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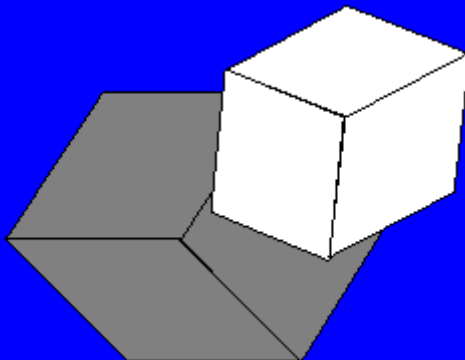
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experiences of good practices at the university level*

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Inequalities of DVT-type—the two-dimensional case

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Abstract. In this note, particular two-dimensional inequalities of Drápal-Valent type in integer numbers are investigated.

Keywords: integer numbers, inequality.

MSC 2020: 11D75.

In [3], A. Drápal and V. Valent proved that in a finite quasigroup Q of order n the number of associative triples $a(Q) \geq 2n - i(Q) + (\delta_1 + \delta_2)$, where $i(Q)$ is the number of idempotents in Q , i.e., $i(Q) = |\{x \in Q \mid xx = x\}|$, $\delta_1 = |\{z \in Q \mid zx \neq x \text{ for all } x \in Q\}|$ and $\delta_2 = |\{z \in Q \mid xz \neq x \text{ for all } x \in Q\}|$ [3, Theorem 2.5]. This important result is an easy consequence of the inequality

$$\sum_{i=1}^n (a_i^2 + b_i^2 + a_i b_i) - \sum_{i=1}^k (a_i + b_i) \geq 3n - 2k + (r + s),$$

where $n \geq k \geq 0$, $a_1, \dots, a_n, b_1, \dots, b_n$ are non-negative integers such that $\sum_{i=1}^n a_i = n = \sum_{i=1}^n b_i$, $a_i \geq 1$ and $b_i \geq 1$ for $1 \leq i \leq k$, r is the number of i with $a_i = 0$ and s is the number of i with $b_i = 0$ ([3, Proposition

*. Corresponding author

2.4(ii)]. The lengthy and complicated proof of this DVT-inequality (inequality of Drápal-Valent type) in [3] is based on highly semantically involved insight.

In [4], a very short elementary arithmetical proof of a more general inequality of this type was found. This inequality is two-dimensional in the sense that it works with two n -tuples of integers. The approach in [4] opens a road to investigation of similar inequalities of DVT-type which could be useful in further investigations of estimates in non-associative algebra and they are also of independent interest. Hence, they deserve a thorough examination, however the research is only at its beginning. In [1] and [2], the one-dimensional case working with one n -tuple of real numbers was investigated. In this note, the two-dimensional case of inequalities of Drápal's type is investigated. Among other results, it is shown that

$$2 \sum_{i=1}^n (a_i^2 + b_i^2 + a_i b_i) \geq 3 \sum_{i=1}^n (a_i + b_i) + 2(r + s),$$

where $a_1, \dots, a_n, b_1, \dots, b_n$ are arbitrary integers with $\sum_{i=1}^n a_i \geq n$, $\sum_{i=1}^n b_i \geq n$, r is the number of i with $a_i = 0$ and s is the number of i with $b_i = 0$. The case when the equality holds is characterized and several other inequalities of this type are investigated.

1. First concepts

Let $n \geq 1$ and let $\alpha = (a_1, \dots, a_n)$, $\beta = (b_1, \dots, b_n)$ be an ordered n -tuples of integers. We put

1. $z(\alpha, a) = |\{i \mid 1 \leq i \leq n, a_i = a\}|$, for every $a \in \mathbb{R}$;
2. $z(\alpha) = z(\alpha, 0)$;
3. $s(\alpha) = \sum_{i=1}^n a_i$;
4. $r(\alpha) = \sum_{i=1}^n a_i^2$;
5. $t(\alpha) = r(\alpha) - s(\alpha) - z(\alpha)$;
6. $p(\alpha, \beta) = \sum_{i=1}^n a_i b_i$;
7. $r(\alpha, \beta) = \sum_{i=1}^n a_i^2 + \sum_{i=1}^n b_i^2 + \sum_{i=1}^n a_i b_i$;
8. $t(\alpha, \beta) = 2 \sum_{i=1}^n a_i^2 + 2 \sum_{i=1}^n b_i^2 + 2 \sum_{i=1}^n a_i b_i - 3 \sum_{i=1}^n a_i - 3 \sum_{i=1}^n b_i - 2z(\alpha) - 2z(\beta)$.

We thus have

- (9) $r(\alpha, \beta) = \sum_{i=1}^n (a_i + b_i)^2 - p(\alpha, \beta) (= r(\alpha + \beta) - p(\alpha, \beta))$;
- (10) $t(\alpha, \beta) = 2r(\alpha, \beta) - 3s(\alpha) - 3s(\beta) - 2z(\alpha) - 2z(\beta) (= 2(r(\alpha, \beta) - z(\alpha) - z(\beta)) - 3s(\alpha + \beta) = 2(r(\alpha, \beta) - s(\alpha) - s(\beta) - z(\alpha) - z(\beta)) - s(\alpha) - s(\beta))$.

If $s(\alpha) = n = s(\beta)$ then

$$(11) \quad t(\alpha, \beta) = 2(r(\alpha, \beta) - 3n - z(\alpha) - z(\beta)).$$

Lemma 1.1. $t(\alpha, \beta) = 2t(\alpha) + 2t(\beta) + 2p(\alpha, \beta) - s(\alpha) - s(\beta)$.

Proof. Easy to check directly. $\square$

Lemma 1.2. Put $\gamma = \alpha + \beta = (a_1 + b_1, \dots, a_n + b_n)$. Then, $t(\alpha, \beta) = t(\alpha) - z(\alpha) + t(\beta) - z(\beta) + t(\gamma) + z(\gamma) - s(\gamma) = t(\alpha) - z(\alpha) + t(\beta) - z(\beta) + r(\gamma) - 2s(\gamma)$.

Proof. By 1.1, $t(\alpha, \beta) = 2t(\alpha) + 2t(\beta) + 2p(\alpha, \beta) - s(\alpha) - s(\beta) = t(\alpha) + t(\beta) + r(\alpha) + r(\beta) + 2p(\alpha, \beta) - 2s(\alpha) - 2s(\beta) - z(\alpha) - z(\beta) = t(\alpha) + t(\beta) + r(\gamma) - 2s(\gamma) - z(\alpha) - z(\beta)$ and the rest is clear. $\square$

Lemma 1.3. Put $\delta = \alpha + \beta - 1 = (a_1 + b_1 - 1, \dots, a_n + b_n - 1)$. Then, $t(\alpha, \beta) = t(\alpha) - z(\alpha) + t(\beta) - z(\beta) + r(\delta) - n$.

Proof. We have $r(\alpha + \beta) - 2s(\alpha + \beta) = \sum_{i=1}^n (a_i + b_i)^2 - 2 \sum_{i=1}^n (a_i + b_i) = \sum_{i=1}^n a_i^2 + \sum_{i=1}^n b_i^2 + 2 \sum_{i=1}^n a_i b_i - 2 \sum_{i=1}^n a_i - 2 \sum_{i=1}^n b_i = \sum_{i=1}^n (a_i + b_i - 1)^2 - n = r(\delta) - n$ and it remains to use 1.2. $\square$

Lemma 1.4. Put $\varepsilon = \alpha + \beta - 2 = (a_1 + b_1 - 2, \dots, a_n + b_n - 2)$. Then, $t(\alpha, \beta) = t(\alpha) - z(\alpha) + t(\beta) - z(\beta) + r(\varepsilon) + 2s(\varepsilon)$.

Proof. We have $r(\varepsilon) + 2s(\varepsilon) = r(\alpha + \beta) - 4s(\alpha + \beta) + 4n + 2s(\alpha + \beta) - 4n = r(\alpha + \beta) - 2s(\alpha + \beta)$ and it remains to use 1.2. $\square$

Lemma 1.5. Put $\alpha_1 = \alpha - 1$ and $\beta_1 = \beta - 1$. Then, $t(\alpha, \beta) = 2r(\alpha_1) + 2r(\beta_1) + 2p(\alpha_1, \beta_1) + 3s(\alpha_1) + 3s(\beta_1) - 2z(\alpha_1, -1) - 2z(\beta_1, -1)$.

Proof. Easy to check directly. $\square$

Lemma 1.6. (i) $r(\alpha) \geq \sum_{i=1}^n |a_i| \geq |s(\alpha)|$.
(ii) $r(\alpha) + s(\alpha) \geq 0$.
(iii) $r(\alpha) = \sum_{i=1}^n |a_i|$ if and only if $a_i \in \{0, 1, -1\}$, for every i .
(iv) $r(\alpha) = s(\alpha)$ if and only if $a_i \in \{0, 1\}$, for every i .

Proof. Easy to see. $\square$

Lemma 1.7. (i) If $s(\alpha) \geq 0$ then $r(\alpha) + 2s(\alpha) \geq r(\alpha) \geq 0$.

(ii) If $\sum_{i=1}^n |a_i + 1| \geq n$ then $r(\alpha) + 2s(\alpha) \geq 0$.

Proof. (i) This is obvious.

(ii) We have $r(\alpha) + 2s(\alpha) = r(\alpha + 1) - n \geq (\sum_{i=1}^n |a_i + 1|) - n \geq 0$. $\square$

Lemma 1.8. If $s(\alpha) \geq 2n$ or $\sum_{i=1}^n |a_i - 1| \geq n$ then $r(\alpha) - 2s(\alpha) \geq 0$.

Proof. We have $r(\alpha) - 2s(\alpha) = r(\alpha - 1) - n \geq \sum_{i=1}^n |a_i - 1| - n \geq \sum_{i=1}^n (a_i - 1) - n = s(\alpha) - 2n$. $\square$

Remark 1.9. First of all, if $a_i \leq -2$ then $a_i^2 - 1 \geq 3$. If $a_i = -1$ then $a_i^2 - 1 = 0$. If $a_i = 0$ then $a_i^2 - 1 = -1$. If $a_i = 1$ then $a_i^2 - 1 = 0$. If $a_i \geq 2$ then $a_i^2 - 1 \geq 3$. Now, if $g = |\{i \mid |a_i| \geq 2\}|$ then $r(\alpha) - n \geq 3g - z(\alpha)$.

Remark 1.10. First of all, if $a_i \leq -1$ then $a_i^2 - 2a_i \geq 3$. If $a_i = 0$ then $a_i^2 - 2a_i = 0$. If $a_i = 1$ then $a_i^2 - 2a_i = -1$. If $a_i = 2$ then $a_i^2 - 2a_i = 0$. If $a_i \geq 3$ then $a_i^2 - 2a_i \geq 3$. Now, if $h = |\{i \mid a_i < 0\}| + |\{i \mid a_i \geq 3\}|$ then $r(\alpha) - 2s(\alpha) \geq 3h - z(\alpha, 1)$.

Remark 1.11. First of all, if $a_i \leq -3$ then $a_i^2 + 2a_i \geq 3$. If $a_i = -2$ then $a_i^2 + 2a_i = 0$. If $a_i = -1$ then $a_i^2 + 2a_i = -1$. If $a_i = 0$ then $a_i^2 + 2a_i = 0$. If $a_i \geq 1$ then $a_i^2 + 2a_i \geq 3$. Now, if $k = |\{i \mid a_i \leq -3\}| + |\{i \mid a_i > 0\}|$ then $r(\alpha) + 2s(\alpha) \geq 3k - z(\alpha, -1)$.

Lemma 1.12. (i) $r(\alpha) \geq 2z(\alpha) - 2n + 3 \sum_{i=1}^n |a_i| \geq 2z(\alpha) - 2n + 3s(\alpha)$.
(ii) $r(\alpha) = 2z(\alpha) - 2n + 3s(\alpha)$ if and only if $a_i \in \{0, 1, 2\}$, for every $i = 1, \dots, n$.

Proof. This result was proved in [1, 6.1] in a more general setting. $\square$

2. Technical results (a)

Let $n \geq 2$ and let $\alpha = (a_1, \dots, a_n)$ and $\beta = (b_1, \dots, b_n)$ be ordered n -tuples of integers. Choose $1 \leq j, k \leq n$, $j \neq k$, and define $\gamma = (c_1, \dots, c_n)$ as $c_j = a_j - 1$, $c_k = a_k + 1$ and $c_i = a_i$ for $i \neq j, k$ (see [1, Section 2]). Clearly, $s(\alpha) = s(\gamma)$.

The following assertions are easily seen:

Lemma 2.1. $p(\alpha, \beta) = p(\gamma, \beta) + (b_j - b_k)$.

Lemma 2.2. $r(\alpha, \beta) = r(\gamma, \beta) + 2(a_j - a_k - 1) + b_j - b_k$.

Lemma 2.3. $t(\alpha, \beta) = t(\gamma, \beta) + 4(a_j - a_k - 1) + 2(b_j - b_k) + 2z(\gamma) - 2z(\alpha)$.

Lemma 2.4. (i) If $a_j = 0$ and $a_k = -1$ then $t(\alpha, \beta) = t(\gamma, \beta) + 2(b_j - b_k)$.

(ii) If $a_j = 1$ and $a_k = 0$ then $t(\alpha, \beta) = t(\gamma, \beta) + 2(b_j - b_k)$.

(iii) If $a_j = 1$ and $a_k = -1$ then $t(\alpha, \beta) = t(\gamma, \beta) + 2(b_j - b_k + 4)$.

(iv) If $a_j = 0 = a_k$ then $t(\alpha, \beta) = t(\gamma, \beta) + 2(b_j - b_k - 4)$.

Lemma 2.5. (i) If $a_j \neq 0, 1$ and $a_k \neq -1, 0$ then $t(\alpha, \beta) = t(\gamma, \beta) + 4(a_j - a_k - 1) + 2(b_j - b_k)$.

(ii) If $a_j \neq 0, 1$ and $a_k = -1$ then $t(\alpha, \beta) = t(\gamma, \beta) + 4a_j + 2(b_j - b_k + 1)$.

(iii) If $a_j = 1$ and $a_k \neq -1, 0$ then $t(\alpha, \beta) = t(\gamma, \beta) - 4a_k + 2(b_j - b_k + 1)$.

(iv) If $a_j \neq 0, 1$ and $a_k = 0$ then $t(\alpha, \beta) = t(\gamma, \beta) + 4a_j + 2(b_j - b_k - 3)$.

(v) If $a_j = 0$ and $a_k \neq -1, 0$ then $t(\alpha, \beta) = t(\gamma, \beta) - 4a_k + 2(b_j - b_k - 3)$.

Lemma 2.6. If $a_j \geq 2$ and $a_k = 0$ then $t(\alpha, \beta) = t(\gamma, \beta) + 2(2a_j - 3 + b_j - b_k) \geq t(\gamma, \beta) + 2(b_j - b_k + 1)$.

Lemma 2.7. If $a_j \geq 2$ and $a_k = -1$ then $t(\alpha, \beta) = t(\gamma, \beta) + 2(2a_j + 1 + b_j - b_k) \geq t(\gamma, \beta) + 2(b_j - b_k + 5)$.

Lemma 2.8. If $a_j \geq 2$ and $a_k \leq -2$ then $t(\alpha, \beta) = t(\gamma, \beta) + 2(2a_j - 2a_k - 2 + b_j - b_k) \geq t(\gamma, \beta) + 2(b_j - b_k + 6)$.

3. Technical results (b)

Let a, b be integers and put $z(0) = 1$, $z(a) = 0$ for $a \neq 0$ and $t(a, b) = 2a^2 + 2b^2 + 2ab - 3a - 3b - 2z(a) - 2z(b)$ ($= t(b, a)$).

Lemma 3.1. *Let $a \neq 0$ and $b \neq 0$. Then:*

- (i) $z(a) = 0 = z(b)$ and $t(a, b) \geq 0$.
- (ii) $t(a, b) = 0$ if and only if $a = 1 = b$.
- (iii) $t(a, b) \neq 1$.
- (iv) $t(a, b) = 2$ if and only if either $a = 1, b = -1$ or $a = -1, b = 1$.
- (v) $t(a, b) = 3$ if and only if either $a = 2, b = -1$ or $a = -1, b = 2$.

Proof. Taking into account that $t(a, b) = t(b, a)$, the proof is divided into seven parts:

- (1) Assume $a \geq 2, b \geq 2$. Then, $2a^2 - 3a \geq 2$, $2b^2 - 3b \geq 2$ and $2ab \geq 8$. Consequently, $t(a, b) \geq 12$.
- (2) Assume $a \geq 2, b = 1$. Then, $2a^2 - 3a \geq 2$, $2b^2 - 3b = -1$ and $2ab \geq 4$. Consequently, $t(a, b) \geq 5$.
- (3) Assume $a = 1 = b$. Then, $t(a, b) = 0$.
- (4) Assume $a + b = 1$. Then, $t(a, b) = 2a^2 - 2a - 1$. Since $a + b = 1$, we have $a \neq 1$. If $a \geq 3$ then $2a^2 - 2a - 1 \geq 11$. If $a = 2$ then $2a^2 - 2a - 1 = 3$ and $b = -1$. If $a = -1$ then $2a^2 - 2a - 1 = 3$ and $b = 2$. If $a \leq -2$ then $2a^2 - 2a - 1 \geq 11$.
- (5) Assume $a + b = 2$. Then, $t(a, b) = 2a^2 - 4a + 2$. Since $a + b = 2$, we have $a \neq 2$. If $a \geq 3$ then $2a^2 - 4a + 2 \geq 8$. If $a = 1$ then $2a^2 - 4a + 2 = 0$ and $b = 1$. If $a \leq -1$ then $2a^2 - 4a + 2 \geq 8$.
- (6) Assume $a < 0, b > 0$ and put $c = 2a^2 + 2b^2 + 4ab - 3a - 3b = 2(a+b)^2 - 3(a+b)$. Then, $t(a, b) > c$. If $a + b \leq -1$ then $c \geq 5$ and $t(a, b) \geq 6$. If $a + b = 0$ then $t(a, b) = 2a^2$. Hence, $t(a, b) \geq 8$ for $a \leq -2$ and $2a^2 = 2$ for $a = -1$ (then $b = 1$). If $a + b = 2$ then (4) applies. If $a + b = 2$ then (5) applies. Finally, if $a + b \geq 3$ then $t(a, b) = (a + b)(a + b - 3) + a^2 + b^2 \geq 17$.
- (7) Assume $a < 0, b < 0$. Then, $t(a, b) \geq 12$. □

Remark 3.2. Let a be a non-zero integer. Then, $t(a, 0) = 2a^2 - 3a - 2$, and hence $t(a, 0) \geq 7$ for $a \geq 3$, $t(a, 0) = 0$ for $a = 2$, $t(a, 0) = -3$ for $a = 1$, $t(-1, 0) = 3$ and $t(a, 0) \geq 12$ for $a \leq -2$. Further, $t(0, 0) = -4$.

Lemma 3.3. *Let $a \geq 2$ and $b \geq 1$. Then, $t(a, b) \geq t(a, -b) + 2$.*

Proof. We have $t(a, b) - t(a, -b) = 4ab - 6b \geq 2b \geq 2$. □

Lemma 3.4. *Let $a \geq 1$ and $b \geq 0$. Then:*

- (i) $t(a + 1, b) > t(a, b)$.
- (ii) If $c \geq a, d \geq b$ and $c + d > a + b$ then $t(c, d) > t(a, b)$.

Proof. We have $t(a + 1, b) - t(a, b) = 4a + 2b - 1 \geq 4a - 1 > 0$ and the rest is clear. □

4. The inequalities

Throughout this section, let $n \geq 1$, a, b be integers, $\alpha = (a_1, \dots, a_n)$ and $\beta = (b_1, \dots, b_n)$ be ordered n -tuples of integers. Put $I = \{1, \dots, n\}$, $A = \{i \in I \mid a_i \geq 0, b_i \geq 0, a_i + b_i \geq 3\}$, $B_1 = \{i \in I \mid (a_i, b_i) = (2, 0)\}$, $B_2 = \{i \in I \mid (a_i, b_i) = (0, 2)\}$, $B_3 = \{i \in I \mid (a_i, b_i) = (1, 1)\}$, $B = B_1 \cup B_2 \cup B_3$, $C_1 = \{i \in I \mid (a_i, b_i) = (2, -1)\}$, $C_2 = \{i \in I \mid (a_i, b_i) = (-1, 2)\}$, $C = C_1 \cup C_2$, $D_1 = \{i \in I \mid (a_i, b_i) = (0, 1)\}$, $D_2 = \{i \in I \mid (a_i, b_i) = (1, 0)\}$, $D = D_1 \cup D_2$ and $E = \{i \in I \mid (a_i, b_i) = (0, 0)\}$. For $X = A, B_1, \dots, E$, denote $x = |X|$. Clearly, $t(\alpha, \beta) = \sum_{i=1}^n t(a_i, b_i)$.

Example 4.1. Taking into account 3.1 and 3.2, it is easy to see that if $I = B \cup C \cup D \cup E$ and $3c = 3d + 4e$ then $t(\alpha, \beta) = 0$.

Theorem 4.2. Let $\sum_{i=1}^n a_i \geq n$ and $\sum_{i=1}^n b_i \geq n$. Put $z_1 = |\{i \in I \mid a_i = 0\}| = z(\alpha)$ and $z_2 = |\{i \in I \mid b_i = 0\}| = z(\beta)$. Then:

- (i) $2 \sum_{i=1}^n (a_i^2 + b_i^2 + a_i b_i) \geq 3 \sum_{i=1}^n (a_i + b_i) + 2(z_1 + z_2)$.
- (ii) $2 \sum_{i=1}^n (a_i + b_i)^2 \geq 2 \sum_{i=1}^n a_i b_i + 3 \sum_{i=1}^n (a_i + b_i) + 2(z_1 + z_2)$.
- (iii) The equalities hold if and only if $I = B$, $b_1 = b_2$, $2b_1 \leq n$ and $b_3 = n - 2b_1$. In this case, $\sum_{i=1}^n a_i = n = \sum_{i=1}^n b_i$.

Proof. Clearly, (i) $\Leftrightarrow$ (ii) $\Leftrightarrow t(\alpha, \beta) \geq 0$. By 1.2, $t(\alpha, \beta) = t(\alpha) - z(\alpha) + t(\beta) - z(\beta) + q$, where $q = r(\alpha + \beta) - 2s(\alpha + \beta)$. By [1, 6.1(i)], $t(\alpha) - z(\alpha) = \sum_{i=1}^n a_i^2 - \sum_{i=1}^n a_i - 2z_1 \geq 2(\sum_{i=1}^n a_i - n) \geq 0$ and $t(\beta) - z(\beta) \geq 0$. By 1.8, $q \geq 0$, and hence $t(\alpha, \beta) \geq 0$.

Now, assume that $t(\alpha, \beta) = 0$. Then, $t(\alpha) = z(\alpha)$, $t(\beta) = z(\beta)$ and $q = 0$. By 1.12(i), $0 = t(\alpha) - z(\alpha) = r(\alpha) - s(\alpha) - 2z(\alpha) \geq 2s(\alpha) - 2n \geq 0$, and hence $s(\alpha) = n$. Then, $\sum_{i=1}^n a_i = n$, $\sum_{i=1}^n b_i = n$ and $a_i, b_i \in \{0, 1, 2\}$, for every $i \in I$ by 1.12(ii). Further, by 1.4, $q = r(\alpha + \beta - 2) + 2s(\alpha + \beta - 2) = 0$. Since $s(\alpha + \beta - 2) = s(\alpha) + s(\beta) - 2n = 0$, we get $\alpha + \beta - 2 = 0$. Thus $a_i + b_i = 2$, for every $i \in I$ and $(a_i, b_i) \in B$. Finally, $s(\alpha) = 2b_1 + b_3 = n = s(\beta) = 2b_2 + b_3$. Hence, $b_1 = b_2$, $2b_1 \leq n$ and $b_3 = n - 2b_1$. Conversely, if $2b_1 \leq n$, $b_2 = b_1$ and $b_3 = n - 2b_1$ then $\sum_{i=1}^n a_i = 2b_1 + b_3 = n = 2b_2 + b_3 = \sum_{i=1}^n b_i$ and $t(\alpha, \beta) = 0$ by 3.1. $\square$

Remark 4.3. By 4.2(iii), the situation $\sum_{i=1}^n a_i \geq n$, $\sum_{i=1}^n b_i \geq n$, $t(\alpha, \beta) = 0$ is completely described. In order to find all such pairs α, β , choose $p \geq 0$ such that $2p \leq n$ and take p pairs $(2, 0)$, p pairs $(0, 2)$ and $n - 2p$ pairs $(1, 1)$.

Remark 4.4. Consider the situation from 4.2. The following inequalities follow from 4.2(i), (ii): $\sum_{i=1}^n (a_i^2 + b_i^2 + a_i b_i) \geq \sum_{i=1}^n (a_i + b_i) + n + z_1 + z_2$, $\sum_{i=1}^n (a_i^2 + b_i^2 + a_i b_i) \geq 3n + z_1 + z_2$, $\sum_{i=1}^n (a_i + b_i)^2 \geq \sum_{i=1}^n (a_i + b_i + a_i b_i) + n + z_1 + z_2$, $\sum_{i=1}^n (a_i + b_i)^2 \geq \sum_{i=1}^n a_i b_i + 3n + z_1 + z_2$ and the equalities hold if and only if the conditions from 4.2(iii) are satisfied.

Remark 4.5. Consider the situation from 4.2 and its proof. Now, by 1.6(i), $r(\alpha + \beta - 1) \geq s(\alpha + \beta - 1) = s(\alpha) + s(\beta) - n$, so that $q \geq s(\alpha) + s(\beta) - 2n$.

Consequently, $2r(\alpha) + 2r(\beta) + 2p(\alpha, \beta) - 4s(\alpha) - 4s(\beta) + 2n - 2z(\alpha) - 2z(\beta) = t(\alpha, \beta) - s(\alpha) - s(\beta) + 2n = t(\alpha) - z(\alpha) + t(\beta) - z(\beta) + q - s(\alpha) - s(\beta) + 2n \geq 0$. From this, $t(\alpha, \beta) \geq s(\alpha) + s(\beta) - 2n$ and $\sum_{i=1}^n (a_i^2 + b_i^2 + a_i b_i) \geq 2 \sum_{i=1}^n (a_i + b_i) - n + z_1 + z_2$. This inequality is a slight improvement of 4.2(i), since $4(s(\alpha) + s(\beta)) - 2n \geq 3(s(\alpha) + s(\beta))$. Of course, if $s(\alpha) = n = s(\beta)$ then we get $\sum_{i=1}^n (a_i^2 + b_i^2 + a_i b_i) \geq 3n + z_1 + z_2$.

Remark 4.6. In view of 4.4, put $t'(\alpha, \beta) = r(\alpha, \beta) - 2s(\alpha) - 2s(\beta) + n - z(\alpha) - z(\beta)$. Now, $t(\alpha, \beta) - 2t'(\alpha, \beta) = s(\alpha) + s(\beta) - 2n$. Thus $t(\alpha, \beta) \geq 2t'(\alpha, \beta)$ if and only if $s(\alpha + \beta) \geq 2n$. Notice also that $t'(\alpha, \beta) = r(\alpha) - 2s(\alpha) - z(\alpha) + r(\beta) - 2s(\beta) - z(\beta) + p(\alpha, \beta) + n = t(\alpha) - s(\alpha) + t(\beta) - s(\beta) + p(\alpha, \beta) + n = r(\alpha + \beta) - 2s(\alpha + \beta) - p(\alpha, \beta) + n - z(\alpha) - z(\beta) = r(\alpha - 1) - z(\alpha) + r(\beta - 1) - z(\beta) + p(\alpha, \beta) - n = r(\alpha + \beta - 2) + 2s(\alpha + \beta + 2) - p(\alpha, \beta) + n - z(\alpha) - z(\beta) = r(\alpha + \beta - 1) - p(\alpha, \beta) - z(\alpha) - z(\beta)$.

Using 1.4, we have $2t'(\alpha, \beta) = t(\alpha, \beta) - s(\alpha) - s(\beta) + 2n = t(\alpha) - z(\alpha) + t(\beta) - z(\beta) + r(\alpha + \beta - 2) + 2s(\alpha + \beta - 2) - s(\alpha) - s(\beta) + 2n = t(\alpha) - z(\alpha) + t(\beta) - z(\beta) + r(\alpha + \beta - 2) + s(\alpha + \beta - 2) \geq t(\alpha) - z(\alpha) + t(\beta) - z(\beta)$ by 1.6(ii).

Theorem 4.7. Let $\sum_{i=1}^n |a_i| \geq n$ and $\sum_{i=1}^n |b_i| \geq n$. Put $z_1 = z(\alpha)$ and $z_2 = z(\beta)$. Then:

- (i) $\sum_{i=1}^n (a_i^2 + b_i^2 + a_i b_i) \geq 2 \sum_{i=1}^n (a_i + b_i) - n + z_1 + z_2$.
- (ii) $\sum_{i=1}^n (a_i + b_i)^2 \geq \sum_{i=1}^n a_i b_i + 2 \sum_{i=1}^n (a_i + b_i) - n + z_1 + z_2$.
- (iii) The equalities hold if and only if the conditions from 4.2(iii) are satisfied.

Proof. Clearly, the inequalities are equivalent to $t'(\alpha, \beta) \geq 0$. By 1.12(i), $t(\alpha) - z(\alpha) = r(\alpha) - \sum_{i=1}^n |a_i| - 2z(\alpha) \geq r(\alpha) - \sum_{i=1}^n |a_i| - 2z(\alpha) \geq r(\alpha) - 3 \sum_{i=1}^n |a_i| + 2n - 2z(\alpha) \geq 0$. Similarly, $t(\beta) - z(\beta) \geq 0$ and $t'(\alpha, \beta) \geq 0$ by 4.6.

If $t'(\alpha, \beta) = 0$ then $t(\alpha) - z(\alpha) = 0 = t(\beta) - z(\beta)$, hence (see the proof of 4.2) $s(\alpha) = n = s(\beta)$ and from 4.6 follows that $t(\alpha, \beta) = 2t'(\alpha, \beta) + s(\alpha) + s(\beta) - 2n = 0$. The rest follows from 4.2. $\square$

Remark 4.8. It follows from 4.6 and 4.7 that $t(\alpha, \beta) \geq 0$, provided that $\sum_{i=1}^n |a_i| \geq n$, $\sum_{i=1}^n |b_i| \geq n$ and $s(\alpha) + s(\beta) \geq 2n$.

Proposition 4.9. Let $\alpha = (a_1, \dots, a_n)$ and $\beta = (b_1, \dots, b_n)$ be n -tuples of non-zero integers. Then:

- (i) $2 \sum_{i=1}^n (a_i^2 + b_i^2 + a_i b_i) \geq 3 \sum_{i=1}^n (a_i + b_i) + 2(z_1 + z_2)$.
- (ii) $2 \sum_{i=1}^n (a_i + b_i)^2 \geq 2 \sum_{i=1}^n a_i b_i + 3 \sum_{i=1}^n (a_i + b_i) + 2(z_1 + z_2)$.
- (iii) The equalities hold if and only if $a_1 = \dots = a_n = b_1 = \dots = b_n = 1$.

Proof. Use 4.1(i),(ii). $\square$

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Diameter estimate for generalized m -quasi-Einstein manifolds

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Abstract. In this paper, we study the diameter estimate for generalized m -quasi-Einstein manifold. Using the Bochner formula and the Hopf maximum principle, we get a gradient estimate for the potential function of the generalized quasi-Einstein manifold. Based on the gradient estimate, we get a diameter estimates for generalized m -quasi-Einstein manifolds under suitable conditions.

Keywords: Generalized m -quasi-Einstein manifold, diameter estimate, gradient estimate, the Hopf maximum principle.

MSC 2020: 53C25, 53C21.

1. Introduction

The diameter estimate is an attractive topic in Riemannian geometry. The upper diameter estimate is mainly focus on the extension of the Bonnet-Myers theorem. Since the diameter of a manifold is related to the volume of this manifold, then the lower diameter estimate can help us to estimate the isoperimetric constant $\inf_{\Omega} \frac{|\partial\Omega|^n}{|\Omega|^{n-1}}$, where Ω ranges over all open submanifolds of M , with compact closure in M , and smooth boundary. The lower diameter estimate can also help us to study the Fraenkel asymmetry $\inf\{\frac{|\Omega\Delta B|}{|\Omega|} : B \text{ is ball}, |B| = |\Omega|\}$, and the Hardy-Littlewood maximal operator $Mf(x) = \sup_{r>0} \frac{\int_{B(x,r)} |f(x)|dx}{|B(x,r)|}$.

In [13], [14], [16], [17], [19], the authors studied the upper diameter estimate and extended the Bonnet-Myers theorem to the Riemannian manifold with Bakry-Emery curvature bounded from below. Futaki and Sano [6] obtained a lower diameter bound for compact shrinking Ricci soliton. Futaki and Li [5] improved the diameter estimate in [6]. Wang [18] got a lower diameter bound for compact τ -quasi-Einstein manifold. Hu, Mao and Wang [8] obtained a lower diameter estimate for compact generalized τ -quasi-Einstein manifold under the strong conditions $\lambda = \lambda(f)$, $\lambda'(t) \geq 0$ and $[t^{\frac{2}{\alpha_0(n-2)}} \lambda(t)]' \geq 0$ (for more details, please see [8]). In [4], the author of this paper got a lower diameter estimate for a class of generalized quasi-Einstein manifolds.

The concept of generalized quasi-Einstein manifold was firstly introduced by Catino [3]. Barros and Ribeiro [2] introduced the concept of generalized

m -quasi-Einstein manifold and obtained some structural equations. Let (M, g) be an n -dimensional Riemannian manifold with $n \geq 3$. If there exist smooth functions f and λ on (M, g) such that the Ricci tensor satisfies

$$(1.1) \quad \text{Ric} + \nabla^2 f - \frac{1}{m} df \otimes df = \lambda g,$$

then (M, g) is called generalized m -quasi-Einstein manifold, where m is a positive integer. The function f in (1.1) is usually called potential function. Suppose that (M, g) is a generalized m -quasi-Einstein manifold with scalar curvature R . If ρ, C are real constants, $\lambda = \rho R + C$, then (M, g) is called (m, ρ) -quasi-Einstein manifold (see [10] for more details).

The classification of generalized m -quasi-Einstein manifolds has been extensively studied. Huang [10] discussed the classification of (m, ρ) -quasi-Einstein manifolds. Barros and Gomes [1] proved that compact generalized m -quasi-Einstein with constant scalar curvature must be isometric to a standard Euclidean sphere. Hu, Li and Zhai [9] proved that some generalized m -quasi-Einstein manifolds with constant Ricci curvature are Einstein manifolds. Jau-regui and Wylie [12] discussed the conformal diffeomorphisms of generalized m -quasi-Einstein manifolds. Neto [15] proved that a 4-dimensional generalized m -quasi-Einstein manifold with harmonic anti-self dual Weyl tensor is locally a warped product with 3-dimensional Einstein fibers provided an additional condition holds. Huang and Zeng [11] discussed the classification for generalized m -quasi-Einstein manifolds under the assumption that the Bach tensor is flat.

As far as we know, the study of the diameter estimate for generalized m -quasi-Einstein manifolds is very few up to now. Motivated by [18] and [8], we study the diameter estimate for generalized m -quasi-Einstein manifolds in this paper. Since λ is a function, the lower diameter estimate of generalized m -quasi-Einstein manifolds is much more difficult than that of τ -quasi-Einstein manifolds. To overcome this difficult, we need to use some new skills. For example, we need to construct some proper auxiliary functions and get proper estimate for these functions.

2. Gradient estimate and some useful Lemmas

To consider the lower diameter estimate for compact generalized m -quasi-Einstein manifolds, we need to get a gradient estimate for $h = e^{-\frac{f}{m}}$. Firstly, we introduce some useful Lemmas.

Lemma 2.1 ([2]). *Let (M, g) be an n -dimensional generalized m -quasi-Einstein manifold satisfying (1.1), then*

$$(2.1) \quad \frac{1}{2} \nabla R = \frac{m-1}{m} \text{Ric}(\nabla f) + \frac{1}{m} [R - (n-1)\lambda] \nabla f + (n-1) \nabla \lambda,$$

where R is the scalar curvature of (M, g) .

Lemma 2.2. *Suppose that (M, g) is an n -dimensional compact generalized m -quasi-Einstein manifold with scalar curvature R . Let $\lambda_{\max} = \max_{x \in M} \lambda(x)$ and $R_{\min} = \min_{x \in M} R(x)$, then $n\lambda_{\max} - R_{\min} \geq 0$. If $n\lambda_{\max} - R_{\min} = 0$, then (M, g) is trivial.*

Proof. Taking trace in both sides of (1.1), we get

$$(2.2) \quad R + \Delta f - \frac{1}{m} |\nabla f|^2 = n\lambda.$$

Let x_1 be the minimum points of $f(x)$ on M . By the Maximum principle, we have

$$(2.3) \quad \Delta f(x_1) \geq 0, \nabla f(x_1) = 0.$$

According to (2.2) and (2.3), we get

$$n\lambda_{\max} - R_{\min} \geq n\lambda(x_1) - R(x_1) \geq 0.$$

If $n\lambda_{\max} - R_{\min} = 0$, then $\Delta f - \frac{1}{m} |\nabla f|^2 \leq 0$. According to the Hopf maximum principle in [7], we conclude that f is a constant. Therefore, (M, g) is trivial. The proof of Lemma 2.2 is complete. $\square$

In the following, we always suppose that M is an n -dimensional compact generalized m -quasi-Einstein manifold with scalar curvature R , $R_{\min} = \min_{x \in M} R(x)$, $\lambda_{\max} = \max_{x \in M} \lambda(x)$, $h = e^{-\frac{f}{m}}$, $D = \{x \in M; \nabla h(x) \neq 0\}$ and

$$(2.4) \quad F(h) = |\nabla h|^2 + \frac{n}{m} \lambda_{\max} h^2 - \frac{1}{m} R_{\min} h^2.$$

Lemma 2.3. *Let $h = e^{-\frac{f}{m}}$. Then, the following equality*

$$(2.5) \quad \begin{aligned} \Delta |\nabla h|^2 &= 2|\nabla^2 h|^2 + (2-m) \frac{\nabla |\nabla h|^2 \nabla h}{h} + \left(\frac{4}{m} - 2\right) \lambda |\nabla h|^2 - \frac{2n}{m} \lambda |\nabla h|^2 \\ &+ \frac{2}{m} R |\nabla h|^2 - \frac{4}{m} [R - (n-1)\lambda] |\nabla h|^2 + \frac{2n-4}{m} h \nabla \lambda \nabla h \end{aligned}$$

holds on generalized m -quasi-Einstein manifolds.

Proof. Direct calculation shows that

$$(2.6) \quad \Delta h = -\frac{n}{m} \lambda h + \frac{1}{m} h R.$$

Therefore,

$$(2.7) \quad 2\nabla \Delta h \cdot \nabla h = -\frac{2n}{m} \lambda |\nabla h|^2 + \frac{2}{m} R |\nabla h|^2 - \frac{2n}{m} h \nabla \lambda \nabla h + \frac{2}{m} h \nabla R \nabla h.$$

Since $\nabla h = -\frac{1}{m}h\nabla f$, by (2.1) we have

$$(2.8) \quad \begin{aligned} \frac{2}{m}h\nabla R\nabla h &= \left(\frac{4}{m} - 4\right)\text{Ric}(\nabla h, \nabla h) - \frac{4}{m}[R - (n-1)\lambda]|\nabla h|^2 \\ &\quad + \frac{4(n-1)}{m}h\nabla\lambda\nabla h. \end{aligned}$$

According to (1.1), we get

$$(2.9) \quad \left(\frac{4}{m} - 2\right)\text{Ric}(\nabla h, \nabla h) = \left(\frac{4}{m} - 2\right)\lambda|\nabla h|^2 + (2-m)\frac{\nabla|\nabla h|^2\nabla h}{h}.$$

By the Bochner formula, we have

$$(2.10) \quad \Delta|\nabla h|^2 = 2|\nabla^2 h|^2 + 2\nabla\Delta h \cdot \nabla h + 2\text{Ric}(\nabla h, \nabla h).$$

Putting (2.7), (2.8) and (2.9) into (2.10), we conclude that (2.5) is true. The proof of Lemma 2.3 is complete. $\square$

Lemma 2.4. *If F is the function defined in (2.4), then there exists a smooth vector field X on M such that*

$$(2.11) \quad \begin{aligned} \Delta F &\geq \nabla F \cdot X + \frac{2n-4}{m}h\nabla\lambda\nabla h + \left(\frac{2n\lambda}{m} - \frac{2R}{m} - 2\lambda\right)|\nabla h|^2 \\ &\quad + (2-2m)\frac{|\nabla h|^2}{h}G(h) \end{aligned}$$

holds on $D = \{x \in M; \nabla h(x) \neq 0\}$.

Proof. Direct calculation shows that

$$(2.12) \quad \nabla F = \nabla|\nabla h|^2 + \frac{2n}{m}\lambda_{\max}h\nabla h - \frac{2}{m}R_{\min}h\nabla h.$$

Therefore, we have

$$(2.13) \quad \begin{aligned} \Delta F &= \Delta|\nabla h|^2 + \frac{2n}{m}\lambda_{\max}|\nabla h|^2 + \frac{2n}{m}\lambda_{\max}h\Delta h \\ &\quad - \frac{2}{m}R_{\min}|\nabla h|^2 - \frac{2}{m}R_{\min}h\Delta h. \end{aligned}$$

For the purpose of convenience, we let

$$(2.14) \quad G(h) = -\frac{n}{m}\lambda_{\max}h + \frac{1}{m}R_{\min}h.$$

Putting (2.5) into (2.13), we obtain

$$(2.15) \quad \begin{aligned} \Delta F &= 2|\nabla^2 h|^2 + (2-m)\frac{\nabla|\nabla h|^2\nabla h}{h} \\ &\quad + \left(\frac{4}{m} - 2\right)\lambda|\nabla h|^2 - \frac{2n}{m}\lambda|\nabla h|^2 + \frac{2}{m}R|\nabla h|^2 \\ &\quad - \frac{4}{m}[R - (n-1)\lambda]|\nabla h|^2 + \frac{2n-4}{m}h\nabla\lambda\nabla h - 2G(h)\frac{|\nabla h|^2}{h} - 2G(h)\Delta h. \end{aligned}$$

Consider a point $O \in D$. Rotating the orthonormal frame at O so that $|\nabla h|(O) = h_1(O) \neq 0$. According to (2.10) in [18], we have

$$(2.16) \quad \begin{aligned} & 2|\nabla^2 h|^2 + (2-m)\frac{\nabla|\nabla h|^2\nabla h}{h} \\ & \geq \frac{2n}{n-1}h_{11}^2 - \frac{4}{n-1}h_{11}\Delta h + \frac{2}{n-1}(\Delta h)^2 + (4-2m)\frac{h_{11}|\nabla h|^2}{h}. \end{aligned}$$

According to (2.12), we get

$$(2.17) \quad h_{11} = \frac{\nabla F \cdot \nabla h}{2|\nabla h|^2} + G(h).$$

Putting (2.17) into (2.16), we conclude that there exists a smooth vector field X on M such that

$$(2.18) \quad \begin{aligned} & 2|\nabla^2 h|^2 + (4-2m)\frac{\nabla|\nabla h|^2\nabla h}{h} \geq \nabla F \cdot X + \frac{2n}{n-1}[G(h)]^2 \\ & - \frac{4}{n-1}G(h)\Delta h + \frac{2}{n-1}(\Delta h)^2 + (4-2m)\frac{|\nabla h|^2}{h}G(h). \end{aligned}$$

From (2.6) and (2.14), we can easily find that $G(h) - \Delta h \leq 0$. According to Lemma 2.2 and (2.14), we have $G(h) \leq 0$. Therefore

$$(2.19) \quad nG(h) - \Delta h \leq G(h) - \Delta h \leq 0.$$

By (2.19), we obtain

$$(2.20) \quad \begin{aligned} & -2G(h)\Delta h + \frac{2n}{n-1}[G(h)]^2 - \frac{4}{n-1}G(h)\Delta h + \frac{2}{n-1}(\Delta h)^2 \\ & = \frac{2n}{n-1}[G(h)]^2 - \frac{2n}{n-1}G(h)\Delta h + \frac{2}{n-1}[L(h)]^2 - \frac{2}{n-1}G(h)\Delta h \\ & = \frac{2n}{n-1}G(h)[G(h) - \Delta h] - \frac{2}{n-1}\Delta h[G(h) - \Delta h] \\ & = \frac{2}{n-1}[G(h) - \Delta h][nG(h) - \Delta h] \geq 0. \end{aligned}$$

According to (2.15), (2.18) and (2.20), we conclude that (2.11) holds. The proof of Lemma 2.4 is complete. $\square$

3. Diameter estimate for compact (m, ρ) -quasi-Einstein manifold

In [9], Huang and Wei introduced the concept of (m, ρ) -quasi-Einstein manifold. Let (M, g) be an n -dimensional Riemannian manifold with scalar curvature R . If $n \geq 3$ and there exist constants ρ, C and $m > 0$ such that

$$(3.1) \quad \text{Ric} + \nabla^2 f - \frac{1}{m}df \otimes df = (\rho R + C)g,$$

then (M, g) is called (m, ρ) -quasi-Einstein manifold.

(m, ρ) -quasi-Einstein manifolds are closely related to the ρ -Einstein flows (see page 844 in [9] or page 270 in [10] for more details). The classification of compact (m, ρ) -quasi-Einstein manifolds was extensively studied by Huang and Wei in [10]. One of the main results in [10] can be stated as

Theorem A. *Let (M, g) be an n -dimensional compact (m, ρ) -quasi-Einstein manifold satisfying (3.1) and $n \geq 3$. Then:*

- (1) *If $\rho < \frac{1}{2(n-1)}$, then either $C > 0$ and $R \geq \frac{n(n-1)C}{m+(n-1)(1-n\rho)}$, or (M, g) is trivial.*
- (2) *If $\frac{1}{2(n-1)} \leq \rho < \frac{1}{n}$, then either $C < 0$ and $R \leq \frac{n(n-1)C}{m+(n-1)(1-n\rho)}$, or (M, g) is trivial.*
- (3) *If $\rho = \frac{1}{2(n-1)}$, then (M, g) is trivial.*
- (4) *If $\rho \geq \frac{1}{n}$ and $m > 1$, then (M, g) is trivial.*

According to Theorem A, we can easily get the following proposition.

Proposition 3.1. *Let (M, g) be an n -dimensional nontrivial compact (m, ρ) -quasi-Einstein manifold satisfying (3.1) and $n \geq 3$. If $\frac{1}{2(n-1)} \leq \rho < \frac{1}{n}$, then $C < 0$ and $R \leq 0$.*

In this section, we only consider the lower diameter estimate for n -dimensional nontrivial compact (m, ρ) -quasi-Einstein manifolds satisfying $\frac{1}{2(n-1)} \leq \rho < \frac{1}{n}$. Using Lemma 2.4, we get

Theorem 3.1. *Suppose that M is an n -dimensional nontrivial compact (m, ρ) -quasi-Einstein manifold satisfying (3.1) and $n \geq 3$. Let $\omega_f = \max_{x \in M} f(x) - \min_{x \in M} f(x)$. If $m > n$, $\frac{1}{2(n-1)} \leq \rho < \frac{1}{n}$ and $\nabla \lambda \nabla f \leq 0$, then the diameter of M satisfies*

$$(3.2) \quad \text{diam} M \geq \frac{\sqrt{m}}{\sqrt{n\rho R_{\max} + nC - R_{\min}}} \left(\frac{\pi}{2} - \arcsin e^{-\frac{\omega_f}{m}} \right).$$

Proof. Since $m > n \geq 3$, $G(h) \leq 0$, then

$$(3.3) \quad (2 - 2m) \frac{|\nabla h|^2}{h} G(h) \geq 0.$$

Since $\nabla \lambda \nabla f \leq 0$, by (2.11) and (3.3) we have

$$(3.4) \quad \Delta F \geq \nabla F \cdot X + \left(\frac{2n\lambda}{m} - \frac{2R}{m} - 2\lambda \right) |\nabla h|^2.$$

It follows from Proposition 3.1 and $\frac{1}{2(n-1)} \leq \rho < \frac{1}{n}$ that $\lambda = \rho R + C < 0$ and $R \leq 0$. Since $m > n$, by (3.4) we conclude that $\Delta F \geq \nabla F \cdot X$ holds on D .

Let x_0 be the maximum point of $F(h(x))$ on M . If $x_0 \in D$, then there exists a neighborhood $\mathcal{U}$ of x_0 so that $\mathcal{U} \subset D$. Moreover, x_0 is the maximum point of $F(h(x))$ on $\mathcal{U}$. It is obvious that $\Delta F \geq \nabla F \cdot X$ holds on $\mathcal{U}$. Therefore, by the Hopf maximum principle in [7] we conclude that F is constant on $\mathcal{U}$. Since $\Delta F(x_0) \leq 0$ and $\nabla F(x_0) = 0$, by (3.4) we conclude that $\nabla h(x_0) = 0$, which is a contradiction with $x_0 \in D$. Therefore, x_0 is not in D , which means that $\nabla h(x_0) = 0$. Thus, we arrive at

$$(3.5) \quad |\nabla h|^2(x) + \mathcal{G}(h(x)) \leq \mathcal{G}(h(x_0)),$$

where $\mathcal{G}(h) = \frac{n}{m}\lambda_{\max}h^2 - \frac{1}{m}R_{\min}h^2$.

According to (3.5), for all $x \in M$, we have

$$\mathcal{G}(h(x)) \leq |\nabla h|^2(x) + \mathcal{G}(h(x)) \leq \mathcal{G}(h(x_0)).$$

Therefore, x_0 is the maximum point of $\mathcal{G}(h(x))$, which means that x_0 is the maximum point of $h(x)$. Let x_1 and x_2 be the maximum and minimum points of $h(x)$ on M respectively. Without loss of generality, we can suppose that $x_1 = x_0$. Similar to the proof of Theorem 1.3 in [18], we choose a minimizing geodesic γ jointing x_1 and x_2 . Let $h_1 = h(x_1)$, $h_2 = h(x_2)$. Since M is a nontrivial generalized quasi-Einstein manifold, by Lemma 2.2 we conclude that $n\lambda_{\max} - R_{\min} > 0$. Therefore, according to (3.5) we have

$$(3.6) \quad \begin{aligned} \text{diam}M &\geq \int_{h_2}^{h_1} \frac{dh}{\sqrt{\mathcal{G}(h(x_1)) - \mathcal{G}(h(x))}} \\ &= \int_{h_2}^{h_1} \frac{dh}{\sqrt{\frac{1}{m}(n\lambda_{\max} - R_{\min})(h_1^2 - h^2)}} \\ &= \frac{1}{\sqrt{\frac{1}{m}(n\lambda_{\max} - R_{\min})}} \int_{h_2}^{h_1} \frac{d\sigma}{\sqrt{1 - \sigma^2}} \\ &= \frac{\sqrt{m}}{\sqrt{n\lambda_{\max} - R_{\min}}} \left(\frac{\pi}{2} - \arcsin e^{-\frac{w_f}{m}} \right). \end{aligned}$$

Since $\lambda_{\max} = \rho R_{\max} + C$, by (3.6) we conclude that (3.2) holds. The proof of Theorem 3.1 is complete. $\square$

4. Diameter estimate for generalized m -quasi-Einstein manifold

In this section, we study the lower diameter estimate for compact generalized m -quasi-Einstein manifold satisfying (1.1). This manifold is more general than (m, ρ) -quasi-Einstein manifold. Using Lemma 2.4, we get

Theorem 4.1. *Suppose that M is an n -dimensional nontrivial compact m -quasi-Einstein manifold satisfying (1.1) and $m \geq 1$. Let $\omega_f = \max_{x \in M} f(x) -$*

$\min_{x \in M} f(x)$. If $\nabla \lambda \nabla f \leq 0$ and $(n - m)\lambda > R$ on M , then the diameter of M satisfies

$$(4.1) \quad \text{diam} M \geq \frac{\sqrt{m}}{\sqrt{n\lambda_{\max} - R_{\min}}} \left(\frac{\pi}{2} - \arcsin e^{-\frac{w_f}{m}} \right).$$

Proof. Since $m \geq 1$, $\nabla \lambda \nabla f \leq 0$ and $G(h) \leq 0$, by (2.11) we conclude that (3.4) holds. It follows from $(n - m)\lambda > R$ that $\frac{2n\lambda}{m} - \frac{2R}{m} - 2\lambda > 0$. Therefore, we have $\Delta F \geq \nabla F \cdot X$. Similar to the proof of Theorem 3.1, we conclude that (3.5) is true. Since M is a nontrivial generalized quasi-Einstein manifold, then $n\lambda_{\max} - R_{\min} > 0$. Let x_1 and x_2 be the maximum and minimum points of $h(x)$ on M , respectively. Set $h_1 = h(x_1)$, $h_2 = h(x_2)$. Similar to the proof of Theorem 3.1, we conclude that (4.1) is true. The proof of Theorem 4.1 is complete. $\square$

Example 4.1. Suppose that $u(x)$ is a smooth function on n -dimensional Riemannian manifold (M, g) with $n \geq 3$, $\tilde{\lambda}$ is a real increasing smooth positive function and $\tilde{f}$ is a real decreasing smooth function. Then, $\lambda = \tilde{\lambda} \circ u$ and $f = \tilde{f} \circ u$ are smooth functions on (M, g) . Obviously, $\nabla \lambda \nabla f \leq 0$. Choose a proper metric g , the Ricci tensor can satisfy (1.1) and the scalar curvature can satisfy $R \leq 0$. If $n > m$, then $(n - m)\lambda > R$. Therefore, the diameter of M satisfies (4.1).

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Relation between ultra matroid and linear decomposition

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Abstract. The notion of linear tangle was initially introduced as an obstacle to mixed searching number and Linear decomposition, both of which have significant connections to game theory and graph theory. In this concise paper, we introduce the concept of ultra matroid on a connectivity system, which combines the matroid concept defined on a set with the idea of an ultrafilter. Finally, we establish the equivalence between linear tangle and ultra matroid under certain conditions.

Keywords: ultra matroid, linear tangle, linear decomposition, matroid, ultrafilter.

MSC 2020: 05B35, 05C65.

1. Introduction

The width parameter is a well-established concept in the field of graph theory. Linear-width, which is one of its variations, has been extensively researched in graph theory as well as in other fields such as network theory, topology, geometry, and combinatorial mathematics. Scholars have conducted extensive research on linear-width and other width parameters (e.g., [1], [2], [3], [4], [12], [13], [14], [17], [18], [19], [20], [22], [23] [24] [25]).

Linear tangle, which obstructs Linear decomposition, was initially introduced in the literature as an obstacle to mixed searching number, a concept deeply related to both graph theory and game theory[1]. Given the significance of studying linear-width as mentioned earlier, studying linear tangles is also crucial.

Filter is widely used in various areas, such as set theory, combinatorial mathematics, and topology. When a filter is ultra, it is well-known that it includes either the set A or its complement. Numerous studies have been conducted in this area(e.g., [5], [6], [7]).

Matroids are concepts that find applications in optimization theory, combinatorial mathematics, topology, algebra, graph algorithm, game theory, geometry, network theory, artificial intelligence, fuzzy theory, and coding theory. Due to their broad range of applications, matroids have gained considerable attention, and several studies have been conducted on this subject(e.g., [8], [9], [10], [11]). Matroids are characterized by the well-known axiom of augmentation property or independent set exchange property.

In this brief paper, we define a connectivity system as a pair (X, f) , where X is a finite set, and f is a symmetric submodular function. We then introduce the new concept of an ultra matroid on the connectivity system (X, f) , which combines the ultrafilter idea with the matroid concept defined on the set X . Finally, we establish the equivalence between linear tangle and ultra matroid under certain conditions. The concept of matroids, which may seem unrelated to graph theory at first glance, can be shown to be very interesting and novel when combined with the properties of ultrafilters. This relationship adds a significant level of relevance, making it a fascinating area of study.

2. Definition and preparation

This section provides mathematical definitions of each concept.

2.1 Symmetric submodular function and connectivity system

The definition of a symmetric submodular function is given below.

Definition 2.1. Let X be a finite set. A function $f : X \rightarrow \mathbb{N}$ is called symmetric submodular if it satisfies the following conditions:

- For all $A \subseteq X$, $f(A) = f(X \setminus A)$.
- For all $A, B \subseteq X$, $f(A) + f(B) \geq f(A \cap B) + f(A \cup B)$.

Lemma 2.2 ([13]). A symmetric submodular function f satisfies

1. For all $A \subseteq X$, $f(A) \geq f(\emptyset) = f(X)$,
2. For all $A, B \subseteq X$, $f(A) + f(B) \geq f(A \setminus B) + f(B \setminus A)$.

In this short paper, a pair (X, f) of a finite set X and a symmetric submodular function f is called a connectivity system. And we use the notation f for a symmetric submodular function, a finite set X , and a natural number k .

Example 2.3. Consider X as the finite set $\{1, 2, 3\}$. For a symmetric submodular function $f : X \rightarrow \mathbb{N}$, with $A = \{1, 2\}$ and $B = \{2, 3\}$, we find that $f(A) + f(B) \geq f(A \cap B) + f(A \cup B)$, demonstrating the concept of symmetric submodularity.

2.2 Matroid on boolean algebra $(X, \cup, \cap)$

The definition of Matroid on boolean algebra $(X, \cup, \cap)$ is shown below.

Definition 2.4. In Boolean algebra $(X, \cup, \cap)$, the set family $M \subseteq 2^X$ is called a matroid if the following axioms hold true:

(MB1) $\emptyset \in M$.

(MB2) If $A \in M$ and $B \subseteq A$ then $B \in M$.

(MB3) If $A, B \in M$ and $|A| < |B|$ then there exists $e \in B \setminus A$ such that $A \cup \{e\} \in M$.

Axiom (MB2) is commonly known as the hereditary property, while axiom (MB3) is often referred to as the augmentation property or independent set exchange property. In addition, the combination of axiom (MB1) and axiom (MB2) defines a combinatorial concept called an independence system, which is also known as an abstract simplicial complex.

2.3 Matroid on a connectivity system (X, f)

In this manuscript, we present a definition of matroids on connectivity systems (X, f) that mimics the concept of matroids on Boolean algebra. By imposing restrictions on submodular functions for each axiom, we ensure a coherent and natural definition.

Definition 2.5. Let X be a finite set and f be a symmetric submodular function. In a connectivity system (X, f) , the set family $M \subseteq 2^X$ is called a matroid of order $k + 1$ on (X, f) if the following axioms hold true:

(M0) For every $A \in M$, $f(A) \leq k$,

(M1) $\emptyset \in M$,

(M2) If $A \in M$, $B \subseteq A$, and $f(B) \leq k$ then $B \in M$,

(M3) If $A, B \in M$, $|A| < |B|$, $e \in X$, $f(\{e\}) \leq k$, and $f(A \cup \{e\}) \leq k$, then there exists $e \in B \setminus A$ such that $A \cup \{e\} \in M$.

Furthermore, let us define an order $k + 1$ matroid $M \subseteq 2^X$ on a connectivity system (X, f) as an Ultra Matroid if it satisfies the following Axiom (M4):

(M4) For any subset $A \subseteq X$, if $f(A) \leq k$, then either $A \in M$ or $X \setminus A \in M$.

Axiom (M4) imitates the idea of Ultrafilters, which states that "either a set A or its complement belongs to the (Ultra)filter." Therefore, ultra matroids can be interpreted as matroids that include only one side.

2.4 Linear tangle: deep relation to linear width and Linear decomposition

The definition of a linear tangle on a connectivity system (X, f) is given below. Linear tangle, which obstructs Linear decomposition, was first introduced in the literature [1].

Definition 2.6 ([1]). Let X be a finite set and f be a symmetric submodular function. A linear tangle of order $k + 1$ on a connectivity system (X, f) is a family $L \subseteq 2^X$, satisfying the following axioms:

- (L1) $\emptyset \in L$,
- (L2) For $A \subseteq X$, if $f(A) \leq k$ then either $A \in M$ or $X \setminus A \in M$,
- (L3) If $A, B \in L$, $e \in X$, and $f(\{e\}) \leq k$, then $A \cup B \cup \{e\} \neq X$.

We propose to use the augmentation property of matroids by considering the following axiom $L3'$ instead of $L3$. We refer to a linear tangle of order $k+1$ on a connectivity system (X, f) that satisfies the axioms $L1$, $L2$, and $L3'$ as a restricted linear tangle of order $k+1$.

- (L3') If $A, B \in L$, $|A| \neq |B|$, $e \in X$, and $f(\{e\}) \leq k$, then $A \cup B \cup \{e\} \neq X$.

3. Main result: equivalence between restricted linear tangle and ultra matroid

We show equivalence between a restricted linear tangle on a connectivity system (X, f) and an ultra matroid on a connectivity system (X, f) under the assumption that $f(\{e\}) \leq k$ for every $e \in X$.

Theorem 3.1. Let X be a finite set and f be a symmetric submodular function. Under the assumption that $f(\{e\}) \leq k$ for every $e \in X$, the family $W \subseteq 2^X$ being an order $k+1$ restricted linear tangle and $W \subseteq 2^X$ being an order $k+1$ ultra matroid are equivalent necessary and sufficient conditions.

Proof. Suppose that W is an ultra matroid of order $k+1$ on the connectivity system (X, f) . We need to show that W is a restricted linear tangle of order $k+1$ on the connectivity system (X, f) .

First, we show that W satisfies the axioms (L1) and (L2) of a restricted linear tangle. Since W is an ultra matroid of order $k+1$, it satisfies Axiom (M4), which implies that for any subset $A \subseteq X$, if $f(A) \leq k$, then either $A \in W$ or $X \setminus A \in W$. In particular, for the empty set $\emptyset$, we have $f(\emptyset) = f(A) \leq k$, and hence $\emptyset \in W$. Moreover, if $A \subseteq X$ and $f(A) \leq k$, then either $A \in W$ or $X \setminus A \in W$ by Axiom (M4), which shows that W satisfies Axiom (L2) as well.

Next, we prove that W satisfies the axiom (L3') of a restricted linear tangle. Let $A, B \in W$ and $e \in X$ be such that $f(\{e\}) \leq k$. We need to show that $A \cup B \cup \{e\} \neq X$. Suppose, on the contrary, that $A \cup B \cup \{e\} = X$. We choose a triple $(A, B, \{e\})$ that minimizes $|A \cap B|$ among such triples. First, we claim that $A \cap B = \emptyset$.

Since $2k \geq f(A) + f(B) \geq f(A \setminus B) + f(B \setminus A)$, at least one of $f(A \setminus B)$ or $f(B \setminus A)$ is at most k . Without loss of generality, assume that $f(A \setminus B)$ is at most k . Hence, by axiom (M2), $A \setminus B \in M$. If $A \cap B \neq \emptyset$, then we have $|A \cap B| > |(A \setminus B) \cap B|$, which contradicts the choice of the triple. Thus, we have shown that $A \cap B = \emptyset$.

Next, we claim that $e \notin A$ and $e \notin B$. Suppose, on the contrary, that $e \in A$ or $e \in B$. If $e \in A$, then $A \cup B = X$, which implies that $X \setminus A = B \in M$, but

this contradicts the axiom (M4). Similarly, we know that $e \notin B$ holds. Now, we know that the triple $(A, B, \{e\})$ consists of a partition of X . Hence, we have $f(A \cup \{e\}) = f(X \setminus B) = f(B) \leq k$.

If $|A| < |B|$, by axiom (M3) there exists $e' \in B \setminus A$ such that $A \cup \{e'\} \in M$. This contradicts that there not exists $e' \in B \setminus A$ such that $A \cup \{e'\} \in M$ because the triple $(A, B, \{e\})$ consists of a partition of X . If $|B| < |A|$, by axiom (M3) there exists $e'' \in A \setminus B$ such that $B \cup \{e''\} \in M$. This contradicts that the triple $(A, B, \{e\})$ consists of a partition of X . So W satisfies the axiom (L3').

Therefore, we have shown that W satisfies all the axioms (L1), (L2), and (L3') of a restricted linear tangle. Hence, W is a restricted linear tangle of order $k + 1$ on the connectivity system (X, f) .

Assume that $f(\{e\}) \leq k$ for every $e \in X$ and let W be a restricted linear tangle of order $k + 1$ on the connectivity system (X, f) . We will show that W satisfies the axioms of an ultra matroid of order $k + 1$ on the connectivity system (X, f) .

(M0) For any $A \in W$, we have by definition of a restricted linear tangle that $f(A) \leq k$, hence (M0) is satisfied.

(M1) Since $\emptyset$ is an element of W by definition of a restricted linear tangle, it is also an element of W , hence the axiom (M1) is satisfied.

(M2) Let $A \in W$, $B \subseteq A$, and $f(B) \leq k$. We need to show that $B \in W$. Suppose, to the contrary, that there exist subsets A and B such that $A \subseteq B$, $f(A) \leq k$, $B \in W$, and $A \notin W$. Then, we have $X \setminus A \in W$ by the axiom (L2), and for any $e \in X$, $(X \setminus A) \cup \{e\} \cup B = X$ holds, but this contradicts the axiom (L3') of a restricted linear tangle. So the axiom (M2) is satisfied.

(M3) Let $A, B \in W$, $|A| < |B|$, $e \in X$, $f(\{e\}) \leq k$, and $f(A \cup \{e\}) \leq k$. We need to show that there exists $e \in B \setminus A$ such that $A \cup \{e\} \in W$. Suppose, to the contrary, that there exists a subset A , $B \in W$ and an element $e \in B \setminus A$ such that $f(A \cup \{e\}) \leq k$ and $A \cup \{e\} \notin W$ hold. Then, we have $(X \setminus A) \cap (X \setminus \{e\}) \in W$ by the axiom (L2). Hence $A, (X \setminus A) \cap (X \setminus \{e\}) \in W$ and $A \cup (X \setminus A) \cap (X \setminus \{e\}) \cup \{e\} = X$ hold. However, this contradicts the axiom (L3'). So the axiom (M3) is satisfied.

Therefore, W satisfies all the axioms of an ultra matroid of order $k + 1$ on the connectivity system (X, f) . This proof is completed. $\square$

4. Conclusion

In this brief manuscript, we have established the equivalence between linear tangles and ultra matroids, subject to certain conditions. Going forward, we intend to further explore the connections between not only matroids, but also antimatroids and greedoids.

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Conflict of Interest Statement

The author declares no conflicts of interest.

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On a Diophantine equation involving Mersenne number

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Abstract. This research work focused on studying an exponential Diophantine equation involving Mersenne numbers. Specifically, it sought to find the nonnegative integer solutions (M_n, x, y, z) of the Diophantine equation $M_n^x + (M_n + 1)^y = z^2$. To obtain the solutions, a combination of modular arithmetic method and factoring method, together with some other results like Mihailescu's theorem, was utilized. Results of the Diophantine analysis revealed that aside from $(3, 2, 2, 5)$ and $(7, 0, 1, 3)$, the equation has infinitely many solutions of the form $(2^{2k} - 1, 1, 0, 2^k)$ where k is a positive integer.

Keywords: exponential Diophantine equation, integer solution, Mersenne number, number theory.

MSC 2020: 11D61, 11D72

1. Introduction

Number theory is regarded as one of the purest areas of mathematics studied because of the intellectual fascination with properties of integers. Recently, it has been an area that also has significant applications to subjects such as cryptography [6].

Number theory researchers are very interested in the search for integer solutions to equations. These equations are termed Diophantine equations. Diophantine equations have a wide range of applications in coordinate geometry, cryptography, trigonometry, and applied algebra. It is also extremely useful in determining the solutions to many puzzle problems [1].

One of the most explored Diophantine equations is the equation of the form

$$(1) \quad a^x + b^y = z^2.$$

Equation (1) has been studied in connection with Mersenne primes by numerous researchers. Some of them focused on the case in which one of a and b in equation (1) is a Mersenne prime. Particularly, some considered the first three Mersenne prime numbers 3, 7, and 31. For instance, Asthana and Singh [2] showed that $3^x + 13^y = z^2$ has four nonnegative integer solutions which are $(1, 0, 2)$, $(1, 1, 4)$, $(3, 2, 14)$ and $(5, 1, 6)$. Also, Rabago [8] proved that the triples $(4, 1, 10)$ and $(1, 0, 2)$ are the only two nonnegative integer solutions to the Diophantine equation $3^x + 19^y = z^2$, and that the triples $(2, 1, 10)$ and $(1, 0, 2)$ are the only nonnegative integer solutions to $3^x + 91^y = z^2$. Furthermore, Sroysang

[10] showed that $(x, y, z) = (0, 1, 3)$ is the unique solution of $7^x + 8^y = z^2$. However, in another work of Sroysang [9], he showed that the equation $31^x + 32^y = z^2$ has no nonnegative integer solution.

In recent years, Gayo and Bacani worked on Diophantine equations that involve Mersenne primes. In 2021, they solved the nonnegative integer solutions (M_p, M_q, x, y, z) of the Diophantine equation $M_p^x + (M_q + 1)^y = z^2$, where M_p and M_q are Mersenne primes using elementary methods in number theory [3]. In 2022, they also showed that the quadruples $(3, 1, 0, 2)$, $(3, 0, 3, 3)$ and $(3, 2, 4, 5)$ are the only nonnegative integer solutions (M, x, y, z) of the exponential Diophantine equation $M^x + (M - 1)^y = z^2$, where M is a Mersenne prime [4]. In 2023, the same authors determined the nonnegative integer solutions (p_M, a, b, c) of the Diophantine equation $(p_M)^a - (p_M + 1)^b = z^2$ and its more generalized form $(p_M)^a - (p_M + 1)^b = c^{2n}$, where p_M is a Mersenne prime number [5]. A variation of the studies of the two aforementioned authors was made for this research study. Instead of Mersenne prime, a larger set of numbers to which Mersenne primes belong is being considered; that is the set of Mersenne numbers. Hence, the present study focused on the Diophantine equation $M_n^x + (M_n + 1)^y = z^2$ where M_n is a Mersenne number.

2. Preliminaries

This part presents the concepts, definitions, lemmas, and theorems necessary to solve the Diophantine equation $M_n^x + (M_n + 1)^y = z^2$.

2.1 Mersenne numbers

Definition 2.1. *Mersenne numbers are numbers of the form $2^n - 1$, where n is a positive integer.*

Two properties needed to solve the Diophantine equation under consideration are given by the next two lemmas.

Lemma 2.1. *The only Mersenne number congruent to 1 (mod 4) is $M_1 = 1$. All others are congruent to 3 (mod 4). In symbols,*

$$M_n \equiv \begin{cases} 1 \pmod{4}, & \text{if } n = 1, \\ 3 \pmod{4}, & \text{if } n > 1 \end{cases}.$$

Proof. Note that $M_n = 2^n - 1$ for positive integer n . If $n = 1$, then $M_n = 1$. This is obviously congruent to 1 modulo 4. If $n > 1$, then $2^n \equiv 0 \pmod{4}$. This means that $2^n - 1 \equiv -1 \pmod{4}$. Thus, $M_n \equiv -1 \pmod{4}$, or equivalently $M_n \equiv 3 \pmod{4}$. $\square$

Lemma 2.2. *Positive even and odd powers of Mersenne numbers greater than 1 are congruent to 1 and 3 modulo 4, respectively.*

Proof. From Lemma 2.1, a Mersenne number greater than 1 is congruent 3 (mod 4). Raising it to a positive integer x leads to

$$M_n^x \equiv 3^x \pmod{4}.$$

If x is even, let $x = 2k, k \in \mathbb{N}$. Then, $M_n^x \equiv 3^{2k} \pmod{4}$ which is equivalent to $M_n^x \equiv 9^k \pmod{4}$. This is also the same as $M_n^x \equiv 1^k \pmod{4}$, or equivalently $M_n^x \equiv 1 \pmod{4}$.

If x is odd, let $x = 2k + 1, k \in \mathbb{N}_0$. Then, $M_n^x \equiv 3^{2k+1} \pmod{4}$ which is equivalent to $M_n^x \equiv 3 \cdot 9^k \pmod{4}$. This is also the same as $M_n^x \equiv 3 \cdot 1^k \pmod{4}$, or equivalently $M_n^x \equiv 3 \pmod{4}$. $\square$

2.2 Square of an integer

The right-hand side of the equation $M_n^x + (M_n + 1)^y = z^2$ is a perfect square. Since we will be utilizing modular arithmetic method as one of the methods, it is just right to find modular properties of the square of an integer.

Lemma 2.3. *The square of an integer is either congruent to 0 or 1 modulo 4. Specifically, the squares of an even integer and odd integer are congruent to 0 modulo 4 and 1 modulo 4, respectively.*

Proof. Let z be an integer. Then, by Division Algorithm, z can be written in the form $2n$ or $2n + 1$. If z is even or of the form $2n$, then $z^2 = 4n^2$, which is obviously congruent to 0 (mod 4). On the other hand, if z is odd or of the form $2n + 1$, then $z^2 = 4n^2 + 4n + 1$, which is congruent to 1 (mod 4). $\square$

2.3 The Mihailescu's theorem

In 1844, Charles Catalan conjectured that if $\min\{a, b, x, y\} > 1$, then the only solution to the Diophantine equation $a^x - b^y = 1$ is the quadruple $(a, b, x, y) = (3, 2, 2, 3)$, which was famously known as the Catalan conjecture. Since then, a number of mathematicians have attempted to prove the conjecture, but it was only in 2002 that the conjecture was finally proven by Mihailescu [7]. Hence, the conjecture was renamed Mihailescu's Theorem. This theorem led to the solvability of many Diophantine equations and will also be used to prove our results. Mihailescu's Theorem is stated below.

Theorem 2.1 ([7], Mihailescu's Theorem). *The triple $(3, 2, 2, 3)$ is the unique solution (a, b, x, y) for the Diophantine equation $a^x - b^y = 1$, where a, b, x and y are integers with $\min\{a, b, x, y\} > 1$.*

3. Main results

This section presents the solutions of the Diophantine equation $M_n + (M_n + 1)^y = z^2$. The cases when $x = 0$ or $y = 0$ will be solved first.

3.1 The equation $1 + (M_n + 1)^y = z^2$

The case when $x = 0$ is solved here. This is when the equation is $1 + (M_n + 1)^y = z^2$. The solution of this equation is given in the following lemma.

Lemma 3.1. *The triple $(7, 1, 3)$ is the unique solution (M_n, y, z) for the Diophantine equation $1 + (M_n + 1)^y = z^2$, where $M_n > 1$ is a Mersenne number and y, z are a nonnegative integer.*

Proof. Let $M_n > 1$ be a Mersenne number, and y, z be nonnegative integers such that $1 + (M_n + 1)^y = z^2$. Then, there exists a positive integer n such that $M_n = 2^n - 1, n > 1$. This results to

$$1 + 2^{ny} = z^2.$$

Three cases for the value of y are considered.

If $y = 0$, then $z^2 = 2$, which has no integer solution.

If $y = 1$, then $1 + 2^n = z^2$, which can be expressed as $z^2 - 1 = 2^n$. This can be factored as $(z - 1)(z + 1) = 2^n$. There exist nonnegative integers u and v such that $u + v = n$ and $u > v$. So, the equation now becomes $(z - 1)(z + 1) = 2^{u+v}$. Since $(z - 1) < (z + 1)$ and $u > v$, it follows that

$$\begin{cases} z + 1 = 2^u, \\ z - 1 = 2^v \end{cases}.$$

Combining the two equations in the system gives $2 = 2^u - 2^v$. Because the greatest common factor of 2^u and 2^v is 2^v , this can be factored out, yielding to $2 = 2^v(2^{u-v} - 1)$. Note that 2^v and $2^{u-v} - 1$ are relatively prime. Equating powers of 2 and non-powers of 2 results to

$$\begin{cases} 2^v = 2, \\ 2^{u-v} - 1 = 1 \end{cases}.$$

The first equation in the system gives the value $v = 1$. So, the value of z is $z = 2^v + 1 = 2^1 + 1 = 3$. Moreover, the second equation in the same system becomes $2^{u-1} = 2$, which has the solution $u = 2$. Computing for the value of n based from $u = 2$ and $v = 1$ results to $n = 3$. This further results to $M_n = 2^3 - 1 = 7$. Thus, $(M_n, x, z) = (7, 1, 3)$ is a solution.

If $y > 1$, then by Mihăilescu's Theorem, the Diophantine equation $1 + 2^{ny} = z^2$ can have a solution if $ny = 3$ and $z = 3$. Since $y > 1$, it follows that $y = 3$ and $n = 1$. This implies that $M_n = 1$, which contradicts the assumption that $M_n > 1$. $\square$

3.2 The equation $M_n^x + 1 = z^2$

The case when $y = 0$ is solved here. This is when the equation is $M_n^x + 1 = z^2$. The solution of this equation is given in the following lemma.

Lemma 3.2. *The solutions (M_n, x, z) of the Diophantine equation $M_n^x + 1 = z^2$, where $M_n > 1$ is a Mersenne number and x, z are nonnegative integers are of the form $(2^{2k} - 1, 1, 2^k)$ where k is a positive integer.*

Proof. Let $M_n > 1$ be a Mersenne number, and x, z be nonnegative integers such that $M_n^x + 1 = z^2$. Then, there exists a positive integer n such that $M_n = 2^n - 1, n > 1$. This results to

$$(2^n - 1)^x + 1 = z^2.$$

Three cases for the value of x are considered. These are $x = 0, x = 1$, and $x > 1$.

When $x = 0$, the equation becomes $z^2 = 2$. This quadratic equation has no integer solution.

For $x = 1$, the equation becomes $z^2 = 2^n$. Let $z = 2^k$, where $k \in \mathbb{N}$. Then, $2^{2k} = 2^n$, and thus, $n = 2k$. It follows that $M_n = 2^{2k} - 1$. Hence, $(2^{2k} - 1, 1, 2^k)$ are solutions of the Diophantine equation.

If $x > 1$, then by Mihăilescu's Theorem, $z = 3, x = 3$, and $2^n - 1 = 2$. However, the last equation does not have an integer solution. $\square$

3.3 The Diophantine equation $M_n^x + (M_n + 1)^y = z^2$

In this part, the Diophantine equation $M_n^x + (M_n + 1)^y = z^2$ is solved. Its solutions are given in the following theorem.

Theorem 3.1. *The solutions (M_n, x, y, z) of the Diophantine equation $M_n^x + (M_n + 1)^y = z^2$ with nonnegative integers x, y, z and Mersenne number $M_n > 1$ are $(7, 0, 1, 3)$, $(3, 2, 2, 5)$ and $(2^{2k} - 1, 1, 0, 2^k)$, where k is a positive integer.*

Proof. Let $M_n > 1$ be a Mersenne number, and x, y, z be nonnegative integers such that $M_n^x + (M_n + 1)^y = z^2$. If $x = 0$, then $1 + (M_n + 1)^y = z^2$. By Lemma 3.1, this equation has the only solution $(7, 1, 3)$. Thus, $(M_n, x, y, z) = (7, 0, 1, 3)$ is a solution to $M_n^x + (M_n + 1)^y = z^2$. On the other hand, if $y = 0$, then $M_n^x + 1 = z^2$. By virtue of Lemma 3.2, this equation has solutions of the form $(2^{2k} - 1, 1, 2^k)$ where k is a positive integer. Hence, $(M_n, x, y, z) \in \{(2^k - 1, 1, 0, 2^k)\}$ is a solution set of $M_n^x + (M_n + 1)^y = z^2$.

Now, suppose $x, y \geq 0$. Since $M_n = 2^n - 1 > 1$, it is clear that $n > 1$. From this, it is easy to observe that $M_n \equiv -1 \pmod{4}$. So, $M_n + 1 \equiv 0 \pmod{4}$. It follows then by Lemma 2.2 that for every positive integer x ,

$$M_n^x \equiv \begin{cases} 1 \pmod{4}, & \text{if } x \text{ is even,} \\ 3 \pmod{4}, & \text{if } x \text{ is odd} \end{cases}.$$

Also, it is easy to see that for every positive integer y , $(M_n + 1)^y \equiv 0 \pmod{4}$. This means that

$$M_n^x + (M_n + 1)^y \equiv \begin{cases} 1 \pmod{4}, & \text{if } x \text{ is even,} \\ 3 \pmod{4}, & \text{if } x \text{ is odd} \end{cases}.$$

Because $z^2 \equiv 1 \pmod{4}$, it must be true that x is even. There exists a positive integer k such that $x = 2k$. So, the equation now becomes $M_n^{2k} + (M_n + 1)^y = z^2$. This can be expressed as $z^2 - M_n^{2k} = (M_n + 1)^y$. Since the left-hand side of the equation is a difference of two squares, it can be factored into $(z - M_n^k)(z + M_n^k) = (M_n + 1)^y$. For the reason that $M_n = 2^n - 1$, the equation can be written as $(z - M_n^k)(z + M_n^k) = 2^{ny}$. There exist nonnegative integers u and v such that $u + v = ny$ and $u > v$. So, the equation now becomes $(z - M_n^k)(z + M_n^k) = 2^{u+v}$. Since $(z - M_n^k) < (z + M_n^k)$ and $u > v$, it follows that

$$\begin{cases} z + M_n^k = 2^u, \\ z - M_n^k = 2^v \end{cases}.$$

Subtracting the two equations in the system results to $2M_n^k = 2^u - 2^v$. Note that the greatest common factor of 2^u and 2^v is 2^v . This can be factored out, resulting to $2M_n^k = 2^v(2^{u-v} - 1)$. Note that 2^v and $2^{u-v} - 1$ are relatively prime. Equating powers of 2 and non-powers of 2 results to

$$\begin{cases} 2 = 2^v, \\ M_n^k = 2^{u-v} - 1 \end{cases}.$$

The first equation implies that $v = 1$, which implies that $u > 1$, $u + 1 = ny$ and $z = M_n^k + 2$. Because $v = 1$ and $M_n = 2^n - 1$, the second equation becomes

$$(2^n - 1)^k = 2^{u-1} - 1.$$

Three cases for the value of k and u are considered.

If $u > 2$ and $k > 1$, by Mihailescu's Theorem, the equation has no integer solution.

If $u = 2$, then $(2^n - 1)^k = 1$, which implies that $k = 0$. This is a contradiction to k being positive.

If $k = 1$, then $x = 2$, $z = M_n + 2$ and $2^{u-1} - 1 = 2^n - 1$. The latter equation is equivalent to $2^{u-1} = 2^n$, which implies that $u - 1 = n$. It is known that $u + 1 = ny$. The equations $u - 1 = n$ and $u + 1 = ny$, when combined, becomes $2 = ny - n = n(y - 1)$. Since $n > 1$, it can be concluded that $n = 2$ and $y = 2$. It follows that $M_n = 3$. This further implies that $z = 5$. Hence, $(M_n, x, y, z) = (3, 2, 2, 5)$. $\square$

4. Concluding remarks

In this research study, we solve the exponential Diophantine equation $M_n^x + (M_n + 1)^y = z^2$ in the set of nonnegative integers. Results show that when $y = 0$, the equation has infinitely many solutions of the form $(2^{2k} - 1, 1, 0, 2^k)$ where k is a positive integer. Also, when $x = 0$, it has only one solution which is $(7, 0, 1, 3)$. Moreover, when x and y are positive integers, the unique solution is the quadruple $(3, 2, 2, 5)$.

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A compression of finite topological spaces based on homomorphism

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Abstract. The task of compressing large topologies into more manageable, smaller ones while leveraging the power of homomorphisms is a pivotal concern. Therefore, this paper focuses on the compression of topologies through the utilization of homomorphisms. In this paper, we devise a subbase-consistent function tailored specifically for topological spaces. To preserve certain properties of the original topological spaces, the concept of topological homomorphism is introduced. Additionally, we delve into the subbase reduction under homomorphism and subsequently construct the corresponding discernibility matrix. An experiment is conducted to validate the feasibility and effectiveness of our approach.

Keywords: discernibility matrix, homomorphism, subbase reduction, topology.

MSC 2020: 22A26, 57N17.

1. Introduction

As a fundamental mathematical discipline, topology theory has many practical applications, such as computer science [15, 19, 20], computational geometry [16], graph theory [4, 23], structural biology [12], chemistry [21], physics [7], data mining [1, 9] and rough sets [8, 13, 14, 22, 27, 28]. For a finite topological space, the base and subbase contribute to improve the efficiency of data analysis.

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Communication between data spaces is a very important topic in the field of artificial intelligence [18, 6, 3, 17]. The primary motivation for examining the communication between data spaces lies in the pursuit of identifying a relatively smaller database, a quest that often leads to techniques like data fusion and compression. These can be explained as a mapping between data spaces in mathematics. Grzymala-Busse first investigated the methodology of homomorphism between information systems [5], and he then considered the conditions that make the data space be selective under endomorphism [6]. A homomorphism serves as a valuable tool for aggregating sets of objects, attributes and descriptors from the original space, and it can be perceived as a distinctive form of communication between various data spaces. According to [3, 17], communications can be elucidated as the process of translating the information embedded in one granular realm into another, bridging the gap between disparate granular worlds. Li et al. [10] investigated homomorphism between information systems. Wang et al. [24, 25, 26] delved into the study of generalized information systems, fuzzy information systems and decision systems, employing the concept of homomorphism as a key analytical tool. Therefore, we will utilize the concept of homomorphism to delve into the comprehensive study of topological spaces.

The discernibility matrix holds a crucial role in the realm of knowledge reductions. Each element within this matrix represents the ensemble of attributes that uniquely differentiate a pair of corresponding objects. In simpler terms, it comprises the set of attributes on which the two objects exhibit contrasting values. Li and Zhang [11] first introduced discernibility matrix into topological space. Each entry within this matrix comprises a collection of topologies, uniquely identifying a pair of points. Furthermore, they delved into the subject of subbase reduction in families of subbases, emphasizing its role in preserving the topological rough membership function.

In this paper, our attention lies in exploring the to the reduction of subbases derived from homomorphism. The remainder of this paper is organized as follows. In Section 2, we provide a review the relevant concepts of topology theory, serving as a foundation for the subsequent discussions. In Section 3, the definition of subbase consistent function in topological space is proposed and its properties are delved into. In Section 4, a subbase discernibility matrix is constructed based on homomorphism to find reduction of subbases. Simultaneously, judgement theorem of reducts is put forward. we introduce a graph approach that facilitates the identification of a reduct of f -induced topological space in Section 5. Section 6 concludes this paper with a summary and discussion of our further work.

2. Preliminaries

For each $i=1, \dots, n$, let τ_i be a topology on X with subbase γ_i , $\mathbf{A}=\{\gamma_1, \gamma_2, \dots, \gamma_n\}$, and $\gamma_{\mathbf{A}} = \bigwedge_{i=1}^n \gamma_i = \{\bigcap_{i=1}^n K_i : K_i \in \gamma_i, i = 1, \dots, n\}$. Then, $\gamma_{\mathbf{A}}$ is a subbase for a topology on X , and $\bigcap(\gamma_{\mathbf{A}})_x = \bigcap_{i=1}^n (\bigcap(\gamma_i)_x)$, for $x \in X$. For $\mathbf{B} \subseteq \mathbf{A}$ and

$\gamma \in \mathbf{A}$, denote $(x)_{\mathbf{B}} = \bigcap (\gamma_{\mathbf{B}})_x$, simply denote $(x)_{\{\gamma\}}$ by $(x)_{\gamma}$. For any $x, y \in X$, we have the following properties[11]:

1. If $y \in (x)_{\mathbf{B}}$, then $(y)_{\mathbf{B}} \subseteq (x)_{\mathbf{B}}$.
2. If $y \notin (x)_{\mathbf{B}}$, there is $\gamma \in \mathbf{B}$ satisfying $y \notin (x)_{\gamma}$.
3. If $(y)_{\mathbf{B}} = (x)_{\mathbf{B}}$, then $(y)_{\gamma} = (x)_{\gamma}$, for each $\gamma \in \mathbf{B}$.

Let γ be a subbase for topology τ on X . $K \in \gamma$ is deemed reducible of γ if the set $\gamma - \{K\}$ is a subbase for τ . In essence, a reducible element K satisfies the condition that there exists $\gamma' \subseteq \gamma - \{K\}$ such that, for any $x \in K$, we have $x \in \bigcap \gamma' \subseteq K$. If $\alpha \subseteq \gamma$ is a subbase for τ and any proper subset of α is not a subbase for τ , then α is said to be a reduct of γ , which is sometimes also called a minimal subbase for τ . Alternatively, if $\alpha \subseteq \gamma$ is a reduct of γ , then $(x)_{\alpha} = (x)_{\gamma}$, for any $x \in X$. As is well-known, a subbase uniquely determines a topology on X and induces a basis for that topology. If an element $K \in \gamma$ is the union of some elements from γ , then K is called a union reducible element of γ ; Conversely, if $K \in \gamma$ is the intersection of some members of γ , then K is called an intersection reducible element of γ . To get a reduct of γ , one can simply eliminate the reducible elements from γ in a prescribed order. (see [11])

Let $\mathbf{A}$ denote a family of subbases representing distinct topologies on X . $\mathbf{B} \subseteq \mathbf{A}$ is called a subbase consistent set if $(x)_{\mathbf{A}} = (x)_{\mathbf{B}}$, for every $x \in X$. If $\mathbf{B}$ is a subbase consistent set and no proper subset of $\mathbf{B}$ satisfies this consistency property, then $\mathbf{B}$ is called a reduct of $\mathbf{A}$. To facilitate the identification of all reducts of $\mathbf{A}$, we introduce the concept of a subbase discernibility matrix, which is defined as follows:[11]:

$$d((x)_{\mathbf{A}}, (y)_{\mathbf{A}}) = \begin{cases} \{\gamma \in \mathbf{A} : y \notin (x)_{\gamma}\}, & y \notin (x)_{\mathbf{A}}, \\ \emptyset, & \text{otherwise.} \end{cases}$$

is called subbase discernibility set for $(x)_{\mathbf{A}}$ and $(y)_{\mathbf{A}}$ and $D = (d((x)_{\mathbf{A}}, (y)_{\mathbf{A}})) : x, y \in X$ is called subbase discernibility matrix.

3. Subbase by consistent function

Definition 3.1. Let X be a finite topological space with subbase $\gamma = \{K_1, \dots, K_t\}$. A surjective map $f : X \rightarrow Y$ satisfies the following property: If for any $x, z \in X$, the condition $f(x) = f(z)$ implies that $(x)_{\gamma} = (z)_{\gamma}$, then f is called a subbase consistent function with respect to γ ; Furthermore, given a subset $U \subseteq X$, if, for any $x, z \in X$ such that $f(x) = f(z)$ and $x \in U$, it follows that $z \in U$, then f is called a subbase consistent function with respect to U .

Generally, when f is a subbase consistent function, for any $x \in X$, there exists $y \in Y$ satisfying $y = f(x')$, for each $x' \in (x)_{\gamma} \setminus \bigcup_{z \in (x)_{\gamma} \wedge (z)_{\gamma} \subset (x)_{\gamma}} (z)_{\gamma}$. For brevity, we denote this as $y = f((x)_{\gamma} \setminus \bigcup_{z \in (x)_{\gamma} \wedge (z)_{\gamma} \subset (x)_{\gamma}} (z)_{\gamma})$. From Definition 3.1, it is easy to see that f is a subbase consistent function with respect

to the subbase γ if and only if f is a subbase consistent function with respect to each $K \in \gamma$. A subbase consistent function is a specific type of mapping between two topological spaces, and this class of functions has the potential to compress a topological space into a relatively smaller one.

Example 1. Let $X = \{1, 2, \dots, 9\}$ be a finite topological space with subbase $\gamma = \{\{1, 2, 4, 8\}, \{3, 4, 6, 7, 8, 9\}, \{3, 5, 7\}\}$. Then, $(1)_\gamma = (2)_\gamma = \{1, 2, 4, 8\}$, $(3)_\gamma = (7)_\gamma = \{3, 7\}$, $(4)_\gamma = (8)_\gamma = \{4, 8\}$, $(5)_\gamma = \{3, 5, 7\}$, $(6)_\gamma = (9)_\gamma = \{3, 4, 6, 7, 8, 9\}$. By Definition 3.1, we define a subbase consistent function $f : X \rightarrow Y$ as follows:

$$y_1 = f(1) = f(2), y_2 = f(3) = f(7), y_3 = f(4) = f(8), y_4 = f(6) = f(9), y_5 = f(5).$$

Then, $f(\gamma) = \{\{y_1, y_3\}, \{y_2, y_3, y_4\}, \{y_2, y_5\}\}$.

In the following, we will delve into the examination of several crucial properties exhibited by the subbase consistent function.

Theorem 1. *Let γ be a subbase for X , f is a subbase consistent function with respect to γ iff $f^{-1}(f(K)) = K$, for any $K \in \gamma$.*

Proof. $\Rightarrow$ Since the inclusion $K \subseteq f^{-1}(f(K))$ is always valid, we only have to prove the reverse inclusion. For any $x \in f^{-1}(f(K))$, it follows that $f(x) \in f(K)$. Consequently, there exists $z \in K$ such that $f(x) = f(z)$. By the definition of $(z)_\gamma$, we know that $(z)_\gamma \subseteq K$. Since f is subbase consistent with respect to γ , it follows that $(x)_\gamma = (z)_\gamma$. This, in turn, implies that $(x)_\gamma \subseteq K$, thereby establishing that $x \in K$.

$\Leftarrow$ To arrive at a contradiction, suppose there exist $x, z \in X$ with $f(x) = f(z)$ satisfying $(x)_\gamma \neq (z)_\gamma$, then $z \in f^{-1}(f((x)_\gamma))$. Without loss of generality, suppose that there exists $\Delta \subseteq \{1, 2, \dots, t\}$ such that $(x)_\gamma = \bigcap_{j \in \Delta} K_j$. Let $K_j = (x)_\gamma \cup (K_j \setminus (x)_\gamma)$, then $K_j = f^{-1}(f(K_j)) = f^{-1}(f((x)_\gamma \cup (K_j \setminus (x)_\gamma))) = f^{-1}(f((x)_\gamma)) \cup f^{-1}(f(K_j \setminus (x)_\gamma))$, hence $f^{-1}(f((x)_\gamma)) \subseteq K_j$. This implies $z \in f^{-1}(f((x)_\gamma)) \subseteq \bigcap_{j \in \Delta} K_j = (x)_\gamma$, that is, $(z)_\gamma \subseteq (x)_\gamma$. Similarly, we can deduce $(x)_\gamma \subseteq (z)_\gamma$. This forces that $(x)_\gamma = (z)_\gamma$, a contradiction. Therefore, f is a subbase consistent function with respect to γ . $\square$

Theorem 2. *Let γ be a subbase, for X . For any $K \in \gamma$, if f is subbase consistent with respect to γ , then $f((x)_\gamma) = \bigcap \{f(K) : K \in \gamma \wedge x \in K\}$.*

Proof. We only need to show that $f(K_i \cap K_j) = f(K_i) \cap f(K_j)$, for any $K_i, K_j \in \gamma$. It is straightforward to verify that $f(K_i \cap K_j) \subseteq f(K_i) \cap f(K_j)$. We now proceed to prove the converse, starting with the case where $K_i \cap K_j = \emptyset$. For contradiction, suppose $f(K_i) \cap f(K_j) = \emptyset$. Let y be an element in this intersection, i.e., $y \in f(K_i) \cap f(K_j)$. Then, there are $x \in K_i$ and $z \in K_j$ such that $f(x) = y = f(z)$, which implies $z \in K_i$. Thus, $z \in K_i \cap K_j$, a contradiction. Next, we consider the case where $K_i \cap K_j \neq \emptyset$. To show

$f(K_i) \cap f(K_j) \subseteq f(K_i \cap K_j)$, let $y \in f(K_i) \cap f(K_j)$. There exist $x \in K_i$ and $z \in K_j$ such that $f(x) = f(z) = y$. By the properties of f and the fact that $K_i \cap K_j$ is non-empty, we can infer that $\{x, z\} \subseteq K_i \cap K_j$. This, in turn, implies $y \in f(K_i \cap K_j)$. $\square$

Corollary 3.1. *For any $K_i, K_j \in \gamma$, if f is subbase consistent with respect to K_i or K_j , then $f(K_i \cap K_j) = f(K_i) \cap f(K_j)$.*

Corollary 3.2. *f is a subbase consistent function with respect to γ iff*

$$f^{-1}(f((x)_\gamma)) = (x)_\gamma,$$

for each $x \in X$.

Lemma 3.1. *Let (X, τ) be a finite topological space endowed with subbase $\gamma = \{K_1, \dots, K_t\}$. Consider a function $f : X \rightarrow Y$ that is subbase-consistent with respect to γ . We define the set $f(\gamma) = \{f(K_i) : i \leq t\}$ and $f(\tau) = \{f(K) : K \in \tau\}$. Then, $f(\gamma)$ constitutes a subbase, for $f(\tau)$ on Y .*

Corollary 3.3. (1) $f((x)_\gamma) = (f(x))_{f(\gamma)}$, for any $x \in X$; (2) $f((\gamma)) = (f(\gamma))$.

We may pose a question regarding the interplay between the reducible elements of γ and $f(\gamma)$. Specifically, can the reduct of γ be converted into the reduct of $f(\gamma)$? We shall proceed with our investigation to address these questions.

Theorem 3. *Let γ be a subbase, for X and f be a subbase consistent function. If $K \in \gamma$ is a reducible element of γ , then $f(K)$ is a reducible element of $f(\gamma)$. If $\alpha \subseteq \gamma$ is a reduct of γ , then $f(\alpha)$ is a reduct of $f(\gamma)$.*

Proof. We will initially establish that, for any reducible element K of γ , its image $f(K)$ under the function f is also a reducible element of $f(\gamma)$. Specifically, if K is a union-reducible element of γ , there exists a collection $\{K_i : i \in \Delta\}$ with $\Delta \subset \{1, 2, \dots, t\}$ and $K \notin \{K_i : i \in \Delta\}$ such that $K = \bigcup_{i \in \Delta} K_i$. On the other hand, if K is an intersection-reducible element, then exists a distinct collection $\{K_j : j \in \Delta'\}$ with $\Delta' \subset \{1, 2, \dots, t\}$ and $K \notin \{K_j : j \in \Delta'\}$ such that $K = \bigcap_{j \in \Delta'} K_j$. It follows from Corollary 3.5 that $f(K) = \bigcap_{j \in \Delta'} f(K_j)$.

Next, we aim to demonstrate that if $\alpha \subseteq \gamma$ is a reduct of γ , then $f(\alpha)$ is also a reduct of $f(\gamma)$. Obviously, $(x)_\alpha = (x)_\gamma$, for any $x \in X$. And there is $y \in Y$ such that $f(x) = y$. Then, $(y)_{f(\alpha)} = \bigcap \{f(K) : f(x) \in f(K) \wedge f(K) \in f(\alpha)\} = f((x)_\alpha) = f((x)_\gamma) = (f(x))_{f(\gamma)} = (y)_{f(\gamma)}$. This shows that $f(\alpha)$ forms a subbase, for $f(\tau)$ on Y . As a matter of fact, any proper subset of α is not a subbase, for τ , indicating that none of the members of α are reducible elements of α . Consequently, the members of $f(\alpha)$ are also not reducible elements of $f(\alpha)$. Therefore, we conclude that $f(\alpha)$ is indeed a reduct of $f(\gamma)$. $\square$

Theorem 4. *Let (X, τ) be a finite topological space with subbase $\gamma = \{K_1, \dots, K_t\}$ and $f : X \rightarrow Y$ is a subbase consistent function with respect to γ . Then, the following statements are equivalent:*

1. K is a reducible element of γ .
2. For $y \in f(K)$, there is $\gamma' \subseteq \gamma \setminus \{K\}$ such that $y \in \bigcap f(\gamma') \subseteq f(K)$.

Proof. (1) $\Rightarrow$ (2) If K is a reducible element of γ , then $(y)_{f(\gamma) \setminus \{f(K)\}} = (y)_{f(\gamma)}$, for any $y \in Y$.

(2) $\Rightarrow$ (1) It is easy to verify that $f(\gamma) \setminus \{f(K)\}$ constitutes a covering of Y . For any $y \in Y$, $(y)_{f(\gamma) \setminus \{f(K)\}} = (y)_{f(\gamma)}$ if $y \notin f(K)$; However, if $y \in f(K)$, there exists a subset $\gamma' \subseteq \gamma \setminus \{K\}$ such that $y \in \bigcap f(\gamma') \subseteq f(K)$. Consequently, $(y)_{f(\gamma)} = (y)_{f(\gamma) \setminus \{f(K)\}} \cap f(K) \supseteq (y)_{f(\gamma) \setminus \{f(K)\}} \cap (\bigcap f(\gamma')) \supseteq (y)_{f(\gamma) \setminus \{f(K)\}}$. This equality ensures that $(y)_{f(\gamma)} = (y)_{f(\gamma) \setminus \{f(K)\}}$, demonstrating that $f(\gamma) \setminus \{f(K)\}$ forms a subbase for $f(\tau)$ on Y , then $f(K)$ is a reducible element of $f(\gamma)$. Next, we shall demonstrate that K is a reducible element of γ . There exists a subset $\Delta \subseteq \{1, 2, \dots, t\}$ satisfying $f(K) = \bigcap_{i \in \Delta} f(K_i)$ or $f(K) = \bigcup_{i \in \Delta} f(K_i)$. Since f is subbase consistent with respect to γ , we have $f^{-1}(f(K_i)) = K_i$, for any $K_i \in \gamma$ by Theorem 3.1. Hence, $K = f^{-1}(f(K)) = f^{-1}(\bigcap_{i \in \Delta} f(K_i)) = \bigcap_{i \in \Delta} K_i$ if $f(K)$ is an intersection reducible element. Conversely, $K = \bigcup_{i \in \Delta} K_i$ if $f(K)$ is a union reducible element. These deductions indicate that K is indeed a reducible element of γ . $\square$

From the preceding discussion, we observe that the subbase consistent function possesses numerous advantageous properties that lend themselves to defining the concepts of homomorphisms in the following.

Definition 3.2. Let (X, τ) be a finite topological space equipped with subbase $\gamma = \{K_1, \dots, K_t\}$. If a function $f : X \rightarrow Y$ is a subbase consistent with respect to γ , then f is designated as a homomorphism from the triplet (X, τ, γ) to the f -induced topological space $(Y, f(\tau), f(\gamma))$ of (X, τ, γ) , and $f(\gamma)$ is called an f -induced subbase.

Utilizing the aforementioned results, the identification of a many-to-one homomorphism allows for the compression of a space into a relatively smaller size. Addressing the challenges posed by mathematical models and theories in data spaces, this compression process simplifies and abstracts the data space while preserving the topological relationships between its objects.

4. Reduction of a family of topologies based on homomorphism

In this section, we define $\mathbf{A} = \{\gamma_1, \gamma_2, \dots, \gamma_n\}$ and $\Gamma = \{\tau_1, \dots, \tau_n\}$, where each τ_i represents a topology on the finite point set X with subbase γ_i . We refer to the family $\mathbf{A}$ of topologies on X paired with Γ as $(X, \Gamma, \mathbf{A})$ and abbreviate it as FTS for brevity.

Definition 4.1. Let $f : X \rightarrow Y$ be a surjective mapping. If f is subbase consistent with respect to each $\gamma_i \in \mathbf{A}$, then f is called a homomorphism on $(X, \Gamma, \mathbf{A})$.

Denote $f(\mathbf{A}) = \{f(\gamma_1), \dots, f(\gamma_n)\}$ and $f(\Gamma) = \{f(\tau_1), \dots, f(\tau_n)\}$. Obviously, for any $y \in Y$, there exists $x \in X$ such that

$$y = f((x)_{\mathbf{A}} \setminus \bigcup_{z \in (x)_{\mathbf{A}} \wedge (z)_{\mathbf{A}} \subset (x)_{\mathbf{A}}} (z)_{\mathbf{A}}).$$

Definition 4.2. Let $f : X \rightarrow Y$ be a homomorphism on $(X, \Gamma, \mathbf{A})$, then $(Y, f(\Gamma), f(\mathbf{A}))$ is called the f -induced FTS of $(X, \Gamma, \mathbf{A})$.

Example 2. Let $(X, \Gamma, \mathbf{A})$ be FTS with $X = \{1, 2, \dots, 20\}$ and $\mathbf{A} = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$, where

$$\begin{aligned} \gamma_1 &= \{\{2, 6, 7, 8, 10, 15, 16, 18\}, \{4, 6, 8, 9, 10, 13, 14, 18, 19\}, \{1, 3, 5, 11, 12, 17, 20\}\}, \\ \gamma_2 &= \{\{1, 2, 6, 7, 8, 10, 11, 12, 15, 16, 18, 20\}, \{2, 7, 15, 16\}, \{3, 4, 5, 9, 13, 14, 17, 19\}\}, \\ \gamma_3 &= \{\{1, 11, 12, 20\}, \{2, 3, 5, 7, 15, 16, 17\}, \{3, 4, 5, 9, 13, 14, 17, 19\}, \{2, 3, 5, 6, 7, 8, \\ &\quad 10, 15, 16, 17, 18\}\}, \\ \gamma_4 &= \{\{2, 3, 5, 7, 15, 16, 17\}, \{3, 4, 5, 9, 13, 14, 17, 19\}, \{1, 3, 5, 6, 8, 10, 11, 12, 17, \\ &\quad 18, 20\}\}. \end{aligned}$$

Hence,

$$\begin{aligned} (1)_{\mathbf{A}} &= (11)_{\mathbf{A}} = (12)_{\mathbf{A}} = (20)_{\mathbf{A}} = \{1, 11, 12, 20\}, \\ (2)_{\mathbf{A}} &= (7)_{\mathbf{A}} = (15)_{\mathbf{A}} = (16)_{\mathbf{A}} = \{2, 7, 15, 16\}, \\ (3)_{\mathbf{A}} &= (5)_{\mathbf{A}} = (17)_{\mathbf{A}} = \{3, 5, 17\}, \\ (4)_{\mathbf{A}} &= (9)_{\mathbf{A}} = (13)_{\mathbf{A}} = (14)_{\mathbf{A}} = \{4, 9, 13, 14\}, \\ (6)_{\mathbf{A}} &= (8)_{\mathbf{A}} = (10)_{\mathbf{A}} = (18)_{\mathbf{A}} = \{6, 8, 10, 18\}. \end{aligned}$$

Next, we define a subbase consistent function $f : X \rightarrow Y$ as follows:

$$f((1)_{\mathbf{A}}) = y_1, f((2)_{\mathbf{A}}) = y_2, f((3)_{\mathbf{A}}) = y_3, f((4)_{\mathbf{A}}) = y_4, f((6)_{\mathbf{A}}) = y_5.$$

Then, $f(\mathbf{A}) = \{f(\gamma_1), f(\gamma_2), f(\gamma_3), f(\gamma_4)\}$, where

$$\begin{aligned} f(\gamma_1) &= \{\{y_2, y_5\}, \{y_4, y_5\}, \{y_1, y_3\}\}, f(\gamma_3) = \{\{y_1\}, \{y_2, y_3\}, \{y_3, y_4\}, \{y_2, y_3, y_5\}\}, \\ f(\gamma_2) &= \{\{y_1, y_2, y_5\}, \{y_2\}, \{y_3, y_4\}\}, f(\gamma_4) = \{\{y_2, y_3\}, \{y_3, y_4\}, \{y_1, y_3, y_5\}\}. \end{aligned}$$

The proposed definitions introduce a methodology for compressing a family of topologies into a smaller-sized representation. Subsequently, we shall delve into the equivalent formulations of subbase consistent functions.

Lemma 4.1. Let $f : X \rightarrow Y$ be a surjective mapping and let γ_1, γ_2 be subbases for different topologies on X . If f is subbase consistent function with respect to γ_1 and γ_2 respectively, then

1. f is a subbase consistent with respect to $\gamma_1 \cup \gamma_2$.
2. f is a subbase consistent with respect to $\gamma_1 \cap \gamma_2$.

Proof. We only prove (1). We observe that, for any $x \in X$ $(x)_{\gamma_1 \cup \gamma_2} = \bigcap \{K \in \gamma_1 \cup \gamma_2 : x \in K\}$. This can be expressed as $(x)_{\gamma_1 \cup \gamma_2} = (x)_{\gamma_1} \cap (x)_{\gamma_2}$. Since f is subbase consistent with respect to γ_1 and γ_2 respectively, it follows that $(x)_{\gamma_i} = (z)_{\gamma_i}$ ($i = 1, 2$), for any $x, z \in X$ such that $f(x) = f(z)$. Therefore, $(x)_{\gamma_1} \cap (x)_{\gamma_2} = (z)_{\gamma_1} \cap (z)_{\gamma_2}$, which implies $(x)_{\gamma_1 \cup \gamma_2} = (z)_{\gamma_1 \cup \gamma_2}$. Consequently, f is a subbase consistent with respect to $\gamma_1 \cup \gamma_2$. $\square$

Lemma 4.2. *Let $f : X \rightarrow Y$ and γ_1, γ_2 be subbases for topologies on X . If f is a subbase consistent with respect to γ_1 or γ_2 , then $f((x)_{\gamma_1} \cap (x)_{\gamma_2}) = f((x)_{\gamma_1}) \cap f((x)_{\gamma_2})$, for any $x \in X$.*

Proof. We only need to prove $f((x)_{\gamma_1}) \cap f((x)_{\gamma_2}) \subseteq f((x)_{\gamma_1} \cap (x)_{\gamma_2})$. Without loss of generality, we assume that the function f is subbase consistent with respect to γ_1 . Let $y \in f((x)_{\gamma_1}) \cap f((x)_{\gamma_2})$. This means that there exist $z_1 \in (x)_{\gamma_1}$ and $z_2 \in (x)_{\gamma_2}$ such that $f(z_1) = f(z_2) = y$, which implies that $(z_1)_{\gamma_1} \subseteq (x)_{\gamma_1}$ and $(z_1)_{\gamma_1} = (z_2)_{\gamma_1}$. Therefore, $(z_2)_{\gamma_1} \subseteq (x)_{\gamma_1}$. It follows that $z_2 \in (x)_{\gamma_1} \cap (x)_{\gamma_2}$. This implies $y \in f((x)_{\gamma_1} \cap (x)_{\gamma_2})$. Consequently, $f((x)_{\gamma_1}) \cap f((x)_{\gamma_2}) \subseteq f((x)_{\gamma_1} \cap (x)_{\gamma_2})$. $\square$

Thus, if f is homomorphism on $(X, \Gamma, \mathbf{A})$, then $f((x)_{\mathbf{B}}) = \bigcap_{\gamma \in \mathbf{B}} f((x)_{\gamma})$ with each $x \in X$, for any $\mathbf{B} \subseteq \mathbf{A}$. Moreover, $f(\bigcap_{\gamma \in \mathbf{A}} \gamma) = \bigcap_{\gamma \in \mathbf{A}} f(\gamma)$.

Theorem 5. *If $f : X \rightarrow Y$ is a homomorphism on $(X, \Gamma, \mathbf{A})$, then $\mathbf{B} \subseteq \mathbf{A}$ is a subbase consistent set of $\mathbf{A}$ if and only if $f(\mathbf{B})$ is a subbase consistent set of $f(\mathbf{A})$.*

Proof. $\Rightarrow$ For each $x \in X$, $f((x)_{\mathbf{B}}) = f((x)_{\mathbf{A}})$ since $(x)_{\mathbf{B}} = (x)_{\mathbf{A}}$. That is, $(y)_{f(\mathbf{B})} = (y)_{f(\mathbf{A})}$, for any $y \in Y$, hence $f(\mathbf{B})$ is a subbase consistent of $f(\mathbf{A})$.

$\Leftarrow$ For any $y \in Y$, $f^{-1}((y)_{f(\mathbf{A})}) = \bigcap_{\gamma \in \mathbf{A}} f^{-1}((y)_{f(\gamma)}) = \bigcap_{\gamma \in \mathbf{A}} \bigcap_{y \in f(K) \in f(\gamma)} K$. By applying this to the definition of $(x)_{\mathbf{A}}$, for any $x \in X$, we obtain $(x)_{\mathbf{A}} = \bigcap_{y \in f((x)_{\mathbf{A}})} (\bigcap_{\gamma \in \mathbf{A}} \bigcap_{y \in f(K) \in f(\gamma)} K) = \bigcap_{y \in f((x)_{\mathbf{A}})} f^{-1}((y)_{f(\mathbf{A})})$.

Analogously, $(x)_{\mathbf{B}} = \bigcap_{y \in f((x)_{\mathbf{B}})} f^{-1}((y)_{f(\mathbf{B})})$, for each $x \in X$. Since $(y)_{f(\mathbf{B})} = (y)_{f(\mathbf{A})}$ holds, for each $y \in Y$, it follows that $(x)_{\mathbf{B}} = (x)_{\mathbf{A}}$, for all $x \in X$. This demonstrates that $\mathbf{B}$ is a subbase consistent set of $\mathbf{A}$. $\square$

Theorem 6. *Let $f : X \rightarrow Y$ be a homomorphism on $(X, \Gamma, \mathbf{A})$ and $(Y, f(\Gamma), f(\mathbf{A}))$ be the f -induced FTS. For any $\mathbf{B} \subseteq \mathbf{A}$, $\mathbf{B}$ is a reduct of $\mathbf{A}$ if and only if $f(\mathbf{B})$ is a reduct of $f(\mathbf{A})$.*

Proof. $\Rightarrow$ Assume that there is a $\gamma \in \mathbf{B}$ such that $(y)_{f(\mathbf{B}) \setminus \{f(\gamma)\}} = (y)_{f(\mathbf{A})}$, for every $y \in Y$, i.e., $(y)_{f(\mathbf{B} \setminus \{\gamma\})} = (y)_{f(\mathbf{A})}$. This implies that $f(\mathbf{B} \setminus \{\gamma\})$ is a subbase consistent of $f(\mathbf{A})$. Invoking Theorem 4.6, we can see that $\mathbf{B} \setminus \{\gamma\}$ is a subbase consistent with $\mathbf{A}$. However, this conclusion contradicts the fact that $\mathbf{B}$ is a reduct of $\mathbf{A}$. This implies $f(\mathbf{B})$ is a reduct of $f(\mathbf{A})$.

$\Leftarrow$ Conversely, if $f(\mathbf{B})$ is a subbase consistent set of $f(\mathbf{A})$, then $\mathbf{B}$ must be also a subbase consistent set of $\mathbf{A}$. To illustrate this, suppose there exists $\gamma \in \mathbf{B}$

such that $(x)_{\mathbf{B} \setminus \{\gamma\}} = (x)_{\mathbf{A}}$, for any $x \in X$. Then, $\mathbf{B} \setminus \{\gamma\}$ is a subbase consistent set of $\mathbf{A}$. That is, $f(\mathbf{B} \setminus \{\gamma\})$ is a subbase consistent set of $f(\mathbf{A})$, this creates a contradiction. $\square$

Definition 4.3. Let $f : X \rightarrow Y$ be a homomorphism on $(X, \Gamma, \mathbf{A})$ and $(Y, f(\Gamma), f(\mathbf{A}))$ be the f -induced FTS. For any $y_i, y_j \in Y$, the subbase discernibility set related to y_i and y_j is defined by

$$d(y_i, y_j) = \begin{cases} \{f(\gamma) : y_j \notin (y_i)_{f(\gamma)}, \gamma \in \mathbf{A}\}, & y_j \notin (y_i)_{f(\mathbf{A})}, \\ \emptyset, & \text{otherwise.} \end{cases}$$

And $D = (d_{ij})_{|Y| \times |Y|}$ is called a subbase discernibility matrix, where $d_{ij} = d(y_i, y_j)$.

Corollary 4.1. For any $x_i, x_j \in X$, $d((x_i)_{\mathbf{A}}, (x_j)_{\mathbf{A}})$ is the subbase discernibility set for $(x_i)_{\mathbf{A}}$ and $(x_j)_{\mathbf{A}}$, then

1. $d((x_i)_{\mathbf{A}}, (x_j)_{\mathbf{A}}) = \emptyset$ if and only if $d(f(x_i), f(x_j)) = \emptyset$.
2. $\gamma \in d((x_i)_{\mathbf{A}}, (x_j)_{\mathbf{A}})$ if and only if $f(\gamma) \in d(f(x_i), f(x_j))$.

Theorem 7. For $\mathbf{B} \subseteq \mathbf{A}$, $f(\mathbf{B})$ is a subbase consistent set iff for $d(y_i, y_j) \neq \emptyset$, $f(\mathbf{B}) \cap d(y_i, y_j) \neq \emptyset$.

Proof. $\Rightarrow$. Suppose $f(\mathbf{B})$ is subbase consistent set for $\mathbf{B} \subseteq \mathbf{A}$. If $d(y_i, y_j) \neq \emptyset$, then $y_j \notin (y_i)_{f(\mathbf{A})}$. If there exists no $\gamma \in \mathbf{B}$ such that $y_j \notin (y_i)_{f(\gamma)}$, that is, $y_j \in (y_i)_{f(\gamma)}$ for any $\gamma \in \mathbf{B}$, then $y_j \in (y_i)_{f(\mathbf{B})} = (y_i)_{f(\mathbf{A})}$, a contradiction. Thus, there is $\gamma \in \mathbf{B}$ satisfying $y_j \notin (y_i)_{f(\gamma)}$, then $f(\gamma) \in f(\mathbf{B}) \cap d(y_i, y_j) \neq \emptyset$.

$\Leftarrow$. For each $y_i \in Y$, it holds that $y_i \in (y_i)_{f(\mathbf{B})}$. Suppose $(y_i)_{f(\mathbf{B})} \neq (y_i)_{f(\mathbf{A})}$. Then, there must exist some $y_j \in (y_i)_{f(\mathbf{B})}$ but $y_j \notin (y_i)_{f(\mathbf{A})}$. Since $d(y_i, y_j) \neq \emptyset$, it follows that $f(\mathbf{B}) \cap d(y_i, y_j) \neq \emptyset$. This means there is some $\gamma \in \mathbf{B}$ such that $f(\gamma) \in f(\mathbf{B}) \cap d(y_i, y_j)$, this implies $y_j \notin (y_i)_{f(\gamma)}$ and consequently, $y_j \notin (y_i)_{f(\mathbf{B})} \subseteq (y_i)_{f(\gamma)}$. This creates a contradiction. Therefore, we must have $(y_i)_{f(\mathbf{B})} = (y_i)_{f(\mathbf{A})}$, indicating that $f(\mathbf{B})$ is a subbase consistent set. $\square$

Definition 4.4. A subbase discernibility function F_Y of $(Y, f(\Gamma), f(\mathbf{A}))$ is defined by $F_Y(f(\gamma_1), \dots, f(\gamma_n)) = \bigwedge \{\bigvee d(y_i, y_j) : y_i, y_j \in Y, d(y_i, y_j) \neq \emptyset\}$ with Boolean variables $f(\gamma_1), \dots, f(\gamma_n)$, where $\bigwedge$ is conjunction operation and $\bigvee$ is disjunction operation. Each conjunction operator of the minimal disjunctive form is called a reduct of $f(\mathbf{A})$.

Example 3. Let $X = \{1, 2, \dots, 15\}$, $\mathbf{A} = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$, where

$$\begin{aligned} \gamma_1 &= \{\{1, 2, 3, 4, 5, 8, 10, 15\}, \{3, 5, 7, 11, 12\}, \{4, 6, 8, 9, 10, 13, 14\}\}, \\ \gamma_2 &= \{\{1, 2, 3, 4, 5, 7, 8, 10, 11, 12, 15\}, \{3, 4, 5, 8, 10\}, \{3, 5, 6, 9, 13, 14\}\}, \\ \gamma_3 &= \{\{1, 2, 3, 4, 5, 8, 10, 15\}, \{6, 9, 13, 14\}, \{4, 7, 8, 10, 11, 12\}\}, \\ \gamma_4 &= \{\{1, 2, 3, 5, 15\}, \{6, 7, 9, 11, 12, 13, 14\}, \{3, 4, 5, 8, 10\}\}. \end{aligned}$$

Then, we have $(1)_{\mathbf{A}} = (2)_{\mathbf{A}} = (15)_{\mathbf{A}} = \{1, 2, 3, 5, 15\}$, $(4)_{\mathbf{A}} = (8)_{\mathbf{A}} = (10)_{\mathbf{A}} = \{4, 8, 10\}$, $(6)_{\mathbf{A}} = (9)_{\mathbf{A}} = (13)_{\mathbf{A}} = (14)_{\mathbf{A}} = \{6, 9, 13, 14\}$, $(7)_{\mathbf{A}} = (11)_{\mathbf{A}} = (12)_{\mathbf{A}} = \{7, 11, 12\}$, $(3)_{\mathbf{A}} = (5)_{\mathbf{A}} = \{3, 5\}$. Let $Y = \{y_1, y_2, y_3, y_4, y_5\}$, define a homomorphism $f : X \rightarrow Y$ as follows: $f((1)_{\mathbf{A}}) = \{y_1, y_2\}$, $f((3)_{\mathbf{A}}) = \{y_2\}$, $f((4)_{\mathbf{A}}) = \{y_3\}$, $f((6)_{\mathbf{A}}) = \{y_4\}$, $f((7)_{\mathbf{A}}) = \{y_5\}$. Hence,

$$\begin{aligned} f(\gamma_1) &= \{\{y_1, y_2, y_3\}, \{y_2, y_5\}, \{y_3, y_4\}\}, \\ f(\gamma_2) &= \{\{y_1, y_2, y_3, y_5\}, \{y_2, y_3\}, \{y_3, y_4\}\}, \\ f(\gamma_3) &= \{\{y_1, y_2, y_3\}, \{y_4\}, \{y_3, y_5\}\}, \\ f(\gamma_4) &= \{\{y_1, y_2\}, \{y_2, y_3\}, \{y_4, y_5\}\}. \end{aligned}$$

Then, $(y_1)_{f(\mathbf{A})} = \{y_1, y_2\}$, $(y_2)_{f(\mathbf{A})} = \{y_2\}$, $(y_3)_{f(\mathbf{A})} = \{y_3\}$, $(y_4)_{f(\mathbf{A})} = \{y_4\}$, $(y_5)_{f(\mathbf{A})} = \{y_5\}$. Thus, we can easy to obtain the discernibility matrix $D =$

$$\begin{pmatrix} \emptyset & \emptyset & \{f(\gamma_4)\} & f(\mathbf{A}) & \{f(\gamma_1), f(\gamma_3), f(\gamma_4)\} \\ \{f(\gamma_1), f(\gamma_2), f(\gamma_4)\} & \emptyset & \{f(\gamma_1), f(\gamma_4)\} & f(\mathbf{A}) & f(\mathbf{A}) \\ f(\mathbf{A}) & \{f(\gamma_1), f(\gamma_2), f(\gamma_3)\} & \emptyset & f(\mathbf{A}) & f(\mathbf{A}) \\ f(\mathbf{A}) & f(\mathbf{A}) & \{f(\gamma_3), f(\gamma_4)\} & \emptyset & \{f(\gamma_1), f(\gamma_2), f(\gamma_3)\} \\ \{f(\gamma_1), f(\gamma_3), f(\gamma_4)\} & \{f(\gamma_3), f(\gamma_4)\} & \{f(\gamma_1), f(\gamma_4)\} & \{f(\gamma_1), f(\gamma_2), f(\gamma_3)\} & \emptyset \end{pmatrix},$$

The discernibility subbase function

$$\begin{aligned} F_Y(f(\gamma_1), f(\gamma_2), f(\gamma_3), f(\gamma_4)) &= f(\mathbf{A}) \wedge f(\gamma_4) \wedge (f(\gamma_1) \vee f(\gamma_3) \vee f(\gamma_4)) \wedge (f(\gamma_1) \\ &\vee f(\gamma_2) \vee f(\gamma_4)) \wedge (f(\gamma_1) \vee f(\gamma_4)) \wedge (f(\gamma_1) \\ &\vee f(\gamma_2) \vee f(\gamma_3)) \wedge (f(\gamma_3) \vee f(\gamma_4)) \\ &= f(\gamma_4) \wedge (f(\gamma_1) \vee f(\gamma_2) \vee f(\gamma_3)) \\ &= (f(\gamma_1) \wedge f(\gamma_4)) \vee (f(\gamma_2) \wedge f(\gamma_4)) \vee (f(\gamma_3) \wedge f(\gamma_4)), \end{aligned}$$

then $\{f(\gamma_1), f(\gamma_4)\}$, $\{f(\gamma_2), f(\gamma_4)\}$ and $\{f(\gamma_3), f(\gamma_4)\}$ are the reducts of $f(\mathbf{A})$. In other words, $\{\gamma_1, \gamma_4\}$, $\{\gamma_2, \gamma_4\}$ and $\{\gamma_3, \gamma_4\}$ are the reducts of $\mathbf{A}$.

Obviously, if $(X, \Gamma, \mathbf{A})$ is a quasi-discrete FTS , then $(Y, f(\Gamma), f(\mathbf{A}))$ is f -induced discrete FTS . And for any $y_i, y_j \in Y$, $d(y_i, y_j) = \{f(\gamma) : \gamma \in \mathbf{A} \wedge y_j \notin (y_i)_{f(\gamma)}\}$.

Example 4. From Example 4.3, we can get the subbase discernibility matrix $D =$

$$\begin{pmatrix} \emptyset & \{f(\gamma_1), f(\gamma_3), f(\gamma_4)\} & \{f(\gamma_2), f(\gamma_3)\} & f(\mathbf{A}) & \{f(\gamma_1), f(\gamma_3)\} \\ f(\mathbf{A}) & \emptyset & \{f(\gamma_1), f(\gamma_2)\} & f(\mathbf{A}) & \{f(\gamma_2), f(\gamma_3), f(\gamma_4)\} \\ \{f(\gamma_2), f(\gamma_3), f(\gamma_4)\} & f(\mathbf{A}) & \emptyset & \{f(\gamma_1), f(\gamma_3), f(\gamma_4)\} & f(\mathbf{A}) \\ f(\mathbf{A}) & f(\mathbf{A}) & \{f(\gamma_1)\} & \emptyset & \{f(\gamma_2), f(\gamma_3), f(\gamma_4)\} \\ \{f(\gamma_1), f(\gamma_3)\} & \{f(\gamma_1), f(\gamma_4)\} & \{f(\gamma_1), f(\gamma_2)\} & f(\mathbf{A}) & \emptyset \end{pmatrix},$$

then $F_Y(f(\gamma_1), f(\gamma_2), f(\gamma_3), f(\gamma_4)) = f(\gamma_1) \wedge (f(\gamma_2) \vee f(\gamma_3))$ and so $\{f(\gamma_1), f(\gamma_2)\}$ and $\{f(\gamma_1), f(\gamma_3)\}$ are all the subbase reducts of $f(\mathbf{A})$. Namely, $\{\gamma_1, \gamma_2\}$, $\{\gamma_1, \gamma_3\}$ are the subbase reducts of $\mathbf{A}$.

To summarize, when dealing with a given FTS , we can utilize the homomorphism technique to efficiently identify a relatively compact representation. Specifically, this approach allows us to swiftly perform equivalent reductions of subbases in a compressed image of the FTS , all without altering the topological structure of the original space.

5. Vertex cover method for the reduction of FTS

As is widely known, computing all possible reducts of subbases based on discernibility matrices or discernibility subbase functions is an NP-hard problem. However, it is often unnecessary to obtain all possible reducts.

Note that the discernibility matrix D can also be represented in set notation, denoted as $D = \{d(y_i, y_j) : (y_i, y_j) \in Y \times Y\}$. Recall that each subset Y' of $Y = \{y_1, y_2, \dots, y_m\}$ can be expressed through a row vector $g(Y')$, where g is the characteristic function. For example, $Y' = \{y_1, y_3\}$ of $Y = \{y_1, y_2, y_3, y_4\}$ can be represented as $g(Y') = (1, 0, 1, 0)$. Consequently, any non-empty subbase discernibility set $d(y_i, y_j) \subseteq f(\mathbf{A})$ can be rewritten as $g(d(y_i, y_j))$. By selecting an ordering for $D = \{d(y_i, y_j) : (y_i, y_j) \in Y \times Y\}$, the subbase discernibility matrix can be represented as a Boolean matrix. We refer to this as the subbase discernibility Boolean matrix, denoted by $M(d(y_i, y_j) : (y_i, y_j) \in Y \times Y)$. In simpler terms, this Boolean matrix corresponds uniquely to the subbase discernibility function $F_Y(f(\gamma_1), \dots, f(\gamma_t))$.

Example 5. According to Example 4.12, $\{f(\gamma_1), f(\gamma_3), f(\gamma_4)\}$, $\{f(\gamma_1), f(\gamma_2), f(\gamma_4)\}$, $\{f(\gamma_1), f(\gamma_4)\}$, $\{f(\gamma_1), f(\gamma_2), f(\gamma_3)\}$, $\{f(\gamma_3), f(\gamma_4)\}$, $\{f(\gamma_4)\}$, $f(\mathbf{A})$ are all the subbase discernibility sets. From the subbase discernibility matrix D , we can get M shown as follows:

$$M^T = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

In the following, we propose a method to derive a reduct.

Firstly, it has been suggested by [2] that the matrix M can be viewed as the incidence matrix of a hypergraph $\mathcal{G} = (f(\mathbf{A}), \{d(y_i, y_j) : (y_i, y_j) \in Y \times Y\})$, in which $d(y_i, y_j)$ is considered as the hyperedge, and $f(\mathbf{A})$ is the collection of vertices. Subsequently, the task of obtaining reduction of subbases can be reframed as the problem of identifying minimal vertex covers in this hypergraph.

Definition 5.1. Let $(Y, f(\Gamma), f(\mathbf{A}))$ be f -induced FTS with subbase discernibility matrix D . For $d(y_i, y_j) \neq \emptyset$ and $d(y_i, y_j) \neq f(\mathbf{A})$, let $w(d(y_i, y_j))$ be equal to the repeated times of $d(y_i, y_j)$ in D . Then, for any $f(\gamma) \in f(\mathbf{A})$, the importance of $f(\gamma)$ is define by:

$$sig(f(\gamma)) = \sum_{f(\gamma) \in d(y_i, y_j) \wedge (y_i, y_j) \in Y \times Y} w(d(y_i, y_j)).$$

Sometimes, we call $sig(f(\gamma))$ the weight of $f(\gamma)$.

In fact, $sig(f(\gamma))$ can be seen as the degree of vertex $f(\gamma)$ in the graph $\mathcal{G}$.

Secondly, the conjunction operation $\wedge$ satisfies absorption law, which states that $a \wedge (a \vee b) = a$. If there exist core elements within $f(\mathbf{A})$, meaning that at

least one subbase discernibility set is a singleton, then any other discernibility sets that contain this core will be absorbed by the conjunction operation. In the graph representation, the edges corresponding to these core elements form cycles.

Thirdly, if there exists no circles in the graph, to get the minimal vertex covers, a greedy strategy can be employed to identify minimal vertex covers. This strategy involves selecting the vertex with the currently maximum degree.

Given a hypergraph $\mathcal{G} = (E, V)$ with vertices V and edges E , let M be its incidence matrix. For any $v \in V$ and $T \subset E$, we define $N(v, T) = \{e \in T : v \in e\}$ and $d(e, V) = \{v \in V : v \in e\}$. Let $M(v)$ be the column vector of M associated with vertex v and $M(N(v, T))$ represent the rows of M corresponding to the edges in $N(v, T)$. Based on the above analysis, we now outline an algorithm to achieve a reduction of subbases using the hypergraph representation:

Algorithm 1 reduct base weighted hypergraph(RWH)

RWH(S, A, δ) $\setminus \setminus$ $S = (X, A)$ is an information system, δ is threshold

Output: One reduct red

```

1: Generating the  $f$ -induced topology  $Y$  and  $base_Y$ 
2: Generateing the subbase discernibility matrix  $D$ 
3: Important vector  $\mathbf{s} \leftarrow \emptyset$ 
4: for  $a \in A$ 
5:    $sig(f(\gamma_a)) \leftarrow \sum_{f(\gamma_a) \in d(y_i, y_j) \wedge (y_i, y_j) \in Y \times Y} w(d(y_i, y_j))$ ,  $\mathbf{s} \leftarrow \mathbf{s} \cup \{sig(f(\gamma_a))\}$ 
6: end for
7: Generating hypergraph  $\mathcal{G} = (V, E)$  and its incident matrix  $M$ , where  $V = \{f(\gamma_a) : a \in A\}$  and
 $E = \{d(y_i, y_j) : (y_i, y_j) \in Y \times Y\}$ 
8:  $red \leftarrow \emptyset$ 
9: while  $M \neq \emptyset$ 
10:   for  $e \in E$ 
11:     if  $|d(e, V)| = 1$ 
12:        $red \leftarrow red \cup \{f(\gamma_a) : f(\gamma_a) \in d(e, V)\}$ 
13:        $M \leftarrow M - M(f(\gamma_a)) - M(N(f(\gamma_a), E))$   $\setminus \setminus$  here  $M$  is incident matrix of sub-graph
14:        $E \leftarrow E - N(f(\gamma_a), E)$ ,  $A \leftarrow A - \{a\}$ ,  $\mathbf{s} \leftarrow \mathbf{s} - sig(f(\gamma_a))$ 
15:     end if
16:   end for
17: end while
18: while  $M \neq \emptyset$ 
19:   Computing  $\mathbf{s}' = \{sig'(f(\gamma_a)) : a \in A\}$ 
20:    $B \leftarrow \{f(\gamma_a) : sig'(f(\gamma_a)) = max(\mathbf{s}')\}$ 
21:    $red \leftarrow red \cup \{f(\gamma_a)\}$ , where  $f(\gamma_a) \in B$  such that  $sig(f(\gamma_a)) = max(\mathbf{s})$ 
22:    $M \leftarrow M - M(f(\gamma_a)) - M(N(f(\gamma_a), E))$ ,  $E \leftarrow E - N(f(\gamma_a), E)$ ,  $A \leftarrow A - \{a\}$  and
 $\mathbf{s} \leftarrow \mathbf{s} - sig(f(\gamma_a))$ 
23: end while

```

In Algorithm 1, both steps 1 and 2 can be done in $\mathbf{O}(|X|^2|A|)$, steps 4 – 6 also require less than $\mathbf{O}(|X|^2|A|)$, while step 7, step 9 – 23 can be finished in less than $\mathbf{O}(|X|^2)$. Therefore, the overall complexity of Algorithm 1 is dominated by steps 1, 2, and 4 through 6, resulting in a time complexity of is $\mathbf{O}(|X|^2|A|)$.

Example 6. From Example 5.1, it is easy to see that

$$\mathbf{s} = (sig(f(\gamma_1)), sig(f(\gamma_2)), sig(f(\gamma_3)), sig(f(\gamma_4))) = (16, 13, 14, 16).$$

By steps 9 to 16, $f(\gamma_4) \in red$ and $M \leftarrow M - M(f(\gamma_a)) - M(N(f(\gamma_a), E)) = (1, 1, 1)$ and $f(\mathbf{A}) = \{f(\gamma_1), f(\gamma_2), f(\gamma_3)\}$. Then, from steps 18 to 23, since $sig'(f(\gamma_1)) = sig'(f(\gamma_3)) = sig'(f(\gamma_3))$, furthermore, $sig(f(\gamma_1)) > sig(f(\gamma_2))$ and $sig(f(\gamma_1)) > sig(f(\gamma_3))$, then $f(\gamma_1) \in red$ and so $M = \emptyset$ by step 22 and the loop end. Hence, $red = \{f(\gamma_1), f(\gamma_4)\}$, then $\{\gamma_1, \gamma_4\}$ is a reduct of $\mathbf{A}$.

6. Conclusion and future work

In this paper, we have introduced a theoretical framework grounded in homomorphism to investigate the reduction of subbases. We demonstrate that the reduct of a subbase for a topological space is equivalent to the reduct of the subbase for the f -induced topological space, effectively transforming a complex topology into a more concise one. Additionally, we show that the reductions of subbases are equivalent to those of the f -induced *FTS*, which may be of a reduced size compared to the original. To facilitate the search for all possible reductions, we propose the subbase discernibility matrix for the f -induced *FTS*. However, it is worth noting that this is an NP-hard problem. As a result, we turn to graph theory to devise a heuristic approach to find a minimal subbase reduction. While we have made significant progress, there remain intriguing problems worthy of further consideration and discussion.

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On $IC\bar{s}$ -subgroups of finite groups

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Abstract. A subgroup H of a group G is said to be an $IC\bar{s}$ -subgroup of G if the intersection of H and $[H, G]$ is contained in $H_{\bar{s}G}$, where $H_{\bar{s}G}$ is the maximal s -semipermutable subgroup of G contained in H . In this paper, we investigate the influence of $IC\bar{s}$ -subgroups on the structure of finite groups. Some new results of p -nilpotency and supersolvability of finite groups are obtained.

Keywords: $IC\bar{s}$ -subgroup, p -nilpotent group, supersoluble group.

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1. Introduction

In this paper, all groups are finite, G always denotes a finite group. $\pi(G)$ denotes the set of all primes dividing $|G|$. $Syl_p(G)$ denotes the set of Sylow p -subgroups of G . $\mathfrak{U}$ is the class of all supersolvable groups. $Z_{\mathfrak{U}}(G)$ denotes the product of all normal subgroups N of G such that every chief factor of G below N has prime order. The $\mathfrak{F}$ -residual of G , denoted by $G^{\mathfrak{F}}$, is the smallest normal subgroup of G with the quotient in $\mathfrak{F}$. We use standard notation as in [1] and [2].

Let H be a subgroup of a group G . It is well known that the normal closure H^G of H in G is the smallest normal subgroup of G containing H and $H^G = H[H, G]$, where $[H, G]$ is the commutator subgroup of H and G . From [3], we know that a subgroup H of a group G has IC -property if the intersection of H and $[H, G]$ satisfies some conditions. It is an interesting question to research the relationship between IC -property of some subgroups and the structure of G . Many results have been obtained. For example, in [3], Gao and Li introduced the concept of $IC\Phi$ -subgroup. A subgroup H of G is said to be an $IC\Phi$ -subgroup of G if the intersection of H and $[H, G]$ is contained in $\Phi(H)$. They proved that a

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group G is p -nilpotent if every cyclic subgroup of G with order p and 4 (if $p = 2$) or every maximal subgroup of P is an $IC\Phi$ -subgroup of G , where $P \in Syl_p(G)$ and p is the smallest prime divisor of $|G|$. Kasperczyk in [16] generalized these results and proved that a group G is p -nilpotent if there is a subgroup D of P with $1 < |D| \leq |P|$ such that every subgroup of P with order $|D|$ and 4 (if $|D| = 2$ and $|P| \geq 8$) is an $IC\Phi$ -subgroup of G . In [17], Gao and Li introduced the concept of $IC\Phi s$ -subgroup. A subgroup H of G is said to be an $IC\Phi s$ -subgroup of G if $(H \cap [H, G])H_G/H_G \leq \Phi(H/H_G)H_{sG}/H_G$, where H_{sG} is the maximal s -permutable subgroup of G contained in H . They proved that a group G is p -nilpotent if there is a subgroup D of P with $1 < |D| < |P|$ such that every subgroup of P with order $|D|$ and 4 (if $|D| = 2$ and P is a non-abelian 2-group) is an $IC\Phi s$ -subgroup of G , where $P \in Syl_p(G)$ and p is the smallest prime divisor of $|G|$. Zhang and Xu in [18] generalized this result and proved the following result. Let N be a normal subgroup of G and P a Sylow p -subgroup of N . Assume that there is a subgroup D of P with $1 < |D| < |P|$ such that every subgroup of P with order $|D|$ and 4 (if $|D| = 2$ and P is a non-abelian 2-group) is an $IC\Phi s$ -subgroup of G , then $N \leq Z_{p\mathfrak{u}}(G)$. Later, in [19], Gao and Li also introduced the concept of ICC -subgroup. A subgroup H of G is said to be an ICC -subgroup of G if the intersection of H and $[H, G]$ is contained in H_{cG} , where H_{cG} is a CAP -subgroup of G contained in H . They proved that a group G is p -nilpotent if there is a subgroup D of P with $1 < |D| < |P|$ such that every subgroup of P with order $|D|$ and 4 (if $|D| = 2$ and P is a non-abelian 2-group) is an ICC -subgroup of G , where $P \in Syl_p(G)$ and p is the smallest prime divisor of $|G|$. In this paper, we continue to study the IC -property of some subgroups.

Recall that a subgroup H is s -semipermutable in G if H permutes with every Sylow q -subgroup of G for prime q not dividing $|H|$. Once the notion of s -semipermutable subgroup was introduced, many authors have been interested in it and have applied it to investigate the structure of groups. For example, Li et al. in [15] proved that a group G is p -nilpotent if P satisfies the following: P has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with order $|H| = |D|$ are s -semipermutable in G . When P is a non-abelian 2-group and $|P : D| > 2$, in addition, H is s -semipermutable in G if there exists $H_1 \trianglelefteq H \leq P$ with $2|H_1| = |D|$ and H/H_1 is cyclic of order 4, where $P \in Syl_p(G)$ and p is the smallest prime divisor of $|G|$. Here, combining IC -property and s -semipermutable subgroup we give a new concept which covers properly s -semipermutable subgroup of G .

Definition 1.1. *Let H be a subgroup of a group G , then H is called $IC\bar{s}$ -subgroup of G if $H \cap [H, G] \leq H_{\bar{s}G}$, where $H_{\bar{s}G}$ is the maximal s -semipermutable subgroup of G contained in H .*

Obviously, a s -semipermutable subgroup of G is an $IC\bar{s}$ -subgroup of G . The converse does not hold in general.

Example 1.1. Let $G = S_4$. Take $H = \langle (34) \rangle$, then $[H, G] \leq A_4$ and $H \cap [H, G] = 1$. Hence, H is an $IC\bar{s}$ -subgroup of G . But, clearly, H is not s -semipermutable subgroup of G because of $\langle (123) \rangle H \neq H \langle (123) \rangle$.

In this paper, we study the structure of a group G under the assumption that some subgroups of G are $IC\bar{s}$ -subgroups and get some characterizations of p -nilpotency and supersolvability of finite groups.

2. Preliminary results

Lemma 2.1 ([4, Lemma 2.2]). *Let G be a group. Suppose that H is an s -semipermutable subgroup of G . Then*

- (1) *If $H \leq K \leq G$, then H is s -semipermutable in K .*
- (2) *Let N be a normal subgroup of G . If H is a p -group for some prime $p \in \pi(G)$, then HN/N is s -semipermutable in G/N .*
- (3) *If $H \leq O_p(G)$, then H is s -permutable in G .*
- (4) *Suppose that H is a p -group for some prime $p \in \pi(G)$ and N is normal in G . Then, $H \cap N$ is also an s -semipermutable subgroup of G .*

Lemma 2.2 ([5, Lemma A]). *If H is an s -permutable subgroup of G and H is a p -group. Then, $O^p(G) \leq N_G(H)$.*

Lemma 2.3. *Let G be a group, $H \leq G$, $N \trianglelefteq G$. Suppose that H is an $IC\bar{s}$ -subgroup of G . Then*

- (1) *If $H \leq K \leq G$, then H is an $IC\bar{s}$ -subgroup of K .*
- (2) *Let $N \leq H$. If H is a p -group for some prime $p \in \pi(G)$, then H/N is an $IC\bar{s}$ -subgroup of G/N .*
- (3) *If H is a p -group and N is a p' -group for some prime $p \in \pi(G)$, then HN/N is an $IC\bar{s}$ -subgroup of G/N .*

Proof. (1) By the hypothesis, $H \cap [H, G] \leq H_{\bar{s}G}$. Then, $H \cap [H, K] \leq H \cap [H, G] \leq H_{\bar{s}G} \leq H_{\bar{s}K}$ by Lemma 2.1(1). Hence, H is an $IC\bar{s}$ -subgroup of K .

(2) By Lemma 2.1(2) and the hypothesis, we have $H/N \cap [H/N, G/N] = (H \cap [H, G]N)/N = (H \cap [H, G])N/N \leq H_{\bar{s}G}/N \leq (H/N)_{\bar{s}(G/N)}$. Hence, H/N is an $IC\bar{s}$ -subgroup of G/N .

(3) By the hypothesis, we have $H \cap [H, G] = H \cap [H, G]N$. Hence, $HN/N \cap [HN/N, G/N] = (HN \cap [H, G]N)/N = (H \cap [H, G]N)N/N = (H \cap [H, G])N/N \leq H_{\bar{s}G}N/N \leq (HN/N)_{\bar{s}(G/N)}$ by Lemma 2.1(2). This shows that HN/N is an $IC\bar{s}$ -subgroup of G/N . □

Lemma 2.4 ([6, Theorem A]). *Let H be an s -semipermutable π -subgroup of G . Then, H^G contains a nilpotent π -complement and all π -complements in H^G are conjugate. Also, if π consists of a single prime, then H^G is solvable.*

Lemma 2.5 ([2, Lemma 1.2]). *Let U, V and W be subgroups of a group G . Then, the following statements are equivalent:*

- (1) $U \cap VW = (U \cap V)(U \cap W)$.
- (2) $UV \cap UW = U(V \cap W)$.

Lemma 2.6 ([7, Lemma 2.4]). *Let N be a normal subgroup of a group G such that G/N is p -nilpotent and let P be a Sylow p -subgroup of N , where p is a prime divisor of $|G|$. If $|P| \leq p^2$ and one of the following conditions holds, then G is p -nilpotent.*

- (1) $(|G|, p-1) = 1$ and $|P| \leq p$.
- (2) G is A_4 -free if $p = \min \pi(G)$.
- (3) $(|G|, p^2-1) = 1$.

Lemma 2.7 ([8, Theorem 10.4.1]). *Let G be a finite group, p be an odd prime and P be a Sylow p -subgroup of G . Then, G is p -nilpotent if and only if $N_G(J(P))$ and $C_G(Z(P))$ are p -nilpotent.*

Lemma 2.8 ([9, Theorem B]). *Let $\mathfrak{F}$ be a formation and let E be a normal subgroup of G . If $F^*(E) \leq Z_{\mathfrak{F}}(G)$, then $E \leq Z_{\mathfrak{F}}(G)$.*

Lemma 2.9 ([10, Lemma 3.3]). *Let $\mathfrak{F}$ be a solubly saturated formation containing all supersoluble groups. Suppose that E is a normal subgroup of G such that $G/E \in \mathfrak{F}$. If $E \leq Z_{\mathfrak{F}}(G)$, then $G \in \mathfrak{F}$. In particular, if E is cyclic, then $G \in \mathfrak{F}$.*

Lemma 2.10 ([11, Lemma 2.12]). *Let $\mathfrak{F}$ be a solubly saturated formation. Suppose that P is a normal p -subgroup of G and C is a Thompson critical subgroup of P . If either $P/\Phi(P) \leq Z_{\mathfrak{F}}(G/\Phi(P))$ or $\Omega(C) \leq Z_{\mathfrak{F}}(G)$, then $P \leq Z_{\mathfrak{F}}(G)$.*

3. New characterization of p -nilpotency of groups

Theorem 3.1. *Let $N \trianglelefteq G$ such that G/N is p -nilpotent, where p is the smallest prime divisor of $|G|$. Suppose that every cyclic subgroup of N with order p and 4 (if $p = 2$) is an $IC\bar{s}$ -subgroup of G , then G is p -nilpotent.*

Proof. Assume that the result is false. Let (G, N) be a minimal counterexample with minimal order $|G| + |N|$.

- (1) G is a minimal nonnilpotent group.

Let L be a proper subgroup of G . Since $L/(L \cap N) \cong LN/N \leq G/N$ and G/N is p -nilpotent, we have $L/(L \cap N)$ is p -nilpotent. It is clear that every cyclic

subgroup of $L \cap N$ with order p and 4 (if $p = 2$) is an $IC\bar{s}$ -subgroup of L by hypothesis and Lemma 2.3(1). Therefore, $(L, L \cap N)$ satisfies the hypothesis of the theorem. The minimal choice of (G, N) implies that L is p -nilpotent. This shows that G is a minimal non- p -nilpotent group. In view of ([1], Proposition 5.4; [2], Theorem 6.18), G is a minimal nonnilpotent group, $G = P \rtimes Q$, where P is a Sylow p -subgroup of G , Q is a Sylow q -subgroup of G and $p \neq q$; $P = G^{\mathfrak{N}}$, that is, P is the intersection of all normal subgroups K of G satisfying G/K is nilpotent; $P/\Phi(P)$ is a chief factor of G ; the exponent of P is p or 4 (when P is a nonabelian 2-group).

(2) Let $x \in P$ such that $x \notin \Phi(P)$, then $\langle x \rangle^G = \langle x \rangle[\langle x \rangle, G] = P$.

Since $P \trianglelefteq G$, we have $\langle x \rangle[\langle x \rangle, G] = \langle x \rangle^G \leq P$ and $\Phi(P)$ are normal subgroups of G , so $\langle x \rangle[\langle x \rangle, G]\Phi(P) \trianglelefteq G$, obviously, $\Phi(P) < \langle x \rangle[\langle x \rangle, G]\Phi(P) \leq P$. Since $P/\Phi(P)$ is a chief factor of G , we have $P = \langle x \rangle[\langle x \rangle, G]\Phi(P) = \langle x \rangle[\langle x \rangle, G]$.

(3) $p = 2$ and P has an element of order 4.

Obviously, $P \leq N$. If P has not an element of order 4, let $x \in P$ such that $x \notin \Phi(P)$, then $|\langle x \rangle| = p$ and $P = \langle x \rangle[\langle x \rangle, G]$. By hypothesis, $\langle x \rangle \cap [\langle x \rangle, G] \leq \langle x \rangle_{\bar{s}G}$. If $\langle x \rangle_{\bar{s}G} = 1$, then $\Phi(P) \leq [\langle x \rangle, G] < P$. Since $P/\Phi(P)$ is a chief factor of G , so $P = \langle x \rangle[\langle x \rangle, G] = \langle x \rangle\Phi(P) = \langle x \rangle$. Since p is the smallest prime divisor of $|G|$, we have G is p -nilpotent, a contradiction. If $\langle x \rangle = \langle x \rangle_{\bar{s}G}$, then $\langle x \rangle Q$ is a subgroup of G . Since $P \neq \langle x \rangle$, we have $\langle x \rangle Q < G$ is nilpotent, so $\langle x \rangle \leq N_G(Q)$. Therefore, $P \leq N_G(Q)$ and so $Q \trianglelefteq G$. This implies that G is nilpotent, a contradiction.

(4) The final contradiction.

By (3), we obtain that P is a nonabelian 2-group and the exponent of $\Phi(P)$ is 2. Let x be an element of P with order 4, then $P = \langle x \rangle[\langle x \rangle, G]$ by (2). If $[\langle x \rangle, G] < P$, then $P/[\langle x \rangle, G]$ is a nontrivial cyclic group. Let $M/[\langle x \rangle, G]$ be the unique maximal subgroup of $P/[\langle x \rangle, G]$. Then, $[\langle x \rangle, G] \leq M < P, M \trianglelefteq G$, obviously, $\Phi(P) \leq M$, hence $M = \Phi(P)$. So P is cyclic. It follows that G is 2-nilpotent, a contradiction. This contradiction shows that $P = [\langle x \rangle, G]$. By hypothesis, $\langle x \rangle = \langle x \rangle \cap [\langle x \rangle, G] \leq \langle x \rangle_{\bar{s}G}$, hence $\langle x \rangle = \langle x \rangle_{\bar{s}G}$. Notice that $\langle x \rangle Q$ is a proper subgroup of G , so $\langle x \rangle Q$ is nilpotent, then $\langle x \rangle \leq N_G(Q)$. Let y be an element of P with order 2. By hypothesis, $\langle y \rangle \cap [\langle y \rangle, G] \leq \langle y \rangle_{\bar{s}G}$. If $\langle y \rangle_{\bar{s}G} = 1$, then $\langle y \rangle \cap [\langle y \rangle, G] = 1$. Suppose that $\langle y \rangle[\langle y \rangle, G] = P$, then $\Phi(P) \leq [\langle y \rangle, G] < P$. Since $P/\Phi(P)$ is a chief factor of G , we get $\Phi(P) = [\langle y \rangle, G]$ and so $P = \langle y \rangle$, a contradiction. Hence, $\langle y \rangle[\langle y \rangle, G]Q$ is a proper subgroup of G and so $\langle y \rangle \leq N_G(Q)$. If $\langle y \rangle_{\bar{s}G} = \langle y \rangle$, then $\langle y \rangle \leq N_G(Q)$. Therefore, $P \leq N_G(Q)$ and so $Q \trianglelefteq G$. This implies that G is nilpotent, a contradiction.

This completes the proof. $\square$

Corollary 3.1. *Let p be the smallest prime divisor of $|G|$. If every cyclic subgroup of G with order p and 4 (if $p = 2$) is an $IC\bar{s}$ -subgroup of G , then G is p -nilpotent.*

Theorem 3.2. *Let G be a group and $P \in \text{Syl}_p(G)$, where p is the smallest prime divisor of $|G|$. Suppose that every maximal subgroup of P is an $IC\bar{s}$ -subgroup of G , then G is p -nilpotent.*

Proof. Assume that the result is false. Let G be a minimal counterexample with minimal order.

(1) G has a unique minimal normal subgroup N , G/N is p -nilpotent and $\Phi(G) = 1$.

Let N be a minimal normal subgroup of G and M/N be a maximal subgroup of PN/N . Then, $M = M \cap PN = (M \cap P)N$. Let $M \cap P = P_1$, then $P_1 \cap N = P \cap N$ and $|P : P_1| = |PN/N : M/N| = p$, this shows that P_1 is maximal in P . Since $P_1 \cap N = P \cap N$ is a Sylow p -subgroup of N , hence

$$|N \cap P_1[P_1, G]|_p = |N|_p = |N \cap P_1| = |(N \cap P_1)(N \cap [P_1, G])|_p$$

and

$$\begin{aligned} |N \cap P_1[P_1, G]|_{p'} &= \frac{|N|_{p'} |P_1[P_1, G]|_{p'}}{|NP_1[P_1, G]|_{p'}} \\ &= \frac{|N|_{p'} |[P_1, G]|_{p'}}{|N[P_1, G]|_{p'}} = |N \cap [P_1, G]|_{p'} = |(N \cap P_1)(N \cap [P_1, G])|_{p'}. \end{aligned}$$

This implies that $N \cap P_1[P_1, G] = (N \cap P_1)(N \cap [P_1, G])$. By Lemma 2.5, we have $P_1N \cap [P_1, G]N = (P_1 \cap [P_1, G])N$, and because P_1 is an $IC\bar{s}$ -subgroup of G , hence $M/N \cap [M/N, G/N] = P_1N/N \cap [P_1N/N, G/N] = (P_1N \cap [P_1, G]N)/N = (P_1 \cap [P_1, G])N/N \leq (P_1)_{\bar{s}G}N/N \leq (P_1N/N)_{\bar{s}(G/N)} = (M/N)_{\bar{s}(G/N)}$ by Lemma 2.1(2). This shows that G/N satisfies the hypothesis of the theorem. The minimal choice of G implies that G/N is p -nilpotent. Since the class of all p -nilpotent groups is a saturated formation, we have N is a unique minimal normal subgroup of G and $\Phi(G) = 1$.

(2) $O_{p'}(G) = 1$.

If $O_{p'}(G) \neq 1$, then $N \leq O_{p'}(G)$ and G/N is p -nilpotent by (1). This implies that G is p -nilpotent, a contradiction.

(3) $O_p(G) \neq 1$.

If $O_p(G) = 1$, then by (2), G is not solvable, so $p = 2$. If $PN < G$, then PN satisfies the hypothesis of the theorem by Lemma 2.3(1), hence PN is 2-nilpotent, so N is also 2-nilpotent. By (1), G/N is 2-nilpotent, hence G is solvable, a contradiction. If $PN = G$, let P_1 be a maximal subgroup of P , by the hypothesis, $P_1 \cap [P_1, G] \leq (P_1)_{\bar{s}G}$. If $[P_1, G] = 1$, then $P_1 \trianglelefteq G$, so $|P| = p$, thus G is 2-nilpotent, a contradiction. Hence, $[P_1, G] \neq 1$. Then, $P_1 \cap N \leq P_1 \cap [P_1, G] \leq (P_1)_{\bar{s}G}$, therefore $P_1 \cap N = (P_1)_{\bar{s}G} \cap N$, obviously, $P_1 \cap N \neq 1$, so $(P_1 \cap N)^G = N$ is soluble by Lemma 2.1(4) and Lemma 2.4, so is G , a contradiction.

(4) The final contradiction.

By (1), $\Phi(O_p(G)) \leq \Phi(G) = 1$, so $O_p(G)$ is an elementary abelian p -subgroup. By (3), $N \leq O_p(G)$. If $N \leq \Phi(P)$, then $N \leq \Phi(G) = 1$, a contradiction. If $N \not\leq \Phi(P)$, let P_1 be a maximal subgroup of P such that $N \not\leq P_1$, then $P = P_1N$. If $N \cap P_1 = 1$, then $|N| = p$, by (1), G/N is p -nilpotent, hence G is p -nilpotent by Lemma 2.6, a contradiction. If $N \cap P_1 \neq 1$, obviously, $[P_1, G] \neq 1$, then $1 \neq P_1 \cap N \leq P_1 \cap [P_1, G] \leq (P_1)_{\bar{s}G}$, hence $P_1 \cap N = (P_1)_{\bar{s}G} \cap N$, so $O^p(G) \leq N_G(P_1 \cap N)$ by Lemma 2.1(3)(4) and Lemma 2.2. Clearly, $P_1 \cap N \trianglelefteq P$, so $P_1 \cap N \trianglelefteq G$, we obtain $P_1 \cap N = N$ by the minimality and uniqueness of N , hence $N \leq P_1$, a contradiction.

This completes the proof. $\square$

Corollary 3.2. *If every maximal subgroup of every Sylow subgroup of G is an $IC\bar{s}$ -subgroup of G , then G is a Sylow tower group of supersolvable type.*

Theorem 3.3. *Let G be a group and $P \in \text{Syl}_p(G)$, where p is a prime divisor of $|G|$. Suppose that $N_G(P)$ is p -nilpotent and every maximal subgroup of P is an $IC\bar{s}$ -subgroup of G , then G is p -nilpotent.*

Proof. If $p = \min \pi(G)$, then G is p -nilpotent by Theorem 3.2. Now, we consider the case when $p \neq \min \pi(G)$, that is, p is an odd prime. Assume that the result is false. Let G be a minimal counterexample with minimal order.

(1) G has a unique minimal normal subgroup N , G/N is p -nilpotent and $\Phi(G) = 1$.

Let N be a minimal normal subgroup of G , then PN/N is a Sylow p -subgroup of G/N . If $N_G(P)N = G$, then $G/N = N_G(P)N/N \cong N_G(P)/N_G(P) \cap N$ is p -nilpotent. If $N_G(P)N \neq G$, then $N_G(P)N$ satisfies the hypothesis of the theorem by Lemma 2.3(1), so $N_G(P)N$ is p -nilpotent by the minimal choice of G , hence $N_{G/N}(PN/N) \cong N_G(P)N/N$ is p -nilpotent. We may obtain that every maximal subgroup of PN/N is an $IC\bar{s}$ -subgroup of G/N by a similar discussion as in Theorem 3.2(1). This shows that G/N satisfies the hypothesis of the theorem. The minimal choice of G implies that G/N is p -nilpotent. Since the class of all p -nilpotent groups is a saturated formation, we have N is a unique minimal normal subgroup of G and $\Phi(G) = 1$.

(2) $O_{p'}(G) = 1$.

If $O_{p'}(G) \neq 1$, then $N \leq O_{p'}(G)$ and G/N is p -nilpotent by (1). This implies that G is p -nilpotent, a contradiction.

(3) G is solvable.

Since G is not p -nilpotent and p is an odd prime, by Lemma 2.7, there exists a nontrivial characteristic subgroup H of P such that $N_G(H)$ is not p -nilpotent, obviously, $P \leq N_G(H)$. If $N_G(H) < G$, then clearly $N_G(H)$ satisfies the hypothesis of the theorem, hence $N_G(H)$ is p -nilpotent, a contradiction. This contradiction shows that $N_G(H) = G$. Therefore, $1 \neq H \trianglelefteq G$, $H \leq O_p(G)$ and so $N \leq O_p(G)$. By (1), $G/O_p(G)$ is p -nilpotent and so G is p -solvable. Then, by [14, Theorem 3.5], there exists a Sylow q -subgroup Q of G such that PQ is a subgroup of G , where $q \nmid |G|$ and $q \neq p$. If $PQ < G$, obviously, PQ satisfies the

hypothesis of the theorem, hence PQ is p -nilpotent. So $O_p(G)Q = O_p(G) \times Q$ and $Q \leq C_G(O_p(G)) \leq O_p(G)$ by [14, Theorem 3.2], a contradiction. Hence, $G = PQ$ is solvable.

(4) The final contradiction.

By (2) and (3), $O_p(G) \neq 1$, then $N \leq O_p(G)$ and $O_p(G)$ is an elementary abelian p -group by (1). Hence, there exists a maximal subgroup M of G such that $G = N \rtimes M$. Obviously, $O_p(G) \cap M \trianglelefteq M$, $O_p(G) \cap M \trianglelefteq O_p(G)$, so $O_p(G) \cap M \trianglelefteq G$. If $O_p(G) \cap M \neq 1$, then $N \leq O_p(G) \cap M$ and so $N \leq M$, a contradiction. Hence, $O_p(G) \cap M = 1$. This induces that $N = O_p(G) = C_G(N)$. Let $M_p = P \cap M$ be a Sylow p -subgroup of M , then $P = NM_p$. It is easy to see that $M_p \neq 1$. Let P_1 be a maximal subgroup of P containing M_p , by the hypothesis, $P_1 \cap [P_1, G] \leq (P_1)_{\bar{s}G}$, obviously, $N \not\leq P_1$ and $N \leq [P_1, G]$, then $P_1 \cap N \leq P_1 \cap [P_1, G] \leq (P_1)_{\bar{s}G}$, so $P_1 \cap N = (P_1)_{\bar{s}G} \cap N$. If $P_1 \cap N \neq 1$, We may obtain $P_1 \cap N \trianglelefteq G$ by a similar discussion as in Theorem 3.2(4), hence $P_1 \cap N = N$ and so $N \leq P_1$, a contradiction. If $P_1 \cap N = 1$, it follows that $|N| = p$ and $P = N \times P_1$. Hence, $P \leq C_G(N) = N$, therefore, $P = N$ and $G = N_G(N) = N_G(P)$ is p -nilpotent, a contradiction.

This completes the proof. $\square$

Remark 3.1. In Theorem 3.1, 3.2, 3.3, the assumptions that p is the smallest prime divisor of $|G|$ and $N_G(P)$ is p -nilpotent are necessary. To illustrate the situation, we consider the semidirect product $G = H \rtimes \langle c \rangle$, where $H = \langle a \rangle \times \langle b \rangle$ with $o(a) = o(b) = 5$ and $o(c) = 2$ defined by $a^c = a^{-1}$ and $b^c = b^{-1}$. Then, the subgroups of G of order 5 are normal in G , but G is not 5-nilpotent.

4. New characterization of supersolubility of groups

Lemma 4.1. *Let P be a normal p -subgroup of G . If every cyclic subgroup of P with order p and 4 (if $p = 2$) is an $IC\bar{s}$ -subgroup of G , then $P \leq Z_{\mathfrak{U}}(G)$.*

Proof. Assume that the result is false. Let (G, P) be a minimal counterexample with $|G| + |P|$ minimal. Let K be a normal subgroup of G such that P/K is a chief factor of G . Obviously, (G, K) satisfies the hypothesis of the theorem and so $K \leq Z_{\mathfrak{U}}(G)$. Suppose that P/L is another chief factor of G which is different from P/K , then $L \leq Z_{\mathfrak{U}}(G)$ and so $P = KL \leq Z_{\mathfrak{U}}(G)$, a contradiction. This contradiction shows that K is the unique normal subgroup of G such that P/K is a chief factor of G . Obviously, $|P/K| \neq p$. Let C be a Thompson critical subgroup of P . If $\Omega(C) < P$, then $\Omega(C) \leq K \leq Z_{\mathfrak{U}}(G)$, so $P \leq Z_{\mathfrak{U}}(G)$ by Lemma 2.10, a contradiction. If $\Omega(C) = P$, then $\exp(P) = p$ or 4. Let G_p be a Sylow p -subgroup of G containing P , obviously, $P/K \cap Z(G_p/K) \neq 1$. Suppose that $M/K \leq P/K \cap Z(G_p/K)$ and $|M/K| = p$. Let $x \in M \setminus K$, then $M = \langle x \rangle K$ and $\langle x \rangle \cap [\langle x \rangle, G] \leq \langle x \rangle_{\bar{s}G}$. If $\langle x \rangle = \langle x \rangle_{\bar{s}G}$, then M/K is s -semipermutable in G/K by Lemma 2.1(2), clearly, $M/K \trianglelefteq G_p/K$, and $O^p(G/K) \leq N_{G/K}(M/K)$ by Lemma 2.1(3) and Lemma 2.2. So $M/K \trianglelefteq G/K$. Hence, $|M/K| = |P/K| = p$, a contradiction. If $\langle x \rangle \neq \langle x \rangle_{\bar{s}G}$, then $1 < [\langle x \rangle, G] < P$, so $[\langle x \rangle, G] \leq K$. It

follows that $K < M = \langle x \rangle K = \langle x \rangle [\langle x \rangle, G] K = \langle x \rangle^G K \leq P$, since K is the unique normal subgroup of G such that P/K is a chief factor of G , we have $M = P$, a contradiction.

This completes the proof. $\square$

Theorem 4.1. *Let $\mathfrak{F}$ be a solubly saturated formation containing $\mathfrak{U}$ and let E be a normal subgroup of G such that $G/E \in \mathfrak{F}$. Suppose that $X = E$ or $X = F^*(E)$. If every cyclic subgroup of every noncyclic Sylow subgroup of X with order p and 4 (if $p = 2$) is an $IC\bar{s}$ -subgroup of G , then $G \in \mathfrak{F}$.*

Proof. We first prove that the theorem is true if $X = E$. Assume that the result is false. Let (G, E) be a minimal counterexample with $|G| + |E|$ minimal.

(1) E is soluble.

If E is not soluble, then $2 \parallel |E|$. Suppose that E has a cyclic Sylow 2-subgroup, then E is 2-nilpotent and so E is soluble, a contradiction. This contradiction shows that every Sylow 2-subgroup of E is noncyclic. By the hypothesis and Corollary 3.1, we have E is 2-nilpotent and so E is soluble, a contradiction again. Thus, (1) holds.

(2) $G^{\mathfrak{F}} \leq E$ and $G^{\mathfrak{F}}$ is a p -group.

Since $G/E \in \mathfrak{F}$, we have $G^{\mathfrak{F}} \leq E$ and so $G^{\mathfrak{F}}$ is soluble by (1). If $G^{\mathfrak{F}} \leq \Phi(G)$, then $G^{\mathfrak{F}} \leq M_G$ for every maximal subgroup M of G and so $G/M_G \in \mathfrak{F}$. Hence, $G^{\mathfrak{F}}$ is a p -group by [13, Theorem 3.4.2]. If $G^{\mathfrak{F}} \not\leq \Phi(G)$, let K be a maximal subgroup of G such that $G^{\mathfrak{F}} \not\leq K$. Then, $K/K \cap E \cong KE/E = G/E \in \mathfrak{F}$ and every cyclic subgroup of every noncyclic Sylow subgroup of $K \cap E$ with order p and 4 is an $IC\bar{s}$ -subgroup of K by Lemma 2.3(1). Thus, $K \in \mathfrak{F}$. So $G^{\mathfrak{F}}$ is a p -group by [13, Theorem 3.4.2].

(3) The final contradiction.

If $G^{\mathfrak{F}}$ is noncyclic, then $G^{\mathfrak{F}} \leq Z_{\mathfrak{U}}(G)$ by the hypothesis and Lemma 4.1. If $G^{\mathfrak{F}}$ is cyclic, obviously, $G^{\mathfrak{F}} \leq Z_{\mathfrak{U}}(G)$. Hence, $G \in \mathfrak{F}$ by Lemma 2.9, a contradiction. This contradiction shows that the theorem holds if $X = E$.

Now, we prove that the theorem holds for $X = F^*(E)$.

We may obtain $F^*(E)$ is soluble by a similar discussion as in (1) and so $F(E) = F^*(E)$. Let P be a Sylow p -subgroup of $F(E)$, then $P \trianglelefteq G$. If P is noncyclic, then $P \leq Z_{\mathfrak{U}}(G)$ by the hypothesis and Lemma 4.1. If P is cyclic, obviously, $P \leq Z_{\mathfrak{U}}(G)$. It follows that $F(E) \leq Z_{\mathfrak{U}}(G)$. Thus, we have $G \in \mathfrak{F}$ by Lemma 2.8 and Lemma 2.9.

This completes the proof. $\square$

Lemma 4.2. *Let P be a normal p -subgroup of G . If every maximal subgroup of P is an $IC\bar{s}$ -subgroup of G , then $P \leq Z_{\mathfrak{U}}(G)$.*

Proof. Assume that the result is false. Let (G, P) be a minimal counterexample with $|G| + |P|$ minimal. Let N be a minimal normal subgroup of G such that $N \leq P$, by Lemma 2.3(2), $(G/N, P/N)$ satisfies the hypothesis of the theorem and so $P/N \leq Z_{\mathfrak{U}}(G/N)$. Obviously, $|N| \neq p$. Suppose that G has another

minimal normal subgroup $K \leq P$, then $P/K \leq Z_{\mathfrak{U}}(G/K)$. It follows that $NK/K \leq Z_{\mathfrak{U}}(G/K)$ and so $|N| = p$, a contradiction. This contradiction shows that N is the unique minimal normal subgroup of G contained in P . By Lemma 2.10, we have $N \not\leq \Phi(P)$, so $\Phi(P) = 1$, then there exists a subgroup T of P such that $P = N \rtimes T$. Let N_1 be a maximal subgroup of N and $N_1 \trianglelefteq P$. It is clear that $P_1 = N_1T$ is maximal in P , so $P_1 \cap [P_1, G] \leq (P_1)_{\bar{s}G}$. Since $P_1[P_1, G] = P_1^G \leq P$, we have $[P_1, G] \leq P$. If $[P_1, G] = 1$, then $P_1 \leq Z(G)$ and $P_1 \cap N \trianglelefteq G$. If $[P_1, G] \neq 1$, then $N \leq [P_1, G]$, we may obtain $P_1 \cap N \trianglelefteq G$ by a similar discussion as in Theorem 3.2(4), hence $N_1 = P_1 \cap N = 1$, that is, $|N| = p$, a contradiction.

This completes the proof. $\square$

Theorem 4.2. *Let $\mathfrak{F}$ be a solubly saturated formation containing $\mathfrak{U}$ and let E be a normal subgroup of G such that $G/E \in \mathfrak{F}$. Suppose that $X = E$ or $X = F^*(E)$. If every maximal subgroup of every Sylow subgroup of X is an $IC\bar{s}$ -subgroup of G , then $G \in \mathfrak{F}$.*

Proof. We first prove that the theorem is true if $X = E$. Assume that the result is false. Let G be a minimal counterexample with minimal order.

(1) G has a minimal normal subgroup $N \leq P$ and N is an elementary abelian p -group, where $P \in \text{Syl}_p(E)$ and p is the largest prime divisor of $|E|$.

By the hypothesis and Lemma 2.3(1), every maximal subgroup of every Sylow subgroup of E is an $IC\bar{s}$ -subgroup of E , so E is a Sylow tower group of supersolvable type by Corollary 3.2. Thus, P is a normal Sylow p -subgroup of E , where p is the largest prime divisor of E , and so $P \trianglelefteq G$. Hence, we have (1).

(2) $G/N \in \mathfrak{F}$ and $N = P$.

Since $(G/N)/(E/N) \cong G/E \in \mathfrak{F}$ and every maximal subgroup of every Sylow subgroup of E/N is an $IC\bar{s}$ -subgroup of G/N by Lemma 2.3(2)(3), we have $G/N \in \mathfrak{F}$. The solvability of E implies that $F(E) = N$ and $C_E(N) \leq F(E)$ by [13, Theorem 1.8.17 and Theorem 1.8.18], so $C_E(N) = N = F(E)$ as N is elementary abelian. Clearly, $P \leq F(E)$, hence $N = P$.

(3) The final contradiction.

By the hypothesis and Lemma 4.2, $P \leq Z_{\mathfrak{U}}(G)$. By (2), $G/P \in \mathfrak{F}$. Hence, $G \in \mathfrak{F}$ by Lemma 2.9, a contradiction. This contradiction shows that the theorem holds if $X = E$.

Now, we prove that the theorem holds for $X = F^*(E)$.

Assume that $F^*(E)$ is not soluble, then $2 \parallel |F^*(E)|$. By the hypothesis and Theorem 3.2, we have $F^*(E)$ is 2-nilpotent and so $F^*(E)$ is soluble, a contradiction. This contradiction shows that $F^*(E)$ is soluble. Hence, $F(E) = F^*(E)$. Let Q be a Sylow q -subgroup of $F(E)$, then $Q \trianglelefteq G$. By the hypothesis and Lemma 4.2, $Q \leq Z_{\mathfrak{U}}(G)$. It follows that $F(E) \leq Z_{\mathfrak{U}}(G)$. Thus, we have $G \in \mathfrak{F}$ by Lemma 2.8 and Lemma 2.9.

This completes the proof. $\square$

5. Conclusion

An interesting question in finite group theory is to determine the influence of the embedding properties of members of some distinguished families of subgroups on the structure of the group. The present paper adds some results to this line of research. We proved that a group G is p -nilpotent if every cyclic subgroup of G with order p and 4 (if $p = 2$) or every maximal subgroup of P is an $IC\bar{s}$ -subgroup of G , where $P \in Syl_p(G)$ and p is the smallest prime divisor of $|G|$. We also proved the following results: Let $\mathfrak{F}$ be a solubly saturated formation containing $\mathfrak{U}$ and let E be a normal subgroup of G such that $G/E \in \mathfrak{F}$. Suppose that $X = E$ or $X = F^*(E)$. If every cyclic subgroup of every noncyclic Sylow subgroup of X with order p and 4 (if $p = 2$) or every maximal subgroup of every Sylow subgroup of X is an $IC\bar{s}$ -subgroup of G , then $G \in \mathfrak{F}$. The $IC\bar{s}$ -subgroup in Theorem 3.1, Theorem 3.2, Theorem 3.3, Theorem 4.1 and Theorem 4.2 can be replaced by s -semipermutable subgroup and the results are also true. Hence, we generalize some results on the basis of s -semipermutability.

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On the SS -supplemented modules over Dedekind domains

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Abstract. A module M is called *ss-supplemented* if every submodule U of M has a supplement V in M such that $U \cap V$ is semisimple. In this paper, we completely determine the structure of (amply) *ss-supplemented* modules over Dedekind domains. In particular, we prove that an abelian group M is *ss-supplemented* (as a $\mathbb{Z}$ -module) if and only if $M \cong (\bigoplus_{p \in I} \mathbb{Z}_p^{(v)}) \oplus (\bigoplus_{q \in J} \mathbb{Z}_{q^2}^{(\nu)})$, where $\mathbb{P}$ is the set of all prime integers, I, J are some subsets of $\mathbb{P}$ and v, ν are any index sets.

Keywords: semisimple module, strongly local module, *ss-supplemented* module, Dedekind domain.

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1. Introduction

In [5], the concept of *ss-supplement* submodules have been introduced as a type of supplement submodules. In the same paper authors have obtained detailed information about modules with the help of *ss-supplement* submodules.

A submodule V of M of an R -module M is called a *supplement* of a submodule U in M if it is minimal with respect to $M = U + V$, equivalently $M = U + V$

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and $U \cap V \ll V$ (see [8]). It follows from [8] that a module M is called *supplemented* if every submodule of M has a supplement in M . A submodule U of M has *ample supplements* in M if every submodule L of M such that $M = U + L$ contains a supplement of U in M . The module M is called *amply supplemented* if every submodule of M has ample supplements in M . As a generalization of semisimple modules, supplemented modules play an important role in module and ring theory, and Abelian groups.

Recall from [3] that a submodule V of a module M is a *Rad-supplement* of a submodule U in M if $M = U + V$ and $U \cap V \subseteq \text{Rad}(V)$. Since $\text{Rad}(V)$ is the sum of all small submodules of V , every supplement submodule of M is a *Rad-supplement* submodule of M . M is called *Rad-supplemented* if all submodules have a *Rad-supplement* in M . Every module M with $M = \text{Rad}(M)$ is *Rad-supplemented*.

There are generally two different ways of working in module classes. The first one is the characterization of some important rings. A ring R is (semi) perfect if and only if every left (finitely generated) R -module is (amply) supplemented (see [8, 42.6§43.9]). $\frac{R}{P(R)}$ is perfect, where $P(R)$ is the sum of all radical left ideals of R if and only if every left R -module is *Rad-supplemented*, that is, every submodule of a module has a *Rad-supplement* in the module ([3, Theorem 6.1]). The other one is determine the structure of given modules classes over a Dedekind domain. It follows from [9, Proposition A.7 and Proposition A.8] and [3, Theorem 7.2 and Theorem 7.4] that the structure of (*Rad*–) supplemented modules over a (*local*) Dedekind domain is given.

For a module M , we consider the following submodule of M as in [4]:

$$\text{Soc}_s(M) = \sum \{N \ll M \mid N \text{ is simple}\}$$

Since $\text{Soc}_s(V) \subseteq \text{Rad}(V)$, it is of interest to investigate the analogue of this notion by replacing “ $\text{Rad}(V)$ ” with “ $\text{Soc}_s(V)$ ”. We will use the same definition and notation as in [5] to call a submodule V of a module M *ss-supplement* of a submodule U in M if $M = U + V$ and $U \cap V \subseteq \text{Soc}_s(V)$. It is shown in [5] that a submodule V of M is *ss-supplement* of some submodule U in M if and only if V is a supplement of U in M and $U \cap V$ is semisimple. Following [5], a module M is said to be *ss-supplemented* if every submodule of M has an *ss-supplement* in M . A submodule U of a module M has *ample ss-supplements* in M if every submodule V of M such that $M = U + V$ contains a *ss-supplement* of U in M . Also, It is called that a module M *amply ss-supplemented* if every submodule of M has ample *ss-supplements* in M . It is clear that every *ss-supplemented* module is supplemented. The basic properties of these modules were obtained in [5]. Clearly, the class of *ss-supplemented* modules is between the class of semisimple modules and the class of supplemented modules. It is shown in [5, Theorem 41] that a ring R is semiperfect with semisimple radical if and only if every left R -module is (amply) *ss-supplemented*.

2. Preliminaries

Throughout this study, all rings are associative with identity and all modules are unitary left modules. Let R be a ring and M be an R -module. The notation $U \subseteq M$ will mean that U is a submodule of M .

Definition 2.1 ([8]). *A submodule N of an R -module M is said to be small in M , denoted by $N \ll M$, if $M \neq N + K$ for every proper submodule K of M .*

Definition 2.2 ([8]). *Let M be an R -module. The sum of simple submodules of M is called the socle of M and is denoted by $\text{Soc}(M)$. If there are no simple submodules in M we put $\text{Soc}(M) = 0$. Also, $\text{Soc}(M)$ is the largest semisimple submodule of M .*

Definition 2.3 ([8]). *Let M be an R -module. The intersection of all maximal submodules of M or the sum of all small submodules of M is called the radical of M and is denoted by $\text{Rad}(M)$. If M has no maximal submodules we set $\text{Rad}(M) = M$.*

Lemma 2.1 ([5, Lemma 2]). *Let M be a module. Then, $\text{Soc}_s(M) = \text{Rad}(M) \cap \text{Soc}(M)$.*

Reduced modules and coatomic modules have a very important place in the theory of supplemented modules. Now we recall:

Definition 2.4 ([6]). *A module M is called reduced if whenever $A \subseteq M$ with $A = \text{Rad}(A)$ implies $A = 0$.*

Definition 2.5 ([6]). *A module M is called coatomic if every proper submodule of M is contained in a maximal submodule of M .*

Note that, a coatomic module has small radical. Semisimple modules or finitely generated modules are coatomic.

Definition 2.6 ([8]). *A non-zero module M is called hollow if every proper submodule of M is small in M and is called local if M has a largest submodule, i.e. a proper submodule which contains all other proper submodules.*

It is obvious that a largest submodule has to be equal to the radical of M and that in this case $\text{Rad}(M)$ is maximal and small in M . It follows from [8, 41.4 (2)] that a module M is hollow and cyclic if and only if it is local. Note that hollow modules are clearly amply supplemented. A ring R is called *local ring* if ${}_R R$ is a local module.

Proposition 2.1 ([9, Proposition A.7 and Proposition A.8]). *Let R be a local Dedekind domain with maximal ideal P , quotient field K , and $Q = \frac{K}{R}$. Let a , b , c and n be natural numbers, and let $B(n_1, \dots, n_s)$ denote the direct sum of arbitrarily many copies of $\frac{R}{P^{n_1}}, \dots, \frac{R}{P^{n_s}}$. Then, an R -module M is supplemented if and only if M is isomorphic to the direct sum $R^a \oplus K^B \oplus Q^c \oplus B(1, 2, \dots, n)$.*

If R is a non-local Dedekind domain, an R -module M is supplemented if and only if every P -primary component (viewed as module over the localization R_P) has the structure described in the above form.

Definition 2.7 ([5]). A module M strongly local if it is local and $\text{Rad}(M) \subseteq \text{Soc}(M)$ and a ring R left strongly local ring if ${}_R R$ is a strongly local module.

It is also state that the following implications on modules:

$$\text{simple} \implies \text{strongly local} \implies \text{local}.$$

The following lemma comes from [5, Lemma 13].

Lemma 2.2 ([5, Lemma 13]). Let M be an ss -supplemented module and N be a small submodule of M . Then, $N \subseteq \text{Soc}_s(M)$.

In this paper, we completely determine the structure of ss -supplemented and amply ss -supplemented modules over Dedekind domains. We show that M is ss -supplemented if and only if it is isomorphic to a direct sum of strongly local R -modules. We prove that over a local Dedekind domain a module M is ss -supplemented if and only if it is amply ss -supplemented if and only if M is a bounded module with semisimple radical. In particular, we show that an abelian group M is ss -supplemented (as a $\mathbb{Z}$ -module) if and only if $M \cong (\bigoplus_{p \in I} \mathbb{Z}_p^{(v)}) \oplus (\bigoplus_{q \in J} \mathbb{Z}_{q^2}^{(\nu)})$, where $\mathbb{P}$ is the set of all prime integers, I, J are some subsets of $\mathbb{P}$ and v, ν are any index sets.

3. SS -supplemented modules over Dedekind domains

In this section, we aim to determine the structure of ss -supplemented modules over Dedekind domains. Unless stated otherwise, here we assume that every ring is a Dedekind domain which is not a field.

Remark 3.1. Let R be a Dedekind domain and K be the quotient field of R . Since $\frac{K}{R}$ is the hollow R -module which is not local, it follows from [5, Proposition 16] that $\frac{K}{R}$ is not ss -supplemented. Hence, again applying [5, Proposition 16], K is also not ss -supplemented.

The following proposition gives the classes of ss -supplemented modules.

Proposition 3.1. Let R be a Dedekind domain with the quotient field K and M be a left R -module. If M is ss -supplemented, then M is reduced, coatomic and $\text{Rad}(M) \subseteq \text{Soc}(M)$.

Proof. Let M be a ss -supplemented module and N be a radical submodule of M . Then, by [7, Lemma 4.4], N is injective and so there exists a submodule L of M such that $M = N \oplus L$. Therefore, N is ss -supplemented by [5, Proposition 26]. Note that $N \cong K^{(I)} \oplus (\frac{K}{R})^{(J)}$ for some index sets I and J . It follows from

[5, Proposition 26] and Remark 3.1 that $I = J = \emptyset$. It means that $N = 0$, that is, M is reduced.

Let U be a proper submodule of M . Then, U has a ss -supplement, say V , in M . Since M is reduced, V has a maximal submodule P . It is easy to see that $U + P$ is a maximal submodule of M and so M is coatomic. Moreover, $\text{Rad}(M) \subseteq \text{Soc}(M)$ by Lemma 2.2. $\square$

Note that a reduced hollow module is local.

Proposition 3.2. *Let R be a Dedekind domain and M be a non-zero left R -module. Then, M is ss -supplemented if and only if it is isomorphic to a direct sum of strongly local R -modules.*

Proof. Let M be a ss -supplemented module. By Proposition 3.1, we have M is reduced. Therefore, it is supplemented and so we can write $M \cong \bigoplus_{i \in I} L_i$, where each L_i is local by Proposition 2.1. Since M is ss -supplemented, it follows from [5, Proposition 26] that each L_i is ss -supplemented. By [5, Proposition 16], each L_i is a strongly local module. This is desired conclusion. The converse follows from [5, Theorem 27]. $\square$

Example 3.1. Consider the left $\mathbb{Z}$ -module $M = \bigoplus_{p \in \mathbb{P}} \mathbb{Z}_{p^2}$. It follows from [5, Theorem 27] that M is ss -supplemented.

A module M over a commutative domain R is said to be *bounded* if $rM = 0$ for some non-zero $r \in R$.

Proposition 3.3. *Let R be a local Dedekind domain and M be a left R -module. Then, M is ss -supplemented if and only if it is a bounded module with semisimple radical.*

Proof. If M is ss -supplemented, then it is coatomic and $\text{Rad}(M) \subseteq \text{Soc}(M)$ by Proposition 3.1. By [6, Lemma 2.1], there exists a nonnegative integer and a bounded submodule N of M such that $M \cong R^{(n)} \oplus N$. By [5, Proposition 26], R is ss -supplemented. It follows from [5, Proposition 16] that R is strongly local and so $n = 0$ by [5, Proposition 11]. It means that M is bounded.

Conversely, let M be a bounded module with $\text{Rad}(M) \subseteq \text{Soc}(M)$. Then, there exists an ideal I of R such that $IM = 0$. So, M can be considered as a $\frac{R}{I}$ -module. Since $\frac{R}{I}$ is an artinian ring, it follows from [8, 43.9] that M is a supplemented $\frac{R}{I}$ -module. Therefore, M is a supplemented R -module. Hence it is ss -supplemented by [5, Lemma 19]. $\square$

Lemma 3.1. *Let R be a local Dedekind domain and M be a left R -module. Then, M is a strongly local module if and only if $M \cong \frac{R}{pR}$ or $M \cong \frac{R}{p^2R}$, where p is the prime element of R .*

Proof. Since M is a strongly local module, it is local and so $M \cong \frac{R}{p^n R}$ for some $n \geq 1$. Therefore, we can write $\text{Rad}(\frac{R}{p^n R}) = \frac{pR}{p^n R} \subseteq \text{Soc}(\frac{R}{p^n R}) \cong \frac{R}{pR}$, and so $n = 1$ or $n = 2$. The converse is clear. $\square$

Theorem 3.1. *Let R be a local Dedekind domain and M be a left R -module. The following statements are equivalent:*

- (1) M is *ss-supplemented*.
- (2) M is bounded and $\text{Rad}(M) \subseteq \text{Soc}(M)$.
- (3) For every submodule N of M , N is bounded and $\text{Rad}(N) \subseteq \text{Soc}(N)$.
- (4) Every submodule of M is *ss-supplemented*.
- (5) M is *amply ss-supplemented*.
- (6) M is isomorphic to a direct sum of $\frac{R}{pR}$'s and $\frac{R}{p^2R}$'s, where p is the prime element of R .

Proof. (1) $\iff$ (2) By Proposition 3.3.

(2) $\implies$ (3) Let $N \subseteq M$. Therefore, N is bounded as a submodule of the bounded module M . By [8, 21.2 (2)], we obtain that $\text{Rad}(N) \subseteq \text{Rad}(M) \cap N \subseteq \text{Soc}(M) \cap N = \text{Soc}(N)$.

(3) $\implies$ (4) It follows from Proposition 3.3

(4) $\implies$ (5) By [5, Corollary 36].

(5) $\implies$ (6) and (6) $\implies$ (1) By Proposition 3.2 and Lemma 3.1. $\square$

Theorem 3.2. *Let R be a non-local Dedekind domain and M be a left R -module. The following statements are equivalent:*

- (1) M is *ss-supplemented*.
- (2) M is *supplemented*, *coatomic* and $\text{Rad}(M) \subseteq \text{Soc}(M)$.
- (3) M is isomorphic to a direct sum of strongly local R -modules.

Proof. (1) $\iff$ (3) By Proposition 3.2.

(1) $\iff$ (2) It follows from By Proposition 3.1 and [5, Lemma 19]. $\square$

Lemma 3.2. *An abelian group M is strongly local (as a $\mathbb{Z}$ -module) if and only if $M \cong \mathbb{Z}_p$ or $M \cong \mathbb{Z}_{p^2}$, where p is a prime integer in $\mathbb{Z}$.*

Proof. Let M be a strongly local group. Then, M is local and so there exists a prime integer p and $1 \leq n \in \mathbb{Z}^+$ such that $M \cong \mathbb{Z}_{p^n}$. Note that $\text{Rad}(M) \cong \mathbb{Z}_{p^{n-1}}$ and $\text{Soc}(M) \cong \mathbb{Z}_p$. Since M is strongly local, we have $\text{Rad}(M) \subseteq \text{Soc}(M)$ and so $n = 1$ or $n = 2$. The converse is clear. $\square$

Combining Proposition 3.2 and Lemma 3.2, we give the structure of *ss-supplemented* abelian groups.

Corollary 3.1. *An abelian group M is *ss-supplemented* (as a $\mathbb{Z}$ -module) if and only if $M \cong (\bigoplus_{p \in I} \mathbb{Z}_p^{(v)}) \oplus (\bigoplus_{q \in J} \mathbb{Z}_{q^2}^{(\nu)})$, where $\mathbb{P}$ is the set of all prime integers, I, J are some subsets of $\mathbb{P}$ and v, ν are any index sets.*

Example 3.2. Given the Abelian group $G = \mathbb{Z}_4^{(\mathbb{N})}$. Following Corollary 3.1, G is ss -supplemented as a $\mathbb{Z}$ -module.

The following lemma is an analogous of [6, Lemma 2.3] for ss -supplement submodules.

Lemma 3.3. *Let R be a local Dedekind domain and M be an R -module. A submodule U of $Rad(M)$ has a ss -supplement in M if and only if $U = Rad(U) \oplus S$, where S is a semisimple submodule of U .*

Proof. Let V be a ss -supplement of U in M . Then, we can write $M = U + V$, $U \cap V \ll V$ and $U \cap V$ is semisimple. By [8, 41.1.(5)], $U \cap V + Rad(U) = (U \cap V) \cap Rad(M) + Rad(U) = U \cap (V \cap Rad(M)) + Rad(U) = U \cap (Rad(V) + Rad(U)) = U \cap (pV + pU) = U \cap pM = U \cap Rad(M) = U$, where p is the prime element of R . Thus, $U = Rad(U) \oplus S$ for some $S \subseteq U \cap V$.

Conversely, since $U = Rad(U) \oplus S$, we get $Rad(U) = Rad(Rad(U))$ and so $Rad(U)$ is injective by [7, Lemma 4.1]. It follows that $M = Rad(U) \oplus V$ for some submodule V of M . By the module law, we can write that $U = U \cap M = U \cap (Rad(U) \oplus V) = Rad(U) \oplus (U \cap V)$. Since R is a local ring, it follows from [2, Theorem 3.5] that $U \cap V \cong \frac{U}{Rad(U)}$ is semisimple. Since $M = U + V$ and $U \cap V \subseteq U \subseteq Rad(M)$, it follows from [8, 19.3 (5)] and [5, Lemma 1] that $U \cap V$ is small in V . It means that V is a ss -supplement of U in M . $\square$

Let R be an arbitrary ring and M be an R -module. Recall from [8] that an injective module E together with essential monomorphism $\Phi : M \rightarrow E$ *injective hull* of M . It is known in [8, 17.9] that every R -module has only one (up to isomorphism) injective hull. We denote by $E(M)$ the injective hull of a module M .

Next, we obtain the following result which determines the structure of a module has a ss -supplement in its injective hull over local Dedekind domains.

Theorem 3.3. *Let R be a local Dedekind domain and M be a left R -module. The following statements are equivalent:*

- (1) M has a ss -supplement in $E(M)$, where $E(M)$ is the injective hull of M ,
- (2) $M \cong K^{(I)} \oplus (\frac{K}{R})^{(J)} \oplus S$, where K is the quotient field of R , I and J denote some index sets and S is a semisimple R -module,
- (3) $Rad(M)$ is a direct summand of M .

Proof. (1) $\implies$ (2) By [7, Lemma 4.1], $E(M)$ is a radical module and so $M \subseteq E(M) = Rad(E(M))$. Suppose that M has a ss -supplement in $E(M)$. Applying Lemma 3.3, we obtain that $M = Rad(M) \oplus S$, where S is semisimple. Therefore, $Rad(M) = P(M) \cong K^{(I)} \oplus (\frac{K}{R})^{(J)}$ for some index sets I and J . Hence $M \cong K^{(I)} \oplus (\frac{K}{R})^{(J)} \oplus S$, where S is a semisimple R -module.

(2) $\implies$ (1) It is clear by Lemma 3.3.

(1) $\iff$ (3) This can be proved by taking $U = \text{Rad}(M)$ in the Lemma 3.3. $\square$

Example 3.3. Put $M = \mathbb{Q} \oplus \mathbb{Z}_3$, where $\mathbb{Q}$ is the fractional field of the domain $\mathbb{Z}$. Therefore, $\text{Rad}(M) = \mathbb{Q} \oplus \{\bar{0}\}$ and thus $M = \text{Rad}(M) \oplus \mathbb{Z}_3$. So, $\text{Rad}(M)$ is a direct summand of M . Hence, by Theorem 3.3, M has a *ss*-supplement in $E(M) = \mathbb{Q} \oplus \mathbb{Z}_3^\infty$.

In [1], a module M is said to be $\oplus$ -*radical supplemented* if $\text{Rad}(M)$ has a supplement that is a direct summand of M . It is shown in [1, Theorem 3.1] that over a local Dedekind domain a module M is $\oplus$ -radical supplemented if and only if $M \cong K^{(I)} \oplus (\frac{K}{R})^{(J)} \oplus R^{(n)} \oplus N$, where K is the quotient field of R , I and J denote some index sets, n is a non-negative integer and N is a bounded R -module. Note that over a local Dedekind domain (which is not field) every semisimple module is bounded. Using this fact, Theorem 3.3 and [1, Theorem 3.1], we get the following result:

Corollary 3.2. *Let R be a local Dedekind domain and M be a left R -module. If $\text{Rad}(M)$ has a *ss*-supplement in M , then it is $\oplus$ -radical supplemented.*

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Intuitionistic fuzzy threshold hypergraphs

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Abstract. A hypergraph is a graph that allows any number of vertices to be connected by an edge. A threshold hypergraph is a hypergraph $\mathbb{H}$ for which there exists a function $w : U(\mathbb{H}) \rightarrow \mathbb{N} \cup \{0\}$ and a non-negative integer s such that for all $V \subseteq U(\mathbb{H})$, V is independent if and only if $\sum_{v \in V} w(v) \leq s$. We introduce Intuitionistic Fuzzy Threshold Hypergraph(IFTHG) along with some of the multiple forms of them such as core, simple, elementary, sectionally elementary IFTHGs and (μ, ν) -tempered IFTHGs, with few of its properties. It is further proven that if an IFTHG $\mathbb{H}_{\mathbb{G}}$ is elementary, support simple and simply ordered, then $\mathbb{H}_{\mathbb{G}}$ is a (μ, ν) -tempered IFTHG. Further, we discuss how this method can be used to recognize water wastage using IFTHG and identify regions where it can be reduced, demonstrating that it is more effective for controlling water management systems compared to other methodologies.

Keywords: Intuitionistic fuzzy threshold hypergraph(IFTHG), Core, Simple, Support Simple, Elementary and Sectionally elementary IFTHGs, (μ, ν) -tempered IFTHG.

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1. Introduction

This research introduces the Intuitionistic Fuzzy Threshold Graphs(IFTHGs) and some types to address the limitations of traditional water management systems, which struggle to model complex real-world water distribution networks. The objective is to enhance the accuracy of detecting water leakage and wastage, leading to more efficient water management. The gap in existing research is the inadequacy of conventional graph-based methods to effectively manage intricate water systems. The novelty of this study is the application of IFTHG, which provides a more precise and effective solution for optimizing water usage.

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In set theory, Zadeh[21] created fuzzy sets as a technique of conveying ambiguity and vagueness. Fuzzy set theory has sparked attention to variety of fields. Atanassov[4, 7] came up with the concept of Intuitionistic Fuzzy Sets(IFS) as a generalisation of fuzzy sets, and Atanassov added an additional module to the concept of fuzzy set (which specifies the degree of non-membership). The concept behind intuitionistic fuzzy graphs were discussed in [5, 6].

Euler introduced the notion of graph theory in 1736. The graph theory is a powerful tool for addressing combinatorial issues in a various fields. To deal with the complexities of application base, the concept of graph was expanded to provide a hypergraph, which is a set of V vertices including a collection of V subsets. The notions of graph and hypergraph were introduced by Berge [8]. Hypergraphs, an extension of classical graphs, are used to model complex relationships among multiple elements by Lee-Kwang & Lee [11]. Muhammad Akram have explored the integration of fuzzy logic into hypergraph structures, known as fuzzy hypergraphs (Akram & Dubek, 2013)[1]. Akram and Luqman (2020)[12] conducted a comprehensive study on fuzzy hypergraphs, making significant contributions to the field. The study of hypergraphs, particularly in fuzzy and soft computing, plays a crucial role in advancing computational intelligence and decision-making methods (Akram et al.,)[2]. These structures are extensively studied in decision-making and computational applications due to their versatility by Akram et al.(2022)[3]. Sarwar et al. (2023)[20] introduced a new group decision-making approach using rough soft approximations of hypergraphs, demonstrating their practical applications. Akram and Nawaz (2023)[13] applied single-valued neutrosophic soft hypergraphs to model the human nervous system, showing the interdisciplinary nature of hypergraph theory. Nawaz, Akram, and Alcantud (2022)[16] developed an algorithm to compute the strength of competing interactions in the Bering Sea using Pythagorean fuzzy hypergraphs.

Intuitionistic fuzzy graphs and Intuitionistic fuzzy hypergraphs and intuitionistic fuzzy directed hypergraphs have been introduced in [17, 18, 19]. Some types of intuitionistic fuzzy directed hypergraphs are discussed in [14]. Lanzhen Yang and Hua Mao [10] introduced intuitionistic fuzzy threshold graph and explained its applications. Chvatal and Hammer were the first to introduce threshold graphs. Historical work by Chvatal and Hammer (1973)[9] on set-packing problems and threshold graphs underscores the longstanding interest in hypergraph theory. Myithili and Nandhini (2024)[15] reviewed intuitionistic fuzzy threshold hypergraphs, contributing to the theoretical expansion of fuzzy hypergraphs.

The new features of the proposed method in the water management system include its ability to accurately model complex structures using IFTHG, which surpasses traditional graph-based methods that often fall short in capturing real-life complexities. The main advantages of the results are the precise detection of water leakage or wastage in specific regions, facilitated by threshold values that maintain optimal water levels, leading to more efficient and targeted water

management. This approach helps prioritize maintenance resources, such as manpower and materials, ensuring they are used where they are most needed, reducing overall costs.

2. Preliminaries

We stepped over some fundamental definitions in this section that are related to our main concept.

Definition 2.1 ([7]). Let a set E be fixed. An *intuitionistic fuzzy set (IFS)* V in E is an object of the form $V = \{\langle v_i, \mu_i(v_i), \nu_i(v_i) \rangle / v_i \in E\}$, where the function $\mu_i : E \rightarrow [0, 1]$ and $\nu_i : E \rightarrow [0, 1]$ determine the degree of membership and the degree of non-membership of the element $v_i \in E$, respectively and for every $v_i \in E$, $0 \leq \mu_i(v_i) + \nu_i(v_i) \leq 1$.

Definition 2.2 ([5]). The six Cartesian products of two IFSs V_1, V_2 of V over E are defined as

$$\begin{aligned} V_1 \times_1 V_2 &= \{\langle (v_1, v_2), \mu_1 \cdot \mu_2, \nu_1 \cdot \nu_2 \rangle | v_1 \in V_1, v_2 \in V_2\}, \\ V_1 \times_2 V_2 &= \{\langle (v_1, v_2), \mu_1 + \mu_2 - \mu_1 \cdot \mu_2, \nu_1 \cdot \nu_2 \rangle | v_1 \in V_1, v_2 \in V_2\}, \\ V_1 \times_3 V_2 &= \{\langle (v_1, v_2), \mu_1 \cdot \mu_2, \nu_1 + \nu_2 - \nu_1 \cdot \nu_2 \rangle | v_1 \in V_1, v_2 \in V_2\}, \\ V_1 \times_4 V_2 &= \{\langle (v_1, v_2), \min(\mu_1, \mu_2), \max(\nu_1, \nu_2) \rangle | v_1 \in V_1, v_2 \in V_2\}, \\ V_1 \times_5 V_2 &= \{\langle (v_1, v_2), \max(\mu_1, \mu_2), \min(\nu_1, \nu_2) \rangle | v_1 \in V_1, v_2 \in V_2\}, \\ V_1 \times_6 V_2 &= \left\{ \left\langle (v_1, v_2), \frac{\mu_1(v_1) + \mu_2(v_2)}{2}, \frac{\nu_1(v_1) + \nu_2(v_2)}{2} \right\rangle | v_1 \in V_1, v_2 \in V_2 \right\}. \end{aligned}$$

It must be noted that $v_i \times_s v_j$ is an IFS, where $s = 1, 2, 3, 4, 5, 6$.

Definition 2.3 ([17]). An *intuitionistic fuzzy graph (IFG)* is of the form $G = (V, E)$, where

- (i) $V = \{v_1, v_2, \dots, v_n\}$ such that $\mu_i : V \rightarrow [0, 1]$ and $\nu_i : V \rightarrow [0, 1]$ denote the degrees of membership and non-membership of the element $v_i \in V$ respectively and $0 \leq \mu_i(v_i) + \nu_i(v_i) \leq 1$ for every $v_i \in V$, $i = 1, 2, \dots, n$.
- (ii) $E \subseteq V \times V$ and $\mu_{ij} : V \times V \rightarrow [0, 1]$ and $\nu_{ij} : V \times V \rightarrow [0, 1]$ are such that $\mu_{ij} \leq \mu_i \wedge \mu_j$, $\nu_{ij} \leq \nu_i \vee \nu_j$ and $0 \leq \mu_i(v_i) + \nu_i(v_i) \leq 1$, where μ_{ij} and ν_{ij} are the membership and non-membership values of the edge (v_i, v_j) . The values of $\mu_i \wedge \mu_j$ and $\nu_i \vee \nu_j$ can be determined by one of the cartesian products $\times_s$, $s = 1, 2, \dots, 6$ for all i and j given in Definition 2.2.

Note: Throughout this paper, it is assumed that the fourth Cartesian product

$$V_{i_1} \times V_{i_2} \times V_{i_3} \times \dots \times V_{i_n} = \left\{ \left\langle (v_1, v_2, \dots, v_n), \min(\mu_1, \mu_2, \dots, \mu_n), \max(\nu_1, \nu_2, \dots, \nu_n) \right\rangle | v_1 \in V_1, v_2 \in V_2, \dots, v_n \in V_n \right\},$$

is used to determine the edge membership μ_{ij} and the edge non-membership ν_{ij} .

Definition 2.4 ([19]). An intuitionistic fuzzy hypergraph (IFHG) is an ordered pair $H = (V, E)$ where

- (i) $V = \{v_1, v_2, \dots, v_n\}$ is a finite set of intuitionistic fuzzy vertices;
- (ii) $E = \{E_1, E_2, \dots, E_m\}$ is a family of crisp subsets of V ;
- (iii) $E_j = \{(v_i, \mu_j(v_i), \nu_j(v_i)) : \mu_j(v_i), \nu_j(v_i) \geq 0 \text{ and } \mu_j(v_i) + \nu_j(v_i) \leq 1\}$,
 $j = 1, 2, \dots, m$;
- (iv) $E_j \neq \emptyset$, $j = 1, 2, \dots, m$;
- (v) $\bigcup_j \text{supp}(E_j) = V$, $j = 1, 2, \dots, m$.

Here, the hyperedges E_j are crisp sets of intuitionistic fuzzy vertices. Moreover, $\mu_j(v_i)$ and $\nu_j(v_i)$ denote the degrees of membership and non-membership of vertex v_i to edge E_j . Thus, the elements of the incidence matrix of IFHG are of the form $(v_{ij}, \mu_j(v_i), \nu_j(v_i))$. The sets (V, E) are crisp sets.

Notation list[14]

- $\langle \mu(u_i), \nu(u_i) \rangle$ or simply $\langle \mu_i, \nu_i \rangle$ denote the degrees of membership and non-membership of the vertex $v_i \in V$, such that $0 \leq \mu_i + \nu_i \leq 1$.
- $\langle \mu(u_{ij}), \nu(u_{ij}) \rangle$ or simply $\langle \mu_{ij}, \nu_{ij} \rangle$ denote the degrees of membership and non-membership of the hyperedge $(u_i, u_j) \in V \times V$, such that $0 \leq \mu_{ij} + \nu_{ij} \leq 1$.
- μ_{ij} and ν_{ij} are the membership and non-membership value of i^{th} vertex in j^{th} hyperedge.
- The support of an IFS V in $\mathcal{E}$ is denoted by $\text{supp}(\mathcal{E}_j) = \{u_i | \mu_{ij}(u_i) > 0 \text{ and } \nu_{ij}(u_i) > 0\}$, where $\mathcal{E}_j$ is a hyperedge in intuitionistic fuzzy threshold hypergraph.

3. Intuitionistic fuzzy threshold hypergraph

Definition 3.1 ([15]). The *Intuitionistic Fuzzy Threshold Hypergraph* (IFTHG) is defined as $\mathbb{H}_G = (U, \mathcal{E}; s_1, s_2)$ where

- (i) $U = \{u_1, u_2, \dots, u_n\}$ is a finite set of intuitionistic fuzzy vertices;
- (ii) $\mathcal{E} = \{\mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_m\}$ is a family of crisp subsets of U ;
- (iii) $\mathcal{E}_j = \{u_i, \mu_j(u_i), \nu_j(u_i) / 0 \leq \mu_j(u_i) + \nu_j(u_i) \leq 1\}$, $j = 1, 2, \dots, m$;
- (iv) $\mathcal{E}_j \neq \emptyset$, $j = 1, 2, \dots, m$;
- (v) $\bigcup_j \text{supp}(\mathcal{E}_j) = U$, $j = 1, 2, \dots, m$;
- (vi) an independent set $V \subseteq U$ has a set of all distinct combinations of a non-adjacent vertices in $\mathbb{H}_G$ if and only if there exists a threshold values $s_1, s_2 > 0$ such that $\sum_{u_i \in V} \mu_j(u_i) \leq s_1$ and $\sum_{u_i \in V} (1 - \nu_j(u_i)) \leq s_2$.

Example. Consider an IFTHG $\mathbb{H}_{\mathbb{G}}$ such that $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ and $\mathcal{E} = \{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4, \mathcal{E}_5, \mathcal{E}_6\}$, where $\mathcal{E}_1 = \{u_1, u_6\}$, $\mathcal{E}_2 = \{u_2, u_6\}$, $\mathcal{E}_3 = \{u_2, u_3\}$, $\mathcal{E}_4 = \{u_3, u_6\}$, $\mathcal{E}_5 = \{u_3, u_4\}$ and $\mathcal{E}_6 = \{u_4, u_5, u_6\}$ with threshold value $\langle 0.4, 0.5 \rangle$ is represented in the following incidence matrix.

$$\begin{array}{c} \begin{matrix} & \mathcal{E}_1 & \mathcal{E}_2 & \mathcal{E}_3 & \mathcal{E}_4 & \mathcal{E}_5 & \mathcal{E}_6 \end{matrix} \\ \begin{matrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{matrix} \left(\begin{array}{cccccc} \langle 0.3, 0.6 \rangle & \langle 0, 1 \rangle & \langle 0, 1 \rangle & \langle 0, 1 \rangle & \langle 0, 1 \rangle & \langle 0, 1 \rangle \\ \langle 0, 1 \rangle & \langle 0.2, 0.6 \rangle & \langle 0.2, 0.6 \rangle & \langle 0, 1 \rangle & \langle 0, 1 \rangle & \langle 0, 1 \rangle \\ \langle 0, 1 \rangle & \langle 0, 1 \rangle & \langle 0.3, 0.5 \rangle & \langle 0.3, 0.5 \rangle & \langle 0.3, 0.5 \rangle & \langle 0, 1 \rangle \\ \langle 0, 1 \rangle & \langle 0, 1 \rangle & \langle 0, 1 \rangle & \langle 0, 1 \rangle & \langle 0.3, 0.7 \rangle & \langle 0.3, 0.7 \rangle \\ \langle 0, 1 \rangle & \langle 0, 1 \rangle & \langle 0, 1 \rangle & \langle 0, 1 \rangle & \langle 0, 1 \rangle & \langle 0.3, 0.5 \rangle \\ \langle 0.4, 0.5 \rangle & \langle 0.4, 0.5 \rangle & \langle 0.4, 0.5 \rangle & \langle 0.4, 0.5 \rangle & \langle 0.4, 0.5 \rangle & \langle 0.4, 0.5 \rangle \end{array} \right) \end{array}$$

The IFTHG $\mathbb{H}_{\mathbb{G}} = (U, \mathcal{E}; 0.4, 0.5)$ has been given below.

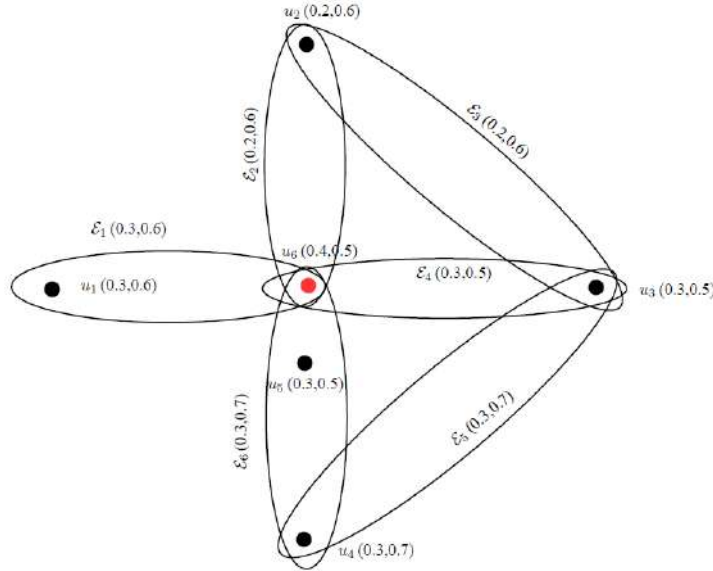


Figure 1: IFTHG

Proposition 3.1. *An IFTHG is an expansion of intuitionistic fuzzy threshold graph (IFTG).*

Proof. Let $\mathbb{H}_{\mathbb{G}} = (U, \mathcal{E}; s_1, s_2)$ be an intuitionistic fuzzy threshold hypergraph. Then by its definition, an independent set $V \subseteq U$ has a set of all distinct combinations of non-adjacent vertices in $\mathbb{H}_{\mathbb{G}}$ if and only if there exists threshold values $s_1, s_2 > 0$ such that $\sum_{u_i \in V} \mu_j(u_i) \leq s_1$ and $\sum_{u_i \in V} (1 - \nu_j(u_i)) \leq s_2$.

We know that, in an IFTG, an edge has only two vertices. As a result, we consider a hyperedge instead of an edge, which contains two or more vertices which is an intuitionistic fuzzy threshold hypergraph. Therefore, an IFTHG is an expansion of IFTG. $\square$

Definition 3.2. Let $\mathbb{H}_{\mathbb{G}} = (U, \mathcal{E}; s_1, s_2)$ be an IFTHG. Then the *height* of $\mathbb{H}_{\mathbb{G}}$ is defined by $h(\mathbb{H}_{\mathbb{G}}) = \{\max(\min(\mu_{ij})), \max(\max(\nu_{ij}))\}$, for which $\sum_{u_i \in V} \mu_j(u_i) \leq s_1$ and $\sum_{u_i \in V} (1 - \nu_j(u_i)) \leq s_2$ for all $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$, where μ_{ij} and ν_{ij} is the membership and non-membership value of i^{th} vertex in j^{th} hyperedge respectively.

Definition 3.3. An IFTHG $\mathbb{H}_{\mathbb{G}} = (U, \mathcal{E}; s_1, s_2)$ is *simple* if there exists an independent set $V \subseteq U$ such that $\mathcal{E}$ has no repeated intuitionistic fuzzy hyperedges and whenever $\mathcal{E}_j, \mathcal{E}_k \in \mathcal{E}$ and $\mathcal{E}_j \subseteq \mathcal{E}_k$ then $\mathcal{E}_j = \mathcal{E}_k$ for which $\sum_{u_i \in V} \mu_{ij}(u_i) \leq s_1$ and $\sum_{u_i \in V} (1 - \nu_{ij})(u_i) \leq s_2$ for all j and k .

Example. Consider an IFTHG $\mathbb{H}_{\mathbb{G}}$ such that $U = \{u_1, u_2, u_3, u_4, u_5\}$, $\mathcal{E} = \{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3\}$ according to the incidence matrix below with threshold value $\langle 0.3, 0.6 \rangle$.

$$\begin{array}{c} \mathcal{E}_1 \quad \mathcal{E}_2 \quad \mathcal{E}_3 \\ \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{array} \left(\begin{array}{ccc} \langle 0.3, 0.4 \rangle & \langle 0, 1 \rangle & \langle 0, 1 \rangle \\ \langle 0, 1 \rangle & \langle 0.2, 0.6 \rangle & \langle 0, 1 \rangle \\ \langle 0, 1 \rangle & \langle 0, 1 \rangle & \langle 0.3, 0.5 \rangle \\ \langle 0, 1 \rangle & \langle 0, 1 \rangle & \langle 0.1, 0.7 \rangle \\ \langle 0.3, 0.6 \rangle & \langle 0.3, 0.6 \rangle & \langle 0.3, 0.6 \rangle \end{array} \right)$$

The IFTHG $\mathbb{H}_{\mathbb{G}} = (U, \mathcal{E}; 0.3, 0.6)$ has been shown below.

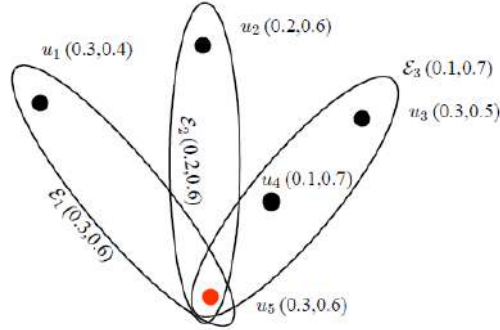


Figure 2: Simple IFTHG

Definition 3.4. An IFTHG, $\mathbb{H}_{\mathbb{G}} = (U, \mathcal{E}; s_1, s_2)$ is *support simple*, if there is an independent set $V \subseteq U$ such that $\mathcal{E}_j, \mathcal{E}_k \in \mathcal{E}$, $\mathcal{E}_j \subseteq \mathcal{E}_k$ and $\text{supp}(\mathcal{E}_j) = \text{supp}(\mathcal{E}_k)$ then $\mathcal{E}_j = \mathcal{E}_k$ for which $\sum_{u_i \in V} \mu_{ij}(u_i) \leq s_1$ and $\sum_{u_i \in V} (1 - \nu_{ij})(u_i) \leq s_2$, for all j and k . Here, the hyperedges $\mathcal{E}_j$ and $\mathcal{E}_k$ are called supporting hyperedges.

Example. Consider an IFTHG, $\mathbb{H}_{\mathbb{G}}$ such that $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$, $\mathcal{E} = \{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4\}$ according to the incidence matrix below with threshold value $\langle 0.3, 0.7 \rangle$.

$$\begin{array}{c} \begin{matrix} & \mathcal{E}_1 & \mathcal{E}_2 & \mathcal{E}_3 & \mathcal{E}_4 \end{matrix} \\ \begin{matrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{matrix} \left(\begin{array}{cccc} \langle 0.1, 0.7 \rangle & \langle 0, 1 \rangle & \langle 0, 1 \rangle & \langle 0, 1 \rangle \\ \langle 0, 1 \rangle & \langle 0.2, 0.5 \rangle & \langle 0, 1 \rangle & \langle 0, 1 \rangle \\ \langle 0, 1 \rangle & \langle 0, 1 \rangle & \langle 0.2, 0.6 \rangle & \langle 0, 1 \rangle \\ \langle 0, 1 \rangle & \langle 0, 1 \rangle & \langle 0, 1 \rangle & \langle 0.3, 0.5 \rangle \\ \langle 0, 1 \rangle & \langle 0, 1 \rangle & \langle 0.3, 0.5 \rangle & \langle 0, 1 \rangle \\ \langle 0.3, 0.7 \rangle & \langle 0.3, 0.7 \rangle & \langle 0.3, 0.7 \rangle & \langle 0.3, 0.7 \rangle \end{array} \right) \end{array}$$

The IFTHG $\mathbb{H}_{\mathbb{G}} = (U, \mathcal{E}; 0.3, 0.7)$ has been illustrated below.

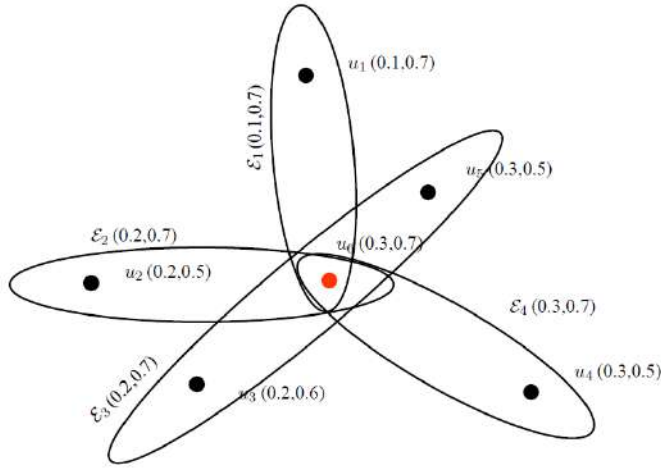


Figure 3: Support simple IFTHG

Here, $\mathcal{E}_2$ and $\mathcal{E}_4$ are supporting hyperedges.

Definition 3.5. Let $\mathbb{H}_{\mathbb{G}} = (U, \mathcal{E}; s_1, s_2)$ be an IFTHG and an independent set $V \subseteq U$ exists, if $\mathcal{E}_j, \mathcal{E}_k \in \mathcal{E}$ and $0 < \alpha, \beta \leq 1$. Then the (α, β) -level is defined by $(\mathcal{E}_j, \mathcal{E}_k)^{(\alpha, \beta)} = \{u_i \in U \mid \min(\mu_{ij}^\alpha(u_i)) \geq \alpha, \max(\nu_{ij}^\beta(u_i)) \leq \beta\}$ for which $\sum_{u_i \in V} \mu_{ij}(u_i) \leq s_1$ and $\sum_{u_i \in V} (1 - \nu_{ij})(u_i) \leq s_2$.

Definition 3.6. Let $\mathbb{H}_{\mathbb{G}} = (U, \mathcal{E}; s_1, s_2)$ be an IFTHG, there exists an independent set $V \subseteq U$ such that $\mathbb{H}_{\mathbb{G}}^{y_i, z_i} = \langle U^{y_i, z_i}, \mathcal{E}^{y_i, z_i} \rangle$ be the (y_i, z_i) -level of $\mathbb{H}_{\mathbb{G}}$. The sequence of real numbers $\{y_1, y_2, \dots, y_n; z_1, z_2, \dots, z_n\}$ such that $0 \leq y_i \leq h_\mu(\mathbb{H}_{\mathbb{G}})$ and $0 \leq z_i \leq h_\nu(\mathbb{H}_{\mathbb{G}})$, satisfying the properties:

- (i) If $y_1 < \alpha \leq 1$ and $0 \leq \beta < z_1$ then $\mathcal{E}^{\alpha, \beta} = \emptyset$;
- (ii) If $y_{i+1} \leq \alpha \leq y_i$; $z_i \leq \beta \leq z_{i+1}$ then $\mathcal{E}^{\alpha, \beta} = \mathcal{E}^{y_i, z_i}$;

(iii) $\mathcal{E}^{y_i, z_i} \sqsubset \mathcal{E}^{y_{i+1}, z_{i+1}}$;

for which $\sum_{u_i \in V} \mu_{ij}(u_i) \leq s_1$ and $\sum_{u_i \in V} (1 - \nu_{ij})(u_i) \leq s_2$ is called the *fundamental sequence* of $\mathbb{H}_{\mathbb{G}}$ and is denoted by $\mathcal{F}(\mathbb{H}_{\mathbb{G}})$.

Example. Consider an IFTHG, $\mathbb{H}_{\mathbb{G}}$ such that $U = \{u_1, u_2, u_3, u_4, u_5\}$, $\mathcal{E} = \{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3\}$ according to the incidence matrix below with threshold value $\langle 0.4, 0.6 \rangle$.

$$\begin{array}{c} \mathcal{E}_1 \quad \mathcal{E}_2 \quad \mathcal{E}_3 \\ \begin{array}{l} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{array} \left(\begin{array}{ccc} \langle 0.2, 0.7 \rangle & \langle 0, 1 \rangle & \langle 0, 1 \rangle \\ \langle 0, 1 \rangle & \langle 0.3, 0.6 \rangle & \langle 0, 1 \rangle \\ \langle 0, 1 \rangle & \langle 0, 1 \rangle & \langle 0.4, 0.5 \rangle \\ \langle 0, 1 \rangle & \langle 0, 1 \rangle & \langle 0.4, 0.4 \rangle \\ \langle 0.4, 0.6 \rangle & \langle 0.4, 0.6 \rangle & \langle 0.4, 0.6 \rangle \end{array} \right)$$

The IFTHG $\mathbb{H}_{\mathbb{G}} = (U, \mathcal{E}; 0.4, 0.6)$ has been given below.

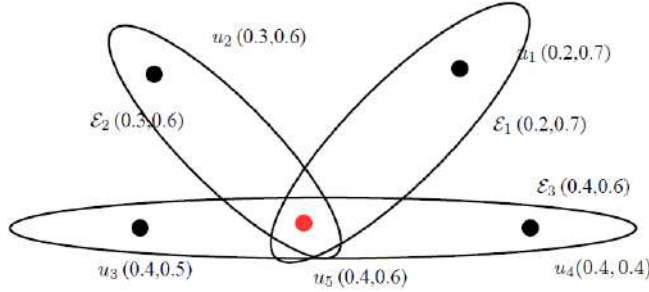


Figure 4: $\mathbb{H}_{\mathbb{G}}^{(\mu_j(y, z), \nu_j(y, z))}$ -level IFTHG

By examining the (y_n, z_n) -level IFTHG of $\mathbb{H}_{\mathbb{G}}$, we see that $\mathbb{H}_{\mathbb{G}}^{\langle 0.2, 0.7 \rangle} = \mathbb{H}_{\mathbb{G}}^{\langle 0.3, 0.6 \rangle} = \mathbb{H}_{\mathbb{G}}^{\langle 0.4, 0.6 \rangle}$ and $\mathcal{F}(\mathbb{H}_{\mathbb{G}}) = \{\langle 0.2, 0.7 \rangle, \langle 0.3, 0.6 \rangle, \langle 0.4, 0.6 \rangle\}$.

Definition 3.7. Let $\mathbb{H}_{\mathbb{G}} = (U, \mathcal{E}; s_1, s_2)$ be an IFTHG, also an independent set $V \subseteq U$ exists. Then, the *core set* of $\mathbb{H}_{\mathbb{G}}$ is denoted by $\mathcal{C}(\mathbb{H}_{\mathbb{G}})$ for $0 < (y_i, z_i) \leq h(\mathbb{H}_{\mathbb{G}})$, and is defined by $\mathcal{C}(\mathbb{H}_{\mathbb{G}}) = \{\mathbb{H}_{\mathbb{G}}^{y_1, z_1}, \mathbb{H}_{\mathbb{G}}^{y_2, z_2}, \dots, \mathbb{H}_{\mathbb{G}}^{y_n, z_n}\}$. The corresponding set of (y_i, z_i) -level hypergraphs is $\mathbb{H}_{\mathbb{G}}^{y_1, z_1} \subset \mathbb{H}_{\mathbb{G}}^{y_2, z_2} \subset \dots \subset \mathbb{H}_{\mathbb{G}}^{y_n, z_n}$ for which $\sum_{u_i \in V} \mu_{ij}(u_i) \leq s_1$ and $\sum_{u_i \in V} (1 - \nu_{ij})(u_i) \leq s_2$ is called the $\mathbb{H}_{\mathbb{G}}$ -induced fundamental sequence and is denoted by $\mathcal{F}(\mathbb{H}_{\mathbb{G}})$. The (y_n, z_n) -level is called the support level of $\mathbb{H}_{\mathbb{G}}$ and the $\mathbb{H}_{\mathbb{G}}^{y_n, z_n}$ is called the support of $\mathbb{H}_{\mathbb{G}}$.

Definition 3.8. Let $\mathbb{H}_{\mathbb{G}} = (U, \mathcal{E}; s_1, s_2)$ be an IFTHG is said to be *elementary*, if an independent set $V \subseteq U$ exists. Then $\mu_{ij} : U \rightarrow [0, 1]$ and $\nu_{ij} : U \rightarrow [0, 1]$ are constant functions or has a range $\{0, r\}$, $r \neq 0$ for which $\sum_{u_i \in V} \mu_{ij}(u_i) \leq s_1$ and $\sum_{u_i \in V} (1 - \nu_{ij})(u_i) \leq s_2$. If $|\text{supp}(\mu_{ij}, \nu_{ij})| = 1$, then it is called a *spike*.

Example. Consider an IFTHG, $\mathbb{H}_{\mathbb{G}}$ such that $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$, $\mathcal{E} = \{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4\}$ according to the incidence matrix below with threshold value $\langle 0.4, 0.5 \rangle$.

$$\begin{array}{c} \mathcal{E}_1 \quad \mathcal{E}_2 \quad \mathcal{E}_3 \quad \mathcal{E}_4 \\ \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \left(\begin{array}{cccc} \langle 0.4, 0.6 \rangle & \langle 0, 1 \rangle & \langle 0, 1 \rangle & \langle 0, 1 \rangle \\ \langle 0, 1 \rangle & \langle 0.4, 0.6 \rangle & \langle 0, 1 \rangle & \langle 0, 1 \rangle \\ \langle 0, 1 \rangle & \langle 0.4, 0.6 \rangle & \langle 0, 1 \rangle & \langle 0, 1 \rangle \\ \langle 0, 1 \rangle & \langle 0, 1 \rangle & \langle 0.4, 0.6 \rangle & \langle 0, 1 \rangle \\ \langle 0.4, 0.6 \rangle & \langle 0.4, 0.6 \rangle & \langle 0.4, 0.6 \rangle & \langle 0, 1 \rangle \\ \langle 0, 1 \rangle & \langle 0, 1 \rangle & \langle 0, 1 \rangle & \langle 0.4, 0.5 \rangle \end{array} \right)$$

The IFTHG $\mathbb{H}_{\mathbb{G}} = (U, \mathcal{E}; 0.4, 0.5)$ has been shown below.

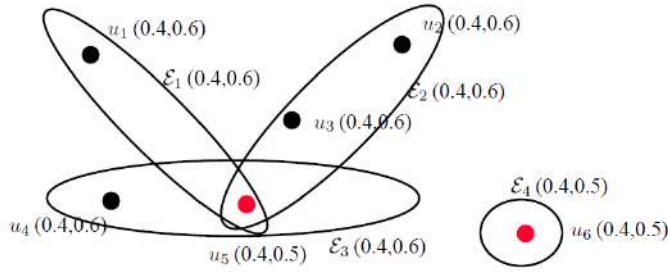


Figure 5: Elementary IFTHG

Theorem 3.1. *The intuitionistic fuzzy threshold hyperedges of an IFTHG are elementary.*

Definition 3.9. Consider $\mathbb{H}_{\mathbb{G}} = (U, \mathcal{E}; s_1, s_2)$ be an IFTHG, if there exists an independent set $V \subseteq U$ such that $\mathbb{H}_{\mathbb{G}}$ be an IFTHG and $\mathcal{F}(\mathbb{H}_{\mathbb{G}}) = \{y_n, y_{n-1}, \dots, y_1, z_1, z_2, \dots, z_n\}$ for which $\sum_{u_i \in V} \mu_{ij}(u_i) \leq s_1$ and $\sum_{u_i \in V} (1 - \nu_{ij})(u_i) \leq s_2$. Then $\mathbb{H}_{\mathbb{G}}$ is called *sectionally elementary* if for each $\mu_{ij}, \nu_{ij} \in \mathcal{E}$ and $y_n, z_n \in \mathcal{F}(\mathbb{H}_{\mathbb{G}})$, $\mu_{ij}^\alpha = \mu_{ij}^{y_i}$ and $\nu_{ij}^\beta = \nu_{ij}^{z_i}$ for all $\alpha, \beta \in (y_{i+1}, z_i]$. (we assume $y_{i+1} = 0$).

Example. Consider an IFTHG, $\mathbb{H}_{\mathbb{G}}$ such that $U = \{u_1, u_2, u_3, u_4, u_5\}$, $\mathcal{E} = \{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3\}$, and it is denoted by the upcoming incidence matrix with threshold value $\langle 0.4, 0.6 \rangle$.

$$\begin{array}{c} \mathcal{E}_1 \quad \mathcal{E}_2 \quad \mathcal{E}_3 \\ \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{array} \left(\begin{array}{ccc} \langle 0.4, 0.5 \rangle & \langle 0, 1 \rangle & \langle 0, 1 \rangle \\ \langle 0.4, 0.5 \rangle & \langle 0, 1 \rangle & \langle 0, 1 \rangle \\ \langle 0, 1 \rangle & \langle 0, 1 \rangle & \langle 0.3, 0.5 \rangle \\ \langle 0, 1 \rangle & \langle 0.3, 0.5 \rangle & \langle 0.3, 0.5 \rangle \\ \langle 0.4, 0.6 \rangle & \langle 0.4, 0.6 \rangle & \langle 0, 1 \rangle \end{array} \right)$$

The IFTHG $\mathbb{H}_G = (U, \mathcal{E}; 0.4, 0.6)$ has been shown below.

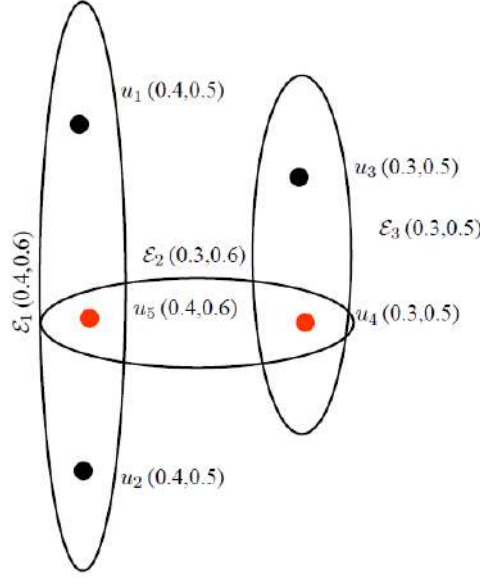


Figure 6: Sectionally Elementary IFTHG

Definition 3.10. Suppose $\mathbb{H}_G = (U, \mathcal{E}; s_1, s_2)$ and $\mathbb{H}_G' = (U', \mathcal{E}'; s_1', s_2')$ are IFTHGs, $\mathbb{H}_G$ is called a *partial IFTHG* of $\mathbb{H}_G'$ if there exists an independent set $V \subseteq U$ such that

$$u' = \begin{cases} \min(\text{supp}(\mu_{ij})) \& \sum_{u_i \in V} \mu_{ij}(u_i) \leq s_1 \mid \mu_{ij} \in \mathcal{E}' \\ \max(\text{supp}(\nu_{ij})) \& \sum_{u_i \in V} (1 - \nu_{ij})(u_i) \leq s_2 \mid \nu_{ij} \in \mathcal{E}' \end{cases}$$

The partial IFTHG generated by $\mathcal{E}'$ and is denoted by $\mathbb{H}_G \subseteq \mathbb{H}_G'$. Also, $\mathbb{H}_G \subset \mathbb{H}_G'$ if $\mathbb{H}_G \subseteq \mathbb{H}_G'$ and $\mathbb{H}_G \neq \mathbb{H}_G'$.

Definition 3.11. Let $\mathbb{H}_G = (U, \mathcal{E}; s_1, s_2)$ be an IFTHG, there exists an independent set $V \subseteq U$ such that $\mathcal{C}(\mathbb{H}_G) = \{\mathbb{H}_G^{y_1, z_1}, \mathbb{H}_G^{y_2, z_2}, \dots, \mathbb{H}_G^{y_n, z_n}\}$ for which $\sum_{u_i \in V} \mu_{ij}(u_i) \leq s_1$ and $\sum_{u_i \in V} (1 - \nu_{ij})(u_i) \leq s_2$. $\mathbb{H}_G$ is said to be *ordered* if $\mathcal{C}(\mathbb{H}_G)$ is ordered. That is $\mathbb{H}_G^{y_1, z_1} \subset \dots \subset \mathbb{H}_G^{y_2, z_2} \subset \mathbb{H}_G^{y_n, z_n}$ for which $\sum_{u_i \in V} \mu_{ij}(u_i) \leq s_1$ and $\sum_{u_i \in V} (1 - \nu_{ij})(u_i) \leq s_2$. The IFTHG is said to be *simply ordered* if the sequence $\{\mathbb{H}_G^{y_i, z_i} \mid i = 1, 2, \dots, n\}$ is simply ordered, that is if it is ordered and if whenever $\mathcal{E} \in \mathbb{H}_G^{y_{i+1}, z_{i+1}} \setminus \mathbb{H}_G^{y_i, z_i}$ then $\mathcal{E} \notin \mathbb{H}_G^{y_i, z_i}$.

Theorem 3.2. If $\mathbb{H}_G$ is an elementary intuitionistic fuzzy threshold hypergraph, then $\mathbb{H}_G$ is ordered. Also, if $\mathbb{H}_G$ is ordered IFTHG with $\mathcal{C}(\mathbb{H}_G) = \{\mathbb{H}_G^{y_1, z_1}, \mathbb{H}_G^{y_2, z_2}, \dots, \mathbb{H}_G^{y_n, z_n}\}$ and if $\mathbb{H}_G^{y_n, z_n}$ is simple, then $\mathbb{H}_G$ is elementary.

Definition 3.12. Let an IFTHG $\mathbb{H}_{\mathbb{G}} = (U, \mathcal{E}; s_1, s_2)$ and an independent set $V \subseteq U$. Then $\mathbb{H}_{\mathbb{G}}$ is called (μ, ν) -tempered intuitionistic fuzzy threshold hypergraph (tIFTHG), if there exists an intuitionistic fuzzy subset, $\mu_{ij} : U \rightarrow [0, 1]$ and $\nu_{ij} : U \rightarrow [0, 1]$ such that $\mathcal{E} = \{(\mu_{ij}(u_i), \nu_{ij}(u_i)) | u_i \in \mathcal{E}_i\}$ where

$$\mu_{ij} = \begin{cases} \wedge \mu_j(a) \text{ and } \sum_{u_i \in V} \mu_{ij}(u_i) \leq s_1 | a \in \mathcal{E}, & \text{if } u_i \in \mathcal{E}_i \\ 0, & \text{otherwise} \end{cases}$$

and

$$\nu_{ij} = \begin{cases} \vee \nu_j(a) \text{ and } \sum_{u_i \in V} (1 - \nu_{ij})(u_i) \leq s_2 | a \in \mathcal{E}, & \text{if } u_i \in \mathcal{E}_i \\ 0, & \text{otherwise} \end{cases}$$

where $\mu_{ij}(u_i)$ and $\nu_{ij}(u_i)$ are degrees of membership and non-membership functions of the element $u_i \in \mathcal{E}$ respectively for every $u_i \in \mathcal{E}$, $0 < \mu_{ij}(u_i) + \nu_{ij}(u_i) \leq 1$.

Example. Consider an IFTHG, $\mathbb{H}_{\mathbb{G}}$ such that $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$, $\mathcal{E} = \{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4\}$ according to the incidence matrix below with threshold value $\langle 0.3, 0.4 \rangle$.

$$\begin{array}{c} \mathcal{E}_1 \quad \mathcal{E}_2 \quad \mathcal{E}_3 \quad \mathcal{E}_4 \\ \begin{array}{l} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \left(\begin{array}{cccc} \langle 0.2, 0.5 \rangle & \langle 0, 1 \rangle & \langle 0, 1 \rangle & \langle 0, 1 \rangle \\ \langle 0, 1 \rangle & \langle 0, 1 \rangle & \langle 0, 1 \rangle & \langle 0.2, 0.5 \rangle \\ \langle 0, 1 \rangle & \langle 0, 1 \rangle & \langle 0.3, 0.4 \rangle & \langle 0, 1 \rangle \\ \langle 0, 1 \rangle & \langle 0, 1 \rangle & \langle 0.3, 0.4 \rangle & \langle 0, 1 \rangle \\ \langle 0, 1 \rangle & \langle 0.3, 0.4 \rangle & \langle 0.3, 0.4 \rangle & \langle 0, 1 \rangle \\ \langle 0.2, 0.5 \rangle & \langle 0, 1 \rangle & \langle 0, 1 \rangle & \langle 0.2, 0.5 \rangle \end{array} \right) \end{array}$$

The following hypergraph depicts the IFTHG $\mathbb{H}_{\mathbb{G}} = (U, \mathcal{E}; 0.3, 0.4)$.

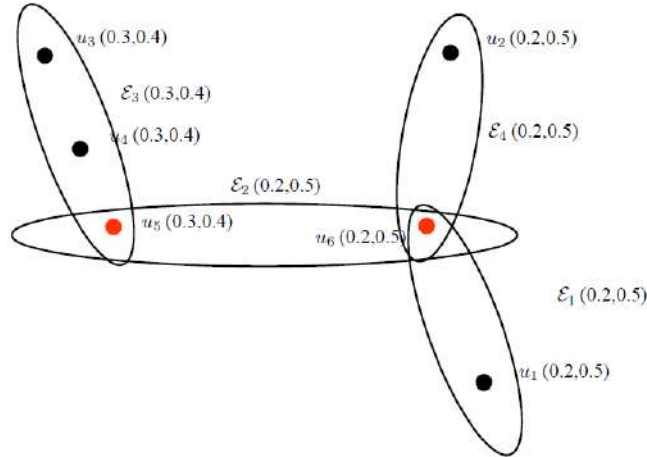


Figure 7: (μ, ν) -tIFTHG

$$\mathcal{E}^{(0.3,0.4)} = \{\{u_3, u_5\}, \{u_4, u_5\}\}$$

$$\mathcal{E}^{(0.2,0.5)} = \{\{u_3, u_5\}, \{u_4, u_5\}, \{u_2, u_6\}, \{u_1, u_6\}\}$$

Define $\mu_{ij} : U \rightarrow [0, 1]$ and $\nu_{ij} : U \rightarrow [0, 1]$ by

$$\mu_{1j}(u_1) = 0.2, \mu_{2j}(u_2) = 0.2, \mu_{3j}(u_3) = 0.3, \mu_{4j}(u_4) = 0.3, \mu_{5j}(u_5) = 0.3, \mu_{6j}(u_6) = 0.2 \text{ and}$$

$$\nu_{1j}(u_1) = 0.5, \nu_{2j}(u_2) = 0.5, \nu_{3j}(u_3) = 0.4, \nu_{4j}(u_4) = 0.4, \nu_{5j}(u_5) = 0.4, \nu_{6j}(u_6) = 0.5$$

Note that

$$\mu_{ij}(\mathcal{E}_1) = \mu_{1j}(u_1) \wedge \mu_{6j}(u_6) = 0.2$$

$$\mu_{ij}(\mathcal{E}_2) = \mu_{5j}(u_5) \wedge \mu_{6j}(u_6) = 0.2$$

$$\mu_{ij}(\mathcal{E}_3) = \mu_{3j}(u_3) \wedge \mu_{4j}(u_4) \wedge \mu_{5j}(u_5) = 0.3$$

$$\mu_{ij}(\mathcal{E}_4) = \mu_{2j}(u_2) \wedge \mu_{6j}(u_6) = 0.2 \text{ and}$$

$$\nu_{ij}(\mathcal{E}_1) = \nu_{1j}(u_1) \vee \nu_{6j}(u_6) = 0.5$$

$$\nu_{ij}(\mathcal{E}_2) = \nu_{5j}(u_5) \vee \nu_{6j}(u_6) = 0.5$$

$$\nu_{ij}(\mathcal{E}_3) = \nu_{3j}(u_3) \vee \nu_{4j}(u_4) \vee \nu_{5j}(u_5) = 0.4$$

$$\nu_{ij}(\mathcal{E}_4) = \nu_{2j}(u_2) \vee \nu_{6j}(u_6) = 0.5$$

Thus, $\mathbb{H}_{\mathbb{G}}$ is (μ, ν) - tIFTHG.

Theorem 3.3. *An IFTHG is (μ, ν) -tIFTHG of some crisp hypergraph $\mathbb{H}_{\mathbb{G}}$, if and only if $\mathbb{H}_{\mathbb{G}}$ is elementary, support simple and simply ordered.*

Proof. Let $\mathbb{H}_{\mathbb{G}} = (U, \mathcal{E}; s_1, s_2)$ be an (μ, ν) -tIFTHG of some crisp hypergraph $\mathbb{H}_{\mathbb{G}}$. There exists an independent set $V \subseteq U$ in $\mathbb{H}_{\mathbb{G}}$. Since, (μ, ν) -tempered the membership values and non-membership values of intuitionistic fuzzy hyperedges with $\sum_{u_i \in V} \mu_{ij}(u_i) \leq s_1$ and $\sum_{u_i \in V} (1 - \nu_{ij})(u_i) \leq s_2$ of $\mathbb{H}_{\mathbb{G}}$ are constant, hence it is elementary.

Clearly if support of two intuitionistic fuzzy hyperedges of the (μ, ν) -tIFTHG are equal then the intuitionistic fuzzy hyperedges with $\sum_{u_i \in V} \mu_{ij}(u_i) \leq s_1$ and $\sum_{u_i \in V} (1 - \nu_{ij})(u_i) \leq s_2$ are also equal. Therefore, it is support simple.

Let $\mathcal{C}(\mathbb{H}_{\mathbb{G}}) = \{\mathbb{H}_{\mathbb{G}}^{y_1, z_1}, \mathbb{H}_{\mathbb{G}}^{y_2, z_2}, \dots, \mathbb{H}_{\mathbb{G}}^{y_n, z_n}\}$. Since $\mathbb{H}_{\mathbb{G}}$ is elementary, it is ordered.

To prove: $\mathbb{H}_{\mathbb{G}}$ is simply ordered.

Let $\mathcal{E} \in \mathbb{H}_{\mathbb{G}}^{y_{i+1}, z_{i+1}} \setminus \mathbb{H}_{\mathbb{G}}^{y_i, z_i}$, then there exists $u_i \in \mathcal{E}$ such that $\mu_{ij}(u_i) = y_{i+1}$ and $\nu_{ij}(u_i) = z_{i+1}$. Since $y_{i+1} > y_i$ and $z_{i+1} > z_i$, it follows that $u_i \notin \mathbb{H}_{\mathbb{G}}^{y_i, z_i}$ and $\mathcal{E} \not\subseteq \mathbb{H}_{\mathbb{G}}^{y_i, z_i}$ for which $\sum_{u_i \in V} \mu_{ij}(u_i) \leq s_1$ and $\sum_{u_i \in V} (1 - \nu_{ij})(u_i) \leq s_2$. Thus, $\mathbb{H}_{\mathbb{G}}$ is simply ordered.

Conversely, suppose $\mathbb{H}_{\mathbb{G}} = (U, \mathcal{E}; s_1, s_2)$ is elementary, support simple and simply ordered.

To prove: (μ, ν) is tIFTHG.

We know that $\mathbb{H}_{\mathbb{G}}^{y_i, z_i} = \mathbb{H}_{\mathbb{G}i} = (U_i, \mathcal{E}_i; s_1, s_2)$. Also define $\mu_{ij} : U_i \rightarrow [0, 1]$ and $\nu_{ij} : U_i \rightarrow [0, 1]$ by

$$\mu_{ij}(u_i) = \begin{cases} y_1, & \text{if } u_i \in \mathcal{E}_1 \\ y_i, & \text{if } u_i \in \mathcal{E}_i \setminus \mathcal{E}_{i-1}, i = 1, 2, \dots, n \end{cases}$$

and

$$\nu_{ij}(u_i) = \begin{cases} z_1, & \text{if } u_i \in \mathcal{E}_1 \\ z_i, & \text{if } u_i \in \mathcal{E}_i \setminus \mathcal{E}_{i-1}, \quad i = 1, 2, \dots, n \end{cases}$$

To prove: $\mathcal{E} = \{(\mu_{ij}(u_i), \nu_{ij}(u_i)) | u_i \in \mathcal{E}_i\}$ where

$$\mu_{ij}(u_i) = \begin{cases} \wedge \mu_i(a) \text{ and } \sum_{u_i \in V} \mu_{ij}(u_i) \leq s_1 | a \in \mathcal{E}, & \text{if } u_i \in \mathcal{E}_i \\ 0, & \text{otherwise} \end{cases}$$

and

$$\nu_{ij}(u_i) = \begin{cases} \vee \nu_i(a) \text{ and } \sum_{u_i \in V} (1 - \nu_{ij})(u_i) \leq s_2 | a \in \mathcal{E}, & \text{if } u_i \in \mathcal{E}_i \\ 0, & \text{otherwise} \end{cases}$$

Let $\mathcal{E}' \in \mathcal{E}_i$.

Since $\mathbb{H}_{\mathbb{G}}$ is elementary and support simple, the intuitionistic fuzzy hyperedge (c_{ij}, d_{ij}) is unique in $\mathcal{E}$ which has support $\mathcal{E}'$. In fact, distinct hyperedges in $\mathcal{E}$ should have distinct supports which lie in $\mathcal{E}_i$ with $\sum_{u_i \in V} \mu_{ij}(u_i) \leq s_1$ and $\sum_{u_i \in V} (1 - \nu_{ij})(u_i) \leq s_2$.

To prove: For each $\mathcal{E}' \in \mathcal{E}_i$, $\mu_{ij}(u_i) = c_{ij}$, $\nu_{ij}(u_i) = d_{ij}$.

Since, all the hyperedges are elementary and different hyperedges have different supports, it follows from the definition of fundamental sequence, that $h(c_{ij}, d_{ij})$ is equal to some number of (y_i, z_i) in $\mathcal{F}(\mathbb{H}_{\mathbb{G}})$.

Consequently, $\mathcal{E}' \subseteq U_i$. Moreover if $i > 1$, then $\mathcal{E}' \in \mathcal{E}' \setminus \mathcal{E}_{i-1}$. Since $\mathcal{E} \subseteq U_i$, it follows from the definition of (μ, ν) -tempered that for each $u_i \in \mathcal{E}_i$, $\mu_{ij}(u_i) \geq y_i$ and $\mu_{ij}(u_i) \leq z_i$ with $\sum_{u_i \in V} \mu_{ij}(u_i) \leq s_1$ and $\sum_{u_i \in V} (1 - \nu_{ij})(u_i) \leq s_2$.

Claim: $\mu_{ij}(u_i) = y_i$ and $\nu_{ij}(u_i) = z_i$ for some $u_i \in \mathcal{E}_i$. By definition of (μ, ν) -tempered $\mu_{ij}(u_i) \geq y_{i-1}$ and $\nu_{ij}(u_i) \leq z_{i-1}$ for all $U_i \in \mathcal{E}_i \Rightarrow \mathcal{E} \subseteq U_{i-1}$ and so $\mathcal{E} \in \mathcal{E}_i \setminus \mathcal{E}_{i-1}$ and since $\mathbb{H}_{\mathbb{G}}$ is simply ordered $\mathcal{E} \not\subseteq U_{i-1}$, which is a contradiction. Hence, it follows from the definition that $\mu_{ij} = c_{ij}$, $\nu_{ij} = d_{ij}$. $\square$

4. Application of IFTHG

Water is required for the survival of all living organisms throughout the universe. Water is, without a doubt, the reason why the earth is still the only planet on which life can survive. This universal solvent is one of the most essential resources we have on this world. Life would be difficult to survive without water. Furthermore, it feeds nearly 70% of the world's population. A water distribution system is a segment of a water supply network that distributes drinking water from a centralised treatment plant or wells to people to meet the needs of residential, commercial, industrial and firefighting users.

The value of other commodities rises as a result of wasting water. Water shortages makes it more difficult to manufacture certain goods that require it. As a result, certain commodities would become scarce or may be priced higher. It's worth noting that there's a critical challenge with leakages in underground

pipelines, which results in water waste in each area. Clearly, we need to know the quantity of actual water usage, estimated water leakage and unexpected water consumption in order to manage and control water supplies in each area. The pictorial representation of water distribution system is shown in Figure 8. The houses in the particular areas and two major suppliers are denoted as vertices and the supply of water to certain areas are considered as hyperedges. Furthermore, the two reservoirs must deliver the appropriate quantity of water to ensure the basic water usage of nine areas, as well as the minimum quantity of water storage required to maintain the suppliers normal water level.

Consider nine areas $e_1, e_2, e_3, \dots, e_9$ and there are two suppliers p and q . During that particular period, water was distributed one by one from major suppliers to all areas of town.

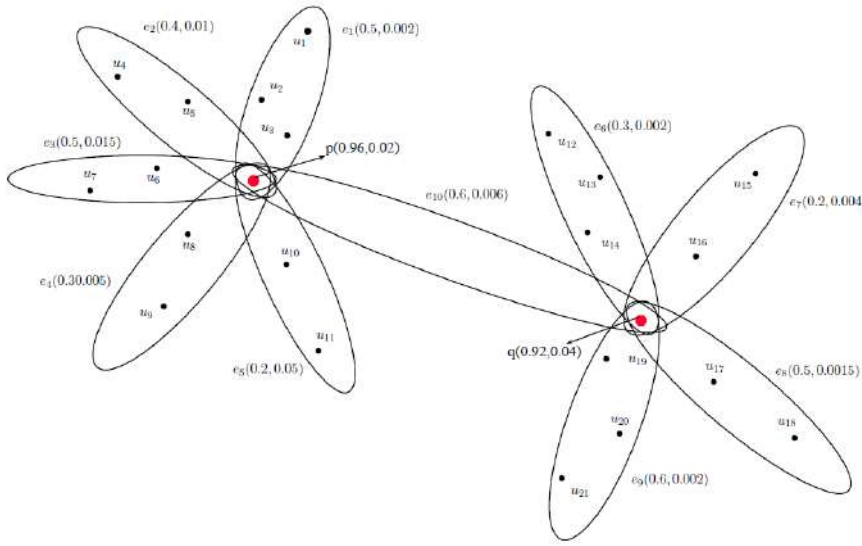


Figure 8: IFTHG linked to water management system

Now, consider a vertex(house) u_1 in the hyperedge(area) e_1 and the supplier p

- $\mu_i(u_1)$ -denotes the actual water usage of the vertex u_1 ;
- $\nu_i(u_1)$ -denotes the quantity of water leakage of the vertex u_1 ;
- $1 - \mu_i(u_1) - \nu_i(u_1)$ -(the index of u_1)- denotes the quantity of unexpected water usage of u_1 ;
- $\mu_i(p)$ -denotes the quantity of water supply of the supplier p ;
- $\nu_i(p)$ -denotes the minimal quantity of water storage of supplier p ;
- $1 - \mu_i(p) - \nu_i(p)$ -(the index of v_1)- denotes the quantity of unexpected water usage of p ;

- $\mu_{ij}(e_1)$ -denotes the total water distributed to the houses u_1, u_2, u_3 ;
- $\nu_{ij}(e_1)$ -denotes the quantity of water leakage of the area e_1 .

The values of vertices are tabulated below:

$u_1 - (0.6, 0.002)$	$u_{12} - (0.5, 0.001)$
$u_2 - (0.5, 0.0015)$	$u_{13} - (0.3, 0.002)$
$u_3 - (0.55, 0.002)$	$u_{14} - (0.4, 0.0015)$
$u_4 - (0.5, 0.001)$	$u_{15} - (0.6, 0.003)$
$u_5 - (0.4, 0.001)$	$u_{16} - (0.2, 0.004)$
$u_6 - (0.5, 0.001)$	$u_{17} - (0.5, 0.001)$
$u_7 - (0.6, 0.0015)$	$u_{18} - (0.7, 0.0015)$
$u_8 - (0.3, 0.004)$	$u_{19} - (0.6, 0.001)$
$u_9 - (0.5, 0.005)$	$u_{20} - (0.7, 0.002)$
$u_{10} - (0.3, 0.05)$	$u_{21} - (0.65, 0.0015)$
$u_{11} - (0.2, 0.045)$	

4.1 Working mechanism

The water distribution system begins by emphasizing the necessity of a raw water source such as a lake, river or groundwater. Water is stored in reservoirs to ensure a consistent supply for the system, with pumps facilitating its transportation from the source to storage facilities. Prior to entering the system, water undergoes filtration to prevent corrosion, followed by purification to meet quality standards.

Main pumping stations near storage facilities supply water directly into the piping system. Reservoirs play a crucial role in system capacity, supporting both consumer consumption and fire protection needs during peak demand periods. Ground-level and raised storage options are common.

Water is then distributed to users through pipes, with each area typically receiving water from reservoirs within a two-hour timeframe. If unexpected water needs arise, interconnected reservoirs can provide additional water via hyperedges. Once all areas are serviced, reservoirs replenish tanks for the next distribution cycle. This process ensures consistent water supply and addresses fluctuating demand effectively.

From the above comparison that using IFTHG in the water distribution system is essential. It can also be used to regulate water resources. Furthermore, in a water distribution system, IFTHGs are more appropriate than fuzzy threshold graph and intuitionistic fuzzy threshold graphs.

4.2 Analysis

Our water management system has two major suppliers, each with a capacity of 1000 litres. And each vertices (houses/industries) has a regular water allocation. Suppose that house 1 takes 60 litres of water per day, which indicates that the

membership value of that vertex is 0.6. Similarly, intuitionistic fuzzy data from a collected data has been generated.

Because reservoirs influence the quantity of water required in each area, also, known that the threshold dimension can simply calculated by the number of reservoirs. As seen in the figures 9 and 10, in which the threshold dimension is 2 and the intuitionistic fuzzy threshold partition number is 2 as considered in [10]. As a result, we could generate two intuitionistic fuzzy threshold subhypergraphs(IFTsHG). It must be observed that as thresholds s_1 and $m - s_2$ refer to the maximum quantity of water replenishment and unexpected water consumption respectively, where m is the number of elements in the maximum independent set of subhypergraphs. For e.g., consider $s_1 = 0.96$ and $s_2 = 4.98$ for first IFTsHG(Fig:9). Here, number of elements in the maximum independent set is $\{u_1, u_4, u_7, u_9, u_{10}\}$ and

$$\mu_i(u_1) + \mu_i(u_4) + \mu_i(u_7) + \mu_i(u_9) + \mu_i(u_{10}) = 2.5.$$

As a conclusion, it is clear that the reservoir p can supply 0.96 quantity of water for the five areas, which requires at least 2.5 quantity of water for actual consumption and 0.02 ($5 - 4.96 = 0.02$) quantity of water goes under unforeseen water consumption/leakage. We can conclude from this that the p supplier can supply 960 litres for five areas, with at least 250 litres for actual consumption and 2 litres for unforeseen water consumption/leakage.

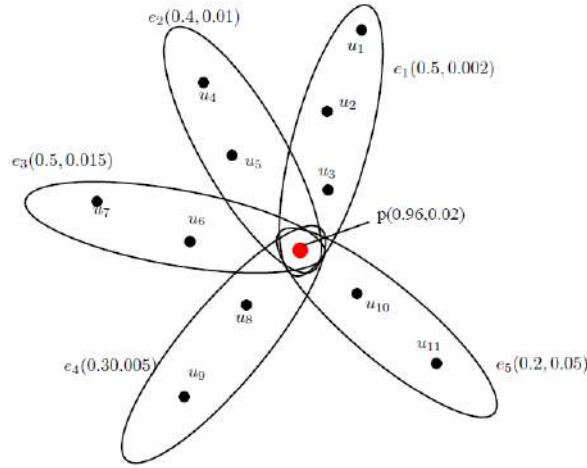


Figure 9: The first IFTsHG linked to water management system

Similarly, consider $s_1 = 0.92$ and $s_2 = 3.96$ for second IFTsHG(Fig:10) and the number of elements in the maximum independent set is $\{u_{12}, u_{15}, u_{18}, u_{20}\}$ which results in

$$\mu_i(u_{12}) + \mu_i(u_{15}) + \mu_i(u_{18}) + \mu_i(u_{20}) = 2.5.$$

Finally, the reservoir q can supply 0.92 quantity of water for the four areas, with at least 2.5 quantity of water for actual consumption and 0.04 ($4-3.96=0.04$) quantity of water goes under unforeseen water consumption/leakage. Now, conclude from this that the q supplier can supply 920 litres for four areas, with at least 250 litres for actual consumption and 4 litres for unforeseen water consumption/leakage.

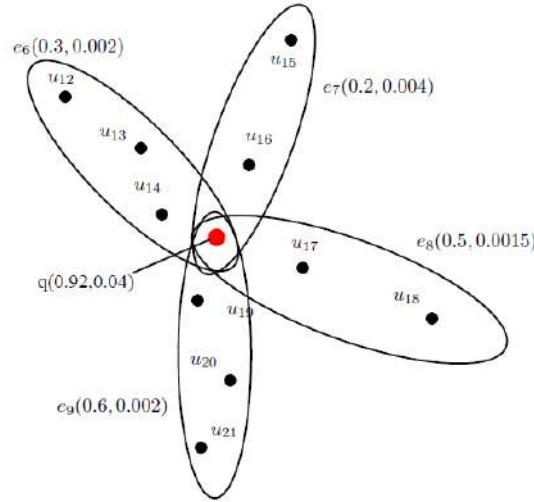


Figure 10: The second IFTsHG linked to water management system

In the first area, there are significant water leakages near houses u_1 , u_4 , u_7 , u_9 , and u_{10} . Similarly, in the second area, leakages are observed around houses u_{12} , u_{15} , u_{18} , and u_{20} , or in the pipes near these locations. By identifying these problem areas, we can take targeted actions to reduce water waste effectively. As a result of above analysis, it is easily seen that the importance of IFTHG as an application part to manage water resources. Furthermore, it is analyzed that IFTHGs are better than fuzzy threshold graphs for controlling water management systems through threshold values. In this application, threshold values are most important to maintain the certain limit of each vertices. Additionally, this method excels in identifying water wastage and, while illustrated here with a smaller example, it can be applied to larger and more complex structures with equal effectiveness.

5. Conclusion

In many real-world problems, data exhibit uncertain behavior and can vary significantly based on different parameters, highlighting the need for IFTHG. The application of fuzzy graph theory in addressing real-time situations is expanding, with IF models offering greater precision and compatibility than traditional methods. IFTHGs, in particular, are better equipped to handle uncertainty

and ambiguity, making them a powerful tool in various applications. In this paper, we introduced the concept and definitions of IFTHGs, including core, fundamental sequence, elementary, and sectionally elementary types, and discussed several key results. Our IFTHG model, designed to reduce water wastage, demonstrates the practical significance of this approach. Future research in this area holds promise for a wide range of applications.

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Some criteria for solvability in finite groups

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Abstract. Let G be a finite group and $m(G) = \sum_{g \in G} \frac{1}{o(g)}$, where $o(g)$ is the order of the element $g \in G$. In this paper, we show that if G is a finite nonsolvable group with $m(G) = m(A_5)$, then $G \cong A_5$. Furthermore, we show that if $m(G) < m(\mathrm{SL}(2, 5))$ and $m(G) \neq m(A_5)$, then G is solvable.

Keywords: order of elements, solvable group, supersolvable group.

MSC 2020: 20D10, 20D20.

1. Introduction

Throughout let G be a finite group. There is a wide body of recent literature exploring the link between structural properties of G and arithmetic functions constructed on the orders of the elements of G . We denote by $o(g)$ the order of an element $g \in G$, and set $\psi(G) = \sum_{g \in G} o(g)$, that is, $\psi(G)$ denote the sum of the orders of the elements of G . H. Amiri, S. M. Jafarian Amiri and I. M. Isaacs in [2] proved if $|G| = n$, then $\psi(G) < \psi(C_n)$, where C_n be the

cyclic group of order n . Hence, the maximum value of ψ on the set of groups of order n will occur at the cyclic group C_n . In [1], the minimum value of ψ on the set of groups of the same order n is investigated. Specifically, it is shown that for nilpotent groups, the minimum value will be obtained for groups where each Sylow subgroup has prime exponent. It is also proved that if there exist nonnilpotent groups of order n , then the minimum value will be obtained for a nonnilpotent group.

Since then, the investigation of the influence of the value of $\psi(G)$ on the structure of the finite group G became one of the main topics in finite group theory. M. Herzog, P. Longobardi and M. Maj [10] proved that if $|G| = n$ and $\psi(G) \geq \psi(C_n)/6.68$, then G is a solvable group, and posed the following conjecture: If $\psi(G) > 211\psi(C_n)/1617$, then G is a solvable group. In [3], M. B. Azad and B. Khosravi prove the validity of this conjecture. Recently, they proved in [4] that If $|G| = n$ and $\psi(G) > 31\psi(C_n)/77$, then G is a supersolvable group. Moreover, they prove that if G is a group of odd order n and $\psi(G) > 271\psi(C_n)/3674$, then G is a supersolvable group.

Let $o(G) = \psi(G)/|G|$, that is, $o(G)$ is the average order of the elements of G . E. I. Khukhro, A. Moret  and M. Zarrin in [14] showed that there is no polynomial lower bound for $o(G)$ in terms of $o(N)$, where $N \trianglelefteq G$, even when G is a prime-power order group and N is abelian. This gives a negative answer to a question of A. Jaikin-Zapira in [13]. M. Herzog, P. Longobardi and M. Maj [11] proved that if $o(G) < o(A_5)$, then G is solvable. This confirm a conjecture posed by E. I. Khukhro, A. Moret  and M. Zarrin in [14]. M. T rn uceanu proved in [16] a sufficient condition for the supersolvability of finite groups in terms of $o(G)$. Specifically, if $o(G) < 211/60 = o(A_4)$, then G is supersolvable. The author in fact classifies all finite groups G such that $o(G) < 211/60$.

M. Garonzi and M. Patassini in [7] introduced the following fuction:

$$R_G(r, s) = \sum_{g \in G} \frac{o(g)^s}{\varphi(o(g))^r},$$

where r, s are two real numbers and φ denote Euler's totient function. The authors of [7] proved several results detecting cyclicity or nilpotency of G in terms of inequalities involving the fuction $R_G(r, s)$.

Note that, $R_G(0, 1) = \sum_{g \in G} o(g) = \psi(G)$ is the sum of the order of elements in G . On the other hand, $R_G(0, -1) = \sum_{g \in G} \frac{1}{o(g)}$ is the sum of the inverses of the element orders in finite groups.

Set $m(G) = \sum_{g \in G} \frac{1}{o(g)}$. Observe that $m(A_5) = 599/30$, and $m(A_4) = 31/6$, where A_n is the alternating group of degree n . B. Azad, B. Khosravi and H. Rashidi in [5] proved that if $m(G) < m(A_5)$, then G is solvable and if $m(G) < m(A_4)$, then G is supersolvable. Furthermore, many interesting results have been given, for example [9, 15].

Following these footsteps, in this paper, we continue to study the relations between $m(G)$ and the structure of G .

The notation is standard and follows that of Isaacs [12]. Let $\pi(G)$ denote the set of all primes of $|G|$, $SL(2, 5)$ denote the special linear group of dimension 2 over the field with 5 elements, and C_n denote a cyclic group of order n .

Since A_5 is not solvable and A_4 is not supersolvable, the bound in the above results [5] are the best possible. Using the idea in [5], we first prove the following two results:

Theorem 1.1. *Let G be a nonsolvable group with $m(G) = m(A_5)$. Then, $G \cong A_5$.*

Theorem 1.2. *Let G be a nonsupersolvable group with $m(G) = m(A_4)$. Then, $G \cong A_4$.*

The following theorems give new criteria for surpersolvability and solvability of a finite group by the sum of the inverses of the element orders.

Theorem 1.3. *Let G be a finite group. If $m(G) \leq |\pi(G)|$, then G is supersolvable.*

Theorem 1.4. *Let G be a finite group. If $m(A_5) \neq m(G) < m(SL(2, 5))$, then G is solvable.*

2. Some known lemmas

In this section, we collect some known results which will be used in the proof of main theorems.

Lemma 2.1 ([5, Lemma 2.1]). *Let G be a finite group with $m(G) \leq t$ for some natural number t . Then, there exists a cyclic subgroup $\langle x \rangle$ such that $|G : \langle x \rangle| \leq t$.*

Lemma 2.2 ([5, Lemma 1.3]). *If G is a finite noncyclic group of order n , then $m(C_n) < m(G)$.*

Lemma 2.3 ([12, Theorem 2.20]). *Let A be a cyclic proper subgroup of a finite group G , and let $K = \text{Core}_G(A)$. Then, $|A : K| < |G : A|$, and in particular, if $|A| > |G : A|$, then $K > 1$.*

Lemma 2.4 ([5, Lemma 2.2]). *Let G be a finite group and $H \leq G$. Then, $m(H) \leq m(G)$, with equality if and only if $H = G$.*

Lemma 2.5 ([5, Lemma 2.3]). *Let G be a finite group and $N \trianglelefteq G$. Then, $m(G/N) \leq m(G)$, with equality if and only if $N = 1$.*

Lemma 2.6 ([5, Lemma 2.6]). *Let G and H be finite groups. Then, $m(G \times H) \geq m(G)m(H)$, with equality if and only if $\gcd(|G|, |H|) = 1$.*

Lemma 2.7 ([12, Lemma 9.1]). *Let G be a finite group, and suppose $G/Z(G)$ is simple. Then, $G/Z(G)$ is nonabelian, and G' is perfect. Also, $G'/Z(G')$ is isomorphic to $G/Z(G)$.*

Lemma 2.8 ([5, Lemma 2.4]). *Let G be a finite group, $P \in \text{Syl}_p(G)$ be a normal and cyclic subgroup of G . Then, $m(Px) \geq m(P)/o(Px)$, where $Px \in G/P$, and the equality holds if and only if x centralizes P . Also, $m(G) \geq m(P)m(G/P)$ with equality if and only if P is central in G .*

Lemma 2.9 ([9, Proposition 2.5]). *Let G be a finite group and suppose that there exists $x \in G$ such that $|G : \langle x \rangle| < 2p$, where p is the maximal prime divisor of $|G|$. Then, one of the following holds:*

- (1) G has a normal cyclic Sylow p -subgroup,
- (2) G is solvable and $\langle x \rangle$ is a maximal subgroup of G of index either p or $p + 1$.

3. Proofs of the main theorems

Proof of Theorem 1.1 Assume that G contains a non-trivial normal subgroup N . Then, by Lemmas 2.4 and 2.5, we have that $m(N) < m(G) = m(A_5)$, and $m(G/N) < m(G) = m(A_5)$. It follows from [5, Theorem 2.7] that N and G/N are solvable, and thus G is solvable, which is a contradiction. Thus, G is a simple group.

Assume that $|G| > 361$. By Lemma 2.1, There exists $x \in G$ such that $|G : \langle x \rangle| \leq 599/30$. Now using Lemma 2.3, we get that $|\langle x \rangle : \text{Core}_G(\langle x \rangle)| < |G : \langle x \rangle| \leq 19$. Thus, $|\langle x \rangle| > \frac{361}{|G : \langle x \rangle|} \geq |G : \langle x \rangle|$. Again by Lemma 2.3, $\text{Core}_G(\langle x \rangle)$ is a non-trivial normal subgroup of G , which is a contradiction. Therefore, $|G| \leq 361$.

By [6], the simple nonabelian groups of order less than 362 are A_5 , A_6 and $\text{PSL}(2, 7)$. If $G \cong A_6$, then the number of Sylow 5-subgroup of G is $n_5(G) = 36$. Therefore, G has 144 elements of order 5. By the definition of $m(G)$, we get that $m(G) > m(A_5)$ which is a contradiction. Similarly, if $G \cong \text{PSL}(2, 7)$, then we get a contradiction. Thus, $G \cong A_5$, as desired. $\square$

Proof of Theorem 1.2. Assume that G contains a nontrivial normal cyclic subgroup N . Then, by Lemma 2.5, we have that $m(G/N) < m(G) = m(A_4)$. It follows from [5, Theorem 2.8] that G/N is supersolvable. Therefore, G is supersolvable, which is a contradiction. Thus, G has no a non-trivial normal cyclic subgroup.

Assume that $|G| > 25$. By Lemma 2.1, there exists $x \in G$ such that $|G : \langle x \rangle| \leq 31/6$. By Lemma 2.3, we get that $|\langle x \rangle : \text{Core}_G(\langle x \rangle)| < |G : \langle x \rangle| \leq 5$. Thus, $|\langle x \rangle| > \frac{25}{|G : \langle x \rangle|} \geq |G : \langle x \rangle|$.

Again by Lemma 2.3, $\text{Core}_G(\langle x \rangle)$ is a nontrivial normal cyclic subgroup of G , which is a contradiction. Thus, $|G| \leq 25$.

Since G is a non-supersolvable group of order less than 26, we have that $G \cong A_4$, $\text{SL}(2, 3)$, S_4 or $C_2 \times A_4$. If $G \cong \text{SL}(2, 3)$, S_4 or $C_2 \times A_4$, then the number of Sylow 3-subgroup of G is 4, and so G has 8 elements order of 3 and at least one element order of 2. Thus, $m(G) > m(A_4)$, which is a contradiction. Therefore, we obtain that $G \cong A_4$, as desired. $\square$

Proof of Theorem 1.3. We prove by induction on $|\pi(G)|$.

Let $|G| = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, where $p_1 < p_2 < \cdots < p_r = p$ are primes, and $\alpha_i > 0$, for each $1 \leq i \leq r$.

If $|\pi(G)| = 1$, then G is a p -group, therefore G is supersolvable. Assume that $|\pi(G)| \geq 2$. We consider the following two cases:

Assume that G has no normal cyclic Sylow subgroups. By Lemma 2.1, there exists $x \in G$ such that $|G : \langle x \rangle| \leq |\pi(G)| < p$ and by Lemma 2.9, we get a contradiction.

Thus, G has some normal cyclic Sylow p -subgroup, say Q . If Q is a prime order cyclic group and $|\pi(G)| = 2$, then G is supersolvable. If not, then by Lemma 2.8, we have that

$$m(G/Q)m(Q) \leq m(G) \leq |\pi(G)|.$$

And thus

$$m(G/Q) \leq m(G)/m(Q) \leq m(G)/2 \leq |\pi(G)|/2 \leq |\pi(G)| - 1 = |\pi(G/Q)|.$$

By the inductive hypothesis, G/Q is a supersolvable group. Therefore, G is supersolvable. $\square$

Proof of Theorem 1.4. Assume that G is nonsolvable. By Lemma 2.1, there exists $x \in G$ such that $|G : \langle x \rangle| \leq 131/5 < 27$. Let $H = \text{Core}_G(\langle x \rangle)$, then H is a normal cyclic subgroup of G . By Lemma 2.3, we get that $|\langle x \rangle : H| < |G : \langle x \rangle| \leq 26$. Hence,

$$|G : H| = |G : \langle x \rangle| |\langle x \rangle : H| \leq 25 \times 26 = 650.$$

Since G is nonsolvable, it follows that G/H is nonsolvable. Let $\overline{G} = G/H$. Then, we have that $|\overline{G}| = |G/H| \leq 650$. By Lemma 2.5,

$$m(\overline{G}) \leq m(G) < m(\text{SL}(2, 5)) = \frac{131}{5}.$$

By GAP [8], we find that the only possibility for $\overline{G}$ of order up to 650 is $\overline{G} \cong A_5$. Therefore, H is a maximal normal subgroup of G and $H \leq C_G(H)$. On the other hand, $C_G(H) \trianglelefteq N_G(H) = G$. Thus, $C_G(H) = H$ or $C_G(H) = G$.

In the case $C_G(H) = H$, then $N_G(H)/C_G(H) = G/H \cong A_5 \lesssim \text{Aut}(H)$, which is a contradiction, since H is cyclic.

In the case $C_G(H) = G$, then $H \leq Z(G)$. Therefore, $H = Z(G)$ and $G/Z(G) = G/H \cong A_5$. Thus, $G' \not\leq Z(G)$ and $G'Z(G) \trianglelefteq G$, by the definition of maximal normal subgroup, we have $G = G'Z(G)$ and

$$G'/(G' \cap Z(G)) \cong G'Z(G)/Z(G) = G/Z(G) \cong A_5.$$

By Lemma 2.8, G' is perfect and $G'/Z(G') \cong G/Z(G) \cong A_5$. Hence G' is a central extension of $Z(G')$ by A_5 . Since the Schur multiplier of A_5 has order 2 and G' is perfect, it follows that $G' \cong \text{SL}(2, 5)$ or $G' \cong A_5$.

If $G' \cong \text{SL}(2, 5)$, then $|G' \cap Z(G)| = 2$ and

$$\begin{aligned} G/(G' \cap Z(G)) &= G'Z(G)/(G' \cap Z(G)) \cong G'/(G' \cap Z(G)) \\ &\times Z(G)/(G' \cap Z(G)) \cong A_5 \times C_n, \end{aligned}$$

where $n = |G|/120$. By Lemmas 2.2 and 2.6,

$$\frac{599}{30}m(C_n) = m(A_5)m(C_n) \leq m(A_5 \times C_n) = m(G/G' \cap Z(G)) \leq m(G) < \frac{131}{5}.$$

Therefore, $m(C_n) < 786/599$. This implies that $n = 1$ and $G \cong \text{SL}(2, 5)$, a contradiction.

If $G' \cong A_5$, then $|G' \cap Z(G)| = 1$ and $G = G' \times Z(G) \cong A_5 \times Z(G)$. Set $|Z(G)| = n$. By Lemmas 2.2 and 2.6,

$$\frac{599}{30}m(C_n) = m(A_5)m(Z(G)) \leq m(A_5 \times Z(G)) = m(G) < \frac{131}{5}.$$

By the above discussion, we have that $n = 1$. Thus $Z(G) = 1$ and $G \cong A_5$, a contradiction. $\square$

4. Conclusions

All the previous conclusions indicate that the sum of the inverses of the order of elements can effectively characterize the solvability and supersolvability of finite groups, and provide us with ideas and directions for future research on similar problems, such as the extension of known important conclusions and the necessary and sufficient conditions for A_4 and A_5 . These will surely be the subject of some further research.

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Conflict of interest

The authors declare that they have no conflict of interest.

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SEP elements and the solution to constructed equations

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Abstract. For the last five years, *SEP* elements in a ring with involution have been discussed by many authors. In this paper, we obtain many new characterizations of *SEP* elements by using group inverse and Moore-Penrose inverse, also by constructing a lot of equations and discussing the expression forms of solution to these equations in certain given set.

Keywords: EP element, partial isometry, SEP element.

MSC 2020: 16E50, 16D30

1. Introduction

The *EP* elements in a ring with involution have been investigated by a lot of authors [7, 13, 14, 16, 20, 21, 23, 25, 26, 28, 31, 32, 33, 34], which originates from the study of *EP* matrices and *EP* linear operators on Banach or Hilbert

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spaces [1, 2, 3, 5, 6, 8, 11, 9, 27]. The study of *SEP* elements originates from [20] and for further research, the readers can refer to [4, 29, 30, 32]. However, the excellent and significant fact is that the people can use generalized inverses of matrices such as *EP* matrices and *SEP* matrices, in the case when ordinary inverses do not exist, in order to solve some matrix equations, operator equations or differential equations.

Let R be a ring. An element $a \in R$ is called the group invertible [10, 12, 17] if there exists $a^\# \in R$ such that

$$a = aa^\#a, a^\# = a^\#aa^\#, aa^\# = a^\#a.$$

According to [17], $a^\#$ is called the group inverse of a , which is unique if it exists. Clearly, $a \in R^\#$ if and only if $a \in a^2R \cap Ra^2$.

We usually write $R^\#$ to denote the set of all group invertible elements in R .

An involution $*$: $a \rightarrow a^*$ in a ring R is an anti-isomorphism of degree 2, that is,

$$(a^*)^* = a, (a + b)^* = a^* + b^*, (ab)^* = b^*a^* \text{ for } a, b \in R.$$

$a \in R$ is called the Moore-Penrose invertible [13, 15, 19, 22] if there exists $a^+ \in R$ such that

$$a = aa^+a, a^+ = a^+aa^+, (aa^+)^* = aa^+, (a^+a)^* = a^+a.$$

We always use R^+ to denote the set of all Moore-Penrose invertible elements in R .

Let $a \in R^\# \cap R^+$. if $a^\# = a^+$, then a is called an *EP* element. We denote the set of all EP elements in R by R^{EP} .

If $a = aa^*a$, then a is said to be a partial isometry (or *PI*) element [20, 24, 25] and we use R^{PI} to denote the set of all *PI* elements in R . It is known that $a \in R^+$ is partial isometry if and only if $a^+ = a^*$.

If $a \in R^{EP} \cap R^{PI}$, then a is said to be a strong *EP* element. We used to write R^{SEP} to represent the set of all strong EP elements in R .

EP elements in rings with involution have been characterized by conditions involving their group inverse and Moore-Penrose inverse, the solution to related constructed equations. Motived by these ways, we discuss the properties of *SEP* elements.

Throughout this paper R is a $*$ -ring with 1.

2. Several properties of SEP elements

In [20], the following equation and theorem are given.

$$(2.1) \quad axa^* = xa^\#a.$$

[20, Theorem 2.2]: Let $a \in R^\# \cap R^+$. Then, $a \in R^{SEP}$ if and only if Eq. (2.1) has at least one solution in $\chi_a = \{a, a^\#, a^+, a^*, (a^\#)^*, (a^+)^*\}$.

In this section, inspired by [20], we discuss some properties of *SEP* elements using the expression of core inverse or dual inverse of certain generalized inverse element.

Let $a \in R^\# \cap R^+$. Then, we have the following formulas on core inverse and dual core inverse

$$\begin{aligned} a \oplus &= a^\# a a^+, \\ a \oplus &= a^+ a a^\#, \\ (a^\#)^+ &= a^+ a^3 a^+, \end{aligned}$$

and

$$(a^+)^{\#} = (a a^{\#})^* a (a a^{\#})^*.$$

Hence, we have the following theorem.

Theorem 2.1. *Let $a \in R^\# \cap R^+$. Then, $a \in R^{SEP}$ if and only if Eq. (2.1) has at least one solution in $\sigma_a = \{a \oplus, a \oplus, (a^\#)^+, (a^+)^{\#}\}$.*

Proof. " $\Rightarrow$ " Assume that $a \in R^{SEP}$. Then, $a^\# = a^+ = a^*$, this gives

$$a^2 a^* = a^2 a^\# = a = a a^\# a.$$

Hence, $x = a$ is a solution.

" $\Leftarrow$ " (1) If $x = a \oplus = a^\# a a^+$ is a solution, then

$$a(a^\# a a^+) a^* = (a^\# a a^+) a^\# a,$$

e.g., $a a^+ a^* = a^\#$. Hence, $a \in R^{SEP}$ by [20, Theorem 1.5.3].

(2) If $x = a \oplus = a^+ a a^\#$, then

$$a(a^+ a a^\#) a^* = (a^+ a a^\#) a^\# a.$$

e.g., $a a^\# a^* = a^+ a a^\#$. Multiplying the equality on the left by a , one has $a a^* = a a^\#$. Hence, $a \in R^{SEP}$ by [20, Theorem 1.5.3].

(3) If $x = (a^\#)^+ = a^+ a^3 a^+$, then $a(a^+ a^3 a^+) a^* = (a^+ a^3 a^+) a^\# a$, e.g.,

$$a^3 a^+ a^* = a^+ a^2.$$

Multiplying the equality on the left by $a^\# a^\#$, one gets $a a^+ a^* = a^\#$. Hence, $a \in R^{SEP}$ by [20, Theorem 1.5.3].

(4) If $x = (a^+)^{\#} = (a a^\#)^* a (a a^\#)^*$, then

$$a(a a^\#)^* a (a a^\#)^* a^* = (a a^\#)^* a (a a^\#)^* a^\# a.$$

Multiplying the equality on the left by $a^+ a^+$, one yields

$$a^* = a^+ a^\# a.$$

It follows that $a^* a = (a^+ a^\# a) a = a^+ a$. Hence, $a \in R^{PI}$ by [20, Theorem 1.5.2], this infers $a^+ = a^* = a^+ a^\# a$. Thus, $a \in R^{EP}$ by [20, Theorem 1.2.1]. Therefore, $a \in R^{SEP}$. $\square$

Observing the proof of Theorem 2.1, we have the following corollary.

Corollary 2.2. *Let $a \in R^\# \cap R^+$. Then, $a \in R^{SEP}$ if and only if $a^* = a^+ a^\# a$.*

Noting that if $a \in R^\# \cap R^+$, then $a \oplus = a^+ a^\# a$. Thus, by Corollary 2.2, we have

Corollary 2.3. *Let $a \in R^\# \cap R^+$. Then, $a \in R^{SEP}$ if and only if $a^* = a \oplus$.*

Since $a \in R^{SEP}$ if and only if $a^* \in R^{SEP}$, use a^* to replace a in Corollary 2.2, we have

Corollary 2.4. *Let $a \in R^\# \cap R^+$. Then, $a \in R^{SEP}$ if and only if $a = (a^+)^*(a^\#)^* a^*$.*

Applying the involution on the equality of Corollary 2.4, we obtain.

Corollary 2.5. *Let $a \in R^\# \cap R^+$. Then, $a \in R^{SEP}$ if and only if $a^* = a \oplus$.*

Theorem 2.6. *Let $a \in R^\# \cap R^+$. Then, $a \in R^{SEP}$ if and only if $aa^+ = (a^\#)^* a^\#$.*

Proof. " $\Rightarrow$ " Assume that $a \in R^{SEP}$. Then, $a^* = a^+ a^\# a$ by Corollary 2.2. This gives $aa^+ = (a^+)^* a^* = (a^+)^* a^+ a^\# a = (a^+)^* a^\# = (a^\#)^* a^\#$.

" $\Leftarrow$ " From the equality $aa^+ = (a^\#)^* a^\#$, one has

$$a^* = a^* aa^+ = a^* ((a^\#)^* a^\#) = (aa^\#)^* a^\#,$$

so

$$a^+ a^* = a^+ a^\#.$$

Hence, $a \in R^{SEP}$ by [20, Theorem 1.5.3]. $\square$

Noting that $a^+ a^\# a = a^+ a^3 a^+ a^\# a^\# = (a^\#)^+ a^\# a^\#$. Then, by Corollary 2.2, we have the following corollary.

Corollary 2.7. *Let $a \in R^\# \cap R^+$. Then, $a \in R^{SEP}$ if and only if $a^* = (a^\#)^+ a^\# a^\#$.*

Since $a \in R^{SEP}$ if and only if $a^\# \in R^{SEP}$, use $a^\#$ to replace a in Corollary 2.7, we have

Corollary 2.8. *Let $a \in R^\# \cap R^+$. Then, $a \in R^{SEP}$ if and only if $(a^\#)^* = a^+ a^2$.*

Noting that $a \oplus \in R^{EP}$ and $a \oplus^+ = a^+ a^2$. Hence, we have.

Corollary 2.9. *Let $a \in R^\# \cap R^+$. Then, $a \in R^{SEP}$ if and only if $(a^\#)^* = a \oplus^+$.*

Since $((a^\#)^*)^+ = ((a^\#)^+)^* = aa^+a^*a^+a$, Corollary 2.9 implies the following corollary.

Corollary 2.10. *Let $a \in R^\# \cap R^+$. Then, $a \in R^{SEP}$ if and only if $aa^+a^* = a \oplus$.*

Remark 2.11. It is well known that $a \in R^{EP}$ if and only if $a \in R^+$ and $aa^+ = a^+a$. However, if $a^+a = (a^\#)^*a^\#$, we can't obtain $a \in R^{SEP}$.

For example, let $R = M_3(Z_2)$ and choose $a = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then, $a^\# = a$ and $a^+ = a^* = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. Clearly, $a^+a = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$; $(a^\#)^*a^\# = a^*a = a^+a$. While $aa^+ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq (a^\#)^*a^\#$. Hence, by Theorem 2.6, a is not SEP . In fact, even $a^\# \neq a^+$

3. Using group inverse and Moore-Penrose inverse to characterize SEP elements

In [20], the group inverse and the Moore Penrose inverse of the product of two generalize inverse elements of an element is represented. Inspired by this, we study in this section SEP elements by constructing the representation forms of the group inverse and the Moore Penrose inverse of the product of several generalized inverse elements of an element.

Lemma 3.1. *Let $a \in R^\# \cap R^+$. Then, $aa^+a^* \in R^{EP}$ and $(aa^+a^*)^+ = aa^+(a^\#)^*$.*

Proof. It is routine. $\square$

Theorem 3.2. *Let $a \in R^\# \cap R^+$. Then, $a \in R^{SEP}$ if and only if $aa^+(a^\#)^* = a^+a^2$.*

Proof. It is an immediate result of Corollary 2.10 and Lemma 3.1. $\square$

Corollary 3.3. *Let $a \in R^\# \cap R^+$. Then, $a \in R^{SEP}$ if and only if $aa^+ = a^+a^2a^*$.*

Proof. " $\implies$ " Assume that $a \in R^{SEP}$. Then, $a \in R^{EP}$ and $aa^+(a^\#)^* = a^+a^2$ by Theorem 3.2. Noting that $a^+ = a^+(aa^\#)^*$ and $(aa^\#)^* = aa^\#$ by [20, Theorem 1.1.3]. Then, $aa^+ = aa^+(a^\#)^*a^* = a^+a^2a^*$.

" $\impliedby$ " From the equality $aa^+ = a^+a^2a^*$, one gets $aa^+ = a^+a^2a^+$. Hence, $a \in R^{EP}$, this induces $(aa^\#)^* = aa^\#$ by [20, Theorem 1.1.3]. It follows that $aa^+(a^\#)^* = a^+a^2a^*(a^\#)^* = a^+a^2(aa^\#) = a^+a^2$. Hence, $a \in R^{SEP}$ by Theorem 3.2. $\square$

Now, we give the following lemma which proof is routine.

Lemma 3.4. *Let $a \in R^\# \cap R^+$. Then*

- (1) $(a^+a^2a^*)^+ = (a^+)^*a^\#$;
- (2) $(a^+a^2a^*)^\# = (a^\#)^*a^+aa^\#(aa^\#)^*$.

Noting that $(aa^+)^+ = (aa^+)^\# = aa^+$. Hence, Corollary 3.3 and Lemma 3.4 induce the following two theorems.

Theorem 3.5. *Let $a \in R^\# \cap R^+$. Then, $a \in R^{SEP}$ if and only if $aa^+ = (a^+)^*a^\#$.*

Theorem 3.6. *Let $a \in R^\# \cap R^+$. Then, $a \in R^{SEP}$ if and only if $aa^+ = (a^\#)^*a^+aa^\#(aa^\#)^*$.*

Noting that $(a \oplus)^* = (a^\#)^*a^+a$. Then, Theorem 3.6 implies the following corollary.

Corollary 3.7. *Let $a \in R^\# \cap R^+$. Then, the followings are equivalent:*

- (1) $a \in R^{SEP}$;
- (2) $aa^+ = (a \oplus)^*a^\#(aa^\#)^*$;
- (3) $aa^+ = (a^\#)^*a \oplus (aa^\#)^*$.

Remark 3.8. Though $a \in R^{EP}$ if and only if $aa^+ = aa^\#$, but Theorem 3.5 can't draw conclusion $a \in R^{SEP}$ under condition $aa^\# = (a^+)^*a^\#$.

In the example of Remark 2.11, $(a^+)^*a^\# = (a^*)^*a = a^2 = aa^\#$. But $a \notin R^{SEP}$.

4. Generalized inverse equations and SEP elements

In this section, we characterize *SEP* elements through the solvability of several constructed generalized inverse equations in a given set.

By observing Theorem 3.5, when $a \in R^{SEP}$, we have

$$aa^+ = (a^+)^*aa^+a^\#.$$

So, we can construct the equation

$$(4.1) \quad xa^+ = (a^+)^*xa^+a^\#.$$

Theorem 4.1. *Let $a \in R^\# \cap R^+$. Then, $a \in R^{SEP}$ if and only if Eq. (4.1) has at least one solution in $\zeta_a = \chi_a \cup \sigma_a$.*

Proof. " $\Rightarrow$ " If $a \in R^{SEP}$, then $aa^+ = (a^+)^*a^\# = (a^+)^*aa^+a^\#$ by Theorem 3.5. Hence, $x = a$ is a solution in ζ_a .

" $\Leftarrow$ " (1) If $x = a$, then $aa^+ = (a^+)^*aa^+a^\# = (a^+)^*a^\#$. Hence, $a \in R^{SEP}$ by Theorem 3.5.

(2) If $x = a^\#$, then $a^\#a^+ = (a^+)^*a^\#a^+a^\#$, this gives

$$a = a^\#a^+a^3 = (a^+)^*a^\#a^+a^\#a^3 = (a^+)^*a^\#a^+a^2 = (a^+)^*.$$

Hence, $a^\#a^+ = aa^\#a^+a^\# = a^\#a^\#$, this infers $a \in R^{EP}$. Thus, $a \in R^{SEP}$.

(3) If $x = a^+$, then $a^+a^+ = (a^+)^*a^+a^+a^\#$. Multiplying the equality on the right by a^+a , one has

$$a^+a^+ = a^+a^+a^+a.$$

By [30, Lemma 2.11], we have $a^+ = a^+a^+a$. Hence, $a \in R^{EP}$, this gives $x = a^+ = a^\#$. Thus, $a \in R^{SEP}$ by (2).

(4) If $x = a^*$, then $a^*a^+ = (a^+)^*a^*a^+a^\# = aa^+a^+a^\#$. Multiplying the equality on the right by a^+a , one gets

$$a^*a^+ = a^*a^+a^+a$$

and

$$a^+ = (a^\#)^*a^*a^+ = (a^\#)^*a^*a^+a^+a = a^+a^+a.$$

Hence, $a \in R^{EP}$, this gives

$$a^*a^\# = a^*a^+ = aa^+a^+a^\# = a^+a^\#.$$

By [20, Theorem 1.5.2], $a \in R^{PI}$. Therefore, $a \in R^{SEP}$.

(5) If $x = (a^+)^*$, then $(a^+)^*a^+ = (a^+)^*(a^+)^*a^+a^\#$. Multiplying the equality on the left by a^* , one gets

$$a^+ = a^+a(a^+)^*a^+a^\# = (a^+a(a^+)^*a^+a^\#)a^+a = a^+a^+a.$$

Hence, $a \in R^{EP}$, this infers

$$\begin{aligned} a^+ &= a^+a(a^+)^*a^+a^\# = a^\#a(a^+)^*a^+a^\# = (a^+)^*a^+a^\#, \\ a^*a^+ &= a^*(a^+)^*a^+a^\# = a^+a^\# = a^\#a^+. \end{aligned}$$

Thus, $a \in R^{SEP}$ by [20, Theorem 1.5.3].

(6) If $x = (a^\#)^*$, then $(a^\#)^*a^+ = (a^+)^*(a^\#)^*a^+a^\#$, it follows that

$$a^+ = a^*(a^\#)^*a^+ = a^*(a^+)^*(a^\#)^*a^+a^\# = (a^\#)^*a^+a^\#$$

and

$$a^*a^+ = a^*(a^\#)^*a^+a^\# = a^+a^\#.$$

Thus, $a \in R^{SEP}$ by [20, Theorem 1.5.3].

(7) If $x = (a^\#)^+ = a^+a^3a^+$, then $a^+a^3a^+a^+ = (a^+)^*a^+a^3a^+a^+a^\#$. Multiplying the equality on the right by a^+a , one has

$$a^+a^3a^+a^+ = a^+a^3a^+a^+a^+a$$

and

$$a^+a^+ = a^+a^\#a^+a^3a^+a^+ = a^+a^\#a^+a^3a^+a^+a^+a = a^+a^+a^+a.$$

By [30, Lemma 2.11], $a^+ = a^+a^+a$. Hence, $a \in R^{EP}$, it follows that

$$x = (a^\#)^+ = (a^+)^+ = a.$$

Thus, $a \in R^{SEP}$ by (1).

(8) If $x = (a^+)^{\#} = (aa^\#)^*a(aa^\#)^*$, then

$$(aa^\#)^*a(aa^\#)^*a^+ = (a^+)^*(aa^\#)^*a(aa^\#)^*a^+a^\#,$$

e.g.,

$$(aa^\#)^* = (a^+)^*(aa^\#)^*a^\#.$$

This gives

$$a^* = a^*(aa^\#)^* = a^*(a^+)^*(aa^\#)^*a^\# = (aa^\#)^*a^\#$$

and

$$a^+a^* = a^+(aa^\#)^*a^\# = a^+a^\#.$$

Thus, $a \in R^{SEP}$ by [20, Theorem 1.5.3].

(9) If $x = a^{\oplus} = a^\#aa^+$, then

$$a^\#aa^+a^+ = (a^+)^*a^\#aa^+a^+a^\# = (a^+)^*a^+a^+a^\#.$$

Multiplying the equality on the right by a^+a , one yields

$$a^\#aa^+a^+ = a^\#aa^+a^+a^+a$$

and

$$a^+a^+ = a^+a(a^\#aa^+a^+) = a^+a(a^\#aa^+a^+a^+a) = a^+a^+a^+a.$$

This implies $a^+ = a^+a^+a$ by [30, Lemma 2.11]. Hence, $a \in R^{EP}$, it follows that $x = a^{\oplus} = a^\#aa^+ = a^\#$. Thus, $a \in R^{SEP}$ by (2).

(10) If $x = a^{\oplus} = a^+aa^\#$, then

$$a^+aa^\#a^+ = (a^+)^*a^+aa^\#a^+a^\# = (a^+)^*a^\#a^+a^\#.$$

Multiplying the equality on the left by $aa^\#$, one obtains

$$a^+aa^\#a^+ = aa^\#a^+aa^\#a^+ = a^\#a^+$$

and

$$a^+a^2 = a^+aa^\#a^+a^3 = a^\#a^+a^3 = a.$$

It follows that $a \in R^{EP}$, this shows $x = a^{\oplus} = a^+aa^\# = a^\#$. Thus, $a \in R^{SEP}$ by (2). $\square$

Now, we revise Eq. (4.1) as follows

$$(4.2) \quad xa^+ = (a^+)^* a^+ xa^\#.$$

Similar to the proof of Theorem 4.1, we have the following theorem.

Theorem 4.2. *Let $a \in R^\# \cap R^+$. Then, $a \in R^{SEP}$ if and only if Eq. (4.2) has at least one solution in ζ_a .*

Multiplying Eq. (4.2) on left by a^* , one gets

$$(4.3) \quad a^* xa^+ = a^+ xa^\#.$$

Theorem 4.3. *Let $a \in R^\# \cap R^+$. Then, $a \in R^{SEP}$ if and only if Eq. (4.3) has at least one solution in ζ_a .*

Now, we revise Eq. (4.2) as follow

$$(4.4) \quad xa^* = aa^+ xa^\#.$$

Theorem 4.4. *Let $a \in R^\# \cap R^+$. Then, $a \in R^{SEP}$ if and only if Eq. (4.4) has at least one solution in ζ_a .*

Clearly, Eq. (4.3) has the same solution as the following equation

$$a^* a^+ a x a^+ = a^+ x a^\#.$$

Hence, we can construct the following equation

$$(4.5) \quad a^* y a x a^+ = y x a^\#.$$

Theorem 4.5. *Let $a \in R^\# \cap R^+$. Then, $a \in R^{SEP}$ if and only if Eq. (4.5) has at least one solution in $\zeta_a^2 =: \{(x, y) | x, y \in \zeta_a\}$.*

Proof. " $\Rightarrow$ " Assume that $a \in R^{SEP}$. Then, $a^* = a^\# = a^+$, it follows that

$$a^* a^3 a^+ = a^\# a^3 a^\# = a = a^2 a^\#.$$

Thus, $(x, y) = (a, a)$ is a solution in ζ_a^2 .

" $\Leftarrow$ " (I) If $y = a$, then $a^* a^2 x a^+ = a x a^\#$ has at least one solution x_0 in ζ_a , one gets

$$(4.6) \quad a^* a^2 x_0 a^+ = a x_0 a^\#.$$

Multiplying the equality on the right by aa^+ , one has $ax_0 a^\# = ax_0 a^\# aa^+$, this gives $x_0^+ a^\# ax_0 a^\# = x_0^+ a^\# ax_0 a^\# aa^+$.

$$\text{Noting that } x_0^+ a^\# ax_0 = \begin{cases} a^+ a, x_0 \in \left\{ a, a^\#, (a^+)^*, a^{\oplus} \right\} \\ aa^+, x_0 \in \left\{ a^+, a^*, (a^\#)^*, a^{\oplus}, (a^\#)^+, (a^+)^{\#} \right\} \end{cases}.$$

Hence, we have

if $x_0 \in \{a, a^\#, (a^+)^*, a^\oplus\}$, then $a^+aa^\# = a^+aa^\#aa^+ = a^+$;
 if $x_0 \in \{a^+, a^*, (a^\#)^*, a^\oplus, (a^\#)^+, (a^+)^#\}$, then $aa^+a^\# = aa^+a^\#aa^+$, that is, $a^\# = a^\#aa^+$.

In any case, we have $a \in R^{EP}$. Hence, $\zeta_a = \{a, a^\#, a^*, (a^+)^*\}$

(1) If $x_0 = a$, we have $a^*a^3a^+ = a^2a^\#$, e.g., $a^*a^2 = a$. Hence, $a \in R^{SEP}$ by [20, Theorem 1.5.3].

(2) If $x_0 = a^\#$, then we have $a^*a^2a^\#a^+ = aa^\#a^\#$, e.g., $a^* = a^\#$. Hence, $a \in R^{SEP}$ by [20, Theorem 1.5.3].

(3) If $x = a^*$, then $a^*a^2a^*a^+ = aa^*a^\#$. Multiplying the equality by $a(a^+)^*$ on the right and remind in heart that $a \in R^{EP}$. We have $a^*a^2 = a$. Hence, $a \in R^{SEP}$ by [20, Theorem 1.5.3].

(4) If $x = (a^+)^*$, then $a^*a^2(a^+)^*a^+ = a(a^+)^*a^\#$. Multiplying the equality on the right by aa^* , one has $a^*a^2 = a$. Hence, $a \in R^{SEP}$ by [20, Theorem 1.5.3].

(II) If $y = a^\#$, then the equation $a^*a^\#axa^+ = a^\#xa^\#$ has at least one solution x_0 in ζ_a , so one gets

$$(4.7) \quad a^*a^\#ax_0a^+ = a^\#x_0a^\#.$$

Multiplying Eq. (4.7) on the right by aa^+ , one has

$$a^\#x_0a^\# = a^\#x_0a^\#aa^+,$$

this gives $ax_0a^\# = ax_0a^\#aa^+$. Hence, $a \in R^{EP}$ by (I) and so

$$\zeta_a = \{a, a^\#, a^*, (a^+)^*\}.$$

(5) If $x_0 = a$, then $a^*a^\#a^2a^+ = a^\#aa^\#$, e.g., $a^* = a^\#$. Hence, $a \in R^{SEP}$;

(6) If $x_0 = a^\#$, then $a^*a^\#aa^\#a^+ = a^\#a^\#a^\#$, that is, $a^*a^\#a^\# = a^\#a^\#a^\#$ because $a \in R^{EP}$. Hence, $a \in R^{SEP}$;

(7) If $x_0 = a^*$, then $a^*a^\#aa^*a^+ = a^\#a^*a^\#$, that is, $a^*a^*a^+ = a^\#a^*a^\#$. Multiplying the equality on the right by $a(a^+)^*$, one gets $a^* = a^\#$. Hence, $a \in R^{SEP}$;

(8) If $x_0 = (a^+)^*$, then $a^*a^\#a(a^+)^*a^+ = a^\#(a^+)^*a^\#$, e.g., $a^+ = a^\#(a^+)^*a^\#$ and $a = aa^+a = aa^\#(a^+)^*a^\#a = (a^+)^*$. Hence, $a \in R^{SEP}$.

(III) If $y = a^+$, then the equation $a^*a^+axa^+ = a^+xa^\#$ has at least one solution x_0 in ζ_a , so we have

$$(4.8) \quad a^*a^+ax_0a^+ = a^+x_0a^\#.$$

Multiplying Eq. (4.8) on the right by aa^+ , one has

$$(4.9) \quad a^+x_0a^\# = a^+x_0a^\#aa^+.$$

Mutiplied the last equation on the left by $x_0^+(aa^\#)^*a$, one gets

$$x_0^+(aa^\#)^*x_0a^\# = x_0^+(aa^\#)^*x_0a^\#aa^+.$$

Noting that

$$x_0^+(aa^\#)^*x_0 = \begin{cases} a^+a, x_0 \in \left\{ a, a^\#, (a^+)^*, a \oplus \right\} \\ aa^+, x_0 \in \left\{ a^+, a^*, (a^\#)^*, a \oplus, (a^\#)^+, (a^+)^\# \right\} \end{cases}.$$

It follows that $aa^+a^\# = aa^+a^\#aa^+$ or $a^+aa^\# = a^+aa^\#aa^+$, e.g., $a^\# = a^\#aa^+$ or $a^+aa^\# = a^+$.

In any case, we have $a \in R^{EP}$ and so $\zeta_a = \{a, a^\#, a^*, (a^+)^*\}$.

(9) If $x_0 = a$, then $a^*a^+a^2a^+ = a^+aa^\#$, that is, $a^* = a^\#$. Hence, $a \in R^{SEP}$;

(10) If $x_0 = a^\#$, then $a^*a^+aa^\#a^+ = a^+a^\#a^\#$, that is, $a^*a^\#a^\# = a^\#a^\#a^\#$. Hence, $a \in R^{SEP}$;

(11) If $x_0 = a^*$, then $a^*a^+aa^*a^+ = a^+a^*a^\#$, that is, $a^*a^*a^+ = a^+a^*a^\#$, $a^*a^* = a^*a^*a^+a = a^+a^*a^\#a = a^+a^*$. Hence, $a \in R^{PI}$ by [30, Corollary 2.10]. Thus, $a \in R^{SEP}$;

(12) If $x_0 = (a^+)^*$, then $a^*a^+a(a^+)^*a^+ = a^+(a^+)^*a^\#$, that is, $a^+ = a^+(a^+)^*a^\#$ and $a = aa^+a = aa^+(a^+)^*a^\#a = (a^+)^*$. Hence, $a \in R^{SEP}$.

(IV) If $y = a^*$, then the equation $a^*a^*axa^+ = a^*xa^\#$ has at least one solution x_0 in ζ_a . So, we have the following equation.

$$(4.10) \quad a^*a^*ax_0a^+ = a^*x_0a^\#.$$

Clearly, we can obtain $a^*x_0a^\# = a^*x_0a^\#aa^+$, this infers

$$x_0^+(aa^\#)^*x_0a^\# = x_0^+(aa^\#)^*x_0a^\#aa^+.$$

Thus, $a \in R^{EP}$ by (III), so $\zeta_a = \{a, a^\#, a^*, (a^+)^*\}$.

Multiplying Eq. (4.10) on the left by $(a^+)^*$, one yeilds

$$(4.11) \quad a^*ax_0a^+ = a^+ax_0a^\#.$$

(13) If $x_0 = a$, then $a^*a^2a^+ = a^+a^2a^\#$, e.g., $a^*a = a^+a$. Hence, $a \in R^{SEP}$;

(14) If $x_0 = a^\#$, then $a^*aa^\#a^+ = a^+aa^\#a^\#$, e.g., $a^*a^+ = a^+a^\#$. Hence, $a \in R^{SEP}$ by [20, Theorem 1.5.3];

(15) If $x_0 = a^*$, then

$$a^*aa^*a^+ = a^+aa^*a^\# = a^*a^\#,$$

and

$$a^*aa^* = a^*aa^*a^+a = a^*a^\#a = a^*.$$

Hence, $a \in R^{SEP}$;

(16) If $x = (a^+)^*$, then

$$a^*a(a^+)^*a^+ = a^+a(a^+)^*a^\# = (a^+)^*a^\#$$

and

$$a^*a(a^+)^* = a^*a(a^+)^*a^+a = (a^+)^*a^\#a = (a^+)^*,$$

this gives

$$a^+ = a^+a^*a.$$

Hence, $a \in R^{SEP}$ by [20, Theorem 1.5.3].

(V) $y = (a^+)^*$, then the equation $a^*(a^+)^*axa^+ = (a^+)^*xa^\#$ has at least one solution x_0 in ζ_a . So, we obtain

$$(4.12) \quad a^+a^2x_0a^+ = (a^+)^*x_0a^\#.$$

We can obtain $(a^+)^*x_0a^\# = a^+a(a^+)^*x_0a^\#$, so

$$(a^+)^*x_0a^\#ax_0^+ = a^+a(a^+)^*x_0a^\#ax_0^+.$$

Noting that

$$x_0a^\#ax_0^+ = \begin{cases} aa^+, & x_0 \in \{a, a^\#, a^*, (a^+)^*, a \oplus\} \\ a^+a, & x_0 \in \{a^+, a^*, (a^\#)^*, a \oplus, (a^\#)^+, (a^+)^\#\} \end{cases}.$$

Then, $(a^+)^*aa^+ = a^+a(a^+)^*aa^+$ or $(a^+)^*a^+a = a^+a(a^+)^*a^+a$.

In the first case, we have

$$(a^+)^* = (a^+)^*aa^\# = (a^+)^*aa^+aa^\# = a^+a(a^+)^*aa^+aa^\# = a^+a(a^+)^*.$$

In the second case, we also have $(a^+)^* = a^+a(a^+)^*$.

Hence, $a \in R^{EP}$. Now, we have $\zeta_a = \{a, a^\#, a^*, (a^+)^*\}$ and Eq. (4.12) changes into $ax_0a^+ = (a^+)^*x_0a^\#$, that is, $ax_0a^\# = (a^+)^*x_0a^\#$. Hence

$$ax_0a^\#ax_0^+ = (a^+)^\#x_0a^\#ax_0^+,$$

this infers

$$a^2a^+ = (a^+)^*aa^+$$

or

$$aa^+a = (a^+)^*a^+a.$$

Noting that $a \in R^{EP}$, then, in any case, we have $a = (a^+)^*$. Hence, $a \in R^{SEP}$.

(VI) If $y = (a^\#)^*$, then the equation $a^*(a^\#)^*axa^+ = (a^\#)^*xa^\#$ has at least one solution x_0 in ζ_a . So, we have

$$(4.13) \quad (aa^\#)^*ax_0a^+ = (a^\#)^*x_0a^\#.$$

Multiplying Eq. (4.13) on the left by a^*a^* , one has

$$(4.14) \quad a^*a^*ax_0a^+ = a^*x_0a^\#.$$

Thus, $a \in R^{SEP}$ by (IV).

(VII) If $y = a^\oplus = a^\#aa^+$, then we have the following equation

$$a^*a^\#axa^+ = a^\#aa^+xa^\#.$$

Multiplying the equation on the right by ax^+ , one obtains

$$\begin{aligned} a^*a^\#axa^+ax^+ &= a^\#aa^+xa^\#ax^+ \\ &= \begin{cases} a^\#aa^+aa^+, & x \in \left\{ a, a^\#, a^*, (a^+)^*, a^\oplus \right\} \\ a^\#aa^+a^+a, & x \in \left\{ a^+, a^*, (a^\#)^*, a^\oplus, (a^\#)^+, (a^+)^\# \right\} \end{cases}. \end{aligned}$$

Multiplying the last equation on the left by a^+a , one gets

$$a^\#aa^+aa^+ = a^+aa^\#aa^+aa^+,$$

or

$$a^\#aa^+a^+a = a^+aa^\#aa^+a^+a.$$

In the first case, we have $a^\#aa^+ = a^+$.

In the second case, we have $a^\#aa^+a^+a = a^+a^+a$, this gives $a^\#aa^+ = a^\#aa^+a^*(a^\#)^* = a^\#aa^+a^+aa^*(a^\#)^* = a^+a^+aa^*(a^\#)^* = a^+$.

Hence, in any case, we have $a^+ = a^\#aa^+$, so $a \in R^{EP}$. Thus, $y = a^\oplus = a^\#$. This shows $a \in R^{SEP}$ by (II).

(VIII) If $y = a^\oplus = a^+aa^\#$, then

$$a^*a^+axa^+ = a^+aa^\#xa^\#.$$

Multiplying the equation on the right by aa^+ , one has

$$a^+aa^\#xa^\# = a^+aa^\#xa^\#aa^+.$$

Multiplying the last equation on the left by x^+a , one gets

$$x^+aa^\#xa^\# = x^+aa^\#xa^\#aa^+.$$

This infers

$$a^+aa^\# = a^+aa^\#aa^+$$

or

$$aa^+a^\# = aa^+a^\#aa^+.$$

Hence, $a^+ = a^+aa^\#$ or $a^\# = aa^\#a^+$. In any case, we have $a \in R^{EP}$, this infers $y = a \oplus = a^\#$. Thus, $a \in R^{SEP}$ by (II).

(IX) If $y = (a^\#)^+ = a^+a^3a^+$, then we have

$$a^*a^+a^3xa^+ = a^+a^3a^+xa^\#.$$

Multiplying the equation on the right by aa^+ , one yields

$$a^+a^3a^+xa^\# = a^+a^3a^+xa^\#aa^+.$$

Multiplying the last equation on the left by $a^+a^\#$, one obtains

$$(4.15) \quad a^+xa^\# = a^+xa^\#aa^+.$$

By the proof of (III), we have $a \in R^{EP}$. It follows that $y = (a^\#)^+ = (a^+)^+ = a$. Hence, $a \in R^{SEP}$ by (I).

(X) If $y = (a^+)^# = (aa^\#)^*a(aa^\#)^*$, then

$$a^*a(aa^\#)^*axa^+ = (aa^\#)^*a(aa^\#)^*xa^\#.$$

Multiplying the equation on the right by aa^+ , one obtains

$$(aa^\#)^*a(aa^\#)^*xa^\# = (aa^\#)^*a(aa^\#)^*xa^\#aa^+.$$

Multiplying the last equation on the left by a^+a^+ , one yields

$$(4.16) \quad a^+xa^\# = a^+xa^\#aa^+.$$

Hence, $a \in R^{EP}$ by (III). It follows that $y = (a^+)^# = (a^\#)^# = a$. Then, by (I), we have $a \in R^{SEP}$. $\square$

5. Conclusions

In this paper, we mainly portray *SEP* elements with the help of representations of nuclear inverse, group inverse and Moore Penrose inverse of the product of several generalised inverse elements of an element, and also construct the corresponding generalised inverse equations through the *SEP* properties of the generalised inverse elements under discussion, and then in turn discuss the solvability of these equations in a given set in order to explore the *SEP* properties of the element.

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The (p, q, r) -generations of the Mathieu sporadic simple group M_{23}

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Abstract. A finite group G is called (l, m, n) -generated, if it is a quotient group of the triangle group $T(l, m, n) = \langle x, y, z | x^l = y^m = z^n = xyz = 1 \rangle$. In [25], Moori posed the question of finding all the (p, q, r) triples, where p , q and r are prime numbers, such that a non-abelian finite simple group G is a (p, q, r) -generated. In this paper we establish all the (p, q, r) -generations of the Mathieu sporadic simple group M_{23} . GAP [16] and the Atlas of finite group representations [30] are used in our computations.

Keywords: conjugacy classes, generation, simple groups, sporadic groups.

1. Introduction

Generations of finite groups by suitable subsets is of great interest and has many applications to groups and their representations. For example, the computations of the genus of simple groups can be reduced to the generations of the relevant simple groups (see Woldar [32] for details). Also Di Martino et al. [23] established a useful connection between generation of groups by conjugates and the existence of elements representable by almost cyclic matrices. Their motivation was to study irreducible projective representations of sporadic simple groups.

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Recently more attention was given to the generation of finite groups by conjugate elements. In his PhD Thesis [29], Ward considered generation of a simple group by conjugate involutions satisfying certain conditions. In this paper we are interested in the generation of the Mathieu sporadic simple group M_{23} by two elements of prime orders not necessary distinct such that the product is an element of a prime order.

A finite group G is said to be (l, m, n) -generated, if $G = \langle x, y \rangle$, with $o(x) = l$, $o(y) = m$ and $o(xy) = o(z) = n$. Here $[x] = lX$, $[y] = mY$ and $[z] = nZ$, where $[x]$ is the conjugacy class of X in G containing elements of order l . The same applies to $[y]$ and $[z]$. In this case G is also a quotient group of the triangular group $T(l, m, n)$ and, by definition of the triangular group, G is also $(\sigma(l), \sigma(m), \sigma(n))$ -generated group for any $\sigma \in S_3$. Therefore we may assume that $l \leq m \leq n$. In a series of papers [18, 19, 20, 21, 22, 24, 25], Moori and Ganief established all possible (p, q, r) -generations, where p, q and r are distinct primes, of the sporadic groups J_1 , J_2 , J_3 , HS , McL , Co_3 , Co_2 and F_{22} . Ashrafi in [3, 4] did the same for the sporadic simple groups He and HN . Also Darafsheh and Ashrafi established in [11, 12, 13, 14], the (p, q, r) -generations of the sporadic simple groups Co_1 , Ru , $O'N$ and Ly . The motivation for this study is outlined in these papers and the reader is encouraged to consult these papers for background material as well as basic computational techniques.

In establishing the (p, q, r) -generations of the group M_{23} , we follow the methods used in [6], [7] and [8], and also methods used in the recent papers [1] and [2] by Ali, Ibrahim and Woldar. Note that, in general, if G is a $(2, 2, n)$ -generated group, then G is a dihedral group and therefore G is not simple. Also by [9], if G is a non-abelian (l, m, n) -generated group, then either $G \cong A_5$ or $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$. Thus for our purpose of establishing the (p, q, r) -generations of $G = M_{23}$, the only cases we need to consider are when $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$. The result on the (p, q, r) -generations of M_{23} can be summarized in the following theorem.

Theorem 1.1. *The sporadic simple group M_{23} is generated by all the triples (p, q, r) , p, q and r primes dividing $|M_{23}|$, except for the cases $(p, q, r) \in \{(2, 3, r), (2, 5, 5), (2, 5, 7), (3, 3, 3), (3, 3, 5), (3, 3, 7)\}$, for all r .*

2. Preliminaries

Let G be a finite group and for $k \geq 3$, suppose $C_1, C_2, \dots, C_k$ (not necessarily distinct) be conjugacy classes of G with $g_1, g_2, \dots, g_k$ being representatives for these classes respectively.

For a fixed representative $g_k \in C_k$ and for $g_i \in C_i$, $1 \leq i \leq k-1$, denote by $\Delta_G = \Delta_G(C_1, C_2, \dots, C_k)$ the number of distinct $(k-1)$ -tuples $(g_1, g_2, \dots, g_{k-1}) \in C_1 \times C_2 \times \dots \times C_{k-1}$ such that $g_1 g_2 \dots g_{k-1} = g_k$. This number is known as *class algebra constant* or *structure constant*. With $\text{Irr}(G) = \{\chi_1, \chi_2, \dots, \chi_r\}$, the number Δ_G is easily calculated from the character table of G through the formula

$$(1) \quad \Delta_G(C_1, C_2, \dots, C_k) = \frac{\prod_{i=1}^{k-1} |C_i|}{|G|} \sum_{i=1}^r \frac{\chi_i(g_1) \chi_i(g_2) \cdots \chi_i(g_{k-1}) \overline{\chi_i(g_k)}}{(\chi_i(1_G))^{k-2}}.$$

Also for a fixed $g_k \in C_k$ we denote by $\Delta_G^*(C_1, C_2, \dots, C_k)$ the number of distinct $(k-1)$ -tuples $(g_1, g_2, \dots, g_{k-1})$ satisfying

$$(2) \quad g_1 g_2 \cdots g_{k-1} = g_k \quad \text{and} \quad G = \langle g_1, g_2, \dots, g_{k-1} \rangle.$$

Definition 2.1. If $\Delta_G^*(C_1, C_2, \dots, C_k) > 0$, the group G is said to be $(C_1, C_2, \dots, C_k)$ -**generated**.

Furthermore if H is any subgroup of G containing a fixed element $h_k \in C_k$, we let $\Sigma_H(C_1, C_2, \dots, C_k)$ be the total number of distinct tuples $(h_1, h_2, \dots, h_{k-1})$ such that

$$(3) \quad h_1 h_2 \cdots h_{k-1} = h_k \quad \text{and} \quad \langle h_1, h_2, \dots, h_{k-1} \rangle \leq H.$$

The value of $\Sigma_H(C_1, C_2, \dots, C_k)$ can be obtained as a sum of the structure constants $\Delta_H(c_1, c_2, \dots, c_k)$ of H -conjugacy classes $c_1, c_2, \dots, c_k$ such that $c_i \subseteq H \cap C_i$.

Theorem 2.1. Let G be a finite group and H be a subgroup of G containing a fixed element g such that $\gcd(o(g), [N_G(H):H]) = 1$. Then the number $h(g, H)$ of conjugates of H containing g is $\chi_H(g)$, where $\chi_H(g)$ is the permutation character of G with action on the conjugates of H . In particular

$$h(g, H) = \sum_{i=1}^m \frac{|C_G(g)|}{|C_{N_G(H)}(x_i)|},$$

where $x_1, x_2, \dots, x_m$ are representatives of the $N_G(H)$ -conjugacy classes fused to the G -class of g .

Proof of Theorem 2.1. See Ganief and Moori [19, 22].

The above number $h(g, H)$ is useful in giving a lower bound for $\Delta_G^*(C_1, C_2, \dots, C_k)$, namely $\Delta_G^*(C_1, C_2, \dots, C_k)$, where

$$(4) \quad \Delta_G^*(C_1, \dots, C_k) = \Delta_G(C_1, \dots, C_k) - \sum h(g_k, H) \Sigma_H(C_1, \dots, C_k),$$

g_k is a representative of the class C_k and the sum is taken over all the representatives H of G -conjugacy classes of maximal subgroups of G containing elements of all the classes $C_1, C_2, \dots, C_k$. Since we have all the maximal subgroups of the sporadic simple groups except for $G = \text{M}$ the Monster group, it is possible to build a small subroutine in GAP to compute the values of $\Theta_G = \Theta_G(C_1, C_2, \dots, C_k)$ for any collection of conjugacy classes and a sporadic simple group.

Lemma 2.1, Theorems 2.2 and 2.3 are in some cases useful in establishing non-generation of finite groups.

Lemma 2.1. *Let G be a finite centerless group. If $\Delta_G^*(C_1, C_2, \dots, C_k) < |C_G(g_k)|$, $g_k \in C_k$, then $\Delta_G^*(C_1, C_2, \dots, C_k) = 0$ and therefore G is not $(C_1, C_2, \dots, C_k)$ -generated.*

Proof of Theorem 2.2. See [5].

Theorem 2.2 ([26]). *Let G be a transitive permutation group generated by permutations $g_1, g_2, \dots, g_s$ acting on a set of n elements such that $g_1 g_2 \dots g_s = 1_G$. If the generator g_i has exactly c_i cycles for $1 \leq i \leq s$, then $\sum_{i=1}^s c_i \leq (s - 2)n + 2$.*

For the Mathieu sporadic simple group $G = M_{23}$ and from the Atlas of finite group representations [30] we have G acting on 23 points, so that $n = 23$ and since our generation is triangular, we have $s = 3$. Hence if G is (l, m, n) -generated, then $\sum c_i \leq 25$.

Theorem 2.3 ([27]). *Let $g_1, g_2, \dots, g_s$ be elements generating a group G with $g_1 g_2 \dots g_s = 1_G$ and $\mathbb{V}$ be an irreducible module for G with $\dim \mathbb{V} = n \geq 2$. Let $C_{\mathbb{V}}(g_i)$ denote the fixed point space of $\langle g_i \rangle$ on $\mathbb{V}$ and let d_i be the codimension of $C_{\mathbb{V}}(g_i)$ in $\mathbb{V}$. Then $\sum_{i=1}^s d_i \geq 2n$.*

With χ being the ordinary irreducible character afforded by the irreducible module $\mathbb{V}$ and $\mathbf{1}_{\langle g_i \rangle}$ being the trivial character of the cyclic group $\langle g_i \rangle$, the codimension d_i of $C_{\mathbb{V}}(g_i)$ in $\mathbb{V}$ can be computed using the following formula ([15]):

$$\begin{aligned} d_i &= \dim(\mathbb{V}) - \dim(C_{\mathbb{V}}(g_i)) = \dim(\mathbb{V}) - \left\langle \chi \downarrow_{\langle g_i \rangle}^G, \mathbf{1}_{\langle g_i \rangle} \right\rangle \\ (5) \quad &= \chi(1_G) - \frac{1}{|\langle g_i \rangle|} \sum_{j=0}^{o(g_i)-1} \chi(g_i^j). \end{aligned}$$

Theorem 2.4 ([18]). *Let G be a $(2X, sY, tZ)$ -generated simple group, then G is $(sY, sY, (tZ)^2)$ -generated.*

Theorem 2.5 ([18]). *Let G be a finite group and let l, m and n be integers that are pairwise coprime. Then for any integer t coprime to n , we have*

$$\Delta(lx, mY, nZ) = \Delta(lX, mY, (nZ)^t).$$

Remark 2.1. Moreover, G is (lX, mY, nZ) -generated if and only if G is $(lX, mY, (nZ)^t)$ -generated.

We see that $(7A)^{-1} = 7B$, $(11A)^{-1} = 11B$ and $(23A)^{-1} = 23B$ in M_{23} . As an application of Theorem 2.5, the group M_{23} is $(p, q, 7A)$ -generated if and only if it is $(p, q, 7B)$ -generated, is $(p, q, 11A)$ -generated if and only if it is $(p, q, 11B)$ -generated and it is also $(p, q, 23A)$ -generated if and only if it is $(p, q, 23B)$ -generated. Therefore, it is sufficient to check the $(p, q, 7A)$ -, $(p, q, 11A)$ - and $(p, q, 23A)$ -generations of M_{23} .

3. The Mathieu sporadic simple group M_{23}

In this section we apply the results discussed in Section 2, to the group M_{23} . We determine all the (p, q, r) -generations of M_{23} , where p , q and r are primes dividing the order of M_{23} .

The group M_{23} is a simple group of order $10200960 = 2^7 \times 3^2 \times 5 \times 7 \times 11 \times 23$. By the Atlas of finite groups [10], the group M_{23} has exactly 17 conjugacy classes of its elements and 7 conjugacy classes of its maximal subgroups. Representatives of these classes of maximal subgroups can be taken as follows:

$$\begin{array}{llll} K_1 = M_{22} & K_2 = L_3(4):2_2 & K_3 = 2^4:A_7 & K_4 = A_8 \\ K_5 = M_{11} & K_6 = 2^4:(3 \times A_5):2 & K_7 = 23:11 & \end{array}$$

In this section we let $G = M_{23}$. From the electronic Atlas of finite group representations [30], we can see that M_{23} has a permutation representation on 23. Generators g_1 and g_2 can be taken as follows $g_1 = (1, 2)(3, 4)(7, 8)(9, 10)(13, 14)(15, 16)(19, 20)(21, 22)$, $g_2 = (1, 16, 11, 3)(2, 9, 21, 12)(4, 5, 8, 23)(6, 22, 14, 18)(13, 20)(15, 17)$, with $o(g_1) = 2$, $o(g_2) = 4$ and $o(g_1 g_2) = 23$.

In Table 1, we list the values of the cyclic structure for each conjugacy of G which containing elements of prime order together with the values of both c_i and d_i obtained from Ree and Scotts theorems, respectively.

Table 2 gives all the values of $d_{nX} = \dim(\mathbb{V}/C_{\mathbb{V}}(nX))$ for nX classes of prime order for the G with $\dim(\mathbb{V}) = 7$. This table will be referred to when we are proving non-generation of a triple for the group G .

In Table 3 we list the representatives of classes of the maximal subgroups together with the orbits lengths of M_{23} on these groups and the permutation characters except for the smallest maximal subgroup of M_{23} .

Table 4 gives us partial fusion maps of classes of maximal subgroups into the classes of M_{23} . These will be used in our computations.

Table 1: Cycle structures of conjugacy classes of M_{23}

nX	Cycle Structure	c_i	d_i
1A	1^{23}	23	0
2A	$1^7 2^8$	15	8
3A	$1^5 3^6$	11	12
4A	$1^3 2^2 4^4$	9	14
5A	$1^3 5^4$	7	16
6A	$1^1 2^2 3^2 6^2$	7	16
7A	$1^2 7^3$	5	18
7B	$1^2 7^3$	5	18
8A	$1^1 2^1 4^1 8^2$	5	18
11A	$1^1 11^2$	3	20
11B	$1^1 11^2$	3	20
14A	$2^1 7^1 14^1$	3	20
14B	$2^1 7^1 14^1$	3	20
15A	$3^1 5^1 15^1$	3	20
15B	$3^1 5^1 15^1$	3	20
23A	23^1	1	22
23B	23^1	1	22

Table 2: $d_{nX} = \dim(\mathbb{V}/C_{\mathbb{V}}(nX))$, nX is a non-trivial class of G and $\dim(\mathbb{V}) = 22$.

nX	2A	3A	5A	7A	7B	11A	11B	23A	23B
Cycle Structure	$1^7 2^8$	$1^5 3^6$	$1^3 5^4$	$1^2 7^3$	$1^2 \bar{7}^3$	$1^1 11^2$	$1^1 11^2$	23^1	23^1
c_i	15	11	7	5	5	3	3	1	1
d_{nX}	8	12	16	18	18	20	20	22	22

Table 3: Maximal subgroups of M_{23}

Maximal Subgroup	Order	Orbit Lengths	Character
M_{22}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	[1,22]	$1a + 22a$
$L_3(4):2_2$	$2^7 \cdot 3^2 \cdot 5 \cdot 7$	[2,21]	$1a + 22a + 230a$
$2^4:A_7$	$2^7 \cdot 3^2 \cdot 5 \cdot 7$	[7,16]	$1a + 22a + 230a$
A_8	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	[8,15]	$1a + 22a + 230a + 253$
M_{11}	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	[11,12]	$1a + 22a + 230a + 1035a$
$2^4:(3 \times A_5):2$	$2^7 \cdot 3^2 \cdot 5$	[3,20]	$1a + 22a + 230a + 253a + 1035a$
23:11	$11 \cdot 23$	[23]	

Table 4: The partial fusion maps into M_{23}

M_{22} -class $\rightarrow M_{23}$ h	2a 2A	3a 3A	5a 5A	7a 7A	7b 7B	11a 11A	11b 11B					
$L_3(4):2_2$ -class $\rightarrow M_{23}$ h	2a 2A	2b 2A	3a 3A	5a 5A	7a 7A	7b 7B	1 1					
$2^4:A_7$ -class $\rightarrow M_{23}$ h	2a 2A	2b 2A	3a 3A	3b 3A	5a 5A	7a 7A	7b 7B	1 1				
A_8 -class $\rightarrow M_{23}$ h	2a 2A	2b 2A	3a 3A	3b 3A	5a 5A	7a 7A	7b 7B	1 2	1 2			
M_{11} -class $\rightarrow M_{23}$ h	2a 2A	3a 3A	5a 5A	11a 11A	11b 11B							
$2^4:(3 \times A_5):2$ -class $\rightarrow M_{23}$ h	2a 2A	2b 2A	2c 2A	3a 3A	3b 3A	3c 3A	5a 5A	1 1				
$23:11$ -class $\rightarrow M_{23}$ h	11a 11A 1	11b 11B 1	11c 11A 1	11d 11A 1	11e 11A 1	11f 11B 1	11g 11B 1	11h 11B 1	11i 11A 1	11j 11B 1	23a 23A 1	23b 23B 1

4. The $(2, q, r)$ -generations of M_{23}

Let pX , $p \in \{2, 3, 5, 7, 11, 23\}$, $X \in \{A, B\}$ be a conjugacy class of $G = M_{23}$ and c_i be the number of disjoint cycles in a representative of pX . For M_{23} with three disjoint cycles, and acting on $n = 23$ points, we get $n(s - 2) + 2 = 23 + 2 = 25$. Also G is not $(2A, 2A, pX)$ -generated, for if G is $(2A, 2A, pX)$ -generated, then G is a dihedral group and thus is not simple. Also we know that if G is (l, m, n) -generated with $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} \geq 1$ and G is simple, then $G \cong A_5$, but $G \cong M_{23}$ and $M_{23} \not\cong A_5$. Hence if G is (p, q, r) -generated, then we must have $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$.

Moreover if G is $(2A, 3A, rX)$ -generated, then we must have $r > 6$ but we show in Theorem 4.1 below that in our case G is not $(2A, 3A, rX)$ -generated for all r .

Now, the $(2, q, r)$ -generations of M_{23} comprises the cases $(2, 3, r)$ -, $(2, 5, r)$ -, $(2, 7, r)$ -, $(2, 11, r)$ - and $(2, 23, r)$ - generations.

4.1 $(2, 3, r)$ -generations

Proposition 4.1. *G is not $(2A, 3A, r)$ -generated for all r .*

Proof. The condition $\frac{1}{2} + \frac{1}{3} + \frac{1}{r} < 1$ shows that $r > 6$. Therefore we have to consider the cases $(2A, 3A, 7X)$, $(2A, 3A, 11X)$ and $(2A, 3A, 23X)$ for all $X \in \{A, B\}$. Theorem 1.1 of [28] implies that G is not a Hurwitz group and hence G is not a $(2A, 3A, 7X)$ -generated for $X \in \{A, B\}$. Generally, if G is $(2A, 3A, r)$ -generated group, then we must have $c_{2A} + c_{3A} + c_p \leq 25$. From Table 1 we see that $c_{2A} + c_{3A} + c_r = 15 + 11 + c_p > 25$ for $p \in \{7A, 7B, 11A, 11B, 23A, 23B\}$. Now, using Ree's Theorem [26], it follows that G is not $(2A, 3A, r)$ -generated. $\square$

Remark 4.1. The above results can be deduced by Scott's Theorem [27], as from Table 2 we can see that $d_{2A} + d_{3A} + d_{nX} = 8 + 12 + d_{nX} < 2 \times 22$ for $nX \in \{7A, 7B, 11A, 11B, 23A, 23B\}$.

4.1.1 $(2, 5, r)$ -generations

The condition $\frac{1}{2} + \frac{1}{5} + \frac{1}{r} < 1$ shows that $r > \frac{10}{3}$. Thus we have to consider the cases $(2A, 5A, 5A)$, $(2A, 5A, 7X)$, $(2A, 5A, 11X)$ and $(2A, 5A, 23X)$ for $X \in \{A, B\}$.

Proposition 4.2. *The group G is neither $(2A, 5A, 5A)$ - nor $(2A, 5A, 7X)$ -generated for $X \in \{A, B\}$.*

Proof. If G is a $(2A, 5A, 5A)$ -generated group, then we must have $c_{2A} + c_{5A} + c_{5A} \leq 25$. From Table 1 we see that $c_{2A} + c_{5A} + c_{5A} = 15 + 7 + 7 = 29 > 25$. Now, using Ree's Theorem, it follows that G is not $(2A, 5A, 5A)$ -generated.

By the same Table 1 we see that $c_{2A} + c_{5A} + c_{7A} = 15 + 7 + 5 = 27 > 25$. Again by Ree's Theorem, it follows that G is not $(2A, 5A, 7A)$ -generated. Since the same holds for $(2A, 5A, 7B)$, it follows that G is not $(2A, 5A, 7X)$ -generated for $X \in \{A, B\}$ and the proof is complete. $\square$

Proposition 4.3. *The group G is $(2A, 5A, 11X)$ -generated for $X \in \{A, B\}$.*

Proof. By Table 4, we see that K_1 , K_5 and K_7 are the maximal subgroups having elements of order 11.

The intersection of the conjugacy classes these three maximal subgroups do not contain elements of order 11. Considering all various pairwise intersections of the conjugacy classes for these three maximal subgroups, we found that the only candidate having elements of order 11 is isomorphic to the group $PSL_2(11)$.

The maximal subgroup K_7 will not have any contributions because it does not contain elements of orders 2 and 5. We obtained that $\sum_{K_1}(2a, 5a, 11b) =$

176 , $\sum_{K_5}(2a, 5a, 11b) = 33$ and $\sum_{PSL_2(11)}(2a, 5x, 11b) = \Delta_{PSL_2(11)}(2a, 5a, 11b) + \Delta_{PSL_2(11)}(2a, 5b, 11b) = 11 + 11 = 22$. By [17, 31], we have $h(11A, K_1) = h(11A, K_5) = h(11A, PSL_2(11)) = 1$. Since by Table 5 we have $\Delta_G(2A, 5A, 11A) = 253$, we then obtain that $\Delta_G^*(2A, 5A, 11A) \geq \Delta_G(2A, 5A, 11A) - \sum_{K_1}(2a, 5a, 11b) - \sum_{K_5}(2a, 5a, 11b) + \sum_{PSL_2(11)}(2a, 5x, 11b) = 253 - 176 - 33 + 22 = 66 > 0$. Hence G is $(2A, 5A, 11A)$ -generated. Since the same holds for $(2A, 5A, 11B)$ (see Remark 2.1), it follows that G is $(2A, 5A, 11X)$ -generated, for all $X \in \{A, B\}$. $\square$

Proposition 4.4. *The group G is $(2A, 5A, 23X)$ -generated for $X \in \{A, B\}$.*

Proof. By Table 4, we see the maximal subgroup K_7 is the only one have elements of order 23. This maximal subgroup will not have any contributions because it does not have elements of orders 2 and 5.

Since by Table 5, we have $\Delta_G(2A, 5A, 23A) = 138$, we then deduce that $\Delta_G^*(2A, 5A, 23A) = \Delta_G(2A, 5A, 23A) = 138 > 0$. Thus G is $(2A, 5A, 23A)$ -generated. Since the same holds for $(2A, 5A, 23B)$, it follows that G is a $(2A, 5A, 23X)$ -generated group, for $X \in \{A, B\}$. $\square$

4.1.2 $(2, 7, r)$ -generations

We check for the generation of G through the triples $(2A, 7X, 7Y)$, $(2A, 7X, 11Y)$ and $(2A, 7X, 23Y)$ for all $X, Y \in \{A, B\}$.

Proposition 4.5. *The group G is $(2A, 7X, 7Y)$ -generated for all $X, Y \in \{A, B\}$.*

Proof. By Table 4 we see that the maximal subgroups of G whose orders are divisible by 7 are K_1 , K_2 , K_3 and K_4 .

The intersection of conjugacy classes from these four maximal subgroups do not contain elements of order 7. The intersection of the conjugacy classes from any three maximal subgroups do not contain elements of order 7. Considering all various intersections of the conjugacy classes for pairwise of these three maximal subgroups, we noticed that the groups $PSL_3(4)$, A_7 (2-copies), $2^3:PLS_3(2)$ (2-copies) and $PSL_3(2)$ are the only ones having elements of order 7.

The group $PSL_3(2)$ has its relevant structure constant zero and as such it will not have any contributions. We obtained that $\sum_{PSL_3(4)}(2a, 7a, 7a) = 42$, $\sum_{A_7}(2a, 7a, 7a) = 7$, $\sum_{2^3:PLS_3(2)}(2a, 7b, 7b) = 7$ and $h(7A, PSL_3(4)) = h(7A, A_7) = h(7A, 2^3:PLS_3(2)) = 2$. For the contributing maximal subgroups, we have $\sum_{K_1}(2a, 7b, 7b) = 147$, $\sum_{K_2}(2x, 7b, 7b) = \Delta_{K_2}(2a, 7b, 7b) + \Delta_{K_2}(2b, 7b, 7b) = 0 + 42 = 42$, $\sum_{K_3}(2a, 7a, 7a) = 7$, $\sum_{K_4}(2x, 7b, 7b) = \Delta_{K_4}(2a, 7b, 7b) + \Delta_{K_4}(2b, 7b, 7b) = 14 + 28 = 42$ and found that $h(7A, K_2) = h(7A, K_3) = 1$ and $h(7A, K_1) = h(7A, K_4) = 2$. Since by Table 5 we have $\Delta_G(2A, 7A, 7A) = 301$, we then obtain that $\Delta_G^*(2A, 7A, 7A) \geq \Delta_G(2A, 7A, 7A) - 2 \cdot \sum_{K_1}(2a, 7b, 7b) - \sum_{K_2}(2x, 7b, 7b) - \sum_{K_3}(2a, 7a, 7a) - 2 \cdot \sum_{K_4}(2x, 7b, 7b) + 2 \cdot \sum_{PSL_3(4)}(2a, 7b, 7b) + 2 \cdot 2 \cdot \sum_{A_7}(2a, 7a, 7a) + 2 \cdot 2 \cdot \sum_{2^3:PLS_3(2)}(2a, 7b, 7b) = 301 - 2(147) - 42 - 7 - 2(42) + 2(42) + 2(2)(7) + 2(2)(7) = 14 > 0$ and it follows that $(2A, 7A, 7A)$ is

a generating triple for G . Since the same holds for $(2A, 7B, 7B)$, it follows that the group G is $(2A, 7X, 7X)$ -generated, for all $X \in \{A, B\}$.

We now investigate the $(2A, 7A, 7B)$ -generations for G . From the intersections, we noticed that the groups $PSL_3(4)$, A_7 (2-copies), $2^3:PLS_3(2)$ (2-copies) and $PSL_3(2)$ will all contribute here. We obtained that $\sum_{PSL_3(4)}(2a, 7a, 7b) = 63$, $\sum_{A_7}(2a, 7a, 7b) = 28$, $\sum_{2^3:PSL_3(2)}(2a, 7b, 7a) = 14$, $\sum_{PSL_3(2)}(2a, 7a, 7b) = 7$ and $h(7B, PSL_3(4)) = h(7B, A_7) = h(7B, 2^3:PSL_3(2)) = h(7B, PSL_3(2)) = 2$.

The maximal subgroup K_3 will not have any contributions because its relevant structure constant is zero. For the contributing maximal subgroups, we have $\sum_{K_1}(2a, 7b, 7a) = 224$, $\sum_{K_2}(2x, 7b, 7a) = \Delta_{K_2}(2a, 7b, 7a) + \Delta_{K_2}(2b, 7b, 7a) = 0 + 63 = 63$, $\sum_{K_4}(2x, 7b, 7a) = \Delta_{K_4}(2a, 7b, 7a) + \Delta_{K_4}(2b, 7b, 7a) = 21 + 42 = 63$ and found that $h(7B, K_2) = 1$ and $h(7B, K_1) = h(7B, K_4) = 2$. Since by Table 5 we have $\Delta_G(2A, 7A, 7B) = 462$, we then obtain that $\Delta_G^*(2A, 7A, 7B) = \Delta_G(2A, 7A, 7B) - 2 \cdot \sum_{K_1}(2a, 7b, 7a) - \sum_{K_2}(2x, 7b, 7a) - 2 \cdot \sum_{K_4}(2x, 7b, 7a) + 2 \cdot \sum_{PSL_3(4)}(2a, 7a, 7b) + 2 \cdot \sum_{A_7}(2a, 7b, 7a) + 2 \cdot \sum_{2^3:PSL_3(2)}(2a, 7b, 7a) + 2 \cdot \sum_{PSL_3(2)}(2a, 7b, 7a) = 462 - 2(224) - 63 - 2(21) + 2(63) + 2(2)(28) + 2(2)(14) + 2(7) = 217 > 0$. Therefore G is $(2A, 7A, 7B)$ -generated. $\square$

Proposition 4.6. *The group G is $(2A, 7X, 11Y)$ -generated for all $X, Y \in \{A, B\}$.*

Proof. Looking at Proposition 4.3, we see that $PSL_2(11)$ is the only group having elements of order 11. This group $PSL_2(11)$ will not have any contributions because it does not contain elements of order 7. With regard to maximal subgroups having elements of order 11, by Table 4 we see that the maximal subgroup K_1 of G is the only one whose order is divisible by 7 and 11. We obtained that $\sum_{K_1}(2a, 7x, 11y) = 176$ and $h(11Z, K_1) = 1$ for $Z \in \{A, B\}$. By Table 5 we have $\Delta_G(2A, 7X, 11Y) = 308$ so that $\Delta_G^*(2A, 7X, 11Y) \geq \Delta_G(2A, 7X, 11Y) - \sum_{K_1}(2a, 7x, 11y) = 308 - 176 = 132 > 0$, implies that G is $(2A, 7X, 11Y)$ -generated for all $X, Y \in \{A, B\}$. $\square$

Proposition 4.7. *The group G is $(2A, 7X, 23Y)$ -generated for all $X, Y \in \{A, B\}$.*

Proof. By Table 4, K_7 is the only maximal subgroup having elements of order 23. This maximal subgroup K_7 does not have elements of order 7. By Table 5 we have $\Delta_G(2A, 7X, 23Y) = 184$. Since there are no contributions from any of the maximal subgroups of G , we then have $\Delta_G^*(2A, 7X, 23Y) = \Delta_G(2A, 7X, 23Y) = 184 > 0$, proving that G is $(2A, 7X, 23Y)$ -generated for all $X, Y \in \{A, B\}$. $\square$

4.1.3 $(2, 11, r)$ -generations

Also here we check for the generation of G through the triples $(2A, 11A, 11A)$ -, $(2A, 11A, 11B)$ -, $(2A, 11A, 23A)$ -, $(2A, 11A, 23B)$ -, $(2A, 11B, 11B)$ -, $(2A, 11B, 23A)$ - and $(2A, 11B, 23B)$ -generation. For this we have the following theorems:

Proposition 4.8. *The group G is $(2A, 11X, 11Y)$ -generated for $X, Y \in \{A, B\}$.*

Proof. Looking at the discussions in Proposition 4.3 for the intersections, we see that the group $PSL_2(11)$ may be involved when proving $(2A, 11X, 11Y)$ -generations. By Table 4 we see that the maximal subgroups of G containing elements of orders 2 and 11 are K_1 and K_5 . The groups K_1 , K_5 and $PSL_2(11)$ have elements of orders 2 and 11. We obtained that $\sum_{K_1}(2a, 11x, 11x) = 99$, $\sum_{K_5}(2a, 11x, 11x) = 11$ and $\sum_{PSL_2(11)}(2a, 11x, 11x) = 11$. We found that $h(11A, K_1) = h(11A, K_5) = h(11A, PSL_2(11)) = 1$. Since by Table 5 we have $\Delta_G(2A, 11X, 11X) = 341$, so that $\Delta_G^*(2A, 11X, 11X) \geq \Delta_G(2A, 11X, 11X) - \sum_{K_1}(2a, 11x, 11x) - \sum_{K_5}(2a, 11x, 11x) + \sum_{PSL_2(11)}(2a, 11x, 11x) = 341 - 147 - 11 + 11 = 194 > 0$, proving that G is $(2A, 11X, 11X)$ -generated for $X \in \{A, B\}$.

Finally, we show that G is $(2A, 11A, 11B)$ -generated. We obtained that $\sum_{K_1}(2a, 11b, 11a) = 132$ and $\sum_{K_5}(2a, 11b, 11a) = 11$. The group $PSL_2(11)$ will not have any contributions because its relevant structure constant is zero. Since by Table 5 we have $\Delta_G(2A, 11A, 11B) = 341$, so that $\Delta_G^*(2A, 11A, 11B) = \Delta_G(2A, 11A, 11B) - \sum_{K_1}(2a, 11b, 11a) - \sum_{K_5}(2a, 11b, 11a) = 341 - 224 - 11 = 106 > 0$, implies that G is $(2A, 11A, 11B)$ -generated. We conclude that G is $(2A, 11Y, 11Z)$ -generated for all $Y, Z \in \{A, B\}$. $\square$

Proposition 4.9. *The group G is $(2A, 11X, 23Y)$ -generated for $X, Y \in \{A, B\}$.*

Proof. By Table 4 we see the K_7 is the only maximal subgroup of G containing elements of order 23. This maximal subgroup will not have any contributions because it does not have elements of order 2. By Table 5 we have $\Delta_G(2A, 11X, 23Y) = 391$, so that $\Delta_G^*(2A, 11X, 23Y) = \Delta_G(2A, 11X, 23Y) = 391 > 0$. Hence the group G is $(2A, 11X, 23Y)$ -generated for all $X, Y \in \{A, B\}$. $\square$

4.1.4 $(2, 23, r)$ -generations

In here we check for the generation of G through the triples $(2A, 23A, 23A)$, $(2A, 23A, 23B)$ and $(2A, 23B, 23B)$. For these we have the following theorems:

Proposition 4.10. *The group G is $(2A, 23X, 23Y)$ -generated for $X, Y \in \{A, B\}$.*

Proof. By Table 4 we see the K_7 is the only maximal subgroup of G containing elements of order 23. This maximal subgroup will not have any contributions because it does not have elements of order 2. By Table 5 we have $\Delta_G(2A, 23X, 23X) = 161$ and $\Delta_G(2A, 23A, 23B) = 230$ for $X \in \{A, B\}$. Since there is no contributing group here, we then obtain that $\Delta_G^*(2A, 23X, 23X) = \Delta_G(2A, 23X, 23X) = 161 > 0$ and $\Delta_G^*(2A, 23A, 23B) = \Delta_G(2A, 23A, 23B) = 230 > 0$ for all $X \in \{A, B\}$. Hence, the group G is a $(2A, 23X, 23Y)$ -generated for $X, Y \in \{A, B\}$. $\square$

4.2 The $(3, q, r)$ -generations

The condition $\frac{1}{3} + \frac{1}{3} + \frac{1}{r} < 1$ shows that $r > 3$. We then handle all the possible $(3, q, r)$ -generations, namely $(3A, 3A, 5A)$ -, $(3A, 3A, 7X)$ -, $(3A, 3A, 11X)$ -

, $(3A, 3A, 23X)$ -, $(3A, 5A, 5A)$ -, $(3A, 5A, 7X)$ -, $(3A, 5A, 11X)$ -, $(3A, 5A, 23X)$ -, $(3A, 7X, 7Y)$ -, $(3A, 7X, 11Y)$ -, $(3A, 7X, 23Y)$ -, $(3A, 11X, 11Y)$ -, $(3A, 11X, 23Y)$ - and $(3A, 23X, 23Y)$ -generations in this section.

4.2.1 $(3, 3, r)$ -generations

Proposition 4.11. *The group G is neither $(3A, 3A, 5A)$ - nor $(3A, 3A, 7X)$ -generated group for $X \in \{A, B\}$.*

Proof. By Table 2, the group G acts on a 22-dimensional irreducible complex module V . By Scott's Theorem applied to this module and using the Atlas of finite groups, we see that $d_{3A} = \dim(V/C_V(3A)) = \frac{2(22-4)}{3} = 12$, $d_{5A} = \dim(V/C_V(5A)) = \frac{4(22-2)}{5} = 16$ and $d_{7A} = d_{7B} = \dim(V/C_V(7A)) = \frac{6(22-1)}{7} = 18$. For the case $(3A, 3A, 5A)$, we get $d_{3A} + d_{3A} + d_{5A} = 2 \times 12 + 16 = 40 < 44$ showing that G is not $(3A, 3A, 5A)$ -generated. We also get that $d_{3A} + d_{3A} + d_{7X} = 2 \times 12 + 16 = 42 < 44$ for $X \in \{A, B\}$ and by Scott's Theorem G is not $(3A, 3A, 7X)$ -generated for all $X \in \{A, B\}$ and the proof is complete. $\square$

Proposition 4.12. *The group G is $(3A, 3A, 11X)$ -generated for $X \in \{A, B\}$.*

Proof. Looking at Proposition 4.3, we notice that the subgroups of G involved here are K_1 , K_5 and $PSL_2(11)$ because both subgroups have their elements of respective orders 3 and 11 which fuse to the elements $3A$ and $11A$ (or $11B$) of the group G . We obtained that $\sum_{K_1}(3a, 3a, 11b) = 209$, $\sum_{K_5}(3a, 3a, 11b) = 11$ and $\sum_{PSL_2(11)}(3a, 3a, 11b) = 11$. We already have $h(11A, K_1) = h(11A, K_5) = h(11A, PSL_2(11)) = 1$. Since by Table 6 we have $\Delta_G(3A, 3A, 11A) = 275$, we then obtain that $\Delta_G^*(3A, 3A, 11A) \geq \Delta_G(3A, 3A, 11A) - \sum_{K_1}(3a, 3a, 11b) - \sum_{K_5}(3a, 3a, 11b) + \sum_{PSL_2(11)}(3a, 3a, 11b) = 275 - 209 - 11 + 11 = 66 > 0$, proving that G is $(3A, 3A, 11A)$ -generated. Since the same holds for $(3A, 3A, 11B)$, it follows that G is $(3A, 3A, 11X)$ -generated, for all $X \in \{A, B\}$. $\square$

Proposition 4.13. *The group G is a $(3A, 3A, 23X)$ -generated for $X \in \{A, B\}$.*

Proof. By Table 4, K_7 is the only maximal subgroup having elements of order 23. This maximal subgroup will not have any contributions because it does not have elements of order 3. By Table 6 we have that $\Delta_G(3A, 3A, 23X) = 138$. Since there is no contributing group, we then obtain that $\Delta^*(3A, 3A, 23X) = \Delta_G(3A, 3A, 23X) = 138 > 0$, so that G is $(3A, 3A, 23X)$ -generated for $X \in \{A, B\}$. $\square$

4.2.2 $(3, 5, r)$ -generations

Proposition 4.14. *The group G is $(3A, 5A, 5A)$ -generated.*

Proof. Looking at Table 4 we see that all the maximal subgroups of G have elements of order 5 except for the seventh maximal subgroup. Let T be the set of all maximal subgroups of G except the seventh one. We look at various

intersections of conjugacy classes for these maximal subgroups. We have the following:

- The groups arising from the intersections of conjugacy classes for any 4, 5 or 6 maximal subgroups in T do not contain elements of order 5.
- The group arising from intersections of the conjugacy classes for any three maximal subgroups in T having elements of orders 3 and 5 is S_5 (2-copies). We obtained that $\sum_{S_5}(3a, 5a, 5a) = 10$ and $h(5A, S_5) = 3$.
- The groups arising from intersections of the conjugacy classes for any two maximal subgroups in T having elements of orders 3 and 5 are $2^4:S_5$ (3-copies), $PSL_3(4)$, A_7 (2-copies), $2^4:A_6$, $PSL_2(11)$, $A_6:2$, A_5 and S_5 (2-copies). We obtained that $\sum_{2^4:S_5}(3a, 5a, 5a) = 160$, $\sum_{PSL_3(4)}(3a, 5x, 5y) = \Delta_{PSL_3(4)}(3a, 5a, 5a) + \Delta_{PSL_3(4)}(3a, 5a, 5b) + \Delta_{PSL_3(4)}(3a, 5b, 5b) = 445 + 445 + 445 = 1335$, $\sum_{A_7}(3x, 5a, 5a) = \Delta_{A_7}(3a, 5a, 5a) + \Delta_{A_7}(3b, 5a, 5a) = 20 + 60 = 80$, $\sum_{2^4:A_6}(3x, 5y, 5z) = \Delta_{2^4:A_6}(3a, 5a, 5a) + \Delta_{2^4:A_6}(3a, 5a, 5b) + \Delta_{2^4:A_6}(3a, 5b, 5b) + \Delta_{2^4:A_6}(3b, 5a, 5a) + \Delta_{2^4:A_6}(3b, 5a, 5b) + \Delta_{2^4:A_6}(3b, 5b, 5b) = 80 + 160 + 80 + 20 + 40 + 20 = 400$, $\sum_{PSL_2(11)}(3a, 5x, 5y) = \Delta_{PSL_2(11)}(3a, 5a, 5a) + \Delta_{PSL_2(11)}(3a, 5a, 5b) + \Delta_{PSL_2(11)}(3a, 5b, 5b) = 20 + 20 + 20 = 60$, $\sum_{A_6:2}(3a, 5a, 5a) = 30$, $\sum_{A_5}(3a, 5x, 5y) = \Delta_{A_5}(3a, 5a, 5a) + \Delta_{A_5}(3a, 5a, 5b) + \Delta_{A_5}(3a, 5b, 5b) = 5 + 5 + 5 = 15$ and $\sum_{S_5}(3a, 5a, 5a) = 10$. We found that the value of h for each of these eight groups is 3.

By Table 6 we have $\Delta_G(3A, 5A, 5A) = 6550$. We obtained that $\sum_{K_1}(3a, 5a, 5a) = 2800$, $\sum_{K_2}(3a, 5a, 5a) = 910$, $\sum_{K_3}(3x, 5a, 5a) = \Delta_{K_3}(3a, 5a, 5a) + \Delta_{K_3}(3b, 5a, 5a) = 320 + 240 = 560$, $\sum_{K_4}(3x, 5a, 5a) = \Delta_{K_4}(3a, 5a, 5a) + \Delta_{K_4}(3b, 5a, 5a) = 25 + 135 = 160$, $\sum_{K_5}(3a, 5a, 5a) = 80$, $\sum_{K_6}(3x, 5a, 5a) = \Delta_{K_6}(3a, 5a, 5a) + \Delta_{K_6}(3b, 5a, 5a) + \Delta_{K_6}(3c, 5a, 5a) = 0 + 0 + 160 = 160$. The value of h for each maximal subgroup is 3 except for K_4 and K_6 . The value of h is 1 for each of these maximal subgroups K_4 and K_6 . It follows that $\Delta_G^*(3A, 5A, 5A) \geq \Delta_G(3A, 5A, 5A) - 3 \cdot \sum_{K_1}(3a, 5a, 5a) - 3 \cdot \sum_{K_2}(3a, 5a, 5a) - 3 \cdot \sum_{K_3}(3x, 5a, 5a) - \sum_{K_4}(3x, 5a, 5a) - 3 \cdot \sum_{K_5}(3a, 5a, 5a) - \sum_{K_6}(3x, 5a, 5a) - 2 \cdot 3 \cdot \sum_{S_5}(3a, 5a, 5a) + 3 \cdot 3 \cdot \sum_{2^4:S_5}(3a, 5a, 5a) + 3 \cdot \sum_{PSL_3(4)}(3a, 5x, 5y) + 2 \cdot 3 \cdot \sum_{A_7}(3x, 5a, 5a) + 3 \cdot \sum_{2^4:A_6}(3x, 5y, 5z) + 3 \cdot \sum_{PSL_2(11)}(3a, 5x, 5y) + 3 \cdot \sum_{A_6:2}(3a, 5a, 5a) + 3 \cdot \sum_{A_5}(3a, 5x, 5y) + 2 \cdot 3 \cdot \sum_{S_5}(3a, 5a, 5a) = 6550 - 3(2800) - 3(910) - 3(560) - 1(160) - 3(80) - 1(160) - 2(3)(10) + 3(3)(160) + 3(1335) + 2(3)(80) + 3(400) + 3(60) + 3(30) + 3(15) + 2(3)(10) = 620 > 0$. It follows that the group G is $(3A, 5A, 5A)$ -generated. $\square$

Proposition 4.15. *The group G is $(3A, 5A, 7X)$ -generated for $X \in \{A, B\}$.*

Proof. As in Proposition 4.5, the groups $PSL_3(4)$, A_7 (2-copies), $2^3:PLS_3(2)$ (2-copies) and $PSL_3(2)$ may have contributions here. The groups $2^3:PLS_3(2)$ and $PSL_3(2)$ will not have any contributions because they do not have elements of order 5. We obtained that $\sum_{PSL_3(4)}(3a, 5x, 7b) = \Delta_{PSL_3(4)}(3a, 5a, 7b) +$

$\Delta_{PSL_3(4)}(3a, 5b, 7b) = 441 + 441 = 882$, $\sum_{A_7}(3x, 5a, 7b) = \Delta_{A_7}(3a, 5a, 7b) + \Delta_{A_7}(3b, 5a, 7b) = 56 + 7 = 63$ and $h(7A, PSL_3(4)) = h(7A, A_7) = 2$.

The maximal subgroups K_1 , K_2 , K_3 and K_4 meet the $3A$, $5A$, $7A$ classes of G . We obtained that $\sum_{K_1}(3a, 5a, 7b) = 2464$, $\sum_{K_2}(3a, 5a, 7b) = 882$, $\sum_{K_3}(3x, 5a, 7a) = \Delta_{K_3}(3a, 5a, 7a) + \Delta_{K_3}(3b, 5a, 7a) = 112 + 224 = 336$, $\sum_{K_4}(3x, 5a, 7b) = \Delta_{K_4}(3a, 5a, 7b) + \Delta_{K_4}(3b, 5a, 7b) = 77 + 7 = 84$. We found that $h(7A, K_1) = h(7A, K_4) = 2$ and $h(7A, K_2) = h(7A, K_3) = 1$.

Since by Table 6 we have $\Delta_G(3A, 5A, 7A) = 5124$, we then obtain that $\Delta_G^*(3A, 5A, 7A) \geq \Delta_G(3A, 5A, 7A) - 2 \cdot \sum_{K_1}(3a, 5a, 7b) - \sum_{K_2}(3a, 5a, 7b) - \sum_{K_3}(3x, 5a, 7a) - 2 \cdot \sum_{K_4}(3a, 5a, 7b) + 2 \cdot \sum_{PSL_3(4)}(3a, 5x, 7b) + 2 \cdot 2 \cdot \sum_{A_7}(3x, 5a, 7b) = 5124 - 2(2464) - 882 - 336 - 2(84) + 2(882) + 2(2)(63) = 826 > 0$. Therefore, the group G is $(3A, 5A, 7A)$ -generated. Since the same holds for $(3A, 5A, 7B)$, it follows that the group G is $(3A, 5A, 7X)$ -generated for $X \in \{A, B\}$. $\square$

Proposition 4.16. *The group G is $(3A, 5A, 11X)$ -generated for $X \in \{A, B\}$.*

Proof. By Table 4 we see that the maximal subgroups of G containing elements of orders 3 and 11 are K_1 and K_5 . The group $PSL_2(11)$ contains elements of orders 3, 5 and 11. We obtained that $\sum_{K_1}(3a, 5a, 11b) = 2112$, $\sum_{K_5}(3a, 5a, 11a) = 99$ and $\sum_{PSL_2(11)}(3a, 5x, 11b) = \Delta_{PSL_2(11)}(3a, 5a, 11b) + \Delta_{PSL_2(11)}(3a, 5b, 11b) = 22 + 22 = 44$. We already have $h(11A, K_1) = h(11A, K_5) = h(11A, PSL_2(11)) = 1$. Since by Table 6 we have $\Delta_G(3A, 5A, 11A) = 4136$, we then have $\Delta_G^*(3A, 5A, 11A) \geq \Delta_G(3A, 5A, 11A) - \sum_{K_1}(3a, 5a, 11b) - \sum_{K_5}(3a, 5a, 11b) + \sum_{PSL_2(11)}(3a, 5x, 11b) = 4136 - 2112 - 99 + 44 = 1969 > 0$, so that G is $(3A, 5A, 11A)$ -generated. Since the same holds for $(3A, 5A, 11B)$, it follows that the group G is $(3A, 5A, 11X)$ -generated for $X \in \{A, B\}$. $\square$

Proposition 4.17. *The group G is $(3A, 5A, 23X)$ -generated group for $X \in \{A, B\}$.*

Proof. By Table 4, K_7 is the only maximal subgroup having elements of order 23. This maximal subgroup will not have any contributions because it does not have elements of order 5. By Table 6 we have that $\Delta_G(3A, 5A, 23X) = 2438$. Since there is no contributing group, we then obtain that $\Delta_G^*(3A, 5A, 23X) = \Delta_G(3A, 5A, 23X) = 2438 > 0$, so that G is $(3A, 5A, 23X)$ -generated for $X \in \{A, B\}$. $\square$

4.2.3 $(3, 7, r)$ -generations

In this subsection we discuss the case $(3, 7, r)$ -generations. It follows that we will end up with 11 cases, namely $(3A, 7A, 7A)$ -, $(3A, 7A, 7B)$ -, $(3A, 7A, 11A)$ -, $(3A, 7A, 11B)$ -, $(3A, 7A, 23A)$ -, $(3A, 7A, 23B)$ -, $(3A, 7B, 7B)$ -, $(3A, 7B, 11A)$ -, $(3A, 7B, 11B)$ -, $(3A, 7B, 23A)$ and $(3A, 7B, 23B)$ -generation.

Proposition 4.18. *The group G is $(3A, 7X, 7Y)$ -generated for all $X, Y \in \{A, B\}$*

Proof. As in Proposition 4.5, the groups $PSL_3(4)$, A_7 (2-copies), $2^3:PSL_3(2)$ (2-copies) and $PSL_3(2)$ have elements of order 7. We obtained that $\sum_{PSL_3(4)}(3a, 7b, 7b) = 357$, $\sum_{A_7}(3x, 7b, 7b) = \Delta_{A_7}(3a, 7b, 7b) + \Delta_{A_7}(3b, 7b, 7b) = 56 + 21 = 77$, $\sum_{2^3:PSL_3(2)}(3a, 7b, 7b) = 28$, $\sum_{PSL_3(2)}(3a, 7a, 7a) = 7$ and $h(7A, PSL_3(4)) = h(7A, A_7) = h(7A, 2^3:PSL_3(2)) = h(7A, PSL_3(2)) = 2$.

The maximal subgroups K_1 , K_2 , K_3 and K_4 meet the $3A$, $7A$ classes of G . We obtained that $\sum_{K_1}(3a, 7b, 7b) = 1792$, $\sum_{K_2}(3a, 7b, 7b) = 357$, $\sum_{K_3}(3x, 7a, 7a) = \Delta_{K_3}(3a, 7a, 7a) + \Delta_{K_3}(3b, 7a, 7a) = 168 + 126 = 294$, $\sum_{K_4}(3x, 7b, 7b) = \Delta_{K_4}(3a, 7b, 7b) + \Delta_{K_4}(3b, 7b, 7b) = 147 + 21 = 168$. We found that $h(7A, K_1) = h(7A, K_4) = 2$ and $h(7A, K_2) = h(7A, K_3) = 1$.

Since by Table 6 we have $\Delta_G(3A, 7A, 7A) = 4886$, we then obtain that $\Delta_G^*(3A, 7A, 7A) \geq \Delta_G(3A, 7A, 7A) - 2 \cdot \sum_{K_1}(3a, 7b, 7b) - \sum_{K_2}(3a, 7b, 7b) - \sum_{K_3}(3x, 7a, 7a) - 2 \cdot \sum_{K_4}(3a, 7b, 7b) + 2 \cdot \sum_{PSL_3(4)}(3a, 7b, 7b) + 2 \cdot 2 \cdot \sum_{A_7}(3x, 7b, 7b) + 2 \cdot 2 \cdot \sum_{2^3:PSL_3(2)}(3a, 7b, 7b) + 2 \cdot \sum_{PSL_3(2)}(3a, 7a, 7a) = 4886 - 2(1792) - 357 - 394 - 2(168) + 2(357) + 2(2)(77) + 2(2)(28) + 2(7) = 1363 > 0$. Therefore, the group G is $(3A, 7A, 7A)$ -generated. Since the same holds for $(3A, 7B, 7B)$, it follows that the group G is $(3A, 7X, 7X)$ -generated for $X \in \{A, B\}$.

We now prove that G is $(3A, 7A, 7B)$ -generated.

We obtained that $\sum_{PSL_3(4)}(3a, 7b, 7a) = 357$, $\sum_{A_7}(3x, 7b, 7b) = \Delta_{A_7}(3a, 7b, 7b) + \Delta_{A_7}(3b, 7b, 7b) = 28 + 14 = 32$, $\sum_{2^3:PSL_3(2)}(3a, 7b, 7b) = 28$, $\sum_{PSL_3(2)}(3a, 7a, 7b) = 7$, $\sum_{K_1}(3a, 7b, 7a) = 1792$, $\sum_{K_2}(3a, 7b, 7a) = 357$, $\sum_{K_3}(3x, 7a, 7b) = \Delta_{K_3}(3a, 7a, 7b) + \Delta_{K_3}(3b, 7a, 7b) = 112 + 70 = 182$, $\sum_{K_4}(3x, 7b, 7a) = \Delta_{K_4}(3a, 7b, 7a) + \Delta_{K_4}(3b, 7b, 7a) = 21 + 147 = 168$, $\sum_{PSL_3(4)}(3a, 7b, 7a) = 357$, $\sum_{A_7}(3x, 7b, 7a) = \Delta_{A_7}(3a, 7b, 7a) + \Delta_{A_7}(3b, 7b, 7a) = 56 + 21 = 77$, $\sum_{2^3:PSL_3(2)}(3a, 7b, 7a) = 28$ and $\sum_{PSL_3(2)}(3a, 7a, 7b)$. Since by the same Table 6 we have $\Delta_G(3A, 7A, 7B) = 4886$, so that $\Delta_G^*(3A, 7A, 7B) \geq \Delta_G(3A, 7A, 7B) - 2 \cdot \sum_{K_1}(3a, 7b, 7a) - \sum_{K_2}(3a, 7b, 7a) - \sum_{K_3}(3x, 7a, 7b) - 2 \cdot \sum_{K_4}(3x, 7a, 7b) + 2 \cdot \sum_{PSL_3(4)}(3a, 7b, 7a) + 2 \cdot 2 \cdot \sum_{A_7}(3x, 7b, 7a) + 2 \cdot 2 \cdot \sum_{2^3:PSL_3(2)}(3a, 7b, 7a) + 2 \cdot \sum_{PSL_3(2)}(3a, 7a, 7b) = 4886 - 2(1792) - 357 - 182 - 2(168) + 2(357) + 2(2)(42) + 2(2)(28) + 2(7) = 1435 > 0$. This proves that G is $(3A, 7A, 7B)$ -generated group. $\square$

Proposition 4.19. *The group G is $(3A, 7X, 11Y)$ -generated for all $X, Y \in \{A, B\}$.*

Proof. Looking at Proposition 4.3, K_1 , K_5 , K_7 and $PSL_2(11)$ are the only groups having elements of order 11. The group $PSL_2(11)$ will not have any contributions because it does not have elements of order 7. Looking at Table 4, we see that K_1 is the only maximal subgroup of G having elements of orders 3, 7 and 11. We obtained that $\sum_{K_1}(3a, 7x, 11y) = 1760$ and $h(11X, K_1) = 1$ for $X \in \{A, B\}$. By Table 6 we have $\Delta_G(3A, 7X, 11Y) = 4136$. We obtained that $\Delta_G^*(3A, 7X, 11Y) \geq \Delta_G(3A, 7X, 11Y) - \sum_{K_1}(3a, 7x, 11y) = 4136 - 1760 = 2376$ and so that the group G becomes is $(3A, 7X, 11Y)$ -generated for all $X, Y \in \{A, B\}$. $\square$

Proposition 4.20. *The group G is $(3A, 7X, 23Y)$ -generated for all $X, Y \in \{A, B\}$.*

Proof. By Table 4, K_7 is the only maximal subgroup having elements of order 23. This maximal subgroup will not have any contributions because it does not have elements of orders 3 and 7. By Table 6 we have that $\Delta_G(3A, 7X, 23Y) = 3312$. Since there is no contributing group, we then obtain that $\Delta_G^*(3A, 7X, 23Y) = \Delta_G(3A, 7X, 23Y) = 3312 > 0$, so that the group G is $(3A, 7X, 23Y)$ -generated for all $X, Y \in \{A, B\}$. $\square$

4.2.4 $(3, 11, r)$ -generations

In this subsection we discuss the case $(3, 11, r)$ -generations.

It follows that we will end up with 7 cases, namely $(3A, 11A, 11A)$ -, $(3A, 11A, 11B)$ -, $(3A, 11A, 23A)$ -, $(3A, 11A, 23B)$ -, $(3A, 11B, 11B)$ -, $(3A, 11B, 23A)$ -, $(3A, 11B, 23B)$ -generation.

Proposition 4.21. *The group G is $(3A, 11X, 11Y)$ -generated for all $X, Y \in \{A, B\}$.*

Proof. Looking at Proposition 4.3, K_1 , K_5 , K_7 and $PSL_2(11)$ are the only groups having elements of order 11. The maximal subgroup K_7 will not have any contributions because it does not have elements of order 3. We obtained that $\sum_{K_1}(3a, 11b, 11b) = 1320$, $\sum_{K_5}(3a, 11b, 11b) = 22$ and $\sum_{PSL_2(11)}(3a, 11b, 11b) = 0$. The value of h for each group is 1. Since by Table 6 we have $\Delta_G(3A, 11A, 11A) = 5126$, it follows that $\Delta_G^*(3A, 11A, 11A) \geq \Delta_G(3A, 11A, 11A) - \sum_{K_1}(3a, 11b, 11b) - \sum_{K_5}(3a, 11b, 11b) + \sum_{PSL_2(11)}(3a, 11b, 11b) = 5126 - 1320 - 22 + 0 = 3784 > 0$. Therefore, the group G is $(3A, 11A, 11A)$ -generated. Since the same holds for $(3A, 11B, 11B)$, the group G becomes $(3A, 11X, 11X)$ -generated for $X \in \{A, B\}$.

We now prove that G is $(3A, 11A, 11B)$ -generated.

We obtained that $\sum_{K_1}(3a, 11a, 11b) = 1276$, $\sum_{K_5}(3a, 11a, 11b) = 77$ and $\sum_{PSL_2(11)}(3a, 11a, 11b) = 22$. By the same Table 6 we have $\Delta_G(3A, 11A, 11B) = 5379$. Then, we obtain that $\Delta_G^*(3A, 11A, 11B) \geq \Delta_G(3A, 11A, 11B) - \sum_{K_1}(3a, 11b, 11a) - \sum_{K_5}(3a, 11b, 11a) + \sum_{PSL_2(11)}(3a, 11b, 11a) = 5379 - 1276 - 77 + 22 = 4048 > 0$, proving that G is $(3A, 11A, 11B)$ -generated. Hence, the group G is $(3A, 11X, 11Y)$ -generated for all $X, Y \in \{A, B\}$. $\square$

Proposition 4.22. *The group G is $(3A, 11X, 23Y)$ -generated for all $X, Y \in \{A, B\}$.*

Proof. By Table 4, K_7 is the only maximal subgroup having elements of order 23. This maximal subgroup will not have any contributions because it does not have elements of order 3. By Table 6 we have that $\Delta_G(3A, 11X, 23Y) = 5129$. Since there is no contributing group, we then obtain that $\Delta^*(3A, 11X, 23Y) = \Delta_G(3A, 11X, 23Y) = 5129 > 0$, so that the group G is $(3A, 11X, 23Y)$ -generated for all $X, Y \in \{A, B\}$. $\square$

4.2.5 $(3, 23, r)$ -generations

In this subsection we discuss the case $(3, 23, r)$ -generations. It follows that we will end up with 3 cases, namely $(3A, 23A, 23A)$ -, $(3A, 23A, 23B)$ -, $(3A, 23B, 23B)$ -generation which will be handled in the following Proposition 4.23.

Proposition 4.23. *The group G is $(3A, 23X, 23Y)$ -generated for all $X, Y \in \{A, B\}$.*

Proof. By Table 4, K_7 is the only maximal subgroup having elements of order 23. This maximal subgroup will not have any contributions because it does not have elements of order 3. By Table 6 we have $\Delta_G(3A, 23A, 23A) = 3082$. Since there is no contributing group, we then obtain that $\Delta^*(3A, 23X, 23X) = \Delta_G(3A, 23A, 23A) = 3082 > 0$, so that the group G is $(3A, 23A, 23A)$ -generated. Since the same holds for $(3A, 23B, 23B)$, the group G will be $(3A, 23B, 23B)$ -generated. Similarly, $\Delta^*(3A, 23A, 23B) = \Delta_G(3A, 23A, 23B) = 2714 > 0$, so that the group G becomes $(3A, 23A, 23B)$ -generated. $\square$

4.3 Other results

In this section we handle all the remaining cases, namely the $(5, q, r)$ -, $(7, q, r)$ -, $(11, q, r)$ - and $(23, q, r)$ -generations.

4.3.1 $(5, 5, r)$ -generations

In this subsection we discuss the case $(5, 5, r)$ -generations. It follows that we will end up with 5 cases, namely $(5A, 5A, 5A)$ -, $(5A, 5A, 11A)$ -, $(5A, 5A, 11B)$ -, $(5A, 5A, 23A)$ -, $(5A, 5A, 23B)$ -generation.

Proposition 4.24. *The group G is $(5A, 5A, 5A)$ -generated.*

Proof. From Table 4 we see that all the maximal subgroups of G have elements of order 5 except for the seventh maximal subgroup. Let T be the set of all maximal subgroups of G except the seventh one. We look at various intersections of conjugacy classes for these maximal subgroups. We have the following:

- The groups arising from the intersections of conjugacy classes for any 4, 5 or 6 maximal subgroups in T do not contain elements of order 5.
- The groups arising from intersections of the conjugacy classes for any three maximal subgroups in T having elements of order 5 are S_5 (2-copies), D_{10} and $5:4$. We obtained that $\sum_{S_5}(5a, 5a, 5a) = 8$, $\sum_{D_{10}}(5x, 5y, 5z) = \Delta_{D_{10}}(5a, 5a, 5a) + \Delta_{D_{10}}(5a, 5a, 5b) + \Delta_{D_{10}}(5a, 5b, 5b) + \Delta_{D_{10}}(5b, 5b, 5b) = 0 + 1 + 1 + 0 = 2$ and $\sum_{5:4}(5a, 5a, 5a) = 3$. We found that the value of h for each of these three groups is 3.
- The groups arising from intersections of the conjugacy classes for any two maximal subgroups in T having elements of order 5 are $2^4:S_5$ (3-copies), $PSL_3(4)$, A_7 (2-copies), $2^4:A_6$, $PSL_2(11)$, $A_6:2$, A_5 and S_5 (2-copies).

We obtained that $\sum_{2^4:S_5}(5a, 5a, 5a) = 128$, $\sum_{PSL_3(4)}(3a, 5x, 5y) = \Delta_{PSL_3(4)}(5a, 5a, 5a) + \Delta_{PSL_3(4)}(5a, 5a, 5b) + \Delta_{PSL_3(4)}(5a, 5b, 5b) + \Delta_{PSL_3(4)}(5b, 5b, 5b) = 845 + 781 + 781 + 845 = 3252$, $\sum_{A_7}(a, 5a, 5a) = \Delta_{A_7}(5a, 5a, 5a) = 108$, $\sum_{2^4:A_6}(5x, 5y, 5z) = \Delta_{2^4:A_6}(5a, 5a, 5a) + \Delta_{2^4:A_6}(5a, 5a, 5b) + \Delta_{2^4:A_6}(5a, 5b, 5b) + \Delta_{2^4:A_6}(5b, 5b, 5b) = 320 + 176 + 176 + 320 = 992$, $\sum_{PSL_2(11)}(5x, 5y, 5z) = \Delta_{PSL_2(11)}(5a, 5a, 5a) + \Delta_{PSL_2(11)}(5a, 5a, 5b) + \Delta_{PSL_2(11)}(5a, 5b, 5b) + \Delta_{PSL_2(11)}(5b, 5b, 5b) = 20 + 31 + 31 + 20 = 102$, $\sum_{A_6:2}(5a, 5a, 5a) = 53$, $\sum_{A_5}(5x, 5y, 5z) = \Delta_{A_5}(5a, 5a, 5a) + \Delta_{A_5}(5a, 5a, 5b) + \Delta_{A_5}(5a, 5b, 5b) + \Delta_{A_5}(5b, 5b, 5b) = 5 + 1 + 1 + 5 = 12$ and $\sum_{S_5}(5a, 5a, 5a) = 8$. We found that the value of h for each of these eight groups is 3.

By Table 7 we have $\Delta_G(5A, 5A, 5A) = 61058$. We obtained that $\sum_{K_1}(5a, 5a, 5a) = 18368$, $\sum_{K_2}(5a, 5a, 5a) = 3188$, $\sum_{K_3}(5a, 5a, 5a) = 1728$, $\sum_{K_4}(5a, 5a, 5a) = 173$, $\sum_{K_5}(5a, 5a, 5a) = 378$, $\sum_{K_6}(5a, 5a, 5a) = 128$. The value of h for each maximal subgroup is 3 except for K_4 and K_6 . The value of h is 1 for each of these maximal subgroups K_4 and K_6 . It follows that $\Delta_G^*(5A, 5A, 5A) \geq \Delta_G(5A, 5A, 5A) - 3 \cdot \sum_{K_1}(5a, 5a, 5a) - 3 \cdot \sum_{K_2}(5a, 5a, 5a) - 3 \cdot \sum_{K_3}(5a, 5a, 5a) - \sum_{K_4}(5a, 5a, 5a) - 3 \cdot \sum_{K_5}(5a, 5a, 5a) - \sum_{K_6}(5a, 5a, 5a) - 2 \cdot 3 \cdot \sum_{S_5}(5a, 5a, 5a) - 3 \cdot \sum_{D_{10}}(5x, 5y, 5z) - 3 \cdot \sum_{5:4}(5a, 5a, 5a) + 3 \cdot 3 \cdot \sum_{2^4:S_5}(5a, 5a, 5a) + 3 \cdot \sum_{PSL_3(4)}(5x, 5y, 5z) + 2 \cdot 3 \cdot \sum_{A_7}(5a, 5a, 5a) + 3 \cdot \sum_{2^4:A_6}(5x, 5y, 5z) + 3 \cdot \sum_{PSL_2(11)}(5x, 5y, 5z) + 3 \cdot \sum_{A_6:2}(5a, 5a, 5a) + 3 \cdot \sum_{A_5}(5x, 5y, 5z) + 2 \cdot 3 \cdot \sum_{S_5}(5a, 5a, 5a) = 61058 - 3(18368) - 3(3188) - 3(1728) - 1(173) - 3(378) - 1(128) - 2(3)(11) - 3(2) - 3(3) + 3(3)(128) + 3(3252) + 2(3)(108) + 3(992) + 3(102) + 3(53) + 3(12) + 2(3)(8) = 6499 > 0$. It follows that the group G is $(5A, 5A, 5A)$ -generated. $\square$

Proposition 4.25. *The group G is $(5A, 5A, 7X)$ -generated for $X \in \{A, B\}$.*

Proof. As in Proposition 4.5, we observe that the groups $PSL_3(4)$, A_7 (2-copies), $2^3:PLS_3(2)$ (2-copies) and $PSL_3(2)$ may have contributions here. The groups $2^3:PLS_3(2)$ and $PSL_3(2)$ will not have any contributions because they do not have elements of order 5. We obtained that $\sum_{PSL_3(4)}(5x, 5y, 7b) = \Delta_{PSL_3(4)}(5a, 5a, 7b) + \Delta_{PSL_3(4)}(5a, 5b, 7b) + \Delta_{PSL_3(4)}(5b, 5b, 7b) = 819 + 819 + 819 = 2457$, $\sum_{A_7}(5a, 5a, 7b) = 84$ and $h(7A, PSL_3(4)) = h(7A, A_7) = 2$.

The maximal subgroups K_1 , K_2 , K_3 and K_4 meet the $5A$, $7A$ classes of G . We obtained that $\sum_{K_1}(5a, 5a, 7b) = 17920$, $\sum_{K_2}(5a, 5a, 7b) = 3276$, $\sum_{K_3}(5a, 5a, 7a) = 1344$, $\sum_{K_4}(5a, 5a, 7b) = 91$. We found that $h(7A, K_1) = h(7A, K_4) = 2$ and $h(7A, K_2) = h(7A, K_3) = 1$.

Since by Table 7 we have $\Delta_G(5A, 5A, 7A) = 54320$, we then obtain that $\Delta_G^*(5A, 5A, 7A) \geq \Delta_G(5A, 5A, 7A) - 2 \cdot \sum_{K_1}(5a, 5a, 7b) - \sum_{K_2}(5a, 5a, 7b) - \sum_{K_3}(5a, 5a, 7a) - 2 \cdot \sum_{K_4}(5a, 5a, 7b) + 2 \cdot \sum_{PSL_3(4)}(5x, 5y, 7b) + 2 \cdot 2 \cdot \sum_{A_7}(5a, 5a, 7b) = 54320 - 2(17920) - 3276 - 1344 - 2(91) + 2(2457) + 2(2)(84) = 18928 > 0$. Therefore, the group G is $(5A, 5A, 7A)$ -generated. Since the same holds for $(5A, 5A, 7B)$, it follows that the group G is $(5A, 5A, 7X)$ -generated for $X \in \{A, B\}$. $\square$

Proposition 4.26. *The group G is $(5A, 5A, 11X)$ -generated for $X \in \{A, B\}$.*

Proof. By Proposition 4.3 we proved that the group G is $(2A, 5A, 11X)$ -generated for $X \in \{A, B\}$. It follows by Theorem 2.4 that G is $(5A, 5A, (11A)^2)$ - and $(5A, 5A, (11B)^2)$ -generated. By GAP, we see that $(11A)^2 = 11B$ and $(11B)^2 = 11A$ and the results follow. $\square$

Proposition 4.27. *The group G is $(5A, 5A, 23X)$ -generated for $X \in \{A, B\}$.*

Proof. By Proposition 4.4 we proved that G is $(2A, 5A, 23X)$ -generated for $X \in \{A, B\}$. It follows by Theorem 2.4 that the group G is $(5A, 5A, (23A)^2)$ - and $(5A, 5A, (23B)^2)$ -generated. Since by GAP we have $(23A)^2 = 23A$ and $(23B)^2 = 23B$, then the results follow. $\square$

4.3.2 $(5, 7, r)$ -generations

In this subsection we discuss the case $(5, 7, r)$ -generations.

It follows that we will end up with 11 cases, namely $(5A, 7A, 7A)$ -, $(5A, 7A, 7B)$ -, $(5A, 7A, 11A)$ -, $(5A, 7A, 11B)$ -, $(5A, 7A, 23A)$ -, $(5A, 7A, 23B)$ -, $(5A, 7B, 7B)$ -, $(5A, 7B, 11A)$ -, $(5A, 7B, 11B)$ -, $(5A, 7B, 23A)$ -, $(5A, 7B, 23B)$ -generation.

Proposition 4.28. *The group G is $(5A, 7X, 7Y)$ -generated for all $X, Y \in \{A, B\}$.*

Proof. As in Proposition 4.5, we observe that the groups $PSL_3(4)$, A_7 (2-copies), $2^3:PLS_3(2)$ (2-copies) and $PSL_3(2)$ may have contributions here. The groups $2^3:PLS_3(2)$ and $PSL_3(2)$ will not have any contributions because they both do not have elements of order 5. We obtained that $\sum_{PSL_3(4)}(5x, 7b, 7b) = \Delta_{PSL_3(4)}(5a, 7b, 7b) + \Delta_{PSL_3(4)}(5b, 7b, 7b) = 567 + 567 = 1134$, $\sum_{A_7}(5a, 7b, 7b) = 84$ and $h(7A, PSL_3(4)) = h(7A, A_7) = 2$.

The maximal subgroups K_1 , K_2 , K_3 and K_4 meet the $5A$, $7A$ classes of G . We obtained that $\sum_{K_1}(5a, 7b, 7b) = 12544$, $\sum_{K_2}(5a, 7b, 7b) = 1134$, $\sum_{K_3}(5a, 7a, 7a) = 672$, $\sum_{K_4}(5a, 7b, 7b) = 189$. We found that $h(7A, K_1) = h(7A, K_4) = 2$ and $h(7A, K_2) = h(7A, K_3) = 1$.

Since by Table 7 we have $\Delta_G(5A, 7A, 7A) = 52584$, we then obtain that $\Delta_G^*(5A, 7A, 7A) \geq \Delta_G(5A, 7A, 7A) - 2 \cdot \sum_{K_1}(5a, 7b, 7b) - \sum_{K_2}(5a, 7b, 7b) - \sum_{K_3}(5a, 7a, 7a) - 2 \cdot \sum_{K_4}(5a, 7b, 7b) + 2 \cdot \sum_{PSL_3(4)}(5x, 7b, 7b) + 2 \cdot \sum_{A_7}(5a, 7b, 7b) = 52584 - 2(12544) - 1134 - 672 - 2(189) + 2(1134) + 2(2)(84) = 27916 > 0$. Therefore, the group G is $(5A, 7A, 7A)$ -generated. Since the same holds for $(5A, 7B, 7B)$, it follows that the group G is $(5A, 7X, 7X)$ -generated for $X \in \{A, B\}$.

We now prove that the group G is $(5A, 7A, 7B)$ -generated. We obtained that $\sum_{PSL_3(4)}(5x, 7b, 7a) = \Delta_{PSL_3(4)}(5a, 7b, 7a) + \Delta_{PSL_3(4)}(5b, 7b, 7a) = 567 + 567 = 1134$, $\sum_{A_7}(5a, 7b, 7b) = 84$, $\sum_{K_1}(5a, 7b, 7a) = 12544$, $\sum_{K_2}(5a, 7b, 7a) = 1134$, $\sum_{K_3}(5a, 7a, 7b) = 672$ and $\sum_{K_4}(5a, 7b, 7a) = 189$. Since by same Table 7 we have $\Delta_G(5A, 7A, 7B) = 52584$, we then obtain that $\Delta_G^*(5A, 7A, 7B) \geq \Delta_G(5A, 7A, 7B) - 2 \cdot \sum_{K_1}(5a, 7b, 7a) - \sum_{K_2}(5a, 7b, 7a) - \sum_{K_3}(5a, 7a, 7b) - 2 \cdot \sum_{K_4}(5a, 7b, 7a) + 2 \cdot \sum_{PSL_3(4)}(5x, 7b, 7a) + 2 \cdot \sum_{A_7}(5a, 7b, 7a) = 52584 - 2(12544) - 1134 - 672 - 2(189) + 2(1134) + 2(2)(84) = 27916 > 0$, proving that the group G is $(5A, 7A, 7B)$ -generated. $\square$

Proposition 4.29. *The group G is $(5A, 7X, 11Y)$ -generated for all $X, Y \in \{A, B\}$.*

Proof. Looking at Proposition 4.3, we see that K_1 , K_5 , K_7 and $PSL_2(11)$ are the only groups having elements of order 11. The groups K_5 , K_7 and $PSL_2(11)$ will not have any contributions because they both do not have elements of order 7. We obtained that $\sum_{K_1}(5a, 7x, 11y) = 12672$ and $h(11Z, K_1) = 1$ for $Z \in \{A, B\}$. By Table 7 we have $\Delta_G(5A, 7X, 11Y) = 48576$ for all $X, Y \in \{A, B\}$. We then obtain that $\Delta_G^*(5A, 7X, 11Y) \geq \Delta_G(5A, 7X, 11Y) - \sum_{K_1}(5a, 7x, 11y) = 48576 - 12672 = 35904 > 0$, so that the group G becomes $(5A, 7X, 11Y)$ -generated for all $X, Y \in \{A, B\}$. $\square$

Proposition 4.30. *The group G is $(5A, 7X, 23Y)$ -generated for all $X, Y \in \{A, B\}$.*

Proof. By Table 4, K_7 is the only maximal subgroup having elements of order 23. This maximal subgroup will not have any contributions because it does not have elements of orders 5 and 7. By Table 7 we have $\Delta_G(5A, 7X, 23Y) = 44160$ for all $X, Y \in \{A, B\}$. Since there is no contributing group, we then obtain that $\Delta_G^*(5A, 7X, 23Y) = \Delta_G(5A, 7X, 23Y) = 44160 > 0$, so that the group G is $(5A, 7X, 23Y)$ -generated for all $X, Y \in \{A, B\}$. $\square$

4.3.3 $(5, 11, r)$ -generations

In this subsection we discuss the case $(5, 11, r)$ -generations. It follows that we will end up with 7 cases, namely $(5A, 11A, 11A)$ -, $(5A, 11A, 11B)$ -, $(5A, 11A, 23A)$ -, $(5A, 11A, 23B)$ -, $(5A, 11B, 11B)$ -, $(5A, 11B, 23A)$ - and $(5A, 11B, 23B)$ -generation.

Proposition 4.31. *The group G is $(5A, 11X, 11Y)$ -generated for all $X, Y \in \{A, B\}$.*

Proof. Looking at Proposition 4.3, we see that K_1 , K_5 , K_7 and $PSL_2(11)$ are the only groups having elements of order 11. The group K_7 will not have any contributions because it does not have elements of order 5. We obtained that $\sum_{K_1}(5a, 11b, 11b) = 8448$, $\sum_{K_5}(5a, 11b, 11b) = 198$ and $\sum_{PSL_2(11)}(5x, 11b, 11b) = \Delta_{PSL_2(11)}(5a, 11b, 11b) + \Delta_{PSL_2(11)}(5b, 11b, 11b) = 11 + 11 = 22$. By Table 7, we have $\Delta_G(5A, 11A, 11A) = 62238$. We already have $h(11A, K_1) = h(11A, K_5) = h(11A, PSL_2(11)) = 1$. We then have $\Delta_G^*(5A, 11X, 11X) \geq \Delta_G(5A, 11A, 11A) - \sum_{K_1}(5a, 11b, 11b) - \sum_{K_5}(5a, 11b, 11b) + \sum_{PSL_2(11)}(5a, 11b, 11b) = 62238 - 8448 - 198 + 22 = 53614 > 0$, showing that the group G is $(5A, 11A, 11A)$ -generated.

Since the same holds for $(5A, 11B, 11B)$ implies that the group G is $(5A, 11A, 11A)$ -generated. For the $(5A, 11A, 11B)$ -generations, we obtained that $\sum_{K_1}(5a, 11b, 11a) = 8448$, $\sum_{K_5}(5a, 11b, 11a) = 99$ and $\sum_{PSL_2(11)}(5a, 11x, 11y) = \Delta_{PSL_2(11)}(5a, 11b, 11b) + \Delta_{PSL_2(11)}(5b, 11b, 11b) = 11 + 11 = 22$. Since by Table 7 we have $\Delta_G(5A, 11A, 11B) = 61479$, we obtain that $\Delta_G^*(5A, 11A, 11B) \geq \Delta_G(5A, 11A, 11B) - \sum_{K_1}(5a, 11b, 11a) - \sum_{K_5}(5a, 11b, 11a) + \sum_{PSL_2(11)}(5a, 11b,$

$11a) = 61479 - 8448 - 99 + 22 = 52954 > 0$, proving that the group G is $(5A, 11A, 11B)$ -generated. $\square$

Proposition 4.32. *The group G is $(5A, 11X, 23Y)$ -generated for all $X, Y \in \{A, B\}$.*

Proof. By Table 4, K_7 is the only maximal subgroup having elements of order 23. This maximal subgroup will not have any contributions because it does not have elements of order 5. By Table 7 we have $\Delta_G(5A, 11X, 23Y) = 61893$ for all $X, Y \in \{A, B\}$. Since there is no contributing group, we then obtain that $\Delta_G^*(5A, 11X, 23Y) = \Delta_G(5A, 11X, 23Y) = 61893 > 0$, so that G is $(5A, 11X, 23Y)$ -generated group for all $X, Y \in \{A, B\}$. $\square$

4.3.4 $(5, 23, r)$ -generations

In this subsection we discuss the case $(5, 23, r)$ -generations. It follows that we will end up with 3 cases, namely $(5A, 23A, 23A)$ -, $(5A, 23A, 23B)$ - and $(5A, 23B, 23B)$ -generation.

Proposition 4.33. *The group G is $(5A, 23X, 23Y)$ -generated for all $X, Y \in \{A, B\}$.*

Proof. By Table 4, K_7 is the only maximal subgroup having elements of order 23. This maximal subgroup will not have any contributions because it does not have elements of order 5. By Table 7 we have $\Delta_G(5A, 23X, 23Y) = 32706$ for all $X, Y \in \{A, B\}$. Since there is no contributing group, we then obtain that $\Delta_G^*(5A, 23X, 23Y) = \Delta_G(5A, 23X, 23Y) = 32706 > 0$, so that the group G is $(5A, 11X, 23Y)$ -generated for all $X, Y \in \{A, B\}$. $\square$

4.3.5 $(7, 7, r)$ -generations

In this subsection we discuss the case $(7, 7, r)$ -generations.

It follows that we will end up with 16 cases, namely $(7A, 7A, 7A)$ -, $(7A, 7A, 7B)$ -, $(7A, 7A, 11A)$ -, $(7A, 7A, 11B)$ -, $(7A, 7A, 23A)$ -, $(7A, 7A, 23B)$ -, $(7A, 7B, 7B)$ -, $(7A, 7B, 11A)$ -, $(7A, 7B, 11B)$ -, $(7A, 7B, 23A)$ -, $(7A, 7B, 23B)$ -, $(7B, 7B, 7B)$ -, $(7B, 7B, 11A)$ -, $(7B, 7B, 11B)$ -, $(7B, 7B, 23A)$ - and $(7B, 7B, 23B)$ -generation.

Proposition 4.34. *The group G is $(7X, 7Y, 7Z)$ -generated for all $X, Y, Z \in \{A, B\}$.*

Proof. By Proposition 4.5 we proved that G is $(2A, 7A, 7X)$ -generated for $X \in \{A, B\}$. It follows by Theorem 2.4 that the group G is $(7A, 7A, (7A)^2)$ - and $(7A, 7A, (B)^2)$ -generated. Since by the power maps, we have $(7A)^2 = 7A$ and $(7B)^2 = 7B$, the group G becomes $(7A, 7A, 7A)$ - and $(7A, 7A, 7B)$ -generated. Since G is $(7A, 7A, 7A)$ -generated, the same will hold for $(7B, 7B, 7B)$.

We are left only to investigate of the $(7A, 7B, 7B)$ generation for the group G . As in Proposition 4.5, we observe that the groups $PSL_3(4)$, A_7 (2-copies),

$2^3:PSL_3(2)$ (2-copies) and $PSL_3(2)$ have contributions. We obtained that $\sum_{PSL_3(4)}(7b, 7a, 7a) = 357$, $\sum_{A_7}(7b, 7a, 7a) = 36$, $\sum_{2^3:PSL_3(2)}(7b, 7a, 7a) = 8$, $\sum_{PSL_3(2)}(7a, 7b, 7b) = 1$ and $h(7A, PSL_3(4)) = h(7A, A_7) = h(7A, 2^3:PSL_3(2)) = h(7A, PSL_3(2)) = 2$.

The maximal subgroups K_1 , K_2 , K_3 and K_4 have elements of order 7. We obtained that $\sum_{K_1}(7b, 7a, 7a) = 8576$, $\sum_{K_2}(7b, 7a, 7a) = 379$, $\sum_{K_3}(7a, 7b, 7b) = 148$, $\sum_{K_4}(7b, 7a, 7a) = 379$. We found that $h(7A, K_1) = h(7A, K_4) = 2$ and $h(7A, K_2) = h(7A, K_3) = 1$.

Since by Table 7 we have $\Delta_G(7A, 7B, 7B) = 51948$, we then obtain that $\Delta_G^*(7A, 7B, 7B) \geq \Delta_G(7A, 7B, 7B) - 2 \cdot \sum_{K_1}(7b, 7a, 7a) - \sum_{K_2}(7b, 7a, 7a) - \sum_{K_3}(7a, 7b, 7b) - 2 \cdot \sum_{K_4}(7b, 7a, 7a) + 2 \cdot \sum_{PSL_3(4)}(7b, 7a, 7a) + 2 \cdot 2 \cdot \sum_{A_7}(7b, 7a, 7a) + 2 \cdot 2 \cdot \sum_{2^3:PSL_3(2)}(7b, 7a, 7a) + 2 \cdot \sum_{PSL_3(2)}(7a, 7b, 7b) = 51948 - 2(8576) - 379 - 148 - 2(379) + 2(379) + 2(2)(36) + 2(2)(8) + 2(1) = 34447 > 0$. Therefore, the group G is $(7A, 7B, 7B)$ -generated. $\square$

Proposition 4.35. *The group G is $(7X, 7Y, 11Z)$ -generated for $X, Y, Z \in \{A, B\}$.*

Proof. By Proposition 4.6 we have proved that G is $(2A, 7X, 11Y)$ -generated for all $X, Y \in \{A, B\}$. It follows by Theorem 2.4 that G is $(7X, 7X, (11Y)^2)$ -generated. It follows that G is $(7X, 7X, (11A)^2)$ - and $(7X, 7X, (11B)^2)$ -generated for $X \in \{A, B\}$. Since by the power maps we have $(11A)^2 = 11B$ and $(11B)^2 = 11A$, it then follows that G is $(7X, 7X, 11B)$ - and $(7X, 7X, 11A)$ -generated group for $X \in \{A, B\}$.

We investigate the $(7A, 7B, 11X)$ generations of G , where $X \in \{A, B\}$. Looking at Proposition 4.3, we see that K_1 , K_5 , K_7 and $PSL_2(11)$ are the only groups having elements of order 11. The groups K_5 , K_7 and $PSL_2(11)$ will not have any contributions because they both do not have elements of order 7. We obtained that $\sum_{K_1}(7b, 7a, 11x) = 9856$ for $x \in \{a, b\}$. We already have $h(11X, K_1) = 1$ for $X \in \{A, B\}$. Since by Table 7 we have $\Delta_G(7A, 7B, 11X) = 56496$ for $X \in \{A, B\}$, we then obtain that $\Delta_G^*(7A, 7B, 11X) \geq \Delta_G(7A, 7B, 11X) - \sum_{K_1}(7b, 7a, 11x) = 56496 - 9856 = 46640 > 0$, proving G is $(7A, 7B, 11X)$ -generated for $X \in \{A, B\}$. $\square$

Proposition 4.36. *The group G is $(7X, 7Y, 23Z)$ -generated for $X, Y, Z \in \{A, B\}$.*

Proof. By Proposition 4.7 we have proved that G is $(2A, 7X, 23Y)$ -generated for all $X, Y \in \{A, B\}$. It follows by Theorem 2.4 that G is $(7X, 7X, (23Y)^2)$ -generated. Since $(23A)^2 = 23A$ and $(23B)^2 = 23B$ then it follows that G is $(7X, 7X, 23A)$ - and $(7X, 7X, 23B)$ -generated for $X \in \{A, B\}$.

We prove that G is $(7A, 7B, 23X)$ -generated for $X \in \{A, B\}$. By Table 4, K_7 is the only maximal subgroup having elements of order 23. This maximal subgroup will not have any contributions because it does not have elements of order 7. By Table 7 we have $\Delta_G(7A, 7B, 23X) = 45264$ for $X \in \{A, B\}$. Since there is no contributing group, we then obtain that $\Delta^*(7A, 7B, 23X) = \Delta_G(7A, 7B, 23X) = 45264 > 0$, so that G is $(7A, 7B, 23X)$ -generated group for $X \in \{A, B\}$. $\square$

4.3.6 $(7, 11, r)$ - and $(7, 23, 23)$ -generations

In this subsection we discuss the cases $(7, 11, r)$ - and $(7, 23, r)$ -generations.

It follows that we will end up with 20 cases, namely $(7A, 11A, 11A)$ -, $(7A, 11A, 11B)$ -, $(7A, 11A, 23A)$ -, $(7A, 11A, 23B)$ -, $(7A, 11B, 11B)$ -, $(7A, 11B, 23A)$ -, $(7A, 11B, 23B)$ -, $(7B, 11A, 11A)$ -, $(7B, 11A, 11B)$ -, $(7B, 11A, 23A)$ -, $(7B, 11A, 23B)$ -, $(7B, 11B, 11B)$ -, $(7B, 11B, 23A)$ -, $(7B, 11B, 23B)$ -, $(7A, 23A, 23A)$ -, $(7A, 23A, 23B)$ -, $(7A, 23B, 23B)$ -, $(7B, 23A, 23A)$ -, $(7B, 23A, 23B)$ - and $(7B, 23B, 23B)$ -generation.

Proposition 4.37. *The group G is $(7X, 11Y, 11Z)$ -generated for $X, Y, Z \in \{A, B\}$.*

Proof. Looking at Proposition 4.3, we see that K_1 , K_5 , K_7 and $PSL_2(11)$ are the only groups having elements of order 11. The groups K_5 , K_7 and $PSL_2(11)$ will not have any contributions because they both do not have elements of order 7. We obtained that $\sum_{K_1}(7X, 11Y, 11Z) = 5632$ and $h(11Z, K_1) = 1$ for all $X, Y, Z \in \{A, B\}$. By Table 7 we have $\Delta_G(7X, 11Y, 11Z) = 64416$. We then obtained that $\Delta_G^*(7X, 11Y, 11Z) \geq \Delta_G(7X, 11Y, 11Z) - \sum_{K_1}(7X, 11Y, 11Z) = 64416 - 5632 = 58784 > 0$ and so G is $(7X, 11Y, 11Z)$ -generated group for all $X, Y, Z \in \{A, B\}$. $\square$

Proposition 4.38. *The group G is $(7X, 11Y, 23Z)$ -generated for all $X, Y, Z \in \{A, B\}$.*

Proof. By Table 4, K_7 is the only maximal subgroup having elements of order 23. This maximal subgroup will not have any contributions because it does not have elements of order 7. By Table 7 we have $\Delta_G(7X, 11Y, 23Z) = 67712$ for all $X, Y, Z \in \{A, B\}$. Since there is no contributing group, we then obtain that $\Delta^*(7X, 11Y, 23Z) = \Delta_G(7X, 11Y, 23Z) = 67712 > 0$, so that G is $(7X, 11Y, 23Z)$ -generated group for all $X, Y, Z \in \{A, B\}$. $\square$

Proposition 4.39. *The group G is $(7X, 23Y, 23Z)$ -generated for all $X, Y, Z \in \{A, B\}$.*

Proof. By Table 4, K_7 is the only maximal subgroup having elements of order 23. This maximal subgroup will not have any contributions because it does not have elements of order 7. By Table 7 we have $\Delta_G(7X, 23Y, 23Z) = 32384$ for all $X, Y, Z \in \{A, B\}$. Since there is no contributing group, we then obtain that $\Delta^*(7X, 23Y, 23Z) = \Delta_G(7X, 23Y, 23Z) = 32384 > 0$, so that the group G is $(7X, 23Y, 23Z)$ -generated for all $X, Y, Z \in \{A, B\}$. $\square$

4.3.7 $(11, 11, r)$ -generations

In this subsection we discuss the case $(11, 11, r)$ -generations.

It follows that we will end up with 10 cases, namely $(11A, 11A, 11A)$ -, $(11A, 11A, 11B)$ -, $(11A, 11A, 23A)$ -, $(11A, 11A, 23B)$ -, $(11A, 11B, 11B)$ -, $(11A, 11B,$

23A)-, (11A, 11B, 23B)-, (11B, 11B, 11B)-, (11B, 11B, 23A)- and (11B, 11B, 23B)-generation.

Proposition 4.40. *The group G is (11X, 11Y, 11Z)-generated for all $X, Y, Z \in \{A, B\}$.*

Proof. By Proposition 4.8 we have proved that G is (2A, 11X, 11Y)-generated for all $X, Y \in \{A, B\}$. Then, by Theorem 2.4 it follows that G is (11X, 11X, (11Y)²)-generated for all $X, Y, Z \in \{A, B\}$. Since (11A)² = 11B and (11B)² = 11A then it follows that G is (11X, 11X, 11Y)-generated for all $X, Y, Z \in \{A, B\}$.

We prove that G is (11A, 11B, 11B)-generated. Looking at Proposition 4.3, we see that K_1 , K_5 , K_7 and $PSL_2(11)$ are the only groups having elements of order 11. The maximal subgroup K_7 have its relevant structure constant zero, so it will not have any contributions. We obtained that $\sum_{K_1}(11b, 11a, 11a) = 3632$, $\sum_{K_5}(11b, 11a, 11a) = 35$ and $\sum_{PSL_2(11)}(11b, 11a, 11a) = 2$. We have found that $h(11B, K_1) = h(11B, K_5) = h(11B, PSL_2(11)) = 1$. By Table 8 we have $\Delta_G(11A, 11B, 11B) = 87485$, we then obtain $\Delta_G^*(11A, 11B, 11B) \geq \Delta_G(11A, 11B, 11B) - \sum_{K_1}(11b, 11a, 11a) - \sum_{K_5}(11b, 11a, 11a) + \sum_{PSL_2(11)}(11b, 11a, 11a) = 87485 - 3632 - 35 + 2 = 83820 > 0$, proving that the group G is (11A, 11B, 11B)-generated. $\square$

Proposition 4.41. *The group G is (11X, 11Y, 23Z)-generated for all $X, Y, Z \in \{A, B\}$.*

Proof. By Proposition 4.9 we have proved that G is (2A, 11X, 23Y)-generated for all $X, Y \in \{A, B\}$. Then, by Theorem 2.4 it follows that G is (11X, 11X, (23Y)²)-generated for all $X, Y \in \{A, B\}$. Since (23A)² = 23A and (23B)² = 23B we then obtained that G is (11X, 11X, 23Y)-generated for all $X, Y \in \{A, B\}$.

We still have to prove the (11A, 11B, 23X)-generations where $X \in \{A, B\}$. By Table 4 we see that K_7 is the only maximal subgroup having elements of orders 11 and 23. We then obtain that $\sum_{K_7}(11x, 11y, 23z) = \Delta_{K_7}(11a, 11j, 23z) + \Delta_{K_7}(11c, 11h, 23z) + \Delta_{K_7}(11d, 11g, 23z) + \Delta_{K_7}(11e, 11f, 23z) + \Delta_{K_7}(11i, 11b, 23x) = 23 + 23 + 23 + 23 + 23 = 115$ for $z \in \{a, b\}$. We have found that $h(23X, K_7) = 1$ for $X \in \{A, B\}$. Since by Table 8 we have $\Delta_G(11A, 11B, 23X) = 79994$, then we obtained that $\Delta_G^*(11A, 11B, 23X) \geq \Delta_G(11A, 11B, 23X) - \sum_{K_7}(11x, 11y, 23z) = 79994 - 115 = 79879 > 0$ for all $Z \in \{A, B\}$. Hence G is (11A, 11B, 23X)-generated group for $X \in \{A, B\}$. $\square$

4.3.8 (11, 23, r)-generations

We will be looking at the cases (11A, 23A, 23A)-, (11A, 23A, 23B)-, (11A, 23B, 23B)-, (11B, 23A, 23A)-, (11B, 23A, 23B)- and (11B, 23B, 23B)-generation.

Proposition 4.42. *The group G is (11X, 23Y, 23Z)-generated for all $X, Y, Z \in \{A, B\}$.*

Proof. By Table 4, K_7 is the only maximal subgroup having elements of orders 11 and 23. This maximal subgroup K_7 will not have any contributions because its relevant structure constants are all zero. By Table 8 we have $\Delta_G(11X, 23Y, 23Z) = 42067$ for all $X, Y, Z \in \{A, B\}$. Since there is no contributing group, we then obtain that $\Delta_G^*(11X, 23Y, 23Z) = \Delta_G(11X, 23Y, 23Z) = 42067 > 0$, showing that G is $(11X, 23Y, 23Z)$ -generated group for all $X, Y, Z \in \{A, B\}$. $\square$

4.3.9 $(23, 23, r)$ -generations

We conclude our investigation on the (p, q, r) -generations of the Mathieu sporadic simple group G by considering the $(23, 23, 23)$ -generations. We will be looking at the cases $(23A, 23A, 23A)$ -, $(23A, 23A, 23B)$ -, $(23A, 23B, 23B)$ - and $(23B, 23B, 23B)$ -generation.

Proposition 4.43. *The group G is $(23X, 23Y, 23Z)$ -generated for all $X, Y, Z \in \{A, B\}$.*

Proof. By Proposition 4.10 we have proved that G is $(2A, 23X, 23Y)$ -generated for all $X, Y \in \{A, B\}$. Then, by Theorem 2.4 it follows that G is $(23X, 23X, (23Y)^2)$ -generated for all $X, Y \in \{A, B\}$. Since $(23A)^2 = 23A$ and $(23B)^2 = 23B$ then it follows that G is $(23X, 23X, 23Y)$ -generated for all $X, Y \in \{A, B\}$. We now check the $(23A, 23B, 23B)$ -generation of G . By Table 4, K_7 is the only maximal subgroup having elements of order 11. We obtained that $\sum_{K_7}(23a, 23b, 23b) = 5$ and $h(23B, K_7) = 1$. Since by Table 8 we have $\Delta_G(23A, 23B, 23B) = 17646$, then we obtained that $\Delta_G^*(23A, 23B, 23B) \geq \Delta_G(23A, 23B, 23B) - \sum_{K_7}(23a, 23b, 23b) = 17646 - 5 = 17641 > 0$. Hence the group G is $(23A, 23B, 23B)$ -generated. $\square$

5. Conclusion

As mentioned at the introduction of this paper that it is natural to ask whether a finite simple group G is (l, m, n) -generated or not. The motivation for this question came from the calculation of the genus of finite simple groups [34]. It can be shown that the problem of finding the genus of a finite simple group can be reduced to one of generations (see [33] for further details). Our aim in this paper was to establish all the (p, q, r) -generations of the Mathieu group M_{23} , where p, q and r are prime numbers dividing $|M_{23}|$. The main result of the paper was to prove Theorem 1.1 and this was done through sequence of propositions. We found that the Mathieu group M_{23} is generated by the following:

- $(2A, 5A, 11X), (2A, 5A, 23X), (2A, 7X, 7Y), (2A, 7X, 11Y), (2A, 7X, 23Y), (2A, 11X, 11Y), (2A, 11X, 23Y)$ and $(2A, 23X, 23Y)$ for all $X, Y \in \{A, B\}$.
- $(3A, 3A, 11X), (3A, 3A, 23X), (3A, 5A, 5A), (3A, 5A, 7X), (3A, 5A, 11X), (3A, 5A, 23X), (3A, 7X, 7Y), (3A, 7X, 11Y), (3A, 7X, 23Y), (3A, 11X, 11Y), (3A, 11X, 23Y)$ and $(3A, 23X, 23Y)$ for all $X, Y \in \{A, B\}$.

- $(5A, 5A, 5A), (5A, 5A, 7X), (5A, 5A, 11X), (5A, 5A, 23X), (5A, 7X, 7Y), (5A, 7X, 11Y), (5A, 7X, 23Y), (5A, 11X, 11Y), (5A, 11X, 23Y)$ and $(5A, 23X, 23Y)$ for all $X, Y \in \{A, B\}$.
- $(7X, 7Y, 7Z), (7X, 7Y, 11Z), (7X, 7Y, 23Z), (7X, 11Y, 11Z), (7X, 11Y, 23Z)$ and $(7X, 23Y, 23Z)$ for all $X, Y, Z \in \{A, B\}$.
- $(11X, 11Y, 11Z), (11X, 11Y, 23Z)$ and $(11X, 23Y, 23Z)$ for all $X, Y, Z \in \{A, B\}$.
- $(23X, 23Y, 23Z)$ for all $X, Y, Z \in \{A, B\}$.

Theorem 1.1 tells us also that the Mathieu group M_{23} is not generated by the following: $(p, q, r) \in \{(2, 3, r), (2, 5, 5), (2, 5, 7), (3, 3, 3), (3, 3, 5), (3, 3, 7)\}$, for all primes r that divides $|M_{23}|$.

Tables : Structure constants of M_{23}

Tables 5 to 8 following here below give the partial structure constants of M_{23} computed using GAP that were used in the calculations above.

Table 5:

pX	2A	3A	5A	7A	7B	11A	11B	23A	23B
$\Delta_{M_{23}}(2A, 2A, pX)$	98	30	5	0	0	0	0	0	0
$\Delta_{M_{23}}(2A, 3A, pX)$	448	180	65	35	35	11	11	0	0
$\Delta_{M_{23}}(2A, 5A, pX)$	896	780	605	364	364	253	253	138	138
$\Delta_{M_{23}}(2A, 7A, pX)$	0	450	390	301	462	308	308	184	184
$\Delta_{M_{23}}(2A, 7B, pX)$	0	450	390	462	301	308	308	184	184
$\Delta_{M_{23}}(2A, 11A, pX)$	0	180	345	392	392	341	341	391	391
$\Delta_{M_{23}}(2A, 11B, pX)$	0	180	345	392	392	341	341	391	391
$\Delta_{M_{23}}(2A, 23A, pX)$	0	0	90	112	112	187	187	161	230
$\Delta_{M_{23}}(2A, 23B, pX)$	0	0	90	112	112	187	187	230	161
$ C_{M_{23}}(pX) $	688	180	15	14	14	11	11	23	23

Table 6:

pX	2A	3A	5A	7A	7B	11A	11B	23A	23B
$\Delta_{M_{23}}(3A, 3A, pX)$	2688	1681	855	511	511	275	275	138	138
$\Delta_{M_{23}}(3A, 5A, pX)$	11648	10260	5490	4886	4886	4136	4136	2438	2438
$\Delta_{M_{23}}(3A, 7A, pX)$	6720	6570	5490	4886	4886	4136	4136	3312	3312
$\Delta_{M_{23}}(3A, 7B, pX)$	6720	6570	5490	4886	4886	4136	4136	3312	3312
$\Delta_{M_{23}}(3A, 11A, pX)$	2688	4500	5175	5264	5264	5126	5379	5129	5129
$\Delta_{M_{23}}(3A, 11B, pX)$	2688	4500	5175	5264	5264	5379	5126	5129	5129
$\Delta_{M_{23}}(3A, 23A, pX)$	0	1080	1590	2016	2016	2453	2453	3082	2714
$\Delta_{M_{23}}(3A, 23B, pX)$	0	1080	1590	2016	2016	2453	2453	2714	3082
$ C_{M_{23}}(pX) $	688	180	15	14	14	11	11	23	23

Table 7:

pX	2A	3A	5A	7A	7B	11A	11B	23A	23B
$\Delta_{M_{23}}(5A, 5A, pX)$	108416	78600	61058	54320	54320	45287	45287	37582	37582
$\Delta_{M_{23}}(5A, 7A, pX)$	69888	65880	58200	52584	52584	48576	48576	44160	44160
$\Delta_{M_{23}}(5A, 7B, pX)$	69888	65880	58200	52584	52584	48576	48576	44160	44160
$\Delta_{M_{23}}(5A, 11A, pX)$	61824	62100	61755	61824	61824	62238	61479	61893	61893
$\Delta_{M_{23}}(5A, 11B, pX)$	61824	62100	61755	61824	61824	61479	62238	61893	61893
$\Delta_{M_{23}}(5A, 23A, pX)$	16128	19080	24510	26880	26880	29601	29601	32706	32706
$\Delta_{M_{23}}(5A, 23B, pX)$	16128	19080	24510	26880	26880	29601	29601	32706	32706
$\Delta_{M_{23}}(7A, 7A, pX)$	88704	62820	56340	51948	60412	48400	48400	52992	52992
$\Delta_{M_{23}}(7A, 7B, pX)$	57792	62820	56340	51948	51948	56496	56496	45264	45264
$\Delta_{M_{23}}(7A, 11A, pX)$	75264	67680	66240	71904	61600	64416	64416	67712	67712
$\Delta_{M_{23}}(7A, 11B, pX)$	75264	67680	66240	71904	61600	64416	64416	67712	67712
$\Delta_{M_{23}}(7A, 23A, pX)$	21504	25920	28800	27552	32256	32384	32384	32384	32384
$\Delta_{M_{23}}(7A, 23B, pX)$	21504	25920	28800	27552	32256	32384	32384	32384	32384
$\Delta_{M_{23}}(7B, 7B, pX)$	88704	62820	56340	60412	51948	48400	48400	52992	52992
$\Delta_{M_{23}}(7B, 11A, pX)$	75264	67680	66240	61600	71904	64416	64416	67712	67712
$\Delta_{M_{23}}(7B, 11B, pX)$	75264	67680	66240	61600	71904	64416	64416	67712	67712
$\Delta_{M_{23}}(7B, 23A, pX)$	21504	25920	28800	32256	27552	32384	32384	32384	32384
$\Delta_{M_{23}}(7B, 23B, pX)$	21504	25920	28800	32256	27552	32384	32384	32384	32384
$ C_{M_{23}}(pX) $	688	180	15	14	14	11	11	23	23

Table 8:

pX	2A	3A	5A	7A	7B	11A	11B	23A	23B
$\Delta_{M_{23}}(11A, 11A, pX)$	83328	88020	83835	81984	81984	87485	88520	81029	81029
$\Delta_{M_{23}}(11A, 11B, pX)$	83328	83880	84870	81984	81984	87485	87485	79994	79994
$\Delta_{M_{23}}(11A, 23A, pX)$	45696	40140	40365	41216	41216	38258	38753	42067	42067
$\Delta_{M_{23}}(11A, 23B, pX)$	45696	40140	40365	41216	41216	38258	38753	42067	42067
$\Delta_{M_{23}}(11B, 11B, pX)$	83328	88020	83835	81984	81984	88520	87485	81029	81029
$\Delta_{M_{23}}(11B, 23A, pX)$	45696	40140	40365	41216	41216	38753	38258	42067	42067
$\Delta_{M_{23}}(11B, 23B, pX)$	45696	40140	40365	41216	41216	38753	38258	42067	42067
$\Delta_{M_{23}}(23A, 23A, pX)$	26880	21240	21330	19712	19712	20119	20119	17646	18222
$\Delta_{M_{23}}(23A, 23B, pX)$	18816	24120	21330	19712	19712	20119	20119	17646	17646
$\Delta_{M_{23}}(23B, 23B, pX)$	26880	21240	21330	19712	19712	20119	20119	18222	17646
$ C_{M_{23}}(pX) $	688	180	15	14	14	11	11	23	23

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Bi-endomorphism induces new types of derivations on BH-algebras

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Abstract. In this paper, we study the concepts of an (l, r) and an (r, l) - τ -derivation on a BH-algebra, which is induced by a left and a right bi-endomorphism and we provide

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important properties. In addition, the relationship among those derivations is also considered.

Keywords: BH-algebra, bi-endomorphism, derivation.

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1. Introduction

A new notion called a BH-algebra, a generalization of BCH/BCI/BCK-algebras, was introduced in 1998 in [13] by Jun, Roh and Kim. They defined the notions of ideals and boundedness in BH-algebras, showed that there is a maximal ideal in bounded BH-algebras, established the construction of the quotient BH-algebras via translation ideals, and obtained the fundamental theorem of homomorphisms for BH-algebras as a consequence. In 2000, Zhang, Jun and Roh [18] introduced a normal BH-algebra. They considered the branch in BH-algebra and investigated some related properties. A theoretical approach to the ideal structure in BH-algebras was established and introduced by Kim and Ahn [14] in 2012. The notions of a falling subalgebra, a falling ideal, a falling strong ideal, a falling n -fold strong ideal and a falling translation ideal of a BH-algebra are introduced. Some fundamental properties are investigated. Relations among a falling subalgebra, a falling ideal, a falling strong ideal, and a falling n -fold strong ideal are stated. A relation between a fuzzy subalgebra/ideal and a falling subalgebra/ideal is provided. In 2014, Abbass and Neamah [3] introduced an implicative ideal with respect to an element of a BH-algebra and some related properties are investigated. The relationships among this notion and other types of ideals of a BH-algebra are given. In 2015, Abbass and Mahdi [2] introduced the notion of a BH-algebra related to a B-algebra as a new class of a BH-algebra. Also, they introduced the notion of an ideal of a BH-algebra related to a B-algebra, and they stated and proved some theorems and examples which determine the relationships between these notions and some types of ideals of a BH-algebra. In 2018, Rajkumar and Kandara [17] introduced some results on derivations on rings and the generalization of BCK and BCI-algebras, and they defined (α, β) derivations on BH-algebras and investigate some important results. The notion of fuzzy BH-subalgebra of a BH-algebra with respect to a triangular norm, and they show that the Cartesian product of two fuzzy BH-subalgebras of a BH-algebra X with respect to a triangular norm is a fuzzy BH-subalgebra of the product of X with respect to the norm introduced by Anitha and Kandaraj [4] in 2019. In 2020, Ganesan and Kandaraj [9] introduced the notion of f -derivations on BH-algebras, the regular f -derivations, and the composition of f -derivations on BH-algebras, and they investigated interesting and elegant results. Next year, they introduced the notion of generalized derivations on BH-algebras and investigated simple, interesting, and elegant results, introduced the notion of t -derivations on BH-algebras and investigated simple, interesting, and elegant results; introduced the notion of generalized (g, h) -derivations on BH-algebras and investigated simple, interesting, and elegant results; and introduced the

notion of composition of f -derivations on BH-algebras and investigated simple, interesting, and elegant results ([11, 6, 10, 7]).

In this paper, we study the concepts of an (l, r) and an (r, l) - τ -derivation on a BH-algebra, which is induced by a left and a right bi-endomorphism and provide important properties. In addition, the relationship among those derivations is also considered.

2. Preliminaries

First, we review some basic definitions and theorems required in our work.

Definition 2.1 ([13]). *Let X be a set with a binary operation $*$ and a constant 0 . Then $(X, *, 0)$ is called a **BH-algebra** if it satisfies the following axioms:*

$$(BH1) \quad (\forall x \in X)(x * x = 0),$$

$$(BH2) \quad (\forall x \in X)(x * 0 = x),$$

$$(BH3) \quad (\forall x, y \in X)(x * y = 0, y * x = 0 \Rightarrow x = y).$$

If there is conditions (BH1), (BH2), and

$$(B) \quad (\forall x, y, z \in X)((x * y) * z = x * (z * (0 * y))),$$

then ([16]) $(X, *, 0)$ is called a B-algebra.

Example 2.1 ([13]). Let $X = \{0, 1, 2, 3\}$ be a BH-algebra with the Cayley table as follows:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	3	1	0

Then $(X, *, 0)$ is a BH-algebra.

Definition 2.2 ([13]). *Let S be a nonempty subset of a BH-algebra X . Then S is called a **BH-subalgebra** of X if $x * y \in S$ for all $x, y \in S$.*

By (BH1), we know that every subalgebra S in any BH-algebra X satisfies the condition $0 \in S$.

Example 2.2 ([13]). In Example 2.4, in X , the set $S_1 = \{0, 1, 2\}$, $S_2 = \{0, 1\}$, $S_3 = \{0, 1, 3\}$ and $S_4 = \{0, 2\}$ are BH-subalgebras of X , while $S_5 = \{0, 2, 3\}$ is not a BH-subalgebra of X .

Definition 2.3 ([1]). *A BH-algebra X is called **medial** if $x * (x * y) = y$ for all $x, y \in X$.*

Definition 2.4 ([5]). A BH-algebra X is called **associative** if $(x * y) * z = x * (y * z)$ for all $x, y, z \in X$.

Theorem 2.1 ([3]). Let X be an associative BH-algebra. Then

- (BH4) $(\forall x \in X)(0 * x = x)$,
- (BH5) $(\forall x, y \in X)(x * y = y * x)$,
- (BH6) $(\forall x, y \in X)(x * (x * y) = y)$, medial
- (BH7) $(\forall x, y, z \in X)((z * x) * (z * y) = x * y)$,
- (BH8) $(\forall x, y \in X)(x * y = 0 \Rightarrow x = y)$,
- (BH9) $(\forall x, y \in X)((x * (x * y)) * y = 0)$,
- (BH10) $(\forall x, y, z \in X)((x * y) * z = (x * z) * y)$.

Definition 2.5. A BH-algebra X corresponds to **BH11-13** if the following conditions are satisfied.

- (BH11) $(\forall x \in X)(0 * (0 * x) = x)$,
- (BH12) $(\forall x, y, z \in X)(z * x = z * y \Rightarrow x = y)$,
- (BH13) $(\forall x, y, z \in X)(x * z = y * z \Rightarrow x = y)$.

The concept of 0-commutativity of B-algebras is developed in [15].

For a associative BH-algebra X , we denote $x \wedge y = y * (y * x)$ for all $x, y \in X$.

We will get the following results as follows:

- (BH14) $(\forall x \in X)(x \wedge 0 = x)$ (by (BH11)).
- (BH15) $(\forall x \in X)(x \wedge x = x)$ if X is an associative BH-algebra (by (BH6)).

Definition 2.6 ([13]). Let X be a BH-algebra and I be a subset of X . Then I is called a **BH-ideal** of X if

- (I1) $0 \in I$,
- (I2) $(\forall x, y \in X)(x * y \in I, y \in I \Rightarrow x \in I)$.

Example 2.3 ([13]). Let $X = \{0, 1, 2\}$ be a BH-algebra with the following table:

$*$	0	1	2
0	0	0	1
1	1	0	2
2	2	2	0

We can easily show that $I = \{0, 1\}$ is a BH-ideal of X .

Example 2.4 ([13]). Let $X = \{0, 1, 2, 3\}$ be a BH-algebra with the Cayley table as follows:

$\ast$	0	1	2	3
0	0	3	0	2
1	1	0	0	0
2	2	2	0	3
3	3	3	1	0

Then $(X, \ast, 0)$ is a BH-algebra. The subset $S = \{0, 2\}$ is a BH-subalgebra of X , but it is not a BH-ideal of X because, for example, it is valid

$$1 \ast 2 = 0 \in S, 2 \in S, \text{ but } 1 \notin S.$$

Definition 2.7 ([8]). Let X be a BH-algebra. A self map $d : X \rightarrow X$ is called

- (1) a **(left-right)-derivation** (**(l, r)-derivation**) of X if

$$(\forall x, y \in X)(d(x \ast y) = (d(x) \ast y) \wedge (x \ast d(y))),$$

- (2) a **(right-left)-derivation** (**(r, l)-derivation**) of X if

$$(\forall x, y \in X)(d(x \ast y) = (x \ast d(y)) \wedge (d(x) \ast y)),$$

- (3) a **derivation** of X if it is both an (l, r) and an (r, l)-derivation of X .

Definition 2.8 ([8]). Let X be a BH-algebra. A self map d of X is said to be **regular** if $d(0) = 0$.

Definition 2.9 ([12]). Let X be a BH-algebra and d, D be two self maps of X with the composition $\circ$, that is, $d \circ D : X \rightarrow X$ as $(d \circ D)(x) = d(D(x))$ for all $x \in X$.

3. Main results

In this section, we introduce bi-endomorphisms on BH-algebras and prove some results of some results of bi-endomorphisms of a BH-algebra X and its derivations as follows:

Definition 3.1. Let X be BH-algebra. A mapping $\tau : X \times X \rightarrow X$ is called

- (1) a **left bi-endomorphism** on X if

$$(\forall x, y, z \in X)(\tau(x \ast y, z) = \tau(x, z) \ast \tau(y, z)),$$

- (2) a **right bi-endomorphism** on X if

$$(\forall x, y, z \in X)(\tau(x, y \ast z) = \tau(x, y) \ast \tau(x, z)),$$

- (3) a **bi-endoromorphism** on X if it is a left and a right bi-endoromorphism on X .

Throughout this section, we assume that τ_l and τ_r are a left and a right bi-endoromorphism of X , respectively.

Let $x \in X$. By (BH2), we have

$$\tau_l(x, x) * 0 = \tau_l(x, x) = \tau_l(x * 0, x) = \tau_l(x, x) * \tau_l(0, x)$$

and

$$\tau_r(x, x) * 0 = \tau_r(x, x) = \tau_r(x, x * 0) = \tau_r(x, x) * \tau_r(x, 0).$$

For an associative BH-algebra X , we have (BH12) which is proved by (BH6). Thus, we get

$$(BH16) \quad (\forall x \in X)(\tau_l(0, x) = 0),$$

$$(BH17) \quad (\forall x \in X)(\tau_r(x, 0) = 0).$$

Example 3.1. In Example 2.4, we define a mapping $\tau_1 : X \times X \rightarrow X$ by

$$\tau_1(x, y) = \begin{cases} 1, & \text{if } (x, y) \in \{(0, 0), (1, 0), (3, 0)\} \\ 0, & \text{otherwise.} \end{cases}$$

Then τ_1 is a left bi-endoromorphism on X .

Definition 3.2. Let X be BH-algebra. A mapping $\tau : X \times X \rightarrow X$ is said to be **symmetric** if $\tau(x, y) = \tau(y, x)$ for all $x, y \in X$.

Remark 3.1. Every symmetric left (right) bi-endoromorphism on a BH-algebra is a bi-endoromorphism.

For any BH-algebra X , define a mapping $0 : X \times X \rightarrow X$ by $0(x, x) = 0$ for all $x \in X$, which is a bi-endoromorphism on X .

Definition 3.3. Let τ be bi-endoromorphism on a BH-algebra X . A BH-ideal I of X is called a **τ -ideal** of X if $\tau(x, y) \in I$ for all $x, y \in I$.

Example 3.2. Let $X = \{0, 1, 2\}$ be as in Example 2.3. Then $I = \{0, 1\}$ is a BH-ideal of X . If we define the self map $\tau_1 : X \times X \rightarrow X$ in the following way $(\forall x, y \in X)(\tau_1(x, y) \in I)$, then τ_1 is a bi-endoromorphism on X and I is a τ_1 -ideal of X .

Next, we generalize derivations on a BH-algebra X with a left (right) bi-endoromorphism $\tau : X \times X \rightarrow X$ from the concept of Definition 2.7.

Definition 3.4. Let X be a BH-algebra. Let $\tau : X \times X \rightarrow X$ be a left (right) bi-endoromorphism on X . A self map d_τ on X is called

(1) an (l, r) - τ -**derivation** of X if

$$(\forall x, y \in X)(d_\tau(x * y) = (d_\tau(x) * \tau(x, y)) \wedge (\tau(x, y) * d_\tau(y))),$$

(2) an (r, l) - τ -**derivation** of X if

$$(\forall x, y \in X)(d_\tau(x * y) = (\tau(x, y) * d_\tau(y)) \wedge (d_\tau(x) * \tau(x, y))),$$

(3) a τ -**derivation** of X if it is both an (l, r) - and an (r, l) - τ -derivation of X .

Example 3.3. Consider the following BH-algebra X with the Cayley table.

$*$	0	1	2
0	0	0	0
1	1	0	1
2	2	1	0

We define $\tau = 0$ and $d_\tau : X \rightarrow X$ by $d_\tau(x) = 0$. Then τ is a bi-endomorphism on X and d_τ is an (l, r) - τ -derivation of X .

Definition 3.5. Let d be a self map on a BH-algebra X . A BH-ideal I of X is said to be d -**invariant** if $d(I) \subseteq I$.

Example 3.4. In Example 2.3, we have $I = \{0, 1\}$ is a BH-ideal of X . From Example 3.3, we have d_τ is an (l, r) - τ -derivation of X . It is obvious that $d_\tau(x) \in I$ for all $x \in I$, that is, $d_\tau(I) \subseteq I$. Hence, I is d_τ -invariant.

Theorem 3.1. Let d_{τ_l} be an (l, r) - τ_l -derivation on an associative BH-algebra X . Then d_{τ_l} is regular if and only if $d_{\tau_l}(0 * x) = 0$ for all $x \in X$ with condition (BH12).

Proof. Suppose that d_{τ_l} is regular. Let $x \in X$. Then

$$\begin{aligned}
 d_{\tau_l}(0 * x) &= (d_{\tau_l}(0) * \tau_l(0, x)) \wedge (\tau_l(0, x) * d_{\tau_l}(x)) \\
 \text{(BH16)} \quad &= (0 * 0) \wedge (0 * d_{\tau_l}(x)) \\
 \text{(BH1)} \quad &= 0 \wedge (0 * d_{\tau_l}(x)) \\
 &= (0 * d_{\tau_l}(x)) * ((0 * d_{\tau_l}(x)) * 0) \\
 \text{(BH2)} \quad &= (0 * d_{\tau_l}(x)) * (0 * d_{\tau_l}(x)) \\
 \text{(BH1)} \quad &= 0.
 \end{aligned}$$

Conversely, suppose that $d_{\tau_l}(0 * x) = 0$ for all $x \in X$. Then

$$\begin{aligned}
 0 &= d_{\tau_l}(0 * 0) \\
 &= (d_{\tau_l}(0) * \tau_l(0, 0)) \wedge (\tau_l(0, 0) * d_{\tau_l}(0)) \\
 \text{(BH16)} \quad &= (d_{\tau_l}(0) * 0) \wedge (0 * d_{\tau_l}(0)) \\
 \text{(BH2)} \quad &= d_{\tau_l}(0) \wedge (0 * d_{\tau_l}(0)) \\
 &= (0 * d_{\tau_l}(0)) * ((0 * d_{\tau_l}(0)) * d_{\tau_l}(0)).
 \end{aligned}$$

By (BH8) and (BH2), we get $(0 * d_{\tau_l}(0)) * 0 = 0 * d_{\tau_l}(0) = (0 * d_{\tau_l}(0)) * d_{\tau_l}(0)$. Using (BH12), we have $d_{\tau_l}(0) = 0$. Hence, d_{τ_l} is regular. $\square$

Theorem 3.2. *Let d_{τ_r} be an (r, l) - τ_r -derivation on an associative BH-algebra X . Then d_{τ_r} is regular and $\tau_r(x, 0) = 0$ for all $x \in X$ if and only if $d_{\tau_r}(x * 0) = 0$ for all $x \in X$.*

Proof. Suppose that d_{τ_r} is regular and $\tau_r(0, x) = 0$ for all $x \in X$. Let $x \in X$. Then

$$\begin{aligned}
 d_{\tau_r}(0 * x) &= (d_{\tau_r}(0) * \tau_r(0, x)) \wedge (\tau_r(0, x) * d_{\tau_r}(x)) \\
 &= (0 * 0) \wedge (0 * d_{\tau_r}(x)) \\
 \text{(BH1)} \quad &= 0 \wedge (0 * d_{\tau_r}(x)) \\
 &= (0 * d_{\tau_r}(x)) * ((0 * d_{\tau_r}(x)) * 0) \\
 \text{(BH2)} \quad &= (0 * d_{\tau_r}(x)) * (0 * d_{\tau_r}(x)) \\
 \text{(BH1)} \quad &= 0.
 \end{aligned}$$

Conversely, suppose that $d_{\tau_r}(0 * x) = 0$ for all $x \in X$. By (BH1), we have $d_{\tau_r}(0) = d_{\tau_r}(0 * 0) = 0$. Thus, d_{τ_r} is regular. Let $x \in X$. Then

$$\begin{aligned}
 0 &= d_{\tau_r}(0 * x) \\
 &= (d_{\tau_r}(0) * \tau_r(0, x)) \wedge (\tau_r(0, x) * d_{\tau_r}(x)) \\
 &= (0 * \tau_r(0, x)) \wedge (\tau_r(0, x) * d_{\tau_r}(x)) \\
 &= (\tau_r(0, x) * d_{\tau_r}(x)) * ((\tau_r(0, x) * d_{\tau_r}(x)) * (0 * \tau_r(0, x))).
 \end{aligned}$$

By (BH8) and (BH2), we get $(\tau_r(0, x) * d_{\tau_r}(x)) * 0 = \tau_r(0, x) * d_{\tau_r}(x) = (\tau_r(0, x) * d_{\tau_r}(x)) * (0 * \tau_r(0, x))$. Using (BH12), we have $0 * \tau_r(0, x) = 0$. Using (BH8) again, we obtain $\tau_r(0, x) = 0$. $\square$

Next, we give a necessary and sufficient condition for (r, l) - τ_l -derivation and (r, l) - τ_r -derivation to be regular with some properties.

Theorem 3.3. *Let d_{τ_l} be an (l, r) - τ_l -derivation on an associative BH-algebra X . Then d_{τ_l} is regular and $\tau_l(0, x) = 0$ for all $x \in X$ if and only if $d_{\tau_l}(x) = 0$ for all $x \in X$.*

Proof. Suppose that d_{τ_l} is regular and $\tau_l(x, 0) = 0$ for all $x \in X$. Let $x \in X$. Then

$$\begin{aligned}
 \text{(BH2)} \quad d_{\tau_l}(x) &= d_{\tau_l}(x * 0) \\
 &= (\tau_l(x, 0) * d_{\tau_l}(0)) \wedge (d_{\tau_l}(x) * \tau_l(x, 0)) \\
 &= (0 * 0) \wedge (d_{\tau_l}(x) * 0) \\
 \text{(BH2)} \quad &= 0 \wedge d_{\tau_l}(x) \\
 &= d_{\tau_l}(x) * (d_{\tau_l}(x) * 0) \\
 \text{(BH2)} \quad &= d_{\tau_l}(x) * d_{\tau_l}(x) \\
 \text{(BH1)} \quad &= 0.
 \end{aligned}$$

Conversely, suppose that $d_{\tau_l}(x) = 0$ for all $x \in X$. Then d_{τ_l} is regular. Let $x \in X$. Then

$$\begin{aligned}
 0 &= d_{\tau_l}(x) \\
 \text{(BH2)} \quad &= d_{\tau_l}(x * 0) \\
 &= (\tau_l(x, 0) * d_{\tau_l}(0)) \wedge (d_{\tau_l}(x) * \tau_l(x, 0)) \\
 &= (\tau_l(x, 0) * 0) \wedge (0 * \tau_l(x, 0)) \\
 \text{(BH2)} \quad &= \tau_l(x, 0) \wedge (0 * \tau_l(x, 0)) \\
 &= (0 * \tau_l(x, 0)) * ((0 * \tau_l(x, 0)) * \tau_l(x, 0)).
 \end{aligned}$$

By (BH8) and (BH2), we get $(0 * \tau_l(x, 0)) * 0 = 0 * \tau_l(x, 0) = (0 * \tau_l(x, 0)) * \tau_l(x, 0)$. Using (BH12), we have $\tau_l(x, 0) = 0$. $\square$

Theorem 3.4. *Let d_{τ_r} be an (r, l) - τ_r -derivation on an associative BH-algebra X . Then d_{τ_r} is regular if and only if $d_{\tau_r}(x) = 0$ for all $x \in X$.*

Proof. Suppose that d_{τ_r} is regular. Let $x \in X$. Then

$$\begin{aligned}
 \text{(BH2)} \quad d_{\tau_r}(x) &= d_{\tau_r}(x * 0) \\
 &= (\tau_r(x, 0) * d_{\tau_r}(0)) \wedge (d_{\tau_r}(x) * \tau_r(x, 0)) \\
 \text{(BH15)} \quad &= (0 * 0) \wedge (d_{\tau_r}(x) * 0) \\
 \text{(BH2)} \quad &= 0 \wedge d_{\tau_r}(x) \\
 &= d_{\tau_r}(x) * (d_{\tau_r}(x) * 0) \\
 \text{(BH2)} \quad &= d_{\tau_r}(x) * d_{\tau_r}(x) \\
 \text{(BH1)} \quad &= 0.
 \end{aligned}$$

The converse shows that d_{τ_r} is regular. $\square$

By (BH16), we define an (l, r) - τ_l -derivation d_{τ_l} (or (r, l) - τ_l -derivation) on a BH-algebra X by $d_{\tau_l}(x) = \tau_l(x, 0)$ for all $x \in X$, and obtain the following proposition.

Proposition 3.1. *Let d_{τ_l} be an (l, r) - τ_l -derivation on an associative BH-algebra X . If $d_{\tau_l}(x) = \tau_l(0, x)$ for all $x \in X$, then d_{τ_l} is the zero self map on X and d_{τ_l} is regular.*

Proof. Suppose that $d_{\tau_l}(x) = \tau_l(x, 0)$ for all $x \in X$. By (BH16), we have $d_{\tau_l}(0) = \tau_l(0, 0) = 0$, that is, d_{τ_l} is regular. Let $x \in X$. Then

$$\begin{aligned}
 \text{(BH2)} \quad d_{\tau_l}(x) &= d_{\tau_l}(x * 0) \\
 &= (d_{\tau_l}(x) * \tau_l(x, 0)) \wedge (\tau_l(x, 0) * d_{\tau_l}(0)) \\
 &= (d_{\tau_l}(x) * d_{\tau_l}(x)) \wedge (d_{\tau_l}(x) * d_{\tau_l}(0)) \\
 \text{(BH1)} \quad &= 0 \wedge (d_{\tau_l}(x) * d_{\tau_l}(0)) \\
 &= (d_{\tau_l}(x) * d_{\tau_l}(0)) * ((d_{\tau_l}(x) * d_{\tau_l}(0)) * 0) \\
 \text{(BH2)} \quad &= (d_{\tau_l}(x) * d_{\tau_l}(0)) * (d_{\tau_l}(x) * d_{\tau_l}(0)) \\
 \text{(BH1)} \quad &= 0.
 \end{aligned}$$

Hence, d_{τ_l} is the zero self map on X . $\square$

Proposition 3.2. *Let d_{τ_l} be an (l, r) - τ_l -derivation on an associative BH-algebra X . If $d_{\tau_l}(x) = \tau_l(0, x)$ for all $x \in X$, then d_{τ_l} is the zero self map on X and d_{τ_l} is regular.*

Proof. Suppose that $d_{\tau_l}(x) = \tau_l(x, 0)$ for all $x \in X$. By (BH16), we have $d_{\tau_l}(0) = \tau_l(0, 0) = 0$, that is, d_{τ_l} is regular. Let $x \in X$. Then

$$\begin{aligned} d_{\tau_l}(0 * x) &= (\tau_l(0, x) * d_{\tau_l}(x)) \wedge (d_{\tau_l}(0) * \tau_l(0, x)) \\ \text{(BH16)} \quad &= (0 * d_{\tau_l}(x)) \wedge (0 * 0) \\ \text{(BH1)} \quad &= (0 * d_{\tau_l}(x)) \wedge 0 \\ \text{(BH11)} \quad &= 0 * d_{\tau_l}(x). \end{aligned}$$

Hence, d_{τ_l} is the zero self map on X . $\square$

By (BH17), we define an (l, r) - τ_l -derivation d_{τ_l} (or (r, l) - τ_l -derivation) on a BH-algebra X by $d_{\tau_r}(x) = \tau_r(0, x)$ for all $x \in X$, and obtain the following proposition.

Proposition 3.3. *Let d_{τ_r} be an (l, r) - τ_r -derivation on an associative BH-algebra X . If $d_{\tau_r}(x) = \tau_r(x, 0)$ for all $x \in X$, then $d_{\tau_r}(0 * x) = 0 * d_{\tau_r}(x)$ for all $x \in X$ and d_{τ_r} is regular with condition (BH11).*

Proof. Suppose that $d_{\tau_r}(x) = \tau_r(0, x)$ for all $x \in X$. By (BH17), we have $d_{\tau_r}(0) = \tau_r(0, 0) = 0$, that is, d_{τ_r} is regular. Let $x \in X$. Then

$$\begin{aligned} d_{\tau_r}(0 * x) &= (d_{\tau_r}(0) * \tau_r(0, x)) \wedge (\tau_r(0, x) * d_{\tau_r}(x)) \\ &= (d_{\tau_r}(0) * d_{\tau_r}(x)) \wedge (d_{\tau_r}(x) * d_{\tau_r}(x)) \\ \text{(BH1)} \quad &= (d_{\tau_r}(0) * d_{\tau_r}(x)) \wedge 0 \\ \text{(BH11)} \quad &= d_{\tau_r}(0) * d_{\tau_r}(x) \\ &= 0 * d_{\tau_r}(x). \end{aligned} \quad \square$$

Proposition 3.4. *Let d_{τ_r} be an (r, l) - τ_r -derivation on an associative BH-algebra X . If $d_{\tau_r}(x) = \tau_r(x, 0)$ for all $x \in X$, then d_{τ_r} is the zero self map on X and d_{τ_r} is regular.*

Proof. Suppose that $d_{\tau_r}(x) = \tau_r(0, x)$ for all $x \in X$. By (BH17), we have $d_{\tau_r}(0) = \tau_r(0, 0) = 0$, that is, d_{τ_r} is regular. Let $x \in X$. Then

$$\begin{aligned} d_{\tau_r}(x) &= d_{\tau_r}(x * 0) \\ &= (\tau_r(x, 0) * d_{\tau_r}(0)) \wedge (d_{\tau_r}(x) * \tau_r(x, 0)) \\ \text{(BH17)} \quad &= (0 * 0) \wedge (d_{\tau_r}(x) * 0) \\ \text{(BH2)} \quad &= 0 \wedge d_{\tau_r}(x) \\ &= d_{\tau_r}(x) * (d_{\tau_r}(x) * 0) \\ \text{(BH2)} \quad &= d_{\tau_r}(x) * d_{\tau_r}(x) \\ \text{(BH1)} \quad &= 0. \end{aligned}$$

Hence, d_{τ_r} is the zero self map on X . $\square$

Theorem 3.5. *Let d_{τ_l} be a regular (r, l) - τ_l -derivation on associative BH-algebra X . Every τ_l -ideal of X is d_{τ_l} -invariant.*

Proof. Let x be an element in a τ_l -ideal I of X . Then

$$\begin{aligned}
 \text{(BH2)} \quad d_{\tau_l}(x) * 0 &= d_{\tau_l}(x) \\
 \text{(BH2)} \quad &= d_{\tau_l}(x * 0) \\
 &= (\tau_l(x, 0) * d_{\tau_l}(0)) \wedge (d_{\tau_l}(x) * \tau_l(x, 0)) \\
 &= (\tau_l(x, 0) * 0) \wedge (d_{\tau_l}(x) * \tau_l(x, 0)) \\
 \text{(BH2)} \quad &= \tau_l(x, 0) \wedge (d_{\tau_l}(x) * \tau_l(x, 0)) \\
 &= (d_{\tau_l}(x) * \tau_l(x, 0)) * ((d_{\tau_l}(x) * \tau_l(x, 0)) * \tau_l(x, 0)) \\
 \text{(BH7)} \quad &= d_{\tau_l}(x) * (d_{\tau_l}(x) * \tau_l(x, 0)).
 \end{aligned}$$

By (BH12), we get $0 = d_{\tau_l}(x) * \tau_l(x, 0)$. Since I is a τ_l -ideal of X and by (BH8), we have $d_{\tau_l}(x) = \tau_l(x, 0) \in I$. Hence, I is d_{τ_l} -invariant. $\square$

Theorem 3.6. *Let d_{τ_r} be a regular (r, l) - τ_r -derivation on an associative BH-algebra X . Every BH-ideal of X is d_{τ_r} -invariant. In particular, every τ_r -ideal of X is d_{τ_r} -invariant.*

Proof. Let x be an element in a BH-ideal I on X . Then

$$\begin{aligned}
 \text{(BH2)} \quad d_{\tau_r}(x) &= d_{\tau_r}(x * 0) \\
 &= (\tau_r(x, 0) * d_{\tau_r}(0)) \wedge (d_{\tau_r}(x) * \tau_r(x, 0)) \\
 \text{(BH17)} \quad &= (0 * 0) \wedge (d_{\tau_r}(x) * 0) \\
 \text{(BH2)} \quad &= 0 \wedge d_{\tau_r}(x) \\
 &= d_{\tau_r}(x) * (d_{\tau_r}(x) * 0) \\
 \text{(BH2)} \quad &= d_{\tau_r}(x) * d_{\tau_r}(x) \\
 \text{(BH1)} \quad &= 0 \in I.
 \end{aligned}$$

Hence, I is d_{τ_r} -invariant. $\square$

Next, we focus on the composition of τ_l and τ_r -derivations.

Theorem 3.7. *Let d_{τ_r} and D_{τ_r} be (l, r) - τ_r -derivations on an associative BH-algebra X . If $\tau_r(x, y) = y$ for all $x, y \in X$, then $d_{\tau_r} \circ D_{\tau_r}$ is also an (l, r) - τ_r -derivation on X .*

Proof. Suppose that $\tau_r(x, y) = y$ for all $x, y \in X$. Then

$$\begin{aligned}
& (d_{\tau_r} \circ D_{\tau_r})(x * y) \\
&= d_{\tau_r}(D_{\tau_r}(x * y)) \\
&= d_{\tau_r}((D_{\tau_r}(x) * \tau_r(x, y)) \wedge (\tau_r(x, y) * D_{\tau_r}(y))) \\
&= d_{\tau_r}((D_{\tau_r}(x) * y) \wedge (y * D_{\tau_r}(y))) \\
\text{(BH15)} \quad &= d_{\tau_r}(D_{\tau_r}(x) * y) \\
&= (d_{\tau_r}(D_{\tau_r}(x)) * \tau_r(D_{\tau_r}(x), y)) \wedge (\tau_r(D_{\tau_r}(x), y) * d_{\tau_r}(y)) \\
\text{(BH15)} \quad &= d_{\tau_r}(D_{\tau_r}(x)) * \tau_r(D_{\tau_r}(x), y) \\
&= d_{\tau_r}(D_{\tau_r}(x)) * y \\
&= d_{\tau_r}(D_{\tau_r}(x)) * \tau_r(x, y) \\
\text{(BH15)} \quad &= (d_{\tau_r}(D_{\tau_r}(x)) * \tau_r(x, y)) \wedge (\tau_r(x, y) * d_{\tau_r}(D_{\tau_r}(y))) \\
&= ((d_{\tau_r} \circ D_{\tau_r})(x) * \tau_r(x, y)) \wedge (\tau_r(x, y) * (d_{\tau_r} \circ D_{\tau_r})(y)).
\end{aligned}$$

Hence, $d_{\tau_r} \circ D_{\tau_r}$ is an (l, r) - τ_l -derivation on X . $\square$

Theorem 3.8. Let d_{τ_l} and D_{τ_l} be (r, l) - τ_l -derivations on an associative BH-algebra X . If $\tau_l(x, y) = x$ for all $x, y \in X$, then $d_{\tau_l} \circ D_{\tau_l}$ is also an (r, l) - τ_l -derivation on X .

Proof. Suppose that $\tau_r(x, y) = x$ for all $x, y \in X$. Then

$$\begin{aligned}
& (d_{\tau_l} \circ D_{\tau_l})(x * y) \\
&= d_{\tau_l}(D_{\tau_l}(x * y)) \\
&= d_{\tau_l}((\tau_l(x, y) * D_{\tau_l}(y)) \wedge (D_{\tau_l}(x) * \tau_l(x, y))) \\
&= d_{\tau_l}((x * D_{\tau_l}(y)) \wedge (D_{\tau_l}(x) * x)) \\
\text{(BH15)} \quad &= d_{\tau_l}(x * D_{\tau_l}(y)) \\
&= (\tau_l(x, D_{\tau_l}(y)) * d_{\tau_l}(D_{\tau_l}(y))) \wedge (d_{\tau_l}(x) * \tau_l(x, D_{\tau_l}(y))) \\
\text{(BH15)} \quad &= \tau_l(x, D_{\tau_l}(y)) * d_{\tau_l}(D_{\tau_l}(y)) \\
&= x * d_{\tau_l}(D_{\tau_l}(y)) \\
&= \tau_l(x, y) * d_{\tau_l}(D_{\tau_l}(y)) \\
\text{(BH15)} \quad &= (\tau_l(x, y) * d_{\tau_l}(D_{\tau_l}(y))) \wedge (d_{\tau_l}(D_{\tau_l}(x)) * \tau_l(x, y)) \\
&= (\tau_l(x, y) * (d_{\tau_l} \circ D_{\tau_l})(y)) \wedge ((d_{\tau_l} \circ D_{\tau_l})(x) * \tau_l(x, y)).
\end{aligned}$$

Hence, $d_{\tau_l} \circ D_{\tau_l}$ is an (r, l) - τ_l -derivation on X . $\square$

Proposition 3.5. Let d_{τ_r} be an (r, l) - τ_r -derivation and D_{τ_r} be an (l, r) - τ_r -derivation on an associative BH-algebra X . Then

- (1) $(\forall x \in X)((d_{\tau_r} \circ D_{\tau_r})(x) = 0)$ if d_{τ_r} is regular,
- (2) $(\forall x \in X)((D_{\tau_r} \circ d_{\tau_r})(x) = D_{\tau_r}(0) * \tau_r(0, d_{\tau_r}(0)))$,

(3) $(\forall x \in X)((d_{\tau_r} \circ D_{\tau_r})(x) = 0 = (D_{\tau_r} \circ d_{\tau_r})(x))$ if d_{τ_r} and D_{τ_r} are regular.

Proof. (1) Suppose that d_{τ_r} is regular. Let $x \in X$. Then

$$\begin{aligned}
 \text{(BH2)} \quad & (d_{\tau_r} \circ D_{\tau_r})(x) = (d_{\tau_r} \circ D_{\tau_r})(x * 0) \\
 & = d_{\tau_r}(D_{\tau_r}(x * 0)) \\
 \text{(BH15)} \quad & = d_{\tau_r}(D_{\tau_r}(x) * \tau_r(x, 0)) \\
 \text{(BH17)} \quad & = d_{\tau_r}(D_{\tau_r}(x) * 0) \\
 \text{(BH15)} \quad & = \tau_r(D_{\tau_r}(x), 0) * d_{\tau_r}(0) \\
 \text{(BH17)} \quad & = 0 * 0 \\
 \text{(BH1)} \quad & = 0.
 \end{aligned}$$

(2) Let $x \in X$. Then

$$\begin{aligned}
 \text{(BH2)} \quad & (D_{\tau_r} \circ d_{\tau_r})(x) = (D_{\tau_r} \circ d_{\tau_r})(x * 0) \\
 & = D_{\tau_r}(d_{\tau_r}(x * 0)) \\
 \text{(BH15)} \quad & = D_{\tau_r}(\tau_r(x, 0) * d_{\tau_r}(0)) \\
 \text{(BH17)} \quad & = D_{\tau_r}(0 * d_{\tau_r}(0)) \\
 \text{(BH15)} \quad & = D_{\tau_r}(0) * \tau_r(0, d_{\tau_r}(0)).
 \end{aligned}$$

(3) It is straightforward by (1) and (2). $\square$

Let $\text{Der}_\tau(X)$ be the set of all τ -derivations on an associative BH-algebra X . For $d_\tau, D_\tau \in \text{Der}_\tau(X)$, we define the binary operation $\wedge$ on $\text{Der}_\tau(X)$ as follows:

$$(\forall x \in X)((d_\tau \wedge D_\tau)(x) = d_\tau(x) \wedge D_\tau(x)).$$

Indeed, let $x \in X$. Then

$$\begin{aligned}
 & (d_\tau \wedge D_\tau)(x) = d_\tau(x) \wedge D_\tau(x) \\
 \text{(BH15)} \quad & = d_\tau(x).
 \end{aligned}$$

Hence, $d_\tau \wedge D_\tau = d_\tau$ for all $d_\tau, D_\tau \in \text{Der}_\tau(X)$.

4. Conclusion and discussion

In this paper, we have introduced the concepts of an (l, r) and an (r, l) - τ -derivation on a BH-algebra, which is induced by a left and a right bi-endomorphism, and provided important properties. The study found that the composition of (l, r) and (r, l) - τ -derivations is also an (l, r) and an (r, l) - τ -derivation on a 0-commutative BH-algebra, respectively. In addition, we can show that the set of all τ -derivations on a 0-commutative BH-algebra X is a left zero semigroup.

Finally, studying bi-endomorphisms in other algebras (d/BF/BG-algebras) is an interesting open problem.

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Students' difficulties with eigenvalues and eigenvectors. An exploratory study

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Abstract. In this paper we study the problem of students' difficulties in interpreting the definition of eigenvalues and eigenvectors, extending the research of other scholars. We analyse the activity of small groups of students, during an optional specifically designed extra class, while trying to make sense of the definitions seen in class. Using Sfard's notions of process/object, we scrutinise students' speech and written productions to detect those aspects in the formal definition of eigenvectors and eigenvalues that might hinder their understanding. We consider also how commognitive conflicts present in students' discourse with their classmates may encourage difficulties in grasping the meaning of these mathematical concepts.

Keywords: Eigentheory, process/object duality, commognitive conflict, undergraduate mathematics education.

MSC 2020: 15A18, 97C70, 97D70, 97H60.

1. Introduction

In recent years, research in mathematics education has been taking an interest in the teaching of linear algebra, recognizing its importance not only in the purely mathematical sphere, but also as a basic subject for the study of STEM subjects, given the many applications this topic has in disciplines such as computer science or engineering. Early studies in the area have focused on the general difficulties university students usually experience in dealing with the “formalism” of linear algebra [5]. Some studies then investigated the teaching and learning processes of basic concepts of the discipline, such as linear dependence, determinant, span, bases, etc. Recently, several researchers have delved into investigating processes relative to more advanced topics, for which a coordination of basic concepts such as those mentioned above is necessary. One such example is the theory of

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eigenvalues which, in our opinion, deserves special attention for several reasons, as pointed out by other researchers who have already investigated educational aspects related to them (i.e. [22, 23]). Firstly, in accordance with what already mentioned regarding linear algebra in general, applications of eigentheory are widespread in STEM subjects, for example in the study of dynamical systems, or of rotating bodies or for modelling quantum mechanical systems [16]. Moreover, the understanding of eigenvalues and eigenvectors implies a good understanding and coordination of other linear algebra concepts such as linear dependence, linear transformation, determinant, kernel, etc. Hence, investigating its learning processes can let difficulties related to the aforementioned notions emerge; lastly, according to both literature and our experience, the topic seems to represent a major obstacle in the study of a subject which is in itself a hindrance for students. Although the literature on eigentheory is still scarce, it is evident how interest in the topic is growing in the research community. Most of the work on the topic is concentrated in the last fifteen years or so and concerns different institutional settings. Some investigate students' difficulties in dealing with such concepts, while others describe the implementation of innovative instructional pathways and their impact on different aspects of student learning - such as a better coordination of algebraic and visual/geometric aspects [2, 3, 8, 15]. Some other researchers [12, 24] have investigated how properly designed task sequences can assist students in understanding eigenvalues and eigenvectors.

In their article "Process-object difficulties in linear algebra: eigenvalues and eigenvectors", Stewart and Thomas [21] showed some difficulties related to the definitions of eigenvector and eigenvalue, faced by students when learning linear algebra. Two main problems related to learning these notions are highlighted by these authors: (i) in a linear algebra course, students are usually given the concept definition in words and then an easy formula to algebraically compute them is soon provided. "In this way the strong visual, or embodied metaphorical, image of eigenvectors is obscured by the strength of this formal and symbolic thrust" ([21], p. 186); (ii) another serious problem with the formula $f(v) = \lambda v$ for students is that the two sides of the equation are quite different processes, since on the left it is a matrix which is multiplied by a vector, and this is equated on the right side to a multiplication of a scalar by the same vector.

Both these difficulties are confirmed by a more recent study conducted by the first author of this paper [10]. Nevertheless, as highlighted also by Stewart and Thomas themselves, the role and genesis of these obstacles, particularly regarding the reading of the formula $f(v) = \lambda v$, requires further investigation. In this paper we will give a possible explanation for the difficulty in dealing with this formula, as emerged from the pilot study of the first author's PhD project, concerning eigentheory teaching and learning processes. This hypothesis is related to the way the formula $f(v) = \lambda v$ is read by students and to the way it, together with the traditional sequence for introducing eigenvalues and eigenvectors, supports what Sfard calls a "pseudostructural" conception of them, to the detriment of their conceptualization as objects.

2. Theoretical framework

2.1 The process/object duality

In different works, Sfard [9, 18] has drawn attention to the operational/structural dual nature of mathematical conceptions. This means that in all fields of mathematics, the same mathematical concepts, or to be more precise, their representations, may at times be interpreted as processes, and other times as objects. Although these two aspects seem to be conflicting, Sfard emphasizes the importance of considering them as complementary, meaning that processes and objects must be seen as two different sides of the same coin, and not as two completely distinct aspects of mathematical activities. Sfard has brought many examples to show how the learning process may follow a pattern which is similar to the way mathematical concepts have developed historically: a process (primary process) slowly becomes an entity in its own right through a process of reification whose outcome is a mathematical object. This new object-like entity serves then as an input for new processes (secondary processes) at a higher level. Let us show an example from the realm of linear algebra to make things clearer. If we consider the notion of determinant, this achieves, both in its historical development and in the learning process, an object-status just after having been considered and manipulated operationally. It is first introduced as the result of a computation, to then become, hopefully also to the students' eyes, a mathematical object that gives important information about a matrix or the linear transformation the matrix represents. The reified object “determinant” becomes then a useful tool to perform new procedures at a higher level, such as computing the inverse of a matrix.

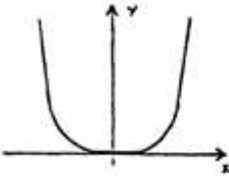
Graph	Algebraic expression	Computer program
	$y = 3x^4$	<pre> 10 INPUT X 20 Y = 1 30 FOR I = 1 TO 4 40 Y = Y * X 50 NEXT I 60 Y = 3 * Y </pre>

Figure 1: Different representations of a function, from (Sfard, 1991, p. 6)

As already mentioned, the dual nature of mathematics can be observed in the different representations used. Although “such property as structurality lies in the eyes of the beholder rather than in the symbols themselves” ([17], p. 5), some of these are more likely to be interpreted as structural, while others as operational. Sfard brings the example of different representations of a function (see Figure 1) stating that the computer program encourages an operational view while the graph supports a structural interpretation.

The algebraic expression can be easily interpreted in both ways and this may represent a benefit for the learning process, or an obstacle, as in the case we will present. It is important to notice, as Sfard points out, that this double interpretability of the algebraic representation also reflects the different ways the “=” sign can be read: “it can be regarded as a symbol of identity, or as a ‘command’ for executing the operations appearing at its right side” (p. 6).

The operational/structural duality of the equality sign as well as its misuse by primary and lower secondary school pupils has been widely discussed [1, 7, 13, 17]. However more recent studies (see for example [6]) have shown that an exclusively operational interpretation of the equal sign can persist in college students. Moreover, Das et al. [4] show how university students can struggle with the bidirectional flexibility in reading an equality, confirming a permanence well beyond elementary school of the difficulty with the structural interpretation of the equals sign.

2.2 Commognitive conflicts

It is crucial to remember that, according to Sfard’s theory of commognition, thinking, and thus thinking mathematically, is a form of communication. Therefore, it is important to consider all the elements related to the development of mathematical discourse to comprehend phenomena such as the persistence of the operational view of mathematical concepts and the challenges associated with transitioning to a structural interpretation of them. In this context, one significant phenomenon to consider is what Sfard refers to as *commognitive conflict* [19]. When a new term is introduced in the discourse, or a new meaning of that term is presented, our impression of understanding it can be misleading. There is no guarantee that other participants in the discourse are using it in the same way, and this potential ambiguity exposes the interlocutors to the risks associated with the use of non-operationalized terms. Specifically, when one encounters a new discourse governed by metarules different from those with respect to which he or she has acted up to that point, a commognitive conflict may arise. This situation involves different participants in the discourse following different metarules. In such cases, the two discourses are compatible but incommensurable, meaning they do not share the same criteria for approving or disapproving the current narrative. According to Sfard, the resolution of these conflicts generates learning. However, this typically occurs when an ‘oldcomer’ in the mathematical discourse participates in the discussion and can provide meaning to the newcomer’s discourse using shared metarules. Unfortunately, this does not always happen, and if a commognitive conflict remains unresolved, it can lead to a failure to achieve a structural interpretation of the mathematical object in question or to the development of a pseudostructural conception of it. Pseudostructural conceptions [14] are the conceptions which develop when the student, unable to think in the terms of abstract objects, uses symbols as things in themselves and, as a result, remains unaware of the rela-

tions between the secondary and primary processes. Consequently, they remain unaware of the relationships between secondary and primary processes.

3. Research questions and methodology

In autumn 2021, as part of the pilot study for the first author's PhD thesis on eigentheory teaching and learning processes, 67 first-year engineering students participated in an experimental activity. This activity took place during tutoring or extra hours a few days after the introduction of eigenvalue and eigenvector notions in a standard lecture by the linear algebra course's teacher.

During this two-hour activity, students were organized into groups of three. The first part involved a collective re-reading of the lecture notes and other study resources to reorganize newly encountered notions and deepen their understanding. To encourage a true comprehension of eigenvectors and eigenvalues, rather than mere repetition of the teacher's or textbook definitions, a guiding question was posed: "How would you explain what eigenvectors and eigenvalues are to someone who only knows basic notions of linear algebra such as linearity, dependence, basis, etc.?" Following this, four problems were presented. Written protocols from all groups were collected, and eight groups were audio and video recorded throughout the activity. A multimodal semiotic analysis [14] was conducted to detect moments where the simultaneous use of different semiotic resources (e.g., utterances, inscriptions, gestures) provided interesting insights into students' understanding of the concepts.

For a more detailed description of the methodology used and some further results, refer to [11]. In this discussion, our focus is on presenting two specific segments of activity from two small groups, highlighting difficulties in reading the formula " $f(v) = \lambda v$ " and various interpretations of the equal sign within it.

Using the described theoretical framework, we address the following research questions: (Q1) In what ways is a commognitive conflict likely to arise in a discourse on eigenvectors, when newcomers participate in that specific discourse? (Q2) What aspects in the formulation of the equation " $f(v) = \lambda v$ " may hinder the development of a structural view of eigenvectors and eigenvalues?

To answer these questions, we present excerpts from discussions between students in two of the observed groups, showcasing their utterances, gestures and drawings produced and used in the discourse. Our analysis seeks elements suggesting the presence of a commognitive conflict, where participants use the same terms or visual mediators but according to different discursive rules. We aim to detect if and how these conflicts produce or reinforce a pseudostructural conception of eigenvectors in, at least, one participant in the discourse. Being our focus on how students interpret the commonly provided definition of eigenvectors and eigenvalues, $f(v) = \lambda v$, we do so examining the segment of the video recording where students respond to the leading question mentioned earlier.

We start presenting and describing the chosen extracts and then we analyze them in order to show the elements that help us answer our research questions.

4. Results

4.1 Description of discourse in Group 1

From now on, we will call the three students in the considered group, Antonio, Bruno and Claudio. Right after having read the assignment, the students agree to start with reading the examples given by the teacher in the lecture:

1. Bruno: We must look at the examples
2. Claudio: Yes, exactly, the definition here is useless.

Their choice of trying to understand the examples, instead of interpreting the definition, seems effective. Indeed, it seems that the chosen example, eigenvectors and eigenvalues of a geometrical transformation in $\mathbb{R}^2$, helps them understand what these actually are.

resume Bruno: Then, basically, when is it that they form... when they overlap?

resume Claudio: When the vector remains itself, so when nothing changes.

resume Bruno: When it is rescaled times λ .

resume Claudio: Mmh...

resume Bruno: Yes, because f of v is equal to λ times v (he points to the formula “ $f(v) = \lambda v$ ” written on his lecture notes) so this means that the image of the vector is the vector rescaled times λ .

resume Claudio: Yes!

resume Bruno: So it means that it remains on the same line basically, when the two end on the same line (while pronouncing this sentence he performs the gesture in Figure 2), so what they do is to increase or decrease.



Figure 2: The gesture produced by Bruno (line 9)

They also re-read another example, an algebraic one, and also in this case it seems that they have grasped the idea of what it means for a vector to be an eigenvector. Nevertheless, a first doubt emerges, expressed by Claudio:

resume Claudio: So, does the eigenvector always coincide with the linear vector?

resume Bruno: Yes, sure, the one that you must have, isn't it? It is the vector of which you take the image.

resume Claudio: I don't know...

This exchange of lines is very brief, and the two students do not dig deeper in this miscomprehension, even if it is clear that Claudio is not satisfied with Bruno's answer. Nevertheless, by analyzing the whole activity, this perplexity emerges at different times and seems to be related to the fact that in the formula " $f(v) = \lambda v$ ", v appears both as the argument of the function and as the vector which is multiplied by λ . After having, almost, convinced themselves of this characterization of eigenvectors and eigenvalues, Bruno tries to write down a definition and he shares it with his group classmates:

resume Bruno: Is it good? If I say that the eigenvector is a vector whose image is defined by the vector itself scaled by the eigenvalue.

resume Antonio: Can you repeat?

resume Bruno: The eigenvector is a vector whose image is defined by the vector itself scaled by the eigenvalue.

The three of them agree on this definition. By reading this, it might seem that they have indeed understood what eigenvectors and eigenvalues are. This is probably right for Bruno, and this can be inferred by the way he solves the exercises in the second part of the activity. Nevertheless, the following part of their discussion makes clear that Antonio has misinterpreted the definition given by Bruno.

resume Antonio: But, I don't understand, this e_1 (they are now looking at an example where e_1 is an eigenvector for the given function f) would be v which finds v ? Bruno and Claudio ask him to explain it with clearer words.

resume Antonio: I mean, f of v is λ times v ? So in this case...

resume Claudio: It is a vector rescaled by its image.

resume Bruno: Because v is 3 times e_1 .

resume Claudio: Basically, the image of e_1 is e_1 rescaled by the eigenvalue.

resume Antonio: Ok!

Are Bruno, Claudio and Antonio's discourses really aligned? When they say " f of v is λ times v " or when they read " $f(v) = \lambda v$ ", are they really interpreting it in the same way?

Let us move a little forward, when the three students are solving the problems. The first one consists of finding the eigenvectors and eigenvalues related to the symmetry with respect to the line $y = x$, in $\mathbb{R}^2$. In this problem-solving phase, students' conceptions of the meaning of " $=$ " appear more evidently. For example, after having realized that all vectors lying on the line $y = x$ are eigenvectors, they cannot find the related eigenvalue. The researcher leading the activity intervenes to help them, by asking which is the image of a generic vector $(3, 3)$. Claudio answers "It is λ times $(3, 3)$ ". He seems convinced of this statement because in the following lines he adds "It is $(3, 3)$ rescaled for whatever eigenvalue—, and then again restates "[the image of $(3, 3)$ is] $(3, 3)$ times λ —. A few lines later, Antonio asks:

resume Antonio: if we had the vector $(3, 3)$, and the vector $(6, 6)$ [as its image], is it wrong to say that it is the vector times λ that is 2?

4.2 Interpretation of Antonio and Claudio's statements

Antonio and Claudio's confusion in solving this problem seems to be related to the way they interpret " $f(v) = \lambda v$ ", as well as the literal definition given by Bruno at the beginning of the activity, namely "The eigenvector is a vector whose image is defined by the vector itself scaled by the eigenvalue". It seems that they are interpreting " $=$ " in the formula as a defining equal sign. This means that they do not recognize the equality in the formula as the fact that the vector obtained computing v 's image is equal to the vector obtained multiplying v by λ . They seem to be reading the formula as "the image of v is given by λ times v ". Bruno's definition unconsciously helps Antonio and Claudio build this interpretation of the formula. When he says "is defined" he is improperly using this verb and it seems that he uses it to intend "is equal to". Nevertheless, the use of the verb "is defined" likely reinforces Antonio and Claudio's interpretation of " $=$ " as a defining equal sign. This way of reading the equality is evident in the fact that, when I ask what is the image of $(3, 3)$, instead of calculating its actual image and checking if it actually is equal to a multiple of $(3, 3)$ itself, Claudio immediately answers " λ times $(3, 3)$ ", as if to calculate a vector's image one had to multiply it times some eigenvalue. Also Antonio seems to think that one is allowed to arbitrarily consider a vector's multiple as its image, without actually applying the transformation considered. This would also shed light on the perplexity shown before by these two students. When Claudio asks "does the eigenvector always coincides with the linear vector?", he seems to be confused by the fact that v appears in both the left and the right side of the formula as if he did not know which one to take to compute the eigenvalue. The same doubt seems to emerge for Antonio when he states that the eigenvector "would be v which finds v ".

4.3 Description of discourse in Group 2

The second observed group is also formed by three students, Davide, Emma, and Flavio. The discussion predominantly involves Davide and Emma, with Flavio primarily listening and contributing minimally. Once they initiate the task, Davide and Emma deliberate on how to approach answering the guiding question. Emma suggests starting “by saying what it [an eigenvalue] is”, while Davide insists on the importance of starting with an example to make it clear what it is. He draws a cartesian plane and provides an example that he recalls the lecturer presenting in class:

1. Davide: There was a rotation. That if this goes this way (drawing a random vector in the first quadrant of the cartesian plane, and showing with an arrows that it rotates anticlockwise, as visible in Figure 3), the other one goes there (drawing a second vector on the second quadrant and show with an arrow that it rotates clockwise – Figure 3). And so, obviously, these two don't have the same direction, but for some specific coordinates, they do have the same direction (drawing a vector lying on the y -axis). For α equal to specific values, these two overlap or we could say that one v is equal to v (and writes $1v = v$).

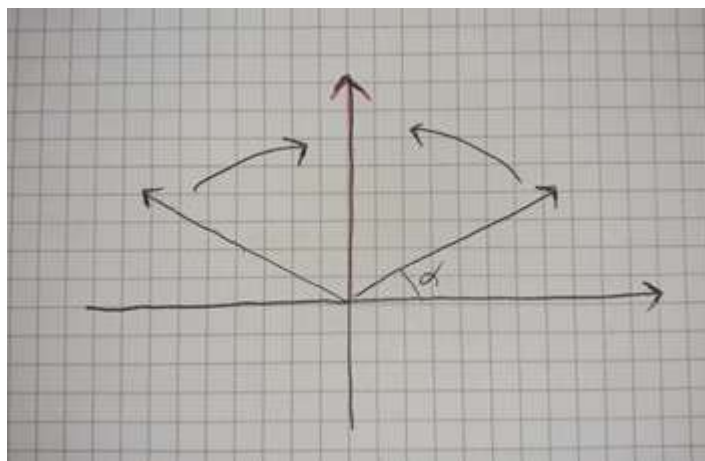


Figure 3: Davide's drawing

Davide talks about a “rotation” but what he is really talking about and representing is in reality a symmetry. The thing that probably confuses him and makes him talk about “rotation” is that he immediately represents a vector and its image, speaking of them as two different vectors. After that, he “rotates” the first vector and consequently the second. What he is doing with this apparent rotation (and probably doing in imitation of what he has seen the teacher do in class) is exploring the various positions that the starting vector and consequently its image can take according to the transformation involved, that is,

its symmetry with respect to the y -axis. He then points out that “for specific values of α ”, by which he means the vector of the same (arbitrarily chosen) length lying on the y -axis, the two vectors (vector and its image) overlap.

At this point, Davide attempts to provide additional examples, but they appear to confuse the other two members of the small group. The first author, who is present during the activity, tries to redirect the focus to the initial example given. She emphasizes that they are exploring a specific transformation; one that, when given a vector, yields its image symmetrical with respect to the y -axis. She then proceeds to inquire about the eigenvalues and eigenvectors of this particular transformation.

resume Flavio: So, I would say that the eigenvalues are 1, because it brings the vector into the multiple of itself and so it is a multiple of 1.

resume Davide: But how can you explain that?

resume Emma: Well...it is all those times in which the vectors overlap, they coincide.

resume Davide: But it is not that they always coincide. You must explain the generic concept of eigenvalue.

resume Emma: But you made that example!

resume Davide: Yes but I would make the example saying, in all these cases, when the direction is the same, if we make the ratio between the lengths of the two vectors, the result is 1. [...]

resume Davide: In this case it is 1.

resume Emma: And this is the scalar that multiplied to the vector...

resume Davide: But if this example was like this...

Davide replicates the previous illustration, this time doubling the length of the image vector. He endeavours to illustrate to his classmates that with the vectors configured in this manner, as they both converge on the y -axis (reiterating the concept from earlier that the vectors rotate towards the y -axis), they share the same direction. However, the crucial distinction lies in the ratio between their lengths, which is now 2 instead of 1. After reflecting on this example, the group tries to synthesize their thoughts.

resume Davide: I would say so: for all those case where a vector is image of itself, this function..

resume Emma: But this is a specific case.

resume Davide: Yes, but I started with the example of this function and then I extend it to all the functions.

resume Emma: And how can you extend it?

resume Davide: It is a concept that I am trying to say. For a generic function. . .

resume Emma: So, any function that has this ratio, I mean, this symmetry. . .

resume Davide: Exactly! But not for a symmetry. I am

resume Emma: In general

resume Davide: For any function where the image of a vector is equal to a multiple of the vector itself, that is a number that is multiplied for the vector itself that in this case (referring to the example in Figure 3) is 1 and in this other -1 (referring to the same example but to the vectors lying on the x -axis), then we will call that value “eigenvalue”.

resume Emma: And the vectors “eigenvectors”.

resume Davide: And all the vectors that respect this criterion “eigenvectors”.

resume Emma: So, eigenvalue and eigenvectors are. . . the eigenvalue, in this case -1, always works for those functions that go into the same direction.

resume Davide: Yes, if you talk about the plane.

4.4 Interpretation of Davide and Emma's interaction

The analysis of this excerpt, while rooted in results, may not be entirely straightforward, and the forthcoming interpretation is acknowledged to be somewhat subjective. Nevertheless, it appears that a commognitive conflict is arising here as well. Upon examining the entire sequence, it becomes apparent that Davide has grasped the concepts of eigenvectors and eigenvalues. However, it seems that the discourse between Davide and Emma is incommensurable. In other words, due to the different representations employed and the terminology used to describe them, it appears that at certain points in their conversation, the two students are using the same terms to refer to different concepts.

Davide's drawing (Figure 3) represents a crucial element in the discussion. In his representation, the distinction that one of the two vectors is independent, while the other is dependent—with its image dictated by the prescribed transformation (symmetry with respect to the y -axis)—is not immediately evident. Specifically, his discourse involves rotation, and he mentions that, for certain values of λ , the two vectors overlap. The commognitive conflict here seems to emerge in two ways. Firstly, the fact that Davide uses the term ‘overlap’ initially makes Emma think that in order to be eigenvectors, the vectors must overlap. Davide recognizes the presence of this misconception by Emma and attempts to resolve it by providing an example in which the two vectors have twice the length of one another. However, in this case, the dependence between the vector and its image is even more tenuous. In fact, it appears that in this

instance, Davide himself is constructing the function arbitrarily, ensuring that one of the two vectors is twice the length of the other. In our opinion, this may further reinforce the notion that eigenvalues exist for functions that are defined by taking a multiple of the initial vector as its image. Furthermore, the fact that it is not evident that the dependence between the two vectors is dictated by the transformation in question, seems to make Emma think that what for Davide are different vector-image configurations for the same transformation, are instead different transformations. In fact, later in the conversation, she explicitly states: “any function that has this ratio ...” and “the eigenvalue [...] always works for those functions that go into the same direction”. It seems, therefore, that for Emma, there are functions for which all vectors have the property of having their image aligned, and in those cases, “the eigenvalue works”—that is, probably, it exists. However, she does not grasp the fact that for every transformation, there can be some vectors with this property. The way in which she and Davide use terms like “exists” or “any” in the dialogue certainly supports this commognitive conflict.

5. Conclusions

In the two excerpts, recurring elements are noticeable. Despite their different manifestations, it appears that in both instances, a participant in the discourse fails to grasp a fundamental aspect of the definition of eigenvalues and eigenvectors: a precise mapping is provided, and for that specific mapping, vectors may exist whose images are multiples of themselves. Instead, it seems that these students (Antonio in the first example and seemingly Emma in the second) are under the impression that the function itself is defined in such a way that each vector has, as its image, a multiple of itself. This misconception is underpinned by a commognitive conflict, wherein the same sentences or representations are employed and interpreted differently by various participants in the discourse based on distinct metarules - in the sense of [19]. In the first example, the conflict arises from the use of the term “is defined” while in the second example, it appears to stem from how the example application is represented and described. Both conflicts are closely tied to the classical formulation of the definition of eigenvalues and eigenvectors. For instance, the definition provided by the teacher of one of the involved courses, which can be taken as a model for the classically given definition, is: “Let $f : V \rightarrow V$ be an endomorphism on the vectorial space V . We say that a vector $v \in V$, $v \neq 0_V$ is an eigenvector of f if there exists a value $\lambda \in K$ (the field of scalars), called its eigenvalue, such that $f(v) = \lambda v$ stands”. In this formulation, two elements are likely to cause confusion. Firstly, the expression $f(v) = \lambda v$ might be misinterpreted as implying that $f(v)$ is defined as $f(v) = \lambda v$, leading to a potential reinforcement of an interpretation as that by Antonio. Secondly, the term ‘exists’ appears to be associated with λ . The way it is formulated assumes, almost taken for granted, the existence of vectors with such a property for that particular application. Conse-

quently, the emphasis is placed on the value λ rather than on what happens to v , considering the specific application. This tendency is observed in both groups, mirroring Antonio's and Emma's interpretations of their peers' examples. In these cases, there seems to be a tendency to overlook the crucial point that a vector's image is determined by the given application: only instances where the image is a multiple of the initial vector qualify this as an eigenvector. This interpretation also stems from a challenge in understanding the inherent logical dependencies in the classical definition through the formulation $f(v) = \lambda v$ (or equivalently $Av = \lambda v$). A correct reading of it would be that there are some linear applications f , for which $\exists v \text{ s.t. } f(v) = \lambda v$ for some $\lambda \in K$ ¹. Contrary to the intended interpretation, students like Antonio or Emma tend to interpret it as: there exist linear applications f such that $\forall v, \exists \lambda [(v, \lambda v) \in f \iff f v = \lambda v]$. In our experience, we have observed this interpretation in many first-year STEM students taking linear algebra courses.

What seems to happen in the two groups is that somehow the definition they have already seen in class guides them in their reasoning. However, the meaning of the definition had remained obscure to them. In their reasoning, they start from an example, which would have the great potential of being able to start them from an operational interpretation of eigenvalues and eigenvectors. However, they have the bias of having already seen the structural definition, which they somehow want to make sense of at the same time as exploring the example. The fact that in both groups, there are those who have actually understood what eigenvalues and eigenvectors are but use representations and terms that are interpreted by their peers according to different metarules, generates commognitive conflicts. As a result, a pseudostructural conception of the concepts involved seems to be reinforced.

These results confirm students' difficulties in dealing with the formula $f(v) = \lambda v$ identified also by Stewart and Thomas [21]. Moreover we have tried to delve into the reasons behind such difficulties, showing how also the way students engage in discourses about these concepts might reveal and reinforce some misconceptions. It seems that, in their attempts at giving an operational meaning to eigenvalues and eigenvectors, through the analysis of examples, students are strongly biased by the fact that they have already encountered their definition. In this way, the passage between process and object is not smooth and results in reaching a pseudostructural conception of eigenvalues and eigenvectors. One potential didactical implication is the importance of starting with an operational interpretation of mathematical concepts, as suggested by Sfard, even at the advanced level of university instruction. Specifically, for eigentheory, instead of immediately presenting the definition of eigenvectors and eigenvalues, it might be beneficial to begin with the exploration of examples. Introduce a linear transformation (in the form of a geometrically known transformation or as a matrix) and ask students to determine if there are vectors whose image

1. Equivalently, in prenex normal form [20], the formula would be: $\exists f \exists v \exists \lambda (f(v) = \lambda v)$.

is a multiple of themselves. Only then could these vectors be designated as eigenvectors, with the ratio between the image and the vector representing the corresponding eigenvalue. This approach can enable students to initiate their understanding from exploring a process before transforming it into a mathematical object. Furthermore, it might reveal to students the inherent logical dependence in the formal definition through concrete examples.

Following this pilot study, we have indeed formulated the subsequent phase of the research project, taking into account these considerations. Specifically, emphasis has been placed on the utilization of dynamic digital representations of linear applications, such as videos or an explorable GeoGebra applet. This approach aims to support the initial process-like conceptualization of eigenvectors, preventing misconceptions likely to arise in a static drawing, as exemplified by Davide in the second example.

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Bifurcation dynamics in a modified Leslie type predator-prey model with predator harvesting and delay

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Abstract. In this article, the bifurcation behaviors of a modified Leslie type predator-prey model with harvesting and gestation delay of predator are discussed. The model takes the form of delayed differential-algebra equations. First, the existence of Hopf bifurcations in the model is studied by choosing the delay as a bifurcation parameter. It reveals that a sequence of stability switches and Hopf bifurcations can occur as the delay increases monotonously from zero. Next, the direction of the Hopf bifurcations and the stability of the bifurcating periodic orbits are also investigated. Moreover, we present several numerical simulations to support the theoretical results with the help of Matlab software. Lastly, the significances of our findings are discussed.

Keywords: predator-prey, differential-algebra, delay, bifurcations, periodic orbits.

MSC 2020: 92D25, 34A09, 37G05.

1. Introduction

Recently, there is a wide-range of interest in analysis and modelling of population interactions, since many complex dynamical behaviors have been discovered in this area, such as coexistence, instability, extinction, persistence, periodic oscillations, various bifurcations, chaos, etc. — cf. Refs. [11, 34, 5, 25, 15, 27, 56, 53, 37, 38, 1, 52, 18, 39, 41, 30, 55, 6, 45, 28, 42]. In this paper, we aim to investigate the bifurcation dynamics of a delayed predator-prey model with predator harvesting, which will be modelled by a differential-algebra system with delay in the following. The basic model that we consider is the famous Leslie's population model [34, 5, 27], which is given by

$$(1.1) \quad \begin{cases} \dot{x}(t) = x(t) (r_1 - ay(t)), \\ \dot{y}(t) = y(t) \left(r_2 - \frac{by(t)}{x(t)} \right), \end{cases}$$

where $x(t)$ and $y(t)$ represent the respective population densities of prey and predator at time t ; the parameters r_1 , a , r_2 , and b are all positive constants, where $r_1 > 0$ is the intrinsic growth rate of prey, a is the catch rate of predator, r_2 is the intrinsic growth rate of predator, and b is the conversion rate of consumed preys into the newborns of predator.

It is well known that the dynamical behaviors of population models depend on the current conditions, as well as the past information. In order to reflect the population dynamics depending on the past history, time delay of one type or another is often incorporated into population models due to gestation period, maturation time, or other reasons [25, 15]. Just as Kuang [25] pointed out: any model of species dynamics and interactions without delay is an approximation at best. Therefore, the introduction of time delay into population models would be more realistic to model population interactions. Moreover, it is notable that time delay has complicated impacts on the dynamics of a system, which can induce various interesting dynamical behaviors, for instances, stability switch, limit cycle, oscillations, Neimark-Sacker bifurcation, Takens-Bogdanov bifurcation, chaotic behavior and so on [25, 15, 1, 52, 51, 54, 50, 49, 46, 29, 42, 17]. Time delay due to gestation is a common situation, since normally the consumption of preys by the predator species throughout its past history governs the present birth rate of predator. Therefore, we introduce the gestation delay τ of predator population to model (1.1) — viz.

$$(1.2) \quad \begin{cases} \dot{x}(t) = x(t) (r_1 - ay(t)), \\ \dot{y}(t) = y(t) \left(r_2 - \frac{by(t-\tau)}{x(t)} \right). \end{cases}$$

Furthermore, we consider human harvesting effort $E(t)$ on predator population for model (1.2), since people usually harvest predators to avoid the extinction of prey species. In fact, biological resources are often harvested to satisfy the material life of humans [9, 8]. Consequently, we have

$$(1.3) \quad \begin{cases} \dot{x}(t) = x(t) (r_1 - ay(t)), \\ \dot{y}(t) = y(t) \left(r_2 - \frac{by(t-\tau)}{x(t)} - E(t) \right). \end{cases}$$

In the light of Refs[9, 8, 31, 12], we can study the harvesting effort $E(t)$ from an economic viewpoint. Let $P(t)$ and $C(t)$ denote the unit selling price of harvested predators, and the unit harvesting cost at time t , respectively. The number of harvested predators can be expressed as $E(t)y(t)$. And we assume that the market has a constant demand for the harvested predators, which means that the catches can always be sold out in the market. Note that the unit selling price $P(t)$ is inversely associated with the market supply $E(t)y(t)$, and the unit harvesting cost $C(t)$ is reversely relative to the population density of predator species $y(t)$. Hence, we take $P(t)$ and $C(t)$ as $p/(l + E(t)y(t))$ and $c/y(t)$ respectively, where p , l and c are positive parameters, p/l can be regarded as the maximum unit selling price. Subsequently,

$$\text{Total Revenue} = P(t)E(t)y(t) = \frac{pE(t)y(t)}{l + E(t)y(t)}, \text{Total Cost} = C(t)E(t)y(t) = cE(t).$$

Thus, the net economic revenue v produced by harvesting can be written as

$$(1.4) \quad E(t)y(t) \left(\frac{p}{l + E(t)y(t)} - \frac{c}{y(t)} \right) = v.$$

Combining (1.3) and (1.4), we can establish the following delayed differential-algebra predator-prey model with predator harvesting:

$$(1.5) \quad \begin{cases} \dot{x}(t) = x(t) (r_1 - ay(t)), \\ \dot{y}(t) = y(t) \left(r_2 - \frac{by(t-\tau)}{x(t)} - E(t) \right), \\ 0 = E(t)y(t) \left(\frac{p}{l + E(t)y(t)} - \frac{c}{y(t)} \right) - v. \end{cases}$$

The initial values of system (1.5) are given by

$$(1.6) \quad x(0) > 0, \quad y(\theta) > 0, \quad \theta \in [-\tau, 0], \quad E(0) > 0,$$

where $y(\theta) \in C([- \tau, 0], \mathbb{R})$ is a continuous bounded function.

In existing literature, most of the predator-prey models with harvesting are modelled by systems of differential equations [37, 38, 1, 52, 18, 39, 41, 30, 43, 33], which have not considered human harvesting from economic perspective. In contrast to such models, our modified Leslie's predator-prey model (1.5) is described by a system of delayed differential-algebra equations due to the economic revenue from predator harvesting is introduced. It should be noted that the analysis of differential-algebra system is more difficult than the corresponding system of differential equations, and normally differential-algebra systems possess more complicated dynamical behaviours compared with the systems of differential equations [40, 22, 23, 24, 3]. In the research of the dynamic behaviors of related models, Refs[53, 55, 6, 45, 28, 36] have studied the issues of the classification of fixed points, Neimark-Sacker bifurcation, invariant closed curve, Flip bifurcation, period windows, maximum Lyapunov exponents, phase portraits, saddle-node bifurcation, etc. However, the formulae for determining the properties of possible Hopf bifurcations in the delayed differential-algebra predator-prey models have not yet appeared. By applying differential-algebra system theory and center manifold argument, we investigate the properties such as stability, direction, and period of Hopf bifurcations in the delayed differential-algebra predator-prey model (1.5) where the delay is chosen as a bifurcation parameter. In some senses, our study supplements and extends the research work of the articles [53, 37, 38, 1, 52, 18, 39, 41, 30, 55, 6, 45, 28, 36, 43, 33].

We organize the rest of this paper as follows. Section 2 is devoted to analyze the stability of equilibrium point and existence of Hopf bifurcations in model (1.5) with gestation delay. Continue with Section 2, Section 3 deals with the stability and direction of the Hopf bifurcations by applying center manifold argument. Some numerical simulations are presented in Section 4 to make the derived findings more complete. Finally, in Section 5 we explain the significances of our results.

2. Stability analysis

In this section, we analyze the asymptotic stability of the delayed predator-prey model (1.5), as well as the occurrence conditions of Hopf bifurcations in the model. At first, we prove the positiveness of the solutions of model (1.5).

Lemma 2.1. *The trajectories of model (1.5) with initial values (1.6) and $v > 0$ stay in $R_+^3 = \{(x(t), y(t), E(t)) \mid x(t) > 0, y(t) > 0, E(t) > 0\}$, for $\forall t > 0$.*

Proof. In view of model (1.5), we have

$$\frac{dx(t)}{x(t)} = (r_1 - ay(t)) \, dt.$$

Due to the initial value $x(0) > 0$, by integrating above equation in the interval $[0, t]$, we obtain

$$x(t) = x(0) \exp \left\{ \int_0^t (r_1 - ay(s)) \, ds \right\} > 0, \text{ for } \forall t > 0.$$

Similarly, we can get

$$y(t) = y(0) \exp \left\{ \int_0^t \left(r_2 - \frac{by(s-\tau)}{x(s)} - E(s) \right) \, ds \right\} > 0, \text{ for } \forall t > 0.$$

Clearly, harvesting effort $E(t)$ is also positive for $\forall t > 0$, since the harvesting profit $v > 0$ here. $\square$

Lemma 2.1 suggests that only the positive equilibrium point of model (1.5) is required to be considered. If $X_0 := (x_0, y_0, E_0)^T$ is an equilibrium point of model (1.5), then we have

$$\begin{cases} r_1 - ay_0 = 0, \\ r_2 - b\frac{y_0}{x_0} - E_0 = 0, \\ \frac{pE_0y_0}{l + E_0y_0} - cE_0 - v = 0. \end{cases}$$

By solving the set of linear equations, model (1.5) has an equilibrium point $X_0 = (x_0, y_0, E_0)^T = (by_0/(r_2 - E_0), r_1/a, E_0)^T$, where $E_0 = \{(py_0 - vy_0 - cl) \pm \sqrt{\Delta}\}/2cy_0$, and $\Delta = (cl + vy_0 - py_0)^2 - 4clvy_0$.

To make such an equilibrium point X_0 is positive, we suppose that

$$(2.1) \quad r_2 > E_0, \, py_0 > cl + vy_0, \, (cl + vy_0 - py_0)^2 \geq 4clvy_0.$$

According to the theory of differential-algebra system [22, 23, 24], model (1.5) can be locally equivalent to the following constrained system near the

point X_0 :

$$(2.2) \quad \begin{cases} \dot{x}(t) = x(t) (r_1 - ay(t)), \\ \dot{y}(t) = y(t) \left(r_2 - \frac{by(t-\tau)}{x(t)} - E(t) \right), \\ \dot{E}(t) = f_3(x(t), y(t), E(t)), \\ 0 = E(t)y(t) \left(\frac{p}{l + E(t)y(t)} - \frac{c}{y(t)} \right) - v, \end{cases}$$

where f_3 is a continuously differentiable function and satisfies $f_3(X_0) = 0$. The explicit expression of f_3 is no need to be defined, which can be seen from Eq. (2.11) below.

For the convenience of discussion, we denote

$$(2.3) \quad \begin{aligned} f(X) &= \begin{pmatrix} f_1(X) \\ f_2(X) \\ f_3(X) \end{pmatrix} = \begin{pmatrix} x(t) (r_1 - ay(t)) \\ y(t) \left(r_2 - \frac{by(t-\tau)}{x(t)} - E(t) \right) \\ f_3(x(t), y(t), E(t)) \end{pmatrix}, \\ g(X) &= E(t)y(t) \left(\frac{p}{l + E(t)y(t)} - \frac{c}{y(t)} \right) - v, \quad X = (x(t), y(t), E(t))^T. \end{aligned}$$

Thus, system (2.2) can be written as

$$(2.4) \quad \begin{cases} \dot{X} = f(X), \\ 0 = g(X). \end{cases}$$

In the light of Refs. [2, 7], we can consider the following parameterisation ψ for system (2.4):

$$(2.5) \quad X = \psi(Y) = X_0 + U_0 Y + V_0 h(Y),$$

$$(2.6) \quad g(\psi(Y)) = 0,$$

where $Y = (y_1, y_2)^T \in \mathbb{R}^2$, $U_0 = \begin{pmatrix} I_2 \\ 0 \end{pmatrix}$, I_2 denotes an identity matrix of dimension

2×2 , $V_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $h : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a smooth mapping. Then substituting $X = \psi(Y)$ into system (2.4) yields

$$(2.7) \quad D_Y \psi(Y) \dot{Y} = f(\psi(Y)),$$

Differentiating Eq. (2.5) regarding Y , and then left multiplying U_0^T to the differentiated equation, which lead to

$$(2.8) \quad U_0^T D_Y \psi(Y) = I_2.$$

Differentiating Eq. (2.6) regarding Y , we derive

$$(2.9) \quad D_X g(X) D_Y \psi(Y) = 0.$$

By Eqs. (2.7)-(2.9), we get

$$(2.10) \quad \begin{pmatrix} D_X g(X) \\ U_0^T \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ I_2 \end{pmatrix} \dot{Y}(t) = f(\psi(Y)).$$

Eqs. (2.7), (2.9) and (2.10) suggest that (2.4) can be locally equivalent to

$$(2.11) \quad \dot{Y} = U_0^T f(\psi(Y)).$$

From the parameterised system (2.11), we can obtain that the Taylor expansions of the parameterised system of system (2.2) at X_0 takes the form

$$(2.12) \quad \dot{Y} = U_0^T D_X f(X_0) D_Y \psi(0) Y + o(|Y|),$$

where D denotes the differential operator, and $D_X f(X)$ represents the Jacobian matrix of function $f(X)$ regarding X . It is notable that the equilibrium point X_0 of model (1.5) corresponds to the equilibrium point $Y = 0$ of parameterised system (2.12).

Next, we calculate the Jacobian matrix $\bar{P}$ of model (1.5) with $\tau > 0$ at its positive equilibrium point X_0 . From the formula (2.12), we can obtain the Jacobian matrix $\bar{P}$, which takes the form

$$\bar{P} = \begin{pmatrix} 0 & -ax_0 \\ \frac{by_0^2}{x_0^2} & -\frac{by_0}{x_0}e^{-\lambda\tau} + \frac{plE_0y_0}{ply_0 - c(l + E_0y_0)^2} \end{pmatrix}.$$

Hence, the characteristic equation of Jacobi matrix $\bar{P}$ is

$$(2.13) \quad \lambda^2 + \left(\frac{by_0}{x_0}e^{-\lambda\tau} - \frac{plE_0y_0}{ply_0 - c(l + E_0y_0)^2} \right) \lambda + \frac{aby_0^2}{x_0} = 0.$$

To investigate the local asymptotic stability of equilibrium point X_0 , in what follows, we shall analyze the roots of characteristic equation (2.13).

For $\tau > 0$, we assume that $\pm i\omega$ (ω is a positive real number) are a pair of purely imaginary roots of Eq. (2.13), then we have the following equation:

$$-\omega^2 + \left[\frac{by_0}{x_0}(\cos \omega\tau - i \sin \omega\tau) - \frac{plE_0y_0}{ply_0 - c(l + E_0y_0)^2} \right] i\omega + \frac{aby_0^2}{x_0} = 0.$$

Separating the real and imaginary parts of the above equation, we get

$$-\omega^2 + \frac{by_0}{x_0}\omega \sin \omega\tau + \frac{aby_0^2}{x_0} + i \left[\frac{by_0}{x_0}\omega \cos \omega\tau - \frac{plE_0y_0\omega}{ply_0 - c(l + E_0y_0)^2} \right] = 0,$$

which gives

$$(2.14) \quad \sin \omega \tau = \frac{x_0 \omega}{by_0} - \frac{ay_0}{\omega},$$

$$(2.15) \quad \cos \omega \tau = \frac{plx_0 E_0}{b(ply_0 - c(l + E_0 y_0)^2)}.$$

Squaring and adding Eqs. (2.14) and (2.15), we have

$$(2.16) \quad \omega^4 + \left[\frac{p^2 l^2 E_0^2 y_0^2}{(ply_0 - c(l + E_0 y_0)^2)^2} - \frac{b^2 y_0^2}{x_0^2} - \frac{2aby_0^2}{x_0} \right] \omega^2 + \frac{a^2 b^2 y_0^4}{x_0^2} = 0.$$

Lemma 2.2. *For model (1.5) with $\tau > 0$,*

(i) *if*

$$\frac{by_0}{x_0} > \frac{plE_0 y_0}{ply_0 - c(l + E_0 y_0)^2}$$

and

$$\frac{p^2 l^2 E_0^2 y_0^2}{(ply_0 - c(l + E_0 y_0)^2)^2} > \frac{b^2 y_0^2}{x_0^2} + \frac{2aby_0^2}{x_0},$$

then all the roots of Eq. (2.13) have negative real parts;

(ii) *if*

$$\left[\frac{p^2 l^2 E_0^2 y_0^2}{(ply_0 - c(l + E_0 y_0)^2)^2} - \frac{b^2 y_0^2}{x_0^2} - \frac{2aby_0^2}{x_0} \right]^2 > \frac{4a^2 b^2 y_0^4}{x_0^2}$$

and

$$\frac{p^2 l^2 E_0^2 y_0^2}{(ply_0 - c(l + E_0 y_0)^2)^2} < \frac{b^2 y_0^2}{x_0^2} + \frac{2aby_0^2}{x_0},$$

then Eq. (2.16) has two positive roots ω^+ and ω^- . Substituting $\omega^\pm$ into Eq. (2.15), we obtain the values of time delay τ corresponding to the pair of purely imaginary roots $\pm i\omega$:

$$\tau_m^\pm = \frac{1}{\omega^\pm} \arccos \left(\frac{plx_0 E_0}{b(ply_0 - c(l + E_0 y_0)^2)} \right) + \frac{2n\pi}{\omega^\pm}, \quad m = 0, 1, 2, \dots$$

Proof. From Eq. (2.16), we know that if

$$\frac{p^2 l^2 E_0^2 y_0^2}{(ply_0 - c(l + E_0 y_0)^2)^2} > \frac{b^2 y_0^2}{x_0^2} + \frac{2aby_0^2}{x_0},$$

then Eq. (2.16) doesn't have positive roots. Thus, Eq. (2.13) with $\tau > 0$ doesn't have purely imaginary roots. Further, if we suppose that

$$\frac{by_0}{x_0} > \frac{plE_0 y_0}{ply_0 - c(l + E_0 y_0)^2},$$

then we can derive from Routh-Hurwitz stability criteria [34, 5, 16] that all the roots of the characteristic equation (2.13) with $\tau = 0$ have negative real parts. Consequently, Eq. (2.13) with $\tau > 0$ also have negative real parts according to Rouché's theorem [32, 47, 48, 19].

For the second case (ii), we have

$$\left[\frac{p^2 l^2 E_0^2 y_0^2}{(p l y_0 - c(l + E_0 y_0)^2)^2} - \frac{b^2 y_0^2}{x_0^2} - \frac{2 a b y_0^2}{x_0} \right]^2 - \frac{4 a^2 b^2 y_0^4}{x_0^2} > 0.$$

Add that to

$$\frac{p^2 l^2 E_0^2 y_0^2}{(p l y_0 - c(l + E_0 y_0)^2)^2} < \frac{b^2 y_0^2}{x_0^2} + \frac{2 a b y_0^2}{x_0},$$

then it is easy to show that Eq. (2.16) has two positive roots:

$$\omega^\pm = \left\{ \frac{1}{2} \left[\frac{b^2 y_0^2}{x_0^2} + \frac{2 a b y_0^2}{x_0} - \frac{p^2 l^2 E_0^2 y_0^2}{(p l y_0 - c(l + E_0 y_0)^2)^2} \right] \pm \sqrt{\left[\frac{p^2 l^2 E_0^2 y_0^2}{(p l y_0 - c(l + E_0 y_0)^2)^2} - \frac{b^2 y_0^2}{x_0^2} - \frac{2 a b y_0^2}{x_0} \right]^2 - \frac{4 a^2 b^2 y_0^4}{x_0^2}} \right\}^{\frac{1}{2}}.$$

Substituting $\omega^\pm$ into (2.15) and solving for τ , then the proof is completed. $\square$

Further, we differentiate Eq. (2.13) with respect to τ , it follows that

$$2\lambda \frac{d\lambda}{d\tau} - \frac{p l E_0 y_0}{p l y_0 - c(l + E_0 y_0)^2} \frac{d\lambda}{d\tau} + \frac{b y_0}{x_0} e^{-\lambda \tau} \frac{d\lambda}{d\tau} + \frac{b y_0}{x_0} \lambda e^{-\lambda \tau} \left(-\lambda - \tau \frac{d\lambda}{d\tau} \right) = 0.$$

As a consequence, we get

$$\left(\frac{d\lambda}{d\tau} \right)^{-1} = \frac{2\lambda e^{\lambda \tau} x_0 - \frac{p l x_0 y_0 E_0}{p l y_0 - c(l + E_0 y_0)^2} e^{\lambda \tau} + b y_0 (1 - \lambda \tau)}{b y_0 \lambda^2}.$$

Consequently, $\left(\frac{d\lambda}{d\tau} \right)^{-1} |_{\lambda=i\omega}$ equals

$$\begin{aligned} & \frac{2i\omega(\cos \omega \tau + i \sin \omega \tau)x_0 - \frac{p l x_0 y_0 E_0}{p l y_0 - c(l + E_0 y_0)^2}(\cos \omega \tau + i \sin \omega \tau) + b y_0(1 - \tau \omega i)}{-b y_0 \omega^2} \\ &= \frac{2x_0}{b y_0 \omega} \sin \omega \tau + \frac{p l x_0 E_0}{b \omega^2 (p l y_0 - c(l + E_0 y_0)^2)} \cos \omega \tau - \frac{1}{\omega^2} \\ &+ i \left[\frac{p l x_0 E_0}{b \omega^2 (p l y_0 - c(l + E_0 y_0)^2)} \sin \omega \tau - \frac{2x_0}{b \omega y_0} \cos \omega \tau + \frac{\tau}{\omega} \right]. \end{aligned}$$

Associated with Eqs. (2.14) and (2.15), we have

$$\begin{aligned} \operatorname{Re} \left(\frac{d\lambda}{d\tau} \right)^{-1} \Big|_{\lambda=i\omega} &= \frac{2x_0}{by_0\omega} \left(\frac{x_0\omega}{by_0} - \frac{ay_0}{\omega} \right) \\ &\quad + \frac{plx_0E_0}{b\omega^2(pl y_0 - c(l + E_0y_0)^2)} \left(\frac{plx_0E_0}{b(pl y_0 - c(l + E_0y_0)^2)} \right) - \frac{1}{\omega^2} \\ &= \frac{x_0^2}{b^2\omega^2y_0^2} \left[2\omega^2 - \frac{2aby_0^2}{x_0} - \frac{b^2y_0^2}{x_0^2} + \frac{p^2l^2E_0^2y_0^2}{(pl y_0 - c(l + E_0y_0)^2)^2} \right]. \end{aligned}$$

Accordingly,

$$\begin{aligned} \operatorname{sign} \left\{ \operatorname{Re} \left(\frac{d\lambda}{d\tau} \right) \right\}_{\lambda=i\omega} &= \operatorname{sign} \left\{ \operatorname{Re} \left(\frac{d\lambda}{d\tau} \right)^{-1} \right\}_{\lambda=i\omega} \\ &= \operatorname{sign} \left\{ 2\omega^2 - \frac{2aby_0^2}{x_0} - \frac{b^2y_0^2}{x_0^2} + \frac{p^2l^2E_0^2y_0^2}{(pl y_0 - c(l + E_0y_0)^2)^2} \right\}. \end{aligned}$$

Hence, the following transversality conditions hold:

$$(2.17) \quad \operatorname{sign} \left\{ \operatorname{Re} \left(\frac{d\lambda}{d\tau} \right) \right\}_{\tau=\tau_m^+, \omega=\omega^+} > 0, \quad \operatorname{sign} \left\{ \operatorname{Re} \left(\frac{d\lambda}{d\tau} \right) \right\}_{\tau=\tau_m^-, \omega=\omega^-} < 0.$$

Combining (2.17) with Lemma 2.2, we can derive the following results:

Theorem 2.1. *For model (1.5) with $\tau > 0$, suppose that*

$$\frac{by_0}{x_0} > \frac{plE_0y_0}{pl y_0 - c(l + E_0y_0)^2},$$

(i) if

$$\frac{p^2l^2E_0^2y_0^2}{(pl y_0 - c(l + E_0y_0)^2)^2} > \frac{b^2y_0^2}{x_0^2} + \frac{2aby_0^2}{x_0},$$

then the equilibrium point X_0 is locally asymptotically stable due to all the roots of Eq. (2.13) have negative real parts;

(ii) if

$$\left[\frac{p^2l^2E_0^2y_0^2}{(pl y_0 - c(l + E_0y_0)^2)^2} - \frac{b^2y_0^2}{x_0^2} - \frac{2aby_0^2}{x_0} \right]^2 > \frac{4a^2b^2y_0^4}{x_0^2}$$

and

$$\frac{p^2l^2E_0^2y_0^2}{(pl y_0 - c(l + E_0y_0)^2)^2} < \frac{b^2y_0^2}{x_0^2} + \frac{2aby_0^2}{x_0},$$

then there exists a positive integer N , such that the equilibrium point X_0 is locally asymptotically stable when the delay $\tau \in [0, \tau_0^+) \cup (\tau_0^-, \tau_1^+) \cup (\tau_1^-, \tau_2^+) \cup \dots \cup (\tau_{N-1}^-, \tau_N^+)$, and it is unstable when the delay $\tau \in (\tau_0^+, \tau_0^-) \cup (\tau_1^+, \tau_1^-) \cup (\tau_2^+, \tau_2^-) \cup \dots \cup (\tau_{N-1}^+, \tau_{N-1}^-) \cup (\tau_N^+, +\infty)$.

Proof. Since the proof of Theorem 2.1 is similar to the Hopf bifurcation theorems for functional differential equations in [19, 10], so it is omitted here. $\square$

Remark 2.1. Theorem 2.1 shows that stability switches would occur at the equilibrium point X_0 when the delay τ takes the critical values $\tau_0^+, \tau_0^-, \tau_1^+, \dots, \tau_{N-1}^-, \tau_N^+$. This means that, around these critical values, Hopf bifurcations can take place in model (1.5) with $\tau > 0$.

3. Stability and direction of Hopf bifurcations

In this section, we further study the properties of the Hopf bifurcations appearing in the previous section, such as the direction, stability and period of the bifurcations. For these issues, the research tool that we will use is the center manifold theorem due to Hassard, Kazarinoff and Wan [21].

Without loss of generality, we use τ_n to represent any one of the critical values in Theorem 2.1, and in the following discussion we always suppose that a Hopf bifurcation can occur in model (1.5) when $\tau = \tau_n$. Besides, $i\omega$ is assumed to be the purely imaginary root of Eq. (2.13) corresponding to τ_n .

If we let $\mu = \tau - \tau_n$, $y_1(t) = x(\tau t) - x_0$, $y_2(t) = y(\tau t) - y_0$, then the second order Taylor expansions of parameterised system (2.12) with $\tau > 0$ can be rewritten as the following functional differential equation in the phase space $C([-1, 0], \mathbb{R}^2)$:

$$(3.1) \quad \dot{Y}(t) = L_\mu(Y_t) + F(\mu, Y_t),$$

where $Y(t) = (y_1(t), y_2(t))^T$, $Y_t = Y(t + \theta) = (y_1(t + \theta), y_2(t + \theta))$, $\theta \in [-1, 0]$.

For $\Phi(\theta) = (\Phi_1(\theta), \Phi_2(\theta)) \in C([-1, 0], \mathbb{R}^2)$, we define

$$\begin{aligned} L_\mu \Phi &= (\tau_n + \mu) \begin{pmatrix} 0 & -ax_0 \\ \frac{by_0^2}{x_0^2} & \frac{plE_0y_0}{ply_0 - c(l + E_0y_0)^2} \end{pmatrix} \Phi^T(0) + (\tau_n + \mu) \begin{pmatrix} 0 & 0 \\ 0 & -\frac{by_0}{x_0} \end{pmatrix} \\ &\times \Phi^T(-1), \text{ and } F(\mu, \Phi(\theta)) = (\tau_n + \mu) \begin{pmatrix} F_{11} \\ F_{22} \end{pmatrix}, \end{aligned}$$

where $F_{11} = -a\Phi_1(0)\Phi_2(0)$,

$$\begin{aligned} F_{22} &= -\frac{by_0^2}{x_0^3}\Phi_1^2(0) + \frac{by_0}{x_0^2}\Phi_1(0)\Phi_2(0) + \frac{by_0}{x_0^2}\Phi_1(0) \times \Phi_2(-1) - \frac{b}{x_0}\Phi_2(0)\Phi_2(-1) \\ &+ \left(\frac{plE_0}{ply_0 - c(l + E_0y_0)^2} + \frac{plE_0y_0[cE_0(l + E_0y_0) - pl]}{[ply_0 - c(l + E_0y_0)^2]^2} \right. \\ &\left. - \frac{p^2cl^2E_0^2y_0^2(l + E_0y_0)}{[ply_0 - c(l + E_0y_0)^2]^3} \right) \Phi_2^2(0) + \dots \end{aligned}$$

It is easy to verify that $L_\mu: C([-1, 0], \mathbb{R}^2) \rightarrow \mathbb{R}^2$ is a continuous linear operator mapping. In the light of Riesz representation theorem in Ref[35], there exists a

2×2 matrix function $\eta(\theta, \mu)$ with elements of bounded variation for $\theta \in [-1, 0]$, such that

$$(3.2) \quad L_\mu \Phi = \int_{-1}^0 [\mathrm{d}\eta(\theta, \mu)] \Phi(\theta), \text{ for } \Phi(\theta) \in C([-1, 0], \mathbb{R}^2).$$

As a matter of fact, if we choose $\eta(\theta, \mu)$ as

$$(\tau_n + \mu) \begin{pmatrix} 0 & -ax_0 \\ \frac{by_0^2}{x_0^2} & \frac{plE_0y_0}{ply_0 - c(l + E_0y_0)^2} \end{pmatrix} \delta(\theta) + (\tau_n + \mu) \begin{pmatrix} 0 & 0 \\ 0 & -\frac{by_0}{x_0} \end{pmatrix} \delta(\theta + 1),$$

where $\delta(\theta)$ denotes delta function, i.e., $\delta(\theta) = \begin{cases} 0, & \theta \in [-1, 0), \\ 1, & \theta = 0 \end{cases}$, then the equation (3.2) is satisfied.

For $\Phi(\theta) \in C^1([-1, 0], \mathbb{R}^2)$, we define the operator $A(\mu)$:

$$A(\mu)\Phi(\theta) = \begin{cases} \frac{\mathrm{d}\Phi(\theta)}{\mathrm{d}\theta}, & -1 \leq \theta < 0, \\ \int_{-1}^0 \mathrm{d}\eta(\theta, \mu)\Phi(\theta), & \theta = 0, \end{cases}$$

and the operator $R(\mu)$:

$$R(\mu)\Phi(\theta) = \begin{cases} 0, & -1 \leq \theta < 0, \\ F(\mu, \Phi(\theta)), & \theta = 0, \end{cases}$$

then system (3.1) can be expressed as

$$(3.3) \quad \dot{Y}(t) = A(\mu)Y_t + R(\mu)Y_t.$$

From the theoretical analysis in the previous section, we can find that the operator $A(0)$ has a pair of simple purely imaginary eigenvalues $\pm i\omega\tau_n$.

For $\Psi(s) \in (C^1([-1, 0], \mathbb{R}^2))^* = C^1([0, 1], (\mathbb{R}^2)^*)$, the formal adjoint of operator A is defined as

$$A^*\Psi(s) = \begin{cases} -\frac{\mathrm{d}\Psi(s)}{\mathrm{d}s}, & 0 < s \leq 1, \\ \int_{-1}^0 \mathrm{d}\eta^T(s, 0)\Psi(-s), & s = 0, \end{cases}$$

under the bilinear pairing

$$(3.4) \quad \langle \Psi(s), \Phi(\theta) \rangle = \bar{\Psi}(0)\Phi(0) - \int_{\theta=-1}^0 \int_{\xi=0}^{\theta} \bar{\Psi}(\xi - \theta) \mathrm{d}\eta(\theta) \Phi(\xi) \mathrm{d}\xi,$$

where $\Psi(s) \in C^1([0, 1], (\mathbb{R}^2)^*)$, $\Phi(\theta) \in C^1([-1, 0], \mathbb{R}^2)$, $\eta(\theta) = \eta(\theta, 0)$. Consequently, the operators $A(0)$ and A^* are a pair of adjoint operators [21], which

means that the purely imaginary roots $\pm i\omega\tau_n$ of operator $A(0)$ are also the eigenvalues of operator A^* .

Next, we should calculate the eigenvectors of operators $A(0)$ and A^* associated with the eigenvalues $i\omega\tau_n$ and $-i\omega\tau_n$, respectively. We assume that the vectors $q(\theta) = (1, q_2)^T e^{i\omega\tau_n\theta}$ ($\theta \in [-1, 0]$) and $q^*(s) = \frac{1}{\bar{D}}(q_2^*, 1)e^{i\omega\tau_n s}$ ($s \in [0, 1]$) respectively denote the eigenvectors of operators $A(0)$ and A^* corresponding to the eigenvalues $i\omega\tau_n$ and $-i\omega\tau_n$. And combining with the definitions of operators $A(0)$ and A^* , we can derive the expressions for q_2 and q_2^* through some simple calculations: $q_2 = -\frac{i\omega}{ax_0}$ and $q_2^* = \frac{i\omega}{ax_0} + \frac{plE_0y_0}{ax_0[ply_0 - c(l + E_0y_0)^2]} - \frac{by_0}{ax_0^2}e^{-i\omega}$.

Moreover, it follows from the bilinear form (3.4) that $\langle q^*(s), q(\theta) \rangle$ equals

$$\begin{aligned} & \frac{1}{\bar{D}}(\bar{q}_2^*, 1)(1, q_2)^T - \int_{\theta=-1}^0 \int_{\xi=0}^{\theta} \frac{1}{\bar{D}}(\bar{q}_2^*, 1)e^{-i\omega\tau_n(\xi-\theta)} d\eta(\theta)(1, q_2)^T e^{i\omega\tau_n\xi} d\xi \\ &= \frac{1}{\bar{D}} \left\{ q_2 + \bar{q}_2^* - \int_{\theta=-1}^0 (\bar{q}_2^*, 1) e^{i\omega\tau_n\theta} \theta d\eta(\theta)(1, q_2)^T \right\} = \frac{1}{\bar{D}} \left(q_2 + \bar{q}_2^* - \frac{bq_2y_0\tau_n e^{-i\omega\tau_n}}{x_0} \right). \end{aligned}$$

Hence, in order to guarantee $\langle q^*(s), q(\theta) \rangle = 1$, we should choose $\bar{D} = q_2 + \bar{q}_2^* - (bq_2y_0\tau_n e^{-i\omega\tau_n}/x_0)$. And we can verify that $\langle q^*(s), \bar{q}(\theta) \rangle = 0$.

In the remainder of this section, we employ the ideas and notations of Hassard et al.[21] to construct the coordinates describing the center manifold C_0 at the Hopf bifurcation value $\mu = 0$ — viz. $\tau = \tau_n$. Define

$$(3.5) \quad z(t) = \langle q^*, Y_t \rangle \quad \text{and} \quad W(t, \theta) = Y_t - 2\text{Re}\{z(t)q(\theta)\},$$

then, on the center manifold C_0 , one has

$$(3.6) \quad W(t, \theta) = W(z(t), \bar{z}(t), \theta) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots,$$

where z and $\bar{z}$ are the local coordinates for manifold C_0 in the directions of q and $\bar{q}^*$.

In view of Eqs. (3.5) and (3.6), we can see that $W(z(t), \bar{z}(t), \theta)$ is real if Y_t is real. For the real solution $Y_t \in C_0$, due to $\mu = 0$, then from Eqs. (3.3)-(3.6), we get

$$(3.7) \quad \dot{z}(t) = i\omega\tau_n z(t) + \bar{q}^*(0)F_0(z, \bar{z}) := i\omega\tau_n z(t) + g(z, \bar{z}),$$

where

$$(3.8) \quad g(z, \bar{z}) = g_{20}(\theta)\frac{z^2}{2} + g_{11}(\theta)z\bar{z} + g_{02}(\theta)\frac{\bar{z}^2}{2} + g_{21}(\theta)\frac{z^2\bar{z}}{2} + \cdots.$$

Furthermore, we can obtain from Eq. (3.7) that

$$(3.9) \quad g(z, \bar{z}) = \bar{q}^*(0)F_0(z, \bar{z}) = \frac{\tau_n}{\bar{D}}(1, \bar{q}_2^*) \begin{pmatrix} F_{11}^0 \\ F_{22}^0 \end{pmatrix} = \frac{\tau_n}{\bar{D}}(F_{11}^0 + \bar{q}_2^* F_{22}^0),$$

where $F_{11} = -ay_{1t}(0)y_{2t}(0)$,

$$\begin{aligned} F_{22} = & -\frac{by_0^2}{x_0^3}y_{1t}^2(0) + \frac{by_0}{x_0^2}y_{1t}(0)y_{2t}(0) + \frac{by_0}{x_0^2}y_{1t}(0) \times y_{2t}(-1) - \frac{b}{x_0}y_{2t}(0)y_{2t}(-1) \\ & + \left(\frac{plE_0}{pl y_0 - c(l + E_0 y_0)^2} + \frac{plE_0 y_0 [cE_0(l + E_0 y_0) - pl]}{[pl y_0 - c(l + E_0 y_0)^2]^2} \right. \\ & \left. - \frac{p^2 cl^2 E_0^2 y_0^2 (l + E_0 y_0)}{[pl y_0 - c(l + E_0 y_0)^2]^3} \right) y_{2t}^2(0) + \dots \end{aligned}$$

According to the literature [21], we have

$$\begin{aligned} (3.10) \quad y_{1t}(0) &= z + \bar{z} + W_{20}^{(1)}(0) \frac{z^2}{2} + W_{11}^{(1)}(0) z\bar{z} + W_{02}^{(1)}(0) \frac{\bar{z}^2}{2} + \dots, \\ y_{2t}(0) &= q_2 z + \bar{q}_2 \bar{z} + W_{20}^{(2)}(0) \frac{z^2}{2} + W_{11}^{(2)}(0) z\bar{z} + W_{02}^{(2)}(0) \frac{\bar{z}^2}{2} + \dots, \\ y_{1t}(-1) &= z e^{-i\omega\tau_n\theta} + \bar{z} e^{i\omega\tau_n\theta} + W_{20}^{(1)}(-1) \frac{z^2}{2} + W_{11}^{(1)}(-1) z\bar{z} \\ &\quad + W_{02}^{(1)}(-1) \frac{\bar{z}^2}{2} + \dots, \\ y_{2t}(-1) &= q_2 z e^{-i\omega\tau_n\theta} + \bar{q}_2 \bar{z} e^{i\omega\tau_n\theta} + W_{20}^{(2)}(-1) \frac{z^2}{2} + W_{11}^{(2)}(-1) z\bar{z} \\ &\quad + W_{02}^{(2)}(-1) \frac{\bar{z}^2}{2} + \dots \end{aligned}$$

Substituting Eq. (3.10) into Eq. (3.9), we gain

$$\begin{aligned} g(z, \bar{z}) &= \frac{\tau_n}{D} \left\{ \left(-aq_2^* + \frac{by_0}{x_0^2} \right) \left[z + \bar{z} + W_{20}^{(1)}(0) \frac{z^2}{2} + W_{11}^{(1)}(0) z\bar{z} + W_{02}^{(1)}(0) \frac{\bar{z}^2}{2} \right] \right. \\ &\quad \times \left[q_2 z + \bar{q}_2 \bar{z} + W_{20}^{(2)}(0) \frac{z^2}{2} + W_{11}^{(2)}(0) z\bar{z} + W_{02}^{(2)}(0) \frac{\bar{z}^2}{2} \right] \\ &\quad - \frac{by_0^2}{x_0^3} \left[z + \bar{z} + W_{20}^{(1)}(0) \frac{z^2}{2} + W_{11}^{(1)}(0) z\bar{z} + W_{02}^{(1)}(0) \frac{\bar{z}^2}{2} \right]^2 \\ &\quad + \frac{by_0}{x_0^2} \left[z + \bar{z} + W_{20}^{(1)}(0) \frac{z^2}{2} + W_{11}^{(1)}(0) z\bar{z} + W_{02}^{(1)}(0) \frac{\bar{z}^2}{2} \right] \\ &\quad \times \left[q_2 z e^{-i\omega\tau_n\theta} + \bar{q}_2 \bar{z} e^{i\omega\tau_n\theta} + W_{20}^{(2)}(-1) \frac{z^2}{2} + W_{11}^{(2)}(-1) z\bar{z} + W_{02}^{(2)}(-1) \frac{\bar{z}^2}{2} \right] \\ &\quad - \frac{b}{x_0} \left[q_2 z + \bar{q}_2 \bar{z} + W_{20}^{(2)}(0) \frac{z^2}{2} + W_{11}^{(2)}(0) z\bar{z} + W_{02}^{(2)}(0) \frac{\bar{z}^2}{2} \right] \\ &\quad \times \left[q_2 z e^{-i\omega\tau_n\theta} + \bar{q}_2 \bar{z} e^{i\omega\tau_n\theta} + W_{20}^{(2)}(-1) \frac{z^2}{2} + W_{11}^{(2)}(-1) z\bar{z} + W_{02}^{(2)}(-1) \frac{\bar{z}^2}{2} \right] \end{aligned}$$

$$+ \left[\frac{plE_0}{ply_0 - c(l + E_0y_0)^2} + \frac{plE_0y_0[cE_0(l + E_0y_0) - pl]}{[ply_0 - c(l + E_0y_0)^2]^2} - \frac{p^2cl^2E_0^2y_0^2(l + E_0y_0)}{[ply_0 - c(l + E_0y_0)^2]^3} \right] \\ \times \left[q_2z + \bar{q}_2\bar{z} + W_{20}^{(2)}(0)\frac{z^2}{2} + W_{11}^{(2)}(0)z\bar{z} + W_{02}^{(2)}(0)\frac{\bar{z}^2}{2} \right]^2 + \cdots \Bigg\},$$

which yields

$$\begin{aligned} & g(z, \bar{z}) \\ &= \frac{\tau_n}{D} \left\{ z^2 \left[-aq_2\bar{q}_2^* + \frac{bq_2y_0}{x_0^2} - \frac{by_0^2}{x_0^3} + \frac{bq_2y_0}{x_0^2}e^{-i\omega\tau_n\theta} - \frac{bq_2^2}{x_0}e^{-i\omega\tau_n\theta} \right. \right. \\ &+ \frac{plq_2^2E_0}{ply_0 - c(l + E_0y_0)^2} + \frac{plq_2^2E_0y_0[cE_0(l + E_0y_0) - pl]}{[ply_0 - c(l + E_0y_0)^2]^2} \\ &- \left. \frac{p^2cl^2q_2^2E_0^2y_0^2(l + E_0y_0)}{[ply_0 - c(l + E_0y_0)^2]^3} \right] + z\bar{z} \left[-2a\bar{q}_2^*\text{Re}(q_2) + \frac{2by_0}{x_0^2}\text{Re}(q_2) \right. \\ &- \left. \frac{2by_0^2}{x_0^3} + \frac{2by_0}{x_0^2}\text{Re}(\bar{q}_2e^{i\omega\tau_n\theta}) - \frac{2b}{x_0}\text{Re}(q_2\bar{q}_2e^{i\omega\tau_n\theta}) \right. \\ &+ \frac{2plq_2\bar{q}_2E_0}{ply_0 - c(l + E_0y_0)^2} + \left. \frac{2plq_2\bar{q}_2E_0y_0[cE_0(l + E_0y_0) - pl]}{[ply_0 - c(l + E_0y_0)^2]^2} \right] \\ &- \frac{2p^2cl^2q_2\bar{q}_2E_0^2y_0^2(l + E_0y_0)}{[ply_0 - c(l + E_0y_0)^2]^3} + \bar{z}^2 \left[-a\bar{q}_2\bar{q}_2^* + \frac{b\bar{q}_2y_0}{x_0^2} - \frac{by_0^2}{x_0^3} + \frac{b\bar{q}_2y_0}{x_0^2}e^{i\omega\tau_n\theta} \right. \\ &+ \frac{pl\bar{q}_2^2E_0}{ply_0 - c(l + E_0y_0)^2} - \frac{b\bar{q}_2^2}{x_0}e^{i\omega\tau_n\theta} + \left. \frac{pl\bar{q}_2^2E_0y_0[cE_0(l + E_0y_0) - pl]}{[ply_0 - c(l + E_0y_0)^2]^2} \right] \\ &- \frac{p^2cl^2\bar{q}_2^2E_0^2y_0^2(l + E_0y_0)}{[ply_0 - c(l + E_0y_0)^2]^3} + z^2\bar{z} \left[\left(-aq_2\bar{q}_2^* + \frac{bq_2y_0}{x_0^2} - \frac{2by_0^2}{x_0^3} \right. \right. \\ &+ \left. \left. \frac{bq_2y_0}{x_0^2}e^{-i\omega\tau_n\theta} \right) W_{11}^{(1)}(0) + \left(-a\bar{q}_2^* + \frac{by_0}{x_0^2} - \frac{bq_2}{x_0}e^{-i\omega\tau_n\theta} \right. \right. \\ &+ \frac{2plq_2E_0}{ply_0 - c(l + E_0y_0)^2} + \left. \frac{2plq_2E_0y_0[cE_0(l + E_0y_0) - pl]}{[ply_0 - c(l + E_0y_0)^2]^2} \right. \\ &- \left. \frac{2p^2cl^2q_2E_0^2y_0^2(l + E_0y_0)}{[ply_0 - c(l + E_0y_0)^2]^3} \right) W_{11}^{(2)}(0) + \left(-\frac{a\bar{q}_2\bar{q}_2^*}{2} + \frac{b\bar{q}_2y_0}{2x_0^2} - \frac{by_0^2}{x_0^3} \right. \\ &+ \left. \frac{b\bar{q}_2y_0}{2x_0^2}e^{i\omega\tau_n\theta} \right) W_{20}^{(1)}(0) + \left(-\frac{a\bar{q}_2^*}{2} + \frac{by_0}{2x_0^2} + \frac{pl\bar{q}_2E_0}{ply_0 - c(l + E_0y_0)^2} \right. \\ &- \left. \frac{b\bar{q}_2}{2x_0}e^{i\omega\tau_n\theta} + \frac{pl\bar{q}_2E_0y_0[cE_0(l + E_0y_0) - pl]}{[ply_0 - c(l + E_0y_0)^2]^2} - \frac{p^2cl^2\bar{q}_2E_0^2y_0^2(l + E_0y_0)}{[ply_0 - c(l + E_0y_0)^2]^3} \right) \\ &\times W_{20}^{(2)}(0) + \left(\frac{by_0}{x_0^2} - \frac{bq_2}{x_0} \right) W_{11}^{(2)}(-1) + \left(\frac{by_0}{2x_0^2} - \frac{b\bar{q}_2}{2x_0} \right) W_{20}^{(2)}(-1) \Bigg] + \cdots \Bigg\}. \end{aligned} \tag{3.11}$$

Comparing the coefficients of Eqs. (3.8) and (3.11) gives

$$\begin{aligned}
g_{20} &= \frac{2\tau_n}{D} \left[-aq_2\bar{q}_2^* + \frac{bq_2y_0}{x_0^2} - \frac{by_0^2}{x_0^3} + \frac{bq_2y_0}{x_0^2} e^{-i\omega\tau_n\theta} \right. \\
&\quad - \frac{bq_2^2}{x_0} e^{-i\omega\tau_n\theta} + \frac{plq_2^2E_0}{ply_0 - c(l + E_0y_0)^2} + \frac{plq_2^2E_0y_0[cE_0(l + E_0y_0) - pl]}{[ply_0 - c(l + E_0y_0)^2]^2} \\
&\quad \left. - \frac{p^2cl^2q_2^2E_0^2y_0^2(l + E_0y_0)}{[ply_0 - c(l + E_0y_0)^2]^3} \right], \\
g_{11} &= \frac{\tau_n}{D} \left[-2aq_2^*\text{Re}(q_2) + \frac{2by_0}{x_0^2}\text{Re}(q_2) - \frac{2by_0^2}{x_0^3} + \frac{2by_0}{x_0^2}\text{Re}(\bar{q}_2e^{i\omega\tau_n\theta}) \right. \\
&\quad - \frac{2b}{x_0}\text{Re}(q_2\bar{q}_2e^{i\omega\tau_n\theta}) + \frac{2plq_2\bar{q}_2E_0}{ply_0 - c(l + E_0y_0)^2} \\
&\quad \left. + \frac{2plq_2\bar{q}_2E_0y_0[cE_0(l + E_0y_0) - pl]}{[ply_0 - c(l + E_0y_0)^2]^2} - \frac{2p^2cl^2q_2\bar{q}_2E_0^2y_0^2(l + E_0y_0)}{[ply_0 - c(l + E_0y_0)^2]^3} \right], \\
g_{02} &= \frac{2\tau_n}{D} \left[-a\bar{q}_2\bar{q}_2^* + \frac{b\bar{q}_2y_0}{x_0^2} - \frac{by_0^2}{x_0^3} + \frac{b\bar{q}_2y_0}{x_0^2} e^{i\omega\tau_n\theta} - \frac{b\bar{q}_2^2}{x_0} e^{i\omega\tau_n\theta} \right. \\
&\quad + \frac{pl\bar{q}_2^2E_0}{ply_0 - c(l + E_0y_0)^2} + \frac{pl\bar{q}_2^2E_0y_0[cE_0(l + E_0y_0) - pl]}{[ply_0 - c(l + E_0y_0)^2]^2} \\
&\quad \left. - \frac{p^2cl^2\bar{q}_2^2E_0^2y_0^2(l + E_0y_0)}{[ply_0 - c(l + E_0y_0)^2]^3} \right], \\
g_{21} &= \frac{2\tau_n}{D} \left[\left(-aq_2\bar{q}_2^* + \frac{bq_2y_0}{x_0^2} - \frac{2by_0^2}{x_0^3} + \frac{bq_2y_0}{x_0^2} e^{-i\omega\tau_n\theta} \right) W_{11}^{(1)}(0) \right. \\
&\quad + \left(-a\bar{q}_2^* + \frac{by_0}{x_0^2} - \frac{bq_2}{x_0} e^{-i\omega\tau_n\theta} + \frac{2plq_2E_0}{ply_0 - c(l + E_0y_0)^2} \right. \\
&\quad \left. + \frac{2plq_2E_0y_0[cE_0(l + E_0y_0) - pl]}{[ply_0 - c(l + E_0y_0)^2]^2} - \frac{2p^2cl^2q_2E_0^2y_0^2(l + E_0y_0)}{[ply_0 - c(l + E_0y_0)^2]^3} \right) W_{11}^{(2)}(0) \\
&\quad + \left(-\frac{a\bar{q}_2\bar{q}_2^*}{2} + \frac{b\bar{q}_2y_0}{2x_0^2} - \frac{by_0^2}{x_0^3} + \frac{b\bar{q}_2y_0}{2x_0^2} e^{i\omega\tau_n\theta} \right) W_{20}^{(1)}(0) \\
&\quad + \left(-\frac{a\bar{q}_2^*}{2} + \frac{by_0}{2x_0^2} - \frac{b\bar{q}_2}{2x_0} e^{i\omega\tau_n\theta} + \frac{pl\bar{q}_2E_0}{ply_0 - c(l + E_0y_0)^2} \right. \\
&\quad \left. + \frac{pl\bar{q}_2E_0y_0[cE_0(l + E_0y_0) - pl]}{[ply_0 - c(l + E_0y_0)^2]^2} - \frac{p^2cl^2\bar{q}_2E_0^2y_0^2(l + E_0y_0)}{[ply_0 - c(l + E_0y_0)^2]^3} \right) W_{20}^{(2)}(0) \\
(3.12) \quad &\left. + \left(\frac{by_0}{x_0^2} - \frac{bq_2}{x_0} \right) W_{11}^{(2)}(-1) + \left(\frac{by_0}{2x_0^2} - \frac{b\bar{q}_2}{2x_0} \right) W_{20}^{(2)}(-1) \right].
\end{aligned}$$

At present, the expressions of g_{20} , g_{11} and g_{02} have already been obtained. However, it is notable that $W_{20}(\theta)$ and $W_{11}(\theta)$ are unknown in g_{21} , therefore we still need to determine them. By using a computational procedure similar to the algorithms presented in Ref[21], we can obtain

$$W_{20}(\theta) = \frac{ig_{20}}{\omega\tau_n} q(0)e^{i\omega\tau_n\theta} + \frac{i\bar{g}_{02}}{3\omega\tau_n} \bar{q}(0)e^{-i\omega\tau_n\theta} + \mathcal{E}_1 e^{2i\omega\tau_n\theta},$$

$$(3.13) \quad W_{11}(\theta) = -\frac{ig_{11}}{\omega\tau_n}q(0)e^{i\omega\tau_n\theta} + \frac{i\bar{g}_{11}}{\omega\tau_n}\bar{q}(0)e^{-i\omega\tau_n\theta} + \mathcal{E}_2,$$

where

$$\mathcal{E}_1 = 2 \begin{pmatrix} 2i\omega & ax_0 \\ -\frac{by_0^2}{x_0^2} & 2i\omega + \frac{by_0}{x_0}e^{-2i\omega\tau_n} - \frac{plE_0y_0}{ply_0 - c(l + E_0y_0)^2} \end{pmatrix}^{-1} \begin{pmatrix} \mathcal{P}_{11} \\ \mathcal{P}_{21} \end{pmatrix},$$

$$\mathcal{E}_2 = 2 \begin{pmatrix} 0 & ax_0 \\ -\frac{by_0^2}{x_0^2} & \frac{by_0}{x_0} - \frac{plE_0y_0}{ply_0 - c(l + E_0y_0)^2} \end{pmatrix}^{-1} \begin{pmatrix} \mathcal{Q}_{11} \\ \mathcal{Q}_{21} \end{pmatrix},$$

with

$$\begin{aligned} \mathcal{P}_{11} &= -aq_2, \quad \mathcal{Q}_{11} = -\text{Re}(aq_2), \\ \mathcal{P}_{21} &= -\frac{by_0^2}{x_0^3} + \frac{bq_2y_0}{x_0^2} + \left(\frac{bq_2y_0}{x_0^2} - \frac{bq_2^2}{x_0}\right)e^{-i\omega\tau_n\theta} + \frac{plq_2^2E_0}{ply_0 - c(l + E_0y_0)^2} \\ &\quad + \frac{plq_2^2E_0y_0[cE_0(l + E_0y_0) - pl]}{[ply_0 - c(l + E_0y_0)^2]^2} - \frac{p^2cl^2q_2^2E_0^2y_0^2(l + E_0y_0)}{[ply_0 - c(l + E_0y_0)^2]^3}, \\ \mathcal{Q}_{21} &= -\frac{by_0^2}{x_0^3} + \frac{by_0}{x_0^2}\text{Re}(\bar{q}_2e^{i\omega\tau_n\theta}) + \frac{by_0}{x_0^2}\text{Re}(q_2) - \frac{b}{x_0}\text{Re}(q_2\bar{q}_2e^{i\omega\tau_n\theta}) \\ &\quad + \frac{plq_2\bar{q}_2E_0}{ply_0 - c(l + E_0y_0)^2} + \frac{plq_2\bar{q}_2E_0y_0[cE_0(l + E_0y_0) - pl]}{[ply_0 - c(l + E_0y_0)^2]^2} \\ &\quad - \frac{p^2cl^2q_2\bar{q}_2E_0^2y_0^2(l + E_0y_0)}{[ply_0 - c(l + E_0y_0)^2]^3}. \end{aligned}$$

Further, we can calculate

$$\mathcal{E}_1 = \begin{pmatrix} 2 \left(2i\omega + \frac{by_0}{x_0}e^{-2i\omega\tau_n} - \frac{plE_0y_0}{ply_0 - c(l + E_0y_0)^2} \right) \mathcal{P}_{11} - 2ax_0\mathcal{P}_{21} \\ \frac{2i\omega \left(\frac{by_0}{x_0}e^{-2i\omega\tau_n} - \frac{plE_0y_0}{ply_0 - c(l + E_0y_0)^2} \right) + \frac{aby_0^2}{x_0} - 4\omega^2}{\frac{2by_0^2}{x_0^2}\mathcal{P}_{11} + 4i\omega\mathcal{P}_{21}} \end{pmatrix}_{2 \times 1},$$

$$\mathcal{E}_2 = \begin{pmatrix} \frac{2x_0}{aby_0^2} \left(\frac{by_0}{x_0} - \frac{plE_0y_0}{ply_0 - c(l + E_0y_0)^2} \right) \mathcal{Q}_{11} - \frac{2x_0^2}{by_0^2} \mathcal{Q}_{21} \\ -\frac{2\text{Re}(q_2)}{x_0} \end{pmatrix}_{2 \times 1}.$$

Then, $W_{20}(\theta)$ and $W_{11}(\theta)$ can be directly worked out by substituting $\mathcal{E}_1$ and $\mathcal{E}_2$ into Eq. (3.13). Subsequently, g_{21} can be determined from Eq. (3.12). So far, the mathematical quantities g_{20} , g_{11} , g_{02} and g_{21} all have been expressed in terms

of known parameters. Hence, we are now able to calculate the following crucial values [21] which determine the properties of the Hopf bifurcation when $\tau = \tau_n$:

$$(3.14) \quad \begin{cases} c_1(0) = \frac{i}{2\omega\tau_n} \left(g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3} \right) + \frac{g_{21}}{2}, \\ \mu_2 = -\frac{\operatorname{Re}\{c_1(0)\}}{\operatorname{Re}\{\lambda'(\tau_n)\}}, \\ \beta_2 = 2\operatorname{Re}\{c_1(0)\}, \\ T_2 = -\frac{\operatorname{Im}\{c_1(0)\} + \mu_2\operatorname{Im}\{\lambda'(\tau_n)\}}{\omega\tau_n}. \end{cases}$$

Associated with the necessary conditions on the emergence of Hopf bifurcations in model (1.5), we can obtain the following results according to the classical Hopf bifurcation theorem in Ref[21]:

Theorem 3.1. *For the model (1.5) with $\tau > 0$, if $\frac{by_0}{x_0} > \frac{plE_0y_0}{ply_0 - c(l + E_0y_0)^2}$,*

$$\left[\frac{p^2l^2E_0^2y_0^2}{(ply_0 - c(l + E_0y_0)^2)^2} - \frac{b^2y_0^2}{x_0^2} - \frac{2aby_0^2}{x_0} \right]^2 > \frac{4a^2b^2y_0^4}{x_0^2},$$

and $\frac{p^2l^2E_0^2y_0^2}{(ply_0 - c(l + E_0y_0)^2)^2} < \frac{b^2y_0^2}{x_0^2} + \frac{2aby_0^2}{x_0}$, then a Hopf bifurcation can occur at the equilibrium point X_0 when time delay τ takes the critical value τ_n . Furthermore,

- (i) the direction of the Hopf bifurcation is supercritical (subcritical) if $\mu_2 > 0$ ($\mu_2 < 0$);
- (ii) the bifurcating periodic orbits are stable (unstable) if $\beta_2 < 0$ ($\beta_2 > 0$);
- (iii) the period of the bifurcating periodic orbits increases (decreases) if $T_2 > 0$ ($T_2 < 0$).

4. Numerical simulations

In this section, we perform several Matlab simulations to complement the analytical results above. In view of the assumptions in (2.1), we choose the coefficients of model (1.5) as follows: $r_1 = 2$, $a = 1$, $r_2 = 7/4$, $b = 3$, $p = 1$, $l = 1$, $c = 4/9$, $v = 2/9$. That is,

$$(4.1) \quad \begin{cases} \dot{x}(t) = x(t)(2 - y(t)), \\ \dot{y}(t) = y(t) \left(\frac{7}{4} - \frac{3y(t-\tau)}{x(t)} - E(t) \right), \\ 0 = E(t)y(t) \left(\frac{1}{1 + E(t)y(t)} - \frac{4}{9y(t)} \right) - \frac{2}{9}, \end{cases}$$

which has a positive equilibrium point $X_0 = (4, 2, 0.25)^T$. Besides, we can calculate that, $by_0/x_0 = 3/2 > plE_0y_0/[ply_0 - c(l + E_0y_0)^2] = 1/2$, $\{p^2l^2E_0^2y_0^2/[ply_0 - c(l + E_0y_0)^2]^2 - b^2y_0^2/x_0^2 - 2aby_0^2/x_0\}^2 = 64 > 4a^2b^2y_0^4/x_0^2 = 36$, $p^2l^2E_0^2y_0^2/[ply_0 -$

$c(l + E_0 y_0)^2]^2 = 1/4 < b^2 y_0^2 / x_0^2 + 2ab y_0^2 / x_0 = 33/4$. Thus, system (4.1) satisfies the conditions of Hopf bifurcation given in Theorems 2.1 and 3.1.

For this example, Eq. (2.16) becomes $\omega^4 - 8\omega^2 + 9 = 0$, which has two positive roots $\omega^+ = 2.5779$ and $\omega^- = 1.1637$. Accordingly, $\tau_0^+ = (1/2.5779) \arccos(7/8) = 0.4775$ and $\tau_0^- = (1/1.1637) \arccos(7/8) = 1.0578$. Moreover, with the aid of Matlab 7.0, we can obtain the values in Eq. (3.14): $c_1(0) = 5.9418 - 21.7397i$, $\lambda'(\tau_0^+) = 2.7466 - 0.4665i$, $\mu_2 = -2.1633 < 0$, $\beta_2 = 11.8836 > 0$, $T_2 = 16.8411 > 0$. From Theorem 2.1, we can conclude that the equilibrium point X_0 is locally asymptotically stable when $\tau \in [0, 0.4775)$ and it is unstable when $\tau \in (0.4775, 1.0578)$. Hence, a Hopf bifurcation takes place at the equilibrium point X_0 when τ takes the critical value $\tau_0^+ = 0.4775$. Furthermore, according to Theorem 3.1, the Hopf bifurcation at τ_0^+ is subcritical, and the corresponding bifurcating periodic orbits are unstable and increase.

In order to verify the theoretical results, we present some numerical simulations of the illustrative example (4.1) at different values of the delay τ . Numerical simulations show that the equilibrium point X_0 is locally asymptotically stable when $\tau = 0.4675 < \tau_0^+$ as illustrated in Fig. 1; periodic orbits bifurcate from the equilibrium point X_0 when τ takes the bifurcation value $\tau_0^+ = 0.4775$ as illustrated in Fig. 2; the bifurcating periodic orbits are unstable and increase when $\tau = 0.4783 > \tau_0^+$ as illustrated in Fig. 3; and the equilibrium point X_0 is unstable when $\tau = 0.4875 > \tau_0^+$ as illustrated in Fig. 4.

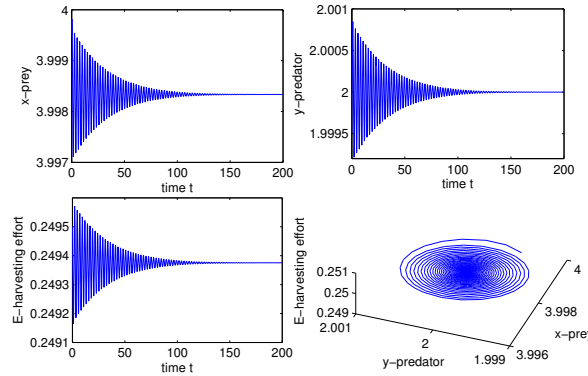


Figure 1: Matlab simulations suggest that the equilibrium point $X_0 = (4, 2, 0.25)^T$ of system (4.1) is locally asymptotically stable when the delay $\tau = 0.4675 < \tau_0^+ = 0.4775$. The initial values $(x(0), y(\theta), E(0)) = (3.9995, 1.9995, 0.2495)$ here, where $\theta \in [-\tau, 0]$.

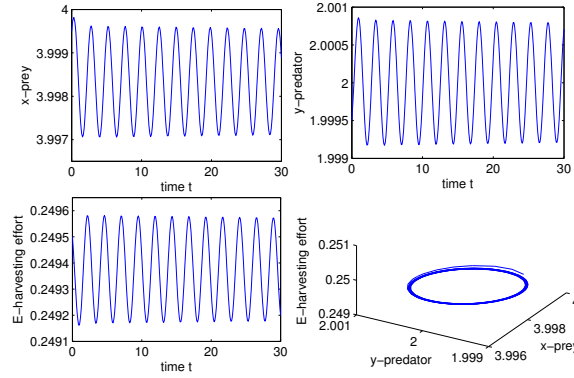


Figure 2: Matlab simulations suggest that periodic orbits bifurcate from the equilibrium point $X_0 = (4, 2, 0.25)^T$ of system (4.1) when the delay $\tau = \tau_0^+ = 0.4775$. The initial values $(x(0), y(\theta), E(0)) = (3.9995, 1.9995, 0.2495)$ here, where $\theta \in [-\tau, 0]$.

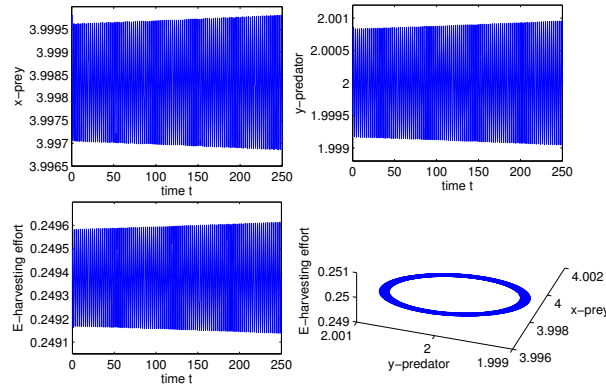


Figure 3: Matlab simulations suggest that the periodic orbits bifurcating from the equilibrium point $X_0 = (4, 2, 0.25)^T$ of system (4.1) are unstable and increase when the delay $\tau = 0.4783 > \tau_0^+ = 0.4775$. The initial values $(x(0), y(\theta), E(0)) = (3.9995, 1.9995, 0.2495)$ here, where $\theta \in [-\tau, 0]$.

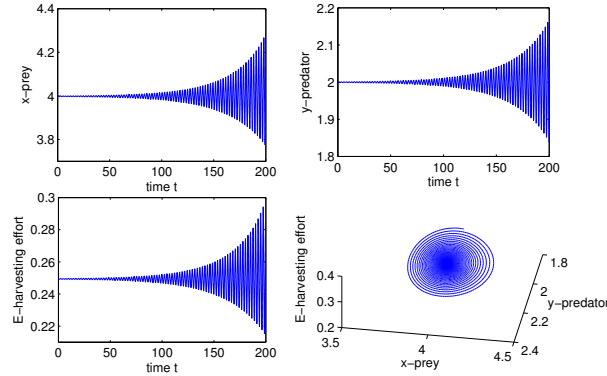


Figure 4: Matlab simulations suggest that the equilibrium point $X_0 = (4, 2, 0.25)^T$ of system (4.1) is unstable when the delay $\tau = 0.4875 > \tau_0^+ = 0.4775$. The initial values $(x(0), y(\theta), E(0)) = (3.9995, 1.9995, 0.2495)$ here, where $\theta \in [-\tau, 0]$.

5. Discussion

The present paper has studied the bifurcation dynamics of the delayed differential-algebra predator-prey model (1.5). We can see from Theorem 2.1 that the stability of the equilibrium point X_0 of model (1.5) can switch finite times when the gestation delay τ is treated as a variable bifurcation parameter. The stability would change from stable to unstable to stable, and eventually becomes unstable as the increasing of the delay τ . And large delays have unstabilizing effect on the stability of the equilibrium point X_0 .

When the delay τ is small and less than the first critical value τ_0^+ , the equilibrium point X_0 is locally asymptotically stable. In this case, the populations of preys and predators, as well as human harvesting are able to coexist in harmony. Hence, the ecological balance can be maintained. However, when the delay τ increases beyond τ_0^+ and within the interval (τ_0^+, τ_0^-) , the equilibrium point becomes unstable. At this moment, the ecological balance would be easily destroyed when it is subject to some external disturbances. Specially, when τ takes the critical value τ_0^+ , periodic orbits would arise from the equilibrium point and alter the dynamics of model (1.5) significantly. The appearance of Hopf-bifurcating periodic orbits means that small amplitude oscillations of population densities are currently in progress, which can be understood as the periodic evolution of the populations of preys and predators.

Similar phenomena will also occur around the other critical values $\tau_0^-, \tau_1^+, \tau_1^-, \dots, \tau_N^+$. That is, a finite number of Hopf bifurcations can take place as the

delay τ increases from zero. Furthermore, the stability of the periodic orbits is directly related to whether the populations of preys and predators as well as the external harvesting can coexist in an oscillatory mode, so the complex bifurcation behaviors have great significance for the delayed predator-prey model. Hence, stability of the Hopf-bifurcating periodic orbits are further analyzed in Section 3.

6. Concluding remarks

Clearly, our delayed differential-algebra predator-prey model (1.5) can exhibit a sequence of Hopf bifurcations as the increase of the delay τ , which can cause potentially dramatic variations in the dynamical behaviors of the model.

Besides, it is notable that the parameterisation method used in section 2 can reduce the delayed differential-algebra predator-prey model (1.5) to the parameterised system (2.12) near the equilibrium point X_0 , which plays an important role in our study. Refs[40, 22, 23, 24, 3] have shown that Differential-Algebraic Equations have widespread applications in constrained dynamical systems, so we expect that the parameterisation method can be employed to investigate the dynamics of more complex constrained systems.

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Finite groups with $124p$ elements of the largest order

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Abstract. We study a finite group with $124p$ elements of the largest order, where p is a prime greater than 5 and not equal to 31, and prove that such group is either a solvable group, or has a normal series like $1 \leq H \leq K \leq G$ such that H is a nilpotent $\{2, 3\}$ -group, $G/H \leq \text{Aut}(S)$, where $S \cong L_2(7), L_2(8), U_3(3)$.

Keywords: group, element, order, solvable group.

MSC 2020: 20D10.

1. Introduction

In this paper, all groups considered are finite. Let G be a finite group, $\pi_e(G)$ denotes the set of orders of elements of G , $M_i(G)$ denotes the set of elements of order i in G . In the whole paper, k always denote the largest element order of G , and write $M_k(G)$ as $M(G)$. For a positive integer n , $\pi(n)$ denotes the set of prime divisors of n , and $\pi(G) = \pi(|G|)$. We use $P_r(G)$ to denote a Sylow r -subgroup, where $r \in \pi(G)$, and $\varphi(x)$ the an Euler function of x . If there is no confusion, $P_r(G)$ is abbreviated as P_r . Sometimes we write the order of a group in the bracket next to the group, for example $A_5(2^2 \cdot 3 \cdot 5)$ means that A_5 has order $2^2 \cdot 3 \cdot 5$.

J. G. Thompson proposed the following famous problem in 1987.

Thompson Problem. *Let G_1 and G_2 be two finite groups. If $|M_i(G_1)| = |M_i(G_2)|$ and G_1 is solvable, where $i = 1, 2, 3, \dots$, is G_2 solvable?*

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Many mathematicians tried to prove Thompson Problem, but not so many beautiful results as a positive answer to this problem were obtained, it is worth to mention that Rulin Shen, Wujie Shi and Feng Tang in 2023 get a positive answer to Thompson Problem for a solvable group with the non-connected prime graph in [16], and Pawel Piwek in 2024 gives a counter example in [15]. Hence it is meaningful to study related topics to Thompson Problem. In the past three decades, several related studies were sponsored, one of them was to study if a finite group is solvable with a given number of elements of the largest order. In fact this topic does be related to Thompson Problem, since if we can prove that a finite group G is solvable with $|M_k(G)| = t$, then we can conclude that if G_1 and G_2 satisfy $|M_k(G_1)| = |M_k(G_2)| = t$, then both G_1 and G_2 are solvable, hence if G_1 and G_2 satisfy conditions of Thompson Problem and $|M_k(G_1)| = t$, then G_2 is solvable, Thompson Problem holds for $M_k(G_1) = t$. The first paper on this topic appeared in 1993, Yang studied finite groups with special number of elements of the largest order in [1], for example, the number is a prime, etc. From 1998 to 2004, Jiang, Du, Liu and Qian etc studied finite groups with the number of the largest order elements is less than 20, or equal to 32, 44, 52, $2p^2$, $2p^3$, $10m$, or $6p$ in [3] -[9], where p is a prime, and m is a positive integer coprime to 10. From 2005 to 2012, Yan, He and Chen studied finite groups with the number of elements of the largest order equal to 42, $68p(p > 7)$, $2p^4(p > 3)$, $52p(p > 5)$, $10p^m(p > 5$ and m is a natural number), $76p(p > 5)$, $4pq$, $10pq(p, q > 5)$, where p and q are prime, see [10] -[13], [20] - [24]. It is worth to mention that because the approaches used in previous articles are not effective for the number of elements of the largest order divided by 15, Chen and Shi studied a finite group with the number of elements of the largest order equal to 30 in [1]. In this paper, we continue this topic to study the finite group having $124p$ elements of the largest order, where p is a prime > 5 and not equal to 31. We shall prove the following theorem.

Main Theorem. *Suppose that G is a finite group with $124p$ elements of the largest order, where p is a prime > 5 and $p \neq 31$, then G is either a solvable group, or has a normal series like $1 \leq H \leq K \leq G$, such that H is a nilpotent $\{2, 3\}$ -group, $G/H \leq \text{Aut}(S)$, where $S \cong L_2(7), L_2(8), U_3(3)$.*

2. Preliminaries

At first we introduce some necessary lemmas.

Lemma 2.1. *Suppose that G is a finite group and has n cyclic subgroups of order l , then $|M_l(G)| = n\varphi(l)$. In particular, $|M(G)| = n\varphi(k)$. If $n = 1$, then G is supersolvable. If k is an odd prime number, then $n \equiv 1 \pmod{k}$.*

Proof. The conclusion is evident from Lemma 2.2 and Theorem 1 in [25], as well as Theorem 2.1 in [19]. $\square$

Lemma 2.2 ([4]). *If $x \in G$ and $|x| = k$, $M(G) \subseteq C_G(x)$, then $\pi_e(C_G(x)) = \pi_e(\langle x \rangle)$, $C_G(x) = \langle M(G) \rangle$ and $C_G(x) \leq G$.*

Lemma 2.3 ([19]). *If $|G| = 2n$ and n is an odd number, then G is solvable.*

For a positive integer n , a finite group is called a K_n -group if its order contains exactly n distinct primes.

Lemma 2.4 ([14]). *Suppose that G is a simple K_3 -group, then G is exactly isomorphic to one of the following simple groups:*

$A_5(2^2 \cdot 3 \cdot 5)$, $A_6(2^3 \cdot 3^2 \cdot 5)$, $L_2(7)(2^2 \cdot 3 \cdot 7)$, $L_2(8)(2^3 \cdot 3^2 \cdot 7)$, $L_2(17)(2^4 \cdot 3^2 \cdot 17)$, $L_3(3)(2^4 \cdot 3^4 \cdot 13)$, $U_3(3)(2^5 \cdot 3^3 \cdot 7)$, $U_4(2)(2^6 \cdot 3^4 \cdot 5)$.

Corollary 2.1. *If $|\pi(G)| = 3$ and $\pi(G) \neq \{2, 3, 5\}, \{2, 3, 7\}, \{2, 3, 13\}, \{2, 3, 17\}$, then G is solvable.*

Lemma 2.5 ([17]). *Let G be a simple K_4 -group with $3 \notin \pi(G)$, then $G \cong Sz(8)(2^5 \cdot 5 \cdot 7 \cdot 13)$ or $G \cong Sz(32)(2^{10} \cdot 5^2 \cdot 31 \cdot 41)$.*

Corollary 2.2. *If G is a K_4 -group, and $(15, |G|) = 1$, then G is solvable.*

Lemma 2.6 ([17]). *Suppose that G is a simple K_4 -group. Then, G can only be isomorphic to one of the following groups:*

- (1) $A_7(2^3 \cdot 3^2 \cdot 5 \cdot 7)$, $A_8(2^6 \cdot 3^2 \cdot 5 \cdot 7)$, $A_9(2^6 \cdot 3^4 \cdot 5 \cdot 7)$, $A_{10}(2^7 \cdot 3^4 \cdot 5^2 \cdot 7)$;
- (2) $M_{11}(2^4 \cdot 3^2 \cdot 5 \cdot 11)$, $M_{12}(2^6 \cdot 3^3 \cdot 5 \cdot 11)$, $J_2(2^7 \cdot 3^3 \cdot 5^2 \cdot 7)$;
- (3) $L_2(16)(2^4 \cdot 3 \cdot 5 \cdot 17)$, $L_2(25)(2^3 \cdot 3 \cdot 5^2 \cdot 13)$, $L_2(49)(2^4 \cdot 3 \cdot 5^2 \cdot 7^2)$, $L_2(81)(2^4 \cdot 3^4 \cdot 5 \cdot 41)$, $L_3(4)(2^6 \cdot 3^2 \cdot 5 \cdot 7)$, $L_3(5)(2^5 \cdot 3 \cdot 5^3 \cdot 31)$, $L_3(7)(2^5 \cdot 3^2 \cdot 7^3 \cdot 19)$, $L_3(8)(2^9 \cdot 3^2 \cdot 7^2 \cdot 73)$, $L_3(17)(2^9 \cdot 3^2 \cdot 17^3 \cdot 307)$, $L_4(3)(2^7 \cdot 3^6 \cdot 5 \cdot 13)$, $S_4(4)(2^8 \cdot 3^2 \cdot 5^2 \cdot 17)$, $S_4(5)(2^6 \cdot 3^2 \cdot 5^4 \cdot 13)$, $S_4(7)(2^8 \cdot 3^2 \cdot 5^2 \cdot 7^4)$, $S_4(9)(2^8 \cdot 3^8 \cdot 5^2 \cdot 41)$, $S_6(2)(2^9 \cdot 3^4 \cdot 5 \cdot 7)$, $O_8^+(2)(2^{12} \cdot 3^5 \cdot 5^2 \cdot 7)$, $G_2(3)(2^6 \cdot 3^6 \cdot 7 \cdot 13)$, $U_3(4)(2^6 \cdot 3 \cdot 5^2 \cdot 13)$, $U_3(5)(2^4 \cdot 3^2 \cdot 5^3 \cdot 7)$, $U_3(7)(2^7 \cdot 3 \cdot 7^3 \cdot 43)$, $U_3(8)(2^9 \cdot 3^4 \cdot 7 \cdot 19)$, $U_3(9)(2^5 \cdot 3^6 \cdot 5^2 \cdot 73)$, $U_4(3)(2^7 \cdot 3^5 \cdot 5 \cdot 7)$, $U_5(2)(2^{10} \cdot 3^5 \cdot 5 \cdot 11)$, $S_z(8)(2^6 \cdot 5 \cdot 7 \cdot 13)$, $S_z(32)(2^{10} \cdot 5^2 \cdot 31 \cdot 41)$, ${}^2D_4(2)(2^{12} \cdot 3^4 \cdot 7^2 \cdot 13)$, ${}^2F_4(2)(2^{11} \cdot 3^3 \cdot 5^2 \cdot 13)$.

(4) $L_2(r)$, where r is a prime satisfying the equation: $r^2 - 1 = 2^a \cdot 3^b \cdot u^c$ for positive integers $a \geq 1$, $b \geq 1$, $c \geq 1$ and a prime $u > 3$.

(5) $L_2(2^m)$, where m satisfies the equation:

$$\begin{cases} 2^m - 1 = u, \\ 2^m + 1 = 3t^b \end{cases}$$

where $m \geq 1$, $t > 3$, $b \geq 1$, u and t are primes.

(6) $L_2(3^m)$, where m satisfies the equation:

$$\begin{cases} 3^m - 1 = 2u^c, \\ 3^m + 1 = 4t \end{cases}$$

or

$$\begin{cases} 3^m - 1 = 2u, \\ 3^m + 1 = 4t^b \end{cases}$$

where $b \geq 1$, $c \geq 1$, u and t are primes.

Definition 1 ([23]). Let k be the largest element order of G , consider the action of G on the set of all cyclic subgroups of order k by conjugation, let $M_1, M_2, \dots, M_t$ be all conjugacy classes of this action, then the lengths of conjugacy classes are uniquely determined. Take $\langle x_i \rangle \in M_i$, then $|M_i| = |G : N_G(\langle x_i \rangle)|$, $1 \leq i \leq t$, are independent of the choice of $\langle x_i \rangle$. For convenience, let $m_i = |M_i|$, we call the array $(m_1, m_2, \dots, m_t)$ the m -type of G and denote it as $m(G) = (m_1, m_2, \dots, m_t)$, and we say that G is of m -type if $m(G) = (m_1, m_2, \dots, m_t)$.

Lemma 2.7. Let G be of m -type, $m(G) = (m_1, m_2, \dots, m_t)$, $n = m_1 + m_2 + \dots + m_t$ the number of all cyclic groups of order k . For any i , if $q \in \pi(m_i)$, then q divides $\frac{|G|}{m_i} \times n \times \varphi(k)$. Further $\pi(G) \subseteq \pi(n) \cup \pi(\varphi(k)) \cup \pi(k)$.

Proof. Reduction to absurdity. Assume there exists some i , without loss of generality, say $i = 1$, and a prime $q \in \pi(m_1)$ such that q does not divide $\frac{|G|}{m_1} \times n \times \varphi(k)$, so $q \nmid n$ and $q \nmid \varphi(k)$, thus there exists some m_j ($j > 1$) not divided by q for n is a sum of m_i , say $q \nmid m_2$. Using notations in Definition 1, we get the following equations:

$$m_1 \cdot |N_G(\langle x_1 \rangle) : C_G(x_1)| |C_G(x_1)| = |G| = m_2 \cdot |N_G(\langle x_2 \rangle) : C_G(x_2)| |C_G(x_2)|.$$

Noticing q does not divide $\frac{|G|}{m_1} = |N_G(\langle x_1 \rangle)|$, we see $q \nmid |C_G(\langle x_1 \rangle)|$. Now, $q \nmid \varphi(k)$ yields q does not divide $|N_G(\langle x_2 \rangle)/C_G(\langle x_2 \rangle)|$, hence q divides $|C_G(\langle x_2 \rangle)|$ for $q \nmid m_2$ and $q \nmid m_1$. Because x_1 and x_2 are two elements of the largest order in G , so are in $C_G(x_1)$ and $C_G(x_2)$ respectively, it forces $\pi(C_G(x_1)) = \pi(|x_1|) = \pi(|x_2|) = \pi(C_G(x_2))$, a contradiction to $q \nmid |C_G(x_2)|$ and $q \nmid |C_G(x_1)|$. The first part of the lemma is proved, i.e., $\pi(m_i) \subseteq \pi(\frac{|G|}{m_i}) \cup \pi(n) \cup \pi(\varphi(k))$. Now, the second part of the lemma follows from the first part of the lemma and $|G| = m_i \times \frac{|G|}{m_i}$. $\square$

Corollary 2.3. Suppose that G is of m -type and $m(G) = (m_1, m_2, \dots, m_t)$, $n = m_1 + m_2 + \dots + m_t$ the number of all cyclic groups of order k , if $\pi(n) \subseteq \{2, r\}$, $\pi(\varphi(k)) \subseteq \{2, p\}$, $\pi(k) \subseteq \{2, 3, q\}$, then G is a $\{2, 3, p, q\}$ -group or $\{2, 3, p, q, r\}$ -group.

Lemma 2.8 ([1]). Let G be a finite group with $|M(G)|$ elements of the largest order k . If k is a prime, then $k \mid (|M(G)| + 1)$.

Lemma 2.9 ([1]). Let G be a finite group with $|M(G)|$ elements of the largest order k . Then, there exists a positive integer α such that $|G| \mid |M(G)| \cdot k^\alpha$.

Lemma 2.10. *Let x be an element of the largest order in G , and r a prime dividing $|x|$, P_r the Sylow r -subgroup of $C_G(x)$. If $P_r \trianglelefteq C_G(x)$, then $P_r \trianglelefteq N_G(\langle x \rangle)$, and $|G : N_G(P_r)|$ divides $|G : N_G(\langle x \rangle)|$. Further, if $mr + 1 \nmid |G : N_G(\langle x \rangle)|$ for any positive integer m , then $P_r \trianglelefteq G$.*

Proof. The lemma follows by trivial calculation and Sylow Theorem. $\square$

Definition 2. *Let G be a finite group, the prime graph $\Gamma(G)$ of G is a graph with the set of vertices $V(G) = \pi(G)$. Let p, q be in $V(G)$, there exists an edge connecting p and q if and only if G has an element of order pq .*

Lemma 2.11 ([18]). *Let G be a finite group whose prime graph has more than one component, then one of the following holds:*

1. G is a Frobenius group or 2-Frobenius group;
2. G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1 -groups and K/H is a simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group and $|G/K| \mid |\text{Out}(K/H)|$.

3. Conclusions

Now, we prove the Main Theorem and give a positive answer to Thompson Problem for a finite group with $124p$ elements of the largest order, where p is a prime > 5 and $p \neq 31$.

Proof of Main Theorem. Take an element $x \in G$ of the largest order k , then $|N_G(\langle x \rangle) : C_G(x)| \mid \varphi(k)$. Let n be the number of cyclic subgroups of order k in G . The following formula is always true

$$(3.1) \quad |G| = |G : N_G(\langle x \rangle)| \cdot |N_G(\langle x \rangle) : C_G(x)| \cdot |C_G(x)|.$$

By Lemma 1, if $|M(G)| = 124p$, where p is prime > 5 and not equal to 31, then all possibilities for n , $\varphi(k)$ and k can be determined as the following Table 1.

Table 1: Values of n , $\varphi(k)$ and k

n	$\varphi(k)$	k
1	$124p$	k with $\varphi(k) = 124p$
2	$62p$	q or $2q$, $q = 62p + 1$ is a prime
31	$4p$	(1) q or $2q$, $q = 4p + 1$ is a prime; (2) $3q$ or $4q$ or $6q$, $q = 2p + 1$ is a prime
62	$2p$	q or $2q$, $q = 2p + 1$ is a prime
$31p$	4	5, 8, 10, 12
$62p$	2	3, 4, 6
$124p$	1	2

In the following, we shall prove that G is solvable case by case upon what n is.

Case 1. If $n = 1$, then G is a supersolvable group by Lemma 1.

Case 2. If $n = 2$, then by Table 1, $\varphi(k) = 62p$, and k equals q or $2q$, where $q = 62p + 1$ is a prime. Hence $C_G(a)$ is a $\{2, q\}$ -group or q -group and $N_G(\langle a \rangle)/C_G(a)$ is a $\{2, 31, p\}$ -group. By Lemma 4 and Corollary 3, we conclude that $C_G(a)$ and $N_G(\langle a \rangle)/C_G(a)$ are solvable, so $N_G(\langle a \rangle)$ is solvable. Moreover, $|G : N_G(\langle a \rangle)| \leq 2$, which implies that G is solvable.

Case 3. If $n = 31$, then $\varphi(k) = 4p$, either $q = 4p + 1$ is a prime and k equals one of q and $2q$, or $q = 2p + 1$ is a prime and k equals one of $3q, 4q$ and $6q$.

Subcase 3.1. If $q = 4p + 1$ is a prime and k equals one of q and $2q$, then $\pi(\varphi(k)) = \{2, p\}$, and $\pi(k) \subseteq \{2, q\}$, thus G is a $\{2, p, q\}$ -group or $\{2, 31, p, q\}$ -group by Corollary 3.

While G is a $\{2, p, q\}$ -group. Since both p and $q = 4p + 1$ are primes, the order of any simple K_3 -group does not contain such kind of prime factors, so G is solvable.

While G is a $\{2, 31, p, q\}$ -group, we have $(15, |G|) = 1$, so G is solvable by Corollary 2.

Subcase 3.2. If $q = 2p + 1$ is a prime and $k = 4q$, then $\pi(\varphi(k)) = \{2, p\}$, and $\pi(k) \subseteq \{2, q\}$, thus G is $\{2, p, q\}$ -group or $\{2, 31, p, q\}$ -group. By the same reasoning as Subcase 3.1, G is solvable.

Subcase 3.3. If $q = 2p + 1$ is a prime and $k = 3q$ or $6q$, this case is much complicated, we divide the proof into two steps.

3.3.1 To prove that G is solvable if $p > 13$, $q = 2p + 1$ is prime and $k = 3q$.

In this case, G is a $\{2, 3, p, q\}$ -group or $\{2, 3, 31, p, q\}$ -group. Let a be an element of the largest order in G , then $C_G(a)$ is a $\{3, q\}$ -group. Suppose that $|C_G(a)| = 3^u \cdot q^v$. Obviously, $C_G(a)$ has no element of order 9 or q^2 . If $u \geq 4$, then there are at least $(3^4 - 1)(q - 1) = 160p$ elements of order $3q$ in $C_G(a)$, a contradiction to $|M(G)| = 124p$. Therefore, $u \leq 3$. If $v > 1$, we get a contradiction by similar reason. Hence $v = 1$, and $|C_G(a)| = 3^u \cdot q$, where $u \leq 3$. Let Q be a Sylow q -group of $C_G(a)$. Obviously, $Q \trianglelefteq C_G(a)$, then $Q \trianglelefteq N_G(\langle a \rangle)$. Since $n = 31$, we have $|G : N_G(\langle a \rangle)| \leq 31$. Notice $q > 31$ in this case, we come to $Q \trianglelefteq G$. Let P_3 be the Sylow p -subgroup of G , and $|P_3| = 3^\beta$, then it follows from $|P_3| = |G|_3 = |G : N_G(\langle a \rangle)|_3 \times |C_G(a)|_3$ that $\beta \leq 6$.

While G is a $\{2, 3, p, q\}$ -group, as G/Q is a $\{2, 3, p\}$ -group, both p and $q = 2p + 1$ are primes and $p > 13$, so G/Q has no section isomorphic to a simple K_3 -group, G/Q is solvable, so is G .

While G is a $\{2, 3, 31, p, q\}$ -group, G/Q is a $\{2, 3, 31, p\}$ -group. Let $\overline{G} = G/Q$, then $|\overline{G}| = |G/Q| = 2^\alpha \cdot 3^\beta \cdot 31 \cdot p^\gamma$, where $\alpha \leq 2, \beta \leq 3, \gamma \leq 1$. By Lemma 3, we may assume that $\alpha = 2, |\overline{G}| = 2^2 \cdot 3^\beta \cdot 31 \cdot p^\gamma$, where $\beta \leq 3, \gamma \leq 1$.

1) If $\gamma = 0$, then $|\overline{G}| = 2^2 \cdot 3^\beta \cdot 31$. By Corollary 1, $\overline{G}$ is solvable, so is G .

2) If $\gamma = 1$, then $|\overline{G}| = |G/Q| = 2^2 \cdot 3^\beta \cdot 31 \cdot p$.

Clearly, no section of $\overline{G}$ is isomorphic to a simple K_3 -group. Suppose $\overline{G}$ is non-solvable, then there exists a section of $\overline{G}$, say $\overline{W}/\overline{S}$, isomorphic to a simple K_4 -group, and $|\overline{W}/\overline{S}| = 2^2 \cdot 3^t \cdot 31 \cdot p$, where $t \leq \beta \leq 3$. Moreover, $\overline{W}/\overline{S}$ is isomorphic to one of K_4 -groups listed in (4), (5) and (6) in Lemma 6.

(a) Assume $\overline{W}/\overline{S} \cong L_2(r)$, where r is a prime such that $r^2 - 1 = 2^a \cdot 3^b \cdot v^c$, where $a \geq 1$, $3 \geq b \geq 1$, $c \geq 1$, and $v > 3$ is a prime. It is easy to see $r \neq 31$, and then $r = p$. Since $(r^2 - 1)/(2, r - 1)$ divides $|L_2(r)|$, we have v^c divides $|\overline{W}/\overline{S}|$, which concludes $v = 31$ and $c = 1$. Comparing orders of Sylow 2-subgroups of G and $L_2(r)$, we get $a = 3$. Solving equations $p^2 - 1 = 2^3 \cdot 3^b \cdot 31$ for $1 \leq b \leq 3$ respectively, we find that there is no prime p satisfying the equation, a contradiction. Therefore $\overline{W}/\overline{S} \not\cong L_2(r)$.

(b) Assume $\overline{W}/\overline{S} \cong L_2(2^m)$, and m satisfies the system of equations:

$$\begin{cases} 2^m - 1 = u, \\ 2^m + 1 = 3t^b \end{cases}$$

where $m \geq 1$, $t \geq 3$, $b \geq 1$, u and t are primes. Since $|L_2(2^m)| = 2^m \cdot (2^m + 1)(2^m - 1) = 2^m \cdot 3t^b \cdot u$, we get $u \neq 31$, $t = 31$, $u = p$, $b = 1$, which is impossible by checking the system of equations, so $\overline{W}/\overline{S} \not\cong L_2(2^m)$.

(c) Assume $\overline{W}/\overline{S} \cong L_2(3^m)$, and m satisfies the system of equations:

$$\begin{cases} 3^m - 1 = 2u^c, \\ 3^m + 1 = 4t \end{cases}$$

or

$$\begin{cases} 3^m - 1 = 2u, \\ 3^m + 1 = 4t^b \end{cases}$$

where $b \geq 1$, $c \geq 1$, u and t are primes, we can prove $\overline{W}/\overline{S} \not\cong L_2(3^m)$ by the same reasoning as (b).

From (a), (b) and (c), G has no section isomorphic to a simple K_4 -group, which concludes that G is solvable.

3.3.2 To prove that G is solvable if $p > 13$, $q = 2p + 1$ is a prime and $k = 6q$.

Clearly, $n = 31$, $\pi(\varphi(k)) = \{2, p\}$, $\pi(k) = \{2, 3, q\}$, and G is a $\{2, 3, p, q\}$ -group or $\{2, 3, 31, p, q\}$ -group.

If G is a $\{2, 3, p, q\}$ -group, we can prove that G is solvable by the same argument as $k = 3q$.

If G is a $\{2, 3, 31, p, q\}$ -group, then $C_G(a)$ is a $\{2, 3, q\}$ -group. Let $|C_G(a)| = 2^\delta \cdot 3^\epsilon \cdot q^\varepsilon$, where $\delta \leq 5$, $\epsilon \leq 3$, $\varepsilon \leq 1$. It is easy to see that $C_G(a)$ has no element of order 9 or q^2 . Let Q be a Sylow q -subgroup of $C_G(a)$, we assert $Q \trianglelefteq G$. In fact, it is enough to show $Q \trianglelefteq C_G(a)$, then by the same reason as Step 3.3.1 we get $Q \trianglelefteq G$. Otherwise, $C_G(a)$ has at least $q + 1$ Sylow q -subgroups, this means that $C_G(a)$ has $3(q^2 - 1)$ elements of order $6q$ since $p > 13$, more than $124p$, a contradiction. Hence $Q \trianglelefteq G$. Suppose that $\overline{G} = G/Q$, then $|\overline{G}| = 2^\alpha \cdot 3^\beta \cdot 31 \cdot p^\gamma$, where $\alpha \leq 7$, $\beta \leq 3$, $\gamma \leq 1$.

While $\gamma = 0$, $|\overline{G}| = 2^\alpha \cdot 3^\beta \cdot 31$, then $\overline{G}$ is a $\{2, 3, 31\}$ -group. By Corollary 1, $\overline{G}$ is solvable.

While $\gamma = 1$, $|\overline{G}| = 2^\alpha \cdot 3^\beta \cdot 31 \cdot p$. Due to the absence of 31 and p in the order of a simple K_3 -group, if $\overline{G}$ is non-solvable, then it should have a section isomorphic to a simple K_4 -group, by the same reasoning as $k = 3q$, we can prove that this is impossible, so $\overline{G}$ is solvable, and then G is solvable.

3.3.3 To prove G is solvable if $p = 7, 11, 13$.

(1) If $p = 7$, then $\varphi(k) = 28$, $k = 29$ or 58 , $n = 31$ by Table 1, so $\pi(\varphi(k)) = \{2, 7\}$, $\pi(k) \subseteq \{2, 29\}$. Hence G is a $\{2, 7, 29\}$ -group or $\{2, 7, 29, 31\}$ -group, so (15, $|G|$) = 1, and then G is solvable.

(2) If $p = 11$, then $\varphi(k) = 44$, $k = 3q, 4q, 6q$ by Table 1, where $q = 23$.

(a) If $k = 4q = 4 \times 23$, then $n = 31$, $\pi(\varphi(k)) = \{2, 11\}$, $\pi(k) = \{2, 23\}$ and G is a $\{2, 11, 23\}$ -group or $\{2, 11, 23, 31\}$ -group. Hence (15, $|G|$) = 1, then G is solvable by Corollary 1.

(b) If $k = 3q = 3 \times 23$, then $n = 31$, $\varphi(k) = 44$, and G is a $\{2, 3, 11, 23\}$ -group or $\{2, 3, 11, 23, 31\}$ -group.

Assume that G is a $\{2, 3, 11, 23\}$ -group. Since the absence of 11 and 23 in the order of a simple K_3 -group, G has no section isomorphic to a simple K_3 -group. Hence if G is non-solvable, then there exists a section of G isomorphic to a simple K_4 -group. By the same reasoning as the case of $k = 3q$ and $p > 13$, we come to a contradiction, so G is solvable.

Now, G is a $\{2, 3, 11, 23, 31\}$ -group. In this case, 31 divides $|G|$, so it follows by Lemma 7 and Formula 3.1 that there exists an element a of the largest order such that $|G : N_G(\langle a \rangle)| = 31$. Then, $C_G(a)$ is a $\{3, 23\}$ -group, and there is no element of order 9 or 23^2 in $C_G(a)$. Suppose that $|C_G(a)| = 3^u \cdot 23^v$, then $3^u \cdot 23, 3 \cdot 23^v \leq 124p = 124 \times 11$, which implies that $u \leq 3, v \leq 1$. By Formula 3.1 we get that $|G|$ divides $31 \cdot 4 \cdot 11 \cdot 3^u \cdot 23$, thus $|G| = 2^\alpha \cdot 3^\beta \cdot 11^\gamma \cdot 23 \cdot 31$ with $\alpha \leq 2, \beta \leq 3, \gamma \leq 1$. Let Q be a Sylow 23-subgroup of $C_G(a)$, which is also a Sylow subgroup of G . Obviously, $Q \trianglelefteq C_G(\langle a \rangle)$, and then $Q \trianglelefteq N_G(\langle a \rangle)$. Hence the number of Sylow q -subgroups of G is $23t + 1$, it divides $|G : N_G(\langle a \rangle)| = 31$, this forces $23t + 1 = 1$, hence $Q \trianglelefteq G$. Now, $\overline{G} = G/Q$ is a $\{2, 3, 11, 31\}$ -group of order $2^\alpha \cdot 3^\beta \cdot 11^\gamma \cdot 31$, where $\alpha \leq 2, \beta \leq 3, \gamma \leq 1$. By the same reasoning as $k = 3q, p > 13$, $\overline{G}$ has no section isomorphic to a simple K_3 -group or K_4 -group, so $\overline{G}$ is solvable, and then G is solvable.

(c) Let $k = 6q$. Since $\varphi(6q) = \varphi(3q)$, we can use the same approach as the case $k = 3q$ to prove that G is solvable.

(3) If $p = 13$, $\varphi(k) = 4p$, then $k = q$ or $2q$, where $q = 4p + 1$ is a prime, $n = 31$, $\pi(\varphi(k)) = \{2, 13\}$, $\pi(k) \subseteq \{2, 53\}$. Hence G is a $\{2, 13, 53\}$ -group or $\{2, 13, 31, 53\}$ -group. Hence (15, $|G|$) = 1, which concludes that G is solvable by Corollary 2.

Case 4. Assume $n = 62$ and $\varphi(k) = 2p$. By Table 1, we have $k = q$ or $2q$, where $q = 2p + 1$ is a prime. Clearly, $\pi(n) = \{2, 31\}$, $\pi(\varphi(k)) = \{2, p\}$, $\pi(k) \subseteq \{2, q\}$. Then, G is a $\{2, p, q\}$ -group or $\{2, 31, p, q\}$ -group.

If G is a $\{2, p, q\}$ -group, then G is solvable by Corollary 1.

If G is a $\{2, 31, p, q\}$ -group, then (15, $|G|$) = 1, and G is solvable by Corollary 2.

Case 5. Assume $n = 31p$ and $\varphi(k) = 4$. By Table 1, k is one of 5, 8, 10 and 12, then $\pi(n) = \{31, p\}$, $\pi(\varphi(k)) = \{2\}$. If $k = 5$ or 8, then $|\pi(G)| \leq 2$, so G is solvable.

If $k = 10$, then $31 \notin \pi(G)$. By Lemma 9, $\pi(G) \subseteq \{2, 5, 31, p\}$, then $\pi(G) \subseteq \{2, 5, 7\}$. By Corollary 1, G is solvable.

If $k = 12$, then $31 \notin \pi(G)$. By Lemma 9, $\pi(G) \subseteq \{2, 3, 31, p\}$, then G is a $\{2, 3, p\}$ -group, where $p = 7, 11$. If $p = 11$, then G is solvable by Corollary 1. If $p = 7$, then G is a $\{2, 3, 7\}$ -group.

By Lemma 9, $7 \parallel |G|$. Suppose that $|G| = 2^\alpha \cdot 3^\beta \cdot 7$, where $\alpha \geq 2, \beta \geq 1$. If G is non-solvable, then by Lemma 11, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $K/H \cong L_2(7)$ or $L_2(8)$ or $U_3(3)$ and $|G/K| \mid |Out(K/H)|$, H is a nilpotent $\{2, 3\}$ -group.

Case 6. Assume $n = 62p$, then $\varphi(k) = 2$. By Table 1, we have $k = 3, 4, 6$, then G is a $\{2, 3\}$ -group, and G is solvable.

Case 7. If $n = 124p$, then $\varphi(k) = 1$. Hence $k = 2$, and G is a 2-group. Clearly, G has $|G| - 1$ elements of order 2, and $|G| - 1$ cannot be equal to $124p$, a contradiction.

Now, Main Theorem follows from cases 1 to 7. $\square$

4. Remark

It is meaningful to study if the non-solvable group in the Main Theorem could be eliminated.

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The MPBT inverse of a complex matrix based on the Hartwig-Spindelböck decomposition

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Abstract. Let A be a square complex matrix. A new generalized inverse of A is introduced by using the Moore-Penrose inverse and B-T inverse of A , named the MPBT inverse of A . The formula of the MPBT inverse can be got by using the Hartwig-Spindelböck decomposition of A . A relationship between the MPBT inverse and the inverse along two given matrix is investigated.

Keywords: Moore-Penrose inverse, B-T inverse, Core-EP decomposition, Hartwig-Spindelböck decomposition.

MSC 2020: 15A09.

1. Introduction

Let $\mathbb{C}$ be the complex field. The symbol $\mathbb{C}^{m \times n}$ denotes the set of all $m \times n$ complex matrices over $\mathbb{C}$. Let $A \in \mathbb{C}^{m \times n}$. The symbol A^* denotes the conjugate transpose of A . The notations $\mathcal{R}(A) = \{y \in \mathbb{C}^m : y = Ax, x \in \mathbb{C}^n\}$ and $\mathcal{N}(A) = \{x \in \mathbb{C}^n : Ax = 0\}$ will be used in the sequel. The smallest positive integer k such that $\text{rank}(A^k) = \text{rank}(A^{k+1})$ is called the index of $A \in \mathbb{C}^{n \times n}$ and denoted by $\text{ind}(A)$.

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Let $A \in \mathbb{C}^{m \times n}$. If a matrix $X \in \mathbb{C}^{n \times m}$ satisfies

$$AXA = A, XAX = X, (AX)^* = AX \text{ and } (XA)^* = XA,$$

then X is called the Moore-Penrose inverse of A ([16, 20]) and denoted by $X = A^\dagger$. If $AXA = A$ and $(AX)^* = AX$ holds, then X is called a $\{1, 3\}$ -inverse of A and the set of all $\{1, 3\}$ -inverse of A is denoted by $A\{1, 3\}$. If $XAX = X$ and $(XA)^* = XA$ holds, then X is called a $\{2, 4\}$ -inverse of A and the set of all $\{2, 4\}$ -inverse of A is denoted by $A\{2, 4\}$. Let $A, X \in \mathbb{C}^{n \times n}$ with $\text{ind}(A) = k$. The definition of the Drazin inverse is as follows:

$$AXA = A, XA^{k+1} = A^k \text{ and } AX = XA,$$

then X is called the Drazin inverse of A . It is unique and denoted by A^D ([9]). If $\text{ind}(A) \leq 1$, X is called the group inverse of A and denoted by $A^\#$.

The core inverse for a complex matrix were introduced by Baksalary and Trenkler [5]. Let $A \in \mathbb{C}^{n \times n}$. A matrix $X \in \mathbb{C}^{n \times n}$ is called a core inverse of A , if it satisfies $AX = P_A$ and $\mathcal{R}(X) \subseteq \mathcal{R}(A)$, where P_A is the orthogonal projector onto $\mathcal{R}(A)$. And if such a matrix exists, then it is unique (and denoted by $A^\oplus$). Baksalary and Trenkler gave several characterizations of the core inverse by using the decomposition of Hartwig and Spindelböck [12]. Manjunatha Prasad and Mohana [17] introduced the core-EP inverse of matrix [17, Definition 3.1]. Let $A \in \mathbb{C}^{n \times n}$. If there exists $X \in \mathbb{C}^{n \times n}$ such that $XAX = X, \mathcal{R}(X) = \mathcal{R}(X^*) = \mathcal{R}(A^k)$, then X is called the core-EP inverse of A . If such inverse exists, then it is unique and denoted by $A^\oplus$. The weak group inverse of a complex matrix was introduced by Wang and Chen in [25], which is the unique matrix X such that $AX^2 = X$ and $AX = A^\oplus A$ and denoted by $X = A^\omega$. The CMP inverse for a complex matrix was introduced by Mehdipour and Salemi [18]. Let $A \in \mathbb{C}^{n \times n}$. Mehdipour and Salemi [18] introduced the CMP inverse of A by using the core part A_1 of A and the Moore-Penrose inverse $A^\dagger$ of A . The *CMP inverse* of A is a matrix $X \in \mathbb{C}^{n \times n}$ such that the following equations hold:

$$XAX = X, AXA = A_1, AX = A_1 A^\dagger \text{ and } XA = A^\dagger A_1.$$

Such matrix X is unique and denoted by $A^{c,\dagger}$. The CMP inverse can be regarded as a tool to study the core part of the core-nilpotent decomposition of a matrix. The concept of the MPCEP-inverse of a Hilbert space operators was initially introduced by Chen, Mosić and Xu [8] and this concept was expanded on quaternion matrices by Kyrchei, Mosić and Stanimirović [13, 14]. Let $A \in \mathbb{C}^{n \times n}$ with $\text{ind}(A) = k$. If there exists a matrix $X \in \mathbb{C}^{n \times n}$ such that

$$XAX = X, AX = AA^\oplus \text{ and } XA = A^\dagger AA^\oplus A$$

then, X is called the MPCEP-inverse of A and denoted by $A^{\dagger,\oplus}$. The MPWC inverse of A was introduced by Liu, Miu and Jin [15] by using the weak group inverse of A and denoted by A° . Moreover, several characterizations of different generalized inverses along the core parts of three matrix decompositions can be found in [7].

2. Some matrix inverses based on the generalized inverses of ΣK

Every matrix $A \in \mathbb{C}^{n \times n}$ of rank r can be represented in the form

$$(1) \quad A = U \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} U^*,$$

where $U \in \mathbb{C}^{n \times n}$ is unitary, $\Sigma = \sigma_1 I_{r_1} \oplus \sigma_2 I_{r_2} \oplus \cdots \oplus \sigma_t I_{r_t}$ is the diagonal matrix of the nonzero singular values of A , where $\sigma_1 > \sigma_2 > \cdots > \sigma_t > 0$, $r_1 + \cdots + r_t = r$, and $K \in \mathbb{C}^{r \times r}$ and $L \in \mathbb{C}^{r \times (n-r)}$ satisfy

$$KK^* + LL^* = I_r.$$

The decomposition in (1) is known as the Hartwig-Spindelböck decomposition [12].

The B-T inverse of A was introduced by Baksalary and Trenkler [6, Definition 1], which is the Moore-Penrose of $A^2 A^\dagger$ and denoted by $A^\diamond$. One can see that the B-T inverse of A is an outer inverse of A ([6, Corollary]). The B-T inverse can be characterized by the Moore-Penrose inverse of ΣK ([6, Lemma 2]), that is the B-T inverse of A is

$$(2) \quad A^\diamond = U \begin{bmatrix} (\Sigma K)^\dagger & 0 \\ 0 & 0 \end{bmatrix} U^*.$$

Lemma 2.1 ([12, Corollary 6(a)]). *Let $A \in \mathbb{C}^{n \times n}$ be represented as in (1). Then A is group invertible if and only if K is nonsingular.*

Lemma 2.2 ([4, p2799 (1.4)]). *Let $A \in \mathbb{C}^{n \times n}$ be represented as in (1). Then*

$$(3) \quad A^\dagger = U \begin{bmatrix} K^* \Sigma^{-1} & 0 \\ L^* \Sigma^{-1} & 0 \end{bmatrix} U^*.$$

By Lemma 2.1 and Lemma 2.2, we have the following proposition.

Proposition 2.1. *Let $A \in \mathbb{C}^{n \times n}$ be represented as in (1) and be group invertible. Then*

$$(4) \quad A^\dagger = U \begin{bmatrix} (K^* K)(\Sigma K)^{-1} & 0 \\ (L^* K)(\Sigma K)^{-1} & 0 \end{bmatrix} U^*.$$

Proof. By Lemma 2.2, we have

$$(5) \quad A^\dagger = U \begin{bmatrix} K^* \Sigma^{-1} & 0 \\ L^* \Sigma^{-1} & 0 \end{bmatrix} U^*.$$

The condition A is group invertible gives K is nonsingular by Lemma 2.1, thus

$$\begin{aligned} A^\dagger &= U \begin{bmatrix} K^* \Sigma^{-1} & 0 \\ L^* \Sigma^{-1} & 0 \end{bmatrix} U^* = U \begin{bmatrix} K^* K K^{-1} \Sigma^{-1} & 0 \\ L^* K K^{-1} \Sigma^{-1} & 0 \end{bmatrix} U^* \\ (6) \quad &= U \begin{bmatrix} (K^* K)(K^{-1} \Sigma^{-1}) & 0 \\ (L^* K)(K^{-1} \Sigma^{-1}) & 0 \end{bmatrix} U^* \\ &= U \begin{bmatrix} (K^* K)(\Sigma K)^{-1} & 0 \\ (L^* K)(\Sigma K)^{-1} & 0 \end{bmatrix} U^*. \end{aligned}$$

□

There are three generalizations of the Proposition 2.1, that is, the CMP inverse, MPWC inverse and MPCEP inverse.

Lemma 2.3 ([18, p.3 (7)]). *Let $A \in \mathbb{C}^{n \times n}$ be represented as in (1) with $\text{ind}(A) = k$. Then*

$$(7) \quad A^{c,\dagger} = U \begin{bmatrix} (K^*K)(\Sigma K)^D & 0 \\ (L^*K)(\Sigma K)^D & 0 \end{bmatrix} U^*.$$

Lemma 2.4 ([15, Lemma 3.1]). *Let $A \in \mathbb{C}^{n \times n}$ be represented as in (1) with $\text{ind}(A) = k$. Then*

$$(8) \quad A^\circ = U \begin{bmatrix} (K^*K)(\Sigma K)^\circ & 0 \\ (L^*K)(\Sigma K)^\circ & 0 \end{bmatrix} U^*.$$

Lemma 2.5 ([30, Theorem 3.2]). *Let $A \in \mathbb{C}^{n \times n}$ be represented as in (1) with $\text{ind}(A) = k$. Then*

$$(9) \quad A^{\dagger,\oplus} = U \begin{bmatrix} (K^*K)(\Sigma K)^\oplus & 0 \\ (L^*K)(\Sigma K)^\oplus & 0 \end{bmatrix} U^*.$$

Lemma 2.6 ([28, Lemma 3.3]). *Let $A \in \mathbb{C}^{n \times n}$ be represented as in (1). Then, $\text{ind}(A) = k$ if and only if $\text{ind}(\Sigma K) = k - 1$.*

In Proposition 2.1, Lemma 2.3 and Lemma 2.4, we present that the Moore-Penrose inverse, CMP inverse and MPWC inverse can relate the inverse, Drazin inverse and weak core inverse of ΣK , respectively. In general, the group inverse and Moore-Penrose inverse are two classical generalized inverses, there are two matrix inverses can relate the group inverse and Moore-Penrose inverse of ΣK , respectively. For the matrix inverse relate the group inverse of ΣK , we can prove this matrix inverse is a very special inverse, which can be showed as follows:

Let

$$(10) \quad X = U \begin{bmatrix} (K^*K)(\Sigma K)^\# & 0 \\ (L^*K)(\Sigma K)^\# & 0 \end{bmatrix} U^*,$$

which says that ΣK is group invertible, then $\text{ind}(\Sigma K) = 0$ by Σ and K are nonsingular, thus $\text{ind}(A) = 1$ by Lemma 2.6. Then, (10) can be written as

$$(11) \quad \begin{aligned} U \begin{bmatrix} (K^*K)(\Sigma K)^\# & 0 \\ (L^*K)(\Sigma K)^\# & 0 \end{bmatrix} U^* &= U \begin{bmatrix} (K^*K)(\Sigma K)^{-1} & 0 \\ (L^*K)(\Sigma K)^{-1} & 0 \end{bmatrix} U^* \\ &= U \begin{bmatrix} (K^*K)(K^{-1}\Sigma^{-1}) & 0 \\ (L^*K)(K^{-1}\Sigma^{-1}) & 0 \end{bmatrix} U^* \\ &= U \begin{bmatrix} K^*KK^{-1}\Sigma^{-1} & 0 \\ L^*KK^{-1}\Sigma^{-1} & 0 \end{bmatrix} U^* \\ &= U \begin{bmatrix} K^*\Sigma^{-1} & 0 \\ L^*\Sigma^{-1} & 0 \end{bmatrix} U^* \\ &= A^\dagger. \end{aligned}$$

In a similar way as in (11), for the core inverse of A , we have

$$\begin{aligned}
 (12) \quad U \begin{bmatrix} (K^*K)(\Sigma K)^{\oplus} & 0 \\ (L^*K)(\Sigma K)^{\oplus} & 0 \end{bmatrix} U^* &= U \begin{bmatrix} (K^*K)(\Sigma K)^{-1} & 0 \\ (L^*K)(\Sigma K)^{-1} & 0 \end{bmatrix} U^* \\
 &= U \begin{bmatrix} K^*\Sigma^{-1} & 0 \\ L^*\Sigma^{-1} & 0 \end{bmatrix} U^* \\
 &= A^\dagger.
 \end{aligned}$$

Lemma 2.7. *Let $A \in \mathbb{C}^{n \times n}$ and be group invertible. If A has the Hartwig-Spindelböck decomposition as given in (1), then*

$$\begin{aligned}
 (13) \quad A^\dagger &= U \begin{bmatrix} (K^*K)(\Sigma K)^{-1} & 0 \\ (L^*K)(\Sigma K)^{-1} & 0 \end{bmatrix} U^* \\
 &= U \begin{bmatrix} (K^*K)(\Sigma K)^\# & 0 \\ (L^*K)(\Sigma K)^\# & 0 \end{bmatrix} U^* \\
 &= U \begin{bmatrix} (K^*K)(\Sigma K)^{\oplus} & 0 \\ (L^*K)(\Sigma K)^{\oplus} & 0 \end{bmatrix} U^*.
 \end{aligned}$$

Proof. It is trivial by Proposition 2.1, equalities (11) and (12). $\square$

Let $A \in \mathbb{C}^{n \times n}$ with $\text{ind}(A) = 2$ and has the Hartwig-Spindelböck decomposition as given in (1). Then, $\text{ind}(A) = 2$ if and only if $\text{ind}(\Sigma K) = 1$ by Lemma 2.6.

In the following example, we will show that $U \begin{bmatrix} (K^*K)(\Sigma K)^\# & 0 \\ (L^*K)(\Sigma K)^\# & 0 \end{bmatrix} U^*$ is different from the $U \begin{bmatrix} (K^*K)(\Sigma K)^{\oplus} & 0 \\ (L^*K)(\Sigma K)^{\oplus} & 0 \end{bmatrix} U^*$.

Example 2.1. Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & \frac{1}{2} \end{bmatrix} \in \mathbb{C}^{3 \times 3}$, then it is easy to check that $\text{ind}(A) = 2$. The Singular Value Decomposition of A is

$$A = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^*,$$

where unitary maytix $U = \begin{bmatrix} -\frac{519}{1025} & \frac{1210}{1593} & -\frac{881}{2158} \\ -\frac{452}{703} & -\frac{197}{304} & -\frac{881}{2158} \\ -\frac{739}{1286} & \frac{270}{4841} & \frac{881}{1079} \end{bmatrix}$, $\Sigma = \begin{bmatrix} \frac{715}{274} & 0 \\ 0 & \frac{683}{1029} \end{bmatrix}$ and

unitary maytix $V = \begin{bmatrix} -\frac{519}{1025} & \frac{1210}{1593} & -\frac{881}{2158} \\ -\frac{452}{703} & -\frac{197}{304} & -\frac{881}{2158} \\ -\frac{739}{1286} & \frac{270}{4841} & \frac{881}{1079} \end{bmatrix}$. By

$$V^*U = \begin{bmatrix} \frac{242}{251} & -\frac{118}{1261} & \frac{410}{1651} \\ \frac{153}{619} & -\frac{244}{10171} & -\frac{402}{415} \\ -\frac{327}{3385} & -\frac{1064}{1069} & 0 \end{bmatrix} \triangleq \begin{bmatrix} K & L \\ M & N \end{bmatrix},$$

where $K = \begin{bmatrix} \frac{242}{251} & -\frac{118}{1261} \\ \frac{133}{619} & -\frac{244}{10171} \end{bmatrix}$, $L = \begin{bmatrix} \frac{410}{1651} \\ -\frac{402}{415} \end{bmatrix}$, $M = \begin{bmatrix} -\frac{327}{3385} & -\frac{1064}{1069} \end{bmatrix}$, $N = 0$. It is easy to check that

$$(\Sigma K)^{\#} = \begin{bmatrix} \frac{316}{785} & -\frac{247}{6322} \\ \frac{178}{6781} & -\frac{61}{23943} \end{bmatrix} \text{ and } (\Sigma K)^{\oplus} = \begin{bmatrix} \frac{2775}{6967} & \frac{421}{16209} \\ \frac{421}{16209} & \frac{16209}{31883} \end{bmatrix}.$$

Thus,

$$U \begin{bmatrix} (K^*K)(\Sigma K)^{\#} & 0 \\ (L^*K)(\Sigma K)^{\#} & 0 \end{bmatrix} U^* = \begin{bmatrix} \frac{195}{1463} & \frac{933}{6998} & \frac{323}{2423} \\ \frac{933}{6998} & \frac{181}{1357} & \frac{2423}{3352} \\ \frac{323}{2423} & \frac{1357}{3352} & \frac{17393}{17393} \end{bmatrix};$$

$$U \begin{bmatrix} (K^*K)(\Sigma K)^{\oplus} & 0 \\ (L^*K)(\Sigma K)^{\oplus} & 0 \end{bmatrix} U^* = \begin{bmatrix} \frac{8}{77} & \frac{12}{77} & \frac{10}{77} \\ \frac{12621}{121477} & \frac{12}{77} & \frac{10}{77} \\ \frac{8}{77} & \frac{12}{77} & \frac{10}{77} \end{bmatrix},$$

which says that the matrix inverse $U \begin{bmatrix} (K^*K)(\Sigma K)^{\#} & 0 \\ (L^*K)(\Sigma K)^{\#} & 0 \end{bmatrix} U^*$ is different from the matrix inverse $U \begin{bmatrix} (K^*K)(\Sigma K)^{\oplus} & 0 \\ (L^*K)(\Sigma K)^{\oplus} & 0 \end{bmatrix} U^*$.

Let $A \in \mathbb{C}^{n \times n}$ with $\text{ind}(A) = 2$ and has the Hartwig-Spindelböck decomposition as given in (1). In the following proposition, we will show that if ΣK is an EP matrix, then the matrix inverse $U \begin{bmatrix} (K^*K)(\Sigma K)^{\#} & 0 \\ (L^*K)(\Sigma K)^{\#} & 0 \end{bmatrix} U^*$ coincides with the matrix inverse $U \begin{bmatrix} (K^*K)(\Sigma K)^{\oplus} & 0 \\ (L^*K)(\Sigma K)^{\oplus} & 0 \end{bmatrix} U^*$.

Proposition 2.2. *Let $A \in \mathbb{C}^{n \times n}$ with $\text{ind}(A) = 2$ and has the Hartwig-Spindelböck decomposition as given in (1). If ΣK is an EP matrix, then the matrix inverse $U \begin{bmatrix} (K^*K)(\Sigma K)^{\#} & 0 \\ (L^*K)(\Sigma K)^{\#} & 0 \end{bmatrix} U^*$ coincides with the matrix inverse*

$$U \begin{bmatrix} (K^*K)(\Sigma K)^{\oplus} & 0 \\ (L^*K)(\Sigma K)^{\oplus} & 0 \end{bmatrix} U^*.$$

Proof. It is trivial by [22, Theorem 3.1]. $\square$

Let $A \in \mathbb{C}^{n \times n}$. A generalized inverse relate the Moore-Penrose inverse of ΣK was introduced by using the the Moore-Penrose inverse and B-T inverse of A , named the MPBT inverse of A . Let $A \in \mathbb{C}^{n \times n}$ be represented as in (1). We will prove that the formula of the MPBT inverse of A is

$$(14) \quad X = U \begin{bmatrix} (K^*K)(\Sigma K)^{\dagger} & 0 \\ (L^*K)(\Sigma K)^{\dagger} & 0 \end{bmatrix} U^*.$$

Note that the formula in (14) can be found in Theorem 3.2.

3. The matrix inverse based on the Moore-Penrose inverse and B-T inverse

Motivated by the definition of the MPWC inverse [15], we introduce the MPBT inverse by using the Moore-Penrose inverse and B-T inverse.

Let $A_1^\diamond = AA^\diamond A$, where $A^\diamond$ is the B-T inverse of A .

Theorem 3.1. *Let $A \in \mathbb{C}^{n \times n}$. The matrix $X = A^\dagger A_1^\diamond A^\dagger$ is the unique matrix that satisfies the following system of equations*

$$(15) \quad XAX = X, \quad XA = A^\dagger A_1^\diamond \quad \text{and} \quad AX = A_1^\diamond A^\dagger.$$

Proof. Let $X = A^\dagger A_1^\diamond A^\dagger$. Note that $A^\diamond AA^\diamond = A^\diamond$ by [6, Corollary 1]. Then

$$\begin{aligned} XAX &= A^\dagger A_1^\diamond A^\dagger AA^\dagger A_1^\diamond A^\dagger = A^\dagger A_1^\diamond A^\dagger A_1^\diamond A^\dagger = A^\dagger AA^\diamond AA^\dagger AA^\diamond AA^\dagger \\ &= A^\dagger AA^\diamond AA^\diamond AA^\dagger = A^\dagger AA^\diamond AA^\dagger = A^\dagger A_1^\diamond A^\dagger = X; \\ XA &= A^\dagger A_1^\diamond A^\dagger A = A^\dagger AA^\diamond AA^\dagger A = A^\dagger AA^\diamond A = A^\dagger A_1^\diamond; \\ AX &= AA^\dagger A_1^\diamond A^\dagger = AA^\dagger AA^\diamond AA^\dagger = AA^\diamond AA^\dagger = A_1^\diamond A^\dagger, \end{aligned}$$

which says that X is a solution of system (15). Let X_1 and X_2 are two candidates of system (15), then

$$X_1 = X_1AX_1 = X_1A_1^\diamond A^\dagger = X_1AX_2 = A^\dagger A_1^\diamond X_2 = X_2AX_2 = X_2,$$

thus X is unique. □

Definition 3.1. *Let $A \in \mathbb{C}^{n \times n}$. The solution of the system (15) is called the Moore-Penrose B-T inverse, the MPBT inverse is short for the Moore-Penrose B-T inverse, denoted by $A^{\dagger, \diamond}$.*

Example 3.1. In general, the MPBT inverse is different from the MPWC inverse, CMP inverse, MPCEP inverse and Moore-Penrose inverse. Let $A =$

$$\begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 3 & 2 & 3 \\ 0 & 3 & 3 & 3 \end{bmatrix} \in \mathbb{C}^{4 \times 4}. \text{ Then}$$

$$A^{\dagger, \diamond} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & -\frac{3}{62} & \frac{15}{62} & \frac{9}{31} \\ 0 & -\frac{1}{62} & \frac{5}{62} & \frac{3}{31} \\ 1 & \frac{1}{31} & -\frac{5}{31} & -\frac{6}{31} \end{bmatrix},$$

however

$$A^\circ = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{163}{13545} & -\frac{10}{387} & -\frac{187}{13545} \\ 0 & \frac{163}{2709} & \frac{50}{387} & \frac{187}{2709} \\ 0 & \frac{326}{4515} & \frac{20}{129} & \frac{374}{4515} \end{bmatrix}, A^{c,\dagger} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{75} & -\frac{2}{75} & -\frac{1}{75} \\ 0 & \frac{1}{15} & \frac{2}{15} & \frac{1}{15} \\ 0 & \frac{2}{25} & \frac{4}{25} & \frac{2}{25} \end{bmatrix},$$

$$A^{\dagger,\oplus} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\frac{8}{903} & \frac{5}{1806} & -\frac{25}{1806} & -\frac{5}{301} \\ \frac{40}{903} & -\frac{25}{1806} & \frac{125}{1806} & \frac{25}{301} \\ \frac{16}{301} & -\frac{5}{301} & \frac{25}{301} & \frac{30}{301} \end{bmatrix}, A^\dagger = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & -\frac{7}{9} & -\frac{2}{9} & \frac{5}{9} \\ 0 & -\frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\ 1 & \frac{4}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}.$$

Theorem 3.2. Let $A \in \mathbb{C}^{n \times n}$. If A has the Hartwig-Spindelböck decomposition as given in (1), then

$$(16) \quad A^{\dagger,\diamond} = U \begin{bmatrix} (K^*K)(\Sigma K)^\dagger & 0 \\ (L^*K)(\Sigma K)^\dagger & 0 \end{bmatrix} U^*.$$

Proof. By Theorem 3.1, we have $A^\dagger A_1^\diamond A^\dagger$ is the MPBT inverse of A , thus

$$\begin{aligned} A^\dagger A_1^\diamond A^\dagger &= A^\dagger A A^\diamond A A^\dagger \\ &= U \begin{bmatrix} K^* \Sigma^{-1} & 0 \\ L^* \Sigma^{-1} & 0 \end{bmatrix} U^* U \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} U^* U \begin{bmatrix} (\Sigma K)^\dagger & 0 \\ 0 & 0 \end{bmatrix} U^* \\ &\quad U \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} U^* U \begin{bmatrix} K^* \Sigma^{-1} & 0 \\ L^* \Sigma^{-1} & 0 \end{bmatrix} U^* \\ &= U \begin{bmatrix} (K^*K)(\Sigma K)^\dagger & 0 \\ (L^*K)(\Sigma K)^\dagger & 0 \end{bmatrix} U^*. \quad \square \end{aligned}$$

Let $A, B, C \in \mathbb{C}^{n \times n}$. We say that $Y \in \mathbb{C}^{n \times n}$ is a (B, C) -inverse of A if we have $YAB = B$, $CAY = C$, $\mathcal{N}(C) \subseteq \mathcal{N}(Y)$ and $\mathcal{R}(Y) \subseteq \mathcal{R}(B)$. If such Y exists, then it is unique (see, [1, Definition 4.1] and [21, Definition 1.2]), we also call the (B, C) -inverse of A is the inverse of A along B and C . Note that the (B, C) -inverse was introduced in the setting of semigroups [10]. The (B, C) -inverse of A will be denoted by $A^{\parallel(B,C)}$. Note that Bapat et al. investigated an outer inverse in [2, Theorem 5] is exactly the same as the (y, x) -inverse, where x and y are element in a semigroup. In [23], Rao and Mitra showed that $A^{\parallel(B,C)} = B(CAB)^-C$, where $(CAB)^-$ stands for arbitrary inner inverse of CAB .

Theorem 3.3. Let $A \in \mathbb{C}^{n \times n}$. The MPBT inverse of A is the $(A^\dagger A_1^\diamond, A_1^\diamond A^\dagger)$ -inverse of A .

Proof. Let $Y = A^\dagger A_1^\diamond A^\dagger$ be the MPBT inverse of A . Then

$$\begin{aligned} YA(A^\dagger A_1^\diamond) &= A^\dagger A_1^\diamond A^\dagger A A^\dagger A_1^\diamond = A^\dagger A_1^\diamond A^\dagger A_1^\diamond = A^\dagger A A^\diamond A A^\dagger A A^\diamond A \\ &= A^\dagger A A^\diamond A A^\diamond A = A^\dagger A A^\diamond A = A^\dagger A_1^\diamond; \\ (A_1^\diamond A^\dagger)AY &= A_1^\diamond A^\dagger A A^\dagger A_1^\diamond A^\dagger = A_1^\diamond A^\dagger A_1^\diamond A^\dagger = A A^\diamond A A^\dagger A A^\diamond A A^\dagger \\ &= A A^\diamond A A^\diamond A A^\dagger = A A^\diamond A A^\dagger = A_1^\diamond A^\dagger. \end{aligned}$$

For any $u \in \mathcal{N}(A_1^\diamond A^\dagger)$, then we have $Yu = A^\dagger A_1^\diamond A^\dagger u = A^\dagger (A_1^\diamond A^\dagger u) = 0$, which gives $\mathcal{N}(A_1^\diamond A^\dagger) \subseteq \mathcal{N}(Y)$. $\mathcal{R}(Y) \subseteq \mathcal{R}(A^\dagger A_1^\diamond)$ is trivial. Thus, the $(A^\dagger A_1^\diamond, A_1^\diamond A^\dagger)$ -inverse of A by the definition of the (B, C) -inverse. $\square$

In the following proposition, we will give some properties of the MPBT inverse and the matrix $A_1^\diamond$.

Proposition 3.1. *Let $A \in \mathbb{C}^{n \times n}$, $A_1^\diamond = AA^\diamond A$ and $A^{\dagger, \diamond}$ be the MPBT inverse of A , where $A^\diamond$ is the B - T inverse of A . If A has the Hartwig-Spindelböck decomposition as given in(1), then*

$$(1) \quad AA^{\dagger, \diamond} = AA^\diamond = A_1^\diamond A^\dagger = U \begin{bmatrix} (\Sigma K)(\Sigma K)^\dagger & 0 \\ 0 & 0 \end{bmatrix} U^*;$$

$$(2) \quad A^{\dagger, \diamond} A = A^\dagger A_1^\diamond = U \begin{bmatrix} K^* K & K^* K(\Sigma K)^\dagger \Sigma L \\ L^* K & L^* K(\Sigma K)^\dagger \Sigma L \end{bmatrix} U^*;$$

$$(3) \quad A_1^\diamond A^\dagger = (A_1^\diamond A^\dagger)^2 = (A_1^\diamond A^\dagger)^*;$$

$$(4) \quad A^\dagger A_1^\diamond = (A^\dagger A_1^\diamond)^2;$$

$$(5) \quad A_1^\diamond A^\dagger A_1^\diamond = A_1^\diamond;$$

$$(6) \quad \mathcal{R}(A^\dagger A_1^\diamond) = \mathcal{R}(A^\dagger AA^\diamond) = \mathcal{R}(A^\dagger A^2(A^\dagger)^2);$$

$$(7) \quad \mathcal{N}(A_1^\diamond A^\dagger) = \mathcal{N}(A^\diamond AA^\dagger) = \mathcal{N}(AA^\diamond).$$

Proof. (1). By Theorem 3.2 and equality (2), note that $KK^* + LL^* = I_r$, then we have

$$\begin{aligned} AA^{\dagger, \diamond} &= U \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} U^* U \begin{bmatrix} (K^* K)(\Sigma K)^\dagger & 0 \\ (L^* K)(\Sigma K)^\dagger & 0 \end{bmatrix} U^* \\ &= U \begin{bmatrix} \Sigma K(K^* K)(\Sigma K)^\dagger + \Sigma L(L^* K)(\Sigma K)^\dagger & 0 \\ 0 & 0 \end{bmatrix} U^* \\ &= U \begin{bmatrix} \Sigma(KK^* + LL^*)K(\Sigma K)^\dagger & 0 \\ 0 & 0 \end{bmatrix} U^* \\ &= U \begin{bmatrix} (\Sigma K)(\Sigma K)^\dagger & 0 \\ 0 & 0 \end{bmatrix} U^*; \\ AA^\diamond &= U \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} U^* U \begin{bmatrix} (\Sigma K)^\dagger & 0 \\ 0 & 0 \end{bmatrix} U^* = U \begin{bmatrix} (\Sigma K)(\Sigma K)^\dagger & 0 \\ 0 & 0 \end{bmatrix} U^*. \end{aligned}$$

Since $AA^{\dagger, \diamond} = AA^\dagger A_1^\diamond A^\dagger = AA^\dagger AA^\diamond AA^\dagger AA^\diamond AA^\dagger = A^\dagger A_1^\diamond$, so

$$A_1^\diamond A^\dagger = U \begin{bmatrix} (\Sigma K)(\Sigma K)^\dagger & 0 \\ 0 & 0 \end{bmatrix} U^*.$$

(2). By Theorem 3.2, we have

$$\begin{aligned}
A^{\dagger, \diamond} A &= U \begin{bmatrix} (K^* K)(\Sigma K)^{\dagger} & 0 \\ (L^* K)(\Sigma K)^{\dagger} & 0 \end{bmatrix} U^* U \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} U^* \\
&= U \begin{bmatrix} (K^* K)(\Sigma K)^{\dagger} \Sigma K & (K^* K)(\Sigma K)^{\dagger} \Sigma L \\ (L^* K)(\Sigma K)^{\dagger} \Sigma K & (L^* K)(\Sigma K)^{\dagger} \Sigma L \end{bmatrix} U^* \\
&= U \begin{bmatrix} (K^* \Sigma^{-1} \Sigma K)(\Sigma K)^{\dagger} \Sigma K & (K^* K)(\Sigma K)^{\dagger} \Sigma L \\ (L^* \Sigma^{-1} \Sigma K)(\Sigma K)^{\dagger} \Sigma K & (L^* K)(\Sigma K)^{\dagger} \Sigma L \end{bmatrix} U^* \\
&= U \begin{bmatrix} (K^* \Sigma^{-1}) \Sigma K (\Sigma K)^{\dagger} \Sigma K & (K^* K)(\Sigma K)^{\dagger} \Sigma L \\ (L^* \Sigma^{-1}) \Sigma K (\Sigma K)^{\dagger} \Sigma K & (L^* K)(\Sigma K)^{\dagger} \Sigma L \end{bmatrix} U^* \\
&= U \begin{bmatrix} (K^* \Sigma^{-1}) \Sigma K & (K^* K)(\Sigma K)^{\dagger} \Sigma L \\ (L^* \Sigma^{-1}) \Sigma K & (L^* K)(\Sigma K)^{\dagger} \Sigma L \end{bmatrix} U^* \\
&= U \begin{bmatrix} K^* K & K^* K (\Sigma K)^{\dagger} \Sigma L \\ L^* K & L^* K (\Sigma K)^{\dagger} \Sigma L \end{bmatrix} U^*.
\end{aligned}$$

Since $A^{\dagger, \diamond} A = A^{\dagger} A_1^{\diamond} A^{\dagger} A = A^{\dagger} A A^{\diamond} A A^{\dagger} A = A^{\dagger} A A^{\diamond} A = A_1^{\diamond} A^{\dagger}$, so

$$A^{\dagger} A_1^{\diamond} = U \begin{bmatrix} K^* K & K^* K (\Sigma K)^{\dagger} \Sigma L \\ L^* K & L^* K (\Sigma K)^{\dagger} \Sigma L \end{bmatrix} U^*.$$

(3) is trivial by item (1).

(4). By the definition of the MPBT inverse, we have $A^{\dagger, \diamond} A A^{\dagger, \diamond} = A^{\dagger, \diamond}$, then by item (2), we have

$$(A^{\dagger} A_1^{\diamond})^2 = (A^{\dagger, \diamond} A)^2 = A^{\dagger, \diamond} A A^{\dagger, \diamond} A = A^{\dagger} A_1^{\diamond}.$$

(5). $A_1^{\diamond} A^{\dagger} A_1^{\diamond} = A A^{\diamond} A A^{\dagger} A A^{\diamond} A = A A^{\diamond} A = A_1^{\diamond}$.

(6). The equality $\mathcal{R}(A^{\dagger} A_1^{\diamond}) = \mathcal{R}(A^{\dagger} A A^{\diamond})$ holds by $A_1^{\diamond} = A A^{\diamond} A$ and $A^{\diamond}$ is an outer inverse of A . Moreover,

$$\begin{aligned}
\mathcal{R}(A^{\dagger} A A^{\diamond}) &= \mathcal{R}(A^{\dagger} A (A^2 A^{\dagger})^{\dagger}) = \mathcal{R}(A^{\dagger} A (A^2 A^{\dagger})^*) = \mathcal{R}(A^{\dagger} A (A A^{\dagger})^* A^*) \\
&= \mathcal{R}(A^{\dagger} A^2 A^{\dagger} A^*) = \mathcal{R}(A^{\dagger} A^2 A^{\dagger} A^{\dagger}) = \mathcal{R}(A^{\dagger} A^2 (A^{\dagger})^2).
\end{aligned}$$

(7) is trivial by item (1) and $A^{\diamond}$ is an outer inverse of A . $\square$

Lemma 3.1 ([3, Remark 2.2 (i)]). *Let $A, B, C, U, V \in \mathbb{C}^{n \times n}$. If $\mathcal{R}(B) = \mathcal{R}(U)$ and $\mathcal{N}(C) = \mathcal{N}(V)$, then A is (B, C) -invertible if and only if A is (U, V) -invertible. In this case, we have $A^{\parallel(B, C)} = A^{\parallel(U, V)}$.*

Theorem 3.4. *Let $A \in \mathbb{C}^{n \times n}$. The MPBT inverse of A is the $(A^{\dagger} A A^{\diamond}, A^{\diamond} A A^{\dagger})$ -inverse of A .*

Proof. It is obvious by Theorem 3.3, Proposition 3.1 and Lemma 3.1. $\square$

Proposition 3.2. *Let $A \in \mathbb{C}^{n \times n}$, then $A_1^{\diamond} \in A^{\dagger} \{2, 4\}$.*

Proof. It is trivial by items (3) and (5) in Proposition 3.1. $\square$

In the next example, we will shows that $A^\dagger A_1^\diamond$ in Proposition 3.1 is not Hermitian and $A^\dagger A_1^\diamond A^\dagger \neq A^\dagger$ in general, that is $A_1^\diamond \notin A^\dagger\{1, 3\}$.

Example 3.2. Let $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 2 \\ 2 & 1 & 1 \end{bmatrix}$. It is easy to check that

$$A^\dagger = \begin{bmatrix} \frac{4}{21} & \frac{1}{21} & \frac{5}{21} \\ \frac{3}{14} & -\frac{1}{14} & \frac{1}{7} \\ -\frac{11}{42} & \frac{13}{42} & \frac{1}{21} \end{bmatrix},$$

$$\text{thus } A^\dagger A_1^\diamond = \begin{bmatrix} \frac{3}{7} & \frac{1}{7} & \frac{3}{7} \\ \frac{3}{28} & \frac{1}{28} & \frac{3}{28} \\ \frac{15}{28} & \frac{5}{28} & \frac{15}{28} \end{bmatrix}, \text{ which says } A^\dagger A_1^\diamond \neq (A^\dagger A_1^\diamond)^* \text{ and } A^\dagger A_1^\diamond A^\dagger = \begin{bmatrix} 0 & \frac{1}{28} & \frac{1}{28} \\ 0 & \frac{1}{28} & \frac{1}{28} \\ 0 & \frac{5}{28} & \frac{5}{28} \end{bmatrix} \neq A^\dagger.$$

In the following two tables, we will collect some facts on the MPBT inverse, MPD inverse, MPCEP inverse, MPWC inverse and MPCEP inverse.

Table 1: Formulae of the MPBT, MPD, CMP, MPWC and MPCEP inverses

generalized inverses	formulae	sources
MPBT inverse	$U \begin{bmatrix} (K^*K)(\Sigma K)^\dagger & 0 \\ (L^*K)(\Sigma K)^\dagger & 0 \end{bmatrix} U^*$	Theorem 3.2
MPD inverse	$U \begin{bmatrix} (K^*K)(\Sigma K)^D & K^*K((\Sigma K)^D)^2 \Sigma L \\ (L^*K)(\Sigma K)^D & L^*K((\Sigma K)^D)^2 \Sigma L \end{bmatrix} U^*$	[19, Remark 2.9]
CMP inverse	$U \begin{bmatrix} (K^*K)(\Sigma K)^D & 0 \\ (L^*K)(\Sigma K)^D & 0 \end{bmatrix} U^*$	Lemma 2.3
MPWC inverse	$U \begin{bmatrix} (K^*K)(\Sigma K)^\oplus & 0 \\ (L^*K)(\Sigma K)^\oplus & 0 \end{bmatrix} U^*$	Lemma 2.4
MPCEP inverse	$U \begin{bmatrix} (K^*K)(\Sigma K)^\oplus & 0 \\ (L^*K)(\Sigma K)^\oplus & 0 \end{bmatrix} U^*$	Lemma 2.5

Table 2: The MPBT, MPD, CMP, MPWC and MPCEP inverses

generalized inverses	column parts	null parts	sources
MPBT inverse	$A^\dagger A A^\diamond$	$A^\diamond A A^\dagger$	Theorem 3.4
MPD inverse	$A^\dagger A^k$	A^k	[11, Theorem 3.2]
CMP inverse	$A^\dagger A^k$	$A^k A^\dagger$	[11, Theorem 3.2]
MPWC inverse	$A^\dagger A^k$	$(A^k)^* A^2 A^\dagger$	[15, Theorem 2.7]
MPCEP inverse	$A^\dagger A^k$	$(A^k)^*$	[29, Theorem 3.11]

Theorem 3.5. *Let $A \in \mathbb{C}^{n \times n}$ with $\text{ind}(A) = k$. We have*

- (1) *If $\mathcal{N}((A^k)^*) = \mathcal{N}((A^k)^* A^2 A^\dagger)$, then $A^\circ = A^{\dagger, \oplus}$;*
- (2) *If $\mathcal{N}((A^k)^* A^2 A^\dagger) = \mathcal{N}(A^k A^\dagger)$, then $A^\circ = A^{c, \dagger}$.*

Proof. It is trivial by Lemma 3.1 and Table 1. $\square$

Corollary 3.1. *Let $A \in \mathbb{C}^{n \times n}$ with $\text{ind}(A) = 2$, then $A^\circ = A^{c, \dagger}$.*

Proof. For $\text{ind}(A)=2$, we have $\mathcal{N}((A^2)^* A^2 A^\dagger) = \mathcal{N}(A^2 (A^2)^\dagger A^2 A^\dagger) = \mathcal{N}(A^2 A^\dagger)$, then, the proof is finished by Theorem 3.5. $\square$

4. Conclusions

For a given complex matrix with a given index, one can get that the computation of the MPBT inverse by using the Hartwig-Spindelsböck decomposition of this matrix. The future perspectives for research are proposed:

Part 1. The MPBT inverse is one of the useful tools to investigate the Hartwig-Spindelsböck decomposition of a complex matrix.

Part 2. The rank properties of a given matrix, such as $\text{rank}(AA^{\dagger, \diamond} - A^{\dagger, \diamond}A)$.

Part 3. The relationships between different generalized inverses relate the generalized inverses of ΣK in Hartwig-Spindelsböck decomposition.

Data availability

All relevant data are within the paper.

Author Contributions

Writing-original draft preparation, Sanzhang Xu; writing-review and editing, Qingyuan Xu and Sanzhang Xu; methodology, Sanzhang Xu and Nan Zhou; supervision, Qingyuan Xu and Nan Zhou.

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Application of new distance and entropy measures for probabilistic dual hesitant fuzzy sets in pattern recognition and VIKOR-based multi-criteria decision-making

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Abstract. Probabilistic dual hesitant fuzzy sets (PDHFSs) are constantly used to handle uncertainty problems. However, measures associated with PDHFSs have yet to be widely studied and may have limitations in application. In this paper, a weak equality is suggested to facilitate the conditional reflexivity of distance measures. Then, the consistency, probabilistic fuzziness and Hausdorff distances are defined to integrate different information in PDHFSs. Based on the above definitions, distance and similarity measures for PDHFSs that are not limited by the number of elements are presented. A simple example indicates that new distance measures can effectively differentiate between PDHFSs. Furthermore, a new entropy measure is built for PDHFSs by combining the sine function. Finally, the practicality and feasibility of development methods are demonstrated through two practical examples: pattern recognition and VIKOR-based multi-criteria decision-making (MCDM). Moreover, parameter analyses and comparative analyses indicate that these measures are superior to their existing counterparts.

Keywords: weak equality, distance measure, entropy measure, pattern recognition, multi-criteria decision-making, probabilistic dual hesitant fuzzy sets.

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1. Introduction

Solving fuzzy phenomena and uncertain events in real life is necessary as science and technology advance. Uncertainty theory brings usefulness and convenience to people's lives and is interconnected with many knowledge systems [17, 20,

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27]. Zadeh [36] explicitly and systematically proposed fuzzy sets (FSs) to solve ambiguity problems. Atanassov described the meanings of non-membership degree and hesitation degree, leading to the development of intuitionistic fuzzy sets (IFSs) [2]. Torra [29] changed the decision rule by hesitant fuzzy sets (HFSs), which permits decision makers (DMs) to take multiple potential values on interval $[0, 1]$. HFSs describe the hesitation of DMs in evaluation. Then, a series of aggregation operators for HFSs that could aggregate the values under each attribute were developed [30]. Alcantud [1] introduced a novel HFS with an inclusive ranking that obeys strict sorting in the stratified assessment. Further, Zhu et al. [41] proposed dual hesitant fuzzy sets (DHFSs), where FSs and IFSs were regarded as particular components of DHFSs. Some scholars systematically explained the operations and basic rules of DHFSs [40, 38, 26]. Xu and Zhou [33] considered the influence of DMs' knowledge level and subjective factors and introduced probability into HFSs to define the probability hesitation fuzzy sets (PHFSs). Krishankumaar [11] proposed a new ranking approach for PHFSs, which made the method more flexible by adding evidence theory. Hao [8] used PDHFSs to translate vague information more appropriately and then studied visualization techniques for PDHFSs to deal with risk evaluation problems. It is worth noting that PDHFSs comprehensively express the fuzziness and hesitancy of human thinking and the importance of different elements.

Measure methods were significant in dealing with uncertainty problems, such as decision analysis, pattern recognition and artificial intelligence [12, 23, 39, 24]. Xu and Xia [31] developed some distance measures to enrich the study of HFSs [32]. Considering DMs' preference, Tong and Yu [28] proposed a new distance measure for HFS to address the defects of measure in [31]. However, when constructing the above measure, assuming that the number of hesitant fuzzy elements isn't equal, DMs needed to increase elements to make them equal. This approach may violate the authenticity of data. Hussain and Yang [9] characterized the connection between HFSs and their complement to construct a new entropy measure. This measure relied on particular data values and had a relatively narrow range of applications. Yang and Hussain [34] complemented the existing measures of HFSs, simplifying the calculation. Farhadinia and Xu [6] extended the Euclidean distance between single-point sets to HFSs and defined a new distance measure. For the problem of unequal-length for elements, Chen et al. [3] described some new distance measures. Zhang [37] also constructed several distance and entropy measures suitable for pattern recognition. To a certain extent, these measures [3, 37] maintained the authenticity of data but did not have triangular inequality property. Su et al. [25] proposed a new entropy called membership entropy based on exponential functions. It provides an innovative method for measuring the uncertainty of PHFSs. In [16] and [15], the characteristics of PHFSs were comprehensively considered and a distance measure was developed. In addition, considering the influence of psychological changes when DMs encounter risk, a TODIM-based multi-criteria decision-making (MCDM)

method was proposed [16]. Fang [5] proposed a series of entropy and cross-entropy with strong discrimination ability and wide application.

PDHFSs have apparent advantages in presenting fuzzy information, and their research value is evident. In [21], the equiprobability distance measure was also an available metric for PDHFSs, but it changed the original structure of PDHFSs and weakened the psychological factor of DMs. Garg and Kau [7] successfully extended distance measures and then brought rationality and loss of DMs into the aggregation operators, underlining the contribution of probability to PDHFSs. Ning et al. [18] proposed some distance measures for PDHFSs from the perspective of averaging and dispersion. However, the study [18] has a defect: the measures cannot distinguish different PDHFSs in exceptional cases, which violates conditional reflexivity. Ning et al. [19] also built some new distance measures that considered the discrete, continuous, ordered and disordered conditions of PDHFSs. Such distances are novel in form but do not satisfy triangular inequality. It is readily apparent from the above analyses that there are few studies and shortcomings on the relevant measures of PDHFS at present, and further studies are needed.

In order to enhance the conditional reflexivity of distance measures, we define weak equality. According to several properties of PDHFSs, distance measures with parameters are constructed. Moreover, a new entropy measure is proposed by combining the sine function. The proposed methods are applied in pattern recognition and VIKOR-based MCDM methods. The contributions of this paper are as follows:

- (1) A weak equivalence between PDHFSs is defined to enhance the conditional reflexivity of distance measures.
- (2) Assuming that all values in probabilistic dual hesitant fuzzy elements (PDHFEs) are in increasing order, we combine the consistency and probabilistic fuzziness of PDHFEs to obtain some distance measures with parameters. New distance measures are more discriminating between PDHFSs. They are suitable for the axioms of distance measure, especially the triangular inequality.
- (3) A new entropy measure with an auxiliary function is presented, enriching the research of entropy measures in the PDHFS environment.

The rest of this paper is arranged as follows: Section 2 introduces the concepts related to with PDHFS and existing research. Section 3 proposes some new distance and similarity measures and discusses their properties. Section 4 defines the entropy measure of PDHFS in combination with the sine function. Section 5 applies the above distance, similarity and entropy measures to two examples. Finally, the conclusion and expectations are in Section 6.

2. Preliminaries

2.1 HFSs and PDHFSs

In this subsection, we introduce the concept of HFSs and PDHFSs and some existing outcomes. The notations used in this section are explained in Table 1.

Table 1: Notations interpretation I

Notations	Details	Notations	Details
$\hat{O}, M$	PDHFSs	$\hat{\partial}$	PDHFE
$\mathcal{G}(x)$	membership degree	$\kappa(x)$	non-membership degree
$p(x)$	membership probabilistic	$q(x)$	non-membership probabilistic
γ, η	possible elements	$\#\mathcal{G}, \#\kappa$	number of elements
$\lambda(\hat{\partial}), \lambda(\hat{O}(x))$	score function	$\ell(\hat{\partial}), \ell(\hat{O}(x))$	accuracy function
$\wp(\hat{\partial}), \wp(\hat{O}(x_i))$	membership means	$\Im(\hat{\partial}), \Im(\hat{O}(x_i))$	non-membership means
$\Phi(\hat{O}(x_i))$	membership standard deviation	$\Psi(\hat{O}(x_i))$	non-membership standard deviation
$\wp(x_i), \Im(x_i)$	means distance	$\Phi(x_i), \Psi(x_i)$	standard deviation distance

Let X be a universal set, $\mathcal{F} = \{\langle x, \hbar_{\mathcal{F}}(x) \rangle : x \in X\}$ is called an HFS, where $\hbar_{\mathcal{F}}(x) \in [0, 1]$ are hesitant fuzzy elements, representing the set of possible memberships of x of X [29].

Definition 1 (Hao et al. [8]). *defined PDHFS $\hat{O}$ on X as:*

$$(1) \quad \hat{O} = \{\langle \mathcal{G}(x)|p(x), \kappa(x)|q(x) \rangle | x \in X\},$$

where $\mathcal{G}(x)$ and $\kappa(x)$ are the the membership and non-membership degrees of $x \in X$, respectively. The probabilities of $\mathcal{G}(x)$ and $\kappa(x)$ are $p(x)$ and $q(x)$ separately. Also, there is

$$(2) \quad \gamma \in [0, 1], \eta \in [0, 1], \gamma^+ + \eta^+ \in [0, 1],$$

$$p_j \in [0, 1], q_j \in [0, 1], \sum_{j=1}^{\#\mathcal{G}} p_j = 1, \sum_{j=1}^{\#\kappa} q_j = 1,$$

where γ, η are some possible elements, $\gamma \in \mathcal{G}(x), \eta \in \kappa(x)$, $\gamma^+ \in \mathcal{G}^+(x) = \bigcup_{\gamma \in \mathcal{G}(x)} \max\{\gamma\}$, $\eta^+ \in \kappa^+(x) = \bigcup_{\eta \in \kappa(x)} \max\{\eta\}$, $p_j \in p(x)$ and $q_j \in q(x)$. $\#\mathcal{G}$ and $\#\kappa$ are the total number of elements in $\mathcal{G}(x)|p(x)$ and $\kappa(x)|q(x)$.

For convenience, the pair $\hat{\partial} = \langle \mathcal{G}(x)|p(x), \kappa(x)|q(x) \rangle$ is called as PDHFE, denoted by $\hat{\partial} = \langle \mathcal{G}|p, \kappa|q \rangle$ ([8]).

Definition 2 ([8]). *The complement definition of the PDHFE $\hat{\partial} = \langle \mathcal{G}|p, \kappa|q \rangle$ is as follows:*

$$(3) \quad \hat{\partial}^c = \bigcup_{\eta \in \kappa, \gamma \in \mathcal{G}} \langle \eta|q, \gamma|p \rangle, \text{ if } \kappa \neq \emptyset \text{ and } \mathcal{G} \neq \emptyset.$$

Definition 3 ([8]). *The score function of PDHFE $\hat{\partial}$ is:*

$$(4) \quad \lambda(\hat{\partial}) = \sum_{j=1, \gamma \in \mathcal{G}}^{\#\mathcal{G}} \gamma_j p_j - \sum_{j=1, \eta \in \kappa}^{\#\kappa} \eta_j q_j.$$

Definition 4 ([33]). *The accuracy function of PDHFE $\hat{\partial}$ is:*

$$(5) \quad \ell(\hat{\partial}) = \sum_{j=1, \gamma \in \mathcal{G}}^{\#\mathcal{G}} \gamma_j p_j + \sum_{j=1, \eta \in \kappa}^{\#\kappa} \eta_j q_j.$$

Let $\hat{\partial}_1, \hat{\partial}_2$ be two PDHFEs. The rules for comparing are as follows [33]:

If $\lambda(\hat{\partial}_1) > \lambda(\hat{\partial}_2)$, then $\hat{\partial}_1 > \hat{\partial}_2$. If $\lambda(\hat{\partial}_1) < \lambda(\hat{\partial}_2)$, then $\hat{\partial}_1 < \hat{\partial}_2$.

If $\lambda(\hat{\partial}_1) = \lambda(\hat{\partial}_2)$, there are the following rules:

- (1) If $\ell(\hat{\partial}_1) > \ell(\hat{\partial}_2)$, then $\hat{\partial}_1 > \hat{\partial}_2$;
- (2) If $\ell(\hat{\partial}_1) < \ell(\hat{\partial}_2)$, then $\hat{\partial}_1 < \hat{\partial}_2$;
- (3) If $\ell(\hat{\partial}_1) = \ell(\hat{\partial}_2)$, then $\hat{\partial}_1 \sim \hat{\partial}_2$.

Definition 5 ([18]). *Let $\hat{\partial} = \langle \mathcal{G}|p, \kappa|q \rangle$ be a PDHFE on X . The means of membership and non-membership for $\hat{\partial}$ are:*

$$\wp(\hat{\partial}) = \sum_{j=1}^{\#\mathcal{G}} \gamma_j p_j,$$

$$\Im(\hat{\partial}) = \sum_{j=1}^{\#\kappa} \eta_j q_j.$$

2.2 An existing distance measure of PDHFSs

In such subsection, we review a distance measure and properties proposed by Ning et al [18]. A simple example explains that the measure has a disadvantage.

Definition 6 ([18]). *Let $\hat{\partial}_1$, $\hat{\partial}_2$ and $\hat{\partial}_3$ be three PDHFEs. Call d a distance measure if the following properties are satisfied:*

- (D1) $0 \leq d(\hat{\partial}_1, \hat{\partial}_2) \leq 1$;
- (D2) $d(\hat{\partial}_1, \hat{\partial}_2) = 0 \Leftrightarrow \hat{\partial}_1 = \hat{\partial}_2$;

$$(D3) \quad d(\hat{\partial}_1, \hat{\partial}_2) = d(\hat{\partial}_2, \hat{\partial}_1);$$

$$(D4) \quad d(\hat{\partial}_1, \hat{\partial}_3) \leq d(\hat{\partial}_1, \hat{\partial}_2) + d(\hat{\partial}_2, \hat{\partial}_3),$$

Definition 7. Ning et al. [18] give a distance measure between PDHFSs $\hat{O}_1$ and $\hat{O}_2$.

$$(6) \quad d_{gpnd}(\hat{O}_1, \hat{O}_2) = \left[\frac{1}{n} \sum_{i=1}^n \left(a^{\frac{\wp(x_i) + \Im(x_i)}{2}} + b^{\frac{\Phi(x_i) + \Psi(x_i)}{2}} \right) \right]^{1/\lambda}$$

where $\lambda > 0$, $\lambda \in R$, $a, b \in [0, 1]$, $a + b = 1$, $\wp(x_i), \Im(x_i)$ are means distances, $\Phi(x_i), \Psi(x_i)$ are standard deviation distances, $\wp(\hat{O}_m(x_i)), \Im(\hat{O}_m(x_i))$ ($m = 1, 2$) are membership and non-membership means, $\Phi(\hat{O}_m(x_i)), \Psi(\hat{O}_m(x_i))$ are membership and non-membership standard deviations, respectively.

$$\wp(x_i) = \left| \wp(\hat{O}_1(x_i)) - \wp(\hat{O}_2(x_i)) \right| = \left| \sum_{j=1}^{\#G_1} \gamma_{1j} p_{1j} - \sum_{t=1}^{\#G_2} \gamma_{2t} p_{2t} \right|,$$

$$\Im(x_i) = \left| \Im(\hat{O}_1(x_i)) - \Im(\hat{O}_2(x_i)) \right| = \left| \sum_{j=1}^{\#\kappa_1} \eta_{1j} q_{1j} - \sum_{t=1}^{\#\kappa_2} \eta_{2t} q_{2t} \right|,$$

$$\begin{aligned} \Phi(x_i) &= \left| \Phi(\hat{O}_1(x_i)) - \Phi(\hat{O}_2(x_i)) \right| \\ &= \left| \left(\sum_{j=1}^{\#G_1} \left(\gamma_{1j} p_{1j} - \wp(\hat{O}_1(x_i)) \right)^2 \right)^{1/2} - \left(\sum_{t=1}^{\#G_2} \left(\gamma_{2t} p_{2t} - \wp(\hat{O}_2(x_i)) \right)^2 \right)^{1/2} \right|, \end{aligned}$$

$$\begin{aligned} \Psi(x_i) &= \left| \Psi(\hat{O}_1(x_i)) - \Psi(\hat{O}_2(x_i)) \right| \\ &= \left| \left(\sum_{j=1}^{\#\kappa_1} \left(\eta_{1j} q_{1j} - \Im(\hat{O}_1(x_i)) \right)^2 \right)^{1/2} - \left(\sum_{t=1}^{\#\kappa_2} \left(\eta_{2t} q_{2t} - \Im(\hat{O}_2(x_i)) \right)^2 \right)^{1/2} \right|. \end{aligned}$$

This distance measure has applicability but violates the conditional reflexivity (D2) of Definition 6. We illustrate this with an example.

Example 1. Let $X = \{x_1\}$, and we give two PDHFSs

$$\hat{O}_1 = \{ \langle x_1, \{0.25|0.35, 0.35|0.25, 0.4|0.4\}, \{0.45|0.6, 0.55|0.4\} \rangle \},$$

$$\hat{O}_2 = \{ \langle x_1, \{0.25|0.35, 0.35|0.25, 0.4|0.4\}, \{0.4|0.55, 0.6|0.45\} \rangle \} \text{ where } a = b = \frac{1}{2},$$

$\lambda \in R$, and $\lambda \geq 1$. For $\hat{O}_1$, we have

$$\wp(\hat{O}_1(x_1)) = 0.25 \times 0.35 + 0.35 \times 0.25 + 0.4 \times 0.4 = 0.3350,$$

$$\Im(\hat{O}_1(x_1)) = 0.45 \times 0.6 + 0.55 \times 0.4 = 0.4900,$$

$$\begin{aligned} \Phi(\hat{O}_1(x_1)) &= ((0.25 \times 0.35 - 0.3350)^2 + (0.35 \times 0.25 - 0.3350)^2 \\ &\quad + (0.4 \times 0.4 - 0.3350)^2)^{1/2} = 0.3913, \end{aligned}$$

$$\Psi(\hat{O}_1(x_1)) = \sqrt{(0.45 \times 0.6 - 0.4900)^2 + (0.55 \times 0.4 - 0.4900)^2} = 0.3483.$$

For $\hat{O}_2$, we have

$$\begin{aligned}\wp(\hat{O}_2(x_1)) &= 0.25 \times 0.35 + 0.35 \times 0.25 + 0.4 \times 0.4 = 0.3350, \\ \Im(\hat{O}_2(x_1)) &= 0.4 \times 0.55 + 0.6 \times 0.45 = 0.4900, \\ \Phi(\hat{O}_2(x_1)) &= \sqrt{(0.25 \times 0.35 - 0.3350)^2 + (0.35 \times 0.25 - 0.3350)^2 + (0.4 \times 0.4 - 0.3350)^2} \\ &= 0.3913, \\ \Psi(\hat{O}_2(x_1)) &= \sqrt{(0.4 \times 0.55 - 0.4900)^2 + (0.6 \times 0.45 - 0.4900)^2} = 0.3483.\end{aligned}$$

Therefore, according to Eq. (6), we can have

$$\begin{aligned}d_{gpm}(\hat{O}_1, \hat{O}_2) &= \left(\frac{1}{2} \times \frac{|0.3350 - 0.3350|^\lambda + |0.4900 - 0.4900|^\lambda}{2} + \frac{1}{2} \times \frac{|0.3913 - 0.3913|^\lambda + |0.3483 - 0.3483|^\lambda}{2} \right)^{1/\lambda} \\ &= 0.\end{aligned}$$

Moreover, score value $\lambda(\hat{O}_1(x_1)) = \wp(\hat{O}_1(x_1)) - \Im(\hat{O}_1(x_1)) = 0.3350 - 0.4900 = -0.1550$, accuracy value $\ell(\hat{O}_1(x_1)) = \wp(\hat{O}_1(x_1)) + \Im(\hat{O}_1(x_1)) = 0.3350 + 0.4900 = 0.8250$.

Similarly, $\lambda(\hat{O}_2(x_1)) = -0.1550$, $\ell(\hat{O}_2(x_1)) = 0.8250$.

Remark 1. Although $d_{gpm}(\hat{O}_1, \hat{O}_2) = 0$ and $\hat{O}_1, \hat{O}_2$ have the same score and accuracy values, $\hat{O}_1$ and $\hat{O}_2$ are different PDHFSs. Therefore, $\hat{\partial}_1 = \hat{\partial}_2$ may be $\hat{\partial}_1 \sim \hat{\partial}_2$ in the conditional reflexivity (D2) of Definition 6. We need to propose new distance measures to improve the above defect.

3. Novel distance measures of PDHFSs

Since the conditions for equality of two PDHFSs are too strict, weak equality is constructed by relaxing these conditions appropriately. Then, we develop some novel notes and four-parameter generalized distance measures. Excellent properties of distance measures are discussed. Note that all values in PDHFSs are assumed to be in ascending order. The notations used in this section are explained in Table 2.

Definition 8. Let $\hat{O}_1$ and $\hat{O}_2$ be two PDHFSs. For any $x_i \in X$, score function $\lambda(\hat{O}_1(x_i)) = \lambda(\hat{O}_2(x_i))$, accuracy function $\ell(\hat{O}_1(x_i)) = \ell(\hat{O}_2(x_i))$ and $\gamma_{1j} = \gamma_{2j}$, $\eta_{1j} = \eta_{2j}$. Then we call $\hat{O}_1$ and $\hat{O}_2$ weak equality, denoted by $\hat{O}_1 =^w \hat{O}_2$.

Definition 9. Let $X = \{x_1, x_2, \dots, x_n\}$ be a fixed set, $\hat{\partial} = \langle \mathcal{G} | p, \kappa | q \rangle$ be a PDHFE on X .

(1) The consistencies of membership and non-membership degrees are defined as:

$$\iota(\hat{\partial}) = \left| \frac{1}{\#\mathcal{G}} \right|$$

Table 2: Notations interpretation II

Notations	Details	Notations	Details
$\iota(\hat{\theta}), \iota(\hat{O}(x_i))$	consistencies of membership degree	$\zeta(\hat{\theta}), \zeta(\hat{O}(x_i))$	consistencies of non-membership degree
$T(\hat{\theta}), T(\hat{O}(x_i))$	probabilistic fuzziness of membership degree	$\Upsilon(\hat{\theta}), \Upsilon(\hat{O}(x_i))$	probabilistic fuzziness of non-membership degree
$\iota(x_i), \zeta(x_i)$	consistency distances	$T(x_i), \Upsilon(x_i)$	probabilistic fuzziness distances
$\theta(x_i), \psi(x_i)$	Hausdorff distances	λ	distance parameter
a, b, c, e	parameters	ω_i	weight

$$\zeta(\hat{\theta}) = \left| \frac{1}{\#\kappa} \right|$$

Our subjective perception is that the more elements in PDHFE, the less consistent. For example, the maximum consistency is achieved when $\#\mathcal{G} = 1$ or $\#\kappa = 1$. As the numbers $\#\mathcal{G}$ and $\#\kappa$ increase, the consistency of $\iota(\hat{\theta})$ and $\zeta(\hat{\theta})$ decreases.

(2) The probabilistic fuzziness for membership and non-membership degrees as:

$$T(\hat{\theta}) = \frac{1}{\#\mathcal{G}} \sum_{j=1}^{\#\mathcal{G}} (1 - 2|p_j - 0.5|)\gamma_j,$$

$$\Upsilon(\hat{\theta}) = \frac{1}{\#\kappa} \sum_{j=1}^{\#\kappa} (1 - 2|q_j - 0.5|)\eta_j,$$

when probability $p_j = 0.5$ or $q_j = 0.5$, the opinions of DMs are most uncertain and have high ambiguity. On the contrary, the probabilistic fuzziness $T(\hat{\theta})$ or $\Upsilon(\hat{\theta})$ decreases when probability moves away from 0.5.

Definition 10. Let $\hat{O}_1 = \{\langle x_i, \mathcal{G}_1(x_i) | p_1(x_i), \kappa_1(x_i) | q_1(x_i) \rangle | x_i \in X\}$ and $\hat{O}_2 = \{\langle x_i, \mathcal{G}_2(x_i) | p_2(x_i), \kappa_2(x_i) | q_2(x_i) \rangle | x_i \in X\}$ be two PDHFSs on $X = \{x_1, x_2, \dots, x_n\}$. We give some definitions between $\hat{O}_1$ and $\hat{O}_2$:

(1) The consistency distances between $\hat{O}_1$ and $\hat{O}_2$:

$$\iota(x_i) = \left| \iota(\hat{O}_1(x_i)) - \iota(\hat{O}_2(x_i)) \right| = \left| \frac{1}{\#\mathcal{G}_1} - \frac{1}{\#\mathcal{G}_2} \right|,$$

$$\zeta(x_i) = \left| \zeta(\hat{O}_1(x_i)) - \zeta(\hat{O}_2(x_i)) \right| = \left| \frac{1}{\#\kappa_1} - \frac{1}{\#\kappa_2} \right|.$$

(2) The probabilistic fuzziness distances between $\hat{O}_1$ and $\hat{O}_2$:

$$\begin{aligned} T(x_i) &= \left| T(\hat{O}_1(x_i)) - T(\hat{O}_2(x_i)) \right| \\ &= \left| \frac{1}{\#\mathcal{G}_1} \sum_{j=1}^{\#\mathcal{G}_1} (1 - 2|p_{1j} - 0.5|)\gamma_{1j} - \frac{1}{\#\mathcal{G}_2} \sum_{t=1}^{\#\mathcal{G}_2} (1 - 2|p_{2t} - 0.5|)\gamma_{2t} \right|, \\ \Upsilon(x_i) &= \left| \Upsilon(\hat{O}_1(x_i)) - \Upsilon(\hat{O}_2(x_i)) \right| \\ &= \left| \frac{1}{\#\kappa_1} \sum_{j=1}^{\#\kappa_1} (1 - 2|q_{1j} - 0.5|)\eta_{1j} - \frac{1}{\#\kappa_2} \sum_{t=1}^{\#\kappa_2} (1 - 2|q_{2t} - 0.5|)\eta_{2t} \right|. \end{aligned}$$

(3) The Hausdorff distances between $\hat{O}_1$ and $\hat{O}_2$:

$$\begin{aligned} \theta(x_i) &= \theta(\hat{O}_1(x_i), \hat{O}_2(x_i)) \\ &= \max\left\{ \max_{\gamma_{1j} \in \mathcal{G}_1} \min_{\gamma_{2t} \in \mathcal{G}_2} \|\gamma_{1j} - \gamma_{2t}\| p_{1j} p_{2t}, \right. \\ &\quad \left. \max_{\gamma_{2t} \in \mathcal{G}_2} \min_{\gamma_{1j} \in \mathcal{G}_1} \|\gamma_{1j} - \gamma_{2t}\| p_{1j} p_{2t} \right\}, \\ \psi(x_i) &= \psi(\hat{O}_1(x_i), \hat{O}_2(x_i)) \\ &= \max\left\{ \max_{\eta_{1j} \in \kappa_1} \min_{\eta_{2t} \in \kappa_2} \|\eta_{1j} - \eta_{2t}\| q_{1j} q_{2t}, \right. \\ &\quad \left. \max_{\eta_{2t} \in \kappa_2} \min_{\eta_{1j} \in \kappa_1} \|\eta_{1j} - \eta_{2t}\| q_{1j} q_{2t} \right\}. \end{aligned}$$

Note that, $\|\cdot\|$ denotes Euclidean distance from one $\gamma_{1j}(\eta_{1j})$ of $\hat{O}_1(x_i)$ to another $\gamma_{2t}(\eta_{2t}) \in \hat{O}_2(x_i)$.

(4) The mean distances between $\hat{O}_1$ and $\hat{O}_2$:

$$\begin{aligned} \wp(x_i) &= \left| \wp(\hat{O}_1(x_i)) - \wp(\hat{O}_2(x_i)) \right| = \left| \sum_{j=1}^{\#\mathcal{G}_1} \gamma_{1j} p_{1j} - \sum_{t=1}^{\#\mathcal{G}_2} \gamma_{2t} p_{2t} \right|, \\ \Im(x_i) &= \left| \Im(\hat{O}_1(x_i)) - \Im(\hat{O}_2(x_i)) \right| = \left| \sum_{j=1}^{\#\kappa_1} \eta_{1j} q_{1j} - \sum_{t=1}^{\#\kappa_2} \eta_{2t} q_{2t} \right|. \end{aligned}$$

We can define a generalized distance measure between PDHFSs based on the above results.

Definition 11. Let $\hat{O}_1$ and $\hat{O}_2$ be two PDHFSs on X , the following four-parameter generalized distance between $\hat{O}_1$ and $\hat{O}_2$ is proposed:

$$\begin{aligned} d_{pgdh}(\hat{O}_1, \hat{O}_2) &= \left[\frac{1}{n} \sum_{i=1}^n \left(a \frac{\iota(x_i) + \zeta(x_i)}{2} \right. \right. \\ (7) \quad &\quad \left. \left. + b \frac{T(x_i) + \Upsilon(x_i)}{2} + c \frac{\theta(x_i) + \psi(x_i)}{2} + e \frac{\wp(x_i) + \Im(x_i)}{2} \right)^\lambda \right]^{1/\lambda}, \end{aligned}$$

where $\lambda \geq 1$, $\lambda \in R$, $0 < a, b, c, e < 1$ and $a + b + c + e = 1$.

Definition 12. Let $\hat{O}_1$, $\hat{O}_2$ and $\hat{O}_3$ be three PDHFSs. d_{pgdh} is a distance measure if d_{pgdh} satisfies the following conditions:

- (D1) $0 \leq d_{pgdh}(\hat{O}_1, \hat{O}_2) \leq 1$ (boundedness);
- (D2) $d_{pgdh}(\hat{O}_1, \hat{O}_2) = 0 \Leftrightarrow \hat{O}_1 =^w \hat{O}_2$ (conditional reflexivity);
- (D3) $d_{pgdh}(\hat{O}_1, \hat{O}_2) = d_{pgdh}(\hat{O}_2, \hat{O}_1)$ (symmetric);
- (D4) $d_{pgdh}(\hat{O}_1, \hat{O}_3) \leq d_{pgdh}(\hat{O}_1, \hat{O}_2) + d_{pgdh}(\hat{O}_2, \hat{O}_3)$ (triangle inequality).

Theorem 1. d_{pgdh} is a distance measure between PDHFSs and satisfies the four properties of Definition 12.

Example 2. We use the data from Example 1, let $a = b = c = e = \frac{1}{4}$, $\lambda=1$, we have $\iota(x_1) = |\frac{1}{3} - \frac{1}{3}| = 0$, $\zeta(x_1) = |\frac{1}{2} - \frac{1}{2}| = 0$,

$$\begin{aligned} T(x_1) &= \left| \frac{1}{3} \begin{pmatrix} (1 - 2|0.35 - 0.5|) 0.25 \\ + (1 - 2|0.25 - 0.5|) 0.35 \\ + (1 - 2|0.4 - 0.5|) 0.4 \end{pmatrix} - \frac{1}{3} \begin{pmatrix} (1 - 2|0.35 - 0.5|) 0.25 \\ + (1 - 2|0.25 - 0.5|) 0.35 \\ + (1 - 2|0.4 - 0.5|) 0.4 \end{pmatrix} \right| = 0, \\ Y(x_1) &= \left| \frac{1}{2} \begin{pmatrix} (1 - 2|0.6 - 0.5|) 0.45 \\ + (1 - 2|0.4 - 0.5|) 0.55 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} (1 - 2|0.55 - 0.5|) 0.4 \\ + (1 - 2|0.45 - 0.5|) 0.6 \end{pmatrix} \right| = 0.0500, \\ \theta(x_1) &= \left\{ \max_{\gamma_{Mj} \in \mathcal{G}_M} \left\{ \min \{0, 0, 0\}, \min \{0, 0, 0\} \right\}, \max_{\gamma_{Nj} \in \mathcal{G}_N} \left\{ \min \{0, 0, 0\}, \min \{0, 0, 0\} \right\} \right\} = 0, \\ \psi(x_1) &= \left\{ \max_{\gamma_{Mj} \in h_M} \left\{ \min \{0.0165, 0.0405\}, \min \{0.033, 0.009\} \right\}, \max_{\gamma_{Nj} \in h_N} \left\{ \min \{0.0165, 0.033\}, \min \{0.0405, 0.009\} \right\} \right\} = 0.0165, \\ \wp(x_1) &= \left| \frac{(0.25 \times 0.35 + 0.35 \times 0.25 + 0.4 \times 0.4)}{(0.25 \times 0.35 + 0.35 \times 0.25 + 0.4 \times 0.4)} \right| = 0, \\ \Im(x_1) &= \left| \frac{(0.45 \times 0.6 + 0.55 \times 0.4) - (0.4 \times 0.55 + 0.6 \times 0.45)}{(0.45 \times 0.6 + 0.55 \times 0.4) - (0.4 \times 0.55 + 0.6 \times 0.45)} \right| = 0, \end{aligned}$$

According to Eq. (7), the distance $d_{pgdh}(\hat{O}_1, \hat{O}_2)$ is

$$\begin{aligned} d_{pgdh}(\hat{O}_1, \hat{O}_2) &= \frac{1}{4} \times \frac{0 + 0.0500}{2} + \frac{1}{4} \times \frac{0 + 0}{2} + \frac{1}{4} \times \frac{0 + 0.0165}{2} + \frac{1}{4} \times \frac{0 + 0}{2} \\ &= 0.0083. \end{aligned}$$

Remark 2. It is clear that $\hat{O}_1$ and $\hat{O}_2$ are distinct PDHFSs, which is confirmed by Example 2. Compared with the distance measure in [18], the new generalized distance measure d_{pgdh} can distinguish PDHFSs better.

If we consider the weight of $x_i \in X$ ($i = 1, 2, \dots, n$), a generalized weighted distance measure with preferences for PDHFS as:

$$(8) \quad d_{gwpdh}(\hat{O}_1, \hat{O}_2) = \left[\sum_{i=1}^n \omega_i \left(a \frac{\iota(x_i) + \zeta(x_i)}{2} + b \frac{T(x_i) + \Upsilon(x_i)}{2} + c \frac{\theta(x_i) + \psi(x_i)}{2} + e \frac{\wp(x_i) + \Im(x_i)}{2} \right)^\lambda \right]^{1/\lambda},$$

where $\lambda \geq 1$, ω_i is the weight of $x_i \in X$, with $\omega_i \in [0, 1]$ and $\sum_{i=1}^n \omega_i = 1$.

Remark 3. If $\omega_i = \frac{1}{n}$, then d_{gwpdh} can be simplified to d_{pgdh} .

Definition 13. Let $\hat{O}_1$ and $\hat{O}_2$ be two PDHFSs on X . The generalized weighted similarity measure is defined as:

$$(9) \quad S_{gwpdh}(\hat{O}_1, \hat{O}_2) = 1 - d_{gwpdh}(\hat{O}_1, \hat{O}_2).$$

Definition 14. Let $\hat{O}_1$ and $\hat{O}_2$ be two PDHFSs, if $S_{gwpdh}(\hat{O}_1, \hat{O}_2)$ satisfies three properties as follows:

- (S1) $0 \leq S_{gwpdh}(\hat{O}_1, \hat{O}_2) \leq 1$ (boundedness);
- (S2) $S_{gwpdh}(\hat{O}_1, \hat{O}_2) = 1 \Leftrightarrow \hat{O}_1 =^w \hat{O}_2$ (conditional reflexivity);
- (S3) $S_{gwpdh}(\hat{O}_1, \hat{O}_2) = S_{gwpdh}(\hat{O}_2, \hat{O}_1)$ (symmetric),

then we call $S_{gwpdh}(\hat{O}_1, \hat{O}_2)$ the similarity measure for PDHFSs.

Theorem 2. S_{gwpdh} is a similarity measure of PDHFS and satisfies the three properties in Definition 14.

4. Entropy measure of PDHFS

Entropy is crucial in measuring the uncertainty of PDHFSs. This section proposes a new entropy measure, and the properties are discussed.

Let $\hat{\partial} = (\mathcal{G}|p, \kappa|q)$ be a PDHFE. [18] defines the mean of hesitant degree.

$$\Theta(\hat{\partial}) = 1 - \wp(\hat{\partial}) - \Im(\hat{\partial}) = 1 - \sum_{j=1}^{\#\mathcal{G}} \gamma_j p_j - \sum_{j=1}^{\#\kappa} \eta_j q_j.$$

Definition 15. Let $\hat{\partial} = (\mathcal{G}|p, \kappa|q)$, $\hat{\partial}_1 = (\mathcal{G}_1|p_1, \kappa_1|q_1)$ and $\hat{\partial}_2 = (\mathcal{G}_2|p_2, \kappa_2|q_2)$ be three PDHFEs. A mapping $\Xi : PDHFEs(X) \rightarrow [0, 1]$, claiming that Ξ is an entropy and satisfies the following properties:

- (E1) $\Xi(\hat{\partial}) = 0$ (minimum) $\Leftrightarrow \hat{\partial} = (\{0|1\}, \{1|1\})$ or $\hat{\partial} = (\{1|1\}, \{0|1\})$;
- (E2) $\Xi(\hat{\partial}) = 1$ (maximum) $\Leftrightarrow \wp(\hat{\partial}) = \Im(\hat{\partial})$;

$$(E3) \quad \Xi(\hat{\partial}) = \Xi(\hat{\partial}^c);$$

$$(E4) \quad \Xi(\hat{\partial}_2) \geq \Xi(\hat{\partial}_1), \text{ if } \wp(\hat{\partial}_2) \geq \Im(\hat{\partial}_2), \text{ there have } \wp(\hat{\partial}_1) \geq \wp(\hat{\partial}_2), \Im(\hat{\partial}_2) \geq \Im(\hat{\partial}_1); \text{ or } \wp(\hat{\partial}_2) \leq \Im(\hat{\partial}_2), \text{ there have } \wp(\hat{\partial}_1) \leq \wp(\hat{\partial}_2), \Im(\hat{\partial}_2) \leq \Im(\hat{\partial}_1).$$

Definition 16. Let $\hat{\partial}$ be a PDHFE, we give the following definition of entropy:

$$(10) \quad \Xi(\hat{\partial}) = 1 - \sin \left| \frac{\wp(\hat{\partial}) - \Im(\hat{\partial})}{2(1 + \Theta(\hat{\partial}))} \pi \right|.$$

Theorem 3. Ξ is an entropy measure for PDHFEs and satisfies the four axioms in Definition 15.

Naturally, let $\hat{O} = \{\langle x_i, \mathcal{G}(x_i) | p(x_i), \kappa(x_i) | q(x_i) \mid x_i \in X \rangle\}$ be a PDHFS on X . The entropy $\Xi(\hat{O})$ for PDHFSs is as follows:

$$(11) \quad \Xi(\hat{O}) = \frac{1}{n} \sum_{i=1}^n \left(1 - \sin \left| \frac{\wp(\hat{O}(x_i)) - \Im(\hat{O}(x_i))}{2(1 + \Theta(\hat{O}(x_i)))} \pi \right| \right).$$

5. Applications and comparisons

Solid theoretical support is involved in solving practical problems. In Subsection 5.1, an example of wood recognition demonstrates the proposed method's effectiveness. In Subsection 5.2, VIKOR-based multi-criteria decision-making technology is developed and applied to the example of a multinational company launching new products. The practicability of the proposed method is explained in parameter analysis and comparative analysis. The studied contributions are summarized in Subsection 5.3.

5.1 Application of the proposed method in pattern recognition

Wood recognition is taken as an example to display the application of new distance measures. The new entropy is used as a technique to derive attribute weights. Moreover, the developed method is compared with the method in [18]. The steps of pattern recognition are as follows:

Step 1. Obtain a PDHFS matrix $L = (\mathcal{L}_{ij})_{n \times m}$ from the evaluation of DMs, then standardize the PDHFS matrix.

$$\mathcal{L}_i = \begin{cases} \langle \mathcal{G} | p, \kappa | q \rangle, & \text{for any benefit criterion } C_i, \\ \langle \kappa | q, \mathcal{G} | p \rangle, & \text{for any cost criterion } C_i. \end{cases}$$

Step 2. Calculate the attribute weights $\omega(x_i)$ by Eq. (12).

$$(12) \quad \omega(C_i) = \frac{1 - \Xi(C_i)}{n - \sum_{i=1}^n \Xi(C_i)},$$

where

$$\Xi(C_i) = \frac{1}{m} \sum_{j=1}^m \Xi(\mathcal{L}_j(C_i)),$$

$\Xi(\mathcal{L}_j(C_i))$, $(i = 1, 2, \dots, n)$, $(j = 1, 2, \dots, m)$ can be determined by Eq. (10).

Step 3. Calculate the distance d_{gwpdh} between PDHFSs by Eq. (8).

Step 4. Sort the patterns based on calculation results. The shorter the distance, the closer the pattern.

Example 3. When enterprises use building wood, they to identify the classification of wood to maximize production benefits. The main characteristics of wood include: C_1 : sound insulation performance, C_2 : processability and C_3 : corrosion resistance. There are three standard classifications of wood: $\hat{O}_j$ ($j = 1, 2, 3$). Now, the classification of wood N needs to be recognized. The PDHFS information for $\hat{O}_j$ ($j = 1, 2, 3$) and M is provided in Table 3.

Table 3: The PDHFSs information

C_1	C_2	C_3
$\hat{O}_1 \left(\begin{array}{c} \{0.2 0.7, 0.4 0.3\}, \\ \{0.4 0.6, 0.5 0.4\} \end{array} \right)$	$\left(\begin{array}{c} \{0.55 0.8, 0.65 0.2\}, \\ \{0.2 0.5, 0.1 0.5\} \end{array} \right)$	$\left(\begin{array}{c} \{0.1 0.45, 0.3 0.55\}, \\ \{0.5 0.5, 0.65 0.5\} \end{array} \right)$
$\hat{O}_2 \left(\begin{array}{c} \{0.6 0.2, 0.7 0.8\}, \\ \{0.2 0.2, 0.3 0.8\} \end{array} \right)$	$\left(\begin{array}{c} \{0.7 1\}, \\ \{0.2 0.8, 0.3 0.2\} \end{array} \right)$	$\left(\begin{array}{c} \{0.1 0.4, 0.4 0.6\}, \\ \{0.4 0.7, 0.6 0.3\} \end{array} \right)$
$\hat{O}_3 \left(\begin{array}{c} \{0.6 1\}, \\ \{0.1 0.4, 0.3 0.6\} \end{array} \right)$	$\left(\begin{array}{c} \{0.5 0.3, 0.6 0.7\}, \\ \{0.3 1\} \end{array} \right)$	$\left(\begin{array}{c} \{0.2 0.3, 0.4 0.3, 0.5 0.4\}, \\ \{0.4 0.5, 0.5 0.5\} \end{array} \right)$
$M \left(\begin{array}{c} \{0.2 0.6, 0.3 0.4\}, \\ \{0.4 0.5, 0.5 0.5\} \end{array} \right)$	$\left(\begin{array}{c} \{0.5 0.4, 0.6 0.6\}, \\ \{0.1 0.45, 0.2 0.55\} \end{array} \right)$	$\left(\begin{array}{c} \{0.2 0.4, 0.3 0.6\}, \\ \{0.6 0.7, 0.65 0.3\} \end{array} \right)$

Step 1. All attributes are beneficial, so there is no need to be standardized.

Step 2. According to Eqs. (12) and (5.1), we have

$$\begin{aligned} \Xi(C_1) &= 0.6601, \quad \Xi(C_2) = 0.6838, \quad \Xi(C_3) = 0.6381, \\ \omega(C_1) &= 0.3339, \quad \omega(C_2) = 0.3106, \quad \omega(C_3) = 0.3555. \end{aligned}$$

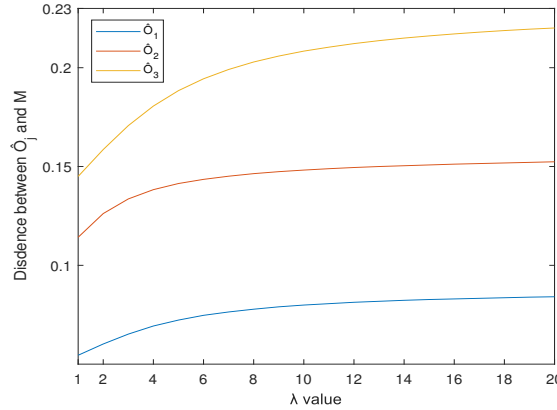
Step 3. The distance between $\hat{O}_j$ ($j = 1, 2, 3$) and M is calculated according to Eq. (8), as shown in Table 4, where $a = b = c = e = \frac{1}{4}$.

Step 4. When $\lambda \in [1, 20]$, all the sorting results are $d(\hat{O}_1, M) < d(\hat{O}_2, M) < d(\hat{O}_3, M)$.

Table 4 and Fig. 1 show that the distance between $\hat{O}_j$ ($j = 1, 2, 3$) and M increases as λ increases from 1 to 20. This result is that M belongs to $\hat{O}_1$. Furthermore, the ranking order remains the same, that is, $d(\hat{O}_1, M) < d(\hat{O}_2, M) < d(\hat{O}_3, M)$.

Table 4: The ranking for different values of λ

	$d(\hat{O}_1, M)$	$d(\hat{O}_2, M)$	$d(\hat{O}_3, M)$	Ranking
$\lambda = 1$	0.0545	0.1142	0.1450	$d(\hat{O}_1, M) < d(\hat{O}_2, M) < d(\hat{O}_3, M)$
$\lambda = 2$	0.0602	0.1262	0.1586	$d(\hat{O}_1, M) < d(\hat{O}_2, M) < d(\hat{O}_3, M)$
$\lambda = 6$	0.0747	0.1435	0.1944	$d(\hat{O}_1, M) < d(\hat{O}_2, M) < d(\hat{O}_3, M)$
$\lambda = 10$	0.0799	0.1482	0.2084	$d(\hat{O}_1, M) < d(\hat{O}_2, M) < d(\hat{O}_3, M)$
$\lambda = 15$	0.0827	0.1508	0.2161	$d(\hat{O}_1, M) < d(\hat{O}_2, M) < d(\hat{O}_3, M)$
$\lambda = 20$	0.0841	0.1524	0.2201	$d(\hat{O}_1, M) < d(\hat{O}_2, M) < d(\hat{O}_3, M)$

Figure 1: Distance between $\hat{O}_j$ and M with $\lambda \in [1, 20]$

Remark 4. The results using Eq. (6) [18] also show that M belongs to $\hat{O}_1$, but the ranking is $d(\hat{O}_1, M) < d(\hat{O}_3, M) < d(\hat{O}_2, M)$. Such a difference can be attributed to our consideration of weak equality. The new measure can identify different PDHFSs better so that the data information expression is comprehensive and valid.

5.2 Application in VIKOR-based multi-criteria decision-making

5.2.1 VIKOR-based multi-criteria decision-making

VIKOR (Serbian name: Vise Kriterijumski Optimizacioni Resenje) is generalized from the ensemble function L_p -metric [35]. The compromise solution obtained by VIKOR considers the situation in which conflicting attributes cannot be satisfied simultaneously, and the decision is made through mutual concessions. The classical TOPSIS technique only considers the distances from alternatives to positive/negative ideal solutions, while VIKOR maximizes group utility between attributes and minimizes individual regret utility. VIKOR technology is extended to fuzzy environments, such as intuitional fuzzy VIKOR [4], hesitant fuzzy VIKOR [13], hesitant fuzzy linguistic VIKOR [14], etc. Ren et al. [22] proposed a decision technique of integrated VIKOR and the Analytic Hier-

archy Process (AHP) for PDHFSs. Based on this, we develop a VIKOR-based MCDM technique. The detailed steps are as follows:

Step 1. Get the MCDM matrix $R = (\rho_{ij})_{n \times m}$. The alternatives $M_j (j = 1, 2, \dots, m)$ are evaluated by DMs, where M_j have n attributes.

Step 2. Determine the ρ_i^* (best evaluation information) and ρ_i^- (worst evaluation information) for attributes C_i adopting Eqs. (4-5) and standardize.

$$\rho_i^* = \begin{cases} \max_{j=1,2,\dots,m} \{\rho_{ij}\}, & \text{for any benefit criterion } C_i \\ \min_{j=1,2,\dots,m} \{\rho_{ij}\}, & \text{for any cost criterion } C_i \end{cases}$$

$$\rho_i^- = \begin{cases} \min_{j=1,2,\dots,m} \{\rho_{ij}\}, & \text{for any benefit criterion } C_i \\ \max_{j=1,2,\dots,m} \{\rho_{ij}\}, & \text{for any cost criterion } C_i \end{cases}$$

Step 3. Calculate the maximum group utility measure G_j and the individual regret measure I_j by Eqs. (13) and (14), where ω_i is the relevant weight of attributes C_i . $d(\rho_i^*, \rho_{ij})$ and $d(\rho_i^*, \rho_i^-)$ is given by Eq. (7).

$$(13) \quad G_j = \sum_{i=1}^n \omega_i d(\rho_i^*, \rho_{ij}) / d(\rho_i^*, \rho_i^-),$$

$$(14) \quad I_j = \max_i [\omega_i d(\rho_i^*, \rho_{ij}) / d(\rho_i^*, \rho_i^-)],$$

where $\omega_i \in [0, 1]$ and $\sum_{i=1}^n \omega_i = 1$.

Step 4. Calculate V_j by Eq. (15).

$$(15) \quad V_j = \theta \frac{(G_j - G^*)}{(G^- - G^*)} + (1 - \theta) \frac{(I_j - I^*)}{(I^- - I^*)},$$

where $G^- = \max_j G_j$, $G^* = \min_j G_j$, $I^- = \max_j I_j$ and $I^* = \min_j I_j$. θ is a parameter, let $\theta = 0.5$.

Step 5. Sort the G_j , I_j and V_j in ascending order.

Step 6. $M^{(1)}$ is an optimal solution if it satisfies the following three conditions:

Condition 1: $M^{(1)}$ has the first minimum in $V_j (j = 1, 2, \dots, m)$.

Condition 2: $V(M^{(2)}) - V(M^{(1)}) \geq \frac{1}{m-1}$, where $M^{(2)}$ has the second minimum in V_j .

Condition 3: $M^{(1)}$ has the first minimum in G_j or I_j .

If conditions 2 and 3 cannot be met simultaneously, go to step 7.

Step 7. If the Condition 2 isn't met. $M^{(\S)}$ satisfies $V(M^{(\S)}) - V(M^{(1)}) < \frac{1}{m-1}$, $M^{(1)}$ is an optimal solution, and $M^{(2)}, M^{(3)}, \dots, M^{(\S)}$ are compromise solutions. If the Condition 3 isn't met. $M^{(1)}$ is an optimal solution, and $M^{(2)}$ is a compromise solution.

5.2.2 Launching new products by a multinational corporation

Example 4. [[7]] A multinational company wants to launch new products according to consumer buying behaviour. Thus, experts from three countries visited the available information about four products $M_j (j = 1, 2, 3, 4)$ based on consumer purchasing behavior. There are four attributes of consumer buying behaviour: C_1 : suitable cultural environment, C_2 : conform to global trends, C_3 : suitable weather and C_4 : excellent after-sales service. The evaluation information of experts is shown in Table 5. Let weights $\omega = (0.4385, 0.1986, 0.1815, 0.1814)$, $\lambda = 1$, and $a = b = c = e = \frac{1}{4}$.

Table 5: The comprehensive PDHFSs matrix

	C_1	C_2	C_3	C_4
M_1	$\left(\begin{array}{c} \{0.2 0.1333, 0.3 0.3667, \\ 0.5 0.1667, 0.75 0.3333\} \\ \{0.2 0.3333, 0.4 0.6667\} \end{array} \right)$	$\left(\begin{array}{c} \{0.2 0.3333, 0.45 0.14, \\ 0.5 0.3333, 0.6 0.1934\} \\ \{0.2 0.3, 0.3 0.2, \\ 0.5 0.1667, 0.7 0.3333\} \end{array} \right)$	$\left(\begin{array}{c} \{0.2 0.3333, 0.3 0.3334, \\ 0.9 0.3333, \\ 0.1 0.3333, 0.4 0.2667, \\ 0.6 0.4\} \end{array} \right)$	$\left(\begin{array}{c} \{0.6 0.9, 0.7 0.1\}, \\ \{0.25 0.3333, 0.3 0.6667\} \end{array} \right)$
M_2	$\left(\begin{array}{c} \{0.2 0.3333, 0.6 0.2334, \\ 0.8 0.4333\} \\ \{0.1 0.6667, 0.7 0.3333\} \end{array} \right)$	$\left(\begin{array}{c} \{0.2 0.5, 0.3 0.5\}, \\ \{0.15 0.1667, 0.2 0.1667, \\ 0.6 0.3333, 0.7 0.3333\} \end{array} \right)$	$\left(\begin{array}{c} \{0.2 0.3334, 0.6 0.3333, \\ 0.9 0.3333 \\ 0.1 0.6667, 0.2 0.1667, \\ 0.6 0.1666\} \end{array} \right)$	$\left(\begin{array}{c} \{0.2 0.4333, 0.3 0.5667\}, \\ \{0.5 0.1333, 0.6 0.5333, \\ 0.8 0.3333\} \end{array} \right)$
M_3	$\left(\begin{array}{c} \{0.05 0.2334, 0.2 0.1, \\ 0.4 0.1333, 0.5 0.2, \\ 0.9 0.3333\} \\ \{0.1 0.3333, 0.5 0.6667\} \end{array} \right)$	$\left(\begin{array}{c} \{0.45 0.3333, 0.5 0.3333, \\ 0.6 0.3334 \\ 0.1 0.1666, 0.2 0.1667, \\ 0.5 0.6667\} \end{array} \right)$	$\left(\begin{array}{c} \{0.6 0.3333, 0.8 0.6667\}, \\ \{0.1 0.1, 0.15 0.3333, \\ 0.2 0.5666\} \end{array} \right)$	$\left(\begin{array}{c} \{0.1 0.3333, 0.12 0.3333, \\ 0.2 0.3334 \\ 0.6 0.2333, 0.7 0.3, \\ 0.8 0.4667\} \end{array} \right)$
M_4	$\left(\begin{array}{c} \{0.3 0.2333, 0.4 0.4, \\ 0.5 0.3667 \\ 0.2 0.1667, 0.3 0.5, \\ 0.4 0.2, 0.5 0.1333\} \end{array} \right)$	$\left(\begin{array}{c} \{0.1 0.3334, 0.2 0.1333, \\ 0.5 0.5333 \\ 0.2 0.1, 0.3 0.2667, \\ 0.4 0.3, 0.8 0.3333\} \end{array} \right)$	$\left(\begin{array}{c} \{0.3 0.6667, 0.4 0.0333, \\ 0.5 0.3 \\ 0.3 0.6667, 0.65 0.3333\} \end{array} \right)$	$\left(\begin{array}{c} \{0.35 0.3334, 0.4 0.3333, \\ 0.5 0.3333 \\ 0.2 0.1, 0.4 0.2334, \\ 0.6 0.6666\} \end{array} \right)$

Step 1. The MCDM matrix $R = (\rho_{ij})_{n \times m}$ is represented in Table 5.

Step 2. All attributes are beneficial. According to Eqs. (4) and (5), we have

$$\rho_i^* = \left(\begin{array}{c} \left\langle \begin{array}{c} \{0.2|0.3333, 0.6|0.2334, 0.8|0.4333\}, \\ \{0.1|0.6667, 0.7|0.3333\} \end{array} \right\rangle, \\ \left\langle \begin{array}{c} \{0.45|0.3333, 0.5|0.3333, 0.6|0.3334\}, \\ \{0.1|0.1666, 0.2|0.1667, 0.5|0.6667\} \end{array} \right\rangle, \\ \left\langle \begin{array}{c} \{0.6|0.3333, 0.8|0.6667\}, \\ \{0.1|0.1, 0.15|0.3333, 0.2|0.5666\} \end{array} \right\rangle, \\ \left\langle \{0.6|0.9, 0.7|0.1\}, \{0.25|0.3333, 0.3|0.6667\} \right\rangle \end{array} \right)$$

$$\rho_i^- = \left(\begin{array}{c} \left\langle \begin{array}{c} \{0.3|0.2333, 0.4|0.4, 0.5|0.3667\}, \\ \{0.2|0.1667, 0.3|0.5, 0.4|0.2, 0.5|0.1333\} \end{array} \right\rangle, \\ \left\langle \begin{array}{c} \{0.2|0.5, 0.3|0.5\} \\ \left\{ \begin{array}{c} 0.15|0.1667, 0.2|0.1667, \\ 0.6|0.3333, 0.7|0.3333 \end{array} \right\} \end{array} \right\rangle, \\ \left\langle \begin{array}{c} \{0.3|0.6667, 0.4|0.0333, 0.5|0.3\}, \\ \{0.3|0.6667, 0.65|0.3333\} \end{array} \right\rangle, \\ \left\langle \begin{array}{c} \{0.1|0.3333, 0.12|0.3333, 0.2|0.3334\}, \\ \{0.6|0.2333, 0.7|0.3, 0.8|0.4667\} \end{array} \right\rangle \end{array} \right).$$

Step 3. Compute the values of $d(\rho_i^*, \rho_i^-)$ and $d(\rho_i^*, \rho_{ij})$ according to Eq. (7). The values of G_j and I_j are derived by Eqs. (13) and (14), and the full results are presented in Table 6.

Take product M_1 as an example, there is

$$\begin{aligned} d(\rho_1^*, \rho_1^-) &= \frac{(|\frac{1}{3} - \frac{1}{3}| + |\frac{1}{2} - \frac{1}{4}|)}{8} + \frac{(|0.3689 - 0.3022| + |0.2666 - 0.1650|)}{8} \\ &\quad + \frac{(0.0477 + 0.0667)}{8} + \frac{(|0.5533 - 0.4133| + |0.3000 - 0.3300|)}{8} \\ &= 0.0878. \end{aligned}$$

Similarly,

$$\begin{aligned} d(\rho_2^*, \rho_2^-) &= 0.1123, & d(\rho_3^*, \rho_3^-) &= 0.1708, & d(\rho_4^*, \rho_4^-) &= 0.2231, \\ d(\rho_1^*, \rho_{11}) &= 0.0876, & d(\rho_2^*, \rho_{21}) &= 0.1481, & d(\rho_3^*, \rho_{31}) &= 0.1772, \\ d(\rho_4^*, \rho_{41}) &= 0. \end{aligned}$$

We can get

$$\begin{aligned} G_1 &= \sum_{i=1}^4 \omega_i d(\rho_i^*, \rho_{i1}) / d(\rho_i^*, \rho_i^-) \\ &= 0.4385 \times \frac{0.0876}{0.0878} + 0.1986 \times \frac{0.1481}{0.1123} + 0.1815 \times \frac{0.1772}{0.1708} + 0.1814 \times \frac{0}{0.2231} \\ &= 0.8875, \\ I_1 &= \max_i [\omega_i d(\rho_i^*, \rho_{i1}) / d(\rho_i^*, \rho_i^-)] = 0.4372. \end{aligned}$$

Step 4. Calculate the values of V_j according to Eq. (15), where $G^- = \max_j G_j = 1.1237$, $G^* = \min_j G_j = 0.4775$, $I^- = \max_j I_j = 0.5168$, $I^* = \min_j I_j = 0.1980$. So,

$$\begin{aligned} V_1 &= \theta \frac{(G_1 - G^*)}{(G^- - G^*)} + (1 - \theta) \frac{(I_1 - I^*)}{(I^- - I^*)} \\ &= 0.5 \times \frac{0.8875 - 0.4775}{1.1237 - 0.4775} + 0.5 \times \frac{0.6924 - 0.1980}{0.5168 - 0.1980} \\ &= 0.6924. \end{aligned}$$

Similarly, $V_2 = 0$, $V_3 = 0.4321$, $V_4 = 1$.

Step 5. Sort G_j , I_j and V_j : $G_2 < G_3 < G_1 < G_4$, $I_2 < I_3 < I_1 < I_4$ and $V_2 < V_3 < V_1 < V_4$.

Step 6. According to the ordering in Step 5, M_2 has the first minimum in V_j , $V(M_3) - V(M_2) = 0.4321 \geq \frac{1}{4-1} = 0.3333$. M_2 has again the minimum in G_j and I_j . Thus, the optimal product is M_2 .

Table 6: The distance of $\rho_i^*, \rho_i^-, \rho_{ij}$

	C_1	C_2	C_3	C_4	G_j	I_j
	$\omega_1 = 0.4385$	$\omega_2 = 0.1986$	$\omega_3 = 0.1815$	$\omega_4 = 0.1814$		
$d(\rho_i^*, \rho_{i1})$	0.0876	0.1481	0.1772	0	0.8875	0.4372
$d(\rho_i^*, \rho_{i2})$	0	0.1123	0.0644	0.1772	0.4775	0.1980
$d(\rho_i^*, \rho_{i3})$	0.0711	0	0	0.0912	0.6203	0.4031
$d(\rho_i^*, \rho_{i4})$	0.0912	0.0854	0.1951	0.1297	1.1237	0.5168
$d(\rho_i^*, \rho_i^-)$	0.0878	0.1123	0.1708	0.2231	—	—

5.2.3 Parameter analysis

We analyze the influence of distance parameters (a, b, c, e) on decision. First, set (a, b, c, e) to $(0.2, 0.3, 0.3, 0.2)$, $(0.2, 0.4, 0.2, 0.2)$, $(0.3, 0.5, 0.1, 0.1)$ and $(0.4, 0.1, 0.1, 0.4)$ randomly without changing the other parameters. The results are shown in Table 7.

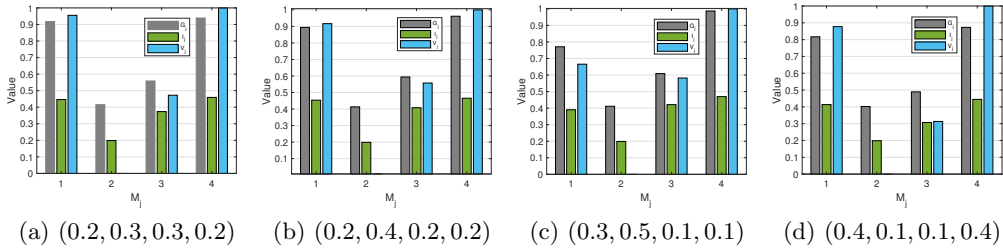
Figure 2: Values of M_j under parameters (a, b, c, e)

Table 7 and Fig.2 illustrate that the optimal product is M_2 . However, when $(a, b, c, e) = (0.4, 0.1, 0.1, 0.4)$, the optimal product is M_2 , followed by M_3 . Suppose the company has the strength to support both products. It may launch M_2 and M_3 . From the actual situation, consumer demand for products may differ due to cultural differences between countries. Therefore, the results obtained in this paper are reasonable. Furthermore, the proposed method can set appropriate parameters according to the actual situations, with stability and practicality.

5.2.4 Comparative analysis

Our method is compared with three methods: the probabilistic dual hesitant fuzzy ordered weighted Einstein averaging (PDHFOWEA) operator [7], the probabilistic dual hesitant fuzzy weighted average (PDHFWA) operator in [8], and distance measure d_{gpnd} in [18]. The results are given in Table 8.

The PDHFWA operator [8] calculates that the best product is M_2 , consistent with our result. The PDHFOWEA operator [7] represents that M_3 is

Table 7: The ranking result with different parameter a, b, c, e

	G_j	I_j	V_j	Ranking
$(a, b, c, e) = (0.2, 0.3, 0.3, 0.2)$				
M_1	0.9177	0.4464	0.9551	$M_2 \succ M_3 \succ M_1 \succ M_4$
M_2	0.4152	0.1986	0	
M_3	0.5585	0.3736	0.4725	
M_4	0.9390	0.4593	1	
$(a, b, c, e) = (0.2, 0.4, 0.2, 0.2)$				
M_1	0.8940	0.4537	0.9165	$M_2 \succ M_3 \succ M_1 \succ M_4$
M_2	0.4132	0.1986	0	
M_3	0.5940	0.4080	0.5574	
M_4	0.9616	0.4654	1	
$(a, b, c, e) = (0.3, 0.5, 0.1, 0.1)$				
M_1	0.7708	0.3903	0.6657	$M_2 \succ M_3 \succ M_1 \succ M_4$
M_2	0.4113	0.1986	0	
M_3	0.6090	0.4212	0.5822	
M_4	0.9872	0.4697	1	
$(a, b, c, e) = (0.4, 0.1, 0.1, 0.4)$				
M_1	0.8167	0.4135	0.8772	$M_2 \succ M_3 \succ M_1 \succ M_4$
M_2	0.4019	0.1986	0	
M_3	0.4893	0.3070	0.3132	
M_4	0.8730	0.4445	1	

Table 8: Comparison of existing methods

References	Method	Ranking
[7]	PDHFOWEA	$M_3 \succ M_1 \succ M_2 \succ M_4$
[18]	d_{gpnd}	$M_1 \succ M_2 \succ M_3 \succ M_4$
[8]	PDHFWA	$M_2 \succ M_3 \succ M_1 \succ M_4$
Our devised method	d_{pgdh}	$M_2 \succ M_3 \succ M_1 \succ M_4$

the best product. M_1 is the optimal product of d_{gpnd} [18]. Different MCDM methods have their advantages in the fuzzy environment, respectively. Our proposed method considers the importance of overall balance and extreme data, but PDHFOWEA and PDHFWA operators pay more attention to the overall balance. d_{gpnd} [18] only considers the mean and standard deviation of PDHFSs. However, our proposed method considers the influence of consistency and probabilistic fuzziness of PDHFSs, expressing the hidden data information. With the support of weak equality, the measurement results are more reasonable. Therefore, compared with existing methods, the novel method is comprehensive and superior in processing MCDM problems.

5.3 The contribution to previous studies

This paper's theoretical results and applications are relatively outstanding compared to previous research.

Theoretically, weak equality is a new definition between similarity and equality. Secondly, the proposed distance measures are taken as a combination of consistency, probabilistic fuzziness, Hausdorff distance and mean. They are an extension of partially existing research. Moreover, the proposed distance measures satisfy triangular inequality, and this property is only found in a few results. New entropy is relatively simple to calculate and provides a tool for determining attribute weights.

For applications, the distance measures are suitable for dealing with unequal-length data. They also have good applicability when compared with existing studies. Weak equality relaxes the constraint on set equality, allowing for a broader application. These comparisons of several examples show that our approach is practical.

6. Conclusion

The different PDHFSs are perfectly equal at zero distance. It is challenging to guarantee this with existing distance measures. Therefore, we defined weak equality. Considering some properties of PDHFSs, new distance measures were proposed and performed well in distinguishing data. We described a similarity measure in conjunction with the correlation between distance and similarity measures. In addition, we developed a new entropy measure and applied it to finding attribute weights. Finally, a pattern recognition method with distance and entropy measures was developed. Moreover, we combined traditional VIKOR methods with the PDHFS environment to construct a new MCDM technology. Through comparative analysis and parameter analysis, we verified the practicality of our proposed methods.

In the future, we will continue to study the weak equality problem of PDHFSs to achieve complete equality. This question will involve investigating more advanced techniques to improve the accuracy of our method. Additionally, we plan to explore the potential of applying distance measures of PDHFSs to other fields, such as stock risk analysis and medical diagnostics.

Appendix A. The proof of Theorem 1

Lemma 1 ([10]). *Let $(\xi_1, \xi_2, \dots, \xi_z), (\eta_1, \eta_2, \dots, \eta_z) \in R^z (i = 1, 2, \dots, z)$, and $1 \leq \lambda < +\infty$, then*

$$\left(\sum_{i=1}^z |\xi_i + \eta_i|^\lambda \right)^{1/\lambda} \leq \left(\sum_{i=1}^z |\xi_i|^\lambda \right)^{1/\lambda} + \left(\sum_{i=1}^z |\eta_i|^\lambda \right)^{1/\lambda}.$$

Lemma 2 ([34]). *Let $\hat{O}_1$, $\hat{O}_2$ and $\hat{O}_3$ be three HFSs, then*

$$\begin{aligned} & \max \left\{ \max_{\gamma_{1j} \in \mathcal{G}_1} \min_{\gamma_{2t} \in \mathcal{G}_2} \|\gamma_{1j} - \gamma_{2t}\|, \max_{\gamma_{2t} \in \mathcal{G}_2} \min_{\gamma_{1j} \in \mathcal{G}_1} \|\gamma_{1j} - \gamma_{2t}\| \right\} \\ & + \max \left\{ \max_{\gamma_{2t} \in \mathcal{G}_2} \min_{\gamma_{3l} \in \mathcal{G}_3} \|\gamma_{2t} - \gamma_{3l}\|, \max_{\gamma_{3l} \in \mathcal{G}_3} \min_{\gamma_{2t} \in \mathcal{G}_2} \|\gamma_{2t} - \gamma_{3l}\| \right\} \\ & \geq \max \left\{ \max_{\gamma_{1j} \in \mathcal{G}_1} \min_{\gamma_{3l} \in \mathcal{G}_3} \|\gamma_{1j} - \gamma_{3l}\|, \max_{\gamma_{3l} \in \mathcal{G}_3} \min_{\gamma_{1j} \in \mathcal{G}_1} \|\gamma_{1j} - \gamma_{3l}\| \right\}. \end{aligned}$$

The proof of Theorem 1 is as follows:

Proof. (D1). Using Eq. (2), know that $\gamma, p, \eta, q \in [0, 1]$. It is easily obtained $0 \leq \iota(x_i), \zeta(x_i), T(x_i), \Upsilon(x_i), \theta(x_i), \psi(x_i), \wp(x_i), \Im(x_i) \leq 1$. Besides, $a + b + c + e = 1$, and $\lambda \geq 1$. So, $0 \leq d_{pgdh}(\hat{O}_1, \hat{O}_2) \leq 1$

(D2). If $d_{pgdh}(\hat{O}_1, \hat{O}_2) = 0$, from Eq. (7), for any $x_i \in X$, we have $\iota(x_i) = 0$, $\zeta(x_i) = 0$, $T(x_i) = 0$, $\Upsilon(x_i) = 0$, $\theta(x_i) = 0$, $\psi(x_i) = 0$, $\wp(x_i) = 0$ and $\Im(x_i) = 0$, then

$$\left\{ \begin{array}{l} \frac{1}{\#\mathcal{G}_1} = \frac{1}{\#\mathcal{G}_2} \\ \frac{1}{\#\kappa_1} = \frac{1}{\#\kappa_2} \end{array} \right\}, \quad \left\{ \begin{array}{l} \sum_{j=1}^{\#\mathcal{G}_1} \gamma_{1j} p_{1j} = \sum_{t=1}^{\#\mathcal{G}_2} \gamma_{2t} p_{2t} \\ \sum_{j=1}^{\#\kappa_1} \eta_{1j} q_{1j} = \sum_{t=1}^{\#\kappa_2} \eta_{2t} q_{2t} \end{array} \right\},$$

$$\max_{\gamma_{1j} \in \mathcal{G}_1} \min_{\gamma_{2t} \in \mathcal{G}_2} \|\gamma_{1j} - \gamma_{2t}\| p_{1j} p_{2t} = 0 \text{ and}$$

$$\max_{\gamma_{2t} \in \mathcal{G}_2} \min_{\gamma_{1j} \in \mathcal{G}_1} \|\gamma_{1j} - \gamma_{2t}\| p_{1j} p_{2t} = 0,$$

where the PDHFEs are sorted in ascending order in this article, thus $\|\gamma_{1j} - \gamma_{2j}\| = 0$, ($j = t$). Similarly, $\|\eta_{1j} - \eta_{2j}\| = 0$, we can get

$$\left\{ \begin{array}{l} \#\mathcal{G}_1 = \#\mathcal{G}_2 \\ \#\kappa_1 = \#\kappa_2 \end{array} \right\}, \quad \left\{ \begin{array}{l} \gamma_{1j} = \gamma_{2j} \\ \eta_{1j} = \eta_{2j} \end{array} \right\}.$$

Besides, for any $x_i \in X$, score function $\lambda(\hat{O}_1(x_i)) = \lambda(\hat{O}_2(x_i))$, accuracy function $\ell(\hat{O}_1(x_i)) = \ell(\hat{O}_2(x_i))$, according to Definition 8, we have $\hat{O}_1 =^w \hat{O}_2$. If $\hat{O}_1 =^w \hat{O}_2$, then $d_{pgdh}(\hat{O}_1, \hat{O}_2) = 0$.

(D3). Straightforward.

(D4). By Lemma 2, we can easily get

$$\begin{aligned} & \max \left\{ \begin{array}{l} \max_{\gamma_{1j} \in \mathcal{G}_1} \min_{\gamma_{2t} \in \mathcal{G}_2} \|\gamma_{1j} - \gamma_{2t}\| p_{1j} p_{2t}, \\ \max_{\gamma_{2t} \in \mathcal{G}_2} \min_{\gamma_{1j} \in \mathcal{G}_1} \|\gamma_{1j} - \gamma_{2t}\| p_{1j} p_{2t} \end{array} \right\} \\ & + \max \left\{ \begin{array}{l} \max_{\gamma_{2t} \in \mathcal{G}_2} \min_{\gamma_{3l} \in \mathcal{G}_3} \|\gamma_{2t} - \gamma_{3l}\| p_{2t} p_{3l}, \\ \max_{\gamma_{3l} \in \mathcal{G}_3} \min_{\gamma_{2t} \in \mathcal{G}_2} \|\gamma_{2t} - \gamma_{3l}\| p_{2t} p_{3l} \end{array} \right\}, \\ & \geq \max \left\{ \begin{array}{l} \max_{\gamma_{1j} \in \mathcal{G}_1} \min_{\gamma_{3l} \in \mathcal{G}_3} \|\gamma_{1j} - \gamma_{3l}\| p_{1j} p_{3l}, \\ \max_{\gamma_{3l} \in \mathcal{G}_3} \min_{\gamma_{1j} \in \mathcal{G}_1} \|\gamma_{1j} - \gamma_{3l}\| p_{1j} p_{3l} \end{array} \right\} \end{aligned}$$

then $\theta_{\hat{O}_1 \hat{O}_2}(x_i) + \theta_{\hat{O}_2 \hat{O}_3}(x_i) \geq \theta_{\hat{O}_1 \hat{O}_3}(x_i)$. Similarly, $\psi_{\hat{O}_1 \hat{O}_2}(x_i) + \psi_{\hat{O}_2 \hat{O}_3}(x_i) \geq \psi_{\hat{O}_1 \hat{O}_3}(x_i)$.

On the other hand, using Lemma 1 and Eq. (7), we have $d_{pgdh}(Q_1, Q_3) \leq d_{pgdh}(Q_1, Q_2) + d_{pgdh}(Q_2, Q_3)$. $\square$

Appendix B. The proof of Theorem 2

Proof. This is similar to the proof of property (D1) - (D3) of Theorem 1. $\square$

Appendix C. The proof of Theorem 3

Proof. Since Eq. (2), we have $\wp(\hat{\partial}), \Im(\hat{\partial}) \in [0, 1]$, thus $\Theta(\hat{\partial}) \in [0, 1]$. Again,

$$\left| \wp(\hat{\partial}) - \Im(\hat{\partial}) \right| \leq \left| 1 + \Theta(\hat{\partial}) \right|, \text{ then } 0 \leq \left| \frac{\wp(\hat{\partial}) - \Im(\hat{\partial})}{1 + \Theta(\hat{\partial})} \right| \leq 1, 0 \leq \left| \frac{\wp(\hat{\partial}) - \Im(\hat{\partial})}{2(1 + \Theta(\hat{\partial}))} \pi \right| \leq \frac{\pi}{2},$$

thus $0 \leq \Xi(\hat{\partial}) \leq 1$.

(E1) If $\Xi(\hat{\partial}) = 0$, then $\sin \left| \frac{\wp(\hat{\partial}) - \Im(\hat{\partial})}{2(1 + \Theta(\hat{\partial}))} \pi \right| = 1 \Leftrightarrow \left| \frac{\wp(\hat{\partial}) - \Im(\hat{\partial})}{1 + \Theta(\hat{\partial})} \right| = 1$, we can get $\wp(\hat{\partial}) - \Im(\hat{\partial}) = 1 + \Theta(\hat{\partial})$ or $\Im(\hat{\partial}) - \wp(\hat{\partial}) = 1 + \Theta(\hat{\partial})$. Because $\Theta(\hat{\partial}) = 1 - \wp(\hat{\partial}) - \Im(\hat{\partial})$, $\wp(\hat{\partial}), \Im(\hat{\partial}), \Theta(\hat{\partial}) \in [0, 1]$, then $\wp(\hat{\partial}) - \Im(\hat{\partial}) = 2 - \wp(\hat{\partial}) - \Im(\hat{\partial})$ or $\Im(\hat{\partial}) - \wp(\hat{\partial}) = 2 - \wp(\hat{\partial}) - \Im(\hat{\partial})$. We have $\wp(\hat{\partial}) = 1, \Im(\hat{\partial}) = 0$ or $\wp(\hat{\partial}) = 0, \Im(\hat{\partial}) = 1$. Therefore, $\Xi(\hat{\partial}) = 0 \Leftrightarrow \hat{\partial} = (\{0|1\}, \{1|1\})$ or $\hat{\partial} = (\{1|1\}, \{0|1\})$.

(E2) If $\Xi(\hat{\partial}) = 1$, then $\sin \left| \frac{\wp(\hat{\partial}) - \Im(\hat{\partial})}{2(1 + \Theta(\hat{\partial}))} \pi \right| = 0 \Leftrightarrow \left| \wp(\hat{\partial}) - \Im(\hat{\partial}) \right| = 0 \Leftrightarrow \wp(\hat{\partial}) = \Im(\hat{\partial})$.

Thus, $\Xi(\hat{\partial}) = 1 \Leftrightarrow \wp(\hat{\partial}) = \Im(\hat{\partial})$.

(E3) By Definition (1-2), then the following conclusions hold: $\wp(\hat{\partial}) = \Im(\hat{\partial}^c)$, $\Im(\hat{\partial}) = \wp(\hat{\partial}^c)$, $\Theta(\hat{\partial}) = \Theta(\hat{\partial}^c)$, which means that

$$\Xi(\hat{\partial}^c) = 1 - \sin \left| \frac{\Im(\hat{\partial}^c) - \wp(\hat{\partial}^c)}{2(1 + \Theta(\hat{\partial}^c))} \pi \right| = 1 - \sin \left| \frac{\wp(\hat{\partial}) - \Im(\hat{\partial})}{2(1 + \Theta(\hat{\partial}))} \pi \right| = \Xi(\hat{\partial}).$$

(E4) Let $\wp(\hat{\partial}_2) \geq \Im(\hat{\partial}_2)$, there have $\wp(\hat{\partial}_1) \geq \wp(\hat{\partial}_2)$, $\Im(\hat{\partial}_2) \geq \Im(\hat{\partial}_1)$, then $\wp(\hat{\partial}_1) \geq \Im(\hat{\partial}_1)$. Here we give a suppose. If $\left| \frac{\wp(\hat{\partial}_1) - \Im(\hat{\partial}_1)}{1 + \Theta(\hat{\partial}_1)} \right| < \left| \frac{\wp(\hat{\partial}_2) - \Im(\hat{\partial}_2)}{1 + \Theta(\hat{\partial}_2)} \right|$, then

$$\begin{aligned} & \left[2 - \wp(\hat{\partial}_2) - \Im(\hat{\partial}_2) \right] \left[\wp(\hat{\partial}_1) - \Im(\hat{\partial}_1) \right] < \left[2 - \wp(\hat{\partial}_1) - \Im(\hat{\partial}_1) \right] \left[\wp(\hat{\partial}_2) - \Im(\hat{\partial}_2) \right] \\ & \Leftrightarrow \wp(\hat{\partial}_2) \Im(\hat{\partial}_1) - \wp(\hat{\partial}_1) \Im(\hat{\partial}_2) + \wp(\hat{\partial}_1) - \wp(\hat{\partial}_2) - \Im(\hat{\partial}_1) + \Im(\hat{\partial}_2) < 0. \end{aligned}$$

We have

$$(16) \quad \left[\wp(\hat{\partial}_2) - 1 \right] \Im(\hat{\partial}_1) + \left[1 - \wp(\hat{\partial}_1) \right] \Im(\hat{\partial}_2) + \wp(\hat{\partial}_1) - \wp(\hat{\partial}_2) < 0.$$

Moreover $1 - \wp(\hat{\partial}_1) \geq 0$, $\Im(\hat{\partial}_2) \geq \Im(\hat{\partial}_1)$, then

$$\begin{aligned} & \left[\wp(\hat{\partial}_2) - 1 \right] \Im(\hat{\partial}_1) + \left[1 - \wp(\hat{\partial}_1) \right] \Im(\hat{\partial}_2) + \wp(\hat{\partial}_1) - \wp(\hat{\partial}_2) \\ & \geq \left[\wp(\hat{\partial}_2) - 1 \right] \Im(\hat{\partial}_1) + \left[1 - \wp(\hat{\partial}_1) \right] \Im(\hat{\partial}_1) + \wp(\hat{\partial}_1) - \wp(\hat{\partial}_2) \\ & = \Im(\hat{\partial}_1) \wp(\hat{\partial}_2) - \Im(\hat{\partial}_1) \wp(\hat{\partial}_1) + \wp(\hat{\partial}_1) - \wp(\hat{\partial}_2) = \left[\wp(\hat{\partial}_1) - \wp(\hat{\partial}_2) \right] \left[1 - \Im(\hat{\partial}_1) \right] \geq 0. \end{aligned}$$

This is contradictory to Eq. (16). Therefore,

$$\left| \frac{\wp(\hat{\partial}_1) - \Im(\hat{\partial}_1)}{1 + \Theta(\hat{\partial}_1)} \right| \geq \left| \frac{\wp(\hat{\partial}_2) - \Im(\hat{\partial}_2)}{1 + \Theta(\hat{\partial}_2)} \right|.$$

When $\wp(\hat{\partial}_2) \leq \Im(\hat{\partial}_2)$, there is $\wp(\hat{\partial}_1) \leq \wp(\hat{\partial}_2)$, $\Im(\hat{\partial}_2) \leq \Im(\hat{\partial}_1)$, similarly

$$\left| \frac{\wp(\hat{\partial}_1) - \Im(\hat{\partial}_1)}{1 + \Theta(\hat{\partial}_1)} \right| \geq \left| \frac{\wp(\hat{\partial}_2) - \Im(\hat{\partial}_2)}{1 + \Theta(\hat{\partial}_2)} \right|,$$

then $0 \leq \left| \frac{\wp(\hat{\partial}_2) - \Im(\hat{\partial}_2)}{2(1 + \Theta(\hat{\partial}_2))} \pi \right| \leq \left| \frac{\wp(\hat{\partial}_1) - \Im(\hat{\partial}_1)}{2(1 + \Theta(\hat{\partial}_1))} \pi \right| \leq \frac{\pi}{2}$. Form the monotonicity of the sine function on the interval $[0, \frac{\pi}{2}]$, we can easily obtain

$$\sin \left| \frac{\wp(\hat{\partial}_2) - \Im(\hat{\partial}_2)}{2(1 + \Theta(\hat{\partial}_2))} \pi \right| \leq \sin \left| \frac{\wp(\hat{\partial}_1) - \Im(\hat{\partial}_1)}{2(1 + \Theta(\hat{\partial}_1))} \pi \right|,$$

then $1 - \sin \left| \frac{\wp(\hat{\partial}_2) - \Im(\hat{\partial}_2)}{2(1 + \Theta(\hat{\partial}_2))} \pi \right| \geq 1 - \sin \left| \frac{\wp(\hat{\partial}_1) - \Im(\hat{\partial}_1)}{2(1 + \Theta(\hat{\partial}_1))} \pi \right|$. So, $\Xi(\hat{\partial}_2) \geq \Xi(\hat{\partial}_1)$. $\square$

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Codes from the incidence matrix of the essential graph

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Abstract. Let R be a commutative ring and M be an R -module. The essential graph over M , denoted by $EG(M)$, is defined as a graph associated to M with vertex set $Z(M) \setminus \text{Ann}_R(M)$, and a pair of distinct vertices x and y are adjacent if and only if $\text{Ann}_M(xy)$ is an essential submodule of M . In this paper, we investigate the linear codes with respect to the Hamming weight from incidence matrix of the essential graphs over M . If $\mathbb{Z}_n$ be the ring of integer module n , then $EG(\mathbb{Z}_n)$ is a linear code. Let p_1 and p_2 be distinct prime numbers. It is shown that if $n = p_1 p_2$, then $C_2(EG(\mathbb{Z}_n)) = [(p_1 - 1)(p_2 - 1), p_1 + p_2 - 2, \min\{p_1 - 1, p_2 - 1\}]_2$. Moreover if $n = p_1^{\alpha_1} p_2^{\alpha_2}$ with $\alpha_i \geq 1$ for $i = 1, 2$, then $C_2(EG(\mathbb{Z}_n)) = [|E|, |V| - 1, \min\{p_1 + 1, p_2 + 1\}]_2$.

Keywords: linear code, incidence matrix, essential graph.

MSC 2020: 94B05, 05C50, 05C38.

1. Introduction

The study of graph theoretic methods and algebraic graph structures over linear codes has attracted a lot of attention in last decades and leads many authors to study and explore their properties, for instance see [1, 5, 6]. Initially, the concept of the zero-divisor graph was studied by I. Beck in [4]. Also, the concept of the zero-divisor graph was studied with a different set of vertices by Anderson et.al

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in [2]. Let R be a commutative ring. The zero divisor graph of R , denoted by $\Gamma(R)$, is the graph with vertex set $Z(R)^* = Z(R) \setminus \{0\}$ and two distinct vertices x and y are adjacent if and only if $xy = 0$. The essential graph of R is a variation of the zero-divisor graph that changed the edge condition, which was studied and introduced in [8]. The essential graph of R is an undirected simple graph, denoted by $EG(R)$, with vertex set $Z(R)^*$, and two distinct vertices x and y are adjacent if and only if $\text{Ann}_R(xy)$ is an essential ideal of R . Recently, the concept of the essential graph and its related topics on modules over commutative rings are studied by many authors (see [10, 11]).

First, we recall some basic properties of coding theory and graph theory which will be used in the sequel. Let $\Gamma(V, E)$ be a simple graph with the vertex set V or $V(\Gamma)$ and the edge set E or $E(\Gamma)$. The order of Γ is $|V|$ and the cardinality of the edge set denoted by $|E|$. A graph with no edge is called the null graph. For every $u, v \in V$, the distance between u and v is defined as the length of a shortest path from u to v and is denoted by $d(u, v)$. We write $u \sim v$ if $d(u, v) = 1$ and $u \not\sim v$ otherwise. The degree of a vertex u , denoted by $\deg(u)$, is the number of edges incident to u . Also, u is called an end-vertex if $\deg(u) = 1$. The minimum and maximum degree of the vertices of Γ are denoted by $\delta(\Gamma)$ and $\Delta(\Gamma)$, respectively. The graph is k -regular if all its vertices have degree k . Assume that u is a vertex of Γ . The open neighborhood of u is defined as $N(u) = \{v \in V(\Gamma) : d(u, v) = 1\}$ and the closed neighborhood of u is $N[u] = N(u) \cup \{u\}$. The graph is connected if there is a path between any two distinct vertices. The diameter of Γ is $\text{diam}(\Gamma) = \sup\{d(x, y) | x, y \in V(\Gamma)\}$, where $d(x, y)$ is the length of shortest path between x and y . The girth of Γ , denoted by $\text{gr}(\Gamma)$, is the length of a shortest cycle contained in the graph; otherwise $\text{gr}(\Gamma) = \infty$. A complete graph is a graph in which each pair of vertices is connected by an edge and a complete graph with n vertices is denoted by K_n . A graph Γ is called bipartite if its vertex set can be partitioned into two parts V_1 and V_2 such that every edge has one end in V_1 and one in V_2 . A complete bipartite graph is a bipartite graph $K_{m,n}$ which consist of an independent set of m vertices of V_1 completely joined to an independent set of n vertices V_2 . A graph H is a subgraph of Γ whose vertex set and edge set are subsets of those of Γ . A dominating set of Γ is a subset D of $V(\Gamma)$ such that every vertex in $V(\Gamma) \setminus D$ is adjacent to some vertex in D . The domination number $\gamma(\Gamma)$ of Γ is the minimum cardinality of a dominating set.

The notation for designs and codes is as in [3]. An incidence structure $\mathfrak{D} = (\mathfrak{P}, \mathfrak{B}, \mathfrak{I})$, with point set $\mathfrak{P}$, block set $\mathfrak{B}$ and incidence $\mathfrak{I}$ is a $t - (\nu, k, \lambda)$ design, if $|\mathfrak{P}| = \nu$, every block $B \in \mathfrak{B}$ is incident with precisely k points, and every t distinct points are together incident with precisely λ blocks. The code $C_F(\mathfrak{D})$ of the design $\mathfrak{D}$ over the finite field F is the space spanned by the incidence vectors of the blocks over F . If Ω is any subset of $\mathfrak{P}$, then we will denote the incidence vector of Ω by n^Ω , and if $\Omega = \{P\}$ where $P \in \mathfrak{P}$, then we will write ν^P instead of $\nu^{\{P\}}$. Thus, $C_F(\mathfrak{D}) = \langle \nu^B | B \in \mathfrak{B} \rangle$, and is a subspace of $F^{\mathfrak{P}}$, the full vector space of functions from $\mathfrak{P}$ to F . For any $w \in F^{\mathfrak{P}}$ and

$P \in \mathfrak{P}$, $\omega(P)$ or ω_P will denote the value of ω at P . If $F = \mathbb{F}_p$, then we write $C_p(\mathfrak{D})$ for $C_F(\mathfrak{D})$. We refer the reader to [1] for undefined terms and conditions.

An incidence matrix $|V| \times |E|$ of Γ is matrix $G = [g]_{ij}$ with $g_{ij} = 1$ if the vertex u_i is on the edge e_j and $g_{ij} = 0$, otherwise. Let W and X be non-empty subsets of V with $W \cap X = \emptyset$ and let $E(W, X)$ be a subset of E which is included one end of W and other end of X . Set $|E(W, X)| = q(W, X)$. An edge-cut of a connected graph Γ , denoted by $\lambda_e(\Gamma)$, is the set S with the property that $\Gamma \setminus S = (V, E \setminus S)$ is disconnected. Also, the edge-connectivity of Γ , denoted by $\lambda(\Gamma)$, is the minimum cardinality of $\lambda_e(\Gamma)$. In the other words, $\lambda(\Gamma) = \min_{\emptyset \neq W \subseteq V} q(W, V \setminus W)$. Clearly, for any connected graph Γ , we have $\lambda(\Gamma) \leq \delta(\Gamma)$, where $\delta(\Gamma)$ is the minimum degree of Γ . The graph Γ is said to be a supper- λ , whenever $\lambda(\Gamma) = \delta(\Gamma)$ and the only edge sets of cardinality $\lambda(\Gamma)$ whose removal disconnected Γ are the set of edges incident with a vertex of degree $\delta(\Gamma)$.

Let $\mathbb{F}_q$ be a finite field with cardinality q and let $A = \{0, 1, \dots, q-1\}$ be an alphabet of order q . The elements of A are called symbols. A q -ary code C of length n and size $|C|$ is a non empty subset of A^n , and any element of C is called a codeword. A q -ary linear code C of length n is a subspace of the vector space $\mathbb{F}_q^n$ over $\mathbb{F}_q$. Let x and y be words of length n over an alphabet A . The Hamming distance from x to y , denoted by $d_H(x, y)$, is defined to be the number of places at which x and y differ. If $x = x_1 \cdots x_n$ and $y = y_1 \cdots y_n$, then $d_H(x, y) = d_H(x_1, y_1) + \cdots + d_H(x_n, y_n)$, where x_i and y_i are words of length 1, and

$$d_H(x_i, y_i) = \begin{cases} 1, & \text{if } x_i \neq y_i \\ 0, & \text{if } x_i = y_i \end{cases}.$$

Note that, $d_H(x, y) = d_H(y, x)$. The minimum Hamming distance of a code C , denoted by $d_H(C)$, is defined by $d_H(C) = \min_{c_1, c_2 \in C} \{d_H(c_1, c_2) | c_1 \neq c_2\}$. Let c be a codeword of $\mathbb{F}_q$. The Hamming weight $wt_H(c)$ is defined by

$$wt_H(c) = \begin{cases} 0, & \text{if } c = 0 \\ 1, & \text{if } c \neq 0 \end{cases}.$$

Let c be a word in $\mathbb{F}_q^n$. The Hamming weight of c , denoted by $wt_H(c)$ (or simply $wt(c)$), is defined to be the number of non zero coordinates in c , that is $wt(c) = d(c, 0)$, where 0 is the zero vector. If $c \in \mathbb{F}_q^n$ with $c = (c_1, c_2, \dots, c_n)$, then $wt(c) = \sum_{i=1}^n w(c_i)$. For any two elements $x, y \in \mathbb{F}_q^n$, we have $d_H(x, y) = wt(x - y)$. The minimum weight of a code C is the smallest among all weights of the non zero codewords of C . For every q -ary linear code C , we have $d_H(C) = wt(C)$. The support of a vector v , denoted by $\text{Supp}(v)$, is the set of coordinate positions where the entry in v is non-zero. Obviously $|\text{Supp}(v)| = |wt(v)|$. A generator matrix G for a linear code C is a matrix whose rows form a basis for C and $C_p(G)$ is a code generated by the matrix G over a finite field F_q and dimension of the code $C_q(G)$ is the rank of the matrix G over the field F_q . A parity-check

matrix for C is a generator matrix for $C^\perp$ where $C^\perp = \{v \in \mathbb{F}_q^n \mid uv = 0, \forall u \in C\}$ is the dual code of C . Here, all the codes are linear codes and the notation $[n, k, d]_q$ will be used for a q -ary code C of length n , dimension k and minimum distance d over the field F_q . For more details we refer the reader to [7].

Let M be an R -module. The annihilator, the set of zero-divisors, the set of associated prime ideals and the minimal set of associated prime ideals of M are denoted by $Z_R(M)$, $\text{Ann}_R(M)$, $\text{Ass}_R(M)$ and $\text{Min Ann}_R(M)$, respectively. Let M be an R -module. Then it is proved that $Z_R(M) = \bigcup_{\mathfrak{p} \in \text{Ass}(M)} \mathfrak{p}$. The submodule N of M is called essential, if $N \cap K \neq \{0\}$ for every non-zero submodule K of M . The radical of an ideal I over commutative ring R , denoted by $r(I)$, is defined to be the set of all elements a of R with $a^n \in I$ for some positive integer n . Then $r(I) = \bigcap_{\substack{\mathfrak{p} \in \text{Spec}(R) \\ \mathfrak{p} \supseteq I}} \mathfrak{p}$, see [9].

Throughout this paper all rings are commutative, all modules are unitary and all codes assumed to be linear codes. Here is a brief summary of the paper. In Theorem 2.2, for Noetherian R -module M with $r(\text{Ann}_R(M)) = \text{Ann}_R(M)$, we show that $C_2(EG(M)) = [|E|, |Z(M) \setminus \text{Ann}_R(M)| - 1, \lambda(EG(M))]_2$, if $|\text{Min Ass}(M)| = 2$. In Theorem 2.3, it is shown that $C_2(EG(\mathbb{Z}_n)) = [(p_1 - 1)(p_2 - 1), p_1 + p_2 - 2, \min\{p_1 - 1, p_2 - 1\}]_2$, whenever $n = p_1 p_2$ where p_1 and p_2 are distinct prime numbers. Also, in Theorem 2.4, it is proved that if $n = p_1^{\alpha_1} p_2^{\alpha_2}$ is an integer, where p_1, p_2 are prime numbers, α_1, α_2 are positive integers and $\alpha_1 \geq 1$ and $\alpha_2 \geq 1$, then $C_2(EG(\mathbb{Z}_n)) = [|E|, |V| - 1, \min\{p_1 + 1, p_2 + 1\}]_2$.

2. Main results

Let R be a commutative ring and let M be an R -module. The essential graph of M , denoted by $EG(M)$, is a graph with vertex set $Z(M) \setminus \text{Ann}_R(M)$ and two distinct vertices $x, y \in Z(M) \setminus \text{Ann}_R(M)$ are adjacent if and only if $\text{Ann}_M(xy)$ is an essential submodule of M , see [10].

The following theorem plays an important role in this paper, so we recall it.

Theorem 2.1 ([5, Theorem 1]). *Let $\Gamma = (V, E)$ be a connected graph and G be a $|V| \times |E|$ incidence matrix for Γ . Then the following statements hold:*

- (i) $C_2(G) = [|E|, |V| - 1, \lambda(\Gamma)]_2$.
- (ii) *If Γ is super- λ , then $C_2(G) = [|E|, |V| - 1, \delta(\Gamma)]_2$, and the minimum words are the rows of G of weight $\delta(\Gamma)$.*

Theorem 2.2. *Let M be a Noetherian R -module with $r(\text{Ann}_R(M)) = \text{Ann}_R(M)$. If $|\text{Min Ass}(M)| = 2$, then*

$$C_2(EG(M)) = [|E|, |Z(M) \setminus \text{Ann}_R(M)| - 1, \lambda(EG(M))]_2.$$

Proof. Assume that c is a non zero codeword in $C_2(EG(M))$ and $\text{Min Ass}(M) = \{\mathfrak{p}_1, \mathfrak{p}_2\}$ where $\mathfrak{p}_i = \text{Ann}_R(m_i)$ with $m_i \in M$ for $i = 1, 2$. Then we conclude that

$c = \sum_{v \in Z(M) \setminus \text{Ann}_R(M)} \mathfrak{p}_i EG_v$, where EG_v is the row of $EG(M)$ corresponding to the vertex v , by [5, Theorem 1]. Let $e = uv$ be an edge of $EG(M)$. Thus, $EG_{u,e}$ and $EG_{v,e}$ are incidence where EG_e is the column corresponding to the edge e . Set $c_e = \mathfrak{p}_1 + \mathfrak{p}_2$, where c_e is a summation of prime ideals $\mathfrak{p}_1$ and $\mathfrak{p}_2$. We consider that $\text{Supp}(c) = \{e = uv | \mathfrak{p}_i \neq \mathfrak{p}_j, \text{ for all } \mathfrak{p}_i, \mathfrak{p}_j \in \text{Min Ass}(M)\}$. Now, suppose that $EG(M)_c$ is a graph with the set of edges $E \setminus \text{Supp}(c)$. Hence, if a, b are adjacent in $EG(M)_c$, then $\mathfrak{p}_1 + \mathfrak{p}_2 = 0$ and so $\mathfrak{p}_1 = \mathfrak{p}_2$ and $|\text{Min Ass}(M)| = 1$. Therefore, $EG(M)$ is a null graph or it has only one vertex, by [10, Theorem 3.2]. Then $EG(M)_c$ is a disconnected graph. Since otherwise, if $EG(M)_c$ is a connected graph, then $c = \mathfrak{p}_1 \sum EG_v = \mathfrak{p}_1(0) = 0$, which is a contradiction. Therefore, $\text{Supp}(c)$ is the set that contained edge connectivity and hence, $wt(c) = |\text{Supp}(c)| \geq \lambda(EG(M))$. Now, we assume that $\text{Min Ass}(M) = \{\mathfrak{p}_1, \mathfrak{p}_2\}$, where $\mathfrak{p}_i = \text{Ann}_R(m_i)$ with $m \in M$, for $i = 1, 2$. Let $u, v \in \mathfrak{p}_1 \setminus \{0\}$. If u and v are adjacent, then $\text{Ann}_R(uv)$ is essential and so $\text{Ann}(uv) \cap Rm \neq 0$ for some $m \in M$. Then there exist $r \in R$ such that $0 \neq rm \in \text{Ann}(uv)$. Thus, $ruv \in \mathfrak{p}_2$. Since $\mathfrak{p}_1 \cap \mathfrak{p}_2 = r(\text{Ann}_R(M)) = 0$. Then $u \in \mathfrak{p}_2$ or $v \in \mathfrak{p}_2$ and so $u = 0$ or $v = 0$, which are contradiction. By similar argument it is shown that the previous result is satisfied for $\mathfrak{p}_1$. Therefore, the vertices of the graph are divided two parts. Assume that W and $V \setminus W$ are two parts which included vertices of the graph with $W \cap (V \setminus W) = \emptyset$. Now, let $S \subseteq E$ where S is the set of edge connectivity and $\lambda(EG(M))$ is the minimum number. Assume that c is a non zero codeword of $\lambda(EG(M))$. Clearly, if $uv \in W$ and so $\text{Ann}(uv)$ is essential, then $\text{Ann}(uv)$ is not essential in S . Now, it is enough to show that the edges of S are edge connectivity between two parts of the graph. Let $u \in \mathfrak{p}_1 \setminus \{0\}$ and $v \in \mathfrak{p}_2 \setminus \{0\}$. Then $uv \in \mathfrak{p}_1 \mathfrak{p}_2 \subseteq \mathfrak{p}_1 \cap \mathfrak{p}_2 = 0$ and so $uvM = 0$, that is, each element of $u \in \mathfrak{p}_1 \setminus \{0\}$ with element of $v \in \mathfrak{p}_2 \setminus \{0\}$ are adjacent. So we assume that $c = \sum_{v \in V} \mu_v EG_v = \sum_{v \in V} \mu_v EG_v$, where if $v \in W$, then $\mu_v = \mathfrak{p}_1$ and if $v \in V \setminus W$, then $\mu_v = \mathfrak{p}_2$. Thus, $uv \in \text{Supp}(c)$ if and only if $\mathfrak{p}_1 \neq \mathfrak{p}_2$ if and only if $uv \in S$. Hence, $wt(c) = |\text{Supp}(c)| = |S| = \lambda(EG(M))$. Therefore, the minimum Hamming weight of incidence matrix over the essential graph equals to the minimum cardinality of the edge connectivity $\lambda(EG(M))$. $\square$

Corollary 2.1. *Let M be a Noetherian R -module with $r(\text{Ann}_R(M)) = \text{Ann}_R(M)$. If $|\text{Min Ass}(M)| = 2$ and $EG(M)$ is a super- λ , then*

$$C_2(EG(M)) = [|E|, |Z(M) \setminus \text{Ann}_R(M)| - 1, \delta(EG(M))]_2.$$

Proof. It follows from Theorems 2.2 and [12, Theorem 2.4]. $\square$

Corollary 2.2. *Let M be a Noetherian R -module with $r(\text{Ann}_R(M)) \neq \text{Ann}_R(M)$ and let $EG(M)$ be k -regular graph. Then $C_2(EG(M))$ is a binary code with minimum weight k .*

Proof. Assume that M is a Noetherian R -module with $r(\text{Ann}_R(M)) \neq \text{Ann}_R(M)$. Then $\text{diam}(EG(M)) \leq 2$ and $gr(EG(M)) \in \{3, \infty\}$, by [10, Theorem 2.6]. Since

$EG(M)$ is k -regular, it is a complete graph. Moreover, G is an incidence matrix $n \times \frac{nk}{2}$. Thus, $C_2(EG(M))$ is a binary code with minimum weight k , by [5, Corollary 1]. $\square$

Example 2.1. Consider the ring $\mathbb{Z}_8$. It is clear that $\text{Ass}(\mathbb{Z}_8) = \{2\mathbb{Z}_8\}$ and $Z(\mathbb{Z}_8)^* = \text{Nil}(\mathbb{Z}_8) = 2\mathbb{Z}_8$. Thus, $EG(\mathbb{Z}_8)$ is a complete graph with 3 vertices, see Figure 1. Moreover, the incidence matrix G is:

$$G = \begin{matrix} & \begin{matrix} e_2 & e_3 & e_1 \end{matrix} \\ \begin{matrix} v_1=2 \\ v_2=4 \\ v_3=6 \end{matrix} & \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{matrix}_{3 \times 3}$$

Hence the minimum weight is 2 and so $C_2(EG(\mathbb{Z}_8))$ is a binary code.

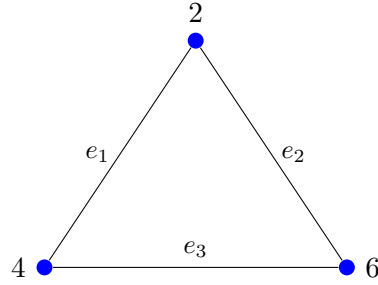


Figure 1: The essential graph $EG(\mathbb{Z}_8)$ with minimum weight 2.

Theorem 2.3. Let $n = p_1 p_2$, where p_1 and p_2 are distinct prime numbers. Let $\mathbb{Z}_n$ be the ring of integers modulo n . Then $EG(\mathbb{Z}_n)$ is a linear code and

$$C_2(EG(\mathbb{Z}_n)) = [(p_1 - 1)(p_2 - 1), p_1 + p_2 - 2, \min\{p_1 - 1, p_2 - 1\}]_2.$$

Proof. Assume that $n = p_1 p_2$, where p_1 and p_2 are distinct prime numbers. Then $Z(\mathbb{Z}_n) = p_1 \mathbb{Z}_n \cup p_2 \mathbb{Z}_n$ and $\text{Nil}(\mathbb{Z}_n) = \{0\}$. It is easy to see that $|Z(\mathbb{Z}_n)^*| = p_1 + p_2 - 1$. Then in view of [10, Theorem 3.7], $EG(\mathbb{Z}_n)$ is a bipartite complete graph. We divided the vertices with two parts:

$$V_1 = \{\text{Ann}(a) \mid \text{Ann}(a) = p\mathbb{Z}/q\mathbb{Z}, a = p_1 k_i, p|q, (p_1, k_i) = 1, 1 \leq i \leq n\};$$

$$V_2 = \{\text{Ann}(b) \mid \text{Ann}(b) = p\mathbb{Z}/q\mathbb{Z}, a = p_2 s_i, p|q, (p_2, s_i) = 1, 1 \leq i \leq n\}.$$

Now, it is enough to find an incidence matrix of $EG(\mathbb{Z}_n)$. Without loss of generality we may assume that $p_1 < p_2$. Then the incidence block matrix of $EG(\mathbb{Z}_n)$ is

$$G = \begin{bmatrix} I_{p_2-1} & I_{p_2-1} & \cdots & I_{p_2-1} \\ \vdots & \vdots & \ddots & \vdots \\ \begin{matrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{matrix} \end{bmatrix}_{|V| \times |E|}$$

where I_{p_2-1} is the $(p_2 - 1) \times (p_2 - 1)$, $1 = [1, 1, \dots, 1]_{1 \times (p_2-1)}$ and $0 = [0, \dots, 0]_{1 \times (p_2-1)}$. Therefore, by Theorem 2.2 and [12, Theorem 3.3], we get that $EG(\mathbb{Z}_n)$ is a code with minimum distance $\lambda(EG(\mathbb{Z}_n))$ and so we conclude that $C_2(EG(\mathbb{Z}_n)) = [(p_1 - 1)(p_2 - 1), p_1 + p_2 - 2, \min\{p_1 - 1, p_2 - 1\}]_2$. $\square$

Example 2.2. Consider the ring $\mathbb{Z}_{15}$. It is clear that $\text{Ass}(\mathbb{Z}_{15}) = \{3\mathbb{Z}_{15}, 5\mathbb{Z}_{15}\}$ and $\text{Nil}(\mathbb{Z}_{15}) = 3\mathbb{Z}_{15} \cap 5\mathbb{Z}_{15} = 0$. In view of [10, Theorem 3.7] and [11, Theorem 2.3], we get that $EG(\mathbb{Z}_{15}) = K_{2,4}$ is a complete bipartite graph with 6 vertices and $D = \{3, 5\}$ is a dominating set for $EG(\mathbb{Z}_{15})$, see Figure 2. In this case

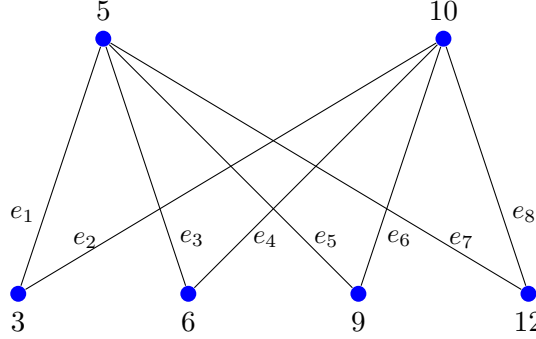


Figure 2: The essential graph $EG(\mathbb{Z}_{15})$ with linear code $[8, 4, 2]_2$.

$$G = \begin{matrix} & \begin{matrix} e_1 & e_3 & e_5 & e_7 & e_2 & e_4 & e_6 & e_8 \end{matrix} \\ \begin{matrix} v_1=3 \\ v_2=6 \\ v_3=9 \\ v_4=12 \\ v_5=5 \\ v_6=10 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \end{matrix}_{6 \times 8}$$

is an incidence matrix for $EG(\mathbb{Z}_{15})$. Then the minimum distance of the code generated by the matrix G is 2 by [12, Theorem 3.3]. Therefore, $C_2(\mathbb{Z}_{15}) = [8, 4, 2]_2$ by Theorem 2.3.

Theorem 2.4. Let $n = p_1^{\alpha_1} p_2^{\alpha_2}$, where p_1, p_2 are prime numbers, α_1, α_2 are positive integers and $\alpha_1 \geq 1$ and $\alpha_2 \geq 1$. Then $EG(\mathbb{Z}_n)$ is a linear code and

$$C_2(EG(\mathbb{Z}_n)) = [|E|, |V| - 1, \min\{p_1 + 1, p_2 + 1\}]_2.$$

Proof. Assume that $n = p_1^{\alpha_1} p_2^{\alpha_2}$, where p_i is prime number and α_i is positive integer with $\alpha_i \geq 1$, for $i = 1, 2$. If $\alpha_1 = \alpha_2 = 1$, then it follows from Theorem 2.3. Without loss of generality we may assume that $\alpha_1 \geq 2$, $\alpha_2 \geq 1$ and $p_1 < p_2$. It is clear that $Z(\mathbb{Z}_n) = p_1\mathbb{Z}_n \cup p_2\mathbb{Z}_n$, $\text{Ass}(\mathbb{Z}_n) = \{p_1\mathbb{Z}_n, p_2\mathbb{Z}_n\}$,

$\text{Nil}(\mathbb{Z}_n) = p_1\mathbb{Z}_n \cap p_2\mathbb{Z}_n$ and $|Z(\mathbb{Z}_n)^*| = p_1^{\alpha_1-1}p_2^{\alpha_2-1}(p_1 + p_2 - 1) - 1$. Now, we divided the vertices by the following subsets:

$$\begin{aligned} V_1 &= \{\text{Ann}(a) \mid \text{Ann}(a) = p\mathbb{Z}/q\mathbb{Z}, a = k_i p_1^{\alpha_1}, p|q, (k_i, p) = 1, 1 \leq i \leq n\}, \\ V_2 &= \{\text{Ann}(b) \mid \text{Ann}(b) = p\mathbb{Z}/q\mathbb{Z}, b = s_i p_2^{\alpha_2}, p|q, (s_i, p) = 1, 1 \leq i \leq n\}, \\ V_3 &= \{\text{Ann}(c) \mid c = s_i p_1 p_2, (p_1, p_2) = 1, 1 \leq i \leq n\}. \end{aligned}$$

where $|V_1| = p_1^{\alpha_1-1}(p_1 - 1)$, $|V_2| = p_2^{\alpha_2-1}(p_2 - 1)$ and $|V_3| = 1$. Then all edges of the graph is $|E| = |V_1||V_2| + |V_1||V_3| + |V_2||V_3|$. In the other words,

$$|E| = \frac{(|V| - 1) - (|V_2| - 1)}{2} - 1,$$

where $|V|$ is the number of vertices. Since $E(V_1)$ is adjacent with $E(V_2)$ and $E(V_3)$ is adjacent with others, we define the incidence matrix of $EG(\mathbb{Z}_n)$ as follows:

$$G = \begin{matrix} & \begin{matrix} E(V_{ij}) & E(V_{ij}) & E(V_{ij}) \end{matrix} \\ \begin{matrix} V_1 \\ V_2 \\ V_3 \end{matrix} & \begin{bmatrix} I & 0 & 0 \\ 1 & 0 & I \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{matrix}_{|V| \times |E|}$$

where I is a $p_1^{\alpha_1} \times p_1^{\alpha_1}$ identity matrix and 0 is a zero block matrix $p_1^{\alpha_1} \times p_2^{\alpha_2}$ in the first row. Also, $1 = [1, 1, \dots, 1]_{1 \times p_1^{\alpha_1}}$ and $0 = [0, \dots, 0]_{1 \times p_1^{\alpha_1}}$, in other rows. In the last column, I is a $p_2^{\alpha_2} \times p_2^{\alpha_2}$ identity matrix and $1 = [1, 1, \dots, 1]_{1 \times p_2^{\alpha_2}}$. Therefore, by Theorem 2.2 and [12, Theorem 3.3], $C_2(EG(\mathbb{Z}_n)) = [n, k, d]_2$, where $n = |E(EG(\mathbb{Z}_n))|$, $k = |V(EG(\mathbb{Z}_n))| - 1$ and $d = \lambda(EG(\mathbb{Z}_n)) = \min_{1 \leq i \leq 2} \{p_i + 1\}$. $\square$

Example 2.3. Consider the ring $\mathbb{Z}_{12}$. It is clear that $\text{Ass}(\mathbb{Z}_{12}) = \{2\mathbb{Z}_{12}, 3\mathbb{Z}_{12}\}$, $\text{Nil}(\mathbb{Z}_{12}) = 2\mathbb{Z}_{12} \cap 3\mathbb{Z}_{12} = 6\mathbb{Z}_{12}$ and $D = \{6\}$ is a dominating set for $EG(\mathbb{Z}_{12})$, see Figure 3. Then

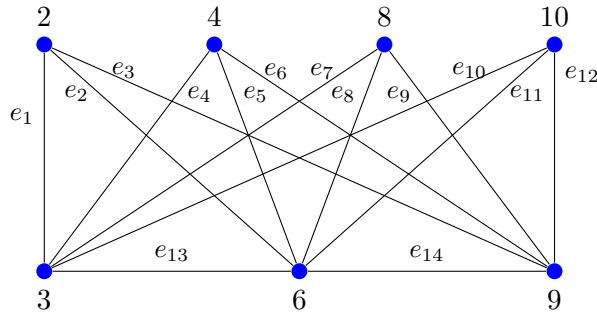


Figure 3: The essential graph $EG(\mathbb{Z}_{12})$ with linear code $[14, 6, 3]_2$.

$$G = \begin{matrix} & \begin{matrix} e_1 & e_4 & e_7 & e_{10} & e_2 & e_5 & e_8 & e_{11} & e_3 & e_6 & e_9 & e_{12} & e_{13} & e_{14} \end{matrix} \\ \begin{matrix} v_1=2 \\ v_2=4 \\ v_3=8 \\ v_4=10 \\ v_5=3 \\ v_6=9 \\ v_7=6 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{matrix} \quad 7 \times 14$$

is an incidence matrix of $EG(\mathbb{Z}_{12})$. Consequently, the minimum distance of the code generated by the matrix G is 3 by [12, Theorem 3.3]. Therefore, $C_2(EG(\mathbb{Z}_n)) = [14, 6, 3]_2$ by Theorem 2.4.

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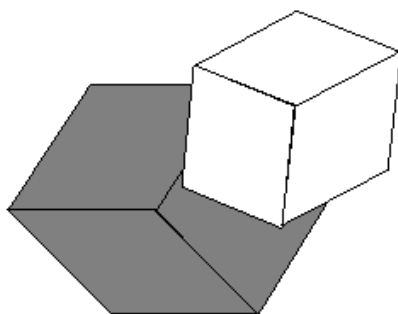
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The role of digital technologies in supporting the teaching of mathematics: experiences of good practices at the university level

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- Engineering students' use of scaffolding elements within digital tasks concerning planar integration domains

Editorial for the Special Issue on “The role of digital technologies in supporting the teaching of mathematics: experiences of good practices at the university level”

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1. Introduction

This Special Issue inaugurates the opening of the Italian Journal of Pure and Applied Mathematics (IJPAM) to contributions in the field of Mathematics Education. The spirit of the IJPAM, since its foundation by Professor Piergiulio Corsini, has been to welcome original articles that range across all sectors of Mathematics, Pure and Applied. In the past there have been some contributions, even if few, in this area. So now, it will be devoted to original research in the fields of pure and applied mathematics and their teaching and learning. The editorial board has also been enriched with experts in Mathematics Education, which is a young research strand compared to other sciences such as Mathematics. Mathematics Education researchers might take the advantage from the development of mathematics and, conversely, mathematicians might take the advantage from the development of didactics. At an institutional level, mathematicians should take an interest in the teaching of mathematics, both to train future generations of mathematicians and to help students to know, understand, and formalize mathematical content. Mathematics educators, on the other hand, play a crucial role in understanding students' ways of thinking

and, building on mathematical contributions, develop methodologies and teaching strategies to transpose these contents. By creating specific situations of discovery, investigation, and institutionalization, educators stimulate students' curiosity, fostering their learning and appropriation of mathematical concepts. Hence the need to welcome contributions in both directions. This dual perspective, where the evolution of mathematics and its didactics intersect, naturally leads to a reflection on the historical and contemporary role of mathematics education in shaping knowledge, as well as the impact of emerging technologies and pedagogical approaches on the teaching and learning process.

Mathematics Education plays a central role in shaping the cultural and scientific knowledge of humanity. As early as antiquity, mathematicians like Euclid and Pythagoras not only developed foundational concepts but also reflected on the methods of teaching them. During the Middle Ages, with the establishment of universities, mathematics became an academic discipline, often linked to logic and philosophy. The Enlightenment and the Industrial Revolution saw mathematics education evolve to meet the practical demands of science and engineering. In the 20th century, the pedagogical theories of Piaget and Vygotsky emphasized the importance of interaction and the active construction of knowledge by the learner. Over time, mathematics teaching has adapted to cultural and technological contexts, transitioning from methods based on memorization and repetition to approaches that prioritize reasoning and problem-solving. The advent of digital technologies has ushered mathematics education into a new era. Tools like GeoGebra and Desmos, along with online platforms and adaptive tutoring systems, have revolutionized how mathematics is taught and learned. These technologies facilitate the visualization of abstract concepts, allow students to explore mathematical relationships interactively and dynamically, and offer opportunities for deeper engagement through multiple pathways ([3]; [5]). They have also redefined the teacher's role, transforming it from a mere transmitter of knowledge to a facilitator and guide in the learning process. However, as highlighted by Noss and Hoyles ([4]), integrating digital tools into education requires careful consideration: it is essential to maintain a focus on deep mathematical thinking, ensuring that digital tools complement rather than replace the development of abstraction, deduction, and argumentation skills.

Today, the primary goal of Mathematics Education is to prepare students to navigate an increasingly complex and interconnected world. Achieving this requires a balance between tradition and innovation, digital tools, and established practices, while fostering critical and creative thinking. As educational research shows, a continuous dialogue between theory and practice remains the cornerstone for building mathematics education that meets the challenges of the 21st century ([1], [2]). On the above premises, the DIGiMATH group (www.digimath.it) was born within the Italian Mathematical Union with the aim of reflecting and understanding on how didactics in presence must be reorganized to integrate and exploit the potential offered by the digital environment,

taking into account that learning processes are strictly dependent on the tools and modes through which they are mediated by the teacher.

The working hypothesis of this group is that specific technological resources can complement face-to-face teaching and help solve the problems identified at the various levels, as long as the integration of digital resources and environments into teaching interventions is carefully designed in relation to the objectives.

2. The role of digital technologies in supporting the teaching of mathematics: experiences of good practices at the university level

This Special Session takes its cue from the workshop of the same name organized in Rome on February 8, 2024 as part of the activities of the DIGiMATH. As a reminder, the purpose of the workshop was to compare and discuss the experiences already underway in Italian universities of practices and research on the use of digital technologies in mathematics education and to promote the development of collaborations between researchers in Mathematics Education and in other areas of Mathematics for the definition of joint practices and research. The themes that emerged, in line with those already identified by DIGiMATH and declined along the three contexts of school/university transition, basic mathematics courses and teacher education courses, were as follows:

- *The use of digital technologies to stimulate meta-cognition*
- *The use of digital technologies to do mathematics*
- *The use of digital technologies to stimulate collaborative processes and foster participation*
- *The use of digital technologies for assessment*

Building on the themes discussed during the workshop, the following sections present a synthesis of the contributions for each of the areas related to the use of digital technologies in mathematics education, offering insights into the practical applications and research findings that emerged in relation to these key aspects.

2.1 The use of digital technologies to stimulate meta-cognition

Cusi and **Telloni** focus their paper on the individualization processes that can be fostered by using digital resources properly designed. They investigate the integration of the expert's scaffolding into the scaffolding provided by the digital resources, with the goal of making this scaffolding more responsive, and the effects of this integration. The presented analysis highlights the usefulness of the role of the expert scaffolding in integrating automatic scaffolding at metacognitive level, both for low-achieving students and for the more competent ones.

The paper of **Albano**, **Mariotti** and **Pierri** describes an experiment in blended learning for university-level mathematics, using a digital platform to

combine quizzes and assignments. The goal is to help students shift from a procedural to a conceptual approach in solving mathematical problems, focusing on the intuitive, algorithmic, and formal aspects of mathematical thinking. The teacher plays a key role in guiding this transition.

2.2 The use of digital technologies to do mathematics

Lepellere, in her article examines, using paper and pencil productions and the answers to a non-anonymous questionnaire, the errors and strategies used in solving three homework tasks regarding visualization and manipulative abilities of boundary surfaces of a three-dimensional region and boundary curves of surfaces. On the basis of the results obtained, four GeoGebra Applets were designed and used during the classroom lessons and the improvements that such use produced were examined. The findings confirm the initial difficulties of the students in drawing sets, expressed through inequalities, in the Cartesian space both for their previous knowledge and for the difficulty in manipulating three-dimensional objects. It shed light on GeoGebra's potential in helping students visualise geometric situations, improve their spatial reasoning, and stimulate the activation of their mental constructions, which are the prelude to knowledge and advanced mathematical thinking.

In her paper, **Taranto** examines the use of the ASYMPTOTE system in a Mathematics course for pre-service teachers, focusing on their involvement in task design. This activity improved both their understanding of mathematical content and exam performance while fostering key teaching skills. Students exceeded expectations by producing more tasks and recognizing task design as a valuable tool for reviewing and deepening course topics. Supported by structured feedback, the process strengthened their disciplinary knowledge and pedagogical abilities. The findings highlight the importance of innovative practices in teacher education to enhance mathematics teaching.

2.3 The use of digital technologies to stimulate collaborative processes and foster participation

In **Dello Iacono** and **Santi** paper, it is described a study involving master's students in a Mathematics Education course, where future teachers used the Quick Chat plugin for group discussions about mathematical misconceptions. The study examines how this tool promotes collaboration and reflection on teaching practices. It shows how Quick Chat helps future teachers deepen their understanding of both mathematical content and teaching methods, integrating online and in-person activities to enhance their professional development.

The paper of **Fiorentino** and **Montone** proposes a new training mode, designed in order to introduce Pre-Service primary Teachers to the pedagogical model of Mathematical Discussion, both from a theoretical and practical point of view. It is based on the combination of a didactical experience lived by the teachers as students in participating to a Mathematical Discussion, and the re-

flective experience involving the same teachers in a new discussion reflecting on their previous lived experience, identifying the key theoretical aspects characterising the experienced pedagogical model of the Mathematical Discussion. The technological environment has been exploited in order to foster collaborative processes and active participation of the teachers.

2.4 The use of digital technologies for assessment

Alessio and **Telloni** propose a sequence of digital tasks involving the representation of subsets of the plane as normal domains in both Cartesian and polar form, starting point for the computation of several categories of double integrals. The tasks contain teacher-driven scaffolding elements, that is immediate and response-specific facilitative feedback, and student-driven scaffolding elements, that are hints of different kinds. Using the combined analysis of video recordings of the students' group interaction with the digital tasks and students' individual answers to a survey concerning their use of the provided scaffolding elements, they identify three categories of behaviors, corresponding to different levels of awareness and ways of grasping the scaffolding.

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Digital resources to promote individualization processes at university level: the role played by the integration between automatic scaffolding and expert scaffolding

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Abstract. This paper reports the results of a study focused on the use of a digital resource designed to promote individualization processes at the university level. The aim of the study was to investigate the interaction between a group of university students enrolled in the first year of a Master's Degree in Management Engineering and the digital resource, with the support of an expert tutor. Specifically, these interactions were analyzed to examine the integration and interplay between the automatic digital scaffolding provided by the digital resource and the scaffolding provided by the expert to help the students overcome the moments of impasse they encountered while solving a series of tasks focused on probability. The analysis we conducted allowed us to highlight the key role of the expert in reinforcing the individualization tools provided by the digital resource.

Keywords: automatic scaffolding, expert scaffolding, digital resources, individualization, metacognitive activities.

MSC 2020: 97K50, 97U50, 97C70.

1. Introduction

The crucial role that digital environments could play in fostering individualized and personalized teaching-learning at the university level has been widely recognized through different studies developed in the last decade that testify the impact that these tools and environments could have on cognitive, metacognitive and affective levels of learning [7, 1, 2, 10].

Within this line of research, we have developed a design-based research to investigate the design and implementation of learning paths within digital environments to promote individualized teaching at the university level [4, 11, 12]. In

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particular, in [11, 12] we presented two educational experiments focused on the design of online teaching-learning paths for first-year Engineering students at the Università Politecnica delle Marche (Italy). Both studies highlighted students' insufficient awareness of their difficulties with mathematical topics and of their needs, a widespread lack of students' metacognitive control, and students' lack of awareness of the individualization of the path in which they are involved. As a result of these deficiencies, students are not able to activate adequate strategies to overcome their difficulties while working in digital environments. For example, they do not use the necessary information to perform tasks, they fail to interpret the feedback provided, or they do not identify appropriate cues that digital environments could provide to scaffold their work.

In this paper we present the results of a study developed from a redesign of one of the tasks of the teaching-learning path presented in [11], aimed at enabling students to activate themselves at the metacognitive level by providing them with feedback that supports their use of digital hints to scaffold their work. The study, which involved a group of university students enrolled in the first year of the Master's program in Management Engineering, aims to investigate the role of the integration between digital scaffolding and expert scaffolding in supporting students to make the most of the individualization tools provided by the digital learning environment.

2. Theoretical background

2.1 The role of scaffolding in promoting individualization at the university level through digital technologies

In the context of university teaching and learning, digital environments have an important role to play in supporting educational processes, both for teachers and students, at different levels. In particular, they allow the promotion of teaching and learning activities tailored to individual needs [3, 7, 20, 25]. Teachers can keep track of students' work in order to characterize their approach in relation to their typical difficulties and learning approaches; students have the possibility to access the resources whenever and wherever they prefer, to deepen mathematical concepts according to multiple representations, and to monitor their own preparation through self-assessment tools.

The present study is framed within a broader research focused on the individualization of teaching-learning paths at the university level [4, 11, 12, 26]. We refer to Baldacci's [5] definition of individualization as the act of differentiating didactic paths in order to enable all students to reach common goals. Individualization is particularly relevant at the university level, where students need to overcome knowledge gaps due to the heterogeneity of their backgrounds and often feel a sense of massification due to a detached relationship with the teacher and classmates [7, 14]. According to Buttler [9], since academic success is based on students' ability to implement strategic approaches and their

development of a positive self-perception, the best way to support students is to provide them with calibrated support that takes into account their existing knowledge, skills, and specific difficulties.

This consideration suggests a shift in focus to the role of scaffolding in supporting the individualization of teaching and learning. Holton and Clarke [16] define scaffolding as "the act of teaching that (i) supports the immediate construction of knowledge by the learner; and (ii) provides the basis for the future independent learning of the individual" (p.131).

Types of scaffolding have been identified in terms of the agents that provide it [16]: expert scaffolding (provided by an expert), reciprocal scaffolding (provided by peers), and self-scaffolding (provided by an individual to herself). Holton and Clarke [16] note that the step from expert to self-scaffolding involves a change in the locus of authority, from external to the learner to internal.

Metacognition is an essential element in students' use of the scaffolding provided and their subsequent development of awareness of the role of scaffolding [16]. This concept is defined by Wilson and Clarke [27] as the awareness that individuals have of their own thinking, their evaluation of that thinking, and their regulation of that thinking. It is important to specify that this definition does not aim to focus on issues that inform metacognitive decision making, but on acts of thinking - acts of metacognition - that operate on cognitive thoughts to assist in the process of learning or solving a problem. According to Wilson and Clarke [27], scaffolding and metacognition rely on the same set of actions: "The outward signs that we see of scaffolding in a learning environment are equivalent to those used by the individual when engaging in metacognition. The external dialogue of scaffolding becomes the inner dialogue of metacognition" (p.141). Meijer et al [18] introduced different categories of metacognitive activities that could support the analysis of the effects of the scaffolding provided to students, particularly in the area of problem solving:

- Orientating refers to utterances focused on identifying prior knowledge to refer to, reflecting on task demands, and sharing feelings about the task;
- Planning refers to utterances that focus on identifying strategies for tackling a particular task;
- Executing refers to utterances focused on carrying out action plans;
- Monitoring refers to utterances related to checking progress in facing and understanding the task;
- Evaluation refers to utterances aimed at evaluating the way the task has been performed;
- Elaboration refers to utterances focused on elaborating on the learning process.

In this study, we focus on computer-based scaffolding [19], which is the “computer-based support that helps students engage in and gain skill at tasks that are beyond their unassisted abilities” (p.26). Belland [8] introduces a number of scaffolding functions that can be activated in digital environments: conceptual scaffolding, strategic scaffolding, metacognitive scaffolding, and motivational scaffolding. In particular, he notes that while conceptual scaffolding guides students in making sense of the data and information they encounter and in identifying things to consider when solving problems, strategic scaffolding aims to get students to develop a strategy for solving problems. Providing this type of scaffolding could produce what Reiser [22] refers to as the fundamental mechanism that digital tools could support, i.e., problematizing aspects of the subject matter rather than simplifying the task, helping students see something as requiring attention and decision making that they might overlook, and getting them to engage with important ideas or processes.

The key-role of metacognitive aspects in guiding the design of scaffolding is emphasized in digital technology-enhanced environments because students’ effective use of the digital scaffolding provided and their subsequent development of awareness about the role of scaffolding requires that they activate themselves at the metacognitive level. Indeed, Sharma and Hannafin [24], in their discussion of the implementation of scaffolding in digital technology-enhanced environments, state that a strength of effective software scaffolding is that it provides consistent support and clarification about basic procedural and metacognitive aspects of a learning task.

2.2 The role of the expert in making digital automatic scaffolding more responsive

The actions of scaffolding are closely related to the notion of responsiveness. In fact, Koole and Elbers [17] suggest that “We argue that the question whether a particular action or utterance by the teacher is an instance of scaffolding cannot be answered in general terms. Scaffolding actions are responsive actions that take the competence the student demonstrates into account. Only by finding out whether a teacher is responsive to previous contributions by the student can we establish if he or she provides the kind of support that can be called scaffolding.” (p.58)

Our previous studies have allowed us to reflect on the importance of focusing on the role that the expert can play in enabling the level of responsiveness of a digital environment. This is in line with Pea’s [23] suggestions that a teacher or other facilitator should support learners through forms of metacognitive scaffolding in selecting appropriate supports when working in digital environments. This requires a balance between the fixed hard scaffolding provided by digital environments and the negotiable soft scaffolding provided by the expert [23].

In line with these ideas, Bakker, Smit and Wegerif [6] note that since intelligent tutoring systems cannot be responsive in the sense that humans can be,

it is crucial to focus not only on the digital tools and their ways of scaffolding students' learning processes, but also on the "whole system in which these tools function as scaffolds" (p.1050).

We therefore adopt the perspective suggested by Reiser [22], who emphasizes that "Scaffolding Requires a System: A final caution to be discussed in exploring models of scaffolding in software tools is that learners, tools, and teachers work together as a system, and it is an over-simplification to consider how tools can scaffold learners without considering the other aspects of this system." (p.298) Therefore, studying the use of computer-based scaffolding in digital environments requires considering the whole system, which consists of three elements: the learner, the digital environment, and the expert who supports the learner's interaction with the digital environment.

Butler [9] suggests that in order to truly support the individualization of learning processes, the expert should design calibrated scaffolding that takes into account students' current levels of performance and what they could achieve with assistance, and also aims to deconstruct supportive scaffolds as students become more independent.

In their analysis of the process by which an expert supports a learner's performance of a particular task, Wood et al. [28] identify six main functions of scaffolding: recruitment (gaining the learner's interest and compliance with the requirements of the task), reduction in degrees of freedom (simplifying the task by reducing the number of constituent acts required to achieve the solution), direction maintenance (keeping the learner on track), marking critical features (highlighting relevant features or parts of the activity), frustration control (reducing the learner's stress without creating too much dependence on the tutor), and demonstration (modeling solutions to a task).

3. Research context and the digital resource used in this study

The context of the research is a Probability and Mathematical Statistics course for Engineering Master's Degree students, taught in the first semester of 2019-20 academic year. Approximately 30 students participated in the course, and 10 of them agreed to participate in our study as volunteers. The study involved the students' individual work with a digital resource within a computer lab, in the presence of an expert, one of the authors, who was in the lab to help the students in case of technical problems or moments of impasse. The students' work with the resource was followed by a semi-structured interview with each of them. These students were familiar with resources similar to the one implemented in the study, although this resource was used by them for the first time during the study.

The digital resource with which the students interacted is the result of a first cycle of design-based research. It consists of a GeoGebra applet designed mainly by exploiting one of its specific features, the conditioned visualization functionality associated with check boxes and insert fields. The choice to use

such functionality is aimed at making the applet provide different feedback and hints according to students' answers.

The redesign of the digital resource was developed on the basis of some critical issues that emerged in a previous study [11], in particular in relation to the students' lack of metacognitive awareness, both in terms of their ability to monitor their processes and their flexibility in using multiple representations. The analysis of the data collected during these studies allowed us to identify a set of criteria for the redesign of the resource [13], which aimed to scaffold students' work at the metacognitive level in order to (a) guide them in the use of the hints provided; (b) support their work by organizing the tasks into subtasks; (c) promote their flexible handling of different representations.

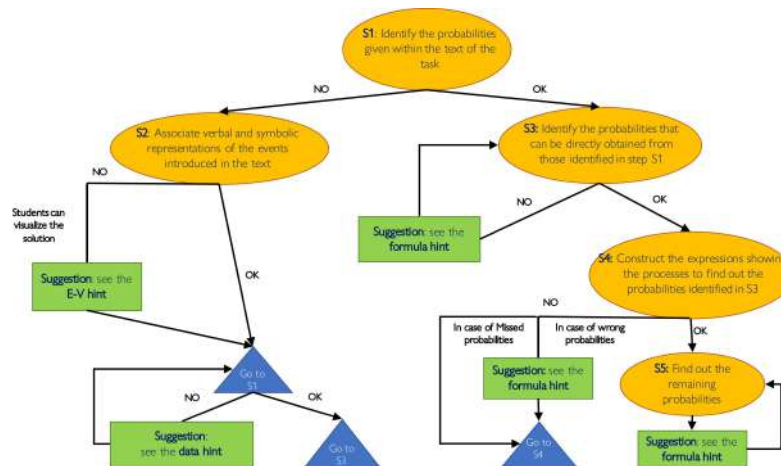


Figure 1: The structure of the digital resource used during the study

The structure of the digital resource is shown in Figure 1. The task that the students have to solve through the resource is to fill in six input fields with elementary, compound and conditional probabilities depending on two events A and B. The digital resource supports the students in solving the task by guiding them through five sub-tasks (corresponding to different steps).

At the beginning (Step S1), a text is provided in verbal language. The text contains three of the six required probabilities. The orange ovals correspond to subtasks that help students fill in the boxes. Students should be able to recognize from the text what the given probabilities are (Step S1) and use probability rules and the definition of conditional probability to find the other probabilities (Steps S3, S4, S5). If the student does not recognize what the given probabilities are, he/she is asked to associate verbal and symbolic representations of the events presented in the text (Step S2).

The arrows and the blue triangles highlight the Steps to which students are addressed according to their answers. The green boxes contain suggestions that are automatically given to students when they fail to complete certain Steps.

Students have five hints available as possible automatic scaffolding for their interaction with the task: a summary of the probabilities given in the text (data hint, suggested when students make mistakes in step S1, after their work on step S2), Eulero-Venn diagrams of compound events (EV hint, suggested in step S2, in case of mistakes), a list of useful formulas (formula hint, suggested in steps S3, S4, and S5, in case of mistakes). In addition, response-specific feedback and scaffolding messages guide students in selecting useful hints and understanding how to use them when they encounter specific difficulties. A screenshot of the digital resource, showing the suggestion given to students when they fail Step S3 (that is to ask for the formula hint) is shown in Figure 2.

Calculate the following events' probabilities, by assuming that

The probability that both A and B verify is $\frac{1}{14}$; the probability that A does not happen is $\frac{3}{4}$. Knowing that B verified, the probability that A realizes is $\frac{2}{3}$.

$P(A)$
 $P(A \cap B)$

$P(\bar{A})$
 $P(\bar{A}|B)$

$P(A|B)$
 $P(B|A)$

$P(B)$
 $P(\bar{B})$

Well done! Now, what probabilities are immediately obtainable by using those available?

☐ $P(A)$
☒ $P(B)$
☐ $P(\bar{A}|B)$

☒ $P(B|A)$
☐ $P(\bar{B})$

submit

If you are in difficulty, choose one or more hints to complete the task.

☐ write the problem data
☐ give me the Eulero-Venn diagrams of the events
☐ give me a calculator
☐ give me the formulae

You could ask for the formula hint to identify the further probability you can determine.

Figure 2: A screenshot of the digital resource

4. Research questions and research method

As mentioned above, the aim of the study documented in this paper is to investigate the ways in which the expert could integrate his/her scaffolding with the automatic scaffolding provided by the digital resource presented in the previous section, especially at the metacognitive level, in order to make this scaffolding more responsive for the students. The researcher who played the role of expert in this study, as the designer of the digital resource, knows the scaffolding elements it provides. At the same time, she is aware of the risk of the automatic scaffolding being unresponsive. However, she had deliberately not defined a priori a set of interventions to scaffold students' work in order to investigate the expert's in-the-moment spontaneous support to students.

We interpret the integration between the expert's and the automatic scaffolding within the scaffolding dynamics, i.e. the interactions within the system constituted by the digital resource, the student and the expert, in terms of co-action [15], which means that two of these actors - the digital resource and the student - alternately play the roles of actors and re-actors during their interaction, while the expert intervenes to support the student's re-reactions when he/she is in a moment of *impasse*. We use the term *impasse* to refer to those

moments when the student is presented with a suggestion to use a particular hint, but gets stuck and may ask for the expert's help.

We investigate both the integration of the expert's scaffolding into the scaffolding provided by the digital resource, with the goal of making this scaffolding more responsive, and the effects of this integration by addressing the following research questions:

1. What kind of difficulties do the students face when they have to react, at the metacognitive level, to the scaffolding provided by the digital resource? What kind of moments of impasse could be observed, during the interaction between the students and the digital resource, in relation to this issue?
2. How does the expert integrate his/her scaffolding with the scaffolding provided by the digital resource in order to help students overcome these moments of impasse? How does the expert's scaffolding take into account the students' specific difficulties at the metacognitive level when carrying out this integration?

As mentioned above, the experiment conducted in this study included the students' individual work with the digital resource and a subsequent a-posteriori interview in which the students were asked to reflect on their interaction with the digital resource and with a tutor (who played the role of expert) who was present in the computer lab to assist the students.

We collected video-recordings of students' screens as they interacted with the digital resource. In addition, to develop an in-depth analysis of students' use of the scaffolding provided by the digital resource, we asked students to think aloud as they completed the tasks, and we audio-recorded their speech. During their interviews, students were shown some excerpts from the video- and audio-recordings of their interaction with the digital resource, asked to read the transcripts of the audio recordings and to reflect on the difficulties they had encountered and on the scaffolding they had received from the digital resource and from the expert.

The data that we analyzed were: the video-recordings of the students' screens, the transcripts of the audio-recordings of the students' work with the digital resource and their a-posteriori interviews. The data analysis was developed in three successive phases.

In the *first phase*, we analyzed the data to identify moments of impasse in their use of the provided scaffolding (suggestion to use specific hints).

In the *second phase*, we focused on each moment of impasse and analyzed the video- and audio-recordings of the students' interaction with the digital resource and with expert to highlight:

- a) the characteristics of the impasses in student's use of the scaffolding provided by the digital resource (to answer research question 1);

- b) the characteristics of the scaffolding provided by the expert to help students overcome their impasse (to answer to research question 2); and
- c) the effects of this scaffolding at the metacognitive level (to answer research question 2).

The characteristics of the impasses in students' use of the scaffolding provided by the digital resource (a) were highlighted by analyzing the data according to the categories of metacognitive activities introduced by Meijer et al. [18] to highlight the interrelationship between specific impasses and the metacognitive activities that were not activated due to the impasses.

The characteristics of the expert's scaffolding (b) and its effects (c) were investigated by analyzing his/her intervention according to both the categories of functions of scaffolding introduced by Wood et al. [28] – to highlight the roles he/she plays – and the categories of metacognitive activities – to highlight the intentions behind his/her interventions and their effects on the students' interaction with the digital resource.

The object of the third phase of analysis were the transcripts of the a-posteriori interviews with students, which were analyzed to find evidence for the results of the previous phase of analysis.

5. Data analysis

In this section we present the results of the data analysis focusing on two students who participated in the study: Maria and Luigi. We chose to focus on these two students because their approaches to using the digital resource and the difficulties they demonstrated were at the extremes of the range of approaches observed.

Each of the following sub-sections focuses on moments of impasse in Maria and Luigi's use of the scaffolding provided by the digital resource in relation to different metacognitive activities:

- scaffolding to support the orientating metacognitive activity (first sub-section);
- scaffolding to support the planning and the executing metacognitive activities (second sub-section);
- scaffolding to support the monitoring metacognitive activity (third sub-section).

In each sub-section, we will present the analysis of excerpts that document the interaction between the students, the digital resource and the tutor. The analysis will be carried out, according to the methodology presented in the previous section, in order to reflect on the ways in which the expert (the tutor) integrates the scaffolding provided by the digital resource to bridge its lack of responsiveness to the specific needs of the students.

The students' comments from the a-posteriori interview will be included in the analysis presented in each sub-section as evidence of the coherence between the results of the analysis and the students' interpretation of the data.

5.1 Moments of impasse related to the scaffolding aimed at supporting the orientating metacognitive activity

5.1.1 Impasse at the orientating level in Maria's interaction with the digital resource

In the following excerpt, Maria is facing S1 (identify the probabilities given in the text of the task), after failing S1 once and then trying to face S2 (associate verbal and symbolic representations of the events introduced in the text of the task).

1. Maria: I try...enter, and then?
2. Tutor: It is recommended to ask for the "data" hint. [Maria clicks on data hint]
3. Tutor: ...uhm, what about it?
4. Maria: It is 4/9...so the first one is wrong...probability that A and B verify...ah, it is not a conditional probability, but the intersection; the second one is ok. The third one...conditional probability...first A than B, instead I put first B than A...
5. Tutor: Well, what do we do? These is the data...how could we interpret it [the data hint]? [Silence]
6. Tutor: Some of the probabilities given are the ones that are required: $P(A \cap B)$ is required...insert the correct value...
7. Maria: Should I correct it?
8. Tutor: Yes, of course
9. Maria: Should I erase here?

The impasse that Maria faces is related to her inability to identify the effective ways to use the data hint that the digital resource automatically suggested to her. In fact, Maria initially focuses only on the need to correct the mistakes she made in S2 when converting from verbal to symbolic representations. The automatic scaffolding provided by the digital resource (consisting of both the suggestion to ask for the data hint and the information contained in the data hint itself) acts at the orientating level, inducing the establishment of task demands and the identification of important information. However, this automatic scaffolding is not responsive enough for Maria, whose metacognitive activity is

blocked at the orientating level. The tutor's scaffolding integrates the one provided by the digital resource through a reformulation of the request in S1 (lines 5 and 6), aiming to suggest how to use the automatic scaffolding to perform this step of the task correctly. This scaffolding turns out to be more responsive for Maria, as can be seen from her a-posteriori interview, in which, referring to this moment, she explains

"I was unsure about how to write in symbols what was in the text...when the tutor intervened, I was more...before I was unsure".

With her intervention, the tutor activates a "direction maintenance" scaffolding function that makes Maria reflect on the role of the data hint; and a "recruitment" scaffolding function that explicitly redirects Maria's attention to the request in S1. Only after the tutor's intervention, which clarifies how to use the automatic scaffolding, does Maria identify, with some difficulty, the first steps she needs to take to complete the task.

5.1.2 Impasse at the orientating level in Luigi's interaction with the digital resource

The excerpt is from the first phase of the activity, when Luigi is facing S1 (identify the probabilities given in the text of the task). The student has inserted some answers that were not requested, so an error message has appeared. Thanks to the a-posteriori reconstruction of Luigi's interaction with subtask S1, we have highlighted that his impasse is due to the fact that he did not read the text of the task carefully:

"I didn't understand what to do, probably I didn't read the task requirement".

1. Luigi: The Bayes' theorem? [Luigi asks the tutor for the Bayes theorem's formula]
2. Tutor: The message says "Are you ready? Let's start! Click here" ... You should click here. [Luigi clicks]
3. Tutor: Now [the tutor reads the message], "What probabilities are given in the text?", so not all of them are required.
4. Luigi: Ok [Luigi deletes some of the answers he entered earlier]
5. Tutor: ...those given in the text, ok.
6. Luigi: And these?
7. Tutor: No, if you want, you can write the probabilities that are not given in the text in a sheet, they can be useful later. Here you can write only the probabilities given in the text. Then you should click "Enter". [Luigi remains silent for a few seconds]

8. Tutor: You should leave here only the probabilities given in the text and delete the others.
9. Luigi: So, I will also delete the probability of the complementary event... ok.

The tutor's interventions act at the orientating level, aiming to focus the student's attention on the requirement of the sub-task. In this case, the problem is that Luigi does not really use the feedback provided because he has not read it. The tutor's interventions (lines 2, 3, 5), aimed at activating the "recruitment" scaffolding function by guiding the careful reading and interpretation of the text, support the orientating metacognitive activity. In this way, the expert scaffolding intervenes and integrates the automatic scaffolding, making it more responsive for Luigi. In his interview, Luigi explains that the tutor's intervention (mainly her suggestion in line 7) was essential to overcome his impasse. In fact, the automatic scaffolding could not support the missed or unfocused reading of the task requirement. In this case, the expert scaffolding integrates the automatic scaffolding in a fruitful way to get the student out of his impasse.

5.2 Moments of impasse related to the scaffolding aimed at supporting the planning and the executing metacognitive activities

5.2.1 Impasse at the planning and executing level in Maria's interaction with the digital resource

The following excerpt refers to the moment when Maria, facing S3 (identify the probabilities that can be directly obtained from those identified in Step S1), receives from the digital resource the suggestion to ask for the formula hint and she decides to follow this suggestion.

1. Maria: ... Well, give me the formula hint... [Silence]
2. Tutor: Well, what can be useful for us?
3. Maria: ... the formula hint?...
4. Tutor: What formula in particular? [Silence]
5. Tutor: For example, how might we calculate $P(A)$?
6. Maria: Is this one? [Maria points to a not useful formula]
7. Tutor: Well, you know $P(\neg A)$... is there a formula that could help you to determine $P(A)$ from $P(\neg A)$?
8. Maria: Is this one? [She indicates the formula for the probability of the complementary event, which is the useful one] [Silence]

9. Tutor: Ok! How can you use this formula? [Silence. Maria selects the box where she should enter $P(A)$. Then, she is blocked again]
10. Tutor: Is there another probability that can be calculated directly? From this formula [She points to the second one] ... You can calculate $P(B)$ from the second formula, right? And then? [Maria is blocked again.]
11. Tutor: $P(B)$ could be calculated from $P(A \cap B)$, so...if you invert this formula, you can find $P(B)$.

Although Maria has asked for the formula hint, she is unable to proceed because she cannot effectively use the hint to identify which probabilities can be obtained directly from the data. Therefore, the tutor's scaffolding is needed to activate the planning metacognitive activity. Indeed, the tutor intervenes by suggesting that Maria focus on one of the probabilities to be found (line 5), then on one of the data available (line 7) and asks Maria to identify a formula that connects them. The tutor's scaffolding aims to "reduce the degree of freedom" of the student's interaction within the resource by identifying a sub-goal with respect to the request in S3. The tutor's intervention also aims to activate the "marking critical features" scaffolding function, allowing Maria to focus on a specific formula for a specific goal. In her a-posteriori reconstruction of this episode, Maria recognizes the key role of the integration of the tutor's scaffolding with the automatic scaffolding, making it more responsive for her:

"I tried to look at all the required probabilities and to compare them with the formulas...the tutor allowed me to overcome my difficulty...the formulas alone were not enough".

5.2.2 Impasse at the planning and executing level in Luigi's interaction with the digital resource

The following excerpt refers to Luigi's work on the sub-task S2 (associate verbal and symbolic representations of the events introduced in the text of the task).

1. Luigi: I do not know $P(B)$, nor $P(\neg B)$, can I find $P(B|A)$? [Then Luigi clicks on "Enter" and a message appears: "To find the probabilities that you need, you could ask for the formula hint to understand which probabilities you can calculate using the ones you have". However, he has already asked for the formula hint.]
2. Luigi [read the message again]: To find the probabilities that you need, you could ask for the formula hint to understand which probabilities you can calculate using the ones you have...
3. Luigi: ...to understand what probabilities you can calculate using the ones you have...so I write $P(A|B)$ is equal to $P(B|A) \dots P(B|A)$ which is... [the tutor walks nearby and Luigi stops her] Can I ask?

4. Tutor: Of course
5. Luigi: What is the formula? I do not remember... $P(B|A)$...
6. Tutor: Well, ... what is written here is enough for you... what is your doubt? [She reads the probabilities that Luigi has inserted]. There is another probability that you can easily find.
7. Luigi: I think it is $P(B)$, but I cannot remember the formula.
8. Tutor: You think $P(B)$...well, what can you use to find $P(B)$?
9. Luigi: Bayes' theorem.
10. Tutor: This is easier. The formula, the definition of conditional probability.
11. Luigi: Uh...
12. Tutor: Which of the formulas in the list can be useful? [Luigi silently reads the list of formulas.]
13. Luigi: The third one is about conditional probability... but I cannot find $P(B)$...
14. Tutor: Exactly. There the conditional probability appears, but you cannot find the probability you are looking for... what is it instead that you can use to calculate $P(B)$?

Luigi explicitly asks for the tutor's help (line 3) after asking for the formula hint. During the a-posteriori interview he explains that he was blocked because he could not remember the formula to use (lines 5, 7). Luigi is therefore blocked at the level of the planning metacognitive activity. According to his a-posteriori reconstruction of his experience, his difficulty was due to a lack of familiarity with the formulas:

"The problem here was that I didn't know the formula... even if I had it [the formula of Bayes' theorem], I might not recognize it immediately".

This lack of familiarity determines that the automatic scaffolding provided by the digital resource is not responsive enough for him. In order to help him overcome this block, the tutor intervenes by "marking critical features", since she suggests looking for a link between the available information, the given formulas and the goal of the sub-task (lines 6, 8, 12, 14). Moreover, when the tutor suggests Luigi to use the conditional probability formula (line 10), she activates the "demonstration" scaffolding function. After the tutor's intervention, Luigi quickly overcomes the impasse and performs sub-task S2 correctly. In this sense, the expert scaffolding makes the automatic scaffolding more responsive

for Luigi, as explained by the student in his a-posteriori interview, where he stresses the crucial role of both the automatic meta-scaffolding provided by the digital resource (on the hints to be required) and the tutor's intervention in helping him to make fruitful use of the hints provided:

“I would be more confused if the suggestions about the hints to ask for were not available”

“If the tutor had not told me that there was an easier way to find the conditional probability I was looking for, I would have spent much more time”.

5.3 Moments of impasse related to the scaffolding aimed at supporting the monitoring metacognitive activity

5.3.1 Impasse at the monitoring level in Luigi's interaction with the digital resource

The following excerpt refers to Luigi's work on sub-task S5 (find the probabilities that were not determined in the previous Steps). Luigi has inserted an incorrect expression for the conditional probability $P(B|A)$, so an automatic feedback appears in the form of an error message. However, the student is not able to recognize his mistake and asks the tutor for help.

1. Luigi: A message told me that these ones are wrong [Luigi indicates the probabilities $P(B|A)$ and $P(A \cap B)$ he has inserted in the input fields]... I don't know... maybe I made a mistake here... maybe I wrote a wrong expression.
2. Tutor: Maybe... maybe some brackets are needed.
3. Luigi: I can try. [Silence] [The tutor talks to another student. Luigi works on the expression of $P(B|A)$, but the error message does not disappear. Then he deletes the expression he has written.]
4. Luigi [reading the expression to find $P(\neg B)$]: One minus...
5. Tutor: Do you need some parentheses? ... Check carefully! There is no division...
6. Luigi [reading]: The product?
7. Tutor: “The product”?
8. Luigi: I should write...?
9. Tutor: Well... how do you find $P(\neg B)$ if you know $P(B)$?
10. Luigi: One minus $P(B)$.

11. Tutor: Well...in the expression of $P(B)$ there is a division...you have to write it.
12. Luigi: Ah, ok. $P(B|A)$ is equal to $P(A|B)$, $10/17$, multiplied by ...in the list of formulas I find the Bayes theorem ... for $P(A)$. $P(A)$ is one minus $2/3$. Everything should be divided by $P(B)$. $P(B)$ is ... [But the message error does not disappear.]
13. Luigi [asking to the tutor]: Is this the correct formula, isn't it? $P(B)$ in the numerator?
14. Tutor: Yes, it is correct. But what is the numerator? What is it equal to?
15. Tutor: ...but it is right [The tutor leaves.]. [Luigi does not follow the tutor's suggestion because his attention was focused on another formula.]
16. Tutor [She approaches Luigi again]: If you look at the list of formulas, what is $P(A|B)$ multiplied by $P(B)$?
17. Luigi: It is...the probability of the intersection of A and B
18. Tutor: Well, ok. It's faster and clearer, isn't it?

The tutor intervenes to help Luigi interpret the feedback provided by the digital resource (lines 2, 5). In particular, she “marks critical features” in order to help Luigi to effectively activate the metacognitive activity of monitoring by making him direct his monitoring focus either to conceptual errors and to possible syntactic errors (lines 2, 5, 9, 11). Finally (lines 14 and 16), the tutor guides the student's monitoring process by trying to help him identify his error by simplifying an expression he has written. When Luigi understands the tutor's suggestion (line 17), he is able to correct his mistake and overcome his impasse.

The student's a-posteriori reconstruction of the episode supports our interpretation, since he declares: “Uhm, at first I didn't understand if I missed the division or the product between the brackets...instead there was a typo... I erased and rewrote several times. I did not know which formula to use, and I wrote it wrong...I did not understand what was wrong, so I tried many times. When the tutor helped me to find the correct formula, I saw the mistake.”

6. Results and final discussion

In this paper we have analyzed the interaction of two students with a digital resource in order to highlight, on the one hand, the students' difficulties in effectively re-acting to the scaffolding provided by the digital resource at the metacognitive level and, on the other hand, the ways in which the expert (the tutor) integrates her scaffolding with the automatic one in order to support the students in overcoming their moments of impasse.

Our analysis of Maria and Luigi's interactions with the digital resource has allowed us to identify three main categories of moments of impasse related to different metacognitive activities. Moreover, the analysis has highlighted the key role played by the tutor's interventions to reinforce or clarify the automatic feedback provided by the digital resource. These interventions are strongly related to the goals of the expert scaffolding that the tutor wants to provide: sometimes her scaffolding is mainly aimed at supporting effective interactions with the digital resource (see section 5.1.2), sometimes the support provided is mainly conceptual (when the student's block is related to unforeseen gaps, as in sections 5.2.2 and 5.3.1), sometimes the scaffolding is aimed at balancing the conceptual level and the level of interaction with the digital resource (for example, when the student needs support to make effective use of the hint received, as in sections 5.1.1 and 5.2.1). Each scaffolding function is primarily associated with one of these goals. For example, while the recruitment function is mainly associated with the goal of supporting the student's interaction with the resource, the demonstration function, when the tutor shows how to use formulas or develop reasoning, is less focused on the level of interaction with the resource.

Despite the identification of these common elements that characterize the interaction within the system constituted by the student, the digital resource and the tutor, our analysis has shown that although some of the impasses that the two students face are similar from a macroscopic point of view (that is, at the level of the whole interaction that characterizes each moment of impasse), their ways of re-acting to the scaffolding provided by the digital resource are different due to their different metacognitive competences. This leads the tutor to re-act to their impasses from a microscopic point of view (i.e. at the level of specific interventions or of short excerpts) by providing them with different forms of scaffolding.

The analysis of the two excerpts in section 5.1, focusing on Maria's and Luigi's impasses at the orientating level, has shown that the tutor's reactions to these impasses are different. This is due to the fact that Maria's impasse is related to her inability to effectively use one of the hints provided, while Luigi's impasse is mainly due to the fact that he did not read carefully the task's request. In fact, the tutor reacts to these impasses in different ways: while she has to activate both the "recruitment" and the "direction maintenance" scaffolding functions to support Maria, she only needs to activate the recruitment function to support Luigi.

The analysis of the two excerpts in Section 5.2, which focus on Maria's and Luigi's impasse at the planning and executing levels, has shown that the tutor's approaches are different because, while Maria again has difficulties in using the provided hint effectively, Luigi's impasse is mainly due to his lack of knowledge about a specific formula. The tutor, therefore, reacts by supporting the two students in different ways. In fact, although in both cases she activates the "marking critical features" scaffolding function to make the students focus on

a specific formula to achieve their goals, she supports Maria by activating the “reduction in degrees of freedom” scaffolding function because the student needs a more structured support, while she activates the “demonstration” scaffolding function to support Luigi because in this case the tutor wants to quickly promote the fading of the scaffolding.

The fact that we have included only one excerpt in section 5.3, related to Luigi’s interaction with the digital resource at the monitoring level, testifies to a further difference between Maria’s and Luigi’s metacognitive competences, since Maria never activated herself at this level. Luigi is supported in overcoming his impasse at the monitoring level by the tutor’s activation of the “marking critical features” scaffolding function.

The results of our analysis highlight the role that expert scaffolding plays in integrating automatic scaffolding for low-achieving students who have serious difficulties at the metacognitive level, especially in interpreting the feedback provided to them (as in Maria’s case). The expert scaffolding also proves to be particularly useful in the case of students who are more competent at the metacognitive level (as in Luigi’s case), when the type of impasse they face requires them to be provided with types of feedback that digital tools alone cannot easily provide.

These reflections suggest that a third cycle of research should be developed with the aim of systematically categorizing the tutor’s interventions in relation to different types of impasse and to different student difficulties at the metacognitive level. This categorization could, in the future, serve as a professional development tool to assist teachers in identifying the most effective expert scaffolding that they could provide to their students in order to make automatic scaffolding more responsive for them.

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A blended teaching and learning environment for developing attitude towards mathematics

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Abstract. This paper reports on the experimentation of a blended learning approach to mathematics at the University level. A first result regards how the teacher's design exploited the didactical potentialities offered by the facilities of a platform, allowing the conceiving specific task based alternatively, on quiz and assignments. Teacher's interventions aim to make students reflect, but mainly to make students focus and interrelate the three fundamental aspects of mathematical thinking: the intuitive, the algorithmic and the formal.

Keywords: digital resources, blended environment, linear algebra, university mathematics education.

MSC 2020: 97C70, 97D40.

1. Introduction

Within the university context, students and teachers encounter difficulties related to some penalising logistical conditions of the university setting, such as the heterogeneity of students' backgrounds and motivations, the high number of students per teacher, and the consequent impossibility of a strong relationship between the learner and the teacher. This implies that students, especially freshmen, experience both learning difficulties at different levels (cognitive, metacognitive and affective) and psychological obstacles (Di Martino & Gregorio, 2019). These obstacles, to be overcome, require the development of autonomy and responsibility of each learner with respect to his or her own learning, as well as a sense of being valued as a person, against the sense of massification that the university context can induce. A number of studies devoted to describing and

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explaining students' difficulties in the transition to University (Clark & Lovric, 2008; Gueudet, 2008; Tall, 2008). They have highlighted as key element the problematic relationship between a procedural approach, privileging calculations and formulas application, and a conceptual approach, based on definitions and proving theorems, i.e dealing with mathematics subjects from a theoretical perspective (Dorier, 2000, Ufer et al., 2017), and as a consequence, the need for a change in students' attitude towards mathematics and mathematics learning. According to the objective of helping students overcome the predominance of procedural knowledge in favour of developing conceptual knowledge (Hiebert and Lefevre, 1986) purposeful teaching experiments were carried out (Albano & Pierri, 2014; Albano, 2017), from which sprouts the study presented in this paper. The main issues in focus were: finding out which Moodle tools offered to support interactivity and designing their use in order to enable students to develop a new attitude towards mathematics knowledge. Pilot studies carried out with university students attending a regular course of "Linear Algebra", led us to design a teaching and learning environment aimed at engaging students outside the classroom and inside the classroom, on the base of the general assumption about the effectiveness of a *blended* approach (Hrastinsk, 2019) combining traditional face-to-face lectures, student's individual interactive purposeful work on platform and the related collective discussions guided by the teacher. In the following, after presenting an outline of our conceptual background and the key elements of the didactical scenario, we will describe the didactical potentialities of the different tools available in the used learning platform, and the general organisation of their use within the blended teaching and learning environment.

2. Theoretical background

As shortly said in the Introduction, there is consensus among researchers that students attending University courses had difficulties to accomplish the requested move from a rote learning of procedures to mathematics as "a scientific discipline based on explicit definitions, deductive proofs, and formal representations" (Ufer et al. 2017, p. 397). This is especially true for these application study courses where mathematics is considered as a "service" subject, that means mathematics aims to provide students with those mathematical competencies required in study courses and their professional practices (Apers, 2020). Let's clarify our assumptions on the requested change.

2.1 Students difficulties and the complexity of mathematical knowledge

The terms procedural and conceptual, referred to mathematical knowledge, have become widespread in the mathematical education literature. A common reference can be the classic definitions given by Hiebert and Lefevre (1986): "*procedural knowledge*" is characterised by managing rules or procedures for solving

mathematical problems; at the same time, because many of the procedures consist in chains of prescriptions for manipulating symbols, procedural knowledge also consists in a managing individual symbols and the syntactic conventions of the representation system” (pp.7-8). Still following these authors, *conceptual knowledge* can be thought of as a connected web of knowledge, a networking in which the linking relationships are as prominent as the discrete pieces of information (pp. 3-4). For this relational character conceptual knowledge is also referred to as *relational knowledge*. It is widely recognized the role that both types (modes) of *knowledge* play in teaching and learning Mathematics. A number of research studies highlight a lack of equilibrium between the two modes, and provide evidence showing that in school practice, often may happen that the procedural prevail over the conceptual (Engelbrecht, Harding & Potgieter, 2005; Baroody, Feil & Johnson, 2007). Applying this distinction showed its effectiveness in describing students’ difficulties and it can also explain those emerging in the transition from high school to university. However, the specific change of focus that students need to get when entering the University, cannot be fully captured by the simple opposition between a mathematics of procedures and a mathematics of concepts. Specifically, what is meant for ‘conceptual knowledge’ requires a more elaborated explanation, overcoming the ambiguity intrinsic of the term knowledge; as a matter of fact, ‘conceptual knowledge’ can be referred both to a formal discipline, organised in a set of properties (axioms, definition, theorems, ...) or to individuals as cognitive agents, for instance, when they solve a mathematical problem. In order to overcome such ambiguity, Fischbein’s discussion (1994) provides an interesting perspective that considers mathematics as a human activity, characterised by the combination of three interrelated aspects: the formal, the algorithmic and the intuitive. All these aspects pertain to Mathematics as human activity and as a formal science, each aspect must have an active part in the mathematical reasoning processes and for this actively used by the student. The formal aspect concerns mathematics as a formal science, that is, the hypothetical deductive construction of theories – axioms, definitions, proofs of theorems – and the feeling of coherence and consistency that must accompany any reasoning or solution process. At the same time, the formal aspect must be related both to the algorithmic aspect, i.e. to the memorised procedures to be applied in problem solving, giving such procedures the reason for application, and to intuition, i.e. that kind of cognition that, immediately accepted without asking any kind of justification, can move problems’ solution processes. According to Fischbein (ibid.) interaction and sometimes conflicts between the three aspects can explain students’ difficulties. Specifically, the difficulty in dealing with mathematics subjects from a theoretical perspective (Gueudet, 2008; Dorier, 2000) can be interpreted as a flaw in the formal aspect, due to its insufficient development at the school level; at the same time a weak or absent relationship between the formal and the algorithmic aspect can explain students’ difficulties in moving from the solution of a single problem, to the solution of a theoretical problem that is a problem

addressing a class of problems. In summary, in the study that we present in this paper we assume that students not only need to develop the interrelationships between the three components but also that students must get awareness of such interrelationships.

2.2 Teaching and learning blended environment

Though the name *blended environment* is used for different educational settings, a general characterization can be shared of blended environment as a combination of face-to-face with distance delivery systems (Osguthorpe & Graham, 2003): asynchronous/synchronous, face-to-face/distance, paper and pencil/digital, students can be engaged in their personal study time/space inside and outside the classroom. Studies show that the online component of a blended environment reduces distractions that are typical in classrooms or lecture halls, increases time-on-task, and improves student performance (Borba et al., 2016). This definition is sufficiently general for describing the nature of what we decided to implement; the various models of blended environment presented in the literature (see Hrastinski (2019) for an overview) differ in the modes of combining and the amount of time devoted to each of the two components, however all of them seem to focus on the students, their personal engagement and the effectiveness of the setting. As anticipated in the Introduction, previous pilot studies make us to develop our fundamental hypothesis concerning the didactical potential offered by combining students online work and traditional face-to-face lectures, as well proposing interactive purposeful work on platform and collective discussions guided by the teacher, focussing on the main difficulties that working on a digital platform might have been evidenced; such hypothesis fittingly resonate with the general hypothesis of combining the effectiveness of a teaching and learning approach that might engage students on both the cognitive and metacognitive level (Schoenfeld, 1985). More specifically, students should be invited not only to solve problems but also to discuss the proposed solutions and their theoretical validation: activities should be proposed that invite students to reflect on their own learning trajectory. Differently, our concern in the design of the blended environment focused on the relationship between the two components, on the basis of the key assumption about the crucial mediation role played by the teacher. In the following section, we are going to describe the main principles driving the design of the architecture of an instructional environment engaging students and teacher with specific activities inside and outside the classroom, exploiting the potentialities of the platform for supporting the teaching and learning process in the way of thinking and reflecting on the necessary change of attitude towards mathematics. In particular, we take into account the Three-dimensional Model for Attitude towards mathematics consisting of the three dimensions: the emotional disposition towards mathematics (I like/don't like), the vision of mathematics (the mathematics is....), the perceived competence in mathematics (I can/cannot do it) (Di Martino &

Zan, 2011). In the following we focus on the vision of mathematics varying from procedural to conceptual knowledge, as specified in the Introduction. Thus, our research goal has been to analyse whether and which features of the implemented blended environment can foster the development and the awareness of interrelation among the formal, the algorithmic and the intuitive aspects.

3. Didactical scenario

The design of the blended environment, above illustrated, sees the involvement of about 250 Computer Engineering freshmen at the University of Salerno, attending a course of Linear Algebra, at the beginning of the second term of the first year. The course develops along 12 weeks, with 3 face-to-face 2-hours classes (both lectures and exercises sessions) per week. The course is mandatory for all students, but participation in online activities is voluntary although strongly recommended. We set a blended instructional environment consisting of traditional face-to-face lectures and a dedicated course space on the e-learning platform Moodle. The Moodle course has been populated by the teacher with didactical material and resources. The didactical material consists of videos showing how to prove some main theorems or how to proceed for solving some typical tasks, books, screenshots of the digital boards used during the lectures, worked-out exercises, slides. The resources consisted of tasks, quizzes, FAQ forum, for reviewing macro-sections of course content, personal wiki. The use of the e-learning platform allows students to have all the course materials available, to use a known environment, and to access their registers of used educational resources. The teacher also has a global view of each student's history. Before each lesson, the students have the theoretical material (a .pdf file). From lecture to lecture he makes available the slides of the lecture he are going to do with the intent that students take only the necessary notes (already having the slides in front of them), not being absorbed in copying the slides students can concentrate on take notes on what the teacher's comments. According to the theoretical framework, we designed the following didactical sequence:

- frontal lecture: introduction of theoretical concepts and results or solving methods, through use of slides or live whiteboard;
- student involvement in interactive activities: the students are required to carry out a quiz or an assignment. The activity is individual in the sense that each student has to hand in his or her own quiz or assignment, but in the classroom they are left free to collaborate (which happens naturally) as well as to consult any teaching materials (from those on the platform, to their own notes, to what is on the web and even to chatGPT);
- classroom discussion: the quiz was made to end with the time break so that the teacher during the break could choose the most meaningful answers for fruitful discussion also based on the a priori analysis that led to the definition of the quiz questions. So, the teacher engineers a collec-

tive discussion, providing formative feedback (Hattie & Timperley, 2007) on possible errors occurred, making constant references to the processes and mathematical theoretical tools underlying the resolution and choices made, supporting the differentiation of the discussion focus by means of different colours. A trace left on on the whiteboard by the development of a discussion and of its interlaced focuses is displayed below (see Figure1).

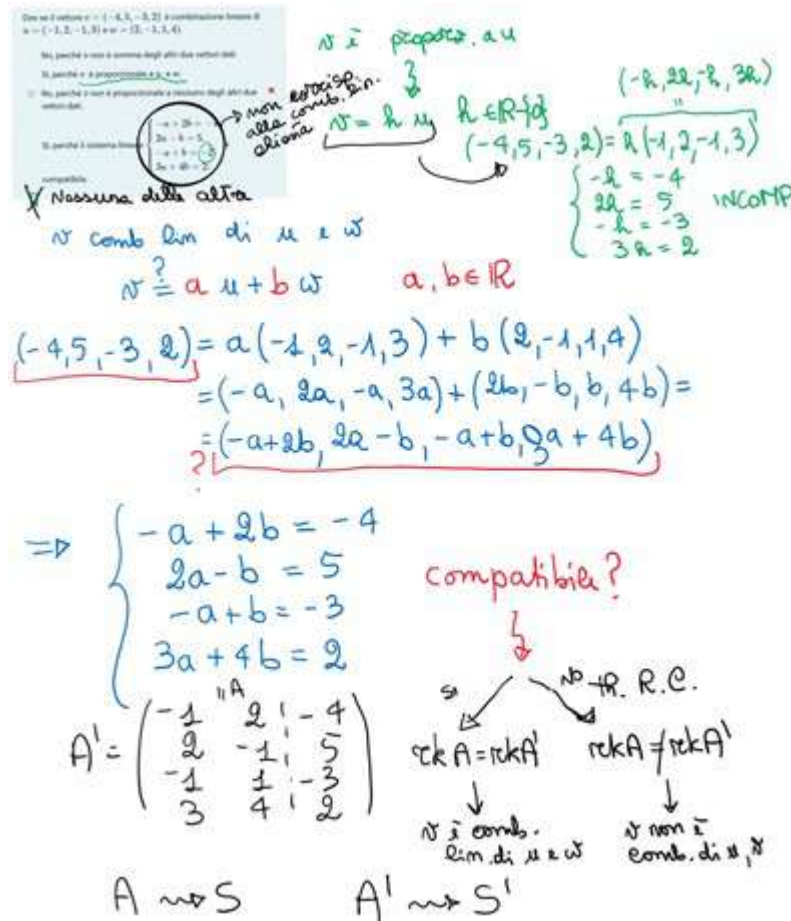


Figure 1: A trace of a collective discussion through a shared whiteboard (e.g. how linearly dependence of vectors are related to resolution of linear systems.)

- the students are required to come back to their performance to reflect on what did or did not work in their solution and why, writing their reflections on their learning portfolio (Albano, et al., 2024), that is implemented as a personal wiki.

4. Moodle resources and their crafting

The core of the teaching approach, as previously mentioned, is based on the interlacement of different types of activities, orchestrated by the teacher; the nature of the activities is related to using different devices designed by purposefully ‘crafting’ different resources made available by the Moodle platform. The main type of activities proposed by the teacher are:

- individual work activities, based on platform resources such as quizzes, assignments,...: these are activities that involve student interaction with a designed teaching resource, and produces for each student an experienced teaching resource (see Figure 2); such experienced teaching resource can be shared among peers, either autonomously, after the suggestion of the teacher.

Figure 2 displays two screenshots of Moodle resources. The left screenshot shows a quiz question about a system of linear equations. The right screenshot shows an assignment question about solving a system of linear equations using Cramer's method and Gaussian elimination.

Left Screenshot (Quiz):

Si consideri il seguente sistema lineare:

$$\begin{cases} x + y - z = -2, \\ 2x - y + 2z = 2, \\ 3x + z = 0. \end{cases}$$

e siano A e A' rispettivamente la matrice dei coefficienti e la matrice completa.

Dire quale delle seguenti affermazioni è vera

Scegli una o più alternative:

- ☒ il sistema ha ∞^1 soluzioni perchè $rkA = 2$ ✗
- ☐ il sistema ha un'unica soluzione perchè ci sono 3 equazioni e 3 incognite
- ☐ Il sistema è incompatibile perchè $|A| = 0$
- ☐ il sistema è compatibile perchè $rkA = rkA'$
- ☒ il sistema è compatibile perchè $|A_{2,3;1,3}|$ è un minore non nullo di ordine massimo sia in A che in A' ✓
- ☐ nessuna delle opzioni è vera

Right Screenshot (Assignment):

Risolvi il seguente sistema lineare sia col metodo di Cramer che col metodo di eliminazione di Gauss. Durante lo svolgimento spiega passo a passo quello che stai facendo e richiama le definizioni e i risultati teorici che giustificano il procedimento che stai usando.

$$\begin{cases} 2x + 4y + 6z = 18 \\ 4x + 5y + 6z = 24 \\ 3x + y - 2z = 4. \end{cases}$$

Applico Gauss:
La matrice iniziale A' è:
(2 4 6 18)
(4 5 6 24)
(3 1 -2 4)
applico le operazioni elementari $r2 \rightarrow r2 - 2r1$ e $r3 \rightarrow r3 - 3r1$ e ottengo la matrice:
(2 4 6 18)
(0 -3 -6 -12)
(0 -10 -22 -44)
applico l'operazione elementare $r3 \rightarrow r3 - 10r2/3$
otengo matrice:
(2 4 6 18)
(0 -3 -6 -12)
(0 0 -2 -40)
il cui sistema associato è:
 $2x + 4y + 6z = 18$
 $-3y - 6z = -12$
 $-2z = -4$
per sostituzione a ritroso ottengo:
 $z = 2$
 $-3y - 12 = -12$ da cui $y = 0$

Figure 2: A Sample of experienced quiz (on the left) and experienced assignment (on the right)

- collective activities, led by the teacher: students are involved in a collective discussion, taking place in the classroom and starting with the experienced teaching resources (see Figure 1). Students are directly challenged by the teacher with questions aiming at soliciting students to come back to the task, rethink their own solution process, and compare it with peers' solutions. Examples of the teacher's questions are: "why?" but also "could we have done - or has anyone done - in another way?", "was there a

possibility of answering without doing any calculations but resorting only to theoretical results?”, “how can we check that what we got is correct?”.

During the discussion the teacher’s interventions aim to make students reflect, but mainly to make students shift the focus from the algorithmic aspects to the intuitive and formal aspects: the objective is to ground the algorithmic aspect on the intuitive meaning but also on its formalisation, to make students connect different macro-topics, to highlight the plurality of possible methods and outcomes, and to set in place control processes.

- intrapersonal activities: these consist of those activities suggested for students to do outside the classroom, such as reviewing slides, whiteboards, videos, working with designed instructional resources, but most importantly they can reflect on what they are learning by drafting, when explicitly requested, their own learning portfolios.

5. Samples of designed resources

In order to illustrate what has been described above, in this section we present individual work activities, based on platform resources such as quizzes and assignments. That in order to show the potential of some specific resources offered by the platform in relation to the goal of shifting the perspective of learners centered on the execution of procedures. We point out the different educational value between quiz and assignment: in the former, the student must understand the solving strategy behind the quiz items; in the latter, the student is free to deploy the solving strategy he or she prefers. In the second part of the section we provide sample of designed resources.

5.1 Quiz vs assignment

Quiz and assignment allow teachers to engage students in individual activities that are the basis of a collective discussion, considered as pivotal to move students towards the development and the awareness of interrelation among the formal, the algorithmic and the intuitive aspects. Let us discuss the differences between the two kinds of activities. While quizzes involve the student selecting options predetermined by the teacher, assignments are questions that the student can solve in the way they see fit. The main difference between the two individual work activities is that quizzes need to manage and coordinate the formal, the intuitive and the algorithmic aspects - often answer items are based on the meaning of the objects or mathematical terms involved in the activity, as distinct from its formalisation in a definition or theorem and from the related algorithms - and are intended to shift students’ attention to the theory or why of calculus/algorithms rather than the calculus/algorithms themselves. Quiz and assignment have a different educational value:

- the assignment is more “traditional”, the student is free to choose his or her resolution strategy (see Figure 2, on the right); he or she takes responsibility for a resolution strategy in which the intuitive and algorithmic aspects intervene, usually well harmonised with each other, circumventing the control of the formal aspect that is not directly called upon by the question;
- the quiz includes questions that present a situation (e.g., a linear system, a vector subspace, etc.) against which scenarios are proposed (through items) that can be true or false (see Figure 2, on the left). The student must identify the true ones. To establish truth, it is not always sufficient to refer to one’s intuitions, much less to the application of procedures. Rather, the student must put in practice some control strategies that refer to theory. The required justifications cannot be based only on the resolution strategy the student would implement if he or she were in the situation presented in the quiz; to answer correctly, the student must understand what the resolution strategy presented is and evaluate its correctness or not.

In the assignment, the student’s knowledge (sometimes only algorithmic) is activated, based on which he or she chooses a solution strategy, implements it, and finds an outcome. In the quiz, solving processes are put in place that are different from those of the task, more complex and mobilizing all three aspects we are interested in. In fact, the student must disengage from what he or she would do and must emphasize with what someone else has done and must check its correctness. Formal control is unavoidable, but even before that, the contribution of the intuitive aspect that guides the interpretation of the proposed situation is essential; the student who moves only with mnemonic procedures is unlikely to be able to govern the quiz. In the architecture of the course, the alternation between assignment and quiz is aimed precisely at giving the student different experiences; on the one hand, the experience of having freedom to move, choosing among one’s own knowledge, those most appropriate for solving a task (working by relating the intuitive and procedural aspects), and on the other hand, to move the student away from resting on having learned only one “way”. Indeed, the request in the quiz to interpret and check the proposed answer is intended to challenge the security of the procedural alone and to relate the intuitive aspect, based on the personal meanings constructed by the learner, and the formal aspect, based on the theory that formalizes them.

5.2 Samples of assignment

In the following we show a sample of assignment and of quiz concerning linear systems. Looking at Figures 3 and 4, we analyse the similarities/differences between the two types of individual work activities.

Stabilisci se il seguente sistema lineare è compatibile o meno; utilizza tutti gli strumenti che conosci per verificare il risultato percorrendo strade diverse. In caso di compatibilità, calcola le soluzioni, utilizzando sia il teorema di Cramer (applicato all'opportuno sistema minimale ridotto equivalente a quello dato) sia il metodo di eliminazione di Gauss. Durante lo svolgimento spiega passo a passo quello che stai facendo e richiama le definizioni e i risultati teorici che giustificano il procedimento che stai usando.

$$\begin{cases} x_1 + 2x_2 - 3x_3 + x_4 &= 3 \\ 2x_1 - x_2 + 2x_3 - x_4 &= 1 \\ -3x_1 + 4x_2 - 7x_3 + 3x_4 &= 1, \end{cases}$$

Figure 3: Sample of assignment on linear system

Let us consider the following linear system:

$$\begin{cases} x + 2y + z + t = 2, \\ 3x + 4y + z + 2t = 5, \\ 2x + 2y + t = 3. \end{cases}$$

Indicate which one(s) of the following statements is true.

- ☐ (2,1,1,-2) is a solution of the linear system
- ☐ $\forall y, z \in \mathbb{R} (z+1, y, z, -2y-2z+1)$ is a solution of the linear system
- ☐ none of the other items is true
- ☐ the system is incompatible
- ☐ the solutions of the linear system are $(x, y, x-1, -2x-2y+3) \forall x, y \in \mathbb{R}$

Figure 4: Sample of quiz on linear system

In the assignment (Figure 3) the student is asked to check compatibility, and to check for correctness with all the ways she knows; compatibility also appears among the items in the quiz question: implicitly where 4-uplet solutions are given; explicitly in item 4. However, the strategies the student implements in the assignment and in the quiz change. In the assignment, the student can use the Rouché-Capelli theorem, then calculate the ranks of coefficient matrix A and of augmented matrix A' and can do so in various ways, top-down (by using rank definition as size of largest non-vanishing minor) and, if she is lucky, find a non-zero minor of order 3 in A which also gives information about the rank of A' ; or bottom-up (by using Kronecker theorem) and even in this case it may happen that the student, starting from a minor of order 2, finds a bordered matrix in A of order 3 that gives information on $\text{rk} A'$. The student may however also decide to reduce the complete matrix of the system in its echelon form so as to check compatibility or otherwise by looking at what the last non-zero row

looks like. In the case of the quiz, the student could use the same strategy as in the assignment if she starts with item 4, but could instead go and check whether one of the quaterns of items 1, 2, 5 is a solution (using the definition of solution, then substituting the 4-uplet in the system and checking the equalities) and if so, she would get the information that item 4 is false for free.

Furthermore, the assignment requires the students to compute, if they exist, the solutions of the system and to do so using two procedures, the first of which assumes the arbitrariness of the choice of the minor that gives the rank on which the reduced minimal system that is solved then depends, and then the description of the solutions. In the case of the quiz, the request is only to verify that certain 4-uplet are solutions, which only implies putting the meaning of system solution into play. Moreover, noting that the numerical quatern is not obtained as a special case of any of the generic 4-uplet provided, in case that one of them is found to be a solution, then one could immediately say that the 4-uplet in item 1 is not a solution. In the case of the quiz, no procedures for computing the solutions of a linear system are required. Figure 5 shows a picture of the whiteboard constructed during the collective discussion related to the quiz of Figure 4.

Si consideri il seguente sistema lineare:

$$\begin{cases} x + 2y + z + t = 2 \\ 3x + 4y + z + 2t = 5 \\ 2x + 2y + t = 3 \end{cases}$$

Dire quale delle seguenti affermazioni è vera

Scegli una o più alternative:

- ☒ le soluzioni del sistema lineare sono $(x, y, z, t) = (2, 1, 1, -2) \forall x, y, z, t \in \mathbb{R}$
- ☒ $(2, 1, 1, -2)$ è soluzione del sistema lineare
- ☐ nessuna delle opzioni è vera
- ☐ $(2, 1, 1, -2)$ è soluzione del sistema lineare
- ☐ il sistema è incompatibile

Handwritten calculations on the whiteboard:

eq 2 = eq 1 + eq 3

$$x + 2y + x - 1 - 2x - 2y + 3 = 2 \quad \checkmark$$

$$3x + 4y + x - 1 + 2(-2x - 2y + 3) = 5 \quad \checkmark$$

$$3x + 4y + x - 1 - 4x - 4y + 6 = 2$$

$$2x + 2y - 2x - 2y + 3 = 3 \quad \checkmark$$

$$2 + 1 + 1 - 2 = 2 \quad \checkmark$$

$$3(2 + 1) + 4y + z + 2(-2y - 2z + 1) = 5 \quad \checkmark$$

$$3 \cdot 2 + 3 + 4y + z - 4y - 4z + 2 = 5$$

$$2(2 + 1) + 2y - 2y - 2z + 1 = 3 \quad \checkmark$$

$$2 \cdot 2 + 2 - 2z + 1 = 3$$

Figure 5: Picture of a whiteboard

Note that both the 4-uplets of items 2 and 4 describe the solutions of the given linear, but most of the students missed both of them or at least one. The teacher started the discussion asking the students how they established the correctness or not of the items. Almost immediately it turned out that the students solved the linear system using Gauss's method and thus they got different solutions, that is the 4-uplet depends on the parameters z and t . After looking for students having acted differently, the whole class arrived at the meaning of solution and thus to the correct algorithm of substituting the 4-uplet in the system for checking the validity of the equalities (Albano et al., 2022).

6. Conclusion

The general issue addressed in our project concerns one of the difficulties, often highlighted in the literature, and emerging at the very beginning of the University courses: overcoming the fixity of a procedural approach and developing a conceptual approach in the solution of mathematical problems. In order to foster in students' change of attitude towards mathematics and mathematics learning, some teaching experiments were carried out, such experiences led to develop a conceptual framework within which to design a complex and articulated structure for a blended learning environment interlacing students' individual work on purposeful online resources and in presence activities, referred to that work and carefully managed by the teacher. As discussed above, the didactical potentialities of a specific blended environment were elaborated on two key assumptions: on the one hand the need of developing the interrelationship between the procedural, the intuitive and the formal aspects that represent the core of mathematical thinking (Fischbein, 1994); on the other hand, the crucial mediating role to be played by the teacher in the development of such interrelationship. Such general principles of design have been consistently articulated in the design of a specific instructional environment experimented with Computer Engineering freshmen at the University of Salerno, attending a course of Linear Algebra, at the beginning of the second term of the first year. What is presented in the previous sections is the discussion of how the didactical potentialities offered by some of the Moodle platform resources can be exploited allowing the conceiving of specific tasks and can be implemented in a specific instructional intervention based alternatively, on quiz or assignments to be accomplished by students, and on collective sessions orchestrated by the teacher. The choice between different types of resources, for instance quiz rather than assignment, is determined by the specific structure of the resource and the educational objective of changing students' attitude towards mathematical knowledge: moving their perspective from just applying a procedure to the theoretical meaning of the answer given by the procedural solution. In particular, in the case of the quiz when it is required to interpret and check the proposed answer, the student is challenged to relate the three aspects: not only the intuitive the procedural but also the intuitive

aspect related to the personal meanings and the formal aspect related to the validity of a statement. During the collective discussion, going back to the reasons that led him/her to answering the quiz, each student has the opportunity to reflect and gain a new point of view. As said, in this ‘reflection’ activities the delicate and fundamental role of the teacher emerges. The analysis developed above illustrates the possible functioning of this dialectics in the specific case of the teaching experiment of a Linear Algebra course. The teaching experiments are still in progress, the first results of students achievements are very encouraging in supporting the arguments presented in this paper. We hope to share these results soon, in a new paper.

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GeoGebra applets and homework tasks to help engineering students with 3D visualization in a multivariable calculus course

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Abstract. In this article, we will examine, for 82 first-year Management Engineering students at Udine University, the difficulties of graphically representing regions in space, surface boundaries of these regions, and curve boundaries of these surfaces, and suggest GeoGebra Applets to overcome them. This included the design, and the analysis of the strategy used, and difficulties encountered in three unfamiliar tasks given as homework, two at the beginning and one during the course, and a non-anonymous Moodle questionnaire. The homework tasks were followed by the design and the use in class of ad-hoc GeoGebra Applets, left available to the students for personal use. The activities will be presented by specifying the aspects indicated in the DiGIMATH good practice guidelines. The potential and limitations of GeoGebra for these subjects will also be illustrated. Since the proposed object of study has not been studied a lot, our research is qualitative and exploratory, based on Duval's theory of registers of semi-otic representation and the Visual-analytical model. Thematic analysis will be used to analyse the responses to the questionnaire.

Keywords: Register of semiotic representation theory, Visual-analytical model, Spatial visualization, Multivariable calculus, GeoGebra, Homework, Questionnaire.

MSC 2020: 97G40, 97I50, 97I60, 97D70, 97D40, 97U70.

1. Introduction

Representation of mathematical concepts is very important in the teaching and learning of mathematics. The ability to represent a mathematical situation in different forms is a very powerful tool in mathematics ([12]). It helps students to understand mathematical concepts, internalise the concepts, and establish connections between the concepts leading to conceptual understanding and creativity. Spatial skills have a vital significance in the acquisition of mathematical knowledge, particularly in the realms of geometry and calculus ([2]). Proficiency in spatial visualization, encompassing the manipulation of figures in three-dimensional space, recognition of patterns, projections, are essential attributes sought in promoting mathematical learning. Research shows that throughout development, there is a consistent, predictive, and strengthening relationship between spatial reasoning and mathematical achievement ([26]).

Particularly, for a comprehensive understanding of the fundamental concepts of calculus in multiple variables, students need spatial visualization abilities that enable them to grasp the relationships between surface transformations in space and the corresponding equations and inequalities ([23]).

Unfortunately Engineering students' incoming knowledge of analytical geometry in three-dimensional space for our experience is very scarce and is often limited to calculating the volumes of solids. During the Linear Algebra course, they begin to see planes and straight lines in space at the end of the course but, due to the lack of time, quadrics are rarely treated.

The purpose of this article (continuing a work started in [20]), is twofold: to investigate difficulties and strategies used by students in drawing and manipulating three-dimensional regions and in identifying their boundary surfaces and boundary curves of these surfaces; to design GeoGebra Applets to help the students to overcome these difficulties. They will be presented by specifying the "good practice" aspects indicated in the Italian Mathematical Union DiGIMATH group guidelines. The group, involving about 50 members from 25 Italian universities and two institutions, develops research projects aimed at a careful integration of technology in teaching and learning at university level (www.digimath.it). For the first purpose of this contribution, we will design and use three unfamiliar homework Tasks, two administered at the beginning and the other during the course, for the second the design of four GeoGebra Applets based on four exercises, linked to the previous three Tasks and presented during the class lessons, have been designed and tested. After the use of the Applets a questionnaire was also proposed to the students asking about the difficulties encountered and strategies used in solving the Tasks, and on the impressions of the use of the GeoGebra Applets. The research questions are the following: 1. What are the students' initial difficulties and strategies used in the representation and manipulation of three-dimensional regions involving quadrics? 2. What graphic perception do they have of the boundary surfaces of a region and boundary curves of a surface? 3. In what way is it possible to obtain improvements thanks to ad hoc GeoGebra Applets?

To design the Tasks and the Applets and to analyse the student productions Duval's theory of register of semiotic representation (TRSR) and the Visual-analytical model (VA) will be considered. Thematic analysis ([5]) will be used to analyse student response to the questionnaires. The potential and weaknesses of using GeoGebra will also be highlighted together with how to take advantage of or overcome them. In this article only the part concerning the drawing of boundary surfaces of a region and boundary curves of a surface, in preparation for Gauss and Stokes' Theorems, will be investigated. The study involved 82 Management Engineering students from the University of Udine, Italy, 56 of them responded to the questionnaire.

2. Literature review

Despite its great relevance for university mathematics courses, mathematics education research on students' understanding and difficulties regarding multivariable calculus and, particularly, vector calculus is still in its early stages. Some past studies have examined students' generalized knowledge of multivariate functions ([18]; [21]; [33]), domain and range ([7]), and derivative ([32]). Some research has touched on student understanding of multivariate integration ([17]; [1]), or on pedagogy related to multivariate integration ([22]).

Research of student understanding in 3D analytic geometry or linear algebra ([8]; [13]; [19]) has shown that, students tend to represent a line in the space by a single implicit linear equation instead of two, or an equation of a circular cylinder with no z -coordinate is interpreted as an equation of a circle. This difficulty can also be observed when analysed in the context of the intersection of surfaces. Manipulation of equations (representing surfaces) that yields a new equation is seen as a solution and it is misinterpreted as a curve of intersection instead of a new surface, or as the projection of that curve to the xy -plane if the equation has no z -coordinate ([4]). Henriques ([16]) in the multivariable integral context considered the volume computations of complex solids and researched students' geometrical and analytical strategies and the role of using Maple in overcoming students' difficulties. He found that students need more support in deriving boundaries of integration and relating them to geometrical representations. He suggested that students should work on activities to parametrize surfaces to produce screen images of volumes and relate these parametrizations to obtain limits of integration. Habre's study ([15]), examines students' difficulties associated with visualizing such surfaces and students' proficiency in using visual imagery when necessary.

The existence of free programs with versatile capabilities and interactive representations helps to improve the presentation of contents, allowing dynamic visualizations. Teaching software packages, such as GeoGebra, Maple, and Mathematica, support multiple representations that can simultaneously demonstrate a function in numeric, algebraic, and graphic models. This feature helps students to understand abstract concepts; therefore, this type of software is often applied in the teaching of calculus. Nobre et al. ([25]), Takaci et. al. ([29]), and Tatar and Zengin ([30]) explored students' learning outcomes in classes that applied GeoGebra to the teaching of calculus. GeoGebra is a software that allows to create different interactive applications that can be used as teaching tools in math classes.

Homework is one of students' opportunities to learn mathematics. Findings from research indicate that undergraduate calculus students spend more time doing homework than they do in class ([11]; [6]). Mathematicians view homework as a critical component of learning ([6]). Moreover, students report that homework and practice exercises are important for their learning, more than other forms of instruction such as lecture. However, undergraduate mathemat-

ics education researchers have only recently begun to study student learning from homework.

3. Theoretical frameworks

Depending on the field of study, visualization has been defined in diverse and multiple ways. A comprehensive and all-embracing definition, that draws together the various aspects of visualization, and serves our purpose in mathematics education, is by Arcavi ([2], p.217):

“Visualization is the ability, the process and the product of creation, interpretation, use of, and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings”.

In mathematics education, to “visualize” means to construct, create, or make connections between an external mathematical object or its representation (a diagram, a table, or a picture) and a mental or internal construct or image and apply analytical methods to develop and advance understanding. Interaction with the mental image can be through physical models, manipulatives, sketches, computer-based static outputs or animations such as simulations.

Comprehension in mathematics, according to Duval, requires the coordination of different registers of representation. For this coordination to be achieved, it is necessary to perform transformations of representations ([9]). Treating a concept within a certain representation (representational mode or register) and successfully converting between different registers of representation, is considered as a prerequisite for conceptual understanding.

Many students’ difficulties can be described and explained by the lack of coordination of different registers of representation ([10]). Research provides examples ([3]; [24]) of students’ difficulties in using the algebraic representation of lines and planes in space, and vice-versa, in recognizing straight lines and planes in space from their equations, and in converting between parametric and implicit viewpoints.

In Duval’s analysis, understanding and learning mathematics require the comparison of similar semiotic representations. According to him, there are two different types of transformations of semiotic representations: treatments, which are transformations of representations that happen within the same representation register, and conversions which consist of changes of representation register without changing the object being denoted. He argues that these transformations are the source of many difficulties in learning mathematics, and that taking them into account helps in overcoming those difficulties: to compare similar representations and treatments within the same register in order to discriminate relevant values of the mathematical object so that the features that

are mathematically relevant and cognitively significant are noticed, and to convert a representation from one register to another to dissociate the represented object and the content of the particular representation introduced so that the register does not remain compartmentalized.

The second theoretical framework used is the visualization-analysis (VA) model proposed by Zazkis' et al. ([34]) to account for flexibility and links between acts of visualization and analysis or analytical thinking in mathematical performance. The model views visual and analytical reasoning as complementing each other in the solution of mathematical problems. The model proposes a series of switches between repeated acts of visualization V1, V2, V3, interspersed with acts of analysis A1, A2, A3 ... following a spiral of steps, each act of analysis leads to a better and richer visual representation followed by more sophisticated analyses.

4. Method

To answer the first two research questions on the difficulties and the strategies used in the representation and manipulation of a three-dimensional region, recognise boundary surfaces and curves, three unfamiliar tasks given as homework, two at the beginning and one during the course were designed and tested together with the answers to a non-anonymous questionnaire inserted as a quiz on the Moodle platform. For all three tasks (Task1, Task2 and Task3), their expected solution in terms of treatment and conversion is discussed in session 4.2 and in 5.1 all the students' solutions were analysed and the strategies used were catalogued; the most representative ones were chosen as examples. To answer the third and final question, four GeoGebra Applets were proposed during class lessons, in relation to four exercises (EX1, EX2, EX3 and EX4) connected with the three previous tasks. After that Task3 was proposed again with the request to draw it again and to write the parametric equations of the surfaces and their intersection curve involved. The Applets will be discussed in 5.2 and in 5.3 the improvements will be examined. We start to summarise the tasks:

Task1. Draw the set

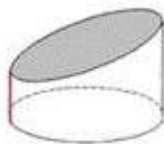
$$A = \{(x, y, z) \in \mathbb{R}^3 | x, y, z \geq 0, x^2 + y^2 \leq 1, z - x \leq 1\}.$$

Moreover, represent graphically what is obtained, respectively, if $x = 0$ or $y = 0$ and $z = 0$ in the respective Cartesian planes.

This task concerns the representation, expressed in the form of inequalities, of a simple quarter of a cylinder with the z -axis cut with a plane parallel to the y -axis. For this cylinder we ask two things: to draw it and to identify and draw the sections on the fundamental Cartesian planes. The choice of the cylinder was made for its simplicity and the inclined plane to have non-trivial sections.

Task2. Imagine cutting a cylinder, obtained by rolling up a sheet of paper, with an oblique plane, as in the figure below. Suppose now cutting the shortest

side dotted in red and unrolling it. Draw the figure that you obtain. Explain how you get it. Do you recognize a function you know on the top of the unrolled figure? If so which one?



Task2 requires, starting from the figure, this time of an entire cylinder, to recognise the function of the boundary curve of the unrolled figure. Task1 and Task2 would make the students reflect on the boundary surfaces of a three-dimensional region and the boundary curves of surfaces, topics that they would address later during the third part of the course which included scalar and vector integral calculus in multiple variables. After having introduced curves, with cartesian and parametric equations, and double integrals with the Gauss-Green formula, which relates this concepts, it was the turn of surface integrals with Stokes' Theorem (which relates integrals of surfaces with the curvilinear integrals of the boundary of the surface). Before this a review of the fundamental conic and quadrics (spheres, cylindroids, cones, paraboloids, hyperboloids, etc.) was given and visualised with the help of GeoGebra. The definition of surface has been given in implicit Cartesian form but above all in parametric form, showing notable examples: graph of two variable functions, surfaces of rotation such as the cylinder, the cone, the sphere and the torus. After that we wanted to test the students' ability to recognise graphically the boundary surfaces of a three-dimensional region and the boundary curve of each surface involved, proposing various tasks. For this article, we chose to analyse the following:

Task3. After drawing the solid between the surfaces $x^2 + y^2 - z = 0$ and $x + y - z + 1 = 0$, graphically identify the boundary surfaces. For each surface found, graphically identify the boundary curves.

This task concerns the graphic representation of the region between a paraboloid and an inclined plane both expressed, this time, in the form of an equality. In this case the two quadrics were known to the students so, that they could concentrate exclusively on the graphic perception of the boundary surfaces of the set and the boundary curves of the two surfaces which was required to be explained.

In summary the exercises and the applets were the following:

EX1. Given the surface $z = 1 - x^2 - y^2$ of parametric equation

$$\begin{cases} x = u \\ y = v \\ z = 1 - u^2 - v^2 \end{cases} \quad \text{with } (u, v) \in T.$$

What happens for different choices of T ?

This is not a real exercise and has the aim of exploring with GeoGebra the effects of the variation of the set T of parameters u and v with the use of Applet1 (<https://www.geogebra.org/m/ebcrctxn>).

EX2. Draw the set $D = \{(x, y, z) \in \mathbb{R}^3 | x, y, z \geq 0, x^2 + y^2 \leq 1, z - x \leq 1\}$ and calculate the total external area.

The set D is the same of Task1, before moving on to the calculation of the area of the external surface of the region, all the surfaces involved were identified graphically using Applet2 (<https://www.geogebra.org/m/nfszf9bk>).

EX3. Given Σ the cylindrical surface $x^2 + y^2 = 1$ between $z = 0$ and $z = 1 + x$ calculate $\iint_{\Sigma} z dS$.

In addition to the calculation of the surface integral on a region like the roll of Task2, attention was then placed with Applet3 (<https://www.geogebra.org/m/wqx2axhc>) on the boundary curve of the surface, anticipating the use of Stokes' Theorem.

EX4. For

$$\Sigma = \{(x, y, z) \in \mathbb{R}^3 | y^2 + z^2 = 4, -1 \leq x \leq z + 1\}$$

oriented outward and the vector field $F = (x^2, z^2, y^2)$ verify Stokes' Theorem.

For the graphic representation of this exercise, the GeoGebra intersection command was used with Applet4 (<https://www.geogebra.org/m/yve3ttgy>).

The students' work on of the three tasks was examined together with the answers to the following three questions of a non-anonymous questionnaire inserted as a quiz on the Moodle platform:

Question1. Task1 required drawing a three-dimensional set and finding the sections in the fundamental planes $x = 0$, $y = 0$ and $z = 0$. Justify your solution, specify whether you used previous knowledge and, if so, where or in what situation you encountered them or, alternatively, what reasoning you used. If you did not do the exercise or part of it, explain why. In the lesson of 22/05, the area of the frontier surface of the set was calculated for the same set. Tell us your impressions of using the GeoGebra Applet <https://www.geogebra.org/m/nfszf9bk> in class on this exercise, if necessary, review it again.

Question2. Task2 required you to imagine what the result of unrolling a cylinder cut by an oblique plane was and whether a function was recognized on the upper part of the resulting plane figure. Explain how you arrived at the indicated

result. If you did not do the exercise, explain why. In the lesson of 22/05, the analytical solution to the exercise was given, describe its effect. Tell us your impressions of using the GeoGebra Applet <https://www.geogebra.org/m/wqx2axhc> in class on this exercise. If necessary, review it.

Question3. Task3 required you to draw the three-dimensional region between a paraboloid and a plane. Tell us what difficulties you had in drawing it and in identifying the boundary surfaces of the set and the boundary curve of the surfaces.

For the questionnaires, thematic analysis was used to identify all the answers in reference to the three main requests of the questions, namely: the strategies, the difficulties and finally their perception of the use of GeoGebra and the Applets in particular. The answers to the first two requests will be analysed within the task results while the last one regarding GeoGebra and Applets will be analysed at the end of their description. Also, in this case, our interest was the treatments and conversions.

4.1 The research context

The research for this contribution was carried out during Mathematical Analysis 2 course in the academic year 23/24 and involved 82 Management Engineering students at the University of Udine, Italy, 56 responded to the questionnaire. Mathematical Analysis 2 is a second course of calculus (9 credits corresponding to 72 hours of lessons) given in the second semester of the first year. It is preceded, among others, by Mathematical Analysis 1 (12 credits), Linear Algebra (6 credits) and it is simultaneous with Physics 1 (9 credits). At the beginning of Mathematical Analysis 2 course, after having introduced the space $\mathbb{R}^3$ as a linear space with scalar (and Fourier series) and vector product, and as a normed space (open, closed sets...), an introduction was made on the various functions treated in the course: scalar-valued and vector-valued functions (curves, surfaces, fields). The initial focus on these functions concerned how to represent them graphically (level curves and surfaces, support for curves and surfaces). The role of the “derivative” is also presented: tangent plane, tangent line, normal vector, curl, and divergence respectively. After this first introduction, the course topics were divided into three macro-sectors, in the order: first and second order differential equations, systems of linear differential equations with simple stability studies; differential calculus in several variables with free and constrained extremes calculations, vector function and implicit functions theorems; the scalar and vector integral calculus with the Gauss-Green formula, Stokes and Divergence theorems. The students’ preparation is constantly monitored through weekly homework tasks, students must photograph their solutions and send them to us via email. They are mandatory for participation to three partial tests on the three macro-topics described previously. The best performances of each task were selected and made available to all students as correction.

4.2 Solution of the tasks and expected reasoning

In Task1 the set

$$A = \{(x, y, z) \in \mathbb{R}^3 | x, y, z \geq 0, x^2 + y^2 \leq 1, z - x \leq 1\}$$

is the internal part of a cylinder with the z -axis cut by a plane of equation $z - x = 1$ parallel to the y -axis. The choice of the cylinder was made both by thinking about the students' previous knowledge, everyone recognizes the cartesian equation in the plane of the circumference with centre the origin and radius 1, and by the simplicity of its sections in the various fundamental planes, obtainable by imagining the figure or by making simple projections. The only element of difficulty should have been the presence of the inclined plane $z - x = 1$. To answer the first request of the task, after the transposition into the three-dimensional space, the students need to convert from the analytic register to the geometric one but also to use their imagination to find the best perspective so that all the elements of the figure are recognizable. To see what happens if $x = 0$, $y = 0$ or $z = 0$, VA could be useful: they could go back to the analytic representation for the three cases, and again to the geometric representation to draw the projections in the planes xy , xz and yz . For example, if $x = 0$, they get in the yz -plane that $y, z \geq 0$, $y^2 \leq 1$, $z \leq 1$, i.e. a square in the graphical register. Finally, to see if the set represented was correct, it is expected that the students could reassemble the projections to reobtain the three-dimensional solid passing again to the geometric register in 3D. For the students this is a complex task, treatments and conversions are needed in VA sequences, as well as spatial orientation, rotation and translation of geometric figures using mental visualization to find the best view. As suggested by DiGIMAT "good practice", we not only want to identify the obstacles encountered by students in carrying out the task but also to see the strategies used to solve it. These latter will be the basis for the design of the Applet2.

The initially perceived figure of Task2 reveals a circular base, an elliptical top, and a curved surface that make the sides of the truncated cylinder. The solution, as seen in Figure 1, is a sinusoid shape.

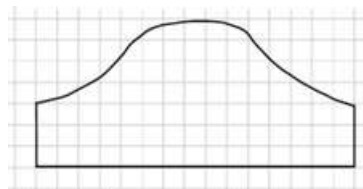


Figure 1: Student solution of Task2

Three possible solutions for the top were expected: a triangle, a semicircle and a sinusoid ([27] p.5). The intersection looks elliptical. One way to find

the solution is, as suggested in the task test, to unfold the truncated cylinder and check its graph, but this has obvious limitations in accuracy. How do we show (analytically) that the third is the correct answer? In Stylianou ([28], p. 314) a mathematics educator proposed the following solution: Suppose that the cylinder has the equation $x^2 + y^2 = 1$ and the plane has the equation $z - x = 1$, passing to the polar coordinates of the base $(\cos(\theta), \sin(\theta))$, the height z has the trigonometric expression $z = 1 + \cos(\theta)$, i.e. a sinusoid. In this case, the previous analytical solution was not expected from the students, and it will be provided later in the course with GeoGebra Applet3. For this task, the treatment going from the 3D to the 2D representation and a conversion to the analytical/graphic recognition of the boundary will be analysed.

Task3 regards the graphical representation of the three-dimensional region delimited by the paraboloid of equation $z = x^2 + y^2$ and the oblique plane $x + y - z + 1 = 0$, both surfaces known to the students. In addition to choosing the Cartesian axes appropriately for a good graphic representation, the expected difficulty was identifying the intersection curve that delimited both the surface given by the plane and that given by the paraboloid. Task3 thus required sequential VA steps that could also be thought of as the achievement of 4 levels. Level1: the two surfaces are recognised and drawn separately. Level2: the two surfaces are drawn on the same graph and highlighted separately. Level3: there is an attempt to identify the boundary curve, but it is not correct. Level4: the intersection curve is correctly identified and consequently also the two surfaces. The latter case is more easily recognisable if a conversion from the geometric to the analytical register was done: by working simultaneously with both equations, the circumference of equation $(x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 = \frac{3}{2}$ is obtained in the xy -plane, and then switching back to the geometric register. Reaching Level4, at least Level3, is desirable to obtain the correct parametric equations of the two surfaces and that of the boundary curve (but also for a possible triple integral calculation if the chord method is chosen). Also, for this task we will present how the students solved it, in particular which level was achieved, together with the strategies used and/or the difficulties encountered.

5. Results

The first paragraph begins with the analysis of students' solutions of Task1, Task2 and Task3 along with the strategies used. The most representative ones have been chosen as examples with the answers to the questionnaire. In the second paragraph, four GeoGebra Applets have been designed in relation to four exercises (EX1, EX2, EX3 and EX4) and linked to the three previous tasks. The answer to the questionnaire on their use will also be analyzed. Finally, in the last paragraph, the results of student work improvements after the use of GeoGebra Applets will be presented.

5.1 Tasks analysis

5.1.1 Task1

Before going into the detail of the analysis of the students' paper and pencil work on Task1, it is necessary to make two premises, both regarding previous results on the same students investigated. The first is about the students' perceptions of their initial knowledge, to the question: "Before the Analysis 2 course, what was your knowledge of three-dimensional space and where did it come from?", 23% wrote that it was absent and 35% basic, 25% wrote that it came from the Linear Algebra course and 12% from high school, 1% from technical drawing lessons and 5% did not answer. Therefore, more than half of the students declare to have very little knowledge and only 1/4 that such knowledge comes from the previous Linear Algebra course. The second regards the results, which are consistent with the research found in the literature, about planes and cylindroids: even 22% of the students were unable to recognize and draw $x = 1$ as a plane in the three-dimensional cartesian space but drew a straight line or even a point (this in four cases of 82), 16% did the same thing for the plane $x + y = 1$ and 24% with the cylindroid $y = -x^2 + 1$. Only 17% managed to do all these three tasks correctly. Therefore, as to Task1, it is not surprising that less than half of them, precisely 7%, managed to correctly draw the whole set and the various sections and 38% tried to draw a part of them, the remaining 62% did not perform the exercise. As we see in Table 1, of these 38% just 24% completed the first part of the task correctly. 12% at least recognized that $x^2 + y^2 = 1$ was a cylinder, but 6% drew a sphere and 3% a paraboloid, even 50% drew instead a circumference cut by a straight line in space. The remaining 5% did not draw or describe it in words.

Correct	Circum. line	Cylinder	Sphere	Paraboloid	No draw
24%	50%	12%	6%	3%	5%

Table 1: Results of first part of Task1

Finally, 26% drew all three sections correctly and 12% forgot the positivity conditions. These results confirm the initial difficulty of students to represent three-dimensional sets in space, partly due to the lack of previous knowledge but also, as we will see from the analysis of some students' work, of coordination between 2D and 3D semiotic registers.

Examining the students' works, two strategies were identified for Task1: starting from drawing the cylinder and the plane and from these identifying either analytically or geometrically which were the sections with the fundamental planes, in Figure 2 there are two examples; or starting first from the sections and then build the three-dimensional figure from those as in Figure 3.

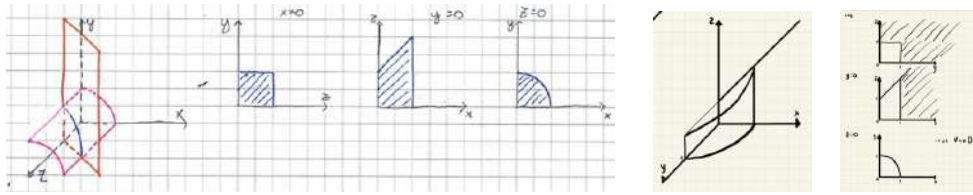


Figure 2: Same strategies but different choice of the axis 2a on the left and 2b on the right

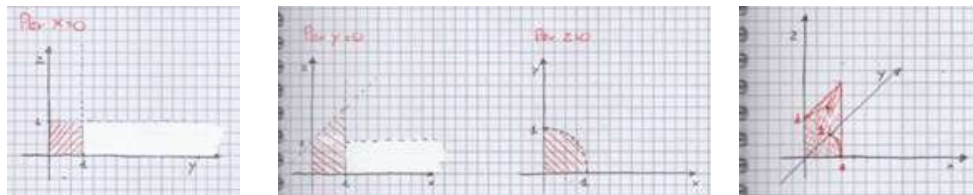


Figure 3: Second strategy

The confirmation of this procedure comes from the answers to Question1 of the questionnaire.

- 2a. *“I set first the constraint $x, y, z \geq 0$ to be able to simplify the work space and have to manage only the space of positive x, y, z instead of working with the whole space, after which I evaluated the other inequalities as equations by drawing then the figures obtained (plane and cylinder) evaluating where they intersected and then colouring the intersection in blue and reporting the projection on the zy , xz and xy planes”*
- 2b. *“The procedure was sequential: first of all, having x, y and z greater than or equal to 0 implies that we only work in one of the eight parts of the space, so, I only represented that; then I imagined the second condition as a cylinder, given that if we reasoned in two dimensions it would represent a circle; finally, I depicted the third condition, given that, if we reasoned in two dimensions, this would be a straight line, like a plane that cuts the previously depicted cylinder”.*

We can first notice how both students followed the order indicated in the analytic description of the set. In either case the sections are already visible from their very good three-dimensional drawing. There were students' VA transitions from geometric to analytical and again to geometric register in order to have a precise drawing of the sections as we expected. In fact, for example, the writing “ $-x \leq 1 \forall x \in D$ ” of 2b in Figure 2, where D is the disk, presupposes that the analytical calculation was done. The difference between the two drawings is the choice of

the position of the three cartesian axes which is fundamental, in fact the choice of representation of 2a was more effective. Moreover, those who chose the xyz sequence for the axes, as they are used to do from high school, obtained an unclear drawing, as can be seen in Figure 3.

Moreover, the student in Figure 3 used the second strategy as can be seen from his work but also from what he wrote in the questionnaire: *"I started drawing the sections: For $x = 0$: Through the first constraint I know that $y \geq 0$ and $z \geq 0$, so I have excluded all quadrants except the first, then from the second constraint I know that $x^2 + y^2 \leq 1$ and that y is a quantity greater than zero, so, the maximum value that y can reach is 1. I did the same reasoning for the third constraint, and I get $z \leq 1$. [...] I drew the respective section on each plane and then I connected them"*. In this answer the student also described all the steps he took to obtain the sections that are those expected.

For drawing the sections, there were two cases in which despite the request that, for example $x = 0$, a three-dimensional set was drawn which is, in this case, a parallelepiped as in the case of Figure 4 (indicated with a red row).

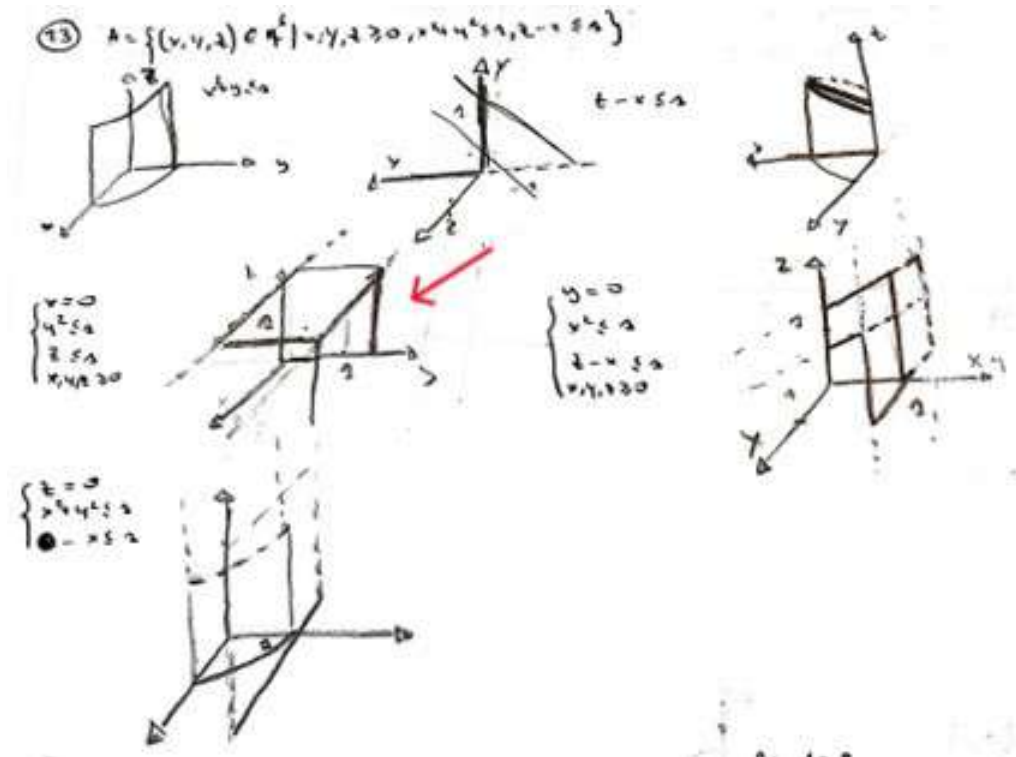


Figure 4: Three dimensional sections

As an explanation of the used method the student wrote in the questionnaire: *"First, I drew the two figures separately and then I joined them by placing them*

in the cartesian space to see the various intersections. I found it easier to draw them in steps so, as not to get confused once I joined them all together. The correction of the exercise helped me to understand that I had made mistakes in drawing the figure and thanks to it I thought better about the exercises proposed in the future” The student noticed the error but did not explain the reason.

We can deduce that the transition from plane to space and vice versa is not so, immediate for students. In these two cases, they continued to think in 3D even when it was explicitly written to make drawings in Cartesian planes. Similarly, the transition from 2D to 3D is not immediate, in fact, the most frequent error was to draw the circle instead of the cylinder and the straight line instead of the plane even though cylindroids and planes were recognized and drawn correctly in the exercises of the initial premise to Task1. In Figure 5 there is an example.

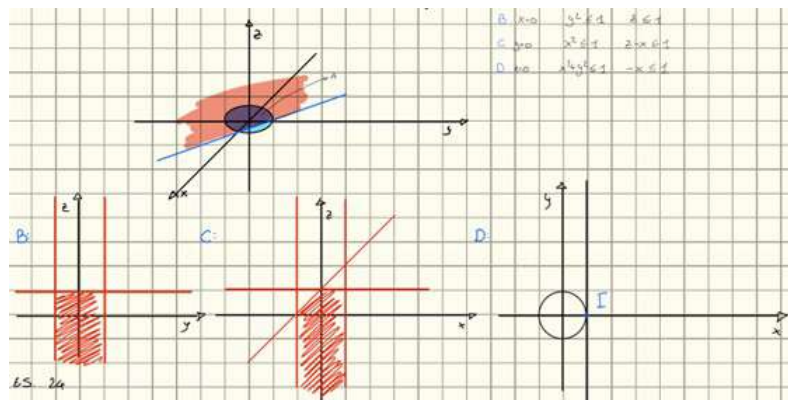


Figure 5: Erroneous cylinder

This student wrote in the questionnaire: “To complete the exercise I started by creating three Cartesian planes, the first $y - z$ cancelling the variable x , the second $x - z$ cancelling the variable y and finally the third $x - y$ cancelling the variable z . By cancelling the variables in turn in the starting equations, three graphs of three surfaces were obtained, in order a plane between 3 lines, a plane between 4 lines and finally a circumference. After having drawn the three graphs I transferred the results on a three-dimensional space; looking at the correction given to me days later I noticed that my mistake was to have transferred the circumference as in the section and not as a cylinder”. In this case the student, as he reported, used the second strategy but not having recognized that he was dealing with a cut cylinder he was unable to obtain the correct set. It is noted in several cases, as in this one, that the request for positivity especially in the sections is often neglected or forgotten. This error was not even highlighted by the student, most likely because he considered it irrelevant.

In two cases a sphere, instead of a cylinder, was drawn thought as the generalization in space of the circle, even though in one case the sections were reported correctly. Furthermore, in the sections, at least in a couple of cases, the presence of x^2 in $x^2 \leq 1$ made one think of the parabola.

Those who did not execute Task1 justified themselves by saying that at the time of the test they did not have the necessary knowledge to answer or that they had never encountered similar requests during high school. In fact, they wrote for example in Question1 of the questionnaire: *“I had a lot of difficulty especially in imagining the required solid, since I have never had to deal with similar exercises, nor with Cartesian space”, “I didn’t do this exercise because I just couldn’t figure out how to graphically represent multiple figures in the same space. Having done little or nothing about three-dimensional spaces in high school, penalized me a lot in this homework”*. Task1 was a challenge for the students given their limited prior knowledge of quadrics but also of simple planes and Cartesian space in general. These difficulties, as we will see, were taken into consideration when the Applet2 were designed.

5.1.2 Task2

It was not difficult for the student to recognise that the plane cut the cylinder in an ellipse (just few said circumference). Five possible solutions for the net of the unrolled cylinder were identified in the students’ homework like those expected and reported in the literature: the correct one of a “Sinusoid”, the two similar ones of a “Parabola” and “Circumference”, a “Cusp”, and a pentagon with a point at the top, that we called the “Tip”.

Sinusoid	Circumference	Parabola	Cusp	Tip
16%	8%	9%	17%	36%

Table 2: Results of Task 2

As we see in Table 2, just 16% correctly draw the unrolled cylinder and recognize that there is a sinusoid at the top, 17% a parabola or a circumference, 17% think that it is a cusp (naming the hyperbola) and even 36% draw a tip with straight sides and name the absolute value as a function on the top.

In Figure 6. below we have reported three representative examples of students work with their respective useful reasoning and observations. The choice was also made with regard to the explanations reported directly on the paper that highlight on the one hand the strategies used and on the other the reasoning to individualize the function.

In fact, they wrote:

- a. *“I tried to hypothesize the shape of the figure in my mind. I drew it and cut out a sheet of paper with that shape. I put the paper in the shape of*

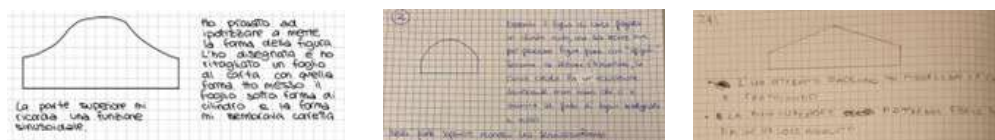


Figure 6: a. Sinusoid, b. Circumference, c. Tip

a cylinder and the shape seemed correct to me. The top reminds me a sinusoid”.

She used one of the most frequent strategies, as we will see shortly, namely the empirical test followed by a subsequent verification of the correctness of the result.

- b. *“Since the folded sheet of paper is an empty cylinder, its section cannot produce flat shapes with “edges”. Since the section is transversal, the curve has a decreasing inclination as it approaches the cutting point dotted in red. In the upper part I recognize a semicircle”.*

Despite the error, the student recognized that the shape could not be flat but must increase and decrease as one approaches the cut. He arrived at the analytical solution even if in a primitive form, which we used to design Applet3.

- c. *“I got this by making a roll of paper and unrolling it. The top could be defined by an absolute value”.*

6c wrote in Question2 of the questionnaire: *“I got it by making a paper model and unrolling it. The top could be an absolute value. I rolled up a sheet of paper and cut with scissors, but in doing so, I crushed the cylinder, the result was wrong. The correction surprised me somewhat, but I understood the error almost immediately”.* The figure offered in Task2 may have been misleading, especially for those who have poor skills in three-dimensional representations. In fact, once it is observed that the required figure must be symmetrical with respect to the highest part of the cut cylinder, one can respond by looking at the front part and obtaining a cusp (or hyperbola) or the rear part obtaining a semicircle (or parabola), or perpendicularly obtaining a segment. In any case, the students somehow had to analytically rework the result obtained and talk about functions and conic sections. 58% have at least verbally explained which function they recognized on the upper part (sinusoid or Gaussian, parabola or circumference or ellipse, hyperbola, absolute value). Moreover, Task2 warned the students that the situation was not so trivial and that a hasty evaluation not supported by empirical evidence, which must also contain a counterevidence, or by an analytical solution could lead to making a mistake. In fact, a student wrote in the questionnaire: *“This was the most stimulating exercise because even if the question seems easy, it is necessary to pay attention. To solve it I*

tried to imagine what the figure would be like in the case of an elliptical cylinder more or less “squashed” compared to the normal cylinder; I also verified the result obtained experimentally by building a rudimentary cylinder and actually imagining the progress of the experiment”.

Let’s now analyse the students’ answers to the second part of Question2 of the questionnaire about the procedures used and the difficulties encountered in solving Task2. As to the procedures, two were identified: the empirical one reproducing the object (32%) and the other using imagination (34%). Moreover, 9% first tried to imagine it and then reconstruct it to see if it was right, while 13% did not give a real explanation but only reported that it was difficult or reported the correct result, 12% did not do the exercise. The most successful procedure was the empirical one of those who arrived at the correct answer, 57% had reconstructed the object and 29% used both their imagination and reconstruction, none of them got there just by imagining it. This result is interesting because it suggests that a future use of physical 3D models, or inviting students to build them, could be an effective strategy. It is interesting that 14% tried to provide a reasoned explanation of how the figure could be. Not much has been written about the difficulties, we just report one: *“In this case I had some difficulty even after seeing the solution”*. In most of the answers, the students referred to the use of GeoGebra and Apple3 as we will see later.

5.1.3 Task3

For Task3 the results of the reached levels 1, 2, 3 and 4 are summarized in Table 3. As we see, just 7% gave no response, nobody confused the paraboloid

No Answer	Wrong	Level1	Level2	Level3	Level4
7%	0%	9%	31%	34%	19%

Table 3: Results of Task 3

with another quadric, just 19% reached “Level4”. But we couldn’t expect a better result as we haven’t yet got to the heart of the topic. The fact that 34% have already achieved Level3 is a good result: it will not be difficult for these students to take a step forward and reach Level4. Three representative examples of student (paper and pencil) homework of Level2, 3 and 4, with the answers to Question3 of the questionnaire, are the following:

- a. *“The greatest difficulty was identifying the lines in which the two three-dimensional figures intersect, since their overlapping in the graph on the paper is not clear. Tools like GeoGebra are essential to truly understand the three-dimensional structure of interlocking solids since it is not easy to recognize them on the two-dimensional sheet”.*

As she also stated, she was only able to represent the two quadrics, reaching Level2, but cannot put them together and identify the boundary curve. To

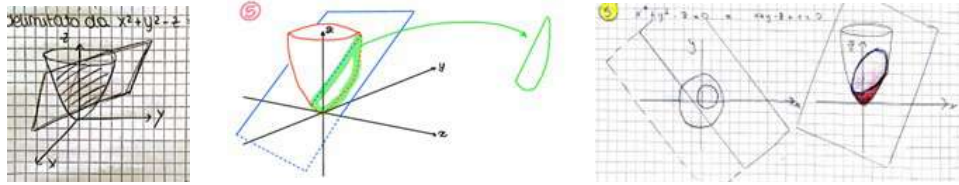


Figure 7: a. Level2, b. Level3, c. Level4

find out how the two superficies intersect it is necessary to find the intersection curve. In case 7b Level 3 is reached, and the student wrote:

- b. *“The difficulties I encountered were how to draw the boundary curves of the involved surfaces”*

In this case the two surfaces, the plane and the paraboloid, have been identified and were also represented in two different colours, red and blue, but they are unbounded. Furthermore, the boundary curve of the surfaces is perceived as the contour of the graphic representation, the green one, and not as the intersection curve between the superficies. The same error was found in 9% of the students. Only 4% put the two surfaces in a system to identify the intersection curve, which however was not explicitly requested in the exercise. Instead, in case c. the student was able to identify both the two surfaces and the boundary curve and wrote:

- c. *In this exercise I did not find any particular difficulties also because I relied on the formulas of the paraboloid present in the slides and therefore I managed more or less to understand and mentally imagine the figure.*

When we asked to explain better how he had obtained the graph, he wrote: *“To get to the result, first of all I recognized the basic equation of the plane $ax+by+cz+d=0$ and the basic equation of the paraboloid, which is $x^2+y^2=z$. Subsequently I tried to mentally imagine a three-dimensional plane so, that I could position the two figures. As for the paraboloid, I had no problems, while for the plane I used GeoGebra only to identify its position in space. Subsequently I set up the drawing and highlighted the intersections both for the top view and for the front view. As for the three-dimensional graphic representations of the boundary surfaces of a set and boundary curves of surfaces in general, let’s say that I had an initial difficulty in conceptually understanding what I was going to do. In Task1 I had no idea what I had to do”.*

The student used a sequential procedure to solve the task and GeoGebra to figure out how the plane was tilted, thus determining the choice of positioning the Cartesian axes. Furthermore, the answer confirms that the two concepts of the boundary surface of a set and even more that of the boundary curve of a surface are not immediately perceptible, even though intuition and the literal

meaning of the used terms should help. Moreover, as the student wrote, the two surfaces must fit together to give rise to the curve. These observations will be taken into account to build GeoGebra Applet4.

Regarding the answers to Question3 on the difficulties encountered in Task3: for 11% they were in identifying the intersection between the paraboloid and the plane, for 20% in highlighting the surfaces and boundary curves, 9% said they did not know what they had to highlight, 27% gave generic expositions. The remaining 34% answered that they had no difficulty: *“I had no difficulty because I used the sections to better understand what the resulting figure could be. I had more difficulty identifying the surfaces and boundary curves, since I based myself on the fact that the surfaces were the areas, and the curves were the boundary lines”*. Another wrote: *“I had no difficulty in carrying out this exercise nor any particular impressions on the correction”*. We can also add that 6% used multiple views like in Figure 7c. As for the difficulties reported in the questionnaire, the main ones were those of drawing the inclined plane and finding where the two surfaces meet. Some students (19%) helped themselves with the graphic representation of some sections.

5.2 Design of interactive GeoGebra Applets

To accompany the students in resolving the difficulties encountered in solving the Tasks and therefore to facilitate the analytical step of identifying the most appropriate parametric equations of surfaces and curves, four GeoGebra Applets were designed and used in class during the lessons as starting point for subsequent integral calculus. In this article we are interested only in the graphical representations of the three-dimensional regions involved in the exercises and therefore the Applets concern only these. Let's start examining the Applets in detail, strategies for overcoming some of GeoGebra's limitations in representing surfaces will also be presented.

5.2.1 Applet1

The first exercise was an investigative one to show the students, during the lesson, the effect of changing the set of parameters in the parametric surface equation. These will help students achieve a “Level4” vision of the mathematical concept “surface”. In fact, as we have seen from the students' solutions to Task3, it was not clear that the plane and paraboloid surfaces cannot be unbounded and that one surface limits the other after the intersection between the two is considered.

Ex 1. Given the surface $z = 1 - x^2 - y^2$ of parametric equation

$$\begin{cases} x = u \\ y = v \\ z = 1 - u^2 - v^2 \end{cases} \quad \text{with } (u, v) \in T.$$

What happens for different choices of T ?

We showed the students Applet1 (<https://www.geogebra.org/m/ebcrctxn>, Figure 8): using different colours and starting with the graph of the function $c(x, y) = 1 - u^2 - v^2$ in GeoGebra command, the entire paraboloid was obtained (brown), reconnecting to what had already been studied in the second part of the course. We remembered that the GeoGebra command “Surface” in general needs that the two parameters vary in a rectangle. Then with the surface command, for T with black border being the square $-1 \leq u \leq 1$ and $-1 \leq v \leq 1$, we obtained, as we see in Figure 8a, the violet part of the previous paraboloid, but this time it was bounded and contained four tips given by the limitations of the parameters.

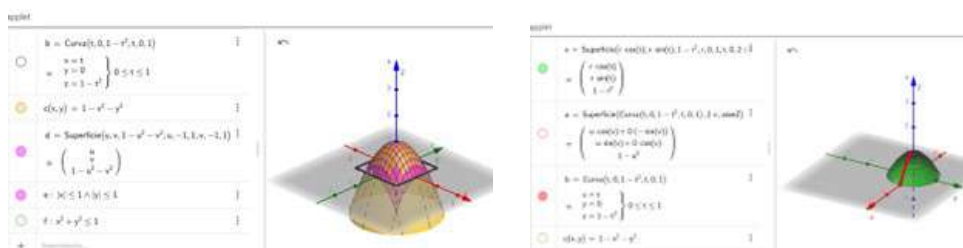


Figure 8: a. On the left the entire paraboloid and with T a square. b. On the right the surface using polar coordinate and solid of revolution

If, however, we are only interested in the part of the paraboloid with $z \geq 0$, we can modify the parametric equation of the surface by resorting, for example, to the change of variables into polar coordinates (Figure 8.b green), in this way we transform the circle $x^2 + y^2 \leq 1$ into the rectangle $0 \leq t \leq \frac{\pi}{2}$ and $0 \leq r \leq 1$ and the parametric equation becomes

$$\begin{cases} x = r \cos(t) \\ y = r \sin(t) \\ z = 1 - r^2 \end{cases} \quad \text{with } 0 \leq t \leq \frac{\pi}{2} \text{ and } 0 \leq r \leq 1.$$

To the students we pointed out that, with this choice of x and y , z become $1 - r^2$. It was also shown that the same result was reached by GeoGebra using the rotate command with respect to the z axis of a parabolic arc. This applet requires VA: several registries change from algebraic to geometric and then back to algebraic due to the need to consider different parts of the paraboloid. It was also emphasized that in practical applications surfaces are bounded and therefore it is important to correctly identify the limiting set T of the parameters.

5.2.2 Applet2

After this first exploration we proposed the calculus of the area of the same set given in Task1 (EX2). Applet2's aim was just to facilitate the student's visual-

ization of the various surfaces, to find the most appropriate parametric equations and to analyse what changes were needed for GeoGebra to work correctly.

EX2. Draw the set

$$D = \{(x, y, z) \in \mathbb{R}^3 | x, y, z \geq 0, x^2 + y^2 \leq 1, z - x \leq 1\}$$

and calculate the total external area.

This exercise involved identifying five surfaces. Three were easy to obtain, by putting $x = 0$, $y = 0$, $z = 0$ on the fundamental planes, we got a square, a trapezoid and a quarter of a circle, respectively. The remaining two were part of the cylinder and part of the oblique plane. The first problem to deal with these was using the GeoGebra Surface command which requires the two parameters to be defined in a rectangle as we said before. This problem has been solved in an easy way by exploiting the change to polar coordinates, as for Applet1, for the quarter circle to be transformed into a rectangle as follows:

$$\begin{cases} x = u \\ y = v \\ z = 0 \end{cases} \quad u, v \geq 0; x^2 + y^2 \leq 1;$$

became

$$\begin{cases} x = u \cos(v) \\ y = u \sin(v) \\ z = 0 \end{cases} \quad 0 \leq u \leq \frac{\pi}{2}; 0 \leq v \leq 1.$$

The other problem consisted in the limitations of one parameter being a function of the other, for example for the trapezium. In this case the problem was solved by rescaling the parameter v in the following way:

$$\begin{cases} x = u \\ y = 0 \\ z = v \end{cases}$$

$0 \leq u \leq 1; 0 \leq v \leq 1 + u$; so,

$$\begin{cases} x = u \\ y = 0 \\ z = (1 + u)v \end{cases} \quad 0 \leq u \leq 1; 0 \leq v \leq 1.$$

With these two tricks, even used at the same time, we can obtain the following parametric representation of the portion of cylinder:

$$(1) \quad \begin{cases} x = \cos(u) \\ y = \sin(u) \\ z = v(1 + \cos(u)) \end{cases} \quad 0 \leq u \leq \frac{\pi}{2}; 0 \leq v \leq 1.$$

This exercise was also an occasion to resume a common error made by the students with the change into polar coordinates of wanting to transform every think into rectangle.

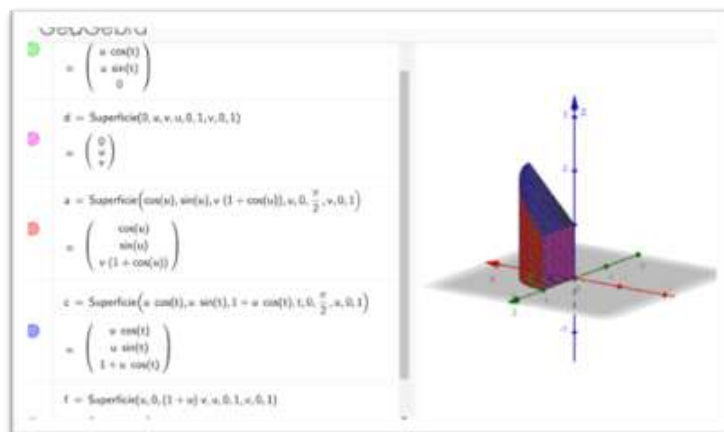


Figure 9: GeoGebra Applet2

Proposing EX2 and the connected Applet2 (<https://www.geogebra.org/m/nfszf9bk>, see Figure 9) had the advantage not only to give a graphic representation of what the boundary of the set was, which the students have already seen in Task1, but also to show the complex meaning of the parameters whose variation profoundly affects the surface appearance. Moreover, the possibility of using different colours for the various surfaces makes visualization easier. This Applet was designed to allow the students to think back to Task1 and correct their errors. By turning the figure all the single parts are shown, making it easier to recognise all the sections that are obtained by intersecting with the fundamental planes. Here too, VA with a sequential change of registry was necessary: analytical-geometric-analytic-geometric, from the analytical representation of the set through inequalities we moved on to the geometric representation of all the parts that make up the boundary, including the sections obtained by intersecting the fundamental planes, whose geometric representation again required the passage to the analytical register including the calculation of the parametric representation of all the various parts. We would still end up with the analytical and geometric register with the calculation of the surface integrals through double integrals.

5.2.3 Applet3

The next exercise (EX3) required calculating a surface integral on a cylinder cut by two planes as in the previous case. This time, however, it was simpler as there were no requests to stay on the first octant, which was proposed with

the aim of strengthening concentration on only the cylindrical surface and its two boundary curves, thus anticipating Stokes' theorem. It also allowed the students to recall Task2.

EX3. Given Σ the cylindrical surface $x^2 + y^2 = 1$ between $z = 0$ and $z = 1 + x$ calculate

$$\iint_{\Sigma} z dS.$$

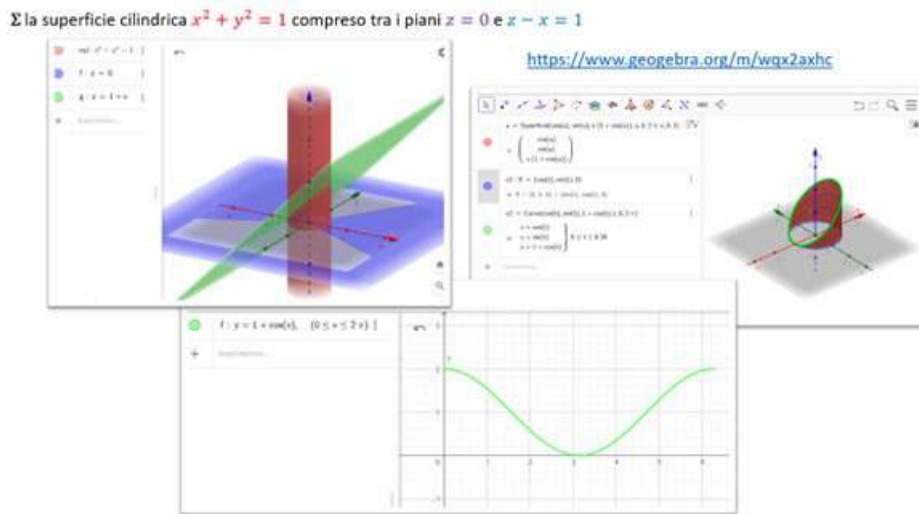


Figure 10: A slide of the lesson

In Figure 10 above, a slide of the lesson, there are the two ways to represent the set using GeoGebra by the overlap of the cartesian equations on the left and using the parametric equations of the surface and of the boundary curves on the right. At the bottom there is also the graphic in green of the unrolled cylinder cut vertically on the higher part of Task2. We took advantage of this example, to observe, using Applet3 (<https://www.geogebra.org/m/wqx2axhc>), what the border curve of the surfaces were and how their parametric equations resulted directly from the parametric equation (1) of the surface of EX2 (but now $0 \leq u \leq 2\pi$) just fixing the parameter u once 0 and once 1. This went into the direction of what the students have reported: it is difficult to see how two surfaces fit together.

In contrast to Task2, this time the cartesian equation of the cylinder $x^2 + y^2 = 1$ is known just as that of the oblique plane $z = 1 + x$. Thus, the students were able to see firsthand that, using the parametric equation in polar form of the circumference in the xy plane, so $z = 1 + \cos(u)$ with $0 \leq u \leq 2\pi$, justifying that, on the upper part of the unrolled cylinder, a sinusoid is obtained and not a parabola or circumference or a cusp or an absolute value type function as many

students have indicated. The analytical solution was highly appreciated by the students. Also, in this one, the possibility of using different colours, (method also used by some students in solving the tasks), for the various surfaces and choosing the same colour for a surface and its boundary curve made visualization easier. It could also be shown that the same curve was simultaneously the boundary of the paraboloid with the plane that delimited it, or the boundary of the plane with the paraboloid that delimited it, the students did not recognize this when they completed Task3. It was also possible from the parametric equations of the boundary curves to reproduce the parametric equation of the surface. Even in this case, to explain the topic, it was necessary to make several changes of register like in the previous case.

5.2.4 Applet4

The last exercise (EX4) was about the verification of Stokes Theorem.

EX4. For $\Sigma = \{(x, y, z) \in \mathbb{R}^3 | y^2 + z^2 = 4, -1 \leq x \leq z + 1\}$ oriented outward and the vector field $F = (x^2, z^2, y^2)$ verify Stokes' Theorem.

Also, in this case, we had a cylinder cut by two planes but this time the strategy for the corresponding adopted Applet4 (<https://www.geogebra.org/m/yve3ttgy>) was different. In fact, it became more difficult to represent the surface in parametric form with GeoGebra, so, we used the “intersection” command which showed the two boundary curves graphically. The advantage was that GeoGebra directly gave the parametric equations of the curves included the choice of the directions, which is fundamental for applying Stokes' theorem. The use of sliders permitted to see how the points move on the curves, making everything more dynamic (see Figure 11). We would like to point out that the intersection command of GeoGebra is limited only to specific cases such as the intersection of a quadric with a plane and to some intersections between quadrics.

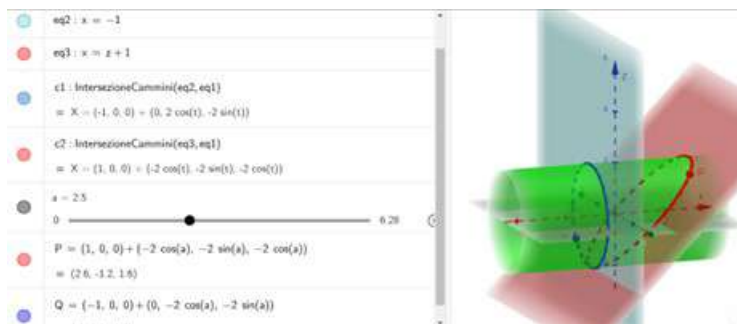


Figure 11: Cylinder cut by two planes

Also, in this case it was necessary to pass from the analytic register to the geometric one with the graph and again move to the algebraic register to find the intersection curves and check with GeoGebra that the two results coincide.

5.3 Effects of Applets use on the students

After the interventions and with the support of the GeoGebra applets, Task3 was represented to the students asking them to find the respective parametric equations. Of the 48 students participating in the revised Task3, as we see in Table 4, none was at Level1, just 4% at Level2 and as many as 71% at Level4, while before that was 19%.

Level1	Level2	Level3	Level4
0%	4%	25%	71%

Table 4: Results of Revised Task3

Analysing the improvements, as we see in Table 5, we can add that no one remained at level2, 15% went from Level2 to Level4 and 25% from Level3 to Level4 (missing 2% who went from Level1 to Level2 and another 2% who went from not having solved the exercise to Level2).

Level3/Level3	Level4/Level4	Level2/Level4	Level3/Level4
23%	31%	15%	25%

Table 5: Results of improvements of revised Task3

In Figure 12 there is a representative example of the procedure carried out, respectively, before and after the intervention.

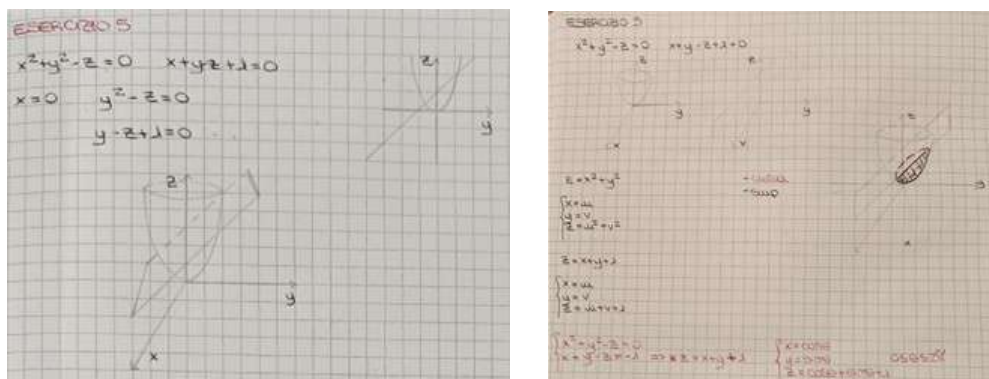


Figure 12: Task3 first (left) and after (right) the intervention

As we can see this student passed from Level2 to Level4 for the graphic representation. The evolution was lighted by the student's comment on this Task

in Question3 of the questionnaire: *“I didn’t find any difficulty in drawing the three-dimensional set as I understood the figure. As regards the second request, I didn’t really understand what I should have highlighted, however, after the week’s explanations I understood how I should have done it”*. The execution of Task3 is not yet complete due to the lack of the set in which the parameters vary for the parametric equations of the surfaces, but it is justified by the fact that the lessons on solving this problem with the use of the Applet took place only a few days before the delivery of the revised Task3. Of the 48 students who returned the test, 54% wrote the parametric equation of the paraboloid but of these only 19% explicitly wrote the set T of variation of the parameters, 90% wrote the parametric equation of the plane but again only 23% of them made T explicit and finally 42% found the analytical representation of the intersection curve of which 25% wrote its parametric equation, 50% found the cartesian equation and finally wrote the corresponding system.

Of these 48 students answering Question1 of the questionnaire, 46% explicitly underlined the importance of GeoGebra and of these 50% referred to Applet2 used in class. Instead, to Question2 of the questionnaire 34% expressed opinions on the use of GeoGebra and of these 74% made explicit reference to Applet3. In all cases they expressed that their use was appreciated. It is interesting to note that of the latter 45% explicitly declared their interest in the analytical solution of the problem contained in Applet3. They wrote for Question2 of the questionnaire: *“To obtain the “unrolled” section of the cylinder I imagined this process in my mind. I tried to sketch out what I thought was the result and then rewind it and verify that the initial figure was formed again. The analytical confirmation is comforting, it is not necessary to be able to carry out a process of imagination, sometimes not obvious, to arrive at the solution, but the calculations help us to confirm or not our “initial theory”. The use of the applet certainly provides us with a clear and useful representation to confirm or not our initial hypotheses”*; *“In this exercise I must say that I had difficulty understanding how once cut perpendicularly as indicated in the text it could unroll, once I saw the correction I must say that it was still not clear to me how it was possible to obtain that figure. These doubts were clarified for me during the lesson of 05/22, as an analytical solution was clearer to me than perhaps a qualitative graphic solution”*; *“I had done that exercise incorrectly because I had not considered that a tip could not appear in the upper part of the graph. But as soon as I saw the correction I immediately understood my mistake and it helped me to see the 3D figures more clearly”*.

6. Discussion and conclusions

The results of our study were based on two sources of data: three paper and pencil homework tasks (Task1, Task2 and Task3) and the answers to a questionnaire. The findings confirm the initial difficulties of the students in drawing sets, expressed through inequalities, in the Cartesian space both for their previ-

ous knowledge and for the difficulty in manipulating three-dimensional objects. Furthermore, this study shows that the concepts of boundary surfaces of a set and boundary curves of a surface, indispensable in vector integral calculus, are difficult for students to understand. To overcome these problems, unusual Tasks and GeoGebra Applets were designed and tested.

Students have difficulties with vector calculus as it involves concepts and problems that require students to think in terms of three-dimensional space and to visualize objects such as curves, surfaces, and volumes. Our Engineering students' incoming knowledge of analytical geometry in three-dimensional space is very poor in fact, out of 82 students, 23% said that it was null and 31% basic. During the Linear Algebra course, they begin to see planes and straight lines in space but, due to lack of time, quadrics are rarely treated. In fact, only 25% of the surveyed students stated that their prior knowledge of three-dimensional Cartesian space was due to the previous Linear Algebra course. The results of Tasks1 and 2 confirm this. The first request of Task1 to draw the set in space was undoubtedly unfamiliar to the students and this perhaps also discouraged them from continuing with the exercise, but in order to draw the sections in the various planes it was sufficient to rewrite the set in the three situations $x = 0$, $y = 0$ and $z = 0$ and draw them in the Cartesian planes. When faced with a new problem, students often give up on tackling it, in fact 63% did not do the exercise because of their poor knowledge of three-dimensional representations, as they wrote in the questionnaire. Moreover, the transition from 2D to 3D is not immediate as we noted in the most common error of drawing a circle and a straight line instead of the cylinder and the plane. But in the same way the transition from 3D to 2D is not obvious as we could see in the three-dimensional representation of the sections. Task2 had, above all, the aim of anticipating the notion of the boundary of a surface and making students appreciate the power of the analytical solution, which removes all doubts about any false concept. However, it was possible to observe also in those who had operationally tried to reconstruct the object a lack of critical evaluation of the obtained result. As regards Task3, after having already had to deal with different types of quadrics in the second part of the course, the students had no problems in graphically representing the paraboloid and the plane. The difficulty, as they stated in the questionnaire, too, was to put them together identifying the intersection curve. These difficulties are probably and mainly due to the poor habit of students to see mathematical objects and problems from different perspectives, and to their tendency to focus their attention on univocal and mechanical methods of treatment. Only a few students immediately managed the analytical calculus of the intersection curve; this number considerably increased after the presentation of the GeoGebra Applets in class. The notion of boundary of a region and boundary of a surface are not as intuitive as one would expect, only 19% were able to correctly draw the boundary surfaces of the sets and the boundary curve of the surfaces in Task3. In 34% of the cases, the latter was confused with the contour of the representation on the sheet, which is necessarily flat (see Figure

7b). But after the use of the GeoGebra Applets, the 48 students who returned the same Task3, but with the request to write the parametric equations of the involved surfaces and curves, had improved from both the analytical and the graphic point of view. Even if there was still some difficulty in describing the set T of variability of the parameters of the two surfaces.

To conclude, we can say that for the teaching of scalar or vector multivariable integral calculus to be graphically effective, it is necessary to consider the students' difficulties in managing three-dimensional representations as could be noted in the results of the three Tasks. To do this, the use of dynamic 3D visualization tools such as GeoGebra could prove to be very effective. The use of GeoGebra facilitates not only the transition from the analytical to the graphic register but also the transition from two different analytical registers, in fact a surface could be expressed in Cartesian or parametric form and even the parametric one is not unique. It is difficult for students to identify a set where the parameters vary and, in the case of multiple intersecting surfaces, to understand where one ends and the other begins. Furthermore, the GeoGebra command intersection between two quadrics can provide them, in at least some cases, "for free", the parametric equation of the resulting curve ([31], [14]).

There are some limitations to the research: the students rarely included all the steps used in their homework and rarely commented on what they were doing even when explicitly asked; the answers to the questionnaires were generally very short. Moreover, too much time had passed between the beginning of the course and the treatment of vector integral calculus.

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Design and use of learning graphs through ASYMPTOTE system

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Abstract. This study investigates the integration of the ASYMPTOTE system into a Mathematics course for pre-service teachers. In particular, the study focuses on pre-service teachers' task design and its impact on their understanding of course topics and exam performance. Pre-service teachers embraced the task design process, producing more tasks than anticipated. They recognized the value of task design for comprehensive topic review and deepening understanding. Collaborative efforts between instructors and students were crucial, resulting in a significant portion of participants passing exams. Initial data show that this direct involvement not only enriches the pre-service teachers' mathematical knowledge but also promises to strengthen their future teaching skills.

Keywords: ASYMPTOTE, pre-service teachers, task design.

MSC 2020: 97B50, 97C70, 97D40, 97D80, 97N80, 97U70.

1. Introduction and theoretical background

It is documented that many pre-service primary school teachers experience difficulties with mathematics, compromising the quality of teaching (e.g. Buxton, 1981; Coppola et al., 2012; Di Martino & Sabena, 2011). To effectively support pre-service teachers (henceforth PT) in overcoming these challenges, one promising approach is integrating the teaching and learning of mathematics with technology (Niess, 2005; Thomas & Hong, 2013). In fact, this combination can offer innovative and accessible methods for enhancing their understanding and delivery of mathematical concepts (Goos & Bennison, 2008). For example, by using mobile phones, students' assimilation of knowledge becomes more accessible, because learning flexibility is fostered, i.e. studying can be done anywhere and anytime (Crompton et al., 2017). Recognising their advantages, students understand that their mobile devices play a crucial role in simplifying and facilitating the teaching-learning process, making it more enjoyable and stimulating. However, using these technological tools in education must be guided by appropriate and well-defined strategies (Mehdi et al., 2020). In this study, the ASYMPTOTE system (<https://www.asymptote-project.eu/en/welcome/>), which is a modern tool for teaching and learning mathematics remotely from

primary education to university level, was considered as a technology to be used to support the study and examination preparation of PT.

From March 2021 to February 2023, the ASYMPTOTE project, which stands for Adaptive Synchronous Mathematics Learning Paths for Online Teaching in Europe, developed and improved the digital system of the same name. It consists of two components, a web portal for teachers and a smartphone application for students (Barlovits et al., 2022). In the web portal, teachers can create tasks or search and choose them from those available in a rich task repository. Tasks must include basic information: a title, the delivery, information about the author who created them; and core information: answer format, sample solution, some hints. The answer format can be: “Exact value”, “Interval”, “Multiple choice”, “Fill in the blanks”, “Exact value vector”, “Interval vector”, “Sets”, “Fractions”, “Matrices”, and “Information station”. Teachers can then group the tasks identified in a sequence with multiple levels of difficulty (by specific topic, but not necessarily) generating what is called a Learning Graph (LG): a veritable succession of nodes (the tasks) and arcs (how one moves from one task to the next). Barlovits and colleagues (2022) define a LG as follow: it “is defined as a directed graph, where each vertex represents a [...] task, based on a learning trajectory as the intended and expected way of learning” (p. 14).

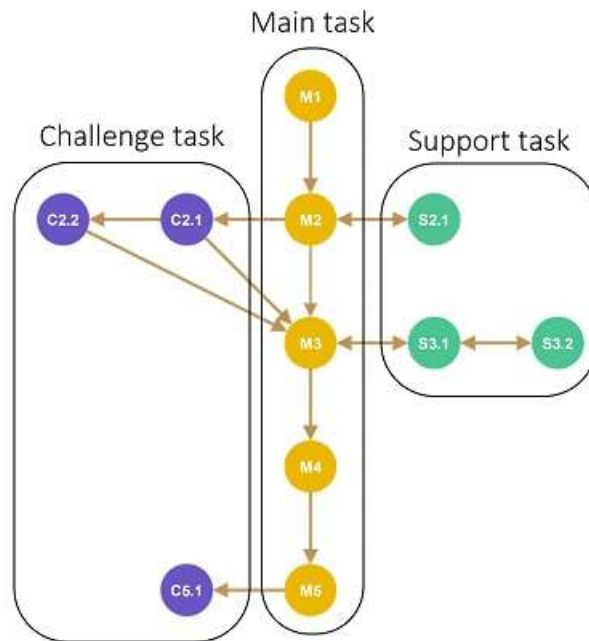


Figure 1: A generic LG

In particular, a LG consists of a sequence of tasks of different typology (Figure 1): the main tasks (yellow), which are mandatory and positioned to form

the backbone of the LG; support tasks (green), positioned to the right of a main task and optional, are at a simpler level and allow support to be given to tackle the corresponding main task; challenge tasks (purple), positioned to the left of a main task and optional, are at a higher level and are designed to challenge learners to solve the corresponding main task straight away or to allow them to explore the topic in more depth. This structure ensures that the entire learning process is self-guided and autonomous (Barlovits et al., 2022).

The students work on the learning graph via the smartphone app: they solve tasks having an immediate validation of the solutions inserted, they get feedback synchronously and systematically and, depending on how well they solve the task, they are directed to the next task, which can be of a simpler, more advanced or equal level to the previous task performed.

This paper will present an exploratory study on the use of the ASYMPOTOTE system in the Mathematics for Primary and Pre-school course of the Master of Education degree course at Kore University of Enna (UKE), Italy. The context in which the study was conducted will be presented and the results of how the primary PT carried out the task design and made subsequent use of it will be illustrated. The aim is to share reflections about their experience in relation to the following research questions:

- i) What personal perceptions of the PT emerge in the task design process with ASYMPOTOTE, in terms of ease and usefulness, and in the enjoyment of the LG created?
- ii) What impact does such task design work with ASYMPOTOTE have on the examination performance of the PT?

2. Methods

2.1 Research condition and process

At UKE, the Master of Education degree course requires PT to attend third-year lessons in Mathematics for Primary and Pre-school (taught by the paper's author). From the first day of class, in the academic year 2023-2024, I explained the ASYMPOTOTE system and proposed to the PT to engage in the design of tasks that would form an LG. The course proceeded by macro-topics: Set Theory, Arithmetic, Geometry, Probability and Statistics. For each macro-topic, the theoretical part is accompanied by a series of exercises. Some are carried out in the classroom: generally, work is done between peers and then a volunteer, at the blackboard, shows the class the exercise. I ask to argue and justify the steps and, if necessary, ask the class if other solving strategies have been considered. Other exercises are left for home. These could be considered by the PT to conceive a task for the LG, taking them as a cue or considering them verbatim. Figure 2 represents the Excel file that was shared with PT. In each line, the topic of the lesson is listed in first column; while the second column shows possible sub-topics for which LG tasks can be designed. The typologies to which each

task can belong are the next three columns, coloured purple for challenge tasks, yellow for main tasks and green for support tasks. The PT had to work in groups (min 3 - max 5 people), on the design of specific tasks listed in the table. They had to choose the task to be designed and its typology by writing their names in the cells of the Excel file (the names of colleagues were public for the students; for privacy reasons, they have been blacked out in Figure 2). For example, for the Arithmetic macro-topic, among the proposed topics was ‘Writing numbers and numbering systems’ - cell A8 in Figure 2. There, I specified two possible sub-topics for generating the task, ‘Writing numbers in positional notation’ - cell B8 of Figure 2; and ‘Basic conversions’ - cell B9 of Figure 2. Most of the groups booked to create a yellow task. If others had already booked this, there were two ways: either a group chose another topic and booked for a yellow task or, if they particularly liked the topic or felt comfortable with it, they could propose a green or purple task, comparing themselves with colleagues who had booked the yellow one. Some groups sometimes asked me for support in identifying the task to be designed. In that case, I suggested that they design a specific type of task according to the current presence of yellow/green/purple tasks, to generate an LG as balanced as possible.

	A	B	C	D	E
4	TASK PER IL LEARNING GRAPH				
5					
6	ARITMETICA (15 task in tutto di cui 6 task principali)				
7					
8	Lezione 5 (scrittura dei numeri e sistemi di numerazione)	Scrittura dei numeri in notazione posizionale		[DEFINITIVO] I Sistemi di numerazione e la scrittura dei numeri	
9		Conversioni di base	[DEFINITIVO] Aiutiamo Claudio!		[DEFINITIVO] Conversione dei numeri tra basi differenti
10					
11	Lezione 6 (i numeri naturali e le operazioni; espressioni; potenze)	Esercizio 2 (espressioni)	[DEFINITIVO] Espressioni di sfida	[DEFINITIVO] Alla scoperta delle espressioni	[DEFINITIVO] Eserciziamoci con le espressioni
12		Esercizio 3 (errori nella risoluzione delle espressioni)			
13		Esercizio 4 (dal linguaggio naturale all'espressione)		[DEFINITIVO] Espressioni con i numeri naturali	
14		Esercizio 5 (Zio Paperone - come sfida)	[DEFINITIVO] Zio Paperone		
15		Moltiplicare e dividere per 100 (come supporto, prendendo spunto dagli esercizi 6, 7, 8, 9, 10, 11)			[DEFINITIVO] Moltiplicare e dividere per potenze di 10

Figure 2: Excel file in which PT choose the topic and the typology of the task they decided to design

The PT did not work on the ASYMPOTOTE web portal: I provided them with a design sheet created on a Word file that exactly traced the entries of the task creation template in ASYMPOTOTE. Once the task had been designed, the PT's group sent this sheet by email to me and I reviewed it. Once the task was deemed suitable (it took between 1 and 3 revision rounds), I digitized the task on the ASYMPOTOTE web portal and created the LG according to the choices

made on the Excel file. I then delivered the code associated with the LG to the PT so that they could use it on their smartphones via the app. In order to foster the PT's involvement in this design work, I stated that I would award an extra point in the exam (it consists of a written test) to all those who would develop a task for the LG.

2.2 Participants and data collection

In the academic year 2023-2024, there were 300 PT enrolled in the third year of the Master of Education degree course. 156 PT have attended my Mathematics for Primary and Pre-school course at least once. 83 PT have voluntarily decided to engage in the design of at least one LG task. For the data collection, the research followed a mixed quantitative and qualitative approach. More specifically, at the end of the lessons and after the first exam call, I administered a questionnaire to investigate the satisfaction index of both the design activity carried out and the user experience with the LGs. The questionnaire consisted of multiple-choice, open-ended and Likert scale questions. It was administered via Google Form. It is available, in Italian, at the following link: <https://lc.cx/iX9B4d>. The data were analyzed using Excel. In particular, inductive coding (Thomas, 2006) was used for the analysis of open-ended questions. The questionnaire was answered by both students who worked on the design and students who only enjoyed the LGs ⁽¹⁾. In the following, I will focus on the data collected among the students involved in the task design work and who then answered the questionnaire.

3. Results

3.1 Observations drawn from the task creation and review process

The interest shown by the PT in task design for the LG is evidenced by the fact that they produced 50 tasks in all (out of 83 PT, 19 groups were formed and each produced between 1 and 5 tasks). As anticipated, it took between 1 and 3 rounds of review before a task was deemed suitable. The most common errors were: calculation errors or improper use of mathematical symbols; terminology errors (e.g. showing only a few numerical explorations cannot be considered a proof). At other times, it was necessary to better direct the PT to a different response format than the one they had thought of (e.g. to study the proof of a theorem, the PT had thought of using the response format "Information station", the purpose of which is only to read the content of a task. Instead, it was suggested that they use the "Fill in the blanks" response format, which

1. The questionnaire consists of several parts and the questions related to task design are only made available to those who answer *Yes* to the question asking: *'Did you contribute to the creation of the Learning Graphs, i.e. did you design at least one task?'* Those who answer *No* are automatically sent to the questions relevant to the use of the LGs.

works exactly like a cloze, i.e. there are blanks in the steps of the proof and it is up to the user to work out how to fill them in).

At other times, more support was needed in the formulation of hints. In the latter case, in almost all cases, I formulated the hints instead of the PT. The formulation of appropriate hints that guide and support, without suggesting, reasoning towards a resolution strategy is a non-trivial practice and it is necessary to have a deep knowledge of the topics in order to be able to adequately support the resolution processes (Perrenet & Groen, 1993). During the course, I worked in the direction of acquiring mathematical knowledge, I did not focus on metacognitive aspects (which are instead the subject of the course that PT would take the following semester).

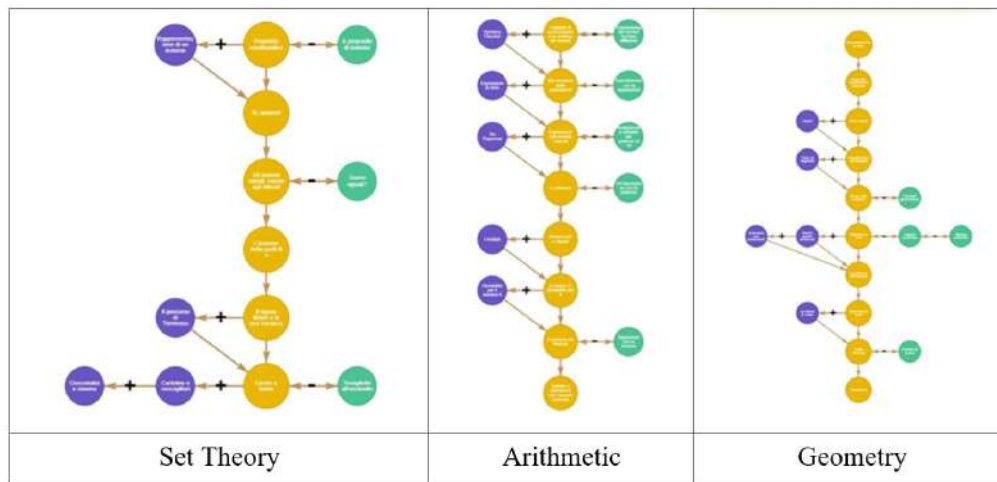


Figure 3: Realized LGs broken down by macro-topics of the course

In the end, the result of their productions culminated in the generation of 3 LGs on the following macro subjects: Set Theory, Arithmetic, Geometry (Figure 3).

3.2 Results drawn from questionnaire

3.2.1 PT's perceptions of task design regarding ease, usefulness and revisions

The questionnaire collected 63 answers, of which 35 were from PT who worked on task design. They were asked, using a Likert scale question (from 1 = absolutely disagree to 4 = absolutely agree), how easy and useful the PT found the task design work, as well as how much they appreciated the revision work on their productions. Considering values 3 and 4 on the scale, it can be observed that ease of use, usefulness and importance of receiving reviews were considered as such by 94%, 92% and 100% respectively (Figure 4).

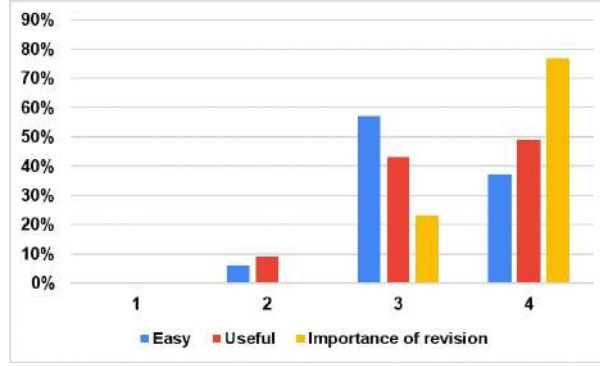


Figure 4: Degree of agreement of PT on the ease, usefulness and importance of receiving reviews

For each of the previous Likert-scale questions, an open question is asked to justify the answer. The Table 1 shows the frequency distribution of the respondents as to why it was perceived as easy or not easy to design tasks for the LG.

Why the task design work was ...			
Easy (#32)		Not easy (#3)	
Thanks to the guidance provided by the teacher	47%	Coordinating with colleagues	34%
The templates to fill was clear and easy to use	19%	Not having good pre-knowledge in mathematics	33%
It just takes a little practice	16%	Not able to formulate suggestions	33%
Having good pre-knowledge in mathematics	6%		
Thanks to group work	6%		
The answer format drives the design	3%		
Not Applicable	3%		
Total	100%		100%

Table 1: Why it was/was not easy for me to design a task for the LG?

It can be observed that the teacher guidance emerges as the most significant factor in making the task easy, with nearly half of the respondents (47%) indicating it. For example, someone who believes this has written: *“It was easy because the professor’s instructions were very clear and the format was quite easy to understand”*. Other reasons the task design was considered easy include the clarity of the templates (e.g. *“The task planning sheet is very clear in its various steps, [...]”*). Still, others felt that task design was easy as soon as one gained some practice (e.g. *“[...] at the beginning with the group we had some difficulties to understand certain things well but then with the others we were faster and faster”*). A minority believes that the ease can be attributed to having worked in a group (e.g. *“Thanks to the support among us colleagues [...]”*) or to the fact that response formats guide the design (e.g. *“I was able to find the appropriate types of answers to structure any task [...]”*). Coordination with colleagues was the main reason for difficulty, with 34% of respondents cit-

ing it. It is a PT who wrote this: *“I do not totally agree because I sometimes had difficulties [...] in coordination with colleagues”*. Mathematical knowledge plays an ambivalent role: while good knowledge helps (6%), the lack of it is one of the main causes of difficulty (33%). In these cases, there were similar responses: *“[it was easy because] I had understood the exercises and their execution”* and *“I often found it difficult to write the solution examples. For me, solving an exercise meant a great achievement. Having to explain it was a double achievement [when I succeeded], because I had internalised the process.”*, respectively. In summary, teacher guidance and the clarity of the templates appear to be the most relevant factors in facilitating the task design work, while the major difficulties are related to coordination and prior mathematical knowledge.

Why the task design work was ...			
Useful (#32)		Not useful (#3)	
It allows you to go over all the topics of the course	38%	The difficulties in mathematics possessed became more evident	33%
It allows you to understand what course topics to study in depth	22%	Not Applicable	67%
It allows you to challenge yourself as a future teacher	13%		
It allows you to reflect on how best to write your own resolution strategy	9%		
It allows you to put into practice what you have studied	9%		
Reading activities created by other peers allows comparison	3%		
Not Applicable	6%		
Total	100%		100%

Table 2: Why it was/was not useful for me to design a task for the LG?

Let's see how the PT expressed themselves with regard to the usefulness of task design work (Table 2). Reviewing all course topics emerges as the main reason the task was perceived as useful, with 38% of respondents indicating this. Here are some of the students' justifications in this respect: *“It is an excellent alternative for revising by finding targeted exercises for each topic.”*; *“It was useful because I could go over all the topics quickly by doing the exercises”*. Mathematical difficulties, instead, are the main reasons the task was perceived as not useful, with 33% of respondents noting this. It is a PT who wrote this: *“For my previous difficulties”*. Other reasons the task design was considered useful include the ability to understand which topics to study in depth (e.g. *“It helped me to understand some topics more deeply”*); the challenge it presents for their future role as teachers (e.g. *“Yes, because it allows you to get involved as a ‘teacher’ ”* ; *“It is also useful in the future”*); reflection on how to write their own resolution strategy (e.g. *“Useful because it allowed me to pause and*

reflect on the way I was explaining a problem”), and putting into practice what they have studied (e.g. *“Because I was able to apply the notions learned during the lessons”*). A minority of respondents found it useful to compare their work with activities created by other peers. It is a PT who expresses himself as follows: *“It is useful to create a task because it puts the student in the game. In my opinion, it is much more useful instead to observe activities constructed by others so that you can compare yourself”*. In summary, the design task was mainly considered useful for its ability to provide a comprehensive review of the course topics and to help identify areas that need further study. On the other hand, mathematical difficulties emerged as one of the obstacles to perceiving the task design’s usefulness.

For all respondents, it was important to receive revisions to their productions.

Why was it important to receive revisions?	
Understanding how to correct errors	22%
Understanding how to improve the task deisng	10%
To be sure that you have worked appropriately	8%
Unders	5%
Not Applicable	3%
Total	100%

Table 3: Why was it important for me to receive lecture’s reviews of the task(s) created for the LG?

In particular, Table 3 shows the motivations that explored PT. The majority (22%) felt that receiving revisions was important to understand and correct errors (e.g. *“It made me realize mistakes in such a way that I would never make them again”*); others to improve the design of the task (e.g. *“I understood how something should be explained and how to help make it understood”*); others to ensure appropriate work (e.g. *“To be sure you have worked correctly”*) and to choose the appropriate response format (e.g. *“To better organize task response according to [system] response formats”*).

Another semi-open question asked “Would you advise a future student who is to start attending/studying the ‘Mathematics for Primary and Pre-school’ course to try their hand at creating a task for a LG?”. 100% answered Yes. And motivations fall into two categories: to review, practice and improve learning (71%) and to put oneself out there for one’s future (17%). Examples of motivations that fall into the first category are: *“because it could help during the study of the subject, it could serve both as an exercise, but also as a way to clarify if there are some”*; *“These tasks allow for a deeper understanding of the topics also thanks to the fact that it is we, the students, who carry them out”*; while examples for the second category are: *“Because we will be future teachers and it is important to try our hand at creating these useful learning*

tasks”; “Because it is good to be able to get involved and have full knowledge of what we should one day be doing in the classroom with our pupils”. Only in the answer of one PT (3%) does the presence of a didactic contract echo (Brousseau, 1988), in fact he writes: “Yes mainly because it is important to the professor”. The remainder (9%) indicate answers that are not relevant, therefore classified as Not Applicable.

LG	# download to 20 December 2023 (first exam call)	# download to 16 January 2024 (second exam call)
Set Theory	141	153
Arithmetic	95	106
Geometry	66	70

Table 4: Number of downloads by PT of the 3 LGs created

3.2.2 The PT’s use of LG

As I created the LGs (as shown in Figure 3), I shared the code associated with each LG on the Google Classroom created for the ‘Mathematics for Primary and Pre-school’ course. The Set Theory one was completed in early November 2023; the Arithmetic one in mid-November 2023; the Geometry one in early December 2023. In this way, any of the PT were in a position to download them onto their smartphones via the ASYMPTOTE app and enjoy them. Table 4 shows the data on the downloads made by the PT of the 3 LGs created ⁽²⁾. Downloads made by the date of the first exam call in December 2023 and downloads made by the date of the second call in January 2024 are recorded. Although the downloads varied in number, depending on the macro-topic related to the LG, this shows that the LGs also benefited other students, not only those who engaged in the task design work.

Although it would be interesting to investigate how LGs were used by those who did not work on the task design, this will not be the subject of this contribution. In the following, we will always focus on the answers of the 35 PT who worked on the task design. The questionnaire asked them to specify which LGs were downloaded. The LG on Set Theory was downloaded by 30 out of 35 PT, the one on Arithmetic by 28 out of 35 PT and the one on Geometry by 26 out of 35 PT. There were 2 PT who did not download any. The reasons why not all downloads were made were not investigated. The 33 PT who downloaded the LGs were asked the following open question: “Have you regularly studied the subject with the use of LG or have you used them only in the run-up to exam?”. Forty-eight per cent stated that they had used them in the run-up to examina-

2. This data is available from the web portal: each user who creates an LG also sees how many downloads associated with it have taken place on the app.

tions, writing for example: *“I used it to revise and practise for the exam”; “[...] in the run-up to the exam to compare the exercises I did with those done by other colleagues and to see alternative ways of solving them to mine”; “I used it close to the exam because I preferred to first fix the theoretical part accompanied by the exercises proposed by the professor in class and use the LGs as a final revision”*. In contrast, 42% stated that they regularly accompanied their study with the use of LGs, writing for example: *“I used the Learning Graph to assess my preparation as soon as it was possible to do so, having studied the subject throughout the semester and not just in the run-up to the exam”; “I regularly accompanied my study of the subject with the use of Learning Graphs because it gradually helped me to understand some aspect that was unclear and served as practice”; “Throughout the course I studied the subject matter with the constant use of Learning Graphs, they were very helpful even when I didn’t quite understand topics”*. The remaining 9% did not report relevant reasons and were classified as Not Applicable.

3.3 The PT’s exam performance

As far as exam performance is concerned, we can observe that at the first exam call out of 225 students, 78 were among those who had worked on the design of at least one LG task. Of these, only 9 failed the examination. At the second exam call, out of 96 students there were 13 PT who had worked on the design of at least one task for a LG and of these only 1 failed the examination.

For both of these exam calls, a chi-square test was performed to examine the relationship between having done the task design for the LGs and having passed the exam. The relationship between these variables was significant (first exam call: $\chi^2(2, N = 225) = 17.49$, $p < .05$; second exam call: $\chi^2(2, N = 96) = 3.62$, $p < .10$). Those who performed at least one task design for an LG were more likely to pass the examination.

4. Discussion and conclusion

This paper intended to answer the two research questions indicated at the end of Section 1. Answers to each of them are given below.

Involving PT in the design of tasks for LGs was a challenge. It is well known, at least in Italy, that in the Educational Sciences courses, mathematics is neither the preferred subject nor a main subject of the curriculum. However, as it can be seen from the data, the attitude of the PT towards this proposed activity was surprisingly positive. The engagement demonstrated exceeded my expectations and I do not consider this to be related to having promised 1 extra point in the examination. Those who worked on the task design did so seriously and showed that they were involved in the design. Designing turned out to be easy for most of them ⁽³⁾, given the instructions they received from me, the opportunity to

3. Excluding the fact that hardly anyone was able to draw up appropriate hints.

work among peers and the willingness to get involved. Sixteen per cent admit that it takes a little practice to understand how to design tasks appropriately. The fact that they produced more than the tasks I expected is also part of the fact that the PT grasped the usefulness of doing such design work. Indeed, they recognised that working on the task design allowed them to make a “general review of the topic” and/or “deepening of certain aspects of it”. Moreover, to be able to write the sample solution was for them a confirmation that they understood the exercise’s solution process and were able to explain it. This also emerges from their comments on the importance of having been able to benefit from the reviews, which accompanied and supported their design work step by step. It was a team effort between me and them, which required dedication and time on my part, but it seems to have paid off. Almost everyone who worked on the task design took the exam in the first session and passed it. Even though the chi-square test revealed statistical significance, I am well aware of the fact that working on the task design is not the guarantee of passing the exam. It is important to attend classes (which are not compulsory) and above all to study.

The preliminary results of this study are in line with similar research that has seen the use of ASYMPTOTE in both school classrooms (Fesakis et al., 2023) and university courses (Caldeira et al., 2023) as well as in-service and PT training courses (Taranto et al., 2023a; 2023b), in which the system was either used to design tasks or to benefit from them or to do both. Indeed, these studies all agree, as does the present paper, in highlighting how the ease of use, usefulness and acceptance of ASYMPTOTE have a positive impact on teaching and high student engagement. The preliminary study conducted here brings out other aspects, not dealt with in previous studies, which I outline below.

The integration of future teachers in the design of mathematical tasks through the ASYMPTOTE system offer an important indication: direct involvement in the creation of teaching materials can have a significant impact on their understanding and mastery of mathematical content. This process, which goes beyond the traditional study of mathematical concepts, requires PT to apply their knowledge in a practical and reflective manner, thereby improving their competence and confidence in teaching mathematics. When PT engage in task design, they are forced to examine concepts from different perspectives, to reflect on how to present material in a way that is accessible and stimulating to students, and to anticipate possible difficulties that students might encounter. This process not only deepens their understanding of mathematical content, but also develops crucial skills such as analytical skills, creativity in problem solving, and the ability to communicate complex concepts clearly and effectively (Kieran et al., 2015). Furthermore, task design offers PT the opportunity to receive constructive feedback on their materials, allowing them to refine their teaching and assessment techniques. This cycle of design, feedback and revision serves not only to improve the quality of the tasks created but also to consolidate the PT’s pedagogical and disciplinary knowledge. In summary, the PT’s involvement in the design of mathematical tasks not only improves their understanding of the

content (e.g., as we have already seen - *“It helped me to understand some topics more deeply”*) but also prepares them to become more effective educators, capable of stimulating interest and understanding of mathematics in their future students (e.g., as we have already seen - *“Because we will be future teachers and it is important to try our hand at creating these useful learning tasks”*). These preliminary results underline the importance of integrating innovative pedagogical practices into teacher education, with the potential to significantly transform mathematics education.

5. Research limitations and future research perspectives

A potential limitation of this study is that the data analyzed pertain to a course I personally taught; therefore, the analysis may be influenced by my own subjectivity and perception of the course. As well as, the questionnaire - as far as it was sent to all PT who attended the course - was filled out voluntarily. This may have biased the results in the sense that the respondents were PT motivated to want to answer.

Additionally, I deprived the students of the opportunity to work on the web portal because a basic knowledge of the LaTeX language was required to write formulas correctly, and this was not a skill possessed by the PT. For the future, I plan to prepare a booklet with the most frequent LaTeX commands for them to work on the web portal. Furthermore, as anticipated, I often drafted the task hints for the students. The aim of another course, the Mathematics Education course (which I always teach in the following semester), is to get the PT to think about metacognitive processes as well. That will be the opportunity to return to these aspects.

For future research, this preliminary study offers different and diverse insights. From the point of view of the ASYMPOTOTE system, since the PT had difficulty creating suggestions of their own, it would be worth investigating how the system could be adapted to help users create appropriate suggestions. From the perspective of mathematics education research, on the other hand, it would be interesting to investigate how the task productions evolved over the various revision rounds; how much and how the practice of writing the sample solution was internalised by the PT and how this practice is reflected in the way the examination questions are carried out. Equally interesting would be to investigate the other questions in the questionnaire, which are more relevant to the use of the LG, especially in PT who did not participate in the task design. Last but not least, PT could be involved in designing not tasks but LGs. It would be worth investigating how they choose main and supporting tasks, e.g. for a specific topic. The design of LGs by PT is an interesting task to give us information about their needs, difficulties and abilities in learning design.

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Prospective mathematics teachers' subjectification processes: joint labor in a Moodle environment through Quick Chat plugin

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Abstract. In this paper, we describe a study involving master's degree students in mathematics taking the "Mathematics Education" course. The students, potential prospective teachers, worked in presence on an activity designed to make them reflect on teaching practices and methodologies. They first responded individually to an online questionnaire on misconceptions in mathematics. Then, in small groups, they discussed in the Moodle environment through multiple communication channels using the Quick Chat plugin. The plugin allowed the teacher to monitor all conversations in real time. We analyzed a specific context of practice characterized by the introduction of the Quick Chat plugin within the framework of Objectification Theory, resorting to the notions of joint labor and subjectification.

Keywords: computer-based environment, prospective teachers education, theory of objectification, joint labor and subjectification, Moodle.

MSC 2020: 97B50, 97C70, 97D40, 97M20, 97U50.

1. Introduction

In the last few years, many mathematics learning activities have been designed and implemented in online platforms on the basis of computer-based collaborative scripts. Such activities, called Vygotskian computer-based learning activities-VCBLAs ([1]; [2]), promote collaboration and discussion among students and foster online exchanges through multiple communication channels (e.g., [3]; [4]; [5]). For collaborating and discussing to lead to effective learning, it is necessary for the technological environment to be accessible to reduce the cognitive load for students to cognitively focus on the task ([6]). Furthermore, it

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is important to increase the degree of user acceptance of the environment itself and hence the usability ([7]). A VCBLA is typically made up of a narrative flow that guides the learning activity, that is, the task, and a set of collaborative tools (e.g., communication channels). Within a VCBLA, students work in small groups and communicate at various levels (peer-to-peer, whole class, and/or with an expert/teacher) through chats. The most common way to implement a VCBLA is with Moodle, which has all of the tools needed to implement both the narrative flow and all of the communication channels. However, when many collaborative tools are needed at the same time, Moodle has some limitations. Users are forced to open multiple browser windows or tabs: one for the flow of the narrative, and one for each chat or additional tool. The problem is not only for the students, but also for the teacher, who may have a hard time keeping track of all the conversations in real time.

The Quick Chat plugin was developed as part of the LIME (Learning Interface for Mathematics Education) project at the University of Campania "L. Vanvitelli" to improve the user experience and make the Moodle environment more accessible. The goal is to overcome the accessibility and usability limitations of the Moodle platform by being able to group both the main course page and all the chats the user is a part of according to their role within the script into a single browser tab. Once installed, the plugin is available from the navigation block on the course home page. It allows students and teachers to manage all the tools and activities provided for a VCBLA, i.e. the task and all the communication channels in which each user (student or teacher) is involved, from a single browser page. Dello Iacono and Fiorentino ([8]; [9]) showed the effectiveness of the plugin in promoting student interactions in the university setting. In particular, they showed that students in all groups interacted using all communication channels, even in parallel, and that the teacher was able to monitor all conversations to intervene effectively in real time.

The aim of this study is to envision teacher training beyond professional development marked by subject matter and pedagogical content knowledge ([24]) whose importance we do not underestimate. We would like to enlarge the approach to teacher training taking into account the production of new identities as individuals establish a reflexive relationship with the sociocultural context of mathematics teaching and learning. This means that in their training trajectory prospective teachers in addition to being affected by their cultural-historical context they react agentially to such a context thereby producing a new identity as a mathematics teacher. Their professional development is related to the production and transformation of subjectivities and to the continuous co-production of singular and unique individuals against the backdrop of their cultural-historical domain. It allows prospective teachers to critically position themselves in the cultural-historical domain of mathematics teaching-learning practices, realizing new ways of thought and action. In this study we continue in the line of Ferretti and Santi ([11]) looking at the role of Quick chat plugin in molding the structure of reflexive activity that accomplishes the relationship between individuals and

the sociocultural context. In light of the TO, the sociocultural context is seen as symbolic superstructures called *semiotic systems of cultural signification* that include cultural conceptions surrounding mathematical objects (their nature, their way of existing, their relation to the concrete world, etc.) and social patterns of meaning production (beliefs about conceptual objects, conceptions about truths, methods of inquiry). In regard to teacher training we have singled out the following: epistemology and ontology of mathematics; systems of truths; forms of rationality; accepted mathematical teaching practices problems and situations; conception of teachers and students; socio-mathematical norms; conceptions about education; language and languages; assessment of learning.

In this paper we describe a study involving 20 undergraduate mathematics students taking a course in "Mathematics Education". The aim of our research is to scrutinize how mathematics education courses for prospective teachers are affected in a mixed learning environment (in person and online) that makes use of the Quick Chat plugin. According to a Vygotskian perspective ([12]), we analyzed the potential of the Quick Chat plugin within the framework of the Theory of Objectification, resorting to the notions of joint labor and subjectification ([13]; [14]; [15]; [16]). Activity is defined by the Theory of Objectification as *joint labor* and it is a dynamic system where individuals produce knowledge and, in doing so, co-produce themselves in processes of *subjectification*. Our research questions are as follows. In a prospective teacher education environment,

- (RQ1) how does the Quick Chat plugin mold the dynamics and the development of joint labor?
- (RQ2) with respect to which cultural symbolic superstructures do prospective teachers position themselves in subjectification processes?

In the following sections we describe: the theoretical framework underlying our work, the research method, and the analysis of the data derived from the study. The paper ends with a section devoted to a discussion of the results and conclusions.

2. Theoretical framework

Asenova ([17]) describes two possible lines in Mathematics Education Research. Research dealing with Topic-Specific (TS) issues that produces usually local results is less broad than research projects dealing with Context-Specific (CS) issues that have more general and abstract scopes. Our study is TS research, in that it provides local results regarding the use of technology in teacher training within the lines of investigation accomplished by the DIGIMATH group that aims at context specific results regarding teacher training with a more extensive impact on this field of research. The DIGIMATH group has identified at least two approaches to using technology for mathematics teaching/learning: to mediate mathematical knowledge, and to foster communication and interaction among students during mathematical practice (e.g., [18]). VCBLAs based

on the Quick Chat plugin implementation can be analyzed using both of these approaches. In this study, we focus on the second approach to the use of technology. In order to deepen the meaning of the activities that take place in VCBLA and according to a Vygotskian perspective, we analyze this specific context of practice characterized by the introduction of digital technology, that is, the Quick Chat's potential, within the framework of Objectification Theory, resorting to the notions of *joint labor* and *subjectification*.

Joint labor is one of the pillars of Objectification Theory in which, continuing along the line of Activity Theory ([19]), mathematical knowledge, thinking and learning are the product of the individual-culture-activity dialectic. Activity is not a series of actions (*Aktivität*) but a *Tätigkeit*, that is, a dynamic system where individuals produce knowledge. Activity is thus a form of life. It is a system ([19]) in continuous development, always influenced by the entities to which it relates. Activity as *Tätigkeit* is defined by the Theory of Objectification as *joint labor*. The notion of *joint labor* allows us to explore the social interactions of individuals engaged in knowledge production in the context of VCBLA. Radford ([13]) states that teachers and students engage in the activity of knowledge production together. Knowledge is produced collectively in *joint labor*. This means that students and teachers work together to propose possible mathematical interpretations related to forms of human action and collaboration shaped by cultural signs and artefacts. These are modes of mathematical enquiry, truth conception, proofs, mathematical argumentation, the use of symbols and the creation of meanings. In the socio-cultural texture of *joint labor*, individuals not only produce knowledge but also co-produce their own identity through processes of *subjectification*. Radford ([14], [15]) defines *subjectification* as the processes in which individuals produce their identity in dialectical interaction with the cultural-historical context by positioning themselves in their social and cultural-historical environment. In the dialogue with the cultural context, through reflexive and mediated activities, the individual is understood as an entity in constant flux, a life project always in progress ([16]). The pillars of the TO are rooted in the dialectical relationship between joint labor and the use of signs and artifacts. In fact, activity bridges the individual to their sociocultural dimension in terms of objectification of knowledge and the ensuing production of subjectivity as the positioning in such a cultural dimension. Subjectification and objectification are two sides of the same coin in our agentic and reflexive encounter with culture as knowing and becoming. As regards the shift of the sociocultural perspective from the realm of the teaching and learning of mathematics to teacher training that we are presenting in this paper, knowing embraces subject and pedagogical content knowledge, becoming involves a positioning with respect to semiotic systems characterizing the school sociocultural environment. What is my positioning towards epistemological stances in mathematics, Platonist or pragmatic? How do I conceive learning in mathematics? How do I use assessment in relation to my conception of the student? How do I use language and languages? What forms of rationality do I accept? In the

subjectification processes teachers express their agentic reaction to the school culture realizing ways of thought and action thereby co-producing in joint labor unique and singular identity.

3. Method

The study reported in this paper involved 20 students in the second year of the master's degree in mathematics (hereafter referred to as participants) attending the Mathematics Education course at the University of Campania "L. Vanvitelli". The study was carried out during a lecture in that course and participants were informed about the conduct of the study. The participants - potential prospective teachers - worked in presence on a learning activity designed to make them reflect on teaching practices and methodologies. The methodology was informed by Educational Design Research in which the iterative development of solutions to practical and complex educational problems provides the setting for scientific inquiry ([20]).

The activity involved 4 phases. In phase 1, the participants responded individually to a questionnaire designed to investigate typical misconceptions in mathematics, administered as Google Form. Figure 1 shows some of the questions in the initial questionnaire. The first two questions are related to radicals in the real number field. The third question, taken from the work of Tall and Vinner [21], refers to typical misconceptions regarding tangents at a point to the graph of a function.

Nel campo dei numeri reali,	Nel campo dei numeri reali,	Quali dei seguenti grafici hanno tangente/i in P?
$\sqrt{49} = \pm 7$ ☐ Vero ☐ Falso	$\sqrt{m^2} = \pm m$ ☐ Vero ☐ Falso	
Giustifica la tua risposta.	Giustifica la tua risposta.	Giustifica la tua risposta.

Figure 1: Some questions from the initial questionnaire

[In the field of real numbers; True, False, Justify your answer; In the field of real numbers; True, False, Justify your answer; Which of the following graphs have tangent(s) in P? Justify your answer].

In phase 2, the participants worked in small random groups (3 participants for each group except for one group consisting of 2 participants) to confront each other in the Moodle environment and to communicate through a group chat. The aim of this phase is for participants to arrive at common and shared answers. Each participant worked on their own laptop. The groups were created by the teacher so that the participants in the same group were physically far

away from each other in the classroom so that they could communicate with their group mates only using Moodle chat.

In phase 3, the responses of each group agreed upon in phase 2 were reported by each group's spokesperson in a General chat. In this chat a discussion started mediated by the teacher. This phase aims to foster a discussion among participants in all groups. The Quick Chat plugin allowed participants to view assignments, Group and General chats on the same webpage. In addition, the teacher had real-time monitoring of all conversations through the plugin (Figure 2). At the end of the activity, the teacher facilitated a class discussion.

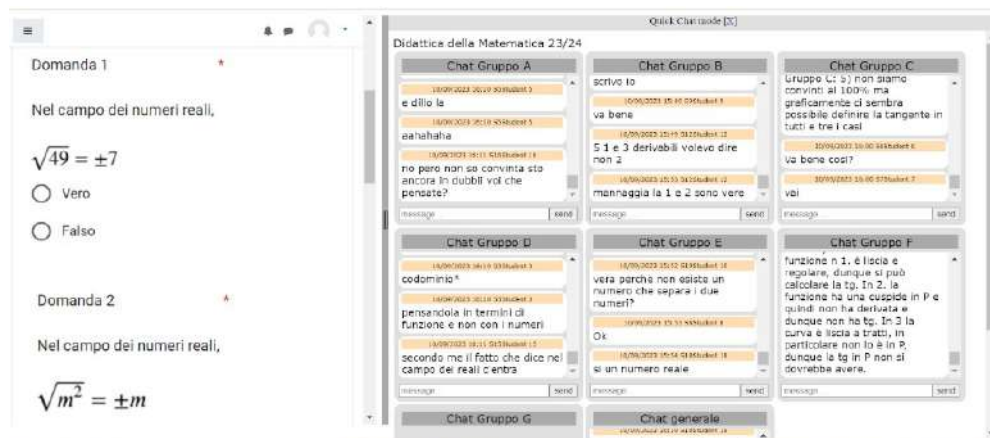


Figure 2: Teacher view of the browser page

In phase 4, an oral discussion (no longer in the Moodle environment) mediated by the lecturer took place.

At the end of the "Mathematics Education" course, participants – prospective teachers – were given a metacognitive task through a Google Form, designed to make them reflect on the whole course and make them relive the most significant moments of the course: *Review the path we took together. Dwell on the moments/activities that were most significant to you and indicate the reasons why these moments/activities were significant.* We collected the following data:

- responses to the initial questionnaire through Google Form;
- discussions in the Group and General chats using the Quick Chat plugin in the Moodle environment;
- the responses to the final metacognitive task using Google Form.

4. Data analysis

In this section, we analyze the collected data from the study in accordance with the theoretical framework described above. In particular, we analyzed discus-

sions in the Group and General chats and the responses to the final metacognitive task. Below, we show a brief excerpt showing a joint labor part related to the use of Quick Chat in the exchange between members of a group that addressed the first two questions of the questionnaire. Specifically, we show the intense relationship between participant P5 and participant P16. We selected the dialogue between P16 and P5 because they are confronted with two meanings of the square root: as an operator to find the number that when squared results in the radicand, and as a real number. In reference to the list of semiotic systems of cultural signification mentioned above, the students' objectification of the notion of square root is entangled with their co-positioning with respect to epistemology and ontology of mathematics, systems of truths and forms of rationality:

P16: My motivation is that the root of a negative number in $\mathbb{R}$ does not exist which by the way is the same answer I also gave to the second question

P5: however to the second one you could put a negative number because there is the square

P16: and then the first one is also a square

P5: yes but of 7 times 7

P16 could be also - 7 squared

P5 that is, yes also -7

P16 but the condition should be that the number under root must be positive and not zero

P5 so what shall we say

P16 so let's say both are true, the first since 49 we can write it as 7 to the quad. . .

P5 First question: the root is a positive value so the answer is unique, that is 7

P16 is the answer I also gave earlier however now I have got the doubt [...] i.e. what is true for the first is not also true for the second?

P5 First question: the answer is false the root is a positive value so the answer is unique, that is 7

P16 i.e. what is true for the first is not also true for the second?

...

P5 First question: the answer is false the root is a positive value so the answer is unique, that is 7. Second question: the answer is true the root is a positive value so the answer is - or + m.

From the excerpt above, we observe a co-positioning of participants with respect to mathematical culture, particularly of the meaning of radicals in the real number field. P16 and P5 are confronted with two meanings of the square root: as an operator to find the number that when squared results in the radical and as a real number. Their co-positioning in *joint labor* expresses the tensions, contradictions, continuous interpretations and reinterpretations that characterize symbolic superstructures and, in this case, the duality between process and object ([22]) that characterizes mathematical concepts. The small group discussion has not overcome the tension between process and object, and the *joint labor* maintains its dynamic and fluid nature in keeping the *subjectification* process open. Indeed, the group concludes that in the first question the answer is false because the root is a positive value so the answer is unique, that is, 7; in the second question the answer is true because the root is a positive value so the answer is - or + m. The contradiction will be overcome in the broader structure of *joint labor* centered on the use of Quick Chat, which systematizes group activity in chat, general discussion also in chat, and in person activities.

As mentioned in the previous section, at the end of the entire course, a final questionnaire was proposed with the following question: Review the path we took together. Dwell on the moments/activities that were most significant to you and indicate the reasons why these moments/activities were significant. The participants' statements show the processes of *subjectification* in which they establish themselves not only as mathematical subjects in mathematical practice but also as mathematics teachers in the training activity of the "Mathematics Education" course. An example is the following response from a participant:

I found the course very interesting because it completely changed my thinking about how to teach math in school. I realized the importance of creating activities and methods to derive mathematical definitions and theorems. The activity on misconceptions was crucial in allowing me to understand what "mistakes" students can be led to make because of explanations that use particular examples. I found the initial activity on grading students and the mistakes we future teachers might make in grading very interesting. It is important to give weight to the process and not to the outcome of the problem. In this course, I also found the group activities very interesting, which then later enabled me to understand how groups should be formed in the classroom. Creating seminars is important for our personal growth and for trying to impart knowledge to our peers. This course really allowed the learning of fundamental notions for us aspiring teachers, and I think we would also need its continuation in order to constantly improve. One thing that I think is very positive is that the lecturer was always available for discussion on all our ideas even if sometimes wrong.

We observe how the symbolic superstructures against which prospective teachers' co-positionings occur are broader and more specific than those with which students are usually confronted: mathematical objects and ways of representing them ("my thinking about how to teach math in school"), systems of truth, forms of rationality; misconceptions and obstacles ("misconceptions was crucial in allowing me to understand what "mistakes" students can be led to make because of explanations that use particular examples"), cognition and metacognition, teaching methodologies ("I also found the group activities very interesting, which then later enabled me to understand how groups should be formed in the classroom"), forms of action and behavior, conceptions of teaching and learning ("It is important to give weight to the process and not to the outcome of the problem"), assessment, affect, communication ("One thing that I think is very positive is that the lecturer was always available for discussion").

5. Discussion and conclusions

The educational problem of mathematics is to provide opportunities for students to encounter mathematical knowledge in rich and meaningful ways, to enter into conversation with culture, and to critically co-position themselves toward mathematical knowledge in and through its practice. Co-positioning refers to the conscious movement of students in knowledge production activities. It refers to the way in which students establish themselves as subjects in the practice of mathematics but, in this context, as future mathematics teachers in training activities. In this paper, we asked how Quick Chat plugin shapes the dynamics and development of *joint labor* in a teacher professional development environment (RQ1), and with respect to which cultural symbolic superstructures future teachers position themselves in *subjectification* processes (RQ2). Concerning RQ1, the Quick Chat plugin fosters the emergence of dynamic and fluid systems of activity that, in the interaction between online and in presence activities, allow for new forms of *joint labor* and the resulting co-positionings of prospective teachers toward mathematics and mathematics teaching/learning. The teacher monitors in real time the performance of the task by the various groups and can intervene in the discussion of individual groups. She also monitors and intervenes in the general discussion that takes place simultaneously with the activity of the individual groups. The student can switch from Group Chat to General Chat when she feels it is appropriate for the performance of the task, both in terms of the mathematical aspects and the emotional and collaborative support offered by the *joint labor*. Student communication and interactions occur at multiple levels and in parallel: at the Group and General chat levels. We also note that in the interaction between oral communication and online activities via Quick Chat, students activate the oral verbal channel during the general oral discussion (in phase 4) and the written channel in Quick Chat communication (in phases 2 and 3). This dual channel can have interesting effects both at the level of objectification of knowledge and *subjectification*. The

introduction of Quick Chat modifies the classroom activity schema introduced by the Theory of Objectification, intensifying the systemic, fluidity and dynamic aspects typical of *joint labor*. Indeed, the relationships between the elements that make up the scheme of activity, take on a higher level of complexity in spatial and temporal terms (see Figure 3). The left side of Figure 3 shows the classroom activity according to the TO based on in presence activity based on subsequent general discussion, group work and group work with the teacher. The right side of Figure 3 shows the enlarged structure of the Classroom activity resulting from the implementation of Quick Chat plugin that blends in a synchronous mode and in presence different channels of communication and activity with a general control on the part of the teacher (on the different chats, via oral discussion, in the big group or in subgroups).

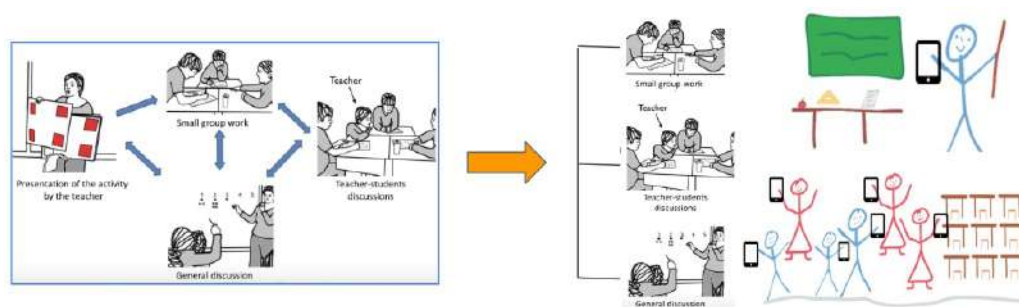


Figure 3: Classroom activity schema modified by Quick Chat plugin

The Quick Chat plugin changes the socio-cultural structure in which teaching-learning processes take place and the specific ways in which knowledge is produced. In this context, the focus shifts from forms of collaboration between teachers and learners in developmental age to those between trainer and trainee teachers. Likewise, the focus shifts from the production of mathematical knowledge to the production of the specialized knowledge of the mathematics teacher ([23]). In the context of teacher education, *subjectification* processes thus enable prospective teachers to position themselves in the cultural and social context of the school characterized by its symbolic superstructures: typically conceptions of the student, conceptions of teaching, conceptions of mathematics related to teaching-learning processes etc. ([11]).

Concerning RQ2, in our study, the effect of the broadened structure and dynamics of *joint labor*, accomplished by the introduction of Quick Chat, was mainly on the prospective teachers' positioning with respect to mathematics and its teaching-learning processes. The individual work on the questionnaire and the subsequent discussion on Quick Chat characterized by the back and forth between the group chat and the general chat triggered a new understanding of the mathematical topics with a more metacognitive outlook that in turn fostered their positioning with respect to the teaching and learning of mathematics as

derives from the answers to the question of the Google Form: *The activity on misconceptions was crucial in allowing me to understand what “mistakes” students can be led to make because of explanations that use particular examples; It is important to give weight to the process and not to the outcome of the problem; In this course, I also found the group activities very interesting, which then later enabled me to understand how groups should be formed in the classroom; This course really allowed the learning of fundamental notions for us aspiring teachers.*

This paper presents a new approach to teacher training that enlarges the understanding of professional development usually marked by subject matter and pedagogical content knowledge ([24]). We take into account the production of new identities as individuals establish a reflexive relationship with the sociocultural context of mathematics teaching and learning. This means that in their training trajectory prospective teachers in addition to being affected by their cultural-historical context they react agentically to such a context thereby producing a new identity as a mathematics teacher. Their professional development is related to the production and transformation of subjectivities and to the continuous co-production of singular and unique individuals against the backdrop of their cultural-historical domain. It allows prospective teachers to critically position themselves in the cultural-historical domain of mathematics teaching-learning practices, realizing new ways of thought and action. We have analyzed the role of Quick Chat plugin in molding the structure of reflexive activity that accomplishes objectification and subjectification processes. The development of our research can be implemented in teacher’s professional development programs where Quick Chat acts as a tool for teacher collaboration in training activities that can be implemented as a tool from teacher collaboration in classroom teaching-learning activities ([25]). A similar approach based on information and communication technology has been implemented in mathematics education courses for prospective primary school teachers at the faculties of Education ([11]).

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A learning experience in and from practice using digital technologies: the mathematical discussion

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Abstract. In this contribution we report on an experimental research developed with Pre-Service primary Teachers (PSTs) in the Project Digimath, with the aim to give PSTs the opportunity to learn in and from the practice. For doing this we foster the Mathematical Discussion (MD) theoretical framework's learning (Bartolini Bussi, 1989). We explored the effectiveness of a new theoretical approach that combines Shulman's (1986) research about the Pedagogical Content Knowledge (PCK), Ball's Mathematical Knowledge for Teaching (MKT) (2008) and further research on how practice can be used for teachers' learning and what is relevant to learn in and from practice (Ball & Even, 2009), in a digital environment.

We noticed that to become aware of professional education and development of mathematics teachers and of specific characteristics of the teachers' actions, playing different roles (student - PST) is fundamental.

Our work was developed with 160 PSTs attending the Mathematics Education Course at the University of Bari. The MD has been at the same time the subject of the lecture and the methodology used during the lecture, both in a digital environment and personally. This choice has been done with two different aims: the construction of the MD's characteristics and the way to manage MD in the future professional practice. The experimental activities have been held using hybrid teaching and were videotaped. Conversations have been transcribed and have been used to analyze the teaching experiments.

As we expected, from the transcriptions' analysis, it emerged that the PTSs became aware of the MD's value. In and from their own practice they learned how to manage MD in class, in a future perspective.

Keywords: mathematical discussion, digital technologies, pre-service teachers.

MSC 2020: 97B50, 97C70.

1. Introduction

This contribution aims to provide new insight on how to train PSTs, focusing on pedagogical models and their implementation in a class. Indeed. we asked

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ourselves how to train the prospective teacher on a specific pedagogical model, and in this contribution, we propose a new training mode, designed in order to introduce PSTs to the pedagogical model of Mathematical Discussion, both from a theoretical and practical point of view. In this study, we propose to let the PSTs experience the pedagogical model at stake in the role of student, then to involve them in a reflective situation guided by the trainer, consisting in a collective discussion focusing on the teacher's role previously played by the trainer. We call such a reflective discussion, Meta Discussion on a Pedagogical model (M-DPm). Our research has involved 160 PSTs attending the undergraduate 'mathematics teaching' course in primary education. Data analysis showed that the training activity carried out by experiencing in first person the pedagogical theories fostered the theory's conceptualisation itself and the future teaching profession's awareness. In the following, we outline the conceptual background, in particular the pedagogical model in focus, then we present the experimental design, the research methodology, the analysis of some of the data collected and the analysis' results.

2. Conceptual background

In last years, several research and theories emerged in the teacher's professional training area. Shulman (1986) claimed that to teach a discipline is necessary to tackle both the specific knowledge and pedagogical knowledge. The specific knowledge is not simply intended as a disconnected set of disciplinary knowledge. Indeed, in order to foster effective teaching-learning processes (e.g., a Pedagogical Content Knowledge, (PCK)), teachers need an integrated view of disciplinary content and issues related to its teaching. Moreover, for how concerns Mathematics, different conceptualisations and models of specialised mathematical knowledge for teaching have been proposed. Among them the Mathematical Knowledge for Teaching (MKT) model (Ball et al., 2008) or the more recent Mathematics Teacher's Specialised Knowledge (MKTS) model (Carrillo-Yanez et al., 2018). For the implementation of educational intervention Ball and Even (2009) provide us an interesting prospective that takes into account the complex articulation of different contents and their relationships.

By doing this they suggest focusing teacher training on practice – and on the problem of doing it effectively. From this perspective, on the one hand, it should be obvious that teaching is a practice and that, therefore, teacher education must provide systematic and lasting opportunities for teachers to learn and develop their effectiveness with that practice – with the complex tasks of their work. It follows the idea of training future teachers through practice and providing them an approach to a given task that can serve as a starting point for forming themselves as teachers through understanding and interpreting the actions of their future students. On the other hand, they highlight the lack of a shared articulation of a "curriculum" that would underlie teachers' opportunities to learn in and from their practice. Therefore increasingly necessary is

the development of ways to support teachers in learning in and from practice, making it a focus of discussion (Ball and Even, 2009).

Our main hypothesis is aligned with these general suggestions, and it concerns the effectiveness of a specific training mode to introduce PSTs to a definite Pedagogical Model (PM). This training mode is characterised by combining the living experience of such a PM and a reflective activity on it. In particular, we introduce a new construct which is the focus of our study that we named Meta Discussion on a Pedagogical model (M-DPm) which is described in the following section.

3. A new theoretical construct: the M-DPm

The M-DPm consists in a collective discussion which involves the PSTs and their instructor on the previous didactical experience of the specific pedagogical model of the mathematical discussion that they lived with the role of students (Fiorentino et al., 2023). In other words, such a reflective experience involves the same PSTs in a discussion, still conducted by their instructor, but with a different motive. That is of reflecting on their previous lived experience and identifying the key aspects characterising the pedagogical model that is at stake. A specific pedagogical model, has been selected in order to explore the proposed training mode, which is the Mathematical Discussion (MD).

MD was firstly elaborated by Bartolini Bussi (1998) and was originally defined as a 'polyphony of voices articulated about a mathematical object (concept, problem, procedure, etc.), which constitutes a motive of the teaching-learning activity'. In this model, the teacher has the responsibility of "orchestrating the polyphony", the voices (represented by the signs produced by the students) have to be coordinated with the voice of the mathematical culture (witnessed by the teacher herself)" (Bartolini Bussi and Mariotti, 2008, p. 763). The teacher plays two fundamental roles during a MD, namely those of mediator and moderator. To play this role, the teacher acts the main following actions: the "back to the task" action (with the aim of reconstructing the context and fostering the (re)emergence of meanings and processes related to the task); "focalizing" action (with the aim of focus on aspects consistent with the didactic objective); "request of synthesis" action (with the aim of supporting students in the process of de-contextualisation and generalisation with respect to specific tasks); "offer of synthesis" action (with the aim of providing a formulation introducing the desired terms; ratifying the acceptability and mathematical status of a specific meaning). A mathematical discussion activity is differentiated according to its objective (motive).

Different types of discussion can be distinguished: the Balance Discussion (MDb) and to a Conceptualisation Discussion (MDc). MDb is defined as the collective process of informing, analysing and evaluating the individual solutions proposed to a given problem. MDc is understood as the collective process of constructing mathematical concepts, building up suitable connections between

already lived experiences and particular mathematical terms (Bartolini Bussi et al., 1995, pp. 11-12). Sometimes it may happen that a Balance Discussion naturally develops into a Conceptualization Discussion, and this is actually what happened in our experiment. Thus, in the following, for clarity reasons, we will simply use the acronym MD to refer to the combination of these two types of discussions.

Our hypothesis is that an effective training mode can be based on the combination of a didactical experience lived by PSTs as the students in participating to a MD (orchestrated by their instructor with the motive of conceptualising a specific mathematical content) and the reflective experience involving the same PSTs in a new discussion, still orchestrated by their instructor. This new experience has a new motive: reflecting on their previous lived experience, identifying the key theoretical aspects characterising the pedagogical model constituted by the MD.

In this new perspective the previous actions characterising the teacher's role, are redefined with new aims in the new construct:

- the "back to the experience" action (with the aim of reconstructing the context and fostering the (re)emergence of characteristics and interactions, among students and with the instructor, related to the PM);
- the "meta-focusing" action (with the aim to make explicit the characterising elements that made it possible to recognise a MD in the experienced activity);
- the "request of synthesis" action (with the aim of supporting students in the process of de-contextualisation and generalisation with respect to the PM, in order to acquire awareness about its practice through distancing process);
- the "offer of synthesis" action (with the aim of providing a formulation introducing the PM's characteristics; ratifying the acceptability and mathematics teachers' role in a specific PM).

Within such a complex conceptual framework the specific hypothesis of this study can be synthesised as follows: Through the combined experience of MD and M-DPm, PSTs can be introduced to the pedagogical model of MD, and develop, in and from practice, both theoretical knowledge and practical experience of this model.

In this combined experience the PSTs shift their experience from living the experience themselves to observe it and recognize the characteristics of the PM, through the actions characterizing the teacher's role.

In summary, our research questions are: can the direct experience of implementing a mathematical discussion make PSTs aware of the functioning of this

teaching methodology/theory? How the combined experience, lived in first person by students, of MD and M-DPm could introduce the PSTs to the PM and to enact it? How technology could support the development of PSTs' training?

According with this hypothesis and with the research questions a teaching experiment has been designed, with the aim of exploring and deepening such hypothesis.

4. The role of technology

The role of technology is fundamental, as the activities are designed to be held online. In this project technology plays a major role, as it allows to support the processes involved in the teaching experiment in its three main functions (Albano et al., 2020): sending and viewing; processing and analysis of the data collected during the lessons; provide an interactive environment, where students can interact to work individually or in groups on a task or to explore mathematical/scientific content (Perry et al., 2021). The use of the digital resources in organizing the activities of the teaching experiment allows to solve the tasks in small online groups; to make a MDb all together to share the solutions and identify a common solution; to make a M-DPm on the MD; to customize both the types of action and the interaction among the PSTs (Fiorentino et al., 2022).

In detail, the environment technology is useful to stimulate collaborative processes and encourage the active participation of PSTs. In particular, it allows each participant to solve the problem individually and to send the solutions in a form shared first in a small group and subsequently, a shared solution in the large group; it encourages listening and reflection; it allows the participation of all students both in small groups (subdivision into rooms) and in large groups with screen sharing where necessary; it freed the students from the embarrassment of intervening; it allows video and audio recording of all discussions; it makes the materials immediately available. (Albano et al., 2020)

In the following section the research methodology will be described.

5. The research methodology and experimental setting

This paper focuses on a specific experimental path of a general project. We intend to foster the PSTs' professional development, in and from practice, of both practical experience and theoretical knowledge on a specific pedagogical model. For doing this, we focus on the design and the implementation of a training mode in which we involve the PSTs into two kinds of activities that develop at two different levels: the practical level where PSTs play the role of students and the instructor applies the pedagogical model at stake; the reflective level where, within the frame of the M-DPm, the PSTs reflect on their previous experience under the guidance of the instructor. From the huge amount of data collected during the experimentation, we consider the M-DPm transcript and the personal reflective diaries. The excerpts we present in the following sections, have been

chosen because they reveal evidence about effectiveness of the training mode and particularly of the construct of the M-DPm. According to the criteria of credibility, dependability, transferability and confirmability (Guba, 1981), to ensure trustworthiness a qualitative analysis of the transcriptions was carried out. The research involved 180 pre-service PSTs in mathematics, fourth-year students in Primary Education, and attending Mathematics Education Course. The training mode and in particular both MD and M-DPm take place synchronously on Microsoft Teams platform. Both the discussions take place with the whole class group. These discussions have been recorded and then transcribed for the analysis. In the following section, we describe the general setting of the specific experimental path in focus.

5.1 The structure of experimented training mode

The training mode is designed by a succession of steps identified by activities informed by our hypothesis on the educational relationship between experiencing a MD and participating in a M-DPm on such an experience.

Step 1: introduction of the MD theoretical model.

The instructor introduces the pedagogical model through a lecture, in which she gives the theoretical elements characterising the MD, in particular the teachers' actions to be performed to orchestrate the discussion.

Step 2: The open-ended problem and the MD (workshop online).

In order to activate a MD, it is necessary to start by solving an open-ended mathematics problem sufficiently elementary to be understood by anyone and included in school curricula. As a matter of fact, the use of an open-ended problem (Pehkonen, 1997) is crucial for triggering a rich and effective mathematical discussion: it is not solved mechanically, but requires different solving strategies, it is sufficiently challenging to make students engaged both cognitively and emotionally, as solvers of the problem. The authors selected the topic of equi-extension which is a complex one. It is generally based on the procedural activity of calculating the measure of equi-extended areas, and not on the idea of extension as a quantity related to equi-decomposability.

The following problem was proposed to be solved in groups, online, through Teams platform: "Two brothers receive a rectangular piece of land as inheritance. In order to divide it into two parts of the same size, one of them suggests planting a stake anywhere on the land and joining it to the four stakes driven into the four vertices of the rectangular land. One of the brothers will take two non-adjacent triangles, the other the remaining part. Are the two parts really equal? Justify your answer."

In the following activity, starting from the solutions of the above problem a MD was developed. The aim of the MD was bringing out all the problems' solutions strategies, converging towards a shared solution, and making explicit the equi-extension's concept. The instructor who orchestrated the discussion performed the actions expected from the Pedagogical model referring to the

patterns described by the MD pedagogical model. In so doing the authors intended to make the prospective teacher experience the effect of these actions as students. In this way, PSTs directly 'experience' the putting into practice of a DM as students and, at the same time, have the opportunity to observe the teacher's actions, recognizing the characteristics of the teacher's role in the practice of this PM.

By doing so they could also observe these actions as performed by the 'teacher'.

The technological tool is fundamental in this activity because the student, standing in front of the screen, can reflect on the development of the activity, have a relaxed time to elaborate answers and listen to the interventions of colleagues. Furthermore, the MD conducted online by the teacher is supported by the technological tool in the management of the intervention by all participants in the discussion and in the possibility of keeping all participants at the same distance.

Step 3: M-DPm on the previous MD (workshop online).

In this step PSTs are asked to reflect on the activity carried out in step 2, by participating to the M-DPm: identify the actions performed by the teacher and characterising the pedagogical model, that is the case of the MD. In this M-DPm Mathematics remains in the background, although it is not ignored by the participants. MD, as a pedagogical model and in particular the teacher's actions, become the subject of discussion. A key aspect for the PSTs' professional development lies in the transition from the role of student to that of teacher, projecting himself/herself in the actions performed by the instructor. Indeed, through the observation and the analysis of the teacher's action, the PST live his/her future role by distancing himself/herself from the student's role.

Even in this phase, technology plays a fundamental role, because through the sharing of observations on the teacher's actions during the DM, the students implement a distancing from the experience lived in practice, recognizing and generalizing the characteristics of the theoretical construct.

6. Analysis of data with respect to the M-DPm

In this section we present the analysis concerning the M-DPm carried out in step 3. The following analysis, according to the main hypothesis inspiring the research study, attempts to highlight the effect of performing such a meta-discussion: how the M-DPm makes the characteristics of the pedagogical model MD emerge and are made explicit by the PSTs.

The transcript of the M-DPm was analysed with the twofold aim: highlighting the PSTs' recognition of MD's characteristics through the recognition of the teacher's actions; and at same time highlighting the importance of experience in practice of PSTs during MD. In the following we report the analysis of some excerpts consistent with this twofold aim.

(0:00) Instructor: What did we do in the last lesson [refers to the lesson in which the MD on solving the mathematical open-ended problem on equi-extension was carried out]. Why did I do this lesson with you? What was the teacher's aim?

(1:42) Stefania: we retrace everything that we did together, [...] so both an analysis from the point of view of Mathematical Discussion, as a pedagogical model and on the solutions of the problem [...] so the discussion's steps, the teacher's role who has to act as ... especially as a mediator within the discussion.

(2:20) Francesca: in my opinion, when the problem has been proposed to us, when we have been divided into groups and when we had to discuss firstly with our peers, and then with the teacher... for all this I suppose we did a Mathematical Discussion, [...] as a pedagogical model and on the solutions of the problem [...] in addition your role was especially that of mediator of the discussion

The instructor starts the M-DPm, with the aim to bring out the MD's characteristics. Indeed, she proposes to discuss the previous lesson in which the PSTs participated. This action carried out by the instructor is the new action: "Back to the experience". It is aimed at going back over the discussion experience in order to recognise the MD as a pedagogical model. The question is formulated with no longer reference to the problem to be solved by the instructor: "Why did I do this lesson with you?". The question is aimed to problematize the actions of the teacher in the attempt to make the characteristics of MD emerge. Stefania's answer shows the effectiveness of the instructor's intervention: she recognises and shares with the others her recognition of the MD pedagogical model in action. Following the question that induces and promotes shifting the discussion's focus from the mathematical content to the pedagogical model Francesca's answer shows a recognition of the MDPm's aim to make PSTs discuss about MD-b-c. Through the Francesca's and Stefania's words to answer to the questions of the instructor, we highlight their recognition of the MD as a PM.

Their answers probably follows a question that induces shifting the discussion's focus from the mathematical content to the pedagogical model. Furthermore, they detect the specific role of the teacher, as mediator. The MDPm proceeds and the teacher invites Eleonora to speak.

(3:22) Instructor: OK, so let's see, Eleonora wanted to intervene.

(3:30) Eleonora: Yes, the first thing that came to my mind was that...ehmmm that mathematics can be discussed, precisely that through reasoning guided by you and... the various ideas emerged, the various hypotheses that we articulated to solve the problem.

Eleonora acknowledges that the activity in the previous lesson was a discussion and in particular a MD because the motive was that of making the "various ideas" emerge and be shared, and this is a characteristic of MD.

In our opinion, already emerges that having experienced in first person the previous MD on the mathematics problem, allows Eleonora, Stefania and Francesca to simpler recognize the PM's characteristics.

A clearer articulation of MD' characteristics is shown in the following excerpt.

(4:16) Instructor: Eleonora said we discussed Mathematics, but can you tell me what were the very elements that allowed you to recognise this activity as a discussion?

(4:45) Ornella: Another thing I'd like to add is that... you had another role, specifically that of moderator. In particular when you gave the opportunity to most of us to intervene during the discussion... you never stopped our speech or you never came to a conclusion and we also came to an uncertainty... Let's say you never said "this is the right solution", "this is another right solution" ... you replicated our words...

(6:22) Daniele: I agree with Ornella, in the last lesson we did a MD and, furthermore, I had the impression that ... that mathematics can be discussed, precisely that through reasoning guided by you teachers...the various ideas emerged...

Here, the instructor performs the "meta-focusing" action, which can be considered a key action for M-DPm, aiming to make explicit the characterising elements that made it possible to recognise a MD in the experienced activity. Ornella refers to the experience related to the concept of moderator as characteristic of the teacher's roles in a MD. From Ornella's words it emerges that, at the moment when the student talks about his/her solution, the teacher intervenes to let most of them intervene during the MD and she never says if the solution is correct/incorrect. Ornella also when she says "you replicated our words", highlight an her personal sign with which she recognize the action's teacher of replicating students' words which we can classify as the action of "mirroring" as a characteristic teacher's action of a MD.

Daniele acknowledges that the activity in the previous lesson was a MD; he also states that a discussion can evolve a mathematical meaning, which is a particular aspect that characterises MD.

(10:21) Instructor: So... synthesising... Daniele said we discussed Mathematics, Francesca said, the role of the teacher is seen to be one of mediator, Ornella added that it is also that of moderator... but can you tell me what were the very elements that allowed you to recognise this activity as a discussion?

(11:57) Rossana: Well, while Daniele was speaking and outlining what was his problem's solution [she refers to the MD], you made an explicit request as to why... you tried to rework that concept several times, so that it wasn't just clear to Daniele but to the whole class. You wanted to make sure that you led everyone on that path.

Here, the instructor makes an "offer of synthesis" action by explaining what emerged from the previous interventions. In particular, referring to Francesca's words, she recalls the teacher's role of mediator, which is another characteristic

of the MD model. In the first part of the intervention, Rossana makes explicit reference to her experience during MD, highlighting specific teacher's actions as mediator in always asking why. In the second part, Rossana recognises another mediator's action related to the teacher's concern about making all the students involved. Referring to the summary intervention and Rossana's answer, we can observe the evolution from direct reference to the lived experience - what was specifically done and said - to reference to the pedagogical model. It seems to us that Rossana shifts from the student's perspective when she refers to the problem's solutions, to the teacher's perspective, when she focuses on the teacher's questions aimed at involving all students in the shared solution.

As in the following intervention (31:55) by Stefania, the PSTs become increasingly aware of the teacher's role through distancing themselves from the lived situation.

(31:55) Stefania: So, in the MD the cognitive conflict emerged... there was a constant constructing and deconstructing of what we were saying, going to investigate why, through your questions "But I didn't understand... but are you sure? But can you make me understand? No, but I don't see it...". At one point when I was describing my solution, I distanced myself from what I was saying; that is, I realised that I was able to treat the subject, the solution... with more detachment; I realised that I was treating it with distance and therefore could be more aware and impartial in the discussion. Distancing oneself from oneself and cognitive conflict are the two fundamental aspects of mathematical discussion realised thanks to the teacher's mediation and moderation. Even now in this discussion, I realise that I distanced myself from the fact that I was involved in the discussion in the previous lesson, thanks to your questions and the interventions of my peers.

Stefania states that she experienced a situation in which it was possible for her, thanks to the teacher's interventions, to become aware of the MD pedagogical model, through a distancing process. Moreover, as she describes, she becomes aware of the teacher's role, that is how the teacher's guidance enabled a process of de-contextualisation of the solution with respect to the problem.

Furthermore, by the analysis of the personal diaries written by the students, has been possible to observe the significant role of technology in this activity. Moreover it is also possible to highlight the importance to have lived in first person the MD to make the PSTs aware of their future professional prospective. In the following sentences we report some expressions:

Daniele: What struck me about today's workshop is that each of us reasoned differently and being at a distance initially it was not possible to influence each other. I also had time to look for a solution, which I couldn't find initially. We identified proposals that referred to completely different ways of reasoning. Thanks to the initial discussion between us, without the teacher, we felt free to express our solution, without the fear of being evaluated.

Francesca: The language used in the discussion within the rooms and during the lesson in which we were all together on Teams was different. In the rooms

the discussion was between peers and therefore the language was less formal, in the discussion of May 6th however the language was more formal, especially when the professor claimed that a certain explanation had not convinced her and therefore, each of us, tried to explain more clearly, acting on language, what he wanted to express... Without the fear of making mistakes!

Rossana: One thing that made me think a lot, even when the lesson was over, was the perception of serenity that was experienced during the discussion. I never expected that everything would be so orderly from a distance and despite the fact that most of us had never taken part in a discussion collective it seemed we were organized in a collaborative dimension perceiving the importance of exchange.

Eleonora: The online discussions made me understand how complicated it is to listen...listening is a serious practice that requires respect. Listening does not mean nodding in silence but understanding, leaving free space to offer it to what the other is communicating and keeping free space for oneself to connect with the other. This was possible because we were at a distance, each in our own space... Perhaps in presence we would not have had the time to reflect and listen that we had while at a distance.

From the reflections that the PSTs wrote freely in the diaries, the value of using technology to carry out the activity seems to emerge. Indeed, Daniele underlines the impossibility of influencing each others by distance, the possibility of having time to develop a solution and to express it freely without the fear of feeling judged. Furthermore, Francesca underlines the importance of discussion between peers, carried out online, without the presence of the teacher, using informal language typical of an environment of discussion among students. Francesca underlines that this language becomes formal with the presence of the teacher, moving from the perspective of a student to that of a future teacher. Rossana also reflects on the value of remote working, declaring that she perceived a serene and well-organised working climate, in which the exchange of opinions was important for a shared choice. Even Rossana, distancing herself from the lesson experienced in first person, observes an unexpected working climate by placing herself in the perspective of a teacher. Finally, Eleonora highlights a potential offered by the technological environment to be able to listen to the other and understand his opinion, "leaving free space" for reflection in order to share the one other's idea. All this, as Eleonora states, is possible because everyone worked remotely in "their own space".

7. Preliminary results and concluding remarks

From the analysis, it seems that our main hypothesis, implemented in the experimented training mode, has been corroborated. The new theoretical construct M-DPm, applied to the MD pedagogical model, made it possible for PSTs to reflect on previous lived experience, identifying the key theoretical aspects characterising the MD's pedagogical model. The instructor's interventions trigger the

recognition of theoretical aspects from the experienced situation, as it arose from the analysis of the transcripts; particularly the identification of the teacher's actions recognised through a detachment process that allows the PSTs to leave the student's role and assume the role of future teacher. Indeed, Eleonora in her intervention (03:30) recognises the MD as an appropriate pedagogical model that is constructed. In particular, the student identifies and makes it own the patterns of the model, recognises the effectiveness of the model with respect to its function and finally, distinguishes two specific functions: 1. solving the problem and constructing the mathematical concept; 2. recognising the nature of mathematics as something that can be discussed.

Furthermore, data analysis seems to show that involving PSTs in training activities could foster the theoretical conceptualisation and the practical implementation of a pedagogical model. This happens because PSTs can experience and reflect in first person and in so doing, they contribute to their effective professional development.

Moreover, also technology has been a fundamental role, since it allows students to collaborate in a stimulative environment, sometimes also without the judger presence of the teacher. Also the tools of the online platform revealed very useful to better understand each others during both the MD and the M-DPm.

In this research study, the implementation of the direct experience provides a specification of what kind of experience has to be lived (activities designed according to the pedagogical model) and how reflection can be promoted (M-DPm). The specific training mode presented here, proposes a specific interpretation of the suggestive expression "in and from practice": through the M-DPm, PSTs, after having personally experienced the teacher's actions as students, recognise them according to the specific pedagogical model.

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Engineering students' use of scaffolding elements within digital tasks concerning planar integration domains

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Abstract. In this paper, we focus on a sequence of digital tasks involving the representation of subsets of the plane as normal domains. We investigate the role of scaffolding elements provided to supports Engineering students' activities with the tasks. The study focuses on 63 volunteer Engineering freshman. Using qualitative methods, we carried on the combined analysis of video recordings of the students' interaction with the tasks in small groups and their individual answers to a survey after the activity. The main result of the data analysis is the identification of three categories of students' behaviours concerning their use of the provided scaffolding elements. These categories reflect different ways of grasping the scaffolding, purposes and levels of awareness by the students.

Keywords: design of digital tasks, scaffolding, feedback, double integrals, engineering students.

MSC 2020: 97I50, 97U50, 97C70.

1. Introduction

This study sets in the stream of research concerning the use of domain-specific software in teaching Mathematics to Engineering students (see [1], [2], [3], [13], [14] and [18]). In particular, in [2] and [3], Alessio et al. presented two sequences of digital tasks concerning the conversion between different representations of subsets of the plane. These sequences of tasks have been designed taking into account the typical difficulties in Multivariable Calculus discussed in the literature, and aiming at the overall educational goal of improving students' understanding of problems involving double integrals. Indeed, to fruitfully face this kind of

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problems, students should be able to a) analytically and graphically represent the integration domain; b) recognize and describe the limits of integration in order to write a double integral as iterated integrals; c) choose the most effective description of the integration domain towards the calculation of a given integral (see [3]).

In this paper, we focus on the sequence of digital tasks about representation of subsets of the plane as normal domains described in [3], thus addressing points a) and b). Precisely, we investigate the role of scaffolding elements provided to support the students' activity with the tasks. In particular, we analyze the students' use of the scaffolding elements provided within the tasks and their perception of the impact of these elements on their learning.

This paper should be a contribution in two directions, both theoretical and practical. On the theoretical side, we aim to deepen how to design effective scaffolding in digital tasks and to grasp the elements allowing or preventing a fruitful use of it by the students. On the practical side, we would like to provide students with a non-human tutor supporting their learning of advanced mathematical concepts in the delicate phase of transition to university.

2. Theoretical framework

Recent research highlighted the essential role of scaffolding in supporting students' learning, especially within digital environments (see [4], [5], [11], [16]).

According to Holton and Clarke [11], scaffolding is a teaching act supporting the immediate construction of knowledge and, at the same time, fostering the students' future independent learning. In tune with [16], each scaffolding element should address two components: a *procedural one*, allowing students to overcome a contingent difficulty to perform a task correctly, by reducing the cognitive load; and a *metacognitive one*, having long-term effects and allowing students to face similar difficulties and tasks in the future.

In some contexts, characterized by large and heterogeneous classes, is very useful to provide scaffolding in digital environments. Indeed, although one-to-one scaffolding is very effective, it cannot be done in university classes with hundreds of students with different learning needs ([5]). So, using digital environments could allow to largely share the scaffolding and adapt it to the individual needs. Computer-based scaffolding (CBS) is defined as "*computer-based support that helps students engage in and gain skill at tasks that are beyond their unassisted abilities*" ([5], p. 26). Differently from face-to-face scaffolding, which can be progressively negotiated, CBS requires a careful design, that considers in advance students' possible difficulties and learning needs ([16]).

CBS is particularly relevant for our study since the careful use of technology is the key aspect for distributing scaffolding, adapting it to the specific difficulties of the learners. In this sense, and according to [9] and [17], scaffolding is intended throughout the study as a form of formative feedback, that is as a piece of

information clarifying the students where they are in the learning process, where they are going, and what following steps should be done to achieve the goal [19].

Hattie and Timperley in [10] discussed the following levels of formative feedback: feedback on the task; feedback on the processing of the task; feedback on self-regulation; feedback on the self as a person.

From the literature (see [16], [17], [21], [22]), some design principles (DP) for scaffolding can be extracted, favouring its effectiveness. In particular:

- (DP_a) its content (information suggested or provided), function (cognitive, meta-cognitive, ...) and form of presentation (timing, semiotic register, ...) should take into account the context and learner characteristics;
- (DP_b) it should be separated from the assessment;
- (DP_c) it should be written or communicated via computer, and provided in a not too formal way so that a dialogue between the sender and the receiver of the scaffolding could be triggered;
- (DP_d) it should contain a limited quantity of information;
- (DP_e) it should be customized to the individual learning needs;
- (DP_f) it should be presented in nested sequences of tasks;
- (DP_g) it should be progressively faded, i.e. gradually reduced.

All the previous design principles have been used for designing the scaffolding elements provided in the sequence of our digital tasks [3] and will be explicitly recalled in the next section.

With regard to (DP_e), CBS presents a critical feature in relation to one-on-one scaffolding provided by an expert tutor: CBS could not be completely individualized to the student's learning needs nor completely adaptively provided ([16]). In this perspective, the challenge of CBS is to be responsive, that is such to induce effective consequences ([12]), to typical difficulties and approaches (that can be revealed by the student's behaviours within the digital environment), rather than to the specific individual's difficulties. As recalled by point (DP_g), a specific feature distinguishing scaffolding from other forms of support is fading. i.e. *"the gradual reduction and eventual elimination of scaffolding"* ([16], p. 29). Thus, each scaffolding element should be such that: *"(a) support learners in the achievement of tasks beyond their unassisted capacity; (b) when the support structure is removed, learners continue to function competently on their own; and (c) removing the support structure does not reduce learning or functioning – instead, learners continue to function at the elevated plane reached via scaffolding"* ([16], p. 29).

Also as concerns fading, in digital environments it could be less responsive with respect to the fading of scaffolding enacted by an expert in a one-to-one

relationship. So, the challenge is to design a responsive fading of scaffolding ([16]) on the basis of typical students' behaviours. From this perspective, scaffolding as formative feedback turns out to be also a crucial tool for improving the individualization of teaching and learning (see [7], [8]).

According to Withney [20], scaffolding, intended as a form of formative feedback ([17]), is afflicted by a paradox: although generally the students consider it helpful, they do not use it. Winstone et al. in [21] investigated the reasons preventing students from using feedback and identified four main barriers: difficulties in decoding feedback content and purpose; poor knowledge of suitable strategies and theoretical elements addressed by the feedback; sense of disempowerment and difficulty in translating feedback in actions; lack of receptiveness and engagement.

In this paper, according to the previous considerations, we investigate whether and how Engineering freshmen use scaffolding elements designed by us to support their interaction with a sequence of digital tasks. Our specific research questions are:

- (1) do students utilize the available scaffolding elements?
- (2) if students use these elements, in what way do they mostly use them?
- (3) does scaffolding in a digital environment influence the subsequent work of students who utilized it?

3. The tasks and the scaffolding elements

We focus on a sequence of digital tasks concerning the representations of subsets of the plane as normal domains. It has been designed within wider research about the teaching-learning of double integrals to Engineering students (see [2], [3] and [8]). The tasks and their scaffolding elements are designed according to the principles listed in Section 2. In particular, the tasks (T1-T6) are nested in a sequence of increasing difficulty (according to (DP_f)): the first three tasks concern normal domains in the x or y -direction; the last three ones are about normal domains in polar coordinates (see [3] for more details about the concept of normal domain). According to (DP_b) , the sequence of tasks is intended for a free use by the students, who can interact with it in their own pace, and it is not envisaged any assessment.

All the tasks have some common elements: they provide a planar region with some points allowing to obtain its analytical description (Figure 1, right side of the screen), and the reminder of the definition of the normal domains (Figure 1, left side of the screen).

Students are expected to establish whether the planar region is a normal domain, in which direction, and identify functions that describe it as a normal domain (as open-ended questions). In the last three tasks, further competence is addressed, i.e. decision-making. Indeed, students are faced with planar regions

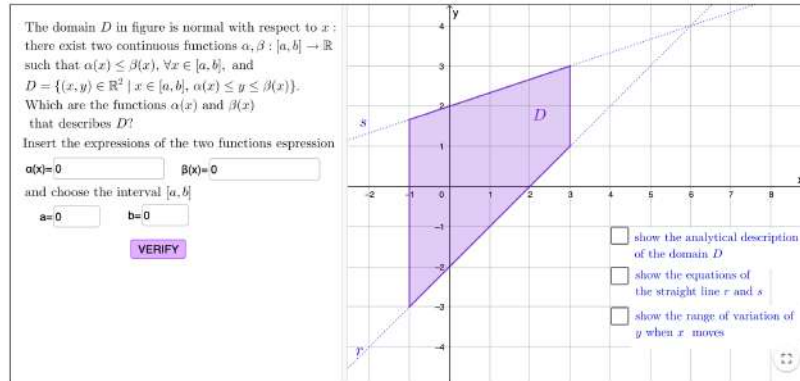


Figure 1: The starting screenshot of the task T1

that can be described in different ways as normal domains; they are required to compare these different descriptions and establish the most convenient description of the domain to set up the calculation of the double integral of a given function (see [3] for more details).

The scaffolding elements provided within the digital tasks are of two kinds: some of them are teacher-driven and others are student-driven. As discussed by Alessio et al. in [3], all the tasks are designed so that automatic, immediate and facilitative feedback in different semiotic registers appears when the student submits an answer, according to (DP_a) . In particular, in case of a wrong answer, a verbal message signals the mistake (feedback at the task level), and the region corresponding to the analytical description submitted by the student, if it is not empty, becomes shaded (feedback at the process level and the self-regulation level). This is a response-specific feedback, designed so that individual learning needs are addressed (according to (DP_e)). It is obviously teacher-driven, communicated via computer and refers to a little piece of information (according to (DP_c)).

The students' activity with the tasks is supported by some other scaffolding elements, also called *hints* throughout the paper, which the students can require if they are in difficulty. These hints scaffold the students' work within the digital environment and represent a form of feedback that makes the students responsible for their learning. Indeed, differently from the automatic and immediate feedback previously described, this form of feedback is student-driven and appears only if the students select it. This choice is in tune with Winston et al. [21] and due to our aim to make the students progressively aware of their learning needs and the kind of support useful for them. In the following, we describe in detail these hints, in relation to the DP listed in Section 2.

In **task T1**, the planar region to be described is a quadrilateral (region D in Figure 2). The student could obtain the expressions of the two functions defining D as a normal domain in the x -direction from the graph, using the

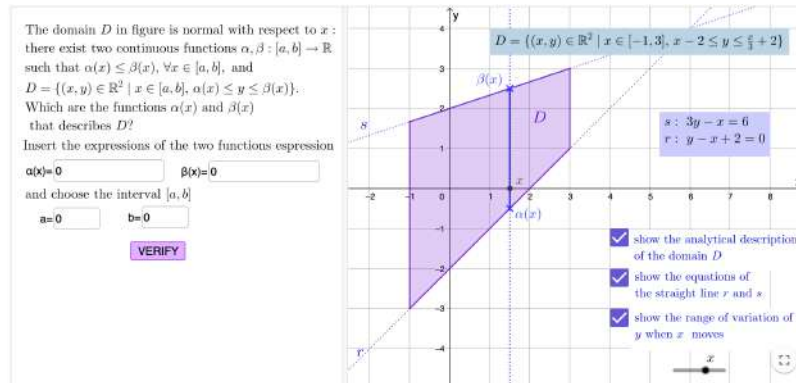


Figure 2: A screenshot of the task T1

coordinates of points in the graphical representation to identify the equations of the straight lines r and s and the interval where x varies.

The hints available for the students are shown on the right side of the screen (Figure 2). Two of them are analytical (“show the analytical description of D ”, which we will call *domain hint*; “show the equations of the straight lines r and s ”, *curves equation hint*) and one is geometrical (“show the range of variation of y when x moves”, *dynamic variation hint*). The hints are listed in the Table 1 and their different features are in tune with (DP_a) .

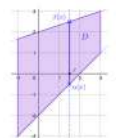
Name of the hint	Description of the hint in the digital task T1	What the hint provides in the digital task T1
<i>domain hint</i>	☐ show the analytical description of the domain D	$D = \{(x, y) \in \mathbb{R}^2 \mid x \in [-1, 3], x - 2 \leq y \leq \frac{3}{5} + 2\}$
<i>curves equation hint</i>	☐ show the equations of the straight line r and s	$s: 3y - x = 6$ $r: y - x + 2 = 0$
<i>dynamic variation hint</i>	☐ show the range of variation of y when x moves 	

Table 1: The available hints in task T1

They have been designed to scaffold the students' work with different degrees of guidance and to take into account different needs, according to (DP_d) and (DP_f) . Indeed, the *domain hint*, provides explicit support to the student's work, giving the range of variation of the x -coordinate and the functions between which the y -coordinate varies. So the student should only recognize in the writing of D an instantiation of the general writing of an x -normal domain shown on the

left side of the screen and suitably substitute the extreme points of the range of x and the expressions of the functions in the insert fields.

The *curves equation hint*, is less guiding since it gives the (implicit) equations of the straight lines. In this case, the student, to correctly perform the task, should first autonomously find the range of variation of the x -coordinate, hence make explicit the equations of the straight lines and recognize the obtained expressions as the lower and upper functions delimiting the y -coordinate of the domain D .

The *dynamic variation hint*, is even less guiding since it does not provide quantitative information, but addresses the geometrical meaning of the concept of normal domain as explained by the teacher during the lessons (i.e. as a domain for which the interception with each vertical straight line is a segment). In this case, the student is induced to focus on the fact that for each fixed value of x in a suitable range, the y -coordinate varies between the straight lines r and s , that is, between the graphs of the functions $y = \alpha(x)$ and $y = \beta(x)$ which identify D as an x -normal domain. Thus, he/she should autonomously find the range of variation of both the coordinates describing D as a normal domain.

All the presented scaffolding elements, either those teacher-driven, i.e. the response-specific feedback, and those student-driven, i.e. the hints, act at both the procedural and metacognitive levels, making the student reflect on the steps to switch from the graphical to the analytical representation of a normal domain. Among the hints, the *dynamic variation hint* is the most metacognitive one, since it does not provide explicit support to perform the task, but creates conditions for significant learning and linking between the theoretical knowledge and the solution of the problem. We expected some students to choose hints with higher level of guidance in the first tasks of the sequence and successively hints with less level of guidance or no hints, having taken advantage of the first ones. This would be in tune with the idea of fading the scaffolding (DP_g).

In **task T2** a y -normal domain is shown (see Figure 3). Also in this case the student can find the equations of the parabola and the two straight lines delimiting D by the graph, and she can ask the same hints previously described. However, according to the idea to progressively fade the scaffolding (see Section 2, (DP_g)), in this case, the *domain hint* does not provide the range of variation of the y -coordinate, leaving the student to find them. Analogously, the *curves equation hint* provides only the equation of the parabola and not the equations of the straight lines. The *dynamic variation hint* is similar to that of task T1.

Differently from the previous task, in T2 when the student describes D correctly, the question “*Is D a normal domain in the x -direction?*” appears. If the student selects the wrong answer “*Yes*”, a verbal feedback is provided, inviting her to dynamically explore the possibility of describing D as an x -normal domain by moving a slider. This is similar to the *dynamic variation hint* but, in this case, the segment linked to the slider becomes red when it splits into

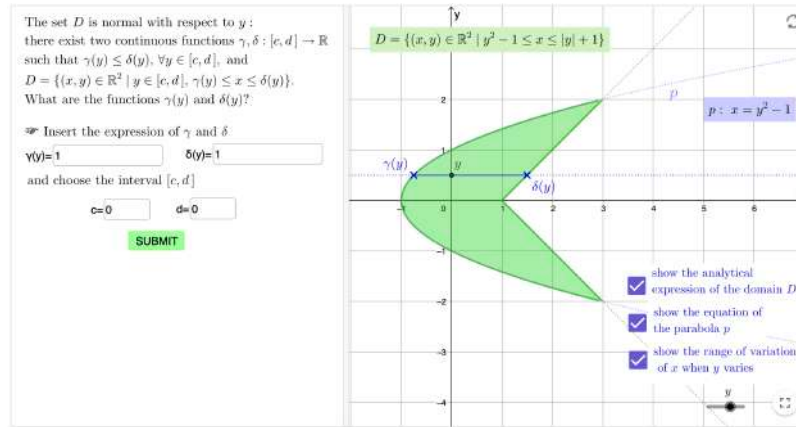


Figure 3: A screenshot of the task T2

two parts, as depicted in Figure 4. This has been designed as another feedback given in the graphical semiotic register and acting at the task level and at the processing of the task level.

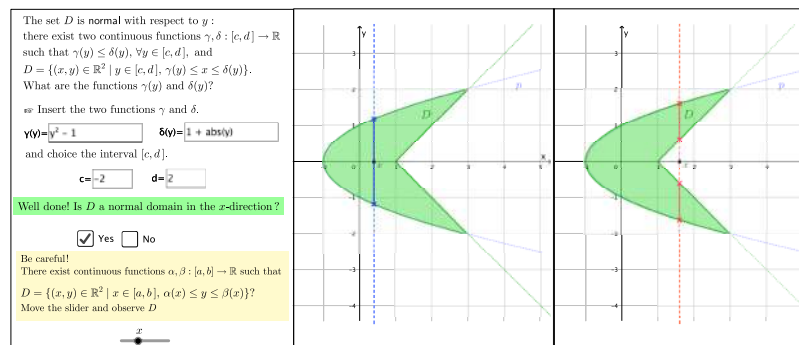


Figure 4: Another screenshot of task T2

In **task T3** a planar region that is x -normal but not y -normal is given. The student is required to choose whether the given planar region is a normal domain in the x or the y -direction (see Figure 5). At this step, only the *domain hint* is available before the student's choice, not revealing the correct one. Indeed, in the analytical description of the domain only the (dis)equations of the circumference and the ellipse delimiting the domain D are provided (see Figure 5).

Since D is x -normal but not y -normal, if the student makes the wrong choice, a piece of feedback information appears, not revealing but inducing the student to understand that the domain is not y -normal, but is the union of two disjoint y -normal domains. Indeed, the student is invited to dynamically explore the possibility of describing D as a normal domain in the y -direction by moving a slider (the segment linked to the slider turns red when it splits into two parts,

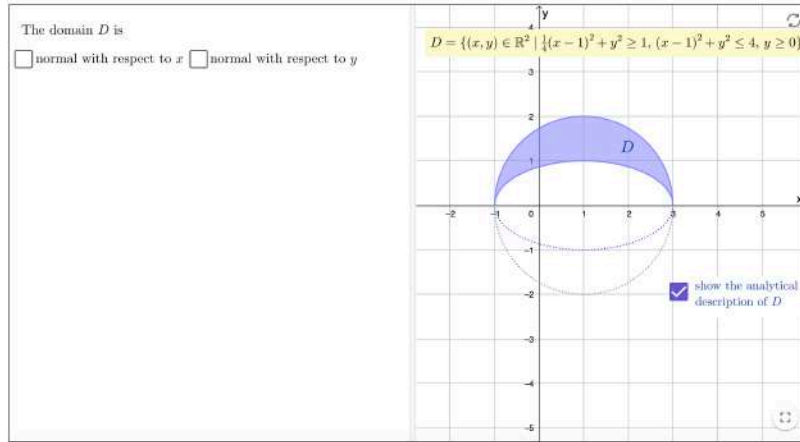


Figure 5: A screenshot of task T3

as depicted in Figure 6). This feedback at the task and process of the task

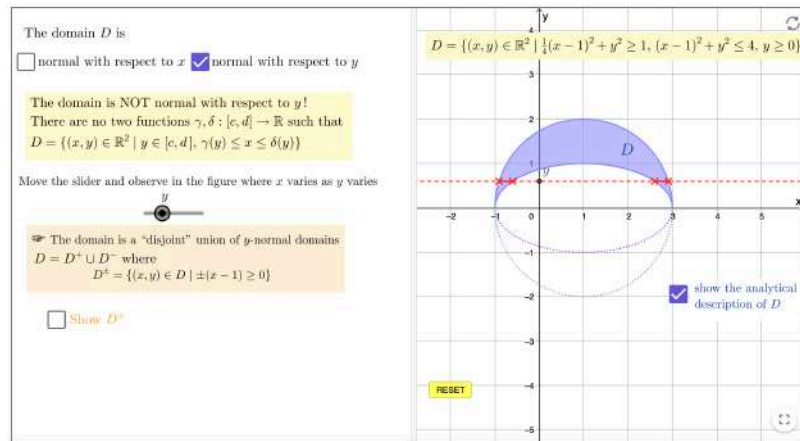


Figure 6: Another screenshot of task T3

levels should make clear that there are more possible ways to describe the given domain, for example, either as an x -normal domain or as a union of y -normal domains.

If the student selects x -normal, the program requires the description of D as an x -normal domain and the other two hints become available (Figure 7). The *curves equation hint* makes the equations of the circumference and the ellipse visible. According to the idea of progressively fading the scaffolding (DP_g), in this case, the *domain hint* and the *curves equation hint* give the equations of the whole circumference and ellipse, from which the student should make explicit the y variable to autonomously find the expression of the functions whose graphs (a semicircle and a semiellipse) delimit D .

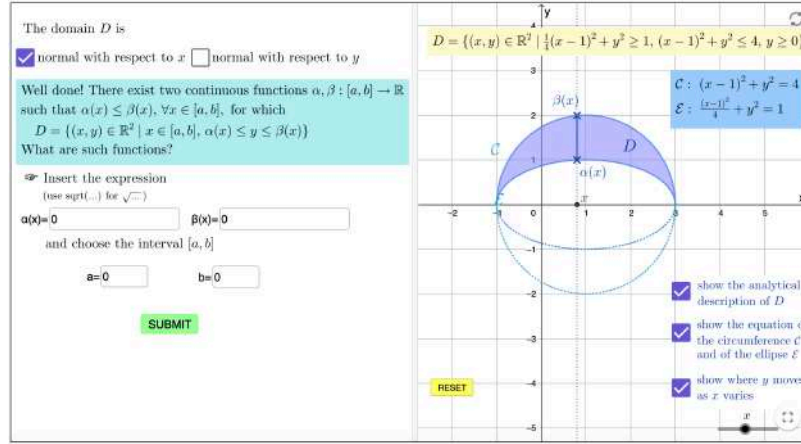


Figure 7: Another screenshot of task T3

In the next three **tasks T4-T6**, the hints proposed are of the same type as the previous ones, and they are provided according to the idea of progressively fading the scaffolding, increasing the activation of the student (see Table 2).

For the sake of completeness and in order to make visible all the potential of

Task T1	Task T2	Task T3
Task T4	Task T5	Task T6

Table 2: The available hints in tasks T1-T6

the *dynamic variation hint*, we briefly illustrate it in **task T6**. Here the domain is normal with respect to both the x and y coordinates, but also with respect to polar coordinates. Considering the domain as x -normal or y -normal, the student has to enter the expression of the functions describing D as piecewise-defined functions. In this case, the *dynamic variation hint* is useful both for identifying

the different function expressions and the point (in this case, $x = 2$) where the function changes expression (see Figure 8).

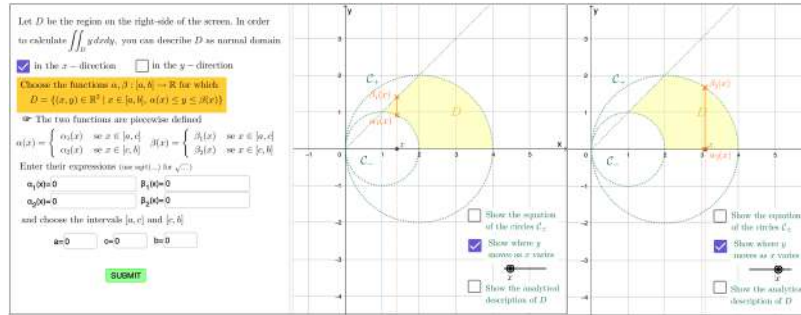


Figure 8: A screenshot of task T6

Finally, to describe the domain using polar coordinates, the same *dynamic variation hint* shows how the variable ρ varies as the angular variable t moves (Figure 9).

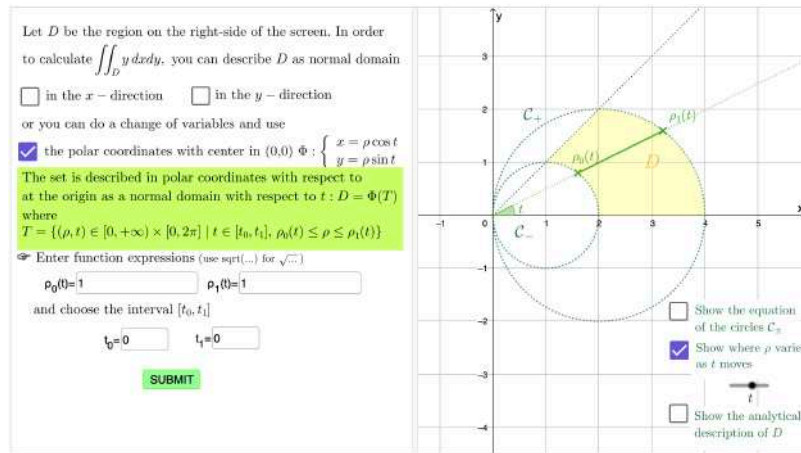


Figure 9: Another screenshot of task T6

4. Research context and methodology

We submitted the sequence of tasks T1-T6 to 63 volunteer Engineering freshmen from the Polytechnic University of Marche (Ancona, Italy), dividing them into small groups of three or four students for the interaction with tasks. They attended the Calculus II course during the 2019-20 academic year and had passed the exam of Calculus I. The first author of the paper was the teacher of the course. Due to the pandemic, the entire course, including the interaction with the sequence of digital tasks, was conducted online. In particular, the students interacted with the sequence of digital tasks for about 2 hours. The authors of

the paper were present in the online room as observers to support the groups of students for possible technical troubles. We want to remark that, at the time of the teaching experiment, the students were already familiar with the concepts of normal domains and reduction formulas for the calculation of double integrals. Specifically, they had already learned how to set up the calculation of double integrals in theoretical and practical lessons and had faced traditional pen-and-paper tasks on this topic.

After the interaction with the sequence of digital tasks, students were asked to individually complete a survey regarding: 1) any difficulties they encountered with the tasks; 2) their use of hints (if they used them, why they did so, which hints they used, and for what purpose); 3) their perception of the usefulness of the hints in solving the tasks and in tackling subsequent ones; and 4) their perception of the impact of using hints on their learning.

To give answers to the research questions, we collected two kinds of data: video recordings of the student groups' interaction with the sequence of tasks and individual students' answers to the survey. A cross-analysis of these data has been conducted, comparing the students' behaviour during the interaction with the tasks, specifically about the use of hints, and their answers to the survey. The researchers individually reviewed and analyzed the video recordings, highlighting the key moments where students required and used the hints. In particular, after a first explorative view of video recordings, the focus has been put on some aspects: the choice of using the hints or not in case of difficulty, the consistency between the kind of possible difficulty and the chosen hint and the effects of the use of the hints on the subsequent work. Moreover, the researchers independently read and coded all the answers to the survey according to the theoretical framework, focusing on the students' motivations for using or not using the hints, their specific goals when requiring a hint and their perception of the usefulness of the hints at the procedural level (to correctly perform the task) and at the metacognitive one (on the subsequent learning). Finally, the researchers discussed their analyses, until an agreement was reached (see [15]).

We notice that after the activities were completed by the participants involved in this research, the tasks were made available to all the students (about 80) enrolled in the course of Calculus II on the Moodle platform of the university, as a self-assessment tool.

5. Results

The analysis of data allowed us to identify three categories of students' behaviours with respect to their use of the scaffolding elements. In the following, we will refer to some excerpts which are paradigmatic for the relative category. The first category (about 17% of the total) includes behaviours of students who did not ask for hints, even in case of need, or did not take advantage of the hints. Students whose behaviour is in this category did not realize the opportunities

offered by the scaffolding elements in the sequence of digital tasks to overcome procedural difficulties nor to deepen their understanding of the topic at stake.

Typically, the focus of students whose behaviors is in this category is on giving the correct answers, putting in the background the theoretical knowledge and understanding. Some of these students preferred to perform the sequence without any scaffolding. One of these students declared *“I did not ask for hints, since the challenge was to solve the tasks only with my own knowledge and skills”*. Other students asked for some hints, but were not able to take advantage of them.

According to the students’ declarations, the non-use of the hints happened for different reasons, which are in tune with those identified by Winston et al. in [21] and recalled in Section 2. Some students had difficulties in decoding the scaffolding elements’ content (*“the hints give too vague suggestions”*, *“the hint was not easily understandable for me”*, *“[the hints] are not enough to understand where I was wrong”*); other students lacked strategies to translate the scaffolding elements into actions to be performed (*“I did not see how to use the hint”*). Other students recognized that their lack of theoretical knowledge made the hints not sufficient for them to perform the task correctly (*“I asked all the hints, but I did not study all the theory, so I was not able to use the hints”*). In our opinion, these behaviours suggest a lack of metacognitive control by these students, who sometimes remained blocked in performing the task rather than asking for hints or did not were able to decode the scaffolding into strategies to be enacted.

A second category (about 40%) includes behaviours of students who used the hints and took advantage of the scaffolding at the procedural level. The analysis of the answers to the survey highlights that some students used the hints to quickly solve the problem, without making calculations. Some of them declared *“the hints make quick the solving process”*, *“the hints saved me to make calculations”*, *“I asked for the equation hint, so that I had yet the equations”*.

Typically, students whose behaviour is in this category required repeatedly the same kind of hint in different tasks (mainly, the domain or curves equation hints) and used it only to overcome their temporary impasse. Indeed, their interaction within the digital environment and their answers to the survey showed that they did not take advantage of the required hint in the subsequent tasks and did not go beyond the task level of feedback. Moreover, their main focus seemed to be to find the correct solution to the tasks, rather than deepen their knowledge. One of them declared *“the hints allowed me to make the calculations quickly [...] the messages on the screen told me if I was wrong”*.

According to the students’ answers, some different reasons induced them to ask for the hints: some students had the goal to check their answers before submitting them (*“to confirm my conclusions”*, *“they helped me to check that my equations were correct”*, *“they avoid calculation mistakes”*) or after the submission, to modify the given answer (*“the hints suggested me where I was wrong”*,

“the hints made me understand what I was wrong and then correct my answer”). We can highlight a fairly superficial level of awareness by these students, who mainly recognize the sequence of tasks as assignments to be correctly performed, but not as an occasion to review the mathematical contents and reflect on them.

Finally, we identified a third category of students' behaviours (about 43%), who displayed of having grasped the scaffolding provided by the hints not only at the procedural level, but at the metacognitive one. During the interaction, typically these students ask different hints according to their specific needs and take advantage of the asked scaffolding element for the subsequent tasks. One student said *“I was in difficulty [...] I needed the hint showing me the equations of the curves to describe D as a normal domain [...]. In the next, thanks to the previous hint, I could recover by myself the equations of the curves. Subsequently, I asked for the geometrical hint [dynamic variation hint], since I could not visualize what the boundaries of the normal domain were”*.

From the answers of students whose behaviour is in this category, it arose that their focus during the interaction in the digital environment was not only on giving the correct answer to the tasks, but also on deepening the knowledge of the topic at stake. Indeed, they used words such as *“understand”*, *“clarify”*, *“reasoning”*, revealing their reflection of the provided scaffolding beyond the task to be performed (*“all the hints have been useful, because they helped me to clarify some concepts”*, *“they [the hints] helped me fully understand the exercise”*). Some of these students clearly linked the scaffolding provided within the digital environment with their general difficulties in learning this topic (*“the geometrical hint allowed me to understand something about my typical difficulties with problems of this kind”*).

The metacognitive level of the scaffolding emerged in many answers of these students, who seemed to project the received scaffolding on their future learning. Some of them declared *“the hints supported me in understanding what I should have found”*, *“the hints supported me in finding by myself the correct reasoning”*, *“the hint induced me to reflect on the given answer”*, *“I will put the same questions to myself in the future”*, *“they directed me towards the correct execution of the exercise and helped me to think”*, *“especially in the last exercise, they [the hints] allowed me to understand what effects the choice I made would have had on the calculation of an integral”*.

6. Conclusive discussion

In this paper, we investigated the use of computer-based scaffolding elements provided in a sequence of digital tasks by Engineering students. We designed teacher-driven scaffolding elements, that is immediate and response-specific facilitative feedback, and student-driven scaffolding elements, that is hints of different kinds. According to [10], we interpreted the scaffolding as a form of formative feedback, that students could grasp at the procedural or metacogni-

tive levels (see [16]). In tune with Winstone et al. in [21], we wanted to explore whether and how students used the scaffolding elements provided within a sequence of digital tasks concerning planar integral domains. Moreover, we wanted to deepen the reasons preventing a fruitful use of these scaffolding elements, if this is the case. For students who took advantages of the provided scaffolding, we investigated the specific purpose inducing them to ask for hints and at what level (procedural or metacognitive, [16]) they interiorized the scaffolding.

The combined analysis of video recordings of the students' group interaction with the digital tasks and their individual answers to a survey concerning their use of the provided scaffolding elements allowed us to identify three categories of behaviours, corresponding to different levels of awareness and ways of grasping the scaffolding. This categorization gives answers to the research questions set in Section 2.

The obtained categories can be put in relation to the functions of a particular form of scaffolding described by Cusi and Telloni ([8]). The authors focused on the use of sliders in some GeoGebra applets, supporting the students' activity within the digital environment. Investigating the instrument-mediated action schemes emerging by students' use of these tools, the authors identified three different functions of the sliders: the *replacement function*, arising when students “*rely completely on the information provided by the digital tools, using them to find out the answers*”; the *diagnostic function*, arising “*when students use the tool to control the correctness of their answers [...], or to detect where the mistake is, and, possibly, how it to be corrected*”; finally, the *elaboration function*, arising “*when the students refer to the tool to deepen their understanding of the theoretical knowledge subtended to the task and their interpretation of the representations involved in it*” (see [8], p. 5).

The second category highlighted by our analysis, characterized by the students' grasping of scaffolding mainly at the procedural level, includes some behaviours linked to the *replacement function* of the scaffolding. They are the behaviours of students who rely on the hints to save time or to not make calculations. Other behaviours in the second category, characterized by the use of hints to check the correctness of the answers, are related to the *diagnostic function* of the scaffolding. Finally, the third category highlighted by our analysis is related to the *elaboration function* of the scaffolding, allowing a fruitful integration between the use of the hints and the deepening of the theoretical knowledge.

The results obtained from this study are expected to contribute theoretically to the design of effective scaffolding elements for digital tasks, as well as to enhance our understanding of the factors that enable or hinder their successful use by university students. Additionally, from an educational standpoint, the sequence of tasks could serve as a tool, acting as a non-human tutor that supports students in learning advanced mathematical concepts.

The outcomes of the study suggest trajectories for ongoing and future research. On the basis of the students' difficulties in fruitfully using the scaffolding

elements and the motivations for this, we would like to exploit the potential of digital environments to eliminate those barriers preventing students from taking advantages of the scaffolding. In particular, to support students who do not know what kind of scaffolding could be useful with respect to their specific difficulties, we are planning to design digital tasks where the students in difficulties are guided by suitable stimulus questions to choose the correct hint or, alternatively, where the hints are presented one by one, from the less guiding to the most guiding. Moreover, to support students who are not able to translate the scaffolding into actions, some guiding messages could be designed, allowing them to take advantage step-by-step of the scaffolding element. In this way, another crucial educational goal, i.e. making the students sensitive to the importance of scaffolding and feedback, could be pursued. The awareness of both the levels at which each scaffolding element acts, i.e. the procedural one and the metacognitive one, is an essential element improving the students' learning, according to [21]. In particular, this should be a key strategy to progressively move the behaviour of the largest possible number of students from the first and second categories before described to the third one.

We conclude by observing that the study presents some clear limitations. Firstly, the interaction with the digital tasks is on volunteer basis and not all students chose to participate. This could be linked to the context of the research, i.e. Engineering degree course, where Mathematics is a service subject. Often in this context Mathematics is viewed as a goal-oriented activity (see [6]) and students may be reluctant to engage in non-mandatory activities. In this perspective, we should foster the participation of more students in the future. Moreover, students' behaviour might have been influenced by factors not investigated in this study, such as potential technical difficulties in interacting with the digital tasks. Another critical issue could concern the pandemic context during which the study has been conducted, which could have afflicted the obtained results. Further theoretical research and experimentation are needed to understand the possible dependence of the obtained results by the context of the study and possibly to validate the obtained classification of the students' behaviours.

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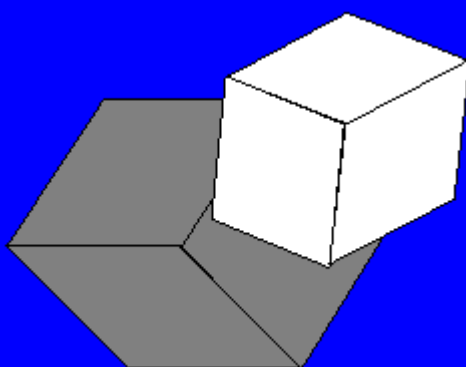
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