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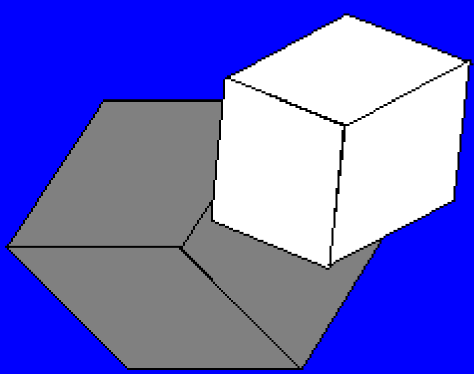

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Characterizing finite groups with mutually N-permutable products and smooth maximal subgroups

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Abstract. Let $G = HK$ be a finite group, where H and K are proper subgroups of G. A group G is called a mutually N-permutable product of H and K if H permutes with every normal subgroup of K, and K permutes with every normal subgroup of H. In this paper, as a next step of some recently studies, we examine the structural properties of finite group G that is a mutually N-permutable product of two subgroups, with the additional assumption that all maximal subgroups of G are generalized smooth groups.
Keywords: generalized smooth groups, permutable subgroups, subgroup lattices.
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1. Introduction

In this paper, only finite groups are considered. For a group G, let $\pi(G)$ stand for the set of primes dividing $|G|$, $L(G)$ the subgroup lattice of G and n the maximal length of $L(G)$.

Let $1 = G_0 < G_1 < G_2 < ... < G_n = G$ be a maximal chain of subgroups of a group G. An interval $[G_{i+j}/G_j] = \{X \leq G : G_j \leq X \leq G_{i+j}\}$ is the set of all subgroups of G_{i+j} which contain G_j. A maximal chain is called smooth if any two intervals have the same length are isomorphic. If all maximal chains from any subgroup of G of prime order to G ($L < ... < G$ where L is any subgroup of G of prime order) are smooth chains, then G is called a GS-group (a generalized smooth group) (see [10]). To clarify the concept of GS-groups more effectively, we present some examples:

- Let $G \cong Q_8$ (the quaternion group of order 8). It is well-known that Q_8 has a unique subgroup of prime order $\langle -1\rangle$, and three maximal chains extending from $\langle -1\rangle$ to Q_8. Specifically, we have $\langle -1\rangle < \langle i\rangle < Q_8$, $\langle -1\rangle < \langle j\rangle < Q_8$ and $\langle -1\rangle < \langle k\rangle < Q_8$. Each of these chains is smooth. Hence, G is a GS-group.

- Let $G \cong Z_{24}$ (cyclic group of order 24). The subgroup $\langle 12\rangle$ has order 2, and the chain $\langle 12\rangle < \langle 4\rangle < \langle 2\rangle < Z_{24}$ forms a maximal chain from $\langle 12\rangle$ to Z_{24}. It is evident that $[Z_{24}/\langle 4\rangle] = \{Z_{24}, \langle 2\rangle, \langle 4\rangle\}$ and $[\langle 2\rangle/\langle 12\rangle] =$

$\{\langle 2\rangle, \langle 4\rangle, \langle 6\rangle, \langle 12\rangle\}$. Clearly, the intervals $[Z_{24}/\langle 4\rangle]$ and $[\langle 2\rangle/\langle 12\rangle]$ are not isomorphic, even though they have the same length. Thus, G is not a GS-group.

A group G is called a P-group if either G is an elementary abelian or $G = G_pG_q$, where G_p is an elementary abelian normal Sylow p-subgroup and G_q is a Sylow q-subgroup of order q which induces a non-trivial power automorphism on G_p, and $q \mid p-1$ (see [16, p. 49]).

Let H and K be subgroups of a group G with $G = HK$. We say that, G is a mutually permutable product of H and K if H permutes with every subgroup of K and K permutes with every subgroup of H (see [6]), also G is called a mutually m-permutable product of H and K if H permutes with every maximal subgroup of K and K permutes with every maximal subgroup of H (see [5]). In this paper, we introduce the following concept:

Definition. Let $G = HK$ be a group with proper subgroups H and K. We say that, G is a mutually N-permutable product of H and K if H permutes with every normal subgroup of K and K permutes with every normal subgroup of H.

Many papers introduced the structure of a group whose maximal subgroups are GS-groups under suitable conditions (see[1]-[3] and [7]-[9]). In [3], the authors studied the structure of a group G which is a mutually permutable product or a mutually m-permutable product of two proper subgroups under the assumption that, all maximal subgroups of G are GS-groups. In this paper, as a next step, we replace these permutability conditions by a new one. More precisely, we prove the following result:

Main theorem. Assume that $G = HK$ is a mutually N-permutable product of its proper subgroups H and K. Suppose further that all maximal subgroups of G are GS-groups with $n \geq 4$. Then, one of the following holds:

(i) $|G| = p_1p_2p_3p_4$, where p_1, p_2, p_3 and p_4 are not necessarily distinct primes.

(ii) G is a P-group.

(iii) G is cyclic of square free order.

(iv) $G = G_{p_1}A$, where $|G_{p_1}| = p_1$, $G_{p_1} \lhd G$ and A is cyclic of order $p_2p_3...p_m$ and operates faithfully on G_{p_1} where p_i are primes with $p_i \neq p_j$ for $i \neq j$ and $i, j \in 1, 2, ..., m$.

(v) $G = G_pG_q$ where G_p is an elementary abelian normal Sylow p-subgroup of order p^2, G_q is cyclic of order q^3 and every subgroup of G_q operates irreducibly on G_p.

(vi) $G = G_pG_q$ where $G_p = P_1P_2$ is an elementary abelian normal Sylow p-subgroup of order p^4 such that P_i $(i = 1, 2)$ is a minimal normal subgroup of G of order p^2, G_q is a Sylow q-subgroup of order q and G_q operates irreducibly on P_i $(i = 1, 2)$.

(vii) $G \cong L_2(11)$.

Now, we list some examples to illustrate the importance of our main theorem:

- Let $G \cong L_2(13)$ (the projective special linear group). All maximal subgroups of G are GS-groups; however, G cannot be expressed as a mutually N-permutable of two subgroups. The same observation holds for $L_2(19)$ and $L_2(29)$. Actually, by the main theorem, A_5 and $L_2(11)$ are the only simple groups that can be expressed as a mutually N-permutable product of two subgroups, with all maximal subgroups being GS-groups.
- Let $G \cong S_5$ (the symmetric group of degree 5). Clearly, $S_5 = A_5P_2$, where P_2 is a Sylow 2-subgroup of S_5. Here, S_5 can be expressed as a mutually N-permutable product of A_5 and P_2. However, S_4 is a maximal subgroup of S_5, is not a GS-group.
- Let G be an abelian group of order $240 = 2^4.3.5$. Clearly, G can be expressed as a mutually N-permutable subgroup of any two maximal subgroups. However, G has a maximal subgroup of order 80 that is not GS-group.
- Let $G \cong Z_{210}$ (the cyclic group of order 210). In this case, all maximal subgroups of G are GS-groups and G can be expressed as a mutually N-permutable subgroup of any two maximal subgroups.

2. Preliminaries

Lemma 2.1 ([10], Main Theorem)**.** *A group G is a GS-group if and only if one of the following holds:*

(i) *$|G| = p_1p_2p_3$, where p_1, p_2, and p_3 are not necessarily distinct primes.*

(ii) *G is cyclic of prime power order.*

(iii) *G is a P-group.*

(iv) *G is cyclic of square free order.*

(v) *$G = G_pG_q$, where G_p is a minimal normal subgroup of order p^2 and G_q is cyclic of order q^2 such that G_q and $\Phi(G_q)$ operate irreducibly on G_p.*

(vi) *$G = G_{p_1}A$, where $|G_{p_1}| = p_1$, $G_{p_1} \lhd G$ and A is cyclic of order $p_2p_3...p_m$ and operates faithfully on G_{p_1} where p_i are primes with $p_i \neq p_j$ for $i \neq j$ and $i, j \in 1, 2, ..., m$.*

(vii) *$G \cong A_5$.*

Lemma 2.2 ([3], Main Theorem)**.** *Assume that G is a mutually m-permutable product of its proper subgroups H and K with $n \geq 4$. Assume further that all maximal subgroups of G are GS-groups. Then, G is one of the following.*

(i) $|G| = p_1p_2p_3p_4$, *where* p_1, p_2, p_3 *and* p_4 *are not necessarily distinct primes.*

(ii) *G is a P-group.*

(iii) *G is cyclic of square free order.*

(iv) $G = G_{p_1}A$, *where* $|G_{p_1}| = p_1$, $G_{p_1} \lhd G$ *and A is cyclic of order* $p_2p_3...p_m$ *and operates faithfully on* G_{p_1} *where p_i are primes with* $p_i \neq p_j$ *for* $i \neq j$ *and* $i, j \in 1, 2, ..., m$.

(v) $G = G_pG_q$ *where G_p is an elementary abelian normal Sylow p-subgroup of order p^2, G_q is cyclic of order q^3 and every subgroup of G_q operates irreducibly on G_p.*

(vi) $G = G_pG_q$ *where* $G_p = P_1P_2$ *is an elementary abelian normal Sylow p-subgroup of order p^4 such that P_i $(i = 1, 2)$ is a minimal normal subgroup of G of order p^2, G_q is a Sylow q-subgroup of order q and G_q operates irreducibly on P_i $(i = 1, 2)$.*

Lemma 2.3 ([3], Lemma 8)**.** *Assume that G is a supersolvable group. Assume further that all maximal subgroups of G are GS-groups with $n \geq 4$. Then, G is one of the following.*

(i) $|G| = p_1p_2p_3p_4$, *where* p_1, p_2, p_3 *and* p_4 *are not necessarily distinct primes.*

(ii) *G is cyclic of prime power order.*

(iii) *G is a P-group.*

(iv) *G is cyclic of square free order.*

(v) $G = G_{p_1}A$, *where* $|G_{p_1}| = p_1$, $G_{p_1} \lhd G$ *and A is cyclic of order* $p_2p_3...p_m$ *and operates faithfully on* G_{p_1} *where p_i are primes with* $p_i \neq p_j$ *for* $i \neq j$ *and* $i, j \in 1, 2, ..., m$.

Corollary 2.1. *Assume that G is a supersolvable group with $|\pi(G)| = 2$. If all maximal subgroups of G are GS-groups with $n > 4$, then G is a nonabelian P-group.*

3. Results

In the following, for simplicity, we will concern that G satisfies structure:

(a) if $G = G_pG_q$, where G_p is a minimal normal subgroup of order p^2 and G_q is cyclic of order q which operates irreducibly on G_p.

(b) if $G = G_pG_q$, where G_p is a minimal normal subgroup of order p^2 and G_q is cyclic of order q^2 such that G_q and $\Phi(G_q)$ operate irreducibly on G_p.

Lemma 3.1. *Assume that G is a mutually N-permutable product of its proper subgroups H and K. Assume further that all maximal subgroups of G are GS-groups. Then, either G is solvable or $G \cong L_2(11)$.*

Proof. A group of odd order is solvable so let G be of even order. Firstly, let both H and K are supersolvable. By ([17]; Lemma 2.4), if G has a Hall $2'$-subgroup, then G is not simple and hence by ([3]; Lemma 6), G is solvable. So, in the following paragraph, in order to prove that G is solvable, we need only to show that it has a Hall $2'$-subgroup.

Let, without loss of generality, that $2 \mid |H|$. As H is supersolvable, it has a normal 2-complement $H_{2'}$. If K is of odd order, then by hypothesis $G_{2'} = H_{2'}K$ is a subgroup of G. So, let both H and K are of even order. Also, K has a normal 2-complement $K_{2'}$ as it is supersolvable. By our hypothesis, $HK_{2'}$ and $H_{2'}K$ are subgroups of G. If $HK_{2'}$ or $H_{2'}K$ has a cyclic Sylow 2-subgroup then it has a normal 2-complement which implies that $G_{2'} = H_{2'}K_{2'}$ is a subgroup of G. Clearly, if H or K is 2-group, then $G_{2'} = K_{2'}$ or $G_{2'} = H_{2'}$ respectively. Now, we can say that, H and K are supersolvable GS-groups with non cyclic Sylow 2-subgroups and $|H|$ and $|K|$ are divided at least by two different primes. According to Lemma 2.1, $|H| = 2^2q$ and $|K| = 2^2r$ where q and r are primes. If $q = r$, then $|\pi(G)| = 2$ and hence G is solvable. So, let $q \neq r$. Say G_2 be the Sylow 2-subgroup of G, if $|G_2| > 2^2$, then by hypothesis HK_r is a proper subgroup of G but not a GS-group, a contradiction. Therefore, $[G : H] = r$ and $[G : K] = q$ and hence by ([4]; Lemma 10), G is solvable. Now, assume that H is not supersolvable. According to Lemma 2.1, either $H \cong A_5$ or $H = H_pH_q$ and satisfies structure (a) or (b). If both H and K are isomorphic to A_5, then by [13], either $G \cong A_5 \times A_5$ or $G \cong A_6$. Since there is no a GS-group contains A_5, we have $G \ncong A_5 \times A_5$. Also A_6 has a subgroup isomorphic to S_4 which is not a GS-group, a contradiction. Thus, we can assume that $K \ncong A_5$. According to Lemma 2.1, K would be a solvable group. It follows that, K has a minimal normal subgroup L, say, which is elementary abelian. By hypothesis HL is a subgroup of G. We handel the following cases:

1. $H \lneqq HL \lneqq G$. If H satisfies structure (b) or $H \cong A_5$, then H would be a maximal subgroup of G as there is no a GS-group contains it. Hence H satisfies structure (a). By hypothesis HL is a GS-group. Applying Lemma 2.1, $HL \cong A_5$ which implies that $H \cong A_4$ and $|L| = 5$ is a Sylow 5-subgroup of A_5. Hence $H_2 \lhd H$ where H_2 is a Sylow 2-subgroup of H and $L \lneqq K$ as $HL \lneqq G$. By hypothesis, H_2K is a proper GS-subgroup of G. Applying Lemma 2.1, H_2K is of order 2^25^2 and satisfies structure (b). This implies that A_5 has a subgroup of order 2^25, a contradiction.

2. $G = HL$. Hence H is of prime power index. If G is not simple then by ([3]; Lemma 6), G is solvable. So, let G be a simple group. By ([14]; Theorem 1), $H \cong A_5$ and $G \cong L_2(11)$. Note that, If $H \cong A_4$ and $G \cong A_5$, then by hypothesis G has a subgroup of order 20, a contradiction.

3. $H = HL$. Hence $L \leq H \cap K$ for each minimal normal subgroup of K. As the previous case if H is of prime index, we are done. So, let $[G : H]$ is divided at least by two different primes and consequently, $|\pi(K)| \geq 2$. Assume that $|\pi(K)| \geq 3$. Applying Lemma 2.1, K is of square free order and hence has a Sylow tower property. Say $K = K_{r_1}K_{r_2}...K_{r_m}$, where K_{r_i} $(i = 1, 2, ..., m)$ is a Sylow r_i-subgroup of K with $r_i > r_{i+1}$. Since each minimal normal subgroup of K is a subgroup of H, we get $K_{r_1} < H$. By hypothesis, $HK_{r_1}K_{r_2} = HK_{r_2}$ is a subgroup of G. If $H \lneqq HK_{r_2} \leq G$, then similar as the previous cases we are done. So, let $H = HK_{r_2}$. If $|\pi(K)| = 3$, then H is of prime index, a contradiction. Thus, $|\pi(K)| > 3$. But in this case, $HK_{r_1}K_{r_2}K_{r_3} = HK_{r_3}$ is a proper subgroup of G but not a GS-group, a contradiction. Thus, $|\pi(K)| = 2$. Since $L < H$ and $[G : H]$ is not prime power, L can't be a Sylow subgroup of K. Say $K = K_{r_1}K_{r_2}$ (it's not necessary that $r_1 > r_2$) and $L \lneqq K_{r_1}$. If $|K_{r_1}| > r_1^2$, then by Lemma 2.1, K is a nonabelian P-group and hence every subgroup of K_{r_1} is a subgroup of H, a contradiction. Thus, $|K_{r_1}| = r_1^2$. Since $L \lneqq K_{r_1}$ and K is a GS-group, we get by Lemma 2.1, K is a supersolvable group of order $r_1^2r_2$. Clearly, either K_{r_1} or K_{r_2} is a normal subgroup of K and hence a subgroup of H which contradicts with our assumption that $[G : H]$ is divided by at least two different primes. By this final contradiction our proof is completed. □

Lemma 3.2. *Assume that G is a solvable group with all maximal subgroups of G are GS-groups. If $|\pi(G)| \neq 2$, then G is supersolvable or $|G| = p^2qr$ where p, q and r are distinct primes.*

Proof. As we know if G is a p-group, then G is supersolvable. And by our hypothesis $|\pi(G)| \neq 2$. So, $|\pi(G)| \geq 3$. Firstly, let $|\pi(G)| > 3$. Since G is solvable, then there exist for each prime $p_i \in \pi(G)$, a maximal subgroup M_i such that $[G : M_i] = p_i^e$ $(e \geq 1)$. By hypothesis, M_i is a GS-group. Applying Lemma 2.1, M_i is of square free order $\forall i$. It's follows that G itself is of square free order and consequently G is supersolvable.

Now, let $|\pi(G)| = 3$. Then, $G = G_{p_1}G_{p_2}G_{p_3}$ where G_{p_i} is a Sylow p_i-subgroup of G. Solvability of G implies that $G_{p_1}G_{p_2}$ is a proper subgroup of G. Let $G_{p_1}G_{p_2}$ is not a maximal subgroup of G. Then, there exist a subgroup M, say, of G such that $G_{p_1}G_{p_2} < M < G$. By hypothesis, M is a GS-group. Applying Lemma 2.1, M is of square free order. It follows that $|G_{p_i}| = p_i$ $(i = 1, 2)$. As $n \geq 4$, $|G_{p_3}| \geq p_3^2$. Clearly, if $|G_{p_3}| = p_3^2$, then $|G| = p_1p_2p_3^2$ and we are done. So, assume that $|G_{p_3}| > p_3^2$. By applying Lemma 2.1, $G_{p_3}G_{p_i}$ is a nonabelian P-group with $p_3 > p_i$ $(i = 1, 2)$. So, every subgroup of G_{p_3} is normal in G. This implies that $G^*_{p_3}G_{p_1}G_{p_2}$ is a proper subgroup of G, where $G^*_{p_3}$ is a maximal subgroup of G_{p_3}. But $G^*_{p_3}G_{p_1}G_{p_2}$ can't be a GS-group, a contradiction. Thus, assume that, $G_{p_i}G_{p_j}$ is a maximal subgroup of G $\forall$ $i, j \in \{1, 2, 3\}$ and $i \neq j$.

Since G is solvable, it has a minimal normal subgroup L, say, which is elementary abelian. Without loss of generality let $L \leq G_{p_1}$. Maximality of $G_{p_2}G_{p_3}$ in G implies that $L = G_{p_1}$. Also by the maximality of $G_{p_1}G_{p_i}$ $(i = 2, 3)$ in G, G_{p_2} and G_{p_3} must be maximal subgroups in $G_{p_2}G_{p_3}$. By hypothesis, $G_{p_2}G_{p_3}$ is a GS-group. Then, either $|G_{p_2}G_{p_3}| = p_2p_3$ or $G_{p_2}G_{p_3}$ satisfies structure (a). Let $|G_{p_2}G_{p_3}| = p_2p_3$. If $|L| > p_1^2$, then $G_{p_1}G_{p_i}$ $(i = 2, 3)$ is a nonabelian P-group and hence every subgroup of L is normal in G, a contradiction. Thus, $|L| = p_1^2$ and hence $|G| = p_1^2p_2p_3$. Now, let $|G_{p_2}G_{p_3}| = p_2^2p_3$ and satisfies structure (a). Once again $|G| = p_1p_2^2p_3$ if $|L| = p_1$. If else, LG_{p_2} is a proper subgroup of G with order divided by $p_1^2p_2^2$. Applying Lemma 2.1, LG_{p_2} satisfies structure (b) with G_{p_2} is cyclic. It follows that $G_{p_2}G_{p_3}$ is supersolvable which contradicts with G_{p_3} is a maximal subgroup of $G_{p_2}G_{p_3}$. By this final contradiction our proof is completed. □

Lemma 3.3. *Assume that $G = HK$ is a mutually N-permutable product of its proper subgroups H and K with $|\pi(G)| = 2$. Suppose further that all maximal subgroups of G are GS-groups with $n > 4$. If H is a nonabelian P-group so as G.*

Proof. Let $H = H_pH_q$ be of order p^jq $(j \geq 1)$. If $|G_q| = q$, then by ([18], Corollary 1.10, p. 6), G is supersolvable and hence by Corollary 2.1, G is a nonabelian P-group. So, let $|G_q| \geq q^2$.

Let $|H_p| \geq p^2$. By hypothesis, $H_pK \leq G$. Assume that $G = H_pK$. Choose H_p^* be a maximal subgroup of H_p such that H_p^*K is a proper subgroup of G. Since $[G : H_p^*K] = p$, $|G_q| \geq q^2$ and $n > 4$, we get by Lemma 2.1 that H_p^*K satisfies structure (b) which contradicts with H is a nonabelian P-group. Thus, H_pK is a proper subgroup of G. Applying Lemma 2.1, H_pK is either a nonabelian P-group with $p > q$ or satisfies structure (b). Then, $H_p \lhd H_pK$ and since $H_p \lhd H$, we get $H_p \lhd G$. If H_pK is a nonabelian P-group, then H_pG_q is a proper subgroup of G of order p^jq^2 $(j \geq 2)$ and by applying Lemma 2.1, H_pG_q satisfies structure (b) which contradicts with H is a nonabelian P-group. Also if H_pK satisfies structure (b), then $H_pG_q^*$ is a proper subgroup of G for each maximal subgroup G_q^* of G_q and by applying Lemma 2.1, $H_pG_q^*$ satisfies structure (b), once again we get a contradiction with H is a nonabelian P-group. This final contradiction shows that $|H_p| = p$. If $|\pi(K)| = 1$, then $|K| \geq q^3$ as $|G_q| \geq q^2$ and $n > 4$. By hypothesis, HK^* is a proper subgroup of G for some maximal subgroup K^* of K. As $pq^3 \mid |HK^*|$ with $p > q$, HK^* can't be a GS-group, a contradiction. Thus, $|\pi(K)| = 2$. Applying Lemma 2.1, we have the following:

1. K satisfies structure (b). If $H \cap K = 1$, then by hypothesis H_pK is a proper subgroup of G but not a GS-group, a contradiction. Thus, either $H_p < K$ or $H_q < K$. Assume that $H_q < K$. Then, $|G| = p^3q^2$ with $p > q$. By ([15], Theorem 6.5.5, p. 147), G has a normal Sylow subgroup. If $G_p \lhd G$, then by Lemma 2.1, $G_pG_q^*$ is a nonabelian P-group which contradicts with K

satisfies structure (b). Thus, $G_q \lhd G$. Choose G_p^* be a maximal subgroup of G_p such that $H_p < G_p^*$. Hence $G_p^* G_q$ is a proper subgroup of G of order p^2q^2 and $H < G_p^* G_q$. Clearly, $G_p^* G_q$ can't be a GS-group, a contradiction. Now, let $H_p < K$. If $K_q \lhd K$, then by hypothesis HK_q is a proper subgroup of G of order pq^3 $(p > q)$. Applying Lemma 2.1, HK_q is not a GS-group, a contradiction. Thus, $K_p \lhd K$. By hypothesis, K_pH is a proper subgroup of G with $|K_pH| = p^2q$. By ([15], Theorem 6.2.11, p. 138), $K_p \lhd K_pH$ and consequently, $K_p \lhd G$. Choose G_q^* be a maximal subgroup of G_q with $H_q < G_q^*$. Then, $K_pG_q^*$ is a proper subgroup of G of order p^2q^2 which contains H. By Lemma 2.1, $K_pG_q^*$ is not a GS-group, a contradiction.

2. K is a nonabelian P-group. Then, $H_q \neq K_q$ as $|G_q| \geq q^2$. By hypothesis, K_pH is a proper subgroup of G. Applying Lemma 2.1, K_pH is a nonabelian P-group. It follows that every subgroup of K_p is normal in K_pH and hence normal in G. By similar, every subgroup of H_p is normal in G. Therefore, every subgroup of G_p is normal in G and hence $G_p^* G_q$ is a proper subgroup of G but not a GS-group, a contradiction.

3. $|K| = p^2q$. Clearly, $H \cap K = 1$ as $n > 4$. By ([15], Theorem 6.2.11, p. 138), $K_p \lhd K$. Our hypothesis and Lemma 2.1 implies that K_pH and H_pK are nonabelian P-groups. Hence every subgroup of G_p is normal in G. Once again $G_p^* G_q$ is a proper subgroup of G but not a GS-group, a contradiction.

4. $|K| = pq^2$. Clearly, $H \cap K = 1$ as $n > 4$. If $K_q \lhd K$, then HK_q is a proper subgroup of G but not a GS-group, a contradiction. Thus, $K_p \lhd K$ and hence K has a subgroup of order pq. Since $H_p \lhd H$, we get by hypothesis that H_pK is a proper subgroup of G of order p^2q^2. But it is not a GS-group, a contradiction. □

Lemma 3.4. *Assume that $G = HK$ is a mutually N-permutable product of its proper subgroups H and K with $|\pi(G)| = 2$. Suppose further that all maximal subgroups of G are GS-groups with $n > 4$. If H satisfies structure (b), then $G = G_pG_q$ where G_p is an elementary abelian normal Sylow p-subgroup of order p^2, G_q is cyclic of order q^3 and every subgroup of G_q operates irreducibly on G_p.*

Proof. Let $H = H_pH_q$, where H_p is a minimal normal subgroup of order p^2 and H_q is cyclic of order q^2 such that H_q and $\Phi(H_q)$ operate irreducibly on H_p. By hypothesis, $H_pK \leq G$. Firstly, let $G = H_pK$. Then, K has a cyclic subgroup of order q^2. If $|\pi(K)| = 2$, then by Lemma 2.1, either K is supersolvable of order pq^2 or K satisfies structure (b). Since $H_p \lhd H$ and $H_p \lhd G_p$, we get $H_p \lhd G$. Then, $H_pK_pK_q^*$ is a proper subgroup of G and by Lemma 2.1, $H_pK_pK_q^*$ is a nonabelian P-group which contradicts with H satisfies structure (b). Thus, $K = G_q$. By Lemma 2.1, K is either cyclic or nonabelian of order q^3. Solvability of G implies that, it has a minimal normal subgroup L, say, which is elementary abelian. Let $L \leq G_q$. Since G_q is either cyclic or nonabelian of order q^3, we have $|L| = q^2$ or $|L| = q$. If $|L| = q^2$ (or $|L| = q$), then H_pL (or $H_pH_q^*L$), respectively, is a proper subgroup of G but not a GS-group, a contradiction. Thus, $L \leq G_p = H_p$. Clearly, if L is a proper subgroup of H_p, we get a contradiction with the structure of H. So, $L = H_p$ and hence H_pK^* is a proper subgroup of G for every maximal subgroup K^* of K. Applying Lemma 2.1, H_pK^* satisfies structure (b). Therefore, every subgroup of K is cyclic of order q^2 and hence by ([12], Satz 8.2, p. 310), either K is cyclic or $K \cong Q_8$ (quaternion group of order 8). If $K \cong Q_8$, then by ([18], Corollary 1.10, p. 6), $H_p\Phi(H_q)$ is supersolvable which contradicts with $\Phi(H_q)$ operates irreducibly on H_p. Thus, K is cyclic and every subgroup of K operates irreducibly on H_p and we are done.

Now, assume that H_pK is a proper subgroup of G. Let $|\pi(H_pK)| = 1$. Clearly, $G_p = H_pK$ and $G_q = H_q$. We will study the structure of $N_G(G_q)$. If $N_G(G_q) = G_q$, then by ([12], Hauptsatz 2.6, p. 419), $G_p \lhd G$ and hence $G_pG_q^*$ is a proper subgroup of G. By Lemma 2.1, $G_pG_q^*$ is a nonabelian P-group which contradicts with H satisfies structure (b). Also $N_G(G_q) \cap H_p = 1$ as G_q operates irreducibly on H_p. Thus, $N_G(G_q)$ is a proper subgroup of G with $p^2 \mid [G : N_G(G_q)]$ and $|N_G(G_q)| = p^iq^2$ $(i \geq 1)$. Applying Lemma 2.1, $N_G(G_q)$ must be of order pq^2. Hence $N_G(G_q)$ has a maximal subgroup N^*, say, of order pq. As $H_p \lhd H$ and $H_p \lhd G_p$, we have $H_p \lhd G$ and hence H_pN^* is a proper subgroup of G and we get the same previous contradiction. This final contradiction implies that $|\pi(H_pK)| = 2$. By Lemma 2.1, either H_pK is a nonabelian P-group $(p > q)$ or $|H_pK| = p^2q$ or H_pK satisfies structure (b).

We argue that $H_p \lhd G$ and $|G_q| = q^3$. Firstly, let H_pK be a nonabelian P-group. Then, $H_p \lhd H_pK$ and hence $H_p \lhd G$. If $K_q < H_q$ then H has a subgroup of order pq which contradicts with $\Phi(H_q)$ operates irreducibly on H_p. Thus, $|G_q| = q^3$. Now, let $|H_pK| = p^2q$. Clearly $|G_q| = q^3$ as $n > 4$. If $q < p$, then $H_p \lhd H_pK$ and hence $H_p \lhd G$. So, let $q > p$. By ([15], Theorem 6.5.5, p. 147), G has a normal Sylow subgroup. If $G_q \lhd G$, then $G_qG_p^*$ is a proper subgroup of G but not a GS-group, a contradiction. Thus, $G_p = H_p \lhd G$. Finally, let H_pK satisfies structure (b). Once again $H_p \lhd H_pK$ and hence $H_p \lhd G$. If $|G_q| > q^3$, then $H_pG_q^*$ is a proper subgroup of G but not a GS-group, a contradiction. Since $n > 4$, we get $|G_q| = q^3$. Therefore, our argument is done. Hence $H_pG_q^*$ is a proper subgroup of G of order p^2q^2. Applying Lemma 2.1, $H_pG_q^*$ satisfies structure (b). Hence every subgroup of G_q is cyclic. Since G_q is not cyclic as $G_q = H_qK_q$ is a factorized group, we have $G_q \cong Q_8$. Once again, by ([18], Corollary 1.10, p. 6), $H_p\Phi(G_q^*)$ is a supersolvable group which contradicts with $H_pG_q^*$ satisfies structure (b). □

Lemma 3.5. *Assume that $G = HK$ is a mutually N-permutable product of its proper subgroups H and K with $|\pi(G)| = 2$. Suppose further that all maximal subgroups of G are GS-groups with $n > 4$. If $|H| = p^2q$, then either G is a nonabelian P-group or $G = G_pG_q$ where $G_p = P_1P_2$ is an elementary abelian normal Sylow p-subgroup of order p^4 such that P_i $(i = 1, 2)$ is a minimal normal* *subgroup of G of order p^2, G_q is a Sylow q-subgroup of order q and G_q operates irreducibly on P_i $(i = 1, 2)$.*

Proof. Let H be abelian. Then, H ia a maximal subgroup of G as there is no a GS-group contains it. Solvability of G implies that H is of prime power index. Firstly, assume that $[G : H] = q^e$ $(e \geq 2$ as $n > 4)$. Then, $G_p = H_p$ and $|G_q| \geq q^3$. Maximality of H in G implies that H_p is not normal in G. Therefore, $N_G(G_p) = C_G(G_p) = H$ and hence $G_q \lhd G$. It follows that $G_p^*G_q$ is a proper subgroup of G but not a GS-group, a contradiction. Thus, $[G : H] = p^e$ $(e \geq 2)$. Then, $G_q = H_q$ and $|G_p| \geq p^4$. By Lemma 2.1, G_p is either cyclic or elementary abelian. Clearly, if G_p is cyclic, then G is supersolvable and we get a contradiction with maximality of H. Thus, G_p is elementary abelian. Since $H_p^* \lhd H$ and $H_p^* \lhd G_p$, we get $H_p^* \lhd G$. By ([15], Theorem 9.3.7, p. 225), H_p^* is complemented in G but this complement can't be a GS-group, a contradiction. This final contradiction shows that either H_p or H_q is not normal subgroup of H. We have the following two cases:

Case (i). $H_q \lhd H$ and $H_p \ntriangleleft H$. Then, H is supersolvable and hence $q > p$. Let $|G_p| \geq p^3$. If $|G_p| = p^3$, then $|G_q| \geq q^2$ as $n > 4$ and hence $[G : H]$ is divided by pq. Solvability of G implies that, H is not a maximal subgroup of G. Then, G has a maximal subgroup contains H but it is not a GS-group, a contradiction. Thus, $|G_p| \geq p^4$. Since $q > p$, we get G_p is a maximal subgroup of G as there is no a GS-group contains it. Hence either $N_G(G_p) = G_p$ or $G_p \lhd G$. If $G_p \lhd G$, then by ([11], Theorem 4.5, p. 253), $N_G(H_q)/C_G(H_q)$ is q-subgroup. But $H_q \lhd H$. Then, $H_p < C_G(H_q)$ and hence H is abelian, a contradiction. Thus, $N_G(G_p) = G_p$. Since G_p is abelian, we have by ([12], Hauptsatz 2.6, p. 419), $G_q \lhd G$ and hence $G_qG_p^*$ is a proper subgroup of G but not a GS-group, a contradiction. This final contradiction shows that $G_p = H_p$ and hence $|G_q| \geq q^3$ as $n > 4$.

By hypothesis $H_qK \leq G$. Let $G = H_qK$. Then, K is a GS-group with $p^2q^2 \mid |K|$ and by Lemma 2.1, K satisfies structure (b). By ([15], Theorem 6.5.5, p. 147), G has a normal Sylow subgroup. Since $G_p = H_p$ is not normal in H, we get $G_q \lhd G$ and hence $G_qG_p^*$ is a proper subgroup of G of order q^3p. Applying Lemma 2.1, $G_qG_p^*$ is a nonabelian P-group which contradicts with K satisfies structure (b). Thus, H_qK is a proper subgroup of G with $[G : H_qK] = p^e$ $(e = 1, 2)$. If $[G : H_qK] = p^2$, then $G_q = H_qK$. By Lemma 2.1, K is either cyclic, elementary abelian or nonabelian of order q^3 and hence each maximal subgroup of K is normal in K. By hypothesis, $HK^* \leq G$ for each maximal subgroup K^* of K with $p^2q^2 \mid |HK^*|$. Choose K^* such that HK^* be a proper subgroup of G. Applying Lemma 2.1, HK^* can't be a GS-group, a contradiction. Thus, $[G : H_qK] = p$ and by Lemma 2.1, H_qK is a nonabelian P-group which implies that $H_q \lhd G$. By ([15], Theorem 9.3.7, p. 225), H_q has a complement U, say. Applying Lemma 2.1, U satisfies structure (b) which contradicts with H_qK is a nonabelian P-group.

Case (ii). $H_p \lhd H$ and $H_q \ntriangleleft H$. By hypothesis $H_pK \leq G$. Assume first that, $G = H_pK$.

Let $|G_q| = q$. Then, $|G_p| \geq p^4$ as $n > 4$. By Lemma 2.1, K is a nonabelian P-group or $|K| = p^2q$. If K is a nonabelian P-group, then by Lemma 3.3, G is a nonabelian P-group and we are done. So, we can assume that, $|K| = p^2q$ with $H_p \cap K_p = 1$ and both of H and K are not nonabelian P-groups. Since $H_p \lhd H$ and $H_p \lhd G_p$, we get $H_p \lhd G$. If K has a subgroup K^*, say, of order pq, then H_pK^* is a proper subgroup of G which contains H. Applying Lemma 2.1, H_pK^* is a nonabelian P-group, a contradiction. Thus, G_q operates irreducibly on H_p and K_p and we are done.

Now, let $|G_q| = q^2$. Since $G = H_pK$, we have $pq^2 \mid |K|$. By Lemma 2.1, either $|K| = pq^2$ or K satisfies structure (b). Assume that $K = K_pK_q$ satisfies structure (b). If $K_p \lhd K$, then by hypothesis, HK_p is a proper subgroup of G. Applying Lemma 2.1, HK_p is a nonabelian P-group which contradicts with $\Phi(K_q)$ operates irreducibly on K_p. If $K_q \lhd K$, then K_q is elementary abelian. By hypothesis, HK_q is a proper subgroup of G and by Lemma 2.1, H_pK_q satisfies structure (b). But $H_p \lhd H$ which follows that K_q is cyclic, a contradiction. Thus, $|K| = pq^2$. It follows that $H \cap K = H_q$ and hence $[G : H] = pq$. Solvability of G implies that, H is not maximal subgroup of G. Then, G has a maximal subgroup G^*, say, contains H. Applying Lemma 2.1, G^* is a nonabelian P-group or satisfies structure (b). If G^* is a nonabelian P-group, then by Lemma 3.3, G is a nonabelian P-group, a contradiction. Thus, $G^* = H_pK_q$ satisfies structure (b)and hence $H_p \lhd G$ and K_q is cyclic. By ([15], Theorem 13.3.1, p. 383), K is supersolvable and consequently has a maximal subgroup K^* of order pq. By Lemma 2.1, H_pK^* is a nonabelian P-group which contradicts with G^* satisfies structure (b).

Finally, let $|G_q| \geq q^3$. If $G_q \lneqq K$, then by Lemma 2.1, K is a nonabelian P-group and by Lemma 3.3, G is a nonabelian P-group, a contradiction. Thus, $G_q = K$. Choose K^* be a maximal subgroup of K which contains H_q. By hypothesis, HK^* is a proper subgroup of G. Applying Lemma 2.1, $HK^* = H_pG_q^*$ satisfies structure (b). Hence G_q^* is cyclic of order q^2 and consequently K is either cyclic or nonabelian with $|K| = q^3$. Clearly if K is cyclic, then our result holds and we are done. So, let K be nonabelian. Since G is solvable, it has a minimal normal subgroup L, say. Let $L \leq G_q$. If $L = G_q$, then LH_p^* is a proper subgroup of but not a GS-group, a contradiction. Also we get a contradiction if $L < G_q$ as HL is a proper subgroup of but not a GS-group. Thus, $L \leq H_p$. If $L < H_p$, then HK^* has a subgroup of order pq, a contradiction with structure (b). Therefore, $L = H_p$ and hence all maximal subgroups of K are cyclic. By ([12], Satz 8.2, p. 310), $K \cong Q_8$. But by ([18], Corollary 1.10, p. 6), $H_p\Phi(G_q^*)$ is supersolvable which contradicts with $\Phi(G_q^*)$ operates irreducibly on H_p.

Now, assume that, H_pK is a proper subgroup of G. If $H_pK = G_p$, then $|G_q| = q$ and $|G_p| \geq p^4$ as $n > 4$. Applying Lemma 2.1, G_p is either cyclic or elementary abelian. If G_p is cyclic, then by ([15], Theorem 13.3.1, p. 383), G is supersolvable and by Corollary 2.1, G is a nonabelian P-group, a contradiction. Thus, G_p is elementary abelian. Clearly, K has a maximal subgroup K^* such that HK^* is a proper subgroup of G with $|HK^*| = p^j q$ $(j \geq 3)$. Applying Lemma 2.1, HK^* is a nonabelian P-group and hence H is a nonabelian P-group. By Lemma 3.3, G is a nonabelian P-group and we are done. So, let $|\pi(H_pK)| = 2$ and hence $|G_q| \geq q^2$. By Lemma 2.1, H_pK is either a nonabelian P-group or satisfies structure (b). Since $p^2q^2 \mid |G|$, G can't be a nonabelian P-group and hence if H_pK is a nonabelian P-group, then we get a contradiction with Lemma 3.3. Thus, H_pK satisfies structure (b). Since H_pK is a proper subgroup of G with $[G : H_pK] = q$, we have $|G_q| = q^3$. Let $H_p \lhd H_pK$. Then, $H_p \lhd G$ and hence $H_pG_q^*$ is a proper subgroup of G for each maximal subgroup G_q^* of G_q. Applying Lemma 2.1, $H_pG_q^*$ satisfies structure (b). Therefore, all maximal subgroups of G_q are cyclic. Similar as we show in the last paragraph of the proof of Lemma 3.4, $G_q \cong Q_8$. This implies that $H_pK_q^*$ is a supersolvable group which contradicts with K_q^* operates irreducibly on H_p. Now, let $K_q \lhd H_pK$. Then, $K_q \lhd G$. Since H_p is cyclic, we get H is supersolvable and hence has a subgroup H^* of order pq. By applying Lemma 2.1, K_qH^* is a nonabelian P-group which contradicts with H_pK satisfies structure (b). □

Lemma 3.6. *Assume that $G = HK$ is a mutually N-permutable product of its proper subgroups H and K with $|\pi(G)| = 2$. Suppose further that all maximal subgroups of G are GS-groups with $n \geq 4$. Then, one of the following holds:*

(i) *$|G| = p^2q^2$ or p^3q, where p and q are distinct primes.*

(ii) *G is a nonabelian P-group.*

(iii) *$G = G_pG_q$ where G_p is an elementary abelian normal Sylow p-subgroup of order p^2, G_q is cyclic of order q^3 and every subgroup of G_q operates irreducibly on G_p.*

(iv) *$G = G_pG_q$ where $G_p = P_1P_2$ is an elementary abelian normal Sylow p-subgroup of order p^4 such that P_i $(i = 1, 2)$ is a minimal normal subgroup of G of order p^2, G_q is a Sylow q-subgroup of order q and G_q operates irreducibly on P_i $(i = 1, 2)$.*

Proof. If both H and K are nilpotent, then G is a mutually m-permutable product of H and K and by Lemma 2.2, we are done. So, let H be not nilpotent. Applying Lemma 2.1, either H is a nonabelian P-group, H satisfies structure (b) or $|H| = p^2q$. Clearly, if $n = 4$, then either $|G| = p^2q^2$ or p^3q and (i) holds. So, let $n > 4$ and hence by the previous three Lemmas, we are done. □

Proof of the main theorem

It is a direct result from Lemma 3.1, Lemma 3.2, Lemma 2.3 and Lemma 3.6.

Conclusion

One of the most important objectives in group theory is to explore the structure of groups through certain properties of their subgroups. This paper deepens that concept by linking the properties of smooth chains and mutually N-permutable subgroups, aiming to determine the structure of a group. To extend this work, we can replace our hypothesis with weaker ones. For instance, we hope to determine the structure of a group G that is mutually N-permutable of two generalized smooth subgroups.

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On the p-biharmonic maps, conformal deformation and the warped product

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Abstract. In this paper we present some constructions of the p-biharmonic maps by conformal deformation, we characterize a p-biharmonicity of the first projection and we give many examples of p-biharmonic maps.
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1. Introduction

Let $\phi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map between two Riemannian manifolds. A map ϕ is called p-harmonic if it is a critical point of the p-energy functional

$$E_p(\phi) = \frac{1}{p}\int_D |d\phi|^p dv_g, \qquad p \geq 1,$$

for every compact domain $D \subset M$ and is characterized by the vanishing of the p-tension field

$$\tau_p(\phi) = |d\phi|^{p-2}\left(\tau(\phi) + (p-2)\, d\phi\,(grad \ln |d\phi|)\right) = 0,$$

where

$$\tau(\phi) = Tr_g \nabla d\phi$$

is the tension field of ϕ. The p-bi-energy of $\phi : (M^m, g) \longrightarrow (N^n, h)$ is defined by

$$E_{p,2}(\phi) = \frac{1}{p}\int_D |\tau(\phi)|^p dv_g.$$

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Then, the map ϕ is called p-biharmonic map if it is a critical point of the p-bi-energy functional, the first variation formula for the p-bi-energy shows that the Euler-Lagrange equation for p-bi-energy is

$$\tau_{p,2}(\phi) = -\frac{2}{p}\Delta\left(|\tau(\phi)|^{p-2}\tau(\phi)\right) - \frac{2}{p}|\tau(\phi)|^{p-2} Tr_g R^N(\tau(\phi), d\phi)\, d\phi = 0.$$

$\tau_{p,2}(\phi)$ is called the p-bi-tension field of the map ϕ. A simple calculation gives

$$\begin{aligned}\Delta\left(|\tau(\phi)|^{p-2}\tau(\phi)\right) &= |\tau(\phi)|^{p-2} Tr_g\left(\nabla^{\phi}\right)^2\tau(\phi)\\ &+ \frac{(p-2)}{2}|\tau(\phi)|^{p-4}\Delta\left(|\tau(\phi)|^2\right)\tau(\phi)\\ &+ \frac{(p-2)(p-4)}{4}|\tau(\phi)|^{p-6}\left|\text{grad}\left(|\tau(\phi)|^2\right)\right|^2\tau(\phi)\\ &+ (p-2)|\tau(\phi)|^{p-4}\nabla^{\phi}_{\text{grad}\left(|\tau(\phi)|^2\right)}\tau(\phi),\end{aligned}$$

then, the p-bi-tension field of the map ϕ is given by

$$\begin{aligned}\tau_{p,2}(\phi) = &\frac{2}{p}|\tau(\phi)|^{p-2}\tau_2(\phi) - \frac{p-2}{p}|\tau(\phi)|^{p-4}\Delta\left(|\tau(\phi)|^2\right)\tau(\phi)\\ &- \frac{(p-2)(p-4)}{2p}|\tau(\phi)|^{p-6}\left|\text{grad}\left(|\tau(\phi)|^2\right)\right|^2\tau(\phi)\\ &- \frac{2(p-2)}{p}|\tau(\phi)|^{p-4}\nabla^{\phi}_{\text{grad}\left(|\tau(\phi)|^2\right)}\tau(\phi) = 0,\end{aligned} \tag{1}$$

where $\tau_2(\phi)$ is the bi-tension field of ϕ defined by

$$\tau_2(\phi) = -Tr_g\left(\nabla^{\phi}\right)^2\tau(\phi) - Tr_g R^N((\phi), d\phi)\, d\phi = 0.$$

The map ϕ is p-biharmonic if and only if

$$\begin{aligned}&|\tau(\phi)|^4\tau_2(\phi) - (p-2)|\tau(\phi)|^2\nabla^{\phi}_{\text{grad}\left(|\tau(\phi)|^2\right)}\tau(\phi)\\ &- \frac{(p-2)(p-4)}{4}\left|\text{grad}\left(|\tau(\phi)|^2\right)\right|^2\tau(\phi)\\ &- \frac{(p-2)}{2}|\tau(\phi)|^2\Delta\left(|\tau(\phi)|^2\right)\tau(\phi) = 0.\end{aligned} \tag{2}$$

The construction of harmonic maps and biharmonic maps has been the subject of several papers. In [1], [3] and [9], the authors present some methods for constructing biharmonic maps by conformally deformation of the metrics and they give several examples of biharmonic non-harmonic maps. In [4], the authors give some properties of the f-biharmonic maps and they characterize the p-biharmonicity of some particular cases. The authors in [5] investigate p-biharmonic maps from a Riemannian manifold into a Riemannian manifold with non-positive sectional curvature. In [6], the authors introduce an intrinsic version of the p-biharmonic energy functional for maps and they prove, by means of the direct method, existence of minimizers of the p-bi-energy within the corresponding intrinsic Sobolev space. This paper is composed of two sections, in the first and by conformal deformation of the metric g, we give a necessary and sufficient condition of the p-biarmonicity for $Id : \left(M^m, \widetilde{g} = e^{2\gamma} g\right) \to (M^m, g)$ and $Id : (M^m, g) \to \left(M^m, \widetilde{g} = e^{2\gamma} g\right)$ ans we construct some examples of p-biharmonic maps.In the last section, we present other examples where we study the p-biharmonicity of some smooth maps.

2. The conformal deformation and the p-biharmonic maps

Let (M^m, g) be a smooth manifold and let $\widetilde{g} = e^{2\gamma} g$ be a metric conformally equivalent to g, $\gamma \in C^\infty(M)$. The relation between $\widetilde{\nabla}$ and ∇ is given by the following equation (see [2])

$$\widetilde{\nabla}_X Y = \nabla_X Y + X(\gamma) Y + Y(\gamma) X - g(X, Y) \operatorname{grad} \gamma, \tag{3}$$

where ∇ et $\widetilde{\nabla}$ are respectively the connections on M associated with g and $\widetilde{g}$. Let us choose$\{e_i\}_{i=1}^m$ to be an orthonormal frame on (M^m, g), then an orthonormal frame on $\left(M^m, \widetilde{g} = e^{2\gamma} g\right)$ is given by $\{\widetilde{e_i} = e^{-\gamma} e_i\}_{i=1}^m$. We have

$$\widetilde{\nabla}_{e_i} e_i = \nabla_{e_i} e_i - (m-2) \operatorname{grad} \gamma$$

and

$$\widetilde{\nabla}_{\widetilde{e_i}} \widetilde{e_i} = e^{-2\gamma} \left(\nabla_{e_i} e_i - (m-1) \operatorname{grad} \gamma\right).$$

Let $\phi : (M^m, g) \to (N^n, h)$ be a smooth map, we have

$$\widetilde{\tau}(\phi) = e^{-2\gamma} \left(\tau(\phi) + (m-2) d\phi(\operatorname{grad} \gamma)\right),$$

where $\widetilde{\tau}(\phi)$ denotes the tension field of the map ϕ with respect to $\widetilde{g}$. In a first result, we will study the p-biharmonicity of $Id : \left(M^m, \widetilde{g} = e^{2\gamma} g\right) \longrightarrow (M^m, g)$ where $m \neq 2$. We obtain the following result

Theorem 1. *The identity map $Id : \left(M^m, \widetilde{g} = e^{2\gamma} g\right) \longrightarrow (M^m, g)\,(m \neq 2)$ is p-biharmonic if and only if*

$$\begin{aligned} & |\operatorname{grad} \gamma|^4 \operatorname{grad} \Delta\gamma + \frac{(m-4p+2)}{2} |\operatorname{grad} \gamma|^4 \operatorname{grad}\left(|\operatorname{grad} \gamma|^2\right) \\ & + \frac{(p-2)(m-4p+2)}{2} |\operatorname{grad} \gamma|^2 d\gamma\left(\operatorname{grad}\left(|\operatorname{grad} \gamma|^2\right)\right) \operatorname{grad} \gamma \\ & + \frac{(p-2)(p-4)}{4} \left|\operatorname{grad}\left(|\operatorname{grad} \gamma|^2\right)\right|^2 \operatorname{grad} \gamma \\ & + (p-2) |\operatorname{grad} \gamma|^2 \nabla_{\operatorname{grad}\left(|\operatorname{grad} \gamma|^2\right)} \operatorname{grad} \gamma \end{aligned} \tag{4}$$

$$
\begin{aligned}
&+\frac{(p-2)}{2}\left|\operatorname{grad}\gamma\right|^{2}\Delta\left(\left|\operatorname{grad}\gamma\right|^{2}\right)\operatorname{grad}\gamma\\
&-2\left(p-1\right)\left(m-2p\right)\left|\operatorname{grad}\gamma\right|^{6}\operatorname{grad}\gamma\\
&-2\left(p-1\right)\left|\operatorname{grad}\gamma\right|^{4}\left(\Delta\gamma\right)\operatorname{grad}\gamma\\
&+2\left|\operatorname{grad}\gamma\right|^{4}Ricci\left(\operatorname{grad}\gamma\right)=0.
\end{aligned}
$$

Proof of Theorem 1. By equation (2), the identity map

$$Id:\left(M^{m},\widetilde{g}=e^{2\gamma}g\right)\longrightarrow\left(M^{m},g\right)$$

is p-biharmonic if and only if

$$
\begin{aligned}
&\left|\widetilde{\tau}\left(Id\right)\right|^{4}\widetilde{\tau}_{2}\left(Id\right)-\left(p-2\right)\left|\widetilde{\tau}\left(Id\right)\right|^{2}\nabla_{\widetilde{\operatorname{grad}}\left(\left|\widetilde{\tau}(Id)\right|^{2}\right)}\widetilde{\tau}\left(Id\right)\\
&-\frac{(p-2)(p-4)}{4}\left|\widetilde{\operatorname{grad}}\left(\left|\widetilde{\tau}\left(Id\right)\right|^{2}\right)\right|_{\widetilde{g}}^{2}\widetilde{\tau}\left(Id\right)\\
&-\frac{(p-2)}{2}\left|\widetilde{\tau}\left(Id\right)\right|^{2}\widetilde{\Delta}\left(\left|\widetilde{\tau}\left(Id\right)\right|^{2}\right)\widetilde{\tau}\left(Id\right)=0,
\end{aligned}\tag{5}
$$

where

$$\widetilde{\tau}\left(Id\right)=\left(m-2\right)e^{-2\gamma}\operatorname{grad}\gamma.$$

A rigorous calculation gives us the following formulas

$$\left|\widetilde{\tau}\left(Id\right)\right|^{2}=\left(m-2\right)^{2}e^{-4\gamma}\left|\operatorname{grad}\gamma\right|^{2},$$

$$
\begin{aligned}
\widetilde{\operatorname{grad}}\left(\left|\widetilde{\tau}\left(Id\right)\right|^{2}\right)&=\left(m-2\right)^{2}\widetilde{\operatorname{grad}}\left(e^{-4\gamma}\left|\operatorname{grad}\gamma\right|^{2}\right)\\
&=\left(m-2\right)^{2}\widetilde{e_{i}}\left(e^{-4\gamma}\left|\operatorname{grad}\gamma\right|^{2}\right)\widetilde{e_{i}}\\
&=\left(m-2\right)^{2}e^{-2\gamma}e_{i}\left(e^{-4\gamma}\left|\operatorname{grad}\gamma\right|^{2}\right)e_{i}\\
&=\left(m-2\right)^{2}e^{-4\gamma}e^{-2\gamma}e_{i}\left(\left|\operatorname{grad}\gamma\right|^{2}\right)e_{i}\\
&\quad+\left(m-2\right)^{2}e^{-2\gamma}\left|\operatorname{grad}\gamma\right|^{2}e_{i}\left(e^{-4\gamma}\right)e_{i}\\
&=\left(m-2\right)^{2}e^{-6\gamma}\operatorname{grad}\left(\left|\operatorname{grad}\gamma\right|^{2}\right)\\
&\quad-4\left(m-2\right)^{2}e^{-6\gamma}\left|\operatorname{grad}\gamma\right|^{2}\operatorname{grad}\gamma,
\end{aligned}
$$

$$
\begin{aligned}
&\left|\widetilde{\operatorname{grad}}\left(\left|\widetilde{\tau}\left(Id\right)\right|^{2}\right)\right|_{\widetilde{g}}^{2}=\widetilde{g}\left(\widetilde{\operatorname{grad}}\left(\left|\widetilde{\tau}\left(Id\right)\right|^{2}\right),\widetilde{\operatorname{grad}}\left(\left|\widetilde{\tau}\left(Id\right)\right|^{2}\right)\right)\\
&=\left(m-2\right)^{4}e^{-10\gamma}g\left(\operatorname{grad}\left(\left|\operatorname{grad}\gamma\right|^{2}\right),\operatorname{grad}\left(\left|\operatorname{grad}\gamma\right|^{2}\right)\right)\\
&+16\left(m-2\right)^{4}e^{-10\gamma}g\left(\left|\operatorname{grad}\gamma\right|^{2}\operatorname{grad}\gamma,\left|\operatorname{grad}\gamma\right|^{2}\operatorname{grad}\gamma\right)\\
&-8\left(m-2\right)^{4}e^{-10\gamma}g\left(\operatorname{grad}\left(\left|\operatorname{grad}\gamma\right|^{2}\right),\left|\operatorname{grad}\gamma\right|^{2}\operatorname{grad}\gamma\right)\\
&=\left(m-2\right)^{4}e^{-10\gamma}\left|\operatorname{grad}\left(\left|\operatorname{grad}\gamma\right|^{2}\right)\right|^{2}+16\left(m-2\right)^{4}e^{-10\gamma}\left|\operatorname{grad}\gamma\right|^{6}\\
&-8\left(m-2\right)^{4}e^{-10\gamma}\left|\operatorname{grad}\gamma\right|^{2}d\gamma\left(\operatorname{grad}\left(\left|\operatorname{grad}\gamma\right|^{2}\right)\right),
\end{aligned}
$$

$$\begin{aligned}\widetilde{\tau}_2(Id) = &-(m-2)\,e^{-4\gamma}\left\{\operatorname{grad}\Delta\gamma + \frac{(m-6)}{2}\operatorname{grad}\left(|\operatorname{grad}\gamma|^2\right)\right\}\\ &+2\,(m-2)\,e^{-4\gamma}\left((\Delta\gamma)+(m-4)\,|\operatorname{grad}\gamma|^2\right)\operatorname{grad}\gamma\\ &-2\,(m-2)\,e^{-4\gamma}Ricci\,(\operatorname{grad}\gamma)\end{aligned}$$

and

$$\begin{aligned}\nabla_{\widetilde{\operatorname{grad}}\left(|\widetilde{\tau}(Id)|^2\right)}\widetilde{\tau}\,(Id) = &\,(m-2)^3\,e^{-6\gamma}\nabla_{\operatorname{grad}\left(|\operatorname{grad}\gamma|^2\right)}e^{-2\gamma}\operatorname{grad}\gamma\\ &-4\,(m-2)^3\,e^{-6\gamma}\,|\operatorname{grad}\gamma|^2\,\nabla_{\operatorname{grad}\gamma}e^{-2\gamma}\operatorname{grad}\gamma,\\ =&\,(m-2)^3\,e^{-8\gamma}\nabla_{\operatorname{grad}\left(|\operatorname{grad}\gamma|^2\right)}\operatorname{grad}\gamma\\ &+8\,(m-2)^3\,e^{-8\gamma}\,|\operatorname{grad}\gamma|^4\operatorname{grad}\gamma\\ &-2\,(m-2)^3\,e^{-8\gamma}d\gamma\left(\operatorname{grad}\left(|\operatorname{grad}\gamma|^2\right)\right)\operatorname{grad}\gamma\\ &-2\,(m-2)^3\,e^{-8\gamma}\,|\operatorname{grad}\gamma|^2\operatorname{grad}\left(|\operatorname{grad}\gamma|^2\right)\end{aligned}$$

Finally, for the term $\widetilde{\Delta}(|\widetilde{\tau}(Id)|^2)$, we have

$$\begin{aligned}&\widetilde{\Delta}\left(|\widetilde{\tau}\,(Id)|^2\right) = (m-2)^2\,\widetilde{\Delta}\left(e^{-4\gamma}\,|\operatorname{grad}\gamma|^2\right)\\ &=(m-2)^2\,e^{-4\gamma}\widetilde{\Delta}\left(|\operatorname{grad}\gamma|^2\right)+(m-2)^2\,|\operatorname{grad}\gamma|^2\,\widetilde{\Delta}\left(e^{-4\gamma}\right)\\ &+2\,(m-2)^2\,\widetilde{g}\left(\widetilde{\operatorname{grad}}\left(|\operatorname{grad}\gamma|^2\right),\widetilde{\operatorname{grad}}\left(e^{-4\gamma}\right)\right)\\ &=(m-2)^2\,e^{-6\gamma}\left(\Delta\left(|\operatorname{grad}\gamma|^2\right)+(m-2)\,d\gamma\left(\operatorname{grad}\left(|\operatorname{grad}\gamma|^2\right)\right)\right)\\ &+(m-2)^2\,e^{-2\gamma}\,|\operatorname{grad}\gamma|^2\left(\Delta\left(e^{-4\gamma}\right)+(m-2)\,d\gamma\left(\operatorname{grad}\left(e^{-4\gamma}\right)\right)\right)\\ &-8\,(m-2)^2\,e^{-6\gamma}g\left(\operatorname{grad}\left(|\operatorname{grad}\gamma|^2\right),\operatorname{grad}\gamma\right)\\ &=(m-2)^2\,e^{-6\gamma}\Delta\left(|\operatorname{grad}\gamma|^2\right)+(m-2)^3\,e^{-6\gamma}d\gamma\left(\operatorname{grad}\left(|\operatorname{grad}\gamma|^2\right)\right)\\ &+(m-2)^2\,e^{-2\gamma}\,|\operatorname{grad}\gamma|^2\,\Delta\left(e^{-4\gamma}\right)\\ &+(m-2)^3\,e^{-2\gamma}\,|\operatorname{grad}\gamma|^2\,d\gamma\left(\operatorname{grad}\left(e^{-4\gamma}\right)\right)\\ &-8\,(m-2)^2\,e^{-6\gamma}d\gamma\left(\operatorname{grad}\left(|\operatorname{grad}\gamma|^2\right)\right)\\ &=(m-2)^2\,e^{-6\gamma}\Delta\left(|\operatorname{grad}\gamma|^2\right)-4\,(m-2)^3\,e^{-6\gamma}\,|\operatorname{grad}\gamma|^4\\ &+(m-2)^2\,e^{-2\gamma}\,|\operatorname{grad}\gamma|^2\left(-4e^{-4\gamma}\,(\Delta\gamma)+16e^{-4\gamma}\,|\operatorname{grad}\gamma|^2\right)\\ &+(m-2)^2\,(m-10)\,e^{-6\gamma}d\gamma\left(\operatorname{grad}\left(|\operatorname{grad}\gamma|^2\right)\right)\\ &=(m-2)^2\,e^{-6\gamma}\Delta\left(|\operatorname{grad}\gamma|^2\right)-4\,(m-2)^3\,e^{-6\gamma}\,|\operatorname{grad}\gamma|^4\\ &-4\,(m-2)^2\,e^{-6\gamma}\,|\operatorname{grad}\gamma|^2\,(\Delta\gamma)+16\,(m-2)^2\,e^{-6\gamma}\,|\operatorname{grad}\gamma|^4\\ &+(m-2)^2\,(m-10)\,e^{-6\gamma}d\gamma\left(\operatorname{grad}\left(|\operatorname{grad}\gamma|^2\right)\right),\end{aligned}$$

which gives us

$$\begin{aligned}\widetilde{\Delta}\left(\left|\widetilde{\tau}\left(Id\right)\right|^{2}\right) &= \left(m-2\right)^{2}e^{-6\gamma}\Delta\left(\left|\operatorname{grad}\gamma\right|^{2}\right)-4\left(m-2\right)^{2}e^{-6\gamma}\left|\operatorname{grad}\gamma\right|^{2}\left(\Delta\gamma\right)\\ &+\left(m-2\right)^{2}\left(m-10\right)e^{-6\gamma}d\gamma\left(\operatorname{grad}\left(\left|\operatorname{grad}\gamma\right|^{2}\right)\right)\\ &-4\left(m-2\right)^{2}\left(m-6\right)e^{-6\gamma}\left|\operatorname{grad}\gamma\right|^{4}.\end{aligned}$$

Substituting these formulas into equation (5), we deduce that

$$Id:\left(M^{m},\widetilde{g}=e^{2\gamma}g\right)\longrightarrow\left(M^{m},g\right)$$

is p-biharmonic if and only if

$$\begin{aligned}&\left|\operatorname{grad}\gamma\right|^{4}\operatorname{grad}\Delta\gamma+\frac{\left(m-4p+2\right)}{2}\left|\operatorname{grad}\gamma\right|^{4}\operatorname{grad}\left(\left|\operatorname{grad}\gamma\right|^{2}\right)\\ &+\frac{\left(p-2\right)\left(m-4p+2\right)}{2}\left|\operatorname{grad}\gamma\right|^{2}d\gamma\left(\operatorname{grad}\left(\left|\operatorname{grad}\gamma\right|^{2}\right)\right)\operatorname{grad}\gamma\\ &+\frac{\left(p-2\right)\left(p-4\right)}{4}\left|\operatorname{grad}\left(\left|\operatorname{grad}\gamma\right|^{2}\right)\right|^{2}\operatorname{grad}\gamma\\ &+\left(p-2\right)\left|\operatorname{grad}\gamma\right|^{2}\nabla_{\operatorname{grad}\left(\left|\operatorname{grad}\gamma\right|^{2}\right)}\operatorname{grad}\gamma\\ &+\frac{\left(p-2\right)}{2}\left|\operatorname{grad}\gamma\right|^{2}\Delta\left(\left|\operatorname{grad}\gamma\right|^{2}\right)\operatorname{grad}\gamma\\ &-2\left(p-1\right)\left(m-2p\right)\left|\operatorname{grad}\gamma\right|^{6}\operatorname{grad}\gamma\\ &-2\left(p-1\right)\left|\operatorname{grad}\gamma\right|^{4}\left(\Delta\gamma\right)\operatorname{grad}\gamma\\ &+2\left|\operatorname{grad}\gamma\right|^{4}Ricci\left(\operatorname{grad}\gamma\right)=0.\end{aligned}$$

Example 1. Let $Id:\left(\mathbb{R}_{+}^{*}\times\mathbb{R}^{m-1},g\right)\rightarrow\left(\mathbb{R}_{+}^{*}\times\mathbb{R}^{m-1},g\right)$, $(m>2)$ the identity map defined by

$$Id\left(t,x_{2},\ldots,x_{m}\right)=\left(t,x_{2},\ldots,x_{m}\right),$$

where g is the Euclidean metric given by

$$g=dt^{2}+dx_{2}^{2}+.....+dx_{m}^{2}.$$

Let $\widetilde{g}=e^{2\gamma}g$ where the function γ depends only on t. By Theorem 1 and by using the fact that $p\geq 2$, we deduce that $Id:\left(\mathbb{R}_{+}^{*}\times\mathbb{R}^{m-1},\widetilde{g}\right)\rightarrow\left(\mathbb{R}_{+}^{*}\times\mathbb{R}^{m-1},g\right)$ is p-biharmonic if and only if $\beta=\gamma'$ satisfies the following differential equation

$$\begin{aligned}&\left(p-1\right)\beta\beta''+\left(p-1\right)\left(p-2\right)\left(\beta'\right)^{2}+\left(m-4p\right)\left(p-1\right)\beta^{2}\beta'\\ &-2\left(p-1\right)\left(m-2p\right)\beta^{4}=0.\end{aligned}$$

We deduce the particular solutions of the form $\beta=\frac{a}{t}$ where $a\neq 0$. We obtain

$$2\left(m-2p\right)a^{2}+\left(m-4p\right)a-p=0.$$

There are two solutions of this type:

1. If $p=\frac{m}{2}$, $m\geq 4$, then $a=-\frac{1}{2}$ and $\gamma(t)=\ln\frac{1}{\sqrt{t}}$, we obtain the metric (up to a constant multiple) $\widetilde{g}=\frac{1}{t}g$. On substituting $u=2\sqrt{t}$, this takes the form
$$\widetilde{g}=du^2+\frac{4}{u^2}dx_2^2+\frac{4}{u^2}dx_3^2+\ldots+\frac{4}{u^2}dx_m^2.$$
So, we conclude that the identity map
$$Id:\left(\mathbb{R}_+^*\times\mathbb{R}^{m-1},\widetilde{g}\right)\to\left(\mathbb{R}_+^*\times\mathbb{R}^{m-1},g\right)$$
is a p-biharmonic map, where $p=\frac{m}{2}$.

2. If $p\neq\frac{m}{2}$, we obtain $a=\frac{p}{m-2p}$ or $a=-\frac{1}{2}$ then $\widetilde{g}=t^{\frac{2p}{m-2p}}g$ or $\widetilde{g}=\frac{1}{t}g$. In this cases, the identity map $Id:\left(\mathbb{R}_+^*\times\mathbb{R}^{m-1},\widetilde{g}\right)\to\left(\mathbb{R}_+^*\times\mathbb{R}^{m-1},g\right)$ is a p-biharmonic map.

Remark 1. If we consider $Id:(\mathbb{R}^m,g)\longrightarrow(\mathbb{R}^m,g)$ $(m\neq 2)$ where we suppose that the function γ is radial ($\gamma=\gamma(r)$, $r=|x|$, $x\in\mathbb{R}^m$), then the p-biharmonicity of $Id:\left(\mathbb{R}^m,\widetilde{g}=e^{2\gamma}g\right)\longrightarrow(\mathbb{R}^m,g)$ is equivalent to an ordinary differential equation:

$$\begin{aligned}&(p-1)\,\beta\beta''+(p-2)\,(p-1)\left(\beta'\right)^2+\frac{(p-1)\,(m-1)}{r}\beta\beta'\\&+(p-1)\,(m-4p)\,\beta^2\beta'-\frac{2\,(p-1)\,(m-1)}{r}\beta^3\\&-\frac{(m-1)}{r^2}\beta^2-2\,(p-1)\,(m-2p)\,\beta^4=0,\end{aligned}\tag{6}$$

where $\beta=\gamma'$.

Example 2. If we consider $Id:(\mathbb{R}^m\setminus\{0\},g)\longrightarrow(\mathbb{R}^m\setminus\{0\},g)$ $(m\neq 2)$ where we suppose $\beta=\frac{a}{r}$, $a\in\mathbb{R}^*$. By equation (9), we deduce that
$$Id:\left(\mathbb{R}^m\setminus\{0\},\widetilde{g}=e^{2\gamma}g\right)\longrightarrow(\mathbb{R}^m\setminus\{0\},g)$$
is p-biharmonic if and only if
$$2\,(p-1)\,(m-2p)\,a^2+(p-1)\,(3m-4p-2)\,a+p\,(m-p)=0.$$
For the resolution of this last equation, we will present several cases ($p\geq 2$ and $m\geq 3$).

- *First case:* $p=m, m\geq 3$, we obtain $a=-\frac{m+2}{2m}$, then
$$\gamma=\ln r^{-\frac{m+2}{2m}}=\ln\frac{1}{r^{\frac{m+2}{2m}}}$$
and
$$\widetilde{g}=\frac{1}{r^{\frac{m+2}{m}}}g.$$

- *Second case:* $p=\frac{m}{2}, m\geq 4$, then $a=-\frac{m^2}{2(m-2)^2}$, which gives us

$$\gamma=\ln\frac{1}{r^{\frac{m^2}{2(m-2)^2}}}$$

and

$$\widetilde{g}=\frac{1}{r^{\frac{m^2}{(m-2)^2}}}g.$$

- *Third case:* $p=\frac{3m-2}{4}, m\geq 4$, then

$$a=\pm\frac{\sqrt{3}}{6}\frac{\sqrt{3m^2+4m-4}}{(m-2)}$$

and

$$\gamma=\ln r^{\pm\frac{\sqrt{3}}{6}\frac{\sqrt{3m^2+4m-4}}{(m-2)}}.$$

It follows that $\widetilde{g}=r^{\frac{\sqrt{3}\sqrt{3m^2+4m-4}}{3(m-2)}}g$ or $\widetilde{g}=\frac{1}{r^{\frac{\sqrt{3}\sqrt{3m^2+4m-4}}{3(m-2)}}}g$.

- *Fourth case:* $p\neq\frac{m}{2}$, then $\widetilde{g}=r^{2a}g$, where $a=\frac{1}{4(m-2p)}(4p-3m+2-\sqrt{H(m,p)})$ or

$$a=\frac{1}{4(m-2p)}(4p-3m+2+\sqrt{H(m,p)})$$

and

$$H(m,p)=\frac{12m-12p+m^2p+12mp-9m^2-4}{p-1}.$$

Now, let's look the p-biharmonicity of the identity map

$$Id:(M^m,g)\longrightarrow\left(M^m,\widetilde{g}=e^{2\gamma}g\right),$$

where $m\neq 2$.

Theorem 2. *The identity map* $Id:(M^m,g)\longrightarrow\left(M^m,\widetilde{g}=e^{2\gamma}g\right)$ $(m\neq 2)$ *is* p*-biharmonic if and only if*

$$\begin{aligned}
&|\text{grad}\,\gamma|^4\,\text{grad}\,(\Delta\gamma)+(p-2)^2\,|\text{grad}\,\gamma|^2\,d\gamma\left(\text{grad}\left(|\text{grad}\,\gamma|^2\right)\right)\text{grad}\,\gamma\\
&+\frac{(p-2)\,(p-4)}{4}\left|\text{grad}\left(|\text{grad}\,\gamma|^2\right)\right|^2\text{grad}\,\gamma+(p-4)\,|\text{grad}\,\gamma|^4\,(\Delta\gamma)\,\text{grad}\,\gamma\\
&-\left(m+2p-p^2-2\right)|\text{grad}\,\gamma|^6\,\text{grad}\,\gamma\\
&-\frac{(m-4p+2)}{2}\,|\text{grad}\,\gamma|^4\,\text{grad}\left(|\text{grad}\,\gamma|^2\right)\\
&+(p-2)\,|\text{grad}\,\gamma|^2\,\nabla_{\text{grad}\left(|\text{grad}\,\gamma|^2\right)}\text{grad}\,\gamma\\
&+\frac{(p-2)}{2}\,|\text{grad}\,\gamma|^2\,\Delta\left(|\text{grad}\,\gamma|^2\right)\text{grad}\,\gamma\\
&+2\,|\text{grad}\,\gamma|^4\,Ricci\,(\text{grad}\,\gamma)=0.
\end{aligned}\tag{7}$$

Proof of Theorem 2. By equation (2), the identity map

$$Id : (M^m, g) \longrightarrow \left(M^m, \widetilde{g} = e^{2\gamma} g\right)$$

is p-biharmonic if and only if

$$\begin{aligned}
&|\widetilde{\tau}(Id)|^4 \widetilde{\tau}_2(Id) - \frac{(p-2)(p-4)}{4} \left|\operatorname{grad}\left(|\widetilde{\tau}(Id)|^2\right)\right|^2 \widetilde{\tau}(Id) \\
&- (p-2)|\widetilde{\tau}(Id)|^2 \widetilde{\nabla}_{\operatorname{grad}\left(|\widetilde{\tau}(Id)|^2\right)} \widetilde{\tau}(Id) \\
&- \frac{(p-2)}{2} |\widetilde{\tau}(Id)|^2 \Delta\left(|\widetilde{\tau}(Id)|^2\right) \widetilde{\tau}(Id) = 0,
\end{aligned} \tag{8}$$

where

$$\widetilde{\tau}(Id) = (2-m)\operatorname{grad}\gamma.$$

A long calculation gives us

$$\widetilde{\tau}(Id) = (2-m)\operatorname{grad}\gamma,$$

$$|\tau(Id)|^2 = (2-m)^2 e^{2\gamma} |\operatorname{grad}\gamma|^2,$$

$$\begin{aligned}
\operatorname{grad}\left(|\widetilde{\tau}(Id)|^2\right) &= (2-m)^2 e^{2\gamma} \operatorname{grad}\left(|\operatorname{grad}\gamma|^2\right) \\
&\quad + 2(2-m)^2 e^{2\gamma} |\operatorname{grad}\gamma|^2 \operatorname{grad}\gamma,
\end{aligned}$$

$$\begin{aligned}
&\left|\operatorname{grad}\left(|\widetilde{\tau}(Id)|^2\right)\right|^2 \\
&= (2-m)^4 e^{4\gamma} \left|\operatorname{grad}\left(|\operatorname{grad}\gamma|^2\right)\right|^2 + 4(2-m)^4 e^{4\gamma} |\operatorname{grad}\gamma|^6 \\
&+ 4(2-m)^4 e^{4\gamma} |\operatorname{grad}\gamma|^2 d\gamma\left(\operatorname{grad}\left(|\operatorname{grad}\gamma|^2\right)\right),
\end{aligned}$$

$$\begin{aligned}
&\Delta\left(|\widetilde{\tau}(Id)|^2\right) \\
&= (2-m)^2 e^{2\gamma} \Delta\left(|\operatorname{grad}\gamma|^2\right) + 4(2-m)^2 e^{2\gamma} d\gamma\left(\operatorname{grad}\left(|\operatorname{grad}\gamma|^2\right)\right) \\
&+ 2(2-m)^2 e^{2\gamma} |\operatorname{grad}\gamma|^2 (\Delta\gamma) + 4(2-m)^2 e^{2\gamma} |\operatorname{grad}\gamma|^4
\end{aligned}$$

$$\begin{aligned}
&\widetilde{\nabla}_{\operatorname{grad}\left(|\widetilde{\tau}(Id)|^2\right)} \widetilde{\tau}(Id) \\
&= (2-m)^3 e^{2\gamma} \nabla_{\operatorname{grad}\left(|\operatorname{grad}\gamma|^2\right)} \operatorname{grad}\gamma + 2(2-m)^3 e^{2\gamma} |\operatorname{grad}\gamma|^4 \operatorname{grad}\gamma \\
&+ 2(2-m)^3 e^{2\gamma} |\operatorname{grad}\gamma|^2 \operatorname{grad}\left(|\operatorname{grad}\gamma|^2\right)
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{\tau}_2(Id) = &-(2-m)\operatorname{grad}(\Delta\gamma) + \frac{(2-m)(m-6)}{2} \operatorname{grad}\left(|\operatorname{grad}\gamma|^2\right) \\
&+ 2(2-m)(\Delta\gamma)\operatorname{grad}\gamma - (2-m)^2\left(|\operatorname{grad}\gamma|^2 \operatorname{grad}\gamma\right) \\
&- 2(2-m)\, Ricci\,(\operatorname{grad}\gamma).
\end{aligned}$$

Substituting these obtained formulas into the equation (8), we conclude that $Id:(M^m,g)\longrightarrow \left(M^m,\widetilde{g}=e^{2\gamma}g\right)$ is p-biharmonic if and only if

$$\begin{aligned}
&\left|\widetilde{\tau}\left(Id\right)\right|^4\widetilde{\tau}_2\left(Id\right)-\frac{\left(p-2\right)\left(p-4\right)}{4}\left|\operatorname{grad}\left(\left|\widetilde{\tau}\left(Id\right)\right|^2\right)\right|^2\widetilde{\tau}\left(Id\right)\\
&-\left(p-2\right)\left|\widetilde{\tau}\left(Id\right)\right|^2\widetilde{\nabla}_{\operatorname{grad}\left(\left|\widetilde{\tau}\left(Id\right)\right|^2\right)}\widetilde{\tau}\left(Id\right)\\
&-\frac{\left(p-2\right)}{2}\left|\widetilde{\tau}\left(Id\right)\right|^2\Delta\left(\left|\widetilde{\tau}\left(Id\right)\right|^2\right)\widetilde{\tau}\left(Id\right)=0,
\end{aligned}$$

Example 3. Let $Id:\left(\mathbb{R}_+^*\times\mathbb{R}^{m-1},g\right)\to\left(\mathbb{R}_+^*\times\mathbb{R}^{m-1},g\right)$, $(m>2)$ the identity map defined by

$$Id\left(t,x_2,\ldots,x_m\right)=\left(t,x_2,\ldots,x_m\right),$$

where g is the Euclidean metric given by

$$g=dt^2+dx_2^2+.....+dx_m^2.$$

Let $\widetilde{g}=e^{2\gamma}g$ where the function γ depends only on t. By Theorem 2 and by using the fact that $p\geq 2$, we deduce that $Id:\left(\mathbb{R}_+^*\times\mathbb{R}^{m-1},g\right)\to\left(\mathbb{R}_+^*\times\mathbb{R}^{m-1},\widetilde{g}\right)$ is p-biharmonic if and only if $\beta=\gamma'$ satisfies the following differential equation

$$\begin{aligned}
&\left(p-1\right)\beta\beta''+\left(p-2\right)\left(p-1\right)\left(\beta'\right)^2\\
&+\left(2p^2-3p-m+2\right)\beta^2\beta'\\
&-\left(m+2p-p^2-2\right)\beta^4=0.
\end{aligned}$$

We deduce the particular solutions of the form $\beta=\frac{a}{t}$ where $a\neq 0$. We obtain

$$\left(m+2p-p^2-2\right)a^2+\left(2p^2-3p-m+2\right)a-p\left(p-1\right)=0.$$

There are two solutions of this type:

1. If $m+2p-p^2-2=0$, then $p=\sqrt{m-1}+1,\ \ m\geq 3$ and $a=1$. It follows that and $\gamma(t)=\ln t$, we obtain the metric $\widetilde{g}=t^2g$. So, we conclude that the identity map $Id:\left(\mathbb{R}_+^*\times\mathbb{R}^{m-1},\widetilde{g}\right)\to\left(\mathbb{R}_+^*\times\mathbb{R}^{m-1},g\right)$ is a p-biharmonic map, where $p=\sqrt{m-1}+1$. For example, if $m=5$, we obtain $p=3$

2. If $m+2p-p^2-2\neq 0$, we obtain $a=\frac{p-p^2}{m+2p-p^2-2}$ or $a=1$ then

$$\widetilde{g}=t^{\frac{2\left(p-p^2\right)}{m+2p-p^2-2}}g\text{ or }\widetilde{g}=t^2g.$$

   In this cases, the identity map $Id:\left(\mathbb{R}_+^*\times\mathbb{R}^{m-1},g\right)\to\left(\mathbb{R}_+^*\times\mathbb{R}^{m-1},\widetilde{g}\right)$ is a p-biharmonic map. For example, if $p=m=4$, we obtain $a=1$ or $a=2$ and if $p=m=3$, we obtain $a=1$ or $a=3$.

Remark 2. If we consider $Id : (\mathbb{R}^m, g) \longrightarrow (\mathbb{R}^m, g)$ $(m \neq 2)$ where we suppose that the function γ is radial $(\gamma = \gamma(r),\ \ r = |x|,\ \ x \in \mathbb{R}^m)$, then the p-biharmonicity of $Id : (\mathbb{R}^m, g) \longrightarrow \left(\mathbb{R}^m, \widetilde{g} = e^{2\gamma} g\right)$ is equivalent to an ordinary differential equation:

$$
\begin{aligned}
&(p-1)\,\beta\beta'' + \left(2p^2 - 3p - m + 2\right)\beta^2\beta' + \frac{(p-1)(m-1)}{r}\beta\beta' \\
&+ (p-2)(p-1)\left(\beta'\right)^2 - \frac{(m-1)}{r^2}\beta^2 + \frac{(p-4)(m-1)}{r}\beta^3 \\
&- \left(m + 2p - p^2 - 2\right)\beta^4 = 0
\end{aligned}
\tag{9}
$$

where $\beta = \gamma'$.

Example 4. If we consider $Id : (\mathbb{R}^m \setminus \{0\}, g) \longrightarrow (\mathbb{R}^m \setminus \{0\}, g)$ $(m \neq 2)$ where we suppose $\beta = \frac{a}{r}$, $a \in \mathbb{R}^*$. By equation (9), we deduce that $Id : (\mathbb{R}^m \setminus \{0\}, g) \longrightarrow \left(\mathbb{R}^m \setminus \{0\}, \widetilde{g} = e^{2\gamma} g\right)$ is p-biharmonic if and only if

$$\left(m + 2p - p^2 - 2\right)a^2 + \left(3m - 2p - mp + 2p^2 - 2\right)a + p\,(m-p) = 0.$$

For the resolution of this last equation, we will present several cases ($p \geq 2$ and $m \geq 3$).

- *First case:* $p = m,\quad m \geq 3$, we obtain $a = \frac{m+2}{m-2}$, then

$$\gamma = \ln r^{\frac{m+2}{m-2}}$$

and

$$\widetilde{g} = r^{\frac{2(m+2)}{m-2}} g.$$

- *Second case:* $p = \sqrt{m-1} + 1,\quad m \geq 3$, then $a = \frac{m-2}{m-2-4\sqrt{m-1}}$, which gives us

$$\widetilde{g} = r^{\frac{2(m-2)}{m-2-4\sqrt{m-1}}} g.$$

For example, if $m = 10$, we obtain $p = 4$ and $a = -4$

- *Third case:* $p = \frac{3m-2}{4},\quad m \geq 4$, then

$$a = \pm\frac{\sqrt{3}}{6}\frac{\sqrt{3m^2 + 4m - 4}}{(m-2)}$$

and

$$\gamma = \ln r^{\pm\frac{\sqrt{3}}{6}\frac{\sqrt{3m^2+4m-4}}{(m-2)}}.$$

It follows that

$$\widetilde{g} = r^{\frac{\sqrt{3}\sqrt{3m^2+4m-4}}{3(m-2)}} g$$

or

$$\widetilde{g} = \frac{1}{r^{\frac{\sqrt{3}\sqrt{3m^2+4m-4}}{3(m-2)}}} g.$$

- *Fourth case:* $p \neq \sqrt{m-1}+1$, then

$$\widetilde{g} = r^{2a} g,$$

where

$$a = \frac{1}{-2m-4p+2p^2+4}\left(3m-2p-mp+2p^2-2+A(m,p)\right)$$

or

$$a = -\frac{1}{-2m-4p+2p^2+4}\left(-3m+2p+mp-2p^2+2+A(m,p)\right)$$

and

$$A(m,p) = \sqrt{-12m+8p+12mp^2-10m^2p+m^2p^2+9m^2-12p^2+4}.$$

Note that, if $m=1$, we obtain the following equation:

$$(p-1)a^2-(2p-1)a+p=0.$$

The solutions for this equation are $a=1$ and $a=\frac{p}{p-1}$, $\quad p \geq 2$.

In a following section, we present other examples where we give the p-biharmonicity condition of some maps.

3. Other examples

Example 5. Let the map $\phi : \left(\mathbb{R}^3, g_{\mathbb{R}^3}\right) \longrightarrow \left(\mathbb{R}^2, g_{\mathbb{R}^2}\right)$ defined by

$$\phi(x_1,x_2,x_3) = \left(\sqrt{x_1^2+x_2^2}, x_3\right).$$

Using cylindrical coordinates (r,θ,x_3) in $\mathbb{R}^3$ and the standard Euclidean coordinates in $\mathbb{R}^2$, the map ϕ is written in the form

$$\phi(r\cos\theta, r\sin\theta, x_3) = (y_1,y_2),$$

where $y_1 = r = \sqrt{x_1^2+x_2^2}$ et $y_2 = x_3$. The metrics on $\mathbb{R}^3$ and $\mathbb{R}^2$ have respectively the expressions

$$g_{\mathbb{R}^3} = dr^2 + r^2 d\theta^2 + dx_3^2$$

and

$$g_{\mathbb{R}^2} = dy_1^2 + dy_2^2.$$

A rigorous calculation give

$$\tau(\phi) = \frac{1}{r}\frac{\partial}{\partial y_1}, \qquad |\tau(\phi)|^2 = \frac{1}{r^2}, \qquad \tau_2(\phi) = -\frac{3}{r^3}\frac{\partial}{\partial y_1},$$

$$\operatorname{grad}\left(|\tau(\phi)|^2\right) = -\frac{2}{r^3}\frac{\partial}{\partial r}, \qquad \left|\operatorname{grad}\left(|\tau(\phi)|^2\right)\right|^2 = \frac{4}{r^6}$$

and

$$\nabla^{\phi}_{\operatorname{grad}\left(|\tau(\phi)|^2\right)}\tau(\phi) = \frac{2}{r^5}\frac{\partial}{\partial y_1}, \qquad \Delta\left(|\tau(\phi)|^2\right) = \frac{4}{r^4}.$$

Then, we conclude the the map $\phi : \left(\mathbb{R}^3, g_{\mathbb{R}^3}\right) \longrightarrow \left(\mathbb{R}^2, g_{\mathbb{R}^2}\right)$ is p-biharmonic if and only if $p = 1$.

Example 6. We consider the inversion $\phi : \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}^n \setminus \{0\}$ $(n \geq 3)$ defined by $\phi(x) = \frac{x}{|x|^2}$. Note that this map is p-harmonic if and only if $p = n$. A long calculation gives us

$$\tau(\phi) = \frac{2(n-2)}{r^3}\frac{\partial}{\partial r}, \quad |\tau(\phi)|^2 = \frac{4(n-2)^2}{r^6}, \quad \tau_2(\phi) = \frac{8(n-2)(n-4)}{r^5}\frac{\partial}{\partial r},$$

$$\operatorname{grad}\left(|\tau(\phi)|^2\right) = -\frac{24(n-2)^2}{r^7}\frac{\partial}{\partial r}, \quad \left|\operatorname{grad}\left(|\tau(\phi)|^2\right)\right|^2 = \frac{576(n-2)^4}{r^{14}}$$

and

$$\nabla^{\phi}_{\operatorname{grad}\left(|\tau(\phi)|^2\right)}\tau(\phi) = \frac{144(n-2)^3}{r^{11}}\frac{\partial}{\partial r}, \quad \Delta\left(|\tau(\phi)|^2\right) = -\frac{24(n-2)^2(n-8)}{r^8},$$

where $r = |x|$, $x \in \mathbb{R}^n$. Then, the inversion $\phi : \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}^n \setminus \{0\}$ is p-biharmonic non-p-harmonic if and only if $p = \frac{n+2}{3}$. A class of solutions is given by the case where $n = 3k+1$, $\quad k \in \mathbb{N}^*$, which gives us $p = k+1$. For example, if we take $n = 7$, we find $p = 3$.

Example 7. Let $\phi : \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R} \times \mathbb{S}^{n-1}$ given in polar coordinates by

$$\phi(r\theta) = (\ln r, \theta), \quad r > 0, \quad \theta \in \mathbb{S}^{n-1} \subset \mathbb{R}^n.$$

The same calculation method gives us

$$\tau(\phi) = \frac{(n-2)}{r^2}\frac{\partial}{\partial r}, \quad |\tau(\phi)|^2 = \frac{(n-2)^2}{r^4}, \quad \tau_2(\phi) = \frac{2(n-2)(n-4)}{r^4}\frac{\partial}{\partial r},$$

$$\operatorname{grad}\left(|\tau(\phi)|^2\right) = -\frac{4(n-2)^2}{r^5}\frac{\partial}{\partial r}, \quad \left|\operatorname{grad}\left(|\tau(\phi)|^2\right)\right|^2 = \frac{16(n-2)^4}{r^{10}}$$

and

$$\nabla^{\phi}_{\operatorname{grad}\left(|\tau(\phi)|^2\right)}\tau(\phi) = \frac{8(n-2)^3}{r^8}\frac{\partial}{\partial r}, \quad \Delta\left(|\tau(\phi)|^2\right) = -\frac{4(n-2)^2(n-6)}{r^6}.$$

We conclude that ϕ is p-biharmonic if and only if $p = \frac{n}{2}$, $\quad n \geq 3$ or $p = 1$.

4. The warped product and the p-biharmonic maps

Let (M^m, g) and (N^n, h) two Riemannian manifolds and let $f \in C^\infty(M)$ be a positive function. The warped product $M \times_f N$ is the product manifolds $M \times N$ endowed with the Riemannian metric G_f defined, for $X, Y \in \Gamma(T(M \times N))$, by

$$G_f(X,Y) = g(d\pi(X), d\pi(Y)) + (f \circ \pi)^2 h(d\eta(X), d\eta(Y)),$$

where $\pi : M \times N \longrightarrow M$ and $\eta : M \times N \longrightarrow N$ are respectively the first and the second projection. The function f is called the warping function function of the warped product. Let $X, Y \in \Gamma(T(M \times N))$, $X = (X_1, X_2)$, $Y = (Y_1, Y_2)$. Denote by ∇ the Levi-Civita connection on the Riemannian product $M \times N$. The Levi-Civita connection $\widetilde{\nabla}$ of the warped product $M \times_f N$ is given by

$$\begin{aligned}\widetilde{\nabla}_X Y = {} & \nabla_X Y + X_1(\ln f)(0, Y_2) + Y_1(\ln f)(0, X_2) \\ & - f^2 h(X_2, Y_2)(\operatorname{grad} \ln f, 0).\end{aligned} \tag{10}$$

There are several methods to give the relation between the curvature tensor fields of G_f and G, we are going to prove one of these methods.

Proposition 1. *The relation between the curvature tensor fields of G_f and G is given by*

$$\begin{aligned}\widetilde{R}(X,Y)Z = {} & R(X,Y)Z - f^2 h(Y_2, Z_2)(\nabla_{X_1} \operatorname{grad} \ln f, 0) \\ & + f^2 h(X_2, Z_2)(\nabla_{Y_1} \operatorname{grad} \ln f, 0) \\ & - f^2 h(Y_2, Z_2) X_1(\ln f)(\operatorname{grad} \ln f, 0) \\ & + f^2 h(X_2, Z_2) Y_1(\ln f)(\operatorname{grad} \ln f, 0) \\ & - f^2 h(Y_2, Z_2) |\operatorname{grad} \ln f|^2 (0, X_2) \\ & + f^2 h(X_2, Z_2) |\operatorname{grad} \ln f|^2 (0, Y_2) \\ & - g(\nabla_{Y_1} \operatorname{grad} \ln f, Z_1)(0, X_2) \\ & + g(\nabla_{X_1} \operatorname{grad} \ln f, Z_1)(0, Y_2) \\ & + X_1(\ln f) Z_1(\ln f)(0, Y_2) \\ & - Y_1(\ln f) Z_1(\ln f)(0, X_2)\end{aligned} \tag{11}$$

for all $X, Y, Z \in \Gamma(T(M \times N))$.

Proof of Proposition 1. By definition, we have

$$\widetilde{R}(X,Y)Z = \widetilde{\nabla}_X \widetilde{\nabla}_Y Z - \widetilde{\nabla}_Y \widetilde{\nabla}_X Z - \widetilde{\nabla}_{[X,Y]} Z.$$

Using equation (10), we obtain

$$\begin{aligned}\widetilde{\nabla}_Y Z = {} & \nabla_Y Z + Y_1(\ln f)(0, Z_2) + Z_1(\ln f)(0, Y_2) \\ & - f^2 h(Y_2, Z_2)(\operatorname{grad} \ln f, 0),\end{aligned}$$

then

$$\widetilde{\nabla}_X \widetilde{\nabla}_Y Z = \widetilde{\nabla}_X \nabla_Y Z + \widetilde{\nabla}_X Y_1 (\ln f) (0, Z_2) + \widetilde{\nabla}_X Z_1 (\ln f) (0, Y_2) \\ - \widetilde{\nabla}_X f^2 h (Y_2, Z_2) (\operatorname{grad} \ln f, 0) .$$

A long calculation gives us

$$\widetilde{\nabla}_X \nabla_Y Z = \nabla_X \nabla_Y Z + X_1 (\ln f) (0, \nabla_{Y_2} Z_2) + (\nabla_{Y_1} Z_1) (\ln f) (0, X_2) \\ - f^2 h (X_2, \nabla_{Y_2} Z_2) (\operatorname{grad} \ln f, 0) ,$$

$$\widetilde{\nabla}_X Y_1 (\ln f) (0, Z_2) = X_1 (\ln f) Y_1 (\ln f) (0, Z_2) + X_1 (Y_1 (\ln f)) (0, Z_2) \\ - f^2 h (X_2, Z_2) Y_1 (\ln f) (\operatorname{grad} \ln f, 0) \\ + Y_1 (\ln f) (0, \nabla_{X_2} Z_2) ,$$

$$\widetilde{\nabla}_X Z_1 (\ln f) (0, Y_2) = X_1 (\ln f) Z_1 (\ln f) (0, Y_2) + X_1 (Z_1 (\ln f)) (0, Y_2) \\ - f^2 h (X_2, Y_2) Z_1 (\ln f) (\operatorname{grad} \ln f, 0) \\ + Z_1 (\ln f) (0, \nabla_{X_2} Y_2)$$

and

$$\widetilde{\nabla}_X f^2 h (Y_2, Z_2) (\operatorname{grad} \ln f, 0) = f^2 h (Y_2, Z_2) (\nabla_{X_1} \operatorname{grad} \ln f, 0) \\ + 2 f^2 X_1 (\ln f) h (Y_2, Z_2) (\operatorname{grad} \ln f, 0) \\ + f^2 h (Y_2, Z_2) |\operatorname{grad} \ln f|^2 (0, X_2) \\ + f^2 h (\nabla_{X_2} Y_2, Z_2) (\operatorname{grad} \ln f, 0) \\ + f^2 h (Y_2, \nabla_{X_2} Z_2) (\operatorname{grad} \ln f, 0) .$$

It follows that

$$\widetilde{\nabla}_X \widetilde{\nabla}_Y Z \\ = \nabla_X \nabla_Y Z - f^2 h (X_2, \nabla_{Y_2} Z_2) (\operatorname{grad} \ln f, 0) - f^2 h (Y_2, Z_2) (\nabla_{X_1} \operatorname{grad} \ln f, 0) \\ - f^2 h (X_2, Y_2) Z_1 (\ln f) (\operatorname{grad} \ln f, 0) - f^2 h (\nabla_{X_2} Y_2, Z_2) (\operatorname{grad} \ln f, 0) \\ - 2 f^2 X_1 (\ln f) h (Y_2, Z_2) (\operatorname{grad} \ln f, 0) - f^2 h (Y_2, \nabla_{X_2} Z_2) (\operatorname{grad} \ln f, 0) \\ - f^2 h (X_2, Z_2) Y_1 (\ln f) (\operatorname{grad} \ln f, 0) - f^2 h (Y_2, Z_2) |\operatorname{grad} \ln f|^2 (0, X_2) \\ + (\nabla_{Y_1} Z_1) (\ln f) (0, X_2) + X_1 (Z_1 (\ln f)) (0, Y_2) + X_1 (\ln f) Z_1 (\ln f) (0, Y_2) \\ + X_1 (\ln f) Y_1 (\ln f) (0, Z_2) + X_1 (Y_1 (\ln f)) (0, Z_2) + X_1 (\ln f) (0, \nabla_{Y_2} Z_2) \\ + Y_1 (\ln f) (0, \nabla_{X_2} Z_2) + Z_1 (\ln f) (0, \nabla_{X_2} Y_2) .$$

A similar calculation gives

$$\begin{aligned}
&\widetilde{\nabla}_Y \widetilde{\nabla}_X Z \\
&= \nabla_Y \nabla_X Z - f^2 h\left(Y_2, \nabla_{X_2} Z_2\right)(\operatorname{grad} \ln f, 0) - f^2 h\left(X_2, Z_2\right)\left(\nabla_{Y_1} \operatorname{grad} \ln f, 0\right) \\
&- f^2 h\left(X_2, Y_2\right) Z_1(\ln f)(\operatorname{grad} \ln f, 0) - f^2 h\left(\nabla_{Y_2} X_2, Z_2\right)(\operatorname{grad} \ln f, 0) \\
&- 2 f^2 Y_1(\ln f)\, h\left(X_2, Z_2\right)(\operatorname{grad} \ln f, 0) - f^2 h\left(X_2, \nabla_{Y_2} Z_2\right)(\operatorname{grad} \ln f, 0) \\
&- f^2 h\left(Y_2, Z_2\right) X_1(\ln f)(\operatorname{grad} \ln f, 0) - f^2 h\left(X_2, Z_2\right)|\operatorname{grad} \ln f|^2\left(0, Y_2\right) \\
&+ \left(\nabla_{X_1} Z_1\right)(\ln f)\left(0, Y_2\right) + Y_1\left(Z_1(\ln f)\right)\left(0, X_2\right) + Y_1(\ln f)\, Z_1(\ln f)\left(0, X_2\right) \\
&+ Y_1(\ln f)\, X_1(\ln f)\left(0, Z_2\right) + Y_1\left(X_1(\ln f)\right)\left(0, Z_2\right) + Y_1(\ln f)\left(0, \nabla_{X_2} Z_2\right) \\
&+ X_1(\ln f)\left(0, \nabla_{Y_2} Z_2\right) + Z_1(\ln f)\left(0, \nabla_{Y_2} X_2\right)
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{\nabla}_{[X,Y]} Z = {} & \nabla_{[X,Y]} Z + \left(\left[X_1, Y_1\right]\right)(\ln f)\left(0, Z_2\right) + Z_1(\ln f)\left(0, \left[X_2, Y_2\right]\right) \\
& - f^2 h\left(X_2, \left[X_2, Y_2\right]\right)(\operatorname{grad} \ln f, 0),
\end{aligned}$$

which leads us to the following formula

$$\begin{aligned}
\widetilde{R}(X, Y) Z = {} & R(X, Y) Z - f^2 h\left(Y_2, Z_2\right)\left(\nabla_{X_1} \operatorname{grad} \ln f, 0\right) \\
& + f^2 h\left(X_2, Z_2\right)\left(\nabla_{Y_1} \operatorname{grad} \ln f, 0\right) \\
& - f^2 h\left(Y_2, Z_2\right) X_1(\ln f)(\operatorname{grad} \ln f, 0) \\
& + f^2 h\left(X_2, Z_2\right) Y_1(\ln f)(\operatorname{grad} \ln f, 0) \\
& - f^2 h\left(Y_2, Z_2\right)|\operatorname{grad} \ln f|^2\left(0, X_2\right) \\
& + f^2 h\left(X_2, Z_2\right)|\operatorname{grad} \ln f|^2\left(0, Y_2\right) \\
& - g\left(\nabla_{Y_1} \operatorname{grad} \ln f, Z_1\right)\left(0, X_2\right) \\
& + g\left(\nabla_{X_1} \operatorname{grad} \ln f, Z_1\right)\left(0, Y_2\right) \\
& + X_1(\ln f)\, Z_1(\ln f)\left(0, Y_2\right) \\
& - Y_1(\ln f)\, Z_1(\ln f)\left(0, X_2\right)
\end{aligned}$$

As consequence, we obtain

Proposition 2.

$$\begin{aligned}
\widetilde{Ricci}(X) = {} & \left(Ricci\left(X_1\right), 0\right) - n\left(\nabla_{X_1} \operatorname{grad} \ln f, 0\right) - n X_1(\ln f)(\operatorname{grad} \ln f, 0) \\
& + \frac{1}{f^2}\left(0, Ricci\left(X_2\right)\right) - (\Delta \ln f)\left(0, X_2\right) - n|\operatorname{grad} \ln f|^2\left(0, X_2\right),
\end{aligned}$$

$$\begin{aligned}
\widetilde{Ric}(X, Y) = {} & Ric\left(X_1, Y_1\right) + Ric\left(X_2, Y_2\right) - n g\left(\nabla_{X_1} \operatorname{grad} \ln f, Y_1\right) \\
& - n X_1(\ln f)\, Y_1(\ln f) - f^2(\Delta \ln f)\, h\left(X_2, Y_2\right) \\
& - n f^2|\operatorname{grad} \ln f|^2 h\left(X_2, Y_2\right)
\end{aligned}$$

and

$$S_{G_f} = S_g + \frac{1}{f^2} S_h - 2n\left(\Delta \ln f\right) - n\left(n+1\right)\left|\operatorname{grad} \ln f\right|^2 .$$

Using the fact that

$$\Delta\left(f^k\right) = k f^k \left(\Delta \ln f + k \left|\operatorname{grad} \ln f\right|^2\right),$$

we obtain the following result. If $S_g = S_h = 0$, then

$$S_{G_f} = 0 \text{ if and only if the function } f^{\frac{n+1}{2}} \text{ is harmonic.}$$

In the first, we study the p-biharmonicity of the first projection, we obtain the following result:

Theorem 3. *The first projection $\pi : M^m \times_f N^n \longrightarrow M^m$ is p-biharmonic if and only if*

$$\begin{aligned}
&\left|\operatorname{grad} \ln f\right|^4 \operatorname{grad}\left(\Delta \ln f\right) + \frac{(p-2)}{2}\left|\operatorname{grad} \ln f\right|^2 \Delta\left(\left|\operatorname{grad} \ln f\right|^2\right) \operatorname{grad} \ln f \\
&+ \frac{n}{2}\left|\operatorname{grad} \ln f\right|^4 \operatorname{grad}\left(\left|\operatorname{grad} \ln f\right|^2\right) + 2\left|\operatorname{grad} \ln f\right|^4 \mathit{Ricci}\left(\operatorname{grad} \ln f\right) \\
&+ (p-2)\left|\operatorname{grad} \ln f\right|^2 \nabla_{\operatorname{grad}\left(\left|\operatorname{grad} \ln f\right|^2\right)} \operatorname{grad} \ln f \\
&+ \frac{(p-2)(p-4)}{4}\left|\operatorname{grad}\left(\left|\operatorname{grad} \ln f\right|^2\right)\right|^2 \operatorname{grad} \ln f = 0.
\end{aligned} \tag{12}$$

Proof of Theorem 3. By equation (2), the first projection

$$\pi : M^m \times_f N^n \longrightarrow M^m$$

is p-biharmonic if and only if

$$\begin{aligned}
&\left|\widetilde{\tau}\left(\pi\right)\right|^4 \widetilde{\tau}_2\left(\pi\right) - (p-2)\left|\widetilde{\tau}\left(\pi\right)\right|^2 \nabla^{\pi}_{\operatorname{grad}\left(\left|\widetilde{\tau}(\pi)\right|^2\right)} \widetilde{\tau}\left(\pi\right) \\
&- \frac{(p-2)(p-4)}{4}\left|\operatorname{grad}\left(\left|\widetilde{\tau}\left(\pi\right)\right|^2\right)\right|^2 \widetilde{\tau}\left(\pi\right) \\
&- \frac{(p-2)}{2}\left|\widetilde{\tau}\left(\pi\right)\right|^2 \Delta\left(\left|\widetilde{\tau}\left(\pi\right)\right|^2\right) \widetilde{\tau}\left(\pi\right) = 0.
\end{aligned}$$

A long and rigorous calculation gives us the following formulas:

$$\widetilde{\tau}\left(\pi\right) = n\left(\operatorname{grad} \ln f\right) \circ \pi, \qquad \left|\widetilde{\tau}\left(\pi\right)\right|^2 = n^2\left|\operatorname{grad} \ln f\right|^2,$$

$$\Delta\left(\left|\widetilde{\tau}\left(\pi\right)\right|^2\right) = n^2 \Delta\left(\left|\operatorname{grad} \ln f\right|^2\right), \qquad \left|\widetilde{\tau}\left(\pi\right)\right|^4 = n^4\left|\operatorname{grad} \ln f\right|^4,$$

$$\operatorname{grad}\left(\left|\widetilde{\tau}\left(\pi\right)\right|^2\right) = n^2 \operatorname{grad}\left(\left|\operatorname{grad} \ln f\right|^2\right) \circ \pi,$$

$$\left|\operatorname{grad}\left(\left|\widetilde{\tau}\left(\pi\right)\right|^2\right)\right|^2 = n^4\left|\operatorname{grad}\left(\left|\operatorname{grad} \ln f\right|^2\right)\right|^2,$$

$$\nabla^{\pi}_{\mathrm{grad}\left(|\widetilde{\tau}(\pi)|^2\right)}\widetilde{\tau}(\pi) = n^3\left(\nabla_{\mathrm{grad}\left(|\mathrm{grad}\ln f|^2\right)}\mathrm{grad}\ln f\right)\circ\pi$$

and

$$\widetilde{\tau}_2(\pi) = -n\left(\mathrm{grad}(\Delta\ln f) + 2\mathrm{Ricci}(\mathrm{grad}\ln f) + \frac{n}{2}\mathrm{grad}\left(|\mathrm{grad}\ln f|^2\right)\right)\circ\pi.$$

Then, we deduce that the first projection π is p-biharmonic if and only if

$$\begin{aligned}
&|\mathrm{grad}\ln f|^4\,\mathrm{grad}(\Delta\ln f) + \frac{(p-2)}{2}|\mathrm{grad}\ln f|^2\,\Delta\left(|\mathrm{grad}\ln f|^2\right)\mathrm{grad}\ln f\\
&+\frac{n}{2}|\mathrm{grad}\ln f|^4\,\mathrm{grad}\left(|\mathrm{grad}\ln f|^2\right) + 2|\mathrm{grad}\ln f|^4\,\mathrm{Ricci}(\mathrm{grad}\ln f)\\
&+(p-2)|\mathrm{grad}\ln f|^2\,\nabla_{\mathrm{grad}\left(|\mathrm{grad}\ln f|^2\right)}\mathrm{grad}\ln f\\
&+\frac{(p-2)(p-4)}{4}\left|\mathrm{grad}\left(|\mathrm{grad}\ln f|^2\right)\right|^2\mathrm{grad}\ln f = 0.
\end{aligned}$$

Example 8. We consider the first projection

$$\pi : \mathbb{R}^m\setminus\{0\}\times_f N^n \longrightarrow \mathbb{R}^m\setminus\{0\}$$

defined by

$$\pi(x = (t, x_2, \ldots, x_2), y) = (t, x_2, \ldots, x_2),$$

where we suppose that the function f depends only on the t. By the Theorem 1, we deduce that π is p-biharmonic if and only if the function $\beta(t) = (\ln f(t))'$ satisfies the following differential equation

$$(p-1)\beta\beta'' + (p-1)(p-2)\left(\beta'\right)^2 + n\beta^2\beta' = 0.$$

To solve this last equation, we will treat two cases:

1. Looking for particular solutions of type $\beta(t) = \frac{a}{t}\,(a\in\mathbb{R}^*)$, then π is p-biharmonic if and only if $a = \frac{p^2-p}{n}$. In this case, we obtain $f(t) = Ct^{\frac{p^2-p}{n}}, C>0$.
2. $\beta = a\,(a\in\mathbb{R}^*)$ is a trivial solution of the differential equation. We obtain $f(t) = Ce^{at}$, $C>0$.

Example 9. In this example, we suppose that the function f is radial ($f = f(r)$). A rigorous calculus give us

- $\mathrm{grad}\left(|\mathrm{grad}\ln f|^2\right) = 2\beta\beta'\frac{\partial}{\partial r}$,
- $\mathrm{grad}\,\Delta\ln f = \left(\beta'' + \frac{(m-1)}{r}\beta' - \frac{(m-1)}{r^2}\beta\right)\frac{\partial}{\partial r}$,
- $\Delta\left(|\mathrm{grad}\ln f|^2\right) = 2\beta\beta'' + 2(\beta')^2 + \frac{2(m-1)}{r}\beta\beta'$,

- $\nabla_{\operatorname{grad}\left(|\operatorname{grad}\ln f|^2\right)} \operatorname{grad}\ln f = 2\beta\left(\beta'\right)^2 \frac{\partial}{\partial r}$,

where $\beta(r) = (\ln f(r))'$. Then, by Theorem 3, we deduce that the first projection

$$\pi : \mathbb{R}^m \setminus \{0\} \times_f N^n \longrightarrow \mathbb{R}^m \setminus \{0\}$$

defined by $\pi(x, y) = x$ is p-biharmonic if and only if the function β satisfies the following differential equation

$$\begin{aligned}&(p-1)\beta\beta'' + (p-1)(p-2)\left(\beta'\right)^2 + n\beta^2\beta' \\ &+ \frac{(p-1)(m-1)}{r}\beta\beta' - \frac{(m-1)}{r^2}\beta^2 = 0.\end{aligned}$$

Looking for particular solutions of type $\beta = \frac{a}{r}$ $(a \in \mathbb{R}^*)$, then π is p-biharmonic if and only if

$$a = \frac{p(p-m)}{n}.$$

We obtain

$$f(r) = Cr^{\frac{p(p-m)}{n}}, \qquad C > 0$$

and the first projection π is p-biharmonic.

Now, we consider two Riemannian manifolds (M^m, g) and (N^n, h) and we will study the p-biharmonicity of the inclusion map $i_{x_0} : N \longrightarrow (M \times_f N, G_f)$ defined by $i_{x_0}(y) = (x_0, y)$. Note that for this map, we have $di_{x_0}(Y) = (0, Y)$ for any $Y \in \Gamma(TN)$. In the case where $f \in C^\infty(M)$, we obtain (see [4])

Proposition 3 ([4]). *The tension and bitension field of the inclusion map $i_{x_0} : N \longrightarrow (M \times_f N, G_f)$ where $f \in C^\infty(M)$ are given by*

$$\widetilde{\tau}(i_{x_0}) = -nf^2(\operatorname{grad}\ln f, 0) \circ i_{x_0}$$

and

$$\widetilde{\tau}_2(i_{x_0}) = -\frac{n^2}{2} f^4 \left(\left(\operatorname{grad}\left(|\operatorname{grad}\ln f|^2\right), 0\right) + 4|\operatorname{grad}\ln f|^2 (\operatorname{grad}\ln f, 0)\right) \circ i_{x_0}.$$

Then, i_{x_0} is biharmonic if and only if

$$\operatorname{grad}\left(|\operatorname{grad}\ln f|^2\right) + 4|\operatorname{grad}\ln f|^2 \operatorname{grad}\ln f = 0.$$

Remark 3. Based on the following two equations:

$$Tr_h\left(\widetilde{\nabla}^{i_{x_0}}\right)^2 (\operatorname{grad}\ln f, 0) \circ i_{x_0} = -nf^2 |\operatorname{grad}\ln f|^2 (\operatorname{grad}\ln f, 0) \circ i_{x_0}$$

and

$$\begin{aligned}Tr_h\widetilde{R}\left(\widetilde{\tau}(i_{x_0}), di_{x_0}\right) di_{x_0} = &\frac{n^2}{2} f^4 \left(\operatorname{grad}\left(|\operatorname{grad}\ln f|^2\right), 0\right) \circ i_{x_0} \\ &+ n^2 f^4 |\operatorname{grad}\ln f|^2 (\operatorname{grad}\ln f, 0) \circ i_{x_0},\end{aligned}$$

we deduce that the p-bitension field is given by

$$\widetilde{\tau}_{p,2}(i_{x_0}) = -\frac{1}{p}n^p f^{2p}|\mathrm{grad}\ln f|^{p-2}\left(\mathrm{grad}\left(|\mathrm{grad}\ln f|^2\right),0\right)\circ i_{x_0} - \frac{4}{p}n^p f^{2p}|\mathrm{grad}\ln f|^p(\mathrm{grad}\ln f,0)\circ i_{x_0}.$$

Then, we conclude that

The inclusion map i_{x_0} is p-biharmonic if and only if it is biharmonic.

Proposition 4. *For the inclusion map $i_{x_0}: N \longrightarrow (M\times_f N, G_f)$, we have*

$$\widetilde{S}_2(i_{x_0}) = -\frac{n(n-4)}{2}f^4|\mathrm{grad}\ln f|^2 h\circ i_{x_0}.$$

Proof of Proposition 4. By definition, we have

$$\widetilde{S}_2(i_{x_0})(X,Y) = \left(\frac{1}{2}|\widetilde{\tau}(i_{x_0})|^2 + \mathrm{Tr}_h G_g\left(\widetilde{\nabla}\widetilde{\tau}(i_{x_0}), di_{x_0}\right)\right)h(X,Y) - G_f\left(\widetilde{\nabla}_X\widetilde{\tau}(i_{x_0}), di_{x_0}(Y)\right) - G_f\left(\widetilde{\nabla}_Y\widetilde{\tau}(i_{x_0}), di_{x_0}(X)\right).$$

Using the fact that $\widetilde{\tau}(i_{x_0}) = -nf^2(\mathrm{grad}\ln f,0)\circ i_{x_0}$, a rigorous calculation gives

$$|\widetilde{\tau}(i_{x_0})|^2 = n^2 f^4|\mathrm{grad}\ln f|^2,$$

$$Tr_h G_f\left(\widetilde{\nabla}\widetilde{\tau}(i_{x_0}), d\phi\right) = -n^2 f^4|\mathrm{grad}\ln f|^2,$$

$$G_f\left(\widetilde{\nabla}_X\widetilde{\tau}(i_{x_0}), di_{x_0}(Y)\right) = -nf^4|\mathrm{grad}\ln f|^2 h(X,Y)$$

and

$$G_f\left(\widetilde{\nabla}_Y\widetilde{\tau}(i_{x_0}), di_{x_0}(X)\right) = -nf^4|\mathrm{grad}\ln f|^2 h(X,Y).$$

It follows that

$$\widetilde{S}_2(i_{x_0})(X,Y) = -\frac{n(n-4)}{2}f^4|\mathrm{grad}\ln f|^2 h(X,Y).$$

Remark 4. Based on the expression of $\widetilde{S}_2(i_{x_0})$, we conclude that

$$\widetilde{S}_2(i_{x_0}) = 0 \text{ if and only if } n = 4.$$

5. Conclusion

In this paper, we have presented some constructions of p-biharmonic maps by conformal deformation and by using the warped product, we have studied the p-biharmonicity in some particular cases. The results obtained have allowed us to construct new examples of p-biharmonic maps. This paper is a generalization of the results obtained for biharmonic maps and our next objective is to study this class of maps by using the doubly warped product manifolds.

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Operator products and algebraic spectral subspace preservers

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Abstract. Let $\mathcal{X}$ be an infinite-dimensional complex Banach space and $\mathcal{B}(\mathcal{X})$ be the algebra of all bounded linear operators on $\mathcal{X}$. For $T \in \mathcal{B}(\mathcal{X})$, and a fixed nonzero complex scalar λ_0, we denote by $E_T(\{\lambda_0\})$, the algebraic spectral subspace of T associated with $\{\lambda_0\}$. In this paper, we characterize maps ϕ on $\mathcal{B}(\mathcal{X})$ for which whose ranges contain all operators of rank at most two (resp. at most four), and that satisfy $E_{TS}(\{\lambda_0\}) = E_{\phi(T)\phi(S)}(\{\lambda_0\})$ (resp. $E_{TST}(\{\lambda_0\}) = E_{\phi(T)\phi(S)\phi(T)}(\{\lambda_0\})$), for all $T, S \in \mathcal{B}(\mathcal{X})$.

Keywords: nonlinear preservers problem, algebraic spectral subspace, algebraic core, rank one idempotent operator.

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1. Introduction

Throughout this note, $\mathcal{X}$ will denote an infinite-dimensional complex Banach space, $\mathcal{X}^*$ the topological dual of $\mathcal{X}$, $\mathcal{B}(\mathcal{X})$ the algebra of all bounded linear operators on $\mathcal{X}$ and $\mathbb{C}$ the field of complex numbers. For any $x \in \mathcal{X}$ and $f \in \mathcal{X}^*$, $x \otimes f$ stands for the operator of rank at most one defined by $(x \otimes f)(y) = f(y)x$ for every $y \in \mathcal{X}$. We denote by $span\{x\}$ the subspace spanned by x. The sets of all finite rank operators, all operators of rank at most n, all rank-one non-nilpotent operators, all rank-one idempotent operators are denoted, respectively, by $\mathcal{F}(\mathcal{X})$, $\mathcal{F}_n(\mathcal{X})$, $\mathcal{F}_1(\mathcal{X}) \setminus \mathcal{N}_1(\mathcal{X})$ and $\mathcal{P}_1(\mathcal{X})$. Note that, $x \otimes f \in \mathcal{P}_1(\mathcal{X})$ if and only if $f(x) = 1$. For $T \in \mathcal{B}(\mathcal{X})$, the kernel and the range of T are denoted, respectively, by $\mathcal{N}(T)$ and $\mathcal{R}(T)$. We define the following set

$$\mathcal{F}_{1,\alpha}(\mathcal{X}) = \{x \otimes f : x \in \mathcal{X}, f \in \mathcal{X}^* \ \ such \ \ that \ \ \ f(x) = \alpha\},$$

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where $\alpha \in \mathbb{C} \setminus \{0\}$.

The algebraic core of T, denoted by $\mathcal{C}(T)$, is defined to be the greatest subspace M of $\mathcal{X}$ for which

$$TM = M.$$

Note that, $y \in \mathcal{C}(T)$ if and only if there exists a sequence $(y_n)_n \subset \mathcal{X}$ such that $y_0 = y$ and $Ty_{n+1} = y_n$, for all $n \in \mathbb{Z}_+$; see [1, Theorem 1.8].

For a vector $x_0 \in \mathcal{X}$, the local resolvent set of an operator $T \in \mathcal{B}(\mathcal{X})$ at x_0, denoted by $\rho_T(x_0)$, is defined as the union of all open subsets $U \subset \mathbb{C}$ for which there exists an analytic function $f : U \to \mathcal{X}$ such that $(T - \lambda I)f(\lambda) = x_0$, for all $\lambda \in U$. The subset $\sigma_T(x_0) = \mathbb{C} \setminus \rho_T(x_0)$ is the local spectrum of T at x_0.

For every subset Ω of $\mathbb{C}$, the local spectral subspace, $X_T(\Omega)$, is defined by

$$X_T(\Omega) = \{x \in \mathcal{X} : \sigma_T(x) \subset \Omega\}.$$

The algebraic spectral subspace of T associated with Ω, denoted by $E_T(\Omega)$, is defined as the algebraic sum of all subspace M of $\mathcal{X}$ with the property that

$$(T - \lambda I)M = M, \text{ for every } \lambda \in \mathbb{C} \setminus \Omega.$$

Evidently, $E_T(\Omega)$ is the largest subspace of $\mathcal{X}$ on which all the restrictions of $\lambda I - T$, $\lambda \in \mathbb{C} \setminus \Omega$, are surjective. In particular,

$$(T - \lambda I)E_T(\Omega) = E_T(\Omega), \quad \text{for all} \quad \lambda \in \mathbb{C} \setminus \Omega. \tag{1}$$

Note that, $E_T(\Omega) \subset \mathcal{C}(T - \lambda I)$, for all $\lambda \notin \Omega$, and for every $\lambda \in \Omega$ we have

$$\mathcal{N}(T - \lambda I) \subset E_T(\Omega) \tag{2}$$

(see, for instance, [1, 9]).

For $T \in \mathcal{B}(\mathcal{X})$, we denote $Lat(T)$ the lattice of T, that is, the set of all invariant subspaces of T. Note that, the subspace $E_T(\Omega)$ is an invariant subspace of T.

In recent decades, a considerable attention has been paid to the nonlinear preserver problems, which demand the characterization of maps between algebras that leave a given set, property or relation invariant without assuming in advance algebraic conditions such as linearity, additivity or multiplicity, but a weak algebraic condition is often imposed through the preserving property.; see for instance [5, 7, 8, 9, 10] and the reference cited there.

In [5], Dolinar, Du, Hou and Legiša characterized the form of maps preserving the lattice of product of operators. They showed that maps (not necessarily linear) $\phi : \mathcal{B}(\mathcal{X}) \to \mathcal{B}(\mathcal{X})$ satisfy $\text{Lat}(\phi(A)\phi(B)) = \text{Lat}(AB)$ (resp. $\text{Lat}(\phi(A)\phi(B)\phi(A)) = \text{Lat}(ABA)$), for all $A, B \in \mathcal{B}(\mathcal{X})$, if and only if there is a map $\varphi : \mathcal{B}(\mathcal{X}) \to \mathbb{K}$ such that $\varphi(A) \neq 0$ if $A \neq 0$ and $\phi(A) = \varphi(A)A$, for all $A \in \mathcal{B}(\mathcal{X})$. These results have motivated several authors to study maps on Banach algebras which preserve invariant subspaces; see for instance [2, 3, 4, 11].

In [2], the authors characterized maps on $\mathcal{B}(\mathcal{X})$ preserving the commutant of the sum $A+B$, the product AB, the Jordan triple product ABA and the Jordan product $AB+BA$, for all $A, B \in \mathcal{B}(\mathcal{X})$.

In the context of local spectral subspace, we cite the results found in [3], in which the authors described surjective maps $\phi : \mathcal{B}(\mathcal{X}) \to \mathcal{B}(\mathcal{X})$ satisfying $X_{\phi(A)\phi(B)}(\{\lambda\}) = X_{AB}(\{\lambda\})$ (resp. $X_{\phi(A)\phi(B)\phi(A)}(\{\lambda\}) = X_{ABA}(\{\lambda\})$,) for all $A, B \in \mathcal{B}(\mathcal{X})$ and $\lambda \in \mathbb{C}$.

For a fixed scalar $\lambda_0 \in \mathbb{C}$, in [4], Bouchangour and Jaatit find a similar result as in [3] by removing the surjectivity condition on ϕ for product or Jordan triple product of operators. They determine the form of all maps on $\mathcal{B}(\mathcal{X})$ which satisfying $X_{TS}(\{\lambda_0\}) = X_{\phi(T)\phi(S)}(\{\lambda_0\})$ (resp.$X_{TST}(\{\lambda_0\}) = X_{\phi(T)\phi(S)\phi(T)}(\{\lambda_0\})$), for all $T, S \in \mathcal{B}(\mathcal{X})$ with λ_0 is a fixed complex scalar.

In this paper, we continue our study of mappings that preserve invariant subspaces. We, therefore, propose to determine the forms of all surjective maps $\phi : \mathcal{B}(\mathcal{X}) \to \mathcal{B}(\mathcal{X})$(not necessarily additive or surjective) which preserve the algebraic spectral subspace of the product or Jordan triple product of operators associated with a fixed singleton $\{\lambda_0\}$ ($\lambda_0 \in \mathbb{C} \setminus \{0\}$).

The paper is organized as follows:
In the second section we give the basic properties of the algebraic spectral subspace and the lemmas necessary of the proofs of the main results.

In the third section, we describe maps ϕ on $\mathcal{B}(\mathcal{X})$, for which their ranges contain all operators of rank at most two, and that preserve the algebraic spectral subspace of product of operators associated with the singleton $\{\lambda_0\}$.

In the fourth section, we determine the structure of maps ϕ on $\mathcal{B}(\mathcal{X})$, for which their ranges contain all operators of rank at most four, and that preserve the algebraic spectral subspace of Jordan triple product of operators associated with the singleton $\{\lambda_0\}$. Note that certain ideas in our main results are inspired by [4].

2. Preliminaries

In this section, we state some useful lemmas needed for the proof of our main results. The first one summarizes some properties of the algebraic spectral subspace which will be used frequently.

Lemma 2.1. *Let $A, B \in \mathcal{B}(\mathcal{X})$ and λ_0 be a fixed nonzero complex scalar. The following statements hold:*

1. $E_{\alpha A}(\{\lambda\}) = E_A(\{\frac{\lambda}{\alpha}\})$, *for all $\alpha \in \mathbb{C} \setminus \{0\}$ and all $\lambda \in \mathbb{C}$.*
2. $E_I(\{1\}) = \mathcal{X}$.
3. *If $E_{AR}(\{\lambda_0\}) = E_{BR}(\{\lambda_0\})$, for all $R \in \mathcal{F}_1(\mathcal{X}) \setminus \mathcal{N}_1(\mathcal{X})$, then $E_{AR}(\{\lambda\}) = E_{BR}(\{\lambda\})$, for all $R \in \mathcal{F}_1(\mathcal{X}) \setminus \mathcal{N}_1(\mathcal{X})$ and all $\lambda \in \mathbb{C} \setminus \{0\}$.*
4. *If $E_{RAR}(\{\lambda_0\}) = E_{RBR}(\{\lambda_0\})$, for all $R \in \mathcal{F}_1(\mathcal{X}) \setminus \mathcal{N}_1(\mathcal{X})$, then $E_{RAR}(\{\lambda\}) = E_{RBR}(\{\lambda\})$, for all $R \in \mathcal{F}_1(\mathcal{X}) \setminus \mathcal{N}_1(\mathcal{X})$ and all $\lambda \in \mathbb{C} \setminus \{0\}$.*

Proof. 1. Let $\alpha \in \mathbb{C} \setminus \{0\}$ and $\lambda \in \mathbb{C}$. For $\mu \in \mathbb{C}$ such that $\mu \neq \frac{\lambda}{\alpha}$, we have $\alpha\mu \neq \lambda$ and

$$(A - \mu I)E_{\alpha A}(\{\lambda\}) = (\alpha A - \alpha\mu I)E_{\alpha A}(\{\lambda\}) = E_{\alpha A}(\{\lambda\}).$$

The fact that $E_A(\{\frac{\lambda}{\alpha}\})$ is the largest subspace M of X for which $(A-\mu I)M = M$ for every $\mu \neq \frac{\lambda}{\alpha}$, implies that

$$E_{\alpha A}(\{\lambda\}) \subset E_A(\{\frac{\lambda}{\alpha}\}).$$

Conversely, let $\mu \in \mathbb{C}$ such that $\mu \neq \lambda$, we have $\frac{\mu}{\alpha} \neq \frac{\lambda}{\alpha}$ and

$$(\alpha A - \mu I)E_A(\{\frac{\lambda}{\alpha}\}) = \alpha(A - \frac{\mu}{\alpha}I)E_A(\{\frac{\lambda}{\alpha}\}) = \alpha E_A(\{\frac{\lambda}{\alpha}\}) = E_A(\{\frac{\lambda}{\alpha}\}).$$

Therefore,

$$E_A(\{\frac{\lambda}{\alpha}\}) \subset E_{\alpha A}(\{\lambda\}).$$

Thus,

$$E_{\alpha A}(\{\lambda\}) = E_A(\{\frac{\lambda}{\alpha}\}).$$

2. We know that $I - \mu I$ is invertible for all $\mu \neq 1$, so $(I - \mu I)\mathcal{X} = \mathcal{X}$, for all $\mu \neq 1$.

Since $E_I(\{1\})$ is the largest subspace M such that $(I - \mu I)M = M$, for all $\mu \neq 1$, it follows that $E_I(\{1\}) = \mathcal{X}$.

3. Let $R \in \mathcal{F}_1(\mathcal{X}) \setminus \mathcal{N}_1(\mathcal{X})$ and $\lambda \in \mathbb{C} \setminus \{0\}$. Using (1), we get

$$\begin{aligned} E_{AR}(\{\lambda\}) &= E_{A(\frac{\lambda_0}{\lambda}R)}(\{\lambda_0\}) \\ &= E_{B(\frac{\lambda_0}{\lambda}R)}(\{\lambda_0\}) \\ &= E_{BR}(\{\lambda\}). \end{aligned}$$

4. Fix $R \in \mathcal{F}_1(\mathcal{X}) \setminus \mathcal{N}_1(\mathcal{X})$, $\lambda \in \mathbb{C}^*$ and μ_0 such that $\mu_0^2 = \frac{\lambda_0}{\lambda}$. Using (1) once again, we have

$$\begin{aligned} E_{RAR}(\{\lambda\}) &= E_{\frac{\lambda_0}{\lambda}RAR}(\{\lambda_0\}) \\ &= E_{(\mu_0 R)A(\mu_0 R)}(\{\lambda_0\}) \\ &= E_{(\mu_0 R)B(\mu_0 R)}(\{\lambda_0\}) \\ &= E_{\frac{\lambda_0}{\lambda}RBR}(\{\lambda_0\}) \\ &= E_{RBR}(\{\lambda\}). \quad \square \end{aligned}$$

The next lemma gives an explicit formula for the algebraic spectral subspace of the operators that has at most rank one associated with a nonzero fixed complex scalar.

Lemma 2.2. *For $x \in \mathcal{X} \setminus \{0\}$, $f \in \mathcal{X}^*$ and $\lambda_0 \in \mathbb{C} \setminus \{0\}$, we have*

$$E_{x\otimes f}(\{\lambda_0\}) = \begin{cases} span\{x\}, & if \;\; f(x) = \lambda_0 \\ \{0\}, & if \;\; f(x) \neq \lambda_0. \end{cases}$$

Proof. Let λ_0 be a fixed nonzero complex scalar, $x \in \mathcal{X} \setminus \{0\}$ and $f \in \mathcal{X}^*$. Since $(x \otimes f)E_{x\otimes f}(\{\lambda_0\}) = E_{x\otimes f}(\{\lambda_0\})$, then

$$E_{x\otimes f}(\{\lambda_0\}) \subset \mathcal{C}(x \otimes f).$$

The description of the algebraic core in terms of sequences shows that $\mathcal{C}(x\otimes f) \subset span\{x\}$. Therefore,

$$E_{x\otimes f}(\{\lambda_0\}) \subset span\{x\}. \tag{3}$$

Now, we shall discuss two cases.

Case 1. $f(x) = \lambda_0$. Let us show first that $(x \otimes f - \mu I)span\{x\} = span\{x\}$, for all $\mu \neq \lambda_0$. Indeed, let $\mu \in \mathbb{C}$ such that $\mu \neq \lambda_0 = f(x)$. We have $(x \otimes f - \mu I)((f(x) - \mu)^{-1}x) = x$. Then, $x \in (x \otimes f - \mu I)span\{x\}$, which proves that $span\{x\} \subset (x \otimes f - \mu I)span\{x\}$. The reverse implication deserves to be added. Indeed, let $y \in span\{x\}$, then $y = \lambda x$ for some $\lambda \in \mathbb{C} \setminus \{0\}$. Hence, we have

$$(x \otimes f - \mu I)(y) = (x \otimes f - \mu I)(\lambda x) = \lambda(\lambda_0 - \mu)x \in span\{x\}.$$

Therefore, for all $\mu \in \mathbb{C} \setminus \{\lambda_0\}$, it follows that

$$(x \otimes f - \mu I)span\{x\} = span\{x\} \;.$$

Since $E_{x\otimes}(\{\lambda_0\})$ is the largest subspace M of $\mathcal{X}$ such that $(x \otimes f - \mu I)M = M$, for all $\mu \neq \lambda_0$, we conclude that

$$span\{x\} \subset E_{x\otimes f}(\{\lambda_0\}).$$

Together with the Eq.(3), we obtain:

$$E_{x\otimes f}(\{\lambda_0\}) = span\{x\}.$$

Case 2. $f(x) \neq \lambda_0$. Suppose that $E_{x\otimes f}(\{\lambda_0\}) = span\{x\}$. Then, for every $\mu \neq \lambda_0$, we have

$$(x \otimes f - \mu I)span\{x\} = span\{x\}.$$

In particular, for $\mu = f(x)$, we obtain

$$span\{x\} = (x \otimes f - f(x)I)span\{x\} = \{0\}$$

which gives a contradiction . By (3), we get that $E_{x\otimes f}(\{\lambda_0\}) = \{0\}$. □

In the following lemma, we give an identity principle that we will use to prove our main results.

Lemma 2.3. *Let $T, S \in \mathcal{B}(\mathcal{X})$, and λ_0 be a fixed nonzero scalar in $\mathbb{C}$. The following statements are equivalent:*

1. $T = S$
2. $E_{TR}(\{\lambda\})) = E_{SR}(\{\lambda\})$, *for all* $R \in \mathcal{F}_1(\mathcal{X}) \setminus \mathcal{N}_1(\mathcal{X})$ *and all* $\lambda \in \mathbb{C} \setminus \{0\}$.
3. $E_{TR}(\{\lambda_0\})) = E_{SR}(\{\lambda_0\})$, *for all* $R \in \mathcal{F}_1(\mathcal{X}) \setminus \mathcal{N}_1(\mathcal{X})$.
4. $E_{RTR}(\{\lambda\}) = E_{RSR}(\{\lambda\})$, *for all* $R \in \mathcal{F}_1(\mathcal{X}) \setminus \mathcal{N}_1(\mathcal{X})$ *and all* $\lambda \in \mathbb{C} \setminus \{0\}$.
5. $E_{RTR}(\{\lambda_0\}) = E_{RSR}(\{\lambda_0\})$, *for all* $R \in \mathcal{F}_1(\mathcal{X}) \setminus \mathcal{N}_1(\mathcal{X})$.

Proof. Obviously, by Lemma 2.1, we have (1) $\Rightarrow$ (2) $\iff$ (3) and (1) $\Rightarrow$ (4) $\iff$ (5). Thus, it remains to show that the implications (2) $\Rightarrow$ (1) and (4) $\Rightarrow$ (1) are true.

(2) $\Rightarrow$ (1). Suppose that (2) holds, and let x be a nonzero vector in $\mathcal{X}$.

Let $f \in \mathcal{X}^*$ such that $f(x) \neq 0$. For $R = x \otimes f$, we have $TR = Tx \otimes f$ and $SR = Sx \otimes f$. Suppose first that $f(Tx) \neq 0$. By Lemma 2.2, we get that

$$span\{Tx\} = E_{TR}(\{f(Tx)\}) = E_{SR}(\{f(Tx)\}).$$

This implies that $f(Tx) = f(Sx)$.

If $f(Tx) = 0$. Suppose, on the contrary, that $f(Sx) \neq 0$. Using Lemma 2.2 once again, we obtain

$$span\{Sx\} = E_{SR}(\{f(Sx)\}) = E_{TR}(\{f(Sx)\}) = \{0\}.$$

This contradiction entails that $f(Tx) = f(Sx) = 0$. Therefore, $f(Tx) = f(Sx)$, for all $f \in \mathcal{X}^*$ for which $f(x) \neq 0$.

Now, if $f(x) = 0$, one can find a functional $g \in \mathcal{X}^*$ such that $g(x) \neq 0$. By applying what has been shown above to g and $f + g$, we get that $(f + g)(Tx) = (f + g)(Sx)$ and $g(Tx) = g(Sx)$, which proves that $f(Tx) = f(Sx)$, for all $f \in \mathcal{X}^*$. Thus, by Hahn-Banach theorem we have $T = S$.

(4) $\Rightarrow$ (1) Is done by the same reasoning. □

The following lemma gives a characterization of rank-one operators in term of the algebraic spectral subspace.

Lemma 2.4. *Let λ_0 be a nonzero fixed scalar of $\mathbb{C}$, and R be a nonzero operator of $\mathcal{B}(\mathcal{X})$. The following statements are equivalent:*

1. *R is a rank-one operator.*
2. $dim E_{TR}(\{\lambda_0\}) \leq 1$, *for all* $T \in \mathcal{B}(\mathcal{X})$ *of rank at most two.*
3. $dim E_{TRT}(\{\lambda_0\}) \leq 1$, *for all* $T \in \mathcal{B}(\mathcal{X})$ *of rank at most four.*

Proof. It is easy to see that 1) $\Rightarrow$ 2) and 1) $\Rightarrow$ 3) are evident. Thus, it remains to show that 2) $\Rightarrow$ 1) and 3) $\Rightarrow$ 1) hold.

Suppose that, R has rank at least two, and let y_1 and y_2 be two linearly independent vectors in the range of R. Let x_1 and x_2 be two vectors of $\mathcal{X}$ such that $Rx_1 = y_1$ and $Rx_2 = y_2$, and note that x_1 and x_2 are linearly independent too. We show that $\dim E_{TR}(\{\lambda_0\}) \geq 2$ for some $T \in \mathcal{F}_2(X)$. Let $f_i \in X^*(i = 1, 2)$ such that $f_i(y_j) = \delta_{ij}$ (δ_{ij} is a Kronecker delta) for $i, j = 1, 2$. Set $T = \lambda_0 x_1 \otimes f_1 + \lambda_0 x_2 \otimes f_2$ and note that $TRx_1 = \lambda_0 x_1$ and $TRx_2 = \lambda_0 x_2$. Which implies that $x_1, x_2 \in \mathcal{N}(TR - \lambda_0 I)$. By (2), we conclude that $span\{x_1, x_2\} \subset \mathcal{N}(TR - \lambda_0 I) \subset E_{TR}(\{\lambda_0\})$. Thus,

$$\dim E_{TR}(\{\lambda_0\}) \geq 2.$$

Consequently, 2) $\Rightarrow$ 1) is established.

Now, we show that $\dim E_{TRT}(\{\lambda_0\}) \geq 2$ for some $T \in \mathcal{F}_4(\mathcal{X})$. Since $\mathcal{X}$ is an infinite dimension space, one can choose y_3 and y_4 in $\mathcal{X}$ such that y_1, y_2, y_3 and y_4 are linearly independent. Let $f_{i,j} \in \mathcal{X}^*(i, j = 1, 2, 3, 4)$ such that $f_i(y_j) = \delta_{i,j}$ for $i, j = 1, 2, 3, 4$. Consider $T \in \mathcal{B}(\mathcal{X})$ such that

$$T = x_1 \otimes f_3 + x_2 \otimes f_4 + \lambda_0 y_3 \otimes f_1 + \lambda_0 y_4 \otimes f_2.$$

We have $TRTy_3 = \lambda_0 y_3$ and $TRTy_4 = \lambda_0 y_4$. This gives $y_3, y_4 \in \mathcal{N}(TRT - \lambda_0 I)$. The Eq.(2), once more, ensures that $span\{y_3, y_3\} \subset E_{TRT}(\{\lambda_0\})$. Thus, $\dim E_{TRT}(\{\lambda_0\}) \geq 2$, which end the proof. □

We close this section with the following lemma, which we will use for the proof of our main theorems.

Lemma 2.5. *Let T and S be two non-scalar operators in $\mathcal{B}(\mathcal{X})$:*

1. *If $TP \in \mathcal{P}(\mathcal{X}) \setminus \{0\}$ implies $SP \in \mathcal{P}(\mathcal{X}) \setminus \{0\}$, for all $P \in \mathcal{P}_1(\mathcal{X})$, then $S = \lambda I + (1 - \lambda I)T$ for some $\lambda \in \mathbb{C} \setminus \{1\}$.*
2. *If $PTP \in \mathcal{P}(\mathcal{X}) \setminus \{0\}$ implies $PSP \in \mathcal{P}(\mathcal{X}) \setminus \{0\}$, for all $P \in \mathcal{P}_1(\mathcal{X})$, then $S = \lambda I + (1 - \lambda I)T$ for some $\lambda \in \mathbb{C} \setminus \{1\}$. Where $\mathcal{P}(\mathcal{X})$ denotes the set of idempotent operators on $\mathcal{X}$.*

Proof. See [6, Proposition 2.3] and [12, Proposition 3.3]. □

3. Maps preserving the algebraic spectral subspace of product of operators

The following theorem is our main result in this section.

Theorem 3.1. *Let λ_0 be a fixed nonzero scalar in $\mathbb{C}$, and $\phi : \mathcal{B}(\mathcal{X}) \longrightarrow \mathcal{B}(\mathcal{X})$ be a map such that its range contains all operators of rank at most two. Then, ϕ satisfies*

$$E_{TS}(\{\lambda_0\}) = E_{\phi(T)\phi(S)}(\{\lambda_0\}), \qquad (T, S \in \mathcal{B}(\mathcal{X})), \tag{4}$$

if and only if there exists $\alpha \in \mathbb{C}$, with $\alpha^2 = 1$, such that $\phi(T) = \alpha T$, for all $T \in \mathcal{B}(\mathcal{X})$.

Proof. The "if" part is easily verified by using Lemma 2.1, so we only need to prove the " only if " part. Indeed, assume that ϕ is a map from $\mathcal{B}(\mathcal{X})$ into itself such that its range contains all operators of rank at most two and satisfies the Eq.(4). We fix a nonzero scalar z_0 in $\mathbb{C}$ such that $z_0^2 = \lambda_0$. We divide the proof into different propositions.

Proposition 3.1. *ϕ is injective and $\phi(0) = 0$.*

Proof. Let $T, S \in \mathcal{B}(\mathcal{X})$ such that $\phi(T) = \phi(S)$. For any $R \in \mathcal{F}_1(\mathcal{X}) \setminus \mathcal{N}_1(\mathcal{X})$, we have

$$\begin{aligned} E_{TR}(\{\lambda_0\}) &= E_{\phi(T)\phi(R)}(\{\lambda_0\}) \\ &= E_{\phi(S)\phi(R)}(\{\lambda_0\}) \\ &= E_{SR}(\{\lambda_0\}). \end{aligned}$$

By Lemma 2.3 we conclude that $T = S$, and ϕ is injective.

Now, let us show that $\phi(0) = 0$. Indeed, for any $T \in \mathcal{B}(\mathcal{X})$, we have $\{0\} = E_{0T}(\{\lambda_0\}) = E_{\phi(0)\phi(T)}(\{\lambda_0\})$ and $\{0\} = E_{0\phi(T)}(\{\lambda_0\})$. Then,

$$E_{\phi(0)\phi(T)}(\{\lambda_0\}) = E_{0\phi(T)}(\{\lambda_0\}),$$

for all $T \in \mathcal{B}(\mathcal{X})$.

It follows from the fact that $\mathcal{F}_2(\mathcal{X}) \subset \phi(\mathcal{B}(\mathcal{X}))$ that $E_{0R}(\{\lambda_0\}) = E_{\phi(0)R}(\{\lambda_0\})$, for all $R \in \mathcal{F}_1(\mathcal{X}) \setminus \mathcal{N}_1(\mathcal{X})$. Lemma 2.3 implies that $\phi(0) = 0$, as desired. □

Proposition 3.2. *ϕ preserves rank-one operators and $\phi(R) = \alpha_R R$ For every $R \in \mathcal{F}_{1,z_0}(\mathcal{X})$, where $\alpha_R \in \mathbb{C} \setminus \{0\}$.*

Proof. First, let $R = x \otimes f$ be a rank-one operator, where $x \in \mathcal{X}$ and $f \in \mathcal{X}^*$. By Proposition 3.1, we have $\phi(R) \neq 0$. For every $T \in \mathcal{B}(\mathcal{X})$, we see that

$$\dim E_{TR}(\{\lambda_0\}) = \dim E_{\phi(T)\phi(R)}(\{\lambda_0\}) \leq 1.$$

The fact that $\mathcal{F}_2(\mathcal{X}) \subset \phi(\mathcal{B}(\mathcal{X}))$ implies that $\dim E_{S\phi(R)}(\{\lambda_0\}) \leq 1$, for all $S \in \mathcal{F}_2(\mathcal{X})$. Therefore, by Lemma 2.4, $\phi(R)$ is an operator of rank one.

Next, let $R = x \otimes f$ be a rank-one operator in $\mathcal{F}_{1,z_0}(\mathcal{X})$ $(x \in \mathcal{X}, f \in \mathcal{X}^*)$. Thus, $\phi(R)$ is a rank-one operator, say $\phi(x \otimes f) = y \otimes g$, where $y \in \mathcal{X}$ and $g \in \mathcal{X}^*$. By hypothesis and Lemma 2.2, we have

$$span\{x\} = E_{f(x)x\otimes f}(\{\lambda_0\}) = E_{(x\otimes f)(x\otimes f)}(\{\lambda_0\}) = E_{g(y)y\otimes g}(\{\lambda_0\}).$$

This implies that $g(y)^2 = \lambda_0$ and $span\{x\} = span\{y\}$. Without loss of generality, we may and shall assume that $x = y$, and therefore $\phi(x \otimes f) = x \otimes g$.

Now, let us prove that f and g are linearly dependent. Indeed, assume, by the way of contradiction, that f and g are linearly independent, and let z be a nonzero vector in $\mathcal{X}$ such that $f(z) = z_0$ and $g(z) = 0$. From what was shown above, there exists $g_{z,f}$ a linear functional on $\mathcal{X}$ such that $\phi(z \otimes f) = z \otimes g_{z,f}$. Note that, $(x \otimes f)(z \otimes f) = f(z)x \otimes f$ and $(x \otimes g)(z \otimes g_{z,f}) = 0$. By hypothesis and Lemma 2.2, we obtain

$$span\{x\} = E_{(x\otimes f)(z\otimes f)}(\{\lambda_0\}) = E_{(x\otimes g)(z\otimes g_{z,f})}(\{\lambda_0\}) = \{0\}.$$

This contradiction shows that there is a nonzero scalar $\alpha_R \in \mathbb{C}$, such that $\phi(R) = \alpha_R R$. □

Proposition 3.3. $\phi((z_0 I)) = \alpha(z_0 I)$, *where* α *be a nonzero scalar of* $\mathbb{C}$ *such that* $\alpha^2 = 1$.

Proof. Assuming that $\phi(z_0 I)$ and $z_0 I$ are linearly independent implies that there exists a nonzero vector x such that $\phi(z_0 I)x$ and $z_0 x$ are linearly independent. Let $f \in \mathcal{X}^*$ such that $f(\phi(z_0 I)x) = 0$ and $f(z_0 x) = z_0^2 = \lambda_0$. For $R = x \otimes f \in \mathcal{F}_{1,z_0}(\mathcal{X})$, we have from Proposition 3.2 $\phi(R) = \alpha_R R$, where $\alpha_R \in \mathbb{C} \setminus \{0\}$. By hypothesis and Lemma 2.2, we arrive at

$$span\{x\} = E_{(z_0 I)R}\{\lambda_0\} = E_{\alpha_R \phi(z_0 I)R}(\{\lambda_0\}) = E_{\alpha_R \phi(z_0 I)x\otimes f}(\{\lambda_0\}) = \{0\}.$$

This contradiction tells us that $\phi((z_0 I)) = \alpha(z_0 I)$, where $\alpha \in \mathbb{C} \setminus \{0\}$.

On the other hand, we have

$$\begin{aligned} \mathcal{X} &= E_I(\{1\}) \\ &= E_{(z_0 I)(z_0 I)}(\{\lambda_0\}) \\ &= E_{\alpha^2 \lambda_0 I}(\{\lambda_0\}) \\ &= E_{\alpha^2 I}(\{1\}). \end{aligned}$$

Then, by the Eq.(1), $(\alpha^2 I - \mu I)\mathcal{X} = \mathcal{X}$, for all $\mu \in \mathbb{C} \setminus \{1\}$, which forces that $\alpha^2 = 1$. □

Proof of Theorem 3.1. First, let us prove that $\phi(R) = \alpha R$, for all $R \in \mathcal{F}_{1,z_0}(\mathcal{X})$. Let R be a rank- one operator in $\mathcal{F}_{1,z_0}(\mathcal{X})$, say $R = x \otimes f$, where $x \in \mathcal{X}$ and $f \in \mathcal{X}^*$ such that $f(x) = z_0$. Note that, by Proposition 3.2, we have $\phi(R) = \alpha_R R$. Since

$$\begin{aligned} span\{x\} &= E_{(z_0 I)(x\otimes f)}(\{\lambda_0\}) \\ &= E_{(z_0 I)R}(\{\lambda_0\}) \\ &= E_{\phi(z_0 I)\phi(R)}(\{\lambda_0\}) \\ &= E_{\alpha\alpha_R(z_0 I)R}(\{\lambda_0\}). \end{aligned}$$

Then, $f(\alpha\alpha_R z_0 x) = \lambda_0$. Thus, $\alpha\alpha_R = 1$, which implies that α_R does not depend on the operator R and we may write α instead of α_R. Therefore,

$$\phi(R) = \alpha R.$$

Now, let us prove that ϕ takes the desired form. Note that, for every $P \in \mathcal{P}_1(\mathcal{X})(z_0 P \in \mathcal{F}_{1,z_0}(\mathcal{X}))$. Then, $\phi(z_0 P) = \alpha(z_0 P)$.

On the other hand, let $R \in \mathcal{F}_1(\mathcal{X}) \setminus \mathcal{N}_1(\mathcal{X})$. For every $P \in \mathcal{P}_1(\mathcal{X})$, we have

$$\begin{aligned} E_{RP}(\{1\}) &= E_{(z_0 R)(z_0 P}(\{\lambda_0\}) \\ &= E_{\phi(z_0 R)\phi(z_0 P)}(\{\lambda_0\}) \\ &= E_{\alpha z_0 \phi(z_0 R) P}(\{\lambda_0\}) \\ &= E_{\frac{\alpha}{z_0}\phi(z_0 R) P}(\{1\}). \end{aligned}$$

It follows by Lemma 2.2 that

$$RP \in \mathcal{P}(\mathcal{X}) \setminus \{0\} \Rightarrow \frac{1}{\alpha z_0}\phi(z_0 R)P \in \mathcal{P}(\mathcal{X}) \setminus \{0\},$$

for all $P \in \mathcal{P}_1(\mathcal{X})$. By Proposition 3.2 once again, $\phi(z_0 R)$ is a rank-one operator, then R and $\frac{1}{\alpha z_0}\phi(z_0 R)$ are non-scalar operators. Lemma 2.5 implies that

$$\frac{1}{\alpha z_0}\phi(z_0 R) = \lambda_R I + (1 - \lambda_R)R,$$

for some $\lambda_R \in \mathbb{C} \setminus \{1\}$. Since $\phi(z_0 R)$ has rank one, then $\lambda_R I$ has rank at most two, which implies that $\lambda_R = 0$. Thus, $\phi(z_0 R) = \alpha z_0 R$. Consequently,

$$\phi(R) = \alpha R, \quad \text{for all} \quad R \in \mathcal{F}_1(X) \setminus \mathcal{N}_1(\mathcal{X}).$$

Finally, let $T \in \mathcal{B}(\mathcal{X})$. For every $R \in \mathcal{F}_1(\mathcal{X}) \setminus \mathcal{N}_1(\mathcal{X})$, we have

$$E_{TR}(\{\lambda_0\}) = E_{\alpha\phi(T)R}(\{\lambda_0\}).$$

Lemma 2.3 implies that $\alpha\phi(T) = T$. Therefore, $\phi(T) = \alpha T$, for all $T \in \mathcal{B}(\mathcal{X})$. The proof is complete. □

4. Maps preserving the algebraic spectral subspace of Jordan triple product of operators

As in the previous section, we determine, using the same approach, the form of a map ϕ on $\mathcal{B}(\mathcal{X})$ that preserves the algebraic spectral subspace of Jordan triple product of operators. Our main result in this section is the following theorem.

Theorem 4.1. *Let λ_0 be a fixed nonzero scalar of $\mathbb{C}$, and $\phi : \mathcal{B}(\mathcal{X}) \longrightarrow \mathcal{B}(\mathcal{X})$ be a map such that its range contains all operators of rank at most four. Then, ϕ satisfies*

$$E_{TST}(\{\lambda_0\}) = E_{\phi(T)\phi(S)\phi(T)}(\{\lambda_0\}), \qquad (T, S \in \mathcal{B}(\mathcal{X})), \tag{5}$$

if and only if there exists $\alpha \in \mathbb{C}$, with $\alpha^3 = 1$, such that $\phi(T) = \alpha T$, for all $T \in \mathcal{B}(\mathcal{X})$.

We only need to show that the "only if" part holds. Let ϕ be a map such that its range contains all operator of rank at most four and satisfies (5). Just as in the proof of Theorem 3.1, we fix a nonzero scalar $\mu_0 \in \mathbb{C}$ such that $\mu_0^3 = \lambda_0$. We break the proof into several propositions.

Proposition 4.1. *$\phi(0) = 0$ and ϕ is injective.*

Proof. For every $T \in \mathcal{B}(\mathcal{X})$, we have

$$\{0\} = E_{T0T}(\{\lambda_0\}) = E_{\phi(T)\phi(0)\phi(T)}(\{\lambda_0\}) = E_{\phi(T)0\phi(T)}(\{\lambda_0\}).$$

It follows from the assumption on range of ϕ that

$$E_{R0R}(\{\lambda_0\}) = E_{R\phi(0)R}(\{\lambda_0\}), \text{ for all } R \in \mathcal{F}_1(\mathcal{X}) \setminus \mathcal{N}_1(\mathcal{X}).$$

Thus, by Lemma 2.3, $\phi(0) = 0$.

Now, let $A, B \in \mathcal{B}(\mathcal{X})$ such that $\phi(A) = \phi(B)$. For every $T \in \mathcal{F}_1(\mathcal{X}) \setminus \mathcal{N}_1(\mathcal{X})$, we have

$$\begin{aligned} E_{TAT}(\{\lambda_0\}) &= E_{\phi(T)\phi(A)\phi(T)}(\lambda_0) \\ &= E_{\phi(T)\phi(B)\phi(T)}(\{\lambda_0\}) \\ &= E_{TBT}(\{\lambda_0\}). \end{aligned}$$

It follows, from Lemma 2.3, that $A = B$, thus ϕ is injective, as desired. □

Proposition 4.2. *For every $R \in \mathcal{F}_{1,\mu_0}(\mathcal{X})$, there exists a nonzero scalar $\alpha_R \in \mathbb{C}$ such that $\phi(R) = \alpha_R R$.*

Proof. Let $R = x \otimes f$ be a rank-one operator, where $x \in \mathcal{X}$ and $f \in \mathcal{X}^*$ such that $f(x) = \mu_0$. Note that, $\phi(R) \neq 0$, since Proposition 4.1. Firstly, for every $T \in \mathcal{B}(\mathcal{X})$, we have

$$\dim E_{TRT}(\{\lambda_0\}) = \dim E_{S\phi(R)S}(\{\lambda_0\}) = \dim E_{\phi(T)\phi(R)\phi(T)}(\{\lambda_0\}) \leq 1.$$

Since $\mathcal{F}_4(\mathcal{X}) \subset \phi(\mathcal{B}(\mathcal{X}))$, then $\dim E_{S\phi(R)S}(\{\lambda_0\}) \leq 1$, for all $S \in \mathcal{F}_4(\mathcal{X})$. Therefore, by Lemma 2.4, $\phi(R)$ is a rank-one operator, say $\phi(R) = y \otimes g$, where $y \in \mathcal{X}$ and $g \in \mathcal{X}^*$.

Now, we show that there exists an $\alpha_{x,f} \in \mathbb{C}$ such that $\phi(x \otimes f) = \alpha_{x,f}(x \otimes f)$. Indeed, by hypothesis and Lemma 2.1, we have

$$span\{x\} = E_{R^3}(\{\lambda_0\}) = E_{\phi(R)^3} = E_{g(y)^2 y \otimes g}(\{\lambda_0\}).$$

This implies that $g(y) \neq 0$ and $span\{y\} = span\{x\}$. Assume without loss of generality that $x = y$. Thus, it remains to show that f and g are linearly dependent. Suppose, for the sake of contradiction, that f and g are linearly independent and let $z \in \mathcal{X}$ such that $f(z) = \mu_0$ and $g(z) = 0$. Then, there exists an $g_{z,f} \in \mathcal{X}^*$ such that $\phi(z \otimes f) = z \otimes g_{z,f}$. Note that, $(x \otimes f)(z \otimes f)(x \otimes f) = \mu_0^2 x \otimes f$ and $(y \otimes g)(z \otimes g_{z,f})(y \otimes g) = 0$. By hypothesis, we have

$$\begin{aligned} span\{x\} &= E_{(x\otimes f)(z\otimes f)(x\otimes f)}(\{\lambda_0\}) \\ &= E_{(y\otimes g)(z\otimes g_{z,f})(y\otimes g)}(\{\lambda_0\}) \\ &= \{0\}. \end{aligned}$$

This contradiction shows that f and g are linearly dependent, then $\phi(x \otimes f) = \alpha_{x,f} x \otimes f$, where $\alpha_{x,f}$ is a scalar in $\mathbb{C}$, as desired. □

Proposition 4.3. *$\phi(\mu_0 I) = \alpha(\mu_0 I)$, where $\alpha \in \mathbb{C}$ is such that $\alpha^3 = 1$.*

Proof. Suppose, by the way of contradiction, that there exists a non zero vector $x \in \mathcal{X}$ such that $\phi(\mu_0 I)x$ and $(\mu_0 I)x$ are linearly independent. Let $f \in \mathcal{X}^*$ such that $f(\phi(\mu_0 I)x) = 0$ and $f((\mu_0 I)x) = \mu_0^2$. For $R = x \otimes f$, it is easy to see that $R \in \mathcal{F}_{1,\mu_0}(\mathcal{X})$. By Proposition 4.2, we have $\phi(R) = \alpha_R R$ where $\alpha_R \in \mathbb{C} \setminus \{0\}$. Then, from Lemma 2.2, we have

$$\begin{aligned} span\{x\} &= E_{R(\mu_0 I)R}(\{\lambda_0\}) \\ &= E_{\alpha_R^2 R\phi(\mu_0 I)R}(\{\lambda_0\}) \\ &= \{0\}. \end{aligned}$$

This gives a contradiction. Thus, $\phi(\mu_0 I) = \alpha(\mu_0 I)$ where $\alpha \in \mathbb{C} \setminus \{0\}$.

On the other hand, since

$$\begin{aligned} \mathcal{X} &= E_I(\{1\}) \\ &= E_{(\mu_0 I)(\mu_0 I)(\mu_0 I)}(\{\lambda_0\}) \\ &= E_{\alpha^3 \mu_0^3 I}(\{\lambda_0\}) \\ &= E_{\alpha^3 I}(\{1\}). \end{aligned}$$

By using (1), we conclude that $(\alpha^3 I - \mu I)\mathcal{X} = \mathcal{X}$, for all $\mu \in \mathbb{C} \setminus \{1\}$, which implies that $\alpha^3 = 1$. □

Proof of Theorem 4.1. First, let us show that $\phi(R) = \alpha R$, for all $R \in \mathcal{F}_{1,\mu_0}(\mathcal{X})$. Let $R = x \otimes f \in \mathcal{F}_{1,\mu_0}(\mathcal{X})$, where $x \in \mathcal{X}$ and $f \in \mathcal{X}^*$. Note that, $\phi(R) = \alpha_R R$ with $\alpha_R \in \mathbb{C} \setminus \{0\}$. Using Proposition 4.2, we obtain

$$span\{x\} = E_{(\mu_0 I)R(\mu_0 I)}(\{\lambda_0\}) = E_{\alpha^2 \alpha_R (\mu_0 I)R(\mu_0 I)}(\{\lambda_0\}).$$

Lemma 2.2 implies that $f(\alpha^2 \alpha_R \mu_0^2 x) = \lambda_0$. Thus, $\alpha^2 \alpha_R = 1$, which implies that α_R does not depend on the operator R, and we my write α instead of α_R. Therefore,

$$\phi(R) = \alpha R.$$

Next, we show that $\phi(R) = \alpha R$, for all $R \in \mathcal{F}_1(\mathcal{X}) \setminus \mathcal{N}_1(\mathcal{X})$. Note that, for every $P \in \mathcal{P}_1(\mathcal{X})$ $(\mu_0 P \in \mathcal{F}_{1,\mu_0}(\mathcal{X}))$, we have

$$\phi(\mu_0 P) = \alpha(\mu_0 P).$$

Let $R \in \mathcal{F}_1(\mathcal{X}) \setminus \mathcal{N}_1(\mathcal{X})$ be a rank one non-nilpotent operator. For every $P \in \mathcal{P}_1(\mathcal{X})$, we have

$$\begin{aligned} E_{PRP}(\{1\}) &= E_{(\mu_0 P)(\mu_0 R)(\mu_0 P)}(\{\lambda_0\}) \\ &= E_{\alpha^2(\mu_0 P)\phi(\mu_0 R)(\mu_0 P)}(\{\lambda_0\}) \\ &= E_{P(\frac{1}{\alpha\mu_0}\phi(\mu_0 R))P}(\{1\}). \end{aligned}$$

It follows from Lemma 2.2 that

$$PRP \in \mathcal{P}(\mathcal{X}) \setminus \{0\} \Rightarrow P(\frac{1}{\alpha\mu_0}\phi(\mu_0 R))P \in \mathcal{P}(\mathcal{X}) \setminus \{0\},$$

for all $P \in \mathcal{P}_1(\mathcal{X})$. Since R is a rank-one operator(non-scalar operator), by what has been shown in Proposition 4.2, $\frac{1}{\alpha\mu_0}\phi(\mu_0 R)$ is also a rank-one operator. Therefore, from lemma 2.5, there exists an $\lambda_R \in \mathbb{C} \setminus \{1\}$ such that

$$\frac{1}{\alpha\mu_0}\phi(\mu_0 R) = \lambda_R I + (1 - \lambda_R)R.$$

Since $\phi(\mu_0 R)$ has rank one, it follows that $\lambda_R = 0$. This implies that

$$\phi(\mu_0 R) = \alpha\mu_0 R.$$

Therefore,

$$\phi(R) = \phi(\mu_0(\frac{1}{\mu_0}R)) = \alpha R.$$

Finally, let $T \in \mathcal{B}(\mathcal{X})$. For all $R \in \mathcal{F}_1(\mathcal{X}) \setminus \mathcal{N}_1(\mathcal{X})$, we have

$$\begin{aligned} E_{RTR}(\{\lambda_0\}) &= E_{\phi(R)\phi(T)\phi(R)}(\{\lambda_0\}) \\ &= E_{\alpha^2 R\phi(T)R}(\{\lambda_0\}) \\ &= E_{R(\frac{1}{\alpha}\phi(T))R}(\{\lambda_0\}). \end{aligned}$$

This implies, by Lemma 2.3, that $\phi(T) = \alpha T$, and the proof is complete.

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Understanding bipartite soft semigraph structures

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Abstract. Soft set theory functions as a flexible mathematical instrument designed to handle uncertain data by aiding in the categorization of universe elements according to predefined parameters. Unlike hypergraphs, semigraphs present a wider interpretation of conventional graphs, allowing for a finer representation of relationships. Through the integration of soft set principles, the notion of soft semigraphs arises, enhancing the adaptability and versatility of semigraphs in addressing uncertainty. This paper sets out to reveal different forms of bipartite soft semigraphs, meticulously examining their varied structures and delving into their inherent characteristics.

Keywords: soft graph, soft semigraph, bipartite soft semigraph.

MSC 2020: 05C99.

1. Introduction

Most conventional approaches in formal modelling, reasoning, and computation are characterized by determinism, clarity, and precision. However, complex challenges in fields like engineering, medicine, economics, and social sciences often involve uncertain data. Various uncertainties in these areas make the application of traditional methods challenging. This led to the emergence of soft set theory in 1999 by Molodtsov [18]. Soft set theory proves more practical than other established theories like probability or fuzzy set theory due to its

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versatility. Authors such as Maji, Biswas, and Roy [16], [17] have expanded on soft set theory, applying it to resolve decision-making problems.

The concept of soft graphs was introduced by Thumbakara and George[24]. George, Thumbakara and Jose [25], [26], [27] investigated some properties of soft graphs. In 2015, Akram and Nawas [1], [2] modified the definition of a soft graph. Further enhancing the field, Akram and Nawas [3], [4] introduced fuzzy soft graphs, strong fuzzy soft graphs, complete fuzzy soft graphs, and regular fuzzy soft graphs, delving into their properties and potential applications. Akram and Zafar [5], [6] pioneered the notions of soft trees and fuzzy soft trees.

Contributions to the study of soft graphs have been made by Thenge, Jain, and Reddy [21], [22], [23]. Owing to their utility in handling parameterization, soft graphs represent a burgeoning domain within graph theory. George, Thumbakara, and Jose introduced soft hypergraphs[7], soft directed graphs [14], [15], and soft directed hypergraphs [13] and studied their properties. The concept of semigraphs, which are an expanded version of graphs, was initially proposed by E. Sampathkumar [19], [20]. Unlike hypergraphs, semigraphs maintain a defined sequence of vertices within their edges. When depicted on a two-dimensional plane, semigraphs bear a visual similarity to traditional graphs. In their research, George, Jose, and Thumbakara [11] introduced soft semigraphs by applying soft set principles to semigraphs and defined some soft semigraph operations. Moreover, they introduced connectedness [10] and various degrees, graphs, and matrices linked to soft semigraphs [12]. This paper introduces different types of bipartite soft semigraphs and investigates their properties.

2. Preliminaries

2.1 Semigraph

The notion of semigraph was introduced by E. Sampathkumar [19], [20] as follows. "A *semigraph* G is a pair (V, X) where V is a nonempty set whose elements are called vertices of G, and X is a set of n-tuples, called edges of G, of distinct vertices, for various $n \geq 2$, satisfying the following conditions.

1. Any two edges have at most one vertex in common;
2. Two edges $(u_1, u_2, \ldots, u_n)$ and $(v_1, v_2, \ldots, v_m)$ are considered to be equal if and only if:
   (a) $m = n$;
   (b) either $u_i = v_i$ for $1 \leq i \leq n$, or $u_i = v_{n-i+1}$ for $1 \leq i \leq n$.

Let $G = (V, X)$ be a semigraph and $E = (v_1, v_2, \ldots, v_n)$ be an edge of G. Then, v_1 and v_n are the *end vertices* of E and $v_i, 2 \leq i \leq n-1$ are the *middle vertices(or m-vertices)* of E. If a vertex v of a semigraph G appears only as an end vertex then it is called an *end vertex*. If a vertex v is only a middle vertex then it is a *middle vertex* or *m-vertex* while a vertex v is called *middle-cum-end vertex or* (m, e)*-vertex* if it is a middle vertex of some edge and an end vertex of some other edge. A *subedge* of an edge $E = (v_1, v_2, \ldots, v_n)$ is a k-tuple $E' = (v_{i_1}, v_{i_2}, \ldots, v_{i_k})$, where $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ or $1 \leq i_k < i_{k-1} < \cdots < i_1 \leq n$. We say that the subedge E' is *induced* by the set of vertices $\{v_{i_1}, v_{i_2}, \ldots v_{i_k}\}$. A *partial edge* of $E = (v_1, v_2, \ldots, v_n)$ is a $(j - i + 1)$-tuple $E(v_i, v_j) = (v_i, v_{i+1}, \ldots, v_j)$, where $1 \leq i < j \leq n$. $G' = (V', X')$ is a *partial semigraph* of a semigraph G if the edges of G' are partial edges of G. Two vertices u and v in a semigraph G are said to be *adjacent* if they belong to the same edge. If u and v are adjacent and consecutive in order then they are said to be *consecutively adjacent*. u and v are said to be *e-adjacent* if they are the end vertices of an edge and *1e-adjacent* if both the vertices u and v belong to the same edge and at least one of them, is an end vertex of that edge".

2.2 Soft set

In 1999, D. Molodtsov [18] initiated the concept of soft sets. Let U be an initial universe set and let A be a set of parameters. A pair (F, A) is called a soft set (over U) if and only if F is a mapping of A into the set of all subsets of the set U. That is, $F : A \to \mathcal{P}(U)$.

2.3 Soft semigraph

B. George, R. K. Thumbakara and J. Jose [11], [12] introduced soft semigraph by applying the concept of soft set in semigraph as follows: "Let $G^* = (V, X)$ be a semigraph having vertex set V and edge set X. Consider a subset V_1 of V. Then, a partial edge formed by some or all vertices of V_1 is said to be a *maximum partial edge* or *mp edge* if it is not a partial edge of any other partial edge formed by some or all vertices of V_1.

Let X_p be the collection of all partial edges of the semigraph G and A be a nonempty set. Let a subset R of $A \times V$ be an arbitrary relation from A to V. We define a mapping Q from A to $\mathcal{P}(V)$ by $Q(x) = \{y \in V | xRy\}, \forall x \in A$, where $\mathcal{P}(V)$ denotes the power set of V. Then, the pair (Q, A) is a soft set over V. Also define a mapping W from A to $\mathcal{P}(X_p)$ by $W(x) = \{\text{mp edges} < Q(x) >\}$, where $\{\text{mp edges} < Q(x) >\}$ denotes the set of all mp edges that can be formed by some or all vertices of $Q(x)$ and $\mathcal{P}(X_p)$ denotes the power set of X_p. The pair (W, A) is a soft set over X_p. Then, we can define a soft semigraph as follows:

The 4-tuple $G = (G^*, Q, W, A)$ is called a *soft semigraph* of G^* if the following conditions are satisfied:

1. $G^* = (V, X)$ is a semigraph having vertex set V and edge set X;
2. A is the nonempty set of parameters;
3. (Q, A) is a soft set over V;
4. (W, A) is a soft set over X_p;

5. $H(a) = (Q(a), W(a))$ is a partial semigraph of $G^*, \forall a \in A$.

Let $G^* = (V, X)$ be a semigraph and $G = (G^*, Q, W, A)$ be a soft semigraph of G^* which is also given by $\{H(x) : x \in A\}$. Then, the partial semigraph $H(x)$ corresponding to any parameter x in A is called a *p-part* of the soft semigraph G. An edge present in a soft semigraph G of G^* is called an *f-edge*. It may be a partial edge of some edge in G^* or an edge in G^*. A partial edge of any f-edge of a soft semigraph G is called a *p-edge* of G. An f-edge is a p-edge of itself. An f-edge or a p-edge of a soft semigraph G is called an *fp-edge* of G."

3. Different types of bipartite soft semigraphs

In this section we extend the concepts of bipartite semigraphs given in [20] to soft semigraphs.

3.1 Bipartite soft semigraph

Let $G^* = (V, X)$ be a semigraph and $G = (G^*, Q, W, A)$ be a soft semigraph of G^* represented by $\{H(x) : x \in A\}$. Then, G is called a bipartite soft semigraph if each of its p-parts $H(x)$ is a bipartite partial semigraph of G^*. That is, $Q(x)$ can be partitioned into sets $\{Q_1(x), Q_2(x)\}$ such that both $Q_1(x)$ and $Q_2(x)$ are independent for all x in A. That is, no f-edge in $W(x)$ is an *mp* edge$< Q_1(x) >$ or an *mp* edge$< Q_2(x) >$ for all x in A. The term *mp* edge$< Q_i(x) >$ denotes a maximum partial edge that can be formed by some or all vertices of $Q_i(x)$.

Example 3.1. Let $G^* = (V, X)$ be a semigraph given in Figure 1, where $V = \{v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9\}$ and $X = \{(v_0, v_1, v_2), (v_2, v_3, v_4), (v_1, v_6, v_5), (v_3, v_9, v_8), (v_7, v_8)\}$.

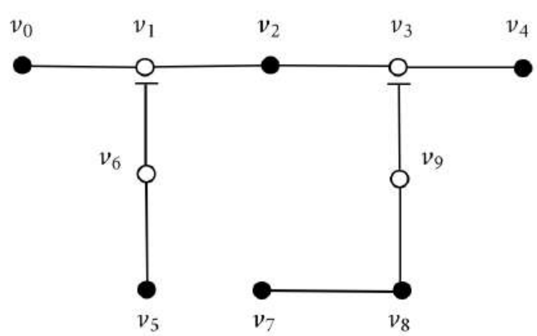

Figure 1: Semigraph $G^* = (V, X)$

Let $A = \{v_1, v_8\} \subseteq V$ be a parameter set. Define Q from A to $\mathcal{P}(V)$ by $Q(x) = \{y \in V | xRy \Leftrightarrow x = y$ or x and y are consecutively adjacent$\}, \forall x \in A$ and W from A to $\mathcal{P}(X_p)$ by $W(x) = \{mp\ edges\langle Q(x)\rangle\}, \forall x \in A$. That is, $Q(v_1) =$

$\{v_0, v_1, v_2, v_6\}$ and $Q(v_8) = \{v_7, v_8, v_9\}$. Also, $W(v_1) = \{(v_1, v_6), (v_0, v_1, v_2)\}$ and $W(v_8) = \{(v_7, v_8), (v_8, v_9)\}$. Then, $H(v_1) = (Q(v_1), W(v_1))$ and $H(v_8) = (Q(v_8), W(v_8))$ are partial semigraphs of G^* as shown below in Figure 2. Hence, $G = \{H(v_1), H(v_8)\}$ is a soft semigraph of G^*. Here, $Q(v_1) = \{v_0, v_1, v_2, v_6\}$

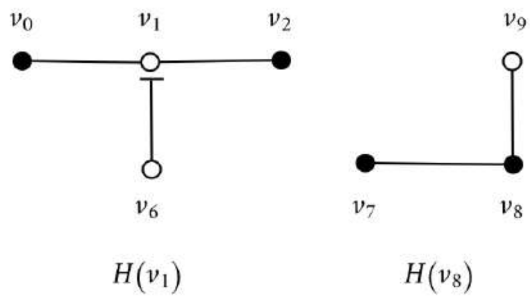

Figure 2: Soft Semigraph $G = \{H(v_1), H(v_8)\}$

can be partitioned into sets $\{Q_1(v_1), Q_2(v_1)\}$, where $Q_1(v_1) = \{v_0, v_2, v_6\}$ and $Q_2(v_1) = \{v_1\}$. Then, $Q_1(v_1)$ and $Q_2(v_1)$ are independent since, no edge in $W(v_1)$ is an *mp* edge$< Q_1(v_1) >$ or an *mp* edge$< Q_2(v_1) >$. Also, $Q(v_8) = \{v_7, v_8, v_9\}$ can be partitioned into sets $\{Q_3(v_8), Q_4(v_8)\}$, where $Q_3(v_8) = \{v_7, v_9\}$ and $Q_4(v_8) = \{v_8\}$. Then, $Q_3(v_8)$ and $Q_4(v_8)$ are independent since, no edge in $W(v_8)$ is an *mp* edge$< Q_3(v_8) >$ or an *mp* edge$< Q_4(v_8) >$. Therefore, $H(v_1)$ and $H(v_8)$ are bipartite partial semigraphs of G^* and hence, $G = \{H(v_1), H(v_8)\}$ is a bipartite soft semigraph.

3.2 e-bipartite soft semigraph

Definition 3.1. *Let $G^* = (V, X)$ be a semigraph and $G = (G^*, Q, W, A)$ be a soft semigraph of G^* represented by $\{H(x) : x \in A\}$. Then, G is called an e-bipartite soft semigraph if each of its p-parts $H(x)$ is an e-bipartite partial semigraph of G^*. That is, $Q(x)$ can be partitioned into sets $\{Q_1(x), Q_2(x)\}$ such that both $Q_1(x)$ and $Q_2(x)$ are e-independent for all x in A. That is, no two end vertices or partial end vertices of an f-edge in $W(x)$ belong to $Q_1(x)$ or $Q_2(x)$ for all x in A.*

Example 3.2. Consider the soft semigraph G given in Figure 2. Here, $Q(v_1) = \{v_0, v_1, v_2, v_6\}$ can be partitioned into sets $\{Q_1(v_1), Q_2(v_1)\}$, where $Q_1(v_1) = \{v_0, v_1\}$ and $Q_2(v_1) = \{v_2, v_6\}$. Then, $Q_1(v_1)$ and $Q_2(v_1)$ are e-independent since no two end vertices or partial end vertices of an f-edge in $W(v_1)$ belong to $Q_1(v_1)$ or $Q_2(v_1)$. Also, $Q(v_8) = \{v_7, v_8, v_9\}$ can be partitioned into sets $\{Q_3(v_8), Q_4(v_8)\}$, where $Q_3(v_8) = \{v_7, v_9\}$ and $Q_4(v_8) = \{v_8\}$. Then, $Q_3(v_8)$ and $Q_4(v_8)$ are e-independent since no two end vertices or partial end vertices of an f-edge in $W(v_8)$ belong to $Q_3(v_8)$ or $Q_4(v_8)$. Therefore, $H(v_1)$ and $H(v_8)$

are e-bipartite partial semigraphs of G^* and hence, $G = \{H(v_1), H(v_8)\}$ is an e-bipartite soft semigraph.

3.3 Strongly bipartite soft semigraph

Definition 3.2. *Let $G^* = (V, X)$ be a semigraph and $G = (G^*, Q, W, A)$ be a soft semigraph of G^* represented by $\{H(x) : x \in A\}$. Then, G is called a strongly bipartite soft semigraph if each of its p-parts $H(x)$ is a strongly bipartite partial semigraph of G^*. That is, $Q(x)$ can be partitioned into sets $\{Q_1(x), Q_2(x)\}$ such that both $Q_1(x)$ and $Q_2(x)$ are strongly independent for all x in A. That is, no two adjacent vertices in $H(x)$ belong to $Q_1(x)$ or $Q_2(x)$ for all x in A.*

Example 3.3. Let be a semigraph given in Figure 3, where $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $X = \{(v_1, v_2), (v_1, v_3, v_6, v_4), (v_4, v_5)\}$.

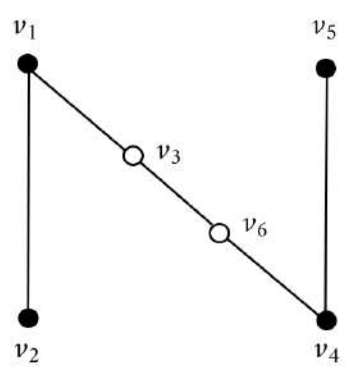

Figure 3: Semigraph $G^* = (V, X)$

Let $A = \{v_2, v_5\} \subseteq V$ be a parameter set. Define Q from A to $\mathcal{P}(V)$ by $Q(x) = \{y \in V | xRy \Leftrightarrow x = y$ or x and y are adjacent$\}, \forall x \in A$ and W from A to $\mathcal{P}(X_p)$ by $W(x) = \{mp\ edges\langle Q(x)\rangle\}, \forall x \in A$. That is, $Q(v_2) = \{v_1, v_2\}$ and $Q(v_5) = \{v_4, v_5\}$. Also, $W(v_2) = \{(v_1, v_2)\}$ and $W(v_5) = \{(v_4, v_5)\}$. Then, $H(v_2) = (Q(v_2), W(v_2))$ and $H(v_5) = (Q(v_5), W(v_5))$ are partial semigraphs of G^* as shown below in Figure 4. Hence, $G = \{H(v_2), H(v_5)\}$ is a soft semigraph of G^*. Here, $Q(v_2) = \{v_1, v_2\}$ can be partitioned into sets $\{Q_1(v_2), Q_2(v_2)\}$, where $Q_1(v_2) = \{v_1\}$ and $Q_2(v_2) = \{v_2\}$. Then, $Q_1(v_2)$ and $Q_2(v_2)$ are strongly independent since, no two adjacent vertices in $H(v_2)$ belong to $Q_1(v_2)$ or $Q_2(v_2)$. Also, $Q(v_5) = \{v_4, v_5\}$ can be partitioned into sets $\{Q_3(v_5), Q_4(v_5)\}$, where $Q_3(v_5) = \{v_4\}$ and $Q_4(v_5) = \{v_5\}$. Then, $Q_3(v_5)$ and $Q_4(v_5)$ are strongly independent since, no two adjacent vertices in $H(v_5)$ belong to $Q_3(v_5)$ or $Q_4(v_5)$. Therefore, $H(v_2)$ and $H(v_5)$ are strongly bipartite partial semigraphs of G^* and hence, $G = \{H(v_2), H(v_5)\}$ is a strongly bipartite soft semigraph.

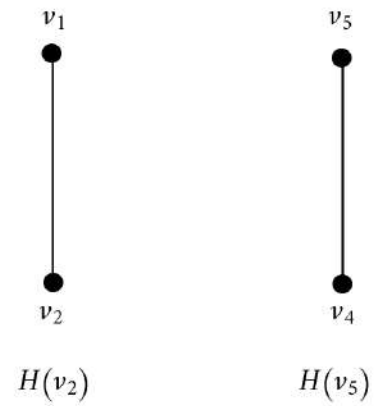

Figure 4: Soft Semigraph $G = \{H(v_2), H(v_5)\}$

Theorem 3.1. *If a soft semigraph G is strongly bipartite then its p-parts are bipartite graphs if we treat both end vertices and partial end vertices as vertices of the bipartite graph.*

Proof. Let $G = (G^*, Q, W, A)$ be a strongly bipartite soft semigraph of G^* represented by $\{H(x) : x \in A\}$. Then, each of its p-parts $H(x)$ is a strongly bipartite partial semigraph of G^*. That is, $Q(x)$ can be partitioned into sets $\{Q_1(x), Q_2(x)\}$ such that both $Q_1(x)$ and $Q_2(x)$ are strongly independent for all x in A. That is, no two adjacent vertices in $H(x)$ belong to $Q_1(x)$ or $Q_2(x)$ for all x in A. So, each $H(x)$ can contain f-edges with 2 vertices only. Therefore, this partition $\{Q_1(x), Q_2(x)\}$ will satisfy all the criteria for a bipartite graph in graph theory, if we treat both end vertices and partial end vertices as vertices and f-edges as edges. □

Theorem 3.2. *If a soft semigraph G is e-bipartite, then it is also a bipartite soft semigraph.*

Proof. Let $G = (G^*, Q, W, A)$ be an e-bipartite soft semigraph represented by $\{H(x) : x \in A\}$. Then, each p-part $H(x)$ of G will be an e-bipartite partial semigraphs of G^*. That is,$Q(x)$ can be partitioned into sets $\{Q_1(x), Q_2(x)\}$ such that both $Q_1(x)$ and $Q_2(x)$ are e-independent for all x in A. That is, no two end vertices or partial end vertices of an f-edge in $W(x)$ belong to $Q_1(x)$ or $Q_2(x)$, for all x in A. Definitely, no edge in $W(x)$ is an mp edge$< Q_1(x) >$ or an mp edge$< Q_2(x) >$, for all x in A. Therefore, G is also a bipartite soft semigraph. □

Remark 3.1. The converse of this theorem need not be true. This is clear from the following example.

Example 3.4. Let be a semigraph given in Figure 5, where $V = \{v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9\}$ and $X = \{(v_0, v_2), (v_0, v_1, v_3), (v_2, v_3), (v_3, v_4, v_5), (v_6, v_5, v_9), (v_6, v_7), (v_5, v_7), (v_7, v_8, v_9)\}$.

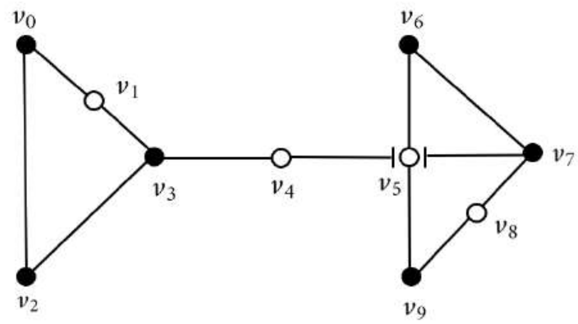

Figure 5: Semigraph $G^* = (V, X)$

Let $A = \{v_0, v_7\} \subseteq V$ be a parameter set.

Define mappings Q from A to $\mathcal{P}(V)$ by $Q(x) = \{y \in V | xRy \Leftrightarrow x = y$ or x and y are adjacent$\}, \forall x \in A$ and W from A to $\mathcal{P}(X_p)$ by $W(x) = \{mp\ edges\langle Q(x)\rangle\}, \forall x \in A$. That is, $Q(v_0) = \{v_0, v_1, v_2, v_3\}$ and $Q(v_7) = \{v_5, v_6, v_7, v_8, v_9\}$. Also, $W(v_0) = \{(v_0, v_2), (v_2, v_3), (v_0, v_1, v_3)\}$ and $W(v_7) = \{(v_6, v_7), (v_6, v_5, v_9), (v_5, v_7), (v_7, v_8, v_9)\}$. Then, $H(v_0) = (Q(v_0), W(v_0))$ and $H(v_7) = (Q(v_7), W(v_7))$ are partial semigraphs of G^* as shown below in Figure 6. Hence, $G = \{H(v_0), H(v_7)\}$ is a soft semigraph of G^*.

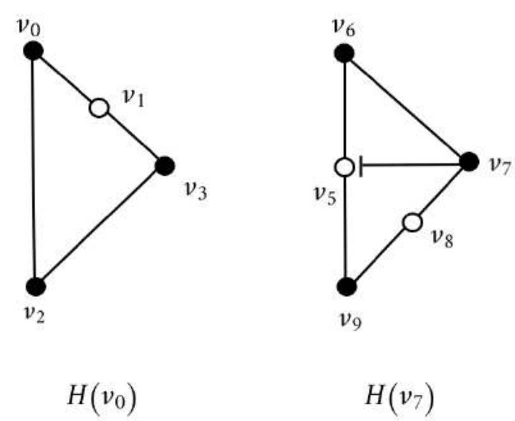

Figure 6: Soft Semigraph $G = \{H(v_0), H(v_7)\}$

Here, $Q(v_0) = \{v_0, v_1, v_2, v_3\}$ can be partitioned into sets $\{Q_1(v_0), Q_2(v_0)$, where $Q_1(v_0) = \{v_0, v_3\}$ and $Q_2(v_0) = \{v_2, v_1\}$. Then, $Q_1(v_0)$ and $Q_2(v_0)$ are independent, since no edge in $W(v_0)$ is an *mp* edge$< Q_1(v_0) >$ or an

mp edge$< Q_2(v_0) >$. Also, $Q(v_7) = \{v_5, v_6, v_7, v_8, v_9\}$ can be partitioned into sets $\{Q_3(v_7), Q_4(v_7)\}$, where $Q_3(v_7) = \{v_5, v_6, v_8\}$ and $Q_4(v_7) = \{v_7, v_9\}$. Then, $Q_3(v_7)$ and $Q_4(v_7)$ are independent, since no edge in $W(v_7)$ is an mp edge$< Q_3(v_7) >$ or an mp edge$< Q_4(v_7) >$. Therefore, $H(v_0)$ and $H(v_7)$ are bipartite partial semigraphs of G^* and hence, $G = \{H(v_0), H(v_7)\}$ is a bipartite soft semigraph. But, $H(v_0)$ and $H(v_7)$ are not e-bipartite partial semigraphs of G^* because, $Q(v_0)$ and $Q(v_7)$ are not e-independent. That is, there exists no partition for $Q(v_0)$ and $Q(v_7)$ such that the two end vertices or partial end vertices of an f-edge in $W(v_0)$ and $W(v_7)$ are in two different sets of the partition.

Theorem 3.3. *A soft semigraph G is e-bipartite if and only if all of its p-part end vertex graphs $H(x)_e$ are bipartite where $H(x)_e$ is a graph having vertex set $Q(x)$ and two vertices u and v in $H(x)_e$ are adjacent if they are the end vertices or a partial end vertices of an f-edge containing these vertices in the p-part $H(x)$.*

Proof. Assume that a soft semigraph $G = (G^*, Q, W, A)$ represented by $\{H(x) : x \in A\}$ is an e-bipartite soft semigraph. Then, all of its p-parts $H(x)$ are e-bipartite partial semigraphs of G^*. That is, $Q(x)$ can be partitioned into sets $\{Q_1(x), Q_2(x)\}$ such that both $Q_1(x)$ and $Q_2(x)$ are e-independent for all x in A. That is, no two end vertices or partial end vertices of an f-edge in $W(x)$ belong to $Q_1(x)$ or $Q_2(x)$ for all x in A. We know that the p-part end vertex graph $H(x)_e$ is $Q(x)$ and two vertices u and v in $H(x)_e$ are adjacent if they are the end vertices or a partial end vertices of an f-edge $W(x)$ for all x in A. So, if we give the same partition $\{Q_1(x), Q_2(x)\}$ to $Q(x)$ in $H(x)_e$ then each edge has one end in $Q_1(x)$ and the other end in $Q_2(x)$. Therefore, $H(x)_e$ is a bipartite graph for all x in A.

Conversely, assume that all p-part end vertex graph $H(x)_e$ of G is bipartite. That is, if $Q(x)$ is the vertex set of $H(x)_e$, then it can be partitioned into two nonempty sets $Q_1(x)$ and $Q_2(x)$ such that each edge in $H(x)_e$ has one end in $Q_1(x)$ and the other end in $Q_2(x)$ for all x in A. In the corresponding p-part $H(x)$, end vertices or partial end vertices of the f-edge are the same as the end vertices of the corresponding edge in $H(x)_e$. Therefore, if we use the same partition of $Q(x)$ in $H(x)$, $Q_1(x)$ and $Q_2(x)$ will be e-independent for all x in A. Therefore, $H(x)$ is an e-bipartite partial semigraph for all x in A. That is, G is an e-bipartite soft semigraph. □

4. Conclusion

This paper has explored the intersection of soft set theory and semigraphs, demonstrating the efficacy of soft set principles in enhancing the adaptability of semigraphs to address uncertainty. The paper focused on different kinds of bipartite soft semigraphs, studying their structures and features closely. This research helps us better understand how to use these tools to tackle real-world problems with uncertain data, which can be important for making better decisions in various fields.

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On semi-Hamilton groups and minimal non-semi-Hamilton groups

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Abstract. A subgroup H of a finite group G is said to be semipermutable in G if it is permutable with every subgroup K of G satisfying that $(|K|, |H|) = 1$. If every subgroup of G is semipermutable in G, then G is said to be a semi-Hamilton group. In this paper, the authors classify the non-semi-Hamilton groups whose proper subgroups are all semi-Hamilton groups.

Keywords: finite groups, semi-Hamilton groups, minimal non-semi-Hamilton groups, power automorphisms.

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1. Introduction

Given a group theoretical property $\mathcal{P}$, a $\mathcal{P}$-critical group or a minimal non-$\mathcal{P}$-group is a group which is not a $\mathcal{P}$-group but all of whose proper subgroups are $\mathcal{P}$-groups. There are many remarkable examples about minimal non-$\mathcal{P}$-groups: minimal non-abelian groups (Miller and Moreno [8]), minimal non-nilpotent groups (Schmidt), minimal non-supersoluble groups ([1]) and minimal non-p-nilpotent groups (Itô), minimal non-MSP-groups([4]) and minimal non-NSN-groups([5]).

In [10], Sastry classified the minimal non-PN-groups.

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Recall that a subgroup H is called quasinormal in a group G, if $HK = KH$ holds for every subgroup K of G, and a group G is called a $\mathcal{QN}$-group if every minimal subgroup of G is quasinormal in G (see [3]). Clearly, a $\mathcal{QN}$-group is a generalization of PN-groups.

In this paper, we consider a generalization of $\mathcal{QN}$-groups, which is called semi-Hamilton groups.

Definition 1.1. *A subgroup H of a group G is said to be s-semipermutable in G if it is permutable with every Sylow p-subgroup of G satisfying that $(p, |H|) = 1$. A subgroup H of a group is said to be semipermutable in G if it is permutable with every subgroup K of G satisfying that $(|K|, |H|) = 1$. If every subgroup of G is semipermutable in G, then G is said to be a semi-Hamilton group.*

By [11, Theorem 1], the following statements are equivalent:

(1) G is a semi-Hamilton group.

(2) every subgroup is semipermutable in G

(3) every Sylow subgroup is semipermutable in G.

(4) every subgroup with prime order is semipermutable in G.

In what follows we will use this result without any declarations.

Definition 1.2. *A group G is said to be a minimal non-semi-Hamilton group if all proper subgroups are all semi-Hamilton groups, but G itself is not a semi-Hamilton group.*

In this paper, we will investigate properties of semi-Hamilton groups, and give the structure of such groups in the first place. Next, by applying the structure of semi-Hamilton groups , we will give a classification of minimal non-semi-Hamilton groups.

Throughout this paper, only finite groups are considered and our notations are all standard. For example, we denote by $[A]P$ the semidirect product of A and P. C_n denotes a cyclic group of order n, and $\pi(G)$ denotes the set of all prime divisors of $|G|$. All unexplained notations can be found in [6] and [9].

2. Some preliminaries

In this section, we collect some lemmas which will be frequently used in the sequel.

Lemma 2.1 ([2, Theorem 7.47]). *Let G be a finite group. If every maximal subgroup of every Sylow subgroup of G is s-semipermutable in G, then G is supersolvable.*

Lemma 2.2 ([13, Lemma 3]). *Let G be a finite group. Then, G possesses a fixed-point-free power automorphism if and only if G is an abelian group of odd order.*

Lemma 2.3 ([12, Theorem 7.2.4]). *Suppose that p'-group H acts on a p-group G. If $\Omega(G)$ is H-reducible, then G is H-decomposable.*

Lemma 2.4 ([6, 8.4.6]). *Suppose that the action of A on an elementary abelian group G is coprime, and H is an A-invariant direct factor of G. Then, H has an A-invariant complement in G.*

Lemma 2.5 ([7]). *Suppose that p'-group H acts on a p-group G. Let*

$$\Omega(G) = \begin{cases} \Omega_1(G), & p > 2, \\ \Omega_2(G), & p = 2. \end{cases}$$

If H acts trivially on $\Omega(G)$, then H acts trivially on G as well.

In classifying the finite semi-Hamilton groups, we need the structure of the minimal non-supersoluble groups, it has been given by Adolfo Ballester-Bolinches and Ramon Esteban-Romero in [1] , We list it as the following Lemma:

Lemma 2.6 ([1, Theorem 10]). *The minimal non-supersoluble groups are exactly the groups of the following types:*

(1) *$G = [P]Q$, where $Q = \langle z \rangle$ is cyclic of order $q^r > 1$, with q a prime not dividing $p-1$, and P is an irreducible Q-module over the field of p elements with kernel $\langle z^q \rangle$ in Q.*

(2) *$G = [P]Q$, where P is a non-abelian special p-group of rank $2m$, the order of p modulo q being $2m$, q is a prime, $Q = \langle z \rangle$ is cyclic of order $q^r > 1$, z induces an automorphism in P such that $P/\Phi(P)$ is a faithful and irreducible Q-module, and z centralises $\Phi(P)$. Furthermore, $|P/\Phi(P)| = p^{2m}$ and $|P'| \le p^m$.*

(3) *$G = [P]Q$, where $P = \langle a_0, a_1, ..., a_{q-1} \rangle$ is an elementary abelian p-group of order p^q, $Q = \langle z \rangle$ is cyclic of order q^r, with q a prime such that q^f is the highest power of q dividing $p-1$ and $r > f \ge 1$. Define $a_j^z = a_{j+1}$ for $0 \le j < q-1$ and $a_{q-1}^z = a_0^i$, where i is a primitive q^f-th root of unity modulo p.*

(4) *$G = [P]Q$, where $P = \langle a_0, a_1 \rangle$ is an extraspecial group of order p^3 and exponent p, $Q = \langle z \rangle$ is cyclic of order 2^r, with 2^f the largest power of 2 dividing $p-1$ and $r > f \ge 1$. Define $a_1 = a_0^z$ and $a_1^z = a_0^i x$, where $x \in \langle [a_0, a_1] \rangle$ and i is a primitive 2^f-th root of unity modulo p.*

(5) $G = [P]E$, *where* E *is a* 2*-group with a normal subgroup* F *such that* $F \leq \Phi(E)$ *and* E/F *is isomorphic to a quaternion group of order 8 and* P *is an irreducible module for* E *with kernel* F *over the field of* p *elements of dimension 2, where* $4|p-1$.

(6) $G = [P]E$, *where* E *is a 2-group with a normal subgroup* F *such that* $F \leq \Phi(E)$ *and* E/F *is isomorphic to a quaternion group of order 8,* P *is an extraspecial group of order* p^3 *and exponent* p*, where* $4|p-1$*, and* $P/\Phi(P)$ *is an irreducible module for* E *with kernel* F *over the field of* p *elements.*

(7) $G = [P]E$, *where* E *is a* q*-group (for a prime* q*) with a normal subgroup* F *such that* $F \leq \Phi(E)$ *and* E/F *is isomorphic to a group* $G_{II}(q, m, 1)$, P *is an irreducible* E*-module of dimension* q *over the field of* p *elements with kernel* F*, and* q^m *divides* $p-1$.

(8) $G = [P]E$,*where* E *is a 2-group with a normal subgroup* F *such that* $F \leq \Phi(E)$ *and* E/F *is isomorphic to a group* $G_{II}(2, m, 1)$, P *is an extraspecial group of order* p^3 *and exponent* p *such that* $P/\Phi(P)$ *is an irreducible* E*-module of dimension 2 over the field of* p *elements with kernel* F*, and* 2^m *divides* $p-1$.

(9) $G = [P]E$, *where* E *is a* q*-group (for a prime* q*) with a normal subgroup* F *such that* $F \leq \Phi(E)$ *and* E/F *is isomorphic to an extraspecial group of order* q^3 *and exponent* q*, with* q *odd,* P *is an irreducible* E*-module over the field of* p *elements with kernel* F *and dimension* q*, and* q *divides* $p-1$.

(10) $G = [P]MC$, *where* C *is a cyclic subgroup of order* r^{s+t}*, with* r *a prime number and* s *and* t *integers such that* $s \geq 1$ *and* $t \geq 0$*, normalising a Sylow* q*-subgroup* M *of* G*,* $M/\Phi(M)$ *is an irreducible* C*-module over the field of* q *elements,* q *a prime, with kernel the subgroup* D *of order* r^t *of* C*, and* P *is an irreducible* MC*-module over the field of* p *elements, where* q *and* r^s *divide* $p-1$*. In this case,* $\Phi(G)_{p'}$*, the Hall* p'*-subgroup of* $\Phi(G)$*, coincides with* $\Phi(M)$ *and centralises* P.

(11) $G = [P]MC$, *where* C *is a cyclic subgroup of order* 2^{s+t}*, with* s *and* t *integers such that* $s \geq 1$ *and* $t \geq 0$*, normalising a Sylow* q*-subgroup* M *of* G*,* q *a prime,* $M/\Phi(M)$ *is an irreducible* C*-module over the field of* q *elements, with kernel the subgroup* D *of order* 2^t *of* C*, and* P *is an extraspecial group of order* p^3 *and exponent* p *such that* $P/\Phi(P)$ *is an irreducible* MC*-module over the field of* p *elements, where* q *and* 2^s *divide* $p-1$*. In this case,* $\Phi(G)_{p'}$*, the Hall* p'*-subgroup of* $\Phi(G)$*, is equal to* $\Phi(M) \times D$ *and centralises* P.

According to [1, Theorem 1], the notations $G_{II}(q, m, 1)$ and $G_{II}(2, m, 1)$ above are the following groups:

$$G_{II}(q, m, n) = \langle a, b | a^{q^m} = b^{q^n} = 1, a^b = a^{1+q^{m-1}} \rangle,$$

where q is a prime number, $m \geq 2, n \geq 1$.

3. Semi-Hamilton groups

In this section, we will classify the finite semi-Hamilton groups. We need some Lemmas before we proceed with our proof.

Lemma 3.1. *Let $G = PQ$, where $P \in Syl_p(G), Q \in Syl_q(G), p < q$. Then, G is a semi-Hamilton-group if and only if one of the following statements is true:*

(1) *G is a nilpotent group.*

(2) *$G = [Q]P$, $P = \langle a, C_P(Q)\rangle$, and a induces a fixed-point-free power automorphism of Q.*

Proof. Let y be any element in P, for any $u \in Q$, we have $\langle y\rangle\langle u\rangle = \langle u\rangle\langle y\rangle$, then $\langle u\rangle \trianglelefteq \langle u, y\rangle$, which implies that y induces a power automorphism of Q.

Choose $v \in Z(Q)$ of order q and $z \in C_P(\langle v\rangle)$, then $v^z = v$. Let $w \in Q$ of order q, and $w^z = w^i$, then $(vw)^z = vw^i$. Since z induces a power automorphism of Q, there is a natural number j such that $(vw)^j = (vw)^z = vw^i$. Thus, we have $1 \equiv j \equiv i (mod\ q)$, that is z acts trivially on w. Hence, z acts trivially on $\Omega_1(Q)$. Now, by Lemma 2.5 we know that z acts trivially on Q. Therefore, we get that $C_P(\langle v\rangle) \leq C_P(Q)$. On the other hand, $C_P(Q) \leq C_P(\langle v\rangle)$, hence $C_P(Q) = C_P(\langle v\rangle)$.

Let $T = C_P(\langle v\rangle)$. If $T = P$, then G is nilpotent. If $T \neq P$, then we have $P/C_P(Q) = P/T \lesssim Aut(\langle v\rangle \lesssim C_{q-1}$ by the so called N/C-Theorem. Thus there exists an element $a \in P$, such that $P = \langle a, C_P(Q)\rangle$. If there exists an element $x \in Q$ satisfying that $a \in C_P(x)$, then there exists an element $w \in Q$ of order q satisfying that $w^a = w$. Since a induces a power automorphism of Q, there exists a natural number k such that $(vw)^k = (vw)^a = v^i w$, where i is also a natural number. Thus, we have $1 \equiv k \equiv i(mod\ q)$, that is a acts trivially on v. Hence, $a \in C_P(v) = C_P(Q)$, a contradiction. Hence, we get that a acts fixed-point-freely on Q. It is easy to check that G is a semi-Hamilton-group in above two cases.

The proof is completed. □

Lemma 3.2. *Suppose that $G = PHQ$, where $P \in Syl_p(G), Q \in Syl_q(G)$, H is a Hall subgroup of G, p is the smallest prime divisor of $|G|$, $P = \langle a, C_P(H)\rangle$, a induces a fixed-point-free power automorphism of H. Then, G is a semi-Hamilton-group if and only if one of the following statements is true:*

(1) *If P acts trivially on Q, then $G = (PH) \times Q$.*

(2) *If P acts non-trivially on Q, then $G = [HQ]P$, $C_P(HQ) = C_P(H)$, a induces a fixed-point-free power automorphism of HQ.*

Proof. The necessity of the theorem is obvious, so we only need to prove the sufficiency.

Suppose that P acts trivially on Q. If the result is not true, then there is a Sylow subgroup $R \in Syl_r(H)$ (without loss of generality, we let $r < q$ here) such that R acts on Q non-trivially, then by Lemma 3.1, there exist three elements $x \in P$, $y \in R$, and $u \in Q$ of order q and integer number i, j, k such that $y^x = y^i$, $u^y = u^j$, $y^i u = (yu)^x = (yu)^k$.

Let $o(y) = r^m$, then $i \not\equiv 1(mod\ r)$, $j^{r^m} \equiv 1(mod\ q)$, and the index of j modulo q is a power of r. By calculations, we have $(yu)^k = y^k u^{\frac{j^k-1}{j-1}}$, then $i \equiv k(mod\ r^m)$, and $1 \equiv \frac{j^k-1}{j-1}(mod\ q)$. Thus, $j^{k-1} \equiv 1(mod\ q)$. Since the index of j modulo q is a power of r, we have $k \equiv 1(mod\ r)$. Now, by $k \equiv i(mod\ r^m)$ we can obtain that $i \equiv 1(mod\ r)$, a contradiction since $i \not\equiv 1(mod\ r)$. Thus, if P acts trivially on Q, then H must acts trivially on Q too. Hence, we obtain that $G = (PH) \times Q$ in this case.

Suppose that P acts non-trivially on Q. We need only to prove that for any $R \in Syl_r(H)$, $C_P(R) = C_P(Q)$ holds. If $z \in C_P(R)$, and $z \notin C_P(Q)$, then by the proof of above paragraph, we know that $PRQ = (PQ) \times R$, a contradiction since a induces a fixed-point-free power automorphism of H. If $z \in C_P(Q)$, and $z \notin C_P(R)$, then we have $PRQ = (PR) \times Q$, a contradiction since a acts on Q non-trivially.

The proof is completed. □

Now, we can prove our main result of this section, this result is a sketchy description of the structure of semi-Hamilton-groups.

Theorem 3.1. *Let G be a finite group. Then, G is a semi-Hamilton-group if and only if $G = G_1 \times G_2 \times \ldots \times G_n$, where G_i are Sylow subgroups of G or Hall subgroups of G satisfying that $G_i = [H_i]P_i$, $P_i \in Syl_{p_i}(G_i)$, where H_i are Hall subgroups of G_i, and p_i is the smallest prime divisor of $|G_i|$, $P_i = \langle a_i, C_{P_i}(H_i)\rangle$, and a_i induces a fixed-point-free power automorphism of H_i.*

Proof. Let $\mathscr{B} = \{P_i \in Syl_{p_i}(G) | 1 \leq i \leq n, p_1 < p_2 < \ldots < p_n\}$ be a Sylow system of G. By Lemma 2.1, G is supersolvable. Then, P_i acts on P_j conjugately if $i < j$.

If P_1 acts trivially on every $P_i (i \geq 2)$, then we may let $G_1 = P_1$, and $S = P_2 P_3 \ldots P_n$. In this case we have that $G = G_1 \times S$.

Suppose that P_1 acts non-trivially on some elements of $\mathscr{B}$ (for instance $Q_1, Q_2, \ldots, Q_r$) and acts some elements of $\mathscr{B}$ (for instance $R_1, R_2, \ldots, R_t$). In this case we can let $H_1 = Q_1 Q_2 \ldots Q_r$, $G_1 = P_1 H_1$, and $S = R_1 R_2 \ldots R_t$. Then, by Lemma 3.1 and Lemma 3.2, we have that $G = G_1 \times S$.

If $S \neq 1$, then we can repeat the way we used above. Thus, we get that $G = G_1 \times G_2 \times \ldots \times G_n$, where G_i are Sylow subgroups of G or Hall subgroups of G satisfying that $G_i = H_i P_i$, $P_i \in Syl_{p_i}(G_i)$, where H_i are Hall subgroups of G_i, and p_i is the smallest prime divisor of $|G_i|$. The rest can be obtained by Lemma 3.2 easily. The proof is completed. □

4. Minimal non-semi-Hamilton groups

In this section, we will give a structure of finite minimal non-semi-Hamilton groups. Our proof will be divided into several parts.

Proposition 4.1. *Suppose that G is a finite group which is not supersolvable. Then G is a minimal non-semi-Hamilton group if and only if one of the following statements is true:*

(1) *$G = [P]Q$, where $Q = \langle z \rangle$ is cyclic of order $q^r > 1$, with q a prime not dividing $p-1$, and P is an irreducible Q-module over the field of p elements with kernel $\langle z^q \rangle$ in Q.*

(2) *$G = [P]Q$, where P is a non-abelian special p-group of rank $2m$, the order of p modulo q being $2m$, q is a prime, $Q = \langle z \rangle$ is cyclic of order $q^r > 1$, z induces an automorphism in P such that $P/\Phi(P)$ is a faithful and irreducible Q-module, and z centralises $\Phi(P)$. Furthermore, $|P/\Phi(P)| = p^{2m}$ and $|P'| \leq p^m$.*

(3) *$G = [P]Q$, where $P = \langle a_0, a_1, ..., a_{q-1} \rangle$ is an elementary abelian p-group of order p^q, $Q = \langle z \rangle$ is cyclic of order q^r, with q a prime such that q^f is the highest power of q dividing $p-1$ and $r > f \geq 1$. Define $a_j^z = a_{j+1}$ for $0 \leq j < q-1$ and $a_{q-1}^z = a_0^i$, where i is a primitive q^f-th root of unity modulo p.*

Proof. By the hypothesis and Lemma 2.1, every proper subgroup of G is supersolvable. Then, G is minimal non-supersolvable, and hence G is isomorphic to one of groups listed in Lemma 2.6.

It easy to check that the groups of type (1) and (3) in Lemma 2.6 are minimal non-semi-Hamilton groups.

Suppose that the groups of type (2) are minimal non-semi-Hamilton groups. Then, we claim that $|Q| = q$. Otherwise $\langle P, z^q \rangle$ is a Hamilton-group, and $P/\Phi(P)$ faithfully, z^q acts on $\Phi(P)$ trivially, which contradicts with Lemma 3.1. Hence, we get that $|Q| = q$ here. On the other hand, if G is a group of type (2) in Lemma 2.6 with $|Q| = q$, we can easily check that G is a minimal non-semi-Hamilton group.

Suppose that G is groups of type (4). Then, z^2 will induces a power automorphism of $\langle a_0 \rangle$, since $\langle P, z^2 \rangle$ is a semi-Hamilton group. On the other hand, by Lemma 2.6, $a_0^{z^2} = a_0^i x$, which implies that $x = 1$. Hence, we have $a_0^{z^2} = a_0^i$, and therefore we get $[a_0, a_1]^{z^2} = [a_0, a_1]^{i^2}$, which contradicts with Lemma 3.1. Thus, G cannot be a minimal non-semi-Hamilton group.

Suppose that G is groups of type (5). If G is a minimal non-semi-Hamilton group, then every element of E induces a power automorphism of P, and hence E acts decomposably on P, a contradiction.

By the same argument we can obtain that groups of type (6) -(11) of Lemma 2.6 are not minimal non-semi-Hamilton groups.

The proof is completed. □

Proposition 4.2. *Suppose that G is a finite supersolvable group. If G is a minimal non-semi-Hamilton group, then $|\pi(G)| \leq 3$.*

Proof. Assume that $|\pi(G)| > 3$. Let $\{P_1, P_2, \ldots, P_n\}(n \geq 4)$ be a Sylow system of G, and p_1 be the smallest prime divisor of $|G|$. If P_1 acts on each $P_i(i > 1)$ trivially, then $G = P_1 \times (P_2P_3 \ldots P_n)$, and hence G itself is a semi-Hamilton group, a contradiction. If P_1 acts on every $P_i(i > 1)$ non-trivially, then for any i, j, then $P_1P_iP_j$ is a semi-Hamilton group. By Lemma 3.2, $G = [P_2P_3 \ldots P_n]P_1$, $P_1 = \langle a, C_{P_1}(P_2)\rangle$, $C_{P_1}(P_2P_3 \ldots P_n) = C_{P_1}(P_2)$, and a induces a fixed-point-free power automorphism of $P_2P_3 \ldots P_n$. That is, G itself is a semi-Hamilton group, a contradiction. Choose a Sylow system $\{P_0, Q_1, \ldots, Q_m, R_{m+1}, \ldots, R_n\}$ of G such that P_0 acts on Q_i non-trivially, and acts on R_j trivially, where $i \in \{1, \ldots, m\}, j \in \{m+1, \ldots, n\}$. Then, for any i, j, $P_1Q_iR_j$ is a semi-Hamilton group. By Lemma 3.2, $G = (P_0Q_1 \ldots Q_m) \times (R_{m+1} \ldots R_n)$, which means that G itself is a semi-Hamilton group also, a contradiction. Hence, $|\pi(G)| \leq 3$. The proof is completed. □

The following Proposition classifies supersolvable minimal non-semi-Hamilton groups which having three prime divisors.

Proposition 4.3. *Suppose that G is a finite supersolvable group and $\pi(G) = \{p, q, r\}, p < q < r$. Then, G is a minimal non-semi-Hamilton group if and only if one of the following statements is true:*

(1) $G = \langle u, v, w | u^p = 1, v^{q^m} = 1, w^r = 1, v^u = v^i, w^u = w, w^v = w^j, i \not\equiv 1(mod\ q), i^p \equiv 1(mod\ q^m), j \not\equiv 1(mod\ r), j^q \equiv 1(mod\ r)\rangle$.

(2) $G = \langle u, v, w | u^p = 1, v^{q^n} = 1, w^r = 1, v^u = v, w^u = w^i, w^v = w^j, i \not\equiv 1(mod\ r), i^p \equiv 1(mod\ r), j \not\equiv 1(mod\ r), j^q \equiv 1(mod\ r)\rangle$.

Proof. The necessity of the theorem is obvious, so we only need to prove the sufficiency.

Suppose that $\{P \in Syl_p(G), Q \in Syl_q(G), R \in Syl_r(G)\}$ is a Sylow system of G. Since PQ, PR are all semi-Hamilton groups, $QR \trianglelefteq G$. If QR is nilpotent, then $Q \trianglelefteq G$ and $R \trianglelefteq G$. Hence, G itself is a semi-Hamilton group, since every subgroup with prime order is semipermutable in G, a contradiction. Therefore, Q acts on R non-trivially.

If P is not cyclic, then we can choose two maximal subgroups P_1, P_2 of P. Assume that there exists one $P_i(i = 1$ or $2)$ acts on Q or R non-trivially, then QR is nilpotent by Lemma 3.2, a contradiction. Hence, both P_1 and P_2 acts on Q and R trivially, which means that $G = P \times (QR)$, that is, G itself is a semi-Hamilton group, a contradiction. Thus, P is cyclic.

If Q is not cyclic, then we can choose two maximal subgroups Q_1, Q_2 of Q, and there exists at least one Q_i(named Q_1 here), such that Q_1 acts on R non-trivially. By hypothesis and Lemma 3.2, $PQ_1R = P \times (Q_1R)$. Hence, P acts on Q trivially by Lemma 3.1 since PQ is a semi-Hamilton group. Thus, we get that $G = P \times (QR)$, which implies that G itself is a semi-Hamilton group, a contradiction. Thus, Q is cyclic.

Now, we claim that $|R| = r$. Let R_1 be a proper subgroup of R, then Q acts on R_1 trivially by Lemma 3.1 since PQ is a semi-Hamilton group. Moreover, we can obtain that P acts on QR_1 trivially by Lemma 3.2 since PQ is a semi-Hamilton group. Thus, we get P acts on R trivially by Lemma 3.1. Hence, $G = P \times (QR)$, that is G itself is a semi-Hamilton group, a contradiction. Thus, we get that $|R| = r$.

Let Q_1 be the maximal subgroup of Q, then Q_1R is nilpotent by Lemma 3.2 since PQ_1R is a semi-Hamilton group. Let P_1 be the maximal subgroup of P, then $P_1QR = P_1 \times (QR)$ by Lemma 3.2 since P_1QR is a semi-Hamilton group.

Let $P = \langle u \rangle$, $Q = \langle v \rangle$, and $R = \langle w \rangle$. If P acts on R trivially, then P must act non-trivially on Q. Otherwise we should obtain that $G = P \times (QR)$, which implies that G itself is a semi-Hamilton group, a contradiction. Hence, we get that:

$$G = \langle u, v, w | u^{p^n} = 1, v^{q^m} = 1, w^r = 1, v^u = v^i, w^u = w, w^v = w^j, i \not\equiv 1(mod\ q), \\ i^p \equiv 1(mod\ q^m), j \not\equiv 1(mod\ r), j^q \equiv 1(mod\ r)\rangle.$$

If P acts on R non-trivially, then $PQ/\Phi(Q)$ induces an automorphism of R of order pq, and hence P acts on Q trivially. Thus, we get that:

$$G = \langle u, v, w | u^{p^n} = 1, v^{q^n} = 1, w^r = 1, v^u = v, w^u = w^i, w^v = w^j, i \not\equiv 1(mod\ r), \\ i^p \equiv 1(mod\ r), j \not\equiv 1(mod\ r), j^q \equiv 1(mod\ r)\rangle.$$ □

The following Proposition classifies supersolvable minimal non-semi-Hamilton groups which order having just two prime divisors.

Proposition 4.4. *Suppose that G is a finite supersolvable group and $\pi(G) = \{p, q\}, p < q$. Then, G is a minimal non-semi-Hamilton group if and only if $G = \langle u, v_1, v_2 | u^{p^m} = v_1^q = v_2^q = 1,\ v_1^u = v_1^i, v_2^u = v_2^j, v_1v_2 = v_2v_1,\ i \not\equiv j(mod\ q), i^p \equiv j^p(mod\ q), i^{p^m} \equiv j^{p^m} \equiv 1(mod\ q)\rangle$.*

Proof. The necessity of the theorem is obvious, so we only need to prove the sufficiency.

Suppose that $P \in Syl_p(G), Q \in Syl_q(G)$. Then, $Q \trianglelefteq G$. If P is not cyclic, choose two different maximal subgroups P_1, P_2 of P, then P_1Q, P_2Q are all semi-Hamilton groups. Hence, for any $Q_1 \leq Q$ we have $Q_1 \trianglelefteq P_1Q_1$ and $Q_1 \trianglelefteq P_2Q_1$, therefore we get $PQ_1 = Q_1P$, which implies that Q_1 is semipermutable in G. On the other hand, each p-subgroup of G is clearly semipermutable in G. Hence, every subgroup with prime order is semipermu- table in G, and thus G itself is a semi-Hamilton group, a contradiction. Thus, P is cyclic.

Now, we have the following conclusions:

(1) Q is a 2-generator group.

Obviously Q cannot be cyclic. Since G is supersolvable, P acts reducibly on $Q/\Phi(Q)$, hence P acts decomposably on $Q/\Phi(Q)$ by Lemma 2.3. Thus, there exist two P-invariant proper subgroups Q_1, Q_2 of Q satisfying $Q = Q_1Q_2$. By hypothesis, both PQ_1 and PQ_2 are semi-Hamilton groups. Let $P = \langle u \rangle$, hence there exists a minimal generating system $\{v_1, v_2, \ldots, v_n\}$ of Q such that $v_i^u = v_i^{m_i}$, where $i = 1, 2, \ldots, n$, and m_1 is a natural number. If $n > 2$, we will prove that $m_i \neq 1$ for each i. Without loss of generality we suppose that $m_1 = 1$, then by Lemma 3.1, u acts trivially on $\langle v_1, v_i \rangle$ for any $i \neq 1$, which means that G is nilpotent, a contradiction. Thus, we get u induces a fixed-point-free power automorphism of $\langle v_i, v_j \rangle$ for any $i \neq j$. Therefore, $v_iv_j = v_jv_i$, that is Q is abelian. Without loss of generality, let $o(v_i) = q^{t_i}$, where $i = 1, 2, \ldots, n$, and v_1 be the element of maximal order in $\{v_1, v_2, \ldots, v_n\}$then for any $i \neq 1$, there exists a natural number k_i such that $(v_1v_i)^u = v_1^{m_1}v_i^{m_i} = (v_1v_i)^{k_i} = v_1^{k_i}v_i^{k_i}$. Hence, $m_1 \equiv k_i(mod\ q^{t_1}), m_i \equiv k_i(mod\ q^{t_i})$, and thus $m_1 \equiv m_i(mod\ q^{t_i})$, that is $v_i^u = v_i^{m_i} = v_i^{m_1}$, which means that u induces a fixed-point-free power automorphism of Q. Hence, G itself is a semi-Hamilton group, a contradiction. Thus, we have already proved that Q is a 2-generator group.

(2) $\Phi(Q) = 1$. In this case Q is an elementary abelian q-group of type (q, q).

Assume that $\Phi(Q) \neq 1$ and let $v \in \Phi(Q)$ be an element of order q and $v^u = v^m$, where m is a natural number. Then, there exists a natural number k such that $(v_1v)^u = v_1^{m_1}v^m = (v_1v)^k = v_1^kv^k$, $m_1 \equiv k(mod\ q^{t_1}), m \equiv k(mod\ q)$, $m_1 \equiv m(mod\ q)$. Hence, $(v_1\Phi(Q))^u = v_1^m\Phi(Q) = (v_1\Phi(Q))^m$. By the same argument we have $(v_2\Phi(Q))^u = (v_2\Phi(Q))^m$, which means that u induces a fixed-point-free power automorphism of $Q/\Phi(Q)$. Thus, we get that u acts trivially on every maximal subgroup of Q , and u induces P-invariant power automorphisms of proper subgroups of Q. Therefore, u induces a fixed-point-free power automorphism of Q, which implies that G itself is a semi-Hamilton group, a contradiction. Thus, Q is an elementary abelian q-group of type (q, q). By above discussion we get that:

$$G = \langle u, v_1, v_2 | u^{p^m} = v_1^q = v_2^q = 1, v_1^u = v_1^i, v_2^u = v_2^j, v_1v_2 = v_2v_1,$$
$$i \not\equiv j(mod\ q), i^p \equiv j^p(mod\ q), i^{p^m} \equiv j^{p^m} \equiv 1(mod\ q)\rangle. \qquad \square$$

From Proposition 4.1, Proposition 4.2, Proposition 4.3 and Proposition 4.4 we can obtain immediately the classification of finite minimal non-semi-Hamilton groups:

Theorem 4.1. *Suppose that G is a finite group. Then, G is a minimal non-semi-Hamilton group if and only if one of the following statements is true:*

(1) *$G = [P]Q$, where $Q = \langle z \rangle$ is cyclic of order $q^r > 1$, with q a prime not dividing $p-1$, and P is an irreducible Q-module over the field of p elements with kernel $\langle z^q \rangle$ in Q.*

(2) *$G = [P]Q$, where P is a non-abelian special p-group of rank $2m$, the order of p modulo q being $2m$, q is a prime, $Q = \langle z \rangle$ is cyclic of order q, z induces*

an automorphism in P such that $P/\Phi(P)$ is a faithful and irreducible Q-module, and z centralises $\Phi(P)$. Furthermore, $|P/\Phi(P)| = p^{2m}$ and $|P'| \leq p^m$.

(3) *$G = [P]Q$, where $P = \langle a_0, a_1, \ldots, a_{q-1}\rangle$ is an elementary abelian p-group of order p^q, $Q = \langle z\rangle$ is cyclic of order q^r, with q a prime such that q^f is the highest power of q dividing $p-1$ and $r > f \geq 1$. Define $a_j^z = a_{j+1}$ for $0 \leq j < q-1$ and $a_{q-1}^z = a_0^i$, where i is a primitive q^f-th root of unity modulo p.*

(4) *$G = \langle u, v, w | u^p = 1, v^{q^m} = 1, w^r = 1, v^u = v^i, w^u = w, w^v = w^j, i \not\equiv 1(mod\ q), i^p \equiv 1(mod\ q^m), j \not\equiv 1(mod\ r), j^q \equiv 1(mod\ r)\rangle$.*

(5) *$G = \langle u, v, w | u^p = 1, v^{q^n} = 1, w^r = 1, v^u = v, w^u = w^i, w^v = w^j, i \not\equiv 1(mod\ r), i^p \equiv 1(mod\ r), j \not\equiv 1(mod\ r), j^q \equiv 1(mod\ r)\rangle$.*

(6) *$G = \langle u, v_1, v_2 | u^{p^m} = v_1^q = v_2^q = 1,\ v_1^u = u^i, v_2^u = v_2^j, v_1v_2 = v_2v_1,\ i \not\equiv j(mod\ q), i^p \equiv j^p(mod\ q), i^{p^m} \equiv j^{p^m} \equiv 1(mod\ q)\rangle$.*

Example. Let $A = \langle a_1\rangle \times \langle a_2\rangle$ and $B = \langle b\rangle$, where $o(a_1) = o(a_2) = 2$ and $o(b) = 9$. Define the group $G = A \rtimes B = (\langle a_1\rangle \times \langle a_2\rangle) \rtimes \langle b\rangle$ with $a_1^b = a_2$ and $a_2^b = a_1a_2$. Clearly, $\langle b^3\rangle = Z(G)$. Therefore, the only proper subgroups of G with prime factors greater than 2 are $\langle a_1, b^3\rangle = \langle a_1\rangle \times \langle b^3\rangle$, $\langle a_2, b^3\rangle = \langle a_2\rangle \times \langle b^3\rangle$, and $\langle a_1a_2, b^3\rangle = \langle a_1a_2\rangle \times \langle b^3\rangle$. All of these are semi-Hamilton groups, while G itself is not a semi-Hamilton group. Thus, G is a minimal non-semi-Hamilton group.

Conclusion

In group theory, subgroups and quotient groups play crucial roles due to their generally simpler structure compared to the original group. Analyzing the properties of the original group through its subgroups and quotient groups is a common and effective method. For finite groups, mathematical induction is especially useful for this "small to large" approach, and when combined with proof by contradiction, it leads to the effective method of minimal counterexamples. This paper aims to clarify the properties of "minimal counterexamples" for Semi-Hamilton Groups under specific conditions and provides classifications for Semi-Hamilton Groups and Minimal Non-semi-Hamilton Groups. These properties not only have intrinsic significance but also offer a powerful tool for studying related groups.

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The influence of $IC\overline{s}$-subgroups on the structure of finite groups

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Abstract. A subgroup H of a group G is said to be an $IC\overline{s}$-subgroup of G if the intersection of H and $[H,G]$ is contained in $H_{\overline{s}G}$, where $H_{\overline{s}G}$ is the maximal s-semipermutable subgroup of G contained in H. Our main result here is the following. Let $\mathfrak{F}$ be a solubly saturated formation containing $\mathfrak{U}$ and E be a normal subgroup of a group G such that $G/E \in \mathfrak{F}$. Let $X = E$ or $X = F^*(E)$. If every non-trivial Sylow subgroup P of X has a subgroup D with $1 < |D| < |P|$ such that every subgroup of P with order $|D|$ and 4 (if $|D| = 2$ and P is a non-abelian 2-group) is an $IC\overline{s}$-subgroup of G, then $G \in \mathfrak{F}$.

Keywords: $IC\overline{s}$-subgroup, p-nilpotent group, p-supersoluble group, saturated formation.

MSC 2020: 20D10, 20D15.

1. Introduction

All groups considered in this paper are finite groups. Let G be a group. $\pi(G)$ denotes the set of all primes dividing $|G|$. $\mathfrak{U}$ denotes the class of all supersoluble groups. $Z_{\mathfrak{U}}(G)$ denotes the product of all normal subgroups N of G such that every chief factor of G below N has prime order. We use standard notation as in [2] and [5].

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Let H be a subgroup of G. It is well known that the normal closure H^G of H in G is the smallest normal subgroup of G containing H and $H^G = H[H, G]$, where $[H, G]$ is the commutator subgroup of H and G. It is an interesting question to research the relationship between $H \cap [H, G]$ and the structure of G. Recall that a subgroup H of G is said to be s-semipermutable in G if H permutes with every Sylow q-subgroup of G for every prime q not dividing $|H|$. In [12], the authors introduced the concept of an $IC\overline{s}$-subgroup of a group.

Definition 1.1. *Let H be a subgroup of G. Then, H is called an $IC\overline{s}$-subgroup of G if $H \cap [H, G] \le H_{\overline{s}G}$, where $H_{\overline{s}G}$ is the maximal s-semipermutable subgroup of G contained in H.*

The main result of [12] is as follows: Let $\mathfrak{F}$ be a solubly saturated formation containing $\mathfrak{U}$ and let E be a normal subgroup of G such that $G/E \in \mathfrak{F}$. Suppose that, $X = E$ or $X = F^*(E)$. If every cyclic subgroup of every noncyclic Sylow subgroup of X with order p and 4 (if $p = 2$) or every maximal subgroup of every Sylow subgroup of X is an $IC\overline{s}$-subgroup of G, then $G \in \mathfrak{F}$. The goal of the present paper is to generalize and extend the result mentioned above by proving the theorems below.

Theorem 1.1. *Let G be a group and $P \in Syl_p(G)$, where p is the smallest prime dividing $|G|$. Suppose that, there is a subgroup D of P with $1 < |D| < |P|$ such that every subgroup of P with order $|D|$ and 4 (if $|D| = 2$ and P is a non-abelian 2-group) is an $IC\overline{s}$-subgroup of G, then G is p-nilpotent.*

Theorem 1.2. *Let G be a group and $P \in Syl_p(G)$, where $p \in \pi(G)$. Suppose that, there is a subgroup D of P with $1 < |D| < |P|$ such that every subgroup of P with order $|D|$ and 4 (if $|D| = 2$ and P is a non-abelian 2-group) is an $IC\overline{s}$-subgroup of G, then G is p-supersoluble.*

Theorem 1.3. *Let $\mathfrak{F}$ be a solubly saturated formation containing $\mathfrak{U}$ and E be a normal subgroup of a group G such that $G/E \in \mathfrak{F}$. Let $X = E$ or $X = F^*(E)$. If every non-trivial Sylow subgroup P of X has a subgroup D with $1 < |D| < |P|$ such that every subgroup of P with order $|D|$ and 4 (if $|D| = 2$ and P is a non-abelian 2-group) is an $IC\overline{s}$-subgroup of G, then $G \in \mathfrak{F}$.*

2. Preliminary results

Lemma 2.1 ([7, Lemma 2.2])**.** *Let G be a group. Suppose that, H is an s-semipermutable subgroup of G. Then:*

(1) *If $H \le K \le G$, then H is s-semipermutable in K.*

(2) *Let N be a normal subgroup of G. If H is a p-group for some prime $p \in \pi(G)$, then HN/N is s-semipermutable in G/N.*

(3) *If $H \le O_p(G)$, then H is s-permutable in G.*

(4) *Suppose that, H is a p-group for some prime $p \in \pi(G)$ and N is normal in G. Then, $H \cap N$ is also an s-semipermutable subgroup of G.*

Lemma 2.2 ([9, Lemma A]). *If H is an s-permutable subgroup of G and H is a p-group. Then, $O^p(G) \leq N_G(H)$.*

Lemma 2.3 ([12, Lemma 2.3]). *Let G be a group, $H \leq G$, $N \trianglelefteq G$. Suppose that, H is an $IC\overline{s}$-subgroup of G. Then*

(1) *If $H \leq K \leq G$, then H is an $IC\overline{s}$-subgroup of K.*

(2) *Let $N \leq H$. If H is a p-group for some prime $p \in \pi(G)$, then H/N is an $IC\overline{s}$-subgroup of G/N.*

(3) *If H is a p-group and N is a p'-group for some prime $p \in \pi(G)$, then HN/N is an $IC\overline{s}$-subgroup of G/N.*

In the following two lemmas, we collect some results related to weakly τ-embedded subgroups. Recall that a subgroup H of G is said to be τ-permutable (τ-quasinormal) in G if H permutes with all Sylow q-subgroups Q of G such that $(q, |H|) = 1$ and $(|H|, |Q^G|) \neq 1$. A subgroup H of G is said to be weakly τ-embedded in G if there exists a normal subgroup T of G such that HT is s-permutable in G and $H \cap T \leq H_{\tau G}$, where $H_{\tau G}$ is the subgroup generated by all those subgroups of H which are τ-permutable (τ-quasinormal) in G. Obviously, $IC\overline{s}$-subgroups are weakly τ-embedded subgroups.

Lemma 2.4 ([8, Theorem 2.1]). *Let p be a prime dividing the order of a group G. Assume that all maximal subgroups of every Sylow p-subgroup of G are weakly τ-embedded in G. Then, either G is a group whose Sylow p-subgroups are of order p or G is a p-supersoluble group.*

Lemma 2.5 ([8, Theorem 2.2]). *Assume that p is a prime dividing the order of a group G. If every cyclic subgroup of G of order p or 4 (if $p = 2$) is weakly τ-embedded in G, then G is p-supersoluble.*

Lemma 2.6. *Let G be a group with an abelian Sylow 2-subgroup, and assume that any subgroup of G with order 2 is weakly τ-embedded in G. Then, G is 2-nilpotent.*

Proof. Assume that the lemma is false and choose G to be a counterexample of the smallest order. Let L be a proper subgroup of G. By the subgroup heritability of weakly τ-embedding, any subgroup of L with order 2 is weakly τ-embedded in L. Hence, L is 2-nilpotent by the minimality of G. It follows that G is minimal non-2-nilpotent. Then, G has an elementary abelian Sylow 2-subgroup P, and P is a minimal normal subgroup of G. Then, let $H = \langle x \rangle$ be a subgroup of P with order 2. Since H is weakly τ-embedded in G, there is a normal subgroup T of G such that HT is s-permutable in G and $H \cap T \leq H_{\tau G}$.

Then, $P \cap HT = H(P \cap T)$ is s-permutable in G and thus normal in G(since P is abelian). If $P \cap T = 1$, it follows that $P = H$, which implies that G is 2-nilpotent, a contradiction. Then, $P \leq T$, and so, $H = H_{\tau G}$. If Q is a non-trivial Sylow subgroup of G different from P, then it follows that HQ is a subgroup of G. Since HQ is nilpotent, Q centralizes H. Then, H is normal in G, a contradiction. □

Lemma 2.7 ([1, Theorem 2.1.6]). *Let G be a p-supersoluble group. Then, the derived subgroup G' of G is p-nilpotent. In particular, if $O_{p'}(G) = 1$, then G has a unique Sylow p-subgroup.*

Lemma 2.8 ([5, VI, 4.10]). *Assume that A and B are two subgroups of a group G and $G \neq AB$. If $AB^g = B^gA$ holds for any $g \in G$, then either A or B is contained in a proper normal subgroup of G.*

Lemma 2.9 ([11, Lemma 2.6]). *Let p be a prime dividing the order of G and P a normal p-subgroup of G. Assume that there is a subgroup D of P with $1 < |D| < |P|$ such that every subgroup of P with order $|D|$ and 4 (if $|D| = 2$ and P is a non-abelian 2-group) is an $IC\Phi_s$-subgroup of G, then $P \leq Z_{\mathfrak{U}}(G)$.*

Lemma 2.10 ([4, Lemma 3.3]). *Let $\mathfrak{F}$ be a solubly saturated formation containing all supersoluble groups. Suppose that, E is a normal subgroup of G such that $G/E \in \mathfrak{F}$. If $E \leq Z_{\mathfrak{U}}(G)$, then $G \in \mathfrak{F}$. In particular, if E is cyclic, then $G \in \mathfrak{F}$.*

Lemma 2.11 ([10, Theorem B]). *Let $\mathfrak{F}$ be a formation and E a normal subgroup of G. If $F^*(E) \leq Z_{\mathfrak{F}}(G)$, then $E \leq Z_{\mathfrak{F}}(G)$.*

3. Proofs of the main theorems

Proof of Theorem 1.1. Assume that the result is false. Let G be a counterexample with minimal order. Obviously, $|P| \geqslant p^2$ since $1 < |D| < |P|$.

(1) $|D| > p$ and $|P : D| > p$.

Assume that $|D| = p$ or $|P : D| = p$. Then, by Lemma 2.5 and Lemma 2.6 or Lemma 2.4, G is p-supersoluble. Since p is the smallest prime dividing $|G|$, we have that G is p-nilpotent, a contradiction.

(2) $O_{p'}(G) = 1$.

It follows from Lemma 2.3(3).

(3) Let L be a proper normal subgroup of G and $L_p \in Syl_p(L)$. If $|L_p| > |D|$, then L is p-nilpotent.

It follows from Lemma 2.3(1).

(4) Let K be a proper normal subgroup of G. Then, $K \leq P$.

If $PK < G$, then PK is p-nilpotent by the hypothesis and Lemma 2.3(1) and so K is p-nilpotent. Hence, $K \leq P$ by (2). If $PK = G$, then $G/K = PK/K \cong P/P \cap K$ is a p-group. Let M/K be a maximal subgroup of G/K. Clearly, $M \trianglelefteq G$, $|G : M| = p$ and $M \cap P$ is a maximal subgroup of P. By (1) and (3), we have M is p-nilpotent. Hence, $K \leq M \leq P$ by (2).

(5) $G = O^p(G)$.

If $O^p(G) < G$, then $O^p(G) \leq P$ by (4). Hence, $G = P$, a contradiction.

(6) G is not a non-abelian simple group.

Assume that G is a non-abelian simple group. Let H be a subgroup of P with order $|D|$ and Q a Sylow q-subgroup of G for some $q \in \pi(G)$ with $q \neq p$. Then, $H \cap [H, G] \leq H_{\overline{s}G}$. If $[H, G] = G$, then $H = H_{\overline{s}G}$. So $HQ^g = Q^g H$ holds for any $g \in G$. This is contrary to the simplicity of G by Lemma 2.8. If $[H, G] = 1$, then $H \leq Z(G) = 1$, so $|D| = 1$, a contradiction.

(7) Let N be a minimal normal subgroup of G. Then, $|N| < |D|$.

By (4) and (6), we have $N \leq P$. Assume that $|N| \geq |D|$. Let H be a subgroup of N with order $|D|$. Then, $H \cap [H, G] \leq H_{\overline{s}G}$. If $[H, G] = 1$, then $H \leq Z(G)$ and so $N = H \leq Z(G)$. It follows that $|N| = |D| = p$, this is contrary to (1). Hence, $[H, G] \neq 1$. Note that $H[H, G] = H^G \leq N$, so $[H, G] = N$. It follows that $H = H \cap N = H \cap [H, G] \leq H_{\overline{s}G}$. Then, $H = H_{\overline{s}G} \leq N \leq O_p(G)$ and so $G = O^p(G) \leq N_G(H)$ by Lemma 2.1(3), Lemma 2.2 and (5). This implies that $H = N$. Let U/N be a normal subgroup of P/N with order p. Since N is non-cyclic, U is non-cyclic, there exists a maximal subgroup H_1 of U such that $U = NH_1$. Obviously, $|H_1| = |N| = |D|$, and so $H_1 \cap [H_1, G] \leq (H_1)_{\overline{s}G}$. It is easy to see that $N \cap H_1 \neq 1$ and $[N \cap H_1, G] \neq 1$, so $1 < [N \cap H_1, G] \leq [N, G] \leq N$. It follows that $N = [N, G] = [N \cap H_1, G] \leq [H_1, G]$ and so $H_1 \cap N \leq H_1 \cap [H_1, G] \leq (H_1)_{\overline{s}G}$. Hence, $H_1 \cap N = (H_1)_{\overline{s}G} \cap N$ is s-permutable in G by Lemma 2.1(3)-(4). Further, $G = O^p(G) \leq N_G(H_1 \cap N)$ by Lemma 2.2 and (5). This implies $H_1 \cap N \trianglelefteq G$ and $H_1 \cap N = N$ for the minimal normality of N, a contradiction.

(8) Let N be a minimal normal subgroup of G. Then, G/N is p-nilpotent.

By (7), $|N| < |D|$. If $p > 2$ or $p = 2$ and P/N is an abelian 2-group or $p = 2$ and $|D/N| > 2$, then G/N satisfies the hypothesis of the theorem by Lemma 2.3(2), so G/N is p-nilpotent by the minimal choice of G. Now suppose that $p = 2$ and P/N is not abelian and $|D/N| = 2$. Then, $|D| = 2|N|$. Obviously, every subgroup of P/N with order 2 is an $IC\overline{s}$-subgroup of G/N. Let U/N be a cyclic subgroup of P/N with order 4. We will prove that U/N is an $IC\overline{s}$-subgroup of G/N.

Firstly, we claim that $|N| > 2$. If $|N| = 2$, then $|D| = 4$. By the hypothesis, all subgroups of P with order 4 are $IC\overline{s}$-subgroups of G. Clearly, N is an $IC\overline{s}$-subgroup of G with order 2. Assume that there is a subgroup $\langle x \rangle$ of P with order 2 such that $\langle x \rangle \neq N$. Then, $T = \langle x \rangle N$ is an elementary abelian 2-group with order 4. If $\langle x \rangle \cap [\langle x \rangle, G] = 1$, obviously, $\langle x \rangle$ is an $IC\overline{s}$-subgroup of G. If $\langle x \rangle \cap [\langle x \rangle, G] = \langle x \rangle$, then $\langle x \rangle = \langle x \rangle \cap [\langle x \rangle, G] \leq T \cap [T, G] \leq T_{\overline{s}G}$. Note that $N \leq T_{\overline{s}G}$, hence $T = T_{\overline{s}G}$. Let $Q \in Syl_q(G)$, where $q \neq 2$. Since $NQ \trianglelefteq TQ$ and NQ is 2-nilpotent, we have $Q \trianglelefteq TQ$ and so $\langle x \rangle Q$ is a subgroup of G. This implies that $\langle x \rangle = \langle x \rangle_{\overline{s}G}$. Hence, $\langle x \rangle$ is an $IC\overline{s}$-subgroup of G. We have proved that every subgroup of P with order 2 and 4 is an $IC\overline{s}$-subgroup of G. Therefore, G is 2-supersoluble by Lemma 2.5 and so G is 2-nilpotent, a contradiction. Hence, $|N| > 2$ and $|D| > 4$.

Suppose that, $N \leq \Phi(U)$, then U is cyclic and N is cyclic, a contradiction.

Hence, $N \nleq \Phi(U)$. Then, there exists a maximal subgroup U_1 of U such that $U = NU_1$. Obviously, $|U_1| = |D|$. Then, $U_1 \cap [U_1, G] \leq (U_1)_{\overline{s}G}$. It is easy to see that $N \cap U_1 \neq 1$ and $[N \cap U_1, G] \neq 1$, then $1 < [N \cap U_1, G] \leq [N, G] \leq N$. It follows that $N = [N, G] = [N \cap U_1, G] \leq [U_1, G]$. So $U \cap [U, G] = NU_1 \cap [NU_1, G] = NU_1 \cap [U_1, G]^N [N, G] = NU_1 \cap [U_1, G] = N(U_1 \cap [U_1, G]) \leq N(U_1)_{\overline{s}G} \leq (NU_1)_{\overline{s}G} = U_{\overline{s}G}$. This shows that U is an $IC\overline{s}$-subgroup of G and so U/N is an $IC\overline{s}$-subgroup of G/N. Hence, G/N is 2-nilpotent by Lemma 2.5.

(9) The final contradiction.

By (8), let K/N be the normal p-complement of G/N. Then, G/K is a p-group. On the other hand, $K \leq P$ by (4). Hence, G is a p-group, the final contradiction.

This completes the proof.

Proof of Theorem 1.2. If $p = 2$, then G is 2-nilpotent by Theorem 1.1. Hence, the theorem holds. Now we consider the case when p is an odd prime.

Assume that the result is false. Let G be a counterexample with minimal order. Obviously, $|P| \geqslant p^2$ since $1 < |D| < |P|$.

(1) $|D| > p$ and $|P : D| > p$.

It follows from Lemma 2.5 and Lemma 2.4.

(2) $O_{p'}(G) = 1$.

It follows from Lemma 2.3(3).

(3) If N is a minimal normal subgroup of G contained in P, then $|N| \leq |D|$.

Assume that $|N| > |D|$. Let H be a subgroup of N with order $|D|$ such that $H \trianglelefteq P$. Then, $H \cap [H, G] \leq H_{\overline{s}G}$. It is easy to see that $[H, G] \neq 1$ and $[H, G] = N$. It follows that $H = H \cap N = H \cap [H, G] \leq H_{\overline{s}G}$. Then, $H = H_{\overline{s}G} < N \leq O_p(G)$ and so $O^p(G) \leq N_G(H)$ by Lemma 2.1(3) and Lemma 2.2. Since $H \trianglelefteq P$, we have $H \trianglelefteq G$ and so $|H| = |D| = 1$, a contradiction.

(4) If N is a minimal normal subgroup of G contained in P, then G/N is p-supersoluble.

By (3), $|N| \leq |D|$. If $|N| < |D|$, then G/N satisfies the hypothesis of the theorem by Lemma 2.3(2), so G/N is p-supersoluble by the minimal choice of G.

If $|N| = |D|$. Now we claim that every subgroup of P/N with order p is an $IC\overline{s}$-subgroup of G/N. Let A/N be a subgroup of P/N with order p. By (1), N is non-cyclic, so A is non-cyclic. Hence, there exists a maximal subgroup T of A such that $A = TN$. Obviously, $|T| = |N| = |D|$. Then, $T \cap [T, G] \leq T_{\overline{s}G}$. It is easy to see that $N \cap T \neq 1$ and $[N \cap T, G] \neq 1$, then $1 < [N \cap T, G] \leq [N, G] \leq N$. It follows that $N = [N, G] = [N \cap T, G] \leq [T, G]$. So $A \cap [A, G] = TN \cap [TN, G] = TN \cap [T, G]^N [N, G] = TN \cap [T, G] = (T \cap [T, G])N \leq T_{\overline{s}G} N \leq (TN)_{\overline{s}G} = A_{\overline{s}G}$. This shows that A is an $IC\overline{s}$-subgroup of G and so A/N is an $IC\overline{s}$-subgroup of G/N. Hence, G/N is p-supersoluble by Lemma 2.5.

(5) $O_p(G) = 1$.

Assume that $O_p(G) \neq 1$. Let N be a minimal normal subgroup of G contained in P. Then, $N \leq O_p(G)$. By (4), it is easy to see that N is the unique minimal normal subgroup of G contained in $O_p(G)$. Moreover, $\Phi(G) = 1$. Hence, $O_p(G)$ is an elementary abelian p-group, and G has a maximal subgroup M such that $G = MN$ and $M \cap N = 1$. It is easy to deduce that $N = O_p(G)$. By (4), obviously, G is p-soluble. Hence, N is the unique minimal normal subgroup of G by (2). By (3), $|N| \leq |D|$.

If $|N| < |D|$. Let $M_p = M \cap P$. Then, $P = NM_p$. Obviously, $M_p \neq 1$ and $|N| > p$. Let P_1 be a maximal subgroup of P containing M_p. Then, $P = NP_1$ and $1 < N \cap P_1 < N$. Let H be a subgroup of P_1 containing $N \cap P_1$ such that $|H| = |D|$ and $H \trianglelefteq P$. Then, $H \cap N = P_1 \cap N \neq 1$. By the hypothesis, $H \cap [H, G] \leq H_{\overline{s}G}$. Obviously, $[H, G] \neq 1$. Hence, $H \cap N \leq H \cap [H, G] \leq H_{\overline{s}G}$. It follows that $H \cap N = H_{\overline{s}G} \cap N$ and so $O^p(G) \leq N_G(H \cap N)$ by Lemma 2.1(3)-(4) and Lemma 2.2. Since $H \cap N \trianglelefteq P$, we have $H \cap N \trianglelefteq G$ and so $H \cap N = N$ by the minimality of N, then $N \leq H \leq P_1$, a contradiction.

If $|N| = |D|$. Let T/N be a normal subgroup of P/N with order p. Then, we can write $T = N\langle x \rangle$, where $x^p \in N$, but $x \notin N$. Assume that $\Phi(T) = N$. Then, T is cyclic, so is N. It follows that $|N| = p$, a contradiction. Hence, $\Phi(T) < N$. Since $T \trianglelefteq P$, we have $\Phi(T) \trianglelefteq P$. Hence, we can choose a maximal subgroup N_1 of N containing $\Phi(T)$ such that $N_1 \trianglelefteq P$. Let $H = N_1\langle x \rangle$. Since $x^p \in \Phi(T) \leq N_1$, we have $|H| = |N| = |D|$. Then, $H \cap [H, G] \leq H_{\overline{s}G}$. Hence, we can obtain $N_1 = H \cap N \trianglelefteq G$ by a similar discussion as in the process of proving $|N| < |D|$. Hence, $N_1 = 1$ and $|N| = p$, a contradiction.

(6) Let A be a minimal normal subgroup of G. Then, A is non-p-supersoluble.

If A is p-supersoluble, then $A_p \trianglelefteq A$ by (2) and Lemma 2.7, where $A_p \in Syl_p(A)$. So $A_p \trianglelefteq G$, but this is contrary to (5).

(7) G is a non-abelian simple group.

Suppose that, G is not a simple group. Let A be a minimal normal subgroup of G. Then, $A < G$. If $|A_p| > |D|$, it easily follows that A is p-supersoluble by Lemma 2.3(1), this is contrary to (6). If $|A_p| \leq |D|$, we can pick a subgroup P_1 of P such that $A_p = A \cap P \leq P_1$ and $|P_1| = p|D|$. Then, P_1 is a Sylow p-subgroup of P_1A. Since every maximal subgroup of P_1 is an $IC\overline{s}$-subgroup of G by the hypothesis, we have every maximal subgroup of P_1 is an $IC\overline{s}$-subgroup of P_1A by Lemma 2.3(1), so P_1A is p-supersoluble by Lemma 2.4. Therefore, A is p-supersoluble, this is contrary to (6) again.

(8) The final contradiction.

Let H be a subgroup of P with $|H| = |D|$. Then, $H \cap [H, G] \leq H_{\overline{s}G}$. By (1), (6) and (7), we have $1 \neq [H, G] = G$, $H = H \cap G = H \cap [H, G] \leq H_{\overline{s}G}$ and $H = H_{\overline{s}G}$. Let Q be a Sylow q-subgroup of G for some $q \in \pi(G)$ with $q \neq p$. Then, $HQ^g = Q^gH$ for any $g \in G$. Since G is a simple group, so $G = HQ$ by Lemma 2.8, the final contradiction.

This completes the proof.

Corollary 3.1. *Let P be a normal p-subgroup of G. Suppose that, there is a subgroup D of P with $1 < |D| < |P|$ such that every subgroup of P with order $|D|$ and 4 (if $|D| = 2$ and P is a non-abelian 2-group) is an $IC\overline{s}$-subgroup of G, then $P \leq Z_{\mathfrak{U}}(G)$.*

Proof. Let $H \leq P$ such that H is an $IC\overline{s}$-subgroup of G. Then, $H \cap [H, G] \leq H_{\overline{s}G}$. By Lemma 2.1(3), $H_{\overline{s}G}$ is equal to H_{sG}, i.e. the subgroup of H generated by all subgroups of H which are s-permutable in G. Thus, H is an $IC\Phi_s$-subgroup of G (see, [11, Definition 1.1]). It follows that any subgroup of P with order $|D|$ and 4 (if $|D| = 2$ and P is a non-abelian 2-group) is an $IC\Phi_s$-subgroup of G. Then, $P \leq Z_{\mathfrak{U}}(G)$ by Lemma 2.9. This completes the proof. □

Proof of Theorem 1.3. We first prove that the theorem is true if $X = E$. Suppose that, this is not the case, and let (G, E) be a counterexample with $|G| + |E|$ minimal.

By the hypothesis and Theorem 1.2, we have E is supersoluble. Let $P \in Syl_p(E)$, where p is the largest prime divisor of $|E|$. Then, $P \trianglelefteq E$ and so $P \trianglelefteq G$. Since $(G/P)/(E/P) \cong G/E \in \mathfrak{F}$ and $(G/P, E/P)$ satisfies the hypothesis of the theorem, we have $G/P \in \mathfrak{F}$. Moreover, $P \leq Z_{\mathfrak{U}}(G)$ by Corollary 3.1. Hence, $G \in \mathfrak{F}$ by Lemma 2.10, and this contradiction completes the proof for the case $X = E$.

Now we prove that the theorem holds for $X = F^*(E)$.

By the hypothesis and Theorem 1.2, we have $F^*(E)$ is supersoluble. Hence, $F(E) = F^*(E)$. Let P be a Sylow p-subgroup of $F(E)$. Then, $P \trianglelefteq G$. By Corollary 3.1, $P \leq Z_{\mathfrak{U}}(G)$. It follows that $F(E) \leq Z_{\mathfrak{U}}(G)$. Thus we have $G \in \mathfrak{F}$ by Lemma 2.10 and Lemma 2.11.

This completes the proof.

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Generalizations of unitarily invariant norm inequalities for matrices

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Abstract. In this article, we begin by deriving a unitarily invariant norm inequality for matrices, which is a generalization of the result due to Cao and Wu. Additionally, we introduce a matrix Cauchy-Schwarz inequality for unitarily invariant norms, further generalizing the inequality proposed by Hu.

Keywords: positive semidefinite matrix, convex function, unitarily invariant norm, Cauchy-Schwarz inequality.

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1. Introduction

Throughout this paper, let M_n denote the space of $n \times n$ complex matrices. A matrix norm $||\cdot||$ is called unitarily invariant norm if $||UAV|| = ||A||$ for all $A \in M_n$ and for all unitary matrices $U, V \in M_n$. Among well-known unitarily invariant norm is the Schatten p-norm, denoted by $||\cdot||_p$ and defined as $\|A\|_p = \left(\sum_{j=1}^{n} s_j^p(A)\right)^{\frac{1}{p}} = (\mathrm{tr}|A|^p)^{\frac{1}{p}}, 1 \le p < \infty$, where $s_j(A) (j = 1, 2, \cdots, n)$ are the singular values of A with $s_1 \ge s_2 \ge \cdots \ge s_{n-1} \ge s_n \ge 0$, that is, the eigenvalues of the positive semidefinite matrix $|A| = (A^*A)^{\frac{1}{2}}$, arranged in decreasing order. The Schatten p-norm for the values $p = 1, p = 2$ and $p = \infty$ represent the

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trace norm, the Hilbert-Schmidt norm or Frobenius norm (sometimes written as $\|A\|_F$ for that reason) and the spectral norm, respectively. Another unitarily invariant norm is the Ky Fan k-norm, denoted by $||\cdot||_{(k)}$ and defined as $\|A\|_{(k)} = \sum_{j=1}^{k} s_j(A), k = 1, \cdots, n$.

Kaur and Singh [1] proved that for $A, B, X \in M_n$, if A and B are positive definite, then for any unitarily invariant norm

$$\frac{1}{2}\|A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu\| \leq \left\|(1-\alpha)A^{\frac{1}{2}}XB^{\frac{1}{2}} + \alpha\left(\frac{AX+XB}{2}\right)\right\|, \tag{1.1}$$

where $\frac{1}{4} \leq \nu \leq \frac{3}{4}$ and $\alpha \in [\frac{1}{2}, \infty)$.

Substituting A, B with A^2, B^2 and taking $u = 2\nu$ in inequality (1.1), we have

$$\frac{1}{2}\|A^u XB^{2-u} + A^{2-u}XB^u\| \leq \left\|(1-\alpha)AXB + \alpha\left(\frac{A^2X+XB^2}{2}\right)\right\|, \tag{1.2}$$

where $\frac{1}{2} \leq u \leq \frac{3}{2}$ and $\alpha \in [\frac{1}{2}, \infty)$.

Let $A, B, X \in M_n$ such that A and B are positive semidefinite. Then, for every unitarily invariant norm, the function

$$\varphi(u) = \|A^u XB^{2-u} + A^{2-u}XB^u\|$$

is convex on $[\frac{1}{2}, \frac{3}{2}]$ and attains its minimum at $u = 1$. Consequently, it is decreasing on $[\frac{1}{2}, 1]$ and increasing on $[1, \frac{3}{2}]$(See[2]). Using the convexity of the function $\varphi(u)$, Cao and Wu [3] obtained an improved version of inequality (1.2) as follows

$$\begin{aligned} &\|A^u XB^{2-u} + A^{2-u}XB^u\| \\ &\leq 2(4r_0-1)\|AXB| + 2(1-2r_0)\|A^{\frac{1}{2}}XB^{\frac{3}{2}} + A^{\frac{3}{2}}XB^{\frac{1}{2}}\| \\ &\leq 2(4r_0-1)\|AXB\| \\ &+4(1-2r_0)\left\|(1-\alpha)AXB + \alpha\left(\frac{A^2X+XB^2}{2}\right)\right\|, \end{aligned} \tag{1.3}$$

where $\frac{1}{2} \leq u \leq \frac{3}{2}$, $\alpha \in [\frac{1}{2}, \infty)$ and $r_0 = \min\{\frac{u}{2}, 1-\frac{u}{2}\}$.

Let $A, B \in M_n$ and $r > 0$. Horn and Mathias proved in [4, 5] that

$$\||A^*B|^r\|^2 \leq \|(AA^*)^r\| \cdot \|(BB^*)^r\|, \tag{1.4}$$

which is a matrix Cauchy-Schwarz inequality for unitarily invariant norms.

Bhatia and Davis [6] (See also [2, p.267, Theorem IX.5.2]) got a stronger version of inequality (1.4) as follows

$$\||A^*XB|^r\|^2 \leq \||AA^*X|^r\| \cdot \||XBB^*|^r\|, \tag{1.5}$$

for $A, B, X \in M_n$ and $r > 0$, (1.5) is equivalent to

$$\||A^{\frac{1}{2}}XB^{\frac{1}{2}}|^r\|^2 \le \||AX|^r\| \cdot \||XB|^r\|, \tag{1.6}$$

for positive semidefinite matrices A, B and arbitrary $X \in M_n$.

For $A, B, X \in M_n$ and A, B are positive semidefinite. Then, for every unitarily invariant norm and $r > 0$, the function

$$\psi(\nu) = \||A^{\nu}XB^{1-\nu}|^r\| \cdot \||A^{1-\nu}XB^{\nu}|^r\|$$

is convex on $[0,1]$ and attains its minimum at $\nu = \frac{1}{2}$. Consequently, it is decreasing on $\left[0, \frac{1}{2}\right]$ and increasing on $\left[\frac{1}{2}, 1\right]$ (See [7]). Using the convexity of the function $\psi(\nu)$, Hiai and Zhan [7] gave a refinement of the inequality (1.6) as follows

$$\||A^{\frac{1}{2}}XB^{\frac{1}{2}}|^r\|^2 \le \||A^{\nu}XB^{1-\nu}|^r\| \cdot \||A^{1-\nu}XB^{\nu}|^r\| \le \||AX|^r\| \cdot \||XB|^r\|. \tag{1.7}$$

Hu [8] utilized the convexity of the function $\psi(\nu)$ to obtain an improvement of the second inequality in (1.7)

$$\begin{aligned} &\||A^{\nu}XB^{1-\nu}|^r\| \cdot \||A^{1-\nu}XB^{\nu}|^r\| \\ &\le 2t_0\||A^{\frac{1}{2}}XB^{\frac{1}{2}}|^r\|^2 + (1-2t_0)\||AX|^r\| \cdot \||XB|^r\|, \end{aligned} \tag{1.8}$$

where $0 \le \nu \le 1$, $t_0 = \min\{\nu, 1-\nu\}$.

The unitarily invariant norm inequalities are widely applied in fields such as quantum mechanics, signal processing, data analysis and optimization theory. For example, in quantum entanglement measures, the unitarily invariant norm inequalities are employed to ensure the consistency of entanglement properties across different reference frames. Therefore, studying the unitarily invariant norm inequalities is of significant theoretical and practical importance. Many authors discussed different proofs, equivalent statements, generalizations, refinements and applications of inequalities for unitarily invariant norms. For more information on this topic, the reader is referred to [9-12] and the references therein.

This note, building on the preceding discussions, focuses on generalizing unitarily invariant norms inequalities. The structure of the note is as follows. In Section 2, we generalize inequalities (1.3) and (1.8) by using the convexity of functions. Finally, Section 3 provides concluding remarks.

2. Main results

We begin this section with the following lemma, which is useful in the proof of our results.

Lemma 2.1([13]). *Let f be a real valued convex function on an interval $[a,b]$ which contains (x_1,x_2). Then for $x_1 \le x \le x_2$, we have*

$$f(x) \le \frac{f(x_2)-f(x_1)}{x_2-x_1}x - \frac{x_1 f(x_2) - x_2 f(x_1)}{x_2-x_1}.$$

Theorem 2.1. *Let $A, X, B \in M_n$ such that A and B are positive semidefinite. Then, for any unitarily invariant norm*

$$\begin{aligned}\|A^{\nu}XB^{2-\nu}+A^{2-\nu}XB^{\nu}\| \le{}& \frac{4(\mu_0-\nu_0)}{2\mu_0-1}\left\|(1-\alpha)AXB+\alpha\left(\frac{A^2X+XB^2}{2}\right)\right\| \\ &+\frac{2\nu_0-1}{2\mu_0-1}\|A^{\mu}XB^{2-\mu}+A^{2-\mu}XB^{\mu}\|,\end{aligned}$$

where $\frac{1}{2} \le \nu \le \frac{3}{2}$, $\frac{1}{2} < \mu < \frac{3}{2}$, $\alpha \in [\frac{1}{2},\infty)$, $\nu_0 = \min\{\nu, 2-\nu\}$ *and* $\mu_0 = \min\{\mu, 2-\mu\}$.

Proof. For $\frac{1}{2} \le \nu \le \mu \le 1$, by the convexity of function $\varphi(\nu) = \|A^{\nu}XB^{2-\nu} + A^{2-\nu}XB^{\nu}\|$ and Lemma 2.1, we get

$$\varphi(\nu) \le \frac{\varphi(\mu)-\varphi(\frac{1}{2})}{\mu-\frac{1}{2}}\nu - \frac{\frac{1}{2}\varphi(\mu)-\mu\varphi(\frac{1}{2})}{\mu-\frac{1}{2}},$$

which is equivalent to

$$\varphi(\nu) \le \frac{2\nu-1}{2\mu-1}\varphi(\mu) + \frac{2(\mu-\nu)}{2\mu-1}\varphi\left(\frac{1}{2}\right), \tag{2.1}$$

combining (1.2) with (2.1), we have

$$\begin{aligned}\|A^{\nu}XB^{2-\nu}+A^{2-\nu}XB^{\nu}\| \le{}& \frac{4(\mu-\nu)}{2\mu-1}\left\|(1-\alpha)AXB+\alpha\left(\frac{A^2X+XB^2}{2}\right)\right\| \\ &+\frac{2\nu-1}{2\mu-1}\|A^{\mu}XB^{2-\mu}+A^{2-\mu}XB^{\mu}\|\end{aligned}$$

which is equivalent to

$$\begin{aligned}&\|A^{\nu}XB^{2-\nu}+A^{2-\nu}XB^{\nu}\| \\ &\le \frac{4(\mu_0-\nu_0)}{2\mu_0-1}\left\|(1-\alpha)AXB+\alpha\left(\frac{A^2X+XB^2}{2}\right)\right\| \\ &+\frac{2\nu_0-1}{2\mu_0-1}\|A^{\mu}XB^{2-\mu}+A^{2-\mu}XB^{\mu}\|.\end{aligned} \tag{2.2}$$

For $1 < \mu \le \nu \le \frac{3}{2}$, by Lemma 2.1, it follows that

$$\varphi(\nu) \le \frac{\varphi(\frac{3}{2})-\varphi(\mu)}{\frac{3}{2}-\mu}\nu - \frac{\mu\varphi(\frac{3}{2})-\frac{3}{2}\varphi(\mu)}{\frac{3}{2}-\mu},$$

which implies

$$\varphi(\nu) \leq \frac{2\nu - 2\mu}{3 - 2\mu}\varphi\left(\frac{3}{2}\right) + \frac{3 - 2\nu}{3 - 2\mu}\varphi(\mu), \tag{2.3}$$

combining (1.2) with (2.3), we have

$$\begin{aligned} \|A^\nu XB^{2-\nu} + A^{2-\nu}XB^\nu\| \leq \frac{4(\nu - \mu)}{3 - 2\mu}&\left\|(1-\alpha)AXB + \alpha\left(\frac{A^2X + XB^2}{2}\right)\right\| \\ &+ \frac{3 - 2\nu}{3 - 2\mu}\|A^\mu XB^{2-\mu} + A^{2-\mu}XB^\mu\|, \end{aligned}$$

which is equivalent to

$$\begin{aligned} &\|A^\nu XB^{2-\nu} + A^{2-\nu}XB^\nu\| \\ &\leq \frac{4(\mu_0 - \nu_0)}{2\mu_0 - 1}\left\|(1-\alpha)AXB + \alpha\left(\frac{A^2X + XB^2}{2}\right)\right\| \\ &+\frac{2\nu_0 - 1}{2\mu_0 - 1}\|A^\mu XB^{2-\mu} + A^{2-\mu}XB^\mu\|. \end{aligned} \tag{2.4}$$

It follows from (2.2),(2.4) and $\frac{1}{2} \leq \nu \leq \frac{3}{2}$, $\frac{1}{2} < \mu < \frac{3}{2}$, $\alpha \in [\frac{1}{2}, \infty)$, $\nu_0 = \min\{\nu, 2-\nu\}, \mu_0 = \min\{\mu, 2-\mu\}$ that

$$\begin{aligned} \|A^\nu XB^{2-\nu} + A^{2-\nu}XB^\nu\| \leq \frac{4(\mu_0 - \nu_0)}{2\mu_0 - 1}&\left\|(1-\alpha)AXB + \alpha\left(\frac{A^2X + XB^2}{2}\right)\right\| \\ &+ \frac{2\nu_0 - 1}{2\mu_0 - 1}\|A^\mu XB^{2-\mu} + A^{2-\mu}XB^\mu\|. \end{aligned}$$

This completes the proof. □

Remark 2.1. Let $\mu = 1$ in Theorem 2.1, we obtain the inequality (1.3).

Remark 2.2. When $\frac{1}{2} \leq \nu \leq \mu \leq 1$ or $1 < \mu \leq \nu \leq \frac{3}{2}$, the inequality in Theorem 2.1 is better than inequality (1.3).

By the convexity of function $\varphi(\nu) = \|A^\nu XB^{2-\nu}+A^{2-\nu}XB^\nu\|$ and Lemma 2.1, we know that inequality (1.3) is equivalent to

$$\varphi(\nu) \leq 2(1-\nu)\varphi\left(\frac{1}{2}\right) + (2\nu - 1)\varphi(1),\ \frac{1}{2} \leq \nu \leq 1$$

and

$$\varphi(\nu) \leq (3 - 2\nu)\varphi(1) + 2(\nu - 1)\varphi\left(\frac{3}{2}\right),\ 1 < \nu \leq \frac{3}{2}.$$

For $\frac{1}{2} \leq \nu \leq \mu \leq 1$, since $\varphi(\nu) = \|A^\nu XB^{2-\nu} + A^{2-\nu}XB^\nu\|$ is convex on $[\frac{1}{2}, 1]$, it follows by a slope argument that

$$\frac{\varphi(\mu) - \varphi(\frac{1}{2})}{\mu - \frac{1}{2}} \leq \frac{\varphi(1) - \varphi(\frac{1}{2})}{1 - \frac{1}{2}}.$$

By a small calculation, we have

$$\begin{aligned} 2(1-\nu)\varphi\left(\frac{1}{2}\right)+(2\nu-1)\varphi(1)-\left[\frac{2(\nu-\mu)}{2\mu-1}\varphi\left(\frac{1}{2}\right)+\frac{2\nu-1}{2\mu-1}\varphi(\mu)\right] \\ =\frac{2\nu-1}{2}\left[\frac{\varphi(1)-\varphi\left(\frac{1}{2}\right)}{\frac{1}{2}}-\frac{\varphi(\mu)-\varphi\left(\frac{1}{2}\right)}{\mu-\frac{1}{2}}\right] \\ \geq 0. \end{aligned}$$

For $1<\mu\leq\nu\leq\frac{3}{2}$, since $\varphi(\nu)=\|A^{\nu}XB^{2-\nu}+A^{2-\nu}XB^{\nu}\|$ is convex on $[1,\frac{3}{2}]$ and $\varphi(\nu)$ is increasing on $[1,\frac{3}{2}]$, we have

$$0\leq\frac{\varphi(\frac{3}{2})-\varphi(1)}{\frac{1}{2}}\leq\frac{\varphi(\mu)-\varphi(\frac{3}{2})}{\mu-\frac{3}{2}}.$$

By a small calculation, we have

$$\begin{aligned} (3-2\nu)\varphi(1)+2(\nu-1)\varphi\left(\frac{3}{2}\right)-\left[\left(\frac{2\nu-2\mu}{3-2\mu}\right)\varphi\left(\frac{3}{2}\right)+\frac{3-2\nu}{3-2\mu}\varphi(\mu)\right] \\ =(\nu-1)\frac{\varphi(\frac{3}{2})-\varphi(1)}{\frac{1}{2}}+\frac{3-2\nu}{2}\frac{\varphi(\mu)-\varphi(\frac{3}{2})}{\mu-\frac{3}{2}} \\ \geq 0. \end{aligned}$$

Obviously, Theorem 2.1 is a generalization of inequality (1.3).

In the following, we utilize the convexity of the function $\psi(\nu)=\||A^{\nu}XB^{1-\nu}|^r\|\cdot\||A^{1-\nu}XB^{\nu}|^r\|$ to obtain a matrix Cauchy-Schwarz inequality for unitarily invariant norms that leads to a generalization of inequality (1.8).

Theorem 2.2. *Let $A,X,B\in M_n$ such that A and B are positive semidefinite. Then, for every unitarily invariant norm*

$$\begin{aligned} &\||A^{\nu}XB^{1-\nu}|^r\|\cdot\||A^{1-\nu}XB^{\nu}|^r\| \\ &\leq(1-r_0)\||AX|^r\|\cdot\||XB|^r\|+r_0\||A^{\mu}XB^{1-\mu}|^r\|\cdot\||A^{1-\mu}XB^{\mu}|^r\|, \end{aligned} \tag{2.5}$$

where $r>0$, $0\leq\nu\leq1$, $0<\mu<1$ and $r_0=\begin{cases}\frac{\nu}{\mu}, & 0\leq\nu\leq\mu,\\ \frac{1-\nu}{1-\mu}, & \mu<\nu\leq1.\end{cases}$

Proof. Inequality (2.5) is obvious for $\nu=0,\ \mu,\ 1$. For $0<\nu<\mu$, since $\psi(\nu)=\||A^{\nu}XB^{1-\nu}|^r\|\cdot\||A^{1-\nu}XB^{\nu}|^r\|$ is convex on $[0,1]$, it follows by a slope argument that

$$\frac{\psi(\nu)-\psi(0)}{\nu-0}\leq\frac{\psi(\mu)-\psi(0)}{\mu-0},$$

then

$$\psi(\nu)\leq\left(1-\frac{\nu}{\mu}\right)\psi(0)+\frac{\nu}{\mu}\psi(\mu).$$

Therefore,

$$\begin{aligned}&|||A^\nu XB^{1-\nu}|^r\| \cdot |||A^{1-\nu}XB^\nu|^r\| \\ &\leq \left(1-\frac{\nu}{\mu}\right)|||AX|^r\| \cdot |||XB|^r\| + \frac{\nu}{\mu}|||A^\mu XB^{1-\mu}|^r\| \cdot |||A^{1-\mu}XB^\mu|^r\|,\end{aligned}$$

which is equivalent to

$$\begin{aligned}&|||A^\nu XB^{1-\nu}|^r\| \cdot |||A^{1-\nu}XB^\nu|^r\| \\ &\leq (1-r_0)|||AX|^r\| \cdot |||XB|^r\| + r_0|||A^\mu XB^{1-\mu}|^r\| \cdot |||A^{1-\mu}XB^\mu|^r\|.\end{aligned} \tag{2.6}$$

For $\mu < \nu < 1$, since $\psi(\nu)$ is convex on $[0,1]$, it follows by a slope argument that

$$\frac{\psi(\nu)-\psi(\mu)}{\nu-\mu} \leq \frac{\psi(1)-\psi(\mu)}{1-\mu},$$

then

$$\psi(\nu) \leq \left(1-\frac{1-\nu}{1-\mu}\right)\psi(1) + \frac{1-\nu}{1-\mu}\psi(\mu).$$

Therefore,

$$\begin{aligned}&|||A^\nu XB^{1-\nu}|^r\| \cdot |||A^{1-\nu}XB^\nu|^r\| \\ &\leq \left(1-\frac{1-\nu}{1-\mu}\right)|||AX|^r\| \cdot |||XB|^r\| + \frac{1-\nu}{1-\mu}|||A^\mu XB^{1-\mu}|^r\| \cdot |||A^{1-\mu}XB^\mu|^r\|,\end{aligned}$$

which is equivalent to

$$\begin{aligned}&|||A^\nu XB^{1-\nu}|^r\| \cdot |||A^{1-\nu}XB^\nu|^r\| \\ &\leq (1-r_0)|||AX|^r\| \cdot |||XB|^r\| + r_0|||A^\mu XB^{1-\mu}|^r\| \cdot |||A^{1-\mu}XB^\mu|^r\|.\end{aligned} \tag{2.7}$$

It follows from (2.6), (2.7) and $r > 0$, $0 \leq \nu \leq 1$, $0 < \mu < 1$,

$$r_0 = \begin{cases} \dfrac{\nu}{\mu}, & 0 \leq \nu \leq \mu, \\ \dfrac{1-\nu}{1-\mu}, & \mu < \nu \leq 1 \end{cases}$$

that

$$\begin{aligned}&|||A^\nu XB^{1-\nu}|^r\| \cdot |||A^{1-\nu}XB^\nu|^r\| \\ &\leq (1-r_0)|||AX|^r\| \cdot |||XB|^r\| + r_0|||A^\mu XB^{1-\mu}|^r\| \cdot |||A^{1-\mu}XB^\mu|^r\|.\end{aligned}$$

The proof is completed. □

Remark 2.3. Let $\mu = \frac{1}{2}$ in Theorem 2.2, we obtain the inequality (1.8).

Remark 2.4. When $0 \le \nu \le \mu \le \frac{1}{2}$ or $\frac{1}{2} < \mu \le \nu \le 1$, the inequality in Theorem 2.2 is better than inequality (1.8).

By the convexity of function $\psi(\nu) = \||A^\nu XB^{1-\nu}|^r\| \cdot \||A^{1-\nu}XB^\nu|^r\|$ and Lemma 2.1, we know that inequality (1.8) is equivalent to

$$\psi(\nu) \le (1-2\nu)\psi(0) + 2\nu\psi\left(\frac{1}{2}\right),\ 0 \le \nu \le \frac{1}{2} \tag{2.8}$$

and

$$\psi(\nu) \le (2\nu-1)\psi(1) + 2(1-\nu)\psi\left(\frac{1}{2}\right),\ \frac{1}{2} \le \nu \le 1. \tag{2.9}$$

For $0 \le \nu \le \mu \le \frac{1}{2}$, compared with inequality (2.8)

$$\begin{aligned}&(1-2\nu)\psi(0)+2\nu\psi\left(\frac{1}{2}\right)-\left[\left(1-\frac{\nu}{\mu}\right)\psi(0)+\frac{\nu}{\mu}\psi(\mu)\right]\\ =&\nu(2(\psi\left(\frac{1}{2}\right)-\psi(0))-\frac{\nu}{\mu}(\psi(\mu)-\psi(0))).\end{aligned}$$

Since $\psi(\nu) = \||A^\nu XB^{1-\nu}|^r\| \cdot \||A^{1-\nu}XB^\nu|^r\|$ is convex on $[0,1]$, it follows by a slope argument that

$$\frac{\psi\left(\frac{1}{2}\right)-\psi(0)}{\frac{1}{2}-0} \ge \frac{\psi(\mu)-\psi(0)}{\mu-0},$$

that is

$$2(\psi\left(\frac{1}{2}\right)-\psi(0)) - \frac{1}{\mu}(\psi(\mu)-\psi(0)) \ge 0,$$

thus, we have

$$(1-2\nu)\psi(0)+2\nu\psi\left(\frac{1}{2}\right) \ge \left(1-\frac{\nu}{\mu}\right)\psi(0)+\frac{\nu}{\mu}\psi(\mu).$$

For $\frac{1}{2} < \mu \le \nu \le 1$, compared with inequality (2.9)

$$\begin{aligned}&(2\nu-1)\psi(1)+2(1-\nu)\psi\left(\frac{1}{2}\right)-\left[\left(1-\frac{1-\nu}{1-\mu}\right)\psi(1)+\frac{1-\nu}{1-\mu}\psi(\mu)\right]\\ &\qquad = (1-\nu)\left(\frac{\psi(1)-\psi(\mu)}{1-\mu}-\frac{\psi(1)-\psi(\frac{1}{2})}{\frac{1}{2}}\right).\end{aligned}$$

Since $\psi(\nu)$ is convex on $[0,1]$, it follows by a slope argument that

$$\frac{\psi(1)-\psi(\mu)}{1-\mu} \ge \frac{\psi(1)-\psi(\frac{1}{2})}{\frac{1}{2}}.$$

Thus, we have

$$(2\nu-1)\psi(1)+2(1-\nu)\psi\left(\frac{1}{2}\right) \ge \left(1-\frac{1-\nu}{1-\mu}\right)\psi(1)+\frac{1-\nu}{1-\mu}\psi(\mu).$$

Obviously, Theorem 2.2 is a generalization of inequality (1.8).

3. Conclusion

In recent years, there has been a growing interest in exploring unitarily invariant norms inequalities. By utilizing the convexity of the functions $\varphi(\nu)$ and $\psi(\nu)$, we introduce two new matrix inequalities for unitarily invariant norms, which generalize several previously known results. The inequalities derived in this work lead to refinements of unitarily invariant norms inequalities under specific conditions. Future research will further explore these topics.

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An overview of hypercompositional algebra applications on graphs

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Abstract. Graphs are fundamental structures in mathematics and computer science for modeling relationships between objects. This paper studies three hypercompositional structures that are derived from graphs, namely the Path hyperoperation, Simple Path hyperoperation, and Ancestry hyperoperation. These hyperoperations capture complex relationships, offering a robust framework for analyzing intricate connections within graphs. We investigate their properties and provide detailed examples to illustrate their applications.

Keywords: hypercompositional algebras, graph theory, hyperstructures.

MSC 2020: 05C20, 05C25, 08Cxx.

1. Introduction

Graphs are ubiquitous in various fields such as computer science, transportation and communication, biology, and social sciences, serving as a crucial model for representing relationships among entities [1], [16]-[17], [29]. Traditionally, graph theory has mainly utilized adjacency matrices or doubly ranked graph operations to represent graphs and analyze their properties [14]-[15], [21]-[22]. This approach allows for the straightforward examination of graph structures and the relationships between vertices. Recently, the introduction of algebraic hyperstructures has provided a powerful alternative for representing and exploring the properties of graph structures. This approach takes advantage of the well-established relationship between hyperstructures and binary relations [9]-[11], [32]-[34], see also [2], [8], [23], [26]-[28], [30]-[31]. Additionally, the inherent symmetry with hyperstructures has been further explored in [3], [12]-[13], [24], [25].

The application of hypercompositional algebra in graph theory has demonstrated its robustness and convenience, providing new insights and methods for exploring complex graph characteristics [18], [19], [35]. By leveraging the rich algebraic framework of hyperstructures, researchers can uncover deeper connections and more intricate properties within graph theory, enhancing the analytical capabilities and expanding the scope of graph analysis.

This paper delves into three such hyperoperations: Path hyperoperation, Simple Path hyperoperation, and Ancestry hyperoperation, exploring their theoretical foundations and practical implications. We demonstrate that the characteristics of these hyperoperations are dictated by the structure of their underlying graphs and that these characteristics define the corresponding classes of graphs. This important connection between hyperoperations and graph properties reveals that we can identify and understand graphs by studying the properties of the related hyperoperations, and vice versa.

2. Preliminaries

A **Hypergroupoid** $(V, \star)$ consists of a nonempty set V and a hyperoperation

$$\star : V \times V \to \mathcal{P}(V),$$

where $\mathcal{P}(V)$ is the powerset of V. A hypergroupoid is called:

Nonpartial if $v \star w \neq \emptyset$ for all $v, w \in V$,

Degenerative if $v \star w = \emptyset$ for all $v, w \in V$,

Total if $v \star w = V$ for all $v, w \in V$.

Given a binary relation $R \subseteq V \times V$, Corsini's hyperoperation [4]-[7], is defined by the below mapping

$$(v, w) \mapsto v *_R w = \{z \in V \mid (v, z),\ (z, w) \in R\}.$$

A **directed graph** G is defined as a pair (V, E) where V is a set of vertices or nodes and $E \subseteq V \times V$ is a set of directed edges, represented as ordered pairs from the set V.

Example 1. Consider the directed graph G_1 of Figure 1 with set of vertices $V_1 = \{v, w, x, y, z\}$ and set of edges

$$E_1 = \{(v, w), (w, x), (w, y), (x, y), (y, z), (z, x)\}.$$

A subgraph of a directed graph $G = (V, E)$ is a graph $H = (V_H, E_H)$ where:

$V_H \subseteq V$ is a subset of the vertices of G,

$E_H \subseteq E$ is a subset of the edges of G such that every edge in E_H has both endpoints in V_H.

Example 2. A subgraph of the graph G_1 of Example 1 is the graph H_1 of figure 2 with set of vertices $V_{H_1} = \{v, w, x\}$ and set of edges $E_{H_1} = \{(v, w), (w, x)\}$.

A **path** P of length n from a node u_1 to a node u_n in a graph $G = (V, E)$ is a sequence of edges

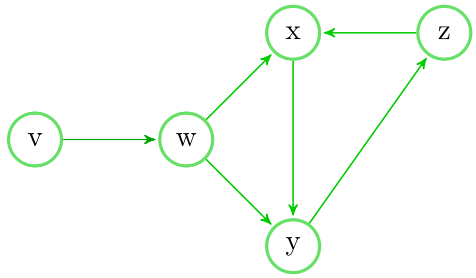

Figure 1: A directed graph G_1

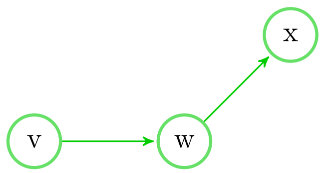

Figure 2: A subgraph H_1 of the graph G_1

$$(u_1, u_2), (u_2, u_3), \ldots, (u_{n-1}, u_n),$$

where each edge $(u_i, u_{i+1}) \in E$. Based on this definition, we say that the nodes $u_1, u_2, \ldots, u_n$ lie on the path P and the same for all the edges in the above sequence. We also note that nodes and edges may appear more than once in a path. If all nodes appear at most once in a path then the path is called **simple**. A **cycle** is a path that starts and ends at the same node. The empty sequence ϵ is a cycle from u to itself, containing only u. The set of all nodes that lie in a path from u_1 to u_n is denoted

$$path(u_1, u_n) = \{x \mid x \text{ lies on a path from } u_1 \text{ to } u_n\}.$$

Example 3. Considering the graph G_1 of Example 1 we see that the sequence $(v, w), (w, x), (x, y)$ is a path from v to y, only the empty path ϵ exists from w to w while there are infinite paths from x to x and there are infinite paths from w to z. Hence we have

$$path(v, w) = \{v, w\},\ path(w, v) = \emptyset,\ path(w, y) = \{w, x, y, z\},$$

and

$$path(w, w) = \{w\},\ path(x, x) = \{x, y, z\}.$$

A directed graph G is called **strongly connected** if, for any two nodes u_1 and u_2, there exists at least one path from u_1 to u_2 and vice versa.

Example 4. The graph S of figure 3 with set of vertices $V_S = \{a, b, c, d\}$ and set of edges

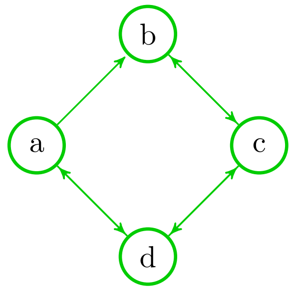

Figure 3: A strongly connected graph S

$$E_S = \{(a,b),(a,d),(d,a),(b,c),(c,b),(c,d),(d,c)\}$$

is strongly connected.

A strongly connected subgraph of a graph G is called **strongly connected component** of G.

Example 5. Considering the graph G_1 of Example 1, we can define a subgraph of it H, depicted in Figure 4, with set of vertices $V_H = \{x, y, z\}$ and set of edges $E_H = \{(x,y),(y,z),(z,x)\}$.

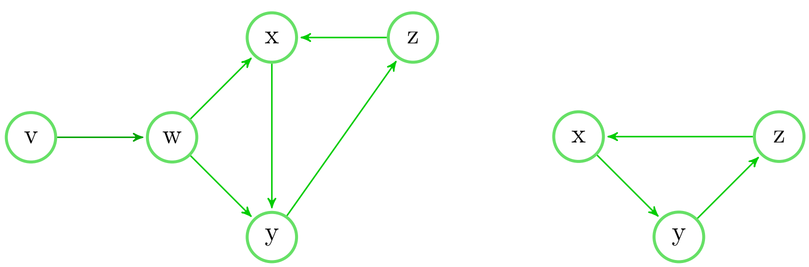

Figure 4: The graph G_1 of Example 1 and a strongly connected subgraph H

3. Hyperoperations on graphs

In this section we are going to introduce three main hyperoperations on graphs. We start with the path hyperoperation $\star_G$ which is defined for a given directed graph G as a mapping that associates a given pair of graph nodes (vertices) to the set composed of all graph nodes that lie on a directed path between the given nodes. This definition extends a well-known hyperoperation introduced by Corsini, where given a relation R on V and u, v elements of V, their product with respect to the Corsini operation is properly included in $u \star_G v$. More formally, given a graph $G = (V, E)$, the path hyperoperation is a mapping

$$\star_G : V \times V \to \mathcal{P}(V)$$

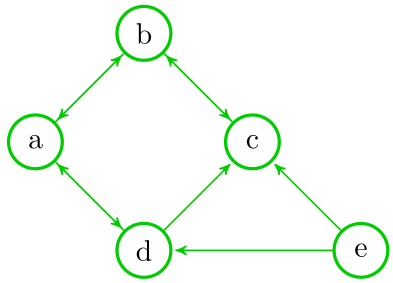

Figure 5: The directed graph G_2 of Example 7

defined by

$$v \star_G w = \{u \in V \mid u \text{ lies on a path between } v \text{ and } w\}.$$

The hyperstructure $(V, \star_G)$ is called the path hypergroupoid corresponding to G.

Example 6. For the graph G_1 of Example 1 the related path hyperoperation is given in the following table.

$\star_{G_1}$	v	w	x	y	z
v	$\{v\}$	$\{v,w\}$	$\{v,w,x,y,z\}$	$\{v,w,x,y,z\}$	$\{v,w,x,y,z\}$
w	$\emptyset$	$\{w\}$	$\{w,x,y,z\}$	$\{w,x,y,z\}$	$\{w,x,y,z\}$
x	$\emptyset$	$\emptyset$	$\{x,y,z\}$	$\{x,y,z\}$	$\{x,y,z\}$
y	$\emptyset$	$\emptyset$	$\{x,y,z\}$	$\{x,y,z\}$	$\{x,y,z\}$
z	$\emptyset$	$\emptyset$	$\{x,y,z\}$	$\{x,y,z\}$	$\{x,y,z\}$

We are now ready to illustrate the relationship between the path hyperoperation and some well known graph properties cf. [20]. The first result illustrates the relationship of the path hyperoperation with the existence of strongly connected components of graphs.

Theorem 1. *For any graph $G = (V, E)$ and nodes $u_1, u_2 \in V$, the following conditions are equivalent*

i) $u_1 \star_G u_2 \neq \emptyset$ *and* $u_2 \star_G u_1 \neq \emptyset$.

ii) There is a strongly connected component of G that includes the nodes u_1 and u_2.

Example 7. The path hyperoperation of the graph G_2 depicted in Figure 5 is given below.

$\star_{G_2}$	a	b	c	d	e
a	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\emptyset$
b	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\emptyset$
c	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\emptyset$
d	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\emptyset$
e	$\{a,b,c,d,e\}$	$\{a,b,c,d,e\}$	$\{a,b,c,d,e\}$	$\{a,b,c,d,e\}$	$\{e\}$

A strongly connected component of G_2 is the graph H_2 depicted in Figure 6. It is easy to check that the conditions of Theorem 1 are satisfied for the nodes a, b, c, d of G_2.

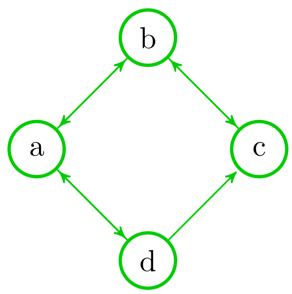

Figure 6: A connected component of the graph G_2

The next result characterizes strong connectivity cf. [18].

Theorem 2. *For any graph $G = (V, E)$, the below conditions are equivalent*

i) The graph G is strongly connected.

ii) The corresponding hypergroupoid $(V, \star_G)$ is nonpartial.

iii) The hyperoperation $\star_G$ is total.

Example 8. It is straightforward to check that the below path hyperoperation table of the strongly connected graph G_3 of Figure 7 satisfies conditions $ii)$ and

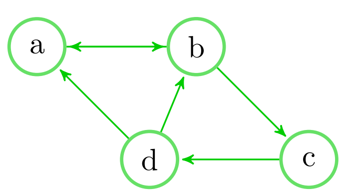

Figure 7: The Strongly connected graph G_3 of Example 8

iii) of Theorem 2.

$\star_G$	a	b	c	d
a	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$
b	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$
c	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$
d	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$

The existence of cycles inside a graph can be also investigated by examining the properties of the related path hyperoperation cf. [19].

Theorem 3. *Given a graph $G = (V, E)$ and nodes $u_1, u_2 \in V$, the following conditions are equivalent.*

i) *It holds $u_1 \star_G u_2 = u_2 \star_G u_1 \neq \emptyset$.*

ii) *There exists a cycle in G that includes both nodes u_1 and u_2.*

Example 9. By examining graph G_1 of Example 1 we can identify a cycle going through the nodes x, y, z. We can verify the validity of Theorem 3 by checking the corresponding path hyperoperation table of G_1 presented in Example 6.

Commutativity of the path hyperoperation is related with strongly connected graph components as it is illustrated in the next theorem cf. [20].

Theorem 4. *The below conditions are equivalent for a given graph $G = (V, R)$.*

i) *The path hyperoperation $\star_G$ is commutative.*

ii) *G can be obtained as the union of disjoint strongly connected graphs.*

Associativity of the path hyperoperation can be obtained as a corollary of the following theorem cf. [19].

Theorem 5. *Given a graph $G = (V, E)$ and nodes $v, w, u \in V$ it holds*

$$(v \star_G w) \star_G u = v \star_G (w \star_G u) = v \star_G w \star_G u.$$

Corollary 1. *The path hyperoperation is associative*

Given a graph G, the **simple path hyperoperation** $\star_G^s$ maps a pair of nodes of the graph G to the set that includes all nodes lying on a directed simple path between the two given nodes. It is evident that the simple path hyperoperation between two nodes of a graph results in a set that is always included in the path hyperoperation between the same nodes. To formally introduce it, given a graph $G = (V, E)$, the simple path hyperoperation is a mapping

$$\star_G^s : V \times V \to \mathcal{P}(V)$$

defined by

$$u_1 \star_G^s u_2 = \{u \in V \mid u \text{ lies on a simple path between } u_1 \text{ and } u_2\}.$$

The hyperstructure $(V, \star_G^s)$ is called the simple path hypergroupoid corresponding to G.

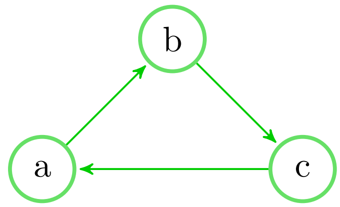

Figure 8: The graph G_4 of Example 10

Example 10. For the graph G_4 of Figure 8, the table of the simple path hyperoperation is given below.

$\star_G^s$	a	b	c
a	$\{a,b,c\}$	$\{a,b\}$	$\{a,b,c\}$
b	$\{a,b,c\}$	$\{a,b,c\}$	$\{b,c\}$
c	$\{a,c\}$	$\{a,b,c\}$	$\{a,b,c\}$

It is clear that $\star_G^s$ is nonpartial and not total. Hence nonpartiality and totality are not equivalent for the simple path hyperoperation as opposed to the case for the path hyperoperation as it is described in Theorem 2.

Discrete graphs can be also characterized by the path and the simple path hyperoperations as follows.

Proposition 1. *For a graph $G = (V, E)$, the following conditions are equivalent*

i) The graph G is discrete.

ii) The path hyperoperation $\star_G$ organizes V into a weakly degenerative hypergroupoid.

iii) The simple path hyperoperation $\star_G^s$ organizes V into of a weakly degenerative hypergroupoid.

The third hyperoperation we will introduce is the **ancestry hyperoperation**, which assigns any two nodes u_1 and u_2 of a graph G to the set of all the nodes of G that have paths going to u_1 and u_2. Formally we have the following definition, given a graph $G = (V, E)$, the ancestry hyperoperation is a mapping

$$\star_G^c : V \times V \to \mathcal{P}(V)$$

defined by

$$u_1 \star_G^c u_2 = \{u \in V \mid \text{path}(u, u_1) \neq \emptyset \text{ and } \text{path}(u, u_2) \neq \emptyset\}.$$

The hyperstructure $(V, \star_G^c)$ is called the ancestry hypergroupoid corresponding to G.

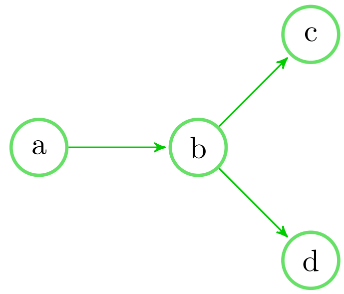

Figure 9: The graph G_5 of Example 11

Example 11. Consider the graph G_5 of Figure 9. In the below tables of the path and the ancestry hyperoperation we can see that the path hyperoperation is partial and non-commutative as opposed to the ancestry hyperoperation which is non-partial and commutative.

$\star_G$	a	b	c	d
a	$\{a\}$	$\{a,b\}$	$\{a,b,c\}$	$\{a,b,d\}$
b	$\emptyset$	$\{b\}$	$\{b,c\}$	$\{b,d\}$
c	$\emptyset$	$\emptyset$	$\{c\}$	$\emptyset$
d	$\emptyset$	$\emptyset$	$\emptyset$	$\{d\}$

$\star_G^c$	a	b	c	d
a	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$
c	$\{a\}$	$\{a,b\}$	$\{a,b,c\}$	$\{a,b\}$
d	$\{a\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b,d\}$

The ancestry hyperoperation of the previous example was commutative and in the next theorem we see that this is a property that holds in general [20].

Theorem 6. *The ancestry hyperoperation $\star_G^c$ is commutative and associative.*

4. Conclusion

By investigating hyperstructures derived from graphs, we obtain a powerful framework to analyze complex relationships within graphs. We explored three key hyperoperations: the path hyperoperation, which maps two vertices to the set of all vertices on paths between them; the simple path hyper-operation, which is similar but only considers simple paths between nodes; and the ancestry hyper-operation, which maps two nodes to the set of their common ancestors, defined by the paths leading to these vertices.

These hyperoperations provide new insights and tools for analyzing the intricate structures of graphs. Future research can extend these concepts to more complex graph structures, such as weighted and dynamic graphs. Additionally, the introduced hyperoperations can be applied to network analysis, aiding in the development of efficient algorithms for large-scale graphs.

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A note on hyperrings and hypermodules

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Abstract. The main purpose of this paper is to study the concept of the hyperring $(\mathbb{N}, \oplus, \cdot)$, where $m \oplus n = \{m + n, k \mid \min\{m, n\} + k = \max\{m, n\}\}$, for all $m, n \in \mathbb{N}$ and the operation $\cdot$ is the usual multiplication in $\mathbb{N}$. In particular, we prove that this hyperring $(\mathbb{N}, \oplus, \cdot)$ is isomorphic to Krasner's quotient hyperring $\frac{\mathbb{Z}}{G}$ in [10]. Moreover, we construct the hyperstructure $(\mathbb{N}_m, \oplus_m, \cdot)$, which is a class of examples of hypermodules and hyperrings.

Keywords: hyperring, hypermodule.
MSC 2020: 20N20, 16D80.

1. Introduction

Let H be a nonvoid set. A mapping from $H \times H$ into H is called a *composition* on H. A composition $\diamond$ on a set H is called *associative* if, for all $x, y, z \in H$, $x \diamond (y \diamond z) = (x \diamond y) \diamond z$, and is called *reproductive* if $x \diamond H = H \diamond x = H$, for all $x \in H$. The pair $(H, \diamond)$ is called *group* if H is a nonvoid set and $\diamond$ is an associative and reproductive composition on H. It follows from [16, Theorem 2

∗. Corresponding author

and Theorem 3] that the pair $(H, \diamond)$ is a group if and only if H is a nonvoid set and $\diamond$ has the following properties:

(1) For all $x, y, z \in H$, $x \diamond (y \diamond z) = (x \diamond y) \diamond z$ (**associative**).

(2) There exists $e \in H$ such that for all $x \in H$, $x \diamond e = e \diamond x = x$ (**existence of an identity**).

(3) For all $x \in H$, there exists y such that $x \diamond y = y \diamond x = e$ (**existence of an inverse**).

In [17], Marty, who is a French mathematician, extends a composition on a set H to a hypercomposition on a set H as follows. A mapping $\uplus : H \times H \longrightarrow P(H)$ is called a *hypercomposition* on a set H, where $P(H)$ is the power set of H. He calls $(H, \uplus)$ *hypergroup* if H is a nonvoid set and $\uplus$ is an associative and reproductive hypercomposition on H.The concept of hypergroups is an algebraic structure, and it is clear that the groups are an example of hypergroups. His French contemporaries continue his ideas to included additional algebraic structures, which they call hypercompositional structures. The nonvoid result of the hypercomposition in hypergroups and in all relevant structures such as hyperfields, hyperrings, hypermodules etc., is a consequence of the associative and reproductive laws ([16, Theorem 12]). Krasner introduces hyperfields, hyperrings, and hypermodules in his papers [10] and [11]. In the literature, the structure hyperring (respectively, hyperfield) is known as Krasner hyperring (respectively, Krasner hyperfield).

The main purpose of this paper is to develop the concept of the hyperring $(\mathbb{N}, \oplus, \cdot)$, where $m \oplus n = \{m + n, k \,|\, \min\{m, n\} + k = \max\{m, n\}\}$, for all $m, n \in \mathbb{N}$ and the operation $\cdot$ is the usual multiplication in $\mathbb{N}$. It follows that the hyperring $(\mathbb{N}, \oplus, \cdot)$ is a principal hyperideal domain. We prove that $(\mathbb{N}, \oplus, \cdot)$ is isomorphic to Krasner's quotient hyperring $\frac{\mathbb{Z}}{G}$ in [10]. Also, we construct the hyperstructure $(\mathbb{N}_m, \oplus_m, \cdot)$, which is a class of examples of hypermodules and hyperrings.

2. Preliminaries

This section briefly recalls the main concepts and results related to types of hyperrings and hypermodules. To better understand the topic, we start with some fundamental definitions of hypercompositional algebra presented in books [5, 7] and overview articles [12, 14, 15, 16, 18].

Let H be a nonvoid set and a mapping $+ : H \times H \longrightarrow \mathcal{P}(H)$ be a hypercomposition on H. Then, $(H, +)$ is said to be a *hypergroupoid*. Moreover, for any nonempty subsets X and Y of H, define

$$X + Y = \bigcup \{z \in x + y \,|\, x \in X \text{ and } y \in Y\} = \bigcup_{(x,y) \in X \times Y} x + y.$$

We simply write $a+X$ and $X+a$ instead of $\{a\}+X$ and $X+\{a\}$, respectively, for any $a \in H$ and any nonvoid subset X of H. A hypergroupoid $(H,+)$ is said to be a

(1) *semihypergroup* if $+$ is an associative hypercomposition on H.

(2) *quasihypergroup* if $+$ is a reproductive hypercomposition on H.

A nonvoid subset S of a hypergroup $(H,+)$ is said to be a *subhypergroup* of H, if for every $a \in S$, $a+S=S=S+a$.

A hypergroup $(H,+)$ is said to be *canonical hypergroup* if

(1) for every $a,b \in H$, $a+b=b+a$, that is, it is commutative;

(2) There exists a unique $0 \in H$ such that for each $a \in H$ there exists a unique element a' in H, denoted by $-a$, such that $0 \in a+(-a)$;

(3) for every $a,b,c \in H$, if $c \in a+b$, then $a \in c+(-b) := c-b$.

As it is proved in [13], if $(H,\,+)$ is a canonical hypergroup, then $a+0=a$, for all $a \in H$.

Let $(R,+,.)$ be a hypercompositional structure. $(R,+,.)$ is said to be a *(Krasner) hyperring* if

(1) $(R,+)$ is a canonical hypergroup;

(2) $(R,.)$ is a semigroup with a bilaterally absorbing element 0, i.e.,

   (a) $a.b \in R$, for all $a,\, b \in R$;
   (b) $a.(b.c)=(a.b).c$, for all $a,\, b,\, c \in R$;
   (c) $a.0=0.a=0$, for all $a \in R$;

(3) The multiplication distributes over the addition on both sides.

If in addition:

(4) $a.b=b.a$, for all $a,\, b \in R$,

then R is said to be a *commutative hyperring*. If $(R,+,.)$ contains an element 1_R such that

(5) $a=a.1_R=1_R.a$ for every $a \in R$,

then R is said to be a hyperring with identity.

Let $(R,+,\cdot)$ be a hyperring and I be a nonvoid subset of R. I is called a *left hyperideal* (respectively, *right hyperideal*) of R provided $(I,\,+)$ is a subhypergroup and $r.a \in I$ (respectively, $a.r \in I$), for all $a \in I$ and $r \in R$. I is said to be *hyperideal* of R if it is both a right and a left hyperideal of R.

A left Krasner hypermodule over a hyperring R with identity is a canonical hypergroup $(M, +)$ together with a map $R \times M \longrightarrow M$ such that to every (r, m), where $r \in R$ and $m \in M$, there corresponds a uniquely determined element $rm \in M$ and the following conditions are satisfied:

(1) $r(m_1 + m_2) = rm_1 + rm_2$;

(2) $(r + s)m = rm + sm$;

(3) $(r.s)m = r(sm)$;

(4) $1_R m = m$ and $r0_M = 0_R m = 0_M$,

for any m, m_1, $m_2 \in M$ and r, $s \in R$.

Throughout this paper, for a simple explanation, when we say hypermodule, we mean the left Krasner hypermodule. A nonvoid subset N of an R-hypermodule M is called a *subhypermodule* of M, denoted by $N \leq M$ if N is an R-hypermodule under the same hyperoperations of M. It is clear that M and $\{0_M\}$ are *trivial subhypermodules* of M. It is known that a non-empty subset N of an R-hypermodule M is a subhypermodule of M if and only if $a - b \subseteq M$ and $ra \in M$, for all a, $b \in M$ and $r \in R$.

Let R be a hyperring. It follows from [3, Lemma 3.1] that R is an R-hypermodule. Then, a nonvoid subset I of R is a left hyperideal of R if and only if it is a subhypermodule of the hypermodule $_R R$.

Let M be a hypermodule over a hyperring R and K be a subhypermodule of M. Consider the set $\frac{M}{K} = \{\, a + K \mid a \in M \,\}$. Then, $\frac{M}{K}$ is a hypermodule over the hyperring R under the hyperoperation $+ : \frac{M}{K} \times \frac{M}{K} \longrightarrow \mathcal{P}(\frac{M}{K})$ and the external operation $\cdot : R \times \frac{M}{K} \longrightarrow \frac{M}{K}$ via $(a+K)+(a'+K) = \{\, b+K \mid b \in a+a' \,\}$ and $r \cdot (a+K) = ra+K$ for every $a, a', b \in M$ and $r \in R$. The hypermodule $\frac{M}{K}$ is called the *quotient hypermodule* of the hypermodule M.

Let M and N be R-hypermodules. A single-valued function $f : M \longrightarrow N$ is called *normal homomorphism* (or briefly, homomorphism) if

(1) $f(m_1 +_M m_2) = f(m_1) +_N f(m_2)$, for all m_1, $m_2 \in M$

(2) $f(rm) = rf(m)$, for all $r \in R$ and $m \in M$.

We denote by $Hom_R(M, N)$ the family of all homomorphisms form M to N.

3. Hyperrings and hypermodules

Let $\mathbb{Z}$ denote the set of all integers. Let $+$ and $\cdot$ denote the usual addition and multiplication. Then, it is well known that the structure $(\mathbb{Z}, +, \cdot)$ is a principal ideal domain. Let $\mathbb{N}$ denote the set of all non-negative integers. However, under the same operations, the structure $(\mathbb{N}, +, \cdot)$ does not have the structure of a ring. We will now construct the hyperring structure on the set $\mathbb{N}$ with the help of the same operations. Then, using this hypercompositional structure, we will give new structures of hyperrings and hypermodules. Note that we will use the following structure of the hyperring we give in this section freely in this article without reference.

Construction. Let $\mathbb{N}$ denote the set of all non-negative integers. Let $+$ and $\cdot$ denote the usual addition and multiplication in $\mathbb{N}$. Define the hypercomposition "$\oplus$" on $\mathbb{N}$ as follows: for any $m, n \in \mathbb{N}$

$$m \oplus n = \{m + n,\, k \mid \min\{m, n\} + k = \max\{m, n\} \text{ for some } k \in \mathbb{N}\}.$$

It is clear that $(\mathbb{N}, \oplus)$ is the hypergroupoid.

(1) Let us, now, show that $(\mathbb{N}, \oplus)$ is the canonical hypergroup. Since $\mathbb{N}$ is well-ordered, we get $m \oplus n = n \oplus m$ and so we can always choose $m \leq n$ whenever $(m, n) \in \mathbb{N} \times \mathbb{N}$.
Let $m, n, p \in \mathbb{N}$. Since $\mathbb{N}$ is well-ordered, we will assume that $m \leq n \leq p$ without restriction of generality. Therefore, $\min\{m, n\} = m$, $\max\{m, n\} = n$, $\min\{n, p\} = n$ and $\max\{n, p\} = p$. It follows that $n = k_1 + m$ and $p = k_2 + n$, for some elements $k_1, k_2 \in \mathbb{N}$. Now,

$$\begin{aligned} m \oplus (n \oplus p) &= m \oplus \{n + p, k_2\} \\ &= m \oplus (n + p) \cup m \oplus k_2 \end{aligned}$$

Case 1. Let $m \leq k_2$. Then, $k_2 = k_3 + m$. So we can write $p = n + k_2 = (n + m) + k_3$. Now

$$\begin{aligned} m \oplus (n \oplus p) &= m \oplus \{n + p, k_2\} \\ &= m \oplus (n + p) \cup m \oplus k_2 \\ &= \{m + (n + p), m + k_1 + k_2\} \cup \{m + k_2, k_3\} \\ &= \{m + (n + p), m + k_1 + k_2, m + k_2, k_3\} \\ &= \{(m + n) + p, k_3\} \cup \{m + k_1 + k_2, m + k_2\} \\ &= \{(m + n) + p, k_3\} \cup \{p + k_1, m + k_2\} \\ &= (m + n) \oplus p \cup k_1 \oplus p \\ &= \{m + n, k_1\} \oplus p = (m \oplus n) \oplus p \end{aligned}$$

Case 2. Let $k_2 \leq m$. Then, we can write $m = k_2 + k_4$, for some element $k_4 \in \mathbb{N}$. Thus,

$$\begin{aligned} m \oplus (n \oplus p) &= m \oplus \{n + p, k_2\} \\ &= m \oplus (n + p) \cup m \oplus k_2 \\ &= \{m + (n + p), m + k_1 + k_2\} \cup \{m + k_2, k_4\} \\ &= \{(m + n) + p, m + 2k_1 + k_2, m + k_2, k_4\} \\ &= \{(m + n) + p, k_1 + p, k_4, m + k_2\} \\ &= \{(m + n) + p, k_4\} \cup \{k_1 + p, m + k_2\} \\ &= (m + n) \oplus p \cup (k_1 \oplus p) \\ &= (m \oplus n) \oplus p. \end{aligned}$$

Let $m \in \mathbb{N}$. Then, $m \oplus m = \{m + m, 0\}$ and so $0 \in m \oplus m$. It means that $-m := m$.

For any elements $m, n, p \in \mathbb{N}$, let $m \in n \oplus p = p \oplus n$. Then, there exists an element $k \in \mathbb{N}$ such that $p = n + k$. It follows that $m \in n \oplus p = \{n + p, k\}$ and so $m = n + p$ or $m = k$. Therefore, $n \in m \oplus p$. Hence, $(\mathbb{N}, \oplus)$ is the canonical hypergroup.

(2) It is obvious that $(\mathbb{N}, \cdot)$ is a commutative monoid and $n \cdot 0 = 0$, for all $n \in \mathbb{N}$, where the operation $\cdot$ is the multiplication in $\mathbb{N}$.

(3) Let $m, n, p \in \mathbb{N}$ and $n \le p$. There, exists an element $k \in \mathbb{N}$ with $n + k = p$. Now

$$\begin{aligned} m \cdot (n \oplus p) = m \cdot \{n + p, k\} &= \{m \cdot (n + p), m \cdot k\} \\ &= \{m \cdot n + m \cdot p, m \cdot k\} \\ &= m \cdot n \oplus m \cdot p \end{aligned}$$

Hence, the structure $(\mathbb{N}, \oplus, \cdot)$ is a hyperdomain.

We will use these conventions $mn = m \cdot n$ for any elements $m, n \in \mathbb{N}$ and the hyperring $(\mathbb{N}, \oplus, \cdot)$ as the hyperring $\mathbb{N}$.

Proposition 3.1. $\mathbb{N}$ *is a principle hyperideal domain.*

Proof. Firstly, note that $a\mathbb{N}$ is a hyperideal of $\mathbb{N}$, for all $a \in \mathbb{N}$. Let I be a hyperideal of the hyperring $\mathbb{N}$. If $I = \{0\}$, then $I = 0\mathbb{N}$. Assume that $I \neq \{0\}$. With the help of the principle of well-ordering, we can show that I contains a smallest positive integer, say $a \in I$. We claim that $I = \{an \,|\, n \in \mathbb{N}\} = a\mathbb{N}$. Clearly, $a\mathbb{N} \subseteq I$. Let $b \in I$. Therefore, we can write $b = aq + r$, $0 \le r < a$, for some elements $q, r \in \mathbb{N}$. Now, $b \oplus aq = \{b + aq, r\} \subseteq I$ and so $r \in I$. Since $0 \le r < a$ and a is the smallest positive integer of I, we get $r = 0$. It implies that $b = aq \in a\mathbb{N}$. Hence, $I = a\mathbb{N}$. □

Proposition 3.2. *Let I be a non-zero hyperideal of* $\mathbb{N}$. *Then, I contains a non-zero hyperideal K of* $\mathbb{N}$ *with $I \neq K$.*

Proof. By Proposition 3.1, we can write $I = a\mathbb{N}$, for some element $a \in I$. Let $0 \neq m \in \mathbb{N}$. Now, we consider the hyperideal $M = (ma)\mathbb{N}$. Then, K is a hyperideal of $\mathbb{N}$ and $I \neq K$. This completes the proof. □

Theorem 3.1. *Let I be a non-trivial hyperideal of* $\mathbb{N}$. *Then, the following statements are equivalent:*

(1) *I is a maximal hyperideal of* $\mathbb{N}$.

(2) *I is a prime hyperideal of* $\mathbb{N}$.

(3) *There exists a prime positive integer $\mathfrak{p} \in \mathbb{N}$ such that $\mathfrak{p}\mathbb{N} = I$*

Proof. (1) $\Rightarrow$ (2). By [7, Proposition 3.3.7].

(2) $\Rightarrow$ (3). By Proposition 3.1, we can write $I = \mathfrak{p}\mathbb{N}$, for some element $\mathfrak{p} \in \mathbb{N}$. Let $a, b \in \mathbb{N}$ be such that $\mathfrak{p} = ab$. Since I is a prime hyperideal of $\mathbb{N}$, it follows from [7, Lemma 3.3.6] that $a \in I$ or $b \in I$. Therefore, either $\mathfrak{p}|a$ or $\mathfrak{p}|b$. Thus, $\mathfrak{p}$ is a prime element of $\mathbb{N}$.

(3) $\Rightarrow$ (1). Let M be a hyperideal of $\mathbb{N}$ such that $I \subseteq M \subset \mathbb{N}$. Again applying Proposition 3.1, there exists an element $a \in M$ with $M = a\mathbb{N}$. Then, $\mathfrak{p} = \mathfrak{p}1 \in \mathfrak{p}\mathbb{N} \subseteq M = a\mathbb{N}$ and so $ab = \mathfrak{p}$, for some $b \in M$. Since $\mathfrak{p}$ is prime, $\mathfrak{p} = a$ or $\mathfrak{p} = b$. Thus, $I = M$. □

Remark 3.1. Krasner gave a method for the construction of hyperrings (see [10, Theorem]). Let $(S, +, .)$ be a commutative ring with unity and $(G, .)$ be a subgroup of the monoid $(S, .)$. Then, $\{aG\}_{a \in S}$ is a partition of S and so this partition defines an equivalence relation on S as follows:

$$\text{“}a \sim b \Longleftrightarrow aG = bG\text{”}.$$

Let $\frac{S}{G}$ be the set of all equivalence classes aG. Define

$$aG + bG = \{cG \mid c = ax + by \text{ for some } x, y \in G\} \subseteq P^*(\frac{S}{G})$$

and

$$aG \cdot bG = abG.$$

Then, $(\frac{S}{G}, +, .)$ is a commutative hyperring. In particular, if $(S, +, .)$ is a field, then $(\frac{S}{G}, +, .)$ is a hyperfield. Krasner calls the hyperring $\frac{S}{G}$ the quotient hyperring of S by G.

Now, we shall show that the hyperring $\mathbb{N}$ is isomorphic to one of Krasner's quotient hyperrings.

Theorem 3.2. *Let* $(\mathbb{Z}, +,)$ *denote the ring of integers and* $H = \{-1, 1\}$. *Then, the hyperring* $(\frac{\mathbb{Z}}{H}, +, .)$ *is isomorphic to the hyperring* $(\mathbb{N}, \oplus, \cdot)$.

Proof. Define $f : \mathbb{N} \longrightarrow \frac{\mathbb{Z}}{H}$ by $f(n) = \overline{n} = \{-n, n\}$, for all $n \in \mathbb{N}$. Let $n, m \in \mathbb{N}$ with $n + k = m$. Now,

$$\begin{aligned} f(n \oplus m) &= \bigcup_{r \in n \oplus m} \{f(r)\} \\ &= \{f(n+m), f(k)\} \\ &= \{\overline{n+m}, \overline{k}\} \\ &= \{\overline{n+m}, \overline{n-m}\} \\ &= f(n) + f(m) \end{aligned}$$

and $f(nm) = \overline{nm} = \overline{n}.\overline{m} = f(n)f(m)$, which implies that f is a homomorphism of hyperrings. Clearly, f is surjective. Let $n \in Ker(f)$. It follows that $f(n) = \overline{n} = 0$ and so $n = 0$. It means that f is injective. Hence, the hyperring $(\frac{\mathbb{Z}}{H}, +, .)$ is isomorphic to the hyperring $(\mathbb{N}, \oplus, \cdot)$ as required. □

Following [19], we construct the fractional hyperrings of the hyperring $\mathbb{N}$. Let S be a multiplicatively closed subset $\mathbb{N}$ such that $0 \notin S$. The relation on the set $\mathbb{N} \times S$ defined by

$$\text{“}(a,\, b) \equiv (c,\, d) \Longleftrightarrow \text{there exists } u \in S \text{ such that } u(ad) = u(bc)\text{”}.$$

This is an equivalence relation on the set $\mathbb{N} \times S$. The equivalence class of $(a,\, b)$ is denoted by $\frac{a}{b}$ and the set of all equivalence classes is denoted by $S^{-1}\mathbb{N}$. Define the hyperoperation $+$ and the operation . on $S^{-1}\mathbb{N}$ as follows:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad \oplus bc}{bd} = \{\frac{e}{bd} | e \in ad \oplus bc\}$$

and

$$\frac{a}{b}.\frac{c}{d} = \frac{ac}{bd}$$

for all $\frac{a}{b}$, $\frac{c}{d} \in S^{-1}\mathbb{N}$. It follows [8, Theorem 3.1] that $(S^{-1}\mathbb{N},\, +,\, .)$ is a hyperring. Here, $\frac{0}{b} := 0$ is the scalar identity element of $(S^{-1}\mathbb{N},\, +)$. Moreover, for all $\frac{a}{b} \in S^{-1}\mathbb{N}$, $\frac{a}{b}.\frac{1}{1} = \frac{a1}{b} = \frac{a}{b}$ and so $\frac{1}{1} := 1$ is an identity element of the hyperring $(S^{-1}\mathbb{N},\, +,\, .)$. Hence, $(S^{-1}\mathbb{N},\, +,\, .)$ is a commutative hyperdomain.

Corollary 3.1. *Let $S = \mathbb{N} \backslash \{0\}$. Then, $(S^{-1}\mathbb{N},\, +,\, .)$ is a hyperfield.*

Proof. Let $0 \neq \frac{a}{b} \in S^{-1}\mathbb{N}$. Therefore, $\frac{a}{b}.\frac{b}{a} = 1$. It means that $(S^{-1}\mathbb{N},\, +,\, .)$ is a hyperfield. □

Observe from Theorem 3.1 that $S^{-1}\mathbb{N} = \{\frac{a}{b} | a,\, b \in \mathbb{N} \textit{ and } b \neq 0\} = \mathbb{Q}^{\geq 0}$. Therefore, $(\mathbb{Q}^{\geq 0},\, +,\, .)$ is a hyperfield, where $\frac{a}{b} + \frac{c}{d} = \frac{ad \oplus bc}{bd}$ and $\frac{a}{b}.\frac{c}{d} = \frac{ad}{bc}$, for all $\frac{a}{b}$, $\frac{c}{d} \in \mathbb{Q}^{\geq 0}$.

Let R be a non-zero hyperring with identity. Recall from [9] that R is *local* if R has the only left maximal hyperideal. Now, we give the following example. Later we shall give other examples of such hyperrings (see Proposition 3.3)

Example 3.1. Let $\mathfrak{p}$ be a prime element of $\mathbb{N}$ and $\mathcal{P} = \mathfrak{p}\mathbb{N}$. Then, by [8, Theorem 3.6-(ii)], $\mathcal{P}_{\mathfrak{p}}$ is the only maximal hyperideal of $\mathcal{P}^{-1}\mathbb{N}$. Therefore, $\mathcal{P}^{-1}\mathbb{N}$ is a local hyperring.

Let R be a hyperring and M be an R-hypermodule. Following [4], M is said to be *divisible* if for every $r \in R$ which is not a zero divisor and every $m \in M$, there exists $m' \in M$ such that $rm' = m$.

Example 3.2. Define $\cdot : \mathbb{N} \times \mathbb{Q}^{\geq 0} \longrightarrow \mathbb{Q}^{\geq 0}$ via $n\frac{a}{b} = \frac{na}{b}$, for all $n \in \mathbb{N}$ and $\frac{a}{b} \in \mathbb{Q}^{\geq 0}$. Let $m,\, n \in \mathbb{N}$ and $\frac{a}{b}$, $\frac{c}{d} \in \mathbb{Q}^{\geq 0}$. Now

(1) $(m \oplus n)\frac{a}{b} = \bigcup_{r \in m \oplus n} r\frac{a}{b} = m\frac{a}{b} + n\frac{a}{b}$;

(2) $m(\frac{a}{b} + \frac{c}{d}) = m(\frac{ad \oplus bc}{bd}) = m\frac{a}{b} + m\frac{c}{d}$;

(3) $mn(\frac{a}{b}) = \frac{(mn)a}{b} = \frac{m(na)}{b} = m\frac{na}{b} = m(n\frac{a}{b})$;

(4) $1\frac{a}{b} = \frac{1a}{b} = \frac{a}{b}$.

Thus, $(\mathbb{Q}^{\geq 0}, +)$ is a $\mathbb{N}$-hypermodule. To show that $(\mathbb{Q}^{\geq 0}, +)$ is a divisible, let $n \in \mathbb{N} \backslash \{0\}$ and $\frac{a}{b} \in \mathbb{Q}^{\geq 0}$. Then, $\frac{a}{b} = n\frac{a}{nb}$, which implies that $\mathbb{Q}^{\geq 0}$ is divisible.

Let $m > 1$. Define the relation "$\equiv$" on $\mathbb{N}$ by for all $x, y \in \mathbb{N}$

$$\text{“}x \equiv y \Longleftrightarrow m|k, \text{ where } \min\{x, y\} + k = \max\{x, y\}\text{”}.$$

It can be seen that "$\equiv$" is an equivalence relation on $\mathbb{N}$. Let $\mathbb{N}_m = \{\overline{x} \,|\, x \in \mathbb{N}\}$, where $\overline{x} = \{0 + x, m + x, 2m + x, ...\} = \{nm + x \,|\, n \in \mathbb{N}\}$. Let $0 \leq x < y < m$. Suppose that $\overline{x} = \overline{y}$. Then, $y \in \overline{x}$ and so $m|k$, $x + k = y$, for some $k \in \mathbb{N}$. This is a contradiction since $0 < k < m$. Hence, the equivalence classes $\overline{0}, \overline{1}, ..., \overline{m-1}$ are distinct. Let $\overline{x}$ be any element of $\mathbb{N}_m$. By the division algorithm, $x = mq + r$, for some elements q and r such that $0 \leq r < m$. Since $m|mq$, we obtain that $\overline{r} = \overline{x}$. Hence, $\mathbb{N}_m = \{\overline{0}, \overline{1}, \overline{2}, ..., \overline{m-1}\}$.

Theorem 3.3. *Let $m > 1$. Define "$\oplus_m$" on $\mathbb{N}_m$ by*

$$\overline{x} \oplus_m \overline{y} = \{\overline{x+y}, \overline{k} |\ \min\{x, y\} + k = \max\{x, y\}\},$$

for all $\overline{x}, \overline{y} \in \mathbb{N}_m$. Then

(1) $(\mathbb{N}_m, \oplus_m)$ is a canonical hypergroup with scalar identity $\overline{0}$.

(2) $(\mathbb{N}_m, \oplus_m, .)$ is a commutative and unitary hyperring, where "." is the usual multiplication.

(3) $(\mathbb{N}^*_m, .)$ is a group, where $\mathbb{N}^*_m = \{\overline{x} \in \mathbb{N}_m \,|\, (x, m) = 1\}$.

(4) $(\mathbb{N}_m, \oplus_m, .)$ is a hyperfield if and only if m is prime.

(5) there exists a isomorphism of hyperrings $f : \frac{\mathbb{N}}{\mathbb{N}_m} \longrightarrow \mathbb{N}_m$.

(6) The canonical hypergroup $(\mathbb{N}_m, \oplus_m)$ is a $\mathbb{N}$-hypermodule.

Proof. (1), (2) and (3) are straightforward.

(4) ($\Rightarrow$) Let $m = ab$, where $1 \leq a < b < m$. Then, $\overline{a}, \overline{b} \in \mathbb{N}_m$ and so $\overline{a}.\overline{b} = \overline{ab} = \overline{0}$, a contradiction.

($\Leftarrow$) Let $\overline{a} \in \mathbb{N}^*_m$. Then, $(a, m) = 1$ and so, we get $1 = ax + my$, for some $x, y \in \mathbb{N}$. It follows that $\overline{1} = \overline{ax + my} = \overline{ax} = \overline{a}.\overline{x}$. Hence, $\mathbb{N}^*_m = \mathbb{N}_m \backslash \{\overline{0}\}$. By (3), $(\mathbb{N}_m, \oplus_m, .)$ is a hyperfield.

(5) Consider the map $\Phi : \mathbb{N} \longrightarrow \mathbb{N}_m$ via $\Phi(x) = \overline{x}$, for all $x \in \mathbb{N}$. Let $x, y \in \mathbb{N}$. Assume that $x + k = y$, for some $k \in \mathbb{N}$. Now,

$$\Phi(x \oplus y) = \bigcup_{r \in x \oplus y} \{\Phi(r)\} = \{\Phi(x+y), \Phi(k)\} = \{\overline{x+y}, \overline{k}\} = \overline{x} \oplus_m \overline{y} = \Phi(x) \oplus_m \Phi(y)$$

and $\Phi(xy) = \overline{xy} = \overline{x}.\overline{y} = \Phi(x)\Phi(y)$. Thus, Φ is a homomorphism of hyperrings. It is clear that Φ is surjective with $Ker(\Phi) = m\mathbb{N}$. Thus, we obtain that $\frac{\mathbb{N}}{\mathbb{N}_m} \cong \mathbb{N}_m$.

(6) Define the map $\cdot : \mathbb{N} \times \mathbb{N}_m \longrightarrow \mathbb{N}_m$ via $n \cdot \overline{x} = \overline{nx}$, for all $n \in \mathbb{N}$ and for all $\overline{x} \in \mathbb{N}_m$. According to the map, it can be checked that N_m is a $\mathbb{N}$-hypermodule. □

The next result gives examples of local hypermodules.

Proposition 3.3. *Let $\mathfrak{p}$ be a prime positive integer. Then, $(\mathbb{N}_{\mathfrak{p}^k}, \oplus, .)$ is a local hyperring, for all $k > 0$.*

Proof. Let $k > 0$. Using Theorem 3.3 (5), we deduce that $\Phi(\mathfrak{p}\mathbb{N})$ is the only maximal hyperideal of the hyperring $(\mathbb{N}_{\mathfrak{p}^k}, \oplus, .)$. Hence, $(\mathbb{N}_{\mathfrak{p}^k}, \oplus, .)$ is a local hyperring. □

Note that the condition "prime positive integer" in the above proposition is necessary. Let's take the following example to see this.

Example 3.3. Given the the hyperring $\mathbb{N}_6$. Using Theorem 3.3, we obtain the following tables:

$\oplus_6$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$
$\overline{0}$	$\{\overline{0}\}$	$\{\overline{1}\}$	$\{\overline{2}\}$	$\{\overline{3}\}$	$\{\overline{4}\}$	$\{\overline{5}\}$
$\overline{1}$	$\{\overline{1}\}$	$\{\overline{0}, \overline{2}\}$	$\{\overline{1}, \overline{3}\}$	$\{\overline{2}, \overline{4}\}$	$\{\overline{3}, \overline{5}\}$	$\{\overline{0}, \overline{4}\}$
$\overline{2}$	$\{\overline{2}\}$	$\{\overline{1}, \overline{3}\}$	$\{\overline{0}, \overline{4}\}$	$\{\overline{1}, \overline{5}\}$	$\{\overline{0}, \overline{2}\}$	$\{\overline{1}, \overline{3}\}$
$\overline{3}$	$\{\overline{3}\}$	$\{\overline{2}, \overline{4}\}$	$\{\overline{1}, \overline{5}\}$	$\{\overline{0}\}$	$\{\overline{1}\}$	$\{\overline{2}\}$
$\overline{4}$	$\{\overline{4}\}$	$\{\overline{3}, \overline{5}\}$	$\{\overline{2}\}$	$\{\overline{1}\}$	$\{\overline{0}, \overline{2}\}$	$\{\overline{1}, \overline{3}\}$
$\overline{5}$	$\{\overline{5}\}$	$\{\overline{0}, \overline{4}\}$	$\{\overline{1}, \overline{3}\}$	$\{\overline{2}\}$	$\{\overline{1}, \overline{3}\}$	$\{\overline{0}, \overline{4}\}$

and

.	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$
$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$
$\overline{1}$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$
$\overline{2}$	$\overline{0}$	$\overline{2}$	$\overline{4}$	$\overline{0}$	$\overline{2}$	$\overline{4}$
$\overline{3}$	$\overline{0}$	$\overline{3}$	$\overline{0}$	$\overline{3}$	$\overline{0}$	$\overline{3}$
$\overline{4}$	$\overline{0}$	$\overline{4}$	$\overline{2}$	$\overline{0}$	$\overline{4}$	$\overline{2}$
$\overline{5}$	$\overline{0}$	$\overline{5}$	$\overline{4}$	$\overline{3}$	$\overline{2}$	$\overline{1}$

Thus, the only maximal hyperideals of the hyperring $\mathbb{N}_6$ are $I_1 = \{\overline{0}, \overline{3}\}$ and $I_2 = \{\overline{0}, \overline{2}, \overline{4}\}$. Also, we have $\mathbb{N}_6 = I_1 \oplus_6 I_2$. It follows that every hyperideal of $\mathbb{N}_6$ is a direct summand of $\mathbb{N}_6$. Hence, the hyperring $\mathbb{N}_6$ is not local.

For hyperstructures the example we will give below is an analogue of $\mathbb{Z}_{\mathfrak{p}^\infty}$, which has a very important place in classical algebra.

Example 3.4. For a prime positive integer $\mathfrak{p} \in \mathbb{N}$, consider the following set:

$$\mathbb{N}[\frac{1}{\mathfrak{p}}] = \{\frac{m}{\mathfrak{p}^s} \in \mathbb{Q}^{\geq 0} \mid m,\, s \in \mathbb{N} \textit{ and } \mathfrak{p} \textit{ is prime}\}.$$

Let $x = \frac{m}{\mathfrak{p}^s}$, $y = \frac{n}{\mathfrak{p}^k} \in \mathbb{N}[\frac{1}{\mathfrak{p}}]$. Then, $x+y = \frac{m}{\mathfrak{p}^s} + \frac{n}{\mathfrak{p}^k} = \frac{m\mathfrak{p}^k \oplus n\mathfrak{p}^k}{\mathfrak{p}^{s+k}} \in \mathbb{N}[\frac{1}{\mathfrak{p}}]$. Thus, $\mathbb{N}[\frac{1}{\mathfrak{p}}]$ is a canonical subhypergroup of $\mathbb{Q}^{\geq 0}$. Now, we consider the quotient canonical hypergroup $\frac{\mathbb{N}[\frac{1}{\mathfrak{p}}]}{\mathbb{N}}$ of the canonical hypergroup $\mathbb{N}[\frac{1}{\mathfrak{p}}]$ by $\mathbb{N}$. Put $\mathbb{N}_{\mathfrak{p}^\infty} := \frac{\mathbb{N}[\frac{1}{\mathfrak{p}}]}{\mathbb{N}}$.

For every $s = 1, 2, ...,$ let $c_s = \frac{1}{\mathfrak{p}^s} + \mathbb{N} \in \mathbb{N}_{\mathfrak{p}^\infty}$. Then

$$\mathfrak{p}c_1 = 0;\ \mathfrak{p}c_2 = c_1;\ ...,\ \mathfrak{p}c_{s+1} = c_s.$$

Therefore, the set $\{c_1, c_2, ..., c_s, ...\}$ generates the canonical hypergroup $\mathbb{N}_{\mathfrak{p}^\infty}$. Let H be any proper canonical subhypergroup of $\mathbb{N}_{\mathfrak{p}^\infty}$. Put $n = sup\{k \mid c_k \in H\}$. If $n = \infty$, then there exists $s \in \mathbb{N}$ such that $k > s$ and $c_k \in H$ for every $a = \frac{m}{\mathfrak{p}^s} + \mathbb{N} \in \mathbb{N}_{\mathfrak{p}^\infty}$. It follows that $a = m\mathfrak{p}^t(\frac{1}{\mathfrak{p}^k} + \mathbb{N}) = m\mathfrak{p}^t c_k + \mathbb{N}$, where $s+t = k$, for some $t \in \mathbb{N}$. So $H = \mathbb{N}_{\mathfrak{p}^\infty}$. This is a contradiction. Thus, $n < \infty$. Next, we show that $H =< c_n >$. Clearly, $< c_n >\subseteq H$. Let $a = \frac{m}{\mathfrak{p}^s} + \mathbb{N} \in H$. We can take $(m, \mathfrak{p}^s) = 1$ without losing generality. Then, we can write $mu + \mathfrak{p}^s v = 1$, for some elements $u, v \in \mathbb{N}$. Thus, $au = \frac{mu}{\mathfrak{p}^s} + \mathbb{N} = \frac{1}{\mathfrak{p}^s} + \mathbb{N} = c_s$ and then $c_s = au \in H$. From this choice of n, we obtain that $s \leq n$. Therefore, $a = m\mathfrak{p}^t c_n \in< c_n >$, where $s + t = n$. It means that $H =< c_n >$. Also, it can be seen that $< c_n >\cong \mathbb{N}_{\mathfrak{p}^n}$. Hence, $\mathbb{N}_{\mathfrak{p}^\infty} = \bigcup_{n\in\mathbb{N}} \mathbb{N}_{\mathfrak{p}^n}$.

Define $\cdot : \mathbb{N} \times \mathbb{N}_{\mathfrak{p}^\infty} \longrightarrow \mathbb{N}_{\mathfrak{p}^\infty}$ by $n \cdot (\frac{m}{\mathfrak{p}^s} + \mathbb{N}) = \frac{nm}{\mathfrak{p}^s} + \mathbb{N}$, for all $n \in \mathbb{N}$ and $\frac{m}{\mathfrak{p}^s} + \mathbb{N} \in \mathbb{N}_{\mathfrak{p}^\infty}$. Thus, it is easily seen that $\mathbb{N}_{\mathfrak{p}^\infty}$ is a normal injective $\mathbb{N}$-hypermodule.

In [6], an R-hypermodule M is said to be *simple* if $RM \neq 0$ and M has no subhypermodules other than $\{0_M\}$ and M. It is shown in [6, Lemma 3.9] that an R-hypermodule M is simple if and only if it is isomorphic to $\frac{R}{I}$, for some maximal left hyperideal I of R. Using this fact and Theorem 3.1, we deduce that a simple $\mathbb{N}$-hypermodule M is of the form $\frac{\mathbb{N}}{\mathfrak{p}\mathbb{N}} \cong \mathbb{N}_{\mathfrak{p}}$, where $\mathfrak{p}$ is a prime positive integer. Then, we have:

Corollary 3.2. *Every simple $\mathbb{N}$-hypermodule can be embedded the normal injective $\mathbb{N}$-hypermodule $\mathbb{N}_{\mathfrak{p}^\infty}$, for some prime positive integer $\mathfrak{p} \in \mathbb{N}$.*

4. Conclusions

In this study, we constructed the hyperring structure on the set $\mathbb{N}$ with the help of the usual operations. Thanks to this construction, we obtain very useful classes of hypermodules and hyperrings. These classes are a resource for researchers working in this category of hypermodules.

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Semihyperlattice regular equivalence relations on ordered semihypergroups

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Abstract. In this paper, we explore the semihyperlattice regular equivalence relations on ordered semihypergroups. To begin with, we present the specific constructions of semihyperlattice regular equivalence relations. This effort is directed towards ensuring the preservation of the hyperalgebraic structure within the quotient hyperalgebra. Subsequently, we obtain a homomorphism theorem from ordered semihypergroups to ordered semihyperlattices. Finally, we discuss some related properties of congruence classes and principal pseudo-hyperfilters.

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Keywords: ordered semihypergroup, semihyperlattice regular equivalence relation, homomorphism, congruence class.

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1. Introduction

The advent of algebraic hyperstructures was pioneered by F. Marty in 1934 [1], signifying a profound expansion into realms such as hyperrings, hyperfields, and hyperlattices. Numerous scholars have explored various dimensions of semihypergroups, for instance, see [2, 3, 4, 5, 6, 7, 8, 9]. Notably, B. Davvaz has significantly propelled the development of semihypergroup since 2000, with a particular emphasis on the congruence theory. The integration of ordered semigroup algebra with hyperstructure theory was further enhanced by the efforts of D. Heidari and B. Davvaz in 2011 [10], culminating in the formulation of the concept of ordered semihypergroups, a domain replete with theoretical and practical possibilities.

A pivotal focus within this field has been the construction of strong regular equivalence relations on ordered semihypergroups by B. Davvaz using pseudoorder. In 2015, B. Davvaz introduced the pseudoorder to induce strong regular equivalence relations and quotient ordered semihypergroups into ordered semigroups, but in the process, the hyperstructure was lost [11]. Then, he raised an open question: Is there a regular equivalence relation ρ on an ordered semihypergroup $(S, \circ, \leq_S)$ for which S/ρ is an ordered semihypergroup? This question prompted subsequent research to find a solution. In 2016, Z. Gu addressed this issue using proper hyperideals [12], and in 2018, X.Y. Feng resolved it through the concept of the weak pseudoorder [13]. These contributions collectively solved the open problem posed by Davvaz, significantly advancing the understanding of regular equivalence relations in ordered semihypergroups and thereby enriching the field.

As we know, semilattice congruences plays an important role in the research of semigroup and ordered semigroup algebraic theory. In 2015, J. Tang et al. generalized the concept of filters to hyperfilters, by using this, he introduced the semilattice strong regular equivalence relations $\mathcal{N}$ on an ordered semihypergroup, resulting in $S/\mathcal{N}$ being an ordered semilattice. Therefore, this paper primarily investigates whether a semihyperlattice regular equivalence relation exists that ensures the quotient structure of any ordered semihypergroup forms an ordered semihyperlattice.

This paper explores the semihyperlattice regular equivalence relations (semihyperlattice congruences) on ordered semihypergroups in detail. After an introduction, in Section 2, we recall some basic definitions and results of ordered semihypergroups which will be used throughout this paper. In Section 3, we construct the semihyperlattice regular equivalence relation on ordered semihypergroups by using pseudo-hyperfilters, ensuring the preservation of the hyperalgebraic structure within the quotient hyperalgebra. In Section 4, we present a homomorphism theorem from ordered semihypergroups to ordered semihyperlattices and we studies the relationship between congruence classes and principal pseudo-hyperfilters.

2. Preliminaries

For the sake of clarity and convenience, the essential definitions are provided first. A mapping $\circ : S \times S \to \mathcal{P}^*(S)$, where $\mathcal{P}^*(S)$ denotes the family of all nonempty subsets of S, is called a hyperoperation on S. A couple $(S, \circ)$ is called a hypergroupoid. For an element $x \in S$ and nonempty subsets $A, B \subseteq S$, the operations are denoted as $A \circ B = \bigcup_{a \in A, b \in B} a \circ b$, $x \circ A = \{x\} \circ A$, and $A \circ x = A \circ \{x\}$. A hypergroupoid $(S, \circ)$ is called a semihypergroup if hyperoperation $\circ$ satisfies $x \circ (y \circ z) = (x \circ y) \circ z$ for all $x, y, z \in S$.

Consider a semihypergroup $(S, \circ)$ and a relation ρ on S. For nonempty subsets $A, B \subseteq S$, we define

$$A\overline{\rho}B \Leftrightarrow (\forall a \in A, \exists b \in B)\ a\rho b \text{ and } (\forall b' \in B, \exists a' \in A) a'\rho b',$$

and

$$A\overline{\overline{\rho}}B \Leftrightarrow (\forall a \in A, \forall b \in B)\ a\rho b.$$

An equivalence relation ρ is classified as regular if

$$(\forall a, b, x \in S)\ a\rho b \Rightarrow a \circ x\overline{\rho} b \circ x \text{ and } x \circ a\overline{\rho} x \circ b,$$

and as strongly regular if

$$(\forall a, b, x \in S)\ a\rho b \Rightarrow a \circ x\overline{\overline{\rho}} b \circ x \text{ and } x \circ a\overline{\overline{\rho}} x \circ b.$$

For a semihypergroup $(S, \circ)$ with an equivalence relation ρ, the equivalence ρ-class containing a is denoted by $(a)_\rho$. As known from [11], if ρ is a regular equivalence relation on S, the quotient S/ρ forms a semihypergroup under the operation $(x)_\rho \star (y)_\rho = \{(z)_\rho \mid z \in x \circ y\}$. Furthermore, if ρ is strongly regular, then S/ρ constitutes a semigroup with the operation $(x)_\rho \star (y)_\rho = (z)_\rho$ for every $z \in x \circ y$.

An ordered semigroup $(S, \cdot, \leq)$ is a semigroup $(S, \cdot)$ accompanied by an order relation $\leq$ such that $a \leq b$ implies $ax \leq bx$ and $xa \leq xb$ for any $a, b, x \in S$. This concept extends to the hyper version as an ordered semihypergroup $(S, \circ, \leq)$, which is a semihypergroup $(S, \circ)$ with an order relation $\leq$ compatible with the hyperoperation $\circ$. That is, $A, B \in \mathcal{P}^*(S)$, $A \preceq B$ if and only if, for all $a \in A$, there exists $b \in B$ such that $a \leq b$. And $a \nsim b$ means that a is not comparable to b. A nonempty subset A of an ordered semihypergroup S is a subsemihypergroup if $A \circ A \subseteq A$. A subsemihypergroup A of S is a hyperfilter if it satisfies: (1) for any $a, b \in S$, $(a \circ b) \cap A \neq \emptyset$ implies $a, b \in A$; (2) $a \in A$, $a \leq b \in S$ implies $b \in A$.

For two ordered semihypergroups $(S, \circ, \leq_S)$ and $(T, \diamond, \leq_T)$, a mapping $f : S \to T$ is called a normal homomorphism if it satisfies: (1) $f(x \circ y) = f(x) \diamond f(y)$

for all $x, y \in S$, where $f(A) = \{f(a) \mid a \in A\}$ for any nonempty subset A of S; (2) f is isotone, i.e., for any $x, y \in S$, $x \leq_S y$ implies $f(x) \leq_T f(y)$. Moreover, a bijective normal homomorphism f from S onto T is called an isomorphism if f satisfies $f(x) \leq_T f(y)$, then $x \leq_S y$ for any $x, y \in S$.

3. Ordered semihyperlattice regular equivalence relations

In this section, we define and study the ordered semihyperlattice regular equivalence relations of an ordered semihypergroup S. Especially, we construct a semihyperlattice regular equivalence relation on ordered semihypergroups in terms of the pseudo-hyperfilter, and discuss its related properties.

Definition 3.1. *Let F be a subset of an ordered semihypergroup S. F is called a* pseudo-hyperfilter *S if it satisfies the following:*

(1) $(a \circ b) \cap F \neq \emptyset$ *if and only if* $a, b \in F$*;*

(2) *If* $(a \circ b) \cap F \neq \emptyset$*, then there exists* $u \in (a \circ b) \cap F$ *such that* $\forall f \in F$*,* $u \leqslant f$ *or* $u \nsim f$*;*

(3) *If* $a \in F$ *and* $a \leqslant b \in S$ *or* $S \ni b \nsim a$*, then* $b \in F$*.*

The following is an example of a pseudo hyperfilter on an ordered semihypergroup.

Example 3.1. Let $S = \{a, b, c, d, e\}$ with the operation $\circ$ and the order relation $\leq$ below:

$\circ$	a	b	c	d	e
a	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$
c	$\{a\}$	$\{a,b\}$	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$
d	$\{a\}$	$\{a,b\}$	$\{a,b,c\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$
e	$\{a\}$	$\{a,b\}$	$\{a,b,c\}$	$\{a,b,c,d\}$	$\{a,b,c,d,e\}$

$$\begin{aligned}\leq := \{&(a,a),(a,c),(a,d),(a,e),\\ &(b,b),(b,c),(b,d),(b,e),\\ &(c,c),(c,d),(c,e),\\ &(d,d),(d,e),\\ &(e,e)\}.\end{aligned}$$

The associativity of $\circ$ is easily verified, and it is also straightforward to confirm that $\leq$ is a partial order that satisfies the compatibility condition. Therefore, we can clearly verified that S is an ordered semihypergroup.

We proceed to prove that $T = \{d, e\}$ is a pseudo-hyperfilter on S.

(1) For $d, e \in T$, we have $d \circ e = e \circ d = \{a, b, c, d\}$. Thus,

$$(d \circ e) \cap T = \{a, b, c, d\} \cap T = \{d\} \neq \emptyset.$$

Moreover,

$$(d \circ d) \cap T = \{a, b, c, d\} \cap T = \{d\} \neq \emptyset$$

and

$$(e \circ e) \cap T = \{a, b, c, d, e\} \cap T = \{d, e\} \neq \emptyset.$$

The condition (1) in Definition 3.1 is satisfied.

(2) By (1), we have

$$(d \circ e) \cap T = (e \circ d) \cap T = (d \circ d) \cap T = \{d\} \neq \emptyset$$

and

$$(e \circ e) \cap T = \{d, e\} \neq \emptyset.$$

Moreover, $d \leq d, d \leq e$. Thus, for any $x, y \in T$, there exists $d \in (x \circ y) \cap T$ such that for all $t \in T, d \leq t$. The condition (2) in Definition 3.1 is satisfied.

(3) For $d, e \in T$, since $d \leq e$ and $e \in \{d, e\}$, the condition (3) in Definition 3.1 is satisfied.

Hence, $\{d, e\}$ is a pseudo-hyperfilter on S.

For any $a \in S$, we denote the principal pseudo-hyperfilter generated by element a as $W(a)$, that is, $W(a)$ is the smallest pseudo-hyperfilter that include a. Let $\mathcal{W} := \{(x, y) \mid W(x) = W(y)\}$. Then, we can obtain the following result.

Lemma 3.1. *Let S be an ordered semihypergroup, the relation $\mathcal{W} := \{(x, y) \in S \times S \mid W(x) = W(y)\}$ is a regular equivalence relation on S.*

Proof. Clearly, $\mathcal{W}$ is an equivalence relation on S. Assume that $x\mathcal{W}y$, then $W(x) = W(y)$. For all $z \in S$, and for any $a \in x \circ z$, we have $(x \circ z) \cap W(a) \neq \emptyset$. Thus, $x, z \in W(a)$ implies $W(x) \subseteq W(a)$, hence $W(y) \subseteq W(a)$. Therefore, $y, z \in W(a)$. Furthermore, there exists $b \in (y \circ z) \cap W(a)$, such that for any $w \in W(a)$, either $b \leq w$ or $b \nsim w$.

If $b \leq a$, and therefore $a \in W(b) \Rightarrow W(a) \subseteq W(b)$. Since $b \in W(a)$, we have $W(b) \subseteq W(a)$. That is, there exists $b \in y \circ z$ such that $W(a) = W(b)$.

If $b \nsim a$, then $a \in W(b) \Rightarrow W(a) \subseteq W(b)$. Since $b \in W(a)$, we have $W(b) \subseteq W(a)$. That is, there exists $b \in y \circ z$ such that $W(a) = W(b)$.

Similarly, for all $b' \in y \circ z$, there exists $a' \in x \circ z$ satisfying $b'\mathcal{W}a'$. The same holds for $z \circ x$ and $z \circ y$. Therefore, $\mathcal{W}$ is a regular equivalence relation on S. $\square$

Definition 3.2. *Let L be a semihypergroup. Then, L is called a* semihyperlattice *if it satisfies the following conditions for all $x, y \in L$:*

(1) $x \in x \circ x$;

(2) $x \circ y = y \circ x$.

In the study of semihyperlattices, the introduction of an order relation is a crucial and indispensable step. Therefore, we are committed to introducing an appropriate order relation on semihyperlattices, thus establishing the concept of the ordered semihyperlattices.

Definition 3.3. *Let L be an ordered semihypergroup. Then, L is called an* ordered semihyperlattice *if it is also a semihyperlattice.*

Remark 3.1. The ordered semihypergroup S in Example 3.1 is an instance of an ordered semihyperlattice.

Furthermore, we proceed to present the concept of ordered semihyperlattice regular equivalence relations.

Definition 3.4. *Let S be an ordered semihypergroup. The relation σ on S is called an ordered semihyperlattice regular equivalence relation if it satisfies the following two conditions:*

(1) *σ is a regular equivalence relation on S;*

(2) *the quotient structure S/σ constitutes an ordered semihyperlattice.*

Let S be an ordered semihypergroup, for each $a \in S$, the $\mathcal{W}$-class containing a is denoted by $(a)_{\mathcal{W}}$.

Theorem 3.1. *Let S be an ordered semihypergroup. Then, $\mathcal{W}$ is an ordered semihyperlattice regular equivalence relation on S.*

Proof. $\mathcal{W}$ is a regular equivalence relation by Lemma 3.1. The conclusion is then proven in two steps.

Step 1. $S/\mathcal{W}$ is an ordered semihypergroup. It is known that $(S/\mathcal{W}, \star_{\mathcal{W}}, \preceq_{\mathcal{W}})$ is an ordered semihypergroup, where $\star_{\mathcal{W}}$ and $\preceq_{\mathcal{W}}$ are defined respectivly as follows:

$$(x)_{\mathcal{W}} \star_{\mathcal{W}} (y)_{\mathcal{W}} := \{(z)_{\mathcal{W}} \in S/\mathcal{W} \mid z \in x \circ y\},$$
$$\preceq_{\mathcal{W}} := \{((x)_{\mathcal{W}}, (y)_{\mathcal{W}}) \in S/\mathcal{W} \times S/\mathcal{W} \mid (x)_{\mathcal{W}} \in (x)_{\mathcal{W}} \star_{\mathcal{W}} (y)_{\mathcal{W}}\}.$$

Let $\pi : S \to S/\mathcal{W}$ be the mapping defined by $\pi(x) = (x)_{\mathcal{W}}$ for all $x \in S$. Let $x, y \in S$, $x \leq y$. Then, $y \in W(x)$. For any $z \in x \circ y$, we have $W(z) \cap (x \circ y) \neq \emptyset$, and thus $x \in W(z)$. Then, $(x)_{\mathcal{W}} \subseteq W(x) \subseteq W(z) \in (x)_{\mathcal{W}} \star_{\mathcal{W}} (y)_{\mathcal{W}}$. Therefore, $(x)_{\mathcal{W}} \preceq_{\mathcal{W}} (y)_{\mathcal{W}}$, that is, $\pi(x) \preceq_{\mathcal{W}} \pi(y)$.

Step 2. $S/\mathcal{W}$ is a semihyperlattice. Take any $(x)_{\mathcal{W}} \in S/\mathcal{W}$. Since $x \in W(x)$, we have $(x \circ x) \cap W(x) \neq \emptyset$. Therefore, there exists $y \in (x \circ x) \cap W(x)$ such that $y \leqslant w$ or $y \nsim w$ for all $w \in W(x)$, which further implies $y \leqslant x$ or $y \nsim x$. Consequently, $x \in W(y)$ leads to $W(x) = W(y)$, then $(x)_{\mathcal{W}} = (y)_{\mathcal{W}} \in (x \circ x)_{\mathcal{W}} = (x)_{\mathcal{W}} \star_{\mathcal{W}} (x)_{\mathcal{W}}$. Moreover, let $x, y \in S$. Take any $(a)_{\mathcal{W}} \in (x)_{\mathcal{W}} \star_{\mathcal{W}} (y)_{\mathcal{W}}$, then there exists

$z \in x \circ y$, such that $(z)_{\mathcal{W}} = (a)_{\mathcal{W}}$. Hence, $(x \circ y) \cap W(a) = (x \circ y) \cap W(z) \neq \emptyset$, then $x, y \in W(a)$. Then, there exists $b \in (y \circ x) \cap W(a) \neq \emptyset$, such that $b \leq a$ or $b \nsim a$. Therefore, $a \in W(b)$, so $W(a) \subseteq W(b)$, then $W(a) = W(b)$. Hence, $(a)_{\mathcal{W}} = (b)_{\mathcal{W}} \in (y \circ x)_{\mathcal{W}} = (y)_{\mathcal{W}} \star_{\mathcal{W}} (x)_{\mathcal{W}}$. Thus $(x)_{\mathcal{W}} \star_{\mathcal{W}} (y)_{\mathcal{W}} \subseteq (y)_{\mathcal{W}} \star_{\mathcal{W}} (x)_{\mathcal{W}}$. Similarly, take any $(b)_{\mathcal{W}} \in (y)_{\mathcal{W}} \star_{\mathcal{W}} (x)_{\mathcal{W}}$, we obtain $(y)_{\mathcal{W}} \star_{\mathcal{W}} (x)_{\mathcal{W}} \subseteq (x)_{\mathcal{W}} \star_{\mathcal{W}} (y)_{\mathcal{W}}$. Thus, $(x)_{\mathcal{W}} \star_{\mathcal{W}} (y)_{\mathcal{W}} = (y)_{\mathcal{W}} \star_{\mathcal{W}} (x)_{\mathcal{W}}$, then $S/\mathcal{W}$ is a semihyperlattice with the hyperoperation $\star_{\mathcal{W}}$. □

Next, we proceed to introduce a more general example, extending beyond the specific instance of the ordered semihypergroup discussed earlier.

Example 3.2. Let $S = \{a, b, c, d, e, f\}$ with the operation $\circ$ and the order relation $\leq$ below:

$\circ$	a	b	c	d	e	f
a	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a,b\}$	$\{a,b\}$
c	$\{a\}$	$\{a,b\}$	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$
d	$\{a\}$	$\{a,b\}$	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$
e	$\{a\}$	$\{a,b\}$	$\{a,b,c\}$	$\{a,b,c,d\}$	$\{a,b,c,d,e\}$	$\{a,b,c,d,e\}$
f	$\{a\}$	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c,d\}$	$\{a,b,c,d,e\}$	$\{a,b,c,d,e,f\}$

$$\begin{aligned} \leq := \{&(a,a),(a,c),(a,d),(a,e),(a,f),(b,b),(b,e),(b,f),\\ &(c,c),(c,d),(c,f),(d,d),(d,f),(e,e),(f,f)\}. \end{aligned}$$

(1) $(S, \circ, \leq)$ is an ordered semihypergroup. By verifying the associativity of the hyperoperation $\circ$ and the compatibility of the partial order $\leq$ on S, we find that $(S, \circ, \leq)$ still forms an ordered semihypergroup.

(2) $(S/\mathcal{W}, \star_{\mathcal{W}})$ is a semihyperlattice. It is easy to derive from the Definition 3.1 of pseudo-hyperfilters that $W(a) = W(b) = S, W(c) = W(d) = W(e) = \{c, d, e, f\}, W(f) = \{e, f\}$. Hence, $(a)_{\mathcal{W}} = (b)_{\mathcal{W}} = \{a, b\}, (c)_{\mathcal{W}} = (d)_{\mathcal{W}} = (e)_{\mathcal{W}} = \{c, d, e\}, (f)_{\mathcal{W}} = \{f\}$. Consequently, We can immediately derive the hyperoperation $\star_{\mathcal{W}}$ and the order $\preceq_{\mathcal{W}}$ for as follows:

$\star_{\mathcal{W}}$	$(a)_{\mathcal{W}}$	$(c)_{\mathcal{W}}$	$(f)_{\mathcal{W}}$
$(a)_{\mathcal{W}}$	$\{(a)_{\mathcal{W}}\}$	$\{(a)_{\mathcal{W}}\}$	$\{(a)_{\mathcal{W}}\}$
$(c)_{\mathcal{W}}$	$\{(a)_{\mathcal{W}}\}$	$\{(a)_{\mathcal{W}}, (c)_{\mathcal{W}}\}$	$\{(a)_{\mathcal{W}}, (c)_{\mathcal{W}}\}$
$(f)_{\mathcal{W}}$	$\{(a)_{\mathcal{W}}\}$	$\{(a)_{\mathcal{W}}, (c)_{\mathcal{W}}\}$	$\{(a)_{\mathcal{W}}, (c)_{\mathcal{W}}, (f)_{\mathcal{W}}\}$

$$\begin{aligned} \preceq_{\mathcal{W}} := \{&((a)_{\mathcal{W}}, (a)_{\mathcal{W}}), ((a)_{\mathcal{W}}, (c)_{\mathcal{W}}), ((a)_{\mathcal{W}}, (f)_{\mathcal{W}}),\\ &((c)_{\mathcal{W}}, (c)_{\mathcal{W}}), ((c)_{\mathcal{W}}, (f)_{\mathcal{W}}),\\ &((f)_{\mathcal{W}}, (f)_{\mathcal{W}}). \end{aligned}$$

It is easy to check that the table above meets the requirements of Definition 3.2. Therefore, $S/\mathcal{W}$ with respect to $\star_{\mathcal{W}}$ forms a semihyperlattice.

(3) $(S/\mathcal{W}, \star_{\mathcal{W}}, \preceq_{\mathcal{W}})$ is an ordered semihyperlattice. It is easy to verify that the order $\preceq_{\mathcal{W}}$ defined by Theorem 3.1 is compatible with $\star_{\mathcal{W}}$, thus $(S/\mathcal{W}, \star_{\mathcal{W}}, \preceq_{\mathcal{W}})$ constitutes an ordered semihyperlattice.

Hereafter, we provide an equivalent characterization of $\mathcal{W}$ through the introduction of the concept of a-maximal.

Definition 3.5. *Let A be a pseudo-hyperfilter of an ordered semihypergroup S and $a \in S$. A pseudo-hyperfilter A is called* a-maximal *if A is maximal in the set of all pseudo-hyperfilters not containing a.*

We define $\mathcal{M}(a)$ to encompass all a-maximal pseudo-hyperfilters within the semihypergroup S. It is conceivable that for some elements a in S, $\mathcal{M}(a)$ may be void, expressed as $\mathcal{M}(a) = \emptyset$. For instance, the case where S includes a supreme element e, in such scenarios, $\mathcal{M}(e)$ is invariably void, indicated by $\mathcal{M}(e) = \emptyset$.

Theorem 3.2. *Let S be an ordered semihypergroup. Then, $x\mathcal{W}y$ if and only if $\mathcal{M}(x) = \mathcal{M}(y)$ for all $x, y \in S$.*

Proof. $(\Rightarrow)$ Assume $x\mathcal{W}y$, which implies $W(x) = W(y)$. Consider any A from $\mathcal{M}(x)$. If $y \in A$, it would suggest $W(y) \subseteq A$, leading to the inclusion $x \in A$, which contradicts the definition of A. Therefore, $y \notin A$, and consequently, $A \in \mathcal{M}(y)$. Contrariwise, if $A \notin \mathcal{M}(y)$, there must exist a y-maximal pseudo-hyperfilter A' in S with $A \subset A'$, implying $x \in A'$ and subsequently $y \in A'$, which is untenable. Thus, we establish that $\mathcal{M}(x) \subseteq \mathcal{M}(y)$. Following a similar argument, we can also deduce $\mathcal{M}(y) \subseteq \mathcal{M}(x)$. Then, $\mathcal{M}(y) = \mathcal{M}(x)$.

$(\Leftarrow)$ If $\mathcal{M}(x) = \mathcal{M}(y)$, it follows that $x \in W(y)$. Suppose the contrary, that is, there is a x-maximal pseudo-hyperfilter A for which $W(y) \subseteq A$. This would necessitate $y \in A$, contradicting the presumption that A is part of $\mathcal{M}(y)$. By symmetry, it can be shown that $y \in W(x)$, leading to the conclusion $W(x) = W(y)$, hence $x\mathcal{W}y$. □

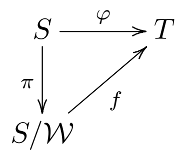
4. The $\mathcal{W}$-classes of ordered semihypergroup

In this section, we first establish the homomorphism theorem from ordered semihypergroups to ordered semihyperlattices, and then investigate some properties of the $\mathcal{W}$-classes of an ordered semihypergroup S and describe the relationship between $(a)_{\mathcal{W}}$ and the principal pseudo-hyperfilter $W(a)$ generated by a.

Lemma 4.1. *Let S be an ordered semihypergroup, and $\mathcal{W}$ be an ordered semihyperlattice regular equivalence relation on S, then the mapping $\pi : S \to S/\mathcal{W}$ given by $\pi(x) = (x)_{\mathcal{W}}$ is an epimorphism.*

Proof. Straightforward. □

Theorem 4.1. *Let $(S, \circ, \leq_S)$ be an ordered semihypergroup, $(T, \diamond, \leq_T)$ be an ordered semihyperlattice. $\varphi : S \to T$ is a homomorphism. Then, if $\mathcal{W}$ defined* *by* $\mathcal{W} = \{(x, y) \mid W(x) = W(y)\}$ *and* $\mathcal{W} \subseteq \ker\varphi$*, there exists the unique homomorphism* $f : S/\mathcal{W} \to T \mid (\alpha)_{\mathcal{W}} \mapsto \varphi(\alpha)$ *such that the diagram*



commutes. Moreover, $Im(\varphi) = Im(f)$*.*

Proof. (1) f is well defined. Indeed, if $(x)_{\mathcal{W}} = (y)_{\mathcal{W}}$, then $(x, y) \in \mathcal{W} \subseteq \ker\varphi$, hence $\varphi(x) = \varphi(y)$. (2) f is a homomorphism and $\varphi = f \circ \pi$. In fact, By Lemma 4.1, there exists an ordered relation $\preceq_{\mathcal{W}}$ on the quotient semihyperlattice $(S/\mathcal{W}, \star_W)$ such that $(S/\mathcal{W}, \star_{\mathcal{W}}, \preceq_W)$ is an ordered semihyperlattice and the mapping π is a homomorphism. Moreover, $(x)_{\mathcal{W}} \preceq_{\mathcal{W}} (y)_{\mathcal{W}}$ means that $(x)_{\mathcal{W}} \in (x)_{\mathcal{W}} \star_{\mathcal{W}} (y)_{\mathcal{W}} = (x \circ y)_{\mathcal{W}}$. Then, there exists $z \in x \circ y$, such that $(x)_{\mathcal{W}} = (z)_{\mathcal{W}}$. Thus, $(x, z) \in \mathcal{W} \subseteq \ker\varphi$, then we have

$$\varphi(x) = \varphi(z) \in \{\varphi(t) \mid t \in x \circ y\} = \varphi(x) \diamond \varphi(y).$$

Hence, $\varphi(x) \leq_T \varphi(y)$, that is, $f((x)_{\mathcal{W}}) \leq_T f((y)_{\mathcal{W}})$

Also, let $(x)_{\mathcal{W}}, (y)_{\mathcal{W}} \in S/\mathcal{W}$. Since φ is a homomorphism from S to T, we have

$$\begin{aligned} f((x)_{\mathcal{W}}) \diamond f((y)_{\mathcal{W}}) &= \varphi(x) \diamond \varphi(y) \\ &= \{\varphi(z) \mid z \in x \circ y\} \\ &= \{f((z)_{\mathcal{W}}) \mid (z)_{\mathcal{W}} \in (x)_{\mathcal{W}} \star_{\mathcal{W}} (y)_{\mathcal{W}}\}. \end{aligned}$$

In addition, for any $\alpha \in S$, $(f \circ \pi)(\alpha) = f((\alpha)_{\mathcal{W}}) = \varphi(\alpha)$, hence $\varphi = f \circ \pi$.

Furthermore, we claim that f is the unique homomorphism from $S/\mathcal{W}$ to T. Let $\tilde{f}$ is a homomorphism from $S/\mathcal{W}$ to T such that $\varphi = \tilde{f} \circ \pi$. Therefore,

$$f((x)_{\mathcal{W}}) = \varphi(x) = (\tilde{f} \circ \pi)(x) = \tilde{f}((x)_{\mathcal{W}}),$$

that is, $f = \tilde{f}$. Finally, we have $Im(f) = \{f((x)_{\mathcal{W}}) \mid x \in S\} = \{\varphi(x) \mid x \in S\} = Im(\varphi)$. □

Based on the homomorphism Theorem 4.1, the subsequent discussion will focus on the study of congruence classes and their properties. Looking ahead, our future research will aim to explore how the properties of quotient semilattices and $\mathcal{W}$-classes can reflect the intrinsic structure of the original ordered semihypergroups.

Lemma 4.2. $(a)_{\mathcal{W}} = (b)_{\mathcal{W}}$ *if and only if* $W(a) = W(b)$*.*

Proof. (⇒) For any $a, b \in S$, Assume $(a)_{\mathcal{W}} = (b)_{\mathcal{W}}$. Then, $a \in (b)_{\mathcal{W}}$ implies $a\mathcal{W}b$, and thus $W(a) = W(b)$.

(⇐) Assume $W(a) = W(b)$. Then, $a \in (b)_{\mathcal{W}}$ and $b \in (a)_{\mathcal{W}}$. For any $c \in (a)_{\mathcal{W}}$, given $c\mathcal{W}a$ and $a\mathcal{W}b$, it follows that $c\mathcal{W}b$, implying $c \in (b)_{\mathcal{W}}$. Therefore, $(a)_{\mathcal{W}} \subseteq (b)_{\mathcal{W}}$. Similarly, $(b)_{\mathcal{W}} \subseteq (a)_{\mathcal{W}}$, hence $(a)_{\mathcal{W}} = (b)_{\mathcal{W}}$. □

Remark 4.1. Since $(a)_{\mathcal{W}}$ and $(b)_{\mathcal{W}}$ are congruence classes of the ordered semihypergroup S under the ordered semihyperlattice equivalence relation $\mathcal{W}$, $(a)_{\mathcal{W}}$ and $(b)_{\mathcal{W}}$ either equal or disjoint. Thus, from $(a)_{\mathcal{W}} \subseteq (b)_{\mathcal{W}}$, one can directly conclude $(a)_{\mathcal{W}} = (b)_{\mathcal{W}}$.

Theorem 4.2. $(a)_{\mathcal{W}} \preceq_{\mathcal{W}} (b)_{\mathcal{W}}$ *if and only if* $W(b) \subseteq W(a)$.

Proof. (⇒) Assume $(a)_{\mathcal{W}} \preceq_{\mathcal{W}} (b)_{\mathcal{W}}$. This is equivalent to $(a)_{\mathcal{W}} \in (a)_{\mathcal{W}} \star_{\mathcal{W}} (b)_{\mathcal{W}} = \{(x)_{\mathcal{W}} \mid x \in a \circ b\}$, implying there exists some $t \in a \circ b$ such that $(a)_{\mathcal{W}} = (t)_{\mathcal{W}}$. Given $t \in a \circ b$, then $(a \circ b) \cap W(t) \neq \emptyset$, leading to $a, b \in W(t)$. Hence $W(b) \subseteq W(t) = W(a)$.

(⇐) Assume $W(b) \subseteq W(a)$. Considering $a, b \in W(a)$, therefore, $(a \circ b) \cap W(a) \neq \emptyset$ and there exists some $t \in (a \circ b) \cap W(a)$, such that $t \leqslant a$ or $t \nsim a$. In either case, we have $W(a) \subseteq W(t)$, and since $t \in W(a)$, it follows that $W(t) \subseteq W(a)$, thus $W(a) = W(t)$. By Lemma 4.2, we obtain $(a)_{\mathcal{W}} = (t)_{\mathcal{W}}$. Lastly, as $t \in a \circ b$, it follows that $(a)_{\mathcal{W}} = (t)_{\mathcal{W}} \in \{(x)_{\mathcal{W}} \mid x \in a \circ b\} = (a)_{\mathcal{W}} \star_{\mathcal{W}} (b)_{\mathcal{W}}$, which means $(a)_{\mathcal{W}} \preceq_{\mathcal{W}} (b)_{\mathcal{W}}$. □

Conclusion

In this paper, we introduced the concept of pseudo-hyperfilters in ordered semihypergroups and by using pseudo-hyperfilters, we constructed the semihyperlattice regular equivalence relation $\mathcal{W}$ which enabled us to establish a homomorphism theorem from ordered semihypergroups to ordered semihyperlattices, preserving the hyperstructure in the quotient process.

This work advances existing theories in two significant aspects. First, it extends the framework of ordered semihypergroup theory by overcoming the limitation where ordered semihypergroups could previously only be mapped to ordered semilattices via semilattice strong regular equivalence relations, resulting in the loss of hyperstructures. By retaining the hyperstructure, our results generalize the classical theory of ordered semigroups. Second, the study enriches the broader field of algebraic hyperstructures by providing a new perspective on semihyperlattice regular equivalence relations and their induced quotient hyperstructures.

Future research could focus on leveraging the properties of quotient semihyperlattices to investigate the internal structure of the original ordered semihypergroups. This approach has the potential to reveal deeper connections between the external quotient hyperstructures and the intrinsic algebraic properties.

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Prime-valent one-regular graphs of order $28p$

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Abstract. A graph is *one-regular* if its full automorphism group acts on its arcs regularly. In this paper, we classify connected one-regular graphs of prime valency and order $28p$ for each prime p, and prove that there is only one sporadic graph: the $\mathbb{Z}_7$-cover CQ_7 of the three dimensional hypercube Q_3 with valency 3.

Keywords: symmetric graph, arc-transitive graph, one-regular graph.
MSC 2020: 05C25, 20B25.

1. Introduction

Throughout this paper graphs are assumed to be finite, simple, connected and undirected. For group-theoretic concepts or graph-theoretic terms not defined here we refer the reader to [22, 25] or [1, 2], respectively. Let G be a permutation group on a set Ω and $v \in \Omega$. Denote by G_v the stabilizer of v in G, that is, the subgroup of G fixing the point v. We say that G is *semiregular* on Ω if $G_v = 1$ for every $v \in \Omega$ and *regular* if G is transitive and semiregular.

For a graph X, denote by $V(X)$, $E(X)$ and $\mathrm{Aut}(X)$ its vertex set, its edge set and its full automorphism group, respectively. A graph X is said to be G-*vertex-transitive* if $G \leq \mathrm{Aut}(X)$ acts transitively on $V(X)$. X is simply called *vertex-transitive* if it is $\mathrm{Aut}(X)$-vertex-transitive. An s-*arc* in a graph is an ordered $(s+1)$-tuple $(v_0, v_1, \cdots, v_{s-1}, v_s)$ of vertices of the graph X such that v_{i-1} is adjacent to v_i for $1 \leq i \leq s$, and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s-1$. In particular, a 1-arc is just an arc and a 0-arc is a vertex. For a subgroup $G \leq \mathrm{Aut}(X)$, a graph X is said to be (G, s)-*arc-transitive* or (G, s)-*regular* if G is transitive or regular on the set of s-arcs in X, respectively. A (G, s)-arc-transitive graph is said to be (G, s)-*transitive* if it is not $(G, s+1)$-arc-transitive. In particular, a $(G, 1)$-arc-transitive graph is called G-*symmetric*. A graph X

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is simply called *s-arc-transitive*, *s-regular* or *s-transitive* if it is $(\mathrm{Aut}(X), s)$-arc-transitive, $(\mathrm{Aut}(X), s)$-regular or $(\mathrm{Aut}(X), s)$-transitive, respectively.

We denote by $\mathbf{C}_n$ and $\mathbf{K}_n$ the cycle and the complete graph of order n, respectively. Denote by $\mathbf{D}_{2n}$ the dihedral group of order $2n$. As we all known that there is only one connected 2-valent graph of order n, that is, the cycle $\mathbf{C}_n$, which is s-regular with full automorphism group $\mathbf{D}_{2n}$. Let p be a prime. Classifying s-transitive and s-regular graphs has received considerable attention. The classification of s-transitive graphs of order p and $2p$ was given in [5] and [6], respectively. Wang [24] characterized the prime-valent s-transitive graphs of order $4p$. The classification of cubic and pentavalent s-transitive graphs of order $28p$ was given in [15] and [18], respectively. For heptavalent symmetric graphs of order $28p$, their characterizations can be obtained in [19].

For 2-valent case, the graph is a cycle, which is s-regular for any integer s, and for cubic case, s-transitivity always means s-regularity by Miller [9]. However, for the other prime-valent case, this is not true, see for example [11] for pentavalent case and [12] for heptavalent case. Thus, characterization and classification of prime-valent s-regular graphs is very interesting and also reveals the s-regular global and local actions of the permutation groups on s-arcs of such graphs. In particular, 1-regular action is the most simple and typical situation. In this paper, we classify prime-valent one-regular graph of order $28p$ for each prime p.

2. Preliminary results

Let X be a connected G-symmetric graph with $G \leq \mathrm{Aut}(X)$, and let N be a normal subgroup of G. The *quotient graph* X_N of X relative to N is defined as the graph with vertices the orbits of N on $V(X)$ and with two orbits adjacent if there is an edge in X between those two orbits. In view of [17, Theorem 9], we have the following:

Proposition 2.1. *Let X be a connected G-symmetric graph with $G \leq \mathrm{Aut}(X)$ and prime valency $q \geq 3$, and let N be a normal subgroup of G. Then, one of the following holds:*

(1) *N is transitive on $V(X)$;*
(2) *X is bipartite and N is transitive on each part of the bipartition;*
(3) *N has $r \geq 3$ orbits on $V(X)$, N acts semiregularly on $V(X)$, the quotient graph X_N is a connected q-valent G/N-symmetric graph.*

To extract a classification of connected prime-valent symmetric graphs of order $2p$ for a prime p from Cheng and Oxley [6], we introduce the graphs $G(2p, q)$. Let V and V' be two disjoint copies of $\mathbb{Z}_p$, say $V = \{0, 1, \cdots, p-1\}$ and $V' = \{0', 1', \cdots, (p-1)'\}$. Let q be a positive integer dividing $p-1$ and $H(p, q)$ the unique subgroup of $\mathbb{Z}_p^*$ of order q. Define the graph $G(2p, q)$ to have vertex set $V \cup V'$ and edge set $\{xy' \mid x - y \in H(p, q)\}$.

Proposition 2.2. *Let X be a connected q-valent symmetric graph of order $2p$ with p, q primes. Then, X is isomorphic to $\mathbf{K}_{2p}$ with $q = 2p - 1$, $\mathbf{K}_{p,p}$ or $G(2p, q)$ with $q|(p-1)$. Furthermore, if $(p, q) \neq (11, 5)$ then $\mathrm{Aut}(G(2p, q)) = (\mathbb{Z}_p \rtimes \mathbb{Z}_q) \rtimes \mathbb{Z}_2$; if $(p, q) = (11, 5)$ then $\mathrm{Aut}(G(2p, q)) = \mathrm{PGL}(2, 11)$.*

Next, by [24, Theorem 3.1], we have the characterization of prime valent symmetric graphs of order $4p$.

Proposition 2.3. *Let p and q be two primes with $q \geq 5$, and let X be a q-valent symmetric graph of order $4p$. Then, X is isomorphic to $\mathbf{K}_{4p}$ with $q = 4p - 1$, $\mathbf{K}_{2p,2p} - 2p\mathbf{K}_2$ with $q = 2p - 1$, or the quotient graph is isomorphic to $\mathbf{K}_{p,p}$ with $q = p$ or $\mathbf{K}_{2p}$ with $q = 2p - 1$.*

The following proposition is about the prime-valent symmetric graphs of order $14p$ with p a prime, which is deduced from [20, Theorem 1.2].

Proposition 2.4. *Let p and q be two primes. If $q \geq 5$, then there is no q-valent symmetric graph of order $14p$ admitting a solvable arc-transitive automorphism group.*

The following proposition is the famous "N/C-Theorem", see for example [14, Chapter I, Theorem 4.5]).

Proposition 2.5. *The quotient group $N_G(H)/C_G(H)$ is isomorphic to a subgroup of the automorphism group $\mathrm{Aut}(H)$ of H.*

From [8, pp.12-14] and [23, Theorem 2], we can deduce the non-abelian simple groups whose orders have at most four different prime divisors.

Proposition 2.6. *Let p and q be two odd primes, and let G be a non-abelian simple group. If the order $|G|$ divides $2^2{\cdot}7{\cdot}p{\cdot}q$ with $p \geq 3$ and $q > 7$, then G is isomorphic to A_5, $\mathrm{PSL}(2, 13)$. If the order $|G|$ has at most three different prime divisors, then G is called K_3-simple group and isomorphic to one of the following groups.*

Table 1: **Non-abelian simple $\{2, 3, p\}$-groups**

Group	Order	Group	Order
A_5	$2^2 \cdot 3 \cdot 5$	$\mathrm{PSL}(2, 17)$	$2^4 \cdot 3^2 \cdot 17$
A_6	$2^3 \cdot 3^2 \cdot 5$	$\mathrm{PSL}(3, 3)$	$2^4 \cdot 3^3 \cdot 13$
$\mathrm{PSL}(2, 7)$	$2^3 \cdot 3 \cdot 7$	$\mathrm{PSU}(3, 3)$	$2^5 \cdot 3^3 \cdot 7$
$\mathrm{PSL}(2, 8)$	$2^3 \cdot 3^2 \cdot 7$	$\mathrm{PSU}(4, 2)$	$2^6 \cdot 3^4 \cdot 5$

3. Classification

This section is devoted to classifying prime-valent one-regular graphs of order $28p$ for each prime p. Let q be a prime. In what follows, we always denote by X a connected q-valent one-regular graph of order $28p$. Set $A = \mathrm{Aut}(X)$, $v \in V(X)$. Then, the vertex stabilizer $A_v \cong \mathbb{Z}_q$ and hence $|A| = 28pq$. Clearly, if $q = 2$, then $X \cong \mathbf{C}_{28p}$ with $A \cong \mathbf{D}_{56p}$, which is s-regular for any positive integer s. If $q = 3$, then by [15, Corollary 4.9] and [10, Theorem 5.1], we have that $X \cong CQ_7$, that is, the $\mathbb{Z}_7$-cover of three dimensional hypercube Q_3. If $q = 5$, then by [18, Theorem 3.1] and [21], there exists no pentavalent one-regular graph of order $28p$. Next we deal with the case $q = 7$.

Lemma 3.1. *Let $q = 7$. Then, there exists no heptavalent one-regular graph of order $28p$.*

Proof. Suppose that $p = 2$. Then, $|V(X)| = 56 = 8{\cdot}7$ and $|A| = 8{\cdot}7^2$. By [13, Theorem 3.1], there is no heptavalent symmetric graphs of order 56, and the statement holds for $p = 2$.

Suppose that $p \geq 3$. Then, by [19, Theorem 1.1], we have that $A \cong \mathrm{PSL}(2,p)$, $\mathrm{PGL}(2,p)$, $\mathrm{PSL}(2,p) \times \mathbb{Z}_2$ or $\mathrm{PGL}(2,p) \times \mathbb{Z}_2$. In particular, A is non-solvable. Note that $|A| = 2^2{\cdot}7^2{\cdot}p$. Thus, A has a non-solvable composition factor H isomorphic to a non-abelian K_3-simple group. By Proposition 2.6, $p = 3$ and $H \cong \mathrm{PSL}(2,7)$. However, $|H| = 2^3{\cdot}3{\cdot}7$, this is contrary to the fact that $|H| \big| 2^2{\cdot}7^2{\cdot}p$. □

Now, we treat with the case $q > 7$. Recall that $|A| = 28pq$, $A_v \cong \mathbb{Z}_q$ and $q > 7$. Let N be a minimal normal subgroup of A. We divide the proof into the following two cases: $p = q$ and $p \neq q$.

Lemma 3.2. *Let $p = q > 7$. Then, there exists no new graph.*

Proof. Suppose that $p = q$. Then, $|A| = 28p^2$ and $A_v \cong \mathbb{Z}_p$ with $p > 7$. Let P be a Sylow p-subgroup of A. Then, $|P| = p^2$. Note that $p = q > 7$. Thus, by Sylow Theorem, the number of Sylow p-subgroups of A is $kp+1 = |A : N_A(P)|$ for some integer k. Since $|A| = 28p^2$, we have that $(kp+1) \div 28$. It is easy to see that either $k = 0$ or $k = 1$ with $p = 13$. If $k = 1$, then P is normal in A. Since $|P| = p^2 > 7^2$, we have that $P_v \cong \mathbb{Z}_p$. However, P acting on $V(X)$ has 28 orbits, and hence P is semiregular by Proposition 2.1. This is contrary to the fact that $P_v \cong \mathbb{Z}_p$. Thus, $k = 1$ and $p = 13$. If A is non-solvable, then a composition factor is isomorphic to a non-abelian simple group and hence this composition factor has order dividing $|A| = 2^2{\cdot}7{\cdot}p^2$. By Proposition 2.6, there is no non-abelian simple K_3-group whose prime divisor does not have 3. It follows that A is solvable with $p = q = 13$ and P is not normal in A.

Let N be a minimal normal subgroup of A. Then, $N \cong \mathbb{Z}_2$, $\mathbb{Z}_2^2$, $\mathbb{Z}_7$ or $\mathbb{Z}_p$ because A is solvable and has no normal Sylow p-subgroup by the above paragraph. By Proposition 2.1, N is semiregular and X_N is a q-valent symmetric graph of order $28p/|N|$. Note that there exists no regular graph of odd order and odd valency. Thus, $N \not\cong \mathbb{Z}_2^2$.

Suppose that $N \cong \mathbb{Z}_2$. Then, X_N is a A/N-symmetric graph of order $2{\cdot}7{\cdot}13$ and valency 13. Since X_N has valency 13 and order $2{\cdot}7{\cdot}13$, we have that $13 \nmid (7-1)$. Note that A/N is solvable. By Proposition 2.4, there exists no symmetric graph of order 14 admitting a solvable arc-transitive automorphism group, a contradiction.

Suppose that $N \cong \mathbb{Z}_7$. Then, X_N is a A/N-symmetric graph of order $4p$ and valency 13. Since $|A/N| = 4{\cdot}13^2$, by Sylow Theorem we have that the Sylow p-subgroup of A/N is nomal in A/N, and by Proposition 2.5, we can easily deduce that A has a normal Sylow p-subgroup, a contradiction.

Suppose that $N \cong \mathbb{Z}_p$. Then, X_N is a A/N-symmetric graph of order 28. By [7], there are two symmetric graphs of order 28, their full automorphism groups are $S_{14} \times \mathbb{Z}_2$ and $\mathrm{PSL}(2,13) \times \mathbb{Z}_2$, respectively. However, these two groups have no subgroup of order $28{\cdot}13$, a contradiction. □

Lemma 3.3. *Let $p \neq q$ and $q > 7$. Then, there exists no new graph.*

Proof. Suppose that $p \neq q$. Then, $|A| = 28pq$ and $A_v \cong \mathbb{Z}_q$ with $p \neq q > 7$. Since $|A| = 28pq$ and $A_v \cong \mathbb{Z}_q$, we have that A_v is a Sylow q-subgroup of A. It follows that the Sylow q-subgroups of A cannot be normal in A. Let N be a minimal normal subgroup of A. Then, N is either a direct product of some isomorphic non-abelian simple groups or an elementary abelian r-group with $r = 2$, 7 or p.

Case 1. Assume that N is non-solvable.

Since $|A| = 2^2{\cdot}3{\cdot}7{\cdot}q$, we have that $|N| \big| 2^2{\cdot}3{\cdot}7{\cdot}q$. By Proposition 2.6, we have that $N \cong \mathrm{A}_5$ or $\mathrm{PSL}(2,13)$.

Let $N \cong \mathrm{A}_5$. Then, $|N| = 2^2{\cdot}3{\cdot}5$. Since $|N| \big| |A|$, we have that $p = 3$ and $q = 5$. This is contrary to our assumption that $q > 7$.

Let $N \cong \mathrm{PSL}(2,13)$. Then, $|N| = 2^2{\cdot}3{\cdot}7{\cdot}13$. Since $q > 7$, we have that $p = 3$ and $q = 13$. It follows that N is arc-transitive on X and $N_v \cong \mathbb{Z}_{13}$ is a Sylow 13-subgroup of N. Thus, X can be viewed as an orbital graph of N acting on N_v. By using the functions `CosetAction` and `OrbitalGraph` in Magma [3], up to graph isomorphism, there is only one connected orbital graph of valence 13 admitting N as an arc-transitive automorphism group. However, this orbital graph has full automorphism group isomorphic to $\mathrm{PSL}(2,13) \times \mathbb{Z}_2$ and so is not one-regular, a contradiction.

Case 2. Assume that A has no non-solvable minimal normal subgroup.

Suppose that $p = 2$. Then, by [16, Theorem 3.3], there is no q-valent one-regular graph of order 56 with $q > 7$. In what follows, we may suppose that $p \geq 3$. Since N is solvable, $N \cong \mathbb{Z}_2$, $\mathbb{Z}_2^2$, $\mathbb{Z}_7$, $\mathbb{Z}_7^2$ with $p = 7$ or $\mathbb{Z}_p$. By Proposition 2.1, N is semiregular and X_N is a q-valent symmetric graphs of order $28p/|N|$ with $A/N \lesssim \mathrm{Aut}(X_N)$. Clearly, $N \ncong \mathbb{Z}_2^2$ because there exists no regular graph of odd order and odd valency.

Let $N \cong \mathbb{Z}_p$. Then, X_N is a q-valent symmetric graph of order 28 with $q > 7$. By [7], there are two symmetric graphs of order 28 and valency q, their full automorphism groups are $S_{14} \times \mathbb{Z}_2$ and $\mathrm{PSL}(2,13) \times \mathbb{Z}_2$, respectively. By Proposition 2.1, A/N can be embedded in $S_{14} \times \mathbb{Z}_2$ or $\mathrm{PSL}(2,13) \times \mathbb{Z}_2$. However, by Magma [3], both $S_{14} \times \mathbb{Z}_2$ and $\mathrm{PSL}(2,13) \times \mathbb{Z}_2$ have no subgroup of order $28{\cdot}13$, a contradiction.

Let $N \cong \mathbb{Z}_7^2$. Then, $p = 7$, $|A| = 2^2{\cdot}7^2{\cdot}q$ and X_N is a A/N-symmetric graph of order 4 and valency q. Note that $q > 7$. This is clearly impossible.

Let $N \cong \mathbb{Z}_7$. Then, X_N is a A/N-symmetric graph of order $4p$ and valency q. By Proposition 2.3, $X_N \cong \mathbf{K}_{4p}$ with $q = 4p-1$, $\mathbf{K}_{2p,2p} - 2p\mathbf{K}_2$ with $q = 2p-1$, or A/N has a normal subgroup $M/N \cong \mathbb{Z}_2$ such that $X_M \cong \mathbf{K}_{p,p}$ with $q = p$ or $\mathbf{K}_{2p}$ with $q = 2p-1$. By our assumption with $p \neq q$, we have that $X_M \ncong \mathbf{K}_{p,p}$.

Assume that $X_N \cong \mathbf{K}_{4p}$ with $q = 4p-1$. Then, $A/N \lesssim S_{4p}$ and $|A/N| = 4pq$. If A/N is non-solvable, then A/N has a composition factor isomorphic to a non-abelian K_3-simple group. By Proposition 2.6, this is impossible because $q > 7$. Thus, A/N is solvable. Since $|A/N| = 4pq$ with $q = 4p-1$, we have that A/N is 2-transitive on $V(X)$. It follows that the vertex stabilizer of A/N is isomorphic to $\mathbb{Z}_q$ and normalizes the subgroup of A/N and order $4p$. By Burnside's Theorem [4, p.192, Theorem IX], A/N is affine. This forces that A/N has a unique minimal normal elementary abelian subgroup and hence $4p = r^k$ for some prime r. This is impossible because $p \geq 3$.

Assume that $X_N \cong \mathbf{K}_{2p,2p} - 2p\mathbf{K}_2$ with $q = 2p-1$. Then, $A/N \lesssim S_{2p} \times \mathbb{Z}_2$. Since $|A/N| = 4pq$ and X_N is a bipartite graph, A/N has a subgroup B/N of index 2, which acting on each bipartition of X_N is 2-transitive. Clearly, $|B/N| = 2pq$ and so B/N is solvable. By Burnside's Theorem, B/N is affine. Similar arguments as the above paragraph, we can deduce that a contradiction because $4p$ can not be a prime power with $p \geq 3$.

Assume that $X_M \cong \mathbf{K}_{2p}$ with $q = 2p-1$. Then, $A/M \lesssim \S_{2p}$ and A/M is 2-transitive on $V(X_M)$. Since $|A/M| = 2pq$, we have that A/M is solvable. By Burnside's Theorem, A/N is affine. Similarly, this is impossible because $2p$ cannot be a prime power with $p \geq 3$.

Let $N \cong \mathbb{Z}_2$. Then, X_N is a A/N-symmetric graph of order $2{\cdot}7{\cdot}p$ and valency q. If $p = 7$, then $|A/N| = 2{\cdot}7^2$. By Sylow Theorem, the Sylow p-subgroup of A/N must be normal, and by Proposition 2.5, A has a normal Sylow p-subgroup P of order 7^2. Thus, X_P is a symmetric graph of order 4 and valency $q > 7$. Clearly, this is impossible. If $p \neq 7$, then X_N has a square free order. By Proposition 2.4, there is no symmetric graph of order $2{\cdot}7{\cdot}p$ admitting a solvable arc-transitive automorphism group, a contradiction. □

Combining the above arguments with the cases $q = 2, 3$ and 5, and by Lemmas 3.1-3.3, we have the following result.

Theorem 3.1. *Let p, q be two primes and let X be a connected q-valent one-regular graph of order $28p$. Then $X \cong CQ_7$ with $p = 7$ and valency 3.*

4. Conclusion

As is known to all, arc-transitive graphs have much higher symmetries and much larger full automorphism groups, and one-regular graphs have smallest full automorphism groups such that the graphs are arc-transitive. Thus, the characterization and classification of one-regular graphs not only reveal the local action but also global action of the full automorphism group acting on vertices and arcs. In the paper, we classify the one-regular graphs of order $28p$ and prime valency for each prime p, and prove that there is only one sporadic such graph. As a natural continuation, could we find and classify one-regular graphs of order $28p$ with more general valencies.

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Hypervaluations on ternary semihyperrings

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Abstract. This article aims to present the concept of hypervaluation on a commutative ternary semihyperring mapped onto an ordered abelian group. It examines various properties of hyperideals within the ternary semihyperring that correspond to the valuation map. Additionally, the article explores results which are similar to those found in classical valuation rings, but within the framework of hypervalued ternary semihyperrings.

Keywords: ternary hypervaluation, ternary semihyperring, ternary semihyperring homomorphism.
MSC 2020: 16W60, 16Y20, 16Y99, 20N20.

1. Introduction

Generalizing the notion of binary operation in groups to hyperoperation, Marty [22] introduced hypergroups in 1934. A hypergroupoid is an ordered pair $(M, \circ)$, where $\circ : M \times M \to P^{\star}(M)$ is a hyperoperation on a nonempty set M. Let L_1 and L_2 be two nonempty subsets of M and $m \in M$, then

$$L_1 \circ L_2 = \bigcup l_1 \circ l_2,\ L_1 \circ m = L_1 \circ \{m\} \text{ and } m \circ L_2 = \{m\} \circ L_2,$$

where $l_1 \in L_1,\ l_2 \in L_2$.

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A hypergroupoid $(M, \circ)$ earns the title of a semihypergroup if, for every $m_1, m_2, m_3 \in M$, the equation $(m_1 \circ m_2) \circ m_3 = m_1 \circ (m_2 \circ m_3)$ holds, indicating that

$$\bigcup_{u \in m_1 \circ m_2} u \circ m_3 = \bigcup_{v \in m_2 \circ m_3} m_1 \circ v.$$

By a hypergroup, we mean a semihypergroup $(M, \circ)$ for which $m_1 \circ M = M \circ m_1 = M$, for all $m_1 \in M$.

The definitions of relevant algebraic structures follow the order: (binary) hyperring (Definition 1.1), ternary hyperring (Definition 1.2) and ternary semihyperring (Definition 1.3).

Krasner [20] initiated the concept of hyperrings and hyperfields, taking addition to be a hypercomposition and retaining multiplication as a binary composition.

Definition 1.1 ([20]). *A hyperring according to Krasner is described as a mathematical structure $(\mathcal{R}, +, \cdot)$ that adheres to the following set of axioms:*

(i) $(\mathcal{R}, +)$ *constitutes a canonical hypergroup, i.e.,*

(1) *for every* $r_1, r_2, r_3 \in \mathcal{R}$, $r_1 + (r_2 + r_3) = (r_1 + r_2) + r_3$,

(2) *for every* $r_1, r_2 \in \mathcal{R}$, $r_1 + r_2 = r_2 + r_1$,

(3) *there exists* $0 \in \mathcal{R}$ *such that* $0 + r_1 = \{r_1\} = r_1 + 0$ *for each* $r_1 \in \mathcal{R}$,

(4) *for each* $r_1 \in \mathcal{R}$, *there is precisely one element* r_1' *in* $\mathcal{R}$, *such that* $0 \in r_1 + r_1'$, *(we will denote* $-r_1$ *as* r_1' *and refer to it as the inverse of* r_1*),*

(5) $r_3 \in r_1 + r_2$ *implies* $r_2 \in -r_1 + r_3$ *and* $r_1 \in r_3 - r_2$.

(ii) $(\mathcal{R}, \cdot)$ *forms a semigroup with zero acting as a bilaterally absorbing element, i.e.,* $r_1 \cdot 0 = 0 \cdot r_1 = 0$.

(iii) *The multiplication distributes with regard to the hyperoperation* $+$.

Example 1.1 ([20]). Consider $\mathcal{R}$ as a ring with identity that satisfies commutativity. We define $\overline{\mathcal{R}} = \{\overline{r} = \{r, -r\}, r \in \mathcal{R}\}$. Then, $\overline{\mathcal{R}}$ becomes a hyperring with respect to the hyperoperation $\overline{r_1} \oplus \overline{r_2} = \{\overline{r_1 + r_2}, \overline{r_1 - r_2}\}$ and multiplication $\overline{r_1} \circ \overline{r_2} = \overline{r_1 \cdot r_2}$.

Let us suppose that $(\mathcal{R}, +, \cdot)$ is a hyperring and A is a nonempty subset of $\mathcal{R}$. We define A as a subhyperring of $\mathcal{R}$ if $(A, +, \cdot)$ forms a hyperring in its own right. A subhyperring A of a hyperring $\mathcal{R}$ is a left (right) hyperideal of $\mathcal{R}$ if $r_1 \cdot r_2 \in A$ $(r_2 \cdot r_1 \in A)$, $\forall\ r_1 \in \mathcal{R}, r_2 \in A$. A is termed a hyperideal if it satisfies the conditions of being both a left and a right hyperideal. A hyperideal P of a hyperring $\mathcal{R}$ is considered prime if, for any pair of hyperideals A and B of $\mathcal{R}$, the inclusion $AB \subseteq P$ implies either $A \subseteq P$ or $B \subseteq P$. In the case of a commutative hyperring $\mathcal{R}$, a hyperideal P is prime if $P \neq \mathcal{R}$ and for every $r_1, r_2 \in \mathcal{R}$, if $r_1 \cdot r_2 \in P$, then either $r_1 \in P$ or $r_2 \in P$.

Significant literature has evolved in the theory of hyperstructures till date, viz., hypergroupoids [16], hypergroups [17, 34, 18], semihypergroups [13], hyperrings [20, 14, 27, 28, 29, 30, 9, 6, 33], semihyperrings [1, 2, 4], ternary hyperrings [8, 7, 32, 31, 15] and ternary semihyperings [11, 26, 25, 5].

Alajbegović and Močkoř [3] studied m-rings with a multivalued addition satisfying certain conditions and commutative associative binary multiplication. In 2010, Davvaz and Mirvakili [12] introduced a new class of multialgebra called (m, n)-hyperring $\mathcal{R}$ in which m-ary addition makes $\mathcal{R}$ into a canonical hypergroup whereas $\mathcal{R}$ with n-ary multiplication is a semigroup.

Krasner ternary hyperrings were investigated by Castillo and Paradero-Vilela [8] in 2014. For a Krasner ternary hyperring $(\mathcal{R}, +, \cdot)$, the symbol '+' refers to a binary hyperoperation, while '$\cdot$' denotes a ternary multiplication.

Definition 1.2 ([8])**.** *A hyperring $(\mathcal{R}, +, \cdot)$ is called a Krasner ternary hyperring if it meets the following conditions:*

(i) $(\mathcal{R}, +)$ *constitutes a canonical hypergroup;*

(ii) $(r_1 \cdot r_2 \cdot r_3) \cdot r_4 \cdot r_5 = r_1 \cdot (r_2 \cdot r_3 \cdot r_4) \cdot r_5 = r_1 \cdot r_2 \cdot (r_3 \cdot r_4 \cdot r_5)$;

(iii) $(r_1 + r_2) \cdot r_3 \cdot r_4 = r_1 \cdot r_3 \cdot r_4 + r_2 \cdot r_3 \cdot r_4$, $r_1 \cdot (r_2 + r_3) \cdot r_4 = r_1 \cdot r_2 \cdot r_4 + r_1 \cdot r_3 \cdot r_4$, $r_1 \cdot r_2 \cdot (r_3 + r_4) = r_1 \cdot r_2 \cdot r_3 + r_1 \cdot r_2 \cdot r_4$,

(iv) $0 \cdot r_1 \cdot r_2 = r_1 \cdot 0 \cdot r_2 = r_1 \cdot r_2 \cdot 0 = 0$,

$\forall\ r_1, r_2, r_3, r_4, r_5 \in \mathcal{R}$.

The concept of ternary semihyperrings was introduced by Davvaz [11] in 2009, extending the idea of semirings.

Definition 1.3 ([11])**.** *A set $\mathcal{R}$ with a binary hyperoperation '+' and a ternary multiplication '$\cdot$' is termed a ternary semihyperring if $(\mathcal{R}, +)$ constitutes a commutative semihypergroup and fulfills the subsequent conditions:*

1. $(r_1 \cdot r_2 \cdot r_3) \cdot r_4 \cdot r_5 = r_1 \cdot (r_2 \cdot r_3 \cdot r_4) \cdot r_5 = r_1 \cdot r_2 \cdot (r_3 \cdot r_4 \cdot r_5)$;
2. $(r_1 + r_2) \cdot r_3 \cdot r_4 = r_1 \cdot r_3 \cdot r_4 + r_2 \cdot r_3 \cdot r_4$;
3. $r_1 \cdot (r_2 + r_3) \cdot r_4 = r_1 \cdot r_2 \cdot r_4 + r_1 \cdot r_3 \cdot r_4$;
4. $r_1 \cdot r_2 \cdot (r_3 + r_4) = r_1 \cdot r_2 \cdot r_3 + r_1 \cdot r_2 \cdot r_4$.

$\forall\ r_1, r_2, r_3, r_4, r_5 \in \mathcal{R}$. *We simply write $r_1 \cdot r_2 \cdot r_3 = r_1 r_2 r_3$.*

Example 1.2 ([11])**.** Consider $\mathbb{Z}$ as the set of all integers. We establish a binary hyperoperation and ternary multiplication on $\mathbb{Z}$ as follows: $r_1 \oplus r_2 = \{r_1, r_2\}$, and $r_1 \cdot r_2 \cdot r_3$ denotes the standard ternary multiplication of integers. Thus, $(\mathbb{Z}, \oplus, \cdot)$ forms a ternary semihyperring.

Example 1.3 ([11]). Consider $(\mathcal{R}, +, \cdot)$ as a semiring. We define a binary hyperoperation by $r_1 \oplus r_2 =< r_1, r_2 >$ (the ideal generated by r_1, r_2), and a ternary multiplication by $r_1 \circ r_2 \circ r_3 = r_1 \cdot r_2 \cdot r_3$. Consequently, $(\mathcal{R}, \oplus, \circ)$ constitutes a ternary semihyperring.

A nonempty subset A of a ternary semihyperring $\mathcal{R}$ is a ternary subsemihyperring of $\mathcal{R}$ if $(A, +)$ is an additive subsemihypergroup of $(\mathcal{R}, +)$, i.e., $r_1 + r_2 \subseteq A$, for any $r_1, r_2 \in A$ and $AAA \subseteq A$, i.e., $r_1r_2r_3 \in A$, for any $r_1, r_2, r_3 \in A$. $\mathcal{R}$ is said to have a zero element if there exists an element $0 \in \mathcal{R}$ such that for all $r_1, r_2 \in \mathcal{R}, 0r_1r_2 = r_10r_2 = r_1r_20 = 0$. An element e of $\mathcal{R}$ is called a unital element if for all $r_1 \in \mathcal{R}$, $eer_1 = er_1e = r_1ee = r_1$. An element $r_2 \in \mathcal{R}$ is called an inverse of $r_1 \in \mathcal{R}$ if $r_1r_2x = xr_1r_2 = r_2r_1x = xr_2r_1 = x$ for all $x \in \mathcal{R}$. A nonempty additive subsemihypergroup I of $\mathcal{R}$ is called a left hyperideal of $\mathcal{R}$ if $\mathcal{RR}I \subseteq I$, lateral hyperideal of $\mathcal{R}$ if $\mathcal{R}I\mathcal{R} \subseteq I$ and right hyperideal of $\mathcal{R}$ if $I\mathcal{RR} \subseteq I$. If I is both a left and right hyperideal as well as a lateral hyperideal of $\mathcal{R}$ then I is called a hyperideal of $\mathcal{R}$. A proper hyperideal P of $\mathcal{R}$ is called prime hyperideal of $\mathcal{R}$ if $I_1I_2I_3 \subseteq P$ implies $I_1 \subseteq P$ or $I_2 \subseteq P$ or $I_3 \subseteq P$, for any three hyperideals I_1, I_2, I_3 of $\mathcal{R}$. A ternary semihyperring $\mathcal{R}$ is called commutative if $r_1r_2r_3 = r_2r_3r_1 = r_3r_1r_2 = r_2r_1r_3 = r_3r_2r_1 = r_1r_3r_2, \ \forall\ r_1, r_2, r_3 \in \mathcal{R}$. Let $\mathcal{R}$ be a commutative ternary semihyperring and P be a hyperideal of $\mathcal{R}$. Then, P is a prime hyperideal if and only if $r_1r_2r_3 \in P$ implies $r_1 \in P$ or $r_2 \in P$ or $r_3 \in P$, for all $r_1, r_2, r_3 \in \mathcal{R}$.

The notion of m-valuation on m-rings was put forward by Alajbegović and Močkoř [3] in 1985. The idea of hypervaluation was explored in 2006 by Davvaz and Salsi [10]. Harijani and Anvariyeh [19] introduced hypervaluation of a hyperfield onto a totally ordered canonical hypergroup. In 2020, Linzi and Stojalowska [21] studied that any hypervaluation from a hyperfield onto an ordered canonical hypergroup is the composition of a hypervaluation onto an ordered abelian group and an order preserving homomorphism of hypergroups. Nikmehr, Nikandish and Yassine [23] studied the notion of hypervaluation hyperideals and then found relations between hypervaluations, integral closure of hyperideals and primary hyperideals.

A group G can be partially ordered [10] by $\leq$ if $(G, \leq)$ is a poset in which $\leq$ is compatible with the binary composition, i.e., if $a_1 < b_1$ then $ga_1 < gb_1$ and $a_1g < b_1g$ for all $g \in G$. If $a_1 < a_1'$ and $b_1 < b_1'$ then we obtain $a_1b_1 < a_1'b_1'$. So $P = \{g \in G : 1 \leq g\}$, called the positive cone of G, is a submonoid of G. Setting $P^{-1} = \{a^{-1} : a \in P\}$ yields

1. $P \cap P^{-1} = \{1\}$, and
2. if $\leq$ is a total order, then $P \cup P^{-1} = G$.

The ordered group G is adjoined with ∞ fulfilling the conditions: $\infty > a$ and $\infty \cdot \infty = \infty = a \cdot \infty = \infty \cdot a$ for all $a \in G$. We write G_∞ to mean $G \cup \{\infty\}$

which is a hyperring with hyperoperation $\oplus$ defined as follows: if $a < b$ then $a \oplus b = \{a\}$ for all $a, b \in G_\infty$ and $a \oplus a = \{g \in G_\infty : a \leq g\}$. The multiplication is given by $a \circ b = a \vee b$ for all $a, b \in G_\infty$.

Definition 1.4 ([10]). *If $\mathcal{R}$ is a hyperring, then a hypervaluation on $\mathcal{R}$ is a mapping $v : \mathcal{R} \to G_\infty = G \cup \{\infty\}$, where G is a totally ordered abelian group, satisfying the following conditions:*

1. $v(0) = \infty$;
2. $v(xy) = v(x) \cdot v(y)$, *for all* $x, y \in \mathcal{R}$;
3. $v(-x) = v(x)$, *for all* $x \in \mathcal{R}$;
4. $z \in x + y \Rightarrow v(x) \geq min\{v(x), v(y)\}$, *for all* $x, y, z \in \mathcal{R}$.

For a hypervaluation v, we say that $(\mathcal{R}, v, G)$ is a hypervalued hyperring. We define two sets $\mathcal{A}_v = \{x \in \mathcal{R} : v(x) \geq 1\}$ and $P_v = \{x \in \mathcal{R} : v(x) > 1\}$. Then, $\mathcal{A}_v$ becomes a subhyperring of $\mathcal{R}$ and P_v becomes a prime hyperideal of $\mathcal{A}_v$. Additionally, the set $v^{-1}(\infty)$ forms a prime hyperideal of $\mathcal{R}$ which is contained in P_v.

Valuation on a ternary semiring has been studied by Pal et. al [24]. In this paper, we introduce and explore the notion of hypervaluation on a ternary semihyperring, finally proving that any hypervaluation from a ternary semihyperring onto an ordered abelian multiplicative group is the composition of a hypervaluation onto an ordered abelian multiplicative group and an order preserving isomorphism of ternary semirings under certain conditions.

2. Hypervaluation on ternary semihyperring

We consider a totally ordered (multiplicative) abelian group G. Throughout this section, we consider G_∞ as $G \cup \{\infty\}$.

Definition 2.1. *Let $\mathcal{R}$ be a commutative ternary semihyperring with a unital element e and zero element 0. By a hypervaluation, we mean a mapping $\mathfrak{v} : \mathcal{R} \to G_\infty$ which satisfies the conditions as follows: for $x, y, z \in \mathcal{R}$,*

(i) $\mathfrak{v}(xyz) = \mathfrak{v}(x) \cdot \mathfrak{v}(y) \cdot \mathfrak{v}(z)$;

(ii) *if* $z \in x + y$ *then* $\mathfrak{v}(z) \geq \min\{\mathfrak{v}(x), \mathfrak{v}(y)\}$;

(iii) $\mathfrak{v}(e) = 1$;

(iv) $\mathfrak{v}(0) = \infty$.

Example 2.1. Consider the ternary semihyperring $\mathcal{R} = (\mathbb{Z}, \oplus, \cdot)$ as shown in Example 1.2. We take $G = (\mathbb{Q}^+, \cdot)$. Define a mapping $\mathfrak{v} : \mathcal{R} \to G_\infty$ by

1. $\mathfrak{v}(x) = |x|, x \neq 0$,

2. $\mathfrak{v}(0) = \infty$.

Then, $\mathfrak{v}$ becomes a hypervaluation on $\mathcal{R}$.

Henceforth, by $\mathcal{R}$, we will mean a ternary semihyperring with a unital element e and a zero element 0.

We consider the set of all hypervaluations of $\mathcal{R}$ which is denoted by $val(\mathcal{R})$. For any hypervaluation $\mathfrak{v} \in val(\mathcal{R})$, we set

$$\mathcal{A}_\mathfrak{v} = \{x \in \mathcal{R} : \mathfrak{v}(x) \geq 1\},$$
$$\mathcal{P}_\mathfrak{v} = \{y \in \mathcal{R} : \mathfrak{v}(y) > 1\},$$
$$\mathcal{I}_\mathfrak{v} = \{z \in \mathcal{R} : \mathfrak{v}(z) = \infty\}.$$

We see that $e \in \mathcal{A}_\mathfrak{v}$ and 0 is in $\mathcal{I}_\mathfrak{v}$ as well as in $\mathcal{P}_\mathfrak{v}$.

Theorem 2.1. *$\mathcal{A}_\mathfrak{v}$ is a ternary subsemihyperring in $\mathcal{R}$.*

Proof of Theorem 2.1. First we show that $\mathcal{A}_\mathfrak{v}$ becomes an additive subsemihypergroup. Let $x, y \in \mathcal{A}_\mathfrak{v}$, then $\mathfrak{v}(x) \geq 1, \mathfrak{v}(y) \geq 1$. Let $z \in x + y$, then $\mathfrak{v}(z) \geq \min\{\mathfrak{v}(x), \mathfrak{v}(y)\} \geq 1$ which gives us $z \in \mathcal{A}_\mathfrak{v} \Rightarrow x + y \subseteq \mathcal{A}_\mathfrak{v}$. Therefore $\mathcal{A}_\mathfrak{v}$ is an additive subsemihypergroup. Again, for $x, y, z \in \mathcal{A}_\mathfrak{v}$, $\mathfrak{v}(xyz) = \mathfrak{v}(x) \cdot \mathfrak{v}(y) \cdot \mathfrak{v}(z) \geq 1 \Rightarrow xyz \in \mathcal{A}_\mathfrak{v}$. Thus, $\mathcal{A}_\mathfrak{v}$ is a ternary subsemihyperring of $\mathcal{R}$.

Theorem 2.2. *$\mathcal{P}_\mathfrak{v}$ is a prime hyperideal of $\mathcal{A}_\mathfrak{v}$.*

Proof of Theorem 2.2. It is clear that $\mathcal{P}_\mathfrak{v}$ is an additive subsemihypergroup of $\mathcal{A}_\mathfrak{v}$. Let $a, b \in \mathcal{A}_\mathfrak{v}$ and $x \in \mathcal{P}_\mathfrak{v}$ then $\mathfrak{v}(a) \geq 1, \mathfrak{v}(b) \geq 1$ and $\mathfrak{v}(x) > 1$. Then, $\mathfrak{v}(abx) = \mathfrak{v}(a) \cdot \mathfrak{v}(b) \cdot \mathfrak{v}(x) > 1 \Rightarrow abx \in \mathcal{P}_\mathfrak{v}$. We can show that $axb \in \mathcal{P}_\mathfrak{v}$ and $xab \in \mathcal{P}_\mathfrak{v}$ in a similar way. Hence $\mathcal{P}_\mathfrak{v}$ is a hyperideal of $\mathcal{A}_\mathfrak{v}$. Again let $a, b, c \in \mathcal{A}_\mathfrak{v}$ and $abc \in \mathcal{P}_\mathfrak{v}$, then $\mathfrak{v}(abc) > 1 \Rightarrow \mathfrak{v}(a) \cdot \mathfrak{v}(b) \cdot \mathfrak{v}(c) > 1$ which further implies either $\mathfrak{v}(a) > 1$ or $\mathfrak{v}(b) > 1$ or $\mathfrak{v}(c) > 1$. Hence $\mathcal{P}_\mathfrak{v}$ is a prime hyperideal of $\mathcal{A}_\mathfrak{v}$.

Theorem 2.3. *$\mathcal{I}_\mathfrak{v}$ is a prime hyperideal of $\mathcal{R}$.*

Proof of Theorem 2.3. Let $x, y \in \mathcal{I}_\mathfrak{v}$ then $\mathfrak{v}(x) = \infty, \mathfrak{v}(y) = \infty$. Let $z \in x+y$, then $\mathfrak{v}(z) \geq \min\{\mathfrak{v}(x), \mathfrak{v}(y)\} \Rightarrow \mathfrak{v}(z) = \infty \Rightarrow z \in \mathcal{I}_\mathfrak{v}$. Therefore $x + y \subseteq \mathcal{I}_\mathfrak{v}$. So $\mathcal{I}_\mathfrak{v}$ is an additive subsemihypergroup. Let $a, b \in \mathcal{R}$ and $x \in \mathcal{I}_\mathfrak{v}$ then $\mathfrak{v}(abx) = \mathfrak{v}(a) \cdot \mathfrak{v}(b) \cdot \mathfrak{v}(x) = \infty \Rightarrow abx \in \mathcal{I}_\mathfrak{v}$. Similarly we can show that $axb \in \mathcal{I}_\mathfrak{v}$ and $xab \in \mathcal{I}_\mathfrak{v}$. Therefore $\mathcal{I}_\mathfrak{v}$ is a hyperideal of $\mathcal{R}$. Now, let $abc \in \mathcal{I}_\mathfrak{v}$ for any $a, b, c \in \mathcal{R}$, then $\mathfrak{v}(abc) = \infty$ and so $\mathfrak{v}(a) \cdot v(b) \cdot \mathfrak{v}(c) = \infty$. Then, at least one of $\mathfrak{v}(a), \mathfrak{v}(b)$ or $\mathfrak{v}(c)$ is ∞. That is, either $a \in \mathcal{I}_\mathfrak{v}$ or $b \in \mathcal{I}_\mathfrak{v}$ or $c \in \mathcal{I}_\mathfrak{v}$. Hence $\mathcal{I}_\mathfrak{v}$ becomes a prime hyperideal of $\mathcal{R}$.

In the results that follow, we present some properties of a hypervalued ternary semihyperring $\mathcal{R}$ where $\mathcal{A}_\mathfrak{v}$, $\mathcal{P}_\mathfrak{v}$ and $v^{-1}(\infty) = \mathcal{I}_\mathfrak{v}$ play a significant role.

Theorem 2.4. *Consider a ternary subsemihyperring A containing e within a commutative ternary semihyperring $\mathcal{R}$. Assume that every nonzero element in $\mathcal{R}$ has a multiplicative inverse and P represents a proper prime hyperideal of A. In this scenario, the following assertions hold true equivalently:*

(I) *for each $x \in \mathcal{R} \setminus A$, there exists $y \in P$ such that $xye \in A \setminus P$,*

(II) *there exists a nontrivial onto ternary hypervaluation $\mathfrak{v} : \mathcal{R} \to G_\infty$ such that $\mathcal{A}_\mathfrak{v} = A$ and $\mathcal{P}_\mathfrak{v} = P$.*

Proof of Theorem 2.4. $(II) \Rightarrow (I)$. Suppose there exists an onto ternary hypervaluation $\mathfrak{v} : \mathcal{R} \to G_\infty = G \cup \{\infty\}$ such that $\mathcal{A}_\mathfrak{v} = A$ and $\mathcal{P}_\mathfrak{v} = P$. If $x \in \mathcal{R} \setminus \mathcal{A}_\mathfrak{v}$, then $\mathfrak{v}(x) < 1$ and therefore $\mathfrak{v}(x) \neq \infty$, i.e., $\mathfrak{v}(x) \in G$. Thus, we get $\mathfrak{v}(x)^{-1} = \mathfrak{v}(y) > 1$ for some $y \in \mathcal{R}$, which implies $y \in \mathcal{P}_\mathfrak{v}$. Again we get $\mathfrak{v}(x) \cdot \mathfrak{v}(y) = 1$. Now, $\mathfrak{v}(xye) = \mathfrak{v}(x) \cdot \mathfrak{v}(y) \cdot \mathfrak{v}(e) = 1 \Rightarrow xye \in \mathcal{A}_\mathfrak{v} \setminus \mathcal{P}_\mathfrak{v}$.

$(I) \Rightarrow (II)$. We see that (I) implies the property: for all $x, y \in \mathcal{R}$, $xye \in P \Rightarrow x \in P$ or $y \in P$.

Next, for every element x belonging to the set $\mathcal{R}$, we set

$$(P : x)_\mathcal{R} = \{z \in \mathcal{R} : xze' \in P, \text{ for some unital } e' \text{ in } \mathcal{R}\}.$$

We define a relation ρ on $\mathcal{R}$ by $x\rho y$ if and only if $(P : x)_\mathcal{R} = (P : y)_\mathcal{R}$. Then, ρ is an equivalence relation on $\mathcal{R}$. Let us denote the equivalence class of an element $x \in \mathcal{R}$ by $x\rho$. As $\mathcal{R}$ is commutative, all the unital elements belong to the same equivalence class, say, $e\rho$. We define multiplication on $\mathcal{R}/\rho$ by $(x\rho)(y\rho) = (xye)\rho$. Consider the set $G = \{x\rho : x \in \mathcal{R}\} \setminus \{0\rho\}$. Then, G is a group with respect to multiplication. In fact, $e\rho$ is the identity element in G and for an inverse y of x in $\mathcal{R}$, $y\rho$ is the inverse of $x\rho$ in G.

Also, G is a totally ordered group where the ordering is given by $x\rho \leq y\rho$ iff $(P : x)_\mathcal{R} \subseteq (P : y)_\mathcal{R}$. Now, we define $\mathfrak{v} : \mathcal{R} \to G \cup \{\infty\}$ by

$$\mathfrak{v}(x) = x\rho, \text{ for } x \in \mathcal{R}, \mathfrak{v}(0) = \infty \text{ and } \mathfrak{v}(e) = e\rho.$$

It is clear that $\mathfrak{v}(xyz) = \mathfrak{v}(x) \cdot \mathfrak{v}(y) \cdot \mathfrak{v}(z)$. Next we prove that for any $z \in x + y, \mathfrak{v}(z) \geq \min\{\mathfrak{v}(x), \mathfrak{v}(y)\}$.

Let $\mathfrak{v}(x) < \mathfrak{v}(y)$ then $(P : x)_\mathcal{R} \subsetneqq (P : y)_\mathcal{R}$ and first we suppose that $\mathfrak{v}(z) < \mathfrak{v}(x)$, then we get $zue \notin P$ and $uxe = xue \in P$ for some $u \in \mathcal{R}$. We now show that $yue \notin P$. Suppose that $yue \in P$ then $zue \in (x + y)ue = xue + yue \subseteq P + P \subseteq P \Rightarrow zue \in P$, this leads to a contradiction, implying that $yue \notin P$. Given the assumption $\mathfrak{v}(x) < \mathfrak{v}(y)$, we deduce that $yte \in P$ and $xte \notin P$ for some $t \in \mathcal{R}$. Consequently, we derive $(xte)(yue)e = (xte)y(uee) = (xue)(yte)e \in PPe \subseteq P$. Hence, we encounter a contradiction, as either $xte \in P$ or $yte \in P$. Similarly, we reach a contradiction for the case $\mathfrak{v}(x) < \mathfrak{v}(z)$. Therefore, we have successfully demonstrated that $\mathfrak{v}(z) = \mathfrak{v}(x), \forall z \in x + y$ if $\mathfrak{v}(x) \neq \mathfrak{v}(y)$. If $\mathfrak{v}(x) = \mathfrak{v}(y)$, we show that $\mathfrak{v}(z) \geq \mathfrak{v}(x)$ for any $z \in x + y$, that is, $\{u \in \mathcal{R} : uxe \in P\} \subseteq \{u \in \mathcal{R} : uze \in P\}$. Let $u \in \mathcal{R}$ and $uxe \in P$. Since $\mathfrak{v}(x) = \mathfrak{v}(y)$, we have $uye \in P$ and $uze \in uxe + uye \subseteq P + P \subseteq P$. Hence $(P : x)_{\mathcal{R}} \subseteq (P : z)_{\mathcal{R}}$, that is, $\mathfrak{v}(z) \geq \mathfrak{v}(x)$. So $\mathfrak{v}$ is a ternary hypervaluation.

Theorem 2.5. *Consider $\mathcal{R}$ as a commutative ternary semihyperring, and let $\mathfrak{v} : \mathcal{R} \to G_\infty$ represent a non-trivial hypervaluation on $\mathcal{R}$. In this context, we observe:*

(*I*) $\mathfrak{v}^{-1}(\infty) = \{x \in \mathcal{R} : xye \in \mathcal{A}_{\mathfrak{v}},\ \forall y \in \mathcal{R} \setminus \mathcal{A}_{\mathfrak{v}}\}$.

(*II*) *If P is a proper prime hyperideal of $\mathcal{A}_{\mathfrak{v}}$ such that $P \subseteq \mathcal{P}_{\mathfrak{v}}$ and $P \nsubseteq \mathfrak{v}^{-1}(\infty)$, then $\mathfrak{v}^{-1}(\infty) \subseteq P$.*

Proof of Theorem 2.5. (I) Let $x \in \mathcal{R}$ be an element such that $\mathfrak{v}(x) = \infty$ and $y \in \mathcal{R} \setminus \mathcal{A}_{\mathfrak{v}}$. Then, we have $\mathfrak{v}(xye) = \mathfrak{v}(x) \cdot \mathfrak{v}(y) \cdot \mathfrak{v}(e) = \infty$. Hence $xye \in \mathcal{A}_{\mathfrak{v}}$.

Now, we show that for any $x \in \mathcal{R}$ such that for every $y \in \mathcal{R} \setminus \mathcal{A}_{\mathfrak{v}}$, whenever $xye \in \mathcal{A}_{\mathfrak{v}}$, then $\mathfrak{v}(x) = \infty$. Suppose $\mathfrak{v}(x) \neq \infty$. If $\mathfrak{v}(x) < 1$, we can take $y = x \in \mathcal{R} \setminus \mathcal{A}_{\mathfrak{v}}$, then $\mathfrak{v}(xye) = \mathfrak{v}(x) \cdot \mathfrak{v}(y) \cdot \mathfrak{v}(e) < 1 \Rightarrow xye \notin \mathcal{A}_{\mathfrak{v}}$, which is a contradiction. Next we consider the case $1 \leq \mathfrak{v}(x) < \infty$. If $\mathfrak{v}(x) > 1$, then for any $y \in \mathcal{R}$ such that $\mathfrak{v}(y) = \mathfrak{v}(x)^{-1} < 1$ implies that $y \in \mathcal{R} \setminus \mathcal{A}_{\mathfrak{v}}$. Now, $\mathfrak{v}(yye) = \mathfrak{v}(y) \cdot \mathfrak{v}(y) \cdot \mathfrak{v}(e) < 1$. Therefore $\mathfrak{v}(x(yye)e) = \mathfrak{v}(x) \cdot \mathfrak{v}(yye) \cdot \mathfrak{v}(e) < 1 \Rightarrow x(yye)e \notin \mathcal{A}_{\mathfrak{v}}$ where $yye \in \mathcal{R} \setminus \mathcal{A}_{\mathfrak{v}}$, which is a contradiction. If $\mathfrak{v}(x) = 1$, let any $y \in \mathcal{R} \setminus \mathcal{A}_{\mathfrak{v}}, \mathfrak{v}(xye) = \mathfrak{v}(x) \cdot \mathfrak{v}(y) \cdot \mathfrak{v}(e) = \mathfrak{v}(y) < 1 \Rightarrow xye \notin \mathcal{A}_{\mathfrak{v}}$, which is a contradiction. Hence we can conclude that $\mathfrak{v}(x) = \infty$. Therefore $\mathfrak{v}^{-1}(\infty) = \{x \in \mathcal{R} : xye \in \mathcal{A}_{\mathfrak{v}}, \forall y \in R \setminus \mathcal{A}_{\mathfrak{v}}\}$.

(II) Let P be a prime hyperideal of $\mathcal{A}_{\mathfrak{v}}$, with $P \subseteq \mathcal{P}_{\mathfrak{v}}$ such that $P \nsubseteq \mathfrak{v}^{-1}(\infty)$. Let $p \in P$ with $1 < \mathfrak{v}(p) < \infty$ and let $z \in \mathfrak{v}^{-1}(\infty) = \{x \in \mathcal{R} : xye \in \mathcal{A}_{\mathfrak{v}}, \forall\ y \in \mathcal{R} \setminus \mathcal{A}_{\mathfrak{v}}\}$. Then, $\mathfrak{v}(x) = \mathfrak{v}(p)^{-1} < 1$, for some $x \in \mathcal{R} \setminus \mathcal{A}_{\mathfrak{v}}$. Now, $\mathfrak{v}(zxe) = \mathfrak{v}(z) \cdot \mathfrak{v}(x) \cdot \mathfrak{v}(e) = \infty \Rightarrow zxe \in \mathcal{P}_{\mathfrak{v}}$. Further we have $xpe \in \mathcal{A}_{\mathfrak{v}} \setminus \mathcal{P}_{\mathfrak{v}} \subseteq \mathcal{A}_{\mathfrak{v}} \setminus P \Rightarrow xpe \notin P$. Now, $(xpe)ze = x(pez)e = x(zep)e = xz(epe) = (xze)pe = (zxe)pe \in \mathcal{P}_{\mathfrak{v}}Pe \subseteq P \Rightarrow z \in P$, since $xpe \notin P$. Therefore $\mathfrak{v}^{-1}(\infty) \subseteq P$.

Suppose $(G, \cdot, \leq)$ is a totally ordered abelian group, it can be made into a ternary semiring in an obvious way by defining

1. $a + b = \max\{a, b\}$,
2. $a \cdot b \cdot c = a \wedge b \wedge c = \min\{a, b, c\}$,

for all $a, b, c \in G$. Then, $(G, +, \cdot)$ becomes a ternary semiring.

Theorem 2.6. *Let $\mathcal{R}$ be a commutative ternary semihyperring, G be a totally ordered abelian group and $\mathfrak{v}_1 : \mathcal{R} \to G_\infty$ be a nontrivial onto ternary hypervaluation on $\mathcal{R}$. Then, for any totally ordered abelian group H and any nontrivial onto ternary hypervaluation $\mathfrak{v}_2 : \mathcal{R} \to H_\infty$ on $\mathcal{R}$, $\mathcal{A}_{\mathfrak{v}_1} = \mathcal{A}_{\mathfrak{v}_2}$ and $\mathcal{P}_{\mathfrak{v}_1} = \mathcal{P}_{\mathfrak{v}_2}$ if and only if there is an order preserving isomorphism $\mathfrak{f} : G_\infty \to H_\infty$ of ternary semirings satisfying $\mathfrak{v}_2 = \mathfrak{f} \circ \mathfrak{v}_1$ and $\mathfrak{f}$ carries identity of G to identity of H.*

Proof of Theorem 2.6. Let us assume that $\mathcal{A}_{\mathfrak{v}_1} = \mathcal{A}_{\mathfrak{v}_2}$ and $\mathcal{P}_{\mathfrak{v}_1} = \mathcal{P}_{\mathfrak{v}_2}$. We define $\mathfrak{f} : G_\infty \to H_\infty$ by $\mathfrak{f}(\mathfrak{v}_1(x)) = \mathfrak{v}_2(x)$, for all $\mathfrak{v}_1(x) \neq \infty_G$ and $\mathfrak{f}(\infty_G) = \infty_H$. This definition is well-defined, in fact, we show that

$$\mathfrak{v}_1(x) = \mathfrak{v}_1(y) \neq \infty_G \Rightarrow \mathfrak{v}_2(x) = \mathfrak{v}_2(y) \neq \infty_H.$$

Now, $\mathfrak{v}_1(y)^{-1} \in G$, so $\mathfrak{v}_1(y)^{-1} = \mathfrak{v}_1(z)$ for some $z \in \mathcal{R}$. Thus, we have $1 = \mathfrak{v}_1(y) \cdot \mathfrak{v}_1(y)^{-1} = \mathfrak{v}_1(x) \cdot \mathfrak{v}_1(z) \cdot \mathfrak{v}_1(e) = \mathfrak{v}_1(xze)$ which implies that $xze \in \mathcal{A}_{\mathfrak{v}_1} \setminus \mathcal{P}_{\mathfrak{v}_1} = \mathcal{A}_{\mathfrak{v}_2} \setminus \mathcal{P}_{\mathfrak{v}_2}$. This gives us $\mathfrak{v}_2(xze) = 1 \Rightarrow \mathfrak{v}_2(x) \cdot \mathfrak{v}_2(z) \cdot \mathfrak{v}_2(e) = 1 \Rightarrow \mathfrak{v}_2(x) \cdot \mathfrak{v}_2(z) = 1 \Rightarrow \mathfrak{v}_2(x) = \mathfrak{v}_2(z)^{-1}$. Also, $\mathfrak{v}_1(yze) = 1$. Proceeding in a similar way, it follows that $\mathfrak{v}_2(yze) = 1 \Rightarrow \mathfrak{v}_2(y) = \mathfrak{v}_2(z)^{-1}$. Thus, $\mathfrak{v}_2(x) = \mathfrak{v}_2(y) \neq \infty_H$. Also, from Theorem 2.5, we have $\mathfrak{v}_1^{-1}(\infty_G) = \mathfrak{v}_2^{-1}(\infty_H)$.

In order to show that $\mathfrak{f}$ is an order preserving isomorphism, first we show that if $\mathfrak{v}_1(x) < \mathfrak{v}_1(y)$ then $\mathfrak{v}_2(x) < \mathfrak{v}_2(y)$. Otherwise $\mathfrak{v}_1(x) < \mathfrak{v}_1(y)$ and $\mathfrak{v}_2(y) \leq \mathfrak{v}_2(x)$ imply that $\mathfrak{v}_1(x) \neq \infty_G$, so $\mathfrak{v}_2(x) \neq \infty_H$ and $\mathfrak{v}_2(y) \neq \infty_H$. Now, there exists $z \in \mathcal{R}$ such that $\mathfrak{v}_2(y)^{-1} = \mathfrak{v}_2(z) \Rightarrow 1 \leq \mathfrak{v}_2(x) \cdot \mathfrak{v}_2(y)^{-1} = \mathfrak{v}_2(x) \cdot \mathfrak{v}_2(y)^{-1} \cdot \mathfrak{v}_2(e) = \mathfrak{v}_2(x) \cdot \mathfrak{v}_2(z) \cdot \mathfrak{v}_2(e) = \mathfrak{v}_2(xze) \Rightarrow xze \in \mathcal{A}_{\mathfrak{v}_2} = \mathcal{A}_{\mathfrak{v}_1} \Rightarrow \mathfrak{v}_1(xze) \geq 1$. But, $yze \in \mathcal{A}_{\mathfrak{v}_2} \setminus \mathcal{P}_{\mathfrak{v}_2} = \mathcal{A}_{\mathfrak{v}_1} \setminus \mathcal{P}_{\mathfrak{v}_1}$. Hence $1 = \mathfrak{v}_1(y) \cdot \mathfrak{v}_1(z) \cdot \mathfrak{v}_1(e) \Rightarrow \mathfrak{v}_1(y) = \mathfrak{v}_1(z)^{-1} \leq \mathfrak{v}_1(x)$, since $1 \leq \mathfrak{v}_1(xze)$. Thus, we get $\mathfrak{v}_1(y) \leq \mathfrak{v}_1(x)$ which is a contradiction.

We can show, in a similar manner, that if $\mathfrak{v}_2(x) < \mathfrak{v}_2(y)$ then $\mathfrak{v}_1(x) < \mathfrak{v}_1(y)$ and so $\mathfrak{f}$ is order preserving.

Now, G and H can be made into ternary semirings. To prove that $\mathfrak{f} : G_\infty \to H_\infty$ is a homomorphism, we need to show that $\mathfrak{f}(a + b) = \mathfrak{f}(a) + \mathfrak{f}(b)$ and $\mathfrak{f}(abc) = \mathfrak{f}(a)\mathfrak{f}(b)\mathfrak{f}(c)$, where $a = \mathfrak{v}_1(x), b = \mathfrak{v}_1(y)$ and $c = \mathfrak{v}_1(z)$ for some $x, y, z \in \mathcal{R}$. Since $a + \infty_G = \infty_G$ for any a in G_∞ and $a_1 + \infty_H = \infty_H$ for any a_1 in H_∞, it is sufficient to consider the case when $a, b \in G$. Let $a, b \in G$ and $x, y \in \mathcal{R}$ such that $a = \mathfrak{v}_1(x), b = \mathfrak{v}_1(y)$. Then, $\mathfrak{f}(a) + \mathfrak{f}(b) = \mathfrak{f}(\mathfrak{v}_1(x)) + \mathfrak{f}(\mathfrak{v}_1(y))$. Then, if $\mathfrak{v}_1(x) < \mathfrak{v}_1(y)$ we have $\mathfrak{v}_2(x) < \mathfrak{v}_2(y)$ and $\mathfrak{v}_1(x) + \mathfrak{v}_1(y) = \mathfrak{v}_1(y)$. Hence $\mathfrak{f}(a+b) = \mathfrak{f}(\mathfrak{v}_1(x) + \mathfrak{v}_1(y)) = \mathfrak{f}(\mathfrak{v}_1(y)) = \mathfrak{v}_2(y) = \mathfrak{f}(\mathfrak{v}_1(x)) + \mathfrak{f}(\mathfrak{v}_1(y)) = \mathfrak{f}(a) + \mathfrak{f}(b)$. If $\mathfrak{v}_1(x) = \mathfrak{v}_1(y)$ then $\mathfrak{v}_2(x) = \mathfrak{f}(\mathfrak{v}_1(x)) = \mathfrak{f}(\mathfrak{v}_1(y)) = \mathfrak{v}_2(y)$. Therefore $\mathfrak{f}(a + b) = \mathfrak{f}(a) + \mathfrak{f}(b)$.

Also, $\mathfrak{f}(abc) = \mathfrak{f}(a)\mathfrak{f}(b)\mathfrak{f}(c)$ and $\mathfrak{f}(1) = 1$ follows from the order preserving property of $\mathfrak{f}$. Obviously $\mathfrak{f}$ is onto.

We now show that $\mathfrak{f}$ is injective. Let $\mathfrak{f}(\mathfrak{v}_1(x)) = \mathfrak{f}(\mathfrak{v}_1(y)) \neq \infty_H$, that is, $\mathfrak{v}_2(x) = \mathfrak{v}_2(y) \neq \infty_H$. If possible, let $\mathfrak{v}_1(x) \neq \mathfrak{v}_1(y)$. Then, either $\mathfrak{v}_1(x) < \mathfrak{v}_1(y)$ or $\mathfrak{v}_1(y) < \mathfrak{v}_1(x)$. Consider the case when $\mathfrak{v}_1(x) > \mathfrak{v}_1(y)$. Since $\mathfrak{v}_2(x) \neq \infty_H$, we must have $\mathfrak{v}_1(x) \neq \infty_G$ and so $\mathfrak{v}_1(y) \neq \infty_G$. Now, there exists $z \in \mathcal{R}$ such that $\mathfrak{v}_1(y)^{-1} = \mathfrak{v}_1(z) \neq \infty_G$. Then, $\mathfrak{v}_1(x) \cdot \mathfrak{v}_1(y)^{-1} > 1 \Rightarrow \mathfrak{v}_1(x) \cdot \mathfrak{v}_1(z) \cdot \mathfrak{v}_1(e) > 1 \Rightarrow xze \in \mathcal{P}_{\mathfrak{v}_1} = \mathcal{P}_{\mathfrak{v}_2} \Rightarrow \mathfrak{v}_2(xze) > 1$. Again $yze \in \mathcal{A}_{\mathfrak{v}_1} \setminus \mathcal{P}_{\mathfrak{v}_1} = \mathcal{A}_{\mathfrak{v}_2} \setminus \mathcal{P}_{\mathfrak{v}_2}$. Hence $\mathfrak{v}_2(yze) = 1$. But, $\mathfrak{v}_2(z) \neq \infty_H$. Now, $\mathfrak{v}_2(y) \cdot \mathfrak{v}_2(z) \cdot \mathfrak{v}_2(e) = 1 \Rightarrow \mathfrak{v}_2(y) = \mathfrak{v}_2(z)^{-1} < \mathfrak{v}_2(x)$, which is a contradiction. Similarly contradiction is obtained for the case $\mathfrak{v}_1(x) < \mathfrak{v}_1(y)$. Hence $\mathfrak{f}$ is injective, making $\mathfrak{f}$ an order preserving isomorphism.

For the converse part, let $x \in \mathcal{A}_{\mathfrak{v}_1} \Rightarrow \mathfrak{v}_1(x) \geq 1 \Rightarrow \mathfrak{f}(\mathfrak{v}_1(x)) \geq \mathfrak{f}(1) \Rightarrow \mathfrak{v}_2(x) \geq 1 \Rightarrow x \in \mathcal{A}_{\mathfrak{v}_2}$. Therefore $\mathcal{A}_{\mathfrak{v}_1} \subseteq \mathcal{A}_{\mathfrak{v}_2}$. Similarly, we can show that $\mathcal{A}_{\mathfrak{v}_2} \subseteq \mathcal{A}_{\mathfrak{v}_1}$, hence $\mathcal{A}_{\mathfrak{v}_1} = \mathcal{A}_{\mathfrak{v}_2}$. In a similar way we can show that $\mathcal{P}_{\mathfrak{v}_1} = \mathcal{P}_{\mathfrak{v}_2}$.

In the above theorem, the hypothesis of surjectivity of the hypervaluations cannot be dispensed with, as supported by Example 2.2 below.

Example 2.2. Consider the ternary semihyperring $\mathcal{R} = (\mathbb{Z}, \oplus, \cdot)$ (as in Example 1.2) and the totally ordered group $G = (\{1\}, \cdot)$. Then, $\mathfrak{v}_1 : \mathcal{R} \to G_\infty$ defined by

1. $\mathfrak{v}_1(x) = 1$, if $x \neq 0$,
2. $\mathfrak{v}_1(0) = \infty$,

is an onto hypervaluation on $\mathcal{R}$.

We take $H = (\mathbb{Q}^+, \cdot)$ and the hypervaluation $\mathfrak{v}_2$ on $\mathcal{R}$ to be the hypervaluation as defined in Example 2.1.

Here, $\mathcal{A}_{\mathfrak{v}_1} = \mathcal{A}_{\mathfrak{v}_2} = \mathbb{Z}$ and $\mathcal{P}_{\mathfrak{v}_1} = \{0\}$, $\mathcal{P}_{\mathfrak{v}_2} = \mathbb{Z} \setminus \{1\}$. We define $\mathfrak{f} : G_\infty \to H_\infty$ by

1. $\mathfrak{f}(1) = 1$,
2. $\mathfrak{f}(\infty) = \infty$,

which is a homomorphism of ternary semirings satisfying $\mathfrak{v}_2 = \mathfrak{f} \circ \mathfrak{v}_1$. But, $\mathfrak{f}$ is not an isomorphism. We also note that $\mathfrak{v}_2$ is not onto.

Conclusion

The impetus for studying hypervaluation on ternary semihyperrings arises from its potential to generalize and expand the theory of hypervaluation on hyperrings. A necessary and sufficient condition for the existence of a ternary hypervaluation is provided in Theorem 2.4. The set $\mathfrak{v}^{-1}(\infty) = \mathcal{I}_{\mathfrak{v}}$ is characterised thereafter. We conclude by proving (in Theorem 2.6) that any hypervaluation is the composition of a hypervaluation and an order preserving isomorphism (of ternary semirings).

This study may be continued with the study of hyperideal theory in ternary semihyperrings. The concept of regularity in these structures along with study in relation to hypervaluation is another area which may be explored. Further, one may investigate the notion of product hypervaluation and attempt to characterise the set of all hypervaluations on ternary semihyperrings into suitable algebraic system, as open issues.

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Non-existence of integer solutions for the Diophantine equation $p^x + p^y + n^z = w^2$, where p is an odd prime number and n is a positive integer

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Abstract. In this research, we investigate some conditions for the non-existence of integer solutions of the Diophantine equation $p^x + p^y + n^z = w^2$, where p is an odd prime number and n is a positive integer. Moreover, numerous examples to illustrate these cases are provided.

Keywords: Diophantine equation, Legendre symbol, congruence.

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1. Introduction

One of the famous Diophantine equations is the exponential Diophantine equation $a^x+b^y = w^2$, where a and b are positive integers. Many authors investigated the non-negative integer solutions of the equation, where a and b are specified as positive integers (c.f. [1], [3] and [15]). Positive integer a or b is studied as variable under certain conditions in various manuscripts. In [5], [11], [14] and [17], either a or b is a fixed number and in [4], [7], [8] [9], [10] both a and b are variables that satisfy some conditions. Moreover, the non-existence of positive integer solutions to the Diophantine equation is studied; see [16].

The Diophantine equation $a^x + b^y + c^z = w^2$, where a, b and c are positive integers, was constructed and studied. In [2], Bacani and Rabago gave all non-negative integer solutions of the equation $3^x + 5^y + 7^z = w^2$. Similarly, a, b and c are considered in other papers as variables satisfying some conditions.

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In [6], all integer solutions of the equation $p^x + (p+1)^y + (p+2)^z = w^2$ are provided, where p is a prime number and $1 \le x, y, z \le 2$. Recently, Pandichelvi and Sandhya [12] showed integer solutions of the equation $p_1^x + p_2^y + p_3^z = M^2$, where p_1, p_2, p_3 are prime numbers and $x, y, z \in \{1, 2\}$.

In this research, the Diophantine equation

$$p^x + p^y + n^z = w^2 \tag{1}$$

is studied, where p is an odd prime number and n is a positive integer. We found some conditions for the contradiction of the existence of solutions of (1) by modulo 4, p, $p-1$ and $p+1$. Thus, many Diophantine equations of the form (1), which have no solutions, are demonstrated.

2. Non-existence of solutions by modulo 4

In this section, we investigate conditions on n modulo 4 for the non-existence of non-negative integer solutions to (1). First, we characterize the conditions for $n^z \equiv 0, 1, 2, 3 \pmod 4$.

Lemma 2.1. *Let n be a positive integer and z be a non-negative integer. Then,*

1. $n^z \equiv 0 \pmod 4$ *if and only if* $n \equiv 0, 2 \pmod 4$*, where* $z \ge 2$*;*
2. $n^z \equiv 1 \pmod 4$ *if and only if* $n \equiv 1 \pmod 4$ *or* $n \equiv 3 \pmod 4$ *and z is even, where* $z \ge 1$*;*
3. $n^z \equiv 2 \pmod 4$ *if and only if* $n \equiv 2 \pmod 4$ *and* $z = 1$*;*
4. $n^z \equiv 3 \pmod 4$ *if and only if* $n \equiv 3 \pmod 4$ *and z is odd.*

Proof. 1. Let $z \ge 2$. First, assume that $n^z \equiv 0 \pmod 4$. Suppose that $n \equiv 1, 3 \pmod 4$. Then, $n^z \equiv 1, 3 \pmod 4$, a contradiction. Thus, $n \equiv 0, 2 \pmod 4$. Conversely, it is easy to show that $n^z \equiv 0 \pmod 4$ if $n \equiv 0, 2 \pmod 4$ and $z \ge 2$.

2. Let $z \ge 1$. First, assume that $n^z \equiv 1 \pmod 4$. Then, $n \equiv 1 \pmod 4$ or $n \equiv 3 \pmod 4$. If $n \equiv 3 \pmod 4$, then $n^z \equiv (-1)^z \pmod 4$ so that z is even. Therefore, $n \equiv 1 \pmod 4$ or $n \equiv 3 \pmod 4$ and z is even. The converse is obtained obviously.

3. Assume that $n^z \equiv 2 \pmod 4$. Then, $n \equiv 2 \pmod 4$. Suppose that $z = 0$ or $z > 1$. Thus, $n^z \equiv 1 \pmod 4$ or $n^z \equiv 0 \pmod 4$, respectively. This contradicts to the assumption. Therefore, $n \equiv 2 \pmod 4$ and $z = 1$. It is obvious for the converse.

4. Assume that $n^z \equiv 3 \pmod 4$. Then, $n \equiv 3 \pmod 4$. Hence, $n^z \equiv (-1)^z \pmod 4$ so that z is odd. Therefore, $n \equiv 3 \pmod 4$ and z is odd. The converse is obtained obviously. □

The following lemma gives the conditions that lead to $w^2 \equiv 2, 3 \pmod 4$ and contradict the existence of non-negative integer solutions of (1).

Lemma 2.2. *Let n be a positive integer. Then,* (1) *has no non-negative integer solution if*

1. $p \equiv 1 \pmod 4$ *and* $n^z \equiv 0, 1 \pmod 4$ *or*
2. $p \equiv 3 \pmod 4$*,* x, y *have same parity and* $n^z \equiv 0, 1 \pmod 4$ *or*
3. $p \equiv 3 \pmod 4$*,* x, y *have opposite parity and* $n^z \equiv 2, 3 \pmod 4$*.*

Proof. 1. Assume that $p \equiv 1 \pmod 4$ and $n^z \equiv 0, 1 \pmod 4$. Then, $p^x \equiv 1 \pmod 4$ and $p^y \equiv 1 \pmod 4$. Thus, $p^x + p^y + n^z \equiv 2, 3 \pmod 4$ so that $w^2 \equiv 2, 3 \pmod 4$. Therefore, (1) has no non-negative integer solution.

2. Assume that $p \equiv 3 \pmod 4$, x, y have same parity and $n^z \equiv 0, 1 \pmod 4$. Then, $p^x \equiv 1 \pmod 4$ and $p^y \equiv 1 \pmod 4$, where x, y are even or $p^x \equiv 3 \pmod 4$ and $p^y \equiv 3 \pmod 4$, where x, y are odd. Hence, $w^2 = p^x + p^y + n^z \equiv 2, 3 \pmod 4$. Therefore, (1) has no non-negative integer solution.

3. Assume that $p \equiv 3 \pmod 4$, x, y have opposite parity and $n^z \equiv 2, 3 \pmod 4$. Then, $p^x \equiv 1 \pmod 4$ and $p^y \equiv 3 \pmod 4$, where x is even and y is odd or $p^x \equiv 3 \pmod 4$ and $p^y \equiv 1 \pmod 4$, where x is odd and y is even. Hence, $w^2 = p^x + p^y + n^z \equiv 2, 3 \pmod 4$. Therefore, (1) has no non-negative integer solution. □

For $p \equiv 1 \pmod 4$, it is easy to show that (1) has no non-negative integer solution, where $z = 0$. Next, we study the case $z \geq 2$.

Theorem 2.1. *Let* $p \equiv 1 \pmod 4$ *and* $z \geq 2$*. Then,* (1) *has no non-negative integer solution if*

1. $n \equiv 0, 1, 2 \pmod 4$ *or*
2. $n \equiv 3 \pmod 4$ *and z is even.*

Proof. 1. Assune that $n \equiv 0, 1, 2 \pmod 4$. By Lemma 2.1 (1) and (2), we obtain that $n^z \equiv 0, 1 \pmod 4$. Thus, (1) has no non-negative integer solution by Lemma 2.2 (1).

2. Assume that $n \equiv 3 \pmod 4$ and z is even. By Lemma 2.1 (2), we obtain that $n^z \equiv 1 \pmod 4$. Thus, (1) has no non-negative integer solution by Lemma 2.2 (1). □

Corollary 2.1. *The Diophantine equation* $p^x + p^y + n^{2z+2} = w^2$ *has no non-negative integer solution, where* $p \equiv 1 \pmod 4$*.*

Proof. It is obvious that $2z + 2$ is even and $2z + 2 \geq 2$ for all non-negative integer z. By Theorem 2.1 (1) and (2), $p^x + p^y + n^{2z+2} = w^2$ has no non-negative integer solution. □

Furthermore, some conditions of n that (1) has no non-negative integer solution are investigated where $z \geq 0$.

Corollary 2.2. *The Diophantine equation* $p^x+p^y+n^z = w^2$ *has no non-negative integer solution, where* $p \equiv 1 \pmod 4$ *and* $n \equiv 0, 1 \pmod 4$*.*

Proof. Since $n \equiv 0, 1 \pmod 4$, we can conclude that $n^z \equiv 0, 1 \pmod 4$. By Lemma 2.2 (1), $p^x + p^y + n^z = w^2$ has no non-negative integer solution. □

Similarly, we obtain the non-existence of non-negative integer solutions for $p \equiv 3 \pmod 4$ and x, y have same parity by Lemma 2.1 (1), (2) and Lemma 2.2 (2).

Theorem 2.2. *Let* $p \equiv 3 \pmod 4$*,* $z \geq 2$ *and* x, y *have same parity. Then,* (1) *has no non-negative integer solution if*

1. $n \equiv 0, 1, 2 \pmod 4$ *or*
2. $n \equiv 3 \pmod 4$ *and* z *is even.*

Proof. 1. Assune that $n \equiv 0, 1, 2 \pmod 4$. By Lemma 2.1 (1) and (2), we obtain that $n^z \equiv 0, 1 \pmod 4$. Thus, (1) has no non-negative integer solution by Lemma 2.2 (2).

2. Assume that $n \equiv 3 \pmod 4$ and z is even. By Lemma 2.1 (2), we obtain that $n^z \equiv 1 \pmod 4$. Thus, (1) has no non-negative integer solution by Lemma 2.2 (2). □

Corollary 2.3. *The Diophantine equation* $p^{2x} + p^{2y} + n^{2z+2} = w^2$ *has no non-negative integer solution, where* $p \equiv 3 \pmod 4$*.*

Proof. It is obvious that $2x$, $2y$ and $2z + 2$ are even. Then, $2x, 2y$ have same parity and $2z + 2 \geq 2$ for all non-negative integer z. By Theorem 2.2 (1) and (2), $p^{2x} + p^{2y} + n^{2z+2} = w^2$ has no non-negative integer solution. □

Corollary 2.4. *The Diophantine equation* $p^{2x+1} + p^{2y+1} + n^{2z+2} = w^2$ *has no non-negative integer solution, where* $p \equiv 3 \pmod 4$*.*

Proof. It is clear that $2x + 1$, $2y + 1$ are odd and $2z + 2$ is even. Then, $2x + 1, 2y + 1$ have same parity and $2z + 2 \geq 2$ for all non-negative integer z. By Theorem 2.2 (1) and (2), $p^{2x+1} + p^{2y+1} + n^{2z+2} = w^2$ has no non-negative integer solution. □

Next, we can confirm the non-existence of non-negative integer solution of (1) where x, y have same parity and $z \geq 0$.

Corollary 2.5. *The Diophantine equation* $p^{2x} + p^{2y} + n^z = w^2$ *has no non-negative integer solution, where* $p \equiv 3 \pmod 4$ *and* $n \equiv 0, 1 \pmod 4$*.*

Proof. It is obvious that $2x$, $2y$ have same parity and $n^z \equiv 0, 1 \pmod 4$. By Lemma 2.2 (2), $p^{2x} + p^{2y} + n^z = w^2$ has no non-negative integer solution. □

Corollary 2.6. *The Diophantine equation* $p^{2x+1} + p^{2y+1} + n^z = w^2$ *has no non-negative integer solution, where* $p \equiv 3 \pmod 4$ *and* $n \equiv 0, 1 \pmod 4$*.*

Proof. It is obvious that $2x+1$, $2y+1$ have same parity and $n^z \equiv 0, 1 \pmod 4$. By Lemma 2.2 (2), $p^{2x+1}+p^{2y+1}+n^z = w^2$ has no non-negative integer solution. □

By Lemma 2.1 (3), (4) and Lemma 2.2 (3), we obtain the following theorem.

Theorem 2.3. *Let $p \equiv 3 \pmod 4$ and x, y have opposite parity. Then,* (1) *has no non-negative integer solution if*

1. *$n \equiv 2 \pmod 4$ and $z = 1$ or*
2. *$n \equiv 3 \pmod 4$ and z is odd.*

Proof. 1. Assune that $n{\equiv}2 \pmod 4$ and $z = 1$. By Lemma 2.1 (3), we obtain that $n^z{\equiv}2 \pmod 4$. Thus, (1) has no non-negative integer solution by Lemma 2.2 (3).

2. Assume that $n{\equiv}3 \pmod 4$ and z is odd. By Lemma 2.1 (4), we obtain that $n^z{\equiv}3 \pmod 4$. Thus, (1) has no non-negative integer solution by Lemma 2.2 (3). □

Corollary 2.7. *The Diophantine equation $p^{2x+1} + p^{2y} + n^{2z+1} = w^2$ has no non-negative integer solution, where $p \equiv 3 \pmod 4$ and $n \equiv 3 \pmod 4$.*

Proof. It is obvious that $2x + 1$, $2y$ have opposite parity and $2z + 1$ is odd. By Theorem 2.3 (2), $p^{2x+1} + p^{2y} + n^{2z+1} = w^2$ has no non-negative integer solution. □

Corollary 2.8. *The Diophantine equation $p^{2x+1} + p^{2y} + n = w^2$ has no non-negative integer solution, where $p \equiv 3 \pmod 4$ and $n \equiv 2 \pmod 4$.*

Proof. It is obvious that $2x + 1$, $2y$ have opposite parity. By Theorem 2.3 (1), $p^{2x+1} + p^{2y} + n = w^2$ has no non-negative integer solution. □

3. Non-existence of solutions by modulo p

In this section, we confirm that (1) has no positive integer solution by modulo p and Legendre symbol $\left(\frac{n}{p}\right)$. Moreover, we gather some forms of an odd prime number p in Legendre symbol $\left(\frac{q}{p}\right)$, $\left(\frac{2q}{p}\right)$ and $\left(\frac{p_1p_2}{p}\right)$, where q, p_1 and p_2 are distinct prime numbers.

Theorem 3.1. *Let x and y be positive integers and z be an odd positive integer. If $\left(\frac{n}{p}\right) = -1$, then* (1) *has no positive integer solution.*

Proof. Assume that (1) has a positive integer solution. Since x and y are positive integers, we have $w^2 \equiv n^z \pmod p$. Then, $\left(\frac{n^z}{p}\right) = 1$. Since z is odd, we can conclude that $\left(\frac{n}{p}\right) = 1$. □

By the above theorem, we found that an odd prime number p with $\left(\frac{n}{p}\right) = -1$ has an important role non-existence of positive integer solutions of (1). In [16], Tadee and Siraworakun investigated the forms of an odd prime number p in Legendre symbols $\left(\frac{q}{p}\right)$ and $\left(\frac{2q}{p}\right)$, where q is an odd prime number.

Theorem 3.2 ([16]). *Let p and q be distinct odd prime numbers with $q \equiv 1 \pmod 4$. Then,*

$$\left(\frac{q}{p}\right) = \begin{cases} 1 & \text{if } p \equiv q + r^{S_1}q + r^{S_1} \pmod{2q} \\ -1 & \text{if } p \equiv q + r^{S_2}q + r^{S_2} \pmod{2q} \end{cases},$$

where $S_1 \in \{2, 4, 6, \dots, q-1\}$, $S_2 \in \{1, 3, 5, \dots, q-2\}$ and r is a primitive root modulo q.

Theorem 3.3 ([16]). *Let p and q be distinct odd prime numbers with $q \equiv 3 \pmod 4$. Then,*

$$\left(\frac{q}{p}\right) = \begin{cases} 1 & \text{if } p \equiv 3q + 4n_0 r^{S_1} \pmod{4q} \text{ or} \\ & \quad\; p \equiv -3q + 4n_0 r^{S_2} \pmod{4q} \\ -1 & \text{if } p \equiv 3q + 4n_0 r^{S_2} \pmod{4q} \text{ or} \\ & \quad\; p \equiv -3q + 4n_0 r^{S_1} \pmod{4q} \end{cases},$$

where $S_1 \in \{2, 4, 6, \dots, q-1\}$, $S_2 \in \{1, 3, 5, \dots, q-2\}$, r is a primitive root modulo q and $n_0 = \frac{q+1}{4}$.

Theorem 3.4 ([16]). *Let p and q be distinct odd prime numbers with $q \equiv 1 \pmod 4$. Then,*

$$\left(\frac{2q}{p}\right) = \begin{cases} 1 & \text{if } p \equiv q^2 + 8n_1 r^{S_1} \pmod{8q} \text{ or} \\ & \quad\; p \equiv -q^2 + 8n_1 r^{S_1} \pmod{8q} \text{ or} \\ & \quad\; p \equiv 3q^2 + 8n_1 r^{S_2} \pmod{8q} \text{ or} \\ & \quad\; p \equiv -3q^2 + 8n_1 r^{S_2} \pmod{8q} \\ -1 & \text{if } p \equiv q^2 + 8n_1 r^{S_2} \pmod{8q} \text{ or} \\ & \quad\; p \equiv -q^2 + 8n_1 r^{S_2} \pmod{8q} \text{ or} \\ & \quad\; p \equiv 3q^2 + 8n_1 r^{S_1} \pmod{8q} \text{ or} \\ & \quad\; p \equiv -3q^2 + 8n_1 r^{S_1} \pmod{8q} \end{cases},$$

where $S_1 \in \{2, 4, 6, \dots, q-1\}$, $S_2 \in \{1, 3, 5, \dots, q-2\}$, r is a primitive root modulo q and if $\frac{q-1}{4}$ is an even number, then $n_1 = \frac{-q+1}{8}$, and if otherwise, then $n_1 = \frac{3q+1}{8}$.

Theorem 3.5 ([16]). *Let p and q be distinct odd prime numbers with $q \equiv 3 \pmod 4$. Then,*

$$\left(\frac{2q}{p}\right) = \begin{cases} 1 & \text{if } p \equiv q^2 + 32n_0n_1r^{S_1} \pmod{8q} \text{ or} \\ & \quad p \equiv -q^2 + 32n_0n_1r^{S_2} \pmod{8q} \text{ or} \\ & \quad p \equiv 3q^2 + 32n_0n_1r^{S_1} \pmod{8q} \text{ or} \\ & \quad p \equiv -3q^2 + 32n_0n_1r^{S_2} \pmod{8q} \\ -1 & \text{if } p \equiv q^2 + 32n_0n_1r^{S_2} \pmod{8q} \text{ or} \\ & \quad p \equiv -q^2 + 32n_0n_1r^{S_1} \pmod{8q} \text{ or} \\ & \quad p \equiv 3q^2 + 32n_0n_1r^{S_2} \pmod{8q} \text{ or} \\ & \quad p \equiv -3q^2 + 32n_0n_1r^{S_1} \pmod{8q} \end{cases},$$

where $S_1 \in \{2,4,6,\ldots,q-1\}$, $S_2 \in \{1,3,5,\ldots,q-2\}$, r *is a primitive root modulo* q, $n_0 = \frac{q+1}{4}$ *and if* $\frac{q-3}{4}$ *is an even number, then* $n_1 = \frac{5q+1}{8}$, *and if otherwise, then* $n_1 = \frac{q+1}{8}$.

Moreover, Siraworakun, Wannaphan and Seesod gave forms of an odd prime number p in Legendre symbol $\left(\frac{p_1p_2}{p}\right)$, where p_1 and p_2 are distinct prime numbers in [13].

Theorem 3.6 ([13])**.** *Let* p_1, p_2 *and* p *be distinct odd prime numbers with* $p_1 \equiv 1 \pmod 4$ *and* $p_2 \equiv 1 \pmod 4$. *Then,*

$$\left(\frac{p_1p_2}{p}\right) = \begin{cases} 1 & \text{if } p \equiv p_1p_2 + 2(n_1r_1^{S_1}p_2 + n_2r_2^{S_2}p_1) \pmod{2p_1p_2} \text{ or} \\ & \quad p \equiv p_1p_2 + 2(n_1r_1^{T_1}p_2 + n_2r_2^{T_2}p_1) \pmod{2p_1p_2} \\ -1 & \text{if } p \equiv p_1p_2 + 2(n_1r_1^{S_1}p_2 + n_2r_2^{T_2}p_1) \pmod{2p_1p_2} \text{ or} \\ & \quad p \equiv p_1p_2 + 2(n_1r_1^{T_1}p_2 + n_2r_2^{S_2}p_1) \pmod{2p_1p_2} \end{cases},$$

where

$$S_1 \in \{2,4,6,\ldots,p_1-1\}, S_2 \in \{2,4,6,\ldots,p_2-1\},$$
$$T_1 \in \{1,3,5,\ldots,p_1-2\}, T_2 \in \{1,3,5,\ldots,p_2-2\}$$

and r_1, r_2 *are primitive roots modulo* p_1 *and* p_2, *respectively and* n_1, n_2 *are integers with* $2p_2n_1 \equiv 1 \pmod{p_1}$ *and* $2p_1n_2 \equiv 1 \pmod{p_2}$.

Theorem 3.7 ([13])**.** *Let* p_1, p_2 *and* p *be distinct odd prime numbers with* $p_1 \equiv 1 \pmod 4$ *and* $p_2 \equiv 3 \pmod 4$. *Then,*

$$\left(\frac{p_1p_2}{p}\right) = \begin{cases} 1 & \text{if } p \equiv -p_1p_2 + 4(n_3r_1^{S_1}p_2 + 4n_0n_4r_2^{S_2}p_1) \pmod{4p_1p_2} \text{ or} \\ & \quad p \equiv p_1p_2 + 4(n_3r_1^{S_1}p_2 + 4n_0n_4r_2^{T_2}p_1) \pmod{4p_1p_2} \text{ or} \\ & \quad p \equiv -p_1p_2 + 4(n_3r_1^{T_1}p_2 + 4n_0n_4r_2^{T_2}p_1) \pmod{4p_1p_2} \text{ or} \\ & \quad p \equiv p_1p_2 + 4(n_3r_1^{T_1}p_2 + 4n_0n_4r_2^{S_2}p_1) \pmod{4p_1p_2} \\ -1 & \text{if } p \equiv -p_1p_2 + 4(n_3r_1^{T_1}p_2 + 4n_0n_4r_2^{S_2}p_1) \pmod{4p_1p_2} \text{ or} \\ & \quad p \equiv p_1p_2 + 4(n_3r_1^{T_1}p_2 + 4n_0n_4r_2^{T_2}p_1) \pmod{4p_1p_2} \text{ or} \\ & \quad p \equiv -p_1p_2 + 4(n_3r_1^{S_1}p_2 + 4n_0n_4r_2^{T_2}p_1) \pmod{4p_1p_2} \text{ or} \\ & \quad p \equiv p_1p_2 + 4(n_3r_1^{S_1}p_2 + 4n_0n_4r_2^{S_2}p_1) \pmod{4p_1p_2} \end{cases},$$

where

$$S_1 \in \{2,4,6,\ldots,p_1-1\}, S_2 \in \{2,4,6,\ldots,p_2-1\},$$
$$T_1 \in \{1,3,5,\ldots,p_1-2\}, T_2 \in \{1,3,5,\ldots,p_2-2\}$$

and r_1, r_2 *are primitive roots modulo* p_1 *and* p_2*, respectively and* n_0, n_3, n_4 *are integers with* $n_0 = \frac{p_2+1}{4}$, $4p_2n_3 \equiv 1 \pmod{p_1}$ *and* $4p_1n_4 \equiv 1 \pmod{p_2}$.

Theorem 3.8 ([13])**.** *Let* p_1, p_2 *and* p *be distinct odd prime numbers with* $p_1 \equiv 3 \pmod 4$ *and* $p_2 \equiv 3 \pmod 4$*. Then,*

$$\left(\frac{p_1p_2}{p}\right) = \begin{cases} 1 & \text{if } p \equiv p_1p_2 + 16(m_0n_3r_1^{S_1}p_2 + n_0n_4r_2^{S_2}p_1) \pmod{4p_1p_2} \text{ or} \\ & \quad p \equiv -p_1p_2 + 16(m_0n_3r_1^{T_1}p_2 + n_0n_4r_2^{T_2}p_1) \pmod{4p_1p_2} \text{ or} \\ & \quad p \equiv p_1p_2 + 16(m_0n_3r_1^{T_1}p_2 + n_0n_4r_2^{T_2}p_1) \pmod{4p_1p_2} \text{ or} \\ & \quad p \equiv -p_1p_2 + 16(m_0n_3r_1^{S_1}p_2 + n_0n_4r_2^{S_2}p_1) \pmod{4p_1p_2} \\ -1 & \text{if } p \equiv p_1p_2 + 16(m_0n_3r_1^{T_1}p_2 + n_0n_4r_2^{S_2}p_1) \pmod{4p_1p_2} \text{ or} \\ & \quad p \equiv -p_1p_2 + 16(m_0n_3r_1^{S_1}p_2 + n_0n_4r_2^{T_2}p_1) \pmod{4p_1p_2} \text{ or} \\ & \quad p \equiv p_1p_2 + 16(m_0n_3r_1^{S_1}p_2 + n_0n_4r_2^{T_2}p_1) \pmod{4p_1p_2} \text{ or} \\ & \quad p \equiv -p_1p_2 + 16(m_0n_3r_1^{T_1}p_2 + n_0n_4r_2^{S_2}p_1) \pmod{4p_1p_2} \end{cases},$$

where

$$S_1 \in \{2,4,6,\ldots,p_1-1\}, S_2 \in \{2,4,6,\ldots,p_2-1\},$$
$$T_1 \in \{1,3,5,\ldots,p_1-2\}, T_2 \in \{1,3,5,\ldots,p_2-2\}$$

and r_1, r_2 *are primitive roots modulo* p_1 *and* p_2*, respectively and* m_0, n_0, n_3, n_4 *are integers with* $m_0 = \frac{p_1+1}{4}$, $n_0 = \frac{p_2+1}{4}$, $4p_2n_3 \equiv 1 \pmod{p_1}$ *and* $4p_1n_4 \equiv 1 \pmod{p_2}$.

Now, we combine Theorem 3.1 with Theorem 3.2-3.8 to prove Theorem 3.9 - 3.15. In addition, many examples of (1), that have no positive integer solution, are demonstrated in Corollary 3.1 - 3.7.

Theorem 3.9. *Let* x *and* y *be positive integers and* z *be an odd positive integer. If* p *and* q *are distinct odd prime numbers with the following conditions:*

1. $q \equiv 1 \pmod 4$ *and*
2. $p \equiv q + r^{S_2}q + r^{S_2} \pmod{2q}$,

where $S_2 \in \{1,3,5,\ldots,q-2\}$ *and* r *is a primitive root modulo* q*, then the Diophantine equation* $p^x + p^y + q^z = w^2$ *has no positive integer solution.*

Proof. By Theorem 3.2, $\left(\frac{q}{p}\right) = -1$. Thus, $p^x + p^y + q^z = w^2$ has no positive integer solution by Theorem 3.1. □

Corollary 3.1. *The Diophantine equation* $p^x + p^y + 5^{2z+1} = w^2$ *has no positive integer solution, where* $p \equiv \pm 3 \pmod{10}$.

Proof. It is obvious that $2z+1$ is odd. Let $q=5$ and $r=2$. Then, $q \equiv 1 \pmod 4$ and r is a primitive root modulo q. Since $p \equiv \pm 3 \pmod{10}$, we have $p \equiv q + r^{S_2} q + r^{S_2} \pmod{2q}$, where $S_2 \in \{1,3,5,\dots,q-2\}$. By Theorem 3.9, $p^x + p^y + 5^{2z+1} = w^2$ has no positive integer solution. □

Theorem 3.10. *Let x and y be positive integers and z be an odd positive integer. If p and q are distinct odd prime numbers with the following conditions:*

1. $q \equiv 3 \pmod 4$ *and*
2. $p \equiv 3q + 4n_0 r^{S_2}, \ -3q + 4n_0 r^{S_1} \pmod{4q}$,

where $S_1 \in \{2,4,6,\dots,q-1\}$, $S_2 \in \{1,3,5,\dots,q-2\}$, r is a primitive root modulo q and $n_0 = \frac{q+1}{4}$, then the Diophantine equation $p^x + p^y + q^z = w^2$ has no positive integer solution.

Proof. By Theorem 3.3, $\left(\frac{q}{p}\right) = -1$. Thus, $p^x + p^y + q^z = w^2$ has no positive integer solution by Theorem 3.1. □

Corollary 3.2. *The Diophantine equation $p^x + p^y + 3^{2z+1} = w^2$ has no positive integer solution, where $p \equiv \pm 5 \pmod{12}$.*

Proof. It is obvious that $2z+1$ is odd. Let $q=3$, $r=2$ and $n_0=1$. Then, $q \equiv 3 \pmod 4$, r is a primitive root modulo q and $n_0 = \frac{q+1}{4}$. Since $p \equiv \pm 5 \pmod{12}$, we have $p \equiv 3q + 4n_0 r^{S_2}, \ -3q + 4n_0 r^{S_1} \pmod{4q}$, where $S_1 \in \{2,4,6,\dots,q-1\}$ and $S_2 \in \{1,3,5,\dots,q-2\}$. By Theorem 3.10, $p^x + p^y + 3^{2z+1} = w^2$ has no positive integer solution. □

Theorem 3.11. *Let x and y be positive integers and z be an odd positive integer. If p and q are distinct odd prime numbers with the following conditions:*

1. $q \equiv 1 \pmod 4$ *and*
2. $p \equiv \ q^2 + 8n_1 r^{S_2}, \ -q^2 + 8n_1 r^{S_2}, \ 3q^2 + 8n_1 r^{S_1}, \ -3q^2 + 8n_1 r^{S_1} \pmod{8q}$,

where $S_1 \in \{2,4,6,\dots,q-1\}$, $S_2 \in \{1,3,5,\dots,q-2\}$, r is a primitive root modulo q and if $\frac{q-1}{4}$ is an even number, then $n_1 = \frac{-q+1}{8}$, and if otherwise, then $n_1 = \frac{3q+1}{8}$, then the Diophantine equation $p^x + p^y + (2q)^z = w^2$ has no positive integer solution.

Proof. By Theorem 3.4, $\left(\frac{2q}{p}\right) = -1$. Thus, $p^x + p^y + (2q)^z = w^2$ has no positive integer solution by Theorem 3.1. □

Corollary 3.3. *The Diophantine equation $p^x + p^y + 10^{2z+1} = w^2$ has no positive integer solution, where $p \equiv \pm 7, \pm 11, \pm 17, \pm 19 \pmod{40}$.*

Proof. It is obvious that $2z+1$ is odd. Let $q=5$, $r=2$ and $n_1=2$. Then, $q \equiv 1 \pmod 4$, r is a primitive root modulo q and $n_1=\frac{3q+1}{8}$. Since $p \equiv \pm 7, \pm 11, \pm 17, \pm 19 \pmod{40}$, we have $p \equiv\ q^2+8n_1r^{S_2},\ -q^2+8n_1r^{S_2},\ 3q^2+8n_1r^{S_1},\ -3q^2+8n_1r^{S_1} \pmod{8q}$, where $S_1 \in \{2,4,6,\ldots,q-1\}$ and $S_2 \in \{1,3,5,\ldots,q-2\}$. By Theorem 3.11, $p^x+p^y+10^{2z+1}=w^2$ has no positive integer solution. □

Theorem 3.12. *Let x and y be positive integers and z be an odd positive integer. If p and q are distinct odd prime numbers with the following conditions:*

1. $q \equiv 3 \pmod 4$ *and*
2. $p \equiv\ q^2+32n_0n_1r^{S_2},\ -q^2+32n_0n_1r^{S_1},\ 3q^2+32n_0n_1r^{S_2},\ -3q^2+32n_0n_1r^{S_1} \pmod{8q}$,

where $S_1 \in \{2,4,6,\ldots,q-1\}$, $S_2 \in \{1,3,5,\ldots,q-2\}$, r is a primitive root modulo q, $n_0=\frac{q+1}{4}$ and if $\frac{q-3}{4}$ is an even number, then $n_1=\frac{5q+1}{8}$, and if otherwise, then $n_1=\frac{q+1}{8}$, then the Diophantine equation $p^x+p^y+(2q)^z=w^2$ has no positive integer solution.

Proof. By Theorem 3.5, $\left(\frac{2q}{p}\right)=-1$. Thus, $p^x+p^y+(2q)^z=w^2$ has no positive integer solution by Theorem 3.1. □

Corollary 3.4. *The Diophantine equation $p^x+p^y+6^{2z+1}=w^2$ has no positive integer solution, where $p \equiv \pm 7, \pm 11, \pmod{24}$.*

Proof. It is obvious that $2z+1$ is odd. Let $q=3$, $r=2$, $n_0=1$ and $n_1=2$. Then, $q \equiv 3 \pmod 4$, r is a primitive root modulo q, $n_0=\frac{q+1}{4}$ and $n_1=\frac{5q+1}{8}$. Since $p \equiv \pm 7, \pm 11, \pmod{24}$, we have $p \equiv\ q^2+32n_0n_1r^{S_2},\ -q^2+32n_0n_1r^{S_1},\ 3q^2+32n_0n_1r^{S_2},\ -3q^2+32n_0n_1r^{S_1} \pmod{8q}$, where $S_1 \in \{2,4,6,\ldots,q-1\}$ and $S_2 \in \{1,3,5,\ldots,q-2\}$. By Theorem 3.12, $p^x+p^y+6^{2z+1}=w^2$ has no positive integer solution. □

Theorem 3.13. *Let x and y be positive integers and z be an odd positive integer. If p_1, p_2 and p are distinct odd prime numbers with the following conditions:*

1. $p_1 \equiv 1 \pmod 4$, $p_2 \equiv 1 \pmod 4$ *and*
2. $p \equiv p_1p_2+2(n_1r_1^{S_1}p_2+n_2r_2^{T_2}p_1),\ p_1p_2+2(n_1r_1^{T_1}p_2+n_2r_2^{S_2}p_1) \pmod{2p_1p_2}$,

where $S_1 \in \{2,4,6,\ldots,p_1-1\}, S_2 \in \{2,4,6,\ldots,p_2-1\}$, $T_1 \in \{1,3,5,\ldots,p_1-2\}$, $T_2 \in \{1,3,5,\ldots,p_2-2\}$ and r_1, r_2 are primitive roots modulo p_1 and p_2, respectively and n_1, n_2 are integers with $2p_2n_1 \equiv 1 \pmod{p_1}$ and $2p_1n_2 \equiv 1 \pmod{p_2}$, then the Diophantine equation $p^x+p^y+(p_1p_2)^z=w^2$ has no positive integer solution.

Proof. By Theorem 3.6, $\left(\frac{p_1p_2}{p}\right)=-1$. Thus, $p^x+p^y+(p_1p_2)^z=w^2$ has no positive integer solution by Theorem 3.1. □

Corollary 3.5. *The Diophantine equation* $p^x+p^y+65^{2z+1}=w^2$ *has no positive integer solution, where* $p \equiv \pm3, \pm11, \pm17, \pm19, \pm21, \pm23, \pm27, \pm31, \pm41, \pm43, \pm53, \pm59 \pmod{130}$.

Proof. It is obvious that $2z+1$ is odd. Let $p_1 = 5$, $p_2 = 13$, $r_1 = 2$, $r_2 = 2$, $n_1 = 1$ and $n_2 = 4$. Then, $p_1 \equiv 1 \pmod 4$, $p_2 \equiv 1 \pmod 4$, r_1, r_2 are primitive roots modulo p_1 and p_2, respectively, $2p_2n_1 \equiv 1 \pmod{p_1}$ and $2p_1n_2 \equiv 1 \pmod{p_2}$. Since $p \equiv \pm3, \pm11, \pm17, \pm19, \pm21, \pm23, \pm27, \pm31, \pm41, \pm43, \pm53, \pm59 \pmod{130}$, we have $p \equiv p_1p_2+2(n_1r_1^{S_1}p_2+n_2r_2^{T_2}p_1),\ p_1p_2+2(n_1r_1^{T_1}p_2+n_2r_2^{S_2}p_1) \pmod{2p_1p_2}$, where $S_1 \in \{2,4,6,\dots,p_1-1\}, S_2 \in \{2,4,6,\dots,p_2-1\}$, $T_1 \in \{1,3,5,\dots,p_1-2\}$ and $T_2 \in \{1,3,5,\dots,p_2-2\}$. By Theorem 3.13, $p^x+p^y+65^{2z+1}=w^2$ has no positive integer solution. $\square$

Theorem 3.14. *Let x and y be positive integers and z be an odd positive integer. If p_1, p_2 and p are distinct odd prime numbers with the following conditions:*

1. $p_1 \equiv 1 \pmod 4$, $p_2 \equiv 3 \pmod 4$ *and*
2. $p \equiv -p_1p_2+4(n_3r_1^{T_1}p_2+4n_0n_4r_2^{S_2}p_1),\ p_1p_2+4(n_3r_1^{T_1}p_2+4n_0n_4r_2^{T_2}p_1),$ $-p_1p_2+4(n_3r_1^{S_1}p_2+4n_0n_4r_2^{T_2}p_1),\ p_1p_2+4(n_3r_1^{S_1}p_2+4n_0n_4r_2^{S_2}p_1)$ $\pmod{4p_1p_2}$,

where $S_1 \in \{2,4,6,\dots,p_1-1\}, S_2 \in \{2,4,6,\dots,p_2-1\}$, $T_1 \in \{1,3,5,\dots,p_1-2\}, T_2 \in \{1,3,5,\dots,p_2-2\}$ and r_1, r_2 are primitive roots modulo p_1 and p_2, respectively and and n_0, n_3, n_4 are integers with $n_0 = \frac{p_2+1}{4}$, $4p_2n_3 \equiv 1 \pmod{p_1}$ and $4p_1n_4 \equiv 1 \pmod{p_2}$, then the Diophantine equation $p^x+p^y+(p_1p_2)^z=w^2$ has no positive integer solution.

Proof. By Theorem 3.7, $\left(\frac{p_1p_2}{p}\right) = -1$. Thus, $p^x+p^y+(p_1p_2)^z=w^2$ has no positive integer solution by Theorem 3.1. $\square$

Corollary 3.6. *The Diophantine equation* $p^x+p^y+15^{2z+1}=w^2$ *has no positive integer solution, where* $p \equiv \pm13, \pm19, \pm23, \pm29 \pmod{60}$.

Proof. It is obvious that $2z+1$ is odd. Let $p_1 = 5$, $p_2 = 3$, $r_1 = 2$, $r_2 = 2$, $n_0 = 1$, $n_3 = 3$ and $n_4 = 2$. Then, $p_1 \equiv 1 \pmod 4$, $p_2 \equiv 3 \pmod 4$, r_1, r_2 are primitive roots modulo p_1 and p_2, respectively, $n_0 = \frac{p_2+1}{4}$, $4p_2n_3 \equiv 1 \pmod{p_1}$ and $4p_1n_4 \equiv 1 \pmod{p_2}$. Since $p \equiv \pm13, \pm19, \pm23, \pm29 \pmod{60}$, we have $p \equiv -p_1p_2+4(n_3r_1^{T_1}p_2+4n_0n_4r_2^{S_2}p_1),\ p_1p_2+4(n_3r_1^{T_1}p_2+4n_0n_4r_2^{T_2}p_1),\ -p_1p_2+4(n_3r_1^{S_1}p_2+4n_0n_4r_2^{T_2}p_1),\ p_1p_2+4(n_3r_1^{S_1}p_2+4n_0n_4r_2^{S_2}p_1) \pmod{4p_1p_2}$, where $S_1 \in \{2,4,6,\dots,p_1-1\}, S_2 \in \{2,4,6,\dots,p_2-1\}$, $T_1 \in \{1,3,5,\dots,p_1-2\}$, and $T_2 \in \{1,3,5,\dots,p_2-2\}$. By Theorem 3.14, $p^x+p^y+15^{2z+1}=w^2$ has no positive integer solution. $\square$

Theorem 3.15. *Let x and y be positive integers and z be an odd positive integer. If p_1, p_2 and p are distinct odd prime numbers with the following conditions:*

1. $p_1 \equiv 3 \pmod 4$, $p_2 \equiv 3 \pmod 4$ *and*
2. $p \equiv p_1p_2+16(m_0n_3r_1^{T_1}p_2+n_0n_4r_2^{S_2}p_1),\ -p_1p_2+16(m_0n_3r_1^{S_1}p_2+n_0n_4r_2^{T_2}p_1),$ $p_1p_2+16(m_0n_3r_1^{S_1}p_2+n_0n_4r_2^{T_2}p_1),\ -p_1p_2+16(m_0n_3r_1^{T_1}p_2+n_0n_4r_2^{S_2}p_1)$ $\pmod{4p_1p_2}$,

where $S_1 \in \{2,4,6,\dots,p_1-1\}, S_2 \in \{2,4,6,\dots,p_2-1\}$, $T_1 \in \{1,3,5,\dots,p_1-2\}, T_2 \in \{1,3,5,\dots,p_2-2\}$ *and* r_1, r_2 *are primitive roots modulo* p_1 *and* p_2, *respectively and and* m_0, n_0, n_3, n_4 *are integers with* $m_0 = \frac{p_1+1}{4}$, $n_0 = \frac{p_2+1}{4}$, $4p_2n_3 \equiv 1 \pmod{p_1}$ *and* $4p_1n_4 \equiv 1 \pmod{p_2}$, *then the Diophantine equation* $p^x + p^y + (p_1p_2)^z = w^2$ *has no positive integer solution.*

Proof. By Theorem 3.8, $\left(\frac{p_1p_2}{p}\right) = -1$. Thus, $p^x + p^y + (p_1p_2)^z = w^2$ has no positive integer solution by Theorem 3.1. □

Corollary 3.7. *The Diophantine equation* $p^x+p^y+21^{2z+1} = w^2$ *has no positive integer solution, where* $p \equiv \pm 11, \pm 13, \pm 19, \pm 23, \pm 29, \pm 31 \pmod{84}$.

Proof. It is obvious that $2z+1$ is odd. Let $p_1 = 3$, $p_2 = 7$, $r_1 = 2$, $r_2 = 3$, $m_0 = 1$, $n_0 = 2$, $n_3 = 1$ and $n_4 = 3$. Then, $p_1 \equiv 3 \pmod 4$, $p_2 \equiv 3 \pmod 4$, r_1, r_2 are primitive roots modulo p_1 and p_2, respectively, $m_0 = \frac{p_1+1}{4}$, $n_0 = \frac{p_2+1}{4}$, $4p_2n_3 \equiv 1 \pmod{p_1}$ and $4p_1n_4 \equiv 1 \pmod{p_2}$. Since $p \equiv \pm 11, \pm 13, \pm 19, \pm 23, \pm 29, \pm 31 \pmod{84}$, we have

$$\begin{aligned} p \equiv\ & p_1p_2 + 16(m_0n_3r_1^{T_1}p_2 + n_0n_4r_2^{S_2}p_1), \quad -p_1p_2 + 16(m_0n_3r_1^{S_1}p_2 + n_0n_4r_2^{T_2}p_1), \\ & p_1p_2 + 16(m_0n_3r_1^{S_1}p_2 + n_0n_4r_2^{T_2}p_1), \\ & -p_1p_2 + 16(m_0n_3r_1^{T_1}p_2 + n_0n_4r_2^{S_2}p_1) \pmod{4p_1p_2}, \end{aligned}$$

where $S_1 \in \{2,4,6,\dots,p_1-1\}, S_2 \in \{2,4,6,\dots,p_2-1\}$, $T_1 \in \{1,3,5,\dots,p_1-2\}$ and $T_2 \in \{1,3,5,\dots,p_2-2\}$ By Theorem 3.15, $p^x + p^y + 21^{2z+1} = w^2$ has no positive integer solution. □

4. Non-existence of solutions by modulo p-1 and p+1

In the last section, we are interested in exploring some conditions of n by modulo $p-1$ and $p+1$ that (1) has no positve integer solution.

Theorem 4.1. *Let* $n \equiv 0 \pmod{p-1}$. *If* $p \equiv 1 \pmod 4$, *then* (1) *has no positive integer solution.*

Proof. Assume that (1) has a positive integer solution. Then, $w^2 \equiv 2 \pmod{p-1}$. There exists an integer k such that $w^2 = (p-1)k+2 = 2((\frac{p-1}{2})k+1)$. Thus, $\frac{p-1}{2}$ is odd. Therefore, $p \equiv 3 \pmod 4$. □

Corollary 4.1. *The Diophantine equation* $p^x + p^y + ((p-1)k)^z = w^2$ *has no positive integer solution, where* $p \equiv 1 \pmod 4$ *and* k *is a positive integer.*

Proof. Since $(p-1)k \equiv 0 \pmod{p-1}$ and $p \equiv 1 \pmod 4$, we can conclude that $p^x + p^y + ((p-1)k)^z = w^2$ has no positive integer solution by Theorem 4.1. □

Theorem 4.2. *Let $n \equiv 0 \pmod{p+1}$ and x, y have the same parity. If $p \equiv 3 \pmod 4$, then (1) has no positive integer solution.*

Proof. Assume that (1) has a positive integer solution. Then, $w^2 = p^x + p^y + n^z \equiv (-1)^x + (-1)^y + 0 \pmod{p+1}$. Since x, y have same parity, we obtain that $w^2 \equiv \pm 2 \pmod{p+1}$. So, there exists an integer k such that $w^2 = (p+1)k \pm 2 = 2((\frac{p+1}{2})k \pm 1)$. Thus, $\frac{p+1}{2}$ is odd. Therefore, $p \equiv 1 \pmod 4$. □

Corollary 4.2. *The Diophantine equation $p^{2x} + p^{2y} + ((p+1)k)^z = w^2$ has no positive integer solution, where $p \equiv 3 \pmod 4$ and k is a positive integer.*

Proof. It is obvious that $2x$, $2y$ have same parity. Since $(p+1)k \equiv 0 \pmod{p+1}$ and $p \equiv 3 \pmod 4$, we can conclude that $p^{2x} + p^{2y} + ((p+1)k)^z = w^2$ has no positive integer solution by Theorem 4.2. □

Corollary 4.3. *The Diophantine equation $p^{2x+1} + p^{2y+1} + ((p+1)k)^z = w^2$ has no positive integer solution, where $p \equiv 3 \pmod 4$ and k is a positive integer.*

Proof. It is obvious that $2x+1$, $2y+1$ have same parity. Since $(p+1)k \equiv 0 \pmod{p+1}$ and $p \equiv 3 \pmod 4$, we can conclude that $p^{2x} + p^{2y} + ((p+1)k)^z = w^2$ has no positive integer solution by Theorem 4.2. □

5. Conclusion

We have studied the Diophantine equations $p^x + p^y + n^z = w^2$, where p is an odd prime number and n is a positive integer. Various conditions are provided to confirm that the Diophantine equations $p^x + p^y + n^z = w^2$ has no non-negative or positive integer solution. Moreover, we obtain numerous examples form all corollaries in this article.

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Precedence hyperstructures and graphs in assembly line design

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Abstract. In this paper we introduce the precedence hyperoperation, which constructs a precedence partial hypergroupoid, i.e. a partial hypergroupoid with some special properties. Given a precedence partial hypergroupoid, a precedence graph can be defined and vice versa. Using the precedence partial hypergroupoid of a precedence graph and the Fewer-Descendants-Vertex First algorithm (FDVF algorithm), a process flow diagram is created, which can be used in mixed-model assembly line design.

Keywords: precedence graph, precedence partial hypergroupoid, process flow diagram, assembly line design, Fewer-Descendants-Vertex First algorithm (FDVF algorithm), More-Descendants-Vertex First algorithm (MDVF algorithm).

MSC 2020: 05C20, 20N20, 05C38, 05C90, 90C35, 68R10.

1. Introduction

Algebraic hyperstructures were introduced in 1934 by Marty [20], as a generalization of the notion of the group and have been studied since by many mathematicians (s. [1]-[17], [19], [21]-[22], [24]-[28]).

Hyperstructures associated with binary relations have a special interest for many researchers, such as I. Rosenberg [22], P. Corsini [1]-[6], Y. Feng [12], V.

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Leoreanu-Fotea [4]-[6], [10], [19], B. Davvaz [10], I. Cristea and M. Stefanescu [7]-[9], M. Konstantinidou-Serafimidou and K. Serafimidis [17], [27], M. De Salvo and J. Lo Faro [11], S. J. Rasovic [21], S. Spartalis [13]-[16], [24]-[27], A. Kalampakas [13]-[16], [26], A. Taouktsoglou [27] and others.

Additionally, binary relations and directed graphs are representing complex information and describing rich structures. Therefore, the connection between hyperstructures and graph theory appear naturally and are studied in [13], [14], [15] and [26]. More precisely, in [14] the *"path hyperoperation"* and the associated *"path hypergroupoids"* were introduced, as a generalization of the C-hypergroupoids [3], [24], [25] and an application to the design and management of mixed-model assembly lines, was presented.

On the other hand, precedence graphs, first developed by M. E. Salveson [23], are traditionally used to visualize the assembly sequence of a model in many industrial environments (see also, The Assembly Line Balancing Problem [18]). A precedence graph is used in order for a product to be assembled in a proper way. Designing an assembly line one has to correspond the tasks needed for each product to workstations standing in a row. All precedence conditions have to be fulfilled and the working time among the workstations has to be balanced.

In this respect, we introduce and study a new hyperoperation, which constructs a *"precedence partial hypergroupoid"*, i.e. a partial hypergroupoid with some special properties. Given a precedence partial hypergroupoid, a precedence graph can be defined and vice versa. Using the precedence partial hypergroupoid of a precedence graph and following the Fewer-Descendants-Vertex First algorithm (FDVF algorithm) that we created, one can design and computer program a process flow diagram, which can be used in mixed-model assembly line designing.

In Section 2 basic concepts on hypergroupoids and directed graphs are presented. In Section 3 we define the precedence hyperoperation and we investigate the properties of the associated precedence hypergroupoid. In Section 4 using a precedence hypergroupoid we define a precedence graph and vice versa, proving that there is a 1-1 correspondence between the two notions. We also, set necessary conditions for a graph to be a precedence graph. In Section 5 we present an algorithm one can use, in order to derive a process flow diagram through a precedence hypergroupoid. The proposed FDVF algorithm used on precedence hypergroupoids can be applied in mixed-model assembly line designing.

2. Preliminaries

A partial hypergroupoid is a pair $(H, *)$, where H is a non empty set and $*$ is a hyperoperation i.e.

$$*: H \times H \to \wp(H), \qquad (x, y) \mapsto x * y.$$

If $A, B \in \wp(H) - \{\emptyset\}$, we set $A * B = \bigcup_{a \in A, b \in B} a * b$. We also, denote $a * B$ (resp. $A * b$) the hyperproduct $A * B$ in case that A is the singleton $\{a\}$ (resp.

B is the singleton $\{b\}$). $(H, *)$ is called a *"hypergroupoid"* if $x * y \neq \emptyset$, for all $x, y \in H$ and it is called *"degenerative* (resp. *"total*) *hypergroupoid"* if $x * y = \emptyset$ (resp. $x * y = H$), for all $x, y \in H$ (s. [27]).

Given a binary relation $R \subseteq H \times H$ hypergroupoids can be defined in many ways. For example, a wide class of partial hypergroupoids, named *"partial C-hypergroupoids"*, are defined by the *"Corsini's hyperoperation"* [3]:

$$*_R : H \times H \to \wp(H) : \ (x, y) \mapsto x *_R y = \{z \in H/(x, z) \in R \text{ and } (z, y) \in R\}.$$

Given a non-empty finite set V and a binary relation $R \subseteq V \times V$ *"a concrete directed graph"* G is defined as the pair (V, R). The elements of V are called *"vertices"* and the elements of R are called *"edges"* (s. [14]). Drawing a graph, a vertex is drawn as a node and an edge as an arrow connecting two vertices, which are called the *"head"* and the *"tail"* of the edge.

A vertex of a graph is called *"isolated"* if there is no edge connected to it. In what follows we consider concrete directed graphs without isolated vertices.

A graph $G' = (V', R')$ is called a *"subgraph"* of the graph $G = (V, R)$ if it holds $V' \subseteq V$ and $R' \subseteq R$. Then we say that "$G' = (V', R')$ *is included in* $G = (V, R)$" and we denote $G' \leq G$.

Given a graph $G = (V, R)$ and $v_1, v_n \in V$, *"a directed path from v_1 to v_n"* (or simply *"a path from v_1 to v_n"*) is defined as a pair $P = (V_P, E_P)$ of a set of vertices $V_P = \{v_1, v_2, \ldots, v_n\} \subseteq V$ and a set of edges $E_P = \{e_1, e_2, \ldots, e_{n-1}\} \subseteq R$, where $e_i = (v_i, v_{i+1})$, for all $i = 1, 2, \ldots, n-1$, where $n \in \mathbb{N}, n > 1$. Then, $v_1, v_2, \ldots, v_n$ are called *"vertices of path P"* and $e_1, e_2, \ldots, e_{n-1}$ are called *"edges of path P"*. Consequently, v_1 is called *"starting vertex"* and v_n *"ending vertex"* of path P. According to the previous definition, the pair $P = (V_P, E_P)$ is a concrete directed graph included in the graph $G = (V, R)$.

Practically, a path from v_1 to v_n is

- an alternating sequence of vertices and edges $v_1, e_1, v_2, e_2, \ldots, e_{n-1}, v_n$ beginning at v_1 and ending at v_n, where all vertices are distinct from one another and $e_i = (v_i, v_{i+1})$, $\forall i = 1, 2, \ldots, n-1$ or equivalently,

- a sequence of vertices $v_1, v_2, \ldots, v_n$, all distinct from one another, beginning at v_1 and ending at v_n, where there exist edges $e_i = (v_i, v_{i+1}) \in R$, $\forall i = 1, 2, \ldots, n-1$.

Example 2.1. In the graph of Figure 1 several paths from vertex 1 to vertex 6 are displayed, i.e.

$$\begin{array}{ll} P_1 : 1, 5, 6 & P_2 : 1, 2, 6 \\ P_3 : 1, 3, 5, 6 & P_4 : 1, 5, 2, 6 \\ P_5 : 1, 2, 4, 6 & P_6 : 1, 2, 5, 6 \end{array}$$

"An induced path" is defined as a path, in which each two adjacent vertices are connected by an edge and each two nonadjacent vertices are not connected by any edge.

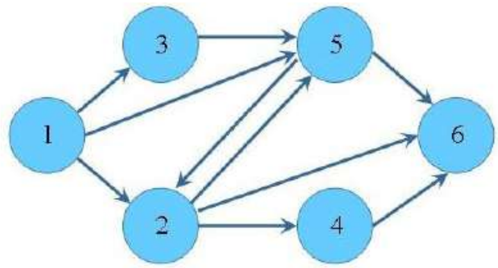

Figure 1: Several paths from vertex 1 to vertex 6

Given a graph $G = (V, R)$, *"a directed circle"* (or simply *"a circle"*) is defined as a pair $C = (V_C, E_C)$ of a set of vertices $V_C = \{v_1, v_2, \dots, v_{n-1}\} \subseteq V$ and a set of edges $E_C = \{e_1, e_2, \dots, e_{n-1}\} \subseteq R$, where $e_i = (v_i, v_{i+1})$, for all $i = 1, 2, \dots, n-2$ and $e_{n-1} = (v_{n-1}, v_1)$, where $n \in \mathbb{N}, n > 2$. Then, $v_1, v_2, \dots, v_{n-1}$ are called *"vertices of the circle C"* and $e_1, e_2, \dots, e_{n-1}$ are called *"edges of the circle C"*.

Practically, a circle is

- a path, whose starting and ending vertices coincide, or equivalently,
- an alternating sequence of vertices and edges $v_1, e_1, v_2, e_2, \dots, e_{n-1}, v_1$ beginning and ending at the same vertex, where all vertices are distinct from one another, except from the starting and the ending vertex, and $e_i = (v_i, v_{i+1})$, $\forall i = 1, 2, \dots, n-2$, $e_{n-1} = (v_{n-1}, v_1)$, where $n \in \mathbb{N}$, $n > 2$.

Given a graph $G = (V, R)$ and $v_1, v_n \in V$, *"a directed walk from v_1 to v_n"* (or simply *"a walk from v_1 to v_n"*) is defined as a union P of consecutive paths, starting at v_1 and ending at v_n, which contains at least one circle. Then, all vertices of the united paths are called *"vertices of walk P"* and all edges of the united paths are called *"edges of walk P"*. Consequently, v_1 is called *"starting vertex"* and v_n *"ending vertex"* of walk P.

3. Precedence hyperoperation

Definition 3.1. *Given a non empty set V a partial hyperoperation*

$$*_P : V \times V \to \wp(V) : \ (x, y) \mapsto x *_P y$$

can be defined, for all $x, y \in V$, in the following way:

i. $x *_P y \neq \emptyset \Rightarrow (x \in x *_P y$ *and* $y \notin x *_P y)$;

ii. $\exists\, a \in V : a *_P x \neq \emptyset$, *for all* $x \in V - \{a\}$;

iii. $\exists\, b \in V : x *_P b \neq \emptyset$, *for all* $x \in V - \{b\}$;

iv. $x *_P y = (a *_P y) \cap (x *_P b)$;

$$v.\ u \in x *_P y \Rightarrow \begin{cases} (x *_P u) \cup (u *_P y) \subseteq x *_P y \\ x \in a *_P u, & \text{if } x \neq u \\ y \in u *_P b, & \text{if } y \neq b \end{cases}$$

Such a hyperoperation is called "precedence hyperoperation" and introduces a partial hypergroupoid $(V, *_P)$ called "precedence partial hypergroupoid". The elements a and b are called "starting" and "ending element" respectively. The results $x *_P u$ and $u *_P y$ are called "complementary to $x *_P y$".

Remark 3.1. It is obvious from Definition 3.1-i that

(1) $$x *_P x = \emptyset, \qquad \text{for all } x \in V.$$

Therefore, conditions i, ii and iii of Definition 3.1 imply that

(2) $$a \in a *_P x, \qquad \text{for all } x \in V - \{a\},$$

(3) $$x \in x *_P b, \qquad \text{for all } x \in V - \{b\}.$$

Moreover, setting $x = y$ in Definition 3.1-iv, we obtain that

(4) $$(a *_P x) \cap (x *_P b) = \emptyset, \qquad \text{for all } x \in V.$$

Example 3.1. Let $V = \{1, 2, 3, 4, 5, 6\}$. We consider the hyperoperation $*_P : V \times V \to \wp(V)$ given by Table 1. One can see that this hyperoperation satisfies

	1	2	3	4	5	6
1	$\varnothing$	$\{1\}$	$\{1,2\}$	$\{1,2,3\}$	$\{1,2\}$	$\{1,2,3,4,5\}$
2	$\varnothing$	$\varnothing$	$\{2\}$	$\{2,3\}$	$\{2\}$	$\{2,3,4,5\}$
3	$\varnothing$	$\varnothing$	$\varnothing$	$\{3\}$	$\varnothing$	$\{3,4\}$
4	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\{4\}$
5	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\{5\}$
6	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$

Table 1: Precedence Partial Hypergroupoid

conditions i-v of Definition 3.1. So, it is a precedence hyperoperation with element 1 as starting element and element 6 as ending element.

Proposition 3.1. *Given a precedence partial hypergroupoid $(V, *_P)$, for all $x, y \in V$ the following hold:*

i. $x *_P y \neq \emptyset \Rightarrow y *_P x = \emptyset$;

ii. $x *_P a = \emptyset = b *_P x$.

Proof. i. Let $x *_P y \neq \emptyset$ and $y *_P x \neq \emptyset$. Then, according to Definition 3.1-i we get $x \in x *_P y$ and $y \in y *_P x$. But $x \in x *_P y$ and Definition 3.1-iv imply that $x \in a *_P y$. Similarly, since $y \in y *_P x = (a *_P x) \cap (y *_P b)$, it holds $y \in a *_P x$ and according to Definition 3.1-v we get either $x \in y *_P b$ or $x = b$.

In case $x \in y *_P b$, it holds $x \in (a *_P y) \cap (y *_P b)$, which is a contradiction according to (4).

In case $x = b$, $x \in x *_P y$ and Definition 3.1-iv imply that $b \in b *_P b$, which is a contradiction according to (1).

ii. For $x = a$ it is obvious that $x *_P a = \emptyset$. Moreover, let $x \in V - \{a\}$. Then, according to (2), $a \in a *_P x$ and from Proposition 3.1-i we obtain that $x *_P a = \emptyset$.

For $x = b$ it is obvious that $b *_P x = \emptyset$. Moreover, let $x \in V - \{b\}$. Then, according to (3), $x \in x *_P b$ and from Proposition 3.1-i we obtain that $b *_P x = \emptyset$.

□

Remark 3.2. Considering the multiplicative table of a precedence partial hypergroupoid $(V, *_P)$ one can see the following:

- All diagonal results of the table are $\emptyset$. Moreover, the starting element a belongs to every result of its row (except the diagonal one) and every element i (except the ending element b) belongs to the result, which appears at its own row and the column of the ending element b (s. Remark 3.1).
- All results in the column of the starting element a, as well as all results in the row of the ending element b are $\emptyset$. Moreover, in case that an (i, j) cell of the multiplicative table is not $\emptyset$, then the diagonal symmetric cell is equal to $\emptyset$ (s. Proposition 3.1).

Proposition 3.2. *Given a precedence partial hypergroupoid* $(V, *_P)$ *the following hold:*

i. *the starting element is unique;*

ii. *the ending element is unique;*

iii. $V *_P y = \bigcup_{x \in V} x *_P y = a *_P y$, *for all* $y \in V$;

iv. $x *_P V = \bigcup_{y \in V} x *_P y = x *_P b$, *for all* $x \in V$;

v. $a \notin x *_P y$, *for all* $x \in V - \{a\}, y \in V$;

vi. $b \notin x *_P y$, *for all* $x, y \in V$.

Proof. i. Let a_1, a_2 be both starting elements of $(V, *_P)$, $a_1 \neq a_2$. Then, (2) implies that $a_1 \in a_1 *_P a_2$ and $a_2 \in a_2 *_P a_1$. So, $a_1 *_P a_2 \neq \emptyset$ and $a_2 *_P a_1 \neq \emptyset$, which is a contradiction, according to Proposition 3.1-i.

ii. Let b_1, b_2 be both starting elements of $(V, *_P)$, $b_1 \neq b_2$. Then, (4) implies that $b_1 \in b_1 *_P b_2$ and $b_2 \in b_2 *_P b_1$. So, $b_1 *_P b_2 \neq \emptyset$ and $b_2 *_P b_1 \neq \emptyset$, which is a contradiction, according to Proposition 3.1-i.

iii. Definition 3.1-iv implies that $x *_P y \subseteq a *_P y$, for all $x, y \in V$. So, $\bigcup_{x \in V} x *_P y \subseteq a *_P y$, for all $y \in V$. Furthermore $a *_P y \subseteq \bigcup_{x \in V} x *_P y$. So, $\bigcup_{x \in V} x *_P y = a *_P y$.

iv. Definition 3.1-iv implies that $x *_P y \subseteq x *_P b$, for all $x, y \in V$. So, $\bigcup_{y \in V} x *_P y \subseteq x *_P b$, for all $x \in V$. Furthermore $x *_P b \subseteq \bigcup_{y \in V} x *_P y$. So, $\bigcup_{y \in V} x *_P y = x *_P b$.

v. Let $x \neq a$. Since $a \in a *_P x$ (s. (3)), according to (4), we get $a \notin x *_P b$. Then, $a \notin x *_P y \subseteq x *_P b$ (s. Proposition 3.2-iv).

vi. Let x, y be two elements of V such that $b \in x *_P y$. Then, Proposition 3.2-iv implies that $b \in x *_P b$, which is a contradiction according to Definition 3.1-i.

□

Remark 3.3. Considering the multiplicative table of a precedence partial hypergroupoid $(V, *_P)$ the unique starting element a (resp. the unique ending element b) may correspond to the first row and the first column (resp. the last row and the last column) of the table. Then one can see the following:

- All results of a column are subsets of the first result of this column (i.e., for all columns except the first column the following holds: every result of a column -except the first result- is either the empty set or a proper subset of the first result of this column).
- All results of a row are subsets of the last result of this row (i.e., for all rows except the last row the following holds: every result of a row -except the last result- is either the empty set or a proper subset of the last result of this row). Furthermore, every element, except the ending element b, belongs to the last result of its own row.
- The starting element a belongs to all results of the first row, except the diagonal one, and to no other result.
- The ending element b belongs to no results.

Definition 3.2. *Let $(V, *_P)$ be a precedence partial hypergroupoid with starting vertex a and ending vertex b. The set $V *_P x$ is called "ancestors' set of x" and the set $(x *_P V - \{x\}) \cup \{b\}$ is called "descendants' set of x". The integer*

$card[(x *_P V - \{x\}) \cup \{b\}] = card(x *_P V)$ is called "number of descendants of x". For every $i, j \in V$ with $card(i *_P V) > card(j *_P V)$ we say that "vertex i has more descendants than vertex j" or equivalently we say that "vertex j has fewer descendants than vertex i". If $card(i *_P V) = card(j *_P V)$ we say that "vertices i and j have both the same number of descendants".

Proposition 3.3. *Given a precedence partial hypergroupoid $(V, *_P)$ the following hold:*

i. $a *_P b = V - \{b\}$;

ii. $x *_P y \neq \emptyset \Rightarrow \begin{cases} x \in a *_P y \\ y \in x *_P b \end{cases}$ *, for all* $x, y \in V,\ y \neq b$;

iii. $x *_P y \neq \emptyset \Rightarrow \begin{cases} x \notin y *_P b \\ y \notin a *_P x \end{cases}$ *, for all* $x, y \in V$;

iv. the starting element and the ending element coincide, only if V is a singleton.

Proof. i. Proposition 3.2-vi implies that $a *_P b \subseteq V - \{b\}$. Let now $x \in V - \{b\}$. Then $x \in x *_P b$ (s. Definition 3.1-iii). According to Proposition 3.2-iii we have $x *_P b \subseteq a *_P b$. So, $x \in a *_P b$, i.e. $V - \{b\} \subseteq a *_P b$. Consequently, $a *_P b = V - \{b\}$.

ii. $x *_P y \neq \emptyset$ implies that $x \in x *_P y$ (s. Definition 3.1-i). Then, according to Definition 3.1-iv we get $x \in a *_P y$. On the other hand, setting $u = x$ in Definition 3.1-v we get $y \in x *_P b$.

iii. It is obvious from (4), Proposition 3.3-ii and Proposition 3.2-vi.

iv. In case $a = b$, since $a *_P b = a *_P a = \emptyset$, but also $a *_P b = V - \{b\}$ (s. Proposition 3.3-i) we obtain that V is a singleton, i.e. $V = \{a\}$.

□

Remark 3.4. Considering the multiplicative table of a precedence partial hypergroupoid $(V, *_P)$ with the starting element a (resp. the ending element b) in the first row and the first column (resp. in the last row and the last column) of the table, one can see the following:

- If the result at an arbitrary (x, y) cell is not $\emptyset$, then the element x belongs to the first result of the y-column and the element y belongs to the last result of the x-row, i.e. x belongs to the ancestors' set of y and y belongs to descendants' set of x.
- If element x precedes element y, then y does not precede x.

Proposition 3.4. *Let V be a non empty set with $card(V) = n$.*

i. For $n = 1$, the only precedence partial hypergroupoid defined on V is the degenerative one.

ii. For $n \in \{2, 3\}$, one precedence partial hypergroupoid is defined on V.

iii. For $n = 4$, three precedence partial hypergroupoids are defined on V.

Proof. i. Let $n = 1$ i.e. $V = \{1\}$. In a precedence partial hypergroupoid $(V, *_P)$ it holds $1 *_P 1 = \emptyset$. So, $(V, *_P)$ is the degenerative one.

ii. Let $n = 2$ i.e. $V = \{1, 2\}$. Let element 1 be the starting element and element 2 be the ending element. According to Proposition 3.1-ii and Proposition 3.3-i the only precedence partial hypergroupoid $(V, *_P)$ defined on V has the multiplicative Table 2.

	1	2
1	$\emptyset$	$\{1\}$
2	$\emptyset$	$\emptyset$

Table 2: Precedence partial hypergroupoid defined on $V = \{1, 2\}$

Similarly, if $n = 3$ i.e. $V = \{1, 2, 3\}$ with element 1 as starting element and element 3 as ending element, the only precedence partial hypergroupoid $(V, *_P)$ defined on V has the multiplicative Table 3.

	1	2	3
1	$\emptyset$	$\{1\}$	$\{1,2\}$
2	$\emptyset$	$\emptyset$	$\{2\}$
3	$\emptyset$	$\emptyset$	$\emptyset$

Table 3: Precedence partial hypergroupoid defined on $V = \{1, 2, 3\}$

iii. Let $n = 4$ i.e. $V = \{1, 2, 3, 4\}$ and $(V, *_P)$ be a precedence partial hypergroupoid defined on V. Let also element 1 be the starting element and element 4 be the ending element of $(V, *_P)$. Then the multiplicative table of $(V, *_P)$ looks like Table 4 because

- since element 1 is its starting element, it belongs to every result of its own row, except the diagonal one, and all results of its own column are $\emptyset$,
- since element 4 is its ending element, every result of its column contains the element, which corresponds to the row of the result, except the diagonal result; furthermore, all results of its own row are $\emptyset$,

	1	2	3	4
1	∅	{1, ?}	{1, ?}	{1, 2, 3}
2	∅	∅	?	{2, ?}
3	∅	?	∅	{3, ?}
4	∅	∅	∅	∅

Table 4: Precedence partial hypergroupoid defined on $V = \{1, 2, 3, 4\}$

- all diagonal results are $\emptyset$,
- $1 *_P 4 = \{1, 2, 3\} = V - \{4\}$,

where

- $\{1, ?\}$ denotes that the corresponding result contains element 1, but it may contain other elements too,
- ? denotes that the corresponding result is totally unknown.

So, we have the following options:

iii-(1) $2 *_P 3 = \emptyset$ and $3 *_P 2 = \emptyset$, which leads to the partial hypergroupoid $(V, *_P)$ given by Table 5.

	1	2	3	4
1	∅	{1}	{1}	{1, 2, 3}
2	∅	∅	∅	{2}
3	∅	∅	∅	{3}
4	∅	∅	∅	∅

Table 5: 1st precedence partial hypergroupoid defined on $V = \{1, 2, 3, 4\}$

iii-(2) $3 *_P 2 \neq \emptyset$, which leads to the partial hypergroupoid $(V, *_P)$ given by Table 6.

We notice that $3 *_P 2 \neq \emptyset$ leads to $2 *_P 3 = \emptyset$, according to Proposition 3.1-i. It also holds $3 \in 1 *_P 2$ and $2 \in 3 *_P 4$, according to Proposition 3.3-ii. Then $3 *_P 2 = (1 *_P 2) \cap (3 *_P 4)$.

iii-(3) $2 *_P 3 \neq \emptyset$, which leads to the partial hypergroupoid $(V, *_P)$ given by Table 7.

We notice that $2 *_P 3 \neq \emptyset$ leads to $3 *_P 2 = \emptyset$, according to Proposition 3.1-i. It also holds $2 \in 1 *_P 3$ and $3 \in 2 *_P 4$, according to Proposition 3.3-ii. Then $2 *_P 3 = (1 *_P 3) \cap (2 *_P 4)$.

	1	2	3	4
1	$\varnothing$	$\{1,3\}$	$\{1\}$	$\{1,2,3\}$
2	$\varnothing$	$\varnothing$	$\varnothing$	$\{2\}$
3	$\varnothing$	$\{3\}$	$\varnothing$	$\{2,3\}$
4	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$

Table 6: 2nd precedence partial hypergroupoid defined on $V = \{1,2,3,4\}$

	1	2	3	4
1	$\varnothing$	$\{1\}$	$\{1,2\}$	$\{1,2,3\}$
2	$\varnothing$	$\varnothing$	$\{2\}$	$\{2,3\}$
3	$\varnothing$	$\varnothing$	$\varnothing$	$\{3\}$
4	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$

Table 7: 3rd precedence partial hypergroupoid defined on $V = \{1,2,3,4\}$

In all three options results $2 *_P 4$ and $3 *_P 4$ are obtained through Proposition 3.2-iv, Proposition 3.3-iii and Remark 3.1-(3).

One can see that the previous three hyperoperations satisfy the conditions of Definition 3.1. So, they are precedence hyperoperations. Consequently, the only precedence partial hypergroupoids that can be defined on $V = \{1,2,3,4\}$ are the precedence partial hypergroupoids described by the previous three tables.

□

4. Precedence graphs

Definition 4.1. *Given a precedence partial hypergroupoid* $(V, *_P)$ *we construct a binary relation* $R \subseteq V \times V$ *in the following way:*
For every $x, y \in V$*, it holds*

$$(x, y) \in R \quad if\ and\ only\ if \quad x *_P y \quad is\ a\ singleton. \tag{5}$$

The above binary relation R *defines a directed graph* $G = (V, R)$ *called a "precedence graph associated to the precedence partial hypergroupoid* $(V, *_P)$*".*

Vice versa, given a precedence graph $G = (V, R)$ we can assume the precedence partial hypergroupoid $(V, *_P)$ that defined the graph $G = (V, R)$. Indeed the hyperoperation $*_P : V \times V \to \wp(V)$ defined as follows:
For every $x, y \in V$

$$x *_P y := \{\text{all vertices of all paths of graph } G \text{ from } x \text{ to } y,\ \text{except } y\} \tag{6}$$

defines a precedence partial hypergroupoid $(V, *_P)$, whose associated precedence graph is $G = (V, R)$.

Remark 4.1. In case $V = \{1, 2, 3, 4\}$ all three precedence partial hypergroupoids $(V, *_P)$ (s. Proposition 3.4-iii) define the precedence graphs shown in Figure 2.

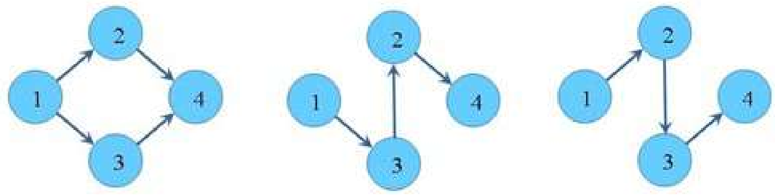

Figure 2: Precedence graphs defined by precedence partial hypergroupoids on $V = \{1, 2, 3, 4\}$

Given a graph $G = (V, R)$ the hyperoperation $*_P$ defined by (6) can check, if the graph is a precedence graph or not. In case the graph is not a precedence graph, the hyperoperation $*_P$ can derive a precedence graph from the given graph.

In the following we give two necessary conditions for a graph to be a precedence graph.

Proposition 4.1. *A precedence graph $G = (V, R)$ has no walks.*

Proof. Let $G = (V, R)$ be a precedence graph defined by a precedence hypergroupoid $(V, *_P)$ according to (5). Let $G = (V, R)$ has a walk. Every walk contains at least one circle. If i is a vertex of this circle, then $i \in i *_P i$ and so, $i *_P i \neq \emptyset$. Consequently, $(V, *_P)$ is not a precedence partial hypergroupoid (s. (1)) and graph $G = (V, R)$ is not a precedence graph. □

Proposition 4.2. *A precedence graph $G = (V, R)$ has only induced paths.*

Proof. Let $G = (V, R)$ be a precedence graph defined by a precedence hypergroupoid $(V, *_P)$ according to (5). Let also i, j be two nonadjacent vertices of a path connected by an edge i.e. $(i, j) \in R$. Then, there is at least one vertex x so that $\{i, x\} \subseteq i *_P j$. So, $i *_P j$ is not a singleton. Then (5) implies that $(i, j) \notin R$, which is a contradiction. □

5. Assembly line design using a precedence hyperoperation

Designing an assembly line one must first set the tasks [1] needed for the product to be assembled and then assign these tasks to workstations standing in a row.

1. Small element of work that cannot be conveniently fragmented further.

The fundamental problem of the assembly line design, the so-called "*assembly line balancing problem (ALB)*", is to manage this assignment to satisfy the precedence relations among the tasks and to minimize the idle time (s. [18]).

To illustrate the need of precedence relations, we usually give the standard example: "A bottle can't be filled, when the cap is already on." The precedence relations are usually visualized by a precedence graph first developed by Salveson [23].

Given a precedence graph and the task times [2] we must work out a process flow diagram, which is the final assignment of the tasks to workstations standing in a row. We assume that every workstation has the same time to complete its own set of tasks ("*Cycle Time*" or "*Task Time*"). If one task needs more time to be completed, more workers can be set at its workstation in order to reduce the working time. Furthermore some tasks with short task times can be assigned to the same workstation in order for the working time among the workstations to be balanced. So, we shall now concentrate on working out the assembly line design based on the precedence relations of the tasks only. For this purpose we shall use the precedence hyperoperation introduced in Section 3.

We consider as an example a "*one-model assembly line*" which consists of 8 tasks. The precedence relations among the tasks are given by the graph $G = (V, R)$ shown in Figure 3. In order for one unit of the model to be assembled all 8 tasks of the graph have to be completed.

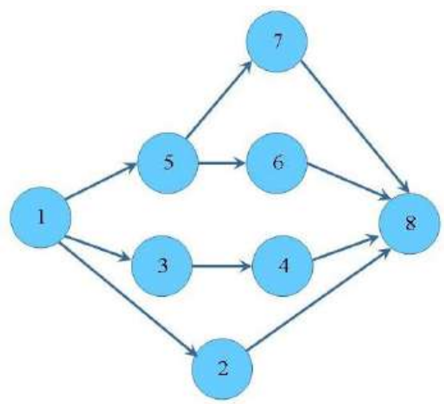

Figure 3: Example of a precedence graph in one-model assembly line

We define the precedence hyperoperation (6) on the set $V = \{1, 2, 3, 4, 5, 6, 7, 8\}$ of the vertices of the graph $G = (V, R)$. So, the associative precedence partial hypergroupoid $(V, *_P)$ is defined by Table 8, where $a = 1$ is the starting element and $b = 8$ the ending element. In what follows the elements of V are also called "*vertices*".

2. Time needed to complete one task by a well trained worker.

	1	2	3	4	5	6	7	8
1	$\varnothing$	$\{1\}$	$\{1\}$	$\{1,3\}$	$\{1\}$	$\{1,5\}$	$\{1,5\}$	$\{1,2,3,4,5,6,7\}$
2	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\{2\}$
3	$\varnothing$	$\varnothing$	$\varnothing$	$\{3\}$	$\varnothing$	$\varnothing$	$\varnothing$	$\{3,4\}$
4	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\{4\}$
5	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\{5\}$	$\{5\}$	$\{5,6,7\}$
6	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\{6\}$
7	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\{7\}$
8	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$

Table 8: $(V, *_P)$ defined by the graph $G = (V, R)$ (s. Figure 3)

In order to construct a process flow diagram we introduce the following algorithm, that we call "*Fewer-Descendants-Vertex First algorithm*" or shorter "*FDVF algorithm*":

- **Step 1.** We check all results $a *_P i$ of the starting vertex row and we collect all singletons. The corresponding vertices will be executed exactly after the starting vertex a. The order of the execution of these vertices will be chosen according to the increasing number of their descendants. So, we collect their complementary results to $a *_P b$ in a set I and we order the elements of I according to their increasing cardinality. If $i *_P b$ is the result we found with the minimum cardinality and $n = card(i *_P b)$, then vertex i is necessary for fewer of the following tasks and this is the vertex we will choose first after the starting vertex. Then, we continue with the rest vertices. If more than one elements of I have the same cardinality, then we choose an arbitrary one of the corresponding vertices.

  In our example, we check all results of the starting vertex row and we collect $1 *_P 2, 1 *_P 3, 1 *_P 5$. So, after vertex 1 we must execute the vertices 2, 3 and 5. To choose their order of execution, we find $card(2 *_P 8) < card(3 *_P 8) < card(5 *_P 8)$ and so, we choose vertex 2, since it is necessary for fewer of the following tasks. After vertex 2 we will execute vertex 3 and then vertex 5. In case $card(2 *_P 8) = card(3 *_P 8) = card(5 *_P 8)$ we would randomly choose the sequence of the vertices 2, 3 and 5. Consequently, our process flow diagram so far is shown in Figure 4.

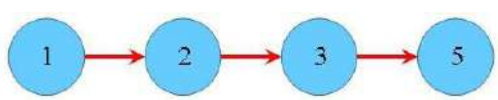

Figure 4: Process flow diagram after step 1 of FDVF algorithm

- **Step 2.** We check the number of descendants of the last executed vertex. If it is equal to 1, then we go to Step 4, otherwise we go to the next Step.

  In our example, we check $card(5 *_P 8) = 3 > 1$, so we go to the next step.

- **Step 3.** Now we check all results $i *_P b, j \neq b$, of the rows of the vertices executed in the previous step and we collect all singletons. The corresponding vertices will be executed exactly after the last executed vertex. The order of the execution of these vertices will be chosen according to the increasing number of their descendants. So, we collect their complementary results to $i *_P b$ in a set I and we order the elements of I according to their increasing cardinality. If more than one elements of I have the same cardinality, then we choose an arbitrary one of the corresponding vertices. Then we go to Step 2.

  In our example, we check all results of the rows 2, 3 and 5 executed in the previous step and we collect $3 *_P 4, 5 *_P 6, 5 *_P 7$. So, after vertex 5 we must execute the vertices 4, 6 and 7. To choose their order of execution, we find $card(4 *_P 8) = card(6 *_P 8) = card(7 *_P 8)$ and so, we randomly choose the sequence of the vertices 4, 6 and 7. Consequently, our process flow diagram so far is shown in Figure 5.

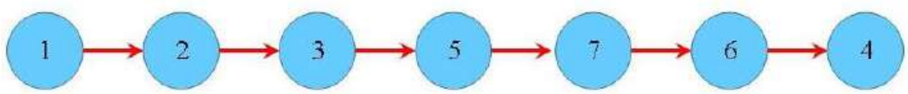

Figure 5: Process flow diagram after step 3 of FDVF algorithm

  We go to Step 2 and we check $card(4 *_P 8)$. Since $card(4 *_P 8) = 1$, we go to Step 4.

- **Step 4.** We execute the last vertex i.e. the ending vertex b.

  In our example, we execute vertex 8.

Consequently, our final process flow diagram is shown in Figure 6.

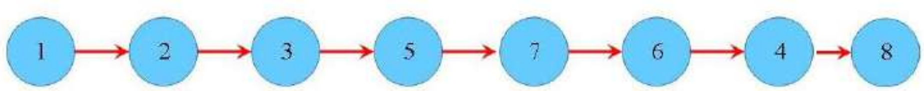

Figure 6: Process flow diagram of FDVF algorithm

Figure 7 shows the initial precedence graph (on the left) and the process flow diagram constructed by the FDVF-algorithm (on the right):

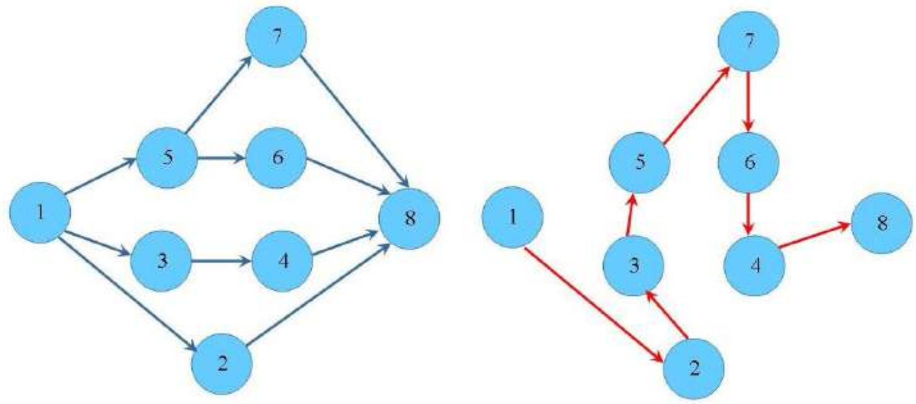

Figure 7: Process flow diagram of FDVF algorithm

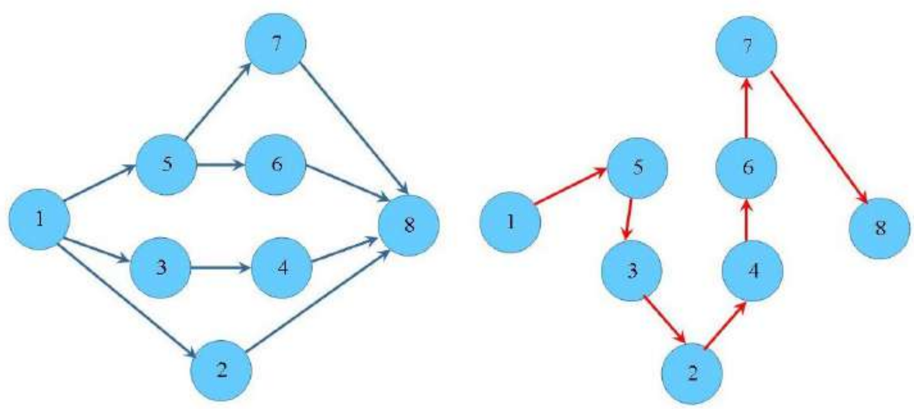

Figure 8: Process flow diagram of MDVF algorithm

Remark 5.1. The "*More-Descendants-Vertex First algorithm*" or shorter "*MDVF algorithm*" that can be defined in a similar way, will construct a similar process flow diagram illustrated in Figure 8.

One can choose FDVF or MDVF process flow diagram considering complementary benefits, such as raw material or skilled workers available at a time.

Remark 5.2. After the process flow diagram design follows the final assignment of the tasks to an ordered sequence of workstations. Some tasks with short task times can be assigned to the same workstation in order for the working time among the workstations to be balanced.

Figure 9 shows an example of a final assignment of the tasks of the process flow diagram of Figure 6 to an ordered sequence of workstations. Every square represents a workstation. In this example task times of 2 and 3 (resp. 6 and 4) supposed to be short enough, so that their sum is almost equal to the task time of the previous and the next workstation. This is a reason why tasks 2 and 3 (resp. 6 and 4) are assigned to the same workstation.

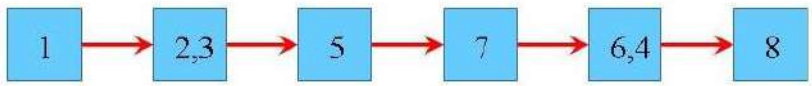

Figure 9: Assignment of the tasks to an ordered sequence of workstations

Remark 5.3. In our case study we examined "*a one-model assembly line*" i.e. in order for one unit of a model to be assembled all tasks of the precedence graph have to be completed. In case of "*a mixed-model assembly line*" we follow the same procedure: Every product unit passes through all tasks of the precedence graph (i.e. all tasks of the process flow diagram) and just skip the tasks that are not required for its specific model.

Discussion

Our research proposes an algorithm of constructing a process flow diagram starting from a precedence partial hypergroupoid of a precedence graph. The examples given deal with sets of tasks with small cardinalities. However, in the real-world implementation we have to deal with sets of tasks with big cardinalities. Then, the multiplicative tables of the associative precedence partial hypergroupoids are big, although they are "half" empty. This fact may put limitations on the application of the algorithm. A solution to the problem would be the partition of the total precedence graph into subgraphs and the application of the algorithm to each subgraph. Then, each subgraph could be considered as a simple task of the total precedence graph and the algorithm could be applied to the simplified form of the total precedence graph.

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On τ-supplemented Krasner hypermodules

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Abstract. In this study we define the radical of the Krasner hypermodules in the subcategory ${}_{R_S}hmod$, then we use short exact sequences in homological algebra for Krasner hypermodules. Besides, by studying the concept of τ-supplements in module theory we will generalize it to the Krasner R-hypermodules by using short exact sequences and a subcategory of ${}_{R_S}hmod$.
Keywords: subcategory of hypermodules, preradical, supplemented hypermodules, τ-supplemented hypermodules.
MSC 2020: 20N20, 16D80.

1. Introduction

In this text, we assume that all hypermodules are left Krasner R-hypermodules, all hyperrings are Krasner hyperrings, and all homomorphisms are strong R-homomorphisms. In order to formalize the theory of hypercomposition algebra, a new branch of abstract algebra began its development in 1934, when F. Marty introduced the concept of hypergroups as a proper generalization of the concept of groups. The combination of two elements is extended by replacing the operation defined in groups with a multivalued operation (or hyperoperation), with the result being a subset of the fundamental set. As a result, new algebraic hypercompositional structures and the properties of classical structures are defined and the properties of classical structures are defined while the properties of classical structures are preserved or not for similar hyperstructures. This also applies to modules and expanded into hypermodules by Krasner [15] today

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known as Krasner hypermodules. Their additive part is canonical hypergroup. Fundamental aspects of hypermodule theory are covered in [3], [4], [5], [7], [8], [9],[18], [22] and [23]. Recently, the concept of smallness in module theory has been carried over and investigated in [13] on the class of hypermodules in the context of similarities and differences. The concept of small subhypermodules was defined in [6]. An R-subhypermodule K of H is said to be *small* in H if $K+T=H$ for each subhypermodule T of H, implies $T=H$, and it is denoted by $K \ll H$. An R-hypermodule H is said to be *hollow* if each subhypermodule is small in H [13]. An R-hypermodule H is said to be *local* if H has a proper subhypermodule that contains all proper subhypermodules of H [12]. A hypermodule H is said to be *supplemented* if for each subhypermodule K of H, there is a subhypermodule T of H provided that $H=K+T$ and $K \cap T \ll T$ [13]. Here T is said to be the*supplement* of K in H.

According to this study the class of τ-supplemented hypermodules has been classified in a category theory. In this way, the functor τ was obtained and the class of τ-supplemented hypermodules was defined. In section 2, the algebraic and categorical properties of Krasner hypermodules will be included and the concept of supplement in subhypermodules will be emphasized. In section 3, (pre)radicals will be presented in the ${}_{R_S}hmod$, which is a special hypermodule subcategory. In section 4, τ-supplemented hypermodules are examined.

2. Preliminaries

This section briefly reviews the basic concepts and results on Krasner hypermodules, which are used throughout this article for clarity. We start with relevant basic definitions of the topic in hypercomposition algebra, which are presented in several books [10], [11] and overview articles [2], [19], [20] and [21]. We also refer the reader to [1] for generalizing the concept of τ-supplement in module theory and the studies related to the category of hypermodules.

Hypermodules. Let H be a non-empty set and $\mathcal{P}^*(H)$ be the set of all non-empty subsets of H. The function $\circ : H \times H \longrightarrow \mathcal{P}^*(H)$ is called a *hyperoperation* on H. The image of the element $(a,\, b) \in H \times H$ under this operation is not a single element, but a non-empty subset of the set H. Thanks to this idea, the theory of hyperstructures was introduced by Marty in [17] as a natural and interesting generalization of the theory of algebraic structures. Following [17], Marty defines hypergroups using the hyperoperation on a set. Let H be a non-empty set and a function $+ : H \times H \longrightarrow \mathcal{P}^*(H)$ be a hyperoperation on H. Then, $(H,\, +)$ is called a *hypergroupoid*. Moreover, for any non-empty subsets X and Y of H, define

$$X+Y=\bigcup\{x+y \,|\, x \in X \text{ and } y \in Y\}.$$

We simply write $a+X$ and $X+a$ instead of $\{a\}+X$ and $X+\{a\}$, respectively, for any $a \in H$ and any non-empty subset X of H. A hypergroupoid $(H,+)$ is called a

(1) *Semihypergroup* if for every $a, b, c \in H$, we have $a + (b + c) = (a + b) + c$;

(2) *Quasihypergroup* if for every $x \in H$, $x + H = H = H + x$.

If the hypergroupoid $(H, +)$ is a semihypergroup and quasihypergroup, then it is called a *hypergroup*. A non-empty subset S of a hypergroup $(H, +)$ is called a *subhypergroup* of H if for every $a \in S$, $a + S = S = S + a$, written as $S \leq H$.

A hypergroup $(H, +)$ is called a *canonical hypergroup* if

(1) for every $a, b \in H$, $a + b = b + a$, that is, it is commutative;

(2) there exists a unique $0 \in H$ such that for each $a \in H$ there exists a unique element a' in H, denoted by $-a$, such that $0 \in a + (-a)$;

(3) for every $a, b, c \in H$, if $c \in a + b$, then $a \in c + (-b) := c - b$.

If $(H, +)$ is a canonical hypergroup, then $a + 0 = a$ for all $a \in H$.
A hyperstructure $(R, +, \cdot)$ is called a *(Krasner) hyperring* if

(1) $(R, +)$ is a canonical hypergroup.

(2) $(R, \cdot)$ is a monoid with a bilaterally absorbing element 0, i.e.,

 (a) $a \cdot b \in R$ for all a, $b \in R$;
 (b) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all a, b, $c \in R$;
 (c) $a \cdot 0 = 0 \cdot a = 0$ for all $a \in R$;
 (d) There exists an *identity element* $1_R \in R$ such that $a = a \cdot 1_R = 1_R \cdot a$ for every $a \in R$.

(3) The multiplication distributes over the addition on both sides.

A hyperring $(R, +, \cdot)$ is called *commutative* if it is commutative concerning the multiplication.

Let $(R, +, \cdot)$ be a hyperring and I be a non-empty subset of R. I is called a *left hyperideal* (respectively, *right hyperideal*) of R provided $(I, +)$ is a subhypergroup and $ra \in I$ (respectively, $ar \in I$) for all $a \in I$, and $r \in R$. I is said to be a *hyperideal* of R if it is both a right and a left hyperideal of R.

A left Krasner hypermodule over a hyperring R (or left Krasner R-hypermodule) is a canonical hypergroup $(M, +)$ together with a map $\cdot : R \times M \longrightarrow M$ such that to every (r, m), where $r \in R$ and $m \in M$, there corresponds a uniquely determined element $rm \in M$ and the following conditions are satisfied:

(1) $r(m_1 + m_2) = rm_1 + rm_2$;

(2) $(r + s)m = rm + sm$;

(3) $(rs)m = r(sm)$;

(4) $1_R m = m$ and $r0_M = 0_R m = 0_M$,

for any m, m_1, $m_2 \in M$ and r, $s \in R$. Throughout this paper, for a simple explanation, when we say hypermodule, we mean the left Krasner hypermodule. A non-empty subset N of an R-hypermodule M is called a *subhypermodule* of M, denoted by $N \leq M$ if N is an R-hypermodule under the same hyperoperations of M. It is clear that M and $\{0_M\}$ are *trivial subhypermodules* of M. It is known that a non-empty subset N of an R-hypermodule M is a subhypermodule of M if and only if $a - b \subseteq N$ and $ra \in N$ for all a, $b \in M$ and $r \in R$.

Let R be a hyperring. It follows that R is an R-hypermodule. Then, a non-empty subset I of R is a left hyperideal of R if and only if it is a subhypermodule of the hypermodule $_RR$.

Let M be a hypermodule over a hyperring R and K be a subhypermodule of M. Consider the set $\frac{M}{K} = \{\, a + K \mid a \in M \,\}$. Then, $\frac{M}{K}$ is a hypermodule over the hyperring R under the hyperoperation defined as $+ : \frac{M}{K} \times \frac{M}{K} \longrightarrow \mathcal{P}^*(\frac{M}{K})$ and the external operation $\odot : R \times \frac{M}{K} \longrightarrow \frac{M}{K}$ defined as $(a + K) + (a' + K) = \{\, b + K \mid b \in a + a' \,\}$ and $r \odot (a + K) = ra + K$ for every $a, a', b \in M$ and $r \in R$. The hypermodule $\frac{M}{K}$ is called the *quotient hypermodule* of the hypermodule M.

Let K be an R-hypermodule and $x \in K$. Then, the subset $Rx = \{\, ux \mid u \in R \,\}$ of K is a subhypermodule of K. Let R be a hyperring and M be an R-hypermodule. For a family of subhypermodules $\{M_i\}_{i \in I}$ of M, the sum of this family is denoted by $\sum_{i\in I} M_i$ and it is the set of these elements $x \in M$ where x is an element of the set $\sum_{i \in I_0} m_i$ with finite subset $I_0 \subseteq I$ for every $i \in I_0$, $m_i \in M_i$. That is,

$$\sum_{i\in I} M_i = \{x \in M \mid x \in \sum_{i\in I_0} m_i,\ m_i \in M_i \ and\ I_0\ is\ a\ finite\ subset\ of\ I\}.$$

It is well known that $\sum_{i\in I} M_i$ is a subhypermodule of M.

Lemma 2.1 (Modular Law [25])**.** *Let H be a Krasner R-hypermodule and K, T, L are subhypermodules of H with $T \leq K$. Then, $K \cap (T + L) = T + (K \cap L)$.*

Let $(K, +_1)$ and $(T, +_2)$ be two R-hypermodules. A function $\psi : K \longrightarrow T$ is said to be an R-homomorphism from K to T if satisfies the following two conditions:

1. $\psi(x +_1 y) \subseteq \psi(x) +_2 \psi(y)$;
2. $\psi(ux) = u\psi(x)$,

for each $u \in R$ and $x, y \in K$. If the inclusion (1) is an equality, then ψ is said to be a *strong* (or good) *R-homomorphism*.

Categories of hypermodules. Throughout the whole text we denoted by $R - hmod$ the category whose objects are whole R-hypermodules and whose morphisms are all R-homomorphisms. The class of all R-homomorphism from

K to T is indicated by $hom_R(K,T)$. Also, $R_S - hmod$ is the category of all R-hypermodules whose morphisms are all strong R-homomorphisms. The class of all strong R-homomorphisms is denoted by $hom_R^S(K,T)$. It is clear that $R_S - hmod$ is a subcategory of $R - hmod$, this situation is indicated by the writing $R_S - hmod \preceq R - hmod$. The hyperset category denoted by $HSets$ is a category with the following:

1. $Ob(HSets) = Ob(Sets)$;
2. $Mor(HSets)$ is the class of all multivalued functions between objects in which the composition $\theta \circ \psi$ is defined as $(\theta \circ \psi)(x) = \bigcup_{y\in\psi(x)} \theta(y)$ for each $x \in K$, and an identity morphism for an object K for all $z \in K$, $id_K(z) = \{z\}$.

Let $(K, +_1)$ and $(T, +_2)$ be R-hypermodules and ψ a multi-valued function from K to T, denoted by $\psi : K \longrightarrow P^*(T)$ providing the following statements:

1. $\psi(x +_1 y) \subseteq \psi(x) +_2 \psi(y)$;
2. $\psi(ux) = u\psi(x)$,

for each $u \in R$, and $x, y \in K$. Then, ψ is said to be a *multi-valued R-homomorphism*, shortly *R_{mv}-homomorphism*. If equality is achieved instead of coverage in the first of the above two properties, then ψ is called a *strong (or good) multi-valued R-homomorphism*, shortly a *R_{smv}-homomorphism*. We denote the class of whole R_{mv}-homomorphisms (resp., R_{smv}-homomorphisms) from K to T as $Hom_R(K,T)$ (resp., $Hom_R^S(K,T)$). $R - Hmod$ (resp., $R_S - Hmod$) specifies a category whose objects are all R-hypermodules and whose morphisms from K to T are whole R_{mv}-homomorphisms (resp., R_{smv}-homomorphisms) from K to T. Obviously, $R_S - Hmod$ is a subcategory of $R - Hmod$, that is, $R_S - Hmod \preceq R - Hmod$.

Let $\psi, \theta \in Hom_R(K,T)$ where $\leq$ is the relation on $Hom_R(K,T)$ defined as $\psi \leq \theta$ if for each $x \in K$, $\psi(x) \subseteq \theta(x)$. Here $(Hom_R(K,T), \leq)$ is a partially ordered set. Let $\psi, \theta, \eta \in Hom_R(K,T)$. Let $+ : Hom_R(K,T) \times Hom_R(K,T) \longrightarrow Hom_R(K,T)$ be an operation on $Hom_R(K,T)$ which is defined by $(\psi, \theta) \mapsto \psi + \theta$ where $(\psi + \theta)(x) = \psi(x) +_2 \theta(x)$ for each $x \in K$ in which $+_2$ is the hyperoperation of the canonical hypergroup $(T, +_2)$. Here $\eta \leq \psi + \theta$ if and only if $\eta(x) \subseteq \psi(x) +_2 \theta(x)$ for each $x \in K$. A hyperoperation on $Hom_R(K,T)$ is defined as follows:

$$\psi \biguplus \theta = \{\, \eta \in Hom_R(K,T) \,|\, \eta \leq \psi + \theta \,\}.$$

The hyperoperation $\uplus$ on $hom_R(K,T)$ is defined by being restricted in the following format

$$\psi \biguplus \theta = \{\, \eta \in hom_R(K,T) \,|\, \eta(x) \in \psi(x) + \theta(x), \text{ for each } x \in K\}.$$

In [3, Theorem 3.11], it has been proved that $(Hom_R(K,T), \uplus)$ is a commutative hypermonoid and $(hom_R(K,T), \uplus)$ is a canonical hypergroup for objects K and T in categories respectively $R - Hmod$ and $R - hmod$.

Supplement subhypermodules of hypermodules. Let K and T be subhypermodules of an R-hypermodule H. Then, both subhypermodules K and T are called *independent* if $K \cap T = \{0_H\}$, so $K + T$ is denoted by $K \oplus T$ and it is called as *internal direct sum*. In addition, a subhypermodule K of H is said to be a *direct summand* of H provided that $H = K \oplus L$ for some subhypermodule L of H ([25]). A non-zero R-hypermodule H is said to be *simple* provided that the only subhypermodules of H are $\{0_H\}$ and H itself. Let R be a hyperring and H be an R-hypermodule. By $Soc(H)$ we denote the sum of all simple subhypermodules of H. An R-hypermodule H is said to be *semisimple* provided that its subhypermodules are direct summands in H ([16]). In [27], it is introduced the concept of semisimple R-hypermodule H as $H = Soc(H)$, that is, it is the sum of simple R-subhypermodules of H. It was shown that the class of semisimple hypermodules was closed under internal direct sums, factor hypermodules, and subhypermodules. It is proven in [27, Proposition 3] that $Soc(T)$ is the largest semisimple subhypermodule of a hypermodule H. The various properties of semisimple hypermodules are given in [27]. In particular, it is shown in [27, Theorem 1] that a hypermodule H is an internal direct sum of subhypermodules K and T, that is, $H = K \oplus T$ if and only if, for every element $m \in H$, there exists a unique element $k \in K$ and a unique element $t \in T$ such that m is an element of the set $k + t$. Let I be a left hyperideal and R be a hyperring. If $I \cap J \neq 0$ for every non-zero left hyperideal J of R, then I is called *essential left hyperideal* of R. Then

$$Z(M) = \{m \in M \mid Im = 0_M \text{ for some essential left hyperideal } I \text{ of } R\}$$

is called a *singular subhypermodule* of M. In [27, Corollary 2], it is proven that every simple hypermodule M is normal projective or $M = Z(M)$.

The following example is given in [27, Example 10].

Example 2.1. Let $R = \{0, 1, 2, 3\}$ with hyperoperation "+" and operation ".":

+	0	1	2	3
0	{0}	{1}	{2}	{3}
1	{1}	{0,1}	{3}	{2,3}
2	{2}	{3}	{0}	{1}
3	{3}	{2,3}	{1}	{0,1}

and

·	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	2	2
3	0	0	2	2

Then, R is an R-hypermodule. It is easy to see that the only proper subhypermodules of R are $M_0 = \{0\}$, $M_1 = \{0, 1\}$, and $M_2 = \{0, 2\}$. Therefore, M_1 and M_2 are simple subhypermodules of M, and so we can write $R = M_1 + M_2$. This means that the R-hypermodule R is semisimple.

In [13], as a generalization class of semisimplicity, the class of supplemented hypermodules was defined. Let H be a Krasner R-hypermodule and K, T be subhypermodules of H. T is said to be a *supplement* of K in H provided that T is a minimal element in the set $\{ L \leq H \mid K + L = H \}$. Then, H is said to be *supplemented* provided that each subhypermodule of H has a supplement in H [13]. A subhypermodule K of a Krasner R-hypermodule H is said to be a *small subhypermodule* of H provided that $K + T \neq H$ for each proper subhypermodule T of H, denoted by $K \ll H$. So T is a supplement of K in H if and only if $K + T = H$ and $K \cap T \ll T$. In addition, K has ample supplements in H if, whenever $K + T = H$, T contains a supplement L of K in H. The Krasner R-hypermodule H is said to be *amply supplemented* provided that each subhypermodule has ample supplements in H (see [13]).

Example 2.2. Let $R = \{0, 1, 2\}$ and $A = \{0, 2\}$. Define the hyperaddition "+" and multiplication "$\cdot$" by the following:

+	0	1	2
0	{0}	{1}	{2}
1	{1}	R	{1}
2	{2}	{1}	A

and

·	0	1	2
0	0	0	0
1	0	1	2
2	0	2	0

Then, R is a hyperring and A is the only left maximal hyperideal of R. It follows that the left R-hypermodule R is (amply) supplemented which is not semisimple.

3. Preradicals in category of ${}_{R_S}hmod$

Our interest in this approach to the structure theory of hypermodules was noticed as a result of an observation mentioned in [1] and it was aimed to generalize to hypermodules by taking advantage of the existing category theory in module structure in [2]. In this context, we study on concept of (pre)radical for using of the concept of τ-supplement subhypermodules in category of ${}_{R_S}hmod$.

Definition 3.1. (1) *A functor τ from the category of $R_s - hmod$ to itself is called a* preradical *if following properties are satisfied:*

(1) *$\tau(H)$ is a subhypermodule of the R-hypermodule H.*

(2) *$\psi(\tau(H)) \subseteq \tau(L)$ for each strong homomorphism $\psi \in Hom_R^S(H, L)$.*

Let τ be a preradical of $R_s - hmod$, where R is a hyperring. Assume that $T \leq H$ are R-hypermodules. Applying the condition (2) of being preradical, it is clear that $\iota(\tau(T)) = \tau(T) \leq \tau(H)$, where $\iota : T \longrightarrow H$ is the inclusion map.

Definition 3.2. *Let τ be a preradical of $R_s - hmod$. Then, τ is called* idempotent *if $\tau(\tau(H)) = \tau(H)$ for every $H \in R_s - hmod$.*

Example 3.1. Let R be a hyperring and M be any R-hypermodule. By $Soc(M)$ we denote the sum of all simple subhypermodules of M as in [27]. It follows from [27, Proposition 3] that $Soc(M)$ is the largest semisimple subhypermodule of M. By [27, Lemma 8], we define the preradical $\tau : R_s - hmod \longrightarrow R_s - hmod$ by $\tau(M) = Soc(M)$ for all $M \in R_s - hmod$. Again applying [27, Proposition 3], we deduce that τ is the idempotent preradical of $R_s - hmod$.

Example 3.2. Let R be a hyperring and M be any R-hypermodule. By [27, Theorem 7], we consider the singular subhypermodule $Z(M)$ of M, that is

$$Z(M) = \{m \in M \mid Im = 0_M \ for\ some\ essential\ left\ hyperideal\ I\ of\ R\}.$$

Here, I is an essential left hyperideal of R if $I \cap J \neq 0$ for every non-zero left hyperideal J of R. For any R-hypermodules M, $N \in R_s - hmod$, let $f : M \longrightarrow N$ be any strong homomorphism and $m \in Z(M)$. Therefore, there exists an essential left hyperideal I of R such that $Im = 0_M$. It follows that $If(m) = f(Im) = 0_N$ and so $f(m) \in Z(N)$. It means that $Z(.)$ defines the preradical in $R_s - hmod$. Since $Z(Z(M)) = Z(M)$ for any hypermodule $M \in R_s - hmod$, the preradical $Z(.)$ is idempotent in $R_s - hmod$.

Example 3.3. Given the hyperring $R = \{0, 1, 2\}$ and the left hyperideal $A = \{0, 2\}$ in Example 2.2, define the preradical $\tau : R_s - hmod \longrightarrow R_s - hmod$ by $\tau(N) = AN$ for all $N \in R_s - hmod$, where $AN = \{m \in N | m \in \sum_{i=1}^{n} r_i m_i,\ m_i \in N,\ r_i \in A,\ 1 \leq i \leq n$ and $n \in \mathbb{Z}^+\}$. It follows that τ is preradical of $R_s - hmod$. Since $AA = \{0\}$, we deduce that τ is not idempotent preradical.

Definition 3.3. *Let τ be a preradical of $R_s - hmod$. Then, τ is called* radical *if $\tau(\frac{H}{\tau(H)}) = \{0\}$ for every $H \in R_s - hmod$.*

Let M be an R-hypermodule. By $Rad(M)$ we denote the intersection of all maximal subhypermodules of M. Now we shall show that $Rad(.)$ induces a radical of $R_s - hmod$. Firstly we need the following lemma.

Lemma 3.1. *Let M be a hypermodule and N be a subhypermodule of M. Assume that $\{M_i\}_{i \in I}$ is a family of subhypermodules of M containing N. Then, $\bigcap_{i \in I} \frac{M_i}{N} = \frac{\bigcap_{i \in I} M_i}{N}$.*

Proof. The proof is straightforward. $\square$

Example 3.4. Let R be any hyperring. Define $\tau : R_s - hmod \longrightarrow R_s - hmod$ by $\tau(M) = Rad(M)$ for all $M \in R_s - hmod$. It follows from [23] that τ is the preradical of $R_s - hmod$.

Next we show that τ is the radical of $R_s - hmod$. Let M be any hypermodule over the hyperring R and $\{M_i\}_{i\in I}$ be the set of all maximal subhypermodules of M. Since $Rad(M)$ is contained in all maximal hypermodules of M, we get that

$$Rad(\frac{M}{Rad(M)}) = \bigcap_{i\in I} \frac{M_i}{Rad(M)} = \frac{\bigcap_{i\in I} M_i}{Rad(M)} = 0$$

by Lemma 3.1. Hence, τ is radical.

Now we give the following lemma and we will use it throughout the paper.

Lemma 3.2. *For a hyperring R, let τ be a preradical of $R_s - hmod$. If $T \leq H$ are R-hypermodules, then $\frac{\tau(H)+T}{T} \leq \tau(\frac{H}{T})$. Moreover, if τ is radical and $T \leq \tau(H)$, $\tau(\frac{H}{T}) = \frac{\tau(H)}{T}$.*

Proof. Consider the canonical strong projection $\Psi : H \longrightarrow \frac{H}{T}$. Since τ is a preradical of $R_s - hmod$, we can write $\Psi(\tau(H)) = \frac{\tau(H)+T}{T} \subseteq \tau(\frac{H}{T})$ as required.

Let τ be a radical of $R_s - hmod$ and $T \leq \tau(H)$. Therefore, $\frac{\tau(H)+T}{T} = \frac{\tau(H)}{T} \subseteq \tau(\frac{H}{T})$. Note that

$$\frac{\frac{H}{T}}{\frac{\tau(H)}{T}} \cong \frac{H}{\tau(H)}.$$

Consider the canonical strong projection $\Phi : \frac{H}{T} \longrightarrow \frac{\frac{H}{T}}{\frac{\tau(H)}{T}}$. It follows that

$$\Phi(\tau(\frac{H}{T})) = \frac{\tau(\frac{H}{T}) + \frac{\tau(H)}{T}}{\frac{\tau(H)}{T}} \subseteq \tau(\frac{\frac{H}{T}}{\frac{\tau(H)}{T}}).$$

Since τ is a radical of $R_s - hmod$, $\tau(\frac{H}{\tau(H)}) = \{0\}$ and so $\tau(\frac{\frac{H}{T}}{\frac{\tau(H)}{T}}) = \{\frac{\tau(H)}{T}\}$. Thus, $\frac{\tau(\frac{H}{T})+\frac{\tau(H)}{T}}{\frac{\tau(H)}{T}} \subseteq \frac{\tau(H)}{T}$. It means that $\tau(\frac{H}{T}) = \frac{\tau(H)}{T}$. $\square$

4. τ-supplemented hypermodules

Recall that an R-module M is *τ-supplemented* if every submodule U of M has a τ-supplement V in M, that is, $M = U + V$ and $U \cap V \subseteq \tau(V)$, where τ is a preradical of $R - Mod$. Now we transfer this definition to hypermodules as follows:

Definition 4.1. *Let R be a hyperring and τ be a preradical of $R_s - hmod$. An R-hypermodule M is called* τ-supplemented *if every subhypermodule U has a τ-supplement V in M, that is, $M = U + V$ and $U \cap V \subseteq \tau(V)$. M is called* amply τ-supplemented *if for every subhypermodules K, T of H with $K + T = M$, there exists a τ-supplement V of K with $V \leq T$.*

It is clear that every semisimple hypermodule is (amply) τ-supplemented. Observe from Example 2.2 that the R-hypermodule R is a Soc-supplemented and Rad-supplemented hypermodule which is not semisimple.

Lemma 4.1. *Let M be an R-hypermodule and τ be a preradical of $R_s - hmod$. Assume that $M = \tau(M)$. Then, M is τ-supplemented.*

Proof. Let $U \leq M$. Therefore, $M = U + M$ and $U \cap M = U \subseteq M = \tau(M)$. It means that M is a τ-supplement of U in M. Hence, M is τ-supplemented. □

Theorem 4.1. *Let τ be a radical of $R_s - hmod$ and T be an R-hypermodule. Then, a subhypermodule K of T is a τ-supplement in T if and only if, for every strong homomorphism $g : K \longrightarrow L$ of hypermodules with $\tau(L) = 0$, there exists a strong homomorphism $h : T \longrightarrow L$ such that $g = h\iota$, where $\iota : K \longrightarrow T$ is the inclusion map.*

Proof. ($\Rightarrow$) Let K be a τ-supplement of some hypermodule U of T. Therefore, we can write $\frac{T}{\tau(K)} = \frac{K}{\tau(K)} \oplus \frac{U+\tau(K)}{\tau(K)}$. Assume that $g : K \longrightarrow L$ is a strong homomorphism of hypermodules, where $\tau(L) = 0$. It follows that there exists a strong homomorphism $h : T \longrightarrow \frac{K}{\tau(K)}$ with $h(x) = x + \tau(K)$ for all $x \in T$. Then, $g(\tau(K)) \subseteq \tau(g(K)) \subseteq \tau(L) = 0$ and thus the strong homomorphism $g : K \longrightarrow L$ factors through the canonical strong epimorphism $\pi : K \longrightarrow \frac{K}{\tau(K)}$ with $g = g^{'}\pi$, where $g^{'} : \frac{K}{\tau(K)} \longrightarrow L$. Put $\alpha = g^{'}h$. Hence, $\alpha\iota = g$.

($\Leftarrow$) Since τ is a radical of $R_s - hmod$, we have $\tau(\frac{K}{\tau(K)}) = 0$ and so, by assumption, there exists a strong homomorphism $h : T \longrightarrow \frac{K}{\tau(K)}$ with $h\iota = \pi$, where $\pi : K \longrightarrow \frac{K}{\tau(K)}$ is the canonical strong homomorphism. It follows that $h(k) = k + \tau(K)$ for all $k \in K$. Thus, h induces a strong homomorphism $f : \frac{T}{\tau(K)} \longrightarrow \frac{K}{\tau(K)}$ such that $f(k + \tau(K)) = h(k)$. So, $\frac{K}{\tau(K)}$ is a direct summand of $\frac{T}{\tau(K)}$ and then we can write $\frac{T}{\tau(K)} = \frac{K}{\tau(K)} \oplus \frac{U}{\tau(K)}$ for some subhypermodule $\frac{U}{\tau(K)}$ of $\frac{T}{\tau(K)}$. Hence, $T = K + U$ and $K \cap U \subseteq \tau(K)$. It means that K is a τ-supplement in T. □

Corollary 4.1. *Let τ be a radical for ${}_{R_S}hmod$ and $K \leq T$, where $T \in_{R_S} hmod$.*

1. *If K is a τ-supplement in T and $\tau(K) = \{0\}$, then K is a direct summand of T.*
2. *If $\tau(T) = \{0_H\}$, then each τ-supplement subhypermodule of T is a direct summand.*

3. *If K is a τ-supplement in T and $W \leq K$, then $\frac{K}{W}$ is a τ-supplement in $\frac{T}{W}$.*

Proof. (1) Let K be a τ-supplement of L in T. Therefore, $T = L + K$ and $L \cap K \subseteq \tau(K) = 0$ and so $T = L \oplus K$, as required.

(2) It follows from (1).

(3) Let $V \leq T$ satisfying conditions $K+V = T$ and $K \cap V \leq \tau(K)$. Then, we can write $\frac{K}{W} + \frac{V+W}{W} = \frac{T}{W}$ and $\frac{K}{W} \cap \frac{(V+W)}{W} = \frac{(K\cap V)+W}{W} \leq \frac{\tau(K)+W}{W} \leq \tau(\frac{K}{W})$. □

Recall from [12] that a subhypermodule K of H is called *essential* in H if $K \cap T = \{0_H\}$ implies $T = \{0_H\}$ for each nonzero subhypermodule T of H, denoted by $K \trianglerighteq H$.

Theorem 4.2. *Let $H \in_{R_S} hmod$ with a τ-supplemented hypermodule. Then, we have:*

1. *Each subhypermodule $K \leq H$ with $K \cap \tau(H) = \{0_H\}$ is a direct summand. Especially, if $\tau(H) = \{0_H\}$, then H is semisimple.*

2. *Each factor hypermodule and each direct summand of H is τ-supplemented.*

3. *$\frac{H}{\tau(H)}$ is a semisimple hypermodule.*

4. *$H = V \oplus T$ where T is semisimple and $\tau(V) \trianglerighteq V$.*

Proof. (1) Since $\tau(K) \leq K \cap \tau(H)$, it follows from [27, Theorem 9] and Corollary 4.1-(1).

(2) and (3) Clear by Corollary 4.1.

(4) Let $T \cap \tau(H) = \{0_H\}$ and $T \oplus \tau(H) \trianglerighteq H$. It follows that $\tau(T) = \{0_H\}$. Since H is τ-supplemented module, there is a subhypermodule V of H such that $T + V = H$ and $T \cap V \leq \tau(V)$. Then, $T \cap V = T \cap (T \cap V) \leq T \cap \tau(V) \leq T \cap \tau(H) = \{0_H\}$, $H = T \oplus V$ and $\tau(H) = \tau(T) \oplus \tau(V) = \tau(V)$. Therefore, $T \oplus \tau(V) \trianglerighteq T \oplus V$. It follows from $\tau(V) \trianglerighteq V$. By (1), T is semisimple. □

Theorem 4.3. *Let $H \in_{R_S} hmod$. Then, the following statements hold.*

1. *Let $K, V \leq H$, where K is τ-supplemented. If $K + V$ has a τ-supplement in H, then V has a τ-supplement in H.*

2. *If K and T are τ-supplemented, then $K + T$ is τ-supplemented.*

3. *Any finite sum of τ-supplemented hypermodules is τ-supplemented.*

Proof. (1) By hypothesis, there is $T \leq H$ provided that $(K+V)+T = H$ and $(K + V) \cap T \leq \tau(T)$. Since $(V + T) \cap K$ has a τ-supplement in K, we have $(V+T) \cap K + W = K$ and $(V+T) \cap W \leq \tau(W)$ for $W \leq K$. As $V+T+W = H$, W is a τ-supplement of $V+T$ in H. To show that $T+W$ is a τ-supplement of V in H, we must prove that $V \cap (T+W) \leq \tau(T+W)$. Note that $W+V \leq K+V$. So $T \cap (W + V) \leq T \cap (K + V) \leq \tau(T)$.

(2) Let $V \leq K+T$. By hypothesis, $K+T+V$ has a trivial τ-supplement in $K+T$ and by using (1), $T+V$ has a τ-supplement in $K+T$. It follows from again (1) that V has a τ-supplement in $K+T$. Hence, $K+T$ is τ-supplemented.

(3) It is obtained directly by (1). □

Theorem 4.4. *Let $H \in_{R_S} hmod$. If H is an amply τ-supplemented hypermodule, then following statements hold.*

1. *Direct summands of H are amply τ-supplemented.*
2. *Factor hypermodules of H are amply τ-supplemented.*

Proof. (1) Let $H = K \oplus T$ and let V, W be subhypermodules of T with $T = V + W$. Since $H = K + V + W$, there is a subhypermodule Z of W such that $Z+K+V = H$ and $Z \cap (K+V) \leq \tau(Z)$. It follows that $Z \cap V \leq Z \cap (V+K) \leq \tau(Z)$ and $T = T \cap H = T \cap (Z + (K \oplus V)) = T \cap (K \oplus V) + Z = V + Z$. Therefore, Z is a τ-supplement of V in T and $Z \leq W$.

(2) Let $V \leq H$ and $\frac{H}{V} = \frac{K}{V} + \frac{T}{V}$ with $V \leq K \leq H$, $V \leq T \leq H$. It follows from $H = K + T$ that there is a subhypermodule $W \leq T \leq H$. It follows from $H = K + T$ that there is a subhypermodule $W \leq T$ provided that $K + W = H$ and $K \cap W \leq \tau(W)$. We must prove that $\frac{W+V}{V}$ is a τ-supplement of $\frac{K}{V}$ in $\frac{H}{V}$. It is clear that $\frac{W+V}{V} + \frac{K}{V} = \frac{H}{V}$ and so $\frac{K}{V} \cap \frac{W+V}{V} = \frac{(K \cap W)+V}{V} \leq \frac{\tau(W)+V}{V} \leq \tau(\frac{W+V}{V})$. □

The following corollary follows from directly using Theorem 4.4.

Corollary 4.2. *Let $H \in_{R_S} hmod$. If H is amply τ-supplemented, then following statements hold.*

1. *If K is a τ-supplement in H and $\tau(K) = \{0_K\}$, then K is amply τ-supplemented.*
2. *If $\tau(H) = \{0\}$, then each τ-supplement subhypermodule of H is amply τ-supplemented.*

Recall from [22] that a hypermodule H is called *normal π-projective* if for each (K, T) of subhypermodules of H providing $K + T = H$, there is a strong homomorphism $\gamma : H \longrightarrow H$ with $Im(\gamma) \leq K$ and $Im(1 - \gamma) \leq T$, where 1 denotes the identity strong homomorphism of H.

Proposition 4.1. *Let $H \in_{R_S} hmod$. Then, following statements hold.*

1. *If each subhypermodule of H is τ-supplemented, then H is an amply τ-supplemented hypermodule.*
2. *If H is normal π-projective and τ-supplemented, then H is amply τ-supplemented.*

Proof. (1) Let $K, T \leq H$ with $H = K + T$. As K is τ-supplemented, there is $W \leq K$ with $(K \cap T) + W = K$ and $K \cap T \cap W \leq \tau(W)$. It follows that $T \cap W = K \cap T \cap W \leq \tau(W)$ and $H = W + T$.

(2) Let $H = K + T$. By hypothesis there exists a strong endomorphism γ of H with $Im(\gamma) \leq K$ and $Im(1-\gamma) \leq T$. As H is τ-supplemented, there is a subhypermodule $V \leq H$ with $V + K = H$ and $V \cap K \leq \tau(V)$. Therefore, $H = Im(\gamma) + Im(1-\gamma) \leq K + (1-\gamma)(K+V) \leq K + (1-\gamma)(V)$. So $H = K + (1-\gamma)(V)$. It can be shown that $(1-\gamma)(V) \leq T$ and $K \cap (1-\gamma)(V) = (1-\gamma)(K \cap V)$. Since $K \cap V \leq \tau(V)$, we have $K \cap (1-\gamma)(V) \leq \tau((1-\gamma)(V))$. Thus, $(1-\gamma)(V)$ is a τ-supplement of K in H and $(1-\gamma)(V) \leq T$. Hence, H is amply τ-supplemented. □

Finally, we give examples of a $\tau = Rad$-supplemented Krasner hypermodule and an amply $\tau = Rad$-supplemented Krasner hypermodule in ${}_{\mathbb{Z}_S}hmod$ without completing the article in the following.

Example 4.1. 1. Consider $\mathbb{Q}$ is quotient field of $\mathbb{Z}$. Then, we take H as a Krasner $\mathbb{Z}$-hypermodule $\mathbb{Q}$. It can be seen clearly from [13] and [26] that the Krasner $\mathbb{Z}$-hypermodule H is Rad-supplemented but not supplemented.

2. Consider the $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty} = \{ \frac{m}{p^n} \,|\, m \in \mathbb{Z},\, n \geq 0,\, 0 \leq \frac{m}{p^n} < 1,\, p \nmid m \}$ and a submodule $H = \langle \frac{1}{p} + \mathbb{Z} \rangle$. By [12, Example 2.5], it can be constructed a $\mathbb{Z}$-hypermodule $(\mathbb{Z}_{p^\infty}, \oplus, \odot)$. Since $\tau(\mathbb{Z}_{p^\infty}) = Rad(\mathbb{Z}_{p^\infty}) = \mathbb{Z}_{p^\infty}$ and each subhypermodule of $\mathbb{Z}_{p^\infty}$ is the form $\langle \frac{1}{p^m} + \mathbb{Z} \rangle$ where $m \geq 1$, $\mathbb{Z}_{p^\infty}$ is amply τ-supplemented, but not local.

5. Conclusions

In essence of this study, the properties of the concept of τ-supplemented module, one of the most basic topics of module theory, in the structure of subcategory hypermodules were investigated. We first characterized the τ-supplemented hypermodules by generalizing the supplemented hypermodules with the help of preradical, which we defined in the subcategory ${}_{R_S}hmod$. The basic properties of τ-supplemented hypermodules have been provided. The connection between the notion of τ-supplemented hypermodules and the notion of supplemented hypermodules can be obtained with the help of the basic algebraic properties provided by the concept of τ-supplement subhypermodule. In particular, we have shown that the class of amply τ-supplemented is closed under direct summands, factor hypermodules and finite sums. Also, we prove that every subhypermodule τ-supplemented hypermodule is τ-supplemented.

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Exchange pre-Hilbert algebras and their connections with other algebras of logic

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Abstract. In the paper, as a generalization of well-known Hilbert algebras, exchange pre-Hilbert algebras are introduced. Their properties and characterizations are investigated. Some important results and examples are given. Moreover, connections between exchange pre-Hilbert algebras, generalized exchange algebras and BE algebras are presented. Finally, implicative and positive implicative algebras are considered. It is shown that implicative (resp. positive implicative) exchange pre-Hilbert algebras are equivalent to implicative BE algebras with (*) (resp. generalized Hilbert algebras).

Keywords: Hilbert algebra, exchange pre-Hilbert algebra, BCK, BE algebra, implicativity, positive implicativity.

MSC 2020: 03G25, 06A06, 06F35.

1. Introduction

L. Henkin [5] introduced the notion of "implicative model", as a model of positive implicative propositional calculus. In 1960, A. Monteiro [11] has given the name "Hilbert algebras" to the dual algebras of Henkin's implicative models. In 1966, K. Iséki [7] introduced the notion of a BCK algebra. It is an algebraic formulation of the BCK-propositional calculus system of C. A. Meredith [10]. In [9], as a generalization of BCK algebras, H. S. Kim and Y. H. Kim introduced BE algebras. A. Rezaei et al. [12] investigated connections between Hilbert algebras and BE algebras. In 2008, A. Walendziak [13] defined commutative BE algebras and proved that they are BCK algebras. Later on, in 2010, D. Buşneag and S. Rudeanu [3] introduced the notion of a pre-BCK algebra. A BCK algebra is just a pre-BCK algebra satisfying also the antisymmetry property. In 2016, A. Iorgulescu [6] introduced new generalizations of BCK and Hilbert algebras (RML, aBE, pi-BE, pimpl-RML algebras and many others). Recently, as a generalization of Hilbert algebras, R. Bandaru et al. [1] introduced GE algebras (generalized exchange algebras) and A. Walendziak [16] introduced pre-Hilbert algebras (the definition of a pre-Hilbert algebra is inspired by Henkin's Positive Implicative Logic [5]). All of the algebras mentioned above are contained in the class of RML algebras (an RML algebra is an algebra $(A, \to, 1)$ of type $(2, 0)$ satisfying the identities: $x \to x = 1 = x \to 1$ and $1 \to x = x$).

In the paper, we introduce and study exchange pre-Hilbert algebras. We give their characterizations and examples. We investigate connections between GE algebras, BE algebras and exchange pre-Hilbert algebras. Moreover, these algebras with the antisymmetry property are considered. Finally, we define and characterize implicative exchange pre-Hilbert algebras. We also define positive implicative exchange pre-Hilbert algebras and prove that they are equivalent to generalized Hilbert algebras.

The motivation of this study consists of algebraic and logical arguments. Namely, exchange pre-Hilbert algebras belong to a wide class of algebras of logic. Furthermore, the results of the paper may have applications for future studies of the relationships between some generalizations of Hilbert algebras. An additional motivation is the fact that the present paper is a continuation of previous papers: [15] on GE algebras and [16] on pre-Hilbert algebras.

2. Preliminaries. GE algebras and pre-Hilbert algebras

Let $\mathcal{A} = (A, \to, 1)$ be an algebra of type $(2, 0)$. We define the binary relation $\leq$ on A by: for all $x, y \in A$, $x \leq y \iff x \to y = 1$.

We consider the following list of properties ([6]) that can be satisfied by $\mathcal{A}$:

(An) (Antisymmetry) $(x \leq y$ and $y \leq x) \Longrightarrow x = y$,

(B) $y \to z \leq (x \to y) \to (x \to z)$,

(C) $x \to (y \to z) \leq y \to (x \to z)$,

(D) $y \leq (y \to x) \to x$,

(Ex) (Exchange) $x \to (y \to z) = y \to (x \to z)$,

(GE) (Generalized exchange) $x \to (y \to z) = x \to (y \to (x \to z))$,

(K) $x \leq y \to x$,

(L) (Last element) $x \leq 1$,

(M) $1 \to x = x$,

(Re) (Reflexivity) $x \leq x$,

(Tr) (Transitivity) $(x \leq y$ and $y \leq z) \Longrightarrow x \leq z$,

(*) $y \leq z \Longrightarrow x \to y \leq x \to z$,

(**) $y \leq z \Longrightarrow z \to x \leq y \to x$,

(pi) $x \to (x \to y) = x \to y$,

(p-1) $x \to (y \to z) \le (x \to y) \to (x \to z)$,

(p-2) $(x \to y) \to (x \to z) \le x \to (y \to z)$,

(pimpl) (Positive implicativity) $x \to (y \to z) = (x \to y) \to (x \to z)$.

Remark 2.1. The properties in the list are the most important properties satisfied by a Hilbert algebra (the properties (An) – (**) are satisfied by a BCK algebra).

From Proposition 2.1 and Theorem 2.7 of [6] we have

Lemma 2.1. *Let* $\mathcal{A} = (A, \to, 1)$ *be an algebra of type* $(2,0)$*. Then, the following hold:*

(i) *(M) + (B) imply (Re), (*), (**);*

(ii) *(M) + (*) imply (Tr);*

(iii) *(M) + (L) + (**) imply (K);*

(iv) *(Re) + (L) + (Ex) imply (D), (K);*

(v) *(An) + (C) imply (Ex).*

Following Iorgulescu [6], we say that $(A, \to, 1)$ is an *RML algebra* if it verifies the axioms (Re), (M), (L). We recall now the following definitions.

Definitions 2.1 ([6])**.** *Let* $\mathcal{A} = (A, \to, 1)$ *be an RML algebra. The algebra* $\mathcal{A}$ *is said to be:*

1. *a BE algebra if it verifies (Ex);*
2. *an aRML algebra if it verifies (An);*
3. *an aBE algebra if it verifies (Ex) and (An), that is, it is a BE algebra with (An);*
4. *a pre-BCK algebra if it verifies (B) and (Ex), that is, it is a BE algebra with (B);*
5. *a BCK algebra if it is a pre-BCK algebra verifying (An).*

Denote by **RML**, **BE**, **aRML**, **aBE**, **preBCK** and **BCK** the classes of RML, BE, aRML, aBE, pre-BCK and BCK algebras, respectively.

By definition, **BE** = **RML** + (Ex), **aRML** = **RML** + (An), **aBE** = **BE** + (An), **preBCK** = **BE** + (B), **BCK** = **preBCK** + (An).

It is known that $\le$ is an order relation in BCK algebras. By definition, in RML and BE algebras, $\le$ is a reflexive relation; in aRML and aBE algebras, $\le$ is reflexive and antisymmetric. Since (M) + (B) imply (Tr), see Lemma 2.1 (i) and (ii), in pre-BCK algebras, $\leq$ is reflexive and transitive (i.e., it is a pre-order relation).

Lemma 2.2. *Let* $(A, \to, 1)$ *be an algebra of type* $(2, 0)$*. Then, the following hold:*

(i) *(Re) + (pi) imply (L);*

(ii) *(Ex) + (pi) imply (GE);*

(iii) *(Re) + (GE) + (L) imply (K);*

(iv) *(M) + (GE) imply (pi);*

(v) *(Re) + (pimpl) imply (B), (L);*

(vi) *(p-1) + (p-2) +(An) imply (pimpl).*

Proof. (i) – (iii) follow from Propositions 2.7 and 3.1 (ii) of [15].
(iv) by the proof of Proposition 2.4 of [15].
(v) follows from Propositions 6.4 and 6.9 of [6].
(vi) is trivial. □

Proposition 2.1. *Let* $\mathcal{A} = (A, \to, 1)$ *be an algebra of type* $(2, 0)$*. We have*

(i) *(Re) + (M) + (GE) + (An) imply (Ex);*

(ii) *(M) + (L) + (p-1) imply (*) and (Tr).*

Proof. (i) Let $\mathcal{A}$ satisfy (Re), (M), (GE) and (An). Using Lemma 2.2 (iv), (i) and (iii), we conclude that $\mathcal{A}$ also satisfies (K). Let $x, y, z \in A$. Applying (GE) and (K), we get $[x \to (y \to z)] \to [y \to (x \to z)] = [x \to (y \to z)] \to [y \to (x \to (y \to z))] = 1$, that is, (C) holds in $\mathcal{A}$. By Lemma 2.1 (v), $\mathcal{A}$ satisfies (Ex).

(ii) Let $x, y, z \in A$ and suppose that $y \leq z$. By (L) and (p-1), $1 = x \to 1 = x \to (y \to z) \leq (x \to y) \to (x \to z)$. From (M) it follows that $x \to y \leq x \to z$. Therefore, (*) holds in $\mathcal{A}$. Using Lemma 2.1 (ii) we see that (Tr) also holds. □

Remark 2.2. Theorem 6.16 (b) of [6] gives

(I) (Re) + (M) + (pimpl) + (An) $\Longrightarrow$ (Ex).

Observe that Proposition 2.1 (i) is a generalization of (I). The property (GE) is just the property (16) from the proof of Theorem 6.16 (a) of [6]. Hence, we obtain: (Re) + (M) + (pimpl) imply (GE). Therefore, Proposition 2.1 (i) implies (I).

Definitions 2.2 ([1])**.** *Let* $\mathcal{A} = (A, \to, 1)$ *be an algebra of type* $(2, 0)$*. We say that* $\mathcal{A}$ *is:*

1. *a GE algebra* (*generalized exchange algebra*) *if it verifies (Re), (M), (GE);*
2. *an antisymmetric GE algebra* (*aGE algebra,* f*or short) if it is a GE algebra verifying (An).*

Denote by **GE** and **aGE** the classes of all GE algebras and aGE algebras, respectively.

Proposition 2.2 ([15], Corollary 3.2)**.** *Any GE algebra satisfies (Re), (M), (L), (C), (D), (K), (GE), (pi).*

Remark 2.3. Since GE algebras satisfy (L), we get **GE** = **RML** + (GE). By definition, **aGE** = **GE** + (An).

GE algebras do not have to satisfy (An), (B), (Tr), (Ex), (p-1), (pimpl); see example below.

Example 2.1 ([15])**.** Consider the set $A = \{a, b, c, d, e, 1\}$ and the operation $\to$ given by the following table:

$\to$	a	b	c	d	e	1
a	1	1	c	c	1	1
b	a	1	d	d	1	1
c	a	1	1	1	1	1
d	a	1	1	1	1	1
e	a	1	1	1	1	1
1	a	b	c	d	e	1

.

We can observe that the properties (Re), (M), (L), (GE) are satisfied. Therefore, $(A, \to, 1)$ is a GE algebra. It does not satisfy (An) for $(x, y) = (c, d)$; (Ex) for $(x, y, z) = (a, b, c)$; (Tr), (B), (p-1), (pimpl) for $(x, y, z) = (a, e, c)$.

Definitions 2.3. *Let $\mathcal{A} = (A, \to, 1)$ be an RML algebra. The algebra $\mathcal{A}$ is called:*

1. *a pi-RML algebra if it verifies (pi),*
2. *a positive implicative RML algebra (for short, a pimpl-RML algebra) if it verifies (pimpl).*

Denote by **pi-RML** and **pimpl-RML** the classes of pi-RML and pimpl-RML algebras, respectively; similarly for the subclasses of the class of all RML algebras. Note that from [6] it follows that in RML algebras, (pimpl) implies (pi). Thus, **pimpl-RML** is a subclass of **pi-RML**. For BCK algebras, (pimpl) and (pi) are equivalent (cf. Theorem 8 of [8]), that is, **pimpl-BCK** = **pi-BCK**.

Recall that an algebra $(A, \to, 1)$ is a *Hilbert algebra* ([4]) if it verifies the axioms (An), (K), (p-1).

Remark 2.4. In [4], A. Diego proved that Hilbert algebras satisfy (Re), (M), (L), (B), (Ex), (pi), (p-2), (pimpl). Moreover, he showed that the class of all Hilbert algebras is a variety. From Remark 6.7 of [6] we see that **H** = **pimpl-BCK** = **pi-BCK**, where **H** denotes the class of all Hilbert algebras.

In [16], we introduced the following notion:

A *pre-Hilbert algebra* is an algebra $(A, \to, 1)$ of type $(2, 0)$ satisfying (M), (K) and (p-1). Let us denote by **preH** the class of pre-Hilbert algebras.

The following example shows that condition (K) cannot be dropped in the definition of pre-Hilbert algebra.

Example 2.2. Let $A = \{a, b, c, d, 1\}$ and $\to$ be defined as follows:

$\to$	a	b	c	d	1
a	1	c	b	d	1
b	1	1	1	d	1
c	1	1	1	d	1
d	b	b	c	1	1
1	a	b	c	d	1

.

We can observe that algebra $\mathcal{A} = (A, \to, 1)$ verifies properties (Re), (M), (L), (p-1). It does not verify (K) for $x = a$, $y = d$.

Proposition 2.3. *Let $\mathcal{A} = (A, \to, 1)$ be an algebra verifying (M), (L) and (p-1). Then*

$$(**) \Longleftrightarrow (K) \Longleftrightarrow (B).$$

Proof. (**) $\Longrightarrow$ (K). By Lemma 2.1 (iii).

(K) $\Longrightarrow$ (B). By Proposition 2.1 (ii), $\mathcal{A}$ satisfies (Tr). To prove (B), let $x, y, z \in A$. From (K) and (p-1) we conclude that

$$y \to z \leq x \to (y \to z) \text{ and } x \to (y \to z) \leq (x \to y) \to (x \to z).$$

Applying (Tr), we have $y \to z \leq (x \to y) \to (x \to z)$. Thus, (B) holds in $\mathcal{A}$.

(B) $\Longrightarrow$ (**). Follows from Lemma 2.1 (i). □

Proposition 2.4 ([16], Theorem 3.9)**.** *Pre-Hilbert algebras satisfy (Re), (M), (L), (B), (C), (K), (Tr), (p-1), (p-2).*

Remark 2.5. By definition and Proposition 2.4, **preH** = **RML** + (K) + (p-1). Since (An) + (K) + (p-1) imply (M) (see [4]), a Hilbert algebra is in fact a pre-Hilbert algebra verifying (An), that is, **H** = **preH** + (An).

Pre-Hilbert algebras do not have to satisfy (An), (Ex), (GE), (pi), (pimpl); see example below.

Example 2.3. Consider the set $A = \{a, b, c, d, 1\}$ and the operation $\to$ given by the following table:

$\to$	a	b	c	d	1
a	1	c	b	d	1
b	a	1	1	d	1
c	a	1	1	d	1
d	a	c	c	1	1
1	a	b	c	d	1

The algebra $\mathcal{A} = (A, \to, 1)$ verifies properties (M), (K), (p-1). Then, $\mathcal{A}$ is a pre-Hilbert algebra. It does not verify (An) for $(x, y) = (b, c)$; (Ex) and (pimpl) for $(x, y, z) = (a, d, b)$; (pi) for $(x, y) = (a, b)$; (GE) for $(x, y, z) = (a, 1, b)$.

Remark 2.6. It is easy to see that in GE algebras, $\leq$ is a reflexive relation; in pre-Hilbert algebras, $\leq$ is a pre-order relation, in aGE algebras, $\leq$ is reflexive and antisymmetric. In Hilbert algebras, $\leq$ is an order relation.

3. Exchange pre-Hilbert algebras

We now introduce a new algebra. We say that an algebra $(A, \to, 1)$ is an *exchange pre-Hilbert algebra* if it is a pre-Hilbert algebra verifying the exchange property (Ex).

Denote by **Ex-preH** the class of exchange pre-Hilbert algebras. By definition, **Ex-preH** = **preH** + (Ex).

Example 3.1. Consider the set $A = \{a, b, c, 1\}$ with the following table of $\to$:

$\to$	a	b	c	1
a	1	c	b	1
b	1	1	1	1
c	1	1	1	1
1	a	b	c	1

The algebra $\mathcal{A} = (A, \to, 1)$ verifies properties (Re), (M), (L), (Ex) (hence (K)), (B), (p-1). It does not verify (An) for $x = b, y = c$; (pi) for $x = a, y = b$. Hence, $\mathcal{A}$ is an exchange pre-Hilbert algebra, without (pi).

Proposition 3.1. *Any exchange pre-Hilbert algebra can be extended to an exchange pre-Hilbert algebra containing one element more.*

Proof. Let $\mathcal{A} = (A, \to, 1)$ be a pre-Hilbert algebra and let $\delta \notin A$. On the set $B = A \cup \{\delta\}$ consider the operation:

$$x \to' y = \begin{cases} x \to y, & \text{if } x, y \in A, \\ \delta, & \text{if } x \in A \text{ and } y = \delta, \\ 1, & \text{if } x = \delta \text{ and } y \in B. \end{cases}$$

Obviously, $\mathcal{B} := (B, \to', 1)$ satisfies the axioms (M), (L) and (K). Further, the axioms (p-1) and (Ex) are easily satisfied for all $x, y, z \in A$. Let at least one of x, y, z be equal to δ. First, let $x = \delta$ and $y, z \in B$. Then, $(x \to' y) \to' (x \to' z) = (\delta \to' y) \to' (\delta \to' z) = 1$ and $x \to' (y \to' z) = \delta \to' (y \to' z) = 1 = y \to' (\delta \to' z) = y \to' (x \to' z)$. Thus, (p-1) and (Ex) hold for $x = \delta$ and $y, z \in B$. Similarly, if $y = \delta$ and $x, z \in A$. Now let $z = \delta$ and $x, y \in A$. We have $x \to' (y \to' z) = x \to' (y \to' \delta) = \delta = (x \to' y) \to' (x \to' \delta) = (x \to' y) \to' (x \to' z)$ and $x \to' (y \to' z) = x \to' (y \to' \delta) = \delta = y \to' (x \to' \delta) = y \to' (x \to' z)$. Therefore, $\mathcal{B}$ satisfies (p-1) and (Ex). Hence, $\mathcal{B}$ is an exchange pre-Hilbert algebra. □

Now, we give some characterizations of exchange pre-Hilbert algebras.

Theorem 3.1. *Let $\mathcal{A} = (A, \to, 1)$ be an algebra of type $(2, 0)$. The following statements are equivalent:*

(i) *$\mathcal{A}$ is an exchange pre-Hilbert algebra*;

(ii) *$\mathcal{A}$ satisfies (M), (L), (B), (Ex), (p-1);*

(iii) *$\mathcal{A}$ is a pre-BCK algebra satisfying (p-1);*

(iv) *$\mathcal{A}$ is a BE algebra satisfying (p-1).*

Proof. (i) $\Longrightarrow$ (ii), (ii) $\Longrightarrow$ (iii) and (iii) $\Longrightarrow$ (iv) are obvious.

(iv) $\Longrightarrow$ (i). By Lemma 2.1 (iv), (Re) + (L) + (Ex) imply (K). Then, $\mathcal{A}$ satisfies (M), (K), (p-1), (Ex). Thus, $\mathcal{A}$ is an exchange pre-Hilbert algebra. □

Lemma 3.1 ([15], Corollary 2.8)**.** *Any GE algebra is a pi-RML algebra.*

From Lemmas 2.2 (ii) and 3.1 we have

Proposition 3.2. *Let $\mathcal{A} = (A, \to, 1)$ be an algebra of type $(2, 0)$. The following statements are equivalent:*

(i) *$\mathcal{A}$ is a pi-BE algebra,*

(ii) *$\mathcal{A}$ is a GE algebra satisfying (Ex).*

Lemma 3.2. *Let $\mathcal{A} = (A, \to, 1)$ be an algebra verifying (B), (Ex) and (pi). Then $\mathcal{A}$ satisfies (p-1).*

Proof. Let $x, y, z \in A$. By (Ex), $x \to (y \to z) = y \to (x \to z)$. Applying (B) and (pi), we get $y \to (x \to z) \leq (x \to y) \to [x \to (x \to z)] = (x \to y) \to (x \to z)$. Then, $x \to (y \to z) \leq (x \to y) \to (x \to z)$, that is, (p-1) holds. □

Remark 3.1. By Proposition 3.2 and Theorem 3.1, **pi-BE** = **GE** + (Ex) and **Ex-preH** = **preBCK** + (p-1) = **BE** + (p-1). Hence **Ex-preH** + (pi) = **preBCK** + (p-1) + (pi) = **pi-preBCK**, since (B) + (Ex) + (pi) imply (p-1), see Lemma 3.2.

Proposition 3.3. *An algebra $\mathcal{A} = (A, \to, 1)$ of type $(2, 0)$ is an antisymmetric GE algebra if and only if it is a pi-aBE algebra.*

Proof. Let $\mathcal{A}$ be an aGE algebra. By Proposition 2.2, $\mathcal{A}$ satisfies (pi) and (C). Since (C) + (An) imply (Ex), see Lemma 2.1 (v), we conclude that $\mathcal{A}$ is a pi-aBE algebra. The converse follows from Proposition 3.2. □

Remark 3.2. By Proposition 3.3, **aGE** = **pi-aBE**. Hence **aGE** + (B) = **pi-aBE** + (B) = **aBE** + (B) + (pi) = **BCK** + (pi) = **pi-BCK**.

The interrelationships between the classes of algebras mentioned before are visualized in Figure 1 (see Remarks 2.3 – 2.5, 3.1 and 3.2).

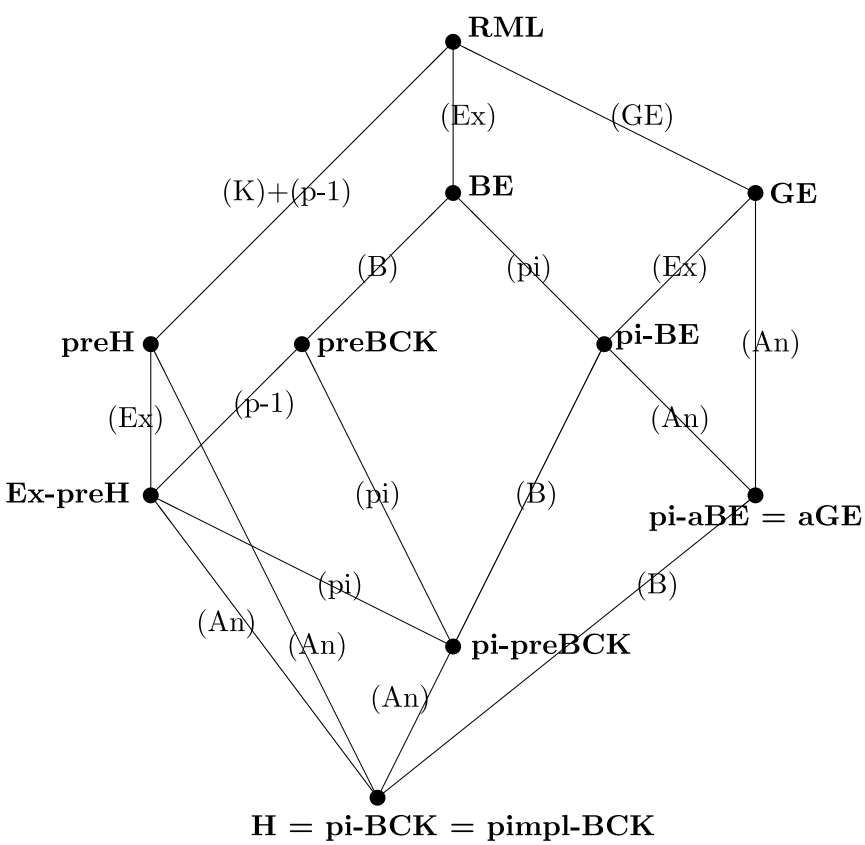

Figure 1: The hierarchy between **RML** and **H**

Consider now the property of positive implicativity for exchange pre-Hilbert algebras. Note that positive implicative GE algebras and pre-Hilbert algebras were studied in [15] and [16], respectively.

Remark 3.3. By Remark 4.9 of [15], **pimpl-RML** = **pimpl-GE** and by Remark 4.9 of [16], **pimpl-preH** = **pimpl-RML**. Therefore, **pimpl-preH** = **pimpl-GE** = **pimpl-RML**.

Theorem 3.2. *Let* $\mathcal{A} = (A, \to, 1)$ *be an algebra of type* $(2, 0)$*. The following statements are equivalent:*

(i) $\mathcal{A}$ *is a positive implicative exchange pre-Hilbert algebra;*

(ii) $\mathcal{A}$ *satisfies (Re), (M), (Ex) and (pimpl);*

(iii) $\mathcal{A}$ *is a pimpl-BE algebra;*

(iv) $\mathcal{A}$ *is a pimpl-pre-BCK algebra.*

Proof. (i) $\Longrightarrow$ (ii). By definitions.

(ii) $\Longrightarrow$ (iii). If $\mathcal{A}$ satisfies (Re), (M), (Ex) and (pimpl), then $\mathcal{A}$ also satisfies (L), since (Re) + (pimpl) imply (L) by Lemma 2.2 (v). Therefore, $\mathcal{A}$ is a pimpl-BE algebra.

(iii) $\Longrightarrow$ (iv). By Lemma 2.2 (v), (Re) + (pimpl) imply (B). Hence $\mathcal{A}$ is a pimpl-pre-BCK algebra.

(iv) $\Longrightarrow$ (i). Obvious. □

Remark 3.4. From Theorem 3.2 it follows that **pimpl-Ex-preH** = **pimpl-BE** = **pimpl-preBCK**.

In 2012, R. A. Borzooei and J. Shohani [2] introduced the notion of a generalized Hilbert algebra. Following [2], a *generalized Hilbert algebra* is an algebra $(A, \to, 1)$ satisfying (Re), (M), (Ex), (pimpl). From Theorem 3.2 we conclude that generalized Hilbert algebras are just pimpl-pre-BCK algebras. Moreover, we have

Corollary 3.1. *An algebra is a positive implicative exchange pre-Hilbert algebra if and only if it is a generalized Hilbert algebra.*

4. The implicative property (im)

The implicative BCK algebras were introduced and investigated by K. Iséki and S. Tanaka [8]. It is well-known that any bounded implicative BCK algebra is a Boolean algebra. Note that the implicative property for some generalizations of BCK algebras were studied in [14].

Let $\mathcal{A} = (A, \to, 1)$ be an algebra of type $(2, 0)$. We consider the following property:

(im) (Implicativity) $(x \to y) \to x = x$.

Lemma 4.1 ([14], Proposition 3.5)**.** *Let* $(A, \to, 1)$ *be an algebra of type* $(2, 0)$*. Then:*

(i) *(Re) + (im) imply (M),*

(ii) *(M) + (im) imply (L),*

(iii) *(im) implies (pi).*

Similarly as in the case of BCK algebras, we say that an RML algebra (in particular, an exchange pre-Hilbert algebra) $(A, \to, 1)$ is *implicative* if it satisfies (im).

Denote by **im-RML** the class of implicative RML algebras; similarly for subclasses of the class of all RML algebras.

Remarks 4.1. (1) By definitions, **im-RML** = **RML** + (im), **im-GE** = **im-RML** + (GE).

(2) By Lemma 2.2 (ii), (Ex) + (pi) imply (GE). Hence (Ex) + (im) imply (GE), because (im) implies (pi). Consequently, **im-BE** = **im-RML** + (Ex) = **im-RML** + (GE) + (Ex) = **im-GE** + (Ex).

Now we give several characterizations of implicative exchange pre-Hilbert algebras.

Theorem 4.1. *Let* $\mathcal{A} = (A, \to, 1)$ *be an algebra of type* $(2, 0)$*. The following are equivalent:*

(i) $\mathcal{A}$ *is an implicative exchange pre-Hilbert algebra;*

(ii) $\mathcal{A}$ *satisfies (Re), (B), (Ex), (im);*

(iii) $\mathcal{A}$ *is an implicative pre-BCK algebra;*

(iv) $\mathcal{A}$ *is an implicative BE algebra satisfying* (*).

Proof. (i) $\Longrightarrow$ (ii). Obvious.

(ii) $\Longrightarrow$ (iii). By Lemma 4.1, $\mathcal{A}$ also satisfies (M) and (L). Thus, $\mathcal{A}$ is an implicative pre-BCK algebra.

(iii) $\Longrightarrow$ (iv). Clearly, $\mathcal{A}$ is an implicative BE algebra. Moreover, $\mathcal{A}$ satisfies (*) by Lemma 2.1 (i).

(iv) $\Longrightarrow$ (i). From Lemma 2.1 (iv) we see that $\mathcal{A}$ satisfies (D) and (K). Observe that $\mathcal{A}$ also satisfies (B). Let $x, y, z \in A$. By (D), $y \leq (y \to z) \to z$. Hence, applying (*) and (Ex), we have

$$x \to y \leq x \to [(y \to z) \to z] = (y \to z) \to (x \to z),$$

that is, $x \to y \leq (y \to z) \to (x \to z)$. From (Ex) we conclude that $y \to z \leq (x \to y) \to (x \to z)$, that is, (B) holds in $\mathcal{A}$. By Lemma 4.1 (iii), (im) implies (pi). Thus, $\mathcal{A}$ satisfies (B), (Ex), (pi). Hence, using Lemma 3.2, we deduce that (p-1) holds. Consequently, $\mathcal{A}$ is an implicative exchange pre-Hilbert algebra. □

Example 4.1 ([15])**.** Let $A = \{a, b, c, d, e, 1\}$ and $\to$ be defined as follows:

$\to$	a	b	c	d	e	1
a	1	1	e	d	e	1
b	1	1	d	d	d	1
c	1	1	1	1	1	1
d	a	b	b	1	1	1
e	a	a	a	1	1	1
1	a	b	c	d	e	1

It is easy to see that the properties (Re), (B), (Ex), (im) are satisfied; (An) is not satisfied for $(x, y) = (a, b)$, (pimpl) is not satisfied for $(x, y, z) = (a, b, c)$, Therefore, $(A, \to, 1)$ is an implicative exchange pre-Hilbert algebra that is not positive implicative.

Lemma 4.2 ([17], Lemma 10)**.** *If* $(A, \to, 1)$ *is an implicative aBE algebra, then* $(x \to y) \to y = x$ *or* $y \to x \neq 1$*, for all* $x, y \in A$*.*

Proposition 4.1. Any *implicative aBE algebra satisfies* (*)*.*

Proof. Let $\mathcal{A} = (A, \to, 1)$ be an implicative aBE algebra. By Lemma 2.1 (iv), $\mathcal{A}$ satisfies (K). Let $x, y, z \in A$ and $y \leq z$. From Lemma 4.2 it follows that $(z \to y) \to y = z$. Hence, by (Ex), we get $x \to z = x \to ((z \to y) \to y) = (z \to y) \to (x \to y)$. Applying (K), we obtain $(x \to y) \to (x \to z) = (x \to y) \to ((z \to y) \to (x \to y)) = 1$. Thus, $x \to y \leq x \to z$, that is, (*) holds in $\mathcal{A}$. □

From Theorem 4.1 and Proposition 4.1 we have

Corollary 4.1. *The class of implicative aBE algebras coincides with the class of implicative BCK algebras.*

Remark 4.2. (1) From Theorem 4.1 we obtain **im-Ex-preH** = **im-preBCK** = **im-BE** + (B).

(2) By definition, **im-BCK** = **im-preBCK** + (An). Hence **im-Ex-preH** + (An) = **im-BCK**. From Proposition 4.17 of [15] and Corollary 4.1 we see that **im-BCK** = **im-aBE** = **im-aGE**.

(3) Since **H** = **pi-BCK** and (im) implies (pi), we obtain **H** + (im) = **pi-BCK** + (im) = **BCK** + (pi) + (im) = **im-BCK** + (pi) = **im-BCK**.

By definitions and Remarks 3.3, 3.4, 4.1 and 4.2 we can draw Figure 2.

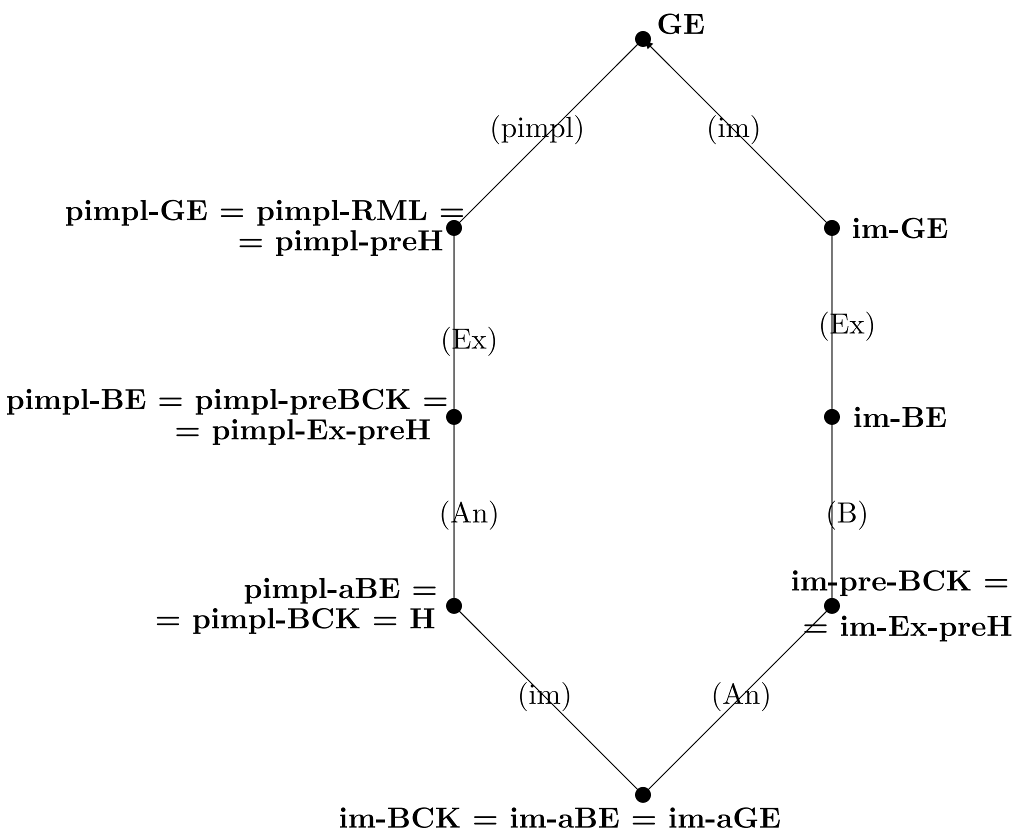

Figure 2: The hierarchy between **GE** and **im-BCK**

Example 4.2. Let $\mathbb{Z}$ be the set of integers and let for $x, y \in \mathbb{Z}$ the symbol $x \mid y$ means that x divides y. Then, the relation $\mid$ is a pre-order on $\mathbb{Z}$ which is not an order (for example, $1 \mid -1$ and $-1 \mid 1$ but $1 \neq -1$). We define the operation $\to$ by

$$x \to y = \begin{cases} 0, & \text{if } x \mid y, \\ y, & \text{otherwise.} \end{cases}$$

Obviously, $x \to x = 0$ and $0 \to x = x$ for each $x \in \mathbb{Z}$. Then, $\mathcal{Z} = (\mathbb{Z}, \to, 0)$ satisfies (Re) and (M). To prove (Ex), let $x, y, z \in \mathbb{Z}$. We will consider three cases:
Case 1. Let $y \mid z$. Then, $x \to (y \to z) = x \to 0 = 0 = y \to (x \to z)$, since $y \mid x \to z$.

Case 2. Let $y \nmid z$ and $x \mid z$. Then, $x \to (y \to z) = x \to z = 0 = y \to 0 = y \to (x \to z)$.

Case 3. Let $y \nmid z$ and $x \nmid z$. We have $x \to (y \to z) = x \to z = z = y \to z = y \to (x \to z)$.
Thus, $\mathcal{Z}$ satisfies (Ex). Similarly, by routine calculation we can show that $\mathcal{Z}$ also satisfies (pimpl). Consequently, $\mathcal{Z}$ is a positive implicative exchange pre-Hilbert algebra. Observe that $\mathcal{Z}$ is not implicative. Indeed, $(2 \to 1) \to 2 = 1 \to 2 = 0 \neq 2$.

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The residuated lattice-orderability of idempotent monoids

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Abstract. In this paper, we study a class of residuated lattices, namely, conical idempotent residuated lattices. We give necessary and sufficient conditions for an idempotent semigroup with an identity to be the semigroup reduct of some conical idempotent residuated lattice.

Keywords: conical idempotent residuated lattice, semilattice, idempotent semigroup.
MSC 2020: 06F05, 20M10.

1. Introduction

Let $(\mathfrak{P}, \leq)$ be a poset. A (binary) operation $\circ$ is called *residuated* if there exist (binary) operations $\rhd$ and $\lhd$ on $\mathfrak{P}$ such that

$$(\forall x, y \in \mathfrak{P})\ x \circ y \leq z \Longleftrightarrow y \leq x \rhd z \Longleftrightarrow x \leq z \lhd y.$$

In this case, the operation pair $(\rhd, \lhd)$ is called a pair of *residuals* of the operation $\circ$. It is well known that an operation $\circ$ on the poset $(\mathfrak{P}, \leq)$ is residuated if and only if $\circ$ is order preserving in each argument and such that, for all $a, b \in \mathfrak{P}$, both $\{p \in \mathfrak{P} \mid a \circ p \leq b\}$ and $\{q \in \mathfrak{P} \mid q \circ a \leq b\}$ contain a greatest element(denote by $a \rhd b$ and $b \lhd a$, respectively). A *residuated lattice* is defined as an algebra $(\mathfrak{L}, \wedge, \vee, \circ, \rhd, \lhd, e)$ satisfying the following conditions:

- $(RL1)$ $(\mathfrak{L}, \wedge, \vee)$ is a lattice;
- $(RL2)$ $(\mathfrak{L}, \circ)$ is a monoid with identity e; and
- $(RL3)$ the operation pair $(\rhd, \lhd)$ is a pair of residuals of the operation $\circ$.

In this case, we call the lattice $(\mathfrak{L}, \wedge, \vee)$ and the semigroup $(\mathfrak{L}, \circ)$ the *lattice reduct* and the *semigroup reduct* of the residuated lattice $(\mathfrak{L}, \wedge, \vee, \circ, \rhd, \lhd, e)$, respectively. Sometime, residuated lattices are also called *residuated lattice-ordered monoids*. A residuated lattice $(\mathfrak{L}, \wedge, \vee, \circ, \rhd, \lhd, e)$ is called *idempotent* if the semigroup reduct of $\mathfrak{L}$ is an idempotent semigroup. Moreover, an idempotent residuated lattice $(\mathfrak{L}, \wedge, \vee, \circ, \rhd, \lhd, e)$ is called *commutative* if the semigroup reduct of $\mathfrak{L}$ is commutative and $\mathfrak{L}$ is called a *commutative idempotent residuated chain* if its lattice reduct is a chain (see [18] and [19]). As in [1] and [8], a residuated lattice $(\mathfrak{L}, \wedge, \vee, \circ, \rhd, \lhd, e)$ is said to be *conical*, if for each $a \in \mathfrak{L}$, $a \leq e$ or $a \geq e$, where $\leq$ is the order on the lattice reduct of $(\mathfrak{L}, \wedge, \vee, \circ, \rhd, \lhd, e)$.

Idempotent residuated lattices play a crucial role in residuated lattice theory. On the one hand they include several important algebraic counterparts of substructural logics, e.g., Brouwerian algebras, i.e. algebras of the positive intuitionistic logic, Heyting algebras, i.e. algebras of the propositional intuitionistic logic and positive Sugihara monoids (see [16]) and on the other hand, the knowledge of idempotent residuated lattices can increase our comprehension of residuated lattices (see [11, 13, 14]). Different kinds of idempotent residuated lattices have been introduced and studied in the literature, first to handle idempotent residuated chains, and then for conical idempotent residuated lattices and for finite idempotent residuated lattices (see[1-6,8-10,12,15-19]). In an earlier paper [1] and [3], we investigated conical idempotent residuated lattices from semigroup perspectives. We established a structure theorem and decomposition theorem for conical idempotent residuated lattices. Recently, Gil-Férez, Jipsen and Metcalfe in [12] have used semigroup reducts to give a complete structural description of finite description of finite idempotent residuated chains. More recently, Fussner and Galatos in [6] have shown that non-isomorphic idempotent residuated chains may have the same semigroup reduct. This paper is a continuation of [1] and [3]. Conical idempotent residuated lattices can be considered as a class of lattice-ordered idempotent monoids. The following question naturally arises : How can we characterize the class of monoids that are semigroup reducts of conical idempotent residuated lattices? The purpose of this paper is to solve this question.

We proceed as follows: In Section 2, we recall some definitions and basic facts needed in later proofs. In Section 3, we provide the necessary and sufficient conditions for an idempotent semigroup with an identity to be the semigroup reduct of some conical idempotent residuated lattice which generalize [2, Theorem 5.2].

2. Preliminaries

In this section, we shall first recall some basic definitions and facts on semigroups. For further information on semigroups, we refer to any standard text book, for example, the book by Howie [7]. After this, we recall the concepts of residuated lattices.

Let $\mathfrak{S}$ be a semigroup and $\mathfrak{S}^1$ the semigroup obtained from $\mathfrak{S}$ by adding an identity if $\mathfrak{S}$ has no identity, otherwise we put $\mathfrak{S}^1 = \mathfrak{S}$. In the theory of semigroups, the Green's relations $\mathcal{L}, \mathcal{R}, \mathcal{J}, \mathcal{H}$ and $\mathcal{D}$ are of fundamental importance. They are defined in the following way:

$$\mathcal{L} = \{(a, b) \in \mathfrak{S} \times \mathfrak{S} | \mathfrak{S}^1 a = \mathfrak{S}^1 b\},$$
$$\mathcal{R} = \{(a, b) \in \mathfrak{S} \times \mathfrak{S} | a\mathfrak{S}^1 = b\mathfrak{S}^1\},$$

$$\mathcal{J} = \{(a,b) \in \mathfrak{S} \times \mathfrak{S} | \mathfrak{S}^1 a \mathfrak{S}^1 = \mathfrak{S}^1 b \mathfrak{S}^1\},$$
$$\mathcal{H} = \mathcal{L} \cap \mathcal{R},$$
$$\mathcal{D} = \mathcal{L} \vee \mathcal{R}.$$

Evidently, $\mathcal{L}$ is a right congruence while $\mathcal{R}$ is a left congruence. Moreover, we have

Lemma 2.1 ([7]). *The relations $\mathcal{L}$ and $\mathcal{R}$ commute and $\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$.*

If $\mathcal{K}$ is one of Green's relations, we shall denote by K_a the $\mathcal{K}$-class of $\mathfrak{S}$ containing $a \in \mathfrak{S}$.

An element x of $\mathfrak{S}$ is called an *idempotent* if $x^2 = x$. $\mathfrak{S}$ is called a *idempotent semigroup* if every element of $\mathfrak{S}$ is an idempotent. idempotent semigroups are also called bands. Moreover, an idempotent semigroup $\mathfrak{B}$ is called a *semilattice* if it is commutative; $\mathfrak{B}$ is called a *rectangular band* if $\mathfrak{B}$ satisfies the identity: $abc \approx ac$.

Now let $\mathfrak{B}$ be an idempotent semigroup. By [7, Proposition 2.1.4 and Theorem 4.1.3], on $\mathfrak{B}$, $\mathcal{D} = \mathcal{J}$ and is a congruence such that $\mathfrak{B}/\mathcal{D}$ is a semilattice, which is usually called *the structure semilattice* of $\mathfrak{B}$. Denote the semilattice $\mathfrak{B}/\mathcal{D}$ by Y. Consider the natural homomorphism $\mathcal{D}^\natural : \mathfrak{B} \to Y$ induced by $\mathcal{D}$. For $\alpha \in Y$, we use D_α to stand for $\alpha {\mathcal{D}^\natural}^{-1}$. Obviously, each D_α is a $\mathcal{D}-$class of $\mathfrak{B}$ and further a rectangular band. It is easy to see that $D_\alpha D_\beta := \{ab \mid a \in D_\alpha, b \in D_\beta\} \subseteq D_{\alpha\beta}$. Thus, we have

Lemma 2.2 ([7]). *Every idempotent semigroup is a semilattice of rectangular bands.*

Let $(\mathfrak{L}, \circ)$ be an idempotent semigroup with an identity. By the arguments before Lemma 2.2, $\mathcal{D}$ is a semilattice congruence on $\mathfrak{L}$. In other words, the quotient semigroup $(\mathfrak{L}/\mathcal{D}, \cdot)$, for short, $\mathfrak{L}/\mathcal{D}$, is a semilattice. For simplicity, we also write the element $a\mathcal{D}^\natural$ of the semigroup $\mathfrak{L}/\mathcal{D}$ as D_a, for $a \in \mathfrak{L}$. On $\mathfrak{L}/\mathcal{D}$, define: for $a, b \in \mathfrak{L}$,

$$D_a \leq^* D_b \text{ if and only if } D_a \cdot D_b = D_a.$$

By [7, Proposition 1.3.2, p. 14], $\leq^*$ is an order on the semilattice $\mathfrak{L}/\mathcal{D}$.

Now, we shall list some basic concepts of residuated lattices used in the sequel. For further information on residuated lattices, we refer to [1] and [10].

Let $(\mathfrak{L}, \wedge, \vee, \circ, \rhd, \lhd, e)$ be a residuated lattice and assume the lattice reduct of $\mathfrak{L}$ is the lattice $(\mathfrak{L}, \leq)$. For convenience, we simply write $a \circ b$ as ab for $a, b \in \mathfrak{L}$. It's well-known that the sets $\{c \in \mathfrak{L} \mid ac \leq b\}$ and $\{c \in \mathfrak{L} \mid ca \leq b\}$ have both a greatest element, in notation, $a \rhd b$ and $b \lhd a$, respectively.

In a residuated lattice term, we assume that multiplication has priority over the division operations, which, in turn, have priority over the lattice operations. So, for example, we write $x \lhd yz \wedge u \rhd v$ for $[x \lhd (yz)] \wedge (u \rhd v)$.

Now, let $(\mathfrak{P}, \leq)$ be a poset and assume $x, y \in \mathfrak{P}$ with $y < x$. We say x *covers* y, in notation, $y \prec x$, if for any $z \in \mathfrak{P}$, $y \leq z \leq x$ implies either $x = z$ or $y = z$. We define $[x, y] = \{u \in \mathfrak{P} \mid x \leq u \leq y\}$ for $x, y \in \mathfrak{P}$ with $x \leq y$. For $a, b \in \mathfrak{P}$, $a \parallel b$ means that a and b are not comparable under $\leq$. Write $P_a^{nc} = \{x \in \mathfrak{P} \mid x \parallel a\}$. b is called a *lower square point* of a if $b < a$ and for any $d \in \mathfrak{P}$ with $b \leq d < a$, there exists $d' \in \mathfrak{P}$ such that $d \parallel d'$. Similarly, we can define *upper square points*. We denote by P_a^l the set of lower square points of a and by P_a^u the set of upper square points of a. Put $P_a = P_a^{nc} \cup P_a^l \cup P_a^u \cup \{a\}$. In what follows, we denote by $P_a^{\top}$ the least upper bound of P_a and by $P_a^{\perp}$ the greatest lower bound of P_a if they exist. (P_a) is obtained by adjoining $P_a^{\top}$ and/or $P_a^{\perp}$, whenever they exist, to P_a. We call the above (P_α) the *square point set* of $\mathfrak{P}$.

Definition 2.1 ([1])**.** *Let $\{\mathfrak{L}^+, \mathfrak{L}^-, \{e\}\}$ be a family of pairwise disjoint subsets of $\mathfrak{L}$ such that $\mathfrak{L} = \mathfrak{L}^+ \cup \mathfrak{L}^- \cup \{e\}$. A partition $\{\mathfrak{L}_\alpha \mid \alpha \in \mathfrak{Y}\}$ on $\mathfrak{L}$ is called a* ***conical semilattice partition*** *on $\mathfrak{L}$ provided*

(1) *$(\mathfrak{Y}, \leq^*)$ is a semilattice with greatest element 1; and*

(2) *The partition satisfies the following conditions:*

- **(CSP1)** *$\mathfrak{L}_1 = \{e\}$, where we let $e = a_1 = b_1$.*
- **(CSP2)** *For each $\alpha \in \mathfrak{Y} \setminus \{1\}$, $|\mathfrak{L}_\alpha \cap \mathfrak{L}^+| \leq 1$ and $|\mathfrak{L}_\alpha \cap \mathfrak{L}^-| \leq 1$. We shall denote the element in $\mathfrak{L}_\alpha \cap \mathfrak{L}^+$ and $\mathfrak{L}_\alpha \cap \mathfrak{L}^-$ by a_α and b_α, respectively.*
- **(CSP3)** *For every $\alpha, \beta \in \mathfrak{Y}$ with $\alpha \parallel^* \beta$, we have $|\mathfrak{L}_\alpha| = |\mathfrak{L}_\beta| = |\mathfrak{L}_{\alpha\wedge\beta}| = 1$ and*
$$\mathfrak{L}_\alpha \cap \mathfrak{L}^+ \neq \emptyset \Leftrightarrow \mathfrak{L}_\beta \cap \mathfrak{L}^+ \neq \emptyset \Leftrightarrow \mathfrak{L}_{\alpha\wedge\beta} \cap \mathfrak{L}^+ \neq \emptyset.$$
- **(CSP4)** *$\alpha \vee \beta$ exists and $\mathfrak{L}_{\alpha\vee\beta} = \mathfrak{L}_{\alpha\vee\beta} \cap \mathfrak{L}^-$ for every $\alpha, \beta \in \mathfrak{Y}$ such that $\alpha \parallel^* \beta$ and $\mathfrak{L}_\alpha \cap \mathfrak{L}^- \neq \emptyset$.*

Definition 2.2 ([1])**.** *Let $\{\mathfrak{L}_\alpha \mid \alpha \in \mathfrak{Y}\}$ be a conical semilattice partition. A subset $\mathfrak{X}$ of $\mathfrak{Y}$ is called a* ***band subset*** *of the partition $\{\mathfrak{L}_\alpha \mid \alpha \in \mathfrak{Y}\}$ if $|\mathfrak{L}_\alpha| = 2$ for every $\alpha \in \mathfrak{X}$.*

Definition 2.3 ([1])**.** *Let $\mathfrak{L}$ be a nonempty set. The conical semilattice partition $\pi = \{\mathfrak{L}_\alpha \mid \alpha \in \mathfrak{Y}\}$ on $\mathfrak{L}$ with band subset $\mathfrak{X}$ is called a* **CLOB-system** *provided the mapping*
$$\Phi : \mathfrak{C} \to \mathfrak{Y}; \quad (\alpha, \beta) \mapsto \Phi(\alpha, \beta),$$
where $\mathfrak{C} = \{(\alpha, \beta) \in \mathfrak{Y} \times \mathfrak{Y} \mid \alpha \parallel^* \beta, \mathfrak{L}_\alpha \cap \mathfrak{L}^+ \neq \emptyset\}$, satisfies the following conditions:

$(B1)$ If $(\alpha, \beta) \in \mathrm{Dom}\Phi$, then $\alpha <^* \Phi(\alpha, \beta)$, $\beta <^* \Phi(\alpha, \beta)$ and $\mathfrak{L}_{\Phi(\alpha,\beta)} \cap (\mathfrak{L}^+ \cup \{e\}) \neq \emptyset$.

$(B2)$ If $(\alpha, \beta) \in \mathrm{Dom}\Phi$ and $\gamma \in \mathfrak{Y}$ such that $\alpha, \beta <^* \gamma <^* \Phi(\alpha, \beta)$, then $\mathfrak{L}_\gamma = \mathfrak{L}_\gamma \cap \mathfrak{L}^-$.

In what follows, we denote this system in Definition 2.3 by $(\mathfrak{L};\pi,\mathfrak{Y};\mathfrak{X},\Phi)$.

Lemma 2.3 ([1]). *Let $(\mathfrak{L};\pi,\mathfrak{Y};\mathfrak{X},\Phi)$ be a CLOB-system. Define an order $\leq$ on $\mathfrak{L}$ as follows: for $a\in\mathfrak{L}_\alpha, b\in\mathfrak{L}_\beta$,*

$$a\leq b \quad \text{if and only if} \quad \begin{array}{l}\text{either condition } (O1)\text{: } \alpha\leq^*\beta \text{ and } a=b_\alpha \text{ or}\\ \text{condition } (O2)\text{: } \beta\leq^*\alpha \text{ and } b=a_\beta.\end{array}$$

Define a multiplication $\circ$ on $\mathfrak{L}$ in the following ways: for $a_\alpha,b_\alpha,a_\beta,b_\beta\in\mathfrak{L}$,

$$a_\alpha\circ a_\beta=a_{\alpha\wedge\beta};$$
$$b_\alpha\circ b_\beta=b_{\alpha\wedge\beta};$$
$$a_\alpha\circ b_\beta=\begin{cases}a_\alpha, & \text{if } \alpha<^*\beta \text{ or } \alpha=\beta\in\mathfrak{X},\\ b_\beta, & \text{if } \beta<^*\alpha \text{ or } \alpha=\beta\notin\mathfrak{X};\end{cases}$$
$$b_\beta\circ a_\alpha=\begin{cases}a_\alpha, & \text{if } \alpha<^*\beta \text{ or } \alpha=\beta\notin\mathfrak{X},\\ b_\beta, & \text{if } \beta<^*\alpha \text{ or } \alpha=\beta\in\mathfrak{X}.\end{cases}$$

Then, $(\mathfrak{L},\circ,\leq)$ is an ordered idempotent semigroup (see [1]).

Definition 2.4 ([1]). *A CLOB-system $(\mathfrak{L};\pi,\mathfrak{Y};\mathfrak{X},\Phi)$ is called a* ***CRLOB-system*** *provided the mappings*

$$\psi:\{\alpha\in\mathfrak{Y}\setminus\{1\}:\mathfrak{L}_\alpha\cap\mathfrak{L}^+\neq\emptyset\}\to\mathfrak{Y};\quad \alpha\mapsto\psi(\alpha),$$
$$\varphi:\{\alpha\in\mathfrak{Y}\setminus\{1\}:\mathfrak{L}_\alpha\cap\mathfrak{L}^-\neq\emptyset\}\to\mathfrak{Y};\quad \alpha\mapsto\varphi(\alpha)$$

and

$$\Psi:\mathfrak{R}\to\mathfrak{Y};(\alpha,\beta)\mapsto\Psi(\alpha,\beta),$$

where $\mathfrak{R}=\{(\alpha,\beta)\in\mathfrak{Y}\times\mathfrak{Y}\mid \alpha\parallel^*\beta,\mathfrak{L}_\alpha\cap\mathfrak{L}^-\neq\emptyset$ or $\beta<^*\alpha,\mathfrak{L}_\beta\cap\mathfrak{L}^-\neq\emptyset$ and $\alpha\in P_\beta\}$, satisfy the following conditions:

$(C1)$ *If $\alpha\in\mathrm{Dom}\psi$, then $\psi(\alpha)<^*\alpha$ and $\mathfrak{L}_{\psi(\alpha)}\cap\mathfrak{L}^-\neq\emptyset$.*
$(C2)$ *If $\alpha\in\mathrm{Dom}\varphi$, then $\alpha<^*\varphi(\alpha)$ and $\mathfrak{L}_{\varphi(\alpha)}\cap(\mathfrak{L}^+\cup\{e\})\neq\emptyset$.*
$(C3)$ *If $\alpha\in\mathrm{Dom}\psi$ and $\beta\in\mathfrak{Y}$ such that $\psi(\alpha)<^*\beta<^*\alpha$, then $\mathfrak{L}_\beta=\mathfrak{L}_\beta\cap\mathfrak{L}^+$.*
$(C4)$ *If $\alpha\in\mathrm{Dom}\varphi$ and $\beta\in\mathfrak{Y}$ such that $\alpha<^*\beta<^*\varphi(\alpha)$, then $\mathfrak{L}_\beta=\mathfrak{L}_\beta\cap\mathfrak{L}^-$.*
$(C5)$ *If $(\alpha,\beta)\in\mathrm{Dom}\Psi$, then $\alpha\wedge\Psi(\alpha,\beta)\leq^*\beta$.*
$(C6)$ *If $(\alpha,\beta)\in\mathrm{Dom}\Psi$ and $\gamma\in\mathfrak{Y}$ such that $\mathfrak{L}_\gamma\cap\mathfrak{L}^-\neq\emptyset$ and $\alpha\wedge\gamma\leq^*\alpha\wedge\beta$, then $\gamma\leq^*\Psi(\alpha,\beta)$.*

We shall denote this system by $(\mathfrak{L};\pi,\mathfrak{Y};\mathfrak{X},\Phi;\psi,\varphi,\Psi)$ and the related ordered idempotent semigroup $(\mathfrak{L},\circ,\leq)$ by $CRLOB(\mathfrak{L};\pi,\mathfrak{Y};\mathfrak{X},\Phi;\psi,\varphi,\Psi)$.

Theorem 2.1 ([1]). *Let $(\mathfrak{L};\pi,\mathfrak{Y};\mathfrak{X},\Phi;\psi,\varphi,\Psi)$ be a CRLOB-system. Then $CRLOB(\mathfrak{L};\pi,\mathfrak{Y};\mathfrak{X},\Phi;\psi,\varphi,\Psi)$ is a conical idempotent residuated lattice. Conversely, any such residuated lattice can be constructed in this manner.*

3. Residuated lattice-orderability of an idempotent semigroup with an identity

In this section we shall give necessary and sufficient conditions for an idempotent semigroup with an identity to be the semigroup reduct of some conical residuated lattice, which generalize [2, Theorem 5.2].

Let $(\mathfrak{Y}, \leq^*)$ be a semilttice. The subset $\mathfrak{U}$ of $\mathfrak{Y}$ is called an *R-sublattice* of $\mathfrak{Y}$ if $\mathfrak{U}$ is a sublattice of $\mathfrak{Y}$ and satisfies the condition (RS): If $\alpha, \beta \in \mathfrak{U}$, $\alpha \nleq^* \beta$, then there exists δ in $\mathfrak{U}$ such that $\alpha \wedge \delta = \alpha \wedge \beta$ and for all $\delta' \in \mathfrak{U}$ such that $\delta' \wedge \alpha \leq^* \alpha \wedge \beta$, $\delta' \leq^* \delta$. The condition (RS) indeed is to ensure that the set $\{c \in \mathfrak{U} \mid a \wedge c \leq^* b\}$ has a greatest element, for all $a, b \in \mathfrak{U}$.

Let (P_α) be the square point set of $\mathfrak{Y}$. Let $(P_\alpha)^+ = \{\beta \in (P_\alpha) \mid (\exists \gamma, \gamma' \in (P_\alpha))\gamma \vee \gamma'$ does not exists and $\beta \leq^* \gamma\}$ and let $(P_\alpha)^- = (P_\alpha) \setminus (P_\alpha)^+$. (P_α) is called a *pre-sublattice* of $\mathfrak{Y}$ if (P_α) satisfies the following conditions:

$(P1)$ (P_α) is not a sublattice of $\mathfrak{Y}$ and $((P_\alpha)^+ \cup \{\beta\}, \leq^*)$ is a lattice for some $\beta \in \mathfrak{Y}$;

$(P2)$ If $\beta \in (P_\alpha)^+$ and $\gamma \in (P_\alpha)^-$, then $\beta <^* \gamma$;

$(P3)$ If $(P_\alpha)^- \neq \emptyset$, then $(P_\alpha)^-$ is an R-sublattice of $\mathfrak{Y}$.

Lemma 3.1. *Let $(\mathfrak{Y}, \leq^*)$ be a semilattice.*

(1) *If $\alpha, \beta, \gamma \in \mathfrak{Y}$ such that $\beta, \gamma \in (P_\alpha)$, then $\beta \wedge \gamma \in (P_\alpha)$ and $\beta \vee \gamma \in (P_\alpha)$ whenever $\beta \vee \gamma$ exists.*

(2) *If $\alpha, \alpha', \gamma \in \mathfrak{Y}$ such that $\alpha \parallel^* \alpha'$ and $\gamma \in P_\alpha$, then $P_\alpha = P_\gamma$.*

(3) *If $\alpha, \beta, \alpha', \beta' \in \mathfrak{Y}$ such that $\alpha \parallel \alpha', \beta \parallel \beta'$ and $P_\alpha \cap P_\beta \neq \emptyset$, then $P_\alpha = P_\beta$ and $(P_\alpha) = (P_\beta)$.*

(4) *If $\alpha, \beta, \gamma \in \mathfrak{Y}$ such that (P_α) is a pre-sublattice of $\mathfrak{Y}$ and $\beta, \gamma \in (P_\alpha)^+$, then $\beta \wedge \gamma \in (P_\alpha)^+$ and $\beta \vee \gamma \in (P_\alpha)^+$ whenever $\beta \vee \gamma$ exists.*

(5) *If $\alpha, \beta, \gamma \in \mathfrak{Y}$ such that (P_α) is a pre-sublattice of $\mathfrak{Y}$ and $\beta, \gamma \in (P_\alpha)^-$, then $\beta \wedge \gamma, \beta \vee \gamma \in (P_\alpha)^-$.*

Proof. (1) Let $\alpha, \beta, \gamma \in \mathfrak{Y}$ such that $\beta, \gamma \in (P_\alpha)$. We consider the following cases:

- $\beta \leq^* \gamma$ or $\gamma \leq^* \beta$. Then, $\beta \wedge \gamma \in \{\beta, \gamma\} \subseteq (P_\alpha)$.
- $\beta \parallel^* \gamma$. Then, $\beta \wedge \gamma <^* \beta$. If $\beta \wedge \gamma \parallel^* \alpha$, then $\beta \wedge \gamma \in (P_\alpha)$. If $\alpha \leq^* \beta \wedge \gamma$, then $\alpha \leq^* \beta \wedge \gamma <^* \beta$. Since $\beta \in (P_\alpha)$, $\beta \wedge \gamma \in (P_\alpha)$. If $\beta \wedge \gamma <^* \alpha$, then we can claim that $\beta \wedge \gamma \in (P_\alpha)$. Otherwise if $\beta \wedge \gamma \notin (P_\alpha)$, then for all $\xi \in (P_\alpha)$, $\beta \wedge \gamma <^* \xi$. If $P_a^\perp$ exists, then $\beta \wedge \gamma <^* P_a^\perp$. Since $P_a^\perp <^* \beta, \gamma$, $P_a^\perp \leq^* \beta \wedge \gamma$. It's a contradiction. If $P_a^\perp$ doesn't exist, then there exists $\delta \in \mathfrak{Y}$ such that $\beta \wedge \gamma <^* \delta <^* \alpha$ and $\delta <^* \eta$, for all $\eta \in (P_\alpha)$. Since $\beta, \gamma \in (P_\alpha)$, $\delta <^* \beta, \gamma$. It follows that $\delta \leq^* \beta \wedge \gamma$, which contrary to $\beta \wedge \gamma <^* \delta$. Thus, $\beta \wedge \gamma \in (P_\alpha)$.

Similarly, if $\beta \vee \gamma$ exists, then $\beta \wedge \gamma \in (P_\alpha)$.

(2) Let $\alpha, \alpha', \gamma \in \mathfrak{Y}$ such that $\alpha \parallel^* \alpha'$ and $\gamma \in P_\alpha$. To prove that $P_\alpha = P_\gamma$, we consider the following cases:

- $\gamma \parallel^* \alpha$. Firstly, we will prove that $P_\gamma \subseteq P_\alpha$. Let $\delta \in P_\gamma$ such that $\delta \parallel^* \gamma$. If $\delta \parallel^* \alpha$, then $\delta \in P_\alpha$. If $\delta <^* \alpha$, then since $\gamma \parallel^* \alpha$, for any $\zeta \in \mathfrak{Y}$ such that $\delta \leq^* \zeta <^* \alpha$, $\zeta \parallel^* \gamma$. It follows that $\delta \in P_\alpha$. Similarly, if $\alpha <^* \delta$, then $\delta \in P_\alpha$. Let $\delta \in P_\gamma$ such that $\delta <^* \gamma$. Then, since $\gamma \parallel^* \alpha$, either $\delta \parallel^* \alpha$, or $\delta <^* \alpha$. If $\delta \parallel^* \alpha$, then $\delta \in P_\alpha$. If $\delta <^* \alpha$, then for any $\zeta \in \mathfrak{Y}$ such that $\delta \leq^* \zeta <^* \alpha$, either $\zeta \parallel^* \gamma$ or $\zeta <^* \gamma$. If $\zeta <^* \gamma$, then since $\delta \leq^* \zeta$ and $\delta \in P_\alpha$, there exists $\zeta' \in \mathfrak{Y}$ such that $\zeta \parallel^* \zeta'$. It follows that $\delta \in P_\alpha$. Similarly, if $\delta \in P_\gamma$ such that $\gamma \leq^* \delta$, then $\delta \in P_\alpha$. It follows that $P_\gamma \subseteq P_\alpha$. Similarly, $P_\alpha \subseteq P_\gamma$. Consequently, $P_\alpha = P_\gamma$.

- $\gamma \leq^* \alpha$. Firstly, we will prove that $P_\gamma \subseteq P_\alpha$. Let $\delta \in P_\gamma$ such that $\delta \parallel^* \gamma$. Then, either $\delta \parallel^* \alpha$ or $\delta <^* \alpha$. If $\delta \parallel^* \alpha$, then $\delta \in P_\alpha$. If $\delta <^* \alpha$, then for any $\zeta \in \mathfrak{Y}$ such that $\delta \leq^* \zeta <^* \alpha$, either $\zeta \parallel^* \gamma$ or $\gamma <^* \zeta$. If $\gamma <^* \zeta$, then $\gamma <^* \zeta <^* \alpha$. Since $\gamma \in P_\alpha$, there exists $\zeta' \in \mathfrak{Y}$ such that $\zeta \parallel^* \zeta'$. It follows that $\delta \in P_\alpha$. Let $\delta \in P_\gamma$ such that $\delta <^* \gamma$. Then, $\delta <^* \gamma \leq^* \alpha$. Suppose that $\zeta \in \mathfrak{Y}$ such that $\delta \leq^* \zeta <^* \alpha$. Then, either $\zeta \parallel^* \gamma$ or $\zeta \nparallel^* \gamma$. If $\zeta <^* \gamma$, then since $\delta \in P_\gamma$, there exists $\zeta' \in \mathfrak{Y}$ such that $\zeta \parallel^* \zeta'$. If $\gamma \leq^* \zeta$, then since $\gamma \in P_\alpha$, there exists $\zeta' \in \mathfrak{Y}$ such that $\zeta \parallel^* \zeta'$. It follows that $\delta \in P_\alpha$. Let $\delta \in P_\gamma$ such that $\gamma \leq^* \delta$. If $\delta \parallel \alpha$ or $\delta = \alpha$, then $\delta \in P_\alpha$. If $\delta <^* \alpha$, then for any $\zeta \in \mathfrak{Y}$ such that $\delta \leq^* \zeta <^* \alpha$, $\gamma \leq^* \zeta <^* \alpha$. Since $\gamma \in P_\alpha$, there exists $\zeta' \in \mathfrak{Y}$ such that $\zeta \parallel^* \zeta'$. It follows that $\delta \in P_\alpha$. If $\alpha <^* \delta$, then for any $\zeta \in \mathfrak{Y}$ such that $\alpha <^* \zeta \leq^* \delta$, $\gamma <^* \zeta \leq^* \delta$. Since $\delta \in P_\gamma$, there exists $\zeta' \in \mathfrak{Y}$ such that $\zeta \parallel^* \zeta'$, which implies that $\delta \in P_\alpha$. It follows that $P_\gamma \subseteq P_\alpha$. Similarly, $P_\alpha \subseteq P_\gamma$. Consequently, $P_\alpha = P_\gamma$.

- $\alpha <^* \gamma$. By similar arguments as in the prior case, we have $P_\alpha = P_\gamma$.

(3) Let $\alpha, \beta, \alpha', \beta' \in \mathfrak{Y}$ such that $\alpha \parallel \alpha', \beta \parallel \beta'$ and $P_\alpha \cap P_\beta \neq \emptyset$. Let $\gamma \in P_\alpha \cap P_\beta$. Then, by (2), $P_\alpha = P_\gamma = P_\beta$ and so $(P_\alpha) = (P_\beta)$.

(4) Let $\alpha, \beta, \gamma \in \mathfrak{Y}$ such that (P_α) is a pre-sublattice of $\mathfrak{Y}$ and $\beta, \gamma \in (P_\alpha)^+$. Since $\beta \wedge \gamma \leq^* \beta$, $\beta \wedge \gamma \in (P_\alpha)^+$ by (1) and $(P2)$. If $\beta \vee \gamma$ exists, then by (1), $\beta \vee \gamma \in (P_\alpha)$. Since $\beta \in (P_\alpha)^+$, there exist $\delta, \delta' \in (P_\alpha)^+$ such that $\delta \vee \delta'$ doesn't exist and $\beta \leq^* \delta$. Suppose that $\beta \vee \gamma \in (P_\alpha)^-$. Then, by $(P2)$, $\delta, \delta' <^* \beta \vee \gamma$ and so there exists $\zeta \in (P_\alpha)$ such that $\delta, \delta' <^* \zeta <^* \beta \vee \gamma$. If $\zeta \in (P_\alpha)^-$, then by $(P2)$, $\beta, \gamma <^* \zeta$, and so $\beta \vee \gamma \leq^* \zeta$, which contrary to $\zeta <^* \beta \vee \gamma$. If $\zeta \in (P_\alpha)^+$, then there exist $\omega, \omega' \in (P_\alpha)^+$ such that $\omega \vee \omega'$ doesn't exist and $\zeta \leq^* \omega$. Since $\delta \vee \delta'$ doesn't exist, there exists $\xi \in (P_\alpha)$ such that $\delta, \delta' <^* \xi <^* \zeta$. Hence, by $(P2)$, $\xi \in (P_\alpha)^+$. Because $((P_\alpha)^+ \cup \{\eta\}, \leq^*)$ is a lattice for some $\eta \in \mathfrak{Y}$, $\delta \vee_{((P_\alpha)^+ \cup \{\eta\})} \delta' = \omega \vee_{((P_\alpha)^+ \cup \{\eta\})} \omega' = \eta$, which implies that $\eta <^* \xi <^* \zeta \leq^* \omega <^* \eta$. It's a contradiction. Thus, $\beta \vee \gamma \in (P_\alpha)^+$.

(5) Let $\alpha, \beta, \gamma \in \mathfrak{Y}$ such that (P_α) is a pre-sublattice of $\mathfrak{Y}$ and $\beta, \gamma \in (P_\alpha)^-$. Then, by (1), $\beta \wedge \gamma, \beta \vee \gamma \in (P_\alpha)$. Since $\beta <^* \beta \vee \gamma$, $\beta \vee \gamma \in (P_\alpha)^-$ by $(P2)$. Assume that $\beta \wedge \gamma \in (P_\alpha)^+$. Then, there exist $\delta, \delta' \in (P_\alpha)^+$ such that $\delta \vee \delta'$ doesn't exist and $\beta \wedge \gamma \leq^* \delta$. Hence, by $(P2)$, $\delta, \delta' <^* \beta, \gamma$. Thus, $\delta, \delta' \leq^* \beta \wedge \gamma$.

It follows that $\delta' \leq^* \beta \wedge \gamma = \delta$. Thus, $\delta \vee \delta' = \delta$, which contrary to $\delta \vee \delta'$ doesn't exist. Consequently, $\beta \wedge \gamma \in (P_\alpha)^-$. □

Lemma 3.2. *Let* $\mathfrak{L} = CRLOB(\mathfrak{L}; \pi_{\mathcal{D}}, \mathfrak{Y}; \mathfrak{X}, \Phi; \psi, \varphi, \Psi)$ *be a conical idempotent residuated lattice.*

(1) *If* $\alpha \in \mathrm{Dom}\psi$*, then for each* $\beta \in \mathfrak{Y}$*,* $\psi(\alpha) \leq^* \beta$ *or* $\beta \leq^* \psi(\alpha)$*. Moreover, in this case,* $\psi(\alpha) \prec^* \varphi(\psi(\alpha))$*.*

(2) *If* $\alpha \in \mathrm{Dom}\varphi$*, then for each* $\beta \in \mathfrak{Y}$*,* $\varphi(\alpha) \leq^* \beta$ *or* $\beta \leq^* \varphi(\alpha)$*. Moreover, if* $\varphi(\alpha) \neq 1$*, then* $\psi(\varphi(\alpha)) \prec^* \varphi(\alpha)$*.*

(3) *If* $(\alpha, \beta) \in \mathrm{Dom}\Phi$ *and* $\alpha \vee \beta$ *doesn't exist, then for each* $\gamma \in \mathfrak{Y}$*,* $\Phi(\alpha, \beta) \leq^* \gamma$ *or* $\gamma \leq^* \Phi(\alpha, \beta)$*.*

(4) *If* $\alpha, \eta, \zeta \in \mathfrak{Y}$ *such that* $\zeta \in (P_\alpha)^+, \eta \in (P_\alpha)^-$*, then* $\zeta <^* \eta$ *and* $D_\zeta = \{a_\zeta\}$*.*

(5) *If* $\alpha, \beta, \gamma \in \mathfrak{Y}$ *such that* $\beta, \gamma \in (P_\alpha)^+$*, then* $\beta \wedge \gamma \in (P_\alpha)^+$ *and* $\beta \vee \gamma \in (P_\alpha)^+$ *whenever* $\beta \vee \gamma$ *exists.*

(6) *If* $\alpha, \beta, \gamma \in \mathfrak{Y}$ *such that* $(P_\alpha)^+ \neq \emptyset$ *and* $\beta, \gamma \in (P_\alpha)^-$*, then* $\beta \wedge \gamma, \beta \vee \gamma \in (P_\alpha)^-$*. Moreover, if* $\beta \neq 1$*, then* $D_\beta = \{b_\beta\}$*.*

Proof. (1) Suppose that there exists $\beta \in \mathfrak{Y}$ such that $\psi(\alpha) \parallel^* \beta$. Since $D_{\psi(\alpha)}$ contains $b_{\psi(\alpha)}$ and by $(CSP3)$ of Definition 2.1, $D_\beta = \{b_\beta\}$, which, together with $a_\alpha \in D_\alpha$ by noting that $\alpha \in \mathrm{Dom}\psi$, derives $\alpha \nparallel^* \beta$. By $(C1)$ of Definition 2.4, $\psi(\alpha) <^* \alpha$, so $\beta <^* \alpha$, hence by $(CSP4)$ of Definition 2.1, $\psi(\alpha) \vee \beta$ exists and $D_{\psi(\alpha) \vee \beta} = \{b_{\psi(\alpha) \vee \beta}\}$. Thus $\psi(\alpha) <^* \psi(\alpha) \vee \beta <^* \alpha$. Therefore, by $(C3)$ of Definition 2.4, $D_{\psi(\alpha) \vee \beta} = \{a_{\psi(\alpha) \vee \beta}\}$, contrary to $D_{\psi(\alpha) \vee \beta} = \{b_{\psi(\alpha) \vee \beta}\}$. We conclude that for each $\beta \in \mathfrak{Y}$, $\varphi(\alpha) \leq^* \beta$ or $\beta \leq^* \varphi(\alpha)$. Suppose that there exists $\beta \in \mathfrak{Y}$ such that $\psi(\alpha) <^* \beta <^* \varphi(\psi(\alpha))$. Then, by $(C4)$ of Definition 2.4, $D_\beta = \{b_\beta\}$. Since $\alpha \in \mathrm{Dom}\psi$, $a_\alpha \in D_\alpha$ and by $(CSP3)$ of Definition 2.1, $\alpha \nparallel^* \beta$. Assume $\alpha <^* \beta$. Then $\psi(\alpha) <^* \alpha <^* \beta <^* \varphi(\psi(\alpha))$ and so by $(C4)$ of Definition 2.4, $D_\alpha = \{b_\alpha\}$, contrary to $\alpha \in \mathrm{Dom}\psi$. This implies $\beta <^* \alpha$ and so by $(C3)$ of Definition 2.4, $D_\beta = \{a_\beta\}$, contrary to $D_\beta = \{b_\beta\}$. Thus, $\psi(\alpha) \prec^* \varphi(\psi(\alpha))$.

(2) It is similar to (1).

(3) Suppose that $(\alpha, \beta) \in \mathrm{Dom}\Phi$ and $\alpha \vee \beta$ doesn't exist. Then, by $(B1)$ of Definition 2.3, $\alpha, \beta <^* \Phi(\alpha, \beta)$ and $D_{\Phi(\alpha,\beta)}$ contains $a_{\Phi(\alpha,\beta)}$. Since $\alpha \vee \beta$ doesn't exist, there exists $\gamma \in \mathfrak{Y}$ such that $\alpha, \beta <^* \gamma <^* \Phi(\alpha, \beta)$. Hence, by $(B2)$ of Definition 2.3, $D_\gamma = \{b_\gamma\}$, so by (2), for each $\delta \in \mathfrak{Y}$, either $\varphi(\gamma) \leq^* \delta$ or $\delta \leq^* \varphi(\gamma)$, thereby $\varphi(\gamma) <^* \Phi(\alpha, \beta)$ or $\Phi(\alpha, \beta) \leq^* \varphi(\gamma)$. If $\varphi(\gamma) <^* \Phi(\alpha, \beta)$, then $\alpha, \beta <^* \gamma <^* \varphi(\gamma) <^* \Phi(\alpha, \beta)$ and so by $(B2)$ of Definition 2.3, $D_{\varphi(\gamma)} = \{b_{\varphi(\gamma)}\}$, contrary to $D_{\varphi(\gamma)}$ contains $a_{\varphi(\gamma)}$. If $\Phi(\alpha, \beta) <^* \varphi(\gamma)$, then $\gamma <^* \Phi(\alpha, \beta) <^* \varphi(\gamma)$ and so by $(C4)$ of Definition 2.4, $D_{\Phi(\alpha,\beta)} = \{b_{\Phi(\alpha,\beta)}\}$, contrary to $D_{\Phi(\alpha,\beta)}$ contains $a_{\Phi(\alpha,\beta)}$. Hence, $\Phi(\alpha, \beta) = \varphi(\gamma)$ and so by (2), for each $\delta \in \mathfrak{Y}$, either $\Phi(\alpha, \beta) \leq^* \delta$ or $\delta \leq^* \Phi(\alpha, \beta)$.

(4) Suppose to the contrary that $\eta \leq^* \zeta$ or $\zeta \parallel^* \eta$. If $\eta \leq^* \zeta$, then since $\zeta \in (P_\alpha)^+$, there exist $\zeta', \delta' \in (P_\alpha)$ such that $\zeta' \vee \delta'$ doesn't exist and $\zeta \leq^* \zeta'$, hence $\eta \leq^* \zeta'$, which implies that $\eta \in (P_\alpha)^+$, contrary to $\eta \in (P_\alpha)^- = (P_\alpha) \setminus (P_\alpha)^+$. If

$\zeta \parallel^* \eta$, then since $\zeta \in (P_\alpha)^+$, there exist $\zeta', \delta' \in (P_\alpha)$ such that $\zeta' \vee \delta'$ doesn't exist and $\zeta \leq^* \zeta'$. If $\eta <^* \zeta'$, then $\eta \in (P_\alpha)^+$, contrary to $\eta \in (P_\alpha)^- = (P_\alpha) \setminus (P_\alpha)^+$. If $\eta \parallel^* \zeta'$, then since $\zeta' \vee \delta'$ doesn't exist, $\zeta' \parallel^* \delta'$, hence by $(CPS3)$ and $(CSP4)$ of Definition 2.1, $D_{\delta'} = \{a_{\delta'}\}$ and $D_{\zeta'} = \{a_{\zeta'}\}$, which imply that $D_\eta = \{a_\eta\}$. Since $\zeta' \vee \delta'$ doesn't exist, there exists $\delta \in \mathfrak{Y}$ such that $\zeta', \delta' <^* \delta <^* \Phi(\zeta', \delta')$, hence by $(B2)$ of Definition 2.3, $D_\delta = \{b_\delta\}$. By $(CSP3)$ of Definition 2.1, $\eta \nparallel^* \delta$. Since $\eta \parallel^* \zeta'$ and $\zeta' <^* \delta$, $\eta <^* \delta$. Note that $\eta \in (P_\alpha)^-$, $\eta \vee \zeta'$ and $(\eta \vee \zeta') \vee \delta'$ exist, hence we have $\zeta', \delta' <^* (\eta \vee \zeta') \vee \delta' \leq^* \delta <^* \Phi(\zeta', \delta')$. Since $\zeta' \vee \delta'$ doesn't exist, there exists $\xi \in \mathfrak{Y}$ such that $\zeta', \delta' <^* \xi <^* (\eta \vee \zeta') \vee \delta' <^* \Phi(\zeta', \delta')$. By $(B2)$ of Definition 2.3, $D_\xi = \{b_\xi\}$. By $(CSP3)$, $\xi <^* \eta$ or $\eta <^* \xi$. Assume that $\xi <^* \eta$. Then, $\zeta', \delta' <^* \xi <^* \eta <^* \delta <^* \Phi(\zeta', \delta')$. By $(B2)$, $D_\eta = \{b_\eta\}$, which contrary to $D_\eta = \{a_\eta\}$. Thus, $\eta <^* \xi$. It follows that $\eta \vee \zeta', \delta' \leq^* \xi <^* (\eta \vee \zeta') \vee \delta'$, contrary to $(\eta \vee \zeta') \vee \delta' \leq^* \xi$. Thus, $\eta \leq^* \zeta$ and $\zeta \parallel^* \eta$ are both impossible. Consequently, $\zeta <^* \eta$. Since $D_{\zeta'} = \{a_{\zeta'}\}$, by $(1, 2)$, $\psi(\zeta') \prec^* \varphi(\psi(\zeta')) \leq^* \zeta \leq^* \zeta'$ and so by $(C3)$ of Definition 2.4, $D_\zeta = \{a_\zeta\}$.

(5) Let $\alpha, \beta, \gamma \in \mathfrak{Y}$ such that $\beta, \gamma \in (P_\alpha)^+$. Then, there exist $\zeta, \zeta', \delta, \delta' \in (P_\alpha)$ such that $\beta \leq^* \zeta$, $\gamma \leq^* \delta$, $\zeta \vee \zeta'$ and $\delta \vee \delta'$ don't exist. Hence, $\beta \wedge \gamma \leq \zeta$ which implies that $\beta \wedge \gamma \in (P_\alpha)^+$ by noting that $\beta \wedge \gamma \in (P_\alpha)$ by Lemma 3.1(1). Suppose that $\beta \vee \gamma$ exists. Then, by Lemma 3.1(1), $\beta \vee \gamma \in (P_\alpha)$. If $\beta \nparallel^* \gamma$, then $\beta \vee \gamma \in \{\beta, \gamma\} \subseteq (P_\alpha)^+$. If $\beta \parallel^* \gamma$, then by Lemma 3.1(1), $\beta \vee \gamma \in (P_\alpha)$. If $\beta \vee \gamma \leq^* \zeta$, then $\beta \vee \gamma \in (P_\alpha)^+$. If $\beta \vee \gamma \parallel^* \zeta$, then by (4), $\beta \vee \gamma \in (P_\alpha)^+$. If $\zeta <^* \beta \vee \gamma$, then $\beta \leq^* \zeta <^* \beta \vee \gamma$. Assume that $\beta \vee \gamma \notin (P_\alpha)^+$. Then, $\beta \vee \gamma \in (P_\alpha)^-$ by Lemma 3.1(1). Since $\zeta, \zeta' \in (P_\alpha)$ and $\zeta \vee \zeta'$ doesn't exist, $\zeta \parallel \zeta'$ and so $\zeta, \zeta' \in (P_\alpha)^+$. Hence, by (4), $\zeta, \zeta' <^* \beta \vee \gamma$ and by $(CSP3)$ and $(CSP4)$ of Definition 2.1, $D_\zeta = \{a_\zeta\}$, $D_{\zeta'} = \{a_{\zeta'}\}$. If $\gamma <^* \zeta'$, then for all $\eta \in \mathfrak{Y}$ such that $\zeta, \zeta' <^* \eta$, $\beta, \gamma <^* \eta$. Hence, $\beta \vee \gamma \leq^* \eta$ and so $\zeta \vee \zeta' = \beta \vee \gamma$, which contrary to $\zeta \vee \zeta'$ doesn't exist. If $\gamma \parallel^* \zeta'$, then by $(CSP2)$, $D_\gamma = \{a_\gamma\}$ and $D_\beta = \{a_\beta\}$. By $(B1)$ of Definition 2.3, $\zeta, \zeta' <^* \Phi(\zeta, \zeta')$. We claim that $\alpha' \leq^* \Phi(\zeta, \zeta')$, for all $\alpha' \in (P_\alpha)$. If $\Phi(\zeta, \zeta') = 1$, then $\alpha' \leq^* \Phi(\zeta, \zeta')$, for all $\alpha' \in (P_\alpha)$. If $\Phi(\zeta, \zeta') \neq 1$, then $\Phi(\zeta, \zeta') <^* 1$. Suppose that there exists $\gamma' \in (P_\alpha)$ such that $\Phi(\zeta, \zeta') \leq^* \gamma'$. Then, by $(B1)$ of Definition 2.3, $\zeta, \zeta' <^* \Phi(\zeta, \zeta')$, hence there exists $\xi \in \mathfrak{Y}$ such that $\zeta, \zeta' <^* \xi <^* \Phi(\zeta, \zeta')$ by noting that $\zeta \vee \zeta'$ doesn't exist. By $(B2)$ of Definition 2.3, $D_\xi = \{b_\xi\}$. It follows from the proof of (3) that $\Phi(\zeta, \zeta') = \varphi(\xi)$. By (2), $\psi(\varphi(\xi)) \prec^* \Phi(\zeta, \zeta') = \varphi(\xi)$. By (1-3), $\alpha \leq^* \psi(\varphi(\xi)) \prec^* \Phi(\zeta, \zeta')$ or $\psi(\varphi(\xi)) \prec^* \Phi(\zeta, \zeta') \leq^* \alpha$. If $\alpha \leq^* \psi(\varphi(\xi)) \prec^* \Phi(\zeta, \zeta')$, then by the definition of (P_α) and (1-3), $\beta' \leq^* \psi(\varphi(\xi)) \prec^* \Phi(\zeta, \zeta')$, for all $\beta' \in (P_\alpha)$, which implies that $\gamma' \notin (P_\alpha)$ by noting that $\Phi(\zeta, \zeta') \leq^* \gamma'$, a contradiction. If $\psi(\varphi(\xi)) \prec^* \Phi(\zeta, \zeta') \leq^* \alpha$, then $\beta, \gamma \notin (P_\alpha)$, a contradiction. Thus, $\alpha' <^* \Phi(\zeta, \zeta')$, for all $\alpha' \in (P_\alpha)$. Consequently, $\alpha' \leq^* \Phi(\zeta, \zeta')$, for all $\alpha' \in (P_\alpha)$. It follows that $\beta \vee \gamma \leq^* \Phi(\zeta, \zeta')$. Since $\zeta \vee \zeta'$ doesn't exist,there exists $\omega \in \mathfrak{Y}$ such that $\zeta, \zeta' <^* \omega <^* \beta \vee \gamma \leq^* \Phi(\zeta, \zeta')$. By $(B2)$ of Definition 2.3, $D_\omega = \{b_\omega\}$. Since $D_\gamma = \{a_\gamma\}$ and $\gamma \parallel^* \zeta'$, $\gamma <^* \omega$ by $(CSP3)$ of Definition 2.1. It follows that $\beta \vee \gamma \leq^* \omega$. It's a contradiction. Consequently, $\beta \vee \gamma \in (P_\alpha)^+$.

(6) Let $\alpha, \beta, \gamma \in \mathfrak{Y}$ such that $(P_\alpha)^+ \neq \emptyset$ and $\beta, \gamma \in (P_\alpha)^-$. Then, there exist $\zeta, \zeta' \in (P_\alpha)^+$ such that $\zeta \vee \zeta'$ doesn't exist and $\beta \wedge \gamma, \beta \vee \gamma \in (P_\alpha)$ by Lemma 3.1(1). Since $\beta \in (P_\alpha)^-$ and $\beta <^* \beta \vee \gamma$, $\beta \vee \gamma \in (P_\alpha)^-$ by (4). If $\beta \nparallel^* \gamma$, then $\beta \wedge \gamma \in \{\beta, \gamma\} \subseteq (P_\alpha)^-$. Suppose that $\beta \parallel^* \gamma$. Then, by the proof of (5), $\zeta, \zeta' <^* \beta \wedge \gamma <^* \beta, \gamma <^* \Phi(\zeta, \zeta')$, and so by $(B2)$ of Definition 2.3, $D_\beta = \{b_\beta\}, D_\gamma = \{b_\gamma\}$ and $D_{\beta\wedge\gamma} = \{b_{\beta\wedge\gamma}\}$. Thus, by (4), $\beta \wedge \gamma \in (P_\alpha)^-$. If $\beta \neq 1$, then by the proof of (5), $\zeta, \zeta' <^* \beta <^* \Phi(\zeta, \zeta')$, and so by $(B2)$ of Definition 2.3, $D_\beta = \{b_\beta\}$. □

Theorem 3.1. *Let $\mathfrak{L}$ be an idempotent semigroup with identity e and $\mathfrak{Y}$ be the structure semilattice with greatest element 1 of $\mathfrak{L}$. Then, $\mathfrak{L}$ is the semigroup reduct of some conical idempotent residuated lattice if and only if $\mathfrak{L}$ satisfies the following conditions:*

$(SR1)$ For each $\alpha \in \mathfrak{Y}$, either (P_α) is a sublattice of $\mathfrak{Y}$ or (P_α) a pre-sublattice of $\mathfrak{Y}$ and $|D_\alpha| = 1$.

$(SR2)$ Each $\mathcal{D}-$class of $\mathfrak{L}$ contains at most two elements.

$(SR3)$ If $a \in D_\alpha$ and $b \in D_\beta$ such that $\alpha <^* \beta$, then $ab = ba = a$.

$(SR4)$ If $\alpha, \beta \in \mathfrak{Y}$ such that $\alpha \parallel^* \beta$, then $|D_\alpha| = |D_\beta| = |D_{\alpha\wedge\beta}| = 1$.

$(SR5)$ If (P_α) is a pre-sublattice of $\mathfrak{Y}$, then α satisfies the following conditions:

$(D1)$ There exists $\alpha^+ \in \mathfrak{Y}$ satisfying

(a) For each $\beta \in \mathfrak{Y}$, either $\alpha^+ \leq^* \beta$ or $\beta \leq^* \alpha^+$;

(b) If $\alpha \neq 1$, then $\alpha <^* \alpha^+$;

(c) If $\alpha^+ \neq 1$, then there exists $\gamma \in \mathfrak{Y}$ such that $\gamma \prec^* \alpha^+$ and for each $\beta \in \mathfrak{Y}$, either $\gamma \leq^* \beta$ or $\beta \leq^* \gamma$;

(d) If $\beta \in \mathfrak{Y}$ such that, for all $\alpha' \in (P_\alpha)$, $\alpha' <^* \beta <^* \alpha^+$, then $|D_\beta| = 1$ and (P_β) is a R-sublattice;

$(D2)$ There exists $\alpha^* \in \mathfrak{Y}$ satisfying

(a) For each $\beta \in \mathfrak{Y}$, either $\alpha^* \leq^* \beta$ or $\beta \leq^* \alpha^*$;

(b) $\alpha^* <^* \alpha$;

(c) There exists $\gamma \in \mathfrak{Y}$ such that $\alpha^* \prec^* \gamma$ and for each $\beta \in \mathfrak{Y}$, either $\gamma \leq^* \beta$ or $\beta \leq^* \gamma$;

(d) If $\beta \in \mathfrak{Y}$ such that, for all $\alpha' \in (P_\alpha)$, $\alpha^* <^* \beta <^* \alpha'$, then $|D_\beta| = 1$ and (P_β) is a sublattice.

$(SR6)$ If $\alpha \in \mathfrak{Y}$ such that $|D_\alpha| = 2$, then $|D_{P_\alpha^\top}| = 1$ whenever $P_\alpha^\top$ exists and $P_\alpha^\top \neq \alpha$, $|D_{P_\alpha^\perp}| = 1$ whenever $P_\alpha^\perp$ exists and $P_\alpha^\perp \neq \alpha$, $\overline{(P_\alpha)} = \{\beta \in (P_\alpha) | \alpha \leq^* \beta\}$ is a R-sublattice of $\mathfrak{Y}$ and α satisfies $(D1)$ and $(D2)$.

$(SR7)$ If $\alpha \in \mathfrak{Y} \setminus \{1\}$ such that $|D_\alpha| = 1$ and (P_α) is a R-sublattice of $\mathfrak{Y}$, then α satisfies $(D1)$ or $(D2)$.

$(SR8)$ If $\alpha \in \mathfrak{Y} \setminus \{1\}$ such that $|D_\alpha| = 1$ and (P_α) isn't a R-sublattice of $\mathfrak{Y}$, then α satisfies $(D2)$.

Proof. Let $\mathfrak{L}$ be a conical idempotent residuated lattice. By Theorem 2.1, $\mathfrak{L} = CRLOB(\mathfrak{L}; \pi, \mathfrak{Y}; \mathfrak{X}, \Phi; \psi, \varphi, \Psi)$. By Lemma 4.7 of [1], $\pi = \{\mathfrak{L}_\alpha \mid \alpha \in \mathfrak{Y}\} =$

$\pi_{\mathcal{D}} = \{D_\alpha \mid \alpha \in \mathfrak{Y}\}$. By $(CSP1)$ and $(CSP2)$ of Definition 2.1, $\mathfrak{L}$ satisfies $(SR2)$. By Lemma 2.3 and Theorem 2.1, condition $(SR3)$ holds. By $(CSP3)$ of Definition 2.1, $\mathfrak{L}$ satisfies $(SR4)$.

Now, we prove that $\mathfrak{L}$ satisfies $(SR1)$. Let $\alpha \in \mathfrak{Y}$ and suppose that (P_α) isn't a sublattice of $\mathfrak{L}$. Then, there exist $\beta, \gamma \in (P_\alpha)$ such that $\beta \vee \gamma$ doesn't exist, hence $(P_\alpha)^+ \neq \emptyset$, so by Lemma 3.2(4,5), for all $\beta', \gamma' \in (P_\alpha)^+$, $D_{\beta'} = \{a_{\beta'}\}$, $\beta' \wedge \gamma' \in (P_\alpha)^+$ and $\beta' \vee \gamma' \in (P_\alpha)^+$ whenever $\beta' \vee \gamma'$ exists. To show that $((P_\alpha)^+ \cup \{\Phi(\beta,\gamma)\}, \leq^*)$ is a lattice, we consider the following cases:

- $\Phi(\beta,\gamma) \neq 1$. By the proof of Lemma 3.2(5), $\alpha' <^* \Phi(\beta,\gamma)$, for all $\alpha' \in (P_\alpha)$. Suppose that $\beta', \gamma' \in (P_\alpha)^+$ such that $\beta' \vee \gamma'$ doesn't exist. Then, by Lemma 3.2(3), either $\Phi(\beta',\gamma') \leq^* \Phi(\beta,\gamma)$ or $\Phi(\beta,\gamma) \leq^* \Phi(\beta',\gamma')$. Assume that $\Phi(\beta',\gamma') <^* \Phi(\beta,\gamma)$. Then, $\Phi(\beta',\gamma') \neq 1$, so by the proof of Lemma 3.2(5), we have $\alpha' <^* \Phi(\beta',\gamma')$, for all $\alpha' \in (P_\alpha)$, hence $\beta, \gamma <^* \Phi(\beta',\gamma') <^* \Phi(\beta,\gamma)$. Thus, by $(B2)$ of Definition 2.3, $D_{\Phi(\beta',\gamma')} = \{b_{\Phi(\beta',\gamma')}\}$, contrary to $D_{\Phi(\beta',\gamma')}$ contains $a_{\Phi(\beta',\gamma')}$ by $(B1)$ of Definition 2.3. If $\Phi(\beta,\gamma) <^* \Phi(\beta',\gamma')$, then $\beta', \gamma' <^* \Phi(\beta,\gamma) <^* \Phi(\beta',\gamma')$, hence by $(B2)$ of Definition 2.3, $D_{\Phi(\beta,\gamma)} = \{b_{\Phi(\beta,\gamma)}\}$, contrary to $D_{\Phi(\beta,\gamma)}$ contains $a_{\Phi(\beta,\gamma)}$ by $(B1)$ of Definition 2.3, which implies $\Phi(\beta,\gamma) <^* \Phi(\beta',\gamma')$ is impossible. Thus, $\Phi(\beta,\gamma) = \Phi(\beta',\gamma')$. If there exists $\delta \in (P_\alpha)$ such that $\beta, \gamma <^* \delta$, then $\beta, \gamma <^* \delta <^* \Phi(\beta,\gamma)$, hence by $(B2)$ of Definition 2.3, $D_\delta = \{b_\delta\}$. Thus, by Lemma 3.1(4), $\delta \notin (P_\alpha)^+$, which means that $((P_\alpha)^+ \cup \{\Phi(\beta,\gamma)\}, \leq^*)$ is a lattice. This proves that $(P1)$ holds.

- $\Phi(\beta,\gamma) = 1$. Then, $\alpha' \leq^* \Phi(\beta,\gamma)$, for all $\alpha' \in (P_\alpha)$ and there exists $\delta \in \mathfrak{Y}$ such that $\beta, \gamma <^* \delta <^* \Phi(\beta,\gamma)$ by noting that $\beta \vee \gamma$ doesn't exist. By $(B2)$ of Definition 2.3, $D_\delta = \{b_\delta\}$. It follows from the proof of Lemma 3.2(3) that $\Phi(\beta,\gamma) = \varphi(\delta)$. Suppose that $\beta', \gamma' \in (P_\alpha)^+$ and $\beta' \vee \gamma'$ doesn't exist. Then, $\Phi(\beta',\gamma') \leq^* \Phi(\beta,\gamma) = 1$. Assume $\Phi(\beta',\gamma') <^* \Phi(\beta,\gamma)$. Then, $\Phi(\beta',\gamma') \neq 1$, so by the proof of the prior case, we have $\alpha' <^* \Phi(\beta',\gamma')$, for all $\alpha' \in (P_\alpha)$, hence $\beta, \gamma <^* \Phi(\beta',\gamma') <^* \Phi(\beta,\gamma)$. Thus, by $(B2)$ of Definition 2.3, $D_{\Phi(\beta',\gamma')} = \{b_{\Phi(\beta',\gamma')}\}$, contrary to $D_{\Phi(\beta',\gamma')}$ contains $a_{\Phi(\beta',\gamma')}$ by $(B1)$ of Definition 2.3. We conclude that $\Phi(\beta',\gamma') = \Phi(\beta,\gamma) = 1$. Furthermore, by Lemma 3.2(4), we have $\delta \notin (P_\alpha)^+$, which means that $((P_\alpha)^+ \cup \{\Phi(\beta,\gamma)\}, \leq^*)$ is a lattice. This proves that $(P1)$ holds.

We conclude that $(P1)$ holds. By Lemma 3.2(4), (P_α) satisfies $(P2)$. We shall show that (P_α) satisfies $(P3)$. By Lemma 3.2(6), $(P_\alpha)^-$ is a sublattice of $\mathfrak{Y}$. Let $\delta, \delta' \in (P_\alpha)^-$ be such that $\delta \parallel^* \delta'$ or $\delta' <^* \delta$. If $\delta = 1$, then $\delta' <^* \delta = 1$ and for all $\zeta \in (P_\alpha)^-$ such that $\zeta \wedge 1 \leq^* 1 \wedge \delta'$, we have $\zeta \leq^* \delta'$. If $\delta \neq 1$, then by Lemma 3.2(6), $D_{\delta'} = \{b_{\delta'}\}$ and $D_\delta = \{b_\delta\}$. So, $(\delta, \delta') \in \mathrm{Dom}\Psi$, hence by $(C5)$ of Definition 2.4, $\delta \wedge \Psi(\delta,\delta') \leq^* \delta'$, which implies that $\delta \wedge \Psi(\delta,\delta') \leq^* \delta \wedge \delta'$. Since $\delta \wedge \delta' \leq^* \delta \wedge \delta'$, by $(C6)$ of Definition 2.4, $\delta' \leq^* \Psi(\delta,\delta')$, hence $\delta \wedge \delta' \leq^* \delta \wedge \Psi(\delta,\delta')$, thus $\delta \wedge \Psi(\delta,\delta') = \delta \wedge \delta'$. Let $\gamma' \in (P_\alpha)^-$ such that $\delta \wedge \gamma' \leq^* \delta \wedge \delta'$. By

Lemma 3.2(6), $D_{\gamma'} = \{b_{\gamma'}\}$, hence by $(C6)$ of Definition 2.4, $\gamma' \leq^* \Psi(\delta,\delta')$. We shall show $\Psi(\delta,\delta') \in (P_\alpha)$. For this, we consider the following cases:

- $\Psi(\delta,\delta') \parallel^* \alpha$. Then $\Psi(\delta,\delta') \in (P_\alpha)$.
- $\Psi(\delta,\delta') \leq^* \alpha$. Then $\delta' \leq^* \Psi(\delta,\delta') \leq^* \alpha$, and so $\Psi(\delta,\delta') \in (P_\alpha)$.
- $\alpha <^* \Psi(\delta,\delta')$ and $\Psi(\delta,\delta') = \delta'$. Then, $\Psi(\delta,\delta') \in (P_\alpha)$.
- $\alpha <^* \Psi(\delta,\delta')$ and $\Psi(\delta,\delta') \neq \delta'$. Then, by $(C6)$ of Definition 2.4, $\delta' <^* \Psi(\delta,\delta')$. If $\Psi(\delta,\delta') \not\parallel^* \delta$, then $\Psi(\delta,\delta') \wedge \delta \in \{\Psi(\delta,\delta'),\delta\}$, contrary to $\Psi(\delta,\delta')\wedge\delta = \delta'\wedge\delta \leq^* \delta'$, thus $\Psi(\delta,\delta') \parallel^* \delta$, which implies that $\Psi(\delta,\delta') \in P_\delta$ and $\delta \in P_\alpha$. By the similar argument as in the proof of Lemma 3.1(2), $P_\delta \subseteq P_\alpha$ and so $\Psi(\delta,\delta') \in P_\alpha \subseteq (P_\alpha)$.

We conclude that $\Psi(\delta,\delta') \in (P_\alpha)$. Since $\delta' \leq^* \Psi(\delta,\delta')$ and $\delta' \in (P_\alpha)^-$, by $(P2)$, $\Psi(\delta,\delta') \in (P_\alpha)^-$, which implies that $(P_\alpha)^-$ satisfies (RS). Thus, $(P_\alpha)^-$ is a R-sublattice of $\mathfrak{Y}$. This proves that $(P3)$ holds. Thereby, (P_α) is a pre-sublattice of $\mathfrak{Y}$. By Lemma 3.2(4,6) and $(CSP1)$ of Definition 2.1, $|D_\alpha| = 1$. Consequently, $\mathfrak{L}$ satisfies $(SR1)$.

Next, we prove that $\mathfrak{L}$ satisfies $(SR5)$. Let (P_α) be a pre-sublattice of $\mathfrak{Y}$. Then, there exist $\beta,\gamma \in (P_\alpha)$ such that $\beta \parallel^* \gamma$ and $\beta \vee \gamma$ doesn't exist, hence by $(CSP3)$ and $(CSP4)$ of Definition 2.1, $D_\beta = \{a_\beta\}, D_\gamma = \{a_\gamma\}$. If $\alpha \neq 1$, then by Lemma 3.2(4,6), $D_\alpha = \{a_\alpha\}$ or $D_\alpha = \{b_\alpha\}$. Let $\alpha^+ = \Phi(\beta,\gamma)$. If $\alpha \neq 1$, then by the proof of $|D_\alpha| = 1$ in the case that $\mathcal{L}$ satisfies $(SR1)$, $\alpha <^* \Phi(\beta,\gamma)$ and by Lemma 3.2(3), for each $\beta' \in \mathfrak{Y}$, either $\beta' \leq^* \Phi(\beta,\gamma)$ or $\Phi(\beta,\gamma) \leq^* \beta'$. This proves that $\Phi(\beta,\gamma)$ satisfies $(D1)(a)$ and (b). Since $\beta \vee \gamma$ doesn't exist, there exists $\delta \in \mathfrak{Y}$ such that $\beta,\gamma <^* \delta <^* \Phi(\beta,\gamma)$ and by $(B2)$ of Definition 2.3, $D_\delta = \{b_\delta\}$. If $\Phi(\beta,\gamma) \neq 1$, then by the proof of Lemma 3.2(3), $\Phi(\beta,\gamma) = \varphi(\delta)$ and by Lemma 3.2(2), $\psi(\Phi(\beta,\gamma)) = \psi(\varphi(\delta)) \prec^* \varphi(\delta) = \Phi(\beta,\gamma)$ and for each $\gamma' \in \mathfrak{Y}$, either $\psi(\Phi(\beta,\gamma)) \leq^* \gamma'$ or $\gamma' \leq^* \psi(\Phi(\beta,\gamma))$. This shows that $\Phi(\beta,\gamma)$ satisfies $(D1)(c)$. If $\beta' \in \mathfrak{Y}$ such that, for all $\alpha' \in (P_\alpha)$, $\alpha' <^* \beta' <^* \Phi(\beta,\gamma)$, then $\beta' \notin (P_\alpha)$. Furthermore, we can claim that there exists $\delta' \in \mathfrak{Y}$ such that $\alpha <^* \delta' <^* \beta'$ and for each $\gamma' \in \mathfrak{Y}$, either $\delta' \leq^* \gamma'$ or $\gamma' \leq^* \delta'$. Otherwise, for any $\delta' \in \mathfrak{Y}$ such that $\alpha <^* \delta' <^* \beta'$, there exists $\gamma' \in \mathfrak{Y}$ such that $\delta' \parallel^* \gamma'$. Assume $\zeta \in \mathfrak{Y}$ such that $\alpha' \leq^* \zeta$, for all $\alpha' \in (P_\alpha)$. If $\beta' \parallel^* \zeta$, then $\beta' \in (P_\alpha)$, contrary to $\beta' \notin (P_\alpha)$. If $\zeta <^* \beta'$, then there exists $\gamma' \in \mathfrak{Y}$ such that $\zeta \parallel^* \gamma'$, so $\gamma' <^* \beta'$. Since $\alpha <^* \beta'$, $\alpha \parallel^* \gamma'$ or $\alpha <^* \gamma' <^* \beta'$, hence $\gamma' \in (P_\alpha)$, which implies that $\gamma' \leq^* \zeta$, contrary to $\zeta \parallel^* \gamma'$. Consequently, $\beta' \leq^* \zeta$, which derives that $\beta' = P_\alpha^\top \in (P_\alpha)$, contrary to $\beta' \notin (P_\alpha)$. We conclude that there exists $\delta' \in \mathfrak{Y}$ such that $\alpha <^* \delta' <^* \beta'$ and for each $\gamma' \in \mathfrak{Y}$, either $\delta' \leq^* \gamma'$ or $\gamma' \leq^* \delta'$, which implies that $\alpha' \leq^* \delta'$, for all $\alpha' \in (P_\alpha)$, and $\delta' \leq^* \xi$, for all $\xi \in (P_{\beta'})$. If $\Phi(\beta,\gamma) \neq 1$, then by the proof of $(P1)$ and Lemma 3.2, $\beta,\gamma <^* \delta' \leq^* \xi \leq^* \psi(\Phi(\beta,\gamma)) \prec^* \Phi(\beta,\gamma)$, for all $\xi \in (P_{\beta'})$, so by $(B2)$ of Definition 2.3, $D_\xi = \{b_\xi\}$, for all $\xi \in (P_{\beta'})$. If $\Phi(\beta,\gamma) = 1$, then $\beta,\gamma <^* \delta' \leq^* \xi \leq^* \Phi(\beta,\gamma)$, for all $\xi \in (P_{\beta'})$, hence by $(B2)$ of Definition 2.3 and $(CSP1)$ of Definition 2.1, $D_\xi = \{b_\xi\}$, for all $\xi \in (P_{\beta'})$. We conclude that $D_\xi = \{b_\xi\}$, for all $\xi \in (P_{\beta'}) \setminus \{1\}$. By similar arguments as in the proof of $(P3)$, $(P_{\beta'})$ is a R-sublattice of $\mathfrak{Y}$, which implies that $(D1)(d)$ holds. We have proved α satisfies $(D1)$. Let $\alpha^* = \psi(\beta)$. Then, by Lemma 3.2(1), $\psi(\beta)$ satisfies $(D2)(a)$ and $\psi(\beta) \prec^* \varphi(\psi(\beta)) \leq^* \alpha'$, for all $\alpha' \in (P_\alpha)$, which implies that $\psi(\beta)$ satisfies $(D2)$ $(b-c)$. If $\gamma' \in \mathfrak{Y}$ such that, for all $\alpha' \in (P_\alpha)$, $\psi(\beta) <^* \gamma' <^* \alpha'$, then $\gamma' \notin (P_\alpha)$ and $\psi(\beta) <^* \gamma' <^* \beta$, so by $(C3)$ of Definition 2.4, $D_{\gamma'} = \{a_{\gamma'}\}$ and $|D_{\gamma'}| = 1$ and by Lemma 3.2(1), $\psi(\beta) \prec^* \varphi(\psi(\beta)) \leq^* \xi$, for all $\xi \in (P_{\gamma'})$. By similar arguments as in the proof of $(D1)(d)$, there exists $\zeta \in \mathfrak{Y}$ with $\gamma' <^* \zeta <^* \alpha$ such that for each $\zeta' \in \mathfrak{Y}$, either $\zeta' \leq^* \zeta$ or $\zeta \leq^* \zeta'$, which implies that $\zeta' \leq^* \alpha'$, for all $\alpha' \in (P_\alpha)$, and $\xi \leq^* \zeta$, for all $\xi \in (P_{\gamma'})$. Suppose that $(P_{\gamma'})$ isn't a sublattice of $\mathfrak{Y}$. Then, there exist $\xi, \xi' \in P_{\gamma'}$ such that $\xi \parallel^* \xi'$ and $\xi \vee \xi'$ doesn't exist. By $(CSP3)$ and $(CSP4)$ of Definition 2.1, $D_\xi = \{a_\xi\}$ and $D_{\xi'} = \{a_{\xi'}\}$. Since $\beta \parallel^* \gamma$, by Lemma 3.2(3), either $\Phi(\xi, \xi') <^* \beta$ or $\beta <^* \Phi(\xi, \xi')$. If $\Phi(\xi, \xi') <^* \beta$, then there exists $\delta \in \mathfrak{Y}$ such that $\psi(\beta) <^* \xi, \xi' <^* \delta <^* \Phi(\xi, \xi') <^* \beta$, hence by $(C3)$ of Definition 2.4 $D_\delta = \{a_\delta\}$, but by $(B2)$ of Definition 2.3, $D_\delta = \{b_\delta\}$, a contradiction. If $\beta <^* \Phi(\xi, \xi')$, then $\xi, \xi' <^* \beta <^* \Phi(\xi, \xi')$, hence by $(B2)$ of Definition 2.3, $D_\beta = \{b_\beta\}$, contrary to $D_\beta = \{a_\beta\}$. Consequently, $(P_{\gamma'})$ is a sublattice of $\mathfrak{Y}$. Thus, α satisfies $(D2)$. This proves that $\mathfrak{L}$ satisfies $(SR5)$.

We shall prove that $\mathfrak{L}$ satisfies $(SR6)$. Let $\alpha \in \mathfrak{Y}$ such that D_α contains two elements. Then, by $(\underline{CSP2})$ of Definition 2.1, $D_\alpha = \{a_\alpha, b_\alpha\}$. By Lemma 3.2(1,2), for all $\gamma \in (P_\alpha) \setminus \overline{(P_\alpha)}$, $\psi(\alpha) \prec^* \varphi(\psi(\alpha)) \leq^* \gamma <^* \alpha$, hence by $(C3)$ of Definition 2.4, $D_\gamma = \{a_\gamma\}$, which implies that $|D_{P_\alpha^\perp}| = 1$ if $P_\alpha^\perp$ exists and $P_\alpha^\perp \neq \alpha$. If $\varphi(\alpha) \neq 1$, then by Lemma 3.2(1,2), for all $\beta \in \overline{(P_\alpha)} \setminus \{\alpha\}$, $\alpha <^* \beta \leq^* \psi(\varphi(\alpha)) \prec^* \varphi(\alpha)$, hence by $(C4)$ of Definition 2.4, $D_\beta = \{b_\beta\}$, which implies that $|D_{P_\alpha^\top}| = 1$ whenever $P_\alpha^\top$ exists and $P_\alpha^\top \neq \alpha$. If $\varphi(\alpha) = 1$, then by Lemma 3.2(2), for all $\beta \in \overline{(P_\alpha)} \setminus \{\alpha, 1\}$, $\alpha <^* \beta <^* \varphi(\alpha) = 1$, hence by $(C4)$ of Definition 2.4, $D_\beta = \{b_\beta\}$, which implies that $|D_{P_\alpha^\top}| = 1$ whenever $P_\alpha^\top$ exists and $\underline{P_\alpha^\top \neq \alpha}$. By similar arguments as in the proof that $(P_\alpha)^-$ is a R-sublattice of $\mathfrak{Y}$, $\overline{(P_\alpha)}$ is a R-sublattice of $\mathfrak{Y}$. Let $\alpha^+ = \varphi(\alpha)$ and $\alpha^* = \psi(\alpha)$. By similar arguments as in the proof of $(SR5)$, α satisfies $(D1)$ and $(D2)$. This proves that $\mathfrak{L}$ satisfies $(SR6)$. Similarly, $\mathfrak{L}$ satisfies $(SR7)$.

Next we shall show that $\mathfrak{L}$ satisfies $(SR8)$. Let $\alpha \in \mathfrak{Y} \setminus \{1\}$ be such that $|D_\alpha| = 1$ and (P_α) isn't a R-sublattice of $\mathfrak{Y}$. If (P_α) is a pre-sublattice of $\mathfrak{Y}$, then by $(SR5)$, α satisfies $(D2)$. Let (P_α) be a sublattice of $\mathfrak{Y}$. If $D_\alpha = \{a_\alpha\}$, then we let $\alpha^* = \psi(\alpha)$, so by similar arguments to those in the case that (P_α) is a pre-sublattice of $\mathfrak{Y}$, α satisfies $(D2)$. If $D_\alpha = \{b_\alpha\}$, then we consider the following cases:

- For all $\beta \in (P_\alpha)$, $D_\beta = \{b_\beta\}$. Then, by similar arguments to the proof of $(D1)(d)$ in the case that (P_α) is a pre-sublattice of $\mathfrak{Y}$, (P_α) is a R-sublattice of $\mathfrak{Y}$, contrary to (P_α) isn't a R-sublattice of $\mathfrak{Y}$.

- There exists $\beta \in (P_\alpha)$ such that D_β contains a_β. Since $D_\alpha = \{b_\alpha\}$, by $(CSP3)$ of Definition 2.1, $\alpha \not\parallel^* \beta$. If $\alpha <^* \beta$, then by Lemma 3.2 and $\beta \in (P_\alpha)$, $\psi(\beta) \prec^* \varphi(\psi(\beta)) \leq^* \alpha <^* \beta$, thus by $(C3)$ of Definition 2.4, $D_\alpha = \{a_\alpha\}$, contrary to $D_\alpha = \{b_\alpha\}$. If $\beta <^* \alpha$, then we let $\alpha^* = \psi(\beta)$, so by similar arguments to those in the case that (P_α) is a pre-sublattice of $\mathfrak{Y}$, α satisfies $(D2)$.

We conclude that $\mathfrak{L}$ satisfies $(SR8)$.

Conversely, suppose that $\mathfrak{L} = \bigcup_{\alpha\in\mathfrak{Y}} D_\alpha$ is an idempotent semigroup with an identity e and satisfies conditions (SR1-8), where $\mathfrak{Y}$ is the structure semilattice with greatest element 1 of $\mathfrak{L}$. By a routine calculation, $D_1 = \{e\}$. We let $e = a_1 = b_1$. Put $\mathscr{D} = \{D_\alpha \mid \alpha \in \mathfrak{Y}\}$, $Y_1 = \{\alpha \in \mathfrak{Y} \mid |D_\alpha| = 2\}$ and $P = \{\alpha \in \mathfrak{Y} \mid |D_\alpha| = 1, (P_\alpha)$ is pre-sublattice of $\mathfrak{Y}$ and there exists $\alpha' \in \mathfrak{Y}$ such that $\alpha \parallel \alpha'\}$. We define a binary relation $\sim$ on P as follows: for $\alpha, \beta \in P$, $\alpha \sim \beta$ if and only if $P_\alpha = P_\beta$. By lemma 3.1(2,3), $\sim$ is an equivalence relation on P. It follows that $P_\alpha = \{\beta \in P \mid \beta \sim \alpha\}$ for every $\alpha \in P$. Let $Y_2 \subseteq P$ be such that for every $\alpha \in P$, $|Y_2 \cap P_\alpha| = 1$. For any $\alpha \in Y_1$, we denote one element of D_α by a_α and the other element of D_α by b_α. If $\beta \in \overline{(P_\alpha)} \setminus \{\alpha, 1\}$, then we let $D_\beta = \{b_\beta\}$. If $\beta' \in (P_\alpha) \setminus \overline{(P_\alpha)}$, then we let $D_{\beta'} = \{a_{\beta'}\}$. For any $\alpha \in Y_2$, let $D_\gamma = \{a_\gamma\}$, for all $\gamma \in (P_\alpha)^+$, and let $D_{\gamma'} = \{b_{\gamma'}\}$, for all $\gamma' \in (P_\alpha)^- \setminus \{1\}$. Let $\alpha \in Y_1 \cup Y_2$. We distinguish two cases.

- For each $\beta \in \mathfrak{Y}$ such that $\alpha' <^* \beta$ for all $\alpha' \in (P_\alpha)$, $|D_\beta| = 1$ and (P_β) is a R-sublattice of $\mathfrak{Y}$. By conditions $(SR5-6)$, we can choose $\alpha^* \in \mathfrak{Y}$ and 1 is interpreted as α^+. Hence, we obtain the closed interval $H_\alpha = [\alpha^*, 1]$ and by $(SR5-6)$, there exists element γ in $\mathfrak{Y}$ such that $\alpha^* \prec^* \gamma$. For any $\beta \in \mathfrak{Y}$ such that $\alpha' <^* \beta <^* 1$, for all $\alpha' \in (P_\alpha)$, let $D_\beta = \{b_\beta\}$. For any $\gamma' \in \mathfrak{Y}$ such that $\alpha^* <^* \gamma' <^* \alpha'$, for all $\alpha' \in (P_\alpha)$, let $D_{\gamma'} = \{a_{\gamma'}\}$. If $|D_{\alpha^*}| = 1$, then we define $D_{\alpha^*} = \{b_{\alpha^*}\}$.

- There exists $\beta \in \mathfrak{Y}$ such that $\alpha' <^* \beta$, for all $\alpha' \in (P_\alpha)$ and $|D_\beta| = 2$ or $\alpha' <^* \beta$, for all $\alpha' \in (P_\alpha)$ and (P_β) isn't a R-sublattice of $\mathfrak{Y}$. By conditions $(SR5-6)$, we can choose α^* and α^+ in $\mathfrak{Y} \setminus \{1\}$. Hence, we obtain the closed interval $H_\alpha = [\alpha^*, \alpha^+]$ and by $(SR5-6)$, there exist elements β, γ in $\mathfrak{Y}$ such that $\beta \prec^* \alpha^+$ and $\alpha^* \prec^* \gamma$. For any $\beta' \in \mathfrak{Y}$ such that $\alpha' <^* \beta' <^* \alpha^+$, for all $\alpha' \in (P_\alpha)$, let $D_{\beta'} = \{b_{\beta'}\}$. If $|D_{\alpha^+}| = 1$, then we define $D_{\alpha^+} = \{a_{\alpha^+}\}$. For any $\gamma' \in \mathfrak{Y}$ such that $\alpha^* <^* \gamma' <^* \alpha'$, for all $\alpha' \in (P_\alpha)$, let $D_{\gamma'} = \{a_{\gamma'}\}$. If $|D_{\alpha^*}| = 1$, then we define $D_{\alpha^*} = \{b_{\alpha^*}\}$.

Let $\alpha, \beta \in Y_1 \cup Y_2$ be such that $\alpha <^* \beta$. If $|H_\alpha \cap H_\beta| > 2$, we choose α^+ to be γ which covers β^*. Then, we obtain a family of closed intervals $\{H_\alpha \mid \alpha \in Y_1 \cup Y_2\}$ such that for $\alpha, \beta \in Y_1 \cup Y_2$ with $\alpha \neq \beta$, $|H_\alpha \cap H_\beta| \leq 2$.

We arbitrarily choose $\alpha \in \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta$ such that $\alpha \neq 1$. We distinguish two cases.

Case A. (P_α) is a R-sublattice. By $(SR7)$, α satisfies $(D1)$ or $(D2)$. We need to consider two subcases:

(1) α satisfies $(D2)$.

- If there exists $\gamma \in Y_1 \cup Y_2$ such that $\gamma <^* \alpha$ and $[\gamma, \alpha] \cap (\bigcup_{\delta \in Y_1 \cup Y_2} H_\delta) \subseteq H_\gamma$, then there exists $\zeta \in H_\gamma$ such that $\zeta \prec^* \gamma^+$. Hence, when ζ is interpreted as α^*, we have the set $H_\alpha = \{\beta \in \mathfrak{Y} \mid [\alpha^*, \beta] \subseteq (\mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta) \cup \{\alpha^*, \gamma^+\}\}$. For each $\beta \in H_\alpha$ such that $\gamma^+ <^* \beta$, let $D_\beta = \{a_\beta\}$.

- If there doesn't exist $\gamma \in Y_1 \cup Y_2$ such that $\gamma <^* \alpha$ and $[\gamma, \alpha] \cap (\bigcup_{\delta \in Y_1 \cup Y_2} H_\delta) \subseteq H_\gamma$, then we can claim that $\alpha^* \in \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta$. Otherwise, if $\alpha^* \in \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta$, then there exists $\gamma' \in Y_1 \cup Y_2$ such that $\alpha^* \in H_{\gamma'}$ and $\gamma' <^* \alpha$, hence there exist $\delta, \delta', \delta'' \in Y_1 \cup Y_2$ such that $\gamma' <^* \delta' <^* \delta <^* \delta'' <^* \alpha$, so $\alpha^* <^* \delta$ and $\delta \notin (P_\alpha)$, which imply that $\delta <^* \alpha'$, for all $\alpha' \in (P_\alpha)$. By $(D2)(d)$, $|D_\delta| = 1$ and (P_δ) is a sublattice of $\mathfrak{Y}$, contrary to $|D_\delta| = 2$ or $|D_\delta| = 1$ and (P_δ) is pre-sublattice of $\mathfrak{Y}$. Consequently, $\alpha^* \in \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta$ and so $[\alpha^*, \alpha] \subseteq \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta$. If the set $\{\beta' \in \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta | \beta'$ satisfies $(D2)(a-d)$ for $\alpha\}$ has the least element β, then by $(D2)(c)$, there exists $\xi \in \mathfrak{Y}$ such that $\beta \prec^* \xi$ and for each $\xi' \in \mathfrak{Y}$, either $\xi \leq^* \xi'$ or $\xi' \leq^* \xi$. Furthermore, we have for any $\zeta \in \mathfrak{Y}$ such that $\zeta <^* \xi$ and $[\zeta, \xi] \subseteq \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta$, (P_ζ) is a R-sublattice; whereas, if (P_ζ) isn't a R-sublattice, then by $(SR8)$, there exists $\zeta^* \in \mathfrak{Y}$ such that $(D2)(a-d)$, by similar arguments as in case: $\alpha^* \in \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta$, $\zeta^* \in \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta$, which, together with $[\zeta, \xi] \subseteq \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta$, implies that $\zeta^* \in \{\beta' \in \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta | \beta'$ satisfies $(D2)(a-d)$ for $\alpha\}$ and $\zeta^* <^* \xi$, contrary to ξ is the least element of the set $\{\beta' \in \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta | \beta'$ satisfies $(D2)(a-d)$ for $\alpha\}$. We conclude that for any $\zeta \in \mathfrak{Y}$ such that $\zeta <^* \xi$ and $[\zeta, \xi] \subseteq \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta$, (P_ζ) is a R-sublattice. Thereby, when ξ is interpreted as β^+, we have the set $H_\beta = \{\beta' \in \mathfrak{Y} \mid [\beta', \beta^+] \subseteq \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta\}$; when β is interpreted as α^*, we have the set $H_\alpha = \{\alpha' \in \mathfrak{Y} \mid [\alpha^*, \alpha'] \subseteq \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta\}$. For each $\beta' \in H_\beta$, let

$$D_{\beta'} = \begin{cases} \{b_{\beta'}\} & \text{if } \beta' \neq \beta^+, \\ \{a_{\beta'}\} & \text{if } \beta' = \beta^+. \end{cases}$$

For each $\alpha' \in H_\alpha$, let

$$D_{\alpha'} = \begin{cases} \{a_{\alpha'}\} & \text{if } \alpha' \neq \alpha^*, \\ \{b_{\alpha'}\} & \text{if } \alpha' = \alpha^*. \end{cases}$$

If the set $\{\beta' \in \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta | \beta'$ satisfies $(D2)(a-d)$ for $\alpha\}$ hasn't the least element, then we choose β in the set $\{\beta' \in \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta | \beta'$ satisfies $(D2)(a-d)$ for $\alpha\}$. Hence, when β is interpreted as α^*, we have the set $H_\alpha = \{\alpha' \in \mathfrak{Y} \mid [\alpha^*, \alpha'] \subseteq \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta\}$. For each $\alpha' \in H_\alpha$, let

$$D_{\alpha'} = \begin{cases} \{a_{\alpha'}\} & \text{if } \alpha' \neq \alpha^*, \\ \{b_{\alpha'}\} & \text{if } \alpha' = \alpha^*. \end{cases}$$

(2) α doesn't satisfy $(D1)$. Then, α satisfies $(D2)$. We can claim that for each $\beta \in \mathfrak{Y}$ such that $[\beta, \alpha] \subseteq \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta$, (P_β) is a R-sublattice. Otherwise, if for some $\beta \in \mathfrak{Y}$ such that $[\beta, \alpha] \subseteq \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta$, (P_β) isn't a R-sublattice, then by condition $(SR8)$, there exists $\beta^* \in \mathfrak{Y}$ satisfying $(D2)$, which, together with $[\beta, \alpha] \subseteq \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta$, implies that we can choose α^* in $\mathfrak{Y}$, a contradiction. Consequently, for each $\beta \in \mathfrak{Y}$ such that $[\beta, \alpha] \subseteq \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta$, (P_β) is a R-sublattice. We consider the following cases:

- If there exists $\beta \in Y_1 \cup Y_2$ such that $\alpha <^* \beta$ and $[\alpha, \beta] \cap (\bigcup_{\delta \in Y_1 \cup Y_2} H_\delta) \subseteq H_\beta$. Hence, there exists $\eta \in H_\beta$ such that $\beta^* \prec^* \eta$. Thus, when η is interpreted as α^+, we have the set $H_\alpha = \{\gamma \in \mathfrak{Y} \mid [\gamma, \alpha^+] \subseteq (\mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta) \cup \{\alpha^+, \beta^*\}\}$. For each $\gamma \in H_\alpha$ such that $\gamma <^* \beta^*$, let $D_\gamma = \{b_\gamma\}$.

- If there doesn't exist $\beta \in Y_1 \cup Y_2$ such that $\alpha <^* \beta$ and $[\alpha, \beta] \cap (\bigcup_{\delta \in Y_1 \cup Y_2} H_\delta) \subseteq H_\beta$, then by similar arguments as in (1), $\alpha^+ \in \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta$. This means that there exists $\gamma \in \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta$ satisfying $(D1)(a-d)$. If $\gamma = 1$, then when 1 is interpreted as α^+, we have the set $H_\alpha = \{\alpha' \in \mathfrak{Y} \mid [\alpha', 1] \subseteq \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta\}$. For each $\alpha' \in H_\alpha$ such that $\alpha' <^* 1$, let $D_{\alpha'} = \{b_{\alpha'}\}$. If $\gamma \neq 1$, then by $(D1)(c)$, there exists $\eta \in \mathfrak{Y}$, such that $\eta \prec^* \gamma$ and for each $\beta' \in \mathfrak{Y}$, either $\eta \leq^* \beta'$ or $\beta' \leq^* \eta$. Hence, when γ is interpreted as α^+, we have the set $H_\alpha = \{\alpha' \in \mathfrak{Y} \mid [\alpha', \alpha^+] \subseteq \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta\}$; when η is interpreted as β^*, we have the set $H_\beta = \{\beta' \in \mathfrak{Y} \mid [\beta^*, \beta'] \subseteq \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta\}$. For each $\alpha' \in H_\alpha$, let

$$D_{\alpha'} = \begin{cases} \{b_{\alpha'}\} & \text{if } \alpha' \neq \alpha^+, \\ \{a_{\alpha'}\} & \text{if } \alpha' = \alpha^+. \end{cases}$$

  For each $\beta' \in H_\beta$, let

$$D_{\beta'} = \begin{cases} \{a_{\beta'}\} & \text{if } \beta' \neq \beta^*, \\ \{b_{\beta'}\} & \text{if } \beta' = \beta^*. \end{cases}$$

Case B. (P_α) isn't a R-sublattice. Then, by $(SR8)$, α satisfies $(D2)$. We need to consider two subcases:

- If there exists $\gamma \in Y_1 \cup Y_2$ such that $\gamma <^* \alpha$ and $[\gamma, \alpha] \cap (\bigcup_{\delta \in Y_1 \cup Y_2} H_\delta) \subseteq H_\gamma$, then there exists $\zeta \in H_\gamma$ such that $\zeta \prec^* \gamma^+$. Hence, when ζ is interpreted as α^*, we have the set $H_\alpha = \{\beta \in \mathfrak{Y} \mid [\alpha^*, \beta] \subseteq (\mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta) \cup \{\alpha^*, \gamma^+\}\}$. For each $\beta \in H_\alpha$ such that $\gamma^+ <^* \beta$, let $D_\beta = \{a_\beta\}$.

- If there doesn't exist $\gamma \in Y_1 \cup Y_2$ such that $\gamma <^* \alpha$ and $[\gamma, \alpha] \cap (\bigcup_{\delta \in Y_1 \cup Y_2} H_\delta) \subseteq H_\gamma$, then we can claim that $\alpha^* \in \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta$. Otherwise, if $\alpha^* \in \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta$, then there exists $\gamma' \in Y_1 \cup Y_2$ such that $\alpha^* \in H_{\gamma'}$ and $\gamma' <^* \alpha$, hence there exist $\delta, \delta', \delta'' \in Y_1 \cup Y_2$ such that $\gamma' <^* \delta' <^* \delta <^* \delta'' <^*$

α, so $\alpha^* <^* \delta$ and $\delta \notin (P_\alpha)$, which imply that $\delta <^* \alpha'$, for all $\alpha' \in (P_\alpha)$. By $(D2)(d)$, $|D_\delta| = 1$ and (P_δ) is a sublattice of $\mathfrak{Y}$, contrary to $|D_\delta| = 2$ or $|D_\delta| = 1$ and (P_δ) is a pre-sublattice of $\mathfrak{Y}$. Consequently, $\alpha^* \in \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta$ and so $[\alpha^*, \alpha] \subseteq \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta$. If the set $\{\beta' \in \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta | \beta'$ satisfies $(D2)(a-d)$ for $\alpha\}$ has the least element β, then by $(D2)(c)$, there exists $\xi \in \mathfrak{Y}$ such that $\beta \prec^* \xi$ and for each $\xi' \in \mathfrak{Y}$, either $\xi \leq^* \xi'$ or $\xi' \leq^* \xi$. Furthermore, we have for any $\zeta \in \mathfrak{Y}$ such that $\zeta <^* \xi$ and $[\zeta, \xi] \subseteq \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta$, (P_ζ) is a R-sublattice; whereas, if (P_ζ) isn't a R-sublattice, then by $(SR8)$, there exists $\zeta^* \in \mathfrak{Y}$ such that $(D2)(a-d)$, by similar arguments as in case: $\alpha^* \in \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta$, $\zeta^* \in \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta$, which, together with $[\zeta, \xi] \subseteq \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta$, implies that $\zeta^* \in \{\beta' \in \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta | \beta'$ satisfies $(D2)(a-d)$ for $\alpha\}$ and $\zeta^* <^* \xi$, contrary to ξ is the least element of the set $\{\beta' \in \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta | \beta'$ satisfies $(D2)(a-d)$ for $\alpha\}$. We conclude that for any $\zeta \in \mathfrak{Y}$ such that $\zeta <^* \xi$ and $[\zeta, \xi] \subseteq \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta$, (P_ζ) is a R-sublattice. Thereby, when ξ is interpreted as β^+, we have the set $H_\beta = \{\beta' \in \mathfrak{Y} \mid [\beta', \beta^+] \subseteq \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta\}$; when β is interpreted as α^*, we have the set $H_\alpha = \{\alpha' \in \mathfrak{Y} \mid [\alpha^*, \alpha'] \subseteq \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta\}$. For each $\beta' \in H_\beta$, let

$$D_{\beta'} = \begin{cases} \{b_{\beta'}\}, & \text{if } \beta' \neq \beta^+, \\ \{a_{\beta'}\}, & \text{if } \beta' = \beta^+. \end{cases}$$

For each $\alpha' \in H_\alpha$, let

$$D_{\alpha'} = \begin{cases} \{a_{\alpha'}\}, & \text{if } \alpha' \neq \alpha^*, \\ \{b_{\alpha'}\}, & \text{if } \alpha' = \alpha^*. \end{cases}$$

If the set $\{\beta' \in \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta | \beta'$ satisfies $(D2)(a-d)$ for $\alpha\}$ hasn't the least element, then we choose β in the set $\{\beta' \in \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta | \beta'$ satisfies $(D2)(a-d)$ for $\alpha\}$. Hence, when β is interpreted as α^*, we have the set $H_\alpha = \{\alpha' \in \mathfrak{Y} \mid [\alpha^*, \alpha'] \subseteq \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta\}$. For each $\alpha' \in H_\alpha$, let

$$D_{\alpha'} = \begin{cases} \{a_{\alpha'}\}, & \text{if } \alpha' \neq \alpha^*, \\ \{b_{\alpha'}\}, & \text{if } \alpha' = \alpha^*. \end{cases}$$

We repeat the proceeding above, replacing $\bigcup_{\delta \in Y_1 \cup Y_2} H_\delta$ by corresponding subsets of $\mathfrak{Y}$. Then, we obtain a family sets $\{H_\alpha \mid \alpha \in Y_3\}$ such that, for all $\beta, \gamma \in Y_1 \cup Y_2 \cup Y_3$ with $\beta \neq \gamma$, $|H_\beta \cap H_\gamma| \leq 2$. We can claim that $\mathfrak{Y} \setminus \{1\} \subseteq \bigcup_{\alpha \in Y_1 \cup Y_2 \cup Y_3} H_\alpha$. Let $\mathscr{H}$ be a subset of power set $\mathscr{P}(\mathfrak{Y} \setminus \{1\})$ such that for $A \in \mathscr{H}$ and $\beta \in A$, there exists $\alpha \in Y_1 \cup Y_2 \cup Y_3$ such that $\beta \in H_\alpha$. Then $(\mathscr{H}, \subseteq)$ is a partial ordered set. Suppose that $\{A_i \mid i \in I\}$ is a totally ordered subset of $\mathscr{H}$. It's easy to see $\bigcup_{i \in I} A_i$ is a upper bound of $\{A_i \mid i \in I\}$ and $\bigcup_{i \in I} A_i \in \mathscr{H}$. By Zorn's Lemma, $\mathscr{H}$ has a maximal element B. Furthermore, we have $B = \mathfrak{Y} \setminus \{1\}$. Otherwise, if $B \neq \mathfrak{Y} \setminus \{1\}$, then there exists $\beta \in \mathfrak{Y} \setminus \{1\}$

such that $\beta \notin B$, so by the procedure above, there exists $\alpha \in Y_1 \cup Y_2 \cup Y_3$ such that $\beta \in H_\alpha$, thus $B \cup \{\beta\} \in \mathscr{H}$ contrary to B is a maximal element of $\mathscr{H}$. Consequently, $\mathfrak{Y} \setminus \{1\} \subseteq \bigcup_{\alpha \in Y_1 \cup Y_2 \cup Y_3} H_\alpha$. Let $\mathfrak{L}^+ = \{a_\alpha \mid \alpha \in \mathfrak{Y} \setminus \{1\}\}$ and $\mathfrak{L}^- = \{b_\beta \mid \beta \in \mathfrak{Y} \setminus \{1\}\}$. Since $\mathcal{D}$ is a congruence on $\mathfrak{L}$, $\pi_{\mathcal{D}} = \{D_\alpha \mid \alpha \in \mathfrak{Y}\}$ is a partition on $\mathfrak{L}$. By $(SR4)$ and the proceeding above, $\pi_{\mathcal{D}} = \{D_\alpha \mid \alpha \in \mathfrak{Y}\}$ is a conical semilattice partition on $\mathfrak{L}$. Let $\mathfrak{X} = \{\alpha \in \mathfrak{Y} \mid |D_\alpha| = 2 \text{ and } (D_\alpha, \cdot) \text{ is a left zero semigroup}\}$ be a subset of $\mathfrak{Y}$. Then, $\mathfrak{X}$ is a band subset of $\{D_\alpha \mid \alpha \in \mathfrak{Y}\}$. Let $\mathfrak{C} = \{(\alpha, \beta) \in \mathfrak{Y} \times \mathfrak{Y} \mid \alpha \parallel^* \beta, a_\alpha \in D_\alpha\}$. We define a mapping Φ from $\mathfrak{C}$ to $\mathfrak{Y}$ by

$$\Phi(\alpha, \beta) = \begin{cases} \alpha \vee \beta & \text{if } \alpha \vee \beta \text{ exists,} \\ \gamma^+ \ (\alpha \in H_\gamma) & \text{otherwise.} \end{cases}$$

By the definition of Φ and Definition 2.3, $\pi_{\mathcal{D}} = \{D_\alpha \mid \alpha \in \mathfrak{Y}\}$ is a $CLOB-$system.

Now we define two mappings ψ and φ. Put Dom$\psi = \{\alpha \in \mathfrak{Y} \setminus \{1\} \mid a_\alpha \in D_\alpha\}$ and Dom$\varphi = \{\alpha \in \mathfrak{Y} \setminus \{1\} \mid b_\alpha \in D_\alpha\}$. For any $\alpha \in$ Domψ, there exists $\beta \in Y_1 \cup Y_2 \cup Y_3$ such that $\alpha \in H_\beta$ and $\alpha \neq \beta^*$. Let

$$\psi(\alpha) = \begin{cases} \beta^* & \text{if } \alpha \neq \beta^+, \\ \gamma \ (\gamma \prec^* \beta^+) & \text{if } \alpha = \beta^+. \end{cases}$$

For any $\alpha \in$ Domφ, there exists $\beta \in Y_1 \cup Y_2 \cup Y_3$ such that $\alpha \in H_\beta$ and $\alpha \neq \beta^+$. Let

$$\varphi(\alpha) = \begin{cases} \beta^+ & \text{if } \alpha \neq \beta^*, \\ \gamma \ (\beta^* \prec^* \gamma) & \text{if } \alpha = \beta^*. \end{cases}$$

It's easy to see that ψ and φ are well defined. Put $\mathfrak{R} = \{(\alpha, \beta) \in \mathfrak{Y} \times \mathfrak{Y} \mid \alpha \parallel^* \beta, b_\alpha \in D_\alpha \text{ or } \beta <^* \alpha, \alpha \in P_\beta, b_\beta \in D_\beta\}$. Now we define a mapping Ψ from $\mathfrak{R}$ to $\mathfrak{Y}$. If $(\alpha, \beta) \in \mathfrak{R}$, then there exists $\gamma \in Y_1 \cup Y_2 \cup Y_3$ such that $\alpha, \beta \in H_\gamma$. We consider the following cases:

- If $\gamma \in Y_1$, then by the definition of $\mathfrak{R}$ and H_γ, $\alpha, \beta \in \{\xi \in H_\gamma | \gamma \leq^* \xi\}$. If $\alpha, \beta \in \overline{(P_\gamma)}$,then since by the definition of $\overline{(P_\gamma)}$, $\overline{(P_\gamma)}$ is a R-sublattice, there exists a unique element δ in $\overline{(P_\gamma)}$ such that $\alpha \wedge \delta = \alpha \wedge \beta$ and for all $\delta' \in \overline{(P_\gamma)}$ such that $\delta' \wedge \alpha \leq \alpha \wedge \beta$, $\delta' \leq^* \delta$. We define $\Psi(\alpha, \beta) = \delta$. If $\alpha, \beta \in \{\xi \in H_\gamma | \gamma \leq^* \xi\} \setminus \overline{(P_\gamma)}$, then by the definition of H_γ, (P_β) is a R-sublattice, hence there exists a unique element δ in (P_β) such that $\alpha \wedge \delta = \alpha \wedge \beta$ and for all $\delta' \in (P_\beta)$ such that $\delta' \wedge \alpha \leq \alpha \wedge \beta$, $\delta' \leq^* \delta$. We define $\Psi(\alpha, \beta) = \delta$.

- If $\gamma \in Y_2$, then by the definition of $\mathfrak{R}$ and H_γ, $\alpha, \beta \in (H_\gamma)^- = \{\alpha' \in H_\gamma | (\exists \beta' \in (P_\gamma)^-) \beta' \leq^* \alpha'\}$. If $\alpha, \beta \in (P_\gamma)^-$, then since by the definition of $(P_\gamma)^-$, $(P_\gamma)^-$ is a R-sublattice, there exists a unique element δ in $(P_\gamma)^-$ such that $\alpha \wedge \delta = \alpha \wedge \beta$ and for all $\delta' \in (P_\beta)^-$ such that $\delta' \wedge \alpha \leq \alpha \wedge \beta$,

$\delta' \leq^* \delta$. We define $\Psi(\alpha, \beta) = \delta$. If $\alpha, \beta \in (H_\gamma)^- \setminus (P_\gamma)^-$, then (P_β) is a R-sublattice by the definition of H_γ. Hence, there exists a unique element δ in (P_β) such that $\alpha \wedge \delta = \alpha \wedge \beta$ and for all $\delta' \in (P_\beta)$ such that $\delta' \wedge \alpha \leq \alpha \wedge \beta$, $\delta' \leq^* \delta$. We define $\Psi(\alpha, \beta) = \delta$.

- If $\gamma \in Y_3$, then by the definition of $\mathfrak{R}$ and H_γ, (P_β) is a R-sublattice. Hence. there exists a unique element δ in (P_β) such that $\alpha \wedge \delta = \alpha \wedge \beta$ and for all $\delta' \in (P_\beta)$ such that $\delta' \wedge \alpha \leq \alpha \wedge \beta$, $\delta' \leq^* \delta$. We define $\Psi(\alpha, \beta) = \delta$.

By Definition 2.4 and a routine computation, we can prove that $(\mathfrak{L}; \pi_{\mathcal{D}}, \mathfrak{Y}; \mathfrak{X}, \Phi; \psi, \varphi, \Psi)$ is a $CRLOB$-system. By Theorem 2.1, $\mathfrak{L} = CRLOB(\mathfrak{L}; \pi_{\mathcal{D}}, \mathfrak{Y}; \mathfrak{X}, \Phi; \psi, \varphi, \Psi)$ is a conical idempotent residuated lattice. Finally, we shall prove that $(\mathfrak{L}, \cdot)$ is the semigroup reduct of $CRLOB(\mathfrak{L}; \pi_{\mathcal{D}}, \mathfrak{Y}; \mathfrak{X}, \Phi; \psi, \varphi, \Psi)$. Let $c \in D_\alpha, d \in D_\beta$. We consider the following cases:

- If $\alpha <^* \beta$ and $c = a_\alpha, d = a_\beta$, then by the definition of $\circ$ in Lemma 2.3, $c \circ d = a_\alpha \circ a_\beta = a_{\alpha\wedge\beta} = a_\alpha = c$ and $d \circ c = a_\beta \circ a_\alpha = a_{\alpha\wedge\beta} = a_\alpha = c$. By $(SR3)$, $c \cdot d = d \cdot c = c$. Thus, $c \cdot d = c \circ d$ and $d \cdot c = d \circ c$.

- If $\alpha <^* \beta$ and $c = b_\alpha, d = b_\beta$, then by the definition of $\circ$ in Lemma 2.3, $c \circ d = b_\alpha \circ b_\beta = b_{\alpha\wedge\beta} = b_\alpha = c$ and $d \circ c = b_\beta \circ b_\alpha = b_{\alpha\wedge\beta} = b_\alpha = c$. By $(SR3)$, $c \cdot d = d \cdot c = c$. Thus, $c \cdot d = c \circ d$ and $d \cdot c = d \circ c$.

- If $\alpha <^* \beta$ and $c = a_\alpha, d = b_\beta$, then by the definition of $\circ$ in Lemma 2.3, $c \circ d = a_\alpha \circ b_\beta = a_\alpha = c$ and $d \circ c = b_\beta \circ a_\alpha = a_\alpha = c$. By $(SR3)$, $c \cdot d = d \cdot c = c$. Thus, $c \cdot d = c \circ d$ and $d \cdot c = d \circ c$.

- If $\alpha <^* \beta$ and $c = b_\alpha, d = a_\beta$, then by the definition of $\circ$ in Lemma 2.3, $c \circ d = b_\alpha \circ a_\beta = b_\alpha = c$ and $d \circ c = a_\beta \circ b_\alpha = b_\alpha = c$. By $(SR3)$, $c \cdot d = d \cdot c = c$. Thus, $c \cdot d = c \circ d$ and $d \cdot c = d \circ c$.

- If $\alpha \parallel^* \beta$ and $c = a_\alpha, d = a_\beta$, then by the definition of $\circ$ in Lemma 2.3, $c \circ d = a_\alpha \circ a_\beta = a_{\alpha\wedge\beta}$ and $d \circ c = a_\beta \circ a_\alpha = a_{\alpha\wedge\beta}$. Since $c = a_\alpha, d = a_\beta$, $c \cdot d, d \cdot c \in D_{\alpha\wedge\beta}$. Since by $(SR4)$, $|D_{\alpha\wedge\beta}| = 1$, $c \cdot d = a_{\alpha\wedge\beta}$ and $d \cdot c = a_{\alpha\wedge\beta}$. Thus, $c \cdot d = c \circ d$ and $d \cdot c = d \circ c$.

- If $\alpha \parallel^* \beta$ and $c = b_\alpha, d = b_\beta$, then by the definition of $\circ$ in Lemma 2.3, $c \circ d = b_\alpha \circ b_\beta = b_{\alpha\wedge\beta}$ and $d \circ c = b_\beta \circ b_\alpha = b_{\alpha\wedge\beta}$. Since $c = b_\alpha, d = b_\beta$, $c \cdot d, d \cdot c \in D_{\alpha\wedge\beta}$. Since by $(SR4)$, $|D_{\alpha\wedge\beta}| = 1$, $c \cdot d = b_{\alpha\wedge\beta}$ and $d \cdot c = b_{\alpha\wedge\beta}$. Thus, $c \cdot d = c \circ d$ and $d \cdot c = d \circ c$.

- If $\alpha = \beta, \alpha \in \mathfrak{X}$ and $c = a_\alpha, d = b_\alpha$, then by the definition of $\circ$ in Lemma 2.3, $c \circ d = a_\alpha \circ b_\alpha = a_\alpha = c$ and $d \circ c = b_\alpha \circ a_\alpha = b_\alpha = d$. Since $c, d \in D_\alpha$, by the definition of $\mathfrak{X}$, $c \cdot d = c$ and $d \cdot c = d$. Thus, $c \cdot d = c \circ d$ and $d \cdot c = d \circ c$.

- If $\alpha = \beta, \alpha \notin \mathfrak{X}$ and $c = a_\alpha, d = b_\alpha$, then by the definition of $\circ$ in Lemma 2.3, $c \circ d = a_\alpha \circ b_\alpha = b_\alpha = d$ and $d \circ c = b_\alpha \circ a_\alpha = a_\alpha = c$. Since $c, d \in D_\alpha$, by the definition of $\mathfrak{X}$, $c \cdot d = d$ and $d \cdot c = c$. Thus, $c \cdot d = c \circ d$ and $d \cdot c = d \circ c$.

We conclude that $(\mathfrak{L}, \cdot)$ is the semigroup reduct of $CRLOB(\mathfrak{L}; \pi_{\mathcal{D}}, \mathfrak{Y}; \mathfrak{X}, \Phi; \psi, \varphi, \Psi)$. $\square$

The following result is an immediate consequence of Theorem 3.1.

Corollary 3.1. *A semilattice $\mathfrak{Y}$ with greatest element 1 is the semigroup reduct of some conical commutative idempotent residuated lattice if and only if $\mathfrak{Y}$ satisfies conditions $(SR1)$, $(SR5)$ and $(SR7-8)$.*

4. Conclusions

In this paper, we have studied conical idempotent residuated lattices. Using square point sets and the structure theorem of conical idempotent residuated lattices, we have obtained necessary and sufficient conditions for an idempotent semigroup with an identity to be the semigroup reduct of some conical idempotent residuated lattice, which generalize [2, Theorem 5.2]. Note that the work presented here heavily relies on the fact that idempotent residuated lattices are conical. In the future, the structure and decomposition of non-conical idempotent residuated lattices may be investigated.

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Recurrent Hopf hypersurfaces in complex 2-plane Grassmannians

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Abstract. In this paper, it is proved that if the shape operator of a Hopf hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ is Reeb recurrent, it is Reeb parallel. Another recurrent hypersurfaces are also classified.

Keywords: recurrent, Hopf hypersurface, complex two-plane Grassmannian, shape operator.

MSC 2020: 53B25, 53C15, 53D15.

1. Introduction

Let A be the shape operator of a hypersurface (M, g) isometrically immersed in a semi-Riemannian manifold $(\bar{M}, \bar{g})$. The hypersurface M is called recurrent if

$$\nabla_X A = \omega(X)A \tag{1}$$

for a certain one-form ω and any vector field $X \in \mathfrak{X}(M)$, where ∇ and $\mathfrak{X}(M)$ denote the Levi-Civita connection of g and the set of all tangent vector fields on M. The recurrence condition for a tensor field T of type (r, s) was first introduced in [8, 21] in which a geometric interpretation of it in terms of the holonomy group was provided. Hamada in [4] considered (1) on a real hypersurface in a complex projective space $\mathbb{C}P^n$ and proved that recurrent hypersurfaces do not exist. The other type of geometric meaning of recurrent shape operator A of a real hypersurface was indicated by Suh in [14], namely (1) implies that the eigenspaces of the shape operator A are invariant with respect to any parallel translation along any curve γ in M.

A complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ is defined as the set of all two-dimensional linear subspaces in the complex Euclidean space $\mathbb{C}^{m+2}$ which is identified with the homogeneous space $SU(m+2)/S(U(2)\times U(m))$. $G_2(\mathbb{C}^{m+2})$ is known as a compact irreducible Hermitian symmetric space of rank two equipped with both a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J}$ with a canonical basis $\{J_1, J_2, J_3\}$ which does not contain J (see [1]). In this paper m is assumed to be an integer greater than two.

Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$ with N and A a unit normal vector field and the shape operator respectively, and g and ∇ be the induced metric and the corresponding Levi-Civita connection on M, respectively. $\xi := -JN$ is called the Reeb vector field and the almost contact metric 3-structure vector fields ξ_ν are defined by $\xi_\nu = -J_\nu N$ for $\nu \in \{1, 2, 3\}$. It is denoted by $\mathfrak{D}^\perp$ the distribution defined by $\mathfrak{D}^\perp = \mathrm{Span}\{\xi_1, \xi_2, \xi_3\}$ and $\mathfrak{D}$ its orthogonal complement distribution satisfying $T_pM = \mathfrak{D}_p \oplus \mathfrak{D}_p^\perp$ at every point $p \in M$. A real hypersurface M in $G_2(\mathbb{C}^{m+2})$ is said to be Hopf if ξ is an eigenvector field of the shape operator at each point, i.e., $A\xi = \alpha\xi$ and $\alpha = g(A\xi, \xi)$ is called the Hopf principal curvature. Classification result for Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ was obtained by Berndt and Suh [2].

Theorem 1.1 ([2])**.** *Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then $\mathfrak{D}^\perp$ is invariant under the shape operator if and only if*

(A) M is an open part of a tube around a totally geodesic

$$G_2(\mathbb{C}^{m+1}) \text{ in } G_2(\mathbb{C}^{m+2}),$$

or

(B) m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic quaternionic projective space $\mathbb{H}P^m$ in $G_2(\mathbb{C}^{m+2})$.

Applying such a theorem, Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ under some other conditions were extensively studied. Here we recall some results related to (1). A real hypersurface in $G_2(\mathbb{C}^{m+2})$ is called parallel if

$$\nabla_X A = 0, \tag{2}$$

for any vector field X. We may regard the parallel condition as a special case of the recurrent condition. Suh in [15] proved that there exist no real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with parallel shape operator. Generalizing this result, some other types of parallelism were introduced. A real hypersurface in $G_2(\mathbb{C}^{m+2})$ is called Reeb (resp. $\mathfrak{D}^\perp$ or $\mathfrak{D}$) parallel if (2) holds only for X belonging to the Reeb distribution (resp. $\mathfrak{D}^\perp$ or $\mathfrak{D}$). Under such weaker conditions, Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ were considered in [5, 7, 9] and main theorem in [15] was extended. In view of the weakness of (1) than (2), it is very natural to consider recurrent hypersurfaces in $G_2(\mathbb{C}^{m+2})$. Suh in [16] first considered this problem and proved that there exist no recurrent real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with $\mathfrak{D}$ (resp. $\mathfrak{D}^\perp$)-invariant shape operator. Later, another nonexistence theorem for recurrent real hypersurfaces was obtained in [7], namely there do not exist any Hopf recurrent hypersurfaces in $G_2(\mathbb{C}^{m+2})$.

Since the distributions generated by the characteristic vector field ξ and $\mathfrak{D}^\perp$ are most important distributions on real hypersurface in $G_2(\mathbb{C}^{m+2})$, in this paper, we consider some new conditions which are much weaker than (1). A real hypersurface in $G_2(\mathbb{C}^{m+2})$ is called Reeb (or $\mathfrak{D}^\perp$) recurrent if (1) is valid for X belonging to the Reeb distribution (or $\mathfrak{D}^\perp$). The relationships between parallelism and recurrence of the shape operator A of real hypersurfaces are given as follows.

Theorem 1.2. *The shape operator of a Hopf hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ is Reeb (or $\mathfrak{D}^\perp$) recurrent if and only if it is Reeb (or $\mathfrak{D}^\perp$) parallel.*

Not all operators on a Hopf hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ is Reeb recurrent if and only if it is Reeb parallel (for example, see the Ricci operators in [18, 19]). The recurrence condition (1) for some other operators on real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ can be seen in [6, 12, 20]. By using results in [5, 9] and Theorem 1.2, Reeb (or $\mathfrak{D}^\perp$) recurrent real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ are classified in this paper.

2. Preliminaries

2.1 Real hypersurfaces in $G_2(\mathbb{C}^{m+2})$

In this section, first we recall some fundamental formulas shown in [1, 2, 3, 17]. Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$ with real codimension one and N be a unit normal vector field. On M there exists an almost contact metric structure (ϕ, ξ, η, g) induced from the Kähler structure J of $G_2(\mathbb{C}^{m+2})$. Let $\{J_1, J_2, J_3\}$ be a canonical local basis of quaternionic Kähler structure $\mathfrak{J}$ of $G_2(\mathbb{C}^{m+2})$. In this paper we put

$$JX = \phi X + \eta(X)N,\ J_\nu X = \phi_\nu X + \eta_\nu(X)N, \tag{3}$$

for any vector field X, $\nu \in \{1, 2, 3\}$. From the first term of (3), it follows that

$$\phi^2 = -\mathrm{id} + \eta \otimes \xi,\ \eta(\xi) = 1,\ \phi\xi = 0,\ \eta(X) = g(X, \xi), \tag{4}$$

where the Reeb vector field ξ is determined by $\xi := -JN$. From the condition

$$J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1} J_\nu$$

we have an almost contact metric 3-structure $(\phi_\nu, \xi_\nu, \eta_\nu, g)$ as the following

$$\begin{aligned} &\phi_\nu^2 = -\mathrm{id} + \eta_\nu \otimes \xi_\nu,\ \eta_\nu(\xi_\nu) = 1,\ \phi_\nu \xi_\nu = 0, \\ &\phi_\nu \xi_{\nu+1} = \xi_{\nu+2},\ \phi_{\nu+1}\xi_\nu = -\xi_{\nu+2}, \\ &\phi_\nu \phi_{\nu+1} = \phi_{\nu+2} + \eta_{\nu+1} \otimes \xi_\nu, \\ &\phi_{\nu+1}\phi_\nu = -\phi_{\nu+2} + \eta_\nu \otimes \xi_{\nu+1}, \end{aligned} \tag{5}$$

where the index is taken modulo three. According to condition $J_\nu J = J J_\nu$, the relationships between two almost contact metric structures are given by

$$\begin{aligned} &\phi\phi_\nu = \phi_\nu \phi + \eta_\nu \otimes \xi - \eta \otimes \xi_\nu, \\ &\phi\xi_\nu = \phi_\nu \xi,\ \eta_\nu(\phi X) = \eta(\phi_\nu X), \end{aligned} \tag{6}$$

for any vector field X. Because J is parallel with respect to the Riemannian connection of $G_2(\mathbb{C}^{m+2})$, we have

$$(\nabla_X\phi)Y = \eta(Y)AX - g(AX,Y)\xi,\ \nabla_X\xi = \phi AX, \tag{7}$$

for any vector fields X and Y, where we have applied the Gauss and Weingarten formulas. Similarly, since J_ν is a quaternionic Kähler structure of $G_2(\mathbb{C}^{m+2})$, we have

$$\begin{aligned} \nabla_X\xi_\nu &= q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \phi_\nu AX, \\ \nabla_X\phi_\nu &= q_{\nu+2}(X)\phi_{\nu+1} - q_{\nu+1}(X)\phi_{\nu+2} + \eta_\nu \otimes AX - g(AX,\cdot)\xi_\nu, \end{aligned} \tag{8}$$

for any vector field X. The Codazzi equation for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ is given by

$$\begin{aligned} (\nabla_X A)Y - (\nabla_Y A)X =& \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X,Y)\xi \\ &+ \sum_{\nu=1}^{3}(\eta_\nu(X)\phi_\nu Y - \eta_\nu(Y)\phi_\nu X - 2g(\phi_\nu X,Y)\xi_\nu) \\ &+ \sum_{\nu=1}^{3}(\eta_\nu(\phi X)\phi_\nu\phi Y - \eta_\nu(\phi Y)\phi_\nu\phi X) \\ &+ \sum_{\nu=1}^{3}(\eta(X)\eta_\nu(\phi Y) - \eta(Y)\eta_\nu(\phi X))\xi_\nu, \end{aligned} \tag{9}$$

for any vector fields X, Y.

2.2 Some key lemmas

We need the following some important results.

Lemma 2.1 ([3]). *If M is a connected and oriented Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$, then we have*

$$\text{grad}\alpha = \xi(\alpha)\xi + 4\sum_{\nu=1}^{3}\eta_\nu(\xi)\phi_\nu\xi, \tag{10}$$

$$\begin{aligned} 2A\phi AX =& \alpha A\phi X + \alpha\phi AX + 2\phi X + 2\sum_{\nu=1}^{3}(\eta_\nu(X)\phi\xi_\nu + \eta_\nu(\phi X)\xi_\nu \\ &+ \eta_\nu(\xi)\phi_\nu X - 2\eta(X)\eta_\nu(\xi)\phi\xi_\nu - 2\eta_\nu(\phi X)\eta_\nu(\xi)\xi), \end{aligned} \tag{11}$$

for any vector field X, where grad *denotes the gradient operator.*

Proposition 2.1 ([2]). *Let M be a connected Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with $A\mathfrak{D} \subset \mathfrak{D}$ and ξ is tangent to $\mathfrak{D}^\perp$. Let $J_1 \in \mathfrak{J}$ be the almost Hermitian structure such that $JN = J_1N$. Then M has three (if $r = \frac{\pi}{4\sqrt{2}}$) or four (otherwise) distinct constant principal curvatures*

$$\alpha = \sqrt{8}\cot(\sqrt{8}r),\ \beta = \sqrt{2}\cot(\sqrt{2}r),\ \lambda = -\sqrt{2}\tan(\sqrt{2}r),\ \mu = 0,$$

with some $r \in (0, \frac{\pi}{\sqrt{8}})$*. The corresponding multiplicities are*

$$m(\alpha) = 1,\ m(\beta) = 2,\ m(\lambda) = m(\mu) = 2m - 2$$

and the corresponding eigenspaces are

$$\begin{aligned} &T_\alpha = \mathbb{R}\xi = \mathbb{R}\xi_1 = \mathbb{R}JN = \mathrm{Span}\{\xi\} = \mathrm{Span}\{\xi_1\}, \\ &T_\beta = \mathbb{C}^\perp\xi = \mathbb{C}^\perp N = \mathbb{R}\xi_2 \oplus \mathbb{R}\xi_3 = \mathrm{Span}\{\xi_2, \xi_3\}, \\ &T_\lambda = \{X : X \perp \mathbb{H}\xi, JX = J_1X\},\ T_\mu = \{X : X \perp \mathbb{H}\xi, JX = -J_1X\}, \end{aligned}$$

where $\mathbb{R}\xi$*,* $\mathbb{C}\xi$ *and* $\mathbb{H}\xi$ *denote the real, complex and quaternionic span of the Reeb vector field* ξ*, respectively, and* $\mathbb{C}^\perp\xi$ *denotes the orthogonal complement of* $\mathbb{C}\xi$ *in* $\mathbb{H}\xi$*.*

Proposition 2.2 ([2])**.** *Let* M *be a connected Hopf hypersurface in* $G_2(\mathbb{C}^{m+2})$ *with* $A\mathfrak{D} \subset \mathfrak{D}$ *and* $\xi \in \mathfrak{D}$*. Then the quaternionic dimension* m *of* $G_2(\mathbb{C}^{m+2})$ *is even, say* $m = 2n$*, and* M *has five distinct constant principal curvatures*

$$\alpha = -2\tan(2r),\ \beta = 2\cot(2r),\ \gamma = 0,\ \lambda = \cot(r),\ \mu = -\tan(r),$$

with some $r \in (0, \frac{\pi}{4})$*. The corresponding multiplicities are*

$$m(\alpha) = 1,\ m(\beta) = m(\gamma) = 3,\ m(\lambda) = m(\mu) = 4n - 4$$

and the corresponding eigenspaces are

$$\begin{aligned} &T_\alpha = \mathbb{R}\xi = \mathrm{Span}\{\xi\},\ T_\beta = \mathfrak{J}J\xi = \mathrm{Span}\{\xi_1, \xi_2, \xi_3\}, \\ &T_\gamma = \mathfrak{J}\xi = \mathrm{Span}\{\phi_1\xi, \phi_2\xi, \phi_3\xi\},\ T_\lambda,\ T_\mu, \end{aligned}$$

where $T_\lambda \oplus T_\mu = (\mathbb{H}\mathbb{C}\xi)^\perp$*,* $\mathfrak{J}T_\lambda = T_\lambda$*,* $\mathfrak{J}T_\mu = T_\mu$*,* $JT_\lambda = T_\mu$*.*

3. Reeb recurrent hypersurfaces

Suppose that M is a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ whose shape operator A is Reeb recurrent. From (1), we have

$$(\nabla_\xi A)X = \omega(\xi)AX \tag{12}$$

for a certain one-form ω and any vector field $X \in \mathfrak{X}(M)$. Using this and setting $Y = \xi$ in the Codazzi equation (9) we get

$$(\nabla_X A)\xi = \omega(\xi)AX - \phi X + \sum_{\nu=1}^{3}(\eta_\nu(X)\phi_\nu\xi - \eta_\nu(\xi)\phi_\nu X - 3\eta_\nu(\phi X)\xi_\nu),$$

for any $X \in \mathfrak{X}(M)$. As M is Hopf, by using $A\xi = \alpha\xi$ and (7) we have

$$(\nabla_X A)\xi = X(\alpha)\xi + \alpha\phi AX - A\phi AX.$$

The subtraction of the above equality from the previous one gives

$$\omega(\xi)AX = X(\alpha)\xi + \alpha\phi AX - A\phi AX + \phi X - \sum_{\nu=1}^{3}(\eta_\nu(X)\phi_\nu\xi - \eta_\nu(\xi)\phi_\nu X - 3\eta_\nu(\phi X)\xi_\nu), \tag{13}$$

for any vector field X. It was proved by Lee and Loo in [13] that $\xi(\alpha) = 0$ on a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$. So, using again $A\xi = \alpha\xi$ and setting $X = \xi$ in (13) we have $\alpha\omega(\xi) = 0$. In view of the above equality, let us consider a subset of M defined as $\Omega = \{p \in M : \omega(\xi) \neq 0 \text{ at } p\}$. In what follows in this section, we work on Ω. It follows directly that $\alpha = 0$ on Ω. Putting this into (10) we obtain

$$\sum_{\nu=1}^{3}\eta_\nu(\xi)\phi_\nu\xi = 0. \tag{14}$$

Simplifying (13) by using $\alpha = 0$ implies

$$A\phi AX = -\omega(\xi)AX + \phi X - \sum_{\nu=1}^{3}(\eta_\nu(X)\phi_\nu\xi - \eta_\nu(\xi)\phi_\nu X - 3\eta_\nu(\phi X)\xi_\nu).$$

Similarly, simplifying (11) by using $\alpha = 0$ and (14) we obtain

$$A\phi AX = \phi X + \sum_{\nu=1}^{3}(\eta_\nu(X)\phi\xi_\nu + \eta_\nu(\phi X)\xi_\nu + \eta_\nu(\xi)\phi_\nu X).$$

The substraction of the above equality from the previous one gives

$$AX = \frac{2}{\omega(\xi)}\sum_{\nu=1}^{3}(\eta_\nu(\phi X)\xi_\nu - \eta_\nu(X)\phi\xi_\nu), \tag{15}$$

for any vector field X.

Lemma 3.1. *On a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with $\alpha = 0$, either $\xi \in \mathfrak{D}^\perp$ or $\xi \in \mathfrak{D}$.*

Proof. Without loss of generality, we assume $\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1$ with X_0 a unit vector field orthogonal to $\mathfrak{D}^\perp$ satisfying $\eta(X_0)\eta(\xi_1) \neq 0$. Using this in (14) gives $\phi_1\xi = 0$ due to $\eta(\xi_1) \neq 0$. It follows directly that

$$0 = g(\phi_1\xi, \phi_1\xi) = \|\xi_1\|^2 - \eta^2(\xi_1) = 1 - \eta^2(\xi_1).$$

But, on the other hand we also have

$$1 = \|\xi\|^2 = g(\eta(X_0)X_0 + \eta(\xi_1)\xi_1, \eta(X_0)X_0 + \eta(\xi_1)\xi_1) = \eta^2(X_0) + \eta^2(\xi_1).$$

By the above two equalities, obviously, $\eta(X_0) = 0$, this contradicts our assumption. Then we obtain the desired result. $\square$

On Ω, as $\alpha = 0$, Lemma 12 is applicable. First, we consider the case $\xi \in \mathfrak{D}$. It has been proved by Lee and Suh in [11] that a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with $\xi \in \mathfrak{D}$ is locally congruent to an open part of a tube around a totally geodesic quaternionic projective space $\mathbb{H}P^m$ in $G_2(\mathbb{C}^{m+2})$. In fact, if $\xi \in \mathfrak{D}$, $\mathfrak{D}$ is invariant under the shape operator A. According to Proposition 2.2 we have $\alpha = -2\tan(2r)$ satisfying $r \in (0, \pi/4)$. Obviously, α is never zero. Therefore, on Ω, by Lemma 3.1, it is necessarily that $\xi \in \mathfrak{D}^\perp$. In this case, without loss of generality we may assume $\xi = \xi_1$. From (15), with the aid of (5), we have $A\xi = 0$, $A\xi_2 = 0$ and $A\xi_3 = 0$.

This means that $\mathfrak{D}$ is invariant under the shape operator and hence now on Ω, M is of type (A) in Theorem 1.1. It has been proved by Lee, Choi and Woo in [9, Remark 4.5] that the shape operator A of real hypersurfaces of type (A) in $G_2(\mathbb{C}^{m+2})$ is necessarily Reeb parallel. This implies $\omega(\xi) = 0$ on Ω, and it contradicts the definition of Ω.

Theorem 3.1. *The shape operator of a Hopf hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ is Reeb recurrent if and only if it is Reeb parallel.*

Reeb parallel Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ were classified in [9]. Applying these results and Theorem 12, the following two theorems are valid.

Corollary 3.1. *The shape operator of a Hopf hypersurface M in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ is Reeb recurrent with $\alpha \neq 0$ if and only if M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$ with radius $r \in (0, \pi/2\sqrt{2})$ but $r \neq \pi/4\sqrt{2}$.*

If $\alpha = 0$, the situation is complex and some additional assumption is needed.

Corollary 3.2. *The shape operator of a Hopf hypersurface M in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ is Reeb recurrent with $\alpha = 0$ and $\|A\|^2 \leq 4m$ if and only if M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$ with radius $r = \pi/4\sqrt{2}$.*

4. $\mathfrak{D}^\perp$-recurrent hypersurfaces

Suppose that M is a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ whose shape operator A is $\mathfrak{D}^\perp$-recurrent. From (1), we have

$$(\nabla_{\xi_\kappa} A)X = \omega(\xi_\kappa)AX, \tag{16}$$

for a certain one-form ω, any vector field X and any $\kappa \in \{1, 2, 3\}$. Setting $Y = \xi_\kappa$ in the Codazzi equation (9) we get

$$\begin{aligned}(\nabla_X A)\xi_\kappa =& \omega(\xi_\kappa)AX + \eta(X)\phi\xi_\kappa - \eta(\xi_\kappa)\phi X - 2\eta_\kappa(\phi X)\xi \\ &+ \sum_{\nu=1}^{3}(\eta_\nu(X)\phi_\nu\xi_\kappa - \eta_\nu(\xi_\kappa)\phi_\nu X - 2\eta_\kappa(\phi_\nu X)\xi_\nu)\end{aligned} \tag{17}$$

$$+\sum_{\nu=1}^{3}(\eta_\nu(\phi X)\phi_\nu\phi\xi_\kappa - \eta_\nu(\phi\xi_\kappa)\phi_\nu\phi X)$$
$$+\sum_{\nu=1}^{3}(\eta(X)\eta_\nu(\phi\xi_\kappa) - \eta(\xi_\kappa)\eta_\nu(\phi X))\xi_\nu,$$

for any vector field X and any $\kappa \in \{1,2,3\}$. Taking the inner product of (4) with ξ and using

$$\eta((\nabla_X A)\xi_\kappa) = g((\nabla_X A)\xi, \xi_\kappa) = X(\alpha)\eta(\xi_\kappa) + \alpha\eta_\kappa(\phi AX) - \eta_\kappa(A\phi AX)$$

we obtain

$$\begin{aligned}
&X(\alpha)\eta(\xi_\kappa) + \alpha\eta_\kappa(\phi AX) - \eta_\kappa(A\phi AX)\\
&= \omega(\xi_\kappa)\eta(AX) - 2\eta_\kappa(\phi X)\\
&+\sum_{\nu=1}^{3}(\eta_\nu(X)\eta(\phi_\nu\xi_\kappa) - \eta_\nu(\xi_\kappa)\eta(\phi_\nu X) - 2\eta_\kappa(\phi_\nu X)\eta(\xi_\nu))\\
&+\sum_{\nu=1}^{3}(\eta_\nu(\phi X)\eta(\phi_\nu\phi\xi_\kappa) - \eta_\nu(\phi\xi_\kappa)\eta(\phi_\nu\phi X))\\
&+\sum_{\nu=1}^{3}(\eta(X)\eta_\nu(\phi\xi_\kappa)\eta(\xi_\nu) - \eta(\xi_\kappa)\eta_\nu(\phi X)\eta(\xi_\nu)),
\end{aligned}$$

for any vector field X and any $\kappa \in \{1,2,3\}$. Setting $X = \xi$ in the above equality and using $A\xi = \alpha\xi$, we obtain

$$\alpha\omega(\xi_\kappa) + 4\sum_{\nu=1}^{3}\eta(\xi_\nu)\eta_\nu(\phi\xi_\kappa) = 0,$$

for any $\kappa \in \{1,2,3\}$, where we applied again the fact that the Hopf condition implies $\xi(\alpha) = 0$ (see [13]). Applying (5), it follows that $\alpha\omega(\xi_\kappa) = 0$, for any $\kappa \in \{1,2,3\}$. This leads us to consider a subset of M defined as follows:

$$\Omega = \{p \in M : \omega(\xi_\mu) \neq 0 \text{ at } p \text{ for some } \mu \in \{1,2,3\}\}.$$

On Ω, we get $\alpha = 0$ and hence from Lemma 3.1 we first consider the case $\xi \in \mathfrak{D}$. In this case, from [11] M is of type (B) in Theorem 1.1. But, from Proposition 2.2, α is never zero and this contradicts with $\alpha = 0$ on Ω. Hence we must have $\xi \in \mathfrak{D}^\perp$ due to Lemma 3.1. Without loss of generality we may assume $\xi = \xi_1$.

Setting $X = \xi_1$ in (4) and using the $\mathfrak{D}^\perp$ recurrent assumption, we get

$$\omega(\xi_1)A\xi_\kappa = \phi_\kappa\xi_1 + \phi_1\xi_\kappa - 4\eta_2(\xi_\kappa)\xi_3 + 4\eta_3(\xi_\kappa)\xi_2,$$

for any $\kappa \in \{1,2,3\}$, where we used (5). We see that $\omega(\xi_1) \neq 0$ at every point in Ω. Otherwise, if there is a point in Ω at which $\omega(\xi_1) = 0$. We have

$$\phi_\kappa\xi_1 + \phi_1\xi_\kappa - 4\eta_2(\xi_\kappa)\xi_3 + 4\eta_3(\xi_\kappa)\xi_2 = 0.$$

Setting $\kappa = 2$ or $\kappa = 3$ in this equality we obtain $\xi_3 = 0$ and $\xi_2 = 0$, respectively, and both these are impossible. Now, taking into account $\omega(\xi_1) \neq 0$, we obtain

$$A\xi_\kappa = \frac{1}{\omega(\xi_1)}(\phi_\kappa \xi_1 + \phi_1 \xi_\kappa - 4\eta_2(\xi_\kappa)\xi_3 + 4\eta_3(\xi_\kappa)\xi_2).$$

This means $A\xi_\kappa \in \mathfrak{D}^\perp$, for any $\kappa \in \{1, 2, 3\}$, or equivalently, $\mathfrak{D}$ is invariant under the shape operator. Consequently, from Theorem 1.1 now M is of type (A) and Proposition 2.1 is valid. Since $\alpha = 0$ on Ω, we have $r = \pi/4\sqrt{2}$ and in this case $\beta = \sqrt{2}$. According to this proposition we get

$$(\nabla_{\xi_2} A)\xi = -A\phi A\xi_2 = -\sqrt{2}A\phi\xi_2 = \sqrt{2}A\xi_3 = 2\xi_3.$$

However, as the shape operator is $\mathfrak{D}^\perp$ recurrent, from (16) we have

$$(\nabla_{\xi_2} A)\xi = \omega(\xi_2)A\xi = 0.$$

This equality contradicts the previous one. Finally, we see that Ω is empty and hence $\omega(\xi_\kappa) = 0$, for any $\kappa \in \{1, 2, 3\}$.

Theorem 4.1. *The shape operator of a Hopf hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ is $\mathfrak{D}^\perp$-recurrent if and only if it is $\mathfrak{D}^\perp$-parallel.*

The following corollary follows immediately from Theorem 4.1 and main theorem in [5].

Corollary 4.1. *There exist no Hopf hypersurfaces in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ with $\mathfrak{D}^\perp$-recurrent shape operator.*

Remark 4.1. *This corollary covers main results in [7, 16].*

5. Conclusion

The study of recurrent condition of some tensor fields on a Riemannian manifold has been an interesting topic for the last sixty years. In this paper the author classified recurrent condition of the shape operator of a Hopf hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ along two special distributions of the tangent bundle. This generalized some recent results in this field (for example see Remark 4.1). Besides this, main results in paper motivate the study of other operators in Hopf hypersurface in complex two-plane Grassmannians.

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New characteristics and applications of the EP, normal and Hermitian matrices

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Abstract. In this paper, we present the EP-matrices, normal, and new features of the Hermitian matrix. We are going to push it to a higher order form. This paper describes EP-matrices, normal matrices and Hermitian matrix equivalent forms by using Core inverse. We also give several special equivalent facts by use decompositions. We also give other special equivalent facts. A new equivalent condition for the reverse order law is also obtained.
Keywords: EP matrix, Hermitian matrix, generalized inverse, decompositions, normal matrix, core inverse.
MSC 2020: 15A10, 15B99, 15A24.

1. Introduction

In this paper, $\mathbb{C}^{m\times n}$ and $\mathbb{R}^{m\times n}$ stand for the sets of all $m \times n$ matrices over the fields of complex numbers and real numbers, respectively. A^* stands for the conjugate transpose of $A \in \mathbb{C}^{m\times n}$ and $r(A)$ stand for the rank of $A \in \mathbb{C}^{m\times n}$. $det(A)$ stands for the determinant of $A \in \mathbb{C}^{m\times m}$, and I_m denotes the identity matrix of order m.

The Core inverse denoted by $A^{\oplus} \in \mathbb{C}^{m\times m}$ of A is the unique matrix satisfying the following matrix equations, $r(A^2) = r(A)$

(1) $AA^{\oplus}A = A$,

(2) $A(A^{\oplus})^2 = A^{\oplus}$,

(3) $(AA^{\oplus})^* = AA^{\oplus}$ ([1]).

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The Moore-penrose inverse denoted by $A^\dagger$ of A is the unique matrix satisfying the following matrix equations

(1) $AA^\dagger A = A$,

(2) $A^\dagger AA^\dagger = A^\dagger$,

(3) $(AA^\dagger)^* = AA^\dagger$,

(4) $(A^\dagger A)^* = A^\dagger A$ ([15]).

Recall that a matrix $A \in \mathbb{C}^{m\times m}$ is said to be EP-matrix if and only if it satisfies the following equality

$$AA^\dagger = A^\dagger A. \tag{1}$$

A matrix $A \in \mathbb{C}^{m\times m}$ is said to be Hermitian matrix if and only if it satisfies the following equality

$$A^* = A. \tag{2}$$

A matrix $A \in \mathbb{C}^{m\times m}$ is said to be normal matrix if and only if it satisfies the following equality

$$AA^* = A^*A. \tag{3}$$

In addition to the definitions in (1), (2) and (3), a Hermitian matrix can be characterized by some other matrix equalities and facts. It can be expressed by the following facts

$$f(A, A^\dagger) = 0 \Leftrightarrow AA^\dagger = A^\dagger A, \tag{4}$$
$$f(A, A^*) = 0 \Leftrightarrow A = A^*, \tag{5}$$
$$f(A, A^*) = 0 \Leftrightarrow AA^* = A^*A, \tag{6}$$

where $f(\cdot)$ is the certain ordinary algebraic operation of A and A^*, A and $A^\dagger$. Due to the arbitrariness of matrix expressions, there does not exist a general and significant method to construct the matrix equations in (4) except some special cases. We already have the following facts from [6][22]:

$$A^2 = AA^* \Leftrightarrow A = A^*, \tag{7}$$
$$A^2 = A^*A \Leftrightarrow A = A^*, \tag{8}$$
$$A^3 = AA^*A \Leftrightarrow A = A^*. \tag{9}$$

Lemma 1.1 ([13]). *Let $A \in \mathbb{C}^{m\times n}$. The singular value decomposition of A is*

$$A = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V,$$

where $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ are two unitary matrices, namely, $UU^* = U^*U = I_m$, $VV^* = V^*V = I_n$, and $\Sigma \in \mathbb{R}^{s\times s}$ is a positive diagonal matrix composed of the singular values of A with $s = r(A)$. In particular, if $A \in \mathbb{C}^{m\times m}$, then A admits the following decomposition:

$$A = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} (VU)\, U^* = U \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} U^*,$$

where VU can be decomposed as $VU = \begin{bmatrix} K & L \\ K_1 & L_1 \end{bmatrix}$ with $(VU)(VU)^* = I_m, K \in \mathbb{C}^{s\times s}, L \in \mathbb{C}^{s\times(m-s)}$, and

$$KK^* + LL^* = I_s, \quad KK^* \leq I_s, \quad LL^* \leq I_s,$$

where $KK^* \leq I_s, LL^* \leq I_s$ mean that $I_s - KK^*$ and $I_s - LL^*$ are positive semi-definite.

So, A is Hermitian if and only if $L = 0$ and $\Sigma K = K^*\Sigma$; A is normal if and only if $L = 0$ and $\Sigma K = K\Sigma$; and finally, A is EP if and only if $L = 0$.

In [2, 6, 7, 10, 13, 15, 21], the properties of the EP-matrix, normal matrix, Hermitian matrix and their equivalence relation are described. Based on this background, we discuss their other equivalent forms. In [9] [18] and [19], the equivalent analysis of different inversion laws of matrix is described. This paper also gives an equivalent form of the reverse order law.

This paper is organized as follows

The second part gives some equivalent forms of the EP-matrix and corollary:

(I) A is EP.

(II) $AA^\dagger A^{\oplus} = A^\dagger A^{\oplus} A$.

(III) $AA^{\oplus} A^* = A^* AA^{\oplus}$.

(IV) $AA^{\oplus} A^\dagger = A^{\oplus} A^\dagger A$.

(V) $A^\dagger AA^{\oplus} = A^{\oplus} A^\dagger A$ and the third part gives some equivalent forms of the normal matrix and corollary.

The fourth part describes the relevant properties of the Hermitian matrix

(I) A is Hermitian,

(II) $AA^{\oplus} = A^* A^{\oplus}$,

(III) $AA^{\oplus} = A^\dagger A^*$,

(IV) $A^\dagger A = A^{\oplus} A^*$,

(V) $AA^{\oplus} = A^* A^\dagger$.

We aim to $A^5 = (AA^*)^2 A$ and $A^7 = (AA^*)^3 A \Leftrightarrow A = A^*$ and higher ones. A new equivalent condition for the reverse order law is also obtained $(AB)^\dagger = B^\dagger A^\dagger \Leftrightarrow (B^*A^*ABB^*A^*)^\dagger = (A^*)^\dagger (BB^*)^\dagger (A^*A)^\dagger (B^*)^\dagger$.

2. Related results of the EP-matrix

In this section we give the equivalent form of the EP-matrix of the form $RS = XY$ and $XY = YX$, where $R, S, X, Y \in A^{\dagger}, A, A^{\oplus}, A^{*}$, and some conclusions are obtained.

Lemma 2.1 ([21]). *Let $A \in \mathbb{C}^{m\times m}$. Then, the following results hold:*

(I) $A^2A^{\dagger} = AA^{\dagger}A \Leftrightarrow AA^{\dagger} = A^{\dagger}A$;

(II) $A^{\dagger}A^2 = AA^{\dagger}A \Leftrightarrow AA^{\dagger} = A^{\dagger}A$.

We present necessary and sufficient conditions for a matrix to be EP by referring to the commutativity property $XY = YX$, where X and Y are transforms of A from the set $A^{\dagger}, A, A^{\oplus}, A^{*}$.

Theorem 2.1. *Let $A \in \mathbb{C}^{m\times m}$. The index is one. Then, the following conditions are equivalent:*

(I) *A is EP;*

(II) $AA^{\dagger}A^{\oplus} = A^{\dagger}A^{\oplus}A$, $r(A) = r(A^2)$;

(III) $AA^{\oplus}A^{*} = A^{*}AA^{\oplus}$, $r(A) = r(A^2)$;

(IV) $AA^{\oplus}A^{\dagger} = A^{\oplus}A^{\dagger}A$, $r(A) = r(A^2)$;

(V) $A^{\dagger}AA^{\oplus} = A^{\oplus}A^{\dagger}A$, $r(A) = r(A^2)$;

(VI) $A^{\dagger}A^{\dagger}A^{\oplus} = A^{\oplus}A^{\dagger}A^{\dagger}$, $r(A) = r(A^2)$;

(VII) $A^{\dagger}A^{\oplus}A^{\dagger} = A^{\oplus}A^{\dagger}A^{\dagger}$, $r(A) = r(A^2)$;

(VIII) $A^{\dagger}A^{\oplus}A^{\oplus} = A^{\oplus}A^{\oplus}A^{\dagger}$, $r(A) = r(A^2)$;

(IX) $AA^{\oplus}A^{\dagger} = A^{\dagger}AA^{\oplus}$, $r(A) = r(A^2)$;

(X) $A^{\dagger}A^{\oplus} = A^{\oplus}A^{\dagger}$, $r(A) = r(A^2)$;

(XI) $A^{\dagger}A^{\oplus}A^{\oplus} = A^{\oplus}A^{\dagger}A^{\oplus}$, $r(A) = r(A^2)$.

Proof. By the Hartwig-Spindelböck decomposition of A, one has

$$A = U\begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} U^{*}.$$

By calculating, we can get the Core inverse of A, that is

$$A^{\oplus} = U\begin{bmatrix} K^{-1}\Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^{*}.$$

On the other hand, the Moore-Penrose inverse of A is

$$A^{\dagger} = U \begin{bmatrix} K^*\Sigma^{-1} & 0 \\ L^*\Sigma^{-1} & 0 \end{bmatrix} U^*.$$

It is easy to figure out that $AA^{\dagger}A^{\oplus}$ and $A^{\dagger}A^{\oplus}A$ are

$$\begin{aligned} AA^{\dagger}A^{\oplus} &= U \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} U^*U \begin{bmatrix} K^*\Sigma^{-1} & 0 \\ L^*\Sigma^{-1} & 0 \end{bmatrix} U^*U \begin{bmatrix} K^{-1}\Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix} \\ &= U \begin{bmatrix} \Sigma KK^*\Sigma^{-1}K^{-1}\Sigma^{-1} + \Sigma LL^*\Sigma^{-1}K^{-1}\Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^* \end{aligned}$$

$$\begin{aligned} A^{\dagger}A^{\oplus}A &= U \begin{bmatrix} K^*\Sigma^{-1} & 0 \\ L^*\Sigma^{-1} & 0 \end{bmatrix} U^*U \begin{bmatrix} K^{-1}\Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*U \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} \\ &= U \begin{bmatrix} K^*\Sigma^{-1} & K^*\Sigma^{-1}K^{-1}L \\ L^*\Sigma^{-1} & L^*\Sigma^{-1}K^{-1}L \end{bmatrix} U^*. \end{aligned}$$

By $AA^{\dagger}A^{\oplus} = A^{\dagger}A^{\oplus}A$, one has

$$\begin{cases} L^*\Sigma^{-1} = 0, \\ K^*\Sigma^{-1}K^{-1}L = 0, \\ L^*\Sigma^{-1}K^{-1}L = 0, \\ K^*\Sigma^{-1} = \Sigma KK^*\Sigma^{-1}K^{-1}\Sigma^{-1} + \Sigma LL^*\Sigma^{-1}K^{-1}\Sigma^{-1}. \end{cases}$$

The condition $AA^{\dagger}A^{\oplus} = A^{\dagger}A^{\oplus}A$ is clearly equivalent to $L = 0$, $K^*\Sigma^{-1} = \Sigma KK^*\Sigma^{-1}K^{-1}\Sigma^{-1}$, multiply the equation to the right by $\Sigma K\Sigma$, can get $K^*K\Sigma = \Sigma KK^*$, because of $KK^* = K^*K = I_r$, so the equation works.

It is easy to figure out $AA^{\oplus}A^*$ and $A^*AA^{\oplus}$ are

$$\begin{aligned} AA^{\oplus}A^* &= U \begin{bmatrix} \Sigma KK^{-1}\Sigma^{-1}K^*\Sigma & 0 \\ 0 & 0 \end{bmatrix} U^*, \\ A^*AA^{\oplus} &= U \begin{bmatrix} K^*\Sigma\Sigma KK^{-1}\Sigma^{-1} & 0 \\ L^*\Sigma\Sigma KK^{-1}\Sigma^{-1} & 0 \end{bmatrix} U^*, \end{aligned}$$

$AA^{\oplus}A^* = A^*AA^{\oplus} \Rightarrow K^*\Sigma = K^*\Sigma$, $L = 0$. So, A is EP.

The other equations prove the same. □

Next, give another equivalent form of the EP-matrix. We present necessary and sufficient conditions for a matrix to be EP by referring to the commutativity property $RS = YX$, where R,S,X,Y are transforms of A from the set $A^{\dagger}, A, A^{\oplus}, A^*$.

Theorem 2.2. *Let $A \in \mathbb{C}^{m\times m}$. Then, the following conditions are equivalent.*

(I) *A is EP;*

(II) $AA^{\oplus} = A^{\dagger}A$, $r(A) = r(A^2)$;

(III) $A^{\dagger}A^{*} = A^{\oplus}A^{*}$, $r(A) = r(A^2)$;

(IV) $A^{\dagger}A^{\dagger} = A^{\oplus}A^{\dagger}$, $r(A) = r(A^2)$;

(V) $A^{\dagger}A^{\dagger} = A^{\oplus}A^{\oplus}$, $r(A) = r(A^2)$;

(VI) $A^{\dagger}A^{\oplus} = A^{\oplus}A^{\oplus}$, $r(A) = r(A^2)$.

Proof. By the singular value decomposition of A, one has

$$A = U \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} U^{*}.$$

By calculating, we can get the Core inverse of A, that is

$$A^{\oplus} = \begin{bmatrix} K^{-1}\Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

On the other hand, the Moore-Penrose inverse of A is

$$A^{\dagger} = U \begin{bmatrix} K^{*}\Sigma^{-1} & 0 \\ L^{*}\Sigma^{-1} & 0 \end{bmatrix} U^{*}.$$

It is easy to figure out that $AA^{\oplus}$ and $A^{\dagger}A$ are

$$AA^{\oplus} = U \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} U^{*}U \begin{bmatrix} K^{-1}\Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^{*} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix},$$

$$A^{\dagger}A = U \begin{bmatrix} K^{*}\Sigma^{-1} & 0 \\ L^{*}\Sigma^{-1} & 0 \end{bmatrix} U^{*}U \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} U^{*} = \begin{bmatrix} K^{*}K & K^{*}L \\ L^{*}K & L*L \end{bmatrix}.$$

The condition $AA^{\oplus} = A^{\dagger}A$ is clearly equivalent to $L = 0$, $K^{*} = K^{-1}$. So, A is EP. It is easy to figure out $A^{\dagger}A^{*}$ and $A^{\oplus}A^{*}$ are

$$A^{\dagger}A^{*} = U \begin{bmatrix} K^{*}\Sigma^{-1}K^{*}\Sigma & 0 \\ L^{*}\Sigma^{-1}K^{*}\Sigma & 0 \end{bmatrix} U^{*}, \quad A^{\oplus}A^{*} = U \begin{bmatrix} K^{-1}\Sigma^{-1}K^{*}\Sigma & 0 \\ 0 & 0 \end{bmatrix} U^{*}.$$

The condition $A^{\dagger}A^{*} = A^{\oplus}A^{*}$ is clearly equivalent to $L = 0$ and $K^{*}\Sigma^{-1}K^{*}\Sigma = K^{-1}\Sigma^{-1}K^{*}\Sigma \Rightarrow K^{*} = K^{-1}$. So, A is EP. It is easy to figure out $A^{\dagger}A^{\dagger}$ and $A^{\oplus}A^{\dagger}$ are

$$A^{\dagger}A^{\dagger} = U \begin{bmatrix} K^{*}\Sigma^{-1}K^{*}\Sigma^{-1} & 0 \\ L^{*}\Sigma^{-1}K^{*}\Sigma^{-1} & 0 \end{bmatrix} U^{*}, \quad A^{\oplus}A^{\dagger} = U \begin{bmatrix} K^{-1}\Sigma^{-1}K^{*}\Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^{*}.$$

The condition $A^{\dagger}A^{\dagger} = A^{\oplus}A^{\dagger}$ is clearly equivalent to $L = 0$ and $K^{*} = K^{-1}$. So, A is EP.

The other equations prove the same. □

The following inference is obtained from Lemma2.1.

Corollary 2.1. *Let $A \in \mathbb{C}^{m\times m}$. Then,*

(I) $A^3A^\dagger = A^2A^\dagger A$ *and* $A^\dagger A^3 = AA^\dagger A^2 \Leftrightarrow AA^\dagger = A^\dagger A$.

(II) $A^4A^\dagger = A^3A^\dagger A$ *and* $A^\dagger A^4 = AA^\dagger A^3 \Leftrightarrow AA^\dagger = A^\dagger A$.

(III) $A^5A^\dagger = A^4A^\dagger A$ *and* $A^\dagger A^5 = AA^\dagger A^4 \Leftrightarrow AA^\dagger = A^\dagger A$.

Proof.

$$\begin{aligned}
&AAG = AAH \Rightarrow AG = AH\\
&A^3A^\dagger = AA^2A^\dagger = A^2A^\dagger A = AAA^\dagger A \Rightarrow A^2A^\dagger = AA^\dagger A,\\
&A^\dagger A^3 = A^\dagger A^2 A = AA^\dagger A^2 = AA^\dagger AA \Rightarrow A^\dagger A^2 = AA^\dagger A.
\end{aligned}$$

From Lemma 2.1, it come to the conclusion $AA^\dagger = A^\dagger A$. On the other hand,

$$AA^\dagger = A^\dagger A \Rightarrow A^3A^\dagger = A^2A^\dagger A \text{ and } A^\dagger A^3 = AA^\dagger A^2$$

is obvious. Hence, (I) is proved.

Similarly, we have

$$\begin{aligned}
&A^4A^\dagger = A^2A^2A^\dagger = A^3A^\dagger A = A^2AA^\dagger A \Rightarrow A^2A^\dagger = AA^\dagger A,\\
&A^\dagger A^4 = A^\dagger A^2A^2 = AA^\dagger A^3 = AA^\dagger AA^2 \Rightarrow A^\dagger A^2 = AA^\dagger A.
\end{aligned}$$

From Lemma 2.1, it come to the conclusion $AA^\dagger = A^\dagger A$. Then, (II) is proved. The third equivalency can be proved in the same way. This completes the proof. □

Corollary 2.2. *Let $A, B \in \mathbb{C}^{m\times m}$. Then,*

(I) $AB^\dagger B = AA^\dagger A$ *and* $B^\dagger AA^\dagger = B^\dagger BB^\dagger \Leftrightarrow AA^\dagger = BB^\dagger$ *and* $B^\dagger B = A^\dagger A$.

(II) $BB^\dagger A = AA^\dagger A$ *and* $A^\dagger AB^\dagger = B^\dagger BB^\dagger \Leftrightarrow AA^\dagger = BB^\dagger$ *and* $B^\dagger B = A^\dagger A$.

Proof. Let $M = \begin{bmatrix} 0 & A \\ B^\dagger & 0 \end{bmatrix}$, then $M^\dagger = \begin{bmatrix} 0 & B \\ A^\dagger & 0 \end{bmatrix}$. In this situation, it is easy to verify

$$M^2 = \begin{bmatrix} AB^\dagger & 0 \\ 0 & B^\dagger A \end{bmatrix}, M^2M^\dagger = \begin{bmatrix} 0 & AB^\dagger B \\ B^\dagger AA^\dagger & 0 \end{bmatrix},\ M^\dagger M^2 = \begin{bmatrix} 0 & BB^\dagger A \\ A^\dagger AB^\dagger & 0 \end{bmatrix}.$$

Moreover,

$$MM^\dagger M = \begin{bmatrix} 0 & AA^\dagger A \\ B^\dagger BB^\dagger & 0 \end{bmatrix}.$$

By Theorem 2.1, one has $M^2M^\dagger = MM^\dagger M \Leftrightarrow MM^\dagger = M^\dagger M$. Hence,

$$AB^\dagger B = AA^\dagger A \text{ and } B^\dagger AA^\dagger = B^\dagger BB^\dagger \Leftrightarrow MM^\dagger = M^\dagger M.$$

It is easy to verify

$$M^{\dagger}M = \begin{bmatrix} BB^{\dagger} & 0 \\ 0 & A^{\dagger}A \end{bmatrix}, \; MM^{\dagger} = \begin{bmatrix} AA^{\dagger} & 0 \\ 0 & B^{\dagger}B \end{bmatrix}.$$

Therefore, $MM^{\dagger} = M^{\dagger}M \Leftrightarrow AA^{\dagger} = BB^{\dagger}$ and $B^{\dagger}B = A^{\dagger}A$, which implies

$$AB^{\dagger}B = AA^{\dagger}A \text{ and } B^{\dagger}AA^{\dagger} = B^{\dagger}BB^{\dagger} \Leftrightarrow AA^{\dagger} = BB^{\dagger} \text{ and } B^{\dagger}B = A^{\dagger}A.$$

On the other hand, by using $M^{\dagger}M^2 = MM^{\dagger}M \Leftrightarrow MM^{\dagger} = M^{\dagger}M$, one has

$$BB^{\dagger}A = AA^{\dagger}A \text{ and } A^{\dagger}AB^{\dagger} = B^{\dagger}BB^{\dagger} \Leftrightarrow MM^{\dagger} = M^{\dagger}M.$$

Conclusions can be drawn

$$BB^{\dagger}A = AA^{\dagger}A \text{ and } A^{\dagger}AB^{\dagger} = B^{\dagger}BB^{\dagger} \Leftrightarrow AA^{\dagger} = BB^{\dagger} \text{ and } B^{\dagger}B = A^{\dagger}A.$$

Thus, we establish the equivalent facts. □

3. Related results of the normal matrix

In this section we give the equivalent form of the normal matrix of the form $RS = XY$ and $XY = YX$, where $R, S, X, Y \in A^{\dagger}, A, A^{\textcircled{\#}}, A^{*}$, and some conclusions are obtained.

Lemma 3.1 ([23])**.** *Let $A \in \mathbb{C}^{m\times m}$. Then, the following results hold*

(I) $A^2A^* = AA^*A \Leftrightarrow AA^* = A^*A$;

(II) $A^*A^2 = AA^*A \Leftrightarrow AA^* = A^*A$.

We present necessary and sufficient conditions for a matrix to be normal by referring to the commutativity property $XY = YX$, where X and Y are transforms of A from the set $A^{\dagger}, A, A^{\textcircled{\#}}, A^{*}$.

Theorem 3.1. *Let $A \in \mathbb{C}^{m\times m}$. Then, the following conditions are equivalent.*

(I) *A is normal;*

(II) $AA^*A^{\textcircled{\#}} = A^*A^{\textcircled{\#}}A$, $r(A) = r(A^2)$;

(III) $AA^{\textcircled{\#}}A^* = A^{\textcircled{\#}}A^*A$, $r(A) = r(A^2)$;

(IV) $A^*A^{\textcircled{\#}} = A^{\textcircled{\#}}A^*$, $r(A) = r(A^2)$;

(V) $A^{\dagger}A^*A^{\textcircled{\#}} = A^{\textcircled{\#}}A^{\dagger}A^*$, $r(A) = r(A^2)$;

(VI) $A^*AA^{\textcircled{\#}} = A^{\textcircled{\#}}A^*A$, $r(A) = r(A^2)$;

(VII) $A^*A^{\dagger}A^{\textcircled{\#}} = A^{\textcircled{\#}}A^*A^{\dagger}$, $r(A) = r(A^2)$;

(VIII) $A^*A^{\oplus}A^* = A^{\oplus}A^*A^*$, $r(A) = r(A^2)$;

(IX) $A^*A^{\oplus}A^{\oplus} = A^{\oplus}A^*A^{\oplus}$, $r(A) = r(A^2)$;

(X) $A^{\dagger}A^{\oplus}A^* = A^{\oplus}A^*A^{\dagger}$, $r(A) = r(A^2)$.

Proof. Following Lemma 1.1, one has

$$A = U \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} U^*.$$

Then,

$$A^* = U \begin{bmatrix} (\Sigma K)^* & 0 \\ (\Sigma L)^* & 0 \end{bmatrix} U^*.$$

Furthermore,

$$\begin{aligned} AA^* &= U \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} U^*U \begin{bmatrix} (\Sigma K)^* & 0 \\ (\Sigma L)^* & 0 \end{bmatrix} U^* \\ &= U \begin{bmatrix} (\Sigma K)(\Sigma K)^* + (\Sigma L)(\Sigma L)^* & 0 \\ 0 & 0 \end{bmatrix} U^*, \\ AA^*A^{\oplus} &= U \begin{bmatrix} \Sigma KK^*\Sigma + \Sigma LL^*\Sigma & 0 \\ 0 & 0 \end{bmatrix} U^*U \begin{bmatrix} K^{-1}\Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^* \\ &= U \begin{bmatrix} \Sigma KK^*\Sigma K^{-1}\Sigma^{-1} + \Sigma LL^*\Sigma K^{-1}\Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*. \end{aligned}$$

Simple computations show that

$$\begin{aligned} A^*A^{\oplus}A &= U \begin{bmatrix} K^*\Sigma & 0 \\ L^*\Sigma & 0 \end{bmatrix} U^*U \begin{bmatrix} K^{-1}\Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*U \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} U^* \\ &= U \begin{bmatrix} K^*\Sigma & K^*\Sigma K^{-1}L \\ L^*\Sigma & L^*\Sigma K^{-1}L \end{bmatrix} U^*. \end{aligned}$$

By $AA^*A^{\oplus} = A^*A^{\oplus}A$, one has

$$\begin{cases} L^*\Sigma = 0 \\ K^*\Sigma K^{-1}L = 0 \\ L^*\Sigma K^{-1}L = 0 \\ K^*L = \Sigma KK^*\Sigma K^{-1}\Sigma^{-1} + \Sigma LL^*\Sigma K^{-1}\Sigma^{-1} \end{cases}.$$

Hence, it is seen that $A^*A^{\oplus}$ and A commute if and only if $L = 0$, $K^*\Sigma = \Sigma KK^*\Sigma K^{-1}\Sigma^{-1}$. However, $L = 0$ implies $K^* = K^{-1}$.

Multiply both sides of $K^*\Sigma = \Sigma KK^*\Sigma K^{-1}\Sigma^{-1}$ by ΣK. Can get the formula

$$K^*\Sigma^2 K = \Sigma^2.$$

Taking square roots,we arrive at $L = 0$ and $K^*\Sigma K = \Sigma$, that is $L = 0$ and $\Sigma K = K\Sigma$. So, A is normal. It is easy to figure out $AA^{\oplus}A^*$ and $A^{\oplus}A^*A$ are

$$AA^{\oplus}A^* = U\begin{bmatrix} K^*\Sigma & 0 \\ 0 & 0 \end{bmatrix}U^*,$$
$$A^{\oplus}A^*A = U\begin{bmatrix} K^{-1}\Sigma^{-1}K^*\Sigma\Sigma K & K^{-1}\Sigma^{-1}K^*\Sigma\Sigma L \\ 0 & 0 \end{bmatrix}U^*.$$

Hence, it is seen that $A^{\oplus}A^*$ and A commute if and only if $L = 0$, $K^*\Sigma = K^{-1}\Sigma^{-1}K^*\Sigma\Sigma K$.

Multiply both sides of this equation ΣK. Then,

$$\Sigma K = K\Sigma.$$

The other equations prove the same. □

Corollary 3.1. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times m}$. Then,*

$$ABB^*A^* = B^*A^*AB \Leftrightarrow (AB)(AB)^* = (AB)^*(AB).$$

Proof. The $(AB)(AB)^* = (AB)^*(AB) \Rightarrow ABB^*A^* = B^*A^*AB$ part is obvious. To show the $ABB^*A^* = B^*A^*AB \Rightarrow (AB)(AB)^* = (AB)^*(AB)$ part, multiply both sides of this equation by AB gives

$$ABABB^*A^* = ABB^*A^*AB \Rightarrow (AB)^2(AB)^* = (AB)(AB)^*(AB).$$

By Lemma 4.3, we can get

$$(AB)(AB)^* = (AB)^*(AB).$$

This proof is complete. □

Corollary 3.2. *Let $A, B \in \mathbb{C}^{m\times n}$. Then*

(I) $AB^*B = AA^*A$ *and* $B^*BB^* = B^*A^*A^* \Leftrightarrow AA^* = BB^*$ *and* $B^*B = A^*A$*;*

(II) $AA^*A = BB^*A$ *and* $A^*AB^* = B^*BB^* \Leftrightarrow AA^* = BB^*$ *and* $B^*B = A^*A$.

Proof. Let $M = \begin{bmatrix} 0 & A \\ B^* & 0 \end{bmatrix}$, then $M^* = \begin{bmatrix} 0 & B \\ A^* & 0 \end{bmatrix}$. In this situation, it is easy to verify

$$M^2 = \begin{bmatrix} AB^* & 0 \\ 0 & B^*A \end{bmatrix}, M^2M^* = \begin{bmatrix} 0 & AB^*B \\ B^*A^*A^* & 0 \end{bmatrix}, M^*M^2 = \begin{bmatrix} 0 & BB^*A \\ A^*AB^* & 0 \end{bmatrix}$$

and

$$MM^*M = \begin{bmatrix} 0 & AA^*A \\ B^*BB^* & 0 \end{bmatrix}.$$

By Theorem 4.3, $M^2M^* = MM^*M \Leftrightarrow MM^* = M^*M$, which implies

$$AB^*B = AA^*A \text{ and } B^*BB^* = B^*A^*A^* \Leftrightarrow MM^* = M^*M.$$

Now, it is easy to verify

$$M^*M = \begin{bmatrix} BB^* & 0 \\ 0 & A^*A \end{bmatrix}, \ MM^* = \begin{bmatrix} AA^* & 0 \\ 0 & B^*B \end{bmatrix}.$$

Hence, $MM^* = M^*M \Leftrightarrow AA^* = BB^*$ and $B^*B = A^*A$. Thus, (I) is established.

Now, prove foemula (II), $M^*M^2 = MM^*M \Leftrightarrow MM^* = M^*M$, which implies

$$AA^*A = BB^*A \text{ and } A^*AB^* = B^*BB^* \Leftrightarrow MM^* = M^*M.$$

Conclusions can be drawn that

$$AA^*A = BB^*A \text{ and } A^*AB^* = B^*BB^* \Leftrightarrow AA^* = BB^* \text{ and } B^*B = A^*A.$$

This proof is complete. □

We present necessary and sufficient conditions for a matrix to be normal by referring to the commutativity property $RS = YX$, where R, S, X, Y are transforms of A from the set $A^\dagger, A, A^{\textcircled{\#}}, A^*$.

Theorem 3.2. *Let $A \in \mathbb{C}^{n\times n}$. Then, the following conditions are equivalent*

(I) *A is normal;*

(II) *$A^*A^\dagger = A^{\textcircled{\#}}A^*$, $r(A) = r(A^2)$;*

(III) *$A^*A^{\textcircled{\#}} = A^\dagger A^*$, $r(A) = r(A^2)$.*

Proof. From Lemma 1.1, it is easy to figure out that $A^*A^\dagger$ and $A^{\textcircled{\#}}A^*$ are

$$A^*A^\dagger = U \begin{bmatrix} K^*\Sigma K^*\Sigma^{-1} & 0 \\ L^*\Sigma K^*\Sigma^{-1} & 0 \end{bmatrix} U^*, \ A^{\textcircled{\#}}A^* = U \begin{bmatrix} K^{-1}\Sigma^{-1}K^*\Sigma & 0 \\ 0 & 0 \end{bmatrix} U^*.$$

The condition $A^*A^\dagger = A^{\textcircled{\#}}A^*$ is clearly equivalent to $L = 0$. Moreover, by $K^* = K^{-1}$, we get $\Sigma K^{-1}\Sigma^{-1} = \Sigma^{-1}K^{-1}\Sigma$, which implies $\Sigma K = K\Sigma$. So, A is normal.

On the other hand, it is easy to figure out that $A^*A^{\textcircled{\#}}$ and $A^\dagger A^*$ are

$$A^*A^{\textcircled{\#}} = U \begin{bmatrix} K^*\Sigma K^{-1}\Sigma^{-1} & 0 \\ L^*\Sigma K^{-1}\Sigma^{-1} & 0 \end{bmatrix} U^*, \ A^\dagger A^* = U \begin{bmatrix} K^*\Sigma^{-1}K^*\Sigma & 0 \\ L^*\Sigma^{-1}K^*\Sigma & 0 \end{bmatrix} U^*.$$

The condition $A^*A^{\textcircled{\#}} = A^\dagger A^*$ is clearly equivalent to $L = 0$. Moreover, by $K^* = K^{-1}$, we get $\Sigma K^{-1}\Sigma^{-1} = \Sigma^{-1}K^{-1}\Sigma$, which implies $\Sigma K = K\Sigma$. So, A is normal. This proof is complete. □

Corollary 3.3. *Let $A \in \mathbb{C}^{m\times m}$. Then*

(I) $A^3A^* = A^2A^*A \Leftrightarrow AA^* = A^*A$;

(II) $A^4A^* = A^3A^*A,\ A^3A^* = A^2A^*A \Leftrightarrow AA^* = A^*A$;

(III) $A^5A^* = A^4A^*A,\ A^4A^* = A^3A^*A \Leftrightarrow AA^* = A^*A$;

(IV) $A^6A^* = A^5A^*A,\ A^5A^* = A^4A^*A \Leftrightarrow AA^* = A^*A$.

Proof. The $AA^* = A^*A \Rightarrow A^3A^* = A^2A^*A$ is obvious. Now, we show the $A^3A^* = A^2A^*A \Rightarrow AA^* = A^*A$ part. in [15]. It is clearly that

$$AAG = AAH \Rightarrow AG = AH$$
$$A^3A^* = A^2A^*A \Rightarrow AA^2A^* = AAA^*A \Rightarrow A^2A^* = AA^*A.$$

By Lemma 4.3, one has

$$A^2A^* = AA^*A \Rightarrow A^*A = AA^*.$$

It is easy to see that

$$A^4A^* = A^3A^*A \Rightarrow A^3A^* = A^2A^*A \Rightarrow A^2A^* = AA^*A \Rightarrow A^*A = AA^*.$$

Hence, $A^4A^* = A^3A^*A \Leftrightarrow AA^* = A^*A$.

The same can be said for the other equations. □

4. Related results of the Hermitian matrix

In this section we give the equivalent form of the Hermitian of the form $RS = XY$, where $R, S, X, Y \in A^\dagger, A, A^{\oplus}, A^*$, and some conclusions are obtained, and we also get a new equivalent form of the reverse order law.

Lemma 4.1. *Let $A \in \mathbb{C}^{m\times n}$ and $B, C \in \mathbb{C}^{n\times p}$. Then, the following results hold*

(I) $A^*AB = A^*AC \Leftrightarrow AB = AC$. *In particular,* $A^*A = 0 \Leftrightarrow AA^*A = 0 \Leftrightarrow A = 0$ ([14]).

(II) *The principal kth root of positive semi-definite matrix exists and is unique* ([16]).

Lemma 4.2. *Let $A \in \mathbb{C}^{m\times m}$. Then, the following results hold*

(I) $AA^*A = A^*AA^* \Leftrightarrow A = A^*$ ([3]).

(II) $A^3 = AA^*A \Leftrightarrow A = A^*$ ([3]).

Lemma 4.3 ([3])**.** *Let $A \in \mathbb{C}^{m\times m}$. Then*

(I) $(AA^*A)^2 = (AA^*)^3 \Leftrightarrow A = A^*$,

(II) $(AA^*A)^2 = (A^*A)^3 \Leftrightarrow A = A^*$,

(III) $(AA^*A)^3 = (AA^*)^2A(A^*A)^2 \Leftrightarrow A = A^*$,

(IV) $A^3 = A^*AA^*$ *and* $A^5 = (A^*A)^2A^* \Leftrightarrow A = A^*$,

(V) $A^5 = (AA^*)^2A$ *and* $A^7 = (AA^*)^3A \Leftrightarrow A = A^*$.

Next,the above results are pushed to higher order form in order to better judge the equivalence conditions of Hermitian.

Theorem 4.1. *Let* $A \in \mathbb{C}^{m\times m}$*. Then,*

(I) $A^7 = (A^*A)^3A^*$ *and* $A^9 = (A^*A)^4A^* \Leftrightarrow A = A^*$*;*

(II) $A^9 = (AA^*)^4A$ *and* $A^{11} = (AA^*)^5A \Leftrightarrow A = A^*$*;*

(III) $A^{11} = (A^*A)^5A^*$ *and* $A^{13} = (A^*A)^6A^* \Leftrightarrow A = A^*$*;*

(IV) $A^{13} = (AA^*)^6A$ *and* $A^{15} = (AA^*)^7A \Leftrightarrow A = A^*$*;*

$\vdots$

(V) $A^{2k-3} = (A^*A)^kA^*$ *and* $A^{2k-1} = (A^*A)^kA^* \Leftrightarrow A = A^*$*;*

(VI) $A^{2k-1} = (AA^*)^kA$ *and* $A^{2k+1} = (AA^*)^kA \Leftrightarrow A = A^*$.

Proof. We are supported by $A = A^*$, it's easy to get the left-hand side. Now, let us derive from the left-hand formula. Since $A^7 = (A^*A)^3A^*$, one has $A^9 = A^7A^2 = (A^*A)^3A^*AA = (A^*A)^4A^*$, which is equivalent to $A^2 = AA^*$ by applying Lemma(4.1)(I)four times. By the formula (7), one has $A = A^*$. By $A^9 = (AA^*)^4A$, one can get $A^{11} = A^9A^2 = (AA^*)^4AAA$. Hence, $(AA^*)^5A = (AA^*)^4AAA$, which is equivalent to $A^3 = AA^*A$ by applying Lemma(4.1) (I) four times. By the Lemma (4.2), one has $A = A^*$.

Notice that $A^{11} = (A^*A)^5A^*$. Then, $A^{13} = A^{11}A^2 = (A^*A)^5A^*A^2$. Hence, which is equivalent to $A^2 = AA^*$ by applying Lemma(4.1)(I)five times. By the formula (7), one has $A = A^*$.

And so on, by the same proofs, we can get

$$A^{2k-3} = (A^*A)^kA^* \text{ and } A^{2k-1} = (A^*A)^kA^* \Leftrightarrow A = A^*,$$
$$A^{2k-1} = (AA^*)^kA \text{ and } A^{2k+1} = (AA^*)^kA \Leftrightarrow A = A^*.$$

This completes the proof. □

Next, an equivalent form of the matrix is given. We present necessary and sufficient conditions for a matrix to be Hermitian by referring to the commutativity property $RS = YX$, where R, S, X, Y are transforms of A from the set $A^\dagger, A, A^{\oplus}, A^*$.

Theorem 4.2. *Let $A \in \mathbb{C}^{n\times n}$. Then, the following conditions are equivalent:*

(I) *A is Hermitian;*

(II) $AA^{\#} = A^*A^{\#}$, $r(A) = r(A^2)$;

(III) $AA^{\#} = A^{\dagger}A^*$, $r(A) = r(A^2)$;

(IV) $A^{\dagger}A = A^{\#}A^*$, $r(A) = r(A^2)$;

(V) $AA^{\#} = A^*A^{\dagger}$, $r(A) = r(A^2)$.

Proof. By the singular value decomposition of A, one has

$$A = U\begin{bmatrix}\Sigma K & \Sigma L\\ 0 & 0\end{bmatrix}U^*.$$

By calculating, we can get the Core inverse

$$A^{\#} = U\begin{bmatrix}K^{-1}\Sigma^{-1} & 0\\ 0 & 0\end{bmatrix}U^*.$$

On the other hand, the Moore-Penrose inverse of A is

$$A^{\dagger} = U\begin{bmatrix}K^*\Sigma^{-1} & 0\\ L^*\Sigma^{-1} & 0\end{bmatrix}U^*.$$

It is easy to figure out that $AA^{\#}$ and $A^*A^{\#}$ are

$$AA^{\#} = U\begin{bmatrix}\Sigma K K^{-1}\Sigma^{-1} & 0\\ 0 & 0\end{bmatrix}U^*,\quad A^*A^{\#} = U\begin{bmatrix}K^*\Sigma K^{-1}\Sigma^{-1} & 0\\ L^*\Sigma K^{-1}\Sigma^{-1} & 0\end{bmatrix}U^*.$$

The condition $AA^{\#} = A^*A^{\#}$ is clearly equivalent to $L = 0$, $I = K^*\Sigma K^{-1}\Sigma^{-1}$, which implies $\Sigma K = K^*\Sigma$, and hence A is Hermitian.

On the other hand, it is easy to figure out that $AA^{\#}$ and $A^{\dagger}A^*$ are

$$AA^{\#} = U\begin{bmatrix}I & 0\\ 0 & 0\end{bmatrix}U^*,\quad A^{\dagger}A^* = U\begin{bmatrix}K^*\Sigma^{-1}K^*\Sigma & 0\\ L^*\Sigma^{-1}K^*\Sigma & 0\end{bmatrix}U^*.$$

The condition $AA^{\#} = A^{\dagger}A^*$ is clearly equivalent to $L = 0$, $I = K^*\Sigma^{-1}K^{\dagger}\Sigma$ which implies $\Sigma K = K^*\Sigma$. So, A is Hermitian.

The other equations can be proved the same. □

By Lemma 4.2, have the following inference.

Corollary 4.1. *Let $A \in \mathbb{C}^{m\times m}$ and $B \in \mathbb{C}^{m\times m}$. Then,*

$$BABBABB = BA(BBA)^*BB \Leftrightarrow A^*B^*B^* = BBA.$$

Proof. If $A^*B^*B^* = BBA$, we can get $BABBABB = BA(BBA)^*BB$. If $BABBABB = BA(BBA)^*BB$, multiply left by B and right by A on both sides of this equation, we can get

$$BBABBABBA = BBA(BBA)^*BBA \Rightarrow (BBA)^3 = BBA(BBA)^*BBA.$$

According to Lemma 4.2, one has $A^3 = AA^*A \Leftrightarrow A = A^*$, by which we can get $BBA = (BBA)^* = A^*B^*B^*$. This completes the proof. □

Theorem 4.3. *Let $A, B \in \mathbb{C}^{m\times n}$. Then,*

(I) $A^*AA^* = B^*BB^* \Leftrightarrow A = B$,

(II) $A^*BA^* = B^*BB^*$ *and* $BB^*B = BA^*B \Leftrightarrow A = B$.

Proof. Let

$$X = \begin{bmatrix} 0 & B \\ A^* & 0 \end{bmatrix} \text{ and } X^* = \begin{bmatrix} 0 & A \\ B^* & 0 \end{bmatrix}.$$

In this situation, it is easy to verify

$$XX^*X = \begin{bmatrix} 0 & BB^*B \\ A^*AA^* & 0 \end{bmatrix}, X^*XX^* = \begin{bmatrix} 0 & AA^*A \\ B^*BB^* & 0 \end{bmatrix},$$
$$X^3 = \begin{bmatrix} 0 & BA^*B \\ A^*BA^* & 0 \end{bmatrix}.$$

From Lemma 4.2, we can get

$$X^3 = XX^*X = X^*XX^* \Rightarrow X = X^* \Rightarrow A = B.$$

So, we get the following:

$$A^*AA^* = B^*BB^* \Rightarrow A = B,$$
$$A^*BA^* = B^*BB^* \text{ and } BB^*B = BA^*B \Rightarrow A = B.$$

This completes the proof. □

Theorem 4.4. *Let $A, B \in \mathbb{C}^{n\times n}$ be two Hermitian matrices. Then, the following seven statements are equivalent:*

(I) $BBA = ABB$;

(II) $(BBA)(ABB)(BBA) = (ABB)(BBA)(ABB)$;

(III) $(BBA)^3 = (BBA)(ABB)(BBA)$;

(IV) $((BBA)(ABB)(BBA))^2 = ((BBA)(ABB))^3$;

(V) $((BBA)(ABB)(BBA))^3 = ((BBA)(ABB))^2(BBA)((ABB)(BBA))^2$;

(VI) $(BBA)^3 = (BBA)(ABB)(BBA)$ *and*
$(BBA)^5 = (BBA)[(ABB)(BBA)^2](BBA)$;

(VII) $(BBA)^5 = (BBA)[(ABB)(BBA)^2](BBA)$ *and*
$(BBA)^7 = (BBA)[(ABB)(BBA)^4](BBA)$.

Proof. If $BBA = ABB$, the other equations are obviously true. Now, let us verify $BBA = ABB$ with something else. Since $(BBA)(ABB)(BBA) = (ABB)(BBA)(ABB)$, one has $(BBA)(BBA)^*(BBA) = (BBA)^*(BBA)(BBA)^*$. It follows from replacing A with BBA in Lemma 4.2. We have that

$$(BBA)(BBA)^*(BBA) = (BBA)^*(BBA)(BBA)^* \Rightarrow (BBA)^* = A^*B^*B^* = ABB = BBA$$

and we can do the same thing with the rest. This completes the proof. □

A typical matrix equality for the case of the product of two matrices of appropriate sizes is

$$(AB)^\dagger = B^\dagger A^\dagger,$$

which is usually called the reverse-order law for the Moore-Penrose inverse of a matrix product. The reverse-order law does not necessarily hold. So, have a fact

$$(AB)^\dagger = B^\dagger A^\dagger \Leftrightarrow (ABB^*A^*AB)^\dagger = B^\dagger (A^*A)^\dagger (BB^*)^\dagger A^\dagger$$

(see [3]). In [9, 12, 13, 18, 22], some equivalent conditions for the reverse-order law are also given. The result of theorem4.3 applies here to the inverse order law. In the following, we will give a new equivalent condition for the reverse-order law.

Theorem 4.5. *Let* $A \in \mathbb{C}^{m\times n}$ *and* $B \in \mathbb{C}^{n\times p}$*. Then,*

$$(AB)^\dagger = B^\dagger A^\dagger \Leftrightarrow (B^*A^*ABB^*A^*)^\dagger = (A^*)^\dagger (BB^*)^\dagger (A^*A)^\dagger (B^*)^\dagger.$$

Proof. The implying

$$(AB)^\dagger = B^\dagger A^\dagger \Rightarrow (B^*A^*ABB^*A^*)^\dagger = (A^*)^\dagger (BB^*)^\dagger (A^*A)^\dagger (B^*)^\dagger$$

is obvious. Now, we will prove the reverse part. Utilization Theorem 4.3 (1), $(AB)^\dagger$ instead of A, $B^\dagger A^\dagger$ instead of B, one has

$$((AB)^\dagger))^*(AB)^\dagger)((AB)^\dagger))^* = ((B^\dagger A^\dagger))^*(B^\dagger A^\dagger)((B^\dagger A^\dagger))^* \Rightarrow (AB)^\dagger = B^\dagger A^\dagger.$$

Hence, $(B^*A^*ABB^*A^*)^\dagger = (A^*)^\dagger (BB^*)^\dagger (A^*A)^\dagger (B^*)^\dagger \Rightarrow (AB)^\dagger = B^\dagger A^\dagger$.

This completes the proof. □

5. Conclusions

Obviously,the results in the above theorem and corollaries offer lots of equivalent facts about EP-matrices, normal matrices, and Hermitian matrices. This fact can be described in the implication form $f(A, A^*) = 0 \Leftrightarrow AA^* = A^*A$ and $f(A, A^\dagger) = 0 \Leftrightarrow AA^\dagger = A^\dagger A$. In this note, applied the core inverse to provide new characterizations. We show servers special cases of the equivalent facts:

(I) $AA^{\oplus} = A^*A^{\oplus} \Leftrightarrow A = A^*$.

(II) $AA^{\oplus} = A^\dagger A^* \Leftrightarrow A = A^*$ and (I) A is EP.

(II) $AA^\dagger A^{\oplus} = A^\dagger A^{\oplus} A$.

(III) $AA^{\oplus} A^* = A^*AA^{\oplus}$.

(IV) $AA^{\oplus} A^\dagger = A^{\oplus} A^\dagger A$.

(V) $A^\dagger AA^{\oplus} = A^{\oplus} A^\dagger A$

without assuming the invertibility of A through the skillful use of decompositions of matrices. Some equivalent forms related to $A^{\oplus}$, $A^\dagger$, A^* can also be obtained. A new equivalent condition for the reverse order law is also obtained:$(AB)^\dagger = B^\dagger A^\dagger \Leftrightarrow (B^*A^*ABB^*A^*)^\dagger = (A^*)^\dagger(BB^*)^\dagger(A^*A)^\dagger(B^*)^\dagger$. We can also use the new generalized inverse to study the equivalent form of EP, normal and Hermitian matrices.

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More on the weakly S-2-prime ideals of commutative rings

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Abstract. Prime ideals and their generalizations are fundamental in various research areas, especially in commutative algebra. The study of weakly prime ideals is marked the beginning of this generalization. Subsequent research has further expanded these concepts, with recent attention on weakly 2-prime and S-2-prime ideals. This study aims for new characterizations of weakly S-2-prime ideals, a generalization that includes both weakly 2-prime and S-2-prime ideals. To achieve this goal, we construct an ideal disjoint with a multiplicatively closed subset of commutative rings. We explore several characterizations concerning weakly S-2-prime ideals and investigate this class of ideals in polynomial and formal power series rings. Besides, we examine several new results regarding the trivial extension and amalgamated algebra along an ideal with respect to a ring homomorphism concerning weakly S-2-prime ideals.

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1. Introduction

In this paper, we suppose that all rings are commutative with a non-zero identity. For any proper ideal Q of a ring R, the radical of Q is defined by intersection of all prime ideals containing Q, denoted by $\sqrt{Q}$ which is equivalent to the set $\{\alpha \in R : \alpha^n \in Q$ for some $n \in \mathbb{N}\}$. In particular, $Nil(R)$ is the set of all nilpotent elements of R called the nilradical of R and described by $Nil(R) := \sqrt{0_R} = \{\alpha \in R : \alpha^n = 0$ for some positive integer $n\}$. Furthermore, a ring R is called a reduced ring if it has no non-zero nilpotent elements (that is $Nil(R) = 0$).

The notion of prime ideals and its generalizations play a central role in commutative algebra, and so this concept has been generalized and investigated in many aspects. In 2003, Anderson and Smith [4] introduced the notion of weakly prime ideals. A proper ideal Q of a ring R is called weakly prime if $0 \neq \alpha\beta \in Q$ for some α, $\beta \in R$ implies that $\alpha \in Q$ or $\beta \in Q$. It is well known that prime ideals are weakly prime ideals but the other statement is not generally true, see [4]. On the other hand, S-prime ideals, which are extentions of prime ideals, were introduced by Hamed and Malek [10]. Remember that a subset S of R is called a multiplicatively closed subset (in briefly m.c.s) if S is closed under multiplication and $1 \in S$. Let S be an m.c.s of R and Q be an ideal with $Q \cap S = \emptyset$. In their work, Hamed and Malek defined an S-prime ideal of R as follows: if there exists an $s \in S$ such that for all α, $\beta \in R$ with $\alpha\beta \in Q$, we have $s\alpha \in Q$ or $s\beta \in Q$. Later, Almahdi et al. [1] defined weakly S-prime ideals, which further generalized S-prime ideals. An ideal Q disjoint with S is considered a weakly S-prime ideal if there exists an $s \in S$ such that for all α, $\beta \in R$ with $0 \neq \alpha\beta \in Q$, we have $s\alpha \in Q$ or $s\beta \in Q$.

Furthermore, Beddani and Messirdi [6] introduced and studied 2-prime ideals, which offer another generalization of prime ideals and this concept has also been investigated by Nikandish et al. [15]. A proper ideal Q of R is called a 2-prime ideal if $\alpha, \beta \in R$ such that $\alpha\beta \in Q$, then either $\alpha^2 \in Q$ or $\beta^2 \in Q$. Additionally, Koç [13] described weakly 2-prime ideals as a generalization of 2-prime ideals and explored this notion in the context of compactly packedness and coprimely packedness in trivial extensions. Moreover, Issoual et al. [12] further examined properties of this class of ideals. In their framework, a proper ideal Q is called a weakly 2-prime ideal of R if $0 \neq \alpha\beta \in Q$ for some $\alpha, \beta \in R$, then either $\alpha^2 \in Q$ or $\beta^2 \in Q$.

As a recent research [16], the concept of S-2-prime ideals, which generalize both S-prime and 2-prime ideals, is introduced. An ideal Q of R with $Q \cap S = \emptyset$ is called an S-2-prime ideal of R if there exists an $s \in S$ such that for all $\alpha, \beta \in R$ with $\alpha\beta \in Q$, we have $s\alpha^2 \in Q$ or $s\beta^2 \in Q$. As a more recent development, in

[17], the notion of weakly S-2-prime ideals, which generalize both S-2-prime and weakly 2-prime ideals, is defined. An ideal Q of R with $Q \cap S = \emptyset$ is called a weakly S-2-prime ideal of R if there exists an $s \in S$ such that for all α, $\beta \in R$ with $0 \neq \alpha\beta \in Q$, we have $s\alpha^2 \in Q$ or $s\beta^2 \in Q$.

In light of the ongoing exploration into generalizations of prime ideals, we endeavor to broaden the scope by incorporating S-2-prime ideals and weakly 2-prime ideals. Motivated by the aforementioned previous studies, we introduce and investigate new characterizations and properties of the concept of weakly S-2-prime ideals in commutative rings in Section 2 (Theorem 2.1, Propositions 2.1, 2.2). Furthermore, we explore weakly S-2-prime ideals in commutative rings whose characteristic is 2 (Theorem 2.2, Corollary 2.1), drawing insights from [3]. Moreover, we examine the behavior of these ideals in polynomial and formal power series rings (Theorems 2.3, 2.4), utilizing references ([2], [8], [9]). Additionally, we delve into new findings concerning weakly S-2-prime ideals in trivial ring extensions, idealizations, and amalgamated algebras along an ideal with regard to a ring homomorphism taking advantage of ([5], [7], [11], [14]) in the subsequent section (Theorems 3.1, 3.2).

As a result, we observe that many of the results established for S-2-prime ideals and weakly 2-prime ideals are analogously obtained by weakly S-2-prime ideals, which encompass a broader scope. Furthermore, in the light of the trivial ring extensions studied in S-2-prime ideals and amalgamated algebra along an ideal with respect to a ring homomorphism studied in weakly 2-prime ideals, we elaborated on the properties of weakly S-2-prime ideals on these algebraic structures.

2. Characterizations and properties of weakly S-2-prime ideals

In this section, we investigate weakly S-2-prime ideals and present their new properties. Unless otherwise stated, R denotes a commutative ring with identity. We recall the following definitions:

Definition 2.1. *Let S be an m.c.s of a ring R and Q be an ideal of R with $Q \cap S = \emptyset$.*

(*i*) *[16], Q is called an S-2-prime ideal of R if there exists an $s \in S$ such that for all α, $\beta \in R$ with $\alpha\beta \in Q$, we have $s\alpha^2 \in Q$ or $s\beta^2 \in Q$. In this case, we say that Q is associated to s.*

(*ii*) *[17], Q is called a weakly S-2-prime ideal of R if there exists an $s \in S$ such that for all α, $\beta \in R$ with $0 \neq \alpha\beta \in Q$, we have $s\alpha^2 \in Q$ or $s\beta^2 \in Q$. In this case, we say that Q is associated to s.*

Clearly, an S-2-prime ideal of R is a weakly S-2-prime ideal of R. However, the converse implication is not true in general see [17, Example 2.4]. Now, we will present a new characterization of weakly S-2-prime ideals.

Theorem 2.1. *Let S be an m.c.s of R and Q be an ideal of R disjoint with S. The following statements are equivalent:*

(i) Q is a weakly S-2-prime ideal of R associated to $s \in S$.

(ii) For every $r \in R$, if $r^2 \notin (Q : s)$, then $(Q : r) \subseteq (0 : r) \cup \{r \in R : sr^2 \in Q\}$.

Proof. $(i) \implies (ii)$: Let Q be a weakly S-2-prime ideal of R associated to $s \in S$. Suppose that $r \in R$ with $r^2 \notin (Q : s)$ and $\beta \in (Q : r)$. Then, we have $\beta r \in Q$. If $0 \neq \beta r \in Q$, then $s\beta^2 \in Q$ since Q is a weakly S-2-prime ideal and $r^2 \notin (Q : s)$. Then, we have $\beta \in \{r \in R : sr^2 \in Q\}$. Now if $\beta r = 0$, then $\beta \in (0 : r)$, so we have $(Q : r) \subseteq (0 : r) \cup \{r \in R : sr^2 \in Q\}$.

$(ii) \implies (i)$: Let $0 \neq \alpha\beta \in Q$ and $s\alpha^2 \notin Q$ for all $\alpha, \beta \in R$. Then, we have $\beta \in (Q : \alpha)$ and $\beta \notin (0 : \alpha)$. From the assumption, we have $\beta \in \{r \in R : sr^2 \in Q\}$. Therefore, $s\beta^2 \in Q$ and Q is a weakly S-2-prime ideal of R associated to s. □

In the following result, we will present another characterization of a weakly S-2-prime ideal. First of all, we need the next definitions.

Definition 2.2. *Suppose that S is an m.c.s of R and Q is a weakly S-2-prime ideal of R associated to $s \in S$.*

(i) Let $\alpha, \beta \in R$. We call (α, β) an S-double-zero of Q if $\alpha\beta = 0$, $s\alpha^2 \notin Q$ and $s\beta^2 \notin Q$.

(ii) Let $AB \subseteq Q$ for some ideals A, B of R. If (α, β) is not an S-double-zero of Q for every $\alpha \in A$ and $\beta \in B$, then we call Q a free S-double-zero with regard to AB.

Note that if Q is a weakly S-2-prime ideal of R without S-double-zeros, then Q is an S-2-prime ideal of R. So, if Q is a weakly S-2-prime ideal which is not an S-2-prime ideal, then there exists an S-double-zero of Q.

Let Q be a proper ideal of R. Recall from [3] that the ideal generated by n^{th} powers of elements of Q is denoted by $Q_{[n]} = \langle\{q^n : q \in Q\}\rangle$. It is easy to see that $Q_{[n]} \subseteq Q^n \subseteq Q$ and also the equality provides if $n = 1$. Moreover, if $n!.1_R$ is a unit of R, then $Q_{[n]} = Q^n$ see [3, Theorem 5].

Proposition 2.1. *Let S be an m.c.s of R, Q be a weakly S-2-prime ideal of R associated to $s \in S$ and P be a proper ideal of R with $\alpha P \subseteq Q$ for some $\alpha \in R$. If (α, p) is not an S-double-zero of Q for all $p \in P$ and $s\alpha^2 \notin Q$, then $sP_{[2]} \subseteq Q$. Furthermore, if 2.1_R is a unit, then $sP^2 \subseteq Q$.*

Proof. Assume that Q is a weakly S-2-prime ideal of R associated to $s \in S$ and $sP_{[2]} \nsubseteq Q$. Then, there exists $p \in P$ such that $sp^2 \notin Q$. Also, we have $\alpha p \in Q$ since $\alpha P \subseteq Q$. If $\alpha p \neq 0$, it contradicts with our assumption that $s\alpha^2 \notin Q$ and $sp^2 \notin Q$. Thus, $\alpha p = 0$. Since (α, p) is not an S-double-zero of Q and $s\alpha^2 \notin Q$, we conclude that $sp^2 \in Q$, which is a contradiction. Therefore, $sP_{[2]} \subseteq Q$. The "furthermore" part is clear because of $P_{[2]} = P^2$. □

Proposition 2.2. *Let S be an m.c.s of R, Q be a weakly S-2-prime ideal of R associated to $s \in S$ and $0 \neq AB \subseteq Q$ for some ideals A and B of R. If Q*

is a free S-double-zero with regard to AB, then either $sA_{[2]} \subseteq Q$ or $sB_{[2]} \subseteq Q$. Furthermore, if 2.1_R is a unit, then either $sA^2 \subseteq Q$ or $sB^2 \subseteq Q$.

Proof. Suppose that Q is a free S-double-zero with regard to AB and $0 \neq AB \subseteq Q$. If $sA_{[2]} \nsubseteq Q$, then there exits $\alpha \in A$ such that $s\alpha^2 \notin Q$. Since Q is a free S-double-zero with regard to AB, we conclude that (α, β) is not an S-double-zero of Q for all $\beta \in B$. From Proposition 2.1, we have $sB_{[2]} \subseteq Q$. The rest of the proof is clear as $A_{[2]} = A^2$ and $B_{[2]} = B^2$. □

Let Q be an ideal of a ring R. Then, we define the set of all elements of R whose square is in Q, that is $\sqrt[2]{Q} = \{\alpha \in R : \alpha^2 \in Q\}$. Also, it is easy to see that $Q \subseteq \sqrt[2]{Q} \subseteq \sqrt{Q}$. Note that $\sqrt[2]{Q}$ may not be an ideal of R. See the next example.

Example 2.1. Suppose that F be a field whose characteristic is not 2 and $R = F[X, Y, Z]$, where X, Y, Z are indeterminates over F. Take the ideal $Q = (X^2, Y^2, Z^2)$ of R. We know that $\sqrt{Q} = (X, Y, Z)$ and also $X, Y, Z \in \sqrt[2]{Q}$. However, $(X + Y + Z)^2 = X^2 + Y^2 + Z^2 + 2XY + 2YZ + 2XZ \notin Q$. Therefore, $\sqrt[2]{Q}$ is not an ideal of R.

We might inquire under what conditions $\sqrt[2]{Q}$ becomes an ideal of R. We provide an answer to this question with the next result.

Proposition 2.3. *Let R be a ring and Q be a proper ideal of R.*

(i) $\sqrt[2]{Q}$ is an ideal of R if and only if $2(\sqrt[2]{Q})^2 \subseteq Q$.

(ii) If $char(R) = 2$, then $\sqrt[2]{Q}$ is an ideal of R.

(iii) Let S be an m.c.s of a ring R, $s \in S$ and $2(\sqrt[2]{Q : s})^2 \subseteq (Q : s)$. Then, for any ideal K of R, $K \subseteq \sqrt[2]{Q : s}$ if and only if $sK_{[2]} \subseteq Q$.

Proof. (i) Let $\sqrt[2]{Q}$ be an ideal of R and α, $\beta \in \sqrt[2]{Q}$. This implies that α^2, $\beta^2 \in Q$. Since $\sqrt[2]{Q}$ is an ideal of R, we have $(\alpha + \beta)^2 = \alpha^2 + 2\alpha\beta + \beta^2 \in Q$. We conclude that $2\alpha\beta \in Q$ or $2(\sqrt[2]{Q})^2 \subseteq Q$.

Conversely, let $2(\sqrt[2]{Q})^2 \subseteq Q$ and α, $\beta \in \sqrt[2]{Q}$. Then, $(r\alpha)^2 = r^2\alpha^2 \in Q$ for all $r \in R$, and thus $r\alpha \in \sqrt[2]{Q}$. Also, from assumption, we have $2\alpha\beta \in 2(\sqrt[2]{Q})^2 \subseteq Q$. This implies $(\alpha + \beta)^2 = \alpha^2 + 2\alpha\beta + \beta^2 \in Q$. Hence, $\alpha + \beta \in \sqrt[2]{Q}$ and $\sqrt[2]{Q}$ is an ideal of R.

(ii) Let $char(R) = 2$. We have $2(\sqrt[2]{Q})^2 = (0) \subseteq Q$. Using (i), proof is clear.

(iii) Let $2(\sqrt[2]{Q : s})^2 \subseteq (Q : s)$. From (i), $\sqrt[2]{Q : s}$ is an ideal of R. Suppose that $K \subseteq \sqrt[2]{Q : s}$ and $sk^2 \in sK_{[2]}$ such that $k \in K$. We have $k \in \sqrt[2]{Q : s}$ that is $sk^2 \in Q$. We conclude $sK_{[2]} \subseteq Q$.

Conversely, let $sK_{[2]} \subseteq Q$ and $k \in K$. We have $sk^2 \in sK_{[2]} \subseteq Q$ and so $k \in \sqrt[2]{Q : s}$. Thus, we conclude $K \subseteq \sqrt[2]{Q : s}$. □

Now, we will present a new characterization of weakly S-2-prime ideals, and also we will benefit from this characterization to examine the weakly S-2-prime ideals on polynomial and formal power series rings.

Theorem 2.2. *Let S be an m.c.s of R and Q be an ideal of R disjoint with S such that $2(\sqrt[2]{Q:s})^2 \subseteq (Q:s)$. The following assertions are equivalent:*

(i) *Q is a weakly S-2-prime ideal of R associated to $s \in S$.*

(ii) *There exists an $s \in S$ such that for all $\alpha \in R - \sqrt[2]{Q:s}$, either $(Q:\alpha) \subseteq ann(\alpha)$ or $(Q:\alpha) \subseteq \sqrt[2]{Q:s}$.*

(iii) *There exists an $s \in S$ such that for all $\alpha \in R$ with $\alpha^2 \notin (Q:s)$, either $(Q:\alpha) \subseteq ann(\alpha)$ or $s(Q:\alpha)_{[2]} \subseteq Q$.*

(iv) *There exists an $s \in S$ such that $0 \neq \alpha K \subseteq Q$ for some $\alpha \in R$ and ideal K of R, either $s\alpha^2 \in Q$ or $sK_{[2]} \subseteq Q$.*

(v) *There exists an $s \in S$ such that $0 \neq JK \subseteq Q$ for some ideals J, K of R, either $sJ_{[2]} \subseteq Q$ or $sK_{[2]} \subseteq Q$.*

Proof. (i)$\Longrightarrow$(ii) Let Q be a weakly S-2-prime ideal of R associated to s and take $\alpha \in R - \sqrt[2]{Q:s}$. Then, $\alpha^2 \notin (Q:s)$. From Theorem 2.1, we have $(Q:\alpha) \subseteq (0:\alpha) \cup \sqrt[2]{Q:s}$. Since $2(\sqrt[2]{Q:s})^2 \subseteq (Q:s)$, $\sqrt[2]{Q:s}$ is an ideal of R. Thus, we have either $(Q:\alpha) \subseteq ann(\alpha)$ or $(Q:\alpha) \subseteq (\sqrt[2]{Q:s})$.

(ii)$\Longleftrightarrow$(iii) Clear from Proposition 2.3 (iii).

(iii)$\Longrightarrow$(iv) Let $0 \neq \alpha K \subseteq Q$ for all $\alpha \in R$ and ideal K of R with $s\alpha^2 \notin Q$. Then, $\alpha^2 \notin (Q:s)$ and from assumption, we conclude either $K \subseteq (Q:\alpha) \subseteq ann(\alpha)$ or $sK_{[2]} \subseteq s(Q:\alpha)_{[2]} \subseteq Q$. The first case is impossible because $\alpha K \neq 0$. So, we have $sK_{[2]} \subseteq Q$.

(iv)$\Longrightarrow$(i) Suppose that $0 \neq \alpha\beta \in Q$ for all $\alpha, \beta \in R$. Put $K = (\beta)$ in (iv).

(v)$\Longrightarrow$(ii) Let $\alpha \in R - \sqrt[2]{Q:s}$ and $\beta \in (Q:\alpha)$. If $\alpha\beta = 0$, then $\beta \in ann(\alpha)$. Suppose that $\alpha\beta \neq 0$. Put $J = (\alpha)$ and $K = (\beta)$. Then, (v) implies that there exists an $s \in S$ such that either $s\alpha^2 \in sJ_{[2]} \subseteq Q$ or $s\beta^2 \in sK_{[2]} \subseteq Q$. Since $\alpha \in R - \sqrt[2]{Q:s}$, we have $\beta \in \sqrt[2]{Q:s}$. Thus, $(Q:\alpha) \subseteq ann(\alpha) \cup \sqrt[2]{Q:s}$, so the claim is clear.

(ii)$\Longrightarrow$(v) Let $0 \neq JK \subseteq Q$ for all ideals J, K of R and $sJ_{[2]} \nsubseteq Q$. Then, there exists $\alpha \in J$ such that $s\alpha^2 \notin Q$ or $\alpha \in R - \sqrt[2]{Q:s}$. If $\alpha K \neq 0$, then by assumption, $K \subseteq (Q:\alpha) \subseteq \sqrt[2]{Q:s}$ which implies that $sK_{[2]} \subseteq Q$. So, suppose $\alpha K = 0$. Since $JK \neq 0$, there exists $\beta \in J$ such that $\beta K \neq 0$. If $\beta \in R - \sqrt[2]{Q:s}$, again by our assumption we have $sK_{[2]} \subseteq Q$. Now, we can suppose that $\beta \in \sqrt[2]{Q:s}$. Since $2(\sqrt[2]{Q:s})^2 \subseteq (Q:s)$, $(\sqrt[2]{Q:s})$ is an ideal of R, $\beta + \alpha \in R - \sqrt[2]{Q:s}$. Moreover, $0 \neq (\beta+\alpha)K = \beta K \subseteq Q$. Then, our assumption yields $K \subseteq (Q:\beta+\alpha) \subseteq \sqrt[2]{Q:s}$ which implies that $sK_{[2]} \subseteq Q$, as desired. □

The following corollary is a direct consequence of Theorem 2.2 and Proposition 2.3.

Corollary 2.1. *Suppose that R is a ring whose characteristic is 2, S is an m.c.s of R and Q is an ideal of R with $Q \cap S = \emptyset$. Then, Q is a weakly S-2-prime ideal of R associated to $s \in S$ if and only if there exists an $s \in S$ such that for all ideals J, K of R with $0 \neq JK \subseteq Q$, either $sJ_{[2]} \subseteq Q$ or $sK_{[2]} \subseteq Q$.*

Let R be a ring and $R[X]$ be a polynomial ring, where X is an indeterminate over R. For any $g(x) = \sum_{j=0}^{n} \alpha_j X^j$, the content ideal of g is denoted by $c(g) = (\alpha_0, \alpha_1, ..., \alpha_n)$ [9]. If Q is an ideal of R, then $Q[X] = \{g \in R[X] : c(g) \subseteq Q\}$ is an ideal of $R[X]$. Moreover, a subset $S[X]$ of $R[X]$ is called an m.c.s of $R[X]$ if $S[X]$ closed under multiplication and $1 \in S[X]$. It is clear to verify that if S is an m.c.s of R, then $S[X]$ is an m.c.s of $R[X]$.

Theorem 2.3. *Let R be a ring with 2.1_R a unit of R and S be an m.c.s of R with $s \in S$. Suppose that Q and $(Q : s)$ are radical ideals of R. Then, Q is a weakly S-2-prime ideal of R associated to s if and only if $Q[X]$ is a weakly $S[X]$-2-prime ideal of $R[X]$ associated to s.*

Proof. Let Q be a weakly S-2-prime ideal of R associated to s. It is clear that $Q[X] \cap S[X] = \emptyset$. Since $\sqrt[2]{Q : s} \subseteq \sqrt{Q : s} = (Q : s)$, this gives $2(\sqrt[2]{Q : s})^2 \subseteq (Q : s)$. Let $0 \neq gh \in Q[X]$ for all $g, h \in R[X]$. This implies $c(gh) \subseteq Q$. Suppose $deg(g) = k$. From Dedekind-Mertens Theorem [9, Theorem 28.1], we have $c(g)c(h)^{k+1} = c(gh)c(h)^k \subseteq Q$. Since Q is a radical ideal, we have $0 \neq c(g)c(h) \subseteq Q$. From Theorem 2.2 (v), $sc(g)_{[2]} \subseteq Q$ or $sc(h)_{[2]} \subseteq Q$. As 2.1_R is a unit of R, we have $sc(g^2) \subseteq sc(g)^2 = sc(g)_{[2]} \subseteq Q$ or $sc(h^2) \subseteq sc(h)^2 = sc(h)_{[2]} \subseteq Q$. Since $sc(g^2) \subseteq Q$, we have $g^2 \in (Q : s)[X]$. That is, $sg^2 \in Q[X]$. Similarly, we can achieve $sh^2 \in Q[X]$. Hence, $Q[X]$ is a weakly $S[X]$-2-prime ideal of $R[X]$ associated to s. The converse part is straightforward by taking constant polynomials. □

From [2], a ring R is called Gaussian ring if $c(gh) = c(g)c(h)$ for every g, $h \in R[X]$. Then, we can remove the condition "Q is a radical ideal of R" in the Theorem 2.3 provided that R is a Gaussian ring.

Let R be a ring and $R[[X]]$ be a ring of formal power series, where X is an indeterminate over R. For any $g = \sum_{j=0}^{\infty} \alpha_j X^j \in R[[X]]$, the content ideal of g is denoted by $c(g) = \langle \{\alpha_j : j \in \mathbb{N} \cup \{0\}\} \rangle$. If Q is an ideal of R, then $Q[[X]] = \{g \in R[[X]] : c(g) \subseteq Q\}$ is an ideal of $R[[X]]$. Note that if S is an m.c.s of R, then $S[[X]]$ is an m.c.s of $R[[X]]$.

In [8], the authors established a version of the Dedekind-Mertens Theorem for Noetherian formal power series rings. We will now examine the weakly S-2-prime ideals in Noetherian formal power series rings.

Theorem 2.4. *Let R be a Noetherian ring, S be an m.c.s of R with $s \in S$ and 2.1_R be a unit of R. Suppose that Q and $(Q : s)$ are radical ideals of R. Then, Q is a weakly S-2-prime ideal of R associated to s if and only if $Q[[X]]$ is a weakly $S[[X]]$-2-prime ideal of $R[[X]]$ associated to s.*

Proof. $\Longleftarrow$: Proof is straightforward by taking constant power series.

$\Longrightarrow$: Let Q be a weakly S-2-prime ideal of R associated to s. It is clear that $Q[[X]] \cap S[[X]] = \emptyset$. Let $0 \neq gh \in Q[[X]]$ for all g, $h \in R[[X]]$. This implies $c(gh) \subseteq Q$. Let $\mu(c(g))$ denotes the minimal number of the generators of $c(g)$. As R is a Noetherian ring, we can choose k as maximum of the numbers

$\mu(c(g)_m)$, taken over all maximal ideals m of R. From [8, Theorem 2.6], we have $c(g)c(h)^k = c(gh)c(h)^{k-1} \subseteq Q$. Since Q is a radical ideal, we have $0 \neq c(g)c(h) \subseteq Q$. Since 2.1_R is a unit of R and $2(\sqrt[2]{Q:s})^2 \subseteq (Q:s)$, by the similar argument in the Theorem 2.3, we conclude that $sc(g^2) \subseteq Q$ or $sc(h^2) \subseteq Q$. Since $sc(g^2) \subseteq Q$, we have $g^2 \in (Q:s)[[X]]$. That is, $sg^2 \in Q[[X]]$. Similarly, we can achieve $sh^2 \in Q[[X]]$. Hence, $Q[[X]]$ is a weakly $S[[X]]$-2-prime ideal of $R[[X]]$ associated to s. □

In the previous theorem, the condition that Q and $(Q:s)$ are radical ideals of R does not ensure that these ideals are weakly S-2-prime ideals of R.

Example 2.2. Let $R = \mathbb{Z}_{12}$ be a ring, $S = \{\overline{1}, \overline{5}\}$ an m.c.s of R, $Q = (\overline{6})$ an ideal of R disjoint with S. We can achieve $\sqrt{Q} = (\overline{6})$ and $(Q:s) = \{\overline{0}, \overline{6}\}$. Q and $(Q:s)$ are radical ideals of R since $Q = \sqrt{Q} = \{\overline{0}, \overline{6}\}$ and $(Q:s) = \sqrt{(Q:s)} = \{\overline{0}, \overline{6}\}$. However, Q and $(Q:s)$ are not weakly S-2-prime ideals of R since $0 \neq \overline{2}.\overline{3} \in Q$ (and $(Q:s)$) but $s.\overline{2}^2 \notin Q$ (and $\notin (Q:s)$) and $s.\overline{3}^2 \notin Q$ (and $\notin (Q:s)$) for all $s \in S$.

3. Idealization and amalgamation properties on the weakly S-2-prime ideals

In this part, we examine the class of weakly S-2-prime ideals of characteristics over $R(+)M$ constructions. Let M be an R-module. The trivial extension or idealization $R(+)M = \{(r,m) : r \in R,\ m \in M\}$ is a commutative ring with componentwise addition and multiplication described by $(\alpha,m)(\beta,m') = (\alpha\beta, \alpha m' + \beta m)$ for all (α,m), $(\beta,m') \in R(+)M$ (see [5, 14]).

Theorem 3.1. *Let S be an m.c.s of a ring R, Q an ideal of R with $Q \cap S = \emptyset$ and M an R-module. Then, the following statements are equivalent:*

(i) $Q(+)M$ is a weakly $(S(+)0)$-2-prime ideal (and weakly $(S(+)M)$-2-prime ideal) of $R(+)M$.

(ii) Q is a weakly S-2-prime ideal of R and for every S-double-zero (α,β) of Q, we have $\alpha M = 0 = \beta M$.

Proof. The proof is clear from [17, Theorem 2.14]. □

Example 3.1. Suppose that R is a reduced ring, M an $R-$module and S an m.c.s of R. The unique ideal of $R(+)M$ which has the form $Q(+)M$ which is weakly $(S(+)0)$-2-prime ideal (resp. weakly $(S(+)M)$-2-prime ideal) and not $(S(+)0)$-2-prime ideal (resp. not $(S(+)M)$-2-prime ideal) is $0(+)M$. Indeed, if $Q(+)M$ is weakly $(S(+)0)$-2-prime ideal (resp. weakly $(S(+)M)$-2-prime ideal) and not $(S(+)0)$-2-prime ideal (resp. not $(S(+)M)$-2-prime ideal), then from [16, Theorem 6] and [17, Theorem 2.14], we have Q is a weakly S-2-prime ideal and not S-2-prime ideal of R. From [17, Theorem 2.6], we know that if R is a reduced ring and Q is a weakly S-2-prime ideal, then either Q is an S-2-prime ideal or $Q = 0$. Then, we have $Q = 0$.

Let $f : A \to B$ be a ring homomorphism and J be an ideal of B. Describe the subring of $A \times B$ as:

$$A \bowtie^f J = \{(\alpha, f(\alpha) + j) : \alpha \in A,\ j \in J\}$$

called the amalgamation of A with B along the ideal J with regard to f. This structure is presented and examined by [7].

We will examine this amalgamation algebra property for weakly S-2-prime ideals.

Theorem 3.2. *Suppose that $f : A \to B$ is a ring homomorphism, J is an ideal of B, Q is an ideal of A and S is an m.c.s of A.*

(1) *If $Q \bowtie^f J$ is a weakly $(S \bowtie^f 0)$-2-prime ideal of $A \bowtie^f J$, then Q is a weakly S-2-prime ideal of A.*

(2) *If Q is a weakly S-2-prime ideal which is not an S-2-prime ideal of A, then the following assertions are equivalent:*

  (i) *$Q \bowtie^f J$ is a weakly $(S \bowtie^f 0)$-2-prime ideal of $A \bowtie^f J$,*

  (ii) *For each S-double-zero $(\alpha, \beta) \in A \times A$ of Q, we have $f(\alpha)J = 0 = f(\beta)J$ and $J^2 = 0$.*

We need the following lemmas to verify the theorem above.

Lemma 3.1. *Let $f : A \to B$ be a ring homomorphism, J be an ideal of B and Q be an ideal of A. Then,*

$$(Q \bowtie^f J)^2 = Q^2 \bowtie^f (f(Q)J + J^2).$$

Proof. See [11, Lemma 3.4]. □

Lemma 3.2. *Let $f : A \to B$ be a ring homomorphism, J be an ideal of B, Q be an ideal of A and S be an m.c.s of A. The following assertions are equivalent:*

(i) *Q is an S-2-prime ideal of A.*

(ii) *$Q \bowtie^f J$ is an $(S \bowtie^f 0)$-2-prime ideal of $A \bowtie^f J$.*

Proof. Let Q be an S-2-prime ideal of A and $(\alpha, f(\alpha) + j_1)(\beta, f(\beta) + j_2) \in Q \bowtie^f J$ for all $(\alpha, f(\alpha)+j_1), (\beta, f(\beta)+j_2) \in A \bowtie^f J$. Then, $(\alpha\beta, f(\alpha\beta)+f(\alpha)j_2+f(\beta)j_1 + j_1j_2) \in Q \bowtie^f J$. We conclude that $\alpha\beta \in Q$. From assumption, there exists an $s \in S$ such that $s\alpha^2 \in Q$ or $s\beta^2 \in Q$. Therefore, $(s, f(s))(\alpha, f(\alpha) + j_1)^2 = (s\alpha^2, f(s\alpha^2) + 2f(s\alpha)j_1 + f(s)j_1^2) \in Q \bowtie^f J$ or $(s, f(s))(\beta, f(\beta) + j_2)^2 = (s\beta^2, f(s\beta^2) + 2f(s\beta)j_2 + f(s)j_2^2) \in Q \bowtie^f J$ for an $(s, f(s)) \in (S \bowtie^f 0)$. Hence, $Q \bowtie^f J$ is an $(S \bowtie^f 0)$-2-prime ideal of $A \bowtie^f J$.
The converse part is similarly verified. □

Proof of Theorem 3.2. (1) Suppose that $Q \bowtie^f J$ is a weakly $(S \bowtie^f 0)$-2-prime ideal of $A \bowtie^f J$. Let $0 \neq \alpha\beta \in Q$ for all $\alpha, \beta \in A$. We have $0 \neq (\alpha, f(\alpha))(\beta, f(\beta)) \in Q \bowtie^f J$. From assumption, there exists an $(s, f(s)) \in (S \bowtie^f 0)$ such that $(s, f(s))(\alpha, f(\alpha))^2 \in Q \bowtie^f J$ or $(s, f(s))(\beta, f(\beta))^2 \in Q \bowtie^f J$. This implies that $s\alpha^2 \in Q$ or $s\beta^2 \in Q$. Hence, Q is a weakly S-2-prime ideal of A.

(2) Suppose that Q is a weakly S-2-prime ideal and which is not an S-2-prime ideal of A. Let $Q \bowtie^f J$ be a weakly $(S \bowtie^f 0)$-2-prime ideal of $A \bowtie^f J$ and $(\alpha, \beta) \in A \times A$ be an S-double-zero of Q. Suppose $f(\alpha) \notin ann(J)$. So, there exists $j \in J$ such that $f(\alpha)j \neq 0$. As a result, $(0, 0) \neq (\alpha, f(\alpha))(\beta, f(\beta) + j) = (0, f(\alpha\beta)+f(\alpha)j) \in Q \bowtie^f J$. From assumption, there exists an $(s, f(s)) \in (S \bowtie^f 0)$ such that $(s, f(s))(\alpha, f(\alpha))^2 \in Q \bowtie^f J$ or $(s, f(s))(\beta, f(\beta) + j)^2 \in Q \bowtie^f J$. This implies that $s\alpha^2 \in Q$ or $s\beta^2 \in Q$. It is a contradiction, so $f(\alpha)J = 0$. Similarly, we conclude $f(\beta)J = 0$. Moreover, from Lemma 3.2, $Q \bowtie^f J$ is not an $(S \bowtie^f 0)$-2-prime ideal of $A \bowtie^f J$. From [17, Theorem 2.6], we know that $(Q \bowtie^f J)^2 = 0$. We have from Lemma 3.1, $(Q \bowtie^f J)^2 = Q^2 \bowtie^f (f(Q)J + J^2) = 0$. We have $J^2 = 0$, because of $Q^2 = 0$.

Conversely, assume that $(\alpha, f(\alpha) + i)$, $(\beta, f(\beta) + j) \in A \bowtie^f J$ such that $(0, 0) \neq (\alpha, f(\alpha) + i)(\beta, f(\beta) + j) \in (Q \bowtie^f J)$.

Case 1. $\alpha\beta \neq 0$: Since Q is a weakly S-2-prime ideal of A, there exists an $s \in S$ such that $s\alpha^2 \in Q$ or $s\beta^2 \in Q$. Hence, $(s, f(s))(\alpha, f(\alpha) + i)^2 \in Q \bowtie^f J$ or $(s, f(s))(\beta, f(\beta) + j)^2 \in Q \bowtie^f J$ for an $(s, f(s)) \in (S \bowtie^f 0)$, as desired.

Case 2. $\alpha\beta = 0$: We know that Q is a weakly S-2-prime ideal and which is not an S-2-prime ideal of A, so we have S-double-zero of Q. Without loss of the generality, we can assume $s\alpha^2 \notin Q$ and $s\beta^2 \notin Q$. Thus, (α, β) is an S-double-zero of Q and from assumption, we have $f(\alpha)J = 0 = f(\beta)J$. Then, $(\alpha, f(\alpha) + i)(\beta, f(\beta) + j) = (\alpha\beta, f(\alpha\beta) + f(\alpha)j + f(\beta)i + ij) = (0, ij) = (0, 0)$ because of $J^2 = 0$. It is a contradiction.

In view of Theorem 3.2 (2) and Lemma 3.2, we conclude the following corollary.

Corollary 3.1. *Suppose that $f : A \to B$ is a ring homomorphism, J is an ideal of B with $J^2 = 0$, Q is a weakly S-2-prime ideal which is not an S-2-prime ideal of A, where S is an m.c.s of A, for each $(\alpha, \beta) \in A \times A$ as an S-double-zero of Q, $(f(\alpha), f(\beta)) \in ann(J) \times ann(J)$. Then, $Q \bowtie^f J$ is a weakly $(S \bowtie^f 0)$-2-prime ideal which is not an $(S \bowtie^f 0)$-2-prime ideal of $A \bowtie^f J$.*

Corollary 3.2. *Suppose that (A, M) is a local ring with a maximal ideal M and S is an m.c.s of A. Let $f : A \to B$ be a ring homomorphism and J be an ideal of B with $f(M)J = 0$. Then, the following assertions are equivalent:*

(i) *Q is a weakly S-2-prime ideal which is not an S-2-prime ideal of A and $J^2 = 0$.*

(ii) *$Q \bowtie^f J$ is a weakly $(S \bowtie^f 0)$-2-prime ideal which is not an $(S \bowtie^f 0)$-2-prime ideal of $A \bowtie^f J$.*

Proof. (i)$\Longrightarrow$(ii) Let Q be a weakly S-2-prime ideal which is not an S-2-prime ideal of A and take $(\alpha, \beta) \in A \times A$ as an S-double-zero of Q. We claim $\alpha, \beta \in M$. Let $\alpha \notin M$. Thus, α is invertible and so $\beta = 0$, which is a contradiction. Hence, $(\alpha, \beta) \in M \times M$ and from hypothesis, we have $f(\alpha)J = 0 = f(\beta)J$. The result is clear from Theorem 3.2 (2) and Lemma 3.2.

(ii)$\Longrightarrow$(i) Let $Q \bowtie^f J$ be a weakly $(S \bowtie^f 0)$-2-prime ideal which is not an $(S \bowtie^f 0)$-2-prime ideal of $A \bowtie^f J$. Then, from Theorem 3.2 (1) and Lemma 3.2, we have Q is a weakly S-2-prime ideal which is not an S-2-prime ideal of A. From Theorem 3.2 (2), we have $J^2 = 0$. □

Corollary 3.3. *Suppose that (A, M) is a local ring with a maximal ideal M and S is an m.c.s of A. Let $f : A \to B$ be a ring homomorphism and J be an ideal of B with $f(M)J = 0$ and $M^2 = 0$. If $J^2 = 0$, then every ideal of $A \bowtie^f J$ disjoint with $(S \bowtie^f 0)$ is a weakly $(S \bowtie^f 0)$-2-prime ideal.*

Proof. It is known that $A \bowtie^f J$ is a local ring with the maximal ideal $M \bowtie^f J$. From Lemma 3.1, $(M \bowtie^f J)^2 = M^2 \bowtie^f (f(M)J + J^2) = 0$. From [13, Lemma 1], every ideal of $A \bowtie^f J$ disjoint with $(S \bowtie^f 0)$ is an $(S \bowtie^f 0)$-2-prime and weakly $(S \bowtie^f 0)$-2-prime ideal. □

Corollary 3.4. *Suppose that $f : A \to B$ is a ring homomorphism, S is an m.c.s of A, J is an ideal of B and Q is an ideal of A with $Nil(A) \not\supseteq Q$. Then, the following assertions are equivalent:*

(i) *$Q \bowtie^f J$ is an $(S \bowtie^f 0)$-2-prime ideal of $A \bowtie^f J$.*

(ii) *$Q \bowtie^f J$ is a weakly $(S \bowtie^f 0)$-2-prime ideal of $A \bowtie^f J$.*

Proof. (i)$\Longrightarrow$(ii) It is clear.

(ii)$\Longrightarrow$(i) Let $Q \bowtie^f J$ be a weakly $(S \bowtie^f 0)$-2-prime ideal of $A \bowtie^f J$. From Theorem 3.2 (1), Q is a weakly S-2-prime ideal of A. From [17, Theorem 2.6], we know that if Q is a weakly S-2-prime ideal of A, then either Q is an S-2-prime ideal of A or $Nil(A) \supseteq Q$. Therefore, Q is an S-2-prime ideal of A. From Lemma 3.2, $Q \bowtie^f J$ is an $(S \bowtie^f 0)$-2-prime ideal of $A \bowtie^f J$. □

4. Conclusion

In this study, we investigate new characterizations and properties of weakly S-2-prime ideals in commutative rings as generalizations of S-2-prime ideals with the help of S-2-prime and weakly 2-prime ideals. Besides, we examine the properties of weakly S-2-prime ideals in rings with characterization 2 and in polynomial and power series rings. Also, we delve into this ideal in idealization and amalgamated algebras along an ideal with regard to a ring homomorphism and we obtain many results. For future work, other generalizations of S-2-prime ideals can be worked on modules with the help of ideal reduction and ideal expansion functions in the light of this paper.

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L-filters and TL-filters in IL-algebras

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Abstract. In this paper, we introduce the concepts of L-filter and TL-filter as two different generalizations of the notion of fuzzy filter in IL-algebras. We investigate some properties with respect to these concepts. We study the relationship between L-filters and TL-filters. We give some characterizations for filters of IL-algebras by using L-filters and TL-filters. We present additional conditions so that the notions of L-filter and TL-filter coincide in an IL-algebra.
Keywords: filter, IL-algebra, L-filter, TL-filter.
MSC 2020: 03G25, 06F35, 08A72.

1. Introduction

BL-algebras were proposed by Hajek [6] in 1998 as algebraic structures formed by left-continuous triangular norms on the unit interval $[0, 1]$ and the residuation operations of these triangular norms. MV-algebras, Gödel algebras are some of the important classes of BL-algebras. Filter theory plays an important role in studying these algebraic structures and their associated logics [15]. The various filters in these algebraic structures correspond to a different class of provable

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formulas in logics related to these algebraic structures. Filters have various applications in logic and topology. After Hajek [6] introduced the concept of filters in BL-algebra, many authors have studied different types of filters in BL-algebras. For example, Haveshki et al. [7] introduced various filter models in BL-algebras such as implicative filter, positive implicative filter, fantastic filter and examined their relationship with other filters in BL-algebras.

Fuzzy logic has been introduced by Zadeh [17]. In fuzzy logic, the real unit interval is used for modelling the set of truth values and minimum is taken for a conjunction connective. But in modern fuzzy logic, more generally, a bounded lattice takes place instead of the real unit interval and t-norms are extensively used as logical conjunction [1]. After fuzzy logic has been presented, the concept of classical mathematics has been reconsidered and fuzzy logic has been applied to the classical algebraic structures. Many authors have studied on fuzzifying different types of filters on several various algebras in recent years [3, 9, 10]. Also, Liu and Li [13] have introduced the notion of fuzzy filter and fuzzy prime filter in BL-algebras and obtained some of their properties. Then, Khorami and Saeid [11] have introduced the concept of TL-filter as a generalization of the concept of fuzzy filter of BL-algebras and investigated some basic properties of TL-filters. They have also presented a method for calculating TL-filters produced by L-subsets. It should be noted that when $L = [0, 1]$ and $T = \wedge$, the concepts of TL-filter and fuzzy filter coincide in BL-algebras.

IL-algebras, which is a generalization of BL-algebras, were introduced by Troelstra [16] in 1992 as the algebraic equivalent of intuitionistic linear logic. Islam et al. [8] presented a different definition for the concept of an IL-algebra than Troelstra's, and introduced the concepts of filter and fuzzy filter in IL-algebras. They studied related properties of these filters. They also presented three concepts of fuzzy prime filters and obtained some results correlating them with each other.

The motivation of this paper is mainly to generalize the notion of fuzzy filter in IL-algebras in a useful way. In this study, we introduce the concepts of L-filters ve TL-filters in IL-algebras, which are two different generalizations of the concept of fuzzy filter of IL-algebras, introduced by Islam et al. [8]. We investigate their related properties. We show that every L-filter is a TL-filter in an IL-algebra. We also prove some other theorems that determine the relationship between these filters.

2. Preliminaries

In this section, we give some known notions and results which will be used throughout this article.

Definition 2.1 ([2]). *Let $(L, \leq)$ be a partially ordered set. If any pair of elements x, y has an infimum and supremum, denoted by $\wedge$ and $\vee$, respectively, then $(L, \leq)$ is said to be a lattice. A lattice L is said to be complete if $\wedge S$ and $\vee S$ exist for any $S \subseteq L$. Obviously, a complete lattice has the least element and* *the greatest element denoted by* 0 *and* 1*, respectively. A lattice* L *is said to be a chain if* $x \leq y$ *or* $y \leq x$ *for any* $x, y \in L$.

Definition 2.2 ([14])**.** *Let* X *be a non-empty set and* L *be a complete lattice. A function from* X *to* L *is called an* L*-subset of* X *(i.e.* $\xi : X \to L$*). The set of all* L*-subsets of* X *is called the* L*-power set of* X *and is denoted by* L^X*. Also, when* L *is* $[0, 1]$*,* L*-subsets of* X *are called fuzzy subsets of* X*. The set of all fuzzy subsets of* X *is called the fuzzy power set of* X *and denoted by* $[0, 1]^X$*.*

Definition 2.3 ([8])**.** *Let* J *be a non-empty set. An* IL*-algebra is an algebraic system* $J = (J, \wedge, \vee, \perp, \to, *, e)$ *which satisfies the following conditions:*

i) $(J, \wedge, \vee, \perp)$ *is a lattice with the least element* $\perp$*.*

ii) $(J, *, e)$ *is a commutative monoid with the identity element* e*.*

iii) For any $x, y, z \in J$*,* $x * y \leq z$ *if and only if* $x \leq y \to z$ *(residuation property).*

Definition 2.4 ([8])**.** *Let* J *be an* IL*-algebra. A non-empty subset* F *of* J *is said to be a filter of* J *if the following conditions are satisfied:*

i) $e \in F$*.*

ii) If $x, y \in F$*, then* $x * y \in F$ *and* $x \wedge y \in F$*.*

iii) If $x \in F$ *and* $x \leq y$*, then* $y \in F$*.*

Theorem 2.1 ([8])**.** *In every* IL*-algebra* J*, the following results hold for all* $x, y, z, w \in J$*:*

i) $(x \to y) * (y \to z) \leq (x \to z)$*.*

ii) If $x \leq z$ *and* $y \leq w$*, then* $x * y \leq z * w$*.*

iii) $x * (x \to y) \leq y$*.*

Definition 2.5 ([5])**.** *Let* X *be a non-empty set and* A *a fixed subset of* X*. For any* x *in* X*, we put*

$$\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}.$$

Then, χ_A *is a function from* X *into* $\{0, 1\}$*. It is called the characteristic function of* A*.*

Definition 2.6 ([8])**.** *Let* J *be an* IL*-algebra,* L *be a complete lattice and* ξ *be an* L*-subset of* J*. Then, for each* $t \in L$*, the set* $\xi_t = \{x \in J \mid \xi(x) \geq t\}$ *is called a level subset of* ξ*.*

Definition 2.7 ([8]). *Let J be an IL-algebra and ξ be a fuzzy subset of J. ξ is said to be a fuzzy filter of J if ξ_t is either an empty set or a filter of J, for all $t \in [0,1]$.*

Definition 2.8 ([12]). *Let L be a complete lattice. A triangular norm (in short t-norm) is a binary operation T on L (i.e. $T : L \times L \longrightarrow L$) satisfying the following conditions:*

i) T is associative, i.e. for all $x, y, z \in L$,

$$T(T(x,y),z) = T(x,T(y,z)).$$

ii) T is symmetric, i.e. for all $x, y \in L$,

$$T(x,y) = T(y,x).$$

iii) T is monotone, i.e. for all $x, y, z, \in L$, if $x \leq z$, then

$$T(x,y) \leq T(z,y).$$

iv) There is a neutral element $1 \in L$ such that $T(1,x) = x$ for all $x \in L$.

Conditions (iii) and (iv) imply that for any t-norm T on L, we have

$$T(x,y) \leq x, \quad T(x,y) \leq y. \tag{1}$$

Definition 2.9 ([4]). *Let T_1 and T_2 be two t-norms on a complete lattice L. T_1 is called smaller than T_2 or T_2 is called greater than T_1 if $T_1(x,y) \leq T_2(x,y)$ for all $x, y \in L$. In this case, we write $T_1 \leq T_2$.*

Remark 2.1 ([4]). The smallest and the greatest t-norm on a complete lattice L are given by the following, respectively:

$$T_D(x,y) = \begin{cases} x \wedge y, & \text{if } x = 1 \text{ or } y = 1 \\ 0, & \text{otherwise} \end{cases},$$

$$T_M(x,y) = x \wedge y.$$

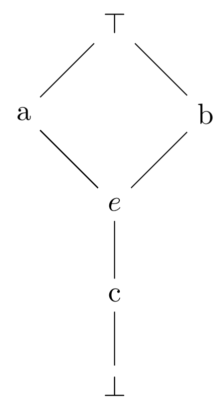
Throughout, this paper, unless otherwise stated, L denotes a complete lattice, T denotes a t-norm on L and J denotes an IL-algebra.

3. L-Filters

In this section, we introduce the notion of L-filter of an IL-algebra J and investigate some properties of L-filters. We illustrate the results with examples to better understand the concept.

Definition 3.1. *Let J be an IL-algebra, L be a complete lattice and $\xi \in L^J$. Then, ξ is said to be an L-filter of J, if for each $t \in L$, the level subset ξ_t is an empty set or a filter of J.*

Example 3.1. Let $J = \{\bot, a, b, c, e, \top\}$ be the lattice given by the following diagram.



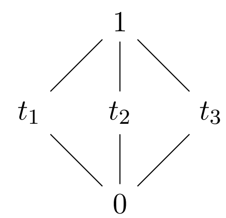
Define the binary operations $*$ and $\rightarrow$ on J by the following tables.

$*$	$\bot$	a	b	e	c	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	a	$\top$	a	a	$\top$
b	$\bot$	$\top$	b	b	b	$\top$
e	$\bot$	a	b	e	c	$\top$
c	$\bot$	a	b	c	c	$\top$
$\top$	$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$

$\rightarrow$	$\bot$	a	b	e	c	$\top$
$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$
a	$\bot$	a	$\bot$	$\bot$	$\bot$	$\top$
b	$\bot$	$\bot$	b	$\bot$	$\bot$	$\top$
e	$\bot$	a	b	e	c	$\top$
c	$\bot$	a	b	e	e	$\top$
$\top$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\top$

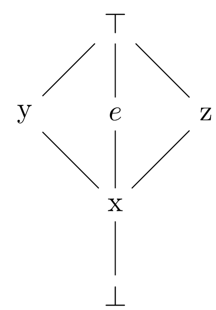
Then, $(J, \wedge, \vee, *, \rightarrow, e)$ is an IL-algebra.

Besides, let $L = \{0, t_1, t_2, t_3, 1\}$ be the complete lattice with the following diagram.



Consider the L-subset of J, namely ξ, defined by $\xi(\top) = \xi(e) = \xi(a) = \xi(b) = 1$, $\xi(c) = t_1$ and $\xi(\bot) = 0$. Then, the level subsets of ξ for each element of L are obtained as $\xi_0 = J$, $\xi_{t_1} = \{c, e, a, b, \top\}$, $\xi_{t_2} = \xi_{t_3} = \xi_1 = \{e, a, b, \top\}$. By using Definition 2.4 and routine verification, we conclude that all level subsets of ξ are filters of J. Therefore, Definition 3.1 shows that ξ is an L-filter of J.

Example 3.2. Let $J = \{\bot, x, y, z, e, \top\}$ be the lattice given by the following diagram.

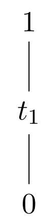


Define the binary operations $*$ and $\to$ on J by the following tables.

$*$	$\bot$	x	y	z	e	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
x	$\bot$	$\bot$	x	x	x	x
y	$\bot$	x	y	y	y	y
z	$\bot$	x	y	e	z	$\top$
e	$\bot$	x	y	z	e	$\top$
$\top$	$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$

$\to$	$\bot$	x	y	z	e	$\top$
$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$
x	x	$\top$	$\top$	$\top$	$\top$	$\top$
y	$\bot$	x	$\top$	x	x	$\top$
z	$\bot$	x	y	e	z	e
e	$\bot$	x	y	z	e	$\top$
$\top$	$\bot$	x	y	x	x	$\top$

Then, $(J, \wedge, \vee, *, \to, e)$ is an IL-algebra.
Besides, let $L = \{0, t_1, 1\}$ be the complete lattice with the following diagram.



Consider the L-subset of J, namely ξ, defined by $\xi(\top) = \xi(e) = 1$, $\xi(y) = \xi(z) = \xi(x) = t_1$ and $\xi(\bot) = 0$. Then, the level subset of ξ for t_1 is obtained as $\xi_{t_1} = \{x, y, z, e, \top\}$. Since $x \in \xi_{t_1}$ but $x * x = \bot \notin \xi_{t_1}$, then ξ_{t_1} is not a filter of J. Consequently, we get that ξ is not an L-filter of J by using Definition 3.1.

Proposition 3.1. *Let J be an IL-algebra and F be a non-empty subset of J. Then, F is a filter of J if and only if χ_F is an L-filter of J.*

Proof. Suppose that F is a filter of J and $\chi_F : J \to \{0, 1\}$ is the characteristic function of F defined by

$$\chi_F(x) = \begin{cases} 1, & x \in F \\ 0, & x \notin F \end{cases}.$$

Then, the level subsets of χ_F are $(\chi_F)_0 = J$ and $(\chi_F)_1 = F$. Thus, for each $t \in \{0, 1\}$, the level subset $(\chi_F)_t$ is a filter of J. Therefore, χ_F is an L-filter of J by Definition 3.1.

Conversely, let χ_F be an L-filter of J. Then, the level subset of χ_F for $t = 1$ is

$$(\chi_F)_1 = \{x \in J : \chi_F(x) \geq 1\} = \{x \in J : \chi_F(x) = 1\} = F.$$

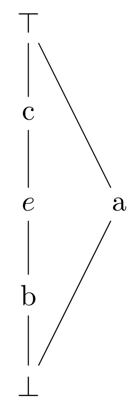
Since F is a non-empty subset of J, using Definition 3.1, we conclude that F is a filter of J. □

Proposition 3.2. *Let ξ be an L-filter of J and x, y be two arbitrary elements of J. If $x \leq y$, then $\xi(x) \leq \xi(y)$.*

Proof. Suppose that $x, y \in J$ and $x \leq y$. Then, $x \wedge y = x$ and $\xi(x \wedge y) = \xi(x)$. Let $\xi(x) = t$, for $t \in L$. Then, clearly, we have that $x \wedge y, x \in \xi_t$. Since ξ is an L-filter of J, by Definition 3.1, we conclude that the level subset ξ_t for $t \in L$ is a filter of J. Since $x \wedge y \leq y$, we have that $y \in \xi_t$ by Definition 2.4 (iii). Therefore, $\xi(x) = t \leq \xi(y)$. □

The following example shows that the converse of Proposition 3.2 does not hold.

Example 3.3. Let $J = \{\bot, a, b, c, e, \top\}$ be the complete lattice given with the following Hasse diagram.



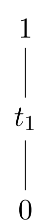
Define the binary operations $*$ and $\to$ on J as follows.

$*$	$\bot$	a	b	c	e	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	a	$\bot$	a	a	a
b	$\bot$	$\bot$	b	b	b	b
c	$\bot$	a	b	$\top$	c	$\top$
e	$\bot$	a	b	c	e	$\top$
$\top$	$\bot$	a	b	$\top$	$\top$	$\top$

$\to$	$\bot$	a	b	c	e	$\top$
$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$
a	b	$\top$	b	b	b	$\top$
b	a	a	b	$\top$	$\top$	$\top$
c	$\bot$	a	b	e	b	$\top$
e	$\bot$	a	b	c	e	$\top$
$\top$	$\bot$	a	b	b	b	$\top$

Then, $(J, \wedge, \vee, *, \to, e)$ is an IL-algebra.

Besides, let $L = \{0, t_1, 1\}$ be the complete lattice with the following diagram.



Consider the L-subset of J, namely ξ, defined by $\xi(\top) = \xi(e) = \xi(z) = 1$, $\xi(y) = \xi(x) = \xi(\bot) = 0$. It is clear that ξ is an L-filter of J. We have that $\xi(x) \leq \xi(e)$, but $x \not\leq e$.

Lemma 3.1. *Suppose that $\xi \in L^J$. If ξ is an L-filter of J, then for each $x \in J$,*

$$\xi(x) \leq \xi(e),$$

where e is the identity element of J.

Proof. Let ξ be an L-filter of J. Then, for any $t \in L$, the level subset ξ_t is a filter of J. Assume that for an arbitrary element $x \in J$, $\xi(x) = t_0$, where $t_0 \in L$. Then, clearly, we get that $x \in \xi_{t_0}$. Since ξ_{t_0} is a filter of J, then $e \in \xi_{t_0}$. Therefore, $\xi(e) \geq t_0 = \xi(x)$. □

Theorem 3.1. *Let ξ be an L-filter of J and $x, y \in J$. If $\xi(x \to y) = \xi(e)$, then $\xi(x) \leq \xi(y)$.*

Proof. Suppose that ξ is an L-filter of J. Then, the level subset ξ_t for each $t \in L$ is a filter of J by Definition 3.1. Let $\xi(x \to y) = \xi(e)$ and $\xi(x) = t$, $t \in L$. Then, clearly, we have that $x \in \xi_t$. On the other hand, since ξ_t is a filter of J, we have that $e \in \xi_t$. Hence we obtain that $t \leq \xi(e) = \xi(x \to y)$. Then, we have that $x \to y \in \xi_t$. Using Definition 2.4 (ii), we get that $x * (x \to y) \in \xi_t$. Since we know that $x * (x \to y) \leq y$ by Theorem 2.1 (iii) and ξ_t is a filter of J, we get that $y \in \xi_t$. So $\xi(x) = t \leq \xi(y)$. □

By the following example, we show that the converse of the above theorem does not hold.

Example 3.4. Consider the L-filter ξ defined in Example 3.3. Note that $\xi(x) \leq \xi(e)$ but $\xi(x \to e) = \xi(y) = 0 \neq \xi(e)$.

4. TL-Filters

Definition 4.1. *Let $\xi \in L^J$ and T be a t-norm on L. Then, ξ is called a TL-filter of J if and only if, for any $x, y \in J$, the following conditions are satisfied.*

i) $x \leq y \Longrightarrow \xi(x) \leq \xi(y)$,

ii) $T(\xi(x), \xi(y)) \leq \xi(x \wedge y)$,

iii) $T(\xi(x), \xi(y)) \leq \xi(x * y)$,

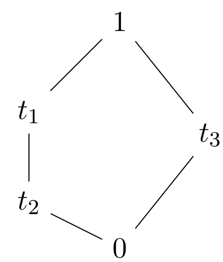
iv) $\xi(x) \leq \xi(e)$.

Theorem 4.1. *In an IL-algebra J, every L-filter is a TL-filter.*

Proof. The conditions (i) and (iv) of Definition 4.1 are clear from Proposition 3.2 and Lemma 3.1, respectively. Now, let ξ be an L-filter of J. Suppose that $T(\xi(x), \xi(y)) = t$ for $t \in L$. Using inequalities (1), we conclude that $T(\xi(x), \xi(y)) \leq \xi(x) \wedge \xi(y)$. Hence $t \leq \xi(x)$ and $t \leq \xi(y)$. Then, clearly, we have that $x, y \in \xi_t$. Since ξ is an L-filter of J, by Definition 3.1, we know that the level subset ξ_t, for any $t \in L$, is a filter of J. Thus, by Definition 2.4 (ii), we have that $x \wedge y \in \xi_t$ and $x * y \in \xi_t$. Thus, $\xi(x \wedge y) \geq t$ and $\xi(x * y) \geq t$. Therefore, $\xi(x \wedge y) \geq t = T(\xi(x), \xi(y))$ and $\xi(x * y) \geq t = T(\xi(x), \xi(y))$. □

The following example shows that the converse of Theorem 4.1 is not true in general.

Example 4.1. Let J be the IL-algebra given in Example 3.3 and $L=\{0, t_1, t_2, t_3, 1\}$ be the complete lattice given with the following diagram.



Now, define the function $\xi : J \to L$ by $\xi(e) = \xi(c) = \xi(\top) = t_1$, $\xi(b) = \xi(a) = t_2$ and $\xi(\bot) = 0$. Since $a, b \in \xi_{t_2}$ but $a \wedge b = \bot \notin \xi_{t_2}$, the level subset ξ_{t_2} is not a filter of J. So, we get that ξ is not an L-filter of J.

Consider the smallest t-norm T_D on L in Remark 2.1 as follows.

$$T_D(x, y) = \begin{cases} y, & x = 1 \\ x, & y = 1 \\ 0, & \text{otherwise} \end{cases} .$$

Then, we can easily verify that ξ is a TL-filter of J.

Theorem 4.2. *Let F be a non-empty subset of J. Then, F is a filter of J if and only if χ_F is a TL-filter of J.*

Proof. Suppose that F is a filter of J. By Proposition 3.1, we have that χ_F is an L-filter of J. Since every L-filter is a TL-filter in an IL-algebra by Theorem 4.1, χ_F is a TL-filter of J.

Conversely, let χ_F be a TL-filter of J. Since $F \subseteq J$ is a non-empty set, then there exists $x \in F$. So, $\chi_F(x) = 1$. Since χ_F is a TL-filter of J, then by Definition 4.1 (iv), $\chi_F(x) \leq \chi_F(e)$. Thus, $1 \leq \chi_F(e)$ which means that $\chi_F(e) = 1$ and whence $e \in F$.

Now, suppose that x, y are two arbitrary elements of F. Then, $\chi_F(x) = \chi_F(y) = 1$ and

$$T(\chi_F(x), \chi_F(y)) = T(1, 1) = 1.$$

Since χ_F is a TL-filter of J, then by Definition 4.1, we have that $1 = T(\chi_F(x), \chi_F(y)) \leq \chi_F(x * y)$. Hence $\chi_F(x * y) = 1$ and therefore $x * y \in F$. Similarly, we can prove that $x \wedge y \in F$ for any $x, y \in F$. Finally, suppose that $x \leq y$ and $x \in F$. Then, $\chi_F(x) = 1$. Because of that χ_F is a TL-filter of J, we get that $\chi_F(x) \leq \chi_F(y)$ which means that $1 = \chi_F(y)$ and $y \in F$. Therefore, F is a filter of J. □

Proposition 4.1. *If ξ is a TL-filter of J, then for any $x, y \in J$, we have that $T(\xi(x), \xi(x \to y)) \leq \xi(y)$.*

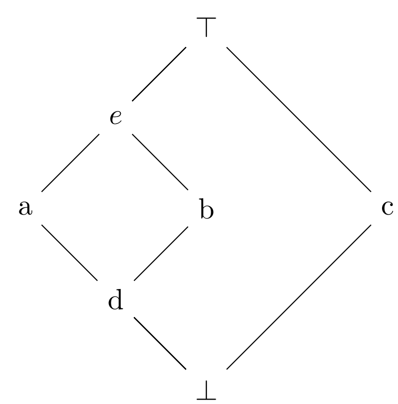
Proof. By Theorem 2.1 (iii), we know that $x * (x \to y) \leq y$. Since ξ is a TL-filter of J, we have that $\xi(x * (x \to y)) \leq \xi(y)$. Therefore, we obtain that

$$T(\xi(x), \xi(x \to y)) \leq \xi(x * (x \to y)) \leq \xi(y).$$

□

In an IL-algebra, some results concerning L-filters are not true for TL-filters. The following example shows that Theorem 3.1 does not hold if ξ is a TL-filter of J.

Example 4.2. Let $J = \{\bot, a, b, c, d, e, \top\}$ be the complete lattice given with the following Hasse diagram.



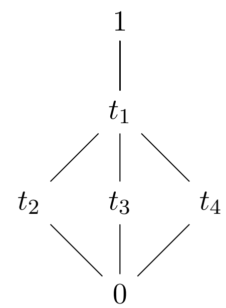
Define the binary operations $*$ and $\to$ on J as follows.

$*$	$\bot$	a	b	c	d	e	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	a	d	c	d	a	$\top$
b	$\bot$	d	b	c	d	b	$\top$
c	$\bot$	c	c	d	c	c	$\top$
d	$\bot$	d	d	c	d	d	$\top$
e	$\bot$	a	b	c	d	e	$\top$
$\top$	$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$

$\to$	$\bot$	a	b	c	d	e	$\top$
$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$
a	$\bot$	e	b	c	b	e	$\top$
b	$\bot$	a	e	c	a	e	$\top$
c	$\bot$	c	c	e	c	c	$\top$
d	$\bot$	e	e	c	e	e	$\top$
e	$\bot$	a	b	c	d	e	$\top$
$\top$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\top$

Then, $(J, \wedge, \vee, \perp \to, *, \mathrm{e})$ is an IL-algebra.

Besides, let $L = \{0, t_1, t_2, t_3, t_4, 1\}$ be the complete lattice given in the following diagram.



Now, consider the function $\xi : J \to L$ defined by $\xi(1) = \xi(\top) = \xi(a) = 1$, $\xi(b) = t_2$, $\xi(c) = t_3$, $\xi(d) = \xi(\perp) = 0$ and the smallest t-norm T_D on L in Remark 2.1 as follows.

$$T_W(x, y) = \begin{cases} y, & x = 1 \\ x, & y = 1 \\ 0, & \text{otherwise.} \end{cases}$$

By routine verification, ξ is a TL-filter of J. Note that

$$\xi(b \to d) = \xi(a) = 1$$

but $\xi(b) = t_2 \not\leq \xi(d) = 0$.

Proposition 4.2. *Let ξ be a TL-filter of J. Then, for all $x, y, z \in J$, we have the following statements.*

i) If $x \wedge y \leq z$, then $T(\xi(x), \xi(y)) \leq \xi(z)$.

ii) If $x * y \leq z$, then $T(\xi(x), \xi(y)) \leq \xi(z)$.

Proof. The proof easily comes from by Definition 4.1 (i) and (ii). □

Proposition 4.3. *Let ξ be a TL-filter of J. Then, for all $x, y, z \in J$, we have the following inequality:*

$$T(\xi(x \to y), \xi(y \to z)) \leq \xi(x \to z).$$

Proof. We know from Theorem 2.1 (i) that $(x \to y) * (y \to z) \leq (x \to z)$. Using Definition 4.1 (i), we obtain that $\xi((x \to y) * (y \to z)) \leq \xi(x \to z)$. Therefore, by using Definition 4.1 (iii), we get that $T(\xi(x \to y), \xi(y \to z)) \leq \xi(x \to z)$. □

Theorem 4.3. *Let ξ be a TL-filter of J. If J is a chain and $T = T_M$, then ξ is an L-filter of J.*

Proof. Suppose that ξ is a TL-filter of J. To show that ξ is an L-filter of J, we should prove that for any $t \in L$, the level subset ξ_t is an empty set or a filter of J. Let ξ_t is a non-empty set. Then, there exists $x \in \xi_t$ and so $\xi(x) \geq t$. Since ξ is a TL-filter, we have that $\xi(x) \leq \xi(e)$. So we obtain that $\xi(e) \geq t$ and whence $e \in \xi_t$. Let $x \in \xi_t$ and $x \leq y$. Then, by Definition 4.1 (i), we get that $\xi(y) \geq \xi(x)$. Therefore, $\xi(y) \geq t$ and $y \in \xi_t$. Since $T(\xi(x), \xi(y)) \leq \xi(x \wedge y)$ by Definition 4.1 (ii) and $T = T_M$, we have that $\xi(x) \wedge \xi(y)) \leq \xi(x \wedge y)$. On the other hand, since J is a chain, we have $x \geq y$ or $y \geq x$. So $\xi(x) \geq \xi(y)$ or $\xi(y) \geq \xi(x)$. Hence $\xi(x) \wedge \xi(y) = \xi(x)$ or $\xi(x) \wedge \xi(y) = \xi(y)$. In both cases, we get that $\xi(x \wedge y) \geq t$. So $x \wedge y \in \xi_t$. By a similar argument and Definition 4.1 (iii), we get that $x * y \in \xi_t$. Consequently, ξ_t is a filter of J. It proves that ξ is an L-filter of J. □

5. Conclusions

We have defined the notions of L-filter and TL-filter in IL-algebras and have studied their properties. We have investigated the relationship between L-filters and TL-filters. We have characterized filters by using L-filters and TL-filters. For future work, characterizations and calculations of these filters may be investigated. Also, other kinds of filters such as implicative TL-filters, positive implicative TL-filters can be studied in IL-algebras.

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On biharmonic surfaces in pseudo-Riemannian 4-dimensional space forms

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Abstract. In this paper, biharmonic pseudo-Riemannian surfaces with diagonalizable shape operetor in pseudo-Riemannian space form $N_s^4(c)$ are studied. We prove that the surfaces with light-like mean curvature vector field are pseudo-umbilical. For non light-like mean curvature vector field, we show that the pseudo-umbilical surfaces is minimal or $H^2 = |c|$. Also, we give some sufficient conditions for such surfaces with parallel mean curvature vector field to be minimal.

Keywords: Pseudo-Riemannian space forms, biharmonic surfaces, minimal surfaces, parallel mean curvature vector field, pseudo-umbilical surfaces.

MSC 2020: 53C50.

1. Introduction

Let $\phi : M_r^n \to N_s^{n+p}$ be the inclusion of a pseudo-Riemannian submanifold M_r^n with index r into a pseudo-Riemannian manifold N_s^{n+p} with index s. We say that M_r^n is a biharmonic submanifold, if its bitension field $\tau_2(\phi)$ vanishes identically, i.e. (see [3, 10, 15])

$$\tau_2(\phi) := -\Delta^\phi \tau(\phi) - \operatorname{tr} R^N(\mathrm{d}\phi, \tau(\phi))\mathrm{d}\phi = 0, \tag{1}$$

where Δ^ϕ is the rough Laplacian defined on sections of $\phi^{-1}(TN)$ and $\tau(\phi) = \operatorname{tr}\nabla^\phi \mathrm{d}\phi$ is the tension field of ϕ that vanishes for ϕ being a harmonic map. R^N, ∇^ϕ and ∇ are the curvature tensor of N_s^{n+p}, the induced connection by ϕ on the bundle $\phi^* TN_s^{n+p}$ and the connection of M_r^n, respectively.

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Notice that $\tau(\phi) = n\overrightarrow{H}$ with $\overrightarrow{H}$ the mean curvature vector field of M_r^n, then it is clear from (1) that a minimal submanifold must be biharmonic, and we call a nonminimal biharmonic submanifold a proper biharmonic submanifold.

During the last decade, a special attention has been paid to the study of biharmonic submanifolds and important progress has been made in the study of this subject. B. Y. Chen and S. Ishikawa in [5, 6] gave full classification of proper biharmonic curves in $\mathbb{E}_q^3$ and proved that there exists no proper biharmonic surface in $\mathbb{E}_q^3$. In [15], T. Sasahara classified proper biharmonic curves and surfaces in nonflat Lorentz 3-space forms.

Now, let us turn to the problem in 4-dimensional pseudo-Riemannian space forms. F. Defever et al. proved in [8] that every biharmonic hypersurface M_r^3 ($r = 0, 1, 2, 3$) of $\mathbb{E}_s^4$ with diagonalizable shape operator is minimal, and the same conclusion holds for Lorentz hypersurfaces in $\mathbb{E}_1^4$ (see [1]).

It seems then natural, as the next step, to study biharmonic surfaces M_r^2 in pseudo-Riemannian space froms $N_s^4(c)$. The structure of the surfaces often appears considerably different from that of the hypersurfaces, the mean curvature vector field $\vec{H}$ of surfaces may be light-like, except space-like and time-like ones. In general, each of them will imply different properties of surfaces. So far, there have been no many developments in this direction.

Surfaces with light-like mean curvature vector field were concerned by many geometers due to the study of trapped surfaces in 4-dimensional Lorentz manifolds in [14], which are related to the presence of a black hole. In [5, 6], B. Y. Chen and S. Ishikawa firstly studied biharmonic surfaces with light-like mean curvature vector field (i.e., marginally trapped biharmonic surfaces) in $\mathbb{E}_s^4$ and gave some examples of such surfaces in $\mathbb{E}_s^4$. Our first goal in this paper continues to study this subject in nonflat cases and obtain

Theorem 1.1. *Let M_r^2 be a biharmonic surface with light-like mean curvature vector in a pseudo-Riemannian space form $N_s^4(c)$. Assume that M_r^2 has diagonalizable shape operator, then it is pseudo-umbilical.*

In [13], C.-Z. Ouyang investigated the minimality of space-like biharmonic surfaces M^2 (i.e., $r = 0$) with parallel mean curvature vector field in pseudo-Riemannian space forms. After that, J.-C. Liu, L. Du and J. Zhang (see [11]) studied the problem for space-like pseudo-umbilical surfaces. Our second goal in this paper is to investigate the minimality of such surfaces M_r^2 with general index r and obtain

Theorem 1.2. *Let M_r^2 be pseudo-umbilical biharmonic surfaces with non light-like mean curvature vector in $N_s^4(c)$. Assume that M_r^2 has diagonalizable shape operator, then one of the following three statements holds:*

(i) *when $c = 0$, then M_r^2 is minimal*;

(ii) *when $c > 0$, then its mean curvature vector $\overrightarrow{H}$ is space-like. Furthermore, either M_r^2 is minimal or $H^2 = c$*;

(iii) *when $c<0$, then either $\overrightarrow{H}=0$, i.e., M_r^2 is minimal, or $\overrightarrow{H}$ is a time-like vector with $H^2=|c|$, where H is the mean curvature of M_r^2.*

Remark 1.1. As a corollary of Theorem 1.2, let M^2 be a pseudo-umbilical biharmonic surface in Riemannian space form $N^4(c)$. When $c \leq 0$, then M^2 must be minimal; when $c>0$, then its mean curvature $H=0$, or $H^2=c$, which has been proved by [9] for $c=0$, [2, Theorem 5.1] for $c>0$ and [4, Theorem 2.4], for $c<0$.

Theorem 1.3. *Let M_r^2 be a biharmonic surface with non light-like mean curvature vector in $N_s^4(c)$. Assume that M_r^2 has parallel mean curvature vector field and diagonalizable shape operator, then M_r^2 is minimal if $c=0$, or $c<0$ and $\vec{H}$ is space-like, or $c>0$ and trace$A_3^2 \neq 2c$, where A_3 is the shape operator with respect to the normal frame field e_3 of M_r^2.*

2. Preliminaries

Let $N_s^{n+p}(c)$ be an $(n+p)$-dimensional pseudo-Riemannian space form with index s of constant curvature c $(0 \leq s \leq n+p)$. Let $x: M_r^n \to N_s^{n+p}(c)$ be an isometric immersion of an n-dimensional manifold M_r^n of signature $(r, n-r)(r \geq 0)$ into $N_s^{n+p}(c)$. Let ∇ and $\widetilde{\nabla}$ denote by the Levi-Civita connections of M_r^n and $N_s^{n+p}(c)$, respectively. For any tangent vector fields X, Y and normal vector field ξ of M_r^n in $N_s^{n+p}(c)$, the Gauss and Weingarten formulas are given by, respectively, (cf. [7] or [12])

$$\widetilde{\nabla}_X Y = \nabla_X Y + B(X,Y), \ \widetilde{\nabla}_X \xi = -A_\xi X + D_X \xi,$$

where B, A_ξ and D are the second fundamental form, the shape operator with respect to ξ and the normal connection, respectively. It is easy to see that B and A_ξ are related by

$$\langle B(X,Y), \xi\rangle = \langle A_\xi X, Y\rangle. \tag{2}$$

The mean curvature vector field $\overrightarrow{H}$ and the mean curvature H of M_r^n in $N_s^{n+p}(c)$ are expressed as $\overrightarrow{H}=\frac{1}{n}\text{trace}B$, and $H=\sqrt{|\langle \overrightarrow{H}, \overrightarrow{H}\rangle|}$, respectively.

We define the covariant derivative of the second fundamental form B by

$$(\widetilde{\nabla}_X B)(Y,Z) = D_X B(Y,Z) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z). \tag{3}$$

Then, the Codazzi equation is given by

$$(\widetilde{\nabla}_X B)(Y,Z) = (\widetilde{\nabla}_Y B)(X,Z).$$

Let $\{e_1, e_2, \ldots, e_{n+p}\}$ be a local orthonormal frame basis on $N_s^{n+p}(c)$ such that $e_1, \ldots, e_n$ are tangent to M_r^n and $e_{n+1}, \ldots, e_{n+p}$ are normal to M_r^n. Then, the connection forms (ω_B^A) are given by (cf. [6])

$$\widetilde{\nabla} e_A = \sum_{B=1}^{n+p} \omega_A^B e_B,\ \omega_B^A = -\varepsilon_A \varepsilon_B \omega_A^B,\ A, B = 1, 2, \ldots, n+p, \tag{4}$$

where and $\varepsilon_A = \langle e_A, e_A \rangle = \pm 1,\ A = 1, 2, \ldots, n+p$. It follows from (2) that

$$\overrightarrow{H} = \frac{1}{n}\text{trace}B = \frac{1}{n}\sum_{i=1}^{n} \varepsilon_i B(e_i, e_i) = \frac{1}{n}\sum_{\alpha=n+1}^{n+p} \varepsilon_\alpha (\text{trace}A_\alpha) e_\alpha, \tag{5}$$

where $A_\alpha = A_{e_\alpha}$.

As it is known that a vector v tangent to $N_s^{n+p}(c)$ is called *space-like* (resp. *time-like*) if $v = 0$ or $\langle v, v \rangle > 0$ (resp. $\langle v, v \rangle < 0$). A vector v is called *light-like* if $v \neq 0$ and $\langle v, v \rangle = 0$.

A submanifold M_r^n is called *minimal* if $\overrightarrow{H} = 0$. M_r^n is called *pseudo-umbilical*, if it is umbilical with respect to the direction of $\overrightarrow{H}$ (cf. [7]), i.e.,

$$\langle B(X, Y), \overrightarrow{H} \rangle = \langle \overrightarrow{H}, \overrightarrow{H} \rangle \langle X, Y \rangle. \tag{6}$$

When $\overrightarrow{H} \neq 0$, (6) becomes $A_{\overrightarrow{H}} = \langle \overrightarrow{H}, \overrightarrow{H} \rangle I$, where I stands for the identity operator. It follows from (6) that every minimal submanifold is pseudo-umbilical.

Using a similar computation as in the proof of Theorem 4.1 in [3] (also see [7]), we obtain the following.

Lemma 2.1. *An isometric immersion $\phi : M_r^n \to N_s^{n+p}(c)$ of an n-dimensional manifold M_r^n into $N_s^{n+p}(c)$ is biharmonic if and only if*

$$\begin{cases} \Delta^D \overrightarrow{H} + \sum\limits_{i=1}^{n} \varepsilon_i B(A_{\overrightarrow{H}} e_i, e_i) - n\overrightarrow{H} c = 0, \\ n\nabla \langle \overrightarrow{H}, \overrightarrow{H} \rangle + 4\sum\limits_{i=1}^{n} \varepsilon_i A_{D_{e_i}\overrightarrow{H}}(e_i) = 0, \end{cases} \tag{7}$$

where

$$\Delta^D = -\sum_{i=1}^{n} \varepsilon_i (D_{e_i} D_{e_i} - D_{\nabla_{e_i} e_i}). \tag{8}$$

Set

$$\begin{cases} \|\omega_{n+1}^\alpha\|^2 = \sum\limits_{i=1}^{n} \varepsilon_i \big(\omega_{n+1}^\alpha (e_i)\big)^2,\ \alpha > n+1, \\ \text{trace}A_{n+1}A_\alpha = \sum\limits_{i=1}^{n} \varepsilon_i \langle A_{n+1}(A_\alpha(e_i)), e_i \rangle,\ \alpha > n+1, \\ \nabla H = \sum\limits_{i=1}^{n} \varepsilon_i (e_i H) e_i. \end{cases}$$

Using Lemma 2.1, we can prove the following lemma.

Lemma 2.2. *Let M_r^n be a biharmonic submanifold in $N_s^{n+p}(c)$ with non light-like mean curvature vector. Then, we have*

$$
\begin{cases}
\Delta H + H\varepsilon_{n+1} \sum\limits_{\alpha=n+2}^{n+p} \varepsilon_\alpha \|\omega_\alpha^{n+1}\|^2 + H\varepsilon_{n+1}\|A_{n+1}\|^2 - nHc = 0, \\
H\varepsilon_\alpha \mathrm{trace} A_{n+1}A_\alpha = 2\omega_{n+1}^\alpha(\nabla H) + H\sum\limits_{i=1}^{n} \varepsilon_i(\nabla_{e_i}\omega_{n+1}^\alpha)(e_i) \\
\qquad\qquad + H\sum\limits_{\beta=n+1}^{n+p}\sum\limits_{i=1}^{n} \varepsilon_i \omega_{n+1}^\beta(e_i)\omega_\beta^\alpha(e_i), \forall \alpha > n+1, \\
n\varepsilon_{n+1}H\nabla H + 2A_{n+1}(\nabla H) + 2H\sum\limits_{\alpha=n+2}^{n+p}\sum\limits_{i=1}^{n} \varepsilon_i\omega_{n+1}^\alpha(e_i)A_\alpha(e_i) = 0.
\end{cases} \tag{9}
$$

Proof. Since $\overrightarrow{H}$ is not light-like, we can choose a local orthonormal frame field $\{e_i\}_{i=1}^{n+p}$ such that $\overrightarrow{H} = H\, e_{n+1}$.

We will calculate each term in (7) individually. First, from (8) we have

$$
\begin{aligned}
\Delta^D \overrightarrow{H} &= -\sum_{i=1}^{n} \varepsilon_i D_{e_i} D_{e_i}(H\, e_{n+1}) + \sum_{i=1}^{n} \varepsilon_i D_{\nabla_{e_i} e_i}(H\, e_{n+1}) \\
&= -\sum_{i=1}^{n} \varepsilon_i \Big(e_i e_i H e_{n+1} + 2 e_i H D_{e_i} e_{n+1} + H D_{e_i} D_{e_i} e_{n+1} \Big) \\
&= -\sum_{i=1}^{n} \varepsilon_i \Big\{ \big[e_i e_i H - \varepsilon_{n+1} H \sum_{\alpha=n+2}^{n+p} \varepsilon_\alpha (\omega_{n+1}^\alpha(e_i))^2 \big] e_{n+1} \\
&\quad + \sum_{\alpha=n+2}^{n+p} \big[2\omega_{n+1}^\alpha(e_i(H)e_i) + H\nabla_{e_i}(\omega_{n+1}^\alpha(e_i)) \\
&\quad + H \sum_{\beta=n+1}^{n+p} \varepsilon_i \omega_{n+1}^\beta(e_i)\omega_\beta^\alpha(e_i) \big] e_\alpha \Big\} \\
&\quad + \sum_{i=1}^{n} \varepsilon_i \Big((\nabla_{e_i} e_i)(H) e_{n+1} + H \sum_{\alpha=n+2}^{n+p} \omega_{n+1}^\alpha(\nabla_{e_i} e_i) e_\alpha \Big).
\end{aligned} \tag{10}
$$

Putting into (10) gives

$$
\begin{aligned}
\Delta^D \overrightarrow{H} =& \Big(\Delta H + \varepsilon_{n+1} H \sum_{\alpha=n+2}^{n+p} \varepsilon_\alpha \|\omega_\alpha^{n+1}\|^2 \Big) e_{n+1} - \sum_{\alpha=n+2}^{n+p} \Big\{ 2\omega_{n+1}^\alpha(\nabla H) \\
& + H \sum_{i=1}^{n} \big[\varepsilon_i (\nabla_{e_i} \omega_\alpha^{n+1}) e_i + \sum_{\beta=n+1}^{n+p} \varepsilon_i \omega_{n+1}^\beta(e_i)\omega_\beta^\alpha(e_i) \big] \Big\} e_\alpha.
\end{aligned} \tag{11}
$$

Using $\overrightarrow{H} = He_{n+1}$ again, a straightforward computation yields

$$\begin{aligned}\sum_{i=1}^{n}\varepsilon_i B(A_{\overrightarrow{H}}(e_i), e_i) =& \sum_{i=1}^{n}\varepsilon_i\Big\{\varepsilon_{n+1}\langle A_{n+1}\big(A_{\overrightarrow{H}}(e_i)\big), e_i\rangle\, e_{n+1} \\ &+ \sum_{\alpha=n+2}^{n+p}\varepsilon_\alpha\langle A_\alpha\big(A_{\overrightarrow{H}}(e_i)\big), e_i\rangle\, e_\alpha\Big\} \\ =&\varepsilon_{n+1}H\|A_{n+1}\|^2 e_{n+1} + H\sum_{\alpha=n+2}^{n+p}\varepsilon_\alpha \mathrm{trace} A_{n+1}A_\alpha e_\alpha.\end{aligned} \tag{12}$$

$$\begin{aligned}\sum_{i=1}^{n}\varepsilon_i A_{D_{e_i}\overrightarrow{H}}(e_i) =& \sum_{i=1}^{n}\varepsilon_i A_{D_{e_i}(H\,e_{n+1})}(e_i) \\ =& A_{n+1}(\nabla H) + H\sum_{\alpha=n+2}^{n+p}\sum_{i=1}^{n}\varepsilon_i\omega_{n+1}^{\alpha}(e_i)A_\alpha(e_i).\end{aligned} \tag{13}$$

Substituting (11), (12) and (13) into (7), we complete the proof of Lemma 2.2.

3. Proof of Main Theorems

Proof of Theorem 1.1. Since A_3 and A_4 are diagonalizable, we choose a local orthonormal frame field $\{e_1, \ldots, e_4\}$ such that $e_1,\ e_2$ are tangent to M_r^2 and

$$A_3 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},\ A_4 = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}. \tag{14}$$

Since $\overrightarrow{H}$ is light-like, we can set

$$\overrightarrow{H} = f(e_3 + e_4) \tag{15}$$

with e_3 being space-like, e_4 being time-like and f being a non-zero function on M_r^2 (in fact, we can assume e_3 be time-like and e_4 be space-like, the proof idea is the same). By a direct computation, we have

$$\Delta^D(fe_3) = (\Delta f)e_3 - 2\omega_3^4(\nabla f)e_4 - f(\mathrm{trace}\nabla\omega_3^4)e_4 - f\|\omega_3^4\|^2 e_3.$$

$$\Delta^D(fe_4) = (\Delta f)e_4 - 2\omega_3^4(\nabla f)e_3 - f(\mathrm{trace}\nabla\omega_3^4)e_3 - f\|\omega_3^4\|^2 e_4.$$

Then, we have

$$\begin{aligned}\Delta^D\overrightarrow{H} =&\Delta^D(fe_3) + \Delta^D(fe_4) \\ =& \big(\Delta f - 2\omega_3^4(\nabla f) - f\mathrm{trace}\nabla\omega_3^4 - f\|\omega_3^4\|^2\big)\, e_3 \\ &+ \big(\Delta f - 2\omega_3^4(\nabla f) - f\mathrm{trace}\nabla\omega_3^4 - f\|\omega_3^4\|^2\big)\, e_4,\end{aligned} \tag{16}$$

where $\text{trace}\nabla\omega_3^4 = \sum_{i=1}^2 \varepsilon_i(\nabla_{e_i}\omega_3^4)(e_i)$. We also have

$$\sum_{i=1}^{2}\varepsilon_i B(A_{\overrightarrow{H}}(e_i), e_i) = f(\text{trace}A_3^2 + \text{trace}A_3A_4)e_3 - f(\text{trace}A_4^2 + \text{trace}A_3A_4)e_4. \tag{17}$$

$$\sum_{i=1}^{2}\varepsilon_i A_{D_{e_i}\overrightarrow{H}}(e_i) = (A_3 + A_4)\Big(\nabla f + f\sum_{i=1}^{2}\varepsilon_i\omega_3^4(e_i)e_i\Big). \tag{18}$$

Substituting (15), (16), (17) and (18) into (7), we get

$$\begin{cases} \Delta f - 2\omega_3^4(\nabla f) - f\text{trace}\nabla\omega_3^4 - f\|\omega_3^4\|^2 - 2fc \\ \qquad +f(\text{trace}A_3^2 + \text{trace}A_3A_4) = 0, \\ \Delta f - 2\omega_3^4(\nabla f) - f\text{trace}\nabla\omega_3^4 - f\|\omega_3^4\|^2 - 2fc \\ \qquad -f(\text{trace}A_4^2 + \text{trace}A_3A_4) = 0, \\ (A_3 + A_4)\Big(\nabla f + f\sum_{i=1}^2 \varepsilon_i\omega_3^4(e_i)e_i\Big) = 0. \end{cases} \tag{19}$$

Using the first and second equations in (19), we have

$$\text{trace}A_3^2 + \text{trace}A_4^2 + 2\text{trace}A_3A_4 = 0, \tag{20}$$

which together with (14), we get

$$\lambda_1 = -\mu_1,\ \lambda_2 = -\mu_2.$$

Note that

$$A_{\vec{H}} = A_{f(e_3+e_4)} = \begin{pmatrix} f(\lambda_1 + \mu_2) & 0 \\ 0 & f(\lambda_2 + \mu_2) \end{pmatrix} = \mathbf{0}. \tag{21}$$

Then, $\langle B(e_i, e_j), \overrightarrow{H}\rangle = \langle A_{\overrightarrow{H}}(e_i), e_j\rangle = 0,\ i, j = 1, 2$. Also, $\langle\overrightarrow{H}, \overrightarrow{H}\rangle\langle e_i, e_j\rangle = 0$ because of $\overrightarrow{H}$ being light-like. So, $\langle B(e_i, e_j), \overrightarrow{H}\rangle = \langle\overrightarrow{H}, \overrightarrow{H}\rangle\langle e_i, e_j\rangle$. We have from (6) that M_r^2 is pseudo-umbilical and complete the proof of Theorem 1.1.

Proof of Theorem 1.2. Choose a local orthonormal frame field $\{e_i\}_{i=1}^4$ such that $\overrightarrow{H} = He_3$ with $\langle e_3, e_3\rangle = \varepsilon_3 = \pm 1$. Then, (9) is simplified as

$$\begin{cases} \Delta H + H\varepsilon_3\varepsilon_4\|\omega_4^3\|^2 + H\varepsilon_3\|A_3\|^2 - 2Hc = 0, \\ \varepsilon_4 H\text{trace}A_3A_4 = 2\omega_3^4(\nabla H) + H\sum\limits_{i=1}^{2}\varepsilon_i(\nabla_{e_i}\omega_3^4)(e_i), \\ \varepsilon_3\nabla H + A_3(\nabla H) + H\sum\limits_{i=1}^{2}\varepsilon_i\omega_3^4(e_i)A_4(e_i) = 0. \end{cases} \tag{22}$$

Since $\overrightarrow{H}$ is non light-like, then either $\overrightarrow{H} = 0$, or $\langle \overrightarrow{H}, \overrightarrow{H} \rangle \neq 0$.

Suppose that $\langle \overrightarrow{H}, \overrightarrow{H} \rangle \neq 0$. Since M_r^2 is pseudo-umbilical, it follows from (2) and (6) that $A_{\overrightarrow{H}} = \langle \overrightarrow{H}, \overrightarrow{H} \rangle I$. On one hand, we obtain

$$\sum_{i=1}^{2} \varepsilon_i (A_{D_{e_i} \overrightarrow{H}})(e_i) = \sum_{i=1}^{2} \varepsilon_i \left(\nabla_{e_i} A_{\overrightarrow{H}} \right)(e_i) - \nabla \langle \overrightarrow{H}, \overrightarrow{H} \rangle. \tag{23}$$

On the other hand, it follows that

$$\sum_{i=1}^{2} \varepsilon_i \left(\nabla_{e_i} A_{\overrightarrow{H}} \right)(e_i) = \nabla \langle \overrightarrow{H}, \overrightarrow{H} \rangle. \tag{24}$$

Putting (23) and (24) into the second equation of (7), we obtain $\nabla \langle \overrightarrow{H}, \overrightarrow{H} \rangle = 0$, i.e., H is a non zero constant. Then, (22) becomes

$$\begin{cases} \varepsilon_4 \varepsilon_3 \|\omega_4^3\|^2 + \varepsilon_3 \|A_3\|^2 - 2c = 0, \\ \varepsilon_4 \text{trace} A_3 A_4 = \sum\limits_{i=1}^{2} \varepsilon_i (\nabla_{e_i} \omega_3^4)(e_i), \\ \sum\limits_{i=1}^{2} \varepsilon_i \omega_3^4(e_i) A_4(e_i) = 0. \end{cases} \tag{25}$$

According to (4), we put

$$\omega_3^4 = -\varepsilon_3 \varepsilon_4 \omega_4^3 = \tilde{f}_1 \omega_1 + \tilde{f}_2 \omega_2, \tag{26}$$

where $\tilde{f}_1$ and $\tilde{f}_2$ are some functions. Note that $\overrightarrow{H} = He_3$, it follows from (5) that trace $A_4 = 0$. Then, we can express the matrix representation of A_4 as

$$A_4 = \begin{pmatrix} \tilde{g}_{11} & 0 \\ 0 & -\tilde{g}_{11} \end{pmatrix}. \tag{27}$$

We claim that $\|\omega_4^3\|^2 = 0$.

Assume on the contrary that $\|\omega_4^3\|^2 \neq 0$, i.e., $\tilde{f}_1^2 + \varepsilon_1 \varepsilon_2 \tilde{f}_2^2 \neq 0$. Making use of the third equation of (25) and (26), we have

$$\varepsilon_1 \varepsilon_2 \tilde{f}_1 A_4(e_1) + \tilde{f}_2 A_4(e_2) = 0, \tag{28}$$

which together with (27), we obtain $\tilde{f}_1 \tilde{g}_{11} = 0$, and $\varepsilon_1 \varepsilon_2 \tilde{f}_2 \tilde{g}_{11} = 0$. So, we have

$$\left(\tilde{f}_1^2 + \varepsilon_1 \varepsilon_2 \tilde{f}_2^2 \right) \tilde{g}_{11} = 0,$$

which shows that $\tilde{g}_{11} = 0$. Thus, $A_4 = 0$.

Since $\overrightarrow{H} = He_3$ and M_r^2 is pseudo-umbilical, $A_3 = \varepsilon_3 H I$. It follows from (2) that

$$\begin{aligned} \varepsilon_1 B(e_1, e_1) &= \varepsilon_1 \varepsilon_3 \langle A_3(e_1), e_1 \rangle \, e_3 + \varepsilon_1 \varepsilon_4 \langle A_4(e_1), e_1 \rangle \, e_4 = \varepsilon_3 H e_3, \\ \varepsilon_1 B(e_1, e_2) &= \varepsilon_1 \varepsilon_3 \langle A_3(e_1), e_2 \rangle \, e_3 + \varepsilon_1 \varepsilon_4 \langle A_4(e_1), e_2 \rangle \, e_4 = 0, \\ \varepsilon_2 B(e_2, e_2) &= \varepsilon_2 \varepsilon_3 \langle A_3(e_2), e_2 \rangle \, e_3 + \varepsilon_2 \varepsilon_4 \langle A_4(e_2), e_2 \rangle \, e_4 = \varepsilon_3 H e_3, \end{aligned}$$

which imply that

$$B(e_1, e_1) = \varepsilon_1\varepsilon_3 H e_3, \quad B(e_1, e_2) = 0, \quad B(e_2, e_2) = \varepsilon_2\varepsilon_3 H e_3.$$

Using these three equations and (3), we compute and get

$$(\widetilde{\nabla}_{e_2} B)(e_1, e_1) = \varepsilon_1\varepsilon_3 H\omega_3^4(e_2)e_4.$$
$$(\widetilde{\nabla}_{e_1} B)(e_2, e_1) = -\varepsilon_3 H\big(\varepsilon_1\omega_2^1(e_1) + \varepsilon_2\omega_1^2(e_1)\big)e_3.$$

Then, the Codazzi equation $(\widetilde{\nabla}_{e_2} B)(e_1, e_1) = (\widetilde{\nabla}_{e_1} B)(e_2, e_1)$ and (26) deduce to $H\tilde{f}_2 = 0$. Similarly, the Codazzi equation $(\widetilde{\nabla}_{e_1} B)(e_2, e_2) = (\widetilde{\nabla}_{e_2} B)(e_1, e_2)$ gives $H\tilde{f}_1 = 0$. These two equations imply that $H = 0$, which is a contradiction.

Since $\|\omega_4^3\|^2 = 0$ and $A_3 = \varepsilon_3 HI$, then the first equation of (25) becomes

$$\varepsilon_3 H^2 - c = 0. \tag{29}$$

When $c = 0$, (29) implies $H = 0$, a contradiction. So, $\overrightarrow{H} = 0$ and M_1^2 is minimal.

When $c > 0$, it follows from (29) that $\varepsilon_3 = 1$. Then, the mean curvature vector $\overrightarrow{H}$ $(= He_3)$ is space-like, and $H^2 = c$.

When $c < 0$, we know from (29) that $\overrightarrow{H}$ is a time-like vector with $H^2 = |c|$.

We complete the proof of Theorem 1.2.

Proof of Theorem 1.3. Choose a orthonormal frame field $\{e_1, \ldots, e_4\}$ such that $\overrightarrow{H} = He_3$. Since A_3 is diagonalizable, we denote by

$$A_3(e_1) = \lambda e_1, \; A_3(e_2) = \mu e_2. \tag{30}$$

Then, $\lambda + \mu = 2H$ and $\mathrm{tr}A_4 = 0$. Since $D_{e_i}\vec{H} = 0$, we easily prove that $e_i(H)e_3 + H\omega_3^4(e_i)e_4 = 0$ for $i = 1, 2$, which means that H is a constant and $\omega_3^4 = 0$. Thus, it follows from (30) that (22) is simplified to

$$H(\varepsilon_3(\lambda^2 + \mu^2) - 2c) = 0. \tag{31}$$

When $c = 0$, we have from (31) that $H(\lambda^2 + \mu^2) = 0$, which implies that $H = 0$ or $\lambda^2 + \mu^2 = 0$. This together with $\lambda + \mu = 2H$ gives that $H = 0$, that is, M_r^2 is minimal.

When $c < 0$, then $(\lambda^2 + \mu^2) - 2c \neq 0$ since $\varepsilon_3 = 1$. Thus, it follows from (31) that $H = 0$.

We claim that $\vec{H}$ is space-like when $c > 0$.

Suppose on contrary that $\vec{H}$ is time-like, then $\varepsilon_3 = -1$ and $H \neq 0$. Then, using (31) leads to $(\lambda^2 + \mu^2) + 2c = 0$, which contradicts the assumption that $(\lambda^2 + \mu^2) + 2c \neq 0$. So, $\vec{H}$ is space-like.

Since $\vec{H}$ is space-like, then $\varepsilon_3 = 1$. This which together with (31) gives

$$H(\lambda^2 + \mu^2 - 2c) = 0, \tag{32}$$

which implies that $H = 0$, and we complete the proof of Theorem 1.3.

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Groups in which every element centralizer is a *TI*-subgroup

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Abstract. Let G be a finite group. Recall that a subgroup H is called a TI-subgroup of G if $H \cap H^g = 1$ or H for every element g of G. We call a group G a CTI-group if its every element centralizer is a TI-group. Clearly S_3, A_5, D_7 and Q_8 are all CTI-groups. In this paper, we investigate the structure of a CTI-group G and prove that a CTI-group G is a nilpotent group or a Frobenius group whose complement is either cyclic or the direct product of a cyclic group of odd order and Q_8, or $G \cong PSL(2, 2^n)$ with $n > 1$.

Keywords: finite groups, CTI-groups, TI-subgroups, solvable groups, centralizers.
MSC 2020: 20D25, 20E99, 20F18.

1. Introduction

All groups considered in this paper are finite. The properties of some element centralizers have a profound influence on the structure of a group G, and there are many results in this regard, see [1, 2, 9, 10, 16, 3, 17, 18, 12, 13, 4]. Among these results, a classic result was derived from Brauer-Fowler ([4]) in 1955, which showed that if G was a group of even order with center of odd order, then there existed an element $x \in G \setminus Z(G)$ such that $|G| < |C_G(x)|^3$. As for the number of element centralizers, Belcastro and Sherman in [3] asserted that there was

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no a finite group with 2 or 3 element centralizers, furthermore, there are also many results relevant to this topic, see [1, 2, 9, 10, 16]. Except for these, [12, 13] concerned the group G whose element centralizers were nilpotent, even abelian, [17, 18] characterized the group G all of whose centralizers were maximal or second maximal, and so on.

On the other hand, in 1979, Walls in [15] introduced the definition of the TI-subgroup(a subgroup H of G is called a TI-subgroup if $H \cap H^g = 1$ or H for every element g of G) and classified groups all of whose subgroups were TI-subgroups. Based on this, some authors investigated the finite groups in which some subgroups were assumed to be TI-subgroups, such as abelian subgroups [6], non-abelian subgroups [11], etc.

Combining the above two aspects of studies, in this paper, we investigate the finite groups in which every element centralizer is a TI-group. For convenience, we call a group G a CTI-group if and only if its every element centralizer is a TI-subgroup of G. In fact, there exists many CTI-groups in finite groups, some examples are as follows:

Example 1. In S_3, for every non-central element $x \in S_3$, it is clear that $|C_{S_3}(x)| = 2$ or 3, so $C_{S_3}(x)$ is a TI-group, and thus S_3 is a CTI-group.

Example 2. In A_5, for every non-central element $x \in A_5$, it is easy to find that $|C_{A_5}(x)| = 3$ or 5 or 4, thus:

$$C_{A_5}(x) \cap C_{A_5}(x)^g = 1 \text{ or } C_{A_5}(x), \text{ for any } g \in A_5.$$

So, A_5 is a CTI-group.

Example 3. The dihedral group D_7 is a CTI-group since $|C_G(x)| = 2$ or 7 for every non-central element $x \in D_7$.

Example 4. The quaternion group Q_8 are CTI-groups since every subgroup of Q_8 is normal.

A natural question is:

What about the structure of a *CTI*-group G?

In this paper, we prove the following result:

Theorem 3.1. *Let G be a CTI-group, then one of the following statements holds:*

(1) *G is nilpotent;*

(2) *G is a Frobenius group whose complement is either cyclic or the direct product of a cyclic group of odd order and Q_8;*

(3) *$G \cong PSL(2, 2^n)$ with $n > 1$.*

2. Preliminaries

Here, we present some useful results needed during the proof of Theorem 3.1.

Lemma 2.1 ([5, Chapter 14, Theorem 1.5])**.** *If G is a solvable CN-group, then one of the following holds:*

1. *G is nilpotent.*
2. *G is a Frobenius group whose complement is either cyclic or the direct product of a cyclic group of odd order and a generalized quaternion group.*
3. *G is a 3-step group*[1].

Lemma 2.2 ([14, Theorem 2])**.** *Let G be a group of even order and its Sylow 2-subgroups are TI-subgroups. Then one of the following three statements is true:*

1. *A Sylow 2-subgroup of G is a normal subgroup;*
2. *A Sylow 2-subgroup of G is either a cyclic group or a generalized (or ordinary) quaternion group;*
3. *G contains normal subgroups G_1 and G_2 such that*

$$G \supseteq G_1 \supset G_2 \supseteq \{1\}$$

where both G/G_1 and G_2 are of odd order and G_1/G_2 is isomorphic to one of the groups $PSL(2, 2^n)$, $U_3(2^n)$ or $Sz(q)$.

The main results of [13] indicate that the class of ZT-groups[2] consists of simple groups $PSL(2, 2^n)$ and simple groups $Sz(q)$, combining [12, Part I, Theorem 4] and [12, Part III, Theorem 5], we have:

Lemma 2.3. *If G is a non-solvable CN-group*[3]*, then the maximal solvable normal subgroup N of G is a 2-group and G/N is one of the following types:*

$$PSL(2, q) \text{ with } q = 2^n \pm 1 \text{ or } 2^n, Sz(q), PSL(3, 4) \text{ or } M_9.$$

where M_9 is the projective group of one variable over near-field of 9 elements.

Lemma 2.4. *Let $G = Sz(q)$ and $S \in Syl_2(G)$. For elements x, y of S with order 4, we have:*

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1. We shall call G a 3-step group (with respect to the prime p) provided: (i) $O_{pp'}(G)$ is a Frobenius group with kernel $O_p(G)$ and cyclic complement of odd order. (ii) $G = O_{pp'p}(G)$ and $O_{pp'}(G) \subset G$. (iii) $G/O_p(G)$ is a Frobenius group with kernel $O_{pp'(G)}/O_p(G)$(see [5]).
2. A permutation group is called a ZT-group if it is a doubly transitive group of odd degree containing no regular normal subgroup, and if no non-identity element leaves more than three letters invariant(see [12]).
3. A group G is a CN-group if the centralizer of every non-identity element is nilpotent(see [12]).

(1) *If* $C_S(x) \neq C_S(y)$*, then* $C_S(x) \cap C_S(y) = Z(S)$.

(2) $C_G(x)$ *is not a TI-subgroup of* G.

Proof. (1) Clearly $|S|=q^2$, in view of [8, Ch. 11, Lemma 5.9 and Lemma 11.7] and [16, Theorem 1.2], we have $|Z(S)| = q$ and $|C_S(x)| = 2q$ for $x \in S - Z(S)$. Let $|\mathrm{Cent}(S)|$ be the number of centralizers of S, clearly $|\mathrm{Cent}(S)|=q$ by [16, Theorem 1.2]. Assume that $C_S(x_i)(0 \leq i \leq q-1)$ are all distinct centralizers, where $C_S(x_0) = S$, we have $S = C_S(x_1) \cup C_S(x_2) \cup \ldots \cup C_S(x_{q-1})$, so

$$\begin{aligned}|S| &\leq |C_S(x_1)| + |C_S(x_2)| + \ldots + |C_S(x_{q-1})| - (q-2) \cdot |Z(S)| \\ &= (q-1) \cdot |C_S(x_1)| - (q-2) \cdot |Z(S)| = q^2 = |S|,\end{aligned}$$

which means

$$C_S(x_i) \cap C_S(x_j) = Z(S), \text{ for } i \neq j.$$

(2) Otherwise, there exists an element u of G with order 4 such that $C_G(u)$ is a TI-subgroup of G. Clearly $u \notin Z(S)$, so there is an $s(\in S)$ such that $u^s \neq u$. By (1), we have $C_S(u) \cap C_S(u^s) = Z(S)$, it follows that $C_G(u) = C_G(u^s)$ since $C_G(u)$ is a TI-subgroup of G, and hence $C_S(u) = C_S(u^s)$, a contradiction. □

3. Main results

In this section, we give some properties about CTI-groups.

Proposition 3.1. *If* G *is a* CTI*-group and* x *a non-identity element of* G*, then* $N_G(\langle y \rangle) \leq N_G(C_G(y)) = N_G(C_G(x))$ *for every* $1 \neq y \in C_G(x)$.

Proof. (1) $N_G(\langle y \rangle) \leq N_G(C_G(y))$ for any $1 \neq y \in C_G(x)$.

For every $1 \neq y \in C_G(x)$ and $g \in N_G(\langle y \rangle)$, we have $y^g \in C_G(y)^g \cap C_G(y)$. Notice that $C_G(y)$ is a TI-subgroup of G, we have $C_G(y) = C_G(y)^g$, consequently $g \in N_G(C_G(y))$, and therefore $N_G(\langle y \rangle) \leq N_G(C_G(y))$ by the arbitrariness of g.

(2) $N_G(C_G(y)) = N_G(C_G(x))$ for every $1 \neq y \in C_G(x)$.

On the one hand, $x^g \in C_G(y)^g = C_G(y)$ for every $1 \neq y \in C_G(x)$ and $g \in N_G(C_G(y))$, and hence $y \in C_G(x) \cap C_G(x)^g$, therefore $C_G(x) = C_G(x)^g$ since $C_G(x)$ is a TI-subgroup of G, so $N_G(C_G(y)) \leq N_G(C_G(x))$. On the other hand, by using a similar argument to the above, we have $N_G(C_G(x)) \leq N_G(C_G(y))$, consequently $N_G(C_G(y)) = N_G(C_G(x))$. □

Proposition 3.2. *If* G *is a* CTI*-group, then* G *is a* CN*-group. Especially, if* $Z(G) \neq 1$*, then* G *is nilpotent.*

Proof. For every $1 \neq x \in G$ and $1 \neq y \in C_G(x)$, it is clear that $\langle y \rangle \trianglelefteq C_G(y)$ and $C_G(y) \trianglelefteq N_G(C_G(y))$. Also, $N_G(C_G(y)) = N_G(C_G(x))$ by Proposition 3.1, so $\langle y \rangle$ is a subnormal subgroup of $C_G(x)$, therefore G is a CN-group. Especially, clearly G is nilpotent if $Z(G) \neq 1$. □

Proposition 3.3. *If G is a CTI-group, p a prime factor of $|G|$ and x a p-element of G, then $P \leq N_G(C_G(x))$ with $x \in P \in Syl_p(G)$. Especially, if $x \in Z(P)$, then $N_G(P) = N_G(C_G(x))$.*

Proof. For every p-element $x \in P \in Syl_p(G)$, it is clear that $Z(P) \leq C_G(x)$. Take $1 \neq y \in Z(P)$, so $P \leq N_G(\langle y \rangle) \leq N_G(C_G(x))$ by Proposition 3.1.

Especially, if $x \in Z(P)$, then $P \leq C_G(x)$. In fact, $P = P^g \leq C_G(x)^g$ for every $g \in N_G(P)$, it shows that $P \leq C_G(x) \cap C_G(x)^g$. Notice that $C_G(x)$ is a TI-subgroup of G, we have $C_G(x) = C_G(x)^g$, it follows that $g \in N_G(C_G(x))$, so $N_G(P) \leq N_G(C_G(x))$ by the arbitrariness of g. On the other hand, obviously $N_G(C_G(x)) \leq N_G(P)$ since $C_G(x)$ is nilpotent by Proposition 3.2. Therefore, $N_G(P) = N_G(C_G(x))$. □

Proposition 3.4. *If G is a CTI-group, p a prime factor of $|G|$ and x a p-element of G, then $x^G \cap C_G(x) \subseteq x^G \cap P$, where $x \in P \in Syl_p(G)$. Especially, $|x^G \cap C_G(x)|\,|\,|x^G|$.*

Proof. Obviously, $x^G \cap C_G(x) = x^G \cap P_1 \subseteq x^G \cap P$ by Proposition 3.2 and Proposition 3.3, where $P_1 \in Syl_p(C_G(x))$. Especially, $x^G \subseteq C_G(x)^G$, while $C_G(x)$ is a TI-subgroup, so $|x^G \cap C_G(x)|\,|\,|x^G|$. □

Proposition 3.5. *If G is a CTI-group, then any two Sylow subgroups of G have a trivial intersection.*

Proof. Let p be a prime factor of $|G|$ and $P \in Syl_p(G)$. Take $1 \neq x \in Z(P)$, clearly $P \leq C_G(x)$. If there exists a $g \in G$ such that $1 \neq u \in P \cap P^g$, then $u \in P \cap P^g \leq C_G(x) \cap C_G(x)^g$, and hence $C_G(x) = C_G(x)^g$ since $C_G(x)$ is a TI-subgroup of G, therefore $P = P^g$ since $C_G(x)$ is nilpotent by Proposition 3.2. □

Now, we would prove Theorem 3.1.

Theorem 3.1. *Let G be a CTI-group, then one of the following statements holds:*

(1) *G is nilpotent;*

(2) *G is a Frobenius group whose complement is either cyclic or the direct product of a cyclic group of odd order and Q_8;*

(3) *$G \cong PSL(2, 2^n)$ with $n > 1$.*

Proof. The proof is divided into two aspects according to the solvablity of G.

(1) If G is solvable, then G is nilpotent or G is a Frobenius group whose complement is either cyclic or the direct product of a cyclic group of odd order and Q_8.

By Proposition 3.2, we have G is a *CN*-group. By Lemma 2.1, we have:

(a) G is nilpotent; or

(b) G is a Frobenius group whose complement is either cyclic or the direct product of a cyclic group of odd order and a generalized quaternion group; or

(c) G is a 3-step group.

For (b), if $G(=K \rtimes H)$ is a Frobenius group and H is the direct product of an odd order cyclic group and a generalized quaternion group Q_{4n}, we assert $n=2$. Otherwise, let $Q_{4n}=\langle a,b\rangle$, where $a^{2n}=1$, $b^2=a^n$, and $b^{-1}ab=a^{-1}$. Notice that $C_{Q_{4n}}(b)=\langle b\rangle \ntrianglelefteq Q_{4n}$. However, $b^2 \in C_{Q_{4n}}(b)\cap C_{Q_{4n}}(b)^g \leq C_G(b)\cap C_G(b)^g$ for any $g\in Q_{4n}$, so $C_{Q_{4n}}(b)=C_{Q_{4n}}(b)^g$ since $C_G(b)$ is a TI-subgroup of G, and hence $C_{Q_{4n}}(b)\trianglelefteq Q_{4n}$, a contradiction. Therefore, $n=2$, and so $Q_{4n}=Q_8$.

For (c), if G is a 3-step group with respect to the prime p, we have $O_{pp'}(G)=O_p(G)\rtimes H$ is a Frobeius group. For every $P_1,P_2\in Syl_p(G)$, clearly $O_p(G)\leq P_1\cap P_2$, and hence P_1 is a normal subgroup of G since G is a CTI-group and Proposition 3.2, in contradiction with the definition of 3-step group.

(2) If G is non-solvable, then $G\cong PSL(2,2^n)$ with $n>1$.

By Proposition 3.2 and Lemma 2.3, we have the maximal solvable normal subgroup N of G is a 2-group and G/N is one of the following types:

$$(*) \qquad PSL(2,q) \text{ with } q=2^n\pm 1 \text{ or } 2^n, Sz(q), PSL(3,4) \text{ or } M_9.$$

It is obvious that the Sylow 2-subgroup of G is not a normal subgroup, a cyclic group or a generalized (or ordinary) quaternion group, so Lemma 2.2 shows that there exists a normal subgroups series

$$G \supseteq G_1 \supset G_2 \supseteq \{1\}$$

such that $G_1/G_2\cong PSL(2,2^n)$, $U_3(2^n)$ or $Sz(q)$, where both G/G_1 and G_2 are of odd order. Applying Lemma 2.3 once again, we have $G_2=1$, and hence

$$(**) \qquad G_1\cong PSL(2,2^n),\ U_3(2^n) \text{ or } Sz(q).$$

Moreover, notice that $N\trianglelefteq G_1$, so $N=1$ by (**), and hence $G\cong PSL(2,2^n)$ or $Sz(q)$ by (*) and (**). By Lemma 2.4(2), we have $G\cong PSL(2,2^n)$ with $n>1$. □

By Lemma 3.1(2) and [8, Chapter 11], we have:

Corollary 3.1. *Let $G=Sz(q)$, $C_G(x)$ is not a TI-subgroup of $Sz(q)$ if and only if $o(x)=4$.*

By Theorem 3.1 and [7, Chapter 2, Theorem 8.2-Theorem 8.5], we have:

Corollary 3.2. *A group G is a simple CTI-group if and only if $G\cong PSL(2,2^n)$ with $n>1$.*

4. Conclusion

The classification of finite groups is one of the important topics in finite group theory. Based on the study of the TI-subgroups and the element centralizers, in this paper, we introduced the definition of the CTI-group, subsequently, studied the properties of CTI-groups, and then finished the classification of the CTI-groups.

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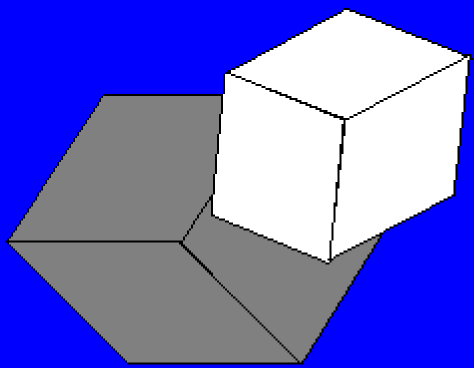

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