasing attention, as it may be more responsive to pedagogical interventions than deeper motivational constructs [25]. Evidence on gender differences in perceived utility is mixed, with some studies suggesting similar or even higher perceived usefulness among girls [17].

Parental beliefs play a central role in shaping children's attitudes. Economic models of skill formation emphasise that parents act as key investors in their children's human capital, influencing both the quantity of educational inputs and the formation of beliefs about ability [10]. When parents hold gender-biased expectations—tending to overestimate boys' mathematical competence relative to girls'—they alter the incentive environment faced by children, affecting their self-efficacy and motivation through differential encouragement, resource allocation, and implicit signals [38, 8].

These parental influences operate both through explicit messages and through less visible channels (i.e. implicit channels), such as role modelling and differential reinforcement, which shape children's beliefs via repeated observation of behavioural patterns within the family environment [4, 37]. The resulting distortions in early belief formation may have lasting consequences, shaping educational choices at secondary school and university well before any performance gap becomes evident. From this perspective, gender stereotypes held by parents operate as a source of misallocation in the early stages of human capital accumulation [29, 22, 30].

In this study, to analyse students' math attitudes we adopt the framework of the Math and Me Survey (M&MS) proposed by [1], adapted to the Italian context and to this age group. M&MS conceptualises students' attitudes toward mathematics along three dimensions: enjoy-

ment (E, i.e., affective dimension, such as "I love math"), self-efficacy (SE, i.e., cognitive/self-belief, such as "I'm very good at math"), and use (U, i.e., behavioural disposition, such as "I use math also in other disciplines at school"). These dimensions are operationalised through the Mathematics Attitudes and Engagement Scale (MATES) [7]. MATES originated from M&MS: while the Enjoyment and Self-efficacy subscales are closely derived from the original framework, the Use subscale was substantially redesigned to capture concrete, age-typical everyday uses of mathematics rather than abstract future-oriented perceived usefulness, making the instrument more developmentally appropriate for early primary school children and for the Italian context. The psychometric properties of MATES have been evaluated on a sample of 2,297 third-grade students. The three-factor structure was supported by both exploratory and confirmatory factor analyses, with the confirmatory model showing good fit (CFI = .968, TLI = .962, RMSEA = .040). Internal consistency is satisfactory across all subscales and for the total score, with Cronbach's alpha ranging from .717 to .867 and McDonald's omega from .718 to .857. Concurrent validity was confirmed through strong correlations between the enjoyment subscale and global liking of mathematics ($r=.719$) and between the self-efficacy subscale and perceived mathematical ability ($r=.623$). Although Boerchi et al. (2026) is currently under review, the validation evidence summarised above provides a sufficient basis for using MATES as the measurement instrument of mathematics attitudes in the present study.

Taken together, these perspectives offer complementary lenses for interpreting the empirical evidence presented below: the human capital framework situates early attitudinal differences within a broader developmental trajectory, highlighting their potential long-run consequences for educational and occupational choices; EVT identifies the motivational mechanisms through which gender gaps emerge and are sustained; and self-efficacy theory specifies the cognitive-affective pathway—namely, beliefs about one's own ability—through which parental stereotypes translate into children's attitudinal differences.

3. MATES project: data, measures, and empirical strategy

MATES (MAtematica per Tutti in Estate) is a project funded under the Italian PRIN programme (<https://progetto-mates.it>). The project's central focus is the magnitude and social distribution of summer learning loss in mathematics among early primary school students. In May 2024, a questionnaire was administered to teachers to identify their practices in teaching mathematics. Later, at the beginning of Grade 3 in 2024, a questionnaire was administered to students to investigate attitudes towards mathematics, perceptions of school, and help-seeking networks in mathematics. In October 2024, a questionnaire was administered to parents to identify summer habits, their personal attitudes towards mathematics, and their usual habits with their children in dealing with mathematics and homework. MATES also included the implementation of an online web-app targeted at parents and pupils, aimed at supporting their reasoning about mathematical problems during the summertime [2]. For the purposes of the current article, we draw on the questionnaire related to students' attitudes towards school and mathematics, and parents' attitudes toward mathematics.

The overall MATES sample comprises 3,312 students enrolled in Grade 3 at the beginning of the 2024–2025 school year, distributed across 180 classes in 73 school buildings and 44 school institutes in the metropolitan areas of Milano and Napoli. Participating schools voluntarily joined the project, which was formally incorporated into their academic activities. The two metropolitan areas are represented as follows: 1,492 students across 75 classes in the Milano metropolitan area, and 1,820 students across 105 classes in the Napoli metropolitan area. Average class size is 20 in Milano and 17 in Napoli. The gender distribution is broadly

balanced across the sample, with a slightly larger proportion of male students (53% male versus 47% female)—a pattern consistent with national demographic data for this age cohort [28]. Parental survey responses were obtained for 1,452 families (659 in Milano and 793 in Napoli), representing approximately 44% of all students participating in the project.

The students' questionnaire, administered in September–October 2024, covers several aspects of young students' lives: families' socio-economic conditions; students' attitudes towards school and mathematics; a psychological scale on mathematics attitudes; summertime lifestyle and extra-curricular activities during the school year; and mathematics help-seeking networks. In this work, we focus on self-reported attitudes towards school and mathematics – overall school enjoyment, liking for mathematics and confidence in mathematics – and on three dimensions of students' mathematics attitudes – enjoyment, self-efficacy, and use – as derived from MATES [7].

In the questionnaire, we assessed three key self-perceptions of 8-year-old students, drawing on self-efficacy theory [5]. Students reported their emotional adjustment to school by indicating how they felt about returning to school after the summer break, using a 5-point Likert scale ranging from “very sad” to “very happy”. This measure captures students' affective response to the school context, which is conceptually related to school-related well-being and may influence subsequent motivation and engagement.

Students rated how much they like mathematics on an 11-point scale (0 = “not at all”, 10 = “very much”). This item represents the degree of liking for mathematics. To facilitate interpretation and account for the limited substantive distinction between adjacent response categories, the original scale was collapsed into three analytically meaningful groups: 0–3, “I do not like math”; 4–6, “I like math”; and 7–10, “I like math a lot”. The last category, “I like math a lot”, is used as a binary outcome in the regression analysis.

Students indicated how well they believe they can perform in mathematics on an 11-point scale (0 = “I cannot do it at all”, 10 = “I can do it very well”). This measure captures students' perceived competence in mathematics, reflecting their expectations about their ability to succeed in this domain. Similarly to the previous item, responses were classified into three categories: 0–3, low confidence; 4–6, moderate confidence; and 7–10, high confidence. The last category, defined as “I am very confident in math”, is used as a binary outcome in the regression analysis.

Beyond these self-reported attitudinal items, the student questionnaire includes MATES items. This scale includes 20 items and uses a five-point Likert response scale ranging from “strongly disagree” to “strongly agree”. For each student ($i = 1, \dots, N$), we computed composite scores for the three attitudinal dimensions—enjoyment (E)², self-efficacy (SE)³, and use (U)⁴—by averaging the relevant items within each dimension.

Let x_{ij}^d denote the response of student i to item j within dimension d , where $d \in \{E, SE, U\}$. This procedure yielded three mean scores per student. Next, we computed the sample mean

²Items are “I love math”; “Math is fun”; “I enjoy studying math”; “Solving math problems is fun”; “I enjoy playing math games”.

³Items are “I am really good at math”; “Doing math is easy for me”; “I can solve difficult math problems”; “I can tell if my answers in math make sense”; “I understand math”.

⁴Items are “I use math in other subjects in school”; “I use math outside school”; “Math is all around us in our everyday life”; “I count the days until my birthday”; “When I play a game, or watch a game, I count the points to understand who is winning”; “If I must buy something by myself (like an ice cream), I check the money I have and the change”; “When there is food on my plate that I don't like, I count the pieces I have left to finish it”; “When I must climb many stairs, I count them to see how many I've done”; “When the teacher gives us homework, I count the number of pages I have to complete”; “When I want to know how much time is left until the end of lessons, I calculate the remaining time by looking at the clock”. These last 7 items are the Italian adaptation of the original scale.

and standard deviation for each dimension across the full sample of observations. Finally, to standardise the scores, we computed z-scores for each student and each dimension using the following formula:

$$z_i^d = \frac{x_i^d - \bar{x}^d}{s^d} \quad (3.1)$$

where \bar{x}^d indicates the mean value of dimension d , while s^d indicates its standard deviation.

To examine the relationship between gender, parental attitudes, stereotype endorsement, and school- and mathematics-related outcomes, we estimate two linear models. The first model examines the average association between the outcomes and parental attitudes towards mathematics, stereotype endorsement, and student's gender controlling for parents' sociodemographic characteristics:

$$Y_i = \alpha + \beta_1 PA_i + \beta_2 PS_i + \beta_3 Female_i + \gamma X_i + \varepsilon_i \quad (3.2)$$

The second model additionally includes an interaction term between gender and stereotype endorsement in order to assess whether the association between parental stereotypes and students' outcomes differs between boys and girls:

$$Y_i = \alpha + \beta_1 PA_i + \beta_2 PS_i + \beta_3 Female_i + \beta_4 (PS_i \times Female_i) + \gamma X_i + \varepsilon_i \quad (3.3)$$

where Y_i represents one of the five outcomes considered in the analysis: enjoyment of mathematics, self-efficacy in mathematics, use of mathematics, self-reported confidence in mathematics, and liking for mathematics. While MATES dimensions are standardised continuous scores (as defined in Equation (3.1)), the last two variables are binary indicators capturing the probability of reporting high confidence in mathematics and strong liking for mathematics. The variable PA_i represents parents' attitudes toward mathematics, measured through a self-reported question asking respondents to evaluate their overall disposition toward the subject on a 5-point Likert scale ranging from "very positive" (1) to "very negative" (5). We define a binary variable equal to 1 if parents report a positive attitude toward mathematics, i.e. values 1 or 2 on the scale. PS_i is a binary indicator of parental stereotype endorsement, equal to 1 if the responding parent agrees with the statement that "males are more naturally suited to mathematics", and 0 otherwise. We distinguish between these two parental dimensions (PA_i and PS_i) because parental attitudes capture a general orientation towards mathematics, whereas stereotypes specifically capture gendered beliefs about mathematical ability. $Female_i$ is a binary indicator equal to 1 for female students and 0 otherwise.

The vector X_i includes a set of parental control variables:

$$X_i = \{\text{education level}_i, \text{age}_i, \text{mother}_i\}$$

education level is a dummy variable equal to 1 if the responding parent holds a university degree, and 0 otherwise; age represents the responding parent's age; mother is a dummy variable equal to 1 if the respondent declares being the student's mother, 0 otherwise.

All models are estimated using ordinary least squares (OLS) with standard errors clustered at the class level, to account for the nested structure of the data whereby students are grouped within the same classes. The first three outcomes are continuous MATES subscale scores, while the last two are binary and estimated as linear probability models.

4. Main results

Across the full MATES sample, students report broadly positive attitudes toward both school and mathematics. However, systematic gender differences emerge across several dimensions.

First, girls report higher levels of general school enjoyment than boys. In the full sample, 73% of girls report being happy or very happy to return to school, compared to 56% of boys. This pattern persists in the subsample matched with parental responses (77% for girls versus 63% for boys), suggesting that the difference is robust. Second, gender differences emerge in self-reported mathematics-specific attitudes. Boys report slightly higher levels of liking for mathematics and self-reported confidence in mathematics. In the full sample, 71% of boys versus 68% of girls report strong liking for mathematics, while in the matched subsample this difference widens (80% versus 70%). Similarly, boys report marginally higher confidence in mathematics: 64% of boys versus 63% of girls report high confidence in mathematics in the full sample, with a larger gap in the matched subsample (76% versus 70%).

These initial statistics show that boys and girls differ in their self-reported attitudes toward mathematics. Turning to the MATES composite measures, mean scores for both mathematics enjoyment and self-efficacy are above the scale midpoint for boys ($\bar{x}_E^B = 0.036$, $\bar{x}_{SE}^B = 0.113$), indicating generally positive attitudes at age 8. Girls show lower values ($\bar{x}_E^G = -0.039$, $\bar{x}_{SE}^G = -0.118$). These differences are statistically significant, consistent with previous findings [7]. By contrast, the use dimension does not display statistically significant gender differences, suggesting that boys and girls report similar scores on the use dimension ($\bar{x}_U^B = -0.018$ and $\bar{x}_U^G = 0.020$).

Figure 1 shows the overall distributions of these scores for both the full sample and the subsample of students matched with parents responding to the questionnaire. Results highlight two main findings, confirmed by Kolmogorov–Smirnov (KS) tests.

First, the matched subsample closely resembles the full sample across almost all attitudinal dimensions. The only statistically significant difference concerns male self-efficacy ($D = 0.086$, $p < 0.01$), suggesting that boys included in the matched parental sample report slightly different levels of self-efficacy compared with the full sample. This pattern may reflect a modest selection effect, whereby families participating in the parental survey are somewhat more positively oriented towards schooling and academic engagement.

Second, gender differences persist in the matched subsample. KS tests reveal significant differences between boys and girls in both mathematics enjoyment and self-efficacy in the full sample and in the matched subsample, while no statistically significant gender differences emerge for the use dimension.

Geographical differences between Milano and Napoli are limited in magnitude and do not substantially alter these patterns, indicating that the observed gender differences are not primarily driven by local socio-economic contexts.

Starting from these results, now we move to a regression framework to estimate how different aspects of students' mathematical attitudes are associated with parents' attitudes towards mathematics and with parental beliefs about gender differences in mathematics.

The first result, based on the estimation of Model 3.2, is that parental attitudes PA toward mathematics are positively and significantly associated with children's attitudes across all MATES dimensions. A more positive parental attitude is associated with higher levels of

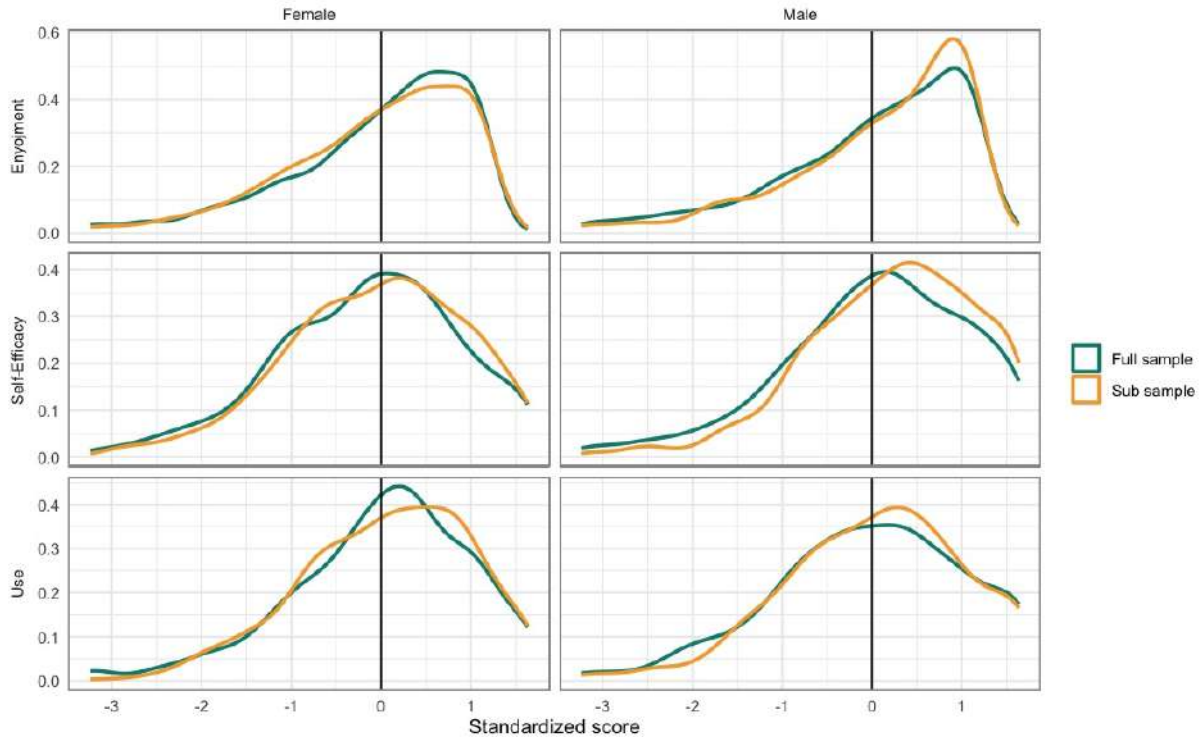


Figure 1. Distribution of standardised scores for the full sample and for the subsample

children’s mathematics enjoyment, self-efficacy, and use of mathematics ($\hat{\beta}_E^{PA} = .217, t = 4.02$; $\hat{\beta}_{SE}^{PA} = .114, t = 2.25$; $\hat{\beta}_U^{PA} = .123, t = 2.44$). In other words, positive parental attitudes toward mathematics are significantly associated with more favourable attitudes toward mathematics among young students, even at this early age.

The second result shows that parental endorsement of the stereotype *PS* that “males are more naturally suited to mathematics” does not exhibit a statistically significant average effect when considered in isolation ($\hat{\beta}_E^{PS} = -.036, t = -0.47$; $\hat{\beta}_{SE}^{PS} = -.064, t = -0.77$; $\hat{\beta}_U^{PS} = .055, t = 0.73$).

The overall explanatory power of these models (over 1,442 observations) remains modest, with R^2 values equal to 3.3%, 4.1%, and 0.9% for enjoyment, self-efficacy, and use, respectively, and standard errors of the regression (SER) equal to 0.963, 0.942, and 0.950. This pattern suggests that, although parental attitudes are systematically associated with children’s mathematical attitudes, a large share of the variability remains explained by other individual and school-related factors, including school environment, peer interactions, individual personality traits, and classroom experiences, not captured in the models.

When considering the binary outcomes, the coefficient associated with parental attitudes is never statistically significant.

When the interaction term between child gender and parental stereotype endorsement is introduced, estimating Model 3.3, a clear asymmetric pattern emerges. Figure 2 reports the predicted values by child gender and parental stereotype endorsement for all five outcomes. For the three continuous MATES subscales, predicted values represent expected standardised scores (z-scores); for the two binary outcomes, predicted values represent predicted probabilities from the linear probability models.

The explanatory power of all these models remains modest but consistent with the literature on attitudinal outcomes among young children. The models explain between 3% and 4%

of the variance for the continuous MATES dimensions related to enjoyment and self-efficacy ($R^2 = 0.0351$, $SER = 0.96231$ for enjoyment; $R^2 = 0.0438$, $SER = 0.94199$ for self-efficacy), while the explanatory power is substantially lower for the use dimension ($R^2 = 0.0094$, $SER = 0.95062$). Similarly, the binary outcomes display limited goodness of fit, with $R^2 = 0.0053$ ($SER = 0.44023$) for the outcome “I like math a lot” and $R^2 = 0.0241$ ($SER = 0.42347$) for “I’m very confident in math”.

Nevertheless, despite the limited overall explanatory power, the interaction coefficients reveal statistically meaningful and theoretically coherent patterns.

Girls exposed to parental stereotypes display significantly lower levels of mathematics enjoyment and self-efficacy compared to boys whose parents do not endorse gender stereotypes (i.e. the reference group). For mathematics enjoyment, girls whose parents endorse stereotypes score 0.332 standard deviations below the baseline, while boys in the same condition score 0.110 standard deviations above it.

The marginal effect of stereotype exposure is -0.206 standard deviations for girls and $+0.110$ for boys, though the latter is not statistically significant. The pattern is clear: parental stereotypes are associated with less favourable attitudes among girls and somewhat more favourable attitudes among boys, widening the gender gap considerably relative to families where no stereotype is endorsed.

Results for self-efficacy follow the same direction and are even stronger in magnitude. Girls whose parents endorse the stereotype score 0.475 standard deviations below the baseline, with a marginal effect of -0.231 standard deviations. Boys in stereotype-endorsing families score slightly higher than the baseline ($+0.079$), though again not significantly so. As shown in Figure 2, the gender gap in self-efficacy is more than twice as large among children exposed to parental stereotypes than among those whose parents do not endorse such beliefs, making self-efficacy the dimension most strongly associated with stereotype endorsement.

By contrast, no comparable effects emerge for the use dimension. Neither the direct effect of stereotypes nor the interaction with gender is statistically significant, confirming that the use dimension appears substantially unaffected by parental stereotypes at this age. Notably, in the absence of parental stereotypes, girls score marginally higher than boys on the use dimension, a reversal of the direction observed for mathematics enjoyment and self-efficacy, further supporting the interpretation that gender differences at this age emerge primarily in affective and self-evaluative dimensions rather than in routine perceptions of the usefulness of mathematics. This finding is particularly relevant because it suggests that stereotypes are not associated with children’s scores on the use dimension, but rather with their mathematics enjoyment and self-efficacy when engaging with it.

Turning to the binary outcomes, when we estimate the probability of reporting strong liking for mathematics (i.e., “I like math a lot”), we observe a gender-specific pattern consistent with the MATES results. For boys, parental endorsement of stereotypes is associated with a higher probability of reporting liking for mathematics (0.865 versus 0.802). For girls, the direction reverses: those whose parents endorse stereotypes are less likely to report strong liking (0.662 versus 0.712). However, the differences between stereotype and no-stereotype groups are not statistically significant for either gender, as indicated by confidence intervals in Figure 2. Although the estimated differences are modest in magnitude, their direction mirrors the patterns observed for enjoyment and self-efficacy, although the estimates remain statistically imprecise, lending additional support to the interpretation that parental stereotypes are associated with less favourable mathematical attitudes among girls and more favourable attitudes among boys.

A similar pattern emerges for confidence in mathematics, measured through the binary item

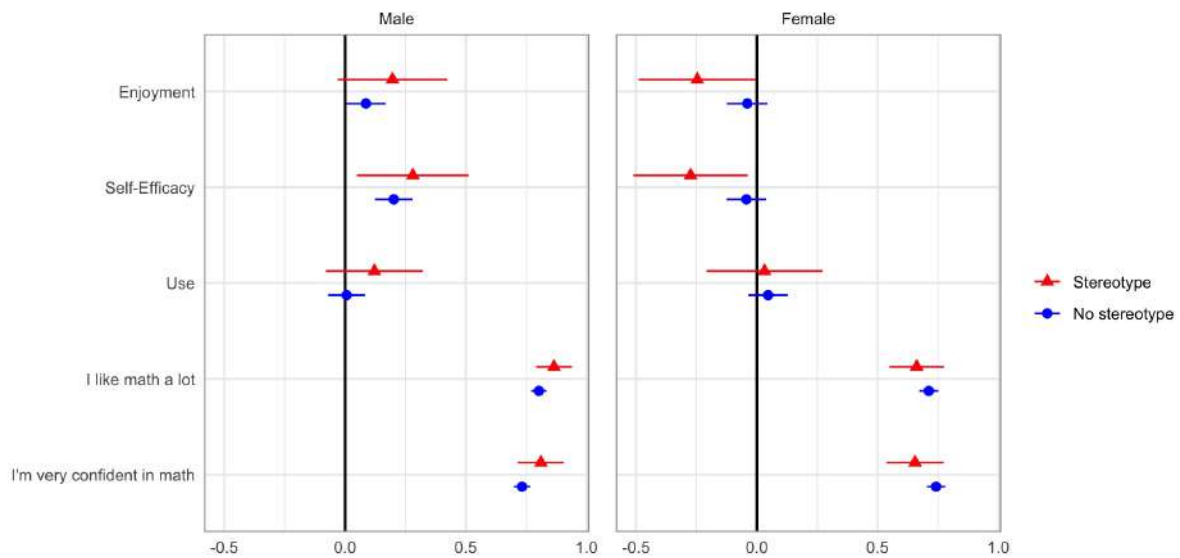


Figure 2. Predicted values and 95% confidence intervals

“I am very confident in math”. In the absence of parental stereotypes, girls report a marginally higher probability of expressing confidence than boys (0.742 versus 0.732), a reversal of the direction observed across all other outcomes. When parents endorse gender stereotypes, however, this pattern inverts substantially: boys increase to 0.810 while girls fall to 0.655, producing a gap of 0.155 in favour of boys. This asymmetric pattern is consistent with the hypothesis that parental stereotypes may contribute to gendered differences in children’s self-evaluations, although the estimates do not reach conventional levels of statistical significance. Notably, this is the largest gender gap observed among all predicted binary outcomes, and it mirrors closely the results obtained for the continuous self-efficacy scale.

5. Discussion

This study provides evidence that gender differences in mathematical attitudes emerge early in primary school. By combining student and parental data, we contribute to the literature by highlighting the role of family-level mechanisms—particularly parental beliefs—in affecting these early attitudinal gaps. The finding that boys report higher mathematics self-efficacy than girls at age 8 is consistent with the well-documented self-efficacy gap [34, 11]. Such differences in self-beliefs are important because they are predictive of subsequent choices, including effort allocation, persistence, and participation in mathematically intensive activities [32].

From an EVT perspective, parental stereotype endorsement effectively undermines the expectancy component of girls’ mathematical motivation—reducing both their self-efficacy and their mathematics enjoyment—while leaving the perceived utility largely unaffected, a pattern fully consistent with the descriptive evidence on the use dimension. This result aligns with evidence that gender differences in perceived utility are generally smaller and less consistent than those observed for affective and self-evaluative dimensions, with some studies reporting similar or even higher perceived usefulness among girls [17, 25].

More broadly, the results suggest that parental stereotypes are associated primarily with children’s self-evaluative and affective orientations toward mathematics rather than with their perceptions of its usefulness [29, 38, 22]. These early attitudinal deficits may function as nega-

tive inputs into a cumulative developmental process, amplifying over time even in the absence of initial performance gaps [23, 10].

While no significant average effect is detected for parental stereotypes, interaction results show that stereotypes are associated with lower attitudinal scores among girls, while the corresponding estimates for boys are generally positive, although not statistically significant. The most consistent pattern across outcomes is not a uniform effect of stereotype endorsement but an amplification of gender differences: parental stereotypes appear to operate primarily as a mechanism that widens the gap between boys' and girls' attitudes toward mathematics, rather than as a general determinant of children's attitudes. This pattern is consistent with models of belief formation in which biased expectations shape individuals' self-efficacy and preferences differently across groups [38, 22].

These findings suggest that early attitudinal differences may contribute to the persistence of gender gaps in STEM fields. If girls internalise lower self-efficacy in mathematics from an early age, this may influence their future educational trajectories, even in the absence of initial performance differences. The fact that the strongest associations emerge for self-efficacy is particularly noteworthy, given the central role of self-beliefs in educational choices and persistence documented in the literature.

At the same time, the absence of significant gender differences in the use dimension is noteworthy. The use dimension captures routine and concrete experiences with mathematics in everyday life—such as counting, games, or school-related activities—which are likely to be similarly experienced by boys and girls at this early stage of schooling. This shared foundation could be leveraged through pedagogical interventions: strengthening connections between mathematics and real-world applications may help sustain engagement and mitigate the emergence of larger gender gaps in later stages of schooling. In contrast, gender differences tend to emerge more strongly in evaluative and affective dimensions, such as self-efficacy and emotional orientation towards mathematics, which are more directly shaped by stereotypes and social expectations [14, 11, 22].

Some limitations of the paper are worth noting. The analysis is primarily descriptive, and results should be interpreted as associations rather than causal effects. Since the study did not include standardised achievement tests, it is not possible to assess whether the observed attitudinal gaps correspond to differences in actual mathematical performance. The role of teachers and peer relationships in shaping students' attitudes also falls outside the scope of the present analysis and will be explored in future work. Finally, because parental participation was voluntary, some degree of self-selection cannot be ruled out.

Notwithstanding these limitations, the findings offer a valuable contribution to the understanding of how the family environment shapes children's attitudes toward mathematics and suggest that parental stereotypes may operate primarily through children's self-efficacy and broader self-evaluative beliefs rather than through beliefs about the usefulness of mathematics. Although the estimated associations are modest in magnitude, their consistency across multiple outcomes and across both continuous and binary measures strengthens confidence in the substantive interpretation of the results. By highlighting the associations between parental beliefs and students' dispositions, this study provides a meaningful empirical basis for future research and for the design of family-oriented educational interventions.

6. Conclusions

This article has presented evidence from the MATES project on gender differences in mathematical attitudes among 8-year-old Italian primary school students. Three main findings emerge.

First, girls report lower levels of liking for mathematics and confidence in mathematics than boys at the beginning of Grade 3, while at the same time reporting higher levels of overall school enjoyment. This suggests that gender differences are already present at an early stage, particularly in domain-specific attitudes rather than in general attitudes toward schooling.

Second, the use dimension of mathematical attitudes—namely the perceived use of mathematics in everyday life—does not differ significantly by gender at this age. This indicates that boys and girls share a similar perception of the returns to mathematical engagement, suggesting that the use dimension may represent the most stable lever for pedagogical interventions—particularly those aimed at making the practical value of mathematics salient—before self-efficacy gaps translate into differential investment decisions with longer-run consequences for educational and labour market trajectories.

Third, parental characteristics play a significant role. In particular, positive parental attitudes toward mathematics are associated with more favourable attitudes among children, while the endorsement of gender stereotypes is associated with a differentiated pattern: it is associated with lower levels of liking for mathematics and confidence in mathematics among girls and, conversely, with more positive attitudes among boys. This highlights the importance of family-level belief transmission in shaping early gender differences.

These findings, while based on non-randomly selected schools, so still contribute to a growing literature emphasising early childhood as a critical period for the formation—and potentially the prevention—of gendered attitudes toward mathematics.

From a policy perspective, interventions aimed at reducing gender gaps should begin early in the educational process and address not only students but also their social environment.

The findings suggest that interventions aimed at reducing early gender gaps in mathematical attitudes should target the belief formation process within the family. Two broad classes of intervention appear particularly promising.

The first concerns information provision: if parental stereotype endorsement partly reflects inaccurate beliefs about boys' and girls' mathematical abilities—rather than deeply held preferences—then low-cost informational interventions providing parents with evidence on gender similarities in early mathematical performance may help shift beliefs and, through them, children's attitudes. This is consistent with a growing literature in economics showing that information provision can update biased parental expectations and improve children's educational outcomes [8]. The second class of interventions concerns school-family engagement initiatives, such as the MATES web app [2], that create structured opportunities for parents and children to interact around mathematical content, potentially reducing the scope for stereotype-consistent behaviour in everyday family dynamics.

Crucially, both types of intervention operate on the belief formation process rather than requiring changes in deep preferences, making them potentially more cost-effective than broader cultural campaigns. The shared foundation identified in the use dimension further suggests that interventions leveraging the perceived practical returns to mathematics may be particularly well-suited to sustaining girls' engagement before attitudinal gaps consolidate into differential investment decisions—with potentially persistent consequences for human capital accumulation and labour market outcomes that extend well beyond the primary school years.

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Conflict of interest

The authors declare that there are no conflicts of interest.

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Children as mediators of home-school continuity in early mathematics: the case of Betta-the-Bee

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Abstract: This paper examines how a mathematically rich card game, *Betta-the-Bee*, supported home-school continuity in early mathematics when it was first introduced in kindergarten and later taken home by children. The study was conducted in a public kindergarten in Turin, Italy, and adopted an exploratory case study design. Data came from three parent questionnaires, an online focus group with three parents, and a supporting interview with two teachers. The analysis focused on parents' initial recognition of mathematics and mathematical play, on the home re-enactment of the game, and on the forms of continuity that emerged through that process. The study contributes to research on home-school continuity in mathematics by identifying children's mediation of a rule-governed mathematical practice as a plausible mechanism through which continuity may be sustained across settings.

2020 Mathematics Subject Classification: Primary 35B65; Secondary 35J70, 35R09.

Keywords: home-school continuity; early childhood mathematics education; educationally rich mathematical games; question formulation.

1. Introduction

As Cahoon et al. [1] recall, a substantial body of research has shown that mathematical competencies developed in the first years of life are related to later educational trajectories and broader life outcomes. Beyond school, children encounter mathematics in everyday routines, in interactions with objects and people, and in the informal situations through which children begin to organise quantity, space, relations, and patterns. For this reason, any account of early mathematics education that focuses exclusively on classroom instruction risks overlooking a decisive part of children's experience. As Phillipson et al. [2] argue, families are children's first mathematics educators, even when the mathematical character of everyday interaction is not fully explicit to the adults involved.

This broad recognition, however, does not resolve a central educational problem. If children's mathematical learning develops across home and school, it is necessary to consider how

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continuity between these contexts can be built. The literature on home-school partnerships in mathematics suggests that this is far from straightforward. Muir [3], for instance, shows that parents are often supportive of school mathematics initiatives while remaining uncertain about what happens in mathematics lessons and about how to engage productively with their children's learning. In a similar vein, Phillipson et al. [2] argue that children's learning can be predicted by the quality of the potential engagement and the extent to which families are supported in recognising meaningful opportunities for mathematical activity. The problem, then, lies not only in participation, but also in fostering intelligibility, mediation, and shared practice.

Several studies have addressed this issue by trying to connect home and school through games, books, resource bags, workshops, or forms of guided family participation. In the chapter "Using Mathematics to Forge Connections Between Home and School", Muir [3] describes initiatives designed to improve parental understanding of contemporary mathematical practices and to foster more regular numeracy-related activity at home. Streit-Lehmann [4], in turn, presents a kindergarten-based project in which families borrowed games and picture books with mathematical content to use at home, showing that structured cooperation between kindergarten and families can have positive effects on children's mathematical learning. Gervasoni [5] reports similarly that when educators and families are brought together around mathematical noticing, discussion, and play, parents can become more aware of the mathematical potential of everyday activities. More recently, Young et al. [6] showed that adding a family mathematics component to a naturalistic, game-based classroom intervention produced positive effects on preschoolers' mathematics knowledge, suggesting that the coordination of mathematics learning opportunities across school and home can add value to classroom-based work. This literature demonstrates that continuity between settings can be cultivated, but it leaves open a question that is especially relevant for early childhood: through what mechanism does that continuity become active within family interaction?

That question is particularly pressing in early mathematics, where both the content and the form of mathematical activity can be difficult for adults to identify. Cahoon et al. [1] insist on the value of qualitative approaches because parents' accounts of mathematics at home are shaped by cultural expectations, everyday routines, and variable conceptions of what counts as mathematics in the first place. Their study also shows that home mathematics environments are structured by parents' views, expectations, and attitudes, and by the extent to which school-related influences become woven into daily life. For these reasons, it seems insufficient to address the issue of continuity simply by providing materials that can be given to children to take home. Instead, opportunities for interaction within the home environment need to be structured – potentially involving materials to take home – which can help to refocus the joint attention of adults and children.

Kinnear et al. [7] emphasise that early mathematics learning gains educational force when children encounter meaningful, purposeful, and connected mathematical experiences rather than fragmented exercises or decontextualised routines. In the same volume, Muir [3] argues that parents need access not only to activities, but also to the mathematical practices that give those activities educational value. This perspective is especially relevant to games: a game can be mathematically productive not simply because it contains numbers or countable elements, but because its rules organise attention, decision-making, comparison, and inference in mathematically consequential ways. From this point of view, some games may be especially well suited to crossing the boundary between school and home, if they are first made intelligible within classroom practice.

The present paper examines one such case, focusing on the card game named *Betta-the-Bee*. *Betta-the-Bee* is a card game developed for players aged from 4 to 7 by a mixed research group composed by researchers in mathematics education, kindergarten and primary school teachers,

and prospective teachers from the University of Torino [8]. In the game, one player hides Betta under a card and the other player must identify the correct card by asking yes/no questions. The game is therefore structured around the recognition of attributes, the formulation of questions, the interpretation of answers, and the elimination of possibilities. In the study discussed here, Betta-the-Bee was first introduced in kindergarten² and then taken home during the Christmas break, with children asked to teach the game to parents or other adults. This design makes it possible to examine continuity in a particularly sharp form: the same game appears in two settings, but it reaches the second setting through children who have already encountered and learned it at school. The study therefore addresses learning beyond the classroom by focusing on the movement of a mathematical practice from kindergarten to the home environment during a period of school interruption.

2. Theoretical background

In early mathematics education, continuity between home and school is often treated as desirable, but it remains difficult to establish in substantive ways. The more demanding question concerns the circulation of mathematical practices across settings: how mathematical activity becomes recognisable, meaningful, and worth engaging with both at school and at home.

This is crucial in the early years, when children's mathematical learning begins well before formal schooling and develops through family routines, play, and everyday interaction. While kindergarten mathematics increasingly takes shape through pedagogically structured activities, home mathematics are generally spontaneous and rely just on parents' views, expectations and attitudes. When home and school experiences remain weakly connected, school mathematics risks appearing bounded within the classroom, while parents may struggle to identify what counts as mathematics in early childhood or how it is being approached.

Research on home-school connections in mathematics has shown that the issue is not a lack of parental interest. Parents often value mathematics and want to support their children, but they may have only partial access to the forms of activity and discourse through which mathematics is organised at school. Continuity therefore depends on the development of shared interpretive ground through which school practice can become more visible and understandable in family life. Muir [3] makes this point clearly in showing that home-school connections become more substantial when parents are helped to understand what children are doing mathematically and why it matters. Similar concerns appear in the volume *Engaging Families as Children's First Mathematics Educators*, where the emphasis falls on awareness, shared language, and meaningful engagement rather than on participation in the abstract [5, 2].

A second relevant strand of literature concerns parents' views of early mathematics and the home mathematics environment. The home mathematics environment has been described as a combination of activities, expectations, interactions, and resources through which children encounter mathematics outside school. Cahoon et al. [1] show why this environment should not be reduced to a checklist of activities. Their qualitative study highlights the cultural and contextual character of the home mathematics environment and shows that parents differ not only in what they do, but also in how they recognise and describe mathematics. A parent may count, compare, sort, or play rule-based games with a child without naming any of this as mathematics. Conversely, mathematics may be understood in narrowly school-based terms and therefore recognised mainly when numbers, calculation, or overt instruction are present. In this respect, the home mathematics environment is shaped as much by recognition as by activity. Cahoon et al. [1] make this explicit when they argue for qualitative approaches capable

²In the Italian context, *scuola dell'infanzia* refers to the educational segment attended by children aged 3 to 6. We are aware that, in different cultural and educational contexts, the terms *school* and *kindergarten* may refer to distinct educational levels. In this paper, however, we use *school* in a broad sense to include kindergarten, in line with the Italian institutional and cultural context in which the study was conducted.

of capturing how parents talk about mathematics, what they notice, and how school-related practices come into daily life. Similar concern emerges in the family-engagement literature. Several contributions in *Engaging Families as Children's First Mathematics Educators* suggest that many parents are surprised to discover how much mathematics is already present in ordinary play and interaction. One educational task then is to help families recognise a broader range of activities as mathematics education, by understanding what counts as mathematics in their eyes [5].

The third conceptual strand concerns educationally rich mathematical games. Russo et al. [9] use the notion of educationally rich mathematical games to distinguish mathematically substantial games from activities that are merely enjoyable or only superficially educational. Their key point is that mathematics must be internal to the activity itself. A game is educationally rich when the player's decisions, actions, and interpretations are organised by mathematically meaningful relations. This is consistent with research on family mathematics showing that games can provide an approachable context for parent-child interaction around mathematical ideas, supporting co-play, mathematical talk, and repeated engagement at home [6]. This view resonates with Kinnear and Wittmann's [10] argument for mathematically founded early childhood activities. Their concern is not to oppose play and learning, but to distinguish activities structured on mathematics from those in which mathematics is little more than decoration. From this perspective, the rules of a game are part of its mathematical substance: they shape what counts as a useful move, a relevant distinction, or an effective strategy. If the mathematical value of a game depends mainly on what an adult later chooses to extract from it, then its educational force may weaken once it moves from school to home. If, instead, the mathematical work is embedded in the activity itself, then the game has a stronger chance of carrying a recognisable practice across settings. We believe that this latter is the case of the Betta-the-Bee game that we briefly present in the following subsection.

3. The mathematical specificity of the Betta-the-Bee game

The game Betta-the-Bee (*Betta l'apetta* in Italian), which can be downloaded free of charge under a Creative Commons licence from the website www.dfedidamath.unito.it, consists of 24 flower cards and one small "Betta" card (Figure 1). At the beginning of the game, the flower cards are placed on a table with the illustrated side facing upwards, and one player (the hiding player), out of sight of the opponent (the seeking player), chooses one card under which to hide the Betta card. By observing the cards displayed on the table, the seeking player must formulate a question about one of the variables represented on the cards, namely the number of petals, the colour of the pot, the position or number of butterflies, the position of the caterpillar, or the position of the leaf (Figure 1 left). The hiding player may respond only "Yes" or "No". On the basis of this answer, the seeking player discards from the table all the cards under which Betta could not possibly be hidden. For instance, if the seeking player asks, "Is Betta-the-Bee hidden under a red pot?" and the hiding player answers "No", all the cards showing a red pot are discarded from the table. Conversely, if the answer is "Yes", all the cards with a red pot remain on the table, while those with a blue or pink pot are discarded. During play, children take turns assuming the roles of hiding player and seeking player. When the game is played in a small group, one child may take on the role of hiding player while a group of children collectively assumes the role of seeking player; within this group, each child takes turns asking a question, and the children work together to discard the cards.

The game requires players to draw on linguistic, numerical, and spatial abilities in each of the three phases that make up a turn of play: formulating questions, answering questions, and discarding cards. As indicated in the National Guidelines for the Italian curriculum in kindergarten, primary and middle school [11], these abilities are among both the expected competences to be developed by the end of preschool and the learning objectives established

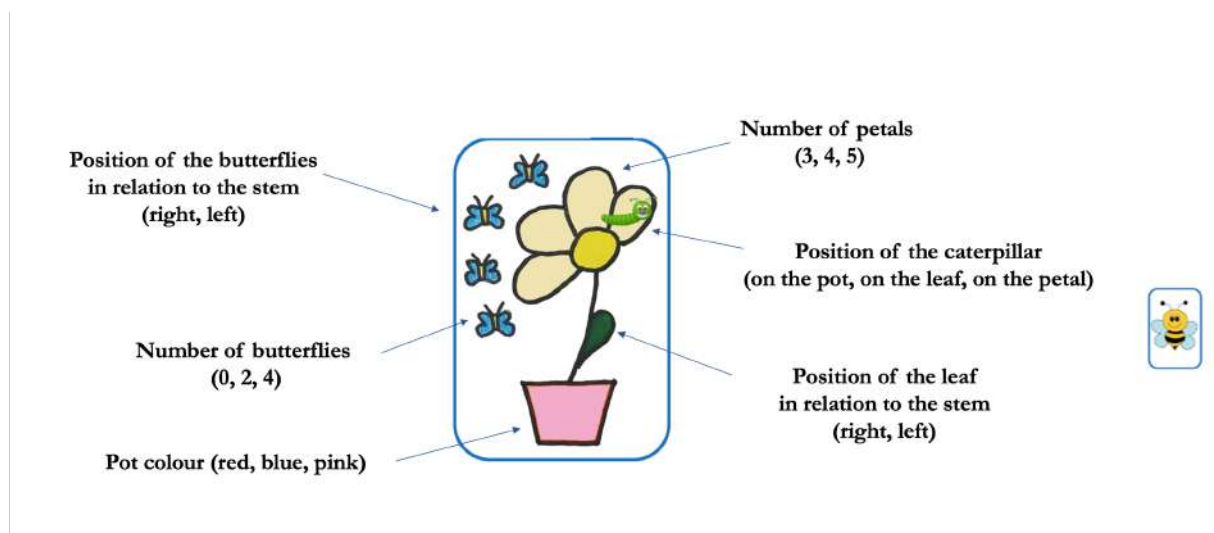


Figure 1. One of the cards of the Betta-the-Bee game with a description of variables and modalities (left); Betta-the-Bee card (right)

for the end of the third year of primary school. In our view, however, the most educationally significant aspect of the game lies in the logical dimensions it can foster from the preschool years onward.

More specifically, the different phases of the game promote forms of reasoning based on the law of excluded middle, the use of negation, material implication, and reasoning by cases (for a more detailed discussion, see [12]). The importance of fostering logical reasoning in early childhood has been highlighted by several studies, including Núñez [13], which points to a causal relationship between logical abilities and mathematical achievement in six-year-old children. Because these logical aspects are embedded within a playful and stimulating context, children can become familiar with them without their being introduced in a formal manner. We referred to this mathematics as Mathematics-in-the-Game (MiG), to distinguish it from the another type of mathematics embedded in Betta-the-Bee: the Mathematics-in-the-Deck (MiD) which refers to the mathematical potential embedded in the design of the cards themselves, independently of the reasoning strategies activated during play. In Betta-the-Bee, MiD includes basic early mathematical content such as number sense and spatial awareness, represented through variables such as the number of petals, the presence or number of butterflies, and the positions of the leaf and the caterpillar. From a mathematics education perspective, however, its main significance lies not only in these specific contents, but also in the fact that the deck presents children with multiple variables at the same time, each one represented with different possible modalities, inviting them to notice, distinguish, and attend to relevant features while playing. In this sense, the mathematical richness of the deck resides in the structured set of attributes through which children engage with comparison, categorisation, and observation in a playful context.

The MiD-MiG distinction serves an analytical purpose in this paper: MiD is certainly the most visible mathematical aspect, even to parents, but it is the discovery of MiG – through the dynamics of play supported by the rules – that enables a structured setting to be recreated at home, one that focuses adults' attention on meaningful mathematical practices.

4. Method

The study adopted an exploratory case study design, appropriate for examining a bounded educational experience in depth and for generating analytic insight [14]. In particular, it examined what happened when Betta-the-Bee, first experienced at school, was later taken into

the home by children. The study was guided by three research questions:

1. How did parents initially recognise mathematics and mathematical play in kindergarten and at home?
2. How was Betta-the-Bee re-enacted in family interaction, and what role did children play in mediating that re-enactment?
3. How did the home experience with Betta-the-Bee broaden what parents recognised as mathematical in the game?

These questions called for a design able to hold together different forms of evidence: parents' initial views of mathematics, the place of games in home life before the intervention, parents' responses after the home experience with Betta-the-Bee, and more detailed accounts of how the game was introduced and played in family settings. For this reason, the study drew on multiple data sources, each serving a complementary analytic function within the exploratory case study design.

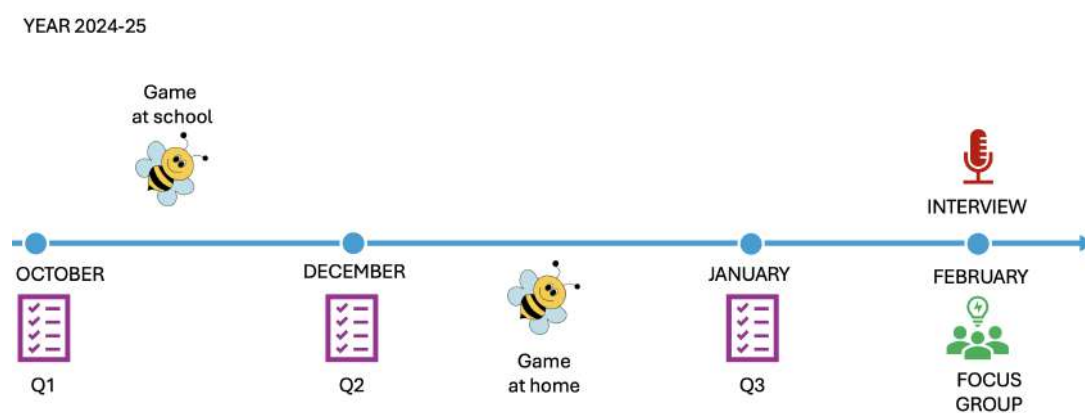


Figure 2. Timeline of the research

The study was carried out in a public kindergarten in Turin between October 2024 and February 2025. The wider school context included six sections. Within this setting, two sections were selected for the pathway centred on Betta-the-Bee. Two teachers were involved, and the core group for the home-school phase of the study consisted of 18 five-year-old children and their parents. In this setting, we collected the following data: three questionnaires for parents (Q1–Q3), one online focus group with three parents, and one semi-structured interview with the two teachers. The parent focus group was used to reconstruct the home experience with Betta-the-Bee in greater detail, whereas the teacher interview provided contextual information on the classroom work that preceded and followed the home phase.

The number of participating parents varied across the different phases of data collection. Q1 was completed by 32 parents from the six kindergarten sections, Q2 by 19 parents from the same six sections, and Q3 by 12 parents from the two sections directly involved in the Betta-the-Bee pathway. The qualitative follow-up involved three volunteer parents in one online focus group and the two kindergarten teachers in one semi-structured interview.

The research developed through a sequence that linked parents' initial views, schoolwork on the game, and the later home experience. Q1 was administered to document how parents described mathematics in kindergarten, what they expected children to learn, and how they viewed the role of games in mathematical learning. The introduction of Betta-the-Bee then took place in the two selected sections through a gradual school pathway in which children played the game, becoming familiar with the deck, the relevant distinctions represented on the cards, and the rules of the game. Before the Christmas break, parents in the two sections

completed Q2, which provided information about games already known and played at home.

During the Christmas holidays, each child took Betta-the-Bee home and was asked to teach it to parents or other adults. This was the central phase of the study, since it made it possible to examine how a mathematical game first experienced at school was later taken up in family interaction. After the holidays, Q3 was administered to the parents of the two sections to capture their views of the game, their child's participation, and the frequency with which the game had been replayed at home. The study was then completed by an online focus group with three parents and a semi-structured interview with the two teachers. These final two sources made it possible to reconstruct the home experience in greater detail and to relate it more closely to the classroom work that had preceded it. The five sources of data served a specific analytic function within the overall design of the study.

Data were analysed thematically across sources. The analysis began with an initial coding of the open responses in Q1 and Q3, the focus group transcript, and the teacher interview. In this first phase, attention was directed to the ways in which parents and teachers described mathematics, mathematical play, the rules of Betta-the-Bee, children's actions, and the home experience with the game. Recurrent expressions and ideas such as counting, logic, explaining the rules, correcting adults, noticing details, and asking questions were therefore retained as initial analytic labels.

A second phase reorganised these initial codes into broader categories aligned with the research questions. This step made it possible to distinguish, on the one hand, parents' initial views of kindergarten mathematics and home play and, on the other, the specific forms of participation and recognition that emerged after the home experience with Betta-the-Bee. The analysis gradually converged on four main claims: parents' initial numerical framing of mathematics, the presence of mathematical play in the home before Betta-the-Bee, children's role in carrying the game home, and the widening of parents' mathematical recognition through the home experience.

The questionnaire data and the data from the focus group and the interview were used in different but complementary ways. Q1 and Q2 were used mainly to identify recurring patterns in how parents described mathematics and play before the home experience. Q3 was used both descriptively and interpretively: descriptively, to make visible compact response patterns concerning parents' recognition of the game and children's participation; interpretively, to identify the themes that required deeper examination in the qualitative material. The focus group and the teacher interview were then examined for episodes in which children were described as explaining the rules, correcting adults, directing attention to relevant variables, or making the structure of the game visible within family interaction.

The findings were constructed through comparison across these sources. Short excerpts were selected as they expressed especially clearly patterns that recurred elsewhere in the dataset, while the tables included in the findings section were used to show the response distributions supporting each claim. The analysis was iterative and involved repeated discussion among the authors to refine category boundaries, test the fit between excerpts and claims, and avoid overly impressionistic interpretations.

5. Findings

The first questionnaire shows that parents do not identify all forms of early mathematics with the same ease. In our data, kindergarten mathematics was recognised above all through visible numerical content: numbers, counting, simple additions and subtractions, quantities, and sequences (see Table 1). Other domains were present, but they appeared less often and with less consistency.

The open responses show this orientation very clearly. Parents wrote "Numbers", "Counting, adding, subtracting", "Numbers from 1 to 10", "Counting at least up to 20", and "Adding

Indicators	n
Counting	27
Operations	10
Magnitudes	6
Sets	4
Logical thinking	4
Space and Geometry	3
Time	3
Grouping objects	2
Problem solving	1
Times tables	1

Note. The thematic counts are based on open responses to the question on what children should learn in mathematics at kindergarten.

Table 1. Initial parental framing of mathematics in kindergarten (Q1, n = 32)

and subtracting”. A smaller cluster referred to “Preparatory to logical-mathematical thinking”, “Logical reasoning”, “Sets”, “Grouping objects”, “Space”, or “Geometry”. Parents were therefore attributing clear educational value to mathematics in kindergarten. They were also doing so through a language strongly centred on numerical content.

Regarding parents’ awareness of school practice, only 11 of the 32 parents reported being aware of mathematical activities proposed at kindergarten, and only 6 mentioned mathematical games used at school. When examples were given, they usually referred to routines in which the numerical dimension was explicit, such as counting the children present in school or using fingers to represent quantity. This pattern is consistent with Muir’s [3] observation that parents may be supportive of school mathematics while still having only partial access to the pedagogical logic of classroom practice.

Q1 also suggests that the conditions for a home-school partnership are present, at least in part (Table 2). This is evident from the group of 19 parents who reported playing mathematical games at home with their children while not knowing which mathematical games were being used at school. This result suggests that the main obstacle is not parental unwillingness to engage, but the lack of a shared view of what happens in the classroom. In other words, many parents are already available to support mathematical activity at home, yet they do so without a clear connection to school practice. This makes them a particularly important group for the present study: they are not outside the educational process, but only partially connected to it. From this perspective, Betta-the-Bee can be seen as a promising mediating object, because it offers a concrete way of aligning home activity with school mathematics without asking parents to become teachers themselves.

Combination of parental awareness of mathematical games played at school and habit of playing mathematical games at home	n
Aware of the mathematical games proposed at school and plays mathematical games at home	6
Aware of the mathematical games proposed at school but does not play mathematical games at home	0
Not aware of the mathematical games proposed at school but plays mathematical games at home	19
Not aware of the mathematical games proposed at school and does not play mathematical games at home	7

Table 2. Parental awareness of mathematical game at school and mathematical game play at home

In Q1, 25 parents reported that they had played mathematical games with their children at home or in other settings (Table 2, rows 1 and 3). Their examples ranged from counting stairs, cars, tiles, fruit, and other everyday objects to small calculations with fingers or household materials, shop and market games, card games, matching activities, and ordering tasks. The home context already included opportunities for counting, comparing, ordering, and reasoning in action. This is very much in line with Cahoon et al. [1], who describe the home mathematics environment as a combination of spontaneous and structured experiences rather than a space of explicit instruction alone.

Q2 sharpens this picture with a closed questionnaire by showing which games were already circulating in the home and which of them parents associated with mathematical development. Table 3 summarises the games that respondents perceived as supporting the development of mathematical skills and/or competences, grouping them into broader game categories.

Game category	Games included	Mentions
Card-based games	UNO/Solo (9), Rubamazzo/Beggar-my-neighbour (4), traditional playing cards (3), Batawaf (2), Scala 40/Rummy (1), Asino/Old Maid (1)	20
Strategy games	Connect Four (5), Checkers/Draughts (5), Tic-tac-toe (3), Chess (3)	16
Dice- and counting-based games	Dice games (8), Goose Game (6), Bingo/Tombola (2), Monopoly (1)	17
Spatial and puzzle-based games	Labyrinth/Maze games (5), Tetris (3), Tangram (2), Rush Hour (1), Battleship (1)	12
Visual attention and speed games	Dobble (1)	1

Table 3. Games played at home that parents identified as having mathematical content

The combined reading of Q1 and Q2 provides a rich overall picture. In Q1, parents begin to identify some playful activities carried out at home as relevant to children's mathematical learning. In Q2, however, when they are presented with a list of games and asked to select those they perceive as having the strongest mathematical content, their responses shift more clearly towards highly structured games characterised by explicit rules, such as card games, strategy games, and dice games. This pattern suggests that parents do recognise the mathematical potential of such activities, even when they are embedded in playful contexts rather than overtly framed as mathematics. At the same time, it shows that this recognition becomes more explicit when parents are guided by a structured closed-ended questionnaire (Q2), whereas it remains less readily accessible in response to a more open-ended prompt (Q1).

Through this background we can assume that the Betta-the-Bee game entered a setting in which rule-based play was already familiar and often appreciated. The home experience with Betta-the-Bee developed from an asymmetrical starting point. By the time the game reached the family, children had already encountered and played it at school, where they had learned its materials, its rules, and the form of attention it requires. Home play therefore began from prior school experience, and children entered the family setting as participants who already knew how the game worked.

Q3 revisits the themes that emerged from parents' open-ended responses in Q1 (Table 1), in which they described the mathematical content they believed should be learned at kindergarten, by asking them whether they recognised these same mathematical aspects in the game Betta-the-Bee.

The responses reported and grouped in Table 4 show a clear pattern in parents' perceptions of the mathematical potential of Betta-the-Bee. The contents most strongly recognised are logical thinking, followed by counting, space and geometry, time, and grouping. By contrast,

More strongly recognised mathematical contents	Occurrences (“very useful”)	Less strongly recognised mathematical contents	Occurrences (“very useful”)
Logical thinking	10/12	Problem solving	6/12
Counting	9/12	Operations	6/12
Space and geometry	8/12	Measures	5/12
Time	8/12	Sets	4/12
Grouping	8/12	Times tables	3/12

Table 4. Mathematical contents more strongly recognized in Betta-the-Bee

times tables, sets, and measures are less frequently identified, while operations and problem solving occupy an intermediate position. Overall, these results suggest that parents primarily associate the game with early forms of mathematical thinking related to logic, classification, and spatial organisation, rather than with more formal or school-based mathematical contents. In other words, parents began to recognise Mathematics-in-the-Game, alongside Mathematics-in-the-Deck, as a valued form of mathematics enacted in kindergarten.

Game process	Occurrences (“very much at ease”)
Explaining the rules	8/12
Remembering where Betta is hidden	10/12
Formulating questions	8/12
Answering questions	9/12
Discarding the cards	9/12

Table 5. Parents’ perceptions of children’s ease with the main game processes in Betta-the-Bee

The responses reported in Table 5 describe children who were already at ease with the structure of the game. Parents generally saw children as comfortable in its practical and verbal dimensions: explaining the rules, managing the phases of play, and keeping the interaction going. The focus group brings what emerged from Table 5 stronger into view: one parent described her daughter as having “asked often” to play Betta-the-Bee, “explaining the rules to different family members”, and added that “the explanation is done by her independently”. Another parent said: “she explained the rules very well”, and after checking the written rules he found that “there was nothing different from what she had explained”. A third parent said that her daughter had explained the rules “in a precise way” and kept proposing the game to “us, friends, or grandparents”. These excerpts show children taking responsibility for starting the activity and shaping it within family interaction. The teacher interview supports this reading from the school side. One teacher described the classroom work on Betta-the-Bee as gradual and underlined how much the children had enjoyed the moment when “we wrote the rules for the parents”. The other reported that the children later “explained very well how to play” even to her. The home competence described by parents grew out of prior classroom work in which the rules, the vocabulary of the game, and its relevant distinctions had already been stabilised.

From the focus group it emerged also that children regulated the game once it had started. One parent said he had been struck by the child’s “precision in telling the rules” and by her “attention to details”, adding that “every now and then I was wrong and I was corrected”. Another recalled her daughter interrupting her with “No, Mum! It isn’t like you said, it’s different”. The teacher interview contains the most compact example of this reversal of expertise: one child reportedly said that “Dad didn’t understand the rules” and that she had had to explain them “many times”. More generally, children explained the rules, restarted the game when needed, corrected adult errors, and kept play aligned with what counted as proper play.

This children's attitude came from the more informed position they have towards the game, since they carried home a rule-governed mathematical practice already established at school. The continuity documented in this study passed through children's capacity to transport that practice into family life and make it workable there.

6. Discussion

The findings allow us to answer the three research questions in a direct way.

Regarding RQ1 – *How did parents initially recognise mathematics and mathematical play in kindergarten and at home?* – the study shows that parents initially recognised mathematics above all through its most visible school-like forms, especially counting and simple operations. Other domains, such as logic, grouping, or spatial reasoning, were present but much less salient in their accounts. At the same time, the data also show that mathematical play was already part of family life before *Betta-the-Bee* was introduced at home. Parents reported a fairly rich repertoire of games and everyday activities involving counting, comparison, ordering, and strategic action. The point, then, is not that mathematics was absent from the home, but that its recognition was selective. Mathematics was more readily identified when it appeared in explicit numerical form, while other mathematically relevant dimensions were less immediately named. This finding is aligned with the literature [3], highlighting that within the involved parents the poor continuity between school mathematics and home mathematics was not the result of lack of parental interest or lack of playful activity, but a partial overlap between what parents recognised as mathematics and the broader range of mathematical activity valued in kindergarten.

Regarding RQ2 – *How was Betta-the-Bee re-enacted in family interaction, and what role did children play in mediating that re-enactment?* – the findings indicate that the home re-enactment of the game was strongly mediated by children. By the time *Betta-the-Bee* reached the home, children had already encountered its materials, rules, and relevant distinctions at school. They therefore entered family interaction from a position of prior familiarity and of greater expertise, in some sense reversing the roles in the family interaction, since it is the child that knows more about a specific topic, while usually it is the parent that has the role of expert. Parents' accounts consistently describe children as introducing the game, explaining the rules, proposing it repeatedly, correcting adult misunderstandings, and keeping play aligned with what counted as proper play. The teacher interview supports this interpretation by showing that this competence did not emerge spontaneously at home, but had been prepared through gradual classroom work on the game. The continuity observed here therefore depended on more than the portability of the material. What children carried home was a rule-governed practice already socialised at school. In this sense, children were not simply participating in a home-school connection organised by adults; they were actively making that connection workable within family interaction.

Regarding RQ3 – *How did the home experience with Betta-the-Bee broaden what parents recognised as mathematical in the game?* – the findings suggest a clear shift in the direction of broader mathematical recognition. After the home experience, parents no longer referred mainly to visible numerical content, but more often identified logical thinking, grouping, spatial organisation, and the role of questioning in the game. This does not mean that parents developed a formal or technical discourse about early mathematics. Rather, the threshold of what became noticeable as mathematics appears to have widened. Through play, parents were drawn not only to the mathematical features visible in the cards, but also to the mathematical organisation generated by the rules: asking informative yes/no questions, attending to relevant variables, interpreting answers, and discarding possibilities. The distinction between *Mathematics-in-the-Deck* and *Mathematics-in-the-Game* helps clarify this point. The home experience seems to have shifted attention from the former alone to a recognition of the latter

as well. This shows that continuity did not concern only the reuse of the same game across settings, but also a broader parental recognition of what counted as mathematical within that activity.

These answers suggest that the educational value of Betta-the-Bee lay not simply in being a game that could travel from school to home, but in being a mathematically structured activity that children could re-enact with others. While large-scale intervention studies have shown the value of adding family mathematics components to classroom-based game interventions [6], the present case study contributes by making visible a possible interactional mechanism through which such continuity may become active: children's mediation of a rule-governed mathematical practice within family life. Continuity was not sustained only through parents' willingness to participate, nor only through the provision of materials, but through children's mediation of a school-shaped mathematical practice within family life. These findings should be treated with a degree of caution, as it must be borne in mind that whilst Betta-the-Bee was indeed carried home by the children, bringing with her a certain perspective on mathematics, the groundwork for this experience had been laid by the teachers and researchers, who had framed the research as a study of children's games with mathematical content. It is therefore possible that this shift in perception, as revealed by the analysis of the children's expertise in play, is underpinned by the awareness raised among parents by teachers and researchers.

7. Conclusions

This study set out to examine three issues: how parents initially recognised mathematics and mathematical play, how Betta-the-Bee was re-enacted at home, and how that experience broadened parental recognition of mathematics. The findings allow a direct answer to each of them.

First, parents initially recognised mathematics mainly through visible numerical content, even though mathematical play was already present in family life through everyday activities and rule-based games. Second, the home re-enactment of Betta-the-Bee was largely mediated by children, who introduced the game, explained and regulated its rules, and sustained it as proper play in family interaction. Third, the home experience broadened what parents recognised as mathematical in the game, shifting attention from visible content alone towards logic, grouping, spatial distinctions, and the strategic role of questioning.

The main contribution of the study lies in the relation between these three findings. What the case makes visible is a specific form of home-school continuity in early mathematics: a mathematically rich activity, first structured at school, can be carried into the home through children's own mediation. In this case, what moved across settings was not only a shared material, but a way of attending, asking, interpreting, and deciding mathematically within play.

The study therefore suggests that children may play a more active role in home-school continuity than is often foregrounded in the literature. Here, they did not simply participate in a connection established by adults but acted as mediators of a rule-governed mathematical practice. This does not reduce the importance of parents or teachers; rather, it shows that continuity may be sustained through a temporary redistribution of expertise within the activity.

These conclusions remain exploratory. The study involved only one kindergarten and a small qualitative post-home dataset. Its value is therefore not statistical generalisation, but the analytic visibility it gives to a specific mechanism. The case suggests that home-school continuity in early mathematics may be strengthened when children are first given enough school experience to appropriate a mathematically structured activity and then are able to re-enact that activity meaningfully in family life. Future research could examine this mechanism in larger samples, in different socio-cultural contexts, and in other beyond-the-classroom environments, including outdoor or community-based settings.

Conflict of interest

The authors declare that there are no conflicts of interest.

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Mortino and the Blue Pearl Heart: a structured-narrative journey through pupils' attitude in mathematics

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Abstract: This paper is situated within the field of research on how mathematics activities embedded in a storytelling setting may influence attitude towards mathematics, in primary school children. It is widely recognized in the literature that the affective dimension has a significant role in mathematical learning, especially when the subject-related prejudices may induce pupils' anxiety. From early schooling, a mnemonic, rigid vision of mathematics risks establishing a didactical contract detached from a critical and relational approach to the discipline. This paper proposes the analysis of an experiment based on an original story, that integrates a triadic narrative arc, story-based problems and gamification dynamics, in order to promote an encouraging and supportive learning environment in a 2nd grade classroom. Grounded in the three-dimensional model of attitude, we investigate how storytelling and gamification might facilitate a positive perception of mathematics for pupils.

2020 Mathematics Subject Classification: Primary 97C70; Secondary 97K50, 97C20.

Keywords: storytelling; gamification; attitude towards mathematics.

1. Introduction

Research in mathematics education has consistently shown that students often perceive the discipline as difficult and problematic, leading them to develop a widespread negative attitude toward the subject [28]. In this regard, the instructional methods teachers use to introduce mathematics to students play a crucial role. The early school years are highly formative for pupils, helping promote a conception that is both positive and stimulating. Mathematics is often perceived as a set of mnemonic rules and abstract concepts far removed from children's daily lives [27]. Such a restricted perspective on mathematics could reinforce misconceptions that, in turn, generate further difficulties. Continuous frustrating experiences might induce such emotional resistance as to lead pupils to develop anxiety [28]. Observing students' attitude towards mathematics is important for understanding what they feel, what they think of

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the discipline and the confidence in their own abilities [9]. To address these prejudices, the present research proposed an experiment based on storytelling in a 2nd-grade class through *Mortino e il Cuore di Perla Blu* (in English: *Mortino and the Blue Pearl Heart*), an original story written by the second author of the paper. The storytelling, inspired by narrative structures commonly used in film studies, was used to design a mathematical learning path aimed at supporting pupils' emotional involvement. In addition, game dynamics were integrated to encourage engagement with the discipline and improve the affective bond between pupils and the main character. Among the teaching tools, the story-based problems proposed by Zan [26] have been adopted to promote an image of mathematics as a personal challenge, fostering pupils' inferential skills and strengthening their own self-confidence in the learning process. Within mathematics education research, the affective domain and, more specifically, the investigation of attitude, has become central to understanding how students approach mathematical activities and construct meaning through them. Nevertheless, the present research suggests that storytelling, gamification and mathematics are often framed as separate strands or used as occasional complementary elements in individual lessons, rather than being integrated within a unified structure over long-term instructional practice. In such a structure, the three elements can be seen as an interconnected system throughout the whole learning experience: the establishment of meaningful connections between the character and the pupils requires time and sustained practice, enabling children to gradually adapt to the educational experience and internalize it as their own. Specifically, narrative structures inspired by cinematic storytelling, together with engagement fostered by gamification, remain relatively underexplored as elements supporting mathematical learning experiences. More specifically, this contribution seeks to examine how such a pedagogical approach may affect the multidimensional nature of pupils' attitude towards mathematics. Beyond their role as pedagogical strategies, in this study storytelling and gamification are conceived as interrelated processes supporting affective participation and meaning-making in mathematics, through identification with the protagonist and involvement in goal-based tasks. Identification with *Mortino* enhances the motivational effects of gamification by increasing a sense of responsibility for participation, while game dynamics reinforce the continuity of narrative engagement. The study therefore addresses the following research question: *how does a gamified narrative structure inspired by cinematic storytelling and narrative-based problem solving influence pupils' attitude towards mathematics?*

2. Literature review

Within educational research in mathematics, attitude is regarded as an essential component of the affective domain, as it influences individuals' decision-making processes and, consequently, shapes their behaviour towards a given subject [2]. Such an attitude is primarily formed through pupils' school experiences and may be influenced by teaching practices and educational contexts [16]. Currently, mathematics learning is often perceived by students as largely rote-based or abstract, preventing them from relating it to real-life situations and reducing the discipline to a mere set of rules [6]. In recent years, however, storytelling has enabled pupils to engage with mathematical activities in a more meaningful and immersive way through the creation of characters and storyworlds closely connected to their own reality [19]. Through the identification process encouraged by a story, children are guided towards the discovery of their emotions and sense of self: narratives suggest that the goals to which pupils aspire can only be achieved by courageously confronting challenges and difficulties involved in the construction of their identity [4]. One of the most widely consumed storytelling forms today can be found in cinematography, owing to its ability to capture and sustain attention over extended periods of time [8]. Visual storytelling in mathematics education promotes the contextualisation of disciplinary concepts by reconstructing a narrative universe responsive to students' educational

needs [21]. It also contributes to developing problem-solving skills [1], while enhancing motivation and engagement process, which are particularly beneficial in contexts where anxiety is likely to surface [14]. Alongside stories, gamification plays a significant role, specifically when associated with pupils' involvement [21]. The creation of a gamified design is based on the use of structured levels, namely progressive challenges that sustain engagement and guide users towards the final goal without inducing cognitive overload [15]. To further strengthen the motivational process, virtual or tangible rewards are employed in order to reinforce pupils' sense of autonomy and progressive advancement within the game. These elements must be carefully balanced to avoid the loss of user interest before the gaming experience is completed.

2.1. Theoretical background

It is well established within the mathematics education literature that the affective domain plays a crucial role in the teaching and learning processes of students, significantly influencing their performance and engagement [18, 29]. In particular, researches confirm how attitude towards mathematics is a relevant component in the affective domain [2]. Extensive research has highlighted how mathematics anxiety and students' beliefs about the nature of the discipline can act as powerful mediators, often shaping their persistence and cognitive approach to problem-solving [23]. Such negative preconceptions often lead to an increasingly complex relationship with the discipline; if consistently experienced throughout their educational path, these feelings can escalate into what has been defined as *Mathophobia* [28, p. 13]. This specific term is the fear of practising mathematics, but it is also the emotional resistance to approach it. As discussed in the Literature review section, storytelling primarily operates at the cognitive and emotional levels through identification with a character, thereby influencing the meaning that pupils attribute to their experience of mathematics [4]. Gamification, by contrast, has a stronger impact on the behavioural and motivational dimensions through reinforcement mechanisms that provide pupils with a sense of progress and an increased perception of control over activities [21]. Within the affective dimension, any anxiety that may emerge during a task is likewise not a unidimensional phenomenon, but rather involves these same cognitive, emotional, and behavioural aspects [12]. The *Three-dimensional Model of Attitude* (TMA) proposed by Di Martino and Zan [9] (Figure 1) captures the multidimensional nature of attitude towards mathematics, relevant in the context of an experiment combining multiple aspects related to both storytelling and gamification.

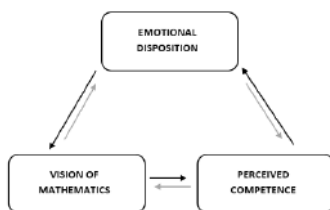


Figure 1. Three-dimensional model of attitude toward mathematics. Adapted from Di Martino and Zan [9].

The TMA model consists of the following dimensions:

- *Emotional Disposition* refers to how we feel when we think about mathematics.
- *Vision of Mathematics* represents what pupils think of the discipline and what characteristics they attribute to it.
- *Perceived Competence* is the individual's perception of being good (or not) when tackling a mathematical task.

Although these are three separate elements, they are closely interconnected, and this must be

considered in the teacher's daily classroom practice. Indeed, the emotional dimension significantly influences a student's vision of mathematics. Research shows as in many cases this element is strongly characterised by fear of failure, anxiety, and a strong sense of frustration when students face a problem to solve. The vision of mathematics is influenced by students' past experiences and has repercussions on their future behaviour and their emotions [9]. Telling a story in mathematics makes concepts easier to understand and remember because it has a strong capacity to connect with our strongest emotions [30]. The ability of narrative to make human experience universal can help students rehabilitate that sterile side of mathematics [5]. As discussed in the Literature review, the present research drew upon one of the most effective tools currently employed for capturing and sustaining pupils' attention, namely visual storytelling in the cinematography field [8]. Popular movies often adopt recurrent narrative structures that may support viewers' emotional engagement and interest throughout the story [24]. The following narrative guidelines were considered to support pupils' engagement: 1) storytelling is expected to present a narrative featuring at least one clearly recognisable main protagonist, characterised through identifiable physical or emotional traits, and driven by a clear narrative goal to be achieved [27]; 2) storytelling is characterised by change: characters are not static entities and, in order to teach us something, they must challenge themselves [25]; 3) storytelling is characterised by a solid narrative structure. The three-act structure proposed by screenwriter Syd Field [13] serves as a classic narrative framework to organise a progression of a story [25]. Act I introduces a character's daily life and its subsequent disruption, portraying the protagonist in his everyday environment; this initial phase is particularly useful to establish an immediate connection with the readers' reality by depicting familiar experiences, such as moments of play, affection from loved ones, or small daily difficulties. Act II describes the development of a strategy to overcome adversity and represents the phase where the narrative action intensifies: although the main character faces a major setback at the end of the first act, often losing what is most precious to him, he is continuously presented with opportunities for growth and positive change. Finally, Act III resolves the narrative arc as the character achieves his goal, overcomes the remaining obstacles, retrieves what was lost, and ultimately restores peace and balance to the storyworld; 4) at Act transitions, stories often include turning points, namely transformative moments that altogether produce significant emotional shifts in the viewer, moving from an initial phase of peace (or equilibrium) to an intermediate phase of disruption and a subsequent return to stability [8]. The narrative structure typically presents a main character living a peaceful life, until an unexpected event disrupts the storyworld (first turning point). The protagonist is then defined by the pursuit of a narrative goal despite an initial defeat (second turning point). Finally, the character emerges as a hero capable of achieving success despite fears, ultimately establishing a renewed equilibrium (third turning point). In mathematics education research, there are specific types of texts that may share common features with narrative acts and that could have a significant impact on students' emotional dimension. In the *C&D Model for the analysis (and reformulation) of text* [26], a narrative-structured problem has its body introducing a sequence of events, followed by a problem-situation just as in Act I. The problem's question invites students to analyse what they have to seek a resolution, as in Act II. Pupils provides an answer, as the protagonist does with his goal in Act III. Therefore it is possible to find elements of similarity between narrative problems and the Acts structure, with the problem context corresponding to Acts I–II and the problem question to Act III. According to Zan [27], a story must include at least one character (whether real or fictional) who pursues a goal over time starting from an initial problem situation. In Acts I and II, a context is established that closely resembles that of a narrative problem: a main character is introduced within an ordinary world that is subsequently disrupted by an unusual and distressing event, thereby creating a situation that calls for resolution. This context also provides relevant logical information, just as Act II presents

a sequence of events in which the protagonist evaluates the tools and resources available, using them to make decisions to resolve the situation. Zan [27] and Virzì [25] also identify the following crucial elements as shared between narrative texts and the three-act structure:

- *temporal progression*, as a linear succession of events;
- *verisimilitude*, which makes the story credible, based on what the reader knows about the world;
- *causality*, which explains why the characters act. If there is a clear goal perceived as true by the protagonist, the student stops seeing the mathematical problem as external and feels it as a personal challenge.

Within the Act III, the problem question finds its resolution. Both share a problem-solving process involving choices and strategies to be evaluated in order to achieve a goal. Zan [26] argues that if teachers intend to elicit an action from a pupil (such as resolving a problem situation), it is necessary to communicate explicitly a request that is meaningful to students. In Act III, the story provides the answers necessary to overcome the problem and determine whether the narrative goal has been achieved. Both involve a problem-solving process based on evaluating choices and strategies. To strengthen the connection between mathematics and game dynamics, different aspects of gamification are considered [22]. Shallow gamification uses physical or intangible rewards (such as points or leaderboards) to capture attention [20]. However, extrinsic motivation may fade quickly in the absence of intrinsic motivation, understood as the learner's own willingness to engage. Mozelius [20] instead refers to deep gamification, characterised by game dynamics that create an affective dimension and a stronger bond between the fictional character and the students. Both types of engagement are necessary, as they make mathematics enjoyable and interactive in the short term, and more humanised over time [7].

3. Methods

3.1. Participants

The research *Mortino and the Blue Pearl Heart* was conducted during the 2024-2025 academic year in a 2nd-grade class of 21 participants, all aged 7. The activities were carried out in collaboration with the class teacher, whose role was as an observer. The second author, a trainee teacher who has been involved with the class since the first year of primary school, acted as both the designer and the facilitator responsible for conducting all classroom activities throughout the intervention. The mathematical core of the investigation included the two fundamental operations of the additive structures, in coherence with the teacher's instructional planning. The prerequisites defined in the documentation included mastery of natural numbers beyond 100 and knowledge of the properties of the arithmetic operations studied. The educational objective for the class was therefore to strengthen pupils' mastery of additive structures. In addition, a second objective that more specifically shaped the research design was the use of gamified storytelling as an innovative way to direct attention and foster engagement, contrasting with practices already familiar to the pupils, and to examine its effects on students' attitudes towards mathematics.

3.2. The design

The research was structured over a 6-week planning period focused on classroom observation and the collection of information to support the development of gamified storytelling, followed by a 6-week period of active experimentation with the pupils. The design was conceived according to a layered and interdependent structure, in which the narrative serves as the outer shell, the gamification dynamics constitute the intermediate level, and mathematics represents

the core of the overall experience. The story provides a unifying narrative thread and serves as the initiating mechanism for the gameplay phases; these, in turn, include mathematical challenges that allow the narrative to progress. Their introduction is also deliberately gradual. Through identification with the protagonist, storytelling acts as the main driver of a different way in which pupils perceive their participation in the activities: it is essential that students feel a voluntary need to use additive structures, not because they are prescribed by the school curriculum, but in order to help the character.

In the second author's story, several narrative design principles were incorporated, including a basic story aligned with pupils' interests, its structural division into Acts describing the temporal progression, and turning points conceived as narrative pivots, used to encourage pupils' emotional involvement in the narrative. The story introduces Mortino, a little spirit king with a great Blue Pearl heart who lives in the magical *Bosco di Cuorgentile* (*Kindheart Forest*). He loves flying freely, playing within nature, and spending time with those he cares about and his heart is always radiant. His peaceful life continues undisturbed until mysterious Black Shadows emerge from the depths of the earth to consume all the light in the world, plunging the forest into coldness and fear. Mortino's precious heart is stolen, the only light capable of preserving the magic of the realm. Frightened yet determined not to give up, he embarks upon a journey, asking the pupils for help in saving his home. To support this narrative, the present study introduced gameplay dynamics in the form of actions that pupils must carry out, with the problems themselves constituting the tools that enable these actions. Storytelling and gamification worked together to create a strong emotional bond between pupils and mathematics. This connection between the students and the storyworld was fostered by enabling a familiar visual identification with Mortino from the outset. Indeed, the research began with an aesthetic study of the narrative universe, based on the participants' graphic productions. The character and his storyworld evolved visually to reflect difficulties and changes throughout the narrative arc. To follow Mortino's adventures with greater engagement, a book-style digital product was created: using PowerPoint software, images that had previously been hand-drawn by the second author were then digitized on the platform. Each image, depicting the most significant scenes of the story, was also accompanied by written text that followed the narrative, as shown in Figure 2.



Figure 2. Extract from the digital book.

Subsequently, the gamified narrative structure was designed according to 3 Phases. In Phase 1, the first activity with pupils was the individual game *What does maths taste like?*, inspired by those reported in [10]. Its educational aim was to establish a bond of trust between pupils and the discipline through a playful task that associated a food (liked or disliked) with maths. The rationale behind this initial task was to enable the observation of pupils' emotional disposition towards mathematics through a playful approach that departs significantly from a more structured assessment context. The activity lasted 15 minutes and aimed to connect mathematics to personal experiences in a fun and creative way. Thereafter, a 30-minute individual questionnaire was administered to the participants, consisting of two semi-structured

questions: “*Maths is pleasing when*” and “*Maths is not pleasing when*” . Students were left the freedom to respond orally to encourage more spontaneous verbalisations. The second activity, by contrast, shifted the focus from merely monitoring pupils’ emotional disposition to exploring their perceived competence. Within the research design, this activity was intentionally positioned after the playful task in order to progressively introduce a more reflective perspective, alongside the previous positive experience, on pupils’ relationship with mathematics, including possible negative attitude. The main objective was to foster an engaging environment that would encourage pupils to express themselves more freely after the game-based activity. It also aimed at investigating how their perceived competence may vary across both more pleasant and more challenging experiences with mathematics. At the end of Phase 1, all the participants were subjected to an individual questionnaire lasting 15 minutes. The first input was structured consisting of a multiple-choice question (one positive and one negative emoticon); the second input was semi-structured, to detect the reasons for the previous choice. The final task reintroduced a multiple-choice format to monitor pupils’ emotional responses. However, the concluding item was designed to go beyond identifying whether pupils sometimes feel uneasy with mathematics, by asking them to explain the reasons behind such feelings. This enabled the study to capture potential changes in both emotional disposition and perceived competence when moving from a game-based context to a more structured task, which pupils often associate with less enjoyable aspects of mathematics.

Phase 2 involved narrating Act I of the story to the pupils, with mathematical problem-situations that unfolded alongside the narrative progression. Starting from an initial equilibrium, the story introduced the breakdown of the equilibrium that formed the basis for the narrative problems. Accordingly, this Phase also included 2 distinct Tasks designed to follow the development of the plot. Within this design, additive structures were embedded as the functional requirements needed to resolve the character’s situation. These two activities shared the same structural design, comprising a problem followed by a subsequent discussion of the solution. This methodological choice was intended to foster multiple experiences with narrative problems, while simultaneously providing a sense of reassurance through the application of previously tested procedures. Task 1 consisted of a narrative problem administered to 4 cooperative groups. Within this design, pupils worked without predefined roles, and the task was structured to require positive interdependence. Each group had to engage with the problem in 30 minutes, being granted maximum freedom in the written formalisation of calculation procedures, with the request to document any divergent solving strategies among members of the same group. The problem was prepared by trying to avoid keywords that typically indicate which operation should be used, as shown in the text below:

Mortino arrives at the Grotta Brulla, abandoned for a very long time. The forest is dark without its colors, and strange noises can be heard among the leaves. The little spirit trembles because the village elders used to say in their stories that the Black Shadows were hidden there. To lift the heavy door, Mortino must follow the instructions on the large sign from the Stone-Diggers and find a total of 268 stones. “Where will I ever find so many stones? And besides, I’m s-s-scared and w-what... what is that?” Mortino cries out when something moves the bushes. Frightened, he starts running through the forest but trips and - poof! - falls into a lake. He is about to cry, angry, when he notices that the water is shining brightly. What a surprise when Mortino sees the many glowing stones at the bottom. Overjoyed, he uses magic to weave a basket and carry it. When he returns to the Grotta Brulla, he realizes too late that some stones have rolled away. Mortino counts them and only 137 are left. He must go back and retrieve the lost stones before night falls. “What a long road ahead, again!!! And how exhausting to have to count again!!! I really need someone to help me,” said the little spirit, very dejected. Can you help Mortino count how many stones he must recover to unlock the door?.

A collective solution discussion [3] was conducted by the second author for the verbalisation of the strategies used and the justification of choices made, in 30 minutes. This educational tool was valuable not only for its capacity to foster critical thinking regarding the choices made by students, but also for detecting the effects that storytelling had produced on the pupils' verbalisation concerning their vision of mathematics. The discussion was structured as follows:

- shared re-reading of the text of the problem;
- verification of the understanding of the problem-situation and its relative requests;
- verbalisation of the processes and solving strategies adopted;
- introduction of teacher's guiding questions inputs aimed at generalising the procedure;
- evaluation of the coherence between the solution provided by the participants and the question posed by the problem, with relative argumentative justification.

The students called into question their traditional practices. The objective of the activity was to observe the influence on the students' vision of mathematics in the presence of activities differing from traditional routines. At the end of Task 1, an individual semi-structured questionnaire was administered in 15 minutes to evaluate pupils' emotional disposition and motivation before and after the activity. The questionnaire was divided into three sections. The first two sections provided two multiple-choice questions to monitor the evolution of the emotional state: the first one referred to the moment before the problem solving phase; the second one related to the post-solving and solution discussion phase; the last section included an open-ended question to detect spontaneous verbalisation regarding individual motivation about the previous selection.

Task 2 replicated the design developed for Task 1 in its problem-solving task and collective solution discussion, using a new narrative text to continue the story as shown below.

Mortino slowly enters the Grotta Brulla; it is very deep and dark. He can hear the sound of his footsteps and his breathing is very fast. How scary! He looks to the right, then turns to the left, but no Dark Shadow can be seen anywhere. Mortino is still alone. He takes a deep breath and thinks that soon he will be back home, with his sweet little spirit Monsi-Lillà. Mortino takes a closer look at the cave and notices that there are 122 lit torches along the walls, but they are not enough to illuminate the entire rocky corridor. Mortino doesn't want to walk in the dark; he is too scared. Other unlit torches have fallen to the ground. Mortino picks up some of them from the dusty floor and knows that at most he could light 150 torches, no more, otherwise he would run out of all his energy. The magic is almost gone, and because of this, Mortino feels a bit dizzy and not very well. The little spirit leans tiredly against the wall and says, "What a mess, I can't count. What should I do? I need help!". Can you help Mortino count how many more torches he needs to light to illuminate the dark corridor?.

To fuel engagement, for every mathematical challenge solved and discussed in Task 1 and 2, the students received a collective bonus: a puzzle piece which, once assembled, would reveal the identity of Mortino's enemy (Figure 3).

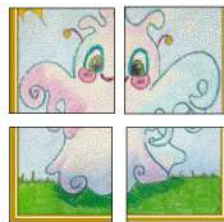


Figure 3. Puzzle reward tiles.

Rather than a dark shadow, as initially suggested by the digital book, the assembled figure

reveals a candid spirit. Within the gamified design, this mechanism was employed to further heighten curiosity regarding the mysterious identity of the true culprit behind the chaos in the Forest, then revealed in the next stage.

Phase 3 involved narrating the end of the story through the digital book. After addressing the mathematical challenges and finally restoring Mortino's heart, the experiment highlighted a significant turning point, described as the moment in which pupils emerge as heroes of the forest's destiny. The semi-structured questionnaire, administered at the end of the intervention, aimed to capture how this moment influenced pupils' emotional disposition and perceived competence. Participants completed the individual questionnaire in 15 minutes. In two multiple-questions, participants had to select one of four emoticons, based on a gradient of intensity from negative to positive: distress, insecurity, ease, and enthusiasm. The final section included an open-ended question designed to capture spontaneous verbalizations regarding participants' individual motivation to continue supporting Mortino.

For the creation of this mathematical-narrative structure, as shown in the image below, the Acts of the narrative arc and the three main Phases were paired according to the following points (Figure 4).

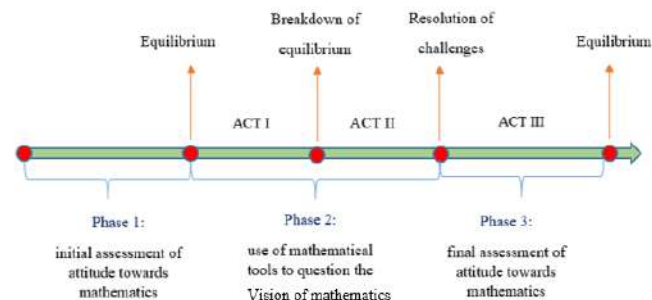


Figure 4 - Sequential integration between Acts and Phases

Figure 4. Sequential integration between Acts and Phases.

To comprehend the overall architecture of the intervention, Figure 4 outlines the sequential integration between narrative dynamics and research phases. Furthermore, it demonstrates how to define the transition between states of equilibrium and moments of crisis within a story arc divided into Acts and gameplay layers.

In Act I, Mortino was presented to the pupils in a situation of calm within his world. This segment begins from an initial state of stability, which frames Phase 1. This phase establishes the students' own initial equilibrium, assessing their emotional and cognitive attitudes toward mathematics before the narrative dynamic changes. At the end of this Act, the story broke this balance to request the direct intervention of the pupils to save Mortino. The disruption of the narrative equilibrium serves to introduce the subsequent mathematical structure

In Act II, Mortino used his skills for chasing his enemy, just as the pupils utilised problem-solving to unlock new clues. This segment represents the core of the project, positioned between Act II (the breakdown of equilibrium) and Phase 2 (the use of narrative problems). In narrative terms, it corresponds to the moment of conflict; from a pedagogical perspective, it is the point at which pupils begin to question their view of mathematics.

In Act III, the narrative structure concluded the journey while the mathematical structure provided solutions to the problem-situation. Once the narrative crisis has been resolved, the third segment marks the beginning of the downward trajectory towards the resolution of the challenges that emerged. This part of the structure (Act III and Phase 3) is dedicated to re-

assembling the overall storytelling framework. The focus shifts to the final assessment of pupils' attitudes towards the discipline, a necessary step to evaluate the impact of the intervention. The narrative process reaches a renewed state of equilibrium. However, this is not a return to the initial state: the visual milestone indicates an evolved form of equilibrium, in which students' attitudes are redefined compared to the first segment.

3.3. Data collection and analysis

Qualitative content analysis, grounded in the Three-Dimensional Model of Attitude [9], was employed to examine the research data through a deductive approach [17]. TMA dimensions are useful for an in-depth examination of how a gamified narrative structure may influence pupils' attitude, particularly in relation to experimental activities where the challenging of established routines may be associated with anxiety-related experiences. The findings were organised according to three predefined categories, coming from TMA model:

1. emotional disposition was assessed by identifying references to pupils' feelings towards mathematics. The analysis distinguished between positive, negative, and ambivalent emotional expressions, as well as shifts in emotional responses across the different phases of the study, drawing on evidence from emoticon selections and written statements completed before and after the tasks;
2. perceived competence was analysed through statements referring to pupils' self-perception in solving mathematical tasks, distinguishing between expressions of confidence and autonomy and of uncertainty especially in relation to their willingness to support the main character after the tasks;
3. vision of mathematics was coded through pupils' explanations and justifications of solutions, distinguishing between procedural reasoning based on task routines and contextual or narrative reasoning linked to the story-based problems.

The data were collected across three phases. Phase 1 involved pupils' individual associations between mathematics and food, recorded on an interactive whiteboard. These data provided initial material for coding emotional disposition towards mathematics in a playful context. In the subsequent activity, the second author transcribed pupils' oral responses based on field notes. Statements expressing liking or disliking of mathematics were then systematically identified and included in the data. This was followed by the collection of individual written protocols. Emoticon selections from a multiple-choice questionnaire were used as indicators to track emotional responses in both playful and non-playful contexts. Open-ended written responses were subjected to qualitative content analysis in order to code expressions of emotional disposition towards mathematics, with particular attention to lexical choices in pupils' productions. Coding also captured evaluative language (positive or negative) associated with the disruption of the storyworld introduced in Act I, which was suggestive of emotional disposition towards mathematics following a key turning point. In Phase 2, Task 1 involved the collection of worksheets containing pupils' written productions in order to investigate the impact of gamified storytelling on pupils' vision of mathematics. The worksheets were reviewed to identify procedural routines in pupils' solution strategies. These analyses formed the basis for Task 2, allowing an exploration of whether such routines remained stable, changed, or were absent. The second author reconstructed and transcribed field notes from group discussions. While worksheet data focused on procedural routines, individual responses were inspected to capture whether pupils established connections between mathematical reasoning and the narrative context. Individual written protocols were then collected. Emoticon selections from two multiple-choice items were used as indicators of emotional disposition across phases of the problem-solving activity. Written statements were interpreted to capture perceived competence in terms of confidence, difficulty, and willingness to engage with mathematical tasks

and support the narrative character. In Phase 3, individual protocols were collected after the conclusion of the narrative. Pupils' selections among the four emoticons were recorded to assess changes in their emotional disposition. These data refer to a significant moment following the final turning point of the storytelling and relate to the effect of the narrative resolution on students' emotional disposition. Open-ended responses were analysed to identify expressions of perceived competence, focusing on whether pupils still supported Mortino. Examination addressed references to responsibility towards the storyworld and lexical choices indicating capability to engage with the mathematical tasks and support the narrative character. This reflected sustained engagement with the narrative task and pupils' perceived ability to assist the character despite emotional or cognitive difficulties during the activities.

4. Results

The present section provides evidence intended to examine in relation to our theoretical framework the effects that the narrative-gamified structure has had on pupils' attitude towards mathematics.

4.1. Emotional Disposition

The emotional disposition was examined in relation to pupils' reported emotional responses to the learning experience. At each phase, the study documented pupils' selected emotional states and their responses before and after the different activities. In Phase 1, in *What does the math taste like*, all participants associated mathematics with terms that only referred to their own favourite foods such as pizza, lasagna, hamburger, ice-cream or fried fish. In this initial activity, responses indicated a generally positive emotional disposition across the class. In the subsequent semi-structured protocol, despite the request to select only one of the opposing options provided, two pupils ticked both the positive and negative option, as shown below.

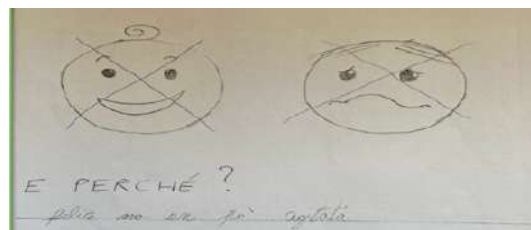


Figure 5. Phase 1 protocol extract .

S10²: *I feel happy but a bit agitated too.*

S10 shows an ambivalent emotional expression. In a task that pupils didn't perceive as a game, but rather as an evaluation of their own feelings, ambivalent emotional responses were observed. The terms *happy* and *agitated* coexisted within the same statement, showing the emergence of negative emotional elements alongside positive ones in pupils' responses.

S16: *I am happy because I feel good in mathematics but I didn't like it when the teacher used to get angry.*

S16 response also shows ambivalent emotional expressions. In this statement as well, the terms *happy* and *I didn't like it* coexist, demonstrating that the pupil also didn't enjoy the subject

²All the label S followed by a numerical value serves as an anonymous identifier for the students.

when the student-teacher relationship was shaken in some didactic situations. In Phase 2, the data derived from students' selections indicated a general tendency towards the consolidation of positive emotional states already present in Phase 1: pupils predominantly selected positive emoticons, without exhibiting emotional fluctuations throughout the experience in this phase. In addition, most pupils selected positive emoticons during the later phases. Whereas ambivalent responses were observed in Phase 1, no ambivalent emoticon selections were recorded in Phases 2 and 3. Positive emoticons were the most frequently selected responses. A limited number of participants continued to express a negative emotional disposition.

In Phase 3, the results of the last semi-structured protocol showed the following changes in relation to emotional disposition. The analysis of responses showed no ambivalent selections, with most pupils marking positive emoticons. A single uncertain response was recorded. The majority of students selected positive emoticons and something changed in their written statements too.

S8: *I'll give him the strength and joy to continue searching for his heart.*

S18: *Don't give up, you can do it, and how would you live without your heart?*

Although in Phase 1 ambivalence had emerged, and in Phases 2 and 3 a partially negative emotional disposition had been observed, responses expressed emotional support towards Mortino, encouraging him not to give up.

4.2. Perceived Competence

As discussed in the Theoretical background, perceived competence refers to pupils' reported sense of autonomy, confidence, and ability to engage with mathematical tasks. This dimension is assessed throughout the experiment to examine pupils' reported sense of competence in relation to assisting Mortino across activities.

In Phase 1, in the questionnaire following the mathematics-food matching game, only 5 participants reported statements in which they disliked mathematics in certain circumstances. Among these, 2 pupils referred specifically to aspects related to their perceived competence :

S10: *I don't like it very much when I have difficulty learning new things.*

S13: *I don't like mathematics when there are new difficult things.*

S10 and S13 stated that there were instructional situations in which they didn't like the subject and provided a reason to support their statements. These two pupils reported terms referring to perceived difficulties in mathematical practice:

S10: *I have difficulty learning new things.*

S13: *new and difficult things.*

In particular, pupils' responses indicate insecurity when they are required to engage with unfamiliar tasks. In Phase 2, changes in pupils' responses were observed. Despite the difficulties reported when pupils faced unfamiliar tasks in Phase 1, students, as shown in the examples below (S10 and S13), reported willingness to assist the main character across activities:

S10: *Yes, I want to help him.*

S13: *I like it because I am learning to reason by myself.*

About this attitude's dimension, S10's and S13's responses showed that both negative and positive evaluations of competence were present in Phase 1, while an increase in positive evaluations was observed in Phase 2.

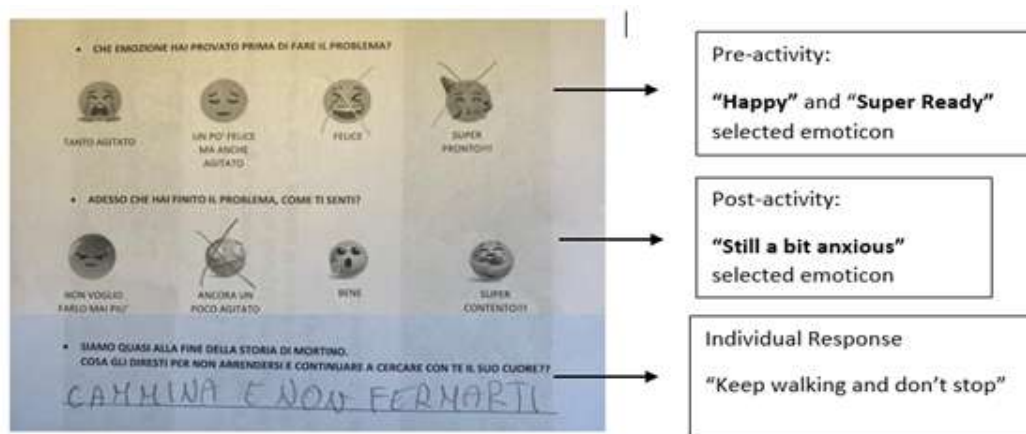


Figure 6. Phase 3 protocol extract.

In Phase 3, all students provided encouragement for Mortino in difficulty, as shown in the extract above (Figure 6).

The following statement was reported by a student.

S4: *I got anxious because I was unsure about the correctness of the operation I had done.*

In Phase 2, not all students reported having overcome anxiety when approaching problem-solving tasks and of feeling confident when facing it. S4's statement reported anxiety related to uncertainty about the correctness of his solution. This example included the expression "I was unsure", indicating uncertainty in Phase 2. The same student revised his statement in Phase 3, as shown below.

S4: *Teacher, why? Don't you help your friends?*

S4 selected an emoticon that still partially reflected insecurity; however, his response to the open-ended question included expressions of encouragement towards Mortino, despite that specific marking.

4.3. Vision of Mathematics

Regarding the vision of mathematics, the present study investigated potential changes in attitude resulting from the combined use of narrative principles and gamified dynamics in conjunction with story-based problems, particularly when children engaged with unfamiliar tasks. In Phase 2, the analysis of the problem-solving protocols highlighted the following results. In Task 1, the groups established similar routines in their written protocols:

- participants structured their problem-solving according to the specified categories: *Data*; *Operation (linear/column)*; *Answer*; *Reasoning*;
- routines also included the use of coloured pens or tables to categorise place values; no attempts at graphic representation to better clarify their problem-solving strategy;
- the adoption of selective reading strategies to identify keywords for operations;
- both *Answer and Reasoning sections* were limited to indicate the numerical value resulting from the calculations, as follows (Figure 7).

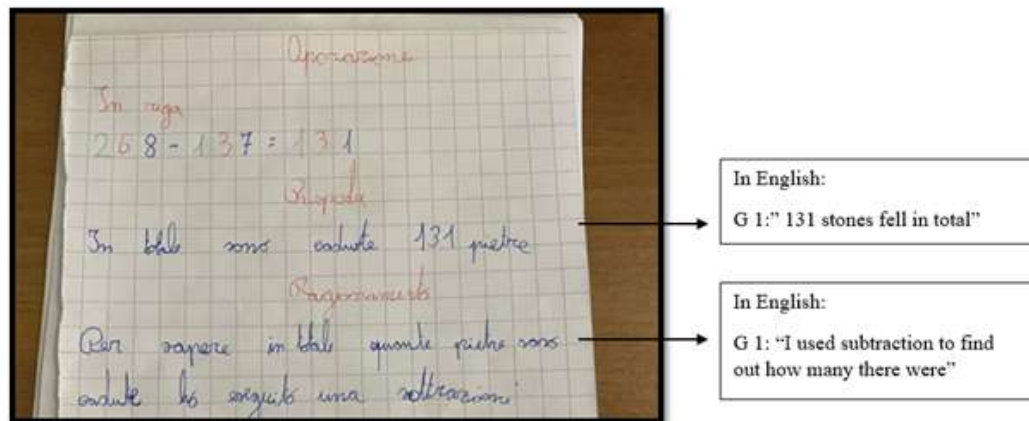


Figure 7. Phase 2 protocol extract.

In Task 1, argumentative interaction during solution-based discussion was characterised by a marked prevalence of teacher scaffolding questions, in contrast to the teacher's direct questions. The students' arguments didn't present explicit connections to the narrative purpose but instead provided justifications based on the routines they used, as shown below:

S4: *No, because 'altogether' is not a subtraction word, teacher, but it's one of addition.*

S20: *I didn't understand, and I did the addition because I didn't see the words for subtraction.*

In Task 2, there were differences compared with Task 1 due to the absence of the following routines within the protocols:

- the four groups no longer circled numerical data or underlined the interrogative clause;
- the pupils no longer used coloured pens to differentiate the terms of the chosen operations;
- in their *Reasoning category*, more frequent references to two types of analysis were observed in relation to the based- story problem in their statements: the narrative purpose and the reasoning behind the chosen operation linked to the narrative elements.

G1³: *To help Mortino pass through the dark corridor, I intended to use 272 torches (122+150).*

G2: *To find out how many torches are lit, I performed an addition because the problem states that the fallen bulbs are then switched back on.*

G3: *To find out how many torches still need to be lit, I performed a subtraction (because) to reach 150, 32 are missing.*

G4: *To find out how many torches are in the cave, I performed an addition because Mortino collects another 20.*

During the collective discussion in Task 2, there has been a more balanced distribution between the two input types; a reduction in external mediation and an increase in spontaneous interventions by the participants were observed.

S11: *With subtraction, we take away the lights and then Mortino is in the dark. We already have 122, and then he collects more, so it's an addition.*

³The label G followed by a numerical value serves as an identifier for each group.

S5: *We help Mortino do the addition because at the beginning we have 122 torches, but then he also picks up the ones on the ground. Teacher, so we do $122 + 20$, which is 142.*

Pupils' arguments presented more explicit connections to the narrative purpose as justification for the chosen operation.

5. Discussion

The present study aimed to investigate how the application of a gamified narrative structure may influence pupils' attitude towards mathematics, through the Three-dimensional Model of Attitude [9]. The findings revealed that changes across the three dimensions didn't occur simultaneously or linearly; rather, the design foregrounded different affective, behavioural and cognitive processes throughout the educational experience.

The analysis highlighted how pupils' emotional disposition towards mathematics showed changes throughout the experiment. Initially, the discipline was associated with positive elements within playful activities, suggesting that the introduction of narrative and game dynamics generated an emotionally reassuring learning environment. However, when pupils were invited to reflect explicitly on their feelings towards mathematics, ambivalent emotional states emerged. As discussed within Literature review, previous school experiences and classroom expectations significantly shape students' emotional responses towards the discipline [16, 9]. When familiar procedural routines were questioned through narrative-based mathematical activities, uncertainty in pupils became more visible. Rather than eliminating fear entirely, the narrative structure appeared to create a context in which pupils could progressively recognise and externalise their emotions towards mathematics. As the gamified experience progressed, students demonstrated greater emotional stability and increasing willingness to engage with activities. Particularly, the narrative structure appeared relevant in supporting this process. As discussed above, storytelling allows children to project emotions and fears onto fictional characters, thereby facilitating emotional elaboration and engagement [4, 5]. By identifying with the main character, pupils interpreted tasks as meaningful actions within a shared story-world, thereby demonstrating empathetic and supportive attitude towards Mortino. Through gamification dynamics, the narrative structure seemed to foster more resilient emotional responses towards challenging tasks. Anxiety therefore did not disappear entirely; however, pupils demonstrated a greater inclination to participate despite experiencing emotional discomfort.

The findings revealed important transformations in the way pupils perceived their own mathematical competence. This dimension was strongly influenced by students' emotional disposition, which initially appeared to depend on the possibility of applying familiar operational routines. Unfamiliar tasks induced insecurity and reduced pupils' initiative, suggesting that their sense of agency was associated more with procedural certainty than with adaptive reasoning processes. Consistent with the theoretical framework, this procedural dependence reflects how students often construct their sense of competence through predictable problem-solving schemes within repetitive school practices [23, 11]. When the gamified layers required interpretation and autonomous reasoning, pupils experienced emotional tension. As the experimentation progressed, however, students demonstrated willingness to participate actively and to persist despite uncertainty. The structured levels sustained their curiosity about subsequent developments and the prospect of receiving reward puzzles. Such results are consistent with studies suggesting that both deep and shallow gamification can reinforce pupils' motivation and control over learning activities [15, 20]. Rather than avoiding difficult tasks, pupils progressively became more inclined to confront them, especially when they perceived their contribution as meaningful within the narrative pathway. The identification process established

with Mortino played a significant role in pupils' motivational engagement; specifically, pupils regarded themselves as responsible for helping the character overcome his setbacks. This form of emotional involvement was found to reinforce their perceived competence, even when anxiety remained present.

Throughout the experiment, pupils' vision of mathematics changed significantly, particularly through the reciprocal interplay between mathematical reasoning and narrative structure. Initially, students approached tasks through routine-based strategies. The findings are coherent with studies suggesting that pupils frequently experience mathematics as highly procedural, abstract and disconnected from meaningful contexts [6, 9]. However, the narrative activities encouraged pupils to gradually challenge these strategies. Mathematical reasoning became increasingly connected to the narrative goals: specifically, pupils began to justify their mathematical choices not only through procedures but also through references to the broader narrative context. This shift suggests that storytelling can support more interpretative and meaningful forms of mathematical reasoning. In addition, the experiment fostered more spontaneous argumentative interactions, with pupils relying less on external guidance and more on shared reasoning processes connected to storytelling. This aspect indicates the emergence of a more participatory vision of mathematics, in which students perceived themselves as active contributors to the problem-solving process rather than passive executors of procedures.

6. Conclusions

The research suggested that the interrelationship between storytelling, gamification and mathematical activities can positively influence pupils' emotional disposition, perceived competence, and vision of the discipline by fostering more meaningful participation in the educational process. In addition, gamified dynamics strengthened distinct forms of motivation and persistence by encouraging responsibility within the learning pathway. In this respect, it became apparent that a gamified narrative structure did not operate as a means of fully eradicating fear. Mathematics anxiety does not disappear entirely; however, pupils can progressively develop greater resilience and willingness to face demanding tasks despite fear. This gradual shift indicates a broader interpretative reading of the role of narrative structure in the learning process. The moments of crisis within the story, namely the emergence of darkness and the loss of Mortino's heart, operated not only as elements of engagement but also as forms of emotional mediation. Mathematical apprehension related to error was partially mediated through the narrative world; rather than being experienced solely as an individual difficulty, it became shared through the collective narrative experience and the relationship established with the main character. The identification process supported pupils in engaging with negative emotions without immediately perceiving them as personal failure. Accordingly, narrative crises provided a supportive context in which mathematical anxiety could be more gradually expressed. Concurrently, these narrative disruptions contributed to a shift in students' mathematical thinking. Pupils who were initially accustomed to rigid routines were gradually exposed to situations in which such strategies were no longer fully sufficient. The narrative problems required operations to be interpreted within the logic of the story and the intentions of the protagonist. In group discussions, pupils increasingly justified their reasoning not only by associating specific words with operations, but also through narrative meaning (considering the consequences of an action within Mortino's world). Thus, through participation in the narrative disruptions, storytelling not only supports pupils' engagement but encourages a gradual transition from procedural to more interpretative approaches to problem solving. A potential limitation of this study resides in its localized context, as the qualitative depth of the observed interactions reflects specific classroom dynamics. Furthermore, while the findings indicate significant transformations, the long-term sustainability of this emotional and cognitive shift remains to be verified, as the re-

sults may be partially influenced by the novelty of the gamified narrative intervention. Further research could explore how this structure might be adapted to different educational contexts, incorporating larger sample sizes and a greater number of classrooms, in order to investigate its long-term impact on pupils' attitude towards mathematics.

Declaration

The contents of this paper stem from Angela Sassone's master's dissertation, under the supervision of the other author of the paper.

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From classroom to outdoors and back: orchestrating the field of experience of sun and shadows in mathematics education

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Abstract: This paper presents a perspective on outdoor mathematics education developed through collaborative inquiry of researchers and Italian in-service teachers from the first cycle of education. Drawing on Priest's notion of Outdoor Education and the construct of fields of experience, we conceptualize our perspective of outdoor mathematics education as a process in which students explore mathematically rich phenomena across classroom and outdoor settings through digital and non digital artefacts. The analysis draws on two data sources: a video-recorded teaching episode in a fourth-grade classroom and schoolyard, and a focus group with teachers after the implementation. The findings highlight the role of relationships, the contribution of digital and material tools, and the challenge of curricular integration. On this basis, the paper proposes the provisional notion of Outfield Education, in which mathematical learning develops in, about, and through a field of experience, connecting school learning, outdoor exploration, and students' lived worlds, within which their actual experiences unfold.

2020 Mathematics Subject Classification: Primary 35B65; Secondary 35J70, 35R09.

Keywords: outdoor education; fields of experience; instrumental orchestration; teacher professional development.

1. Days of a future past

In 2019, Bakker, Cai, and Zenger asked mathematics education researchers, through a survey, the following question: "What themes should research in mathematics education focus on in the coming decade?" [1, p. 2]. The survey, which involved a significant number of researchers from diverse countries, identified many topics. Among these, one theme in particular concerned the proposal to teach and learn mathematics in settings other than the classroom, situated in contexts that stimulate the exploration and understanding of mathematics within them:

"Though methodologically and theoretically challenging, it is of great importance to study learning and teaching mathematics across contexts. After all, students do

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not just learn at school; [...] Mathematics learning could also be situated on streets or in museums, homes, and other informal settings.” [1, p. 16]

As the authors point out, teaching mathematics in real-world settings outside the classroom presents methodological challenges. Wolf et al. [2] highlight that teachers often feel ill-prepared to teach outdoors, which is one reason why they make limited use of this approach in their lessons. This difficulty is confirmed by several literature reviews [3, 2], which show that Outdoor Education as a teaching and learning methodology is rarely addressed in teacher education programs, both pre-service and in-service. Exceptions can be found in contexts where Outdoor Education has long been recognized as a legitimate teaching practice, such as in Swedish universities, where pre-service teachers study outdoor activities as part of their teacher education programs [4]. The Italian context, in which this research is situated, is no exception. Outdoor Education receives very little attention in teacher education programs and, in practice, remains limited to a small number of schools that choose to specialize in these methodologies, even though, over recent years, awareness of this issue has been growing, encountering both resistance and interest, as well as a willingness to change [5, p. 21].

Bakker et al. [1] also mention a theoretical challenge, which is further emphasized by Wolf et al. [2] in their discussion of Outdoor Education. The understanding and practice of Outdoor Education are highly diverse and influenced by the sociocultural contexts in which they are enacted [6]. As a result, defining Outdoor Education and positioning it within a theoretical framework has also proved difficult [7, 8]. In the Italian context, the National Guidelines for the Curriculum of the first cycle of education (from early childhood to grade 8) suggest that a tradition of outdoor teaching is scarcely present, as it is mentioned only as a desirable practice in early childhood education or for science activities in lower secondary school (grades 6 to 8) that may specifically require fieldwork [9]. In the new proposal for the Italian National Guidelines of the first cycle of education, Outdoor Education is cited for geography and physical education as a methodology that should be promoted [10]. In conducting our research, we actually found researchers in education that, over the years, may have contributed to defining what Outdoor Education is in the Italian context [5]. On this matter of definition, we turned our attention to one encountered in various contributions to Outdoor Education research and that appears relatively free from specific sociocultural influences. We decided to proceed in this direction since we wanted to adopt a definition that could serve as a starting point for developing our own perspective on Outdoor Education for mathematics. Such definition is provided by Priest (Figure 1):

“outdoor education is an experiential process of learning by doing, which takes place primarily through the exposure to the out-of-doors. In outdoor education the emphasis for the subject of learning is placed on RELATIONSHIPS, relationships concerning people and natural resources.” [11, p. 13]

In this definition, we find several connections to our own field of research, namely mathematics education. A first theme that is particularly significant for us is the idea that learning takes place primarily outdoors, but not exclusively. This point is important because it opens the discussion on one of the critical issues of outdoor or informal experiences, which often risk remaining isolated, wonderful activities from which only a fond memory endures over time [12], rather than becoming part of a broader educational trajectory. It also raises the issue of how to create bridges between in-school and out-of-school learning [13]. In this regard, Rickinson et al. [3] argue for the need to rethink school curricula by providing more space for outdoor experiences, as well as guidelines and resources for teachers to integrate meaningful outdoor activities into classroom curricula.

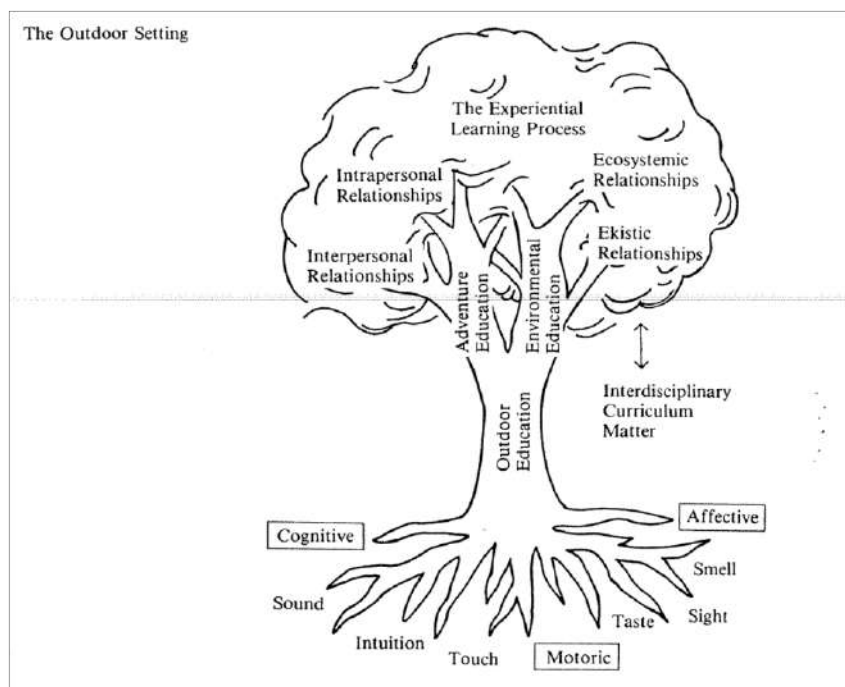


Figure 1. A drawing representing a tree as a metaphor to describe Outdoor Education. In the soil the cognitive domain and the senses through the learning happens, the outdoor setting as the Sun interacting with the leaves of the tree in which is situated the experiential learning process and with it the relationships between people and the environment, in the air the interdisciplinarity matter [11, p. 15].

The definition also emphasizes relationships: between individuals who learn together and between individuals and the environment in which learning takes place. Learning occurs through meaningful experiences, with an approach that resonates with Dewey’s notion of “learning by doing” and, more broadly, with the tradition of pedagogical activism. This focus on relationships and on the surrounding environment is also emphasized by some of the researchers who responded to the survey by Bakker et al. [1], highlighting the societal goals of mathematics education:

“Apart from instrumental goals of mathematics education, some emphasized goals related to developing as a human being, for instance learning to see the mathematics in the world and develop a relation with the world.” [1, p. 7]

Learning to see mathematics in the surrounding reality thus appears to be a key theme. However, from the perspective developed in this section, the main question that emerges and that we wish to address is the following: What is our idea, as an Italian community of inquiry composed of mathematics education researchers and mathematics teachers, of outdoor mathematics education activities? To better understand our vision, we sought to design and implement mathematics programs that included both outdoor and classroom activities, with the aim of creating continuity between traditional curricular work and outdoor experiences. In reflecting on the feasibility of such integration, we as researchers also created opportunities for discussion and reflection with teachers after their implementations, in order to gain a deeper understanding of the work carried out and to frame it also theoretically, so that these activities could be meaningfully shared within both the worlds of mathematics education research and teaching.

This paper should therefore be read as a pilot and exploratory contribution, somewhat unusual in genre. Rather than presenting a primarily empirical study aimed at systematic

generalization, it offers a theoretically oriented reflection emerging from an empirical experience conducted with teachers. The data are used to illustrate and refine a conceptual proposal for outdoor mathematics education. The paper first situates this proposal within the literature on Outdoor Education, Fields of Experience, and the Instrumental Approach; it then presents the collaborative inquiry with teachers, analyzes one implementation episode and a post-implementation focus group, and finally introduces the provisional notion of Outfield Education, discussing its limitations and possible future developments.

2. You are here: our perspective on outdoor mathematics education

Just as maps of public places help people orient themselves by clearly marking “you are here” with an ideolocator, we too seek to position our work and theoretical perspective within the existing literature on outdoor mathematics education. Our approach to outdoor mathematics education stems from bringing together Priest’s [11] definition of Outdoor Education (OE) — understood as an experiential process of learning by doing, in which emphasis is placed on relationships (see Figure 1) — with the notion of fields of experience (FoE), understood as “an area of human culture identifiable by learners as part of their (actual or potential) experience, with specific characteristics that make it (under the guidance of the teacher) suitable for activities in mathematical modeling, mathematical problem solving, and the construction and development of mathematical concepts” [14]. In this respect, Priest’s definition differs from earlier ones, such as that of Donaldson and Donaldson [15], who defined OE as education “in, about, and for the outdoors”. With Priest, OE begins to be framed more explicitly as a resource for teaching and learning in school: the focus is no longer only on the outdoor context itself as the object of study, but also on the educational possibilities it opens up, including interdisciplinary ones. From this perspective, the outdoor setting can offer meaningful insights for learning across domains, including mathematics. Similarly, as we will discuss later, the FoE can become the context within which learning unfolds, both outdoors and in continuity with classroom activity, offering a space for the exploration and development of mathematical ideas.

This choice of mixing OE and FoE is grounded in the fact that the concept of FoE serves as a reference point in the National Curriculum Guidelines [9], where it is defined in terms that are less specifically oriented toward mathematics and is presented as an approach to teaching in early childhood education. Moreover, when discussing teaching practices based on FoEs, outdoor education is explicitly suggested [10]. Within the Italian tradition, studies on FoEs already exist, such as those focused on sun shadows [14], which can retrospectively be interpreted as proposals for OE, since they included some outdoor activities aimed at the exploration of a FoE. Seeking a FoE that is (or could be) consistently part of the learner’s lived experience creates strong connections with Priest’s view of OE, which is defined first and foremost as a process of experiential learning and, in this case, unfolds through the experiential study of the FoE.

We believe that artifacts can play a crucial role in the experiential process of exploring and understanding a FoE and the mathematics embedded within it. To define these artifacts and the ways in which they are used, we draw on the Instrumental Approach [16], which distinguishes between an artifact, that is, the given object, and an instrument, a psychological construct that exists “when the subject has been able to appropriate it for himself, has integrated it with his activity”. The instrument thus becomes an extension of the body, and its construction is described as a process of instrumental genesis. According to Rabardel [16], for an artifact to become an instrument, it is necessary to develop utilization schemes, that are schemes that organize activities with the artifact in order to solve specific tasks. He distinguishes between usage schemes, oriented toward the management of artifacts, and instrumented action schemes, oriented toward carrying out specific tasks. Going deeper in the theory, instrumental genesis can be also described as the combination of two processes: instrumentalization, directed toward

the artifact, and instrumentation, directed toward the subject.

A theory rooted in the instrumental approach that explains how teachers can guide their students processes of instrumental genesis has been defined as instrumental orchestration [17]. This refers to the teacher's intentional and systematic organization and use of available artefacts within a given learning environment and mathematical task situation, with the aim of guiding students' instrumental genesis [17, 16]. Three key components define instrumental orchestration (IO): (1) the didactic configuration, referring to the arrangement of artefacts within the learning environment; (2) the exploitation mode, which concerns how the teacher chooses to make use of a given configuration in order to serve specific didactic intentions; and (3) the didactical performance, encompassing the spontaneous, in-the-moment decisions made by the teacher during instruction [18]. The ways in which the teacher guides this process of instrumental genesis can be further understood through the notion of semiotic mediation [19], which describes how the teacher, by building on students' artefact-mediated activity, directs attention toward mathematical signs and meanings in a collective setting. The guidance of students' instrumental genesis is closely related to OE, since artifacts always carry a social dimension: "Tools are not passive, they are active elements of the culture into which they are inserted" [20, p. 58]. In this direction, it has been argued that schemes always have both an individual and a social aspect, since they can be social schemes, "elaborated and shared in communities of practice and can give rise to appropriation by the subjects, even come under training processes" [21]. From this perspective, one of the main actors are the relationships that develop among a group of individuals and within a specific environment while exploring a FoE and its mathematics through the use of tools.

Our conceptualisation of outdoor mathematics education, which emerges from the networking of these theoretical perspectives through combining [22], seeks to describe and understand a scenario in which mathematics teachers attempt to orchestrate artefacts, tasks, social interactions, and learning environments in support of the exploration and comprehension of a specific FoE. Particular attention is given to the teacher's action, which also involves orchestrating different settings — both in-school and out-of-school — in order to pursue specific educational goals while supporting the experiential learning process. In this sense, the challenge lies in recognizing what is best developed within the classroom and what, instead, is more effectively learned through direct experience in the outside world [23].

2.1. The outdoor mathematics education horizon

In mathematics education, out-of-school learning is a relatively new field compared with science education, although awareness of its crucial role in a range of contexts — from museums and outdoor settings to everyday life — is steadily increasing [24]. In science education, Eshach [13] is one of the scholars who addresses an issue that we also in the introduction considered central: establishing connections between in-school and out-of-school learning. His approach distinguishes between non-formal learning, which is structured and takes place in institutions outside school, and informal learning, which refers to situations that arise spontaneously in everyday life, such as within the family, the neighborhood, and similar contexts [13]. This distinction also aims to bring greater clarity to the use of the terms out-of-school, informal, and non-formal, which are often used interchangeably, thereby generating considerable confusion.

In mathematics education as well, the concept of informal mathematics education has been defined as a voluntary educational experience characterized by fluid boundaries between disciplines and by the absence of traditional forms of assessment [25]. Nemirovsky et al. [25] introduced the term informal mathematics education in order to distinguish it from the everyday, spontaneous ways in which people encounter mathematics in daily life. For example, they considered museums to be intentionally designed informal mathematics learning settings because of their structured schedules, the presence of educators, and the availability of technolo-

gies and tools to support mathematical learning. Given these characteristics, their perspective appears closer to Eshach's concept of non-formal learning than to his definition of informal learning.

The work that we describe in the next section seeks to address how to connect out-of-school mathematics learning experiences with classroom curricular experiences. Since the proposed activities are carried out by students during school hours, in mathematics lessons, together with their teacher, and are integrated as much as possible into the classroom curriculum, they are, on paper, quite distant from the notion of informal learning described by Nemirovsky et al. [25], particularly with regard to the voluntary nature of students' participation. Students are assigned a task to solve, which guides them toward specific learning goals; the pathway is therefore clearly planned both in the classroom and outdoors. Consequently, according to Eshach's [13] categories, our idea of outdoor mathematics education, as we will also see in a teaching episode discussed in the analysis, is more appropriately classified as a form of formal education, but one that also makes use of a setting out-of-school in order to foster a less formal reflection on the mathematics of the explored FoE and to prompt a more genuinely experiential learning process [11]. One of the aims of this approach is also to create stronger connections between experiences conducted in informal settings and classroom lessons, thus resembling explorations of the ways in which informal and formal learning experiences can be connected, a topic that is already prominent in mathematics education research [26, 27].

Working with FoEs can also open up a dialogue among different scientific disciplines, an aspect that has likewise been identified as a topic of interest in mathematics education research:

“Many responses can be characterized as highlighting boundary crossing [28] with disciplines or communities outside mathematics education, such as in science, technology, engineering, art, and mathematics education (STEM or STEAM); parents or families; the workplace; and leisure (e.g., drama, music, sports).” [1, p. 7-8]

The interdisciplinary nature of STEM education, in which different disciplines are intertwined within a broader academic strand, also presents mathematics teachers with a new challenge: mathematics is no longer approached as an isolated subject, but in dialogue with other disciplines. Among these challenges, digital technologies are especially pressing, as they create new learning environments with countless possibilities, but also with risks that must be discussed and addressed. From our perspective, since today's mathematics teacher is required to face these new challenges in both primary and secondary education [1, 10], a convincing response lies in seeking a balance between the educational goals associated with natural and digital learning environments (Figure 1). According to some studies, the combination of technology use and outdoor teaching practices has a positive impact on both dimensions. Students who are accustomed to spending substantial time outdoors do not seem to reach alarming levels of media use, as they tend to choose certain tools only when they are truly necessary [29]. Moreover, OE engages directed attention through successive action planning, which helps inhibit the impulse toward distraction that technology can intensify; for this reason, it has beneficial effects on cognitive and emotional functions, as well as on memory and learning [30]. At the same time, the possibility of enriching an already multisensory outdoor experience through technological mediation is encouraged, provided that the experience remains stimulating in terms of whole-body engagement [31].

Within mathematics education research, Cahyono and Ludwig [32] proposed an activity that can be implemented outside the classroom through the use of digital technology. Students were asked to solve math trail tasks distributed throughout a city using a mobile application and a map; for example, they were asked to estimate the base area of a historical building, a task that required them to identify the geometric shape of the base and measure its dimensions in order to calculate its area. Similar examples can also be found in the activities proposed

through MathCityMap and presented on its website, a powerful application that also offers opportunities for teacher education activities [33]. These studies represent forms of outdoor mathematics education in which digital technologies are involved, and, in this respect, they show similarities with the work we conducted. In our study, we view mathematics as guiding a broader interdisciplinary discussion capable of integrating multiple perspectives and insights from different fields, as in a STEM approach, aimed at uncovering how a specific FoE works.

Bakker et al. [1] also make clear how boundaries with other learning environments — such as the personal sphere and leisure time — can be crossed, fostering connections between mathematics education and other practices. In this regard, an approach such as Outdoor Education — which places strong emphasis on relationships — encourages the involvement of multiple stakeholders, including actors beyond the classroom, as will be shown in the examples discussed in the following sections (see Figure 2).

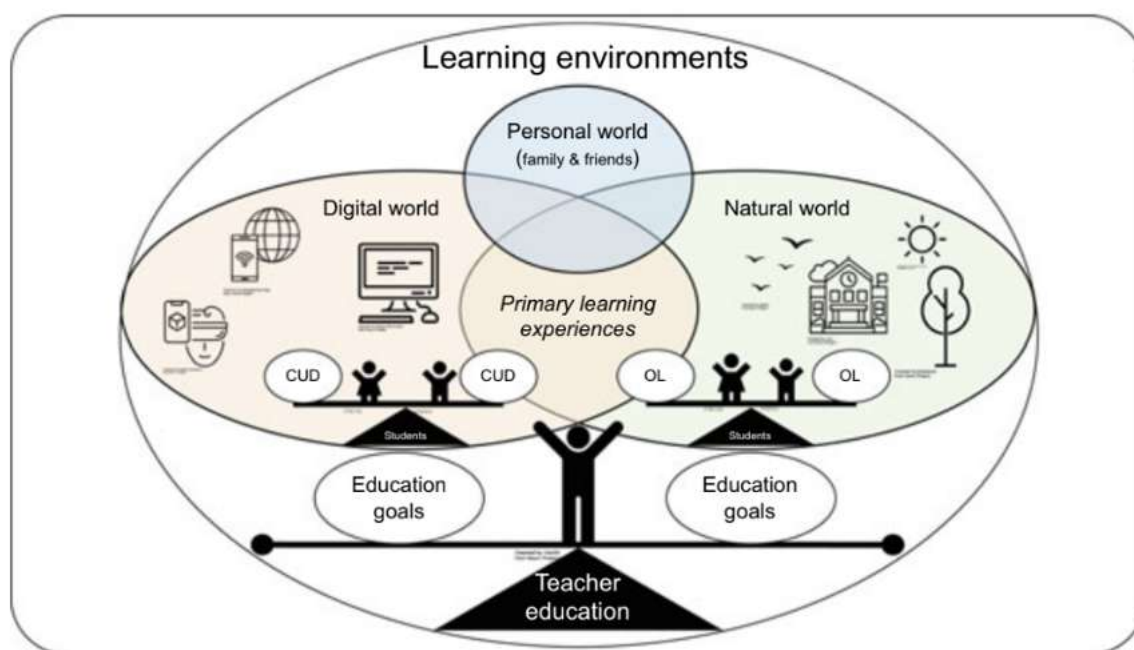


Figure 2. The teacher, supported by teacher education, has to find the right balance between the education goals for the natural and the digital learning environments. Together and in interaction with these two must be considered also the personal world, which includes family, friends and leisures [2, p. 260]

3. How our inquiry started

The first two sections of this paper outlined our perspective on outdoor mathematics education and situated it within the existing literature. Drawing on specific theoretical frameworks, we proposed an approach to mathematics education that emphasizes continuity between in-school and out-of-school settings. These theoretical considerations led us to formulate two research questions:

- RQ1 How can the networking of Fields of Experience, Outdoor Education, and Instrumental Orchestration characterize and interpret outdoor mathematics education activities across classroom and outdoor settings?
- RQ2 What do classroom implementation and teachers' post-implementation reflections reveal about the affordances and challenges of this approach?

To explore this theoretical proposal in practice, we worked with a group of Italian in-service

teachers from the first cycle of education, spanning Grades 1 to 8. More specifically, the group consisted of 16 mathematics teachers from the same school, working across both primary education (Grades 1 to 5) and lower secondary education (Grades 6 to 8). These teachers had been collaborating for several years with researchers from the University of Eastern Piedmont, together forming a community of inquiry in the sense described by Jaworski [34]. The collaborative processes through which teachers and researchers learn from their own practice and from shared reflection constitute a fundamental aspect of such communities of inquiry. Through self-reflection and the exchange of perspectives, participants have the opportunity to develop new knowledge and to discuss both theoretical issues emerging from mathematics education research and questions arising from teaching practice, thereby fostering critical reflection on their professional work.

Methodologically, the study is framed within an Educational Design Research perspective [35], as it investigates a designed educational intervention in a real school context while also contributing to the refinement of both theoretical understanding and educational practice. In line with this approach, teachers and researchers engaged in dialogue around two closely related issues: how outdoor mathematics activities can be designed and implemented within the FoE of sun shadows, and how such work can contribute both to outlining a theoretical conception of outdoor mathematics education and to informing the development of teacher education pathways in outdoor mathematics education. Teachers designed outdoor mathematics education activities for their classes and implemented them within a framework of continuous dialogue and exchange with the researchers. From this perspective, implementation was not considered merely a context for data collection, but rather a site in which theoretical assumptions, didactical choices, artefacts, and classroom practices were tested and progressively refined.

The study therefore pursues a twofold contribution. At the theoretical level, it examines how the networking [22] of Fields of Experience, Outdoor Education, and Instrumental Orchestration can support the interpretation of mathematical activity across indoor and outdoor learning environments. At the practical-design level, the broader research project aims to develop principles for organizing teacher education pathways that support teachers in designing, enacting, and revising outdoor mathematics activities. While this second level belongs to the wider Educational Design Research trajectory, the present paper focuses specifically on the implementation and post-implementation reflections, showing how the analysis of classroom events and teachers' accounts can inform the progressive refinement not only of the educational design, but also of theoretical perspectives on education. To this end, after the implementation phase, focus groups with volunteer teachers were proposed by the researchers in order to reflect more deeply on what had concretely taken place during the classroom and outdoor implementation. Within this ongoing collaborative work with teachers, aimed at fostering richer opportunities for students' mathematical learning, the researchers can be described as didacticians²:

“people from the university with knowledge of research and theory in the didactics of mathematics, interested to work with teachers to promote better opportunities for mathematics learning in classrooms.” [41, p. 623]

For this inquiry, we chose to work within the FoE of sun and shadows. As highlighted by previous studies [14], this context offers rich potential for the emergence of mathematical concepts. A concept in mathematics as defined by Vergnaud [36] may be conceived and described as a dynamic system made up of three components: *reference situations*, which give meaning to a concept and can be recalled and used to reason or solve problems (e.g., noticing

²What we have so far referred to as researchers are, in light of our theoretical framework, now termed didacticians; however, although the two terms are sometimes used interchangeably in this paper, they refer to the same people within the community of inquiry.

that a shadow changes length during the day); *invariants*, that is, implicit ideas such as rules or relationships (e.g., the shadow depends on the position of the sun and the object); and *representations* (e.g., drawings, gestures, words). For instance, proportionality and ratio can be explored by examining the relationship between the height of an object and the length of its shadow; similar triangles can be studied through comparisons of shadow triangles; angle measurement becomes relevant when calculating the sun's elevation or tracking the movement of shadows throughout the day; and the concept of parallelism can also be meaningfully introduced. Moreover, this FoE provides a suitable context for science education activities in dialogue with mathematics, with mathematics acting as an instrument for developing a deeper understanding of these natural phenomena. In this way, it becomes possible to explore how changes in shadows throughout the day and across the seasons are related to the position of our planet and its movement around the sun. This aspect resonates with the interdisciplinary nature typically associated with OE, as well as with the idea that OE "is a matter of relationships", particularly through the exploration of ecosystemic relationships connected, in this case, to the Earth's rotation and revolution, and to the ways in which our lives and the environment in which we live change across the seasons in relation to this Sun–Earth relationship, thereby affecting our relationships with other people and with the environment.

Compared with earlier studies [14], in which teachers and researchers also worked together but with a particular focus on students' perspectives and responses to mathematics activities on sun and shadows, this research explores the same FoE with a specific focus on the perspectives of in-service teachers who had never previously engaged either with such teaching methodologies or with this particular FoE. More specifically, we seek to discuss with them both the potential and the challenges of teaching through experiential learning and, in particular, the difficulties teachers may encounter when teaching mathematics through the shadows cast by the sun, both in the classroom and outdoors. In this article, we focus on the questions highlighted above, but as said earlier this research work within a community of inquiry has also led to a further reflection that extends beyond previous studies: how to design effective teacher education pathways on these topics.

In the next section, we present two types of collected data: a teaching experiment and a focus group that involved both teachers and didacticians. The teaching experiment was led by PT1 (Primary Teacher 1), a teacher from the community of inquiry who volunteered to be filmed with her fourth-grade class, together with the researcher present to observe and support the teaching and not only to record the lessons. The teaching experiment comprised approximately 18 hours of lessons in total. From these video recordings, we selected a number of excerpts for this article that are particularly useful for highlighting key aspects of the three theoretical frameworks — Instrumental Orchestration, Fields of Experience, and Outdoor Education — as they emerge in synergy and in ways that complement one another during the teaching process. We provide a narrative description and a few excerpts accompanied by photographs from one selected lesson of this teaching experiment, intended to illustrate key aspects of our theoretical proposal in a specific moment rather than to exhaust the empirical material on which the broader research draws.

In analysing this episode, we focus on aspects that are particularly significant in relation to our vision of outdoor mathematics education. More specifically, we highlight the three characteristic elements of instrumental orchestration — *didactic configuration*, *exploitation mode*, and *didactic performance* [18]. Since instrumental orchestration concerns the ways in which a set of instruments is organized and exploited within a learning environment, we examine how these elements change across different learning environments and how relationships between students and the environment are fostered [11] through specific orchestration choices aimed at supporting understanding of the phenomenon of sun shadows through particular activities. Consequently, we focus on the ways in which the FoE of sun shadows, together with some of the

mathematical concepts embedded within it and their components [36], are explored through the instruments orchestrated by the teacher, as well as on the relationships among all the agents involved. This episode is also intended to provide an overview of the types of activities proposed and of how they were carried out both inside and outside the classroom.

The analysis of this episode is conducted through qualitative content analysis [37], which is used to identify the three characteristic elements of orchestration [18], the components of the mathematical concepts explored within the FoE [36], and the relationships among the agents involved in the activities [11]. The aim is not merely to categorize these elements with a deductive coding approach, but to show how they interact with one another. In particular, we seek to show how the teacher's orchestration is guided by an understanding of the mathematics embedded in the FoE, how it engages different components of the concepts being explored, and how it fosters relationships among all the people involved in the activity and with the environment in which it takes place, thereby supporting a collective learning-by-doing approach — constructed together with the whole class in interaction with the environment — that is characteristic of Outdoor Education [11]. Going deeper in the analysis searching for the interactions between these three theories we highlighted two categories that will be discussed in “Analysis I”: *the relational mathematization of the phenomenon* and *the role of artefacts across environments*.

In the “Analysis II” section we analyze excerpts (transcribed by the author) from reflections shared by teachers during a video-recorded focus group (one of the two focus groups implemented), involving two didacticians and six teachers (4 of primary school, 2 of low-secondary school) of the community of inquiry that proposed themselves as volunteers. The focus group has been proposed and conducted by the didacticians at the end of the implementation phase. The focus group was organized as a structured collective reflection, guided by three broad questions concerning the unfolding of the implementation, teachers' and students' experiences, the use of tools inside and outside the classroom, and possible modifications of the proposal. The discussion of the group was guided by three questions:

1. How did the implementation unfold in the classes? How did you experience it as teachers, and how did your students respond to this Outdoor Mathematics Education initiative?
2. We used various tools in the activities. Some were suggested by us, such as the app, while others were selected and used by you. We would like you to tell us about the reasons that guided your choices regarding the types of tools adopted and the ways in which you used them inside and outside the classroom. Would you make the same choices again, or do you think some aspects should be changed?
3. What would you keep unchanged, and what would you modify in the proposal?

The didactician who organized the focus group (the author of this paper) had also prepared a checklist for himself including all the topics he intended to address during the discussion. If any of these topics did not emerge spontaneously, he intervened beyond the three main questions by providing more direct prompts.

The analysis of the excerpts of the focus group was conducted through an abductive thematic qualitative content analysis [38, 37, 39]. The transcript was first divided into meaning units, corresponding to teacher turns or short sequences in which participants reflected on a specific aspect of the experience. These units were initially coded descriptively, remaining close to the content of teachers' accounts. In a second step, related codes were grouped into broader thematic areas (see Table 1). This process led to the construction of three main areas: (1) relationships among participants and with the environment; (2) the digital world and the use of instruments; and (3) difficulties related to teaching practice and curriculum integration. The analysis was abductive because these themes were not treated as purely inductive findings, nor as deductive applications of pre-existing theory. Rather, they were constructed

through a movement between teachers’ reflections, the two categories identified in “Analysis I” — *relational mathematization of the phenomenon and role of artefacts across environments* — and the networking through combining [22] of Outdoor Education, Fields of Experience, and Instrumental Orchestration. In this sense, the focus group was not used to validate the classroom analysis, but to examine how teachers expanded, problematized, and recontextualized the issues that had emerged during implementation.

Focus-group thematic area	First-cycle codes	Link with Analysis I	Theoretical interpretation
Relationships among participants and with the environment	Student collaboration; mutual support; autonomy; increased observation; familiar places; family involvement; school community; local territory	Expands <i>relational mathematization of the phenomenon</i>	OE: relationships and environment. FoE: continuity between lived experience and mathematical phenomenon. IO: teacher organization to foster social and spatial relations
Digital world and use of instruments	App engagement; construction of gnomons; physical tools; digital tools; students’ different strengths; participation through tools	Expands <i>artefact-mediated continuity across environments</i>	IO: orchestration of tools. FoE: artefacts mediate access to the sun-shadow phenomenon. OE: tools connect school activity with students’ lived worlds
Difficulties and curriculum matter	Teacher uncertainty; need for colleagues; need for didacticians; weather constraints; time pressure; re-design; longer duration; vertical curriculum	Emerges from teachers’ reflection as a third dimension, only partly visible in Analysis I	IO: flexibility and didactical performance. FoE: progressive development of concepts across grades. OE: feasibility and sustainability of outdoor practice

Table 1. Description of the process of abductive thematic qualitative content analysis

4. Analysis I: the teaching episode

The first episode we wish to present began in the classroom, where PT1 implemented the activities previously outlined during the collaborative design process carried out within the community. In this opening session, she introduced the gnomon³ and encouraged students to share their prior knowledge and conjectures about its function. To do so, she organized a didactic configuration that included a photograph of the gnomon and its shadows projected on the interactive whiteboard, a tangible model consisting of a straw positioned perpendicularly on a paper plate serving as its base, which students were later asked to replicate, and two drafting triangles, later used to check that the straw was indeed perpendicular to the base.

³namely, a nail or stylus whose shadow is used to indicate the hours on a sundial



PT1: (after two groups have aligned the shadows of their gnomons to form a what they called “highway of shadows”) Shall we try to join them all together? (involves the third group as well. The children join the shadows of all the gnomons together)

PT1: I see that you put these three gnomons side by side. What can you observe? How are those three shadows?

S1&S2: teacher we understood that it forms a drafting triangle. (S2 shows to the teacher first with his hand then with a real drafting triangle) [...]

PT1: So how is the shadow formed?

S2: The sun’s rays are blocked by the gnomon and create the shadow

Figure 3. The Figure is divided in three rows. In each row you see a photo of the activity and selected excerpts of teachers and students. First row (A) the students putting gnomons side by side; in the middle row (B) the “highway of shadows”; on the last row (C) a student using a drafting triangle to represent the shadow triangle

Although some moments may be interpreted as instances of didactic performance, insofar as she shifted orchestration types in ways not anticipated during the design phase, this became particularly evident when students first tested the instruments in the classroom by opening the curtains to let sunlight in and then asked whether they could go outside to test their gnomons. At that point, the teacher took up this stimulus and modified the lesson plan as it had been defined during the design phase, anticipating the outdoor component already in the first lesson.

As a result of this change, the teacher had to redefine the didactic configuration on the spot. The students and the teacher moved outdoors to the school courtyard — a new learning environment — bringing with them the personal gnomon each student had constructed and the two drafting triangles provided by the teacher. PT1 divided the class into three working groups and assigned the following task: “Working with your group, position the gnomon and observe the shadows to understand what happens”. The groups were then free to explore the phenomenon outdoors and to use the tools in ways they considered most appropriate for addressing the task. The teacher’s initial exploitation mode consisted of circulating among the three groups to offer prompts and suggestions, such as using her own shadow to cover that of the gnomon or redirecting students’ attention to their own shadows and to those cast by objects in the courtyard, such as the gate, the tree, and a wall. In this way, the class was guided in exploring the FoE by reflecting on the relationships among the elements present in the environment at that moment [11], particularly on the shadows they cast and on the relationship between those shadows and their source — the Sun. The following excerpts illustrate how the teacher supported students’ explorations, thought in the moment to be of interest for all the students, through a series of guiding questions.

We can observe how, starting from students’ spontaneous ways of exploring sun shadows through these relationships, the teacher drew on representations that she considered signifi-

cant for understanding the phenomenon. For example, when some students coordinated their gnomons to produce a single continuous shadow (Figure 3A), thereby enacting a spontaneous utilization scheme, this became a pivotal moment of didactic performance: prompted by students' actions, she shifted from monitoring small-group exploration to orchestrating a whole-class discussion. She may have recognized an ongoing process of instrumentation as students coordinated their instruments, which in turn created a further opportunity to direct their attention to the direction of the shadows.

Subsequently, when she observed that some students were placing their gnomons side by side to compare the shadows (Figure 3B), she took up this utilization scheme and reoriented the whole class's attention. In doing so, she not only extended the reflection on shadow length by inviting comparisons across different gnomons, but also appeared to shift the focus from the direction of each individual shadow to the recognition of the parallelism of shadows at the same moment of the day, thus continuing the observation of invariant elements. At first, this idea did not seem readily available to the students. However, by building on their spontaneous actions, the teacher was able to make this mathematical feature more visible within the collective discussion. In this action, the teacher worked as a semiotic mediator [19]: she built on what some students had independently begun to represent, seeking to involve the entire class in reconstructing that representation together and thereby extending an understanding of the parallelism of shadows — which initially seemed to be grasped only by a few students — to the rest of the class. In this way, the teacher created a network of connections among the ideas generated by students during their exploration, putting these interpersonal ideas [11] into relation with one another in order to synthesize, through dialogue around shared observations, an understanding consistent with what the phenomenon reveals from a mathematical perspective.

A similar situation occurred when two students began constructing the shadow triangle, also using the drafting triangle (Figure 3C) as a representation capable not only of modelling the phenomenon, but also of offering insight into mathematical concepts such as angles, triangles, and similarity. This process can be interpreted both as instrumentalization — in the ways the students acted on the drafting triangle, adapting its use beyond what is typically expected for this artifact — and as instrumentation, insofar as this use shaped their activity by supporting the visualization of a model of the shadow triangle.

The episode can therefore be interpreted through two main cross-theoretical categories. The first is the relational mathematization of the phenomenon: students' outdoor exploration of sun shadows became mathematical as they established relationships among bodies, gnomons, environmental objects, shadows, and the sun [11]. These relationships were made available by the outdoor environment, belonged to the FoE of sun shadows, and were progressively oriented by the teacher's orchestration, which acted as a form of semiotic mediation by selecting the representations most suited to the educational goals she had in mind and directing attention toward mathematical meanings such as direction, shadow length, parallelism, and the modelling of the phenomenon through the shadow triangle.

The second category concerns the role of artefacts across environments: the gnomon, the drafting triangle, classroom representations, students' bodies, and the outdoor shadows formed an artefact ecology that connected the classroom introduction with the outdoor investigation, later culminating, within the teaching experiment, in a process of formalization back in the classroom. Through this ecology, artefacts changed function, supported students' spontaneous utilization schemes, and mediated the transition from perceptual exploration to mathematical representation.

Taken together, these two categories show that the implementation was not simply a movement from inside to outside, nor a straightforward enactment of a planned task. Rather, the teacher's didactic performance consisted in recognizing students' exploratory actions and using them to connect outdoor experience, artefact use, and the construction of mathematical

meaning. For this reason, these categories provided the basis for the author of this paper in structuring the subsequent focus group, in which teachers and researchers were invited to reflect on relationships, tools, and the actual integration of the proposal into the curriculum.

5. Analysis II: inquiry over the implementation

Following the implementation, teachers were invited to participate voluntarily in a structured reflection session organized as focus groups. The two categories that emerged from the analysis of the implementation — namely, *the relational mathematization of the phenomenon* and *the role of artefacts across environments* — were subsequently used to structure the post-implementation focus group, allowing us to examine whether teachers' broader reflections on the experience also recalled these categories. Their reflections were ultimately organized into three macro-areas: (1) relationships among participants and with the environment; (2) the digital world and the use of instruments; and (3) difficulties related to teaching practice and curricular integration. The first two macro-areas echo the categories that emerged from the implementation analysis, while the third introduces a further issue of interest, namely the teachers' perspective on the feasibility of integrating such practices into their teaching and into the class curriculum. These macro-areas do not replace the theoretical categories developed from the implementation; rather, they provide a second interpretive layer through which the networking of Outdoor Education, Fields of Experience, and Instrumental Orchestration can be further specified.

With regard to the participants involved in the focus group, the term PT stands for Primary Teacher and ST for Secondary Teacher. PT1 is the same teacher featured in the teaching episode described in the previous section. The excerpts are drawn from the broader discussion and are organized according to the three main categories.

5.1. Relationships among participants and with the environment

PT2: They worked very well together. [...] Just as PT1 mentioned earlier, while one person might be taking measurements, another would step away — but only to explore our yard and observe. [...] I've noticed that since we started this project, they've become more observant.

Based on the reflections shared by the teachers in the focus group, it appears that, in their interpretation of the classroom experience, the proposed activities helped strengthen relationships among the children [11, 3] by fostering spontaneous collaboration and mutual support during exploration. Again from the teachers' perspective, this process seems to have been intertwined with the development of greater student autonomy, not only within the context of the educational activities themselves, but also in broader patterns of behaviour. In particular, the teachers reported observing in the children a stronger inclination to attend to their surroundings and a more active and curious attitude extending beyond structured work sessions.

PT3: As for the environment and the children, it's a setting they're already familiar with. Sometimes, in the afternoons, they climb up to the park to observe and engage in hands-on activities there.

The role of the environment emerges as a crucial dimension of the experience, extending beyond the schoolyard alone and becoming something more than just a space in which to organize artifacts [18]. In the teachers' view, the value of the explored contexts lies not only in their availability as outdoor spaces, but also in their capacity to foster a meaningful relationship with places that are aligned with the FoE under investigation and, at the same time, already embedded in the children's everyday lives. From this perspective, a particularly significant role

was attributed to the park, described by the teachers as a themed park located near the school and equipped with instruments for measuring time through shadows. The choice of this location appears especially meaningful because it constitutes a space closely connected to the topic being explored while also remaining familiar to the students, who visit it outside school hours as well. According to the teachers, the exploration of the park, further enriched by the explanations provided by the park experts about the various instruments present, strengthened the link between educational experience and lived experience [14], reinforcing continuity between school learning and students' everyday relationship with their local area. Similarly, the selection of other spaces, such as the town square, is described by the teachers as a choice that, although at times motivated by practical constraints — such as limited sun exposure or the small size of the schoolyard — maintains a strong connection with places that form part of the children's actual daily experience [14]. In this sense, their reflections suggest that the environment is not treated merely as a logistical backdrop for the activity, but as a constitutive component of the FoE, capable of supporting forms of exploration that are richer in meaning precisely because they are rooted in spaces that already belong to the students' lives.

ST1: I can say that I was particularly impressed by some students who aren't the usual high-achievers who typically participate in class; I actually saw them actively participating, taking charge, and making a meaningful contribution to the activity that had been proposed. [...] At first, I was worried that if I took the students outside, not everyone might pay attention. [...] Most of them understood that this was an educational activity, so their behavior outdoors was just like it was in the classroom.

PT1: I felt the same way whenever I saw some students wandering off... I'd already be giving them the death stare, but maybe they were just exploring another part of the courtyard.

These comments also appear to echo some of the concerns that often accompany teachers' initial perceptions of activities conducted outside the classroom [3]. Their words reveal a tension between, on the one hand, the initial impression that children may be less engaged or less visibly "at work" than in more traditional school settings and, on the other hand, the subsequent recognition that meaningful learning processes are nevertheless taking place throughout the experience [11]. From the teachers' perspective, what may initially appear to be reduced concentration or more dispersed participation seems instead to point to different forms of engagement — freer and less controlled, yet no less relevant in terms of learning. In this sense, their reflections, grounded in students' actual behaviour during the proposed activities, challenge the assumption that learning must necessarily manifest itself through forms that are immediately recognizable as academic. Rather, they suggest that, in outdoor settings, learning can also take shape through exploratory and participatory dynamics that are less rigidly structured.

PT1: There were parents from the preschool coming out with their children, and they often stopped to ask what we were doing.

PT3: It happened to me, too — a mom asked me why we were doing this activity, and I didn't think the child had mentioned it at home because he's a bit reserved. So when she told me, I was really happy to hear that he'd opened up to.

PT4: As for the families, the children told their parents all about what they were doing. In our school complex, we usually invite families to visit two or three times a year to show them the projects we're working on. They came to see these activities

once we had finished them, and the principal — who is always there — joined them, and the families were truly shocked. They hadn't expected anything like that.

The experience also appears to have extended beyond the school setting, involving students' family lives as well (Figure 2)[2]. According to the teachers, the children tended to speak at home about what they had experienced during the activities and, in some cases, spontaneously recreated similar experiences outside school, a tendency encouraged by the fact that the FoE of sun shadows lends itself to exploration across many contexts of daily life. From the teachers' perspective, this continuity between school and out-of-school life generated growing interest among parents, who sought more information both about how the proposed activities were carried out and about the educational rationale underpinning them. In this sense, the teachers view the project as an opportunity to bridge students' personal and school lives, fostering stronger connections among lived experience, learning, and family involvement.

ST1: And then we asked for the help of other teachers of the class who basically assisted during the shadow measurements, at times when we math teachers couldn't be present[...] I can say that we involved the entire school, every single person. . .

ST2: Even the school staff who told us, "Look, teacher, it's starting to rain, you should remove the gnomon. . ."

PT2: The staff member helped us a lot with the activities, and everyone was talking about the gnomon. A colleague went on vacation to the mountains and sent me a photo of a meridian.

PT4: And of course, my colleagues helped me a lot because even when I wasn't in class, they took the kids outside to take measurements, and the other classes got interested too. Once, the third and fourth graders came out with us to see what their classmates were doing and asked a few questions.

As these accounts show, support among colleagues — including those working at the same grade level but in different subject areas — appears to be a key factor in the success of certain activities. The teachers also emphasize how the project strengthened relationships among multiple school stakeholders by involving school staff and teachers from other classes. Finally, according to their reports, the outdoor activities also sparked curiosity among students who were not directly involved and who, on some occasions, asked to take part.

5.2. The digital world and the use of instruments

ST1: I'd like to echo what ST2 said earlier: for secondary school students, it's definitely been a positive experience. It really brought out some of the students' abilities that don't always come to the surface in math class. One girl who doesn't usually speak up surprised me with her analysis of the app.

ST2: I agree because, in fact, for middle school students, digital technology serves as the kind of stimulus that they might have experienced through hands-on activities in elementary school.

The use of the digital app appears to have been a factor that fostered student engagement, particularly among secondary school students. According to the teachers, the use of a digital tool helped draw students into the proposed activities by activating a dimension of their actual experience that is especially meaningful to them [14]. From their perspective, digital technology

is not only an environment that has become pervasive in contemporary life, but also a significant component of students' lived worlds (Figure 2), capable of shaping how they participate in, observe, and relate to the proposed experience. In this sense, the use of the app seems to have strengthened the connection between school activities and domains of experience that students perceive as familiar and relevant.

ST1: The project involved various aspects and, consequently, practical activities such as building our gnomon, the digital component, and the use of various tools. Many students felt capable; in other words, throughout the process, everyone discovered their strengths.

It appears that the variety of activities orchestrated [18] — in particular, the alternation between the use of digital applications and the construction of physical objects within a STEM-oriented framework — encouraged the participation of a large portion of the class. In the teachers' view, this diversification allowed many students to find forms of participation more closely aligned with their interests and strengths. In this way, each student was able to identify an area in which they felt particularly competent and, consequently, to perceive themselves as a valuable resource both in group work and in whole-class discussions. From the teachers' perspective, this variety of approaches to the experience therefore seems to have fostered broader and more inclusive forms of participation.

5.3. Difficulties and curriculum matter

ST1: I have to admit that at many points I didn't feel quite up to the task. I've always needed to consult with my colleagues. [...] I found a great deal of cooperation from my colleagues; I got to know many of them on a professional level — people I hadn't known so well before — so it was certainly very important in terms of internal collaboration among the faculty as well.

PT2: To be honest, I don't usually teach right outside, and even though the approach is, let's say, different from the one I use — which is a bit more mechanical — it was still helpful for me because I learned to teach in a different way. It was obviously difficult.

The teachers report that they did not always feel fully confident in managing practices that were largely new to them [2], and that they often turned to colleagues and didacticians for support. At the same time, the experience appears to have prompted a transformation in their teaching practices. In particular, planning outdoor activities required a high degree of flexibility, both in responding to students' contributions as they emerged and in adapting the work to unforeseen constraints, such as weather conditions.

PT1: It was also very important to adjust our plans, because we had originally drawn up a plan that underwent a lot of changes and revisions along the way. [...] At first, we had planned to do it only in the classroom, but then the sun was out, so we went outside and took advantage of it.

ST1: I think, based on how it went in my class, that the last part was rushed a bit, so I would devote a little more time to the final phase, which involves reflecting on the data collected. [...] I realized I had rushed through it too quickly. [...] As for feasibility, I believe it is repeatable. And if I had to think of a way to improve it, I would make it a multi-year project spanning the three years of lower secondary school.

ST2: before ST1 talking I was thinking, I would do it again but from grade 6, and not in grade 8. And throughout the three years. That would certainly be wonderful, because, as ST1 said, it's precisely those mathematical concepts that evolve.

PT2: I also agree with what my colleagues said — you should make the course a little longer. This year we started in March, so time was a bit tight toward the end, between the INVALSI⁴ tests and the holidays. That part was perhaps covered a bit too hurriedly. So I'd extend it a little bit too, partly because those mathematical concepts can be covered in all grade levels, with some variation depending on the students' age.

These excerpts suggest that integrating the program into the curriculum requires, first of all, a flexible design capable of adapting both to organizational and environmental contingencies and to the actual pace of students' learning. The teachers emphasize that the work required continuous adjustments during implementation and that certain phases — particularly the final phase devoted to reflecting on the collected data — would have benefited from more time. At the same time, the reflections indicate that the program is regarded as sustainable and replicable, but only if it is situated within a broader and less episodic framework. In particular, the idea emerges of integrating it vertically into the lower secondary curriculum, beginning as early as grade 6 and developing it across multiple school years, so as to support the progressive development of the mathematical concepts involved.

6. Discussing the Outfield Education

Through the account of a teaching episode and the reflections shared by the teachers within the community of inquiry, we sought to address RQ1 by examining more deeply the different dimensions of our proposal for outdoor mathematics education, which we present at the end of this paper under the provisional name of Outfield Education (Figure 4). “Outfield” is a term derived from baseball, referring to the outermost part of the playing field, in contrast to the “infield”, which contains the bases. Just as the ball is hit by the batter toward the outfield and then thrown back by the opposing team toward the infield, so too does our educational approach unfold both inside and outside the classroom, through a continuous yet carefully planned alternation intended to support specific educational aims [23]. This term appeals to us because it is a portmanteau of “outdoor education” and “field of experience”.

While each framework contributes essential tools, none alone is sufficient: FoE identifies what is worth exploring mathematically but does not specify how learning unfolds across environments; OE foregrounds experiential and relational dimensions but lacks analytical precision for artefact-mediated orchestration; and IO accounts for the teacher's organization of tools but was developed primarily for classroom settings. Outfield Education names the specific configuration that emerges from their combination: one in which the field of experience itself functions as the primary learning environment, connecting formal inquiry, outdoor exploration, and students' lived worlds into a coherent educational trajectory.

By Outfield Education, we therefore mean an education that takes place out-in-the-field, that is, within the FoE itself and through its direct exploration. In this sense, the aim is not simply to propose mathematical tasks situated in a FoE while still preserving the structure of traditional classroom exercises, such as conventional word problems typically associated with the *infield*. Rather, our goal is to explore the FoE and understand how it works, using mathematics as a guiding discipline in dialogue with other relevant areas of knowledge, including not only STEM disciplines, but also more humanistic ones such as history or art.

⁴National Institute for the Evaluation of the Education and Training System

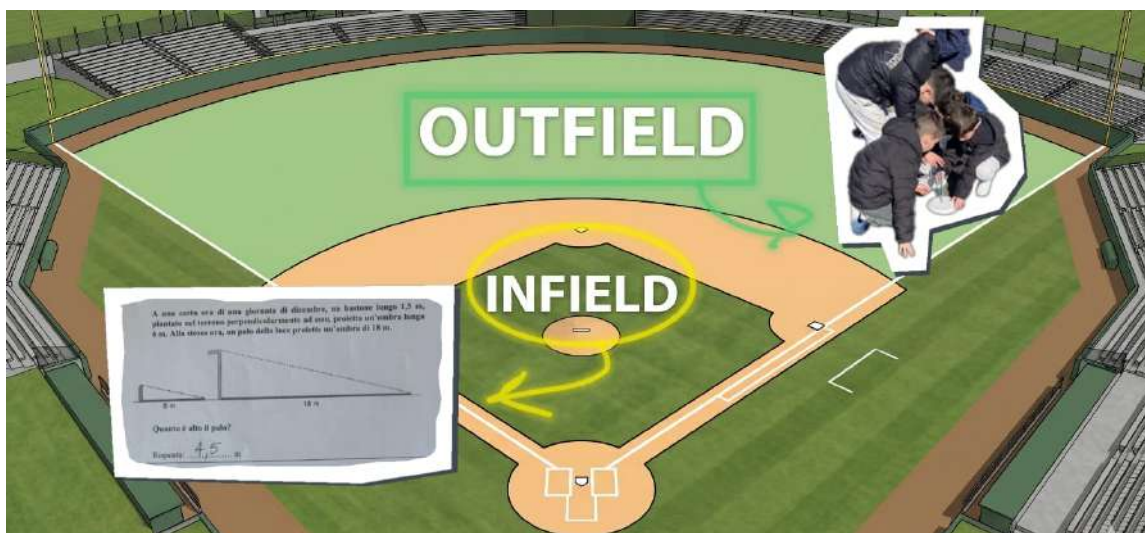


Figure 4. Image that serve as an example to show what we intend with the baseball metaphor used to mint “Outfield Education” (the background is generated through Gemini Flash 3.5 and then edited by the author)

In summarizing the different perspectives on out-of-school mathematics education, Kuş [24] identifies three categories of settings in which learning takes place: (a) everyday life environments, (b) designed learning environments, and (c) activities outside the classroom within schools. In the approach developed by the teachers, designed learning environments were certainly used — understood as informal settings intentionally structured through schedules, the presence of educators, and the provision of technologies and tools to support mathematics learning [25] — given the use of tools and technologies integrated into the learning environment [18]. However, everyday life environments, such as theme parks, and outdoor spaces adjacent to the school were also involved. Some of these choices were made precisely in order to make use of spaces already familiar to the students, where they had previously lived experiences that could intertwine with the FoE exploration activity and further enrich this body of place-related experience. From the perspective of Outfield Education, it therefore seems less important to define one specific learning environment among these three categories than to ensure that the environments serve the purpose of exploring the FoE. The FoE itself thus appears to function as the learning environment that connects the natural world we explore, the classroom space in which we engage in more formal inquiry into our understanding of the FoE, the digital world in which applications allow us to simulate that FoE, and finally the world of the personal sphere of teachers and students, which manifests itself in relationships with people and environments and in life experiences, whether actual or potential [14].

This approach to the exploration and study of the FoE is grounded precisely in relationships and collaboration. With regard to RQ2, the teachers who participated in the focus group highlighted the affordances of working through different approaches and in an outdoor context, which loosens the constraints of the classroom and fosters relationships both within and beyond it. At the same time, they also emphasized the challenges teachers encounter. Recalling the teaching episode, in the more improvisational phases in which the teacher, outdoors, is called upon to engage in didactical performance oriented toward educational aims that had not been planned for that lesson, but perhaps for subsequent ones, the lesson relies heavily on the social construction of knowledge [40]. During the outdoor sequence of the teaching episode, the teacher becomes a semiotic mediator [19]: she observes the students’ work with the tools, identifies meaningful representations through which to explore a mathematical concept [36], directs the whole class’s attention to that representation by asking them to reproduce it, thus

working on the construction of social schemes [16], and then reflects together with them on the mathematics conveyed by that sign. We seek to explore these aspects more deeply by establishing a theoretical connection with the Instrumental Approach [16], which constitutes one of the components of our initial conceptualization of Outfield Education.

If we wish to synthesize an initial definition of what Outfield Education is, we may do so by paraphrasing Donaldson and Donaldson's [15] definition, "Outdoor education is in, about and for the outdoors", which, in our case, becomes: "Outfield Education is in, about, and through the field of experience". It refers to the way in which the teacher situates the educational process *in* the field of experience; learning is *about* the field of experience, and it is *through* education conducted within the field of experience that mathematics is learned.

7. Where are we heading now? Limitations and future perspectives

This paper presents the more theoretical contribution that emerged from a collaborative effort carried out by a very specific group of teachers who had been working for years with researchers from the University of Eastern Piedmont. Throughout the 2024/2025 academic year and into the following one, teachers and researchers worked together by revisiting previous studies on teaching mathematics through the FoE of the sun shadows [14], exploring what it means to teach through an approach that was new to them, such as Outdoor Education, and discussing for the first time the integration of digital technologies into activities and teaching. Such in-depth work was made possible by the particular nature of this group, but its structure and intensity are clearly not easily replicable.

For this reason, the present study is intended as an initial pilot and exploratory investigation, to be followed by further discussions with other groups of teachers, for whom accessible and replicable teacher education pathways will be developed on the basis of the observations gathered from this first pilot group. The aim is therefore to identify, through ongoing dialogue with the teachers who will gradually take part in these cycles, design principles [35] for teacher education in a practice such as Outfield Education, while also using this joint work and collaboration to deepen the more theoretical perspective pursued through RQ1. Since this is only an initial conceptualization, it will need to be progressively refined and clarified over time, always in collaboration with teachers. This reflects the nature of Educational Design Research [35], which offers both a practical contribution — in this case, concerning the design of a possible teacher education pathway — and a theoretical one, aimed at clarifying the teaching and learning approach on which this work is based.

Regarding RQ2, the teachers' reflections in the focus group highlighted two issues that will need to be addressed in the near future. The first concerns teacher education on this instructional approach, which differs from more traditional classroom practices while also building on them, particularly because some teachers in the community felt that they did not yet have sufficient knowledge or confidence to lead such activities. Within these teacher education pathways, Outfield Education will be studied, implemented, and further explored, since it is still at an early stage of development and requires additional refinement. Moreover, because the present research focuses primarily on teachers, the further development of Outfield Education will also require greater attention to students and their learning, which should be examined directly rather than only through teachers' accounts.

The second issue concerns curricular integration, which the teachers identified as a central challenge and one that could be addressed by distributing such activities across more than one school year. This will involve considering multi-year teacher education pathways capable of supporting teachers in the implementation of these practices, while also creating opportunities for discussion with them in order to further refine and improve teaching practice and to engage with theoretical issues and questions emerging from scientific research [41].

A further development will concern the use of video-stimulated reflection after the second

year of implementation. Whereas the focus group discussed in this paper was organized around teachers' general reflections on the experience, future discussions will be grounded in selected classroom episodes. This will make it possible for teachers and didacticians to examine more closely how mathematical opportunities emerge during outdoor activity, how teachers recognize and respond to students' actions, and how artefacts and environments are orchestrated in real time.

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Conflict of interest

The authors declare that there are no conflicts of interest.

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Mathematical modelling in outdoor learning: exploring analog strategies and cooperative processes in a primary school context

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Abstract: This study explores the design and implementation of an outdoor mathematics learning experience aimed at fostering mathematical modelling, problem solving, and student engagement in a primary school context. The intervention, conducted in a fifth-grade class, was structured around a sequence of outdoor tasks designed by the MathCityMap platform, but adapted to a non-digital format due to school constraints on smartphone use. Grounded in the modelling cycle for the outdoor context, the study examines how alternative instructional strategies can support mathematical learning processes typically mediated by digital tools, as well as the role of cooperative learning in shaping students’ participation and understanding. Data were collected through audio recordings, students’ written work, and classroom observations, with a qualitative focus on a case study group. The findings show that analog strategies, such as treasure-hunt activities for spatial orientation and structured paper-based hints, can effectively sustain key processes of modelling and problem solving. At the same time, cooperative group work emerged as a crucial factor in promoting engagement and shared reasoning, although it did not always ensure correct interpretation and validation of results. The study highlights the potential of outdoor mathematics education, even in the absence of technology, while also emphasizing the complementary role of digital tools in supporting feedback and assessment processes.

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Keywords: outdoor mathematics; mathematical modelling; MathCityMap; problem solving; cooperative learning.

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1. Introduction

The teaching and learning of mathematics are increasingly evolving in response to the challenges of a globalized and technologically mediated society. Traditionally perceived as an abstract discipline, mathematics is now considered in a more dynamic and concrete way, thanks to approaches that emphasize authentic experiences, situated learning, and active student participation. Within this perspective, outdoor learning has emerged as promising pedagogical approach capable of connecting mathematical concepts with meaningful real-world context (Ford, 1986; García-González & Schenetti, 2022).

Outdoor mathematics education goes beyond simply relocating lessons outside the classroom. Rather, it represents a pedagogical vision centered on direct experience, interaction with the environment, and embodied learning processes (Kelton & Ma, 2018). Through activities situated in natural or urban settings, students are encouraged to explore mathematical ideas through movement, observation, measurement, and collaboration, fostering a more concrete and experiential understanding of mathematics (Taranto et al., 2021). In these contexts, mathematics becomes a tool for interpreting and describing reality, supporting the development of problem-solving abilities, critical thinking, and collaborative/cooperative skills.

In recent years, growing attention has been devoted to outdoor mathematics activities such as math trails, in which students solve contextualized mathematical problems connected to elements of the surrounding environment. Digital tools have further expanded the possibilities for designing and implementing these activities, supporting exploration, immediate feedback, and situated problem solving (Buchholtz et al., 2020; Fessakis et al., 2018; Ludwig & Jesberg, 2015). At the same time, research has increasingly highlighted the educational potential of mathematical modelling processes developed within authentic contexts. However, the literature also presents some limitations. First, many studies focus primarily on technology-supported implementations, while less attention has been devoted to contexts in which digital tools are unavailable or restricted. Second, although outdoor mathematics is frequently associated with increased engagement and meaningful learning, findings are not always unequivocally positive, as the effectiveness of outdoor activities strongly depends on task design, teacher guidance, and the balance between exploration and mathematical focus. Finally, limited research has investigated how cooperative learning dynamics and non-digital scaffolding strategies support students' mathematical modelling processes during outdoor activities in primary education.

To address these gaps, the present study investigates the implementation of non-digital math trails in a fifth-grade primary school context. In particular, the study examines how alternative teaching strategies can replace digital tools while maintaining the educational value of outdoor mathematics activities, and how cooperative learning influences students' participation and mathematical understanding during the modelling process.

2. Literature review

2.1. Outdoor mathematics education

Outdoor learning has increasingly attracted attention within mathematics education research as a way of promoting meaningful, experiential, and embodied learning processes. By engaging directly with natural and urban environments, students can explore mathematical concepts through observation, movement, and interaction with real-world contexts (Zender et al., 2020). Mathematics, experienced in this way, becomes a dynamic activity connected to authentic and everyday contexts, fostering a deeper understanding of mathematical concepts and supporting the construction of more solid knowledge (Jablonski, 2025).

Several studies highlight how outdoor mathematics activities may positively influence students' engagement, motivation, and participation. Activities such as measuring objects in the environment, estimating distances, identifying geometric configurations, or collecting and representing

data allow students to connect abstract concepts with concrete experiences (Barbosa, 2006; Schukajlow et al., 2022; Wiehe et al., 2025). Outdoor contexts may also encourage collaborative problem solving and peer interaction, contributing to the development of communication and cooperation skills (Fägerstam & Blom, 2013; Johnson & Johnson, 1999). From a pedagogical perspective, outdoor mathematics is often associated with constructivist and inquiry-based approaches in which students actively participate in the construction of knowledge. In these settings, the teacher assumes the role of facilitator, supporting exploration, discussion, and reflection rather than simply transmitting information (Hmelo-Silver et al., 2007). Furthermore, embodied and multisensory experiences may support students' concentration, emotional involvement, and mathematical understanding by integrating cognitive, physical, and social dimensions of learning (Ferrari & Taranto, 2024).

Outdoor mathematics education also promotes interdisciplinary connections between mathematics and other subjects such as science, geography, and art, helping students perceive mathematics as meaningful and relevant within broader real-world contexts (Beames et al., 2012). At the same time, the literature also reports some challenges and mixed findings concerning outdoor mathematics education. Some studies suggest that outdoor settings alone do not automatically increase students' interest, enjoyment, or mathematical understanding. Rather, the effectiveness of outdoor activities appears to depend strongly on factors such as task authenticity, the quality of scaffolding, and the organization of collaborative work (Hartmann & Schukajlow, 2021). Similarly, recent research comparing different modelling settings highlights that outdoor environments may foster specific modelling processes, while also introducing additional challenges related to data collection, interpretation, and cognitive management (Jablonski, 2025).

2.2. Math trails: digital and non-digital approaches

Among the different approaches developed within outdoor mathematics education, outdoor mathematics trails, or math trails, represent one of the most widespread and established formats. Math trails are introduced as structured itineraries in natural or urban environments where students engage with authentic problem situations closely connected to the context. These trails transform real space into a learning environment in which mathematics emerges through observation, measurement, and interpretation of elements in the surrounding environment, fostering processes of modeling, argumentation, and problem solving (Jablonski, 2025). Traditionally, math trails have been implemented through paper-based tasks and teacher-guided activities (Blane & Clarke, 1984). More recently, digital technologies have introduced new possibilities for designing and managing outdoor mathematical experiences. One of the most prominent examples is MathCityMap (<https://mathcitymap.eu/en/>), widely used for designing and implementing the math trails. It is a geolocation-based digital platform that guides students along a trail composed of situated mathematical "tasks": using mobile devices, students reach specific points in physical space and, once there, activate the problem to be solved. Geolocation thus directly links mathematical content to the real context, making the experience highly situated and interactive. Another distinctive feature is the presence of progressive hints, which support students when they encounter difficulties without revealing the solution, thereby fostering scaffolding processes and autonomy. Furthermore, the system includes mechanisms for immediate validation of answers: students receive automatic feedback, often based on numerical tolerances, which allows them to verify the correctness of their results and reflect on the procedures used (Jablonski, 2024).

From a pedagogical perspective, math trails generally develop according to a cooperative methodology. Students work in small groups, within which specific roles are assigned. For example, one student manages the device and navigation, another is responsible for measurements, and another coordinates calculations and discussion, thus fostering active participation

and shared responsibility among all members. This organization promotes not only the learning of mathematical content but also the development of transversal skills such as cooperation, communication, and critical thinking. In this way, math trails, supported by digital tools such as MathCityMap, create a dynamic learning environment in which experience, technology, and social interaction are integrated, making mathematics more accessible, meaningful, and deeply connected to reality.

Several studies report positive effects of digital math trails on students' engagement and motivation (Ludwig et al., 2021; Jablonski, 2024). However, scholars also underline that the educational effectiveness of these tools depends not only on technology itself but also on the pedagogical quality of task design and classroom orchestration (Ratnayake et al., 2020). Furthermore, recent restrictions on the use of mobile devices in some school contexts raise the need to investigate alternative ways of implementing math trails without losing their exploratory, cooperative, and contextualized nature. From this perspective, non-digital adaptations may represent a relevant area for further research.

Within this framework, the present study explores the implementation of a non-digital outdoor math trail in a primary school context. While inspired by the MathCityMap approach to task design and outdoor mathematical exploration, the activities were redesigned in an analog format due to restrictions on the use of mobile devices from the students. In particular, the study investigates how key features of math trails, such as contextualized problem solving, cooperation, and scaffolding through discussion and comparison, can be maintained in the absence of digital technologies. In doing so, the study aims to contribute to current discussions on the adaptation of outdoor mathematics activities to different educational contexts and practical constraints.

3. Theoretical framework and research questions

Mathematical modeling represents a central element in mathematics education, as it makes it possible to connect abstract concepts with real-world situations, fostering meaningful and contextualized learning (Blum et al., 2007; Kaiser & Sriraman, 2006). Through modeling, students are required to interpret authentic problems, select relevant information, and translate situations into mathematical terms, thereby developing problem-solving skills and critical thinking.

Unlike traditional exercises, modeling problems are characterized by their open and unstructured nature, requiring students to actively construct solution strategies. This process involves moving from a real situation to a mathematical representation through the identification of variables, relationships, and constraints, which can be expressed through formulas, graphs, or diagrams (Kaiser, 2007). A distinctive feature of modeling is its iterative nature: students formulate hypotheses, build models, and compare them with reality, revising and adapting them progressively. This dynamic fosters not only conceptual understanding but also cognitive flexibility and the ability to reflect on one's own thinking processes.

Over the past two decades, research on mathematical modelling has considerably expanded, moving beyond predominantly cognitive interpretations of modelling processes toward socio-cultural, embodied, and situated perspectives (Borromeo Ferri, 2018; Greefrath et al., 2023). Recent studies increasingly highlight the importance of authentic contexts, collaboration, multi-modality, and inquiry-based environments in fostering students' modelling competencies (Maass et al., 2019; González-Martín et al., 2021).

Within the classical modelling tradition, the modelling cycle proposed by Blum and Leiß (2007) represents one of the most influential theoretical frameworks. The model conceptualizes modelling as a cyclical process involving several interconnected phases: understanding and simplifying a real-world situation, mathematization, mathematical work, interpretation, and validation of results. The cyclical nature of the process emphasizes that modelling is not linear,

but rather characterized by continuous transitions between reality and mathematics through revisions, reinterpretations, and adaptations.

More recent research has further emphasized that modelling processes are deeply shaped by the environments in which they take place. In outdoor mathematics activities, students do not simply apply mathematical concepts to external situations; rather, they interact bodily, socially, and materially with the surrounding environment while constructing mathematical meaning (Ferrari & Taranto, 2024). Outdoor contexts therefore transform modelling into an embodied and situated activity in which movement, perception, spatial orientation, and interaction with real objects become integral parts of the learning process. From this perspective, modelling is no longer interpreted solely as a transfer between a reality domain and a mathematical domain, but also as a process mediated by bodily engagement, collaborative interactions, and the ways students orient themselves within the environment (Ferrari & Taranto, 2024).

For this reason, the present study adopts the adaptation of the modelling cycle for outdoor contexts (Figure 1) proposed by Jablonski and Bakos (2022), which explicitly emphasizes the transitions between reality and mathematics during outdoor mathematical activities. In this model, students continuously move between understanding and structuring the outdoor context, mathematizing real situations, interpreting mathematical results, and validating solutions within the physical environment itself. The framework is particularly suitable for analysing math trails because it highlights how modelling processes emerge through direct interaction with space, measurements, estimation, and contextual constraints.

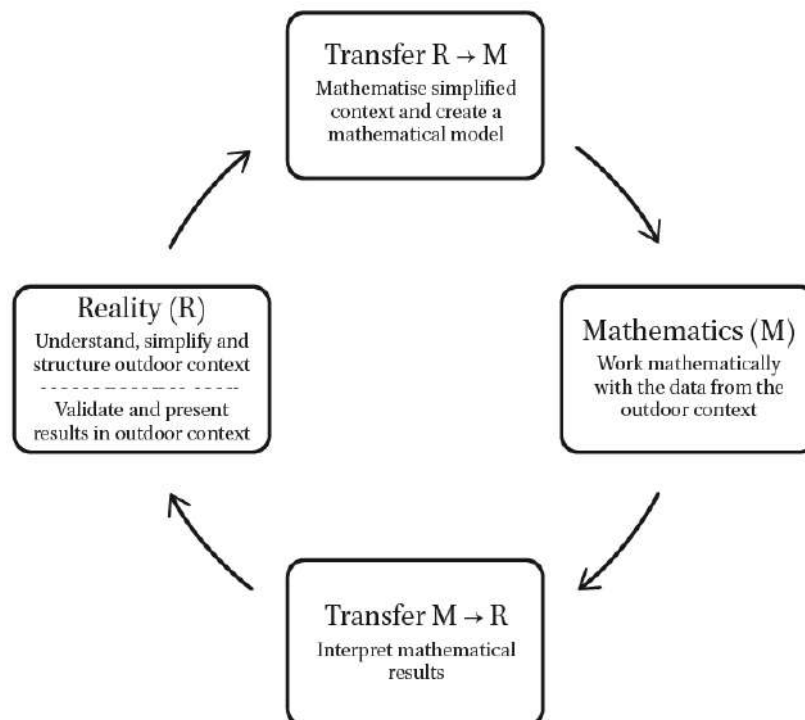


Figure 1. Modelling cycle for the outdoor context (from Jablonski & Bakos, 2022, p. 4)

At the same time, modeling activities in outdoor contexts often take place through cooperation among students, fostering cooperative learning dynamics. Peer interaction allows students to articulate their reasoning, negotiate meanings, and construct shared solutions. In this sense, the social dimension of the modeling process intertwines with the cognitive one, contributing to the development of transversal skills such as communication, cooperation, and argumentation (Jablonski, 2025). Furthermore, the introduction of problem-solving support systems, such as

structured hints, plays a significant role. These tools can be interpreted as forms of scaffolding that support students through the different phases of the modeling cycle, particularly in moments of difficulty, without replacing the process of constructing the solution (Taranto & Jablonski, in press). In the absence of digital tools, the adoption of alternative strategies still allows this support function to remain active, promoting students' autonomy and reflection.

In the present study, mathematical modelling is therefore interpreted as a situated, cooperative, and embodied process. The integration of outdoor learning, cooperative interactions, and non-digital scaffolding strategies provides a particularly meaningful context for investigating how students construct mathematical understanding while engaging with authentic problems in real environments.

In light of these theoretical considerations, the present study aims to investigate the following research questions:

- i) Which teaching strategies can effectively replace digital tools without reducing the quality of mathematical learning in math trails?
- ii) How does cooperative learning influence students' participation and mathematical understanding during math trails activities?

4. Research context and method

The design of the instructional intervention is part of a math trail, with the aim of integrating concrete experience and mathematical reflection, promoting active, cooperative, and contextualized learning. The classroom context is understood not only as a physical space, but as a dynamic learning environment, constructed through relationships, interactions, and shared experiences, which also extends to the school's outdoor spaces. From this perspective, the activities were designed to enhance the connection between field exploration and classroom formalization, fostering the development of both disciplinary and transversal skills.

The study was conducted in a fifth-grade class at the "Via Puglia" primary school of the "Leonardo Sciascia" Comprehensive Institute in Misterbianco (Catania), located in a peripheral area of the municipality. The school involved offers numerous spaces and resources, including a gym, an outdoor sports field, a library, laboratories, and environments dedicated to innovative activities: elements that made it possible to implement a structured math trail. The second author of the article carried out a teaching internship in this class during the fifth year of the Primary Education degree program; the contribution presented here derives from the results of her master's thesis, conducted under the supervision of the first author.

The participating class, a fifth-grade group of 23 students, ranges in ages from 10 to 11; 10 of the students are girls and 13 are boys. The class shows an overall average level of mathematical proficiency and is characterized by a certain heterogeneity, which allowed for observing the effectiveness of the proposed activities across different levels of competence. One student with Special Educational Needs is present, supported by a special education teacher, which required particular attention in designing inclusive and accessible activities. The students had already acquired arithmetic, geometric, and logical reasoning skills, which were further consolidated through the proposed math trail. The intervention took place during regular mathematics lessons over three days, between late April and early May 2024.

The activities are structured as a math trail composed of authentic tasks designed through the MathCityMap platform; however, due to the school regulation prohibiting the use of smartphones during instructional activities, it was not possible to use the application in its digital form. It was therefore necessary to adopt alternative approaches to compensate for the absence of features such as geolocation and hints, while maintaining the structure and educational objectives of the trail. To this end, a printed booklet containing the entire math trail was downloaded from the MathCityMap platform (available at the following link: <https://url-shortener.me/KJG4>, including a map, task descriptions, images of reference objects, instruc-

tions, and spaces for carrying out calculations and recording solutions, with the aim of connecting disciplinary content to real-life situations.

The tasks cover the main thematic areas outlined in the Italian national curriculum (Numbers; Space and figures; Relationships, data and predictions) and address concepts such as area, perimeter, angles, length, volume, fractions, and time measurement. Presented as authentic tasks, they require students to observe, measure, compare, and calculate using concrete tools (i.e., measuring tape, protractor, . . .), fostering modeling and problem-solving processes in real contexts. The entire math trail is oriented toward the development of transversal skills such as logical reasoning, critical observation, communication, and cooperation.

4.1. Design of the instructional intervention

The design is structured into three main phases.

4.1.1. Design and implementation of Phase 1

Attention was given to the initial activities, designed to prepare students both from an organizational and a relational perspective. Given the prohibition on smartphone use and the resulting impossibility of accessing MathCityMap's geolocation features, it was necessary to design alternative activities aimed at fostering spatial orientation. To this end, the introductory phase was structured into two distinct sub-phases: an initial treasure hunt activity aimed at developing orientation skills and familiarization with the environment, followed by a cooperative physical game, "Untangle the Knot", intended to strengthen group relational dynamics. In the treasure hunt, through riddles and multidisciplinary challenges distributed across the school spaces, students were guided to orient themselves within the school environment and to become familiar with the locations where the mathematical tasks would later take place. The stops of the treasure hunt did not coincide exactly with the locations of the MathCityMap tasks, but were situated in nearby areas, so as to encourage gradual familiarization with the spaces involved in the math trail. Organized into two teams, the children followed different routes, solving puzzles and completing challenges to progress in the game. This activity fostered not only spatial orientation but also active engagement and early collaborative dynamics among peers. Furthermore, this phase allowed the second author (i.e., the experimenter) to observe the interactions among the students and to form heterogeneous group suitable for the subsequent phase involving the mathematics trail.

The "Untangle the Knot" game aimed instead to introduce students to cooperative work. In this activity, children, divided into groups, were involved in a physical problem-solving task: untying a human knot without letting go of each other's hands. Within each group, 5 children stood in a circle holding hands and served as the "inner" members, while 1 or 2 classmates remained outside the circle as observers and facilitators. To create the knot, the children inside the circle turned their arms and hands in a "whirling" motion (Figure 2). At the experimenter's signal, the hands had to be extended towards the center of the circle along an imaginary vertical line (Figure 3). Starting from the top, the hands were to be held in pairs until a human tangle was formed. The participants outside the circle were responsible for carefully observing the configuration of the tangles and providing strategic instructions to their classmates inside the circle, while the latter carried out the necessary actions to untangle the knots in their hands. The instructions mainly involved spatial and topological movements, such as "raise your right arm", "turn to the left" or "pass under your arm". This experience made it possible to bring out group dynamics, roles, and communication patterns, preparing students for the cooperative work required in the following activities and fostering the development of shared strategies.



Figure 2. "Whirling" motion



Figure 3. Arms extended toward the center of the circle

4.1.2. Design and implementation of Phase 2

The second phase represents the core of the pathway and involves the implementation of a math trail structured into 12 tasks. The tasks can be accessed through the printed booklet <https://url-shortener.me/KJG4> or via the free MathCityMap app by entering the code "3920683".

Students, divided into groups of 3-4 members, take on specific roles: the secretary, responsible for completing the printed booklet; the measurer, in charge of using measurement tools (rigid ruler, measuring tape, protractor, calculator); the navigator, who guides the group using the map; and, in groups of 4 members, the timekeeper, whose task is to monitor the duration of the activities and signal when it is time to move on to the next task after approximately 10–15 minutes of work.

To address the impossibility of using the MathCityMap application and accessing digital hints, an alternative system based on envelopes containing paper slips was introduced. In case of difficulty, each group could request a limited number of hints by turning to a designated reference figure: the experimenter, acting as the researcher, or one of the classroom teachers (the mathematics teacher and the support teacher), who primarily acted as observers and, in this phase, as facilitators of the activity. The hints, provided one envelope at a time for a maximum of three per task, corresponded to those originally designed within the tasks on MathCityMap. This system served as a mediating tool in the problem-solving process, supporting students' autonomy without oversimplifying the task and fostering the development of solution strategies.

The structure of the math trail, with staggered starting points for each group, ensured effective space management and avoided overlap, allowing each group to work independently. Although the MathCityMap platform offers useful tools for managing flows, even in the absence of technology the experimenter organized the start of the activities by assigning each group a different initial task and planning a cyclical progression along the trail. This solution made it possible to optimize movement and prevent multiple groups from concentrating on the same task at the same time, maintaining a good balance between operational autonomy and overall activity management.

4.1.3. Design and implementation of Phase 3

The third phase takes place in the classroom and is dedicated to reflection and the re-elaboration of the experience. Through a whole-class discussion, the completed tasks are revisited, with particular attention to those that posed greater difficulties. The experimenter guides the discussion through prompting questions, encouraging students to articulate the strategies they adopted, compare solutions, and reflect on errors. This moment makes it possible to consolidate learning and to make explicit the mathematical processes activated during the experience. Finally, an evaluation of the experience is proposed, in which students are invited to express their perspectives on both the emotional and cognitive aspects of the trail. This phase values the subjective dimension of learning and provides useful insights for assessing the effectiveness of the intervention.

4.2. Data analysis

In analyzing the data related to Phase 2, a qualitative case study approach was adopted, combining thematic analysis with theory-driven coding informed by the modelling cycle for outdoor contexts (Jablonski & Bakos, 2022). One group was selected as a case study in order to examine in depth the interaction dynamics and problem-solving processes enacted during the math trail. In particular, Group A (one of the five groups involved), composed of three students with heterogeneous levels of competence, was selected because its balanced composition was considered meaningful for observing processes of negotiation, cooperation, and shared knowledge construction in an authentic context.

Within the data corpus, two tasks were identified as particularly significant in relation to the research questions (Table 1): Task 2 (Cultural exchange), in which difficulties emerged, and Task 6 (The time of the swings), which was completed successfully.

Both tasks, like all tasks that constitute a math trail, required not only the use of the data provided in the instructions, but also the direct collection of certain information through observation and measurement of the surrounding environment. To make this point clearer and assist the reader, the table below indicates in square brackets which data must be measured on-site and which must instead be gathered through direct observation, as they are intentionally not provided in the task instructions, in line with the design principles of outdoor math trail tasks.

The analysis aimed at examining students' situated mathematical activity within the outdoor learning environment. In line with the theoretical framework presented above, particular attention was devoted not only to the modelling processes activated during task resolution, but also to the cooperative, contextual, and embodied dimensions of students' interactions. The selection of the episodes was guided by the presence of rich interactions, explicit negotiation of modelling strategies, and the use of the hint system, which represented a central element of the instructional design. In particular, Task 6 makes it possible to observe how cooperation and scaffolding through hints supported the successful completion of the modeling process, whereas Task 2 allows for the analysis of a situation in which, despite the presence of cooperative strategies and support, the modeling cycle remains incomplete, especially in the phases of interpretation and validation.

The data corpus consisted of video recordings, field notes collected during the activities, and students' written productions contained in the printed booklets. The analysis focused on episodes characterized by negotiation of strategies, use of hints, cooperative interactions, and moments of difficulty or revision during problem solving. An iterative thematic analysis was conducted through repeated observation of the selected episodes. The coding process was informed by the modelling cycle for outdoor contexts (Jablonski & Bakos, 2022), with particular attention to transitions between understanding the real situation, mathematization, interpretation, and validation. The coding and interpretation of the episodes were discussed



Task	Instruction	Sample Solution	Hints
<p>Cultural Exchange</p> 	<p>A school welcomes a group of 10 foreign children for a cultural exchange. Consider the sofa and the pouf as a single seating space, and note that there are two sofas and two poufs. Knowing that each seated child occupies 40 cm of length, how many children remain standing?</p>	<p>Sofa length (obtained through direct measurement) = 90 cm. Pouf length (obtained through direct measurement) = 45 cm. Total seating space:</p> $90 + 45 = 135 \text{ cm}$ <p>Number of children who can sit:</p> $135 \div 40 = 3$ <p>Since there are two sofas and two poufs:</p> $3 \times 2 = 6$ <p>Therefore, 6 children can sit and</p> $10 - 6 = 4$ <p>children remain standing.</p>	<ol style="list-style-type: none"> 1. Measure the length of the sofa and the pouf. 2. To determine how many children can sit, divide the available seating space by the space occupied by one seated child.
<p>The Time of the Swings</p> 	<p>A group of 24 children goes to the playground to use the swings. At 4:40 p.m., all swings are occupied and each child uses a swing for 10 minutes. At exactly 4:50 p.m., they take turns with other children. How many more children will be able to use the swings until 5:30 p.m.?</p>	<p>There are 4 swings in the playground (obtained through direct observation). From 4:40 p.m. to 5:30 p.m., 50 minutes pass.</p> $50 \div 10 = 5$ <p>groups of children can take turns. Therefore,</p> $5 \times 4 = 20$ <p>children can use the swings.</p>	<ol style="list-style-type: none"> 1. Count the number of swings in the playground. 2. Pay attention to the elapsed time.

Table 1. Examples of mathematical tasks proposed in the playground.

repeatedly by the authors in order to ensure coherence between the analytical categories and the theoretical framework.

5. Implementation of the study

The instructional intervention took place between late April and early May 2024, for a total of 6 hours divided into three sessions over the course of one week. As mentioned above, two sessions were conducted outdoors, promoting experiential learning, while the final session took place in the classroom, with the aim of revisiting and consolidating the knowledge acquired.

5.1. Phase 1

The first day was designed to familiarize students with the school environment and to prepare them for cooperative work required in the subsequent mathe trail activities. Since digital devices could not be used and students therefore had no access to MathCityMap geolocation tools, a treasure hunt was organized to foster spatial orientation skills.

Students were divided into two teams and followed different routes through the school spaces by solving riddles and completing multidisciplinary tasks. The activity required students to interpret clues, identify landmarks within the environment, and move through the school using spatial references. In this way, students progressively developed orientation skills while collaboratively negotiating hypotheses and solutions. The riddles guided the groups toward locations such as doors, stairs, windows, and gates, encouraging direct interaction with the surrounding environment (e.g., Figure 4).



Figure 4. One of the teams searching for the window

The second part of the first day focused more explicitly on cooperation and spatial communication through the “Untangle the Knot!” game. Students worked in groups, with some children forming a human knot and others acting as external guides. The task required the group to untangle the structure without releasing their hands.

During the activity, students continuously used topological and directional language such as “go under,” “turn to the left,” “raise your arms,” and “step over,” demonstrating the active use of spatial references to coordinate collective actions (see Figures 5 and 6). The effectiveness of the task depended on the group’s ability to communicate clearly, negotiate strategies, and cooperate in solving the problem.

The activity therefore represented an authentic cooperative problem-solving situation in which spatial orientation, verbal communication, and mutual coordination were strictly interconnected. Consistent with cooperative learning principles (Johnson & Johnson, 1999), students’ success depended on the group’s capacity to act collectively rather than on individual performance alone.

Overall, the first phase allowed students to become familiar with the school environment, strengthen spatial orientation skills, and develop cooperative strategies that were fundamental for the subsequent math trail activities.



Figure 5. Movements of the first group during the “Untangle the Knot” game



Figure 6. Movements of the second group during the “Untangle the Knot” game

5.2. Phase 2

The second day of the intervention was dedicated to carrying out a math trail designed using MathCityMap. The session began in the classroom with an introductory explanation, during which the experimenter illustrated the operational procedures of the activity, with the aim of actively and consciously engaging the students. The intention was to promote dynamic learning, capable of combining theoretical knowledge with concrete applications in real contexts. Subsequently, the students were divided into five groups (A, B, C, D, E), each consisting of 3 or 4 members. Within each group, specific roles were assigned: secretary, measurer, navigator, and, in groups of four, the timekeeper. This organization encouraged active and responsible participation from all students, structuring the work in a cooperative way.

As digital devices could not be used, the groups oriented themselves using a printed map of the tasks (Figure 7), following the math trail with its 12 activities distributed across the school spaces. In case of difficulty, each group could request hints from a previously assigned reference figure (i.e., the experimenter or one of the teachers). In case of difficulty, each group could request hints from a previously assigned reference figure (i.e., the experimenter or one of the teachers).

The following analysis focuses on the implementation of Tasks 2 and 6 by Group A, composed



Figure 7. MathCityMap map with the tasks to be completed

of three students with well-defined roles: Giuly, the secretary; Alex, the measurer; Viky, the navigator.

5.2.1. Task 2 – Cultural exchange

During the implementation of the task “Cultural exchange”, Group A immediately showed intense peer interaction, characterized by attempts at shared interpretation and negotiation of strategies. After reading the instructions, the students correctly identified the need to take measurements, activating the measurer:

G.: “You have to measure the sofa”

V.: “You have to measure everything, so the pouf as well, I think”

Once the data had been collected, the group began discussing the operations to be carried out, but showed uncertainty in structuring the problem mathematically. Proposals followed one another rapidly, without a clearly shared direction:

A.: “The total length is 136 cm. . . so now we have to do $136 \div 40$ ”

V.: “But we have to consider that there are two”

G.: “Then we have to do $136 + 136$ ”

V.: “So 136×2 ”

This phase highlights how the students began to construct a model of the situation but were unable to stabilize a coherent representation of the problem. Peer interaction remained active but did not immediately lead to an effective solution strategy:

A.: “Should we do $400 \div 272$?”

G.: “Maybe the other way around. . . $272 \div 400$ ”

A turning point occurred with the use of hints. After recognizing the limited usefulness of the first hint (see Table 1), the group accessed the second one (see Table 1), which oriented their reasoning toward a correct relationship between the quantities:

V.: “It was complicated. . . now I understand what we have to do”

Despite this, a significant issue emerged in the final phase. The students correctly performed the calculations but accepted a result that was not consistent with the real context:

V.: “3.2 remain standing”

This episode is particularly relevant from a theoretical perspective. The group was able to activate several phases of the modeling process, from initial understanding to mathematization and mathematical work, but showed difficulties in the interpretation and validation phase. The acceptance of a non-integer value when referring to a number of people reveals a lack of reflection on the plausibility of the solution, interrupting the modeling cycle.

At the same time, the task highlights the ambivalent role of cooperation: while peer interaction supports the construction of reasoning and helps overcome moments of impasse, it does not

automatically ensure metacognitive control over the final result. In this sense, the hint system appears to be an effective support for the mathematization phase, but insufficient to sustain validation processes, which require a more advanced level of mathematical awareness.

5.2.2. Task 6 – *The time of the swings*

In the task “The time of the swings”, Group A engages with a problem situation that requires the integration of spatial and temporal information. In an initial phase, the students correctly identify some elements of the context, such as the number of swings available, but struggle to coordinate the data to construct a solution strategy:

G.: “There are 4 swings”

V.: “Should we do $24 \div 4$?”

A.: “But what is the question?”

This uncertainty leads the group to request a hint. The first hint (see Table 1), related to counting the swings, does not provide new information, while the second (see Table 1) introduces a decisive element:

V.: “So 4 children every 10 minutes can use the swings”

From this point, a significant change in group dynamics can be observed. The students begin to progressively construct the solution, making each step explicit and coordinating their reasoning:

G.: “In 10 minutes 4 children use the swings [...] if another 10 minutes pass, that’s 8 children”

A.: “And it’s 5:10 p.m.”

G.: “At 5:20 p.m. we reach 16 children”

G., V., A.: “At 5:30 p.m. it’s 20 children!”

The process develops through continuous interaction among group members, in which each student contributes to building the solution, integrating and completing each other’s reasoning. In this case, cooperation goes beyond simply sharing ideas, becoming a genuine process of co-construction of knowledge.

From a theoretical perspective, the group is able to complete all phases of the modeling cycle. After an initial difficulty in understanding and structuring the problem, the hint helps guide the mathematization correctly, transforming the real situation into an organized temporal sequence. The mathematical work develops through an iterative progression, while the interpretation of the result appears consistent with the context.

In this case, the hint system performs a particularly effective scaffolding function, supporting the group in overcoming a cognitive obstacle without directly providing the solution. Peer cooperation also contributes to distributing the cognitive load and making the reasoning steps explicit, fostering a shared understanding of the problem.

The comparison with the previous task highlights how, despite having the same tools and instructional organization, the outcome of the process depends on the group’s ability to integrate the different phases of modeling, particularly those related to interpretation and validation, which are crucial for successful problem solving.

5.3. Phase 3

The intervention continued the following day with the third session, held in the classroom and dedicated to reflection and discussion. In this concluding phase, the experimenter guided the students in analyzing the completed tasks, following a chronological order and focusing in particular on those that had presented greater difficulties. This work was made possible through audio recordings of Group A’s conversations during the activities and the review of the booklets completed by all groups. Based on these materials, targeted questions were prepared, also differentiated according to the roles assumed by the students during the math trails.

Within this reflective phase, it was possible to interpret the processes enacted by the students through the modeling cycle for outdoor contexts, which proved useful for analyzing the degree of completeness of the problem-solving process.

The analysis also highlighted how, in the absence of the immediate validation typical of the digital MathCityMap environment, errors tend to emerge mainly during the subsequent review phase. This made it possible to reflect on the value of the adopted hint system (namely, the use of envelopes) which proved effective in supporting the problem-solving process, although it did not fully replace the feedback functionalities offered by the digital platform.

These findings also resonate with research highlighting the complexity of modelling processes for learners, particularly in authentic and outdoor contexts. Although the present study adopts the outdoor adaptation of the modelling cycle proposed by Jablonski and Bakos (2022), this framework still reflects the challenges originally discussed within the modelling tradition developed by Blum and Leiß (2007), especially regarding students' difficulties in coordinating interpretation, validation, and transitions between the outdoor context and mathematical representations. From this perspective, the hint system adopted in the present study can be interpreted as a form of scaffolding that supported students in managing the complexity of the modelling activity while preserving their autonomy during problem solving.

The final phase was enriched by an open discussion, during which students expressed their perceptions of the experience. A high level of engagement and appreciation for the activities emerged, particularly for those involving logic, movement, and cooperation. The session concluded with the awarding of participation certificates and a collective debriefing moment, which acknowledged and valued the students' effort and active participation.

6. Discussion, implications and conclusions

The findings of the present study are consistent with recent research on outdoor mathematics education and situated mathematical modelling, which emphasizes the importance of authentic contexts, collaboration, and embodied interaction in supporting students' construction of mathematical meaning (Ferrari & Taranto, 2024; Jablonski, 2025; Kelton & Ma, 2018). At the same time, the study extends previous research by exploring how these processes can also be supported in the absence of digital technologies through alternative instructional and scaffolding strategies.

Considering the research questions, the results allow for several relevant considerations.

Regarding the first question, "Which teaching strategies can effectively replace digital tools without reducing the quality of mathematical learning in math trails?", the data show math trails can be effectively implemented even in the absence of the direct use of technology. Some instructional choices proved effective in compensating for the main features of the MathCityMap application. The initial treasure hunt played a key role in replacing digital geolocation, enabling students to become familiar with the space and develop orientation skills. Similarly, the system of envelopes containing hints represented a valid alternative to digital hints, supporting the problem-solving process through forms of gradual scaffolding.

These findings are in line with previous studies highlighting the importance of orientation, contextualization, and task design in outdoor mathematics activities (Fessakis et al., 2018; Ratnayake et al., 2020). However, while much of the existing literature focuses on technology-supported implementations, the present study suggests that some key educational affordances of math trails (such as exploration, situated problem solving, and scaffolding) can also be preserved through non-digital adaptations. More broadly, the findings support situated and embodied perspectives on mathematical modelling, according to which interaction with physical space and direct engagement with the environment play an important role in the construction of mathematical meaning (Ferrari & Taranto, 2024; Kelton & Ma, 2018).

A particularly relevant point of difference concerns the management of validation and feed-

back processes. The whole-class discussion phase, in fact, represents an essential moment in math trails, as it allows students to reflect on the processes activated, compare the strategies adopted, and consolidate learning. However, in the absence of immediate feedback, typical of the digital environment, the experimenter was not able to obtain an immediate overview of the progress of the activities and the groups' results. This led to a greater workload during the analysis phase, requiring a detailed review of the booklets to reconstruct the processes and identify the difficulties encountered by the students.

Regarding the second question, "How does cooperative learning influence students' participation and mathematical understanding during math trails activities?", the results highlight the central role of the cooperative dimension. Working in small groups, structured through the assignment of roles, fostered the active participation of all students and the shared construction of solutions. In particular, the initial "Untangle the Knot" game helped make the meaning of cooperation explicit, supporting students in understanding the importance of their role within the group and the need to coordinate with others to achieve a common goal.

The analysis of the tasks shows how cooperation supports the modeling process, allowing students to articulate their reasoning, engage in discussion, and negotiate strategies. However, it also emerges that cooperation alone does not automatically ensure successful completion of the modelling cycle: as shown in Task 2, the group, despite working cooperatively, was not able to correctly validate the result. This suggests that cooperation must be accompanied by careful instructional guidance, capable of supporting even the more complex phases of the modeling cycle, particularly those related to interpretation and validation.

This result confirms previous research on cooperative learning and outdoor mathematics education, according to which peer interaction may support reasoning processes, negotiation of meanings, and shared construction of mathematical understanding (Johnson & Johnson, 1999; Fägerstam & Blom, 2013). At the same time, the findings also extend existing research by showing that cooperation alone does not automatically ensure successful completion of the modelling cycle, particularly in the phases of interpretation and validation.

A further relevant aspect concerns assessment, understood not only as the evaluation of the final product, but as the analysis of the process. Observing interactions, adopted strategies, and encountered difficulties made it possible to gain a deeper understanding of learning processes, in line with the model proposed by Jablonski and Bakos (2022). In this sense, assessment takes on a formative function, guiding both instructional action and students' reflection.

Despite the positive results, the study presents some limitations. First, the analysis focused on a single case study, limiting the possibility of generalizing the findings. Moreover, the absence of digital tools prevented the automatic collection of data and the immediate validation of responses, making the analysis process more complex and less immediate. Another limitation concerns the short duration of the intervention, which does not allow for the observation of long-term effects on students' learning.

In light of these considerations, future research could further explore the comparison between digital tools and analog teaching strategies, in order to combine the advantages of both approaches.

In conclusion, the study contributes to current research on outdoor mathematics education and mathematical modelling by showing that meaningful math trails can also be implemented in contexts where digital technologies are unavailable or restricted. The findings highlight the importance of contextualized task design, cooperative interaction, and scaffolding strategies in supporting modelling processes within outdoor learning environments. More broadly, the study extends existing research on digital math trails by demonstrating how some of their core educational features can be maintained through carefully designed non-digital adaptations. In this sense, the findings contribute to current discussions on situated and embodied approaches to mathematical modelling in outdoor learning environments.

Conflict of interest

The authors declare that there are no conflicts of interest.

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Weaving mathematics across contexts: a theoretical–methodological framework for researching classroom–museum continuity

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Abstract: Museums are increasingly recognised as relevant settings for mathematics education beyond school, but the relation between classroom and museum learning remains difficult to describe. This theoretical–methodological paper addresses classroom–museum continuity as a research problem: how can mathematical work remain recognisable and meaningful when it moves across contexts shaped by different objects, spaces, forms of mediation, institutional purposes, and ways of participating? To address this question, the paper develops a framework that brings together informal mathematics education, the museum experience as contextual configuration, double didactical continuity, and continuity across sociocultural difference. Organised through Falk and Dierking’s Contextual model of learning, the framework shifts attention from the museum visit as an isolated event to the broader learning opportunity woven across classroom and museum. A worked example involving circumferences, spirals, bodily movement, and mathematical machines illustrates how the framework can support analysis. The paper closes by discussing methodological implications for studying continuity across contexts, including the need to follow mathematical meanings, mediations, and forms of work before, during, and after the museum visit.

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Keywords: informal mathematics education; museum mathematics education; classroom–museum continuity; out–of–school mathematics education; didactical continuity.

1. Introduction

Research in mathematics education has increasingly acknowledged that mathematical work exceeds the institutional boundaries of the classroom. Bakker et al. [1] identify the relation between mathematics education and other practices as one of the major directions for future research, and recent work on out–of–school mathematics education has begun to give sharper visibility to the environments in which such activity is organised beyond school [2]. This

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development leaves an important question open. Once mathematical work is recognised across multiple settings, how should mathematics education research describe the relations among those settings?

Museums make that question unusually demanding. Within informal mathematics education, they have emerged as environments in which mathematically rich activity can be intentionally designed beyond formal instruction [3]. At the same time, museum learning is shaped by a configuration of objects, spaces, mediations, and temporalities that has its own educational logic. Falk and Dierking [4] locate museum learning in the interplay of personal, sociocultural, and physical contexts, and thus place the visit within a wider process of meaning making. For mathematics education, this has an immediate consequence: what is at stake in museums cannot be described through a generic opposition between school and outside school. The museum enters the field as a distinctive context in which mathematical work takes form under conditions that differ from those of classroom practice.

The kind of continuity addressed in this paper can be illustrated through the mathematical work later discussed in the example. In that path, students move between classroom explorations of circumferences and a museum workshop on spirals, where distance from a centre, rotation, and variation of distance are encountered through architectural details, bodily movement, and mathematical machines such as the spiralograph and the helicograph. These examples show that what travels across classroom and museum is richer than a simple mathematical topic, providing a set of ways of seeing, moving, describing, conjecturing, and using artefacts mathematically.

Research on field trips, much of it developed in science education, has repeatedly shown that the educational significance of a visit depends on what prepares it and on what takes it up afterwards [5, 6]. Within museum learning, the same point has been developed through a temporal understanding of experience that exceeds the moment of the visit itself [4]. This body of work provides an important basis for studying school–museum relations, especially by making visible the role of preparation, mediation, and follow–up. However, a mathematics education perspective raises a more specific question: not only whether the visit is well integrated into a broader educational sequence, but how mathematical meanings, representations, gestures, artefacts, and forms of inquiry remain available when they move across contexts. In this sense, classroom–museum continuity concerns the intelligibility of mathematical work across different environments, whose materials, rhythms, mediations, and institutional purposes do not coincide. What a mathematics education perspective adds, then, is an analytic concern with the continuity of mathematical meanings and forms of mathematical work. The central question is whether the mathematical work developed in one setting remains intelligible in another, so that ideas explored during the visit can be taken up, reinterpreted, and further developed back in the classroom. This requires attention to how mathematical meaning is produced through objects, inscriptions, bodily movement, dialogue, and documentation, and to how resources become available again when the mathematics returns to the classroom.

Against this background, this theoretical–methodological paper treats classroom–museum continuity as a specific problem for mathematics education research. The focus is therefore on how mathematical meanings and forms of mathematical work remain available when they are reconfigured across classroom tasks, museum encounters, mediated interactions, and later classroom discussions. The paper is organised around three guiding questions:

1. In what terms can classroom–museum continuity be conceptualised as a distinct problem for mathematics education research?
2. Which analytical dimensions are needed to study the continuity of mathematical activity across classroom and museum as contexts shaped by different material, institutional, and sociocultural conditions?

3. What does the proposed framework make visible in an integrated classroom–museum path?

The contribution of the paper does not lie simply in bringing together adjacent literatures on informal mathematics education, museum learning, and continuity across settings. Rather, it lies in coordinating these perspectives into a single theoretical–methodological framework that makes classroom–museum continuity more precisely researchable as a mathematics education problem. More specifically, the paper shifts attention from the visit as an isolated event to the *distributed learning environment* woven across classroom and museum, and from the simple recurrence of mathematical topics to the ways mathematical work is sustained, reformulated, or weakened across contexts. In these terms, continuity becomes analytically visible without collapsing classroom and museum into a single pedagogical logic. A related implication concerns outdoor mathematics education: rather than classifying museums as outdoor settings, the paper shows the analytic value of an outdoor lens for foregrounding place, materiality, movement, mediation, and contextual difference. This paper is conceived as a conceptual and theoretical–methodological contribution rather than as a report of a new empirical study. It first develops the conceptual bases of the proposal, then illustrates the framework through a worked example, and finally draws out its methodological implications for research on continuity across contexts.

2. Informal Mathematics Education as a Conceptual Entry Point

A conceptual challenge arises from the expression *outside school*. Taken on its own, it names a location more readily than an educational form. Eshach [7] showed the limits of such spatial shorthand in distinguishing formal, non–formal, and informal education through differences of structure, mediation, participation, and intentionality rather than through place alone. Indeed, museum does not enter this argument merely as what lies beyond the classroom, but as part of a field of educational activity whose relation to school mathematics requires a more exact vocabulary.

Informal mathematics education (IME) can provide that vocabulary. Nemirovsky et al. [3] use the term for mathematically intentional environments organised beyond formal instruction and marked by forms of participation that are not governed by the ordinary logics of curriculum, compulsion, and assessment. This stance makes thinkable the connection between school mathematics and museum mathematics by creating an educational context through deliberate design in which the activities remain open to exploratory movement, shifting disciplinary boundaries, and forms of participation usually unavailable in routine classroom settings.

Its contribution becomes sharper once it is placed alongside research on mathematics in everyday practices. Nunes, Schliemann, and Carraher [8] established that sophisticated mathematics may arise outside school while remaining embedded in the goals and representations of specific practices. Pattison, Rubin, and Wright [9] later gave this distinction a useful formulation by separating everyday mathematics from designed informal learning environments. Museums belong to the latter configuration: their mathematical activity is intentionally arranged, institutionally framed, and educationally mediated. This places them in a position that is neither that of school mathematics nor that of mathematics embedded in ordinary practice.

This intermediate position also helps clarify what distinguishes museum mathematics from school mathematics. The difference lies in the conditions under which mathematical meaning is produced. Mathematical objects such as spirals, symmetries, measures, or spatial relations remain central, but they are encountered and explored through forms of activity that are characteristic of the museum setting. In school, mathematical work is usually organised through structured curricular sequences and opportunities for collective discussion and written representation, together with the possibility of returning to the same ideas over time. In museums, mathematical work is more strongly shaped by spatial exploration and encounters with objects,

often within brief interactions and culturally mediated experiences, that begin from noticing, questioning, and interpreting what is materially present. Participation is therefore organised differently: students or visitors may engage mathematically by looking, moving, gesturing, comparing, following a guide's prompt, manipulating an artefact, or connecting a form to the cultural setting in which it appears. For this reason, classroom–museum continuity cannot be understood as the simple transfer of school tasks into a different venue, but requires attention to how mathematical meanings are reconfigured through museum–specific forms of participation and mediation.

This distinction gives the present paper its starting point. The problem of continuity arises between school mathematics and another form of mathematically intentional work, one organised through different modes of participation, materials, mediations, temporalities, and expectations. IME opens that space conceptually; it does not yet explain the museum as such. The next step, therefore, concerns the museum's own specificity as an educational and cultural configuration for mathematics education.

3. Museums as Specific Contexts for Mathematics Education

Museum specificity begins at an institutional level. According to the International Council of Museums [10], a museum is a not–for–profit, permanent institution in the service of society that researches, collects, conserves, interprets, and exhibits tangible and intangible heritage, offering experiences of education, enjoyment, reflection, and knowledge sharing. As Walz [11] argues, the museum definition functions less as a description exhausted by any single institution than as a collective norm for a heterogeneous field. A museum is therefore never just a site in which educational activity happens to take place. Curatorial decisions, architectural organisation, public mission, and interpretive regimes already shape the forms of attention, movement, and mediation available within it. Mathematical work, when it takes place in museums, is formed within these conditions rather than simply placed inside them. For example, a spiral encountered in a staircase, a shell–shaped decoration, or an architectural detail does not first appear as an already isolated mathematical object. It becomes mathematically available through the way students are invited to look at it, move around it, compare it with other forms, describe its variation, and relate it to ideas such as rotation, distance from a centre, or movement in space.

Falk and Dierking [4] give this specificity its most productive articulation. Their contextual model locates museum learning in the interplay of personal, sociocultural, and physical contexts, and in the temporal relations that extend beyond the visit itself. Thus, mathematics in museums does not unfold in an abstract setting; it takes shape within a context that distributes attention, organises encounters, and gives force to what may later be recalled, resumed, or reinterpreted. The museum therefore belongs to the activity under study, conditioning what can be noticed, how it can be talked about, and under what forms it may continue.

This becomes clearer once objects and artefacts are brought into focus. The object–centred perspective developed in the work collected by Paris [12] shifts attention away from exhibits as carriers of preformed content and towards the interpretive work through which objects become meaningful in relation to narrative, inquiry, and use. De Kluis et al. [13] sharpen the point further in showing that the educational force of museum objects depends on context, presentation, and opportunities for interaction. For mathematics education, the decisive issue can be the mediation: artefacts become mathematically productive through the signs and shared meanings that emerge in activity, as Bartolini Bussi and Mariotti [14] have shown in their work on semiotic mediation. Gesture enters the same field as a semiotic resource rather than as a marginal accompaniment to speech [15]. In museum settings, then, objects, spaces, and bodily movement participate directly in the formation of mathematical work.

The museum specificity at stake is also cultural. Museum encounters are organised through

historical, artistic, scientific, and civic meanings that remain active in the activity itself. Kéfi et al. [16] describe mediation in museums as the production of relations among participants, cultural objects, and institutional structures. From this point of view, it is necessary to prevent mathematics from losing its connection with the museum environment in which it occurs. By way of example, which will be discussed later, a spiral encountered in a museum is not only a geometrical form: it belongs simultaneously to material display, cultural interpretation, and mathematical inquiry. Research on continuity therefore cannot treat the museum as a backdrop against which mathematical work takes place. It must recognise the museum as one of the conditions through which that work acquires its form. The next section considers what research gains when this specificity is approached through an outdoor lens.

4. Museums and the Analytic Gain of an Outdoor Lens

For the purposes of this paper, it appears appropriate to adopt the analytical framework of outdoor education to examine specifically the continuity in mathematics between the classroom and the museum. This approach, as noted above, is not intended to be classificatory; rather, it brings into focus dimensions of experience that outdoor education has taught research to treat with greater precision: place, movement, materiality, mediation, and the pedagogical consequences of contextual difference. In Priest's [17] classic formulation, outdoor education is structured through relations rather than through a simple change of venue. More recent work has given that intuition a sharper shape. Lloyd, Truong, and Gray [18] describe place-based outdoor learning as an integrated, place-responsive approach that interconnects place, curriculum, and learners; Zanato Orlandini [19] similarly treats outdoor education through the conjunction of place, mediation, intentionality, and reflective planning. In this light, the outdoor lens provides a perspective for examining continuity by directing attention to how environments participate in the formation of educational activity.

Applied to museums, this lens makes contextual specificity more analytically productive. The museum appears as a place in which mathematics is shaped by routes of movement, perceptual orientation, spatial density, object encounters, and culturally sedimented forms of mediation. Students looking for spirals across a staircase, a ceiling decoration, or the movement generated by a spiralograph are not simply applying a classroom definition in a new place; they are learning to coordinate perception, movement, artefacts, and mathematical language under conditions that the museum itself helps to organise. Although these dimensions have already been partially introduced in the previous section regarding Falk and Dierking's contextual model, this section proposes a shift in focus through the outdoor lens. They no longer belong only to a description of museum experience; they become part of the explanation of how mathematical work is configured within the museum. Place-responsive outdoor research asks what the environment contributes to the activity that unfolds within it. The same question becomes decisive here, pointing out that mathematical noticing in a museum is inseparable from the way bodies move through rooms, from how objects arrest attention, from the relation between visual search and interpretive framing, and from the rhythms through which the institution organises encounter. Casi and Sabena [20] bring this point directly into mathematics education by extending outdoor mathematics beyond nature-based environments and towards culturally mediated settings such as museums.

A second gain concerns continuity itself. An outdoor lens foregrounds the fact that continuity unfolds across environments whose affordances are not interchangeable. Research on classroom-museum continuity therefore becomes more sensitive to the ways in which mathematics is sustained, transformed, or interrupted under altered conditions of place, movement, mediation, and perception. The question is no longer exhausted by whether classroom work and museum activity can be connected. It extends to how mathematical meanings remain recoverable across settings that reorganise attention and participation differently. The outdoor

lens strengthens the research focus, supporting a more precise formulation of the distributed learning environment constituted through the weaving together of classroom and museum. The next section develops continuity itself as a research problem internal to that distributed environment.

5. Continuity as a Research Problem

Continuity becomes a research problem at the point where mathematics can no longer be in a single setting. Museum experience may be mathematically rich, perceptually intense, and educationally memorable, yet remain detached from the longer movement of classroom work. Casi [21] names this fragility through the *parenthesis effect*: the visit acquires local significance without entering a broader didactical trajectory. Continuity becomes visible precisely where mathematical meanings are required to survive a change in setting.

Casi et al. [22] propose *double didactical continuity* to account for a more demanding form of persistence across settings. Continuity depends on the passage of mathematical content together with the didactical forms through which that content becomes thinkable, discussable, and workable. The point resonates with broader discussions of pedagogical continuity, where common subject matter may coexist with marked discontinuities in forms of activity, expectations, and participation [23]. In the present argument, continuity concerns the continued recognisability of mathematical work across contexts, not only the reappearance of mathematical themes.

The problem deepens further once classroom and museum are treated as distinct socio-cultural practices. Akkerman and Bakker [24] show that boundaries do not simply separate domains; they make visible differences in action, interaction, and participation. Continuity therefore takes shape within difference rather than beyond it. The museum contributes its own materials, rhythms, mediations, and forms of attention; the classroom contributes others. Relations between them are neither automatic nor neutral: they must be educationally sustained. This is also why the problem of continuity cannot be resolved simply through pedagogical equivalence. Continuity is sustained when mathematics remains recoverable across differentiated conditions, when it can be resumed without erasing the difference that gave each context its educational force.

These concepts help in adding precision to the object of inquiry: the parenthesis effect names a recurrent vulnerability, the double didactical continuity specifies what must persist, and the sociocultural difference defines the condition under which continuity becomes educationally demanding.

6. Toward a Theoretical–Methodological Framework for Researching Classroom–Museum Continuity

Recent work on mathematics beyond the classroom has widened the range of settings treated as relevant for mathematics education without producing, in parallel, an equally robust language for relating them. As noted by Memiş and Özkale [25], this gap is especially visible in the case of museums, where the relations among museum education, mathematics education, and informal education remain only partially theorised. The framework proposed here is intended as more than a synthetic overview of relevant concepts. Its specific contribution is to provide an analytic structure for describing classroom–museum continuity with greater precision within mathematics education research. The contribution of the framework lies in its focus on the continuity of mathematical meaning across contexts. The framework is designed to follow how mathematical objects and relations are developed and taken up again across settings. In doing so, it extends museum–learning concerns with experience, mediation, and follow–up toward the analysis of mathematical intelligibility: how something encountered in the museum can become part of a continuing mathematical trajectory rather than remaining an isolated experi-

ential episode. By coordinating four lenses that are often mobilised separately, the framework makes it possible to examine not only whether classroom and museum are connected, but how mathematical work is carried forward, reworked, or interrupted across them. In this sense, its value lies in turning continuity into an object that can be analysed through trajectories, mediations, forms of mathematical work, and relations across distinct educational practices, rather than through a generic appeal to connection, transfer, or coherence. The framework coordinates four intertwined lenses, already introduced in the previous sections: IME, museum experience as a contextual configuration, double didactical continuity, and continuity across sociocultural difference. The articulation of these four lenses is organised through Falk and Dierking’s [4] Contextual model of learning, whose personal, sociocultural, and physical contexts, together with its temporal dimensions, provide the framework’s operative architecture. Table 1 summarises the analytic contribution of the four lenses.

Analytical lens	What it foregrounds	Analytic contribution to researching classroom–museum continuity
Informal mathematics education	Mathematically intentional activity beyond formal instruction	Identifies the museum as an educational environment in which mathematical activity is deliberately organised under conditions different from those of ordinary classroom instruction
Museum experience as a contextual configuration	Objects, spaces, mediations, temporalities, institutional and cultural framing	Makes visible the museum’s specificity as part of the mathematical activity itself, rather than as a neutral venue in which that activity happens
Double didactical continuity	Mathematical content together with didactical forms	Allows continuity to be analysed through the joint persistence of conceptual nuclei and recognisable forms of mathematical work across contexts
Continuity across sociocultural difference	Relations across distinct educational practices	Frames continuity as an educational relation sustained through difference rather than through equivalence, alignment, or transfer
<p><i>Note. The four lenses do not function as sequential stages or independent layers. Their articulation is organised through Falk and Dierking’s [4] Contextual model of learning.</i></p>		

Table 1. The four intertwined lenses of the framework and their analytic contribution

In what follows, I present an example of how the framework can be used. To do so, I draw on an integrated learning path already discussed in Casi et al. [22], with the aim of showing how the framework can operate at different levels of analysis. The example is therefore used in an illustrative, heuristic sense, rather than as the object of a fresh empirical re-analysis. The learning path was implemented with two sixth-grade classes in lower secondary schools in Turin, and involved their classroom teachers, who also acted as teacher-researchers. It centred on a mathematically coherent but contextually differentiated trajectory linking classroom work and a visit-workshop at Palazzo Madama around the themes of circumferences and spirals. The path took shape in response to a difficulty that had already emerged in earlier museum-based mathematics experiences: the visit was often positively evaluated, yet it did not continue within the educational life of the class, thereby producing the very parenthesis effect the project sought to avoid. The integrated path was designed precisely to address that fragility, embedding the museum visit-workshop into a series of activities that begin in the classroom, move on to the museum, and then return to the classroom.

The museum visit-workshop at the centre of the path was *Swirls of Ideas*, developed at Palazzo Madama in Turin, a museum whose architectural and decorative forms offered a par-

ticularly rich environment for work on spirals. In the Juarrian staircase and elsewhere in the building, students could encounter spirals as ornamental motifs, shell-shaped decorations, staircases, and curvilinear forms distributed across the museum space. From a mathematical point of view, the workshop treated spirals as kinetic objects, generated through the combined movement of rotation around a centre and displacement toward or away from that centre. The workshop unfolded through three connected activities: first, (1) a perceptual-motor choreography in which students physically enacted a spiral by walking a path of gradually changing distance from a centre; then an exploration of two mathematical machines, (2) a spiralograph, which generated spirals on the plane through coordinated rotation and translation; and (3) a helicograph, which generates three-dimensional helices by coordinating rotation and translation on orthogonal planes. The museum phase of the integrated path therefore organised mathematical work through bodily movement, guided observation, and interaction with specifically designed mathematical machines in relation to a culturally dense environment.

The classroom work before and after the visit was designed to hold that museum phase within a broader trajectory. Before the visit, students worked on circumferences through exploratory activities aimed at making distance from a point, constancy and variation of distance, and rotational movement mathematically available. These activities included work with standard and non-standard tools and with a rotating plane designed to connect classroom exploration to the logic of the spiralograph later encountered in the museum. After the visit, the trajectory returned to the classroom and resumed the museum experience analytically. Students were invited to recall the guide’s instructions for using the spiralograph, discuss why those instructions mattered, and conjecture what would happen if they were modified. In this way, the visit re-entered classroom mathematics as material for renewed discussion, reformulation, and conceptual work. A documentation booklet accompanied the whole path. Students used it across the different phases to record titles, drawings, observations, key words, and reflections. The booklet preserved traces of what had been noticed, selected, and reformulated from one context to the next, and made continuity more readable both pedagogically and analytically. Figure 1 gives an illustrative scheme of the integrated path.

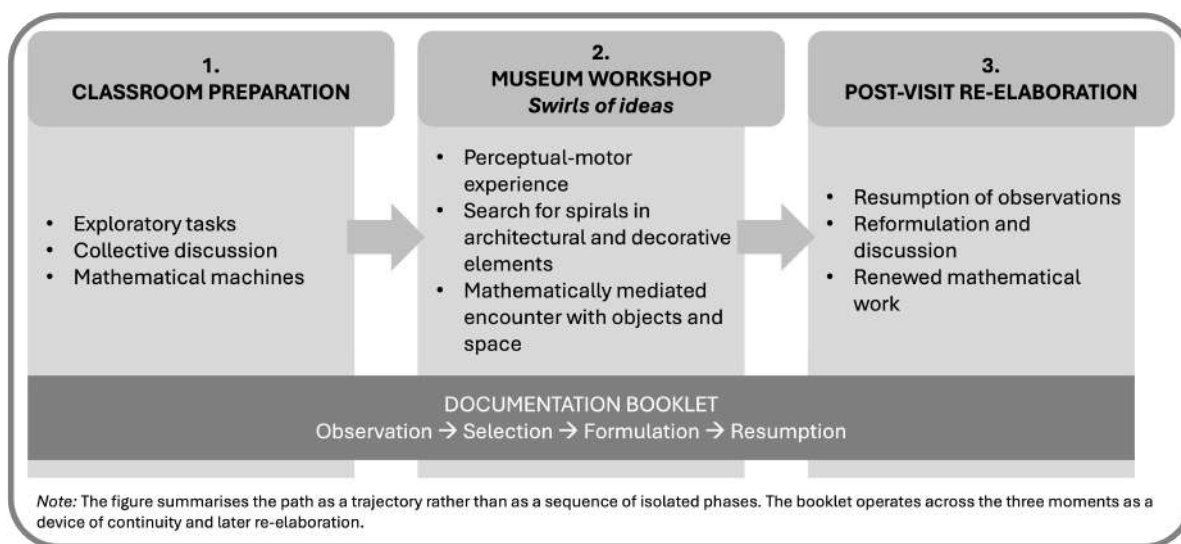


Figure 1. Structure of the integrated classroom–museum path discussed in Casi et al. [22]

Table 2 shows more explicitly how the proposed framework renders the integrated path analytically legible by specifying what each lens brings into view.

Through the framework, the integrated path becomes legible as more than a sequence of

Analytical lens	What it brings into view	Concrete example(s) from the integrated path	Relevance for researching continuity
Informal mathematics education	The museum phase as a form of mathematically intentional activity organised beyond ordinary classroom instruction	The visit–workshop <i>Swirls of Ideas</i> is not treated as a generic school trip, but as a designed mathematical experience in which students explore spirals through guided activity, bodily movement, and interaction with mathematical machines; the path also replaces traditional assessment with documentation and formative feedback	Clarifies that continuity is at stake between school mathematics and another form of mathematically intentional activity, rather than between school and a generic “outside”
Museum experience as a contextual configuration	The constitutive role of space, objects, movement, mediation, and cultural framing in shaping mathematical activity	At Palazzo Madama, students encounter spirals in the Juvarrian staircase, shell decorations, spiral staircases, and architectural details; they then experience spirals through a bodily choreography, the spiralograph, and the helicograph	Shows that continuity cannot be studied independently of the museum conditions through which mathematical activity takes form
Double didactical continuity	The persistence across phases of both mathematical content and forms of mathematical work	Before the visit, classroom activities on the rope, set square, everyday artefacts, and rotating plane make fixed distance, variable distance, and rotation mathematically available; after the visit, students recall the guide’s instructions for the spiralograph, discuss why they matter, and conjecture what changes if they are modified	Makes visible that continuity depends not only on the recurrence of conceptual nuclei, but also on the continued circulation of exploratory work, discussion, conjecturing, and reformulation across classroom and museum
Continuity across sociocultural difference	The educational work required to sustain mathematical intelligibility across contexts with different norms, temporalities, and forms of mediation	In the museum, the spiralograph activity is shaped by brief, guided, and relatively directive interaction; back in class, teachers reopen that experience through discussion and “what if...?” questions, allowing students to re-elaborate aspects that remained implicit or only partially explored during the visit	Frames continuity as an achievement across distinct educational practices, rather than as simple alignment, transfer, or equivalence

Table 2. Analytic visibility of the integrated classroom–museum path through the proposed framework

connected activities. It emerges instead as a trajectory in which mathematics is organised across classroom and museum under different but related conditions. In the present case, the museum contributes more than a setting: it shapes how spirals become mathematically available through architecture, movement, guided observation, and interaction with artefacts. Paris's [12] object-centred perspective clarifies how objects acquire educational force through interpretive work. Bartolini Bussi and Mariotti [14] show how this force becomes mathematically productive through semiotic mediation, while Arzarello et al. [15] make visible the role of gesture as a semiotic resource within that process. The classroom, in turn, prepares some of these conditions in advance and later re-elaborates the visit through discussion, conjecture, documentation, and reformulation. This becomes visible when the guide's instructions for using the spiralograph, initially encountered in the museum as situated prompts for action, are taken up again in the classroom as material for explanation, discussion, and "what if...?" conjecture. Continuity, then, lies not in simple topic alignment, but in the continued recoverability of mathematical work across contexts whose norms, rhythms, and mediations do not coincide.

Double didactical continuity becomes visible at another level. Circumferences and spirals recur across classroom and museum, and the coherence of the path also depends on recognisable forms of work: exploratory attention, mediated discussion, collective interpretation, documentation, and later reformulation. The activity moves forward because mathematical meanings and didactical forms remain in circulation across differently configured environments. The documentation booklet embodies the double didactical continuity in two ways: on the one hand, it records the students' mathematical work; on the other, it accompanies and follows the students throughout the whole process, functioning as a cross-context semiotic trace through which observations, words, drawings, and emerging mathematical meanings can be carried from one setting to the other. In this sense, it does not serve merely as a record of activity, but as a mediating artefact of continuity, supporting the reactivation and reformulation of museum experience in subsequent classroom work.

7. Methodological Implications

The framework proposed here redefines the object of inquiry for research on classroom–museum continuity. It cannot be organised around the visit as a self-contained event, because continuity becomes empirically legible only within a broader educational trajectory that is time sensitive. Falk and Dierking [4] already place museum learning within a temporal field that exceeds the physical and chronological bounds of the visit; Casi [21] sharpens the point for mathematics education through the parenthesis effect, which captures the contraction of mathematical meaning into the local intensity of the museum experience when later classroom uptake is weak or absent.

Evidence of continuity is temporally dispersed and unequally distributed across the trajectory. Preparatory classroom activity matters because it establishes tasks, vocabularies, expectations, and modes of attention through which the museum can later become mathematically legible. The visit matters because it concentrates encounters with objects, spaces, movement, and museum mediation under conditions unavailable elsewhere. Post-visit activity matters because continuity acquires empirical density only when observations, gestures, descriptions, or documents are resumed, reformulated, selected, or allowed to dissipate in subsequent classroom work. Tal and Steiner [26] make this especially clear by following school–museum relations across planning, visit, and wrap-up; Rennie and McClafferty [27] similarly treat before, during, and after as educationally consequential phases. A continuity-sensitive corpus therefore takes longitudinal form.

The same logic applies to mediation. Teachers prepare and later reactivate the museum mathematically; museum educators organise access to objects, spaces, and tasks; students appropriate, select, and transform what becomes available in activity; artefacts and documents

stabilise some threads of the trajectory while allowing others to disappear. Fredricks [28] gives this point broader support in showing that engagement is shaped by relations with teachers or staff, by peers, by structure, and by task characteristics. Kelton [29] adds a crucial interactional refinement: continuity becomes visible in situated practices of naming, linking, and framing across school and museum. For these reasons, a suitable corpus of researchable data may combine preparatory classroom video, museum interaction data, student documentation, post-visit classroom discussion, teacher planning notes, and interviews with both teachers and museum educators. The point is not exhaustive coverage of the trajectory, but the possibility of reconstructing it analytically.

A further consequence concerns what counts as evidence. Continuity leaves its traces in uptake, reformulation, delayed reuse, and selective disappearance. It appears when museum-based observations re-enter classroom discourse, when perceptual noticing becomes mathematised description, when an artefact or a page of documentation is reused as a semiotic resource, and when recognisable forms of mathematical work persist under changed conditions. Some formulations remain local to the visit, some discoveries lose didactical future, and some links are never resumed. Continuity research therefore depends on a notion of evidence that includes persistence, transformation, and loss within the same analytic frame.

Documentation occupies a particularly strong place within this methodological design. In the integrated path discussed in Casi et al. [22], the booklet accompanied the movement between classroom and museum and supported later re-elaboration. More generally, documentation preserves transitions: what was noticed, selected, named, reformulated, or carried forward can be followed there with a precision that fleeting interaction rarely affords on its own. Krechevsky, Rivard, and Burton [30] treat documentation as a practice of observing, recording, interpreting, and sharing learning processes. Within the present framework, documentation becomes one of the sites in which continuity leaves a readable mark. Its methodological value lies in making longitudinal reconstruction possible. Continuity becomes researchable where transitions can be followed across documents, interactions, and later resummptions of activity.

8. Discussion and Conclusions

This paper has treated classroom-museum continuity as a research object internal to mathematics education across contexts. Bakker et al. [1] identify relations between mathematics education and other practices as a major direction for the field. The contribution developed here gives that direction a more precise form by focusing on the intelligibility of mathematical work across educational practices that do not share the same material, institutional, or sociocultural conditions. The proposed framework makes this problem analytically tractable, relocating inquiry at the level of the distributed learning environment constituted through the weaving together of classroom and museum, where mathematical meanings, didactical forms, and mediations may persist, weaken, or acquire new educational force. Framing continuity as a distributed relation has consequences for how research on mathematics beyond the classroom may be framed. If continuity is treated only as coherence between activities, the analysis risks privileging planned connections while overlooking how mathematical meaning is reconstituted across settings. The framework proposed here invites a different reading: continuity becomes visible in the relations among tasks, mediations, gestures, artefacts, documents, and later classroom uptake. It therefore directs attention to the work through which mathematical ideas are made available again after a change of context, and to the moments in which this work remains partial, fragile, or unsuccessful.

The rationale for bringing the four lenses together lies in the fact that none of them, taken alone, is sufficient to account for classroom-museum continuity as a mathematics education problem. IME identifies the museum as a deliberately designed environment for mathematical engagement beyond ordinary instruction, but it does not by itself specify how a classroom-

museum trajectory can be analysed. The museum–learning perspective, organised here through the Contextual model of learning, foregrounds the personal, sociocultural, physical, and temporal configuration of experience, but it needs to be connected to the specificity of mathematical meaning making. Double didactical continuity indicates what must remain in circulation across contexts: not only mathematical content, but also recognisable forms of mathematical work. Finally, the lens of sociocultural difference prevents continuity from being reduced to alignment or equivalence, by treating classroom and museum as distinct practices whose relation must be educationally sustained. Their combination therefore makes visible a level of analysis that would remain difficult to grasp otherwise: the way mathematical meanings are carried, transformed, stabilised, or lost across a distributed learning environment.

Within this combined framework, the role of museums in mathematics education research can be stated more precisely. Museums are not simply the second setting in a classroom–museum trajectory; they intensify the problem of continuity because they configure mathematical work through objects, spaces, rhythms, and mediations that heighten contextual difference and make educational threading more demanding. The outdoor lens strengthens this analysis by foregrounding place, materiality, movement, and mediation as dimensions of mathematical work. Its contribution is analytic rather than classificatory: research becomes more capable of following how mathematical work travels across contexts while remaining shaped by the environments through which it passes.

This way of framing classroom–museum continuity also has methodological consequences. If mathematical work is distributed across settings and shaped by the environments through which it passes, the unit of inquiry can no longer be confined to the visit, nor can it be reduced to the teacher’s planning intentions. Research requires trajectories, heterogeneous sources, and an evidentiary frame capable of following uptake, reformulation, delayed reuse, and loss. The integrated path discussed in this paper shows why these aspects matter. The same mathematical theme may appear before, during, and after the museum visit, but continuity becomes analytically convincing only when the researcher can show how meanings are prepared, transformed, documented, and reactivated across these moments. What becomes visible is neither a simple extension of classroom mathematics into the museum nor an isolated enrichment episode, but a distributed learning environment constituted through the weaving together of classroom and museum, whose mathematical coherence must be reconstructed through documents, interactions, mediations, and later resumptions of work.

The framework also has limitations that need to be made explicit. It has been developed from work on classroom–museum relations in culturally mediated museum settings and illustrated through a geometrical trajectory involving circumferences, spirals, bodily movement, and mathematical machines. For this reason, it should not be treated as a general model of all forms of mathematics education beyond the classroom. It is likely to be most productive in contexts where mathematical meanings are distributed across several phases, where material and spatial mediation play a strong role, and where researchers can follow how ideas, gestures, artefacts, and documents are taken up over time. Its applicability may be more limited in short, weakly documented visits, in museum experiences with little post-visit classroom elaboration, or in settings where mathematical work is only loosely connected to the institutional and material features of the place.

Further empirical work is therefore needed to test and refine the framework across different museum settings, mathematical domains, age groups, and forms of collaboration between teachers and museum educators. Comparative studies would be especially valuable: for instance, analyses of trajectories in which continuity is strongly sustained alongside trajectories in which the visit remains a parenthesis. Such comparisons could clarify which forms of preparation, mediation, documentation, and classroom re–elaboration support the persistence and transformation of mathematical meanings across contexts. They could also help examine how

the framework operates beyond geometry, for example in relation to measurement, modelling, statistics, or algebraic thinking in museum and heritage environments.

At this stage, the paper offers a way of studying how mathematical work may continue across contexts without erasing the differences that make those contexts educationally productive. Its contribution is not to stabilise a general model of all classroom–museum relations, but to make such relations more open to analysis in mathematics education. The proposed framework helps distinguish between superficial connection and didactical continuity, between the recurrence of a topic and the reactivation of mathematical meanings, and between the museum as a venue and the museum as a condition of mathematical work. Its value lies in making the weave of continuity visible where research has too often seen only the visit.

Conflict of interest

The author declare that there are no conflicts of interest.

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Exploring empathy in mathematics feedback: a comparative study of human and AI-generated responses in informal learning contexts

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Abstract: The aim of this study is to analyze feedback perception of empathy generated by humans and Large Language Models (LLMs) in informal mathematics learning contexts. Using the dimensions of Emotion Recognition (ER), Perspective -Taking (PT), and Emotional Contagion (EC), we conducted a comparative evaluation on a dataset of formal logic problems sourced from the Reddit online community. Findings indicate that feedback generated by LLMs, when supported by well - structured prompts, is rated as significantly more empathetic than human feedback, which tends to focus more on procedural accuracy. While ER and EC show the most pronounced gaps in favor of AI, PT emerges as the most complex and least differentiated dimension. Finally, the study suggests that LLMs can effectively integrate effective support into informal mathematics education.

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1. Introduction

Recently, education has increasingly been considered a process that extends beyond the boundaries of formal educational settings. Learning in a non - formal education environment is mainly characterized by a high level of students' autonomy [1] and is no longer considered a phenomenon restricted to the school environment alone, but rather a continuous process that unfolds across a variety of contexts, interactions, and mediating tools. From this perspective, non - formal learning contexts contribute to building continuous learning trajectories, in which diverse experiences are deeply interconnected [2]. Mathematical learning can also take place through informal practices, that is, outside the school environment, often mediated by digital technologies [3] or everyday social interactions [4]. Due to students' apprehension toward math learning environments involving assessment [5], they are inclined to seek support in various non - formal settings to learn math without the pressure of assessment. Technology - Related informal mathematics Learning Activities (TRLA) are not merely isolated episodes, but rather essential elements for ensuring educational continuity, thereby lending further validity to technology - mediated learning [6].

Within this context, online forums and digital peer - support communities emerge as privileged spaces for non - formal learning, where solving mathematical problems becomes a collaborative and public activity. In such environments, students not only access content but also seek support on specific problems, transforming individual doubts into opportunities for discussion. Learning environments characterized by a supportive atmosphere for seeking help are not only a means of resolving a specific task but also a strategy for active self - regulation. Students, released from the pressures of formal assessment, are able to independently identify their own gaps in knowledge and proactively seek out resources [6].

In such environments, feedback plays a crucial role. Feedback is expressed through various shapes and forms. The literature defines feedback based on its objectives and ways of interacting. For example, formative feedback is distinguished from summative feedback, where the summative feedback gives an assessment of how well students completed a specific task such as might justify an overall grade for a piece of work, while formative feedback gives specific guidance on how to improve their performance highlighting, for example, what is good or what appears to be missing or mis - understood [7]. Furthermore, there is peer feedback: such feedback has the potential to foster collaborative learning among students, making the entire learning process more meaningful [8]. Lastly, there is affective feedback, which recognizes and supports students' emotional states [9]. Hence, it is essential to recognize the inherent diversity of feedback in order to avoid oversimplified interpretations that view it as a form of communication devoid of nuance. Developments in Artificial Intelligence have led to the introduction of automated methods for providing adaptive, process - oriented, and rapid feedback [10]. However, significant issues arise regarding their ability to address not only students' cognitive needs but also their affective and emotional well - being, especially important in learning processes. A significant turning point was reached with the development of Large Language Models (LLMs) which are capable of identifying the context and relevant sections of students' problems. They provide the technological framework necessary to achieve advanced levels of interaction and to deliver feedback that is as efficient as possible and, importantly, in real time [10]. These capabilities are essential for ensuring the relevance of the feedback. Unfortunately,

this technological process brings with it new challenges in terms of socio - emotional nuances, especially in contexts where emotional support plays a fundamental role in sustaining learning processes. Although these systems are extremely effective at generating content, their empathetic and motivational capabilities are often very limited for effective tutoring. [11] showed that feedback generated by LLMs tends to satisfy objective criteria to a greater extent than emotional ones. This aspect helps us understand how, from a cognitive standpoint, it is possible to compare LLMs and humans, whereas from a purely emotional relational standpoint, there is always a weakness in automated systems. [12] evaluated the performance of LLMs and showed that, from a pedagogical perspective, they are not optimal when assuming the role of educational tutors. For these reasons, it is essential to conduct an in-depth analysis of the empathy in the feedback and examine whether these systems can simulate or modulate the empathy required to address the complex affective needs of students facing formal logic challenges. As part of this study, our research focuses specifically on formative and affective feedback within asynchronous peer feedback dynamics (the providers of feedback are not giving grades but are constructively responding to questions asked by those seeking feedback), exploring how the affective component of feedback can be tailored to students' needs.

2. Theoretical Background

In informal digital environments, interaction plays a pivotal role in supporting learning processes. In these contexts, feedback emerges through interactions among peers or with more experienced members of the community, forming a socially mediated and context - specific process [13, 14]. Feedback is one of the most influential factors supporting learning across various contexts. In mathematics, it not only supports conceptual understanding and problem - solving processes but also impacts the affective dimension of learning, influencing motivation, engagement, and self - perception [9]. Considering feedback as a teaching tool, it can influence students' learning outcomes [15]. However, in order to be effective, feedback must be timely and process - focused to stimulate self - regulation and metacognition. Its function in supporting conceptual understanding and problem - solving, feedback also has a significant affective dimension that influences students' motivation, self - efficacy, and attitudes [9, 16, 17]. From this perspective, the quality of feedback cannot be fully understood without considering its emotional and relational dimensions.

In informal settings, the affective dimension of feedback is particularly significant, due to the self - directed nature of learning, which is strongly linked to intrinsic motivation and self - regulation processes [18, 19]. From this perspective, affective feedback, understood as the set of elements that acknowledge and support students' emotional states, helps shape how students interpret and use the information they receive [9, 17]. In mathematics education, these factors are particularly noteworthy, as emotions such as anxiety can hinder performance and reduce students' willingness to tackle complex tasks [16, 20, 21]. Feedback that integrates both affective and cognitive components can alleviate anxiety and foster perseverance. In informal digital environments, however, asynchronous interactions, heterogeneous participation, and the absence of structured pedagogical guidance make feedback highly variable: alongside constructive responses, there are comments that are unclear or lack sensitivity [22]. In the informal context of online math and logic forums, active help - seeking is a crucial moment in student learning. Whenever they hit a cognitive roadblock, students look not only for a technical solution but also for feedback to validate their efforts. However, the human responses in these forums can be inconsistent both within individual responses and across responses to distinct questions in terms of accuracy, in depth analysis, tone and timeliness. The responses vary from detailed and helpful explanations to brief, ambiguous or even discouraging answers. This inconsistency can affect not only learning outcomes but also student motivation and en-

gement, making the affective dimension of feedback particularly fragile in informal contexts.

The empathetic aspect of feedback can greatly influence how students interpret and apply it. Empathy is a broader term, encompassing both cognitive and behavioural processes as well as related concepts such as sympathy and compassion [23, 24]. It can be defined in a multidimensional way. On the one hand, there is cognitive empathy, such as the system's ability to accurately identify the specific source of the student's struggle or logical mistake. On the other hand, there is affective empathy, such as the ability to generate helpful, judgment-free responses that validate the student's emotions. [25] list three fundamental components of empathy in artificial systems:

- **Emotion recognition:** the system is able to identify the user's emotional state (or, more broadly, their experience), deriving it from inputs such as text, speech, or multimodal signals, which are also necessary for successfully performing its assigned task.
- **Perspective - taking:** the agent interprets the user's thoughts and inferential processes by linking the recognized emotion to their personal context (preferences, personality, goals) and to the communicative situation.
- **Emotional contagion:** the system simulates an emotion that is consistent with or appropriate to that of the user, generating responses that reflect and integrate emotional cues within their context.

Based on the above components of empathy, empathetic feedback is not limited to correcting mistakes. It takes into account the student's level of competence and the strategies they used, providing gradual support toward the solution. Moreover, it involves recognizing the student's emotional state and responding with understanding and sensitivity and conveys trust, encouragement, and respect, helping to create a positive atmosphere that supports motivation and the willingness to take on new challenges. Adopting the student's perspective, it identifies the exact point at which the student deviated from the correct process, showing that mistakes are part of learning and highlighting what was done correctly. Empathetic feedback thus requires a balance between educational rigor and relational sensitivity. Integrating empathy with cognitive criteria makes instructional feedback more comprehensive and realistic. To be effective, it must be clear, complete, and relevant, while also recognizing emotions, adopting the student's perspective, and responding with emotional contagion, thus supporting learners cognitively, motivationally and relationally.

3. Research aim and question

This study aims to investigate not only the ability of LLMs to generate empathetic feedback but also to recognize and evaluate feedback empathy itself through a cross - evaluation process in non - formal mathematics learning contexts, comparing human feedback with feedback generated by large language models. The aim is to investigate the role and effectiveness of LLMs and human, generated feedback's affective element in the domain of Formal Logic within purely informal educational contexts, such as online forums. Specifically, this paper aims to analyze the socio - emotional dimension of feedback, with a specific focus on the expression and assessment of empathy. The paper focuses on understanding how empathy is expressed in feedback and how it is recognized and assessed within informal mathematical contexts that are characterized by high procedural accuracy. To this end, we adopt an analytical framework that considers the main components of empathy, such as emotional recognition, perspective - taking, and emotional contagion [25]. Specifically, this study is motivated by the following research question: (RQ) How is empathy evaluated in terms of emotional recognition, perspective - taking and emotional contagion across human and LLM - generated feedback in informal mathematics learning contexts?

4. Methodology

In this study, we adopted a comparative research methodology aimed to investigate the affective dimension of feedback in informal mathematics learning contexts. In accordance with the research question, this methodology was designed to analyze how empathy is expressed and evaluated in feedback provided by humans and large language models (LLMs), considering these agents as providers of feedback.

The research process was organized into four sequential phases:

1. Dataset collection and selection from Reddit discussions on formal logic problems;
2. Generation of LLM - based feedback using a standardized emotionally supportive prompt;
3. Affective evaluation of feedback through a cross - evaluation procedure involving both human and AI evaluators;
4. Statistically grounded analysis of the empathy - related dimensions.

These phases are described each in turn in this section; the final phase extends in expanded form with discussion of the results that emerge from our analysis.

To address the research question, we conducted a comparative analysis using a dataset made up of answers to Formal Logic problems collected from an online community site. The answers provided by humans were compared with those generated by three different LLMs (GPT-4.1, Gemini 2.5 Flash, and Mistral non-Large), all specifically trained with a specific prompt to provide emotional support. This study adopted a cross - evaluation methodology, in which both humans and AI models take on the role of evaluators for feedback analysis, based on an analytical framework of empathy that includes *emotional recognition*, *perspective - taking* and *emotional contagion*.

4.1. Dataset Collection

The dataset underlying the study consists of textual data extracted from public online discussions regarding problems in formal logic and reasoning, sourced from the Reddit platform (<https://www.reddit.com>). The selected posts ($n = 50$) are publicly accessible and were collected without retaining user identifiers. We collected all the interactions analyzed in their original language, which is English. We selected the problems through a systematic search using a predefined Boolean string: ("*symbolic logic*" OR "*formal logic*" OR "*natural deduction*") AND ("*my answer*" OR "*proof*" OR "*exercise*") AND ("*feedback*" OR "*help*" OR "*explanation*"). We also filtered the results based on the following inclusion criteria:

- a) presence of a clearly formulated logic problem;
- b) student - type question;
- c) presence of at least one meaningful answer.

After removing duplicates and posts lacking sufficient context, we structured the final set of items. Each item includes the problem text, the student's attempt or question, and one or more pieces of feedback provided by other users.

4.2. LLMs Feedback Generation

For each problem, feedback generated by various large language models (LLMs) was integrated into the dataset to enable a direct comparison with human feedback. To ensure consistency and comparability among the generated feedback, all models were trained using a standardized prompt designed to elicit responses that were not only educationally effective but also emotionally supportive. The prompt used for this process was as follows (Table 1):

You are an **expert logic tutor** helping a student who is struggling with a formal logic proof. Your goal is to provide a **clear, pedagogically effective, and emotionally supportive** feedback. Be **concise but helpful**. Do not give the full solution but guide the student toward the next correct step.

Problem: [formal logic problem]

Student question: [student's message]

Please provide your feedback.

Table 1. The standardized prompt for large language models (LLMs), designed to ensure educational effectiveness and emotional support

Notably, the prompt refers to "*emotionally supportive feedback*" instruction as a key element to foster the development of empathetic skills, encouraging recognition of the student's state of mind, attention to their thought process, and the use of an encouraging tone. This approach helped reduce variability related to the wording of the prompt, ensuring consistent conditions in the generation of feedback.

4.3. Evaluation Criteria and Procedure

Following the data collection phase, we developed a feedback corpus drawn from both human and Artificial Intelligence sources, which then enabled us to conduct a rigorous evaluation of the feedback. The primary aim of this evaluation was to quantify the quality and effectiveness of the feedback across three specific dimensions. We carefully selected the socio - emotional dimensions of the feedback (*Emotion Recognition (ER)*, *Perspective - Taking (PT)*, *Emotional Contagion (EC)*), in accordance with current practices in educational research and the evaluation of AI - assisted learning [25]. Each of these dimensions was then evaluated using a 5 - point Likert Scale. The evaluation was conducted using a cross - tabulation approach, in which each item was analyzed comparatively across the entire set of feedback (both human and LLMs - generated) regarding the same Logical problem. This allowed us to ensure consistency in evaluation criteria, minimize inter - rater variability, and facilitate a systematic and direct cross - comparison between the different types of feedback. In the context of human evaluation of the feedback, this was performed by the first author.

The prompt used to perform the affective process evaluation was as follows (Table 2):

<p><i>You are evaluating the empathy of a "XLLMs or Human" feedback message given to a student about a math/logic problem.</i></p> <p><i>Input will be structured as follows:</i></p> <p><i>Problem:</i> <i>[formal logic problem]</i></p> <p><i>Student question:</i> <i>[student's message]</i></p> <p><i>Feedback to evaluate:</i> <i>[XLLMs or Human feedback]</i></p> <p><i>Evaluate the feedback according to the three dimensions below. For each dimension, assign a score from 1 to 5 using the operational descriptions.</i></p> <p><i>Dimension 1 - Emotion Recognition</i></p> <p><i>Does the feedback show awareness of the student's emotional state (e.g., frustration, confusion)?</i></p> <p><i>1 = No emotional recognition.</i></p> <p><i>2 = Minimal or unclear recognition.</i></p> <p><i>3 = Generic acknowledgment (e.g., "I understand").</i></p> <p><i>4 = Clear and context - based recognition.</i></p> <p><i>5 = Specific, accurate, and sensitive recognition.</i></p> <p><i>Dimension 2 - Perspective -Taking</i></p> <p><i>Does the feedback consider how the student reasoned and where the mistake originated?</i></p> <p><i>1 = Generic feedback that could apply to anyone.</i></p> <p><i>2 = Minimal reference to the student's process.</i></p> <p><i>3 = Partial attempt to connect to the student's reasoning.</i></p> <p><i>4 = Clearly identifies the origin of the mistake and uses it constructively.</i></p> <p><i>5 = Reconstructs the student's mental path and guides them step - by - step.</i></p> <p><i>Dimension 3 - Emotional Contagion (Appropriate Answers)</i></p> <p><i>Is the tone supportive and encouraging, without being judgmental or patronizing?</i></p> <p><i>1 = Judgmental, cold, or harsh tone.</i></p> <p><i>2 = Neutral tone without encouragement.</i></p> <p><i>3 = Respectful tone but not particularly motivating.</i></p> <p><i>4 = Warm and supportive tone.</i></p> <p><i>5 = Strongly encouraging tone that conveys confidence in the student.</i></p> <p><i>Return the evaluation in the following format (do not add anything else):</i></p> <p><i>Emotion Recognition (ER):</i> <i>[score]</i></p> <p><i>Perspective - Taking (PT):</i> <i>[score]</i></p> <p><i>Emotional Contagion (EC):</i> <i>[score]</i></p>
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Table 2. The prompt used to perform the affective process evaluation

We systematically examined the differences in the scores assigned to the affective dimensions of empathy (*Emotion Recognition, Perspective - Taking, Emotional Contagion*) across the feedback generated by the various evaluators (humans and LLMs), using a comparative quantitative approach.

4.4. Statistical Analysis

Analysis was conducted using R Studio Software. The final Dataset included a total of 693 observations. Feedback distribution by provider involved Human responses ($n = 243$) and responses for each of the three LLMs ($n = 150$). These were evaluated by a group of Evaluators (Gemini, ChatGPT-4, Mistral, with $n = 181$ while Humans, with $n = 150$).

Given the nonparametric nature of the data, we conducted the comparisons using nonparametric statistical tests between groups (*Wilcoxon test*), distinguishing between comparisons of independent groups (feedback providers) and paired evaluations of the same items (evaluators). Furthermore, to control for the Type I error rate resulting from performing multiple group pairwise tests, we applied the Bonferroni correction to all resulting p-values ($p < 0.05$).

5. Results

Table 3 shows a perceived quality generated by each source, as assessed by the group of evaluators. Combining the scores by feedback provider, we can identify which source (Human or LLMs generated feedback) demonstrates a greater ability to express empathy in an informal mathematical learning context.

EMPATHY DIMENSION	GEMINI	GPT-4	HUMAN	MISTRAL
Perspective Taking (PT)	4.05	4.29	3.77	3.89
Emotion Recognition (ER)	3.83	3.75	2.22	3.28
Emotional Contagion (EC)	4.15	4.15	2.95	3.86

Table 3. Average of Empathy Dimensions scores

Data highlights a clear ranking of performance and shows significant differences in how emotional indicators are used by humans in comparison to large language models. Specifically, in the domain of *Perspective - Taking*, we can see that AI systems emerge as the primary providers of feedback. Regarding the ability to adapt to the student's point of view and understand their cognitive state, both GPT-4 (4.29) and GEMINI (4.05) demonstrate an advantage over Mistral (3.89) and the Human (3.77). The low score for human feedback suggests that, in this context, it is more task - focused rather than centered on the empathetic perspective necessary for understanding the student's subjective experience.

Meanwhile, regarding *Emotion Recognition*, there is a significant difference in the ability of the systems to identify and recognize the student's emotional state. Once again, AI systems outperform humans. Specifically, GEMINI (3.83) and GPT-4 (3.75) are capable of understanding emotional nuances and reflecting them in the feedback they generate. The feedback generated by MISTRAL (3.28) receives a lower score than the other two LLMs, indicating a more moderate but less precise ability to recognize emotions. Significantly, the score for human feedback (2.22) is markedly lower than that of the LLMs: this suggests that human feedback appears less focused on addressing students' emotional states.

Finally, the analysis of *Emotional Contagion* shows that both GEMINI and GPT-4 (4.15) have the highest scores. This indicates that their feedback is highly aligned with students' needs, showing a high level of emotional contagion. MISTRAL (3.86) has also achieved a high

score, demonstrating that LLMs exhibit a high level of emotional accuracy. Again, human feedback scores lower (2.95) in this domain: this indicates that the feedback provided may appear less emotionally aligned and fails to offer students the emotional resonance they need.

The significance analysis, performed using the Wilcoxon test for pairedwise comparisons of groups, also confirms this trend (see Table 4, Table 5, Table 6):

	GEMINI	GPT-4	HUMAN
GPT-4	0.0786	-	-
HUMAN	0.1879	1.7e-05	-
MISTRAL	1.0000	0.0067	1.0000

Table 4. Perspective Taking variable p_value matrix

Table 4 shows that GPT-4 is statistically more effective than HUMANS ($p = 1.7e-05$) and MISTRAL ($p = 0.0067$). There was no significant difference between GPT-4 and GEMINI ($p = 0.0786$), neither between GEMINI, MISTRAL, and HUMANS. These data suggest that GPT-4 is clearly more effective in the domain of PT, while the other systems perform within a statistically similar range.

	GEMINI	GPT-4	HUMAN
GPT-4	1.0000	-	-
HUMAN	< 2e-16	< 2e-16	-
MISTRAL	5e-05	0.0016	< 2e-16

Table 5. Emotion Recognition variable p_value matrix

Table 5 shows a highly divergent pattern. Notably, the difference between HUMANS and LLMs is extremely significant ($p < 2e-16$). Furthermore, while GEMINI and GPT-4 show no significant differences between themselves ($p = 1000$), they significantly outperform MISTRAL (with $p = 5e-05$ and $p = 0.0016$, respectively). This confirms that GEMINI and GPT-4 show significantly higher performance in emotion recognition.

	GEMINI	GPT-4	HUMAN
GPT-4	1.0000	-	-
HUMAN	< 2e-16	< 2e-16	-
MISTRAL	0.0019	0.0037	< 2e-16

Table 6. Emotional Contagion variable p_value matrix

Table 6 shows that the difference between HUMAN feedback and all LLMs is highly significant ($p < 2e-16$). Again, while GEMINI and GPT-4 show no significant differences between themselves ($p = 1.000$), they significantly outperform MISTRAL (with $p = 0.0019$ and $p = 0.0037$, respectively). All of this confirms that the LLMs (GEMINI and GPT-4) are perceived as significantly more effective on an emotional level than both HUMAN and MISTRAL.

6. Discussion and Conclusion

This study explored how empathy is evaluated by humans and LLMs feedback in informal mathematics contexts. Specifically, RQ was addressed by investigating the dimensions of *Emotion Recognition (ER)*, *Perspective - Taking (PT)*, and *Emotional Contagion (EC)*, as reported by [25], are perceived and evaluated by comparing human - generated feedback with that generated by various language models. While the data explored here was solely anchored in English,

we expect that similar findings would emerge for other languages - this is an exploration that we leave for the future.

Findings highlight statistically significant differences in how empathetic quality is rated for different feedback providers. Across all dimensions GPT-4 and GEMINI achieved the highest scores while HUMAN showed the lowest scores, specifically in emotion recognition and emotional contagion.

Feedback generated by LLMs is rated as more empathetic than human feedback across all three dimensions considered. This is particularly noteworthy in light of previous studies in Literature. Specifically, the studies by [11] and [12] highlight how LLMs are very effective from a cognitive perspective but less so in managing the emotive dimension. On the other hand, our study suggests that, if LLMs generate feedback using well - structured prompts [26], they are perceived as capable of producing feedback that adequately addresses students' emotional needs.

We can look at this apparent discrepancy in light of how empathy is assessed. As [17] and [9] have pointed out, the affective aspect of feedback significantly influences how students perceive and interpret it. Therefore, it is likely that LLMs, through the use of systematic language that provides emotional support and validation, may be perceived as empathetic. Specifically, our data analysis showed that the most significant differences emerge in the dimensions of Emotion Recognition and Emotional Contagion, where human feedback scores are significantly lower than LLMs. One interpretation of these findings is that, in mathematical contexts, human feedback is more likely to focus on procedural correctness and task resolution. On the other hand, LLMs, designed to generate complete and appropriate answers, are able to incorporate affective elements, making them more aligned with the components of *emotion recognition* and *emotional contagion* [25]. With regard to the dimension of *perspective - taking*, findings reveal a more nuanced scenario. Although GPT-4 outperforms humans significantly, the differences between it and other LLMs are less notable. Such findings suggest that the ability to adopt the student's perspective is the most complex dimension of empathy, which aligns with the study by [23], who distinguish between cognitive and affective empathy, noting that the former requires more complex and profound inferential processes and is therefore less easily simulated than affective empathy. Overall, these results help redefine the role of LLMs in informal educational contexts. While the literature tends to highlight the limitations of such systems in terms of empathy, the data from this study suggest that they can be highly effective in producing feedback perceived as emotionally appropriate. Nevertheless, we must highlight a significant aspect: within the educational context, the role of the expert differs from that provided by LLMs. Specifically, although in this study LLMs demonstrate a strong capacity to simulate immediate and supportive empathy, human support remains the only source of comprehensive pedagogical guidance. However, the scores achieved by human answers do not indicate a shortcoming but rather a focus on the educational process that takes into account the student's entire long - term progress, unlike LLMs, which are artificially programmed to be immediately effective.

Results suggest that LLMs have the potential not only to be cognitive tutors but also to provide effective support in the context of informal mathematics education. LLMs' added value lies in their ability to incorporate an emotional and relational dimension alongside their technical capabilities. In informal mathematics contexts, where anxiety toward the subject often acts as a learning obstacle, the use of LLMs can foster a learning environment where feedback is not only corrective but also emotionally effective and validating. This is not inherent in LLMs but

is mediated by the quality of the prompt. According to [26], the structuring of the prompt is important but, above all, effective. Therefore, it is essential that teachers and developers collaborate to "train" LLMs not to limit themselves to mere procedural correctness but to also be able to incorporate elements of emotional and affective support. As a result, the use of these technologies in educational settings should not be viewed as an alternative but rather as a perfectly complementary tool, with humans serving as guides and supporters.

Another aspect to consider regarding the analysis of these findings is the nature of LLMs. Namely, these systems should not be viewed as entirely neutral. Their high scores on affective dimensions do not necessarily imply that they are truly capable of understanding the student's emotional status. On the other hand, these results show that LLMs primarily operate by following a set prompt. Consequently, the empathy demonstrated by LLMs risks being merely "ideal" empathy. Hence, it is essential that future studies not only measure the effectiveness of feedback but also, and above all, analyze the cultural and expressive biases hidden within these systems.

Concluding, future research could explore which specific components of feedback influence the perception of empathy by conducting a qualitative analysis aimed at identifying the most influential text segments. Specifically, it might be interesting to classify how certain linguistic elements (i.e. the use of modal auxiliaries, expressions acknowledging mistakes, or the explicit expression of emotional support) actually affect the dimensions of *emotion recognition* and *emotional contagion*.

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Conflict of interest

The authors declare that there are no conflicts of interest.

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Formative feedback and artificial intelligence: new perspectives for pre-service teacher education

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Abstract: This contribution investigates the role of generative Artificial Intelligence in formative feedback processes in mathematics education, with particular reference to pre-service teacher education. The study interprets an Artificial Agent, developed through ChatGPT 5.2, as a possible dynamic semiotic mediator in light of the Theory of Semiotic Mediation and the paradigm of Formative Assessment. The experimentation was carried out with 300 students enrolled in the Primary Education Sciences degree programme at the University of Bari Aldo Moro, who worked in pairs on the solution of an open-ended mathematical problem concerning the equi-extension of parts of a rectangle. The activity was structured into three phases: the production of an initial solution, interaction with the Artificial Agent through a structured prompt aimed at generating non-solution-providing feedback, and a subsequent collective discussion. The qualitative analysis of 150 protocols made it possible to observe the evolution of students' responses before and after the interaction with AI, with particular attention to solution strategies, the quality of argumentation, and processes of cognitive and metacognitive self-regulation. The results show a tendency to move from local, incomplete, or poorly argued responses towards broader, more structured, and more generalised responses, especially in cases characterised by extended dialogic interactions, reconsideration, or contestation. At the same time, some critical issues emerge, related to possible misleading feedback, overly verbose outputs, and the risk of a mechanical use of the tool. These aspects, when appropriately discussed and mediated by the teacher, may become formative opportunities for developing a

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critical and informed use of AI. The study therefore highlights that the Artificial Agent cannot replace the role of the teacher, but may constitute a significant resource for supporting revision, argumentation, and the shared construction of mathematical meanings.

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Keywords: formative assessment; semiotic mediation; artificial intelligence; feedback; mathematics education.

1. Introduction

In the contemporary landscape of educational research, feedback has progressively become a central element in pedagogical and didactic reflection, particularly in the field of mathematics education [7]. Numerous studies highlight that timely, structured, and meaningful feedback can have a significant impact on learning processes, supporting the development of metacognitive skills, self-regulation processes, and argumentative and problem-solving abilities [11, 4].

Despite the broad consensus on the formative value of feedback, its implementation in everyday school practice still presents several critical issues. Large class sizes, organisational constraints, and limited instructional time often make it difficult to provide personalised and timely feedback capable of promoting in-depth discussions on students' productions [10]. These difficulties are particularly evident in mathematics teaching, a discipline in which feedback does not concern only the correctness of the result, but also involves the quality of the solution processes, the strategies adopted, and the arguments developed by students.

In this scenario, the spread of generative Artificial Intelligence (AI) has introduced new possibilities for transforming educational and teaching practices [6, 13]. Systems based on Large Language Models (LLMs), thanks to their capacity to support articulated and adaptive dialogic interactions, can provide personalised feedback, requests for clarification, progressive hints, and reformulations consistent with users' responses [1]. These characteristics make AI a potentially relevant tool in teaching-learning processes, moving beyond a purely technical-operational conception of technology.

The integration of AI into educational contexts, however, requires a critical and informed approach that avoids mechanical or passive uses of the tool. This implies the need to develop effective forms of AI literacy among both teachers and students, so that interaction with AI—and specifically with an Artificial Agent (AA) that is clearly defined in relation to the activity carried out—can become an opportunity for reflection, re-elaboration, and the shared construction of meanings [18].

In light of these considerations, the present study aims to investigate the role of the AA within the formative feedback process in mathematics education, interpreting it as a dynamic semiotic mediator in the light of the Theory of Semiotic Mediation [2]. From this perspective, the AA is not considered merely as a support tool, but as an artefact capable of actively participating in the construction of mathematical meanings through dialogic interactions that guide the evolution of solution strategies and argumentative processes.

The aim of this contribution is to analyse the evolution of the solutions produced by pre-service teachers in solving a mathematical problem, as well as their progressive awareness of the role of the AA as a mediator in the feedback process. In particular, the study focuses on the interactions developed between pre-service teachers and ChatGPT 5.2, with the purpose of observing how the feedback generated by the AA may influence the re-elaboration of solution strategies, the development of mathematical argumentation, and self-regulation processes.

Through this perspective, the study is positioned at the intersection of mathematics education, formative assessment, and studies on AI in education, contributing to the debate on the potential of the AA as a mediating device in processes of mathematical knowledge construction.

2. Theoretical Framework

The activity under study was designed and analysed with reference to two main theoretical frameworks: the Theory of Semiotic Mediation (TSM) [2] and the paradigm of Formative Assessment (FA) [4, 11]. The integration of these perspectives makes it possible to interpret the role of generative AI and specifically that of an AA defined within AI, not only as a technological support, but as an artefact involved in the construction of mathematical meanings and in formative feedback processes.

2.1. Theory of Semiotic Mediation and the role of AI

Developed within the Vygotskian tradition, the Theory of Semiotic Mediation considers central the semiotic relationships established among artefacts, activity, signs produced by students and the construction of mathematical knowledge [2]. From this perspective, mathematical learning emerges through processes mediated by the use of appropriately selected artefacts, employed within specific teaching practices. According to TSM, an artefact incorporates knowledge and, precisely for this reason, can offer those who use it, according to a scheme of use related to an assigned task, a way of accessing the mathematical meanings it conveys [19]. An artefact therefore assumes a semiotic mediation function when it is used by students to carry out a task and, simultaneously, by the teacher with the intention of promoting specific mathematical meanings. A central element of TSM is the concept of semiotic potential, defined as the dual link that the artefact has: on the one hand with the personal meanings produced by students during the activity, on the other hand with mathematical meanings. Through the teacher's mediation and moderation actions, particularly in the phases of collective discussion [3], individual signs can progressively evolve into shared and mathematically recognised signs. In the present contribution, generative AI makes it possible to characterise an AA interpreted as an artefact endowed with a specific semiotic potential. Unlike traditional manipulative or digital artefacts, the AA is characterised by the ability to support dynamic dialogic interactions, progressively adapting its responses to users' productions. This characteristic makes the AA a particularly relevant semiotic mediator, since it is capable of accompanying the student in the explicit formulation of their reasoning, in the reformulation of solution strategies and in the progressive construction of mathematical meanings. Dialogic interaction with the AA can in fact foster processes of clarification, argumentation and revision, stimulating the student to reflect on their own procedures and choices. From this perspective, the AA does not merely provide information or corrections, but actively participates in the construction of mathematical meaning, orienting students' actions and productions.

2.2. Formative Assessment and Feedback in Mathematical Learning Processes

Alongside the perspective of semiotic mediation, the present study is grounded in the paradigm of Formative Assessment, understood as a set of practices aimed at collecting and interpreting information on students' learning processes in order to guide subsequent teaching decisions [4, 5, 8].

Within Formative Assessment, feedback assumes a central role. Previous studies in pre-service teacher education have highlighted how feedback, when embedded in structured formative assessment paths, can support the development of communicative skills, reflective processes and awareness of one's own mathematical work [9]. In the field of mathematics, it cannot be limited to the identification of errors or to the validation of the correct result, but must support the explicit formulation of the strategies adopted, highlight possible conceptual difficulties and promote processes of revision and self-regulation. Effective feedback in fact contributes to the development of metacognitive awareness and supports the construction of argumentative and problem-solving competences [11].

According to Black and Wiliam [4], Formative Assessment takes place when the evidence col-

lected during the activity is used by teachers, students or peers to make teaching decisions that are more effective than those that would have been made in the absence of such information. Wiliam and Thompson [20] identify five key strategies of Formative Assessment: clarifying learning goals and success criteria; promoting effective discussions; providing feedback that guides improvement; activating students as instructional resources for one another; supporting processes of self-regulation and responsibility.

Within this framework, peer feedback assumes particular relevance, since it fosters processes of comparison, negotiation and shared revision of mathematical productions. In order for feedback to be genuinely formative, it is necessary that it enables students to recognise critical issues, gaps or inconsistencies and to identify possible strategies for improvement [17].

Hattie and Timperley [11] distinguish four levels of feedback: feedback on the task, related to the correctness of the activity carried out; feedback on the process, referring to the strategies used; self-regulation feedback, oriented towards the student's ability to monitor and control their own learning; feedback on the self, related to personal and emotional aspects. In the present study, particular attention is devoted to feedback on the process and on self-regulation, considered fundamental for supporting the evolution of the solution strategies and argumentative levels of pre-service teachers.

From this perspective, Formative Assessment can be interpreted as a mediating device, through which the teacher, and, in the present study, also the AA, contributes to the construction of meaning from the signs produced by students, accompanying them in the processes of re-elaboration, revision and construction of mathematical thinking. In their central role in designing the activity, defining the tasks, choosing the ways in which the AA is used and structuring the collective discussions aimed at the negotiation of meanings, the teacher ensures that interaction with the AA can effectively promote an evolution of mathematical meanings.

In light of these theoretical references, the present work aims to analyse to what extent interaction with an AA, interpreted as a dynamic semiotic mediator within the formative feedback process, can affect the re-elaboration of solutions to a mathematical problem and the evolution of the argumentative strategies of pre-service teachers. In particular, the research questions that guided the study are:

RQ1. How does dialogic interaction with an AA, interpreted as an artefact and dynamic semiotic mediator within a formative feedback process, affect the evolution of the solution strategies and argumentative levels adopted by pre-service teachers in solving an open-ended mathematical problem?

RQ2. What processes of cognitive and metacognitive self-regulation emerge in the interactions between pre-service teachers and the AA during the revision of mathematical solutions?

3. Methodology

3.1. Choice of the problem

The experimentation was conducted with students enrolled in the Primary Education Sciences degree programme at the University of Bari, during the Mathematics Education course. The sample consisted of a total of 300 pre-service teachers. The activity was carried out in pairs, this choice allowed peer comparison and fostered the sharing of solution strategies.

With regard to the assigned task, the following open-ended problem was chosen [16]: *“Two brothers inherit a rectangular plot of land. In order to divide it into two parts of the same area, an acquaintance suggests that they place a pole at any point on the land and connect it to the four small posts fixed at the four vertices of the rectangular plot. One of the brothers will take the part coloured grey in the drawing, the other will take the remaining part. Are the two parts really equal? Justify your reasoning.”*

This choice is related to the plurality of applicable solution strategies: the separate or

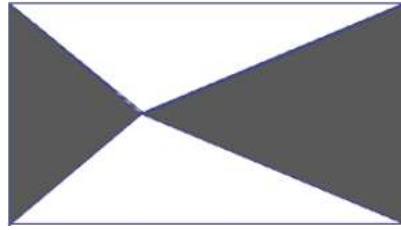


Figure 1. The image of the land

combined application of algebraic, geometric and empirical strategies makes it possible to propose, or re-propose, the problem at different school levels. The importance of these problems lies in the possibility of stimulating logical-critical thinking, argumentation and creativity in solution methods. Unlike closed problems, this type of problem allows teachers to analyse and understand different mathematical competences, different approaches to problem solving and possible misconceptions, information that is useful for creating personalised pathways to guide the teaching-learning process from a formative perspective.

3.2. Structure of the experimentation

The phases of the experimentation are divided as follows:

- Phase 1: Problem solving: the pairs of students produced a first solution to the problem and entered their answers in a Google Form. The form was prepared in order to keep track of the different strategies adopted and arguments made explicit.
- Phase 2: Interaction with AI: in this phase, students were given the opportunity to interact with ChatGPT 5.2. The interaction began with the insertion of a prompt structured in advance, through which it was possible to instruct the Chatbot to provide indications that did not solve the problem, but acted as catalysts for critical thinking.

The choice of a prompt structured in advance is related to the importance of providing the chatbot with the necessary information to construct a high-quality and meaningful interaction [12].

The structured prompt for Primary Education Sciences students is:

I am a Primary Education Sciences student, and I have solved the following mathematics problem. I would like you to help me reflect on the steps of my solution without providing me with answers, but by giving me formative feedback. The text of the problem is “Two brothers inherit a rectangular plot of land. In order to divide it into two parts of the same area, an acquaintance suggests that they place a pole at any point on the land and connect it to the four small posts fixed at the four vertices of the rectangular plot. One of the brothers will take the part coloured grey in the drawing, the other will take the remaining part.

Are the two parts really equal? Justify your reasoning. My solution is -insert your own solution with the image-.

Following the exchange with ChatGPT 5.2, the students entered both the interactions with the chatbot and the reformulated answers in a further Google Form.

In the analysis of the responses, it was possible to highlight the evolution of the solutions, derived from the dialogic interactions with the AA: the continuous reciprocal feedback was a crucial element in the transformation of the initial answers.

- Phase 3: Collective discussion: the solutions returned by the students were then the object of collective discussion [3]. This choice is also consistent with previous research on Mathematical Discussion in pre-service teacher education, which has shown the importance of learning in and from practice through digitally mediated discussion environments

[15]. In this discussion, issues of both a mathematical nature and concerning the unprecedented role of the AA were addressed. The teacher, in this phase, played a central role as a bridge for the mathematical institutionalisation of the contents and for the critical comparison on the new practices in which the AA intervened.

4. Analysis of the Interaction with the AA: Two Case Studies

The analysed sample consists of 150 protocols (300 students divided into pairs), structured as follows: initial answer (before the introduction of the AA), interaction with the AA and reformulated answer (after the use of the AA). In the following analysis, the solutions to the problem and the different interactions with the AA were classified. The types of answers identified are the following:

- Local answer: the answer is correct but refers only to one or more particular cases, without grasping or making explicit the general structure of the problem.
- Incomplete answer: the answer provides a correct result, but without an explicit argument or with omitted logical steps.
- Broad answer: the answer is correct, argued and formulates a general solution and includes multiple solution strategies.
- Incorrect answer: the answer contains conceptual or procedural errors that lead to a false conclusion.

Data analysis was conducted according to a qualitative exploratory approach, through the classification of the protocols and interactions on the basis of categories consistent with the theoretical framework of reference. As recalled in the theoretical framework, the AA is an active mediator in the construction of meaning: this construction takes place through feedback that provides continuous support to the user. This mode of interaction, guided by the teacher's a priori design, makes it possible to frame ChatGPT, in version 5.2, as a possible dynamic semiotic mediator, capable of transposing knowledge and orienting the learning of the interlocutor. From the analysis of the data, multiple modes of interaction with the AA emerge: the types of conversations observed present differences in language, in the quality of the dialogue and in the breadth of the exchanges. The effects of the suggestions provided by the chatbot produced different reactions both at the cognitive level and at the metacognitive level. Indeed, during the dialogue phase with the AA, students were able to ask for in-depth explanations of the mathematical concepts involved and for a revision of their own solution processes useful for the reformulation of the solutions. Following a more or less argued reformulation of the solution strategy, the answers were placed at a new level with respect to the initial classification. In addition to the classification of the answers mentioned above, it was possible to highlight the different modes of interaction between the AA and the user, grouped into five operational categories:

- Rapid interaction: sequence of 1–2 conversational turns, oriented exclusively towards verifying the correctness of the answer, without requests for clarification or in-depth analysis.
- Minimal interaction: substantially passive acceptance of the feedback generated by the AA, in the absence of reformulations, explanatory requests or revision of the initial solution.
- Broad interaction: structured and prolonged dialogue (several turns), aimed at conceptual exploration, understanding errors and analysing solution strategies.
- Rethinking interaction: interactive process that leads to an explicit modification of the initial solution, with reorganisation of the procedure or reformulation of the answer.
- Contestation interaction: interaction characterised by the critical questioning of the AA's feedback, through arguments, counterexamples or requests for justification.

The analysis of two case studies of protocols produced by pre-service teachers is reported below.

4.1. The first case

The protocol presented in this first case was produced by a pair of Primary Education Sciences students during the Mathematics Education course.

The initial answer was classified as local and incomplete, since the invariance of area was only partially recognised, with reference to a particular position of the point, and the strategy presented was empirical-visual in nature.

The students' initial answer is reported below:

We measured the land with a ruler and reproduced it on paper with measurements of 10 cm and 5.5 cm, also drawing the 4 triangles inside. Then we cut out the various parts, thus obtaining the 4 separate triangles. By joining the two pairs of triangles, we obtained two equal figures. Therefore yes, the two parts are really equal. In addition, we noticed that by joining the two white triangles and the two black triangles, the resulting figure is a rhombus, whose diagonals are the base and the height of the original rectangle.



Figure 2. Examples of procedures

From the first lines of the solution, it emerges that the students used measurement to reproduce the figure on paper, taking the measurements directly from the screen of the pc used during the activity. The use of measurement in this way suggests the need to make concrete and to “mathematise” the proposed problematic situation through numbers. Subsequently, they cut out and superimposed the different parts of the rectangle, trying to demonstrate the invariance of area through a visual proof. This strategy inevitably entails a lack of abstraction and generalisation, which makes the proof provided not sufficiently complete.

After the students entered the initial answer in ChatGPT, the first feedback received was the following:

Your work shows:

- *good modelling ability;*
- *conscious use of concrete representations;*
- *active exploration of the geometric figure;*
- *first steps towards generalisation.*

The reflections I suggested could help you make your argument even stronger and better understand which mathematical ideas are involved. If you want, we can also explore together one of the steps to clarify how to transform your intuition into a more general justification

The analysis shows that the AA, thanks to the indications included in the initial prompt, does not directly provide the solution but invites students to review, re-elaborate and deepen their argumentation. Moreover, in the continuation of the conversation, reported in part below, the chatbot proposes guiding questions and suggestions for alternative solution strategies

in order to improve the deep understanding of the problem and allow for more structured argumentation.

Students: *From a theoretical point of view, what suggestions could you provide us?*

ChatGPT: *Here are some theoretical suggestions that can help you strengthen your reasoning without replacing your empirical path with a “ready-made” solution. The aim is to show you which mathematical ideas can support and make more general what you have already discovered through practical work [...]*

Students: *So in your opinion what is the step that is missing in our reasoning and that we should therefore add?*

ChatGPT: *I would say that the step that is really missing in your reasoning, the one that would allow you to transform an excellent empirical verification into a theoretical justification, is to make explicit why the two pairs of triangles necessarily have the same area independently of the position of the chosen point. In other words: You need to theoretically justify the equality of the areas of the two pairs of triangles, without relying only on the physical superimposition obtained through cutting out.*

The exchange highlights that the students do not settle for the first feedback received from the AA, but ask for further clarification in order to improve their answer.

With reference to the previous classification, we interpreted this interaction as broad and rethinking.

Indeed, in the third phase, that is, the reformulation of the answer, an understanding of the invariance of area independently of the experimental verification emerges, as can be inferred from the following:

[...] To this concrete verification we added a geometric reflection in order to understand why this equality of area does not depend on the specific drawing but always holds, whatever point is chosen inside the rectangle. The opposite sides of a rectangle are parallel and have the same length: this means that, if we consider for example the upper side and the lower side, the two triangles that have them as their bases have as their respective heights two distances whose sum always gives the total height of the rectangle. Since the area of a triangle is always half the product of base and height, the sum of the areas of these two triangles is exactly half of the area of the whole rectangle. The same reasoning can be applied to the other pair of opposite sides, again obtaining half of the total area. Therefore, both by observing the cut-outs and recomposing them, and by reasoning geometrically, we concluded that the two parts obtained by connecting the internal point to the vertices really have the same area, independently of where the chosen point is located.

The reformulated answer highlights a relevant change in the nature of the argumentation: from empirical verification through measurement, cutting out and recomposition, the students move to a justification based on the geometric properties of the rectangle and the triangles. This transition emerges when the students state that they added to the “concrete verification” a “geometric reflection” in order to explain why the equality of area does not depend on the specific drawing, but holds for any position of the internal point. The reference to the opposite sides of the rectangle, parallel and congruent, allows them to shift their attention from the measured figure to the geometric structure of the problem. In particular, the sentence “Since the area of a triangle is always half the product of base and height, the sum of the areas of these two triangles is exactly half of the area of the whole rectangle” shows the achievement of a more structured and complete level of argumentation. The students no longer rely only on visual superimposition, but use the relationship between bases, heights and areas to justify invariance. Moreover, the statements “The same reasoning can be applied to the other pair of opposite sides” and “independently of where the chosen point is located” indicate an evolution of mathematical thinking towards generalisation. Therefore, there is a transition from a local and incomplete answer to a complete answer, based on geometric properties and on the

understanding of area invariance.

Focusing on the role of the AA, several formative aspects can be recognised, such as the dialogic and encouraging climate, the stimulus to reflect on and revise the reasoning. From what emerged, the choice to analyse this specific case of interaction derives from the possibility of clearly observing the constructive and critical use of the AA, which allowed a semiotic evolution of the mathematical meanings related to the assigned problem.

From the perspective of TSM, the students' initial empirical and visual productions can be interpreted as personal signs emerging from the use of concrete actions such as measuring, cutting and recomposing. Through the dialogic feedback provided by the AA and the subsequent teacher-led collective discussion, these signs progressively evolved towards more explicit mathematical signs, grounded in the properties of the rectangle, the relationship between bases and heights, and the invariance of area.

4.2. The second case

The second protocol analysed falls within the category of an incomplete and broad answer. The pair of students involved provided the following initial solution:

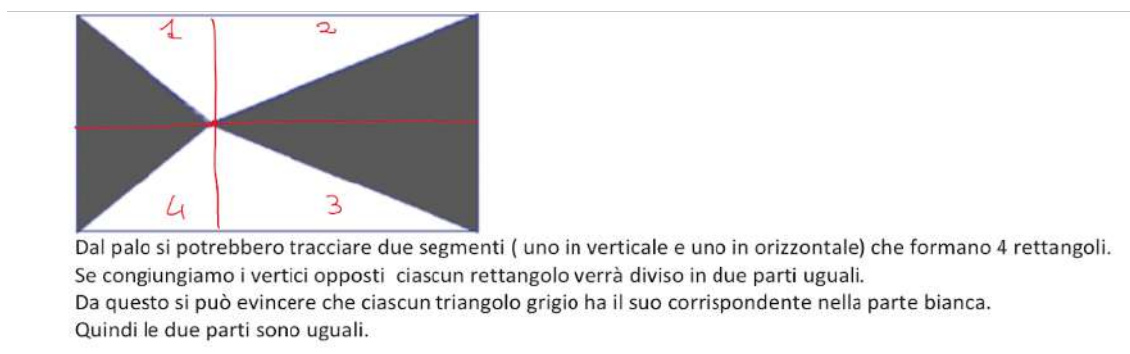


Figure 3. Initial solution produced by a pair of pre-service teachers.

Translation of the students' written response: “From the pole, it would be possible to draw two segments, one vertical and one horizontal, which form 4 rectangles. If we join the opposite vertices, each rectangle will be divided into two equal parts. From this, it can be inferred that each grey triangle has its corresponding part in the white region. Therefore, the two parts are equal.”

From the image, it can be observed that the strategy used includes both geometric arguments, deducible from the presence of words such as “segments”, “rectangles”, “vertices”, and visual ones: the demonstration of equal area is accompanied by the representation of the figure in which there is the subdivision and correspondence of the parts coloured grey and white. Moreover, the particular position of the point leads to the demonstration of a specific case and therefore indicates a lack of generality. The proposed argument is rich in implicit elements, and despite the presence of geometric terms, it does not present a rigorous theoretical basis. To this formulation the chatbot provided the following feedback: *Good job, this is a nice attempt, it is clear that you have already intuited the right direction. I will give you specific feedback on the steps, without “spoiling” the final answer, with suggestions to make the reasoning more precise and rigorous [...] 2. Correct the description of the figure Instead of speaking of “4 rectangles”, it is more accurate to say that by connecting the internal point to the four vertices, 4 triangles are obtained. This clarifies the nature of the areas that you need to compare.* It can be noted that ChatGPT's feedback is not entirely exhaustive or clear, but it provides implicit suggestions about the geometric figures to be considered, triangles instead of rectangles, and it does not describe with precision the logical steps necessary for the students' understanding. Indeed, they ask for further explanations: Students: *Can you explain point 2 to me with a*

drawing? ChatGPT:

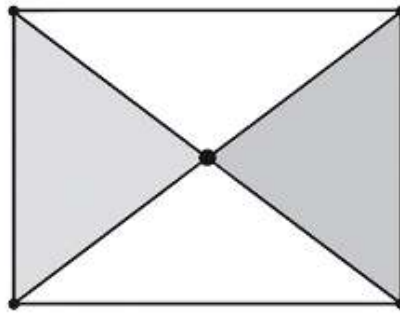


Figure 4. Visual representation provided by ChatGPT in response to the students' request for clarification.

The AI feedback, in this case, is misleading for the user: the figure provided shows the pole at the centre of the rectangle, thus bringing the problem back to a particular case.

The possibility of incorrect suggestions from ChatGPT constitutes a real risk when the AA is used mechanically [14]. Moreover, the subsequent dialogue was characterised by very extensive feedback, for which the students explicitly asked for a reduction and simplification of the information provided.

From our point of view, the presence of hallucinations, large outputs and the various possible critical issues becomes a potential resource in the phase of collective discussion, since reflection can stimulate critical thinking and a critical use of the AA. Indeed, the students involved in this conversation were not diverted by this error; on the contrary, they took the opportunity to ask for further explanations, creating an interaction of contestation, which step by step led to the reformulation of the answer:

Let $ABCD$ be the rectangle and let P be any point inside it. By connecting P to the four vertices, the rectangle is divided into four triangles. In the drawing, two triangles are coloured grey and form the part assigned to one brother. To compare the areas, we observe that each triangle has as its base one side of the rectangle and as its height the distance of point P from that side. Let h_1 and h_2 be the heights of the two grey triangles with respect to their bases, which are two opposite sides of length b . Since the heights add up to the whole distance between the opposite sides, we have:

$$h_1 + h_2 = H,$$

where H is the total height of the rectangle with respect to those sides. The total area of the two grey triangles is therefore:

$$A_{grey} = \frac{b \cdot h_1}{2} + \frac{b \cdot h_2}{2} = \frac{b \cdot (h_1 + h_2)}{2} = \frac{b \cdot H}{2}.$$

Similarly, the other two non-coloured triangles together have the same area. The two parts are therefore always equal, independently of the position of point P , because each includes two triangles whose sum of the areas is half the area of the rectangle.

The final answer to the problem therefore becomes complete and broad: it presents a dual algebraic and geometric strategy expressed through multiple argumentative registers.

The possibility of showing the critical issues, as well as the potentialities of this tool, motivates the choice of this protocol. It provides several points for reflection regarding the conscious

use of AI in teaching-learning processes, actively involving pre-service teachers in effective training.

5. Conclusions

The analysis of the protocols reveals a tendency to move from local and incomplete answers to broader and more argued answers following the interaction with the AA, particularly in cases characterised by broad and rethinking interactions.

The present work explored the role of the AA as a dynamic mediator in the feedback process in mathematics, in particular the effects on the solutions to the problem proposed to the students were analysed.

With structured didactic design, it is possible to include the AA as a significant support in revision and argumentation processes. The results seem to highlight that AI cannot replace the teacher; on the contrary, it requires a critical and situated use, which must be designed and planned in advance by the teacher.

In this sense, the activity also functioned as an opportunity for developing AI literacy among pre-service teachers. Rather than using the AA as a source of ready-made answers, students were guided to interrogate, evaluate and, when necessary, contest the feedback received, thus developing a more critical awareness of the educational use of AI.

Thanks to the methodology identified and to the structuring of the phases of the experimentation, as well as to the decision to instruct the chatbot with a specific prompt, it was possible to observe the evolution of the answers. The comparison between the productions before the AA and after the AA made it possible to observe changes in solution strategies and the evolution of the levels of argumentation.

The choice of the open-ended problem seems to be essential for highlighting the different solution strategies, bringing out the main difficulties encountered and making explicit the effect of feedback. Indeed, from the analysis of the protocols, the transition from local and incomplete answers to broad answers emerged. Despite the positive evidence presented, AI showed several critical issues such as misleading suggestions and verbose outputs. These critical issues provided material for reflection in the third phase of the experimentation, namely the collective discussion, in which several beliefs concerning a mechanical and little aware use of AI emerged. Through the alternative methodology used in the experimentation, in which the AA assumed a role different from the usual one, the students modified their schemes of use of this tool, recognising its scaffolding potential when used analytically. Contestation interactions therefore assumed a particularly relevant role, since they activate forms of control and regulation of the process.

In conclusion, the integration of the AA into the feedback process can represent a significant resource for mathematics education. It can support problem solving and foster the development of cognitive and metacognitive autonomy if designed and used critically by all the actors involved in the didactic activity.

Conflict of interest

The authors declare that there are no conflicts of interest.

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Reducing students learning gap in multicultural classrooms through mathematics teacher training: an Italian case study

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Abstract: This paper examines how mathematics teacher training can reduce the learning gap experienced by students from diverse cultural backgrounds in Italian multicultural classrooms. Drawing on a reinterpretation of learning loss through a cultural lens, we argue that students' mathematical knowledge is not lost but rather unrecognized within the dominant school culture. Survey data from 560 in-service teachers enrolled in the professional development course "Cultures Count" (2025-2026) reveal significant gaps in preparation for plurilingual and multicultural teaching contexts. An inductive content analysis of teachers' conceptions of mathematics further highlights prevalent epistemological positions. In the final part of the article, a section of the teacher training course phase is presented, aimed at discussing a didactic design from an intercultural perspective. Building on Villegas and Lucas's framework of culturally responsive teaching, the paper presents a co-designed planning scheme as a practical tool for developing teachers' roles as cultural brokers, capable of recognizing and valuing students' diverse mathematical knowledge as a resource rather than a deficit.

2020 Mathematics Subject Classification: Primary 97B50; Secondary 97C50.

Keywords: learning gap; multicultural classrooms; mathematics teacher education; teacher training course; intercultural competence.

1. Introduction

The phenomenon of multiculturalism in education has been studied for several years from pedagogical, psychological, and general didactic perspectives. Few works, however, address the

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issues of subject-specific teaching and learning in non-monocultural contexts. Even fewer are the studies in mathematics, both nationally and internationally. The topic discussed here falls within the field of research in mathematics education and may also be of interest to other areas in dialogue with it. The article, in discussing the theme, frames in its second section the issue of multiculturalism and multilingualism as a new challenge for teachers. In this section, examples are provided of different processes enacted by students from "other" cultures, which are often complex for teachers to analyze. Subsequently, the reported TALIS and OECD data highlight a gap in teacher training on these issues, thereby giving greater meaning to the teacher training innovation work we propose. The following sections discuss the theoretical framework (Section 3), the methodological aspects (Section 4), and the results (Sections 5 and 6) of the research described here. We believe that the article may be of interest to the research community in mathematics education, but not only there, also because we propose a shift in perspective in framing the phenomenon of *learning loss* from a cultural standpoint, such that it should not be interpreted merely as an effect of a temporal gap, but rather as a consequence of a change in the implicit or explicit frame of reference. This particular aspect is described in the section on theoretical frameworks.

2. Teaching mathematics in multicultural contexts: a "new" challenge for teachers

In contemporary educational research, the persistence of social and economic disparities among groups within the same context has long been one of the most widely debated issues. The social and cultural gap is often experienced on a daily basis as a form of "dissonance" that unfolds within school environments. Research shows that this gap develops along three main dimensions: the epistemology of the discipline, language (or discourse), and teachers' expectations, and that this cultural distance frequently manifests itself in learning contexts [1, 2]. Below, we recall some examples related to these three dimensions in order to prompt a didactic reflection.

In [3], examining the difficulties faced by Ukrainian refugee students in the UK, several challenges are identified that stem from differences in symbolic systems (digits, decimal point, multiplication and division signs), syntactic aspects of mathematical language (such as parentheses), and the algorithms used for computation and for solving quadratic equations and linear systems, which are often not aligned with those commonly used in the UK. The implication is clear: what was valid and meaningful for a Ukrainian student up until a few weeks before arriving in the UK becomes something to be "set aside", as it is no longer readily understood or easily assessed by teachers in the new context.

In [4], starting from a different setting—namely, two multicultural classes totaling 50 first-year technical high school students (25 from Italian families and 25 from non-Italian families)—the author describes several forms of "distance" emerging in relation to epistemological aspects and the use of more structured algorithms, such as those for division. The context outlined in the study is highly diverse: among the students with non-Italian family backgrounds, nine had arrived in Italy less than three years earlier. Figure 1 presents two division algorithms discussed in the study that differ from those typically used in Italy. According to the author, in many educational settings these alternative algorithms are not adequately recognized or valued and are sometimes even "dismissed" by teachers, as they are not aligned with the institutionalized "standard" knowledge of the classroom—that is, the knowledge associated with the dominant culture. [4] examines this issue also through a historical lens, which helps explain the origins and rationale behind these algorithms, suggesting that the ability to recognize and flexibly engage with multiple approaches can foster greater pedagogical "openness" and support a broader, more multidimensional understanding of the discipline (historically, critically and geographically flexible).

From the two contributions discussed above, it clearly emerges that the issue at stake con-

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Figure 1. Examples of spontaneous algorithms taken from figure 2 in [4].

cerns both the student and their way of learning mathematics in the classroom, as well as the teacher, who is embedded in a personal and often deeply ingrained frame of reference that is difficult to move beyond [5], despite the strong demands posed by multicultural educational contexts. *When confronted with bodies of knowledge that may be culturally "distant" from one another, how should they be addressed? How can the distance between them be reduced? What happens, at both a cognitive and affective level, to a student who, when placed in a different context, "distances" themselves from an algorithm or from knowledge that is part of their own (culturally grounded) background? Is this knowledge lost? Can this be considered a case of learning loss?*

Providing definitive answers to these questions is complex; however, with regard to the last question, we argue that the answer is "no". In our view, this is not a matter of "forgetting," but rather of "setting aside," thereby creating a space of *learning gap* between a "before" and an "after"—a distance between prior knowledge and what one is now required to learn. For this reason, in the present paper, we will conceptually reframe the notion of learning gap with an empirical and design-oriented account of teacher education in multicultural mathematics classrooms. In the landscape of multicultural literature, our work, due to this hybrid nature between theoretical and practical, represents a novel element. In the next section we set the research problem of learning gap in multicultural context and in section 3 we address this problem through different complementary lenses.

This complex situation requires teachers—who are called to educate in the broadest sense within such contexts—to adopt an open, dialogic approach. In multicultural educational settings, this complexity is further intensified by linguistic and social dissonances. For this reason, it appears promising to approach the issue not primarily from the perspective of the individual student and their knowledge (acquired and then lost), as in the case of learning loss, but rather from the perspective of the environment in which the student is situated. This involves examining the relationship between what the student knows and what, within the dominant culture of the host classroom, they are normatively expected to know.

Within this second set of dimensions—more appropriately framed in terms of the learning gap—it becomes possible to disentangle and analyze structural inequalities. In this context, mathematics can become a space for narration and for grounding knowledge in deep, lived experiences. Viewing mathematics as a bridge between different bodies of knowledge—distinct yet connected to diverse cultural references—rests on a pluralistic stance: cultural and linguistic differences, as they emerge in mathematical practice, are not only accepted and supported, but actively sought as resources for the construction of richer, shared knowledge.

From this perspective, a relativistic stance—free from hierarchical value judgments about different forms of knowledge—becomes a necessary condition for genuine inclusion, grounded in concrete diversity. This allows for moving beyond a naïve and rhetorical notion of equality

toward an equity-oriented perspective: we are different, yet we have equal value and equal rights. It is therefore essential to begin with differences, to understand them, and to engage with them relationally. Culture, in this sense, becomes a lens through which we interpret and act in the world—an opportunity rather than a fixed inheritance or a burden tied to an immutable tradition [5, 6].

Data from TALIS (2018) indicate that engaging in these issues constitutes, in many cases, a clear professional development need for teachers [7]. Only 26.3% of lower secondary teachers report that elements of "teaching in plurilingual and multicultural contexts" were included in their formal education or training, compared to an average of 38.5% across TALIS countries (see Table 2.a in TALIS 2018). Furthermore, only 18.7% consider themselves "well prepared" or "very well prepared" in this area, compared to an international average of 31.4% (see Table 2.b in TALIS 2018). Figure 2 presents data from TALIS 2018 regarding the "high level of need for professional development in the area of teaching in a multicultural or multilingual setting" among lower secondary teachers: within this international framework, Italy ranks third, after Mexico and Brazil².

Figure 2. Teachers' needs for professional development for teaching in a multicultural setting

Percentage of lower secondary teachers who indicated that they have a high level of need for professional development in the area of teaching in a multicultural or multilingual setting

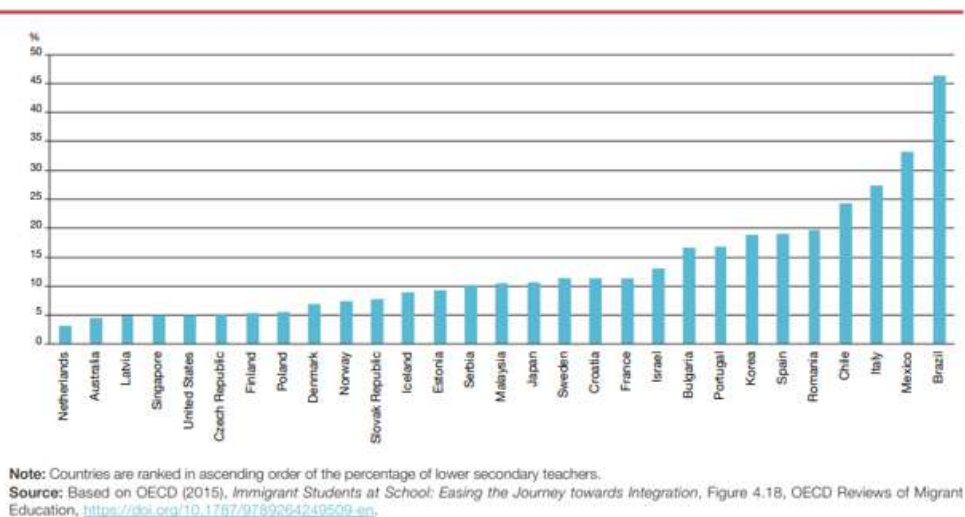


Figure 2. TALIS 2018 regarding "high level of need for professional development in the area of teaching in a multicultural or multilingual setting" among lower secondary teachers.

In the most recent TALIS 2024 survey [8], the data show slight improvement but, in our view, still remain "below threshold": 35.7% of Italian teachers identify teaching in a multicultural or multilingual setting as a component of their professional learning activities (see Table 4.22 in [8]), while 39.6% consider themselves prepared in this area (see Table 4.5 in [8]).

3. Theoretical frameworks

The phenomenon discussed here is very complex from a didactic point of view and therefore requires multiple lenses of analysis. For us, a possible perspective, albeit an initial one, has referred to the possibility of identifying a gap in knowledge related to student learning in multicultural classrooms that is very difficult to detect, analyze, and interpret. In this work, we attempt to establish a connection between the construct of *learning loss* and that of *learning gap*, the latter being more widely discussed in literature. The learning gap is mentioned in this

²https://www.oecd.org/content/dam/oecd/en/publications/reports/2019/02/how-education-systems-respond-to-cultural-diversity-in-schools_3359b650/1baa285c-en.pdf

sense as an initial step and an opportunity for developing a teaching practice attentive to students' needs, in terms of "recovering" and valuing their knowledge, from a perspective of inclusion and cultural transposition. The theoretical framework described here further focuses on the role of the teacher working within these classrooms, referring to the teacher as a *cultural broker*. The theoretical framework, shifting the focus from the student to the context and finally to the teacher, discusses the relationship among these constructs (the triad of learning loss, learning gap, and "teaching gap"), which finds its natural application in the design of the training course (Section 5) and in the analysis of the data collected from the teachers (Section 6).

In recent years, numerous studies have examined the phenomenon of *learning loss*. These studies primarily analyze the relationship between the level of a given competence (expressed through measures such as degree of mastery or speed of use), periods of inactivity related to specific mathematical content, or changes in the learning context. Typically, research focuses on the effects of summer breaks (or exceptional disruptions such as the pandemic [9]), as well as on phases in which, within a spiral curriculum, certain topics are revisited and reworked in new ways.

From this perspective, learning loss can be understood less as the effect of a temporal interruption (before vs. after a break) and more as the consequence of a (prolonged) change in context (school vs. out-of-school). Framed more broadly, if a piece of knowledge (a competence, a concept, or even simple information) holds meaning only within the teacher's cultural framework but not within the student's, the likelihood that it will leave no lasting trace is very high. There is, in fact, a strong relationship between what remains in our minds—what becomes permanently "ours," even beyond schooling—and what is meaningful to us, that is, what makes sense within our cultural framework [10]. Positioning itself within this line of research, the present contribution proposes a broader interpretation of learning loss by opening it up to a cultural perspective. Within this framework, we connect the phenomenon of learning loss to the broader concept of learning gap.

The concept of *learning gap* is often mistakenly interpreted solely through a "deficit model," which attributes failure to intrinsic shortcomings of students or their families [11]. However, research has repeatedly shown that this gap is more accurately understood as the result of a lack of "alignment" between the learner's culture of origin and the cultural context of schooling, with culture understood in a broad sense. In line with [11], the key shift involves moving away from a deficit-oriented view—where the student is seen as lacking—toward a culturally responsive approach that treats students' cultural backgrounds and prior knowledge as "cognitive hooks" for expanding their intellectual capacity [12].

In the educational field, [13] further emphasize that the learning gap is a multidimensional phenomenon arising from the interaction between a school system that is often rigid and hegemonic and a culturally diverse student population. Along similar lines, [14] highlights a certain "predictability" of failure based on intersecting demographic variables such as gender, class, and language. From this broader cultural perspective, the gap is not merely a difference in performance or scores—a simple learning loss—but rather an *opportunity gap*, from the very beginning of the learning trajectory, within school systems embedded in a dominant culture [15]. In many educational contexts, as noted by [16], a systematic gap can be observed between an increasingly diverse, multiethnic, multicultural, and multilingual student population and a teaching body that often remains anchored in homogeneous, outdated, and "Eurocentric" pedagogical models, far removed from the actual needs of the classroom.

Within classroom practice, this translates into several recurring dynamics:

- Implicit bias. Teachers may unintentionally hold lower expectations for students from certain minority groups, a phenomenon described as part of the "social construction

of failure" [17]. By neglecting the temporal dimension in which learning unfolds, and by confining students' mathematical competences—even future ones—to a fixed cultural identity, mathematics becomes a gatekeeper, also for subsequent career choices [18].

- Cultural invisibility. When mathematical problems are framed in contexts that are unfamiliar to students, cognitive load increases—not because of computational difficulty, but because of the unfamiliarity of the situation. The lack of coherence between the cultural backgrounds—understood in a broad sense as family culture or students' socio-economic background—of students and those of teachers and institutions can lead teachers to partially misinterpret students' modeling choices, for example in problem-solving contexts: real-life problems are not necessarily inclusive [19].

In addressing issues of equity, the goal should not be to provide identical treatment to all students (equality), but rather to ensure that each student has access to the resources necessary to achieve shared goals (equity), with attention to the learning process itself. In this sense, [17] advocates for a culturally grounded approach to equity that can have a transversal impact on the learning gap experienced by students. However, this would require a substantial restructuring of the educational system—particularly in terms of teacher education—so as to include intervention models that build upon and diverse values rather than marginalizing them.

The research question guiding this contribution is therefore the following: *how can the learning gap experienced by students from different cultural backgrounds in Italian classrooms be reduced through teacher education?* In the following sections, we explore what we consider to be key moments (initial questionnaire and collaborative planning) within an in-service teacher training course that the authors have organized over the past three years (here we refer to the 2025-2026 edition). In our view—supported by the data presented in Sections 3 and 4—the issue brought by the research question has a structural nature that requires a strong shift in awareness, starting from reflection on the symptom or effect (the student's learning gap) and grounding itself in its underlying cause: the open, complex, and dialogic relationship between teacher and learner in contexts marked by significant cultural distance, as is often the case in multicultural classrooms.

In our view, the central pillar for reducing the cultural learning gap necessarily lies in teacher education. This means preparing teachers who are not only subject-matter experts, but also "cultural mediators," capable of challenging the hegemonic roots of traditional education. Within this framework, the teacher's role is that of a *cultural broker*, who does not conceive knowledge as a static entity to be transmitted, but as a mediated construct [20]. As argued by Villegas and Lucas [20], teachers should act as bridges, helping students interpret the codes of the dominant school culture without requiring them to renounce their own identities. It is therefore not sufficient to simply add a module on diversity to teacher education programs (which, moreover, is often absent in the Italian context). What is needed is a comprehensive restructuring of the curriculum in order to prepare culturally responsive teachers [20], capable of:

1. Developing *sociocultural awareness*: understanding that one's own ways of thinking and acting are culturally shaped, and that multiple worldviews exist.
2. Adopting an *affirming stance*: viewing diversity as a resource rather than as a problem to be solved.
3. Acting as an *agent of change* within the educational context: taking responsibility for fostering a more equitable school system and addressing structural injustices.
4. Fostering a *constructivist orientation*: understanding how students build knowledge based on their prior experiences.
5. *Knowing students' lives*, interests, and the traditions of their communities.
6. Designing *inclusive practices* that build bridges between what students previously knew,

what they currently know, and what they are expected to learn.

Without such awareness and preparation, there is a significant risk that teachers will unintentionally reproduce "pedagogies, curricula, and assessment practices that better serve a homogeneous, dominant group within the context rather than a diverse and heterogeneous student population composed of minorities" [16]. The choice to refer to the theoretical framework described here is consistent with what has been discussed in the literature on the same subject, as it is capable of linking the design of the training course to the methodological aspects of data collection and interpretation, as well as to the outputs related to the training needs of teachers.

Clearly, implementing such an approach is complex. In the absence of top-down, centralized, and well-structured initiatives—unfortunately a characteristic of the Italian context—our approach is to work alongside teachers, fostering an intercultural sensitivity toward the ideas, perspectives, and language of those who come from different contexts and bring with them backgrounds that are often perceived as "distant." Within this perspective, the aim is to promote a form of intercultural competence among teachers that does not consist in knowing the language of newly arrived students, nor in mastering their specific cultures. Rather, it is a transversal capacity: the ability to remain open, to learn from one's students, and to value their out-of-school cultural knowledge [5]. Cultural Transposition—working on disciplinary and epistemological aspects—and culturally responsive teaching—working on a broader pedagogical direction—become complementary, as we will see when analysing the data in Sections 5 and 6.

4. Methods

The overall limited preparation—and the resulting perception of low competence—mentioned regarding teaching in plurilingual and multicultural contexts is consistent with the data collected by the authors in relation to the teacher training course "Cultures Count", launched in winter 2026. The program involved more than 500 in-service mathematics teachers from across Italy, spanning educational levels from kindergarten education to adult education ($N = 560$), and consisted of 24 hours of online training distributed over three months. Teachers voluntarily participated in this teacher training course by completing an online form distributed nationwide in the weeks before the start of the course. Participants also agreed to videorecord each meeting, complete the questionnaires, and allow the data to be used for research purposes, as long as they were appropriately anonymized. In the present section we quantitatively analyse the data that emerged during registration for the teacher training course; in the next section, using an inductive content analysis, we refer to the first two meetings of the course.

During registration teachers were asked: "*Have you ever addressed, during your studies or through professional development activities, the topic of teaching in plurilingual and multicultural contexts?*" The 560 responses were as follows: 50.4% answered "no," 37.7% answered "yes, but without a focus on mathematics," and only 12.0% answered "yes, with a focus on mathematics." Even though the majority of teachers were interested in the topic, only a small part of them were already competent or trained, which is natural for those attending a teacher training course. Due to the voluntary participation, this makes the sample not representative of all Italian teachers, but only of a small minority of teachers who are sensitive and interested, yet not yet trained.

These data points in several directions:

Practical level. Among the variables collected—teaching level (kindergarten, primary school, lower secondary school, upper secondary school, adult education center, other), role (mathematics teacher, support teacher, other subjects), and years of service (1–5, 5–10, 10–20, over 20)—the only variable significantly correlated with the question above is the school level ($\chi^2 = 46.66$, $p < .001$, Cramér's $V = .204$, small-to-medium effect) (Figure 4). The pattern is clear: lower

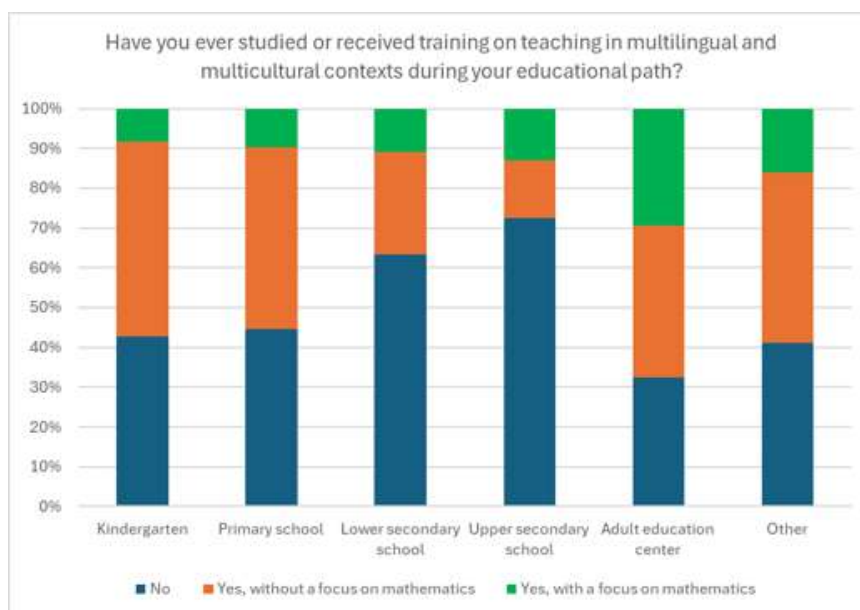


Figure 3. Responses from the 560 teachers to the question: "Have you ever addressed, during your studies or through professional development activities, the topic of teaching in plurilingual and multicultural contexts?"

and upper secondary school teachers are the least trained (63% and 73%, respectively), while adult education centers, kindergartens, and primary school teachers show the highest levels of training. The adult education center stands out in particular for its focus on mathematics (29%), nearly three times higher than in other school levels. There is no evident generational divide (experienced vs. novice teachers). In other words, there is no significant in-service training trajectory leading to a linear increase in competence with experience, nor is there evidence of systematic improvement over time due to top-down initiatives. This suggests weak long-term planning and a general lack of change in approach over recent decades, despite the fact that the issue has become structurally significant at the statistical level [7].

Theoretical level. Teachers report a lack specifically related to the articulation of teaching practices for mathematics within plurilingual and multicultural contexts. This gap is not only pedagogical and related to training, but also theoretical. At both national and European levels, there is a noticeable difficulty—if not an outright inability—to integrate an intercultural perspective into disciplinary teaching, despite it being formally promoted as a flexible and transdisciplinary approach for effective education in such contexts. As a result, intercultural pedagogy remains misaligned with mathematics education, in contrast with European policy documents that emphasize the importance of valuing students' linguistic and cultural backgrounds, particularly those rooted in family contexts [21]. This theoretical gap, in turn, generates a corresponding didactic gap.

This analysis can be further refined by considering additional variables: the weight of plurilingual and multicultural educational contexts—as suggested by the adult education center data in our sample—and the educational background of teachers, whether oriented toward mathematics (upper secondary), science (lower secondary and adult education center), or more pedagogically grounded approaches (kindergarten and primary education). This preliminary overview of teachers' perceptions and beliefs helped us refine our teacher training course.

Given the need for an opening introductory training, in the first phase of the teacher training course, a general framework was provided across four sessions, with particular attention to cultural and linguistic dimensions, aiming to highlight, explore, and problematize key issues in teaching and learning mathematics with students from diverse linguistic and cultural

backgrounds. This was followed by five hands-on workshops organized by broad educational groupings—students aged 3–11 (kindergarten and primary), 12–19 (lower and upper secondary), and adults—with the goal of developing synergistic and concrete approaches to lesson design, in line with [7]. The final two sessions focused on sharing lesson plans, challenges, and open questions, thereby strengthening the community of teachers and researchers (see Figure 3).

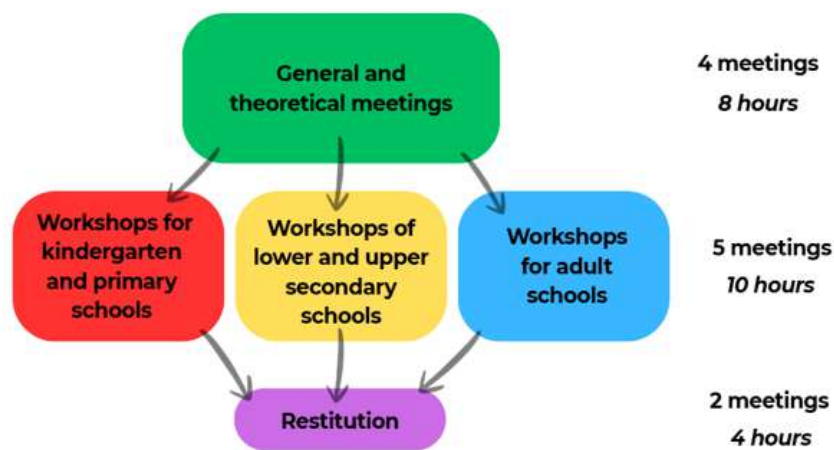


Figure 4. Roadmap of the teacher training course.

5. Results: teachers' conceptions and learning gaps

A fundamental obstacle to reducing the learning gap observed among students often lies in teachers' own conception of mathematics. In this regard, [22] highlights how, in some cases, teachers view mathematics as a single, unified body of knowledge, characterized by an ideology of an objectivist epistemology and a logic of certainty. While already problematic in itself, this view, when applied in multicultural contexts, leads to a form of "culturally decontextualized" instruction that tends to alienate students whose backgrounds do not align with the dominant culture of the classroom. To overcome this limitation, recent research has proposed reconceptualizing mathematics as a "system of fluid and interacting bodies of knowledge" [23, 6]. In line with [24], it becomes desirable to shift mathematics from a "monologue of universal truths" to a "dialogue of human practices," emphasizing the discursive nature of school mathematical activity and decentering the teacher from their own system of reference and perceived certainties [25, 26].

Within this perspective, we report findings from an inductive content analysis of responses provided by in-service mathematics teachers participating in the training program "Cultures Count" (school year 2025–2026), which will be described in the next section. The analysis focused on two questions: "Is mathematics universal?" and "What are the two most significant difficulties in teaching students with a migrant background?" The categorization was carried out by identifying keywords (e.g., "algorithm") and grouping them into higher-order themes (e.g., "Didactic"). This approach preserves the participants' voices while enabling a structured, quantitative overview of qualitative data.

Regarding the first question (Table 1), 114 in-service teachers replied. The results show a clear distribution: the majority of teachers are divided between an absolutist, Platonist view of mathematics (47.8%)—"mathematics is universal"—and a more nuanced universalist view (46.9%)—"mathematics is universal in its results but particular in its practices." In the latter case, teachers demonstrate a clear awareness that mathematical practices vary significantly across cultural contexts (Table 1). Only a small minority adopts a fully relativist perspective,

Table 1. "Is mathematics universal?" Responses from 114 in-service teachers.

Category	Weight (%)	Justification
Culturally mediated universal	47.8%	Responses acknowledging that while logic is universal, the methods, approaches, teaching styles, and algorithms are culture-bound.
Absolutist	46.9%	Responses viewing math as a "universal language" or "unquestionable logic" that transcends human culture.
Relativist	4.4%	Responses defining math as a purely human/cultural construct or a social product.
Other	0.9%	Ambiguous or incomplete responses.

in which culture plays a primary role. This emerging awareness aligns with the ethnomathematical perspective advanced by Ubiratan D'Ambrosio [27], according to which each educational community develops its own mathematical knowledge within its specific context. Such knowledge is not merely a subset of academic mathematics (the "knowledge to be taught") recognized within a given national system, but rather a set of themes, concepts, procedures, and practices that are locally meaningful—and not always recognized as valid by other communities, including those that produce academic mathematics. Thus, one classroom may favor problem-solving procedures that another does not recognize, and such differences frequently emerge across schools operating in different social, geographical, or educational contexts.

Table 2. "What are the two most significant difficulties in teaching students with a migrant background?" Responses from 137 in-service teachers.

Category	Mentions (%)	Justification
Language barriers	61.3%	Explicit mentions of lack of Italian proficiency, communication gaps, and second language learning difficulties.
Cultural differences	19.7%	Mentions of different worldviews, cultural practices, or classroom behavioral norms.
Didactic issues	13.1%	Issues with different algorithms, varying levels of prior schooling, and difficulty in individualization.
Institutional and resource constraints	8.8%	Lack of time, lack of linguistic mediators, and classroom management pressure.
Lack of family involvement	5.8%	Difficulty in establishing a "pact" with families or lack of support at home.

In the second question (Table 2), we see the replies by the 137 in-service teachers. A particularly strong cross-cutting theme concerns language: more than half of the teachers (61.3%)

identify it as a key issue for effective teaching in multicultural contexts. Linguistic difficulties are therefore seen as a *gatekeeper* for the inclusion of all students. Some teachers also explicitly refer to the specialized language of the discipline. Slightly more than 10% highlight more strictly disciplinary dimensions—curricular, epistemological (e.g., algorithms), and cognitive (e.g., logical structures)—related to how mathematics is learned. Finally, another group of responses points to the importance of both the classroom context (motivation, participation, peer interaction) and the out-of-school environment (family). In the following section, we outline—more schematically—the structure of the teacher training program designed on the basis of these questions and the corresponding responses.

6. Designing a planning scheme to reduce the learning gap: a dialogic process

In this section we present a synthesis of the co-design process of a planning scheme, developed through the interaction between teachers and one of the article's authors, an upper secondary school teacher acting as a teacher training facilitator within the "Cultures Count" training course, during a workshop session. The figure shows the scheme produced after approximately 40 minutes of dialogue with around 70 lower and upper secondary school teachers connected online. This initial phase was followed by 30 minutes of small-group design work. What follows is a discursive reconstruction of the key moments in the development of the shared planning scheme, organized by macro-themes (Figure 5) and highlighting the roles played by both teachers and the facilitator. The different phases are interpreted through the six dimensions identified by Villegas and Lucas [20], introduced in Section 3, which characterize the development of teachers' awareness of their role as cultural brokers operating at the intersection of disciplinary knowledge and the heterogeneous, plural contexts of multicultural classrooms. In this sense, the dialogue reported here can be understood both as a moment of rupture and as a step toward the emergence of a collective and critical awareness of the discipline and its teaching.

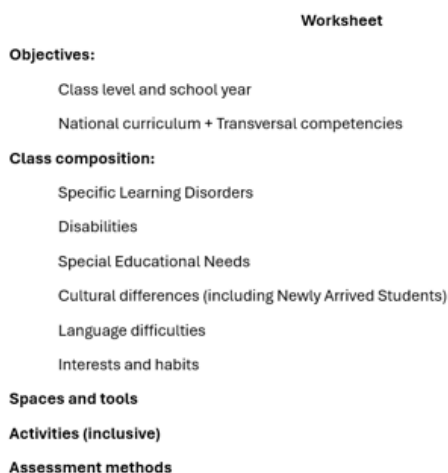


Figure 5. Shared planning scheme for lower and upper secondary school teachers.

Selection of objectives

After defining the class level and school year and aligning with the core thematic areas of the *national curriculum*—with particular attention to vertical coherence—participants, following a suggestion from one teacher, emphasizing the importance of *transversal competences* applied to mathematics.

Class composition

The discussion then moved to issues central to the course. Teacher A remarked: "We also need to consider the students—what they are like, how many have *special educational needs*, how many have *disabilities*, and so on." The facilitator summarized: "So, let's include *specific learning disorders*, possible disabilities—and what else?" Teacher A added: "Foreign students, *newly arrived* students." The facilitator proposed: "Let's include *cultural differences*." Another teacher, B, suggested: "General *special educational needs* categories?" The facilitator responded by noting that while useful, such broad categories risk obscuring important distinctions. Other participants refined the categories (e.g., *linguistic special educational needs*, *language difficulties*, dysgraphia). Ultimately, the facilitator emphasized the importance of explicitly identifying the issues: "So, in this context I would first give a brief overview of what we are talking about: difficulties, cultural differences—*cultural differences* in a broad sense, including also *newly arrived* students".

The inclusion of students' *interests and habits* was also proposed. This aspect emerged as crucial both for understanding the class and for fostering mutual knowledge among students. It was framed in multiple ways: formal entry assessments, small-group work, and especially informal whole-class discussions, which were considered preferable—particularly for students with difficulties—over static, normative documentation inherited from previous schooling. To gain deeper insight into each student, teachers suggested assigning a reflective text on "one's relationship with mathematics and with school," to be revisited during the year. Teacher C noted that revisiting such reflections often reveals meaningful developments and helps reopen a dialogue: "Then perhaps we revisit it halfway through the year, go over it again, and then return to it once more at the end of the year—it really depends on how the situation develops. Very interesting insights always emerge. It's usually the primary school teacher who made them hate mathematics, so from there we need to find a way to establish a channel of communication".

This multiplicity of perspectives led to flexible uses of this component during lesson planning in groups, often tailored to specific classes. These elements align with Dimension 5—*Knowing students' lives*—of [20].

Affective dimensions

Taking emotional aspects into consideration, the facilitator explicitly opened this line of inquiry: "Let me take this opportunity—why do you think mathematics teachers sometimes lead students to dislike mathematics? I know this is not directly related to the workshop, but I'd really like to hear your thoughts, because I have my own answer." Teacher D responded: "In my opinion, it's quite simply because students have different learning speeds. I have a student now in third year who told me that between fourth and fifth grade [primary] she was placed in a small group with newly arrived students and others with significant difficulties because she was slower in mathematics. She was separated from the rest [of the class], and she began to feel different, unable to keep up, and eventually to dislike the subject."

As the discussion evolved, Teacher E added: "The first question I always ask in class is: how were you in mathematics? Who is afraid of mathematics? And I always say: I don't know whether I will be able to teach it well, whether you will learn it, understand it, or even like it—but what I do not want is for you to be afraid of mathematics. Face it without fear. Make mistakes as many times as you need, but do it without fear."

This awareness—that students' complex relationship with mathematics often originates in their interactions with previous teachers—leads to a sense of responsibility on the part of the current teacher, who becomes a mediator of knowledge to be rediscovered and reconstructed, and thus an *agent of change* (Dimension 3 of [20]).

Spaces, structures, materials, tools

This dimension emerged collectively as a key necessity. Teacher F stated: "Based on the spaces, structures, and tools available, our way of acting follows—how we manage the teaching activity, how we design and propose activities, as was mentioned earlier, that are engaging and grounded in students' real contexts, especially not only in cases of difficulty but also in situations of cultural difference."

The context thus shapes educational possibilities from the ground up, particularly where classroom diversity does not allow for rigid, pre-defined planning. An *affirming stance* (Dimension 2 of [20]) encourages teachers to work from within the given context, promoting and valuing diversity as a resource.

Misconceptions

Teacher G observed: "In my opinion, in primary school they focus too much on formulas—too many formulas—so students retain mostly memorized notions that they then struggle to apply. We need to deconstruct those memorized learnings and then apply them concretely, whether to geometric figures or to real situations that can be modeled geometrically. Let me give a simple example: I always tell my students that to find an area, you multiply two perpendicular segments. Then we may call them base and height, or diagonals, depending on the figure—but they struggle, because sometimes they think: the area of a rectangle is base times height. I don't even call them base and height anymore, I call them the two dimensions of the rectangle. But they still struggle." The facilitator added: "For example, when a student arrives from another educational system and performs operations using different algorithms, some teachers do not accept those procedures simply because they are written or structured differently."

This reflection—rooted within the discipline itself—on typical misconceptions (e.g., base vs. height, verticality/horizontality, perpendicularity) opened up both disciplinary and cultural perspectives, in line with the development of *sociocultural awareness* (Dimension 1 of [20]).

Inclusive activities and assessment methods

Teachers referred to dialogic, interdisciplinary, playful, laboratory-based, and experiential elements, as well as to the possibility of co-designing activities with students, as key components to be integrated into classroom practice.

The practical and engaging dimension emerged as fundamental (*constructivist orientation*, Dimension 4 of [20]), although it must be carefully adapted to account for the plurality of students' "experiences"—for instance, in the design of real-world problems [19]. Finally, different forms of assessment were discussed, variably interpreted and implemented by teachers within their own planning practices.

The meaning of planning

The facilitator concluded the co-design session with the following reflection: "The real planning that happens in the classroom actually emerges in the classroom. The written plan we prepared at the beginning of the year—I don't know if this happens to you, but it often happens to me—I end up forgetting it. What really determines what we do and how far we go is what happens in the classroom. The issue is that the system still requires certain outcomes and targets, quite rightly. So we return to the usual tension between the two aims of schooling: on the one hand, educating a citizen who has the competencies to make informed choices—so someone who knows enough mathematics to participate, for example, in civic life—and on the other hand, the technical school, which has the function of providing skills that can be used later [work]. These are two important and complementary functions, but without critical reflection we risk losing our way."

This reflection points critically to the need for an open form of literacy, capable of avoiding

what [17] defines as the "social construction of failure," and of moving toward the more dynamic and intercultural direction described in Dimension 6 of [20]: designing *inclusive practices* that connect students' prior knowledge with new learning, creating shared meaning in the classroom without generating gaps.

7. Conclusions

Building on the wide range of heterogeneous situations already observed in multicultural classrooms, both in Italy and in other contexts (Section 1), we have argued that reframing the issue of teaching and learning mathematics in multicultural classrooms through the notion of a cultural learning gap offers a broader and more accurate lens for interpreting learning processes than the traditional concept of learning loss (Section 2). We prefer the learning gap perspective because, when a cultural distance exists between teacher and student, what is often interpreted as "missing knowledge" is not actually lost; rather, it remains unrecognized, culturally situated, and therefore not directly accessible to the teacher's interpretation.

Drawing on the idea of cultural transposition, which shifts attention toward the teacher's own unexamined assumptions through encounters with other disciplinary traditions—such as those emerging in the design dialogue discussed in Section 6 or through engagement with emerging algorithmic tools in Section 1—we have stressed the idea to reduce the students learning gap enacting to teachers training through a cultural lens. According to this perspective we think that the teacher has to become a cultural mediator, a bridge-builder, and a *cultural broker* of meanings and traditions [20]. In this view, the teacher has to be prepared not merely as a transmitter of content but someone who actively engages in intercultural communication, capable of recognizing, interpreting, and valuing mathematical knowledge even when it is produced in distant or unfamiliar contexts.

The evidence emerging from our training course "Cultures Count" (Sections 4 and 6) highlights, within the case study, both the inherent complexity of this process (Section 5) and the pressing need (Section 3) to co-design educational resources and practices collaboratively. These outcomes concern not only teaching materials but also more structured and sustainable pedagogical approaches. The training experience has thus supported a reflective process that begins with decentering—understood through the lens of cultural transposition—prompted by engagement with alternative mathematical traditions (Sections 1), with the goal of fostering intercultural awareness. In a world still marked by cultural conflict and, at times, irrational but persistent forms of violence, the teacher's role as mediator between home and school cultures, between institutional and informal knowledge systems, becomes particularly critical. It is around this role that the efforts of all stakeholders involved in students' educational trajectories should converge. The dialogical account of designing flexible teaching materials presented in Section 6 represents only an initial step, which must evolve into fully developed teaching designs and practices that can be collected, adapted, tested, and iteratively refined, in continuity with [7].

With the aim to reduce students learning gap in multicultural classrooms through mathematics teacher training, intercultural sensitivity, consistent with an understanding of intercultural as a non-hierarchical space of dialogue and communication between cultural systems, has to naturally aligned with this bottom-up approach, which is deeply grounded in specific classroom realities [23, 28]. From this perspective, intercultural can help bridge the gap between teacher and student cultures, particularly addressing teachers' difficulty in recognizing the reasoning processes underlying students' actions. The cases discussed in Section 1 are reinterpreted in Section 6 as opportunities for professional reflection and transformation for in-service teachers as (future) cultural brokers [20]. Reducing the distance between classroom and home cultures, or between dominant and subaltern (though not necessarily minority) cultures, requires the development of intercultural competence and the abandonment of a one-dimensional view of

mathematics education in which diversity is interpreted primarily as lack or deficiency relative to a fixed standard—whether that standard is in time (as in traditional notions of learning loss) or in context (as in comparisons between host-country schooling and educational practices from students' countries of origin).

What has been done with the teachers made it possible to respond to the needs of teachers highlighted by TALIS and OECD data, starting from the teachers' daily didactic problems, for example by asking them open-ended questions and using a strongly dialogic approach focused on cultural and linguistic considerations in the teaching of mathematics. The contents addressed and the methodological aspects appropriately chosen for this training course represent a novelty in the national and, in some ways, international context of teacher education. They explicitly contrast our 'Cultures Count' model with standard top-down professional development courses in Italy, highlighting how the collaborative negotiation between researchers and teachers represents a methodological innovation for culturally responsive education – a topic we will explore further in future publications. Secondly, thanks to the work we did with teachers during the training, rather than a simple "loss", multicultural classrooms were better understood as spaces where experiences are fragmented and communication is constrained by what is "common": not all linguistic expressions are available, not all algorithms are equally meaningful, and not all symbolic systems are equally accessible [28]. Of course, within such classrooms, power relations are always present, shaped by norms, habits, the didactic contract, curricular structures (both explicit and implicit), and textbooks. However, by reversing the analytical perspective it becomes possible to learn from students themselves, especially those with diverse educational and cultural trajectories. This shift places mathematical dialogue and lived classroom mathematics at the center of teaching practice. Ultimately, knowing, recognizing and valuing students' knowledge is fundamental to moving from a deficit-based interpretation of diversity (equality) toward one that understands and enacts diversity as a resource and an opportunity for learning (equity) [29].

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Conflict of interest

The authors declare that there are no conflicts of interest.

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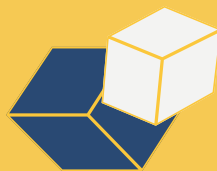
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