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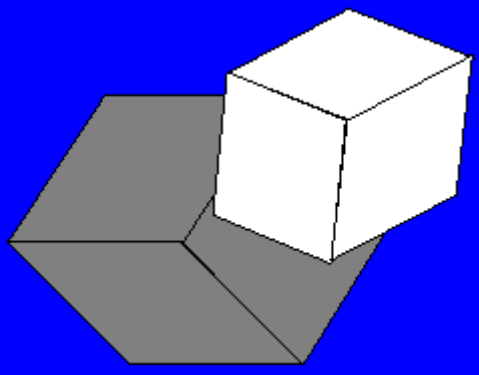
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Implicit hybrid block methods for solving second, third and fourth orders ordinary differential equations directly

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Abstract. Strong numerical algorithm for simultaneously solving second, third and fourth orders of ordinary differential equations with given initial conditions is considered in this paper. Linear Multi-step collocation approach was applied in the derivation of the new method with the use of power series approximate solution as interpolation polynomial. The numerical integrators that formed the block were derived by evaluating the continuous scheme along with its derivatives at the non-interpolating points within the selected interval of integration. The basic properties of the method were investigated properly. In order to examine the effectiveness of the block, the methods were tested on some second, third and fourth orders ordinary differential equations and the results generated proved its effectiveness over existing methods in terms of accuracy.

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Keywords: hybrid block method, collocation, higher order ordinary differential equations, interpolation, modeled higher order, off-step points and power series.

1. Introduction

The numerical solution of second, third and fourth orders initial value problems of ordinary differential equations (ODEs) directly using a linear multistep hybrid block method is considered in this study.

Higher order ODEs have extensive usage in engineering and physical sciences. Very often, these problems do not have an analytical solution, and this has necessitated the use of numerical method to approximate their solutions. Previously, higher order ODEs were solved by reducing them to their equivalent system of first order ordinary differential equations and thereafter suitable numerical method for first order ODE would be applied to solve the systems. However, it is observed in Lambert and Watson [10] and Awoyemi [1] that this reduction approaches has serious problems which include wastage of human effort, computational burden and non-economization of computer time. A P-stable linear multistep method for direct solution of second order ODEs was developed by Awoyemi [2] which was implemented in predictor-corrector mode. It was observed that the predictors have reducing order of accuracy which affects the efficiency of the predictor-corrector method. In addition, the implementation requires much human effort and computer time.

To cater for these setbacks encountered in the reduction and predictor-corrector approaches, and also bring about improvement on the accuracy of numerical method Jator [9], Ogunware et al. [13], Skwame et al. [18] amongst others developed block methods for solving higher order ordinary differential equations directly in which the accuracy is better than the reduction of order to system of first order ordinary differential equations.

Also, various authors such as Omole and Ogunware [15], Adeyeye and Omar [3], Akinfewa et al. [5] respectively developed hybrid block methods for the direct solution of second, third and fourth order ODEs. These hybrid block methods, while retaining certain characteristics of the continuous linear multistep method, share with Runge-Kutta's Methods the property of utilizing data at further points, other than the step point. The block method was found to be efficient in handling higher order ODEs and it was also cost effective.

Therefore, in this research, the development and application of a two-step linear multistep method with four hybrid points implemented in block mode for the solution of second, third and fourth orders ODEs concurrently which is rare in numerical analysis works will be our target.

2. Research methodology

We considered power series as an approximate solution to higher order ODEs to be of the form

$$y(x) = \sum_{j=0}^{\eta+\varphi} a_j x^j \tag{1}$$

where $a_j^{'s}$ are parameters to be determined, η and ψ are the respective number of distinct collocation and interpolation points.

The fourth, fifth and sixth derivatives of (1) are obtained as

$$y^{'v}(x) = \sum_{j=4}^{\eta+\varphi} j(j-1)(j-2)(j-3)a_j x^{j-4}, \tag{2}$$

$$y^{v}(x) = \sum_{j=5}^{\eta+\varphi} j(j-1)(j-2)(j-3)(j-4)a_j x^{j-5}, \tag{3}$$

$$y^{v'}(x) = \sum_{j=6}^{\eta+\varphi} j(j-1)(j-2)(j-3)(j-4)(j-5)a_j x^{j-6} \tag{4}$$

The approximate power series is being interpolated at $x = x_{n+j}, j = 0, \frac{1}{2}, 1, \frac{3}{2}$ While the fourth derivative is being collocated at all the grid and off grid points, $x = x_{n+j}$, where $j = 0, \frac{1}{4}, \frac{1}{2}, 1, \frac{5}{4}, \frac{3}{2}, 2$. The fifth and sixth derivatives are collocated at $x = x_{n+u}$ where $u = 2$. These equations are then combined to generate a structure of non-linear system of equations which are solved using Gaussian Elimination Method to obtain $a_j, j = 0, 1, 2, ..., 12$. The obtained values of $a_j^{'s}$ are then substituted to the power series to give a continuous hybrid formula of the form:

$$\begin{aligned} &\alpha_0(z)y_n + \alpha_{\frac{1}{2}}(z)y_{n+\frac{1}{2}} + \alpha_1(z)y_{n+1} + \alpha_{\frac{3}{2}}(z)y_{n+\frac{3}{2}} \\ &= h^4\left(\sum_{j=0}^{k}\beta_j(z)f_{n+j} + \beta_v(z)f_{n+v}\right) + h^5\gamma_2(z)g_{n+2} + h^6\psi_2(z)m_{n+2}, \end{aligned} \tag{5}$$

where $v = \frac{1}{4}, \frac{1}{2}, \frac{5}{4}, \frac{3}{2}$ is the hybrid points.

Using the transformation

$$z = \frac{(x - x_{n+1})}{h}, dz = \frac{1}{h}dx \tag{6}$$

(see, Adeyefa and Kuboye [4]).

The coefficients of y_{n+j}, g_{n+j} and m_{n+j} are obtained in terms of z as follows:

$$\alpha_0(z) = \left(-\frac{11}{3}z - \frac{4}{3}z^3 + 4z^2 + 1\right),$$

$$\alpha_{\frac{1}{2}}(z) = (4z^3 - 10z^2 + 6z),$$
$$\alpha_1(z) = (-4z^3 + 8z^2 - 3z),$$
$$\alpha_{\frac{3}{2}}(z) = \left(\frac{4}{3}z^3 - 2z^2 + \frac{2}{3}z\right),$$

$$\begin{aligned}
\beta_0(z) &= h^4(\frac{304987}{127733760}z^2 + \frac{10471}{201600}z^8 - \frac{7}{9900}z^11 \\
&- \frac{9113}{100800}z^7 + \frac{17}{160}z^6 - \frac{1811}{90720}z^9 + \frac{149}{30240}z^{10} + \frac{1}{22275}z^12 \\
&+ \frac{1}{24}z^4 - \frac{52411}{212889600}z - \frac{372689}{29030400}z^3 - \frac{299}{3600}z^5), \\
\beta_{\frac{1}{4}}(z) &= h^4(\frac{1474751}{4862025}z^2 + \frac{10496}{231525}z^{11} - \frac{1008128}{231525}z^6 + \frac{76736}{64827}z^9 \\
&- \frac{1024606}{1620675}z^3 + \frac{2430208}{540225}z^7 - \frac{495616}{1620675}z^{10} \\
&- \frac{2141}{46305}z - \frac{1555136}{540225}z^8 - \frac{2048}{694575}z^{12} + \frac{11264}{5145}z^5), \\
\beta_{\frac{1}{2}}(z) &= h^4(\frac{5579221}{107775360}z^2 - \frac{589397}{35925120}z - \frac{5356}{8505}z^7 + \frac{3757}{8505}z^8 - \frac{64}{8019}z^{11} \\
&- \frac{16}{81}z^5 + \frac{1912}{3645}z^6 - \frac{2957}{15309}z^9 - \frac{126881}{4898880}z^3 + \frac{64}{120285}z^{12} + \frac{38}{729}z^{10}), \\
\beta_1(z) &= h^4(\frac{189881}{3193344}z^2 - \frac{5599}{5040}z^8 - \frac{593}{3780}z^{10} + \frac{38}{1485}z^{11} - \frac{269}{270}z^6 + \frac{29}{54}z^9 \\
&- \frac{8}{4455}z^{12} + \frac{1}{3}z^5 + \frac{251}{180}z^7 - \frac{94799}{5322240}z - \frac{667}{10080}z^3), \\
\beta_{\frac{5}{4}}(z) &= h^4(-\frac{9472}{18225}z^{11} - \frac{11264}{2025}z^5 + \frac{865216}{42525}z^8 + \frac{61952}{3645}z^6 \\
&- \frac{40507}{76545}z^2 + \frac{2048}{54675}z^{12} + \frac{299266}{382725}z^3 \\
&+ \frac{2641}{18225}z - \frac{149248}{6072}z - \frac{149248}{6075}z^9 + \frac{11264}{3645}z^{10}), \\
\beta_{\frac{3}{2}}(z) &= h^4(\frac{576599}{19958400}z^2 - \frac{1457}{190080}z - \frac{41981}{907200}z^3 - \frac{1058}{4725}z^{10} - \frac{2183}{1575}z^8 \\
&- \frac{64}{22275}z^{12} + \frac{2564}{1575}z^7 + \frac{32}{825}z^{11} - \frac{248}{225}z^6 + \frac{407}{567}z^9 + \frac{16}{45}z^5), \\
\beta_2(z) &= h^4(\frac{676023449}{53770106880}z^3 - \frac{158927665}{21508042752}z^2 + \frac{12739553}{40007520}z^6 \\
&+ \frac{155082617}{373403520}z^8 - \frac{45187}{444528}z^5 + \frac{19774007}{280052640}z^{10} - \frac{37038373}{168031584} \\
&+ \frac{3581}{3750705}z^{12} - \frac{691067}{55010340}z^{11} - \frac{89199919}{186701760}z^7 + \frac{107543549}{56330588160}), \\
\gamma_2(z) &= h^5(\frac{17275933}{9388431360}z^2 - \frac{34}{130977}z^{12} - \frac{21372}{197568}z^8 - \frac{26023}{317520}z^6
\end{aligned}$$

$$+\frac{491}{145530}z^{11}+\frac{25745}{444528}z^{9}+\frac{61043}{493920}z^{7}$$
$$-\frac{42187}{89413632}z-\frac{452701}{142248960}z^{3}+\frac{23}{882}z^{5}-\frac{1543}{82320}z^{10}),$$
$$\psi_2(z)=h^6(\frac{2}{93555}z^{12}-\frac{181429}{1341204480}z^{2}-\frac{865}{190512}z^{9}+\frac{473}{317520}z^{10}-\frac{2011}{211680}z^{7}$$
$$+\frac{3551}{423360}z^{8}+\frac{439}{12773376}z+\frac{14537}{60963940}z^{3}+\frac{71}{11340}z^{6}-\frac{17}{62370}z^{11}-\frac{1}{504}z^{5}).$$

Evaluating the continuous method at the end point i.e. at $z=3$ gives equation (7), the discrete scheme.

$$\begin{aligned}y_{n+2}&=4y_{n+\frac{1}{2}}-y_n-6y_{n+1}+4y_{n+\frac{3}{2}}\\&+h^4(\frac{199}{4064256}h^2m_{n+2}+\frac{18595}{28449792}hg_{n+2}+\frac{353}{1935360}f_n\\&+\frac{10349}{241920}f_{n+1}-\frac{45059093}{17923368960}f_{n+2}+\frac{2741}{233280}f_{n+\frac{1}{2}}\\&-\frac{326}{324135}f_{n+\frac{1}{4}}+\frac{1049}{60480}f_{n+\frac{3}{2}}-\frac{22}{3645}f_{n+\frac{5}{4}}).\end{aligned}\tag{7}$$

The evaluation of the first, second and third derivatives of the continuous scheme at all the points give:

$$\begin{aligned}y'_n&=\frac{1}{281652940800h}(9679950h^6m_{n+2}-132889050h^5g_{n+2}\\&-69339753h^4f_n-5016763080h^4f_{n+1}+537717745h^4f_{n+2}\\&-4620872480h^4f_{n+\frac{1}{2}}-1183887360h^4f_{n+\frac{1}{4}}-2158924320h^4f_{n+\frac{3}{2}}\\&+3710414848h^4f_{n+\frac{5}{4}}-1032727449600y_n-844958822400y_{n+1}\\&+1689917644800y_{n+\frac{1}{2}}+187768627200y_{n+\frac{3}{2}}),\end{aligned}\tag{8}$$

$$\begin{aligned}y'_{n+\frac{1}{4}}&=\frac{1}{1514166209740800h}(5049383850h^6m_{n+2}-70104884850h^5g_{n+2}\\&-20273486625h^4f_n-3169618739280h^4f_{n+1}+286356165745h^4f_{n+2}\\&-4468957482080h^4f_{n+\frac{1}{2}}-1454169860352h^4f_{n+\frac{1}{4}}\\&-1188753517728h^4f_{n+\frac{3}{2}}+2129134577920h^4f_{n+\frac{5}{4}}\\&-2902151902003200y_n-37854155243500y_{n+1}\\&-2649790867046400y_{n+\frac{1}{2}}+126180517478400y_{n+\frac{3}{2}}),\end{aligned}\tag{9}$$

$$\begin{aligned}y'_{n+\frac{1}{2}}&=\frac{1}{11829423513600h}(193391730h^6m_{n+2}-2656692990h^5g_{n+2}\\&-691972659h^4f_n-101125304280h^4f_{n+1}+10754937355h^4f_{n+2}\\&-64309810880h^4f_{n+\frac{1}{2}}-1210871808h^4f_{n+\frac{1}{4}}\end{aligned}\tag{10}$$

$$- 43218197568h^4 f_{n+\frac{3}{2}} + 74156314624h^4 f_{n+\frac{5}{4}} + 7886282342400y_n$$
$$- 23658847027200y_{n+1} + 11829423513600y_{n+\frac{1}{2}} + 3943141171200y_{n+\frac{3}{2}}),$$

$$y'_{n+1} = \frac{1}{5914711756800h}(95075190h^6 m_{n+2} - 1307370330h^5 g_{n+2}$$
$$- 332553249h^4 f_n - 60445065240h^4 f_{n+1} + 5297215865h^4 f_{n+2}$$
$$- 22146439840h^4 f_{n+\frac{1}{2}} - 1735299072h^4 f_{n+\frac{1}{4}} - 21367275552h^4 f_{n+\frac{3}{2}} \tag{11}$$
$$+ 35647238144h^4 f_{n+\frac{5}{4}} + 1971570585600y_n + 5914711756800y_{n+1}$$
$$+ 11829423513600y_{n+\frac{1}{2}} + 3943141171200y_{n+\frac{3}{2}}),$$

$$y'_{n+\frac{5}{4}} = \frac{1}{378541552435200h}(1605581775h^6 m_{n+2} - 21968673300h^5 g_{n+2}$$
$$- 5893116957h^4 f_n - 1559150194770h^4 f_{n+1} + 88585440655h^4 f_{n+2}$$
$$- 377189979320h^4 f_{n+\frac{1}{2}} - 33117526080h^4 f_{n+\frac{1}{4}} - 350423274600h^4 f_{n+\frac{3}{2}} \tag{12}$$
$$+ 199383013312h^4 f_{n+\frac{5}{4}} + 31545129369600y_n + 662447716761600y_{n+1}$$
$$- 94635388108800y_{n+\frac{1}{2}} + 725537975500800y_{n+\frac{3}{2}}),$$

$$y'_{n+\frac{3}{2}} = \frac{1}{563305881600h}(19029150h^6 m_{n+2} - 261497250h^5 g_{n+2} - 67281165h^4 f_n$$
$$- 14884437960h^4 f_{n+1} + 1059057365h^4 f_{n+2} - 4365108160h^4 f_{n+\frac{1}{2}}$$
$$+ 366391296h^4 f_{n+\frac{1}{4}} - 4337153856h^4 f_{n+\frac{3}{2}} \tag{13}$$
$$+ 4625223680h^4 f_{n+\frac{5}{4}} + 375537254400y_n + 3379835289600y_{n+1}$$
$$- 1689917644800y_{n+\frac{1}{2}} - 2065454899200y_{n+\frac{3}{2}}),$$

$$y'_{n+2} = \frac{1}{1182942351360h}(184130730h^6 m_{n+2} - 2251516302h^5 g_{n+2}$$
$$- 787768443h^4 f_n - 191311144920h^4 f_{n+1} + 7060477075h^4 f_{n+2}$$
$$- 51022210400h^4 f_{n+\frac{1}{2}} - 4350799872h^4 f_{n+\frac{1}{4}} - 94177998432h^4 f_{n+\frac{3}{2}} \tag{14}$$
$$+ 17829941248h^4 f_{n+\frac{5}{4}} + 4337455288320y_n + 22475904675840y_{n+1}$$
$$- 16561192919040y_{n+\frac{1}{2}} - 10252167045120y_{n+\frac{3}{2}}),$$

$$y''_n = \frac{1}{2957355878400h}(800101890h^6 m_{n+2} - 10883837790h^5 g_{n+2}$$
$$- 14122423035h^4 f_n - 351697588200h^4 f_{n+1} + 43705107875h^4 f_{n+2}$$
$$- 306187648480h^4 f_{n+\frac{1}{2}} - 163095662592h^4 f_{n+\frac{1}{4}} - 170876266848h^4 f_{n+\frac{3}{2}} \tag{15}$$

$$+ 284547092480h^4 f_{n+\frac{5}{4}} - 23658847027200y_n - 47317694054400y_{n+1}$$
$$+ 59147117568000y_{n+\frac{1}{2}} + 11829423513600y_{n+\frac{3}{2}}),$$

$$y''_{n+\frac{1}{4}} = \frac{1}{15141662097408000h}(139595784930h^6 m_{n+2} - 1925300342070h^5 g_{n+2}$$
$$- 281037000951h^4 f_n - 77587505561160h^4 f_{n+1}$$
$$+ 7819909232575h^4 f_{n+2} - 66405992542400h^4 f_{n+\frac{1}{2}}$$
$$- 2454354155520h^4 f_{n+\frac{1}{4}} - 31790742943680h^4 f_{n+\frac{3}{2}}$$
$$+55383061866496h^4 f_{n+\frac{5}{4}} - 9084997258444800y_n - 1514166209740800y_{n+1}$$
$$- 21198326936371200y_{n+\frac{1}{2}} + 3028332419481600y_{n+\frac{3}{2}}), \tag{16}$$

$$y''_{n+\frac{1}{2}} = \frac{1}{844958822400h}(6804630h^6 m_{n+2} - 94852170h^5 g_{n+2}$$
$$- 11032497h^4 f_n - 3341315880h^4 f_{n+1} + 388600025h^4 f_{n+2}$$
$$- 13894659520h^4 f_{n+\frac{1}{2}} - 2141835264h^4 f_{n+\frac{1}{4}} - 1624262976h^4 f_{n+\frac{3}{2}}$$
$$+ 3021197312h^4 f_{n+\frac{5}{4}} + 3379835289600y_n - 3379835289600y_{n+1}$$
$$- 6759670579200y_{n+\frac{1}{2}}), \tag{17}$$

$$y''_{n+1} = \frac{1}{422479411200h}(4493790h^6 m_{n+2} - 61552890h^5 g_{n+2} - 15361353h^4 f_n$$
$$- 8183707560h^4 f_{n+1} + 248347825h^4 f_{n+2} - 519172640h^4 f_{n+\frac{1}{2}}$$
$$- 135254016h^4 f_{n+\frac{1}{4}} - 974193696h^4 f_{n+\frac{3}{2}}$$
$$+ 507179008h^4 f_{n+\frac{5}{4}} + 3379835289600y_{n+1}$$
$$+ 1689917644800y_{n+\frac{1}{2}} + 31689917644800y_{n+\frac{3}{2}}), \tag{18}$$

$$y''_{n+\frac{5}{4}} = \frac{1}{15141662097408000h}(15072008130h^6 m_{n+2} - 2069280738630h^5 g_{n+2}$$
$$- 533157946335h^4 f_n - 110021504229000h^4 f_{n+1}$$
$$+8370764537575h^4 f_{n+2} - 34837218234560h^4 f_{n+\frac{1}{2}} - 2868160444416h^4 f_{n+\frac{1}{4}}$$
$$- 33290701327296h^4 f_{n+\frac{3}{2}} + 57035703961600h^4 f_{n+\frac{5}{4}}$$
$$+ 3028332419481600y_n + 21198326936371200y_{n+1}$$
$$- 1514166209740800y_{n+\frac{1}{2}} + 9084997258444800y_{n+\frac{3}{2}}), \tag{19}$$

$$
\begin{aligned}
y''_{n+\frac{3}{2}} = \frac{1}{5914711756800h^2}(&1248748830h^6m_{n+2} - 17218844370h^5g_{n+2}\\
&- 4341936501h^4f_n - 1010022811560h^4f_{n+1} + 70006820125h^4f_{n+2}\\
&- 278412276800h^4f_{n+\frac{1}{2}} + 23978004480h^4f_{n+\frac{1}{4}}\\
&- 31031028840h^4f_{n+\frac{3}{2}} + 153638711296h^4f_{n+\frac{5}{4}} + 23658847027200y_n\\
&+118294235136000y_{n+1} - 94635388108800y_{n+\frac{1}{2}} - 47317694054400y_{n+\frac{3}{2}}),
\end{aligned} \tag{20}
$$

$$
\begin{aligned}
y''_{n+2} = \frac{1}{591471175680h^2}(&77010066h^6m_{n+2} - 212236290h^5g_{n+2}\\
&- 833110299h^4f_n - 221571499464h^4f_{n+1}\\
&- 11786692445h^4f_{n+2} - 55440752864h^4f_{n+\frac{1}{2}}\\
&- 4585697280h^4f_{n+\frac{1}{4}} - 154370645856h^4f_{n+\frac{3}{2}}\\
&+ 8135938048h^4f_{n+\frac{5}{4}} + 4731769405440y_n + 18927077621760y_{n+1}\\
&- 16561192919040y_{n+\frac{1}{2}} - 7097654108160y_{n+\frac{3}{2}}),
\end{aligned} \tag{21}
$$

$$
\begin{aligned}
y'''_n = \frac{1}{44808422400h^3}(&64108170h^6m_{n+2} - 855604890h^5g_{n+2}\\
&- 3451472829h^4f_n - 17790010560h^4f_{n+1} + 3380117245h^4f_{n+2}\\
&- 6963229280h^4f_{n+\frac{1}{2}} - 15451803648h^4f_{n+\frac{1}{4}}\\
&- 12441153312h^4f_{n+\frac{3}{2}} + 19111235584h^4f_{n+\frac{5}{4}}\\
&- 358467379200y_n - 1075402137600y_{n+1}\\
&+ 1075402137600y_{n+\frac{1}{2}} + 358467379200y_{n+\frac{3}{2}}),
\end{aligned} \tag{22}
$$

$$
\begin{aligned}
y'''_{n+\frac{1}{4}} = \frac{1}{1911826022400h^3}(&366131430h^6m_{n+2} - 5234701290h^5g_{n+2}\\
&- 2607159807h^4f_n - 336774289320h^4f_{n+1} + 21890686505h^4f_{n+2}\\
&- 584711715840h^4f_{n+\frac{1}{2}} - 151059723264h^4f_{n+\frac{1}{4}}\\
&- 98085004416h^4f_{n+\frac{3}{2}} + 190219875328h^4f_{n+\frac{5}{4}}\\
&- 15294608179200y_n - 37854155243500y_{n+1}\\
&- 45883824537600y_{n+\frac{1}{2}} + 15294608179200y_{n+\frac{3}{2}}),
\end{aligned} \tag{23}
$$

$$
\begin{aligned}
y'''_{n+\frac{1}{2}} = \frac{1}{44808422400h^3}(&26411490h^6m_{n+2} - 361080090h^5g_{n+2}\\
&- 109437237h^4f_n - 11780732880h^4f_{n+1} + 1455672965h^4f_{n+2}
\end{aligned}
$$

$$
\begin{aligned}
&- 6030488800h^4 f_{n+\frac{1}{2}} - 1295308800h^4 f_{n+\frac{1}{4}} - 5756637600h^4 f_{n+\frac{3}{2}} \\
&+ 9724209152h^4 f_{n+\frac{5}{4}} + 358467379200y_n - 1075402137600y_{n+1} \\
&+ 1075402137600y_{n+\frac{1}{2}} + 358467379200y_{n+\frac{3}{2}}),
\end{aligned} \tag{24}
$$

$$
\begin{aligned}
y'''_{n+1} = \frac{1}{14936140800h^3}(&7466130h^6 m_{n+2} - 103046370h^5 g_{n+2} \\
&- 24742305h^4 f_n - 4351435200h^4 f_{n+1} + 418983905h^4 f_{n+2} \\
&- 1481595360h^4 f_{n+\frac{1}{2}} - 144175104h^4 f_{n+\frac{1}{4}} - 1719928224h^4 f_{n+\frac{3}{2}} \\
&+ 3280506880h^4 f_{n+\frac{5}{4}} + 119489126400y_n + 358467379200y_{n+1} \\
&- 358467379200y_{n+\frac{1}{2}} + 119489126400y_{n+\frac{3}{2}}),
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
y'''_{n+\frac{5}{4}} = \frac{1}{5735478067200h^3}(&2210022990h^6 m_{n+2} - 30091755330h^5 g_{n+2} \\
&- 8412998013h^4 f_n - 2305452973320h^4 f_{n+1} + 120717845765h^4 f_{n+2} \\
&- 545465052160h^4 f_{n+\frac{1}{2}} + 46980946944h^4 f_{n+\frac{1}{4}} \\
&- 453102056064h^4 f_{n+\frac{3}{2}} + 276995253248h^4 f_{n+\frac{5}{4}} + 45883824537600y_n \\
&+137651473612800y_{n+1} - 137651473612800y_{n+\frac{1}{2}} + 45883824537600y_{n+\frac{3}{2}}),
\end{aligned} \tag{26}
$$

$$
\begin{aligned}
y'''_{n+\frac{3}{2}} = \frac{1}{44808422400h^3}(&23051070h^6 m_{n+2} - 324208710h^5 g_{n+2} \\
&- 71189307h^4 f_n - 17196565680h^4 f_{n+1} + 1347045995h^4 f_{n+2} \\
&- 4363344160h^4 f_{n+\frac{1}{2}} + 406923264h^4 f_{n+\frac{1}{4}} \\
&- 9354943584h^4 f_{n+\frac{3}{2}} - 4374243328h^4 f_{n+\frac{5}{4}} \\
&+ 375537254400y_n + 1075402137600y_{n+1} \\
&- 1075402137600y_{n+\frac{1}{2}} - 358467379200y_{n+\frac{3}{2}}),
\end{aligned} \tag{27}
$$

$$
\begin{aligned}
y'''_{n+2} = \frac{1}{44808422400h^3}(&39271050h^6 m_{n+2} - 1095912090h^5 g_{n+2} \\
&- 52593219h^4 f_n - 19005053760h^4 f_{n+1} \\
&+ 11447376125h^4 f_{n+2} - 4051406240h^4 f_{n+\frac{1}{2}} \\
&- 276811776h^4 f_{n+\frac{1}{4}} - 22760722656h^4 f_{n+\frac{3}{2}} - 1029812224h^4 f_{n+\frac{5}{4}} \\
&+ 358467379200y_n + 1075402137600y_{n+1} \\
&+ 1075402137600y_{n+\frac{1}{2}} + 358467379200y_{n+\frac{3}{2}}).
\end{aligned} \tag{28}
$$

These schemes (the discrete scheme and its derivatives) are combined together in matrix form and by using the matrix inversion, a block method of the following form is produced (see, Lee et al. [12])

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n+\frac{1}{4}} \\ y_{n+\frac{1}{2}} \\ y_{n+1} \\ y_{n+\frac{5}{4}} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n-\frac{1}{4}} \\ y_{n-\frac{1}{2}} \\ y_{n-1} \\ y_{n-\frac{5}{4}} \\ y_{n-\frac{3}{2}} \\ y_n \end{bmatrix}$$

$$+ h \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \frac{5}{4} \\ 0 & 0 & 0 & 0 & 0 & \frac{3}{2} \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} y'_{n-\frac{1}{4}} \\ y'_{n-\frac{1}{2}} \\ y'_{n-1} \\ y'_{n-\frac{5}{4}} \\ y'_{n-\frac{3}{2}} \\ y'_n \end{bmatrix}$$

$$+ h^2 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{32} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{8} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & \frac{25}{32} \\ 0 & 0 & 0 & 0 & 0 & \frac{9}{8} \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} y''_{n-\frac{1}{4}} \\ y''_{n-\frac{1}{2}} \\ y''_{n-1} \\ y''_{n-\frac{5}{4}} \\ y''_{n-\frac{3}{2}} \\ y''_n \end{bmatrix} + h^3 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{384} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{48} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & \frac{125}{384} \\ 0 & 0 & 0 & 0 & 0 & \frac{9}{16} \\ 0 & 0 & 0 & 0 & 0 & \frac{4}{3} \end{bmatrix} \begin{bmatrix} y'''_{n-\frac{1}{4}} \\ y'''_{n-\frac{1}{2}} \\ y'''_{n-1} \\ y'''_{n-\frac{5}{4}} \\ y'''_{n-\frac{3}{2}} \\ y'''_n \end{bmatrix}$$

$$+ h^4 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{25610353}{249141657600} \\ 0 & 0 & 0 & 0 & 0 & \frac{41273}{36495360} \\ 0 & 0 & 0 & 0 & 0 & \frac{71161}{6652800} \\ 0 & 0 & 0 & 0 & 0 & \frac{646970875}{29896998912} \\ 0 & 0 & 0 & 0 & 0 & \frac{37772}{985600} \\ 0 & 0 & 0 & 0 & 0 & \frac{38}{405} \end{bmatrix} \begin{bmatrix} f_{n-\frac{1}{4}} \\ f_{n-\frac{1}{2}} \\ f_{n-1} \\ f_{n-\frac{5}{4}} \\ f_{n-\frac{3}{2}} \\ f_n \end{bmatrix}$$

$$+ h^4 \begin{bmatrix} \frac{9485369}{7969417600} & -\frac{6131177}{63063982080} & \frac{569345}{37371248864} & -\frac{2246441}{9853747200} & \frac{206153}{1297612800} & -\frac{437315029541}{96906637423411120} \\ \frac{127942}{53482275} & -\frac{53929}{35925120} & \frac{13423}{5806080} & -\frac{967}{280665} & \frac{95581}{39916800} & -\frac{59482693}{87625359360} \\ \frac{1823872}{53482275} & -\frac{1138}{120285} & \frac{2447}{99792} & -\frac{152192}{4209975} & \frac{62}{2475} & -\frac{262184651}{36966948480} \\ \frac{2610574375}{35050143755} & -\frac{28121875}{2675441664} & \frac{1473284375}{26159874048} & -\frac{72754625}{919683072} & \frac{25285625}{467140608} & -\frac{3278324715625}{215348083163136} \\ \frac{182547}{1320550} & -\frac{439}{98560} & \frac{2619}{22528} & -\frac{53}{350} & \frac{25299}{246400} & -\frac{7753787}{270448640} \\ \frac{3817472}{10696455} & \frac{1792}{40095} & \frac{1648}{4455} & -\frac{16384}{40095} & \frac{256}{891} & -\frac{497456}{6417873} \end{bmatrix}$$

$$\cdot \begin{bmatrix} f_{n+\frac{1}{4}} \\ f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+\frac{5}{4}} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \end{bmatrix} + h^5 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{80778437}{6991820881920} \\ 0 & 0 & 0 & 0 & 0 & \frac{652247}{3755372544} \\ 0 & 0 & 0 & 0 & 0 & \frac{532249}{293388480} \\ 0 & 0 & 0 & 0 & 0 & \frac{59798346875}{15382005940224} \\ 0 & 0 & 0 & 0 & 0 & \frac{282357}{38635520} \\ 0 & 0 & 0 & 0 & 0 & \frac{90292}{4584195} \end{bmatrix} \begin{bmatrix} g_{n+\frac{1}{4}} \\ g_{n+\frac{1}{2}} \\ g_{n+1} \\ g_{n+\frac{5}{4}} \\ g_{n+\frac{3}{2}} \\ g_{n+2} \end{bmatrix}$$

$$+ h^6 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -\frac{9634379}{10987147100160} \\ 0 & 0 & 0 & 0 & 0 & -\frac{1963}{149022720} \\ 0 & 0 & 0 & 0 & 0 & -\frac{1153}{8382528} \\ 0 & 0 & 0 & 0 & 0 & -\frac{71834375}{2441588244448} \\ 0 & 0 & 0 & 0 & 0 & -\frac{6093}{11038720} \\ 0 & 0 & 0 & 0 & 0 & -\frac{4}{2695} \end{bmatrix} \begin{bmatrix} m_{n+\frac{1}{4}} \\ m_{n+\frac{1}{2}} \\ m_{n+1} \\ m_{n+\frac{5}{4}} \\ m_{n+\frac{3}{2}} \\ m_{n+2} \end{bmatrix} \tag{29}$$

writing out the block explicitly, we have

$$\begin{aligned} y_{n+\frac{1}{4}} = y_n + \frac{1}{4}hy'_n + \frac{1}{32}h^2y''_n + \frac{1}{384}h^3y'''_n + h^4(\frac{25610353}{249141657600}f_n \\ + \frac{9485369}{79659417600}f_{n+\frac{1}{4}} - \frac{6131177}{63063982080}f_{n+\frac{1}{2}} + \frac{569345}{3737124864}f_{n+1} \\ - \frac{2246441}{9853747200}f_{n+\frac{5}{4}} + \frac{206153}{1297612800}f_{n+\frac{3}{2}} \\ - \frac{437315029541}{9690663742341120}f_{n+2} + \frac{80778437}{6991820881920}hg_{n+2} \\ - \frac{9634379}{10987147100160}h^2m_{n+2}), \end{aligned} \tag{30}$$

$$\begin{aligned} y_{n+\frac{1}{2}} = y_n + \frac{1}{2}hy'_n + \frac{1}{8}h^2y''_n + \frac{1}{48}h^3y'''_n + h^4(\frac{41273}{36495360}f_n + \frac{127942}{53482275}f_{n+\frac{1}{4}} \\ - \frac{53929}{35925120}f_{n+\frac{1}{2}} + \frac{13423}{5806080}f_{n+1} - \frac{967}{280665}f_{n+\frac{5}{4}} + \frac{95581}{39916800}f_{n+\frac{3}{2}} \\ - \frac{59482693}{87625359360}f_{n+2} + \frac{652247}{3755372544}hg_{n+2} - \frac{1963}{149022720}h^2m_{n+2}), \end{aligned} \tag{31}$$

$$\begin{aligned} y_{n+1} = y_n + hy'_n + \frac{1}{2}h^2y''_n + \frac{1}{6}h^3y'''_n + h^4(\frac{71161}{6652800}f_n + \frac{1823872}{53482275}f_{n+\frac{1}{4}} \\ - \frac{1138}{120285}f_{n+\frac{1}{2}} + \frac{2447}{99792}f_{n+1} - \frac{152192}{4209975}f_{n+\frac{5}{4}} \\ + \frac{62}{2475}f_{n+\frac{3}{2}} - \frac{262184651}{36966948480}f_{n+2} \\ + \frac{532249}{293388480}hg_{n+2} - \frac{1153}{8382528}h^2m_{n+2}), \end{aligned} \tag{32}$$

$$
\begin{aligned}
y_{n+\frac{5}{4}} &= y_n + \frac{5}{4}hy'_n + \frac{25}{32}h^2y''_n + \frac{125}{384}h^3y'''_n \\
&+ h^4\Big(\frac{646970875}{29896998912}f_n + \frac{2610574375}{35050143744}f_{n+\frac{1}{4}} \\
&- \frac{28121875}{2675441664}f_{n+\frac{1}{2}} + \frac{1473284375}{26159874048}f_{n+1} - \frac{72754625}{919683072}f_{n+\frac{5}{4}} \\
&+ \frac{25285625}{467140608}f_{n+\frac{3}{2}} - \frac{3278324715625}{215348083163136}f_{n+2} \\
&+ \frac{59798346875}{15382005940224}hg_{n+2} - \frac{71834375}{244158824448}h^2m_{n+2}\Big),
\end{aligned} \tag{33}
$$

$$
\begin{aligned}
y_{n+\frac{3}{2}} &= y_n + \frac{3}{2}hy'_n + \frac{9}{8}h^2y''_n + \frac{9}{16}h^3y'''_n + h^4\Big(\frac{37773}{985600}f_n + \frac{182547}{1320550}f_{n+\frac{1}{4}} \\
&- \frac{439}{98560}f_{n+\frac{1}{2}} + \frac{2619}{22528}f_{n+1} - \frac{53}{350}f_{n+\frac{5}{4}} + \frac{25299}{246400}f_{n+\frac{3}{2}} \\
&- \frac{7753787}{270448640}f_{n+2} + \frac{282357}{38635520}hg_{n+2} - \frac{6093}{11038720}h^2m_{n+2}\Big),
\end{aligned} \tag{34}
$$

$$
\begin{aligned}
y_{n+2} &= y_n + 2hy'_n + 2h^2y''_n + \frac{4}{3}h^3y'''_n + h^4\Big(\frac{38}{405}f_n + \frac{3817472}{10696455}f_{n+\frac{1}{4}} \\
&+ \frac{1792}{40095}f_{n+\frac{1}{2}} + \frac{1648}{4455}f_{n+1} - \frac{16384}{40095}f_{n+\frac{5}{4}} + \frac{256}{891}f_{n+\frac{3}{2}} \\
&- \frac{496456}{6417873}f_{n+2} + \frac{90292}{4584195}hg_{n+2} - \frac{4}{2695}h^2m_{n+2}\Big).
\end{aligned} \tag{35}
$$

Substituting the above equations (30)-(35) into the equations (8)-(28) yield equation (36)-(53):

$$
\begin{aligned}
y'_{n+\frac{1}{4}} &= y'_n + \frac{1}{4}hy''_n + \frac{1}{32}h^2y'''_n + h^3\Big(\frac{338231}{22889889792}h^2m_{n+2} + \frac{156033833}{801146142720}hg_{n+2} \\
&+ \frac{80269559}{54499737600}f_n + \frac{1758583}{681246720}f_{n+1} - \frac{6983495549}{9176764907520}f_{n+2} \\
&- \frac{3071041}{1839366144}f_{n+\frac{1}{2}} + \frac{157195609}{73021132800}f_{n+\frac{1}{4}} \\
&+ \frac{4567571}{1703116800}f_{n+\frac{1}{4}} - \frac{4567571}{1703116800}f_{n+\frac{3}{2}} - \frac{22153493}{5748019200}f_{n+\frac{5}{4}},
\end{aligned} \tag{36}
$$

$$
\begin{aligned}
y'_{n+\frac{1}{2}} &= y'_n + \frac{1}{2}hy''_n + \frac{1}{8}h^2y'''_n + h^3\Big(-\frac{42151}{447068160}h^2m_{n+2} + \frac{486287}{391184640}hg_{n+2} \\
&+ \frac{803171}{10644480}f_n + \frac{175921}{10644480}f_{n+1} - \frac{87079903}{17923368960}f_{n+3} - \frac{754387}{71850240}f_{n+\frac{1}{2}} \\
&- \frac{349981}{17827425}f_{n+\frac{1}{4}} + \frac{227993}{13305600}f_{n+\frac{1}{2}} - \frac{34589}{1403325}f_{n+\frac{5}{4}},
\end{aligned} \tag{37}
$$

$$y'_{n+1} = y'_n + hy''_n + \frac{1}{2}h^2y'''_n + h^3(-\frac{647}{1397088}h^2m_{n+2} + \frac{299153}{48898080}hg_{n+2} + \frac{112859}{3326400}f_n + \frac{725}{8316}f_{n+1} - \frac{2683189}{112021056}f_{n+2} - \frac{3697}{280665}f_{n+\frac{1}{2}} - \frac{2710816}{17827425}f_{n+\frac{1}{4}} + \frac{4423}{51975}f_{n+\frac{3}{2}} - \frac{24896}{200475}f_{n+\frac{5}{4}}, \tag{38}$$

$$y'_{n+\frac{5}{4}} = y'_n + \frac{5}{4}hy''_n + \frac{25}{32}h^2y'''_n + h^3(-\frac{9266875}{11444944896}h^m_{n+2} + \frac{214931875}{20028653568}hg_{n+2} + \frac{29669125}{544997376}f_n + \frac{47745625}{272498688}f_{n+1} - \frac{19333410625}{458838245376}f_{n+2} + \frac{13608125}{1839366144}f_{n+\frac{1}{2}} + \frac{598077625}{1839366144}f_{n+\frac{1}{4}} + \frac{10316125}{68124672}f_{n+\frac{3}{2}} - \frac{51867125}{229920768}f_{n+\frac{5}{4}}, \tag{39}$$

$$y'_{n+\frac{3}{2}} = y'_n + \frac{3}{2}hy''_n + \frac{9}{8}h^2y'''_n + h^3(-\frac{351}{275968}h^2m_{n+2} + \frac{652833}{38635520}hg_{n+2} + \frac{629667}{7884800}f_n + \frac{123201}{394240}f_{n+1} - \frac{3269271}{49172480}f_{n+2} + \frac{123}{2816}f_{n+\frac{1}{2}} + \frac{203877}{660275}f_{n+\frac{1}{4}} + \frac{118791}{492800}f_{n+\frac{3}{2}} - \frac{687}{1925}f_{n+\frac{5}{4}}, \tag{40}$$

$$y'_{n+2} = y'_n + 2hy''_n + 2h^2y'''_n + h^3(-\frac{548}{218295}h^2m_{n+2} + \frac{50738}{1528065}hg_{n+2} + \frac{7558}{51975}f_n + \frac{7648}{10395}f_{n+1} - \frac{226126}{1750329}f_{n+2} + \frac{45824}{280665}f_{n+\frac{1}{2}} + \frac{10338304}{17827425}f_{n+\frac{1}{4}} + \frac{27392}{51975}f_{n+\frac{3}{2}} - \frac{966656}{1403325}f_{n+\frac{5}{4}}, \tag{41}$$

$$y''_{n+\frac{1}{4}} = y''_n + \frac{1}{4}hy'''_n + h^2(-\frac{1865797}{10404495360}h^2m_{n+2} + \frac{172242689}{72831467520}hg_{n+2} + \frac{72668371}{4954521600}f_n + \frac{1303619}{41287680}f_{n+1} - \frac{84837118099}{9176764907520}f_{n+2} - \frac{8706799}{418037760}f_{n+\frac{1}{2}} + \frac{1628609}{55319040}f_{n+\frac{1}{4}} + \frac{72169}{2211840}f_{n+\frac{3}{2}} - \frac{3069139}{65318400}f_{n+\frac{5}{4}}, \tag{42}$$

$$y''_{n+\frac{1}{2}} = y''_n + \frac{1}{2}hy'''_n + h^2(-\frac{5971}{13547520}h^2m_{n+2} + \frac{1637243}{284497920}hg_{n+2} + \frac{130541}{3870720}f_n + \frac{7319}{96768}f_{n+1} - \frac{268600651}{11948912640}f_{n+2} - \frac{2557}{60480}f_{n+\frac{1}{2}} - \frac{185951}{1620675}f_{n+\frac{1}{4}} + \frac{23927}{302400}f_{n+\frac{3}{2}} - \frac{193}{1701}f_{n+\frac{5}{4}}, \tag{43}$$

$$y''_{n+1} = y''_n + hy'''_n + h^2(-\frac{73}{63504}h^2m_{n+2} + \frac{33937}{2222640}hg_{n+2} + \frac{10919}{151200}f_n + \frac{163}{630}f_{n+1} - \frac{16822367}{280052640}f_{n+2} + \frac{1292}{25515}f_{n+\frac{1}{2}} - \frac{17408}{60025}f_{n+\frac{1}{4}} + \frac{1028}{4725}f_{n+\frac{3}{2}} - \frac{41984}{127575}f_{n+\frac{5}{4}}, \tag{44}$$

$$
\begin{aligned}
y''_{n+\frac{5}{4}} &= y''_n + \frac{5}{4}hy'''_n + h^2(-\frac{1121875}{693633024}h^2m_{n+2} + \frac{313973125}{14566293504}hg_{n+2} \\
&+ \frac{2600725}{28311552}f_n + \frac{11148125}{24772608}f_{n+1} - \frac{52028243125}{611784327168}f_{n+2} + \frac{3169375}{27869184}f_{n+\frac{1}{2}} \\
&+ \frac{12413875}{33191424}f_{n+\frac{1}{4}} + \frac{963875}{3096576}f_{n+\frac{3}{2}} - \frac{45925}{96768}f_{n+\frac{5}{4}},
\end{aligned} \tag{45}
$$

$$
\begin{aligned}
y''_{n+\frac{3}{2}} &= y''_n + \frac{3}{2}hy'''_n + h^2(-\frac{1047}{501760}h^2m_{n+2} + \frac{97899}{3512320}hg_{n+2} \\
&+ \frac{79923}{716800}f_n + \frac{11601}{17920}f_{n+1} - \frac{16257883}{147517440}f_{n+2} + \frac{1187}{6720}f_{n+\frac{1}{2}} \\
&+ \frac{5499}{12005}f_{n+\frac{1}{4}} + \frac{921}{2240}f_{n+\frac{3}{2}} - \frac{299}{525}f_{n+\frac{5}{4}},
\end{aligned} \tag{46}
$$

$$
\begin{aligned}
y''_{n+2} &= y''_n + 2hy'''_n + h^2(-\frac{2}{735}h^2m_{n+2} + \frac{4744}{138915}hg_{n+2} \\
&+ \frac{712}{4725}f_n + \frac{992}{945}f_{n+1} - \frac{112958}{972405}f_{n+2} + \frac{512}{1701}f_{n+\frac{1}{2}} \\
&+ \frac{1015808}{1620675}f_{n+\frac{1}{4}} + \frac{512}{675}f_{n+\frac{3}{2}} - \frac{32768}{42525}f_{n+\frac{5}{4}},
\end{aligned} \tag{47}
$$

$$
\begin{aligned}
y'''_{n+\frac{1}{4}} &= y'''_n + h(-\frac{1611667}{1300561920}h^2m_{n+2} + \frac{29782007}{1820786688}hg_{n+2} \\
&+ \frac{48548753}{619315200}f_n + \frac{3419713}{15482880}f_{n+1} - \frac{73396589569}{1147095613440}f_{n+2} - \frac{70189}{466560}f_{n+\frac{1}{2}} \\
&+ \frac{55145099}{207446400}f_{n+\frac{1}{4}} + \frac{1095161}{4838400}f_{n+\frac{3}{2}} - \frac{762857}{2332800}f_{n+\frac{5}{4}},
\end{aligned} \tag{48}
$$

$$
\begin{aligned}
y'''_{n+\frac{1}{2}} &= y'''_n + h(-\frac{2137}{2540160}h^2m_{n+2} + \frac{2453}{222264}hg_{n+2} \\
&+ \frac{45109}{604800}f_n + \frac{8111}{60480}f_{n+1} - \frac{48111107}{1120210560}f_{n+2} + \frac{607}{29160}f_{n+\frac{1}{2}} \\
&- \frac{605726}{1620675}f_{n+\frac{1}{4}} + \frac{5639}{37800}f_{n+\frac{3}{2}} - \frac{3818}{18225}f_{n+\frac{5}{4}},
\end{aligned} \tag{49}
$$

$$
\begin{aligned}
y'''_{n+1} &= y'''_n + h(-\frac{613}{317520}h^2m_{n+2} + \frac{11555}{444528}hg_{n+2} \\
&+ \frac{11897}{151200}f_n + \frac{1301}{1890}f_{n+1} - \frac{28981681}{280052640}f_{n+2} + \frac{928}{3645}f_{n+\frac{1}{2}} \\
&- \frac{543232}{1620675}f_{n+\frac{1}{4}} + \frac{1856}{4725}f_{n+\frac{3}{2}} - \frac{11776}{18225}f_{n+\frac{5}{4}},
\end{aligned} \tag{50}
$$

$$
\begin{aligned}
y'''_{n+\frac{5}{4}} &= y'''_n + h(-\frac{472375}{260112384}h^2m_{n+2} + \frac{44320375}{1820786688}hg_{n+2} \\
&+ \frac{1944505}{224772608}f_n + \frac{2474125}{3096576}f_{n+1} - \frac{22134914125}{229419122688}f_{n+2} + \frac{23375}{93312}f_{n+\frac{1}{2}} \\
&+ \frac{2793475}{8297856}f_{n+\frac{1}{4}} + \frac{69025}{193536}f_{n+\frac{3}{2}} - \frac{44305}{93312}f_{n+\frac{5}{4}},
\end{aligned} \tag{51}
$$

$$
\begin{aligned}
y'''_{n+\frac{3}{2}} = y'''_n + h(&-\frac{61}{31360}h^2 m_{n+2} + \frac{289}{10976}hg_{n+2} \\
&+ \frac{1761}{22400}f_n + \frac{1749}{2240}f_{n+1} - \frac{1459001}{13829760}f_{n+2} + \frac{91}{360}f_{n+\frac{1}{2}} \\
&+ \frac{20154}{60025}f_{n+\frac{1}{4}} + \frac{681}{1400}f_{n+\frac{3}{2}} - \frac{74}{225}f_{n+\frac{5}{4}},
\end{aligned} \tag{52}
$$

$$
\begin{aligned}
y'''_{n+2} = y'''_n + h(&-\frac{11}{19845}h^2 m_{n+2} - \frac{149}{27783}hg_{n+2} \\
&+ \frac{739}{9450}f_n + \frac{776}{945}f_{n+1} + \frac{3151273}{17503290}f_{n+2} + \frac{896}{3645}f_{n+\frac{1}{2}} \\
&+ \frac{548864}{1620675}f_{n+\frac{1}{4}} + \frac{3712}{4725}f_{n+\frac{3}{2}} - \frac{8192}{18225}f_{n+\frac{5}{4}}.
\end{aligned} \tag{53}
$$

3. Analysis of the properties of the block method

3.1 Order and error constant of the block

Following Fatunla [6] and Lambert [11], we define the local truncation error associated with linear multistep method to be the linear difference operator

$$L[y(x);h] = \sum_{j=0}^{k} \alpha_j y(x+jh) - h^4 \beta_j j'^v(x+jh). \tag{54}$$

Assuming that $y(x)$ is sufficiently differentiable, we can expand the terms in (7) above as a Taylor series about x to obtain the expression

$$L[y(x);h] = C_0 y(x) + C_1 hy' + ..., C_q h^q y^q(x) + ..., \tag{55}$$

where the constant coefficient $C_q, q = 0, 1, ...$ are given as follows:

$$C_0 = \sum_{j=0}^{k} \alpha_j, C_1 = \sum_{j=0}^{k} j\alpha_j, ..., C_q = \frac{1}{q!}\left[\sum_{j=1}^{k} j^q \alpha_j - (q-4)! \sum_{j=1}^{k} j^{q-4}\beta_j\right].$$

According to Henrici [8], we say that our block is of uniform order $p = [9, 9, 9, 9, 9, 9]$ and error constants given by the vector

$$
\begin{aligned}
C_{13} = [&\frac{138668197}{7521989255862681600}, \frac{503747}{1836423158169600}, \frac{71801}{25107347865600}, \\
&\frac{12690929375}{2106156991641550848}, \frac{1333}{119957094400}, \frac{163}{5064318720}]^T.
\end{aligned}
$$

3.2 Zero stability of the block method

Given the general form of block method

$$A^{(0)}Y_m = A^{(i)}Y_{m-1} + h^\mu[B^{(i)}F_m + B^{(0)}F_{m-1}].$$

A block method is said to be zero stable, if the roots

$$det[\lambda A^{(0)} - A^{(i)}] = 0$$

of the first characteristic polynomial satisfy $|\lambda| \leq 1$ and for the roots with $|\lambda| \leq 1$, the multiplicity must not exceed the order of the differential equation (see, Adeyefa and Kuboye [4] and Omar and Kuboye [15]). For our method

$$\left[z \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} - \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} \right] = 0,$$

$$A = z^6 - z^6 = 0, z = 0, 0, 0, 0, 0, 0$$

Hence, the block is zero stable.

3.3 Region of absolute stability

The stability nature of the method is found in the footstep of Awoyemi [2] as shown in Figure 1.

The region of absolute stability of our method is A-stable, because the region consists of the complex plane outside the enclosed figure (see, Yakubu et al. [19]).

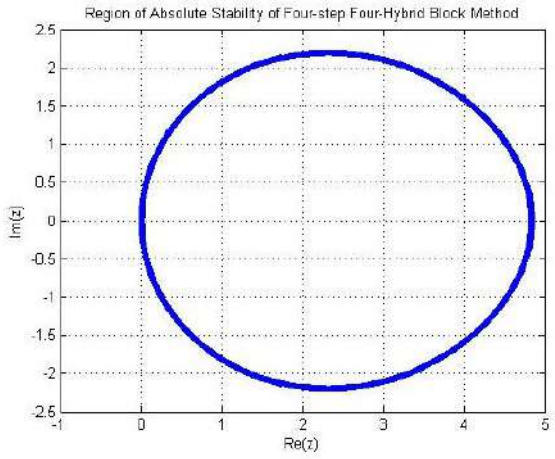


Figure 1: Showing the region of absolute stability of the method

3.4 Consistency and convergence

Our new hybrid block method is consistent since the order of each of the method is greater than 1.

Theorem 1: Convergence

According to Lambert [11], the necessary and sufficient condition for a linear multistep method to be convergent is for it to be consistent and zero stable. Also, Akinfewa et al. [5], corroborated Lambert's stanceon convergence as thus; convergence = consistency + zero-stability. Hence the new hybrid block method is convergent.

4. Numerical results

Here in this segment, the performance of the new hybrid method is seen on modeled second, third and fourth order ODEs. The results obtained from the test Problems are shown in tabular form. All the test problems are solved using our written code in Maple.

$Error = |Exact\ solution - Computed\ solution|$.

Problem 1: Real-life Problem (Cooling of a body)

The temperature y degrees of a body, t minutes after being placed in a certain room, satisfies the differential equation $3\frac{d^2y}{dt^2} + \frac{dy}{dt} = 0$. By using the substitution $z = \frac{dy}{dt}$ or the otherwise, find y in terms of t given that $y = 60$ when $t = 0$ and $y = 35$ when $t = 6In4$. Find after how many minutes the rate of cooling of the body will have fallen below one degree per minute, giving your answer correct to the nearest minute. How cool does the body get?

Formulation of the Problem

$$y'' = \frac{(-y')}{3}, y(0) = 60, y'(0) = -\frac{80}{9}, h = 0.1$$

Analytical Solution

$$y(x) = \frac{80}{3}e^{-(\frac{1}{3})x} + \frac{100}{3}$$

Table 1. Showing the comparison of the result for problem 1 with the error in Obarhua [14]

X	Exact	Computed	Error	Error in Obarhua[14]
0.1	59.125762679520157387	59.125762679520157388	1.10E-18	2.344791 E -13
0.2	58.280186267509806339	58.280186267509806339	0.000000	2.202682 E -13
0.3	57.462331147625588617	57.462331147625588619	2.10E-18	3.935749 E -12
0.4	56.671288507811932106	56.671288507811932107	1.10E-18	2.704951 E -12
0.5	55.906179330416375307	55.906179330416375308	1.10E-18	7.599112 E -12
0.6	55.166153415412849564	55.166153415412849565	1.10E-18	1.569518 E -12
0.7	54.450388435647511050	54.450388435647511050	0.000000	2.756872 E -12
0.8	53.758089023057298472	53.758089023057298472	0.000000	4.375392 E -11
0.9	53.088485884845809762	53.088485884845809760	2.10E-18	6.474571 E -11
1.0	52.440834948634380011	52.440834948634380010	1.10E-18	9.100178 E -11

In this result displayed above, the new two-step implicit hybrid block method produced more accurate result for solving problem 1 than the method of Obarhua [14].

Problem 2. Consider the third order oscillatory differential equation

$$y'''(t) = -4y'(t) + t,\ y(0) = y'(0) = 0,\ y''(0) = 1$$

with the exact solution given by:

$$y(t) = \left(\frac{3}{16}\right)(1 - cos2t) + \left(\frac{1}{8}\right)t^2.$$

Table 2. Showing the comparison of the result for problem 2 with the error in Sunday [17]

x	Exact	Computed	Error	Error in Sunday[17]
0.1	0.0049875166547671941600	0.0049875166547671940759	8.4100E-20	8.3209 E-13
0.2	0.019801063624459046980	0.019801063624459045988	9.9200E-19	3.4752 E -12
0.3	0.043999572204435319270	0.043999572204435312050	7.2200E-18	7.8178 E -12
0.4	0.076867491997406483580	0.076867491997406460536	2.3044E-17	1.3681 E -11
0.5	0.11744331764972380299	0.11744331764972374178	6.1210E-17	2.0825 E -11
0.6	0.16455792103562370419	0.16455792103562357206	1.3213E-16	2.8962 E -11
0.7	0.21688116070620482401	0.21688116070620456653	2.5748E-16	3.7764 E -11
0.8	0.27297491043149163616	0.27297491043149118204	4.5412E-16	4.6879 E -11
0.9	0.33135039275495382287	0.33135039275495307253	7.5034E-16	5.5941 E -11
1.0	0.39052753185258919756	0.39052753185258803045	1.16711E-15	6.4592 E -11

Our new method generated better result than the method of Sunday [17] in solving third order oscillatory differential equation as seen in Table 2 above.

Problem 3. The new method is used to solve a physical problem that arising from ship dynamics. In particular, this problem describes how the sinusoidal wave of frequency Ω passes along a ship or an offshore structure as stated by Familua and Omole [7]. This leads to a fourth order ordinary differential equation defined as follows:

$$y^4 + 3y'' + y(2 + \epsilon\cos(\Omega t)) = 0,\ t\ f\ 0.$$

Is subjected to the following initial conditions:

$$y(0) = 1, y'(0) = 0, y''(0) = 0, y'''(0) = 0,\ h = \frac{1}{320},$$

where $\epsilon = 0$ for the existence of the theoretical solution: $y(t) = 2\cos t - \cos(t\sqrt{2})$.

Table 3. Showing the comparison of the result for problem 3 with the error in Familua and Omole [7]

X	Exact	Computed	Error	Error in Familua and Omole [7]
0.003125	0.99999999999205272181	0.99999999999205272179	2.10E-20	6.685763E-13
0.006250	0.99999999987284392123	0.99999999987284392118	5.10E-20	1.458489E-11
0.009375	0.99999999935627549414	0.99999999935627549419	5.10E-20	1.082968E-10
0.001250	0.99999999796552658062	0.99999999796552658062	0.00000	3.917803E-10
0.015625	0.99999999503306753347	0.99999999503306753346	1.10E-20	1.025145E-09
0.018750	0.99999998970067947569	0.99999998970067947573	4.10E-20	2.217319E-09
0.021875	0.99999998091947944412	0.99999998091947944411	1.10E-20	4.226068E-09
0.025000	0.99999996744995111889	0.99999996744995111890	1.10E-20	7.358019E-09
0.028125	0.99999994786198113959	0.99999994786198113956	3.10E-20	1.196868E-08
0.031250	0.99999992053490100516	0.99999992053490100515	1.10E-20	1.846249E-08

In solving fourth order ODE in Problem 3, our new hybrid block method yielded far better result than the method of Familua and Omole [7] as shown in Table 3.

5. Discussion of results

In Table 1, it is observed that our new two-step hybrid block method performs far better than the result generated by Obarhua [14]. In Problem 1, it was observed that the body gets cooler and the temperature falling below one degree Celsius as the step length of our scheme is being reduced which make the temperature of the body in the room to satisfy our differential equation. The result generated by our new scheme when applied to problem 2, an oscillatory third order ODE as displayed in table 2 reveals that our method produces better performance over that of Sunday [17]. And table 3 shows the supremacy of our method over that of Familua and Omole [7] in solving a fourth order ODE arising from ship dynamics in Problem 3.

6. Conclusion

The new multi-derivative block method for the numerical solution of second, third and fourth orders ODEs was developed in this paper. The method is stable, convergent and consistent. The main benefit of the method over the existing numerical approaches is its knack for solving directly and effectively three different orders of differential equations namely: second, third and fourth-orders IVPs of ordinary differential equations. To prove the effectiveness of the new hybrid block method, it is used to solve some ordinary differential equations of order two,three and four. The generated results confirm the supremacy of our new block method over the existing methods in relations to error as shown in Tables 1-3 above.

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Modules with quasi-continuous submodules

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Abstract. The notion of left q-ring was generalized to the big class of left π-rings. Furthermore, q-rings extended to module theory under the concept of Q-module (modules all of whose submodules are qausi-injective). In this article, the concept of QC-modules (modules all of whose submodules are qausi-continuous) are introduced as a wider class of both π-rings and Q-modules. Numerous characterizations of QC-modules have been obtained. Also, properties and results of these modules are investigated. Moreover, strong forms of QC-modules are given and studied.

Keywords: C_1-condition, C_2-condition, QC-modules, qausi-continuous modules, continuous modules.

1. Introduction

Rings with left ideals are qausi-injective (named as left q-rings) were identified by S.K. Jain et al. [6] and were extended to the modules case by the authors in [8], that is, a module M with qausi-injective submodules is called Q-module. In 1999, S.K. Jain et al. [5] introduced π-rings as a generalization of q-rings, that is, a ring R is called π-ring if every right ideal of is qausi-continuous (or π-injective).

In this work, we generalize π-rings to the category of modules and moreover we study QC-modules as a proper bigger class than the class of Q-modules. We call a module M is QC-module if every sub module of M is qausi-continuous (π-injective). Also, we introduce strong forms of QC property. A module is C-continuous if M whose submodule are continuous as well as we call a module M is strictly QC-module if M is continuous and every sub module is qausi-continuous.

In the whole of this work R will denote an associative ring with unity and M will denote a unitary left R-module. From [14], a submodule A is called Large (or essential) in M if $A \cap B \neq (0)$ where B is a non-zero sub module of M. So, C is called closed if there is a sub module B of M with C is large in B, then $C = B$. By using Zorn's lemma, every submodule of M is contained essentially in closed sub module of M. By $E(M)$ we mean that the injective hull (or envelope) of the

module M and the set of all endomorphisms of M is denoted by S =End (M). Given two modules K and H, K is called H-injective if each $f \in Hom(C, K)$ where $C$is a sub module of H can be extended to $g \in Hom(H, K)$ [3]. A module K is called injective if for each module H, K is H- injective. In particular, A is called qausi-injective if A is A -injective.

Let M be a module and regard the next conditions:

C_1-condition: Let A be a submodule of M. Then there is a direct summand B of M such that $A$is large in B.

C_2- condition: If $A \cong B$(where A be a submodule of M and B is direct summand of M), then B is direct summand of M.

C_3- condition: The direct sum of any two direct summands of M with zero intersection is direct summand.

A module M is called (qausi-)continuous, if it satisfies C_1-condition and C_2-condition (C_1-condition and C_3-condition) [9]. From [14], the concepts of qausi-continuity and π-injectivity are the same. M is π-injective whenever A_1, A_2 are submodules of M with$A_1 \cap A_2 = 0$, each projection $\pi_i : A_1 \oplus A_2 \longrightarrow A_i$ $(i = 1, 2)$ lifts to an endomorphism of M. Moreover, a module N is qausi-continuous iff whenever $E(N) = \oplus_{\alpha\in\Lambda} E_\alpha$ (where E_α are submodules of $E(N)$ $\forall\alpha \in \Lambda$, then $N = (N \cap E_\alpha)$ [4]. From [12], a sub module F of M is called fully invariant if $\alpha(F) \subseteq F$ for any $\alpha \in End(M)$. Furthermore, M is called duo if it whose submodules are fully invariant [12].

2. QC-modules

Definition 2.1 A module M is called QC-module if every submodule of M is qausi-continuous (equivalently, π-injective).

In particular, a ring R is right π-ring [5] if R as right R-module is QC-module.

Remark 2.2. 1. Precisely, Q-module is QC-module. The contrary is not true generally. In fact, the set of rational numbers Q_Z is QC-module but not Q-module.

2. QC-module implies qausi-continuous. The contrary is not valid. Absolutely, $Z_{2^\infty} \oplus Z_{2^\infty}$ is qausi-continuous (it is an injective Z-module since injective class is closed under finite direct sum), although $M = Z_{2^\infty} \oplus Z_{2^\infty}$ is not QC-module since $Z_2 \oplus Z_8$ is sub module of M which is not qausi-continuous (since it is not satisfy the C_1-condition).

3. Each uniform module implies QC-module (because every uniform module is qausi-continuous and every submodule of uniform module is uniform [9], while the contrary is not valid generally. $Z/6Z$ as Z-module is QC-module and not uniform.

4. Each semi-simple module is QC-module (because semisimple module implies π-injective (qausi-continuous) and qausi-continuous property is closed

under direct summands [9]), but the contrary is not valid. Instance, $Q_Z$is not semisimple while it is QC-module.

5. It is explicit that the QC-module property is closed under submodules.

6. One can investigate QC-modules property is not closed under direct sum. Consider Z_2 and Z_4 as Z-modules, thus they are QC -modules (since they are uinform modules). But $Z_2 \oplus Z_4$ is not π-module. In fact, $A = \{(\overline{0},\overline{0}),(\overline{1},\overline{0})\}$ and $B = \{(\overline{0},\overline{0}),(\overline{1},\overline{2})\}$ are direct summands of $Z_2 \oplus Z_4$ with $A \cap B = 0$. Easy calculation shows that$A \oplus B = \{(\overline{0},\overline{0}),(\overline{1},\overline{2}),(\overline{1},\overline{0}),(\overline{0},\overline{2})\}$ is not direct summand of $Z_2 \oplus Z_4$. Thus, $Z_2 \oplus Z_4$ is not qausi-continuous Z-module. So, $Z_2 \oplus Z_4$ is not QC-module.

7. An indecomposable module is qausi-continuous iff it is QC -module.

8. Directly, by (6) and Theorem 1.10 in [4], we have an analogue of Matlis theorem [7] as follows: If R is Notherian ring and M be a QC -module, then M is a direct sum of indecomposable QC -modules.

9. The classes of QC -modules and continuous modules are distinct. For instance, Z_Z is not continuous (since it is not satisfying C_2-condition) whereas Z_Z is QC-module. The contrary, one can use the same example in (2).

The next proposition generalizes Theorem 1.4 in [5] to the category of modules.

Theorem 2.3. *M is QC-module iff M whose sub modules are satisfy -condition and if C and D are sub modules of M with $C \cap D = 0$, then C and D are mutually injective.*

Proof. For the necessary part, let M be a QC -module. Directly, the submodules of M are satisfy C_1-condition. Let C, D are sub module of M with $C \cap D = 0$. By hypothesis, $C \oplus D$ is qausi-continuous and by [9] C is D –injective and D is C –injective (i.e)C and D are mutually injective. Conversely, Given a sub module T in M and $E(T) = E_1 \cap E_2$ and put $C = E_1 \cap T$ and $C = E_2 \cap T$. It is easy to inquiry that$C = E_1 \cap T$ is closed of T and by hypothesis T satisfies C_1-condition, so $T = C \oplus H$ for some sub module H of T. Let π_C and π_D be the corresponding projections maps. $D \approxeq \pi_H(D) \subseteq X$. Define $\alpha : \pi_H(D) \longrightarrow C$ by $\alpha(\pi_H(d)) = \pi_C(d)$ for each d in D. The map is well defined since $\pi_H : D \longrightarrow \pi_H(D)$ is an isomorphism. Now, since C is H-injective, so α extended for $\beta : H \longrightarrow C$. Put $H^* = \{h + \beta(h) \,|h \in H\}$ Then $T = C$. Then $T = C \oplus H*$. Also, D is essential in H^* and D is closed. Thus $D = H^*$ and therefore, $T = C \oplus D$. Hence, T is qausi-continuous and therefore M is QC -module. □

Corollary 2.4 ([5])**.** *A ring is a left π-ring iff R whose left ideals are satisfy C_1-condition and if C and D are left ideals of R with $C \cap D = 0$, then C and D are mutually injective.*

The next result asserts that we need a very restricted class of sub modules to get QC -modules.

Theorem 2.5. *M is QC-module iff each large submodule of M is qausi-continuous.*

Proof. ($\Rightarrow$). The proof id direct by the definition of QC -modules.

($\Leftarrow$) Given a submodule T in M, thus $T \oplus K$ is large submodule in M, where K a complement of N. Hence, from assumption, $T \oplus K$ is qausi-continuous and moreover T is qausi-continuous [9]. So, M is QC-module. □

Corollary 2.6 ([5])**.** *A ring R is left π -ring iff each large left ideal in R is qausi-continuous.*

By using Theorem 2.4 and Theorem 2.3, the next result is concluded.

Corollary 2.7. *M is QC-module iff any large submodule in M satisfy C_1-condition and if C and D are large submodules of M with $C \cap D = 0$, then C and$D$ are mutually injective.*

Forthcoming Theorem 1.1.6 in [8], Q-modules is equivalent to qausi-injective modules whose essential sub modules are fully invariant. Here, we have the next result as a characterization of QC-modules.

Firstly, we need the following well-known result for qausi-continuous modules.

Lemma 2.8 ([9])**.** *A module W is qausi-continuous if and only if $e(W) \subseteq W$ for each idempotent endomorphism e of $E(W)$.*

Theorem 2.9. *M is QC-module iff M is qausi-continuous with any essential submodule is fully invariant in M relative to idempotent elements of $S = End(M)$.*

Proof. ($\Rightarrow$) Directly, by QC-module property, we have M is qausi-continuous module. Consider A as an essential sub module of M and let $g : M \longrightarrow M$ be idempotent homomorphism. But the injective envelope $E(M)$ is qausi-continuous module, so g extended to an idempotent endomorphism of $E(M)$ [14]. Now, since N is essential, thus by [14] $E(A) = E(M)$ and A is qausi-continuous (since M is QC-module), thus $g(N) \subseteq N$.

($\Leftarrow$) Given A a large sub module in M. Let τ be an idempotent endomorphism of A. But A is large sub module in M, so $E(M) = E(M)$ and moreover τ is idempotent endomorphism of $E(M)$. But, by assumption, M implies qausi-continuous module, thus $\tau(M) \subseteq (M)$ (i.e) $\tau \in End(M)$. But A (by hypothesis) is fully invariant sub module of M relative to idempotent elements. Hence, $\tau(A) \subseteq (A)$. Thus, A is qausi-continuous module. So M satisfy QC –module (from Theorem 2.6). □

Following [1], sub module T is called projection-invariant in M if $\alpha(T) \subseteq T$ for all idempotent endomorphism α of M as well as if every sub module of M is projection-invariant, then M is said to satisfy $(*)$ condition.

Corollary 2.10. *A module M satisfy $(*)$ condition is qausi-continuous iff M is QC -module.*

Since every Duo module satisfy $(*)$ condition [1], so the next result arises.

Corollary 2.11. *A duo module is qausi-continuous iff it is QC-module.*

The next corollary provide a rich source of examples of QC-modules.

Corollary 2.12. *A commutative ring is a π-ring iff it is a qausi-continuous.*

Following [9], if a module N isomorphic to $N \oplus N$, then N is called purely infinite. Here, we call M is fully purely infinite if its sub modules are purely infinite.

Proposition 2.13. *A fully purely infinite module is Q-module iff it is QC -module.*

Proof. It is obvious by corollary 2.12 in [9]. □

3. Strong forms of QC-modules

Definition 3.1. A module M is called is C-module whenever any submodule in M is continuous.

Definition 3.2. M is called strictly QC-module whenever M is continuous and their sub modules of M are qausi-continuous.

Remarks 3.3. 1. The following implications are direct by using the above definitions: C-modules $\Rightarrow$ Strictly QC - modules $\Rightarrow$ QC -modules. The opposite of these implications are not true generally. Indeed, Z_Z is QC-module while Z_Z is not strictly QC-module because it is not continuous. Moreover, Q_Z is strictly QC-module which it is not satisfy C-module.

2. Precisely, each semi-simple module is C-module.

3. Unlike QC -modules, uniform modules need not be C-modules. For example, $Q$is uniform Z-module which it is not C-module.

4. The class of C-modules is need not be closed under direct sum property. $Z_2 \oplus Z_8$ as Z-module is not C-module while Z_2 and Z_8 as Z-modules are C-modules.

5. As like QC -modules, the C-module property is closed under sub modules without auxiliary conditions.

By using the similar criteria in the proof of Theorem 2.5, we get a characterization of C-modules.

Proposition 3.4. *M is QC-module if and only if every large submodule of M continuous.*

Observe which Z_Z is large of Q_Z and Q_Z is continuous which it is not C-module.

Definition 3.5. (1) A module M is said to be fully C_1-module if sub modules of M are satisfy C_1-condition.

(2) A module M is said to be fully C_2-module if sub modules of M are satisfy C_2-condition.

Remarks 3.6. 1. It is obvious that$M$ is C-module if and only if it is fully C_1 -module and fully C_2 -module.

2. Each QC-module is fully C_1-module while the contrary is not valid generally. Let $T = \begin{bmatrix} Z_2 & Z_2 \\ 0 & Z_2 \end{bmatrix}$ where Z_2 is the set of integers modulo 2 . Then, T as T-module is fully C_1-module but it is not QC-module since it is not qausi-continuous. Note that T_T does not satisfy fully C_2 module.

3. There is a QC-module which it is not fully C_2-module. For instance,Z_Z is not fully C_2- module whereas it achieve QC-module. Recall that an module M is direct-injective if for each direct summand D in M with inclusion mapping $i_D : D \longrightarrow M$ and any monomorphism $\lambda : D \longrightarrow M$ there exist an endomorphism $\psi : M \longrightarrow M$ such that$\psi \circ \lambda = i_D$[11]. Following Theorem 7.13 in [11], direct-injective modules and C_2-modules are equivalent. So the next result is direct.

Proposition 3.7. *A module M is fully C_2-module if and only M whose sub modules are direct-injective.*

Proposition 3.8. *A module $M$is fully C_2-module if and only if M whose essential sub modules are direct-injective.*

Proof. The necessary part is immediate. For the sufficient part, let T be a submdodule in M. Thus, a complement submodule H of T in M is exist such that $T \oplus H$ is large sub module of M (by Zorn,s Axiom). By hypothesis, $T \oplus H$ is direct-injective. By [11], T is direct-injective. So, by proposition (3.7), M is fully C_2-module. □

By using the similar attitude of the proof of proposition (2.6), we get a useful equivalent statement of fully C_1-modules.

Proposition 3.9. *A module M is fully C_1-module iff large sub modules of M are satisfy C_1-condition.*

On can obtain another express of Theorem 1.3 such the following:

Theorem 3.10. *A module M is QC-module if and only if M is fully C_1-module and if C and D are sub modules of M with $C \cap D = 0$, then C and D are mutually injective.*

We mentioned that every Q-module is C-module. In the next result, we give a condition under it the converse is true.

Proposition 3.11. Over commutative Noetherian, the concepts of Q-modules and C-modules are equivalent.

Proof. Since over a commutative Noetherian ring qausi-injectivity and continuity for modules are equivalent [13]. □

From [10], recall that a sub module A of M is stable if $\alpha(A) \subseteq A$ for any $\alpha \in Hom(A, M)$. Moreover, if every submodule of M is stable, then M is called fully stable.

Following [2], a module M is called has SC_1-condition if for each submodule A of M there is a stable direct summand B such that A is large in B. Also, M is called has SC_3-condition if whenever S_1 and S_2 are summands of S_2 with $S_1 \cap S_2$, $S_1 \oplus S_2$ is summand of M. Moreover, if M satisfy both SC_1-condition and SC_3-condition, then $M$is called strongly qausi-continuous.

As strong form of QC-modules arises as follows:

Definition 3.12. A module M is called amply QC-module if every sub module of M is strongly qausi-continuous. A ring R is called amply QC-ring if R as R-module is amply QC-module.

Remarks 3.13. 1. Directly, amply QC-module is QC-module while the contrary is not true generally. The vector space$V = R \oplus R$ as R-module is QC-module since it is semisimple whereas it is not amply QC-module because Vdoesn't have SC_2-condition since $0 \oplus R \cong R$ but $0 \oplus R$ is not stable.

2. Precisely, uniform module is amply QC-module. The contrary is not valid. $Z/10Z$ is amply QC-module whereas it not uniform.

3. Contrary QC-modules, semisimple modules need not be amply QC-modules. One can take the same example in (1).

4. Q-modules and amply QC-modules are various classes. Q_Z is amply QC-module whilst Q_Z is not satisfy Q-module. Also, $V = R \oplus R$ as R-module is Q-module whereas it is not amply QC-module.

5. Similar to QC-modules, amply QC-modules have hereditary property.

6. Identical to QC-modules, the direct sum of amply QC-modules not necessary amply QC-module. $Z_{p^\infty} \oplus Z_{p^\infty}$ as Z-module is not amply QC-module whereas Z_{p^∞} as Z-module is amply QC-module.

Proposition 3.14. *M is amply QC-module if and only if every large sub module of M is strongly qausi-continuous.*

Proof. $\Rightarrow$ Straightforward.

$\Leftarrow$ Given a submodule A of M. Then there a complement sub module B with $A \oplus B$ is large in M. From assumption, $A \oplus B$ is strongly qausi-continuous. From [2], A is strongly qausi-continuous and so M is amply QC-module. □

From [2], a module whose direct summand are stable is called SS-module. We use this concept to introduce the next concept:

Definition 3.15. *A module M is called fully SS-module if every sub module of M is SS-module. Uniform modules, fully stable modules and commutative rings are explicitly examples of fully SS-modules.*

Proposition 3.16. *A module M is fully SS-module if and only if any sub module T in M with $T = S \oplus F$,$Hom(S, F) = 0$.*

Proof. Given a direct summand T of a sub module N in M. Thus $N = D \oplus C$where C be a sub module of N. By assumption, $Hom(D, C) = 0$. Take the projection homomrphism $\rho : N = D \oplus C \longrightarrow C$ with $ker(\rho) = S$ .Let $\eta \in Hom(D, N)$. Then, $\rho\eta(D) = \rho(\eta(D)) = 0$ and so $\eta(D) \subseteq D$. Therefore, D is stable as well as N is SS-module and it is fully SS-module. Conversely, suppose T is a sub module of M such that $T = S \oplus F$. By hypothesis, T is SS-module and so S is stable sub module of T. Let $\xi \in Hom(S, F)$. Then, ξ can be extended to a homomrphism $\psi : T \longrightarrow F$ such that $\psi(s + f) = \xi(s)$. Since S is stable sub module of T, thus $\psi(s) = \xi(s) \in S$ for each $s \in S$. But $\xi(s) \in F$ and $S \cap F = 0$, thus $\xi = 0$. Therefore, $Hom(S, F) = 0$. □

Definition 3.17. (a) If sub modules of M are satisfy SC_1-condition, then M is called fully SC_1-condition.

(b) A module M is called fully SC_2-condition if every sub module of M has SC_2-condition.

Clearly, M is amply QC-module if and only if M is fully SC_1-condition and M is fully SC_3-condition.

Lemma 3.18. (1) *M is fully SC_1 -condition if and only if M is fully SS-module and M is fully C1-condition.*

(2) *M is fully SC_3-condition if and only if M is fully SS-module and M is fully C_3-condition.*

Proof. (1) $\Rightarrow$ Given a direct summand K of a sub module N in M. By hypothesis, N has SC_2-condition and snice K is closed so K is stable. By, definitions, fully SC_1-condition implies fully C_1-condition.

$\Leftarrow$ From Lemma 2.3.4 in [2], SC_1-condition and C_1-condition are equivalent under SS-module. So the necessary part is immediate.

(2) $\Rightarrow$ Given a direct summand C of a sub module N in M. Consider a direct summand $D = (0)$ of N. So $C \cap D = 0$. But N has SC_3-condition (because M is fully SC_2-module), so $C \oplus D = C$ is stable direct summand of N. So, M is fully SS-module.

$\Leftarrow$ It is obvious from Lemma (2.3.6) in [2]. □

From Lemma (3.18), the next result is achieved directly.

Proposition 3.19. *M is amply QC-module if and only if M is fully SS-module and M is QC-module.*

Corollary 3.20. *Let R commutative ring. Then the next statements are equivalent:*

I. *R is amply QC-module.*
II. *R is strongly qausi-continuous ring.*
III. *R is qausi-continuous.*
IV. *R is QC-module.*

Corollary 3.21. *Every commutative q-ring is amply QC-ring.*

Corollary 3.22. *The following are equivalent:*

I. *M is amply QC-module:*
II. *M is fully SC_1-condition and M is fully C_3-condition.*
III. *M is fully C_1-condition and M is fully SC_3-condition.*

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Dual hesitant fuzzy Bonferroni means and its applications in decision-making

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Abstract. The Bonferroni mean (BM) was initially presented by Bonferroni and afterward more as of late summed up by Yager. The exceptionally solid characteristic of the BM is its ability to catch the interrelationship between input datum. Nevertheless, it appears that the previous work just considers the BM for aggregating crisp information rather than some other kinds of contentions. In this paper, we explore the BM under dual hesitant fuzzy datum. The dual hesitant fuzzy set (DHFS) is the all-encompassing type of intuitionistic fuzzy sets (IFSs) and hesitant fuzzy sets (HFSs). We build up an dual hesitant fuzzy BM (DHFBM) and talk about its assortment of extraordinary cases. At that point, we apply the weighted DHFBM to multicriteria dynamic. Some numerical models are given to show our outcomes.

Keywords: Bonferroni mean, dual hesitant fuzzy set, dual hesitant fuzzy Bonferroni mean (DHFBM), decision-making.

1. Introduction

In recent decades, the aggregation operators received much attention from many researchers and practitioners [1]-[4]. The aggregation operators are generally characterized into two groups. In the first group, the aggregated arguments behaves dependently, whereas, the aggregated arguments behaves independently in second group. For the first type of aggregation operators, Yager [5] developed the ordered weighted averaging (OWA) operator to reorder the arguments before the aggregation process. Later on, Chiclana *et al.* [6] and Xu and Da [7] developed the ordered weighted geometric (OWG) operator. Yager [8] also proposed the continuous ordered weighted averaging (C-OWA) operator. Moreover, Yager and Xu [9] defined the ordered weighted geometric (OWG) opera-

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tor. Many researchers explored the second type of aggregation operators and developed many novel aggregation operators, such as, the Chouqet [10] integral operator, the power aggregation operator [11] and the Heronian mean (HM) [12] operator. Yager [13] defined the power average (PA) to develop an aggregation operator. Xu and Yager [14] introduced the power geometric (PG) operator, the power ordered geometric (POG) operator and the power ordered weighted geometric (POWG). Basically, Bonferroni explored the BM (Bonferroni, [15] which can provide for the aggregation lying among the max and min operators and the logical "or" and "and" operators. The desirable characteristic of the BM is that captures the communicated interrelationship of the individual arguments. It has wide application in MADM under different fuzzy environments. Liu *et al*. [16] presented an interpretation of HM operator and proposed some new generalization instead of simple averaging operators defined earlier. Beliakov *et al*. [17] suggested some new generalized Bonferroni mean operators and applied them to multi-attribute aggregation under different environments. Xu and Yager [18] developed BM and weighted BM operators under intuitionistic fuzzy environment. Zhu *et al*. [19] introduced the BM and weighted BM aggregation operators, Zhu *et al*. [20] also developed the geometric BM combining with hesitant fuzzy information. Yu *et al*. [21] introduced the generalized hesitant fuzzy BM operator and its importance in multi-attribute group decision making.

So as to present dual hesitant fuzzy information, Zhu *et al*. [22] presented the idea of dual hesitant fuzzy sets (DHFSs), it covers the arguments with two sets of possible values, that is the membership (favorable) hesitancy function and the non-membership (unfavorable) hesitancy function, It is very much powerful appliance to represent and collect information under uncertain environment, exclusively in the decision making process, Fuzzy sets [23], intuitionistic fuzzy sets [24], hesitant fuzzy sets [25] and fuzzy multi sets [26] are the special cases of DHFSs. Many researchers developed different aggregation operators for other fuzzy information, such as IFSs, HFSs, etc. A very little research have been developed on the MADM problems for dual hesitant fuzzy environment. Therefore, we have explored the dual hesitant fuzzy Bonferroni mean (DHFBM) operator and discussed its several cases. We have also explored the weighted dual hesitant fuzzy Bonferroni mean (WDHFBM) operator. In this connection, the remainder of the paper is composed as follows. Section 2 consist of definitions and some basic concepts and special cases of BM. In Section 3, we discuss the properties and operations of DHFSs. Further, We have developed the DHFBM and the dual hesitant Bonferroni element (DHFBE), we have discussed its main properties such as, Idempotency, Monotonocity, Boundedness and Commutativity. In, Section 4, we introduce weighted dual hesitant fuzzy Bonferroni mean (WDHFBM) operator. A numerical example of real life related to decision-making is presented to demonstrate our discussion. The paper finishes with some ending remarks in Section 5.

2. Preliminaries

2.1 Bonferroni mean

The Bonferroni [15] introduced the BM as given below:

Definition 1. Suppose $\rho, \ell \geq 0$, where both of the parameters are not simultaneously 0 and $r_i (i = 1, 2, \ldots, \tau)$ be an assortment of nonnegative numbers. If

$$B^{\rho,\ell}(r_1, r_2, \ldots, r_\tau) = \left(\frac{1}{\tau(\tau-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{\tau} r_i^{\rho} r_j^{\ell} \right)^{\frac{1}{\rho+\ell}}, \tag{1}$$

then the BM is denoted as $B^{\rho,\ell}$.

Clearly, the BM saves the accompanying properties:

1. $B^{\rho,\ell}(0, 0, \ldots, 0) = 0$.
2. $B^{\rho,\ell}(r, r, \ldots, r) = r$, if $r_i = r$, for all i.
3. $B^{\rho,\ell}(r_1, r_2, \ldots, r_\tau) \geq B^{\rho,\ell}(c_1, c_2, \ldots, c_\tau)$ i.e. $B^{\rho,\ell}$ is monotonic, if $r_i \geq c_i$ all i.
4. $\min_i\{r_i\} \leq B^{\rho,\ell}(r_1, r_2, \ldots, r_\tau) \leq \max_i\{r_i\}$.

Additionally, we discuss several cases of BM, if $\ell = 0$, then Eq.1 becomes

$$B^{\rho,0}(r_1, r_2, \ldots, r_\tau) = \left(\frac{1}{\tau} \sum_{i=1}^{\tau} r_i^{\rho} \left(\frac{1}{\tau - 1} \sum_{j=1}^{\tau} r_j^{0} \right) \right)^{\frac{1}{\rho+0}} = \left(\frac{1}{\tau} \sum_{i=1}^{\tau} r_i^{\rho} \right)^{\frac{1}{\rho}}.$$

Dyckhoff and Pedrycz [27] called this the generalized mean operator or power-root operator. Some more case are given below.

1) When $\rho = 2$ and $\ell = 0$, then Eq.1 turns into square mean

$$B^{2,0}(r_1, r_2, \ldots, r_\tau) = \left(\frac{1}{\tau} \sum_{i=1}^{\tau} r_i^{2} \right)^{\frac{1}{2}}.$$

2) When $\rho = 1$ and $\ell = 0$, then Eq.1 turns into usual average

$$B^{1,0}(r_1, r_2, \ldots, r_\tau) = \frac{1}{\tau} \sum_{i=1}^{\tau} r_i.$$

3) When $\rho \to +\infty$ and $\ell = 0$, then Eq.1 turns into max operator

$$\lim_{\rho \to +\infty} B^{\rho,0}(r_1, r_2, \ldots, r_\tau) = \max_i\{r_i\}.$$

4) When $\rho \to 0$ and $\ell = 0$, then Eq.1 turns into geometric mean

$$\lim_{\rho \to 0} B^{\rho,0}(r_1, r_2, \ldots, r_\tau) = \left(\prod_{i=1}^{\tau} r_i \right)^{\frac{1}{\tau}}.$$

5) When $\rho = 1$ and $\ell = 1$, then Eq.1 turns into square mean

$$B^{1,1}(r_1, r_2, \ldots, r_\tau) = \left(\frac{1}{\tau(\tau-1)} \sum_{i,j=1}^{\tau} r_i r_j \right)^{\frac{1}{2}}.$$

2.2 Dual hesitant fuzzy sets

In this part, we illustrate dual hesitant fuzzy set as a combination of two sets, the membership values and non-membership values, corresponding to each element in the crisp set as given below:

Definition 2. Suppose X a crisp set, Zhu *et al.*[22] presented the idea of dual hesitant fuzzy set (DHFS) on X as follows:

$$\text{Ç} = \{< x, r(x), s(x) > | x \in X\}$$

with $r(x) = \cup\{\sigma | \sigma \in r(x)\}$ and $s(x) = \cup\{\kappa | \kappa \in s(x)\}$ are two sets of few values in [0,1], $r(x), s(x)$ represents the possible membership degree and possible non-membership degree of the element $x \in X$ in the set Ç, respectively, satisfying the criteria:

$$0 \leq \sigma, \kappa \leq 1, \quad \sigma^+ + \kappa^+ \leq 1,$$

where $\sigma^+ \in r^+(x) = \bigcup_{\sigma \in r(x)} \max\{\sigma\}$, and $\kappa^+ \in s^+(x) = \bigcup_{\kappa \in s(x)} \max\{\kappa\}$ for all $x \in X$. In general, the pair $d(x) = (r(x), s(x))$ represents the dual hesitant fuzzy element (DHFE) simply written as $d = (r, s)$, with the conditions: $\sigma \in r, \kappa \in s, \sigma^+ \in r^+ = \bigcup_{\sigma \in r} \max\{\sigma\}, \kappa^+ \in s^+ = \bigcup_{\kappa \in s} \max\{\kappa\}, 0 \leq \sigma, \kappa \leq 1$, and $\sigma^+ + \kappa^+ \leq 1$.

Now, we describe some special types of DHFEs. For a DHFE, d,
(1) full vulnerability: $d = [0, 1]$; (complete ill-known, all is possible)
(2) full assurance: $d = (1, 0)$;
(3) full hesitancy: $d = (r, s)(0 < r < 1, 0 < s < 1)$;
(4) empty element: $d = \emptyset(r = \emptyset, s = \emptyset)$.

Definition 3. The complement $\overline{d}$, of a DHFE, d and $d \neq \emptyset$, is defined as:

$$\overline{d} = \begin{cases} \bigcup_{\substack{\sigma \in r \\ \kappa \in s}} \{\{\kappa\}, \{\sigma\}\}, & \text{if } s \neq \emptyset, \quad r \neq \emptyset, \\ \bigcup_{\sigma \in r} \{\{1-\sigma\}, \{\emptyset\}\}, & \text{if } s = \emptyset, \quad r \neq \emptyset, \\ \bigcup_{\kappa \in s} \{\{\emptyset\}, \{1-\kappa\}\}, & \text{if } r = \emptyset, \quad s \neq \emptyset. \end{cases}$$

Obviously, the complement function involutes as $\overline{\overline{d}} = d$.

Definition 4. Let d_1 and d_2 be two DHFEs with lower and upper bounds corresponding to r and s are r^-, r^+, s^- and s^+, respectively, where $r^- = \bigcup_{\sigma \in r} \min\{\sigma\}, r^+ = \bigcup_{\sigma \in r} \max\{\sigma\}, s^- = \bigcup_{\kappa \in s} \min\{\kappa\}$, and $s^+ = \bigcup_{\kappa \in s} \max\{\kappa\}$.

Now using the above notations, the union and intersection on d_1 and d_2 is defined as follows:

(1) $d_1 \cup d_2 = \{r \in (r_1 \cup r_2) | r \geq \max(r_1^-, r_2^-), s \in (s_1 \cap s_2) | s \leq \min(s_1^+, s_2^+)\}$;

(2) $d_1 \cap d_2 = \{r \in (r_1 \cap r_2) | r \leq \min(r_1^+, r_2^+), s \in (s_1 \cup s_2) | s \geq \max(s_1^-, s_2^-)\}$,

Furthermore, the following basic operations are valid and will be helpful in the next sections.

(1) $\oplus$-union: $d_1 \oplus d_2 = \{r_{d_1} \oplus r_{d_2}, s_{d_1} \oplus s_{d_2}\} = \bigcup_{\substack{\sigma_{d_1} \in r_{d_1} \\ \kappa_{d_1} \in s_{d_1} \\ \sigma_{d_2} \in r_{d_2} \\ \kappa_{d_2} \in s_{d_2}}} \{\{\sigma_{d_1} + \sigma_{d_2} - \sigma_{d_1}\sigma_{d_2}\}, \{\kappa_{d_1}\kappa_{d_2}\}\}$;

(2) $\oplus$-intersection: $d_1 \otimes d_2 = \{r_{d_1} \otimes r_{d_2}, s_{d_1} \oplus s_{d_2}\} = \bigcup_{\substack{\sigma_{d_1} \in r_{d_1} \\ \kappa_{d_1} \in s_{d_1} \\ \sigma_{d_2} \in r_{d_2} \\ \kappa_{d_2} \in s_{d_2}}} \{\{\sigma_{d_1}\sigma_{d_2}\}, \{\kappa_{d_1} + \kappa_{d_2} - \kappa_{d_1}\kappa_{d_2}\}\}$;

(3) $\tau d = \bigcup_{\substack{\sigma_d \in r_d \\ \kappa_d \in s_d}} \{1 - (1 - \sigma_d)^\tau, (\kappa_d^\tau)\}$;

(4) $d^\tau = \bigcup_{\substack{\sigma_d \in r_d \\ \kappa_d \in s_d}} \{(\sigma_d)^\tau, 1 - (1 - \kappa_d)^\tau\}$.

here τ is non-negative integer and above results are also $DHFEs$.

Definition 5. ([22]). Let d_1=$\{r_{d_1}, s_{d_1}\}$ and d_2=$\{r_{d_2}, s_{d_2}\}$ be any two $DHFEs$, the score and accuracy function of $d_i(i$=$1, 2)$ can be defined as S_{d_i}=$(\frac{1}{\#r}) \sum_{\sigma \in r} \sigma$ $-(\frac{1}{\#s}) \sum_{\kappa \in s} \kappa$, and $P_{d_i} = (\frac{1}{\#r}) \sum_{\sigma \in r} \sigma + (\frac{1}{\#s}) \sum_{\kappa \in s} \kappa$ is the accuracy function of $d_i(i = 1, 2)$, where $\#r$ and $\#s$ are the number of elements in r and s, respectively, then:

(i) If $S_{d_1} > S_{d_2}$, this means, d_1 greater than d_2, denoted as $d_1 \succ d_2$;

(ii) If $S_{d_1} = S_{d_2}$, then find accuracy function;

(1) If $P_{d_1} = P_{d_2}$, then d_1 and d_2 are equal, denoted as $d_1 = d_2$;

(2) If $P_{d_1} > P_{d_2}$, then d_1 is greater then d_2, written as $d_1 \succ d_2$.

3. Dual hesitant fuzzy Bonferroni mean

Firstly, we determine the relationship between the operations of $DHFEs$ listed in Theorem 1:

Theorem 1. Suppose d_1 and d_2 be two $DHFEs$, and $\tau \geq 0$, then

(1) $d_1 \oplus d_2 = d_2 \oplus d_1$;

(2) $d_1 \otimes d_2 = d_2 \otimes d_1$;

(3) $(d_1 \otimes d_2)^\tau = d_1^\tau \otimes d_2^\tau$;

(4) $\tau(d_1 \oplus d_2) = \tau d_1 \oplus \tau d_2$.

Proof. Properties (1) and (2) are evident from Definition (4), we just produce the verification of properties (3) and (4)

(3) $(d_1 \otimes d_2)^\tau = (\bigcup_{\substack{\sigma_{d_1}\in r_{d_1}\\ \kappa_{d_1}\in s_{d_1}\\ \sigma_{d_2}\in r_{d_2}\\ \kappa_{d_2}\in s_{d_2}}} \{\{\sigma_{d_1}\sigma_{d_2}\}, \{\kappa_{d_1}+\kappa_{d_2}-\kappa_{d_1}\kappa_{d_2}\}\})^\tau$

$= \bigcup_{\substack{\sigma_{d_1}\in r_{d_1}\\ \kappa_{d_1}\in s_{d_1}\\ \sigma_{d_2}\in r_{d_2}\\ \kappa_{d_2}\in s_{d_2}}} \{\{\sigma_{d_1}\sigma_{d_2}\}, \{\kappa_{d_1}+\kappa_{d_2}-\kappa_{d_1}\kappa_{d_2}\}\}^\tau$

$= \bigcup_{\substack{\sigma_{d_1}\in r_{d_1}\\ \kappa_{d_1}\in s_{d_1}\\ \sigma_{d_2}\in r_{d_2}\\ \kappa_{d_2}\in s_{d_2}}} \{\{\sigma_{d_1}\sigma_{d_2}\}^\tau, \{\kappa_{d_1}+\kappa_{d_2}-\kappa_{d_1}\kappa_{d_2}\}^\tau\}$

$= \bigcup_{\substack{\sigma_{d_1}\in r_{d_1}\\ \kappa_{d_1}\in s_{d_1}\\ \sigma_{d_2}\in r_{d_2}\\ \kappa_{d_2}\in s_{d_2}}} \{\{\sigma_{d_1}^\tau\sigma_{d_2}^\tau\}, \{\kappa_{d_1}+\kappa_{d_2}-\kappa_{d_1}\kappa_{d_2}\}^\tau\}$

$= \bigcup_{\substack{\sigma_{d_1}\in r_{d_1}\\ \kappa_{d_1}\in s_{d_1}\\ \sigma_{d_2}\in r_{d_2}\\ \kappa_{d_2}\in s_{d_2}}} \{\{\sigma_{d_1}^\tau\sigma_{d_2}^\tau\}, \{1-(1-\kappa_{d_1}-\kappa_{d_2}+\kappa_{d_1}\kappa_{d_2})^\tau\}\}$

$= \bigcup_{\substack{\sigma_{d_1}\in r_{d_1}\\ \kappa_{d_1}\in s_{d_1}\\ \sigma_{d_2}\in r_{d_2}\\ \kappa_{d_2}\in s_{d_2}}} \{\{\sigma_{d_1}^\tau\sigma_{d_2}^\tau\}, \{1-(1-\kappa_{d_1})^\tau(1-\kappa_{d_2})^\tau\}\}$

and

$d_1^\tau \otimes d_2^\tau = \bigcup_{\substack{\sigma_{d_1}\in r_{d_1}\\ \kappa_{d_1}\in s_{d_1}}} \{(\sigma_{d_1}^\tau), 1-(1-\kappa_{d_1})^\tau\} \otimes \bigcup_{\substack{\sigma_{d_2}\in r_{d_2}\\ \kappa_{d_2}\in s_{d_2}}} \{(\sigma_{d_2}^\tau), 1-(1-\kappa_{d_2})^\tau\}$

$= \bigcup_{\substack{\sigma_{d_1}\in r_{d_1}\\ \kappa_{d_1}\in s_{d_1}\\ \sigma_{d_2}\in r_{d_2}\\ \kappa_{d_2}\in s_{d_2}}} \{\{\sigma_{d_1}^\tau\sigma_{d_2}^\tau\}, \{(1-(1-\kappa_{d_1})^\tau)+(1-(1-\kappa_{d_2})^\tau)-(1-(1-\kappa_{d_1})^\tau)(1-$

$(1-\kappa_{d_2})^\tau)\}\} = \bigcup_{\substack{\sigma_{d_1}\in r_{d_1}\\ \kappa_{d_1}\in s_{d_1}\\ \sigma_{d_2}\in r_{d_2}\\ \kappa_{d_2}\in s_{d_2}}} \{\{\sigma_{d_1}^\tau\sigma_{d_2}^\tau\}, \{1-(1-\kappa_{d_1})^\tau(1-\kappa_{d_2})^\tau\}\}.$

Therefore, we have $(d_1 \otimes d_2)^\tau = d_1^\tau \otimes d_2^\tau$.

(4) Since

$\tau(d_1 \oplus d_2) = \tau(\bigcup_{\substack{\sigma_{d_1}\in r_{d_1}\\ \kappa_{d_1}\in s_{d_1}\\ \sigma_{d_2}\in r_{d_2}\\ \kappa_{d_2}\in s_{d_2}}} \{\{\sigma_{d_1}+\sigma_{d_2}-\sigma_{d_1}\sigma_{d_2}\}, \{\kappa_{d_1}\kappa_{d_2}\}\})$

$= \bigcup_{\substack{\sigma_{d_1}\in r_{d_1}\\ \kappa_{d_1}\in s_{d_1}\\ \sigma_{d_2}\in r_{d_2}\\ \kappa_{d_2}\in s_{d_2}}} \{1-(1-\sigma_{d_1}-\sigma_{d_2}+\sigma_{d_1}\sigma_{d_2})^\tau, (\kappa_{d_1}\kappa_{d_2})^\tau\}$

$= \{1-(1-\sigma_{d_1})^\tau(1-\sigma_{d_2})^\tau, (\kappa_{d_1})^\tau(\kappa_{d_2})^\tau\}$

and

$\tau d_1 \oplus \tau d_2$

$= \bigcup_{\substack{\sigma_{d_1}\in r_{d_1}\\ \kappa_{d_1}\in s_{d_1}\\ \sigma_{d_2}\in r_{d_2}\\ \kappa_{d_2}\in s_{d_2}}} \{1-(1-\sigma_{d_1})^\tau+1-(1-\sigma_{d_2})^\tau-(1-(1-\sigma_{d_1})^\tau)(1-(1-\sigma_{d_2})^\tau), (\kappa_{d_1}\kappa_{d_2})^\tau\}$

$= \bigcup_{\substack{\sigma_{d_1}\in r_{d_1}\\ \kappa_{d_1}\in s_{d_1}\\ \sigma_{d_2}\in r_{d_2}\\ \kappa_{d_2}\in s_{d_2}}} \{1-(1-\sigma_{d_1})^\tau(1-\sigma_{d_2})^\tau, (\kappa_{d_1})^\tau(\kappa_{d_2})^\tau\}$

Therefore, we have $\tau(d_1 \oplus d_2) = \tau d_1 \oplus \tau d_2$.

With the help of Definition 1, we have combined the BM and dual hesitant fuzzy information expressed as $DHFEs$ and constructed the $DHFBM$ as below:

Definition 6. Suppose $d_i = (r_{d_i}, s_{d_i})(i = 1, 2, \ldots, \tau)$ be a family of $DHFEs$. For any $\rho, \ell > 0$, if

$$DHFB^{\rho,\ell}(d_1, d_2, \ldots, d_\tau) = \left(\frac{1}{\tau(\tau-1)} \left(\bigoplus_{\substack{i,j=1 \\ i \neq j}}^{\tau} (d_i^\rho \otimes d_j^\ell) \right) \right)^{\frac{1}{\rho+\ell}}, \tag{2}$$

then we can call $DHFB^{\rho,\ell}$ as $DHFBM$.

Using the Theorem 1 and Definition 6, we have the following important result:

Theorem 2. *Suppose $\rho, \ell > 0$, $d_i = (r_{d_i}, s_{d_i})(i = 1, 2, \ldots, \tau)$ be a family of $DHFEs$, then by applying $DHFBM$, their aggregated value is a $DHFE$, and*

$$DHFB^{\rho,\ell}(d_1, d_2, \ldots, d_\tau) = \bigcup_{\substack{\sigma_{d_i} \in r_{d_i} \\ \kappa_{d_i} \in s_{d_i} \\ \sigma_{d_j} \in r_{d_j} \\ \kappa_{d_j} \in s_{d_j} \\ i \neq j}} \left\{ \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{\tau} (1 - \sigma_{d_i}^\rho \sigma_{d_j}^\ell)^{\frac{1}{\tau(\tau-1)}} \right)^{\frac{1}{\rho+\ell}}, \right.$$

$$\left. 1 - \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{\tau} (1 - (1 - \kappa_{d_i})^\rho (1 - \kappa_{d_j})^\ell)^{\frac{1}{\tau(\tau-1)}} \right)^{\frac{1}{\rho+\ell}} \right\}. \tag{3}$$

Proof. Applying operations (2)-(4) and as discussed in Definition 4 and Theorem 1, we have

$d_i^\rho = \bigcup_{\substack{\sigma_{d_i} \in r_{d_i} \\ \kappa_{d_i} \in s_{d_i}}} \{\sigma_{d_i}^\rho, 1 - (1 - \kappa_{d_i})^\ell\}$,

$d_j^\ell = \bigcup_{\substack{\sigma_{d_j} \in r_{d_j} \\ \kappa_{d_j} \in s_{d_j}}} \{\sigma_{d_j}^\ell, 1 - (1 - \kappa_{d_j})^\ell\}$,

$$d_i^\rho \otimes d_j^\ell = \bigcup_{\substack{\sigma_{d_i} \in r_{d_i} \\ \kappa_{d_i} \in s_{d_i} \\ \sigma_{d_j} \in r_{d_j} \\ \kappa_{d_j} \in s_{d_j}}} \left\{ \sigma_{d_i}^\rho \sigma_{d_j}^\ell, 1 - (1 - \kappa_{d_i})^\rho (1 - \kappa_{d_j})^\ell \right\}. \tag{4}$$

We first prove the following expression

$$\bigoplus_{\substack{i,j=1 \\ i \neq j}}^{\tau} (d_i^\rho \otimes d_j^\ell) = \bigcup_{\substack{\sigma_{d_i} \in r_{d_i} \\ \kappa_{d_i} \in s_{d_i} \\ \sigma_{d_j} \in r_{d_j} \\ \kappa_{d_j} \in s_{d_j}}} \left\{ 1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{\tau} (1 - \sigma_{d_i}^\rho \sigma_{d_j}^\ell), \prod_{\substack{i,j=1 \\ i \neq j}}^{\tau} (1 - (1 - \kappa_{d_i})^\rho (1 - \kappa_{d_j})^\ell) \right\} \tag{5}$$

by applying induction method on τ as follows:

Step 1. When $\tau = 2$, we have

$$\bigoplus_{\substack{i,j=1\\ i\neq j}}^{2}(d_i^{\rho}\otimes d_j^{\ell}) = (d_1^{\rho}\otimes d_2^{\ell})\oplus(d_2^{\rho}\otimes d_1^{\ell}) = \bigcup_{\substack{\sigma_{d_1}\in r_{d_1}\\ \kappa_{d_1}\in s_{d_1}\\ \sigma_{d_2}\in r_{d_2}\\ \kappa_{d_2}\in s_{d_2}}}\Big\{1-(1-\sigma_{d_1}^{\rho}\sigma_{d_2}^{\ell})(1-\sigma_{d_2}^{\rho}\sigma_{d_1}^{\ell}),$$
$$(1-(1-\kappa_{d_1})^{\rho}(1-\kappa_{d_2})^{\ell})\times(1-(1-\kappa_{d_2})^{\rho}(1-\kappa_{d_1})^{\ell})\Big\}.$$

Step 2. Suppose Eq. 5 is true for $\tau = k$, i.e.,

$$\bigoplus_{\substack{i,j=1\\ i\neq j}}^{k}(d_i^{\rho}\otimes d_j^{\ell}) = \bigcup_{\substack{\sigma_{d_i}\in r_{d_i}\\ \kappa_{d_i}\in s_{d_i}\\ \sigma_{d_j}\in r_{d_j}\\ \kappa_{d_j}\in s_{d_j}}}\Big\{1-\prod_{\substack{i,j=1\\ i\neq j}}^{k}(1-\sigma_{d_i}^{\rho}\sigma_{d_j}^{\ell}),\ \prod_{\substack{i,j=1\\ i\neq j}}^{k}(1-(1-\kappa_{d_i})^{\rho}(1-\kappa_{d_j})^{\ell})\Big\} \tag{6}$$

then, if $\tau = k+1$, and using the operations $(1)-(4)$ as defined in Definition 4, we have

$$\bigoplus_{\substack{i,j=1\\ i\neq j}}^{k+1}(d_i^{\rho}\otimes d_j^{\ell}) = \Big(\bigoplus_{\substack{i,j=1\\ i\neq j}}^{k}(d_i^{\rho}\otimes d_j^{\ell})\Big)\oplus\Big(\bigoplus_{i=1}^{k}(d_i^{\rho}\otimes d_{k+1}^{\ell})\Big)$$
$$\oplus\Big(\bigoplus_{j=1}^{k}(d_{k+1}^{\rho}\otimes d_j^{\ell})\Big). \tag{7}$$

After applying induction on k and some simple calculations we can get

$$\bigoplus_{i=1}^{k}(d_i^{\rho}\otimes d_{k+1}^{\ell})$$
$$= \bigcup_{\substack{\sigma_{d_i}\in r_{d_i}\\ \kappa_{d_i}\in s_{d_i}\\ \sigma_{d_{k+1}}\in r_{d_{k+1}}\\ \kappa_{d_{k+1}}\in s_{d_{k+1}}}}\Big\{1-\prod_{i=1}^{k}(1-\sigma_{d_i}^{\rho}\sigma_{d_{k+1}}^{\ell}),\prod_{i=1}^{k}(1-(1-\kappa_{d_i})^{\rho}(1-\kappa_{d_{k+1}})^{\ell})\Big\}. \tag{8}$$

In the same way, we can have

$$\bigoplus_{j=1}^{k}(d_{k+1}^{\rho}\otimes d_j^{\ell})$$

$$= \bigcup_{\substack{\sigma_{d_{k+1}}\in r_{d_{k+1}}\\ \kappa_{d_{k+1}}\in s_{d_{k+1}}\\ \sigma_{d_j}\in r_{d_j}\\ \kappa_{d_j}\in s_{d_j}}} \left\{1-\prod_{j=1}^{k}(1-\sigma_{d_{k+1}}^{\rho}\sigma_{d_j}^{\ell}), \prod_{j=1}^{k}(1-(1-\kappa_{d_{k+1}})^{\rho}(1-\kappa_{d_j})^{\ell})\right\}. \tag{9}$$

Thus, from Eq. 6, Eq. 8, and Eq. 9, we can further transform Eq. 9 as

$$\begin{aligned}
&\bigoplus_{\substack{i,j=1\\ i\neq j}}^{k+1}(d_i^{\rho}\otimes d_j^{\ell}) = \bigcup_{\substack{\sigma_{d_i}\in r_{d_i}\\ \kappa_{d_i}\in s_{d_i}\\ \sigma_{d_j}\in r_{d_j}\\ \kappa_{d_j}\in s_{d_j}}} \left\{1-\prod_{\substack{i,j=1\\ i\neq j}}^{k}(1-\sigma_{d_i}^{\rho}\sigma_{d_j}^{\ell}),\right.\\
&\left.\prod_{\substack{i,j=1\\ i\neq j}}^{k}(1-(1-\kappa_{d_i})^{\rho}(1-\kappa_{d_j})^{\ell})\right\}\\
&\oplus \bigcup_{\substack{\sigma_{d_i}\in r_{d_i}\\ \kappa_{d_i}\in s_{d_i}\\ \sigma_{d_{k+1}}\in r_{d_{k+1}}\\ \kappa_{d_{k+1}}\in s_{d_{k+1}}}} \left\{1-\prod_{i=1}^{k}(1-\sigma_{d_i}^{\rho}\sigma_{d_{k+1}}^{\ell}), \prod_{i=1}^{k}(1-(1-\kappa_{d_i})^{\rho}(1-\kappa_{d_{k+1}})^{\ell})\right\}\\
&\oplus \bigcup_{\substack{\sigma_{d_{k+1}}\in r_{d_{k+1}}\\ \kappa_{d_{k+1}}\in s_{d_{k+1}}\\ \sigma_{d_j}\in r_{d_j}\\ \kappa_{d_j}\in s_{d_j}}} \left\{1-\prod_{j=1}^{k}(1-\sigma_{d_{k+1}}^{\rho}\sigma_{d_j}^{\ell}), \prod_{j=1}^{k}(1-(1-\kappa_{d_{k+1}})^{\rho}(1-\kappa_{d_j})^{\ell})\right\}\\
&= \bigcup_{\substack{\sigma_{d_i}\in r_{d_i}\\ \kappa_{d_i}\in s_{d_i}\\ \sigma_{d_j}\in r_{d_j}\\ \kappa_{d_j}\in s_{d_j}}} \left\{1-\prod_{\substack{i,j=1\\ i\neq j}}^{k+1}(1-\sigma_{d_i}^{\rho}\sigma_{d_j}^{\ell}), \prod_{\substack{i,j=1\\ i\neq j}}^{k+1}(1-(1-\kappa_{d_i})^{\rho}(1-\kappa_{d_j})^{\ell})\right\}.
\end{aligned}$$

This implies that, Eq. 5 is true for $\tau = k+1$. Therefore, Eq. 5 is true for all τ. Then, combining Eq. 5 and operation (3), we can have

$$\begin{aligned}
&\frac{1}{\tau(\tau-1)}\left(\bigoplus_{\substack{i,j=1\\ i\neq j}}^{\tau}(d_i^{\rho}\otimes d_j^{\ell})\right)\\
&= \bigcup_{\substack{\sigma_{d_i}\in r_{d_i}\\ \kappa_{d_i}\in s_{d_i}\\ \sigma_{d_j}\in r_{d_j}\\ \kappa_{d_j}\in s_{d_j}}} \left\{1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-\sigma_{d_i}^{\rho}\sigma_{d_j}^{\ell})\right)^{\frac{1}{\tau(\tau-1)}}, \prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-(1-\kappa_{d_i})^{\rho}(1-\kappa_{d_j})^{\ell})^{\frac{1}{\tau(\tau-1)}}\right\},
\end{aligned} \tag{10}$$

when we apply operation (4) on Eq. 11, it yields the form

$$DHFB^{\rho,\ell}(d_1, d_2 \ldots, d_\tau) = \left(\frac{1}{\tau(\tau-1)}\left(\bigoplus_{\substack{i,j=1\\ i\neq j}}^{\tau}(d_i^\rho \otimes d_j^\ell)\right)\right)^{\frac{1}{\rho+\ell}}$$

$$= \bigcup_{\substack{\sigma_{d_i}\in r_{d_i}\\ \kappa_{d_i}\in s_{d_i}\\ \sigma_{d_j}\in r_{d_j}\\ \kappa_{d_j}\in s_{d_j}}} \left\{\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-\sigma_{d_i}^\rho\sigma_{d_j}^\ell)^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}}, \right. \tag{11}$$

$$\left. 1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-(1-\kappa_{d_i})^\rho(1-\kappa_{d_j})^\ell)^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}}\right\}.$$

In addition, since

$$0 \leq \left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-\sigma_{d_i}^\rho\sigma_{d_j}^\ell)^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}} \leq 1,$$

$$0 \leq 1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-(1-\kappa_{d_i})^\rho(1-\kappa_{d_j})^\ell)^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}} \leq 1$$

from Definition 2, we have

$$\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-\sigma_{d_i}^\rho\sigma_{d_j}^\ell)^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}} + 1$$

$$-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-(1-\kappa_{d_i})^\rho(1-\kappa_{d_j})^\ell)^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}}$$

$$\leq 1+\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-(1-\kappa_{d_i})^\rho(1-\kappa_{d_j})^\ell)^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}}$$

$$-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-(1-\kappa_{d_i})^\rho(1-\kappa_{d_j})^\ell)^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}} = 1$$

hence, the proof completes.

Now, further we explore some important properties of $DHFBM$.

(1) **Idempotency:** If all d_i's $(i = 1, 2, \ldots, \tau)$ are same, i.e., $d_i = d = (r_d, s_d)$,

for all i, then

$$
\begin{aligned}
&DHFB^{\rho,\ell}(d_1, d_2 \ldots, d_\tau) = DHFB^{\rho,\ell}(d, d, \ldots, d)\\
&= \bigcup_{\substack{\sigma_{d_i}\in r_{d_i}\\ \kappa_{d_i}\in s_{d_i}}} \left\{ \left(1 - \prod_{\substack{i,j=1\\ i\neq j}}^{\tau} (1 - \sigma_{d_i}^{\rho}\sigma_{d_i}^{\ell})^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}},\right.\\
&\left. 1 - \left(1 - \prod_{\substack{i,j=1\\ i\neq j}}^{\tau} (1 - (1-\kappa_{d_i})^{\rho}(1-\kappa_{d_i})^{\ell})^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}} \right\}\\
&= \bigcup_{\substack{\sigma_{d_i}\in r_{d_i}\\ \kappa_{d_i}\in s_{d_i}}} \left\{ (1 - (1 - \sigma_{d_i}^{\rho+\ell}))^{\frac{1}{\rho+\ell}}, 1 - (1 - (1 - (1-\kappa_{d_i})^{\frac{1}{\rho+\ell}}))^{\frac{1}{\rho+\ell}} \right\}\\
&= \bigcup_{\substack{\sigma_{d_i}\in r_{d_i}\\ \kappa_{d_i}\in s_{d_i}}} \left\{ (\sigma_{d_i}^{\rho+\ell})^{\frac{1}{\rho+\ell}}, 1 - ((1-\kappa_{d_i})^{\rho+\ell})^{\frac{1}{\rho+\ell}} \right\}\\
&= \bigcup_{\substack{\sigma_{d_i}\in r_{d_i}\\ \kappa_{d_i}\in s_{d_i}}} \{\sigma_{d_i}, \kappa_{d_i}\}\\
&= \{h_d, g_g\} = d.
\end{aligned}
$$

(2) **Monotonicity:** Let $d_i = (r_{d_i}, s_{d_i})(i = 1, 2, \ldots, \tau)$ and $\tilde{d} = (r_{\tilde{d}_i}, s_{\tilde{d}_i})(i = 1, 2, \ldots, \tau)$ be two accumulations of $DHFNs$. If for any $\sigma_{d_i} \in r_{d_i}, \kappa_{d_i} \in s_{d_i}, \sigma_{d_j} \in r_{d_j}, \kappa_{d_j} \in s_{d_j}$, we have $\sigma_{d_i} \leq \sigma_{\tilde{d}_i}$ and $\kappa_{d_i} \geq \kappa_{\tilde{d}_i}$, for all i, then

$$DHFB^{\rho,\ell}(d_1, d_2 \ldots, d_\tau) \leq DHFB^{\rho,\ell}(\tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_\tau).$$

Proof. Since $\sigma_{d_i} \leq \sigma_{\tilde{d}_i}$ and $\kappa_{d_i} \geq \kappa_{\tilde{d}_i}$, for all i, then $\sigma_{d_i}\sigma_{d_j} \leq \sigma_{\tilde{d}_i}\sigma_{\tilde{d}_j}$, for all i, j

$$
\begin{aligned}
&\prod_{\substack{i,j=1\\ i\neq j}}^{\tau} (1 - \sigma_{d_i}\sigma_{d_j})^{\frac{1}{\tau(\tau-1)}} \geq \prod_{\substack{i,j=1\\ i\neq j}}^{\tau} (1 - \sigma_{\tilde{d}_i}\sigma_{\tilde{d}_j})^{\frac{1}{\tau(\tau-1)}}\\
&1 - \prod_{\substack{i,j=1\\ i\neq j}}^{\tau} (1 - \sigma_{d_i}\sigma_{d_j})^{\frac{1}{\tau(\tau-1)}} \leq 1 - \prod_{\substack{i,j=1\\ i\neq j}}^{\tau} (1 - \sigma_{\tilde{d}_i}\sigma_{\tilde{d}_j})^{\frac{1}{\tau(\tau-1)}}\\
&\left(1 - \prod_{\substack{i,j=1\\ i\neq j}}^{\tau} (1 - \sigma_{d_i}\sigma_{d_j})^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}} \leq \left(1 - \prod_{\substack{i,j=1\\ i\neq j}}^{\tau} (1 - \sigma_{\tilde{d}_i}\sigma_{\tilde{d}_j})^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}}
\end{aligned}
$$

similarly, for non membership function, if $\kappa_{d_i} \geq \kappa_{\tilde{d}_i}$, for all i, then

$$(1 - \kappa_{d_i})^{\rho}(1 - \kappa_{d_j})^{\ell} \leq (1 - \kappa_{\tilde{d}_i})^{\rho}(1 - \kappa_{\tilde{d}_j})^{\ell}$$

$$\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-(1-\kappa_{d_i})^{\rho}(1-\kappa_{d_j})^{\ell})^{\frac{1}{\tau(\tau-1)}} \geq \prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-(1-\kappa_{\tilde{d}_i})^{\rho}(1-\kappa_{\tilde{d}_j})^{\ell})^{\frac{1}{\tau(\tau-1)}}$$

$$\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-(1-\kappa_{d_i})^{\rho}(1-\kappa_{d_j})^{\ell})^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}}$$

$$\leq \left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-(1-\kappa_{\tilde{d}_i})^{\rho}(1-\kappa_{\tilde{d}_j})^{\ell})^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}}$$

$$1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-(1-\kappa_{d_i})^{\rho}(1-\kappa_{d_j})^{\ell})^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}} \tag{12}$$

$$\geq 1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-(1-\kappa_{\tilde{d}_i})^{\rho}(1-\kappa_{\tilde{d}_j})^{\ell})^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}}$$

$$\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-\sigma_{d_i}\sigma_{d_j})^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}}$$

$$-\left(1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-(1-\kappa_{d_i})^{\rho}(1-\kappa_{d_j})^{\ell})^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}}\right)$$

$$\leq \left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-\sigma_{\tilde{d}_i}\sigma_{\tilde{d}_j})^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}}$$

$$-\left(1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-(1-\kappa_{\tilde{d}_i})^{\rho}(1-\kappa_{\tilde{d}_j})^{\ell})^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}}\right).$$

Let $d = DHFB^{\rho,\ell}(d_1, d_2, \ldots, d_\tau)$ and $\tilde{d} = DHFB^{\rho,\ell}(\tilde{d_1}, \tilde{d_2}, \ldots, \tilde{d_\tau})$, and let S_d and $S_{\tilde{d}}$ be the scores of d and $\tilde{d}$, respectively. Then, inequality (3.12) says $S_d \leq S_{\tilde{d}}$. Presently, the accompanying cases emerge.

Case 1. If $S_d < S_{\tilde{d}}$, then, we have from Definition 3

$$DHFB^{\rho,\ell}(d_1, d_2, \ldots, d_\tau) < DHFB^{\rho,\ell}(\tilde{d_1}, \tilde{d_2}, \ldots, \tilde{d_\tau}). \tag{13}$$

Case 2. If $S_d = S_{\tilde{d}}$, then

$$\begin{aligned}
&\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-\sigma_{d_i}\sigma_{d_j})^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}}\\
&-\left(1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-(1-\kappa_{d_i})^{\rho}(1-\kappa_{d_j})^{\ell})^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}}\right)\\
&=\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-\sigma_{\tilde{d}_i}\sigma_{\tilde{d}_j})^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}}\\
&-\left(1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-(1-\kappa_{\tilde{d}_i})^{\rho}(1-\kappa_{\tilde{d}_j})^{\ell})^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}}\right).
\end{aligned}$$

Since $\sigma_{d_i} \leq \sigma_{\tilde{d}_i}$ and $\kappa_{d_i} \geq \kappa_{\tilde{d}_i}$, for all i, then

$$\begin{aligned}
&\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-\sigma_{d_i}\sigma_{d_j})^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}}=\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-\sigma_{\tilde{d}_i}\sigma_{\tilde{d}_j})^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}}\\
&1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-(1-\kappa_{d_i})^{\rho}(1-\kappa_{d_j})^{\ell})^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}}\\
&=1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-(1-\kappa_{\tilde{d}_i})^{\rho}(1-\kappa_{\tilde{d}_i})^{\ell})^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}}.
\end{aligned}$$

and thus

$$\begin{aligned}
P_d&=\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-\sigma_{d_i}\sigma_{d_j})^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}}\\
&+\left(1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-(1-\kappa_{d_i})^{\rho}(1-\kappa_{d_j})^{\ell})^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}}\right)\\
&=\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-\sigma_{\tilde{d}_i}\sigma_{\tilde{d}_j})^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}}
\end{aligned}$$

$$+\left(1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-(1-\kappa_{\tilde{d}_i})^{\rho}(1-\kappa_{\tilde{d}_j})^{\ell})^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}}\right)=P_{\tilde{d}}.$$

Then, by Definition 3, we get

$$DHFB^{\rho,\ell}(d_1,d_2,\ldots,d_\tau)=DHFB^{\rho,\ell}(\tilde{d_1},\tilde{d_2},\ldots,\tilde{d_\tau}) \tag{14}$$

and hence, Eq. 13 and Eq. 14 guarantees the Monotonocity.

(3) **Commutativity:** Let $d_i=(r_{d_i},s_{d_i})(i=1,2,\ldots,\tau)$ be a family of $DHFNs$. Then $DHFB^{\rho,\ell}(d_1,d_2,\ldots,d_\tau)=DHFB^{\rho,\ell}(\dot{d_1},\dot{d_2},\ldots,\dot{d_\tau})$ where $(\dot{d_1},\dot{d_2},\ldots,\dot{d_\tau})$ is any alteration of $(d_1,d_2,\ldots,d_\tau)$.

Proof. Since $(\dot{d_1},\dot{d_2},\ldots,\dot{d_\tau})$ is any alteration of $(d_1,d_2,\ldots,d_\tau)$, then

$$\begin{aligned}DHFB^{\rho,\ell}(d_1,d_2,\ldots,d_\tau)&=\left(\frac{1}{\tau(\tau-1)}\left(\bigoplus_{\substack{i,j=1\\ i\neq j}}^{\tau}(d_i^{\rho}\otimes d_j^{\ell})\right)\right)^{\frac{1}{\rho+\ell}}\\&=\left(\frac{1}{\tau(\tau-1)}\left(\bigoplus_{\substack{i,j=1\\ i\neq j}}^{\tau}(\dot{d_i}^{\rho}\otimes \dot{d_j}^{\ell})\right)\right)^{\frac{1}{\rho+\ell}}\\&=DHFB^{\rho,\ell}(\dot{d_1},\dot{d_2},\ldots,\dot{d_\tau}).\end{aligned}$$

(4) **Boundedness:** Let $d_i=(r_{d_i},s_{d_i})$ $(i=1,2,\ldots,\tau)$ be a family of $DHFNs$, and let

$$\begin{aligned}d^-&=(\min_i\{\sigma_{d_i}\},\max_i\{\kappa_{d_i}\}),\\d^+&=(\max_i\{\sigma_{d_i}\},\min_i\{\kappa_{d_i}\}).\end{aligned}$$

Then

$$d^-\leq DHFB^{\rho,\ell}(d_1,d_2,\ldots,d_\tau)\leq d^+. \tag{15}$$

Proof. Since $\min_i\{\sigma_{d_i}\}\leq\sigma_{d_i}\leq\max_i\{\sigma_{d_i}\}$ and $\min_i\{\kappa_{d_i}\}\leq\kappa_{d_i}\leq\max_i\{\kappa_{d_i}\}$, for all i, then

$$\begin{aligned}&(\min_i\{\sigma_{d_i}\})^{\rho+\ell}\leq\sigma_{d_i}^{\rho}\sigma_{d_j}^{\ell}\leq(\max_i\{\sigma_{d_i}\})^{\rho+\ell}\\&\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-\sigma_{d_i}^{\rho}\sigma_{d_j}^{\ell})^{\frac{1}{\tau(\tau-1)}}\leq\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-(\min_i\{\sigma_{d_i}\})^{\rho+\ell})^{\frac{1}{\tau(\tau-1)}}=1-(\min_i\{\sigma_{d_i}\})^{\rho+\ell}\\&\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-\sigma_{d_i}^{\rho}\sigma_{d_j}^{\ell})^{\frac{1}{\tau(\tau-1)}}\geq\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-(\max_i\{\sigma_{d_i}\})^{\rho+\ell})^{\frac{1}{\tau(\tau-1)}}=1-(\max_i\{\sigma_{d_i}\})^{\rho+\ell}\end{aligned}$$

and thus

$$\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-\sigma_{d_i}^{\rho}\sigma_{d_j}^{\ell})^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}}$$

$$\leq \left(1-\left(1-(\max_i\{\sigma_{d_i}\})^{\rho+\ell}\right)\right)^{\frac{1}{\rho+\ell}}=\max_i\{\sigma_{d_i}\}, \tag{16}$$

$$\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-\sigma_{d_i}^{\rho}\sigma_{d_j}^{\ell})^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}}$$

$$\geq \left(1-\left(1-(\min_i\{\sigma_{d_i}\})^{\rho+\ell}\right)\right)^{\frac{1}{\rho+\ell}}=\min_i\{\sigma_{d_i}\}. \tag{17}$$

Similarly, for non-membership function, we can have

$$(1-\max_i\{\kappa_{d_i}\})^{\rho+\ell}\leq(1-\kappa_{d_i})^{\rho}(1-\kappa_{d_j})^{\ell}\leq(1-\min_i\{\kappa_{d_i}\})^{\rho+\ell}$$

$$\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-(1-\kappa_{d_i})^{\rho}(1-\kappa_{d_j})^{\ell})^{\frac{1}{\tau(\tau-1)}}$$

$$\geq \prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-(1-\min_i\{\kappa_{d_i}\})^{\rho+\ell})^{\frac{1}{\tau(\tau-1)}}=1-(1-\min_i\{\kappa_{d_i}\})^{\rho+\ell}$$

$$\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-(1-\kappa_{d_i})^{\rho}(1-\kappa_{d_j})^{\ell})^{\frac{1}{\tau(\tau-1)}}$$

$$\leq \prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-(1-\max_i\{\kappa_{d_i}\})^{\rho+\ell})^{\frac{1}{\tau(\tau-1)}}=1-(1-\max_i\{\kappa_{d_i}\})^{\rho+\ell}$$

thus

$$1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-(1-\kappa_{d_i})^{\rho}(1-\kappa_{d_j})^{\ell})^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}}$$

$$\geq 1-(1-(1-(1-\min_i\{\kappa_{d_i}\}))^{\rho+\ell})^{\frac{1}{\rho+\ell}}=\min_i\{\kappa_{d_i}\}, \tag{18}$$

$$1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-(1-\kappa_{d_i})^{\rho}(1-\kappa_{d_j})^{\ell})^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}}$$

$$\leq 1-(1-(1-(1-\max_i\{\kappa_{d_i}\}))^{\rho+\ell})^{\frac{1}{\rho+\ell}}=\max_i\{\kappa_{d_i}\}. \tag{19}$$

Let $d = DHFB_{\rho,\ell}(d_1, d_2, \ldots, d_\tau) = (r_d, s_d)$. Then

$$S_d = \frac{1}{\#r}\sum_{\sigma\in r}\sigma - \frac{1}{\#s}\sum_{\kappa\in s}\kappa \leq \max_i\{\sigma_{d_i}\} - \min_i\{\kappa_{d_i}\} = S_d^+, \tag{20}$$

$$S_d = \frac{1}{\#r}\sum_{\sigma\in r}\sigma - \frac{1}{\#s}\sum_{\kappa\in s}\kappa \geq \min_i\{\sigma_{d_i}\} - \max_i\{\kappa_{d_i}\} = S_d^-. \tag{21}$$

As a result, following of the three cases arise.

Case 1. If $S_d < S_d^+$ and $S_d > S_d^-$, then, from Definition 3, this implies that $d^- < DHFB^{\rho,\ell}(d_1, d_2, \ldots, d_\tau) < d^+$.

Case 2. If $S_d = S_d^+$, then, by Eq. 18 and Eq. 19, we have $\sigma_d = \max_i\{\sigma_{d_i}\}$, $\kappa_d = \min_i\{\kappa_{d_i}\}$. Thus

$$P_d = \frac{1}{\#r}\sum_{\sigma\in r}\sigma + \frac{1}{\#s}\sum_{\kappa\in s}\kappa = \max_i\{\sigma_{d_i}\} + \min_i\{\kappa_{d_i}\} = P_d^+,$$
$$DHFB^{\rho,\ell}(d_1, d_2, \ldots, d_\tau) = d^+.$$

Case 3. If $S_d = S_d^-$, then, by Eq. 20 and Eq. 21, we have $\sigma_d = \min_i\{\sigma_{d_i}\}, \kappa_d = \max_i\{\kappa_{d_i}\}$. Therefore, we can have

$$P_d = \frac{1}{\#r}\sum_{\sigma\in r}\sigma + \frac{1}{\#s}\sum_{\kappa\in s}\kappa = \min_i\{\sigma_{d_i}\} + \max_i\{\kappa_{d_i}\} = P_d^-,$$
$$DHFB^{\rho,\ell}(d_1, d_2, \ldots, d_\tau) = d^-.$$

Therefore, in all the aforementioned cases, Eq. 15 holds clearly.

For various estimations of the parameters ρ and ℓ some unique cases of the $DHFBM$ arises which are discussed as given below.

Case 1. If $\ell \to 0$, then, we have

$$\lim_{\ell\to 0} DHFB^{\rho,0}(d_1, d_2, \ldots, d_\tau) = \lim_{\ell\to 0}\left(\frac{1}{\tau}\left(\bigoplus_{\substack{i,j=1\\ i\neq j}}^{\tau}(d_i^\rho \otimes d_j^\ell)\right)\right)^{\frac{1}{\rho+\ell}} \tag{22}$$

$$= \bigcup_{\substack{\sigma_{d_i}\in r_{d_i}\\ \kappa_{d_i}\in s_{d_i}}}\left\{\left(1 - \prod_{i=1}^{\tau}(1-\sigma_{d_i}^\rho)^{\frac{1}{\tau}}\right)^{\frac{1}{\rho}}, 1 - \left(1 - \prod_{i=1}^{\tau}(1-(1-\kappa_{d_i})^\rho)^{\frac{1}{\tau}}\right)^{\frac{1}{\rho}}\right\}. \tag{23}$$

This is called generalized dual hesitant fuzzy mean ($GDHFM$).

Case 2. When $\rho = 2$ and $\ell \to 0$, we get dual hesitant fuzzy square mean $(DHFSM)$.

$$\lim_{\ell\to 0} DHFB^{2,0}(d_1, d_2, \ldots, d_\tau) = \left(\frac{1}{\tau}\left(\bigoplus_{i=1}^{\tau}(d_i^2 \otimes d_j^0)\right)\right)^{\frac{1}{2}} \tag{24}$$

$$= \bigcup_{\substack{\sigma_{d_i}\in r_{d_i}\\ \kappa_{d_i}\in s_{d_i}}} \left\{\left(1 - \prod_{i=1}^{\tau}(1-\sigma_{d_i}^2)^{\frac{1}{\tau}}\right)^{\frac{1}{2}}, 1 - \left(1 - \prod_{i=1}^{\tau}(1-(1-\kappa_{d_i})^2)^{\frac{1}{\tau}}\right)^{\frac{1}{2}}\right\}. \tag{25}$$

Case 3. If $\rho = 1$ and $\ell \to 0$, we get dual hesitant fuzzy averaging operator $(DHFA)$.

$$\lim_{\ell\to 0} DHFB^{1,0}(d_1, d_2, \ldots, d_\tau) = \frac{1}{\tau}\left(\bigoplus_{i=1}^{\tau}(d_i^1 \otimes d_j^0)\right) \tag{26}$$

$$= \bigcup_{\substack{\sigma_{d_i}\in r_{d_i}\\ \kappa_{d_i}\in s_{d_i}}} \left\{\left(1 - \prod_{i=1}^{\tau}(1-\sigma_{d_i})^{\frac{1}{\tau}}\right), \left(\prod_{i=1}^{\tau}(\kappa_{d_i})^{\frac{1}{\tau}}\right)\right\}. \tag{27}$$

Case 4. If $\rho = 1$ and $\ell = 1$, we get dual hesitant fuzzy interrelated square mean $(DHFISM)$.

$$DHFB^{1,1}(d_1, d_2, \ldots, d_\tau) = \left(\frac{1}{\tau}\left(\bigoplus_{i=1}^{\tau}(d_i^1 \otimes d_j^1)\right)\right)^{\frac{1}{2}}$$

$$= \bigcup_{\substack{\sigma_{d_i}\in r_{d_i}\\ \kappa_{d_i}\in s_{d_i}\\ \sigma_{d_j}\in r_{d_j}\\ \kappa_{d_j}\in s_{d_j}\\ i\neq j}} \left\{\left(1 - \prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-\sigma_{d_i}^1\sigma_{d_j}^1)^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{2}}, \right. \tag{28}$$

$$\left. 1 - \left(1 - \prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-(1-\kappa_{d_i})^1(1-\kappa_{d_j})^1)^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{2}}\right\}.$$

4. The WDHFBM and its application in MADM

In multiple attribute decision-making process, usually, the degree of importance of attributes may not be same, therefore they should be assigned different weights. To underscore the significance of collected information, we have constructed a $WDHFBM$ as given below:

Definition 5. Let $\rho, \ell > 0$, $d_i = (r_{d_i}, s_{d_i})(i = 1, 2, \ldots, \tau)$ be a family of $DHFEs$, $\omega = (\omega_1, \omega_2, \ldots, \omega_\tau)^T$ be the weight vector of $d_i = (i = 1, 2, \ldots, \tau)$,

where ω_i represents the importance degree of d_i, satisfying $\omega_i \in [0,1]$ $(i = 1,2,\ldots,\tau)$ and $\sum_{i=1}^{\tau} \omega_i = 1$, if

$$WDHFB_{\omega}(d_1, d_2, \ldots, d_\tau) = \left(\frac{1}{\tau(\tau-1)} \left(\bigoplus_{\substack{i,j=1 \\ i \neq j}}^{\tau} (\omega_i d_i)^{\rho} \otimes (\omega_j d_j)^{\ell}) \right) \right)^{\frac{1}{\rho+\ell}} \tag{29}$$

then we can say $DHFB_{\omega}^{\rho,\ell}$ a $WDHFBM$.

Using Eq. 29, Next Theorem is there:

Theorem 4. *Let $\rho, \ell > 0$, $d_i = (r_{d_i}, s_{d_i})$ $(i = 1,2,\ldots,\tau)$ be a family of $DHFEs$, and $\omega = (\omega_1, \omega_2, \ldots, \omega_\tau)^T$ be the weight vector of $d_i (i = 1,2,\ldots,\tau)$, satisfying $\omega_i \in [0,1]$ $(i = 1,2,\ldots,\tau)$ and $\sum_{i=1}^{\tau} \omega_i = 1$, then according to the $WDHFB$, the collected value of the d_i's also a $DHFN$,*

$$\begin{aligned} WDHFB^{\rho,\ell}(d_1, d_2, \ldots, d_\tau) = & \bigcup_{\substack{\sigma_{d_i} \in r_{d_i} \\ \kappa_{d_i} \in s_{d_i} \\ \sigma_{d_j} \in r_{d_j} \\ \kappa_{d_j} \in s_{d_j} \\ i \neq j}} \\ & \left\{ \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{\tau} (1 - (1 - (1 - \sigma_{d_i})^{\omega_i})^{\rho} \times (1 - (1 - \sigma_{d_j})^{\omega_j})^{\ell})^{\frac{1}{\tau(\tau-1)}} \right)^{\frac{1}{\rho+\ell}}, \right. \\ & \left. 1 - \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{\tau} (1 - (1 - \kappa_{d_i}^{\omega_i})^{\rho} (1 - \kappa_{d_j}^{\omega_j})^{\ell})^{\frac{1}{\tau(\tau-1)}} \right)^{\frac{1}{\rho+\ell}} \right\}. \end{aligned} \tag{30}$$

Now, for the usual application of weighted aggregation operators, we describe some basic steps for $MADM$ based on the $WDHFBM$.

Step 1. For a $MADM$ problem, let $Y = \{y_1, y_2, \ldots, y_n\}$ and $C = \{c_1, c_2, \ldots, c_m\}$ be a set of n alternatives with m attributes respectively, having weight vector $\omega = \{\omega_1, \omega_2, \ldots, \omega_m\}^T$, which satisfy the condition $\omega_j > 0, j = 1,2,\ldots,m$, and $\sum_{j=1}^{m} \omega_j = 1$, where ω_j expresses the importance degree of the attribute c_j. The decision maker measure the performance of the alternative y_i corresponding to the attribute c_j by the $DHFN$ $k_{ij} = (r_{ij}, s_{ij})$, $r_{ij} = \bigcup_{\sigma_{ij} \in r_{ij}} \{\sigma_{ij}\}$ denotes the satisfaction degree of an alternative y_i corresponding the attribute c_j and $s_{ij} = \bigcup_{\kappa_{ij} \in s_{ij}} \{\kappa_{ij}\}$ denotes the dissatisfaction degree of an alternative y_i corresponding the attribute c_j such that $r_{ij}, s_{ij} \in [0,1]$, and $0 \leq \sigma_{ij}^{+} + \kappa_{ij}^{+} \leq 1$ where $\sigma_{ij}^{+} \in r_{ij}^{+} = \bigcup_{\sigma_{ij} \in r_{ij}} \max\{\sigma_{ij}\}$, and $\kappa_{ij}^{+} \in s_{ij}^{+} = \bigcup_{\kappa_{ij} \in s_{ij}} \max\{\kappa_{ij}\}$,
The dual hesitant fuzzy decision matrix $K = (k_{ij})_{n \times m}$ contains all information

Table 1: **The dual hesitant fuzzy decision matrix**

	c_1	c_2	...	c_n
y_1	(r_{11}, s_{11})	(r_{12}, s_{12})	...	(r_{1m}, s_{1m})
y_2	(r_{21}, s_{21})	(r_{22}, s_{22})	...	(r_{2m}, s_{2m})
...	...	...	...	...
y_n	(r_{n1}, s_{n1})	(r_{n2}, s_{n2})	...	(r_{nm}, s_{nm})

$k_{ij} = (r_{ij}, s_{ij})$ $(i = 1, 2, \ldots, n; j = 1, 2, \ldots, m)$. (see Table1). During the aggregation process, there may exist two types of attributes, that is, (1) the profit type attribute; and (2) the cost type attribute. If this happens, we should convert the cost type attribute preference values in terms of the benefit type attribute preference values. Then, $K = (k_{ij})_{n\times m}$ can be convert ed into the normalized matrix $R = (g_{ij})_{n\times m}$, where $g_{ij} = (t_{ij}, f_{ij}) = \begin{cases} g_{ij}, & \text{for benefit attribute} \quad c_j \\ \overline{g_{ij}}, & \text{for cost attribute} \quad c_j \end{cases}$

$i = 1, 2, \ldots, n \quad j = 1, 2, \ldots, m$

where $\overline{g_{ij}}$ is the complement of g_{ij} such that $\overline{g_{ij}} = (\kappa_{ij}, \sigma_{ij})$.

Step 2. Utilize the $WGHFBM$ (generally, we will use $\rho = \ell = 1$). Let r_i denotes the aggregated value of all the preference values $r_{ij}(j = 1, 2, \ldots, m)$ of the ith line and corresponding to the alternative y_i.

Step 3. Utilize the steps of Definition 3 for ranking the inclusive performance values $r_i(i = 1, 2, \ldots, n)$.

Step 4. Arrange all the alternatives $y_i(i = 1, 2, \ldots, n)$ in descending manner according to $r_i(i = 1, 2, \ldots, n)$, and then accept the optimal alternative having inclusively high performance value.

We apply our developed $WDHFBM$ aggregation operator to an $MADM$ problem.

Example. Let us examine a real life example of decision making. A Mayor of the town is planning to construct a hospital. Four feasible contractors (alternatives) are available $y_i(i = 1, 2, 3, 4)$. The decision makers examines four attributes, c_1 (economic), c_2 (functional), c_3 (operational) and c_4 (time period). Let $\omega = (0.2, 0.4, 0.1, 0.3)^T$ the weight vector of the attribute $c_j(j = 1, 2, 3, 4)$. Suppose that the preference values of the alternatives $y_i(i = 1, 2, 3, 4)$ corresponding to the attributes $c_j(j = 1, 2, 3, 4)$ are given by the $DHFEs$ $d_{ij} = (r_{ij}, s_{ij}) = (\bigcup_{\sigma_{ij}\in h_{ij}}\{\sigma_{ij}\}, \bigcup_{\kappa_{ij}\in g_{ij}}\{\kappa_{ij}\})$ where σ_{ij} illustrates the satisfaction level of the alterative y_i with attribute c_j and κ_{ij} illustrates the dissatisfaction level of the alternative y_i with attribute c_j. All d_{ij}'s $(i = 1, 2, 3, 4; j = 1, 2, 3, 4)$ are involved in the dual hesitant fuzzy decision matrix $K = (k_{ij})_{4\times 4}$ (see Table 2). Now, we apply $WDHFBM$ (here, we take $\rho = \ell = 1$)

Table 2: **The dual hesitant fuzzy decision matrix**

	c_1	c_2	c_3	c_4
y_1	$\langle\{0.5, 0.6, 0.7\}, \{0.6, 0.7\}\rangle$	$\langle\{0.3, 0.4, 0.5\}, \{0.1, 0.2\}\rangle$	$\langle\{0.1, 0.2, 0.3\}, \{0.1, 0.2\}\rangle$	$\langle\{0.1, 0.2, 0.3\}, \{0.4, 0.5\}\rangle$
y_2	$\langle\{0.6, 0.7\}, \{0.1, 0.2\}\rangle$	$\langle\{0.6, 0.7\}, \{0.1, 0.2\}\rangle$	$\langle\{0.1, 0.2\}, \{0.1, 0.2\}\rangle$	$\langle\{0.4, 0.5, 0.6\}, \{0.3, 0.4\}\rangle$
y_3	$\langle\{0.4, 0.5\}, \{0.1, 0.2\}\rangle$	$\langle\{0.3, 0.4\}, \{0.1, 0.2\}\rangle$	$\langle\{0.6, 0.7\}, \{0.4, 0.5\}\rangle$	$\langle\{0.3, 0.4\}, \{0.1, 0.2\}\rangle$
y_4	$\langle\{0.2, 0.3, 0.4\}, \{0.1, 0.2\}\rangle$	$\langle\{0.2, 0.3\}, \{0.3, 0.4\}\rangle$	$\langle\{0.3, 0.4\}, \{0.3, 0.4, 0.5\}\rangle$	$\langle\{0.7, 0.8\}, \{0.2, 0.3\}\rangle$

$$d_i = (r_{d_i}, s_{d_i}) = DHFB_w^{1,1}(d_{i1}, d_{i2}, d_{i3}, d_{i4})$$

to total all the inclination esteems $d_{ij}(i = 1, 2, 3, 4)$ of the ith line and get the inclusive performance value d_i of the alternative y_i as

$$d_1 = (\{0.0624, 0.0714, \ldots, 0.1079\}, \{0.7326, 0.7487, \ldots, 0.7817\}),$$
$$d_2 = (\{0.1176, 0.1265, \ldots, 0.1693\}, \{0.6369, 0.6623, \ldots, 0.7219\}),$$
$$d_3 = (\{0.1249, 0.1446, \ldots, 0.1644\}, \{0.6181, 0.6444, \ldots, 0.6814\}),$$
$$d_4 = (\{0.1241, 0.1102, \ldots, 0.1529\}, \{0.6924, 0.7123, \ldots, 0.7608\}).$$

Score function of all the alternatives is given as below

$$S_{d_1} = -0.6802,\ S_{d_2} = -0.5327,\ S_{d_3} = -0.5146,\ S_{d_4} = -0.5886,$$

Since $S_{d_3} > S_{d_2} > S_{d_4} > S_{d_1}$, then according to Definition 5, we can get the ranking of the $DHFNs$, which is $d_3 > d_2 > d_4 > d_1$, and then arrangements of the alternatives $y_i(i = 1, 2, 3, 4)$ in descending manner is given as $y_3 > y_2 > y_4 > y_1$. Hence, y_3 is the optimal choice.

5. Conclusion

In this manuscript, we have extended the BM under the dual hesitant fuzzy datum. We have defined some novel dual hesitant fuzzy aggregation operators, such as $DHFBM$, the $WDHFBM$, and the several special cases of the $DHFBM$. We have also discuss the basic properties, such as, idempotency, monotonicity, commutativity and boundedness in detail. The basic idea of the $WDHFBM$ in multi-attribute decision-making is that it captures the importance of attribute individually and reflects the interrelationship of the each attribute and thus gives the information about decision-making into account as much as possible. In future research, we will try to explore our BM aggregation operators to more comprehensive models like Dual Extended HFSs [27] or Expanded Dual HFSs [28]. They complement the ideas of Extended HFSs with duality.

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Gumbel-exponential distribution

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Abstract. In this paper we introduce a new distribution denoted (GE) which depends on the Exponential and Gumbel distributions. Some of its mathematical properties like moment generated function, hazard function, quantile functions, mode, median, variance, the r-th moment about the mean and origin of skewness and kurtosis, a simulation study is carried out to examine the bias and mean square error of the maximum likelihood estimators of the parameters. A real data is adapted to illustrate the importance of the new distribution.
Keywords: Gumbel distribution, exponential distribution, T-X family, moments, estimation, real data.

1. Introduction

Statistical distributions are commonly applied to describe real world phenomena. There are always ways for developing it, they are either more flexible or fitting specific real world scenarios. This has researchers are motivated for seeking and developing new and more flexible distributions. Many classes of generalized distributions have been developed and applied to describe various phenomena. In statistical literature, researchers have introduced new distributions by adding one or more parameters.

Can we use other distributions with support as a generator to derive different classes of distributions?

This question is to introduce a new technique to derive families of distributions using any probability density function as a generator. A new method is bing proposed to generate these families of continuous distributions. They consider a random variable X called "the transformer" whith is used to transform another variable T "the transformed", this result is the family of distributions

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$T - X$. The well-known generators are the Exponentiated-exponential family introduced by Gupta and Kundu [14]. Nadarajah (2006) enunciated the Exponentiated Gumbel distribution . Cordeiro and al (2011) have suggested the Kumaraswamy Gumbel distribution [22]. the Weibull-G family of probability distribution has been advanced by Bourguignon and al [19]. to generate new families, Alzaatreh and al (2013) has introduced a method [3], in wich they have proposed an Exponential $T - X$ family of distribution [4] and Weibull-Pareto distribution with applications [2].

2. Method of the transform-transformer

Alzaatreh and al [3], have introduced a method to generate new families labelled $T - X$ that satisfies some conditions.

Definition 2.1. *Let X be a random variable with density f, F would function accumulated, let T be a continuous random variable with the use of the probability density $r(x)$ defined on $[a, b]$, $G(x)$ the function of the accumulated distribution of the new family of distribution defined as*

$$G(x) = \int_a^{W(F(x))} r(t)dt, \tag{1}$$

where $W(F(x))$ satisfies the following conditions

$$\begin{cases} W(F(x)) \in [a, b] \\ W(F(x)) \text{ is differentiable and monotonically non decreasing} \\ \text{if } x \to -\infty, W(F(x)) \to a \text{ and } x \to \infty, W(F(x)) \to b \end{cases} \tag{2}$$

The probability density function $g(x)$ is related to the accumulated function distribution $G(x)$ connected to the previous given by the relation

$$g(x) = \left\{ \frac{d}{dx} W(F(x)) \right\} r\left\{ w(F(x)) \right\}, \tag{3}$$

where $R(t)$ is the function of accumulated distribution of the a random variable.

3. Gumbel-exponential distribution "GE"

In this section we introduce a new family of the distribution generated by a random variable T. We have $T \in (-\infty, \infty)$ a random variable that follows the Gumbel distribution [25].

Theorem 3.1. *The PDF for Gumbel exponential(GE) distribution is defined by*

$$g(x) = \frac{e^{-\frac{\mu}{\delta}}}{(x-\mu)^2} \left(\frac{\delta}{x-\mu} \right)^{\frac{1}{\delta}-1} \exp\left(-e^{-\frac{\mu}{\delta}} \left(\frac{\delta}{x-\mu} \right)^{\frac{1}{\delta}} \right), \mathbf{x} > \mu. \tag{4}$$

Proof. Let be $T \in (-\infty, \infty)$ a random variable that follows the Gumbel distribution, so that $r(t)$ the density probability function

$$r(t) = \frac{1}{\delta} \exp(-\frac{t-\mu}{\delta}) \exp(-\exp(-\frac{t-\mu}{\delta})),\ t \in \mathbb{R},$$
$$R(x) = \exp\left(-\exp(-\frac{x-\mu}{\sigma})\right), si\ x \in \mathbb{R}.$$

Let the random variable X follow the exponential distribution, that the distribution function is

$$F(x) = 1 - e^{-\frac{x-\mu}{\delta}},\quad \mathbf{x} \geq \mu.$$

Therefore,

$$w(F(x) = \log\left[-\log(1-F(x))\ \right] = \log(\frac{x-\mu}{\delta}).$$

We find

$$g(x) = \left\{\frac{d}{dx} W(F(x))\right\} r\left\{w(F(x))\right\}$$

and

$$G(x) = \int_{\mu}^{W(F(x))} r(t)dt = R\left\{W(F(x))\right\},$$
$$G(x) = \int_{\mu}^{W(F(x))} r(t)dt = R\left\{W(F(x))\right\}$$
$$= \int_{\mu}^{W(F(x))} \frac{1}{\delta} \exp(-\frac{t-\mu}{\delta}) \exp(-\exp(-\frac{t-\mu}{\delta}))dt$$
$$= \exp\left(-(\frac{\delta}{x-\mu})^{\frac{1}{\delta}} \exp(-\frac{\mu}{\delta})\right) - \frac{1}{e}$$

we drift $G(x)$, we find, the density function of probability $g(x)$ is given by

$$g(x) = \frac{e^{-\frac{\mu}{\delta}}}{(x-\mu)^2} \left(\frac{\delta}{x-\mu}\right)^{\frac{1}{\delta}-1} \exp\left(-e^{-\frac{\mu}{\delta}} \left(\frac{\delta}{x-\mu}\right)^{\frac{1}{\delta}}\right), \mathbf{x} > \mu. \tag{5}$$ □

3.1 Quantile function

The quantile function of X is $G_x^{-1}(\gamma),\ 0 < \gamma < 1$. The quantile function of the distribution GE is

$$G^{-1}(\gamma) = \frac{\delta}{\left(-\ln\left(\gamma + \frac{1}{e}\right) / \exp(\frac{-\mu}{\delta})\right)^{\delta}} + \mu, \tag{6}$$

for $\gamma = 0.25, \gamma = 0.50$, and $\gamma = 0.75$, the three quartiles of the distribution GE are

$$\varphi_1 = G_x^{-1}(\frac{1}{4}) = \frac{\delta}{\left(-\ln\left(\frac{1}{4} + \frac{1}{e}\right) / \exp(\frac{-\mu}{\delta})\right)^{\delta}} + \mu, \tag{7}$$

$$\text{median } \varphi_2 = G_x^{-1}\left(\frac{1}{2}\right) = \frac{\delta}{\left(-\ln\left(\frac{1}{2}+\frac{1}{e}\right)/\exp(\frac{-\mu}{\delta})\right)^{\delta}} + \mu, \tag{8}$$

$$\varphi_3 = G_x^{-1}\left(\frac{3}{4}\right) = \frac{\delta}{-\left(\ln\left(\frac{3}{4}+\frac{1}{e}\right)/\exp(\frac{-\mu}{\delta})\right)^{\delta}} + \mu. \tag{9}$$

3.2 Hazard function

The hazard function of GE distribution is

$$h(x) = \frac{g(x)}{1-G(x)} = \frac{\frac{e^{-\frac{\mu}{\delta}}}{(x-\mu)^2} z^{\frac{1}{\delta}+1}}{\left(1+\frac{1}{e}\right)\exp(e^{-\frac{\mu}{\delta}} z^{\frac{1}{\delta}})}. \tag{10}$$

By simple computation, we obtain

$$h(x) = \frac{g(x)}{1-G(x)} = \frac{\frac{e^{-\frac{\mu}{\delta}}}{(x-\mu)^2}\left(\frac{\delta}{x-\mu}\right)^{\frac{1}{\delta}-1} \exp\left(-e^{-\frac{\mu}{\delta}}\left(\frac{\delta}{x-\mu}\right)^{\frac{1}{\delta}}\right)}{1-\exp\left(-e^{-\frac{\mu}{\delta}}\left(\frac{\delta}{x-\mu}\right)^{\frac{1}{\delta}}\right)+\frac{1}{e}} \tag{11}$$

Mutiplying by $\exp(e^{-\frac{\mu}{\delta}}(\frac{\delta}{x-\mu})^{\frac{1}{\delta}})$, we obtain

$$h(x) = \frac{\frac{e^{-\frac{\mu}{\delta}}}{(x-\mu)^2}\left(\frac{\delta}{x-\mu}\right)^{\frac{1}{\delta}-1}}{(1+\frac{1}{e})z-1}, \tag{12}$$

where $z = \exp(e^{-\frac{\mu}{\delta}}(\frac{\delta}{x-\mu})^{\frac{1}{\delta}})$.

3.3 Moments and function generating moments

3.3.1 Function generating moments

The moment Generating function of a continuous random variable X is given by

$$M_X(t) = \int_{\mu}^{\infty} \exp(tx) g(x) dx = \int_{\mu}^{\infty} \left(\sum_{i=0}^{\infty} \frac{t^i x^i}{i!} g(x)\right) dx, \tag{13}$$

$$M_X(t) = \sum_{i=0}^{\infty} \frac{t^i}{i!} E(X^i) = \frac{t^i}{i!} k \sum_{k=0}^{n} C_k^n \delta^k \mu^{n-k} \Gamma\left(1-n\delta\right). \tag{14}$$

3.3.2 Moments

The first moment of X can be written as

$$E(X) = \int_{\mu}^{\infty} x g(x) dx = k\left(\mu + \delta\Gamma\left(1-\delta\right)\right), \quad k = \exp\left(-\frac{\mu}{\delta} + \exp(-\frac{\mu}{\delta})\right) \tag{15}$$

with

$$g(x) = \frac{e^{-\frac{\mu}{\delta}}}{(x-\mu)^2}\left(\frac{\delta}{x-\mu}\right)^{\frac{1}{\delta}-1}\exp\left(-e^{-\frac{\mu}{\delta}}\left(\frac{\delta}{x-\mu}\right)^{\frac{1}{\delta}}\right), \mathbf{x} > \mu.$$

By variable change

$$y = \left[\frac{\delta}{x-\mu}\right]^{\delta^{-1}}, \tag{16}$$

$$y^{\delta} = \frac{\delta}{x-\mu}. \tag{17}$$

We obtain

$$x = \delta y^{-\delta} + \mu, \tag{18}$$

$$dx = -\delta^2 y^{-\delta-1} dy. \tag{19}$$

We obtain a new $g(x)$

$$g(x) = \frac{e^{-\frac{\mu}{\delta}}}{\delta^2} y^{\delta}\exp(-ye^{-\frac{\mu}{\delta}}) \tag{20}$$

We can calculate the moment of ordre1 E(x)

$$E(X) = \int_{\mu}^{\infty} xg(x)dx = k\delta\int_0^{\infty} y^{-\delta}\exp(-y)dy + k\mu\int_0^{\infty}\exp(-y)dy, \tag{21}$$

where

$$\int_0^{\infty} y^{-\delta}\exp(-y)dy = k\int_0^{\infty} y^{(1-\delta)-1}\exp(-y)dy = \Gamma(1-\delta). \tag{22}$$

By consequence

$$E(X) = k\left(\mu + \delta\Gamma\left(1-\delta\right)\right). \tag{23}$$

The second order moment of the GE distribution is

$$E(X^2) = k\left(\mu + 2\delta\mu\Gamma(\delta-1) + \delta^2\Gamma(1-2\delta)\right). \tag{24}$$

The nth moment on the origin

$$E(X^n) = \int_{\mu}^{\infty} x^n g(x)dx = k\int_0^{\infty}(\delta y^{-\delta}+\mu)^n\exp(-y)dy, \tag{25}$$

$$E(X^n) = \sum_{k=0}^{n} C_k^n \delta^k \mu^{n-k}\int_0^{\infty} y^{-k\delta}dy = k\sum_{k=0}^{n} C_k^n\delta^k\mu^{n-k}\Gamma\left(1-n\delta\right), \tag{26}$$

for $k = 1, \ldots, n$.

The variance of the GE distribution is

$$V(X) = \Gamma(1-\delta)\left(2\delta\mu - \Gamma(1-\delta) - 2\mu\right) + \mu(1-\mu) + \delta^2\Gamma(1-2\delta), \tag{27}$$

$$\begin{aligned} V(X) &= E(x^2) - E(x)^2 \\ &= k\left(\left[\mu + 2\delta\mu\Gamma(\delta-1) + \delta^2\Gamma(1-2\delta)\right] - \left[\mu + \Gamma(1-\delta)\right]^2\right) \\ &= k\left(\Gamma(1-\delta)\left(2\delta\mu - \Gamma(1-\delta) - 2\mu\right) + \mu(1-\mu) + \delta^2\Gamma(1-2\delta)\right). \end{aligned}$$

The coefficients of skewness and kurtosis of the GE distribution are

$$\begin{aligned} skewness = E(x^3) &= \sum_{k=0}^{n} C_3^k \delta^k \mu^{3-k} \int_0^\infty y^{-3\delta} dy \\ &= k\sum_{k=0}^{n} C_3^k \delta^k \mu^{3-k} \Gamma(1-k\delta), \end{aligned} \tag{28}$$

$$skewness = k\left(\mu^3 + 3\delta\mu^2\Gamma(1-\delta) + 3\delta^2\mu\Gamma(1-2\delta) + \delta^3\Gamma(1-3\delta)\right),$$

$$kurtosis = E(x^4) = k\sum_{k=0}^{n} C_4^k \delta^k \mu^{4-k} \Gamma(1-4\delta), \tag{29}$$

$$\begin{aligned} kurtosis = k\big(&\mu^4 + 4\delta\mu^3\Gamma(1-\delta) + \delta^2\mu^2\Gamma(1-2\delta) \\ &+4\delta^3\mu\Gamma(1-3\delta) + \delta^4\Gamma(1-4\delta)\big). \end{aligned} \tag{30}$$

3.4 Entropy

The entropy (Rényi, 1961),of a random variable X is a measure of variation of uncertainty , that of the GE distribution is given by

$$\begin{aligned} I_R(s) &= \frac{1}{1-s}\left\{\ln(g^s(x)dx\right\}, \quad s>0, s\neq 1 \\ &= \frac{ke^{-\frac{s\mu}{\delta}}}{1-s}\ln\left\{\delta^{-2s+2}s^{-\delta s+\delta-s}\Gamma(\delta s+s-\delta)\right\}, \ s>0, s\neq 1. \end{aligned}$$

Moreover, the Shannon entropy (1948) defined by $E\left\{-\ln(g(x))\right\}$, this is a special case derived from $\lim_{s\to 1} I_R(s)$.

3.5 Reliability

The measure of reliability of industrial components has many applications especially in the area of lifetime testing and engineering,The component fails at the instant that the random stress X_2 applied to it exceeds the random strength $X1$, and the component will function satisfactorily whenever $X_1 > X_2$. Hence, $R = P[X_2 < X_1]$ is a measure of component reliability. We derive the reliability

R when X_1 and X_2 have independent $GE(\delta_1;\mu)$ and $GE(\delta_2;\mu)$ distributions with the same shape parameter μ and fixed parameter δ. The reliability is defined

$$R=\int_0^{\infty} g_1(x)G_2(x)dx=-k(2-\delta_1)\left(-\exp\left((\frac{\delta_2}{x-\mu})^{\frac{1}{\delta}}*\exp(\frac{-\mu}{\delta_2})\right)-\frac{1}{e}\right),$$

$k=\exp\left(-\frac{\mu}{\delta_1}+\exp(-\frac{\mu}{\delta_1})\right).$

3.6 Mean deviations

The amount of scatter in a population is measured to some extent by the totality of deviations from the mean m and median M, the mean deviation from the mean $D(m)$ and the mean deviation from the median $D(M)$, can be written as

$$D(m)=\int_{\mu}^{\infty}|x-m|\,g(x)dx=2mG(x)-2\int_{\mu}^{m}xg(x)dx,$$
$$D(M)=\int_{\mu}^{\infty}|x-M|\,g(x)dx=m-2\int_{\mu}^{M}xg(x)dx,$$

where $\int_{\mu}^{m}xg(x)dx=\delta\left[\Gamma\left((1-\delta),m\right)-\Gamma\left((1-\delta),\mu\right)\right]+\mu\left[\exp(m)-\exp(\mu)\right]$

We obtain, where $k=e^{-\frac{\mu}{\delta_1}+\exp(-\frac{\mu}{\delta_1})}$

$$D(m)=2mG(x)-2\delta k\left[\Gamma\left((1-\delta),m\right)-\Gamma\left((1-\delta),\mu\right)\right]+\mu k\left[\exp(m)-\exp(\mu)\right],$$
$$D(M)=m-2\delta k\left[\Gamma\left((1-\delta),M\right)-\Gamma\left((1-\delta),\mu\right)\right]+\mu k\left[\exp(M)-\exp(\mu)\right].$$

3.7 Maximum likelihood estimates

Let X_i follows $GE(\delta,\mu), i=1,....,n$, random variables, the maximum likelihood function $l(x,\mu,\delta)$ is

$$L(\theta)=lnl(x,\mu,\delta)=\sum(\frac{-\mu}{\delta})-2n\log(x-\mu)+n(\frac{1}{\delta}-1)\log\delta$$
$$-(\frac{1}{\delta}-1)\sum\log(x_i-\mu)+\sum\left((\frac{1}{x_i-\mu})*\exp(\frac{\mu}{\delta})\right). \tag{31}$$

The derivatives of $L(\theta)$ with respect to δ and μ are

$$\frac{\partial L(\theta)}{\partial\delta}=\sum(\frac{\mu}{\delta^2})-\frac{n}{\delta^2}\log\delta+n\frac{1}{\delta}(\frac{1}{\delta}-1)-\frac{\mu}{\delta^2}\sum_{i=1}^{n}\left(\frac{\mu}{\delta^2}(\frac{\delta}{x_i-\mu})*\exp(\frac{\mu}{\delta})\right), \tag{32}$$

$$\frac{\partial L(\theta)}{\partial\mu}=-\sum(\frac{1}{\delta})\frac{2n}{(x-\mu)}+(\frac{1}{\delta}-1)\sum\frac{1}{(x_i-\mu)}$$
$$-\sum\frac{1}{(x_i-\mu)^2}*\exp(\frac{\mu}{\delta})-\frac{1}{\delta^2}\sum(\frac{1}{x_i-\mu})*\exp(\frac{\mu}{\delta}). \tag{33}$$

The two equations can not be solved directly, we must used the Fisher scoring method. We have

$$\begin{bmatrix} \frac{\partial L(\theta)}{\partial \delta^2} & \frac{\partial^2 L(\theta)}{\partial \delta \partial \mu} \\ \frac{\partial^2 L(\theta)}{\partial \mu \partial \delta} & \frac{\partial L(\theta)}{\partial \mu^2} \end{bmatrix}_{\substack{\hat{\delta}=\delta_0 \\ \hat{\mu}=\mu_0}} \begin{bmatrix} \hat{\delta} = \delta_0 \\ \hat{\mu} = \mu_0 \end{bmatrix} = \begin{bmatrix} \frac{\partial L(\theta)}{\partial \delta} \\ \frac{\partial L(\theta)}{\partial \mu} \end{bmatrix}_{\substack{\hat{\delta}=\delta_0 \\ \hat{\mu}=\mu_0}} . \tag{34}$$

The equation (42) can be solved iteratively where δ_0, μ_0 are the initial values of δ, μ.

3.8 Existence and uniquess of ML parameters

Lemma 3.1. *there exists a compact subset* $K \equiv K\ (\eta) \subset (0,\infty)\ (0,\infty)$.*such that*

$$\{(\delta,\mu) : L(\theta) \geq -\eta\} \subset K. \tag{35}$$

Theorem 3.2. *Suppose that* $X_i \backsim GE\ (\delta,\mu), i = 1,....,n$, *then the MLEs of parameters* δ *and* μ *of Gumbel Exponential distribution exist and are unique.*

Proof. We need only to show that the MLEs of parameters δ and μ uniquely exist. According to the results of Màkelàinen et al. (1981), in order to show the existence and uniqueness of the MLEs of δ and μ , it is sufficient to verify the following two conditions:

(i) For any given $\eta > 0$, (49) holds

(ii) The Hessian matrix of $L(\theta)$

$$H = \begin{bmatrix} \frac{\partial L(\theta)}{\partial \delta^2} & \frac{\partial^2 L(\theta)}{\partial \delta \partial \mu} \\ \frac{\partial^2 L(\theta)}{\partial \mu \partial \delta} & \frac{\partial L(\theta)}{\partial \mu^2} \end{bmatrix}$$

is negative definite at every point $(\delta,\mu) \in (0,\infty)$. □

Condition i is certainly satisfied by Lemma 1. Therefore, to prove the theorem, we need only to show ii. Then, $x^t H x < 0$ and H is negative definite, where $x^t = (x_1\ x_2)$.

4. Simulation

In this section, we investigate the behaviour of the ML estimators for a finite sample size (n). A simulation study consisting of the following steps is being carried out for each (δ,μ,n),where n=100, 500, 1000.

- Choose the initial values of δ_0, μ_0 for the corresponding elements of the parameter vector $\theta = (\delta,\mu)$ to specify GE(δ,μ) distribution.

- Choose sample size n.
- Generate N independent samples of size n from $GE(\delta, \mu)$ distribution.
- Compute the ML estimate $\hat{\theta}_n$ of θ_0 for each of the N samples.
- Compute the mean of the obtained estimators over all N samples

$$Averagebias = \frac{1}{N}\sum_{i=1}^{N}\left(\hat{\theta}_i - \theta_0\right) \tag{36}$$

and the average square error

$$MSE\left(\theta\right) = \frac{1}{N}\sum_{i=1}^{N}\left(\hat{\theta}_i - \theta_0\right)^2 \tag{37}$$

Table 03

N	$biais(\delta)\quad \delta = 0.01$	$biais(\mu)\quad \delta = 0.01$	$biais(\delta)\ \delta = 0.3$	$biais(\mu)\ \delta = 0.3$
$n = 100$	-0.004220959	-0.0495	-0.294221	-0.00425
$n = 500$	-0.005392097	-0.0499	-0.2953921	-0.0499
$n = 1000$	-0.005784495	-0.04995	-0.2957845	-0.04995

Table 04

N	$biais(\delta)\ \delta = 0.8$	$biais(\mu)\delta = 0.8$	$biais(\delta)\delta = 0.25$	$biais(\mu)\ \delta = 0.25$
$n = 100$	-0.794221	-0.099	-0.195784	-0.099
$n = 500$	-0.7953921	-0.0998	-0.1953921	-0.0998
$n = 1000$	-0.7957845	-0.0999	-0.2457845	-0.0999

Table 05

N	$biais(\delta)\delta = 0.5$	$biais(\mu)\delta = 0.5$	$biais(\delta)\delta = 0.75$	$biais(\mu)\delta = 0.75$
$n = 100$	-0.494221	-0.7425	-0.744221	-0.7425
$n = 500$	-0.4953921	-0.7485	-0.7453921	-0.7485
$n = 1000$	-0.4957845	-0.74925	-0.7457845	-0.74925

Tables 03,04 and 05 : represent average bias of the estimations feigned with $\mu = 0.05, \mu = 0.1, 0.75$

N	$MSE(\delta)\delta = 0.01$	$MSE(\mu)\delta = 0.01$	$MSE(\delta)\quad \delta = 0.3$	$MSE(\mu)\quad \delta = 0.3$
$n = 100$	$1.78165e-05$	0.00245025	0.08656597	0.00245025
$n = 500$	$0.907471e-05$	0.002419001	0.08125649	0.00241001
$n = 1000$	$0.346038e-05$	0.00241000	0.0118847	0.00140003

Table 06

N	$MSE(\delta)\ \delta = 0.8$	$MSE(\mu)\ \delta = 0.8$	$MSE(\delta)\delta = 0.25\ \ )$	$MSE(\mu)\delta = 0.25$
$n = 100$	0.6307869	0.009801	0.03833157	0.00998001
$n = 500$	0.6316486	0.009196004	0.03817807	0.00996004
$n = 1000$	0.00630273	0.00098001	0.001041002	0.008001

Table 07

N	$MSE(\delta)\quad \delta = 0.5$	$MSE(\mu)\quad \delta = 0.5$	$MSE(\delta)\quad \delta = 0.75$	$MSE(\mu)\quad \delta = 0.75$
$n = 100$	0.2442544	0.5513063	0.5561945	0.5613756
$n = 500$	0.12454133	0.25602523	0.35556094	0.45602523
$n = 1000$	0.02458023	0.05613756	0.05561945	0.05613756

Table 08

Tables 06,07 and 08: Represent MSE of the feigned estimations with $\mu = 0.05, \mu = 0.1, \mu = 0.75$

Discussion

Previous tables shows how the bias and mean squared errors vary with respect to n. The mean squared errors for each parameter decrease to zero as $n \to \infty$. These numerical results coincide with the established theoretical results.

5. Application

In this section, we show present that the GE distribution may be a best model for some distributions, for example Nadarajah [22] presented the exponential distribution of Gumbel. Persson and Ryden [24] have examined this before distribution. in oceanographic application. In addition, Pinheiro and Ferrari [23], compare several generalizations of the Gumbel distribution.

We provied an applicability of GE distribution by considering real data used by different researchers. Application to waiting times in a queue, and compare them with different distribution.In each case, the parameters are estimated by maximum likelihood

Illustration. Application to waiting times in a queue.

We consider 100 observations on waiting time as a real example that happens before the customer received service in a bank. The data set represents the waiting time (mins) of one hundred (100) bank customers before service is being rendered, This data has previously been used by Ghitany et al (2008). These data are

$$0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.2, 2.23, 0.26, 0.31, 0.73,$$
$$0.52, 4.98, 6.97, 9.02, 13.29, 0.4, 2.26, 3.57, 5.06, 7.09, 11.98, 4.51, 2.07,$$
$$0.22, 13.8, 25.74, 0.5, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 19.13, 6.54, 3.36,$$
$$0.82, 0.51, 2.54, 3.7, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 1.76, 8.53, 6.93,$$
$$0.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 3.25, 12.03, 8.65,$$
$$0.39, 10.34, 14.83, 34.26, 0.9, 2.69, 4.18, 5.34, 7.59, 10.66, 4.5, 20.28, 12.63,$$
$$0.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 6.25, 2.02, 22.69,$$
$$0.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 8.37, 3.36, 5.49,$$
$$0.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 12.02, 6.76,$$
$$0.4, 3.02, 4.34, 5.71, 7.93, 11.79, 18.1, 1.46, 4.4, 5.85, 2.02, 12.07$$

The distribution of GE is adjusted to the data,In each case, the parameters are estimated by maximum likelihood, we adapt (GE) distributions, Exponential Pareto (EP), Exponential (E), Lindley Exponential (LE) and Lomax-Gumbel (LG). The required calculations are carried out using the software R-softwar.

Distribution	*Parameters*
Gumbel- Exponential	$\hat{\delta} = 0.0004061848, \hat{\mu} = 0.009$
Exponential- Pareto	$\hat{\alpha} = 0.801,\ \hat{k} = 1.5137,\ \hat{\lambda} = 0.0183$
Exponential	$\hat{\theta} = 0.101$
Lindley -Exponential	$\hat{\theta} = 2.6501, \hat{\lambda} = 0.152$
Lomax-Gumbel	$\hat{\gamma} = 0.577, \hat{\alpha} = 0.125$

We used statistical tools to compare models and choose the best possible distribution for the dataset among other distributions such as (AIC) Akaike Information Criterion, (CAIC) information criterion corrected Akaike and BIC (Bayesian InformationCriterion) are described as follows

$$AIC = -2LL + 2p$$

$$BIC = -2LL + p\log(n)$$

$$CAIC = 2LL + \frac{2np}{n-p-1}$$

Distribution	$-LL$	AIC	$CIAC$	BIC
Gumbel- Exponential	74.20096	152.4019	152.5256	157.6122
Exponential -Pareto	312.1154	628.2308	628.372	633.9192
Exponential	329.00	660.00	662.1237	660.00
Lindley -Exponential	317.005	638.01	638.1337	643.2203
Lomax-Gumbel	416.379	836.757	836.8817	842.446

Discussion

After a visual reading of the numerical results of estimated parameters and statistical tools from CIA CAIC and BIC, we note that the GE model more powerful than other popular models, we can conclude that the GE model is simple to manipulate and hence best fits the data among all the models considered.

Conclusion

We have proposed the new distribution based on the Gubmel and exponential distribution with discussion some properties including moment generated function, hazard functions, quantil functions, mode, median, mean ,variance, entropy, maximum Likelihood Estimates. A simulation study is carried out to examine the bias and mean square error of the maximum likelihood estimators of the parameters. We have also illustrated the application of GE distribution to real data used by researchers earlier. By comparing GE distribution with other popular models, we conclude GE is distribution best model for the prediction and may be the most flexible for generated this data. Thereafter, it would be interesting to use other data, also we can in our future research propose other distribution has more parameters.

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Optimality criterion for three proposed methods to unbalansed transportation models

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Abstract. The idea of balancing the transportation table starts by adding a quantity to the supply or demand to fill the shortage and achieve the requirement of equality between the quantities of demand and supply with making the costs of the row (column) added zeroes, on the other hand what is the profit desired from this addition as it cannot display a need made or created effort and money without Price or for any zero cost and vice versa as no need or commodity can be ordered without a small price, and accordingly has been proposed three methods to address the unbalanced transportation table and agencies without resorting to adding a zero-cost column (row) with application work to comparing the three proposed methods with the classical methods.

Keywords: transportation problem, linear programming, optimization problems, unbalanced transportation model, proposed methods.

1. Introduction

In the life, the facility usually encounters a scarcity of the resources it requires in its production process. Or it seems a decline in the demand for its products, which creates surplus to it and this leads to difficulties on the ability of the marketing device in the discharge of this surplus or, conversely, an increase in demand for products above their capacity or production lines.

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In order to deal with this phenomenon, the transport models tend to carry out a kind of balance, in light of which the excess quantities of supply or demand are distributed, and the costs or profits are calculated at a later stage.

This balance between supply and demand requires a dummy column to address the amount of shortfall in demand by a certain amount by finding the difference between total supply and demand. In the case where the demand is more than the supply, it is necessary to add a dummy row to address this imbalance, and in terms of costs that are placed within the small squares must be zero. But from another perspective, in the practical and life side, we cannot make the costs of the dummy column or row zeroes, if so what is the desired profit of this addition, as well as when the demand for a particular commodity cannot be promoted without any cost zero, and also cannot offer a need made or From this point, we proposed three methods to balance the transportation model without resorting to adding a row or column with zero costs, but we used some calculation methods to balance the model and then compare these methods with the classic methods.

2. Transportation model [3]

In a transportation problem the points may represent factories to produce items, and to supply a required quantity of the products to a certain number of destinations. This process must be done successfully in such a way as to maximize the profit or minimize the cost transportation. Therefore, the places of production and supply are collected as the original points and the destinations respectively. Sometimes the original and destinations points are also termed as sources and sinks. However, to illustrate a typical transportation model, suppose that m factories supply certain items to n warehouses. As well as, let factory $i, (i = 1, 2, \ldots, m)$ produces ai units, and the warehouse $j, (j = 1, 2, \ldots, n)$ requires bj units. Furthermore, suppose the cost of transportation from factory i to warehouse j is c_{ij}. The decision variables x_{ij} is being the transported amount from the factory i to the warehouse j.

3. Unbalanced transportation problem [1, 2, 4, 5]

There are two different forms of imbalance in the transportation table

Increase in supply $\sum a_i > \sum b_j$;

$$Min(Z) = \sum_{i=1}^{m} \sum_{j=1}^{n} C_{ij} X_{ij},$$

$$\sum_{j=1}^{n} X_{ij} < ai; \quad i = 1, 2, \ldots, m,$$
$$\sum_{i=1}^{m} X_{ij} = bj; \quad j = 1, 2, \ldots, n \quad \forall X_{ij} \geqslant 0.$$

Increase in demand $\sum a_i < \sum b_j$;

$$Min(Z) = \sum_{i=1}^{m} \sum_{j=1}^{n} C_{ij} X_{ij},$$
$$\sum_{j=1}^{n} X_{ij} = ai; \quad i = 1, 2, \ldots, m,$$
$$\sum_{i=1}^{m} X_{ij} < bj; \quad j = 1, 2, \ldots, n \quad \forall X_{ij} \geqslant 0.$$

4. Proposed methods

4.1 Added difference method (1)

Calculate the difference between total supply and demand or vice versa.

$$\sum_{j=1}^{n} D_j - \sum_{i=1}^{m} S_i = Q. \tag{1}$$

Add this difference to the quantity of the row that contains the least cost if the total supply is less than the total demand.

$$S_i = a_{i(min\ cost)} + Q. \tag{2}$$

Or add the difference of the column that has the least cost if the total demand is less than the total supply.

$$D_j = b_{j(min\ cost)} + Q. \tag{3}$$

We complete the solution depending on the methods adopted in finding the first basic solution.

4.2 Added difference method (2)

Calculate the difference between total supply and demand or vice versa.

$$\sum_{j=1}^{n} D_j - \sum_{i=1}^{m} S_i = Q. \tag{4}$$

Add this difference to the quantity of the row that contains the highest cost if the total supply is less than the total demand.

$$S_i = a_{i(max\ cost)} + Q. \tag{5}$$

Or add the difference of the column that has the highest cost if the total demand is less than the total supply.

$$D_j = b_{j(max\ cost)} + Q. \tag{6}$$

We complete the solution depending on the methods adopted in finding the primary solution.

4.3 Supply Max, demand Min method

Calculate the difference between total supply and demand or vice versa

$$\sum_{j=1}^{n} D_j - \sum_{i=1}^{m} S_i = Q. \tag{7}$$

Determine which row has the highest cost (if the total demand is the lowest) to achieve the highest profit to demand.

$$S_i = a_{i(max\ cost)} + Q. \tag{8}$$

Determine which column has the lowest cost (if the total supply is the lowest) to achieve the highest profit to supply.

$$D_j = b_{j(max\ cost)} + Q. \tag{9}$$

The difference is added to the selected row (column) to achieve equilibrium status in the transportation table.

We complete the solution depending on the methods adopted in finding the primary solution.

5. Empirical side

Introduction

Many algorithms have been developed to solve transportation problem because of their wide application in logistics and supply chain when cost and supply and demand quantities are well known to reduce total cost. Transportation problem is said to be balanced if total supply from all sources equals total demand in all destinations It is called otherwise unbalanced. The following is an unbalanced digital example containing three sources of supply and three demand sites in order to show the application of the idea of the proposed methods

Table (1) unbalance transportation table

		Destinations			Supply
		b_1	b_2	b_3	
Sources	a_1	6	7	8	7
	a_2	6	10	9	5
	a_3	5	4	3	10
Demand		10	20	30	22 60

Based on the steps to solve the Added Difference Method (1) is calculated the difference between supply and demand based on equation (1), since the total supply quantities are the lowest is determined row that has the lowest cost and then add the difference resulting from equation (1) to the amount of that row on equation (2)

$$\sum_{j=1}^{n} D_j - \sum_{i=1}^{m} S_i = 60 - 22 = 38 = Q$$

$S_3 = a_{3(min\ cost)} + Q = 10 + 38 = 48.$

Table (2) balance transportation table

		Destinations			Supply
		b_1	b_2	b_3	
Sources	a_1	6	7	8	7
	a_2	6	10	9	5
	a_3	5	4	3	10+38=48
Demand		10	20	30	60 60

Based on the steps of the solution of the Added Difference Method (2)is calculated the difference between supply and demand based on equation (4), since the total supply quantities are the lowest is determined row with the highest cost and then add the difference resulting from equation (4) to the quantity of that row on equation (5)

$$\sum_{j=1}^{n} D_j - \sum_{i=1}^{m} S_i = 60 - 22 = 38 = Q,$$

$S_2 = a_{2(max\ cost)} + Q = 5 + 38 = 43.$

Based on the steps of solving the method of Supply quantity is larger than demand quantity is calculated the difference between supply and demand based

Table (3) balance transportation table

		Destinations			Supply
		b_1	b_2	b_3	
Sources	a_1	6	7	8	7
	a_2	6	10	9	5+38=43
	a_3	5	4	3	10
Demand		10	20	30	60
					60

on equation (7), since the total supply quantities are the lowest is determined row with the highest cost and then add the difference resulting from equation (7) to the quantity of that row on equation (8). Otherwise, we move to a situation where the sum of the demand is the least to determine which column has the least cost to add the difference to which to achieve the equilibrium state in the transportation table.

6. Applied side

Testing example

The above three methods were applied to (Brisbol) company as this company distributes medicines from its main locations located in four governorates (Baghdad, Diwaniyah, Najaf, Karbala) by transferring them to a group of Medicine repository cooperating with the company located in the four governorates. Quantities of supply and demand and transportation costs were measured in Iraqi dinars.

Table (4) unbalance transportation table

		Destinations				Supply
		Medicine repository1	Medicine repository2	Medicine repository3	Medicine repository4	
sources	Baghdad	109	209	150	129	125
	Diwaniyah	200	119	145	134	205
	Najaf	151	110	166	116	134
	Karbala	108	130	149.5	140	150
Demand		125	175	109	168	614
						577

By using Added Difference Method (1). Based on the steps to solve the first proposed method is calculated the difference between supply and demand based on equation (1), since the total demand quantities are the lowest is determined column that has the lowest cost and then add the difference resulting from equation (1) to the amount of that column on equation (2), If there is more

than one column that has the lowest cost, the selection will be randomly

$$\sum_{i=1}^{m} S_i - \sum_{j=1}^{n} D_j = 614 - 577 = 37 = Q$$

$D_1 = b_{1(min\ cost)} + Q = 125 + 37 = 162.$

Table (5) balance transportation table

		Destinations				Supply
		Medicine repository1	Medicine repository2	Medicine repository3	Medicine repository4	
Sources	Baghdad	109	209	150	129	125
	Diwaniyah	200	119	145	134	205
	Najaf	151	110	166	116	134
	Karbala	108	130	149.5	140	150
Demand		125+37=162	175	109	158	614

By using Added Difference Method (2) and back to Table 4. Based on the steps of the solution of the second proposed method is calculated the difference between supply and demand based on equation (4), since the total demand quantities are the lowest is determined column with the highest cost and then add the difference resulting from equation (4) to the quantity of that column on equation (5)

$$\sum_{i=1}^{m} S_i - \sum_{j=1}^{n} D_j = 614 - 577 = 37 = Q,$$

$D_2 = b_{2(max\ cost)} + Q = 175 + 37 = 212.$

Table (6) balance transportation table

		Destinations				Supply
		Medicine repository1	Medicine repository2	Medicine repository3	Medicine repository4	
sources	Baghdad	109	209	150	129	125
	Diwaniyah	200	119	145	134	205
	Najaf	151	110	166	116	134
	Karbala	108	130	149.5	140	150
Demand		125	175+37=212	109	168	614

By using method of Supply max, demand min and back to Table 4.

Based on the steps of the solution of the third proposed method is calculated the difference between supply and demand based on equation (7), since the total supply quantities are the lowest is determined row with the highest cost and then add the difference resulting from equation (7) to the quantity of that row on

equation (8)

$$\sum_{i=1}^{m} S_i - \sum_{j=1}^{n} D_j = 614 - 577 = 37 = Q,$$

$$D_1 = b_{1(min\ cost)} + Q = 125 + 37 = 162.$$

Table (7) balance transportation table

		Destinations				*Supply*
		Medicine repository1	Medicine repository2	Medicine repository3	Medicine repository4	
Sources	Baghdad	109	209	150	129	**125**
	Diwaniyah	200	119	145	134	**205**
	Najaf	151	110	166	116	**134**
	Karbala	108	130	149.5	140	**150**
Demand		**125+37=162**	**175**	**109**	**168**	**614**

After balancing the transportation table by using the three proposed solution methods, we will use the software WINQSP and compare the results of the proposed methods with the classic Vogel approximation method.

Table (8) Results of the Added Difference Method (1)

12-25-2019	From	To	Shipment	Unit Cost	Total Cost	Reduced Cost
1	Baghdad	Repository 1	12	109	1308	0
2	Baghdad	Repository 4	113	129	14577	0
3	Diwaniyah	Repository 2	96	119	11424	0
4	Diwaniyah	Repository 3	109	145	15805	0
5	Najaf	Repository 2	79	110	8690	0
6	Najaf	Repository 4	55	116	6380	0
7	Karbala	Repository 1	150	108	16200	0
	Total	Objective	Function	Value =	74384	

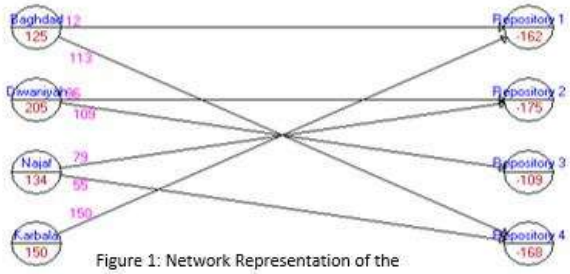


Figure 1: Network Representation of the Transportation Problem

Table (9) Limits of Optimization and Sensitivity Analysis

12-25-2019 22:20:04	From	To	Unit Cost	Reduced Cost	Basis Status	Allowable Min. Cost	Allowable Max. Cost
1	Baghdad	Repository 1	109	0	basic	108	164
2	Baghdad	Repository 2	209	86	at bound	123	M
3	Baghdad	Repository 3	150	1	at bound	149	M
4	Baghdad	Repository 4	129	0	basic	125	130
5	Diwaniyah	Repository 1	200	95	at bound	105	M
6	Diwaniyah	Repository 2	119	0	basic	118	123
7	Diwaniyah	Repository 3	145	0	basic	-4	146
8	Diwaniyah	Repository 4	134	9	at bound	125	M
9	Najaf	Repository 1	151	55	at bound	96	M
10	Najaf	Repository 2	110	0	basic	106	111
11	Najaf	Repository 3	166	30	at bound	136	M
12	Najaf	Repository 4	116	0	basic	115	120
13	Karbala	Repository 1	108	0	basic	-M	109
14	Karbala	Repository 2	130	8	at bound	122	M
15	Karbala	Repository 3	149.50	1.50	at bound	148	M
16	Karbala	Repository 4	140	12	at bound	128	M

It is noted that from the results of table no. (8)special of the Added Difference Method (1) that the required quantities of (614) tons from the cities equipped with the transport to the sub repositories need for the orders at the lowest possible transport cost, for example was equipped medicine repository4 from the cities of Baghdad and Najaf, where the quantity supplied from the city of Baghdad 113 The cost of transporting 14,577 and the quantity supplied from Najaf city was 55 and the cost of transporting 6,380, which was divided

the quantities transported on the basis of reducing the total costs of transportation. But the extent of optimization and sensitivity analysis and the possibility of change in demand and supply and the cost of transporting per unite this is shown in table (9).

It is noted that there is a reduction in costs on some of the variables that were not transported from the cities to the repository of medicines containing low costs as they increase the value of the objective function, for example, the main site located in Baghdad city was not transported to repository of medicines (2, 3) due to reduced costs by (86, 1) respectively and so on for the other non-basic variables. With regard to the upper and lower limits that make the transportation problem within the limits of optimization, for example, the main site in Najaf city has been transported to the repositories (2, 4) at a transport cost per unit of (110, 116) respectively which are within Upper and lower limits of that province.

Table (10) Results of the Added Difference Method (2)

12-25-2019	From	To	Shipment	Unit Cost	Total Cost	Reduced Cost
1	Baghdad	Repository 4	125	129	16125	0
2	Diwaniyah	Repository 2	121	119	14399	0
3	Diwaniyah	Repository 3	84	145	12180	0
4	Najaf	Repository 2	91	110	10010	0
5	Najaf	Repository 4	43	116	4988	0
6	Karbala	Repository 1	125	108	13500	0
7	Karbala	Repository 3	25	149.50	3,737.50	0
	Total	Objective	Function	Value =	74,939.50	

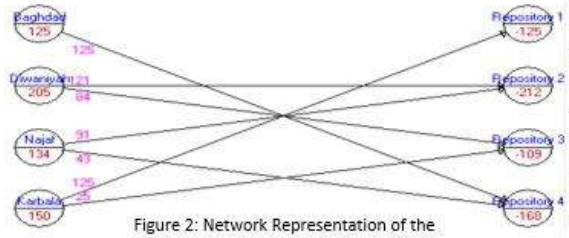


Figure 2: Network Representation of the Transportation Problem

Table (11) Limits of Optimization and Sensitivity Analysis

12-25-2019 22:23:41	From	To	Unit Cost	Reduced Cost	Basis Status	Allowable Min. Cost	Allowable Max. Cost
1	Baghdad	Repository 1	109	1.50	at bound	107.50	M
2	Baghdad	Repository 2	209	86	at bound	123	M
3	Baghdad	Repository 3	150	1	at bound	149	M
4	Baghdad	Repository 4	129	0	basic	-M	129.50
5	Diwaniyah	Repository 1	200	96.50	at bound	103.50	M
6	Diwaniyah	Repository 2	119	0	basic	118.50	125.50
7	Diwaniyah	Repository 3	145	0	basic	138.50	145.50
8	Diwaniyah	Repository 4	134	9	at bound	125	M
9	Najaf	Repository 1	151	56.50	at bound	94.50	M
10	Najaf	Repository 2	110	0	basic	101	110.50
11	Najaf	Repository 3	166	30	at bound	136	M
12	Najaf	Repository 4	116	0	basic	115.50	125
13	Karbala	Repository 1	108	0	basic	0	109.50
14	Karbala	Repository 2	130	6.50	at bound	123.50	M
15	Karbala	Repository 3	149.50	0	basic	149	156
16	Karbala	Repository 4	140	10.50	at bound	129.50	M

It is noted from the results of table no. (10) special of the Added Difference Method (2) that the required quantities of (614) tons from the cities equipped with the transport to the sub repositories need for the orders at the lowest possible transport cost, for example was equipped medicine repository2 from the cities of Diwaniyah and Najaf, where the quantity supplied from the city of Diwaniyah 121 The cost of transporting 14399 and the quantity supplied from Najaf city was 91 and the cost of transporting 10010, which was divided the quantities transported on the basis of reducing the total costs of transportation. As for the extent of optimization and sensitivity analysis and the possibility of change in demand and supply and the cost of transporting, this is shown in table (11).

It is noted that there is a reduction in costs on some of the variables that were not transported from the cities to the repository of medicines containing low costs as they increase the value of the objective function, for example, the

main site located in Najaf city was not transported to repository of medicines (1, 3) due to reduced costs by (56.50, 30) respectively and so on for the other non-basic variables.

With regard to the upper and lower limits that make the transportation problem within the limits of optimization, for example, the main site in Karbala city has been transported to the repositories (1, 3) at a transport cost per unit of (108, 149.50) respectively which are within Upper and lower limits of that province.

Table (12) Results of Supply max demand min method

12-25-2019	From	To	Shipment	Unit Cost	Total Cost	Reduced Cost
1	Baghdad	Repository 1	12	109	1308	0
2	Baghdad	Repository 4	113	129	14577	0
3	Diwanipah	Repository 2	96	119	11424	0
4	Diwanipah	Repository 3	109	145	15805	0
5	Najaf	Repository 2	79	110	8690	0
6	Najaf	Repository 4	55	116	6380	0
7	Karbala	Repository 1	150	108	16200	0
	Total	Objective	Function	Value =	74384	

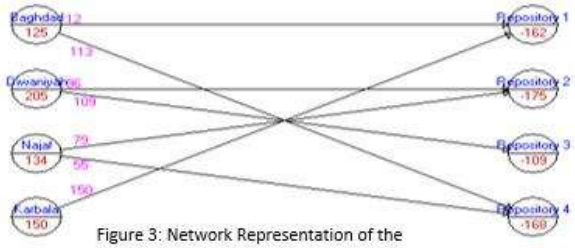


Figure 3: Network Representation of the Transportation Problem

Table (13) Limits of Optimization and Sensitivity Analysis

12-25-2019 22:20:04	From	To	Unit Cost	Reduced Cost	Basis Status	Allowable Min. Cost	Allowable Max. Cost
1	Baghdad	Repository 1	109	0	basic	108	164
2	Baghdad	Repository 2	209	86	at bound	123	M
3	Baghdad	Repository 3	150	1	at bound	149	M
4	Baghdad	Repository 4	129	0	basic	125	130
5	Diwaniyah	Repository 1	200	95	at bound	105	M
6	Diwaniyah	Repository 2	119	0	basic	118	123
7	Diwaniyah	Repository 3	145	0	basic	-4	146
8	Diwaniyah	Repository 4	134	9	at bound	125	M
9	Najaf	Repository 1	151	55	at bound	96	M
10	Najaf	Repository 2	110	0	basic	106	111
11	Najaf	Repository 3	166	30	at bound	136	M
12	Najaf	Repository 4	116	0	basic	115	120
13	Karbala	Repository 1	108	0	basic	-M	109
14	Karbala	Repository 2	130	8	at bound	122	M
15	Karbala	Repository 3	149.50	1.50	at bound	148	M
16	Karbala	Repository 4	140	12	at bound	128	M

It is noted from the results of table no. (12) special of Supply max demand min method that the required quantities of (614) tons from the cities equipped with the transport to the sub repositories need for the orders at the lowest possible transport cost, for example was equipped medicine repository1 from the cities of Baghdad and Karbala, where the quantity supplied from the city of Baghdad 12 The cost of transporting 1308 and the quantity supplied from Karbala city was 150 and the cost of transporting 16200, which was divided the quantities transported on the basis of reducing the total costs of transportation. As for the extent of optimization and sensitivity analysis and the possibility of change in demand and supply and the cost of transporting the unit and one, this is shown in Table 13.

It is noted that there is a reduction in costs on some of the variables that were not transported from the cities to the repository of medicines containing low costs as they increase the value of the objective function, for example, the main site located in Baghdad city was not transported to repository of medicines (2, 3) due to reduced costs by (86, 1) respectively and so on for the other non-basic variables. With regard to the upper and lower limits that make the transportation problem within the limits of optimization, for example, the main site in Diwaniyah city has been transported to the repositories (2, 3) at a transport cost per unit of (119, 145) respectively which are within Upper and lower limits of that province.

Table (14) Results of the Vogel approximation method

12-26-2019	From	To	Shipment	Unit Cost	Total Cost	Reduced Cost
1	Baghdad	Repository 4	113	129	14577	0
2	Baghdad	Repository 5	12	0	0	0
3	Diwaniyah	Repository 2	96	119	11424	0
4	Diwaniyah	Repository 3	109	145	15805	0
5	Najaf	Repository 2	79	110	8690	0
6	Najaf	Repository 4	55	116	6380	0
7	Karbala	Repository 1	125	108	13500	0
8	Karbala	Repository 5	25	0	0	0
	Total	Objective	Function	Value =	70376	

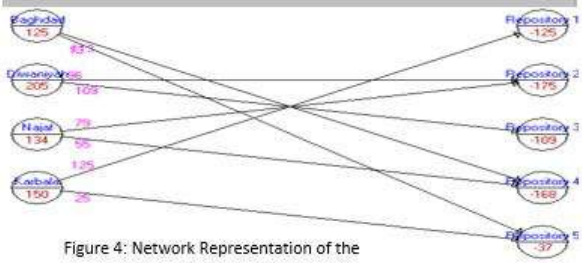


Figure 4: Network Representation of the Transportation Problem

Table (15) Limits of Optimization and Sensitivity Analysis

12-26-2019 00:34:33	From	To	Unit Cost	Reduced Cost	Basis Status	Allowable Min. Cost	Allowable Max. Cost
1	Baghdad	Repository 1	109	1	at bound	108	M
2	Baghdad	Repository 2	209	86	at bound	123	M
3	Baghdad	Repository 3	150	1	at bound	149	M
4	Baghdad	Repository 4	129	0	basic	125	129.50
5	Baghdad	Repository 5	0	0	basic	0	1
6	Diwaniyah	Repository 1	200	96	at bound	104	M
7	Diwaniyah	Repository 2	119	0	basic	118.50	123
8	Diwaniyah	Repository 3	145	0	basic	-4	145.50
9	Diwaniyah	Repository 4	134	9	at bound	125	M
10	Diwaniyah	Repository 5	0	4	at bound	-4	M
11	Najaf	Repository 1	151	56	at bound	95	M
12	Najaf	Repository 2	110	0	basic	106	110.50
13	Najaf	Repository 3	166	30	at bound	136	M
14	Najaf	Repository 4	116	0	basic	115.50	120
15	Najaf	Repository 5	0	13	at bound	-13	M
16	Karbala	Repository 1	108	0	basic	0	109
17	Karbala	Repository 2	130	7	at bound	123	M
18	Karbala	Repository 3	149.50	0.50	at bound	149	M
19	Karbala	Repository 4	140	11	at bound	129	M
20	Karbala	Repository 5	0	0	basic	-1	0

It is noted from the results of table no. (14) special of Vogel approximation method that the required quantities of (614) tons from the cities equipped with the transport to the sub repositories need for the orders at the lowest possible transport cost, for example was equipped medicine repository2 from the cities of Diwaniyah and Najaf, where the quantity supplied from the city of Diwaniyah 96 The cost of transporting 11424 and the quantity supplied from Najaf city was 79 and the cost of transporting 8690, which was divided the quantities transported on the basis of reducing the total costs of transportation. As for the extent of optimization and sensitivity analysis and the possibility of change in demand and supply and the cost of transporting the unit and one, this is shown in table (15).

It is noted that there is a reduction in costs on some of the variables that were not transported from the cities to the repository of medicines containing low costs as they increase the value of the objective function, for example, the main site located in Baghdad city was not transported to repository of medicines (1, 2, 3) due to reduced costs by (1, 86, 1) respectively and so on for the other non-basic variables. With regard to the upper and lower limits that make the transportation problem within the limits of optimization, for example, the main site in Diwaniyah city has been transported to the repositories (2, 3) at a transport cost per unit of (119, 145) respectively which are within Upper and lower limits of that province.

7. Discussion

Based on the use and application of the proposed methods, the total transportation costs are presented in the following table:

No	Methods	Total Transportation Costs
1.	Added Difference (1)	74 384
2.	Added Difference (2)	74 939.50
3.	Supply max demand min	74 384
4.	Vogel Approximation Method	70 376

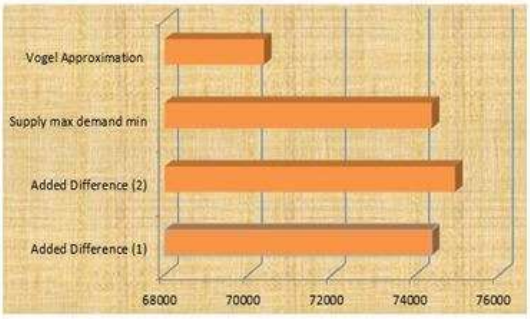


Figure 5. Total Transportation Cost

Note from the Empirical example after balancing the transportation problem using the three solution methods proposed for the above example, and using WINQSP program we have shown the total cost for the added difference methods (1) and the Supply quantity is larger than demand quantity by (236) and the total transportation cost of the added difference method (2) by (449). And the Vogel approximation method had shown the total cost of transportation (104).

1. As for the application side, we find that the added difference (1)has been equal to the Supply max demand min method in terms of the total cost equivalent of 74384.

2. As for the added difference (2) method, it recorded the total costs equivalent 74939.50.

3. The best method is the normal method, which used the method of approximation Vogel, where the total cost of 70376, which is the lowest total transportation costs of these methods.

4. Point (3) shows us that the total transportation costs reached (70376) according to the Vogel approximation method, which is less than the methods remained by a large difference because of the balance of the transportation problem by adding a row or column with an amount equal to the amount of difference between supply and demand at zero costs, while In the other proposed methods, the transportation problem was balanced without adding a row or column with zero costs.

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A matrix expansion solution for a hyperbolic system of time-fractional PDEs with variable coefficients

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Abstract. In this research, we introduce a series solution to a hyperbolic system of time-fractional partial differential equations with variable coefficients in the sense of Caputo fractional derivative. An appropriate expansion of matrix functions is derived and used to create a series solution for the target problem and the residual power series method is also used to determine the coefficients of the series solution. To test our proposed method, we discuss four interesting and important applications. The first three applications are set up so that the exact solution is already known whereas the last application is set up without knowing the solution in advance to test the predictability of the solution or obtain a suitable approximate solution. Numerical results are analyzed to confirm the ability of the used method and to verify the solution obtained. The surface graphs of the solution are plotted to illustrate the behavior of the solution in various conditions. Mathematica 7 software is used to calculate the numerical and symbolic quantities.
Keywords: fractional partial differential equations, hyperbolic systems, power series.

1. Introduction

Many natural phenomena have been modeled through partial differential equations (PDEs), especially in physics, engineering, chemistry, and biology, as well as in humanities [5, 25]. A wide range of PDEs can be classified under the name of hyperbolic PDEs which has the following general form [25, 21, 15, 34, 29]:

$$u_t(x,t) = a(x,t)\,u_x(x,t) + b(x,t)\,u(x,t) + f(x,t), x \in S, t > 0, \tag{1}$$

subject to

$$u(x,0) = u_0(x). \tag{2}$$

The equations of compressible fluid flow and the Euler equations are examples of PDEs that can be reduced to hyperbolic PDEs when the effects of viscosity and heat conduction are neglected [34]. In addition, many mathematical models are appearing as hyperbolic systems of PDEs that have the general

form:

$$U_t(x,t) = A(x,t)\,U_x(x,t) + B(x,t)\,U(x,t) + F(x,t), x \in S, t \geq 0, \tag{3}$$

subject to

$$U(x,0) = U_0(x), \tag{4}$$

where $U(x,t)$, $F(x,t) \in M_{n\times 1}$, $n \in \mathbb{N}$ are vector functions of two variables x and t, $U_0(x) \in M_{n\times 1}$ is vector function of x, $A(x,t)$, $B(x,t) \in M_{n\times n}$ are matrix functions of two variables x and t, and $A(x_0,t_0)$ is diagonalizable with real eigenvalues for every $(x_0,t_0) \in S \times [0,\infty)$. The system in Eqs. (3) and (4) is said to be strictly hyperbolic if the eigenvalues of $A(x_0,t_0)$ are all distinct whereas it is said to be elliptic at a point (x_0,t_0) if none of the eigenvalues of $A(x_0,t_0)$ are real for every $(x_0,t_0) \in S \times [0,\infty)$.

In recent decades, many mathematical models have been reformulated using the concept of fractional calculus because they are found to reflect the phenomenon that has been modeled in a more precise and realistic way by replacing the ordinary derivative with a fractional derivative of the model. The concept of fractional calculus is derivatives and integrals of fractional order and dates back to the seventeenth century [24, 6] and has recently gained considerable interest because of its extensive use in widespread fields, for instance, biological, chemical, engineering and applied physics such as nonlinear oscillation, waves, and diffusion as we mentioned [24, 6, 33, 18, 23, 35]. In fact, from that date until now, there are many definitions of fractional differential and integral operators, the most popular definition is the Caputo fractional differential operator, $\mathfrak{D}_t^{\alpha}$, which is defined by the following formula [24, 6]:

$$\mathfrak{D}_t^{\alpha} u(x,t) = \begin{cases} J_t^{m-\alpha} \frac{\partial^m}{\partial t^m} u(x,t), & m-1 < \alpha < m, t > t_0 \geq 0, \\ \frac{\partial^m}{\partial t^m} u(x,t), & \alpha = m, \end{cases} \tag{5}$$

where $m \in \mathbb{N}$ and J_t^{β} is the Riemann-Liouville fractional integral operator of order $\beta > 0$ with respect to $t \geq t_0 \geq 0$ which is defined by the following formula:

$$J_t^{\beta} u(x,t) = \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} (t-\tau)^{\beta-1} u(x,\tau)\, d\tau,\ t > \tau > t_0 \geq 0. \tag{6}$$

For more details about the properties of the two previous definitions, you can refer to the references [24, 6, 33, 18, 23, 35]. The most useful properties that we need in this research are:

1. $J_t^{\alpha}(t-t_0)^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}(t-t_0)^{\mu+\alpha}, \mu > -1, t \geq t_0 \geq 0.$
2. $J_t^{\alpha}\lambda = \frac{\lambda}{\Gamma(\alpha+1)}(t-t_0)^{\alpha}$, λ is a constant.

3. $\mathfrak{D}_t^{\alpha}(t-t_0)^{\mu} = \begin{cases} \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}(t-t_0)^{\mu-\alpha}, & \mu \notin \{0,1,2,\ldots,m-1\}, \\ 0, & \mu \in \{0,1,2,\ldots,m-1\} \end{cases}$.

4. $\mathfrak{D}_t^{\alpha}\lambda = 0, \lambda$ is a constant.

It is difficult to find exact solutions for fractional differential and integral equations, for this reason, it is usually used analytical and numerical methods to find approximate solutions to those equations. In recent decades, many methods have been used to find analytical and numerical solutions for fractional differential and integral equations such as the Adomian decomposition method [7], the homotopy perturbation method [28], the variational iteration method [28], the homotopy analysis method [8], and other methods [3, 9, 10].

In the last five years, the residual power series method (RPSM) has achieved an advanced rank among the methods used to find approximate solutions for many fractional differential and integral equations. It has been used in finding exact and approximate solutions for many equations such as homogeneous and nonhomogeneous time and space-fractional telegraph equation [12], linear and nonlinear neutral fractional pantograph Equations [14], space and time-fractional linear and nonlinear KdV-Burgers equation [11], multi-energy groups of neutron diffusion equations [31], fractional multi-pantograph system [13] and other equations. RPSM is characterized by its ease and speed in finding solutions for equations in the form of a power series. In fact, RPSM is a technique for finding the coefficients of the power series without having to find a recurrence relation which we normally obtain from equating the corresponding coefficients in the series. The RPSM is a good alternate proceeding for gaining analytic solutions for fractional PDEs.

Several articles are interested in providing analytical and numerical solutions to fractional PDEs of hyperbolic type, such as Caputo time fractional-order hyperbolic telegraph equation [32], time-fractional hyperbolic PDEs [17, 4, 27, 2, 1], time-fractional diffusion equation [22], fractional advection-dispersion flow equations [26] and other hyperbolic equations. However, a limited number of researches provided analytical as well as numerical solutions for hyperbolic systems of time-fractional PDEs. Kochubei [20] presented a numerical analytical solution for homogenous hyperbolic fractional systems, Hendy et al. [16] introduced a solution for two-dimensional fractional systems was provided by a separate contrast scheme. Therefore, more research is needed in providing analytical and numerical solutions for such systems due to their importance in many applications as mentioned above.

The present work aims to extend the application of RPSM to construct approximate solutions of a hyperbolic system of time-fractional partial differential equations with variable coefficients in the sense of the Caputo's fractional derivative which are given in the form of the following model:

$$U_t^{(\alpha)}(x,t) = A(x,t)\,U_x^{(\beta)}(x,t) + B(x,t)\,U(x,t) + F(x,t), x \in S, t \geq 0, \tag{7}$$

subject to the following initial condition:

$$U(x,0) = U_0(x), \tag{8}$$

where $0 < \alpha, \beta \leq 1$, $U_t^{(\alpha)}(x,t) = \mathfrak{D}_t^{\alpha} U(x,t)$ refers to the Caputo's time-fractional derivative of order α of the multivariable vector function $U(x,t)$, $U_x^{(\beta)}(x,t) = \mathfrak{D}_x^{\beta} U(x,t)$ refers to the Caputo's space-fractional derivative of order β of the multivariable vector function $U(x,t)$ and the definitions of all terms and variables in the Eqs. (7) and (8) are the same as those in Eqs. (3) and (4).

This paper is arranged as follows: In Section 2, the analysis of matrix fractional power series is prepared. In Section 3, the construction of fractional power series solution to a hyperbolic system of time-fractional partial differential equations with variable coefficients in the sense of the Caputo's fractional derivative is presented using RPSM. In Section 4 application models, graphical and numerical simulations are performed in order to illustrate the capability and the simplicity of the proposed method. Finally, conclusions are presented in Section 5.

2. Analysis of matrix fractional power series

In this section, we present some definitions and theories regarding matrix analysis and the matrix power series which are playing a central role in constructing the RPSM solution to a hyperbolic system of time-fractional partial differential equations with variable coefficients.

Definition 2.1 ([19]). *The Riemann-Liouville fractional integral operator of order $\alpha = 0$ of a matrix functions $U(x,t) = [u_{ij}(x,t)] \in M_{r\times k}$ is defined as:*

$$J_t^{\alpha} U(x,t) = \left[J_t^{\alpha} u_{ij}(x,t)\right]_{r\times k},\ 1 \leq i \leq r,\ 1 \leq j \leq k. \tag{9}$$

Definition 2.2 ([19]). *The Caputo's time-fractional derivative operator of order α of a matrix function $U(x,t) = [u_{ij}(x,t)] \in M_{r\times k}$ is:*

$$\mathfrak{D}_t^{\alpha} U(x,t) = \left[\mathfrak{D}_t^{\alpha} u_{ij}(x,t)\right]_{r\times k},\ 1 \leq i \leq r,\ 1 \leq j \leq k. \tag{10}$$

Lemma 2.1. *If $m-1 < \alpha = m$ and $m \in \mathbb{N}$, then*

$$(i) \qquad \mathfrak{D}_t^{\alpha} J_t^{\alpha} U(x,t) = U(x,t),$$

$$(ii) \qquad J_t^{\alpha} \mathfrak{D}_t^{\alpha} U(x,t) = U(x,t) - \sum_{j=0}^{m-1} \frac{\partial^j U(x,0^+)}{\partial t^j} \frac{(t-t_0)^j}{j!},\ t > t_0.$$

Definition 2.3. *Let $A_k \in M_{m\times n}$. We say that a sequence $\{A_k\}$ converges to a matrix $A \in M_{m\times n}$ with respect to a matrix norm $\|\bullet\|$ on $M_{m\times n}$ if and only if $\lim_{k\to\infty} \|A_k - A\| = 0$. If $\{A_k\}$ converges to A with respect to $\|\bullet\|$, we write $\lim_{k\to\infty} A_k = A$.*

Definition 2.4. *For $0 < \alpha \leq 1$, $x \in S$, and $t \geq t_0$ a matrix power series (MPS) of the form:*

$$\sum_{m=0}^{\infty} H_m(x)(t-t_0)^{\ m\alpha} = H_0(x) + H_1(x)(t-t_0)^{\alpha} + H_2(x)(t-t_0)^{2\alpha} + \ldots, \tag{11}$$

is called a bivariate fractional MPS around t_0, where t is a variable and $H_m(x) \in M_{r\times k}$ are matrix functions called series coefficients.

Theorem 2.1. *Assume that $U(x,t) = [u_{ij}(x,t)] \in M_{r\times k}$, $1 \leq i \leq r$, $1 \leq j \leq k$ such that $u_{ij}(x,t) \in C(S \times [t_0, t_0+T))$ and $\mathfrak{D}_t^{w\alpha} u_{ij}(x,t) \in C(S \times (t_0, t_0+T))$ for $1 \leq i \leq r$, $1 \leq j \leq k$, $w = 0,1,2,\ldots,n+1$ where $\mathfrak{D}_t^{w\alpha} = \mathfrak{D}_t^{\alpha} \ . \ \mathfrak{D}_t^{\alpha} \ ... \ \mathfrak{D}_t^{\alpha}$ (w-times) and $\alpha > 0$. Then*

$$J_t^{(n+1)\alpha} \mathfrak{D}_t^{(n+1)\alpha} U(x,t) = \frac{\mathfrak{D}_t^{(n+1)\alpha} U(x,\xi)}{\Gamma((n+1)\alpha+1)} (t-t_0)^{(n+1)\alpha}, t_0 \leq \xi \leq t < t_0+T. \tag{12}$$

Proof. Of the operator definition in Eqs. (6) and (9), we have

$$\begin{aligned} J_t^{(n+1)\alpha} \mathfrak{D}_t^{(n+1)\alpha} U(x,t) &= \frac{1}{\Gamma((n+1)\alpha)} \int_{t_0}^{t} (t-y)^{(n+1)\alpha-1} \mathfrak{D}_y^{(n+1)\alpha} U(x,y)\, dy \\ &= \frac{\mathfrak{D}_t^{(n+1)\alpha} U(x,\xi)}{\Gamma((n+1)\alpha)} \int_{t_0}^{t} (t-y)^{(n+1)\alpha-1} dy \\ &\quad \text{(Based on the second mean value theorem for integral)} \\ &= \frac{\mathfrak{D}_t^{(n+1)\alpha} U(x,\xi)}{\Gamma((n+1)\alpha+1)} (t-t_0)^{(n+1)\alpha}, t_0 \leq \xi \leq t < t_0+T. \quad \square \end{aligned}$$

Theorem 2.2. *Assume that $U(x,t) = [u_{ij}(x,t)] \in M_{r\times k}$, $1 \leq i \leq r$, $1 \leq j \leq k$ such that $u_{ij}(x,t) \in C(S \times [t_0, t_0+T))$ and $\mathfrak{D}_t^{w\alpha} u_{ij}(x,t) \in C(S \times (t_0, t_0+T))$ for $1 \leq i \leq r$, $1 \leq j \leq k$, $w = 0,1,2,\ldots,n+1$ where $\alpha \in (0,1]$. Then*

$$\begin{aligned} U(x,t) = &\sum_{m=0}^{n} \frac{\mathfrak{D}_t^{m\alpha} U(x,t_0)}{\Gamma(m\alpha+1)} (t-t_0)^{m\alpha} \\ &+ \frac{\mathfrak{D}_t^{(n+1)\alpha} U(x,\xi)}{\Gamma((n+1)\alpha+1)} (t-t_0)^{(n+1)\alpha}, \ t_0 \leq \xi \leq t \leq t_0+T. \end{aligned} \tag{13}$$

Proof. From Theorem 2.1, it suffices to demonstrate that:

$$J_t^{(n+1)\alpha} \mathfrak{D}_t^{(n+1)\alpha} U(x,t) = U(x,t) - \sum_{m=0}^{n} \frac{\mathfrak{D}_t^{m\alpha} U(x,t_0)}{\Gamma(m\alpha+1)} (t-t_0)^{m\alpha}.$$

According to Lemma 2.1, it is easy to show that the formula is correct for $n = 0$ and $n = 1$. Thus, inductively, we prove the theorem as follows:

$$\begin{aligned}
J_t^{(n+2)\alpha}\mathfrak{D}_t^{(n+2)\alpha}U(x,t) &= J_t^{\alpha}\left(\left(J_t^{(n+1)\alpha}\mathfrak{D}_t^{(n+1)\alpha}\right)\mathfrak{D}_t^{\alpha}U(x,t)\right)\\
&= J_t^{\alpha}\left(\mathfrak{D}_t^{\alpha}U(x,t) - \sum_{m=0}^{n}\frac{\mathfrak{D}_t^{(m+1)\alpha}U(x,t_0)}{\Gamma(m\alpha+1)}(t-t_0)^{m\alpha}\right)\\
&= J_t^{\alpha}\mathfrak{D}_t^{\alpha}U(x,t) - \sum_{m=0}^{n}\frac{\mathfrak{D}_t^{(m+1)\alpha}U(x,t_0)}{\Gamma((m+1)\alpha+1)}(t-t_0)^{(m+1)\alpha}\\
&= U(x,t) - U(x,t_0) - \sum_{m=1}^{n+1}\frac{\mathfrak{D}_t^{m\alpha}U(x,t_0)}{\Gamma(m\alpha+1)}(t-t_0)^{m\alpha}\\
&= U(x,t) - \sum_{m=0}^{n+1}\frac{\mathfrak{D}_t^{m\alpha}U(x,t_0)}{\Gamma(m\alpha+1)}(t-t_0)^{m\alpha}.
\end{aligned}$$

Thus, the proof of Theorem 2.2 is completed. □

Let's call the series (13); bivariate fractional matrix Taylor's formula (BFMTF) of the matrix function $U(x,t)$. As any series, the tail of the series (13), $R_n(x,t) = \frac{\mathfrak{D}_t^{(n+1)\alpha}U(x,\xi)}{\Gamma((n+1)\alpha+1)}(t-t_0)^{(n+1)\alpha}$, is called the nth remainder for the Taylor series of $U(x,t)$. The function $P(x,t) = U(x,t) - R_n(x,t)$ is an approximate function for $U(x,t)$ and the accuracy of the approximation increases as $R_n(x,t)$ decreases. Finding a bound for $R_n(x,t)$ gives an indication of the accuracy of the approximation $P(x,t) \approx U(x,t)$. The following corollary provides such a bound.

Corollary 2.1 (The Remainder Estimation)**.** *Assume that* $\mathfrak{D}_t^{(n+1)\alpha}U(x,t)$, $\alpha \in (0,1]$ *is defined on* $(S \times (t_0, t_0+d))$. *If* $\left\|\mathfrak{D}_t^{(n+1)\alpha}U(x,t)\right\| \le M(x)$ *on* $t_0 \le t \le d$ *and fixed* x *for some matrix norm* $\|\bullet\|$, *then the reminder* $R_n(x,t)$ *of the BFMTF of* $U(x,t)$ *satisfies:*

$$\left\|\mathfrak{D}_t^{(n+1)\alpha}U(x,t)\right\| \le \frac{M(x)}{\Gamma((n+1)\,\alpha+1)}(t-t_0)^{(n+1)\alpha}, t_0 \le t \le d. \tag{14}$$

Note that when $n \to \infty$, the Taylor's formula (13) becomes of the form:

$$U(x,t) = \sum_{m=0}^{\infty}\frac{U_t^{(m\alpha)}(x,t_0)}{\Gamma(m\alpha+1)}(t-t_0)^{m\alpha},\ x\in S, t_0\le t<t_0+T, \tag{15}$$

which can be applied directly throughout this work.

Finally, it is worth to mention that if $\alpha = 1$, then the BFMTF (15) becomes as follows:

$$U(x,t) = \sum_{m=0}^{\infty}\frac{\partial^m U(x,t_0)}{m!\partial t^m}(t-t_0)^{m},\ t_0\le t<t_0+T, \tag{16}$$

which is the bivariate classical matrix Taylor's formula of a matrix function.

3. Construction of the RPSM solution for a hyperbolic system

In this section, we construct an approximate solution to the hyperbolic system of time-fractional PDEs with variable coefficients given in Eqs. (7) and (8) by using RPSM. To achieve it, we rewrite the Eqs. (7) and (8) as follow:

$$U_t^{(\alpha)}(x,t) - A(x,t)\,U_x^{(\beta)}(x,t) - B(x,t)\,U(x,t) - F(x,t) = 0, \tag{17}$$

$0 < \alpha, \beta \leq 1,\ x \in S, t \geq 0$, subject to the initial condition:

$$U(x,0) = U_0(x). \tag{18}$$

Suppose that the solution of the IVP (17) and (18) has the following expansion:

$$U(x,t) = \sum_{m=0}^{\infty} \frac{H_m(x)}{\Gamma(1+m\alpha)} t^{m\alpha},\ 0 < \alpha \leq 1, x \in S, \quad 0 \leq t < R, \tag{19}$$

where $H_m(x) = U_t^{(m\alpha)}(x,0) \in M_{r\times 1},\ n = 0,1,2,\ldots,$ and $A(x,t), B(x,t)$ & $F(x,t)$ have the following BFMTF:

$$\begin{aligned} A(x,t) &= \sum_{m=0}^{\infty} \frac{A_t^{(m\alpha)}(x,0)}{\Gamma(1+m\alpha)} t^{m\alpha},\ 0 < \alpha \leq 1, x \in S,\ 0 \leq t, \\ B(x,t) &= \sum_{m=0}^{\infty} \frac{B_t^{(m\alpha)}(x,0)}{\Gamma(1+m\alpha)} t^{m\alpha},\ 0 < \alpha \leq 1, x \in S,\ 0 \leq t, \\ F(x,t) &= \sum_{m=0}^{\infty} \frac{F_t^{(m\alpha)}(x,0)}{\Gamma(1+m\alpha)} t^{m\alpha},\ 0 < \alpha \leq 1, x \in S,\ 0 \leq t. \end{aligned} \tag{20}$$

Of course, dealing with a finite series is easier than dealing with an infinite series. For this reason, the RPSM deals with a finite series while calculating coefficients of the series solution. Therefore, we define the kth-truncated series of $U(x,t)$ as follows:

$$U_k(x,t) = \sum_{m=0}^{k} \frac{H_m(x)}{\Gamma(1+m\alpha)} t^{m\alpha},\ 0 < \alpha \leq 1, x \in S,\ 0 \leq t < R. \tag{21}$$

$U(x,t)$ satisfies the initial conditions (18), so from Eq. (19), we obtain $U(x,0) = H_0(x) = H(x)$. On the other hand, from Eq. (21) the initial guess approximation of $U(x,t)$ should be $U_0(x,t) = H_0(x)$. As a result, we can reformulate the expansion of Eq. (21) in the following form:

$$\begin{aligned} U_k(x,t) = H(x) + \sum_{m=1}^{k} \frac{H_m(x)}{\Gamma(1+m\alpha)} t^{m\alpha}, \\ 0 < \alpha \leq 1, x \in S,\ 0 \leq t < R,\ k = 1,2,3,... \end{aligned} \tag{22}$$

To apply the RPSM for finding the value of coefficients $H_m(x), m{=}1,2,3,\ldots,k$ in the series expansion of Eq. (22), we define the residual matrix function (RMF) for Eq. (17) as:

$$\text{RMF}\,(x,t) = U_t^{(\alpha)}\,(x,t) - A\,(x,t)\,U_x^{(\beta)}\,(x,t) - B\,(x,t)\,U\,(x,t) - F\,(x,t)\,, \tag{23}$$

and the k-th residual matrix function (RMF_k) of the style form:

$$\begin{aligned}\text{RMF}_k\,(x,t) &= (U_k)_t^{(\alpha)}\,(x,t) - A\,(x,t)\,(U_k)_x^{(\beta)}\,(x,t)\\ &\quad - B\,(x,t)\,U_k\,(x,t) - F\,(x,t)\,,\end{aligned} \tag{24}$$

where $0<\alpha,\beta\le 1, x\in S,\ 0\le t<t_0+R$.

Substitute the expansions of $U\,(x,t)\,, A\,(x,t)\,, B\,(x,t)$ and $F\,(x,t)$ into Eqs. (23) and (24) to obtain the following form of the RMF and RMF_k functions:

$$\begin{aligned}\text{RMF}\,(x,u) &= \sum_{m=0}^{\infty}\mathcal{H}_{(m+1)}\,(x)t^{m\alpha} - \sum_{m=0}^{\infty}\mathcal{A}_m\,(x)\,t^{m\alpha}\\ &\quad - \sum_{m=0}^{\infty}\mathcal{B}_m\,(x)\,t^{m\alpha} - \sum_{m=0}^{\infty}\mathcal{F}_m\,(x)\,t^{m\alpha},\end{aligned} \tag{25}$$

and

$$\text{RMF}_k\,(x,t) = \sum_{m=0}^{k}\big(\mathcal{H}_{(m+1)}\,(x) - \mathcal{A}_m\,(x) - \mathcal{B}_m\,(x) - \mathcal{F}_m(x)\big)t^{m\alpha}, \tag{26}$$

where

$$\begin{aligned}\mathcal{H}_{(m+1)}\,(x) &= \frac{H_{(m+1)}(x)}{\Gamma\,(1+m\alpha)},\\ \mathcal{A}_m\,(x) &= \sum_{i=0}^{m}\frac{A_t^{(i\alpha)}\,(x,0)}{\Gamma\,(1+i\alpha)}\frac{H_{m-i}^{(\beta)}\,(x)}{\Gamma\,(1+(m-i)\alpha)},\\ \mathcal{B}_m\,(x) &= \sum_{i=0}^{m}\frac{B_t^{(i\alpha)}\,(x,0)}{\Gamma\,(1+i\alpha)}\frac{H_{m-i}(x)}{\Gamma\,(1+(m-i)\alpha)},\\ \mathcal{F}_m\,(x) &= \frac{F_t^{(m\alpha)}\,(x,0)}{\Gamma\,(1+m\alpha)}.\end{aligned} \tag{27}$$

The main idea of the RPSM can be shown in the following facts:

1. $\text{RMF}(x,t){=}\,0$.
2. $\lim_{k\to\infty}\text{RMF}_k\,(x,t)$ =RMF (x,t) for each $x\in S$ and $0\le t<R$.
3. $\mathfrak{D}_t^{(j-1)\alpha}\text{RMF}_k\,(x,0)=0,\ j=1,2,3,\ldots$ (by (4)).

4. $\mathfrak{D}_t^{(j-1)\alpha}\mathrm{RMF}\,(x,0)=\mathfrak{D}_t^{(j-1)\alpha}\mathrm{RMF}_k\,(x,0)$, $j=1,2,3,\ldots$.

For the form of the coefficients formulas $H_m\,(x)$, $m=1,2,3,\ldots,k$ in Eq. (22), we apply the operator $\mathfrak{D}_t^{(j-1)\alpha}$ on the Eq. (26) to get the following formula:

$$\mathfrak{D}_t^{(j-1)\alpha}\mathrm{RMF}_k\,(x,t)=$$

$$\sum_{m=j-1}^{k}\left(\mathcal{H}_{(m+1)}\,(x)-\mathcal{A}_m\,(x)-\mathcal{B}_m\,(x)-\mathcal{F}_m(x)\right)\frac{\Gamma\,(1+m\alpha)\,t^{(m-j+1)\alpha}}{\Gamma\,(1+(m-j+1)\alpha)}. \tag{28}$$

For $j=1$, the fact $\mathfrak{D}_t^{(j-1)\alpha}\mathrm{RMF}_k\,(x,0)=0$ and formula (28) lead to the algebraic equation $\mathcal{H}_1\,(x)-\mathcal{A}_0\,(x)-\mathcal{B}_0\,(x)-\mathcal{F}_0\,(x)=0$. According to the relations in Eq. (27), we have the second coefficient in the series (21) which is given in the following formula:

$$H_1\,(x)=A\,(x,0)\,H^{(\beta)}\,(x)+B\,(x,0)\,H\,(x)+F(x,0). \tag{29}$$

Similarly, for $j=2$, the third coefficient of the series (21) has the following form:

$$\begin{aligned}H_2\,(x)&=\left(\mathcal{A}_1\,(x)+\mathcal{B}_1\,(x)+\mathcal{F}_1\,(x)\right)\ \Gamma\,(1+\alpha)\\&=A\,(x,0)\,H_1^{(\beta)}\,(x)+A_t^{(\alpha)}\,(x,0)\,H^{(\beta)}\,(x)+B\,(x,0)\,H_1\,(x)\\&\quad+B_t^{(\alpha)}\,(x,0)\,H\,(x)+F_t^{(\alpha)}\,(x,0)\,.\end{aligned} \tag{30}$$

This procedure can be repeated up to arbitrary order coefficients of the bivariate fractional MPS solution of Eqs. (17) and (18) that are obtained. So, it is clear that for $j=k$, the coefficient $H_k(x)$ of the series (21) will be given in the following recurrence relation:

$$\begin{aligned}&H_0\,(x)=H\,(x)\\&H_k(x)=\ \Gamma(1+(k-1)\alpha)(\mathcal{A}_{k-1}(x)+\mathcal{B}_{k-1}(x)+\mathcal{F}_{k-1}(x)),k=1,2,\ldots,\end{aligned} \tag{31}$$

where $\mathcal{A}_{m-1}$, $\mathcal{B}_{m-1}$ and $\mathcal{F}_{m-1}$ are given in Eq. (27). Therefore, the exact solution of the IVP (17) and (18) will be given in the following series form:

$$\begin{aligned}U(x,t)=&H(x)+\sum_{m=1}^{\infty}(\mathcal{A}_{m-1}(x)+\mathcal{B}_{m-1}(x)\\&+\mathcal{F}_{m-1}(x))\frac{\Gamma(1+(m-1)\alpha)}{\Gamma\,(1+m\alpha)}t^{m\alpha},\end{aligned} \tag{32}$$

and the kth-approximation of the solution of the IVP (17) and (18) will be as follows:

$$\begin{aligned}U_k\,(x,t)=&H\,(x)+\sum_{m=1}^{k}(\mathcal{A}_{m-1}\,(x)+\mathcal{B}_{m-1}\,(x)\\&+\mathcal{F}_{m-1}\,(x))\frac{\Gamma\,(1+(m-1)\,\alpha)}{\Gamma\,(1+m\alpha)}t^{m\alpha},\end{aligned} \tag{33}$$

where $x\in S,\ 0\leq t<R$.

4. Applications and numerical simulations

To test our proposed method, we present in this section four interesting and important applications. The first three applications are set up so that the exact solution is already known whereas the last application is set up without knowing the solution in advance to test the predictability of the solution or obtain a suitable approximate solution. All symbolic and numerical calculations were performed using Mathematica 7 software.

Application 4.1. *Consider the following homogeneous hyperbolic system of time-fractional PDEs with variable coefficients:*

$$U_t^{(\alpha)}(x,t) - A(x,t)\,U_x^{(1)}(x,t) - B(x,t)\,U(x,t) = 0,\ 0 < \alpha \le 1, x \in \mathbb{R}, t \ge 0, \tag{34}$$

subject to the following initial condition:

$$U(x,0) = \begin{pmatrix} x \\ 1 \end{pmatrix}, \tag{35}$$

where

$$A(x,t) = \begin{pmatrix} t^{\alpha}x & \frac{1}{2x} \\ x^2 & -\frac{t^{3\alpha}}{2x} \end{pmatrix}, \quad B(x,t) = \begin{pmatrix} t^{4\alpha}x - t^{\alpha} & -t^{2\alpha} \\ \frac{t^{5\alpha}+t^{2\alpha}x^4}{x} + \frac{t^{\alpha}x\Gamma(1+2\alpha)}{\Gamma(1+\alpha)} & -x^2 \end{pmatrix},$$

and the exact solution is $U(x,t) = \begin{pmatrix} x \\ 1 + t^{2\alpha}x^2 \end{pmatrix}$.

To obtain a matrix expansion solution for this application using RPSM, assume that the solution takes the form of the series in Eq. (19). According to the initial condition in Eq. (35), the first coefficient of the expansion Eq. (19) $H_0(x) = H(x) = \begin{pmatrix} x \\ 1 \end{pmatrix}$. The coefficients of expansion are computed by recurrence relation in Eq. (31) and the formulas in Eq. (27). The functions of Eq. (27) for this application has the following expressions

$$\begin{aligned}
&\mathcal{A}_0(x) = \begin{pmatrix} 0 \\ x^2 \end{pmatrix}, \quad \mathcal{B}_0(x) = \begin{pmatrix} 0 \\ -x^2 \end{pmatrix}, \\
&\mathcal{A}_1(x) = \begin{pmatrix} x \\ 0 \end{pmatrix}, \quad \mathcal{B}_1(x) = \begin{pmatrix} -x \\ \frac{x^2\Gamma(1+2\alpha)}{\Gamma(1+\alpha)} \end{pmatrix}, \\
&\mathcal{A}_2(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathcal{B}_2(x) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \\
&\mathcal{A}_m(x) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \mathcal{B}_m(x) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad m = 3,4,5,\ldots.
\end{aligned} \tag{36}$$

So, by Eq. (31), the coefficients of the expansion in Eq. (19) are given by the following vector functions:

$$
\begin{aligned}
H_0(x) &= \begin{pmatrix} x \\ 1 \end{pmatrix}, \\
H_1(x) &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\
H_2(x) &= \begin{pmatrix} 0 \\ x^2 \Gamma(1+2\alpha) \end{pmatrix}, \\
H_m(x) &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad m = 4, 5, 6, \dots .
\end{aligned} \tag{37}
$$

Therefore, the matrix expansion solution of the IVP (34) and (35) would be as follows:

$$
U(x,t) = \begin{pmatrix} x \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ x^2 \end{pmatrix} t^{2\alpha}, \tag{38}
$$

which coincides with the exact solution.

Application 4.2. *Consider the following non-homogeneous hyperbolic system of time-fractional PDEs with variable coefficients:*

$$
\begin{aligned}
&U_t^{(\alpha)}(x,t) + A(x,t)\, U_x^{(1)}(x,t) + B(x,t)\, U(x,t) \\
&= F(x,t), 0 < \alpha \leq 1, x \in \mathbb{R},\ t \geq 0,
\end{aligned} \tag{39}
$$

subject to the following initial condition:

$$
U(x,0) = \begin{pmatrix} 0 \\ x^2 \end{pmatrix}, \tag{40}
$$

where

$$
A(x,t) = \begin{pmatrix} x & t^{\alpha} \\ 2x + t^{2\alpha} & 0 \end{pmatrix}, \quad B(x,t) = \begin{pmatrix} t^{\alpha} & x \\ 0 & 2x + t^{2\alpha} \end{pmatrix},
$$

$$
F(x,t) = \begin{pmatrix} x^3 + e^x \Gamma(1+\alpha) + (2+e^x)\, x t^{\alpha} + (x+e^x)\, t^{2\alpha} \\ 2x^3 + \left(2x e^x + \frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}\right) t^{\alpha} + (2+x)\, x t^{2\alpha} + e^x t^{3\alpha} + t^{4\alpha} \end{pmatrix}.
$$

According to the construction in Section 3, assume that the IVP (39) and (40) has a matrix expansion solution as in Eq. (19). The initial coefficient of the expansion in Eq. (19) is determined from the initial condition Eq. (40) which is equal to $H_0(x) = H(x) = \begin{pmatrix} 0 \\ x^2 \end{pmatrix}$. By the coefficients of Eq. (39), $A(x,t)$,

$B(x,t)$ and $F(x,t)$, the functions in Eq. (27) can be determined as follows:

$$\begin{aligned}
&\mathcal{A}_0(x)=\begin{pmatrix}0\\0\end{pmatrix},\quad \mathcal{B}_0(x)=\begin{pmatrix}-x^3\\-2x^3\end{pmatrix},\quad \mathcal{F}_0(x)=\begin{pmatrix}x^3+\mathrm{e}^x\Gamma(1+\alpha)\\2x^3\end{pmatrix},\\
&\mathcal{A}_1(x)=\begin{pmatrix}-(2+\mathrm{e}^x)x\\-2\mathrm{e}^x x\end{pmatrix},\quad \mathcal{B}_1(x)=\begin{pmatrix}0\\0\end{pmatrix},\quad \mathcal{F}_1(x)=\begin{pmatrix}(2+\mathrm{e}^x)x\\2\mathrm{e}^x x+\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}\end{pmatrix},\\
&\mathcal{A}_2(x)=\begin{pmatrix}0\\0\end{pmatrix},\quad \mathcal{B}_2(x)=\begin{pmatrix}-\mathrm{e}^x-x\\-x(2+x)\end{pmatrix},\quad \mathcal{F}_2(x)=\begin{pmatrix}\mathrm{e}^x+x\\x(2+x)\end{pmatrix},\\
&\mathcal{A}_3(x)=\begin{pmatrix}0\\-\mathrm{e}^x\end{pmatrix},\quad \mathcal{B}_3(x)=\begin{pmatrix}0\\0\end{pmatrix},\quad \mathcal{F}_3(x)=\begin{pmatrix}0\\\mathrm{e}^x\end{pmatrix},\\
&\mathcal{A}_4(x)=\begin{pmatrix}0\\0\end{pmatrix},\quad \mathcal{B}_4(x)=\begin{pmatrix}0\\-1\end{pmatrix},\quad \mathcal{F}_4(x)=\begin{pmatrix}0\\1\end{pmatrix},\\
&\mathcal{A}_m(x)=\begin{pmatrix}0\\0\end{pmatrix},\quad \mathcal{B}_m(x)=\begin{pmatrix}0\\0\end{pmatrix},\quad \mathcal{F}_m(x)=\begin{pmatrix}0\\0\end{pmatrix},\quad m=5,6,7,\ldots.
\end{aligned}\tag{41}$$

These functions are the basic components of the expansion coefficients in Eq. (19) and by compensating them in recurrence relation in Eq. (31), we get those coefficients which are given as follows:

$$\begin{aligned}
&H_0(x)=\begin{pmatrix}0\\x^2\end{pmatrix},\\
&H_1(x)=\begin{pmatrix}\mathrm{e}^x\Gamma(1+\alpha)\\0\end{pmatrix},\\
&H_2(x)=\begin{pmatrix}0\\\Gamma(1+2\alpha)\end{pmatrix},\\
&H_m(x)=\begin{pmatrix}0\\0\end{pmatrix},\quad m=3,4,5,\ldots.
\end{aligned}\tag{42}$$

Thus, the matrix expansion solution of the IVP (39) and (40) would be as follows:

$$U(x,t)=\begin{pmatrix}0\\x^2\end{pmatrix}+\begin{pmatrix}\mathrm{e}^x\\0\end{pmatrix}t^{\alpha}+\begin{pmatrix}0\\1\end{pmatrix}t^{2\alpha},\tag{43}$$

which is coinciding with the exact solution $U(x,t)=\begin{pmatrix}\mathrm{e}^x t^{\alpha}\\x^2+t^{2\alpha}\end{pmatrix}$.

Application 4.3. *Consider the following non-homogeneous hyperbolic system of time-Space-fractional PDEs with variable coefficients:*

$$\begin{aligned}
&U_t^{(\alpha)}(x,t)+A(x,t)\,U_x^{(\beta)}(x,t)+B(x,t)\,U(x,t)\\
&=F(x,t),\,0<\alpha,\beta\le 1,x,t\ge 0,
\end{aligned}\tag{44}$$

subject to the following initial condition:

$$U(x,0)=\begin{pmatrix} 0 \\ x^{\beta} \end{pmatrix}, \tag{45}$$

where

$$A(x,t)=\begin{pmatrix} x^{\beta} & 0 \\ 0 & x^{\beta} \end{pmatrix}, \quad B(x,t)=\begin{pmatrix} 1 & t^{\alpha} \\ t^{\alpha} & 1 \end{pmatrix},$$
$$F(x,t)=\begin{pmatrix} t^{\alpha}x^{\beta}E_{\alpha}(t^{\alpha})+\left(t^{\alpha}+t^{\alpha}x^{\beta}+\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}\right)t^{\alpha}E_{\beta}(x^{\beta}) \\ (2+\Gamma(1+\beta))\,x^{\beta}E_{\alpha}(t^{\alpha})+t^{3\alpha}E_{\beta}(x^{\beta}) \end{pmatrix},$$

and the exact solution is

$$U(x,t)=\begin{pmatrix} t^{2\alpha}E_{\beta}(x^{\beta}) \\ x^{\beta}E_{\alpha}(t^{\alpha}) \end{pmatrix}, \tag{46}$$

where $E_{\alpha}(t)$ is the Mittag-Leffler function defined by the following expansion [30]:

$$E_{\alpha}(t)=\sum_{m=0}^{\infty}\frac{t^{m}}{\Gamma(1+m\alpha)}. \tag{47}$$

All symbolic and numerical calculations were done using Mathematica 7 through a low-Ram PC. Since the Mittag-Leffler function is an infinite expansion, it was difficult to perform the calculations using the Mittag-Leffler function as it is. For this the 5th-truncated of the expansion in Eq. (47) was used through the calculations.

Like the previous applications, assume that the solution of the IVP (44) and (45) has a matrix expansion as in Eq. (19). According to the initial conditions in Eq. (45), $H_0(x)=H(x)=\begin{pmatrix} 0 \\ x^{\beta} \end{pmatrix}$. To determine the rest coefficients of the series Eq. (19), we need to compute the basic functions in Eq. (27) and use them in the recurrence relation in Eq. (31). The first few coefficients are given in the following vectors:

$$H_0(x)=\begin{pmatrix} 0 \\ x^{\beta} \end{pmatrix},$$
$$H_1(x)=\begin{pmatrix} 0 \\ x^{\beta} \end{pmatrix},$$
$$H_2(x)=\begin{pmatrix} \Gamma(1+2\alpha)\,E_{\beta}(x^{\beta}) \\ x^{\beta} \end{pmatrix},$$

$$H_3(x) = \begin{pmatrix} 0 \\ x^\beta \end{pmatrix}, \tag{48}$$
$$H_4(x) = \begin{pmatrix} 0 \\ x^\beta \end{pmatrix},$$
$$H_5(x) = \begin{pmatrix} 0 \\ x^\beta \end{pmatrix}.$$

So, the 5th-approximation of the solution of the IVP (44) and (45) can be expressed as follows:

$$\begin{aligned} U_5(x,t) = &\begin{pmatrix} 0 \\ x^\beta \end{pmatrix} + \begin{pmatrix} 0 \\ x^\beta \end{pmatrix}\frac{t^\alpha}{\Gamma(1+\alpha)} + \begin{pmatrix} \Gamma(1+2\alpha)E_\beta(x^\beta) \\ x^\beta \end{pmatrix}\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \\ &+ \begin{pmatrix} 0 \\ x^\beta \end{pmatrix}\frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + \begin{pmatrix} 0 \\ x^\beta \end{pmatrix}\frac{t^{4\alpha}}{\Gamma(1+4\alpha)} + \begin{pmatrix} 0 \\ x^\beta \end{pmatrix}\frac{t^{5\alpha}}{\Gamma(1+5\alpha)}. \end{aligned} \tag{49}$$

Obvious, there is a pattern between the terms of Eq. (49) which gives us the exact solution as in Eq. (46).

The mathematical behavior of the solution of the IVP (44) and (45) are illustrated next by plotting the 3-dimensional space figures of the 5th-approximation of the two components of the vector solution in Eq. (49) for different cases. Figures 1 (a), (b) and (c) show the 5th-approximate solution, $(U_1)_5(x,t)$ and $(U_2)_5(x,t)$, when $(\alpha,\beta) = (0.7, 0.4)$, $(\alpha,\beta) = (0.4, 0.7)$ and $(\alpha,\beta) = (0.9, 1)$, respectively on the square $[0,1] \times [0,1]$. Figure 1 (d) shows the exact solution expressed by Eq. (46) for $(\alpha,\beta) = (0.9, 1)$.

Figure 1 (c) and (d) show that the 5th-approximation of the solution of the IVP (44) and (45) is excellent compared to the exact solution, as well as in previous cases, which have not been documented in order not to increases the numbers of graphs. It is known that by increasing the number of terms in the series, the accuracy of the solution increases and thus the error of solution reduces, therefore we can reduce the error of the solution by calculating more coefficients of the matrix expansion solution in Eq. (19).

In the next application, the exact solution is unknown. Therefore, we are trying to find the exact solution or an appropriate approximation of the solution.

Application 4.4. *Consider the following non-homogeneous hyperbolic system of time-fractional PDEs with variable coefficients:*

$$\begin{aligned} &U_t^{(\alpha)}(x,t) - A(x,t)\,U_x^{(1)}(x,t) - B(x,t)\,U(x,t) \\ &= F(x,t), 0 < \alpha \leq 1, x \in \mathbb{R}, t \geq 0, \end{aligned} \tag{50}$$

subject to the following initial condition:

$$U(x,0) = \begin{pmatrix} x+1 \\ e^x \end{pmatrix}, \tag{51}$$

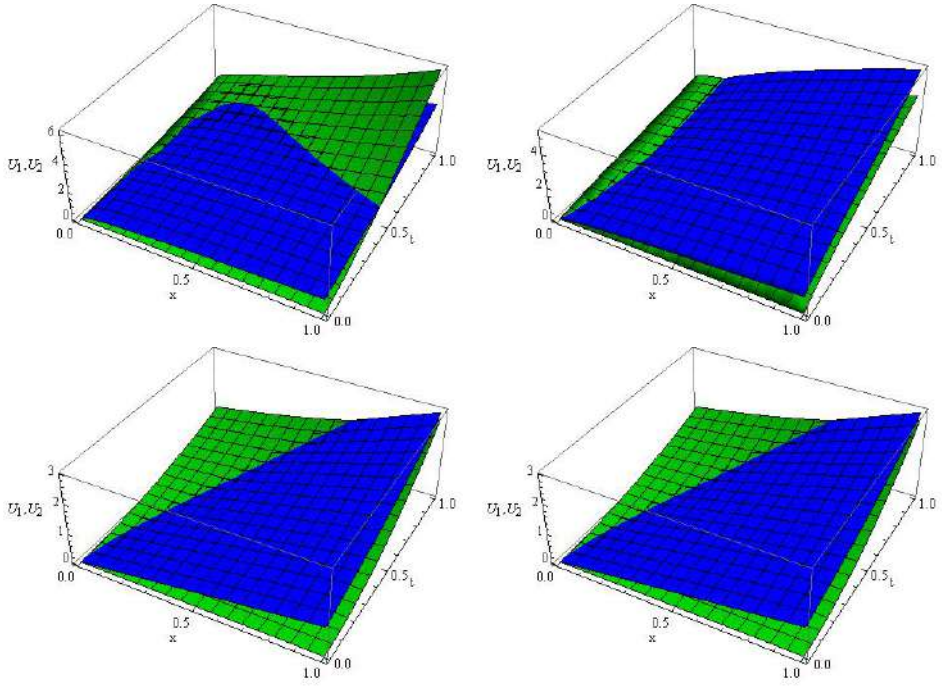


Figure 1: The surface graphs of the 5th-approximate solutions of $U_1(x,t)$ and $U_2(x,t)$ in Eq. (49) and the exact solution of $U_1(x,t)$ and $U_2(x,t)$ in Eq. (46): (a) $(\alpha,\beta)=(0.7,0.4)$, (b) $(\alpha,\beta)=(0.4,0.7)$, (c) $(\alpha,\beta)=(0.9,1)$, (d) exact solution when $(\alpha,\beta)=(0.9,1)$.

where

$$A(x,t)=\begin{pmatrix} 0 & -e^{-x}\left(e^{x}+(x-1)E_{\alpha}(t^{\alpha})+(1+x)t^{\alpha}\right) \\ \Gamma(1+\alpha)-e^{x}x-t^{\alpha}x & 0 \end{pmatrix},$$

$$B(x,t)=\begin{pmatrix} t^{\alpha} & 1 \\ 0 & x \end{pmatrix},\quad F(x,t)=\begin{pmatrix} xE_{\alpha}(t^{\alpha}) \\ 0 \end{pmatrix}.$$

Similar to the previous applications, the recurrence relation in Eq. (31) gives us the coefficients of the matrix expansion solution of the IVP (50) and (51). The first seven coefficients of the expansion Eq. (19) for this application are given by the following vector functions:

$$H_0(x)=\begin{pmatrix} x+1 \\ e^{x} \end{pmatrix},$$

$$H_1(x)=\begin{pmatrix} 1 \\ \Gamma(1+\alpha) \end{pmatrix},$$

$$H_2(x)=\begin{pmatrix} 1+\Gamma(1+\alpha) \\ 0 \end{pmatrix},$$

$$
\begin{aligned}
(52)\qquad H_3(x) &= \begin{pmatrix} 1+\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)} \\ 0 \end{pmatrix},\\
H_4(x) &= \begin{pmatrix} 1+\frac{(1+\Gamma(1+\alpha))\Gamma(1+3\alpha)}{\Gamma(1+2\alpha)} \\ 0 \end{pmatrix},\\
H_5(x) &= \begin{pmatrix} 1+\frac{(\Gamma(1+\alpha)+\Gamma(1+2\alpha))\Gamma(1+4\alpha)}{\Gamma(1+\alpha)\Gamma(1+3\alpha)} \\ 0 \end{pmatrix},\\
H_6(x) &= \begin{pmatrix} 1+\frac{(\Gamma(1+2\alpha)+(1+\Gamma(1+\alpha))\Gamma(1+3\alpha))\Gamma(1+5\alpha)}{\Gamma(1+2\alpha)\Gamma(1+4\alpha)} \\ 0 \end{pmatrix},\\
H_7(x) &= \begin{pmatrix} 1+\frac{(\Gamma(1+\alpha)\Gamma(1+3\alpha)+(\Gamma(1+\alpha)+\Gamma(1+2\alpha))\Gamma(1+4\alpha))\Gamma(1+6\alpha)}{\Gamma(1+3\alpha)\Gamma(1+5\alpha)} \\ 0 \end{pmatrix}.
\end{aligned}
$$

Thus, the 7th-approximate solution of the IVP (50) and (51) can be expressed as follows:

$$
\begin{aligned}
(53)\qquad U_7(x,t) &= H_0(x) + H_1(x)\frac{t^{\alpha}}{\Gamma(1+\alpha)} + H_2(x)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\\
&\quad + H_3(x)\frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + H_4(x)\frac{t^{4\alpha}}{\Gamma(1+4\alpha)} + H_5(x)\frac{t^{5\alpha}}{\Gamma(1+5\alpha)}\\
&\quad + H_6(x)\frac{t^{6\alpha}}{\Gamma(1+6\alpha)} + H_7(x)\frac{t^{7\alpha}}{\Gamma(1+7\alpha)}.
\end{aligned}
$$

In order to examine the approximate solution as in Eq. (53), we need to find the norm of residual error vector ($\|\mathrm{RES}(x,t)\|$) for different values of t and x in the region $[0,1]\times[0,1]$, where the residual error vector is defined by

$$
\begin{aligned}
\mathrm{RES}_k(x,t) = &(U_k)_t^{(\alpha)}(x,t) - A(x,t)\,(U_k)_x^{(1)}(x,t)\\
&- B(x,t)\,U_k(x,t) - F(x,t).
\end{aligned}
\qquad (54)
$$

Tables 1 and 2 show that the values of $\|\mathrm{RES}_6(x,t)\|$ and $\|\mathrm{RES}_7(x,t)\|$, respectively, for different values of α. The data in the tables indicate that the norm of the residual error of the obtained approximate solution decreases as $(x,t)\to(0,0)$ as well as when $\alpha\to 1$. This indicates that the convergence of the bivariate fractional MPS Eq. (19) depends on t, x and α as illustrated in Corollary 2.1. As we know, we can reduce the error in the matrix expansion solution Eq. (19) as we increase the number of terms of the expansion. As we can see from the data in Tables 1 and 2, the 7th approximation gave us a solution better than the 6th approximation. Anyway, it can be said that the RPSM is good at providing an accurate approximate solution of a hyperbolic system of time-fractional PDEs with variable coefficients.

Table 1: The values of $\|\mathrm{RES}_6(x,t)\|$ for different values of α.

$(\boldsymbol{x},\boldsymbol{t})$	$\alpha=0.6$	$\alpha=0.8$	$\alpha=1.0$
$(0.0,0.0)$	0.000000	0.000000	0.000000
$(0.2,0.2)$	3.5563×10^{-3}	1.6882×10^{-4}	7.6648×10^{-6}
$(0.4,0.4)$	4.8228×10^{-2}	5.1980×10^{-3}	5.3168×10^{-4}
$(0.6,0.6)$	2.2559×10^{-1}	3.9524×10^{-2}	6.5249×10^{-3}
$(0.8,0.8)$	6.8021×10^{-1}	1.6888×10^{-1}	3.9295×10^{-2}
$(1.0,1.0)$	1.6103×10^{0}	5.2523×10^{-1}	1.5995×10^{-1}

Table 2: The values of $\|\mathrm{RES}_7(x,t)\|$ for different values of α.

$(\boldsymbol{x},\boldsymbol{t})$	$\alpha=0.6$	$\alpha=0.8$	$\alpha=1.0$
$(0.0,0.0)$	0.000000	0.000000	0.000000
$(0.2,0.2)$	9.6895×10^{-4}	2.6675×10^{-5}	6.82732×10^{-7}
$(0.4,0.4)$	1.9432×10^{-2}	1.3932×10^{-3}	9.2535×10^{-5}
$(0.6,0.6)$	1.1395×10^{-1}	1.4361×10^{-2}	1.6690×10^{-3}
$(0.8,0.8)$	4.0283×10^{-1}	7.5982×10^{-2}	1.3162×10^{-2}
$(1.0,1.0)$	1.0780×10^{0}	2.7858×10^{-1}	6.5901×10^{-2}

5. Conclusions

We have found that the exact solution for the hyperbolic system of time-fractional partial differential equations with variable coefficients is available if the solution was a linear combination of power functions or if it a composite of an elementary function and a power function. In case the exact solution is not available, a good approximation of the solution can be obtained. The RPSM is an effective and accurate technique in determining the values of coefficients for the series solution. By this work, we have presented a solution that may be missing for this kind of problem and we have opened the way for researchers to provide other ways to solve this class of equations. Moreover, we can use the proposed method to solve other types of systems of Differential equations such as the temporal-fractional Drinfeld–Sokolov–Wilson System, the epidemiological model, and others.

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Near BE-semigroups

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Abstract. In this paper, we introduce a new algebraic structure the so-called near BE-semigroup which is a generalization of a BE-semigroup. We provide some examples to show the existence of the structure. Furthermore, some properties of this structure are investigated.
Keywords: near BE-semigroups, unit divisors, deductive systems.

1. Introduction

Iseki and Tanaka defined BCK-algebra in [2] while Iseki defined BCI-algebra in [7]. It is well known that every BCK-algebra is a BCI-algebra, i.e. a BCI-algebra is a generalization of a BCK-algebra. Hu and Li defined another class of abstract algebras which is known as a BCH-algebra [6, 15]. They showed that

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every BCI-algebra is a BCH-algebra. For general information on BCK-algebras we refer to [3]. Neggers and Kim [12] gave the idea of a d-algebra which is a generalization of BCK-algebras, and they gave the idea of a B-algebra in [1]as well. They showed that it is equivalent to the idea of a group. Moreover, in [5] they discussed homomorphisms of B-algebras and explored some useful properties. Furthermore, the author in [10] introduced a new idea, known as a BH-algebra, which is another generalization of BCH, BCI and BCK-algebras. B-algebra was deeply studied by Walendziak in [16] in which he obtained other equivalent set of axioms for a B-algebra. The authors in [4] gave the idea of a (pre-) Coxeter algebra and showed that a Coxeter algebra is equivalent to a Boolean group. Moreover, in [11] the idea of a BM-algebra has been given which is a special case of B-algebras and they showed that a BM-algebra is equivalent to a 1-commutative B-algebra. In [8], the authors gave the idea of a BE-algebra which is a generalization of a BCK-algebra. In [13, 14], the authors gave the idea of ideals and upper sets in BE-algebras and obtained several properties of such ideals and upper sets. By using the idea of upper sets, they gave an equivalent condition for filters in BE-algebras. By combining BE-algebra and semigroup, Ahn and Kim gave the idea of a BE-semigroup and explored some elegant and interesting properties [9]. In the near past, a new generalization of BE-algebra, termed as PSRU-algebra, has been defined in [17] while in the paper [18] the author has worked on the generalizations of BCK-algebras and explored the implicative property of the generalized algebras.

2. Preliminaries

In this section, we define *near BE-semigroup* which is a generalization of a BE-semigroup. We give some examples and then some of their properties are discussed. Firstly, we recall the definition of BE-algebra [8].

Definition 2.1. *An algebraic structure ($\boldsymbol{W}$; $\diamond$, 1) is said to be a BE-algebra if the following conditions are satisfied:*

(i) $(\forall x \in \boldsymbol{W}), x \diamond x = 1$,

(ii) $(\forall x \in \boldsymbol{W}), x \diamond 1 = 1$,

(iii) $(\forall x \in \boldsymbol{W}), 1 \diamond x = x$,

(iv) $(\forall x, y, z \in \boldsymbol{W}), x \diamond (y \diamond z) = y \diamond (x \diamond z)$.

The property (iv) *is known as exchange law. Also note that if ($\boldsymbol{W}$; $\diamond$, 1) is a BE-algebra, then $u \diamond (v \diamond u) = 1$ for any $u, v \in \boldsymbol{W}$. Let we define a relation "$\leq$" in the BE-algebra ($\boldsymbol{W}$; $\diamond$,1) as follows:*

Let $a, b \in \boldsymbol{W}$, then $a \leq b$ if and only if $a \diamond b = 1$.

Let us give some examples. For the following examples we refer to [8].

Example 2.1. (i) Let $\boldsymbol{W} = \{1, 2, 3, 4, 5, 6\}$ and define "$\diamond$" in $\boldsymbol{W}$ in the table given below:

$\diamond$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	1	1	2	4	4	5
3	1	1	1	4	4	4
4	1	2	3	1	2	3
5	1	1	2	1	1	2
6	1	1	1	1	1	1

Then $(\boldsymbol{W}; \diamond, 1)$ is a BE-algebra.

(ii) Let $\boldsymbol{W} = \{1, 2, 3, 4, 5\}$. Let "$\diamond$" be a binary operation in $\boldsymbol{W}$ defined in the following table:

$\diamond$	1	2	3	4	5
1	1	2	3	4	5
2	1	1	1	1	1
3	1	2	1	1	1
4	1	1	1	1	1
5	1	2	3	4	1

Then $(\boldsymbol{W}; \diamond, 1)$ is a BE-algebra.

A BE-algebra $(\boldsymbol{W}; \diamond, 1)$ is said to be self-distributive if $\forall\ u, v, w \in \boldsymbol{W}$, $u \diamond (v \diamond w) = (u \diamond v) \diamond (u \diamond w)$.

Let us state some properties of a self-distributive BE-algebra $(\boldsymbol{W}; \diamond, 1)$ which will be used later. For the following result we refer to [8].

Proposition 2.1. *Let $(\boldsymbol{W}; \diamond, 1)$ be a self-distributive BE-algebra and further let $u, v \in \boldsymbol{W}$ be such that $u \leq v$. Then $u \diamond w \leq v \diamond w$ for all $w \in \boldsymbol{W}$.*

Proof. It is straightforward. □

In the following definition, we introduce the new structure of *near BE-semigroup*.

Definition 2.2. *An algebraic system $(\boldsymbol{W}; \otimes, \diamond, 1)$ is said to be a near BE-semigroup if the following conditions are satisfied:*

(i) $(\boldsymbol{W}; \otimes)$ is a semigroup,

(ii) $(\boldsymbol{W}; \diamond, 1)$ is a BE-algebra,

(iii) The operation "$\otimes$" is left distributive over the operation "$\diamond$" .

Note that it will be more accurate to call what we have just defined a near left BE-semigroup. There is a similar definition of a near right BE-semigroup. Throughout in this paper, we shall consider a near left BE-semigroup and simply call it a near BE-semigroup. Also note that we shall use NBE-semigroup for a near BE-semigroup.

Let us give some examples in order to show the existence of the above structure.

Example 2.2. Let $\boldsymbol{W} = \{1, s, t, q\}$ and define two operations "$\otimes$" and "$\diamond$" in $\boldsymbol{W}$ as follows:

$\diamond$	1	s	t	q
1	1	1	1	1
s	1	1	1	1
t	1	1	1	s
q	1	1	1	s

$\otimes$	1	s	t	q
1	1	s	t	q
s	1	1	t	q
t	1	s	1	q
q	1	1	1	1

Then, it can be easily verified that $(\boldsymbol{W}; \otimes, \diamond, 1)$ is an NBE-semigroup.

Example 2.3. Let $\boldsymbol{W} = \{1, s, t, r\}$ and define two operations "$\otimes$" and "$\diamond$" in $\boldsymbol{W}$ as follows:

$\diamond$	1	s	t	r
1	1	1	1	1
s	1	1	1	1
t	1	1	1	1
r	1	s	1	r

$\otimes$	1	s	t	r
1	1	s	t	r
s	1	1	t	r
t	1	s	1	r
r	1	1	1	1

Then, it can be easily verified that $(\boldsymbol{W}; \otimes, \diamond, 1)$ is an NBE-semigroup.

In the following, we provide some properties of NBE-semigroups. These properties are true for BE-semigroups as well.

Proposition 2.2. *Let $(\boldsymbol{W}; \otimes, \diamond, 1)$ be an NBE-semigroup. Then, the following statements hold:*

(i) $a \otimes 1 = 1, \forall a \in \boldsymbol{W}$,

(ii) $a \leq b \Rightarrow c \otimes a \leq c \otimes b, \forall a, b, c \in \boldsymbol{W}$.

Proof. (i) Let $a \in \boldsymbol{W}$, then

$a \otimes 1 = a \otimes (1 \diamond 1)$

$= (a \otimes 1) \diamond (a \otimes 1)$ ($\because$ by left distributive law)

$= 1$ ($\because$ by first property of a BE-algebra)

(ii) Let $a, b, c \in \boldsymbol{W}$ be such that $a \leq b$. Then

$(c \otimes a) \diamond (c \otimes b) = c \otimes (a \diamond b)$ ($\because$ by left distributive law)

$= c \otimes 1$ ($\because$ $a \diamond b = 1$)

$= 1$ ($\because$ by part (i))

Thus, $c \otimes a \leq c \otimes b$. □

3. Unit divisors

In this section, we define the unit divisors and characterize them by some of properties.

Definition 3.1. *In an NBE-semigroup ($\boldsymbol{W}$; $\otimes$, $\diamond$, 1), an element $p \neq 1$ is called a left unit divisor if their exists $q \neq 1 \in \boldsymbol{W} \ni p \otimes q = 1$. Similarly we can define a right unit divisor. An element $p \neq 1$ of $\boldsymbol{W}$ is said to be a unit divisor if it is both left and right unit divisor.*

In the following, we state and prove some properties for NBE-semigroups. These properties are true in case of BE-semigroups as well.

Theorem 3.1. *Let us suppose that the left cancellation law holds for "$\otimes$" in an NBE-semigroup ($\boldsymbol{W}$; $\otimes$, $\diamond$, 1), that is, $s \otimes g = s \otimes h \Rightarrow g = h \; \forall \; s \neq 1$, $g, h \in \boldsymbol{W}$, then $\boldsymbol{W}$ contains no left unit divisors.*

Proof. Let us suppose that the left cancellation law holds for "$\otimes$" in $\boldsymbol{W}$. Let us suppose that $s, g \in \boldsymbol{W}$ such that $s \otimes g = 1$ and $s \neq 1$.

Then $s \otimes g = 1 = s \otimes 1$ ($\because$ by Proposition 2.2)

$\Rightarrow g = 1$ ($\because$ by left cancellative law)

Thus, $\boldsymbol{W}$ contains no left unit divisors. □

Theorem 3.2. *Let ($\boldsymbol{W}$; $\otimes$, $\diamond$,1) be an NBE-semigroup which contains no left unit divisors. Then, the left cancellation law holds for the operation "$\otimes$".*

Proof. Suppose that $g, h, k \in \boldsymbol{W}$ are such that $g \otimes h = g \otimes k$ and $g \neq 1$.

Therefore,

$g \otimes (h \diamond k) = (g \otimes h) \diamond (g \otimes k) = 1$ and

$g \otimes (k \diamond h) = (g \otimes k) \diamond (g \otimes h) = 1$.

As there is no left unit divisor in $\boldsymbol{W}$, so it follows that

$h \diamond k = 1 = k \diamond h \Rightarrow h = k$. □

4. Deductive systems

In this section, we define deductive systems and give some examples. Furthermore, we discuss some of their properties.

Definition 4.1. *A non-empty subset $\boldsymbol{G}$ of an NBE-semigroup ($\boldsymbol{W}$; $\otimes$, $\diamond$, 1) is said to be a deductive system (which is shortly denoted by DS) if the following conditions are satisfied:*

(i) $\boldsymbol{W} \otimes \boldsymbol{G} \subseteq \boldsymbol{G}$,

(ii) $\forall \; a \in \boldsymbol{G}$ *and* $\forall \; p \in \boldsymbol{W} \ni a \diamond p \in \boldsymbol{G} \Rightarrow p \in \boldsymbol{G}$.

Let us explain deductive systems by giving some examples.

Example 4.1. Let $\boldsymbol{W} = \{1, s, t, r\}$ and define the two binary operations "$\otimes$" and "$\diamond$" in $\boldsymbol{W}$ in the following tables:

$\otimes$	1	s	t	r
1	1	1	1	1
s	1	1	1	1
t	1	1	1	s
r	1	1	1	s

$\diamond$	1	s	t	r
1	1	s	t	r
s	1	1	t	r
t	1	1	1	r
r	1	1	1	1

It can be easily verified that $(\boldsymbol{W}; \otimes, \diamond, 1)$ is an NBE-semigroup. Let us assume $\boldsymbol{G} = \{1, s\}$. Then $\boldsymbol{G}$ is a DS, but $\boldsymbol{E} = \{1, r\}$ is not a DS of $\boldsymbol{W}$, because $r \otimes r = s \notin \boldsymbol{E}$ or $r \diamond s = 1 \in \boldsymbol{E}$, $r \in \boldsymbol{E}$, but $s \notin \boldsymbol{E}$

Example 4.2. Let $\boldsymbol{W} = \{1, s, t, q\}$ and define the two binary operations "$\otimes$" and "$\diamond$" in $\boldsymbol{W}$ as follows:

$\otimes$	1	s	t	q
1	1	1	1	1
s	1	s	1	1
t	1	1	1	1
q	1	1	1	t

$\diamond$	1	s	t	q
1	1	s	t	q
s	1	1	t	q
t	1	1	1	q
q	1	1	1	1

It can be easily shown that $(\boldsymbol{W}; \otimes, \diamond, 1)$ is an NBE-semigroup. Let $\boldsymbol{G} = \{1, t\}$, then $\boldsymbol{G}$ is a DS, but $\boldsymbol{E} = \{1, q\}$ is not an DS of $\boldsymbol{W}$, because $q \otimes q = t \notin \boldsymbol{E}$ or $q \diamond t = 1 \in \boldsymbol{E}$, $q \in \boldsymbol{E}$ but $t \notin \boldsymbol{E}$.

Let us state and prove some properties of deductive systems for NBE-semigroups. These properties hold in case of BE-semigroups as well. Firstly we define a special subset of a BE-algebra which will be used later.

Definition 4.2. *Let us suppose that $(\boldsymbol{W}; \diamond, 1)$ is a BE-algebra, and let $p, q \in \boldsymbol{W}$. Then, we define: $\boldsymbol{A}(p, q) = \{z \in \boldsymbol{W} : p \diamond (q \diamond z) = 1\}$.*

It should be noted that $\boldsymbol{A}(p, q)$ is clearly non-empty because we have $1, p, q \in \boldsymbol{A}(p, q)$. Let us give an example in order to understand the above definition.

Example 4.3. In Example 2.1 (i), we have $\boldsymbol{A}(2, 3) = \{1, 2, 3, 4, 5\}$ while in Example 2.1 (ii), we have $\boldsymbol{A}(4, 5) = \{1, 2, 4, 5\}$.

Proposition 4.1. *Let $(\boldsymbol{W}; \otimes, \diamond, 1)$ be an NBE-semigroup and $\boldsymbol{G}$ be a DS. Then $\boldsymbol{A}(p, q) \subseteq \boldsymbol{G}, \forall p, q \in \boldsymbol{G}$.*

Proof. Suppose that $x \in \boldsymbol{A}(p, q)$, then $p \diamond (q \diamond x) = 1$ and $1 \in \boldsymbol{G}$. By the second property of a DS, it follows that $x \in \boldsymbol{G}$. This completes the proof. □

Theorem 4.1. *Let us assume that $\{\boldsymbol{G}_i : i \in \boldsymbol{I}\}$ is a family of DSs of an NBE-semigroup $(\boldsymbol{W}; \otimes, \diamond, 1)$. Then $\cap \boldsymbol{G}_i$ is a DS of $\boldsymbol{W}$.*

Proof. It is straightforward. □

In the following definition, we introduce a particular subset of an NBE-semigroup.

Definition 4.3. *Let $(\boldsymbol{W}; \otimes, \diamond, 1)$ be an NBE-semigroup and let $\boldsymbol{G}$ be a subset of $\boldsymbol{W}$. The intersection of all DSs of $\boldsymbol{W}$ containing $\boldsymbol{G}$ is said to be the DS generated by $\boldsymbol{G}$, and it is represented by the symbol $<\boldsymbol{G}>$.*

It is very clear from the above definition that if $\boldsymbol{G}$ and $\boldsymbol{E}$ are subsets of an NBE-semigroup $(\boldsymbol{W}; \otimes, \diamond, 1)$ such that $\boldsymbol{G} \subseteq \boldsymbol{E}$, then $<\boldsymbol{G}> \subseteq <\boldsymbol{E}>$. Furthermore, if $\boldsymbol{G}$ is an DS of $\boldsymbol{W}$, then one can have $<\boldsymbol{G}>= \boldsymbol{G}$.

Before the next result, note that an NBE-semigroup $(\boldsymbol{W}; \otimes, \diamond, 1)$ is said to be self-distributive if $(\boldsymbol{W}; \diamond, 1)$ is a self-distributive BE-algebra.

Theorem 4.2. *Let $(\boldsymbol{W}; \otimes, \diamond, 1)$ be a self-distributive NBE-semigroup and let $\boldsymbol{G}$ be a non-empty subset of $\boldsymbol{W}$ such that $\boldsymbol{W} \otimes \boldsymbol{G} \subseteq \boldsymbol{G}$. Then, we have*
$<\boldsymbol{G}>= \{a \in \boldsymbol{W} : q_n \diamond (\ldots \diamond (q_1 \diamond a)\ldots) = 1$ *for some* $q_1,\ldots,q_n \in \boldsymbol{G}\}$.

Proof. Let $\boldsymbol{B} = \{a \in \boldsymbol{W} : q_n \diamond (\ldots \diamond (q_1 \diamond a)\ldots) = 1$ for some $q_1,\ldots,q_n \in \boldsymbol{G}\}$. Assume that $b \in \boldsymbol{B}$, then we have some $q_1,\ldots,q_n \in \boldsymbol{G} \ni q_n \diamond (\ldots \diamond (q_1 \diamond b)\ldots) = 1$. Now, let $g \in \boldsymbol{W}$, then $1 = g \otimes 1 = g \otimes (q_n \diamond (\ldots \diamond (q_1 \diamond b)\ldots)) = (g \otimes q_n) \diamond (\ldots \diamond ((g \otimes q_1) \diamond (g \otimes b)) \ldots)$.

As $g \otimes q_i \in \boldsymbol{G}$ for $i = 1,\ldots,n$, it follows that $g \otimes b \in \boldsymbol{B}$. Take $g, a \in \boldsymbol{W}$ such that $a \diamond g \in \boldsymbol{B}$ and $a \in \boldsymbol{B}$. Then, by definition of $\boldsymbol{B}$, we have some $q_1,\ldots,q_n$, $r_1,\ldots, r_m \in \boldsymbol{G}$ such that $q_n \diamond (\ldots \diamond (q_1 \diamond (a \diamond g))\ldots) = 1$ and $r_m \diamond (\ldots \diamond (r_1 \diamond a)\ldots) = 1$.

From the fourth property of BE-algebra and from the above first equation it follows that $a \diamond (q_n \diamond (\ldots \diamond (q_1 \diamond g)\ldots)) = 1$. In other words, we have $a \leq q_n \diamond (\ldots \diamond (q_1 \diamond g)\ldots)$. Furthermore, from the above second equation and from Proposition 2.2, we have $1 \leq r_m \diamond (\ldots \diamond (r_1 \diamond (q_n \diamond (\ldots \diamond (q_1 \diamond g)\ldots)))\ldots)$.

Therefore, $r_m \diamond (\ldots \diamond (r_1 \diamond (q_n \diamond (\ldots \diamond (q_1 \diamond g)\ldots)))\ldots) = 1$. This implies that $g \in \boldsymbol{B}$. Thus $\boldsymbol{B}$ is a DS of $\boldsymbol{W}$. Clearly $\boldsymbol{G} \subseteq \boldsymbol{B}$. Let $\boldsymbol{M}$ be another DS which contains $\boldsymbol{G}$. In order to show $\boldsymbol{B} \subseteq \boldsymbol{M}$, let us suppose that $a \in \boldsymbol{B}$. Then, we have some $q_1,\ldots,q_n \in \boldsymbol{G}$ such that $q_n \diamond (\ldots \diamond (q_1 \diamond a)\ldots) = 1$. Thus, by definition of deductive systems, we have $a \in \boldsymbol{M}$ and thus $\boldsymbol{B} \subseteq \boldsymbol{M}$. This completes the proof. □

Furthermore, we have the following important fact.

Remark 4.1. The union of two DSs $\boldsymbol{G}$ and $\boldsymbol{E}$ of a self-distributive NBE-semigroup $(\boldsymbol{W}; \otimes, \diamond, 1)$ is not necessary to be a DS.

Example 4.4. Let $\boldsymbol{W} = \{1, s, t, u, v\}$ and define two binary operations in $\boldsymbol{W}$ in the following tables:

$\otimes$	1	s	t	u	v
1	1	1	1	1	1
s	1	1	1	1	1
t	1	1	1	1	1
u	1	1	1	1	s
v	1	1	1	1	s

$\diamond$	1	s	t	u	v
1	1	s	t	u	v
s	1	1	t	t	v
t	1	s	1	s	v
u	1	1	1	1	v
v	1	1	t	t	1

It can be easily verified that $(\boldsymbol{W}; \otimes, \diamond, 1)$ is a self-distributive NBE-semigroup. Let $\boldsymbol{G} = \{1, s\}$ and $\boldsymbol{E} = \{1, t\}$. Then $\boldsymbol{G}$ and $\boldsymbol{E}$ are DSs of $\boldsymbol{W}$. Although, we have $\boldsymbol{G} \cup \boldsymbol{E} = \{1, s, t\}$ is not a DS of $\boldsymbol{W}$, because $t \diamond u = s \in \boldsymbol{G} \cup \boldsymbol{E}$ but $u \notin \boldsymbol{G} \cup \boldsymbol{E}$.

In the following, we prove another property of self-distributive NBE-semigroups.

Theorem 4.3. *Let ($\boldsymbol{W}$; $\otimes$, $\diamond$, 1) be a self-distributive NBE-semigroup and further assume that $\boldsymbol{G}$ and $\boldsymbol{E}$ are DSs of ($\boldsymbol{W}$; $\otimes$, $\diamond$, 1). Then, we have*

$<\boldsymbol{G}\cup\boldsymbol{E}>=\{a \in \boldsymbol{W} : p \diamond(q \diamond a) = 1$ *for some* $p \in \boldsymbol{G}$, $q \in \boldsymbol{E}\}$.

Proof. Let $\boldsymbol{T} = \{a \in \boldsymbol{W} : p \diamond(q \diamond a) = 1$ for some $p \in \boldsymbol{G}$, $q \in \boldsymbol{E}\}$. Clearly, $\boldsymbol{T} \subseteq <\boldsymbol{G}\cup\boldsymbol{E}>$. Let us suppose that $g \in <\boldsymbol{G}\cup\boldsymbol{E}>$. Then, by Theorem 4.2, we have some $q_1, \ldots, q_n \in \boldsymbol{G}\cup\boldsymbol{E} \ni q_n \diamond (\ldots \diamond (q_1 \diamond g)\ldots) = 1$. Let $q_i \in \boldsymbol{G}$ $\forall$ $i = 1, \ldots, n$, then $g \in \boldsymbol{G}$. Therefore $g \in \boldsymbol{T}$ as $g \diamond (1 \diamond g) = 1$. Similarly, if $q_i \in \boldsymbol{E}$ $\forall$ $i = 1, \ldots, n$, then $g \in \boldsymbol{E}$. Therefore $g \in \boldsymbol{T}$ as $1 \diamond (g \diamond g) = 1$. Let some of $q_1, \ldots, q_n \in \boldsymbol{G}$ and others belongs to $\boldsymbol{E}$, then we may suppose $q_1, \ldots, q_t \in \boldsymbol{G}$ and $q_{t+1}, \ldots, q_n \in \boldsymbol{E}$ for $1 \leq t \leq n$ without loss of generality. Let us assume that $u = q_t \diamond (\ldots \diamond (q_1 \diamond g)\ldots)$. Then, we have $q_n \diamond (\ldots \diamond (q_{t+1} \diamond u)\ldots) = q_n \diamond (\ldots \diamond (q_{t+1} \diamond (q_t \diamond (\ldots \diamond (q_1 \diamond g)\ldots)))\ldots) = 1$. Hence $u \in \boldsymbol{E}$. Now, let $v = u \diamond g = (q_t \diamond (\ldots \diamond (q_1 \diamond g)\ldots)) \diamond g$, then we have $q_t \diamond (\ldots \diamond (q_1 \diamond v)\ldots) = q_t \diamond (\ldots \diamond (q_1 \diamond ((q_t \diamond (\ldots \diamond (q_1 \diamond g)\ldots)) \diamond g)) \ldots)$

$= (q_t \diamond (\ldots \diamond (q_1 \diamond g) \ldots)) \diamond (q_t \diamond (\ldots \diamond (q_1 \diamond g) \ldots) = 1$.

It follows that $v \in \boldsymbol{G}$. As $u \diamond (v \diamond g) = v \diamond (u \diamond g) = v \diamond v = 1$, then $g \in \boldsymbol{T}$. This completes the proof. □

5. Conclusion

In this paper, we have studied near BE-semigroups. We have defined it in the usual way by excluding one of the distributive laws. It is a generalization of a BE-semigroup. We have seen that the said structure exists by constructing some non-trivial examples. Moreover, we have generalized some properties of BE-semigroups to near BE-semigroups. At the end, we have discussed different classes of near BE-semigroups and investaged some of their properties. We have defined near left BE-semigroup and named it near BE-semigroup but we have seen that we can define near right BE-semigroup as well by excluding left distributive law from a BE-semigroup. Furthermore, analogues of the results which are true for near left BE-semigroups can be proved.

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Deformation problem for a double porous viscoelastic medium using state space approach

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Abstract. A deformation problem for a homogeneous, isotropic, double porous, viscoelastic medium has been studied by considering a new mathematical model. To solve the problem, state space approach has been used. A normal force has been applied to describe the problem as an illustration. The solution is obtained in the frequency domain and hence the deformation described by the components of normal stress and equilibrated stress has been found in the form of frequency. Then, these components have been converted into physical domain and computed numerically for a particular material. The numerical results thus obtained have been discussed graphically to show the viscous effect on deformation. A Particular case has also been given to show the generalization of the model taken for the problem.

Keywords: double porous, deformation, equilibrated stress, normal force, normal stress, state space approach, viscoelasticity.

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1. Introduction

The porous medium plays a significant role in different fields of science, engineering and technology. A general three-dimensional consolidation problem for this material has been given by Biot [1] and this is known as Biot theory of poroelasticity. Bowen [2] de Boer and Ehlers [3] have described another theory for these materials with incompressible constituents. Abbas [4] employed Biot's theory to obtain the frequency equation for radial vibrations of a poroelastic cylinder for a porous solid. Aizahrani and Abbas [5] provided a method to study the effect of porosity in a porothermoelastic medium by the finite element technique. A generalization of Biot's theory of poroelasticity describing double porosity model has been given by Barenblatt et al. [6] to express the fluid flow in hydrocarbon reservoirs and aquifers. The investigation of double porosity model has given the chance to represent many important aspects of different problems related to civil engineering. Thus, many researchers have applied this model to solve different kinds of research problems.

The linear theory of viscoelasticity describes the linear conduct of both elastic and an elastic materials. Bland [7] has given description of three dimensional linear viscoelasticity theory. According to Bland, in a perfectly elastic isotropic medium, under the assumption of small displacements, two types of waves can propagate in an isotropic, viscoelastic medium in absence of body forces. Iesan and Scalia [8] has given and solved some theorems in the theory of thermoviscoelasticity. Svanadze [9] described the problem of plane waves and steady vibrations in the theory of viscoelasticity for Kelvin-Voigt materials with double porosity. Kumar et al. [10] studied the wave propagation in anisotropic viscoelastic medium in the context of the theory of three phase lag model of thermoelasticity. Sharma et al. [11] studied the wave propagation in anisotropic thermoviscoelastic medium in the context of Green-Naghdi theories of type-II and type-III. Sharma et al. [12] considered electro–microstretch viscoelastic solids for investigating the propagation of plane waves and fundamental solution. A few problems with double porosity model in elastic solids and thermoelastic solids have been studied by Svanadze [13, 14, 15].

The state space description of linear systems has been used extensively during recent past to avoid the difficulties of the traditional linear model approach. Bahar and Hetnarski [16, 17] described the state space approach to thermoelasticity. Sharma [18] used the state space approach to study the one dimensional problem for generalized theories of thermoelasticity subjected to heat source and body force. Ezzat et al. [19] used state space approach for a problem of generalized thermoviscoelasticity with two relaxation times. Palmeri et al. [20] used state space formulation for the study of linear viscoelastic dynamic system with memory. Menon and Tang [21] applied state space approach to the dynamic analysis of viscoelastic system. Youssef and Al-Lehaibi [22] with the help of state space approach, obtained the general solution for any set of boundary conditions for a half-space filled with an elastic material. Youssef and Harby

[23] considered an infinite elastic body with a spherical cavity and applied state space technique to obtain the general solution for any set of boundary conditions. Ezzat and Karamany [24] established state space approach for two temperature magneto-viscoelasticity theory with thermal relaxation in a medium of perfect conductivity. Kumar et al. [25] used state space approach to discuss a problem for thermoelastic material with double porosity. Kumar and Vohra [26] used the state space approach to look into the plane deformation problem in elastic materials with double porosity. Ezzat and EL-Bary [27] applied state space approach to two-dimensional magneto-thermoelasticitywith fractional order heat transfer in a medium of perfect conductivity.

The problems have been solved by researchers by taking different mathematical models, methods and materials. Some of them are listed here for reference purposes. Abbas and Youssef [28] proposed a general finite element method (FEM) to analyze transient phenomena in a thermoelastic model in the context of the theory of generalized thermoelasticity with one relaxation time. Kumar and Abbas [29] investigated two dimensional problems of micropolarthermoelastic material with two temperature in the context of Lord-Shulman theory. Marin and Vlase [30] studied the effect of internal state variables in thermoelasticity of microstretch bodies. Vlase et al. [31] studied the vibration of mechanical bars systems with symmetries. Marin et al. [32] proposed a mixed initial-boundary value problem in modeling a three-phaselag dipolar thermoelastic body.

In this paper, a boundary value problem of a double porous viscoelastic medium has been formulated and solved using state space approach. The components of deformation, normal stress and equilibrated stresses, have been found in closed form, after getting the expressions in frequency domain. The values for these components have been obtained numerically for a particular medium. These are depicted graphically to show the effect of viscosity. A particular case of interest has also been deduced.

2. Basic equations

The basic equations in absence of body forces and extrinsic equilibrated body forces for a double porous homogeneous isotropic elastic medium has been taken, following Ieşan and Quintanilla [33], as Constitutive Relations:

$$t_{ij} = \lambda e_{rr}\delta_{ij} + 2\mu e_{ij} + b\delta_{ij}\varphi + d\delta_{ij}\psi, \tag{1}$$

$$\sigma_i = \alpha\varphi_{,i} + b_1\psi_{,i}, \tag{2}$$

$$\tau_i = b_1\varphi_{,i} + \gamma\psi_{,i}\,. \tag{3}$$

Equation of motion:

$$\mu\nabla^2\vec{u} + (\lambda+\mu)\nabla\nabla\cdot\vec{u} + b\nabla\varphi + d\nabla\psi = \rho\frac{\partial^2 u}{\partial t^2}\,, \tag{4}$$

Equilibrated stress equations of motion:

$$\alpha\nabla^2\varphi + b_1\nabla^2\psi - b\nabla\vec{u} - \alpha_1\varphi - \alpha_3\psi = \kappa_1\frac{\partial^2\varphi}{\partial t^2}\,, \tag{5}$$

$$b_1\nabla^2\varphi + \nabla^2\psi - d\nabla\vec{u} - \alpha_3\varphi - \alpha_2\psi = \kappa_2\frac{\partial^2\psi}{\partial t^2}\,, \tag{6}$$

where λ and μ are the Lame's constants; ρ is the mass density; u_i $(i = 1, 2, 3)$ are the displacement components; t_{ij} is the stress tensor; κ_1 and κ_2 are the coefficients of equilibrated inertia; φ and ψ are the volume fraction fields corresponding to pores v_1 and fissures v_2, respectively; σ_i is the equilibrated stress corresponding to pores; τ_i is the equilibrated stress corresponding to fissures; b, d, b_1, α, γ, α_1, α_2, α_3 are the constitutive coefficients; δ_{ij} is the Kronecker's delta; Δ is the Laplacian operator; and a superposed dot represents differentiation with respect to time variable 't'.

Following the Voigt model of linear viscoelasticity, the porous elastic constants λ, μ, b, d, α_1, α_2, α_3, α, γ, b_1 are replaced by the following constants for viscoelastic nature of the material λ_ε, μ_ε, b_ε, d_ε, $\alpha_{1\varepsilon}$, $\alpha_{2\varepsilon}$, $\alpha_{3\varepsilon}$, α_ε, γ_ε, $b_{1\varepsilon}$, respectively.

Therefore, equations (1)-(6) for viscoelastic nature of the medium take the following form of equations:

$$t_{ij} = \lambda_\varepsilon e_{rr}\delta_{ij} + 2\mu_\varepsilon e_{ij} + b_\varepsilon\delta_{ij}\varphi + d_\varepsilon\delta_{ij}\psi, \tag{7}$$

$$\sigma_i = \alpha_\varepsilon\varphi_{,i} + b_{1\varepsilon}\psi_{,i}, \tag{8}$$

$$\tau_i = b_{1\varepsilon}\varphi_{r_i} + \gamma_\varepsilon\psi_{,i}, \tag{9}$$

$$\mu_\varepsilon\nabla^2\vec{u} + (\lambda_\varepsilon + \mu_\varepsilon)\,\nabla\nabla\vec{u} + b_\varepsilon\nabla\varphi + d_\varepsilon\nabla\psi = \rho\frac{\partial^2\vec{u}}{\partial t^2}, \tag{10}$$

$$\alpha_\varepsilon\nabla^2\varphi + b_{1\varepsilon}\nabla^2\psi - b_\varepsilon\nabla\vec{u} - \alpha_{1\varepsilon}\varphi - \alpha_{3\varepsilon}\psi = \kappa_1\frac{\partial^2\varphi}{\partial t^2}, \tag{11}$$

$$b_{1\varepsilon}\nabla^2\varphi + \gamma_\varepsilon\nabla^2\psi - d_\varepsilon\nabla\vec{u} - \alpha_{3\varepsilon}\varphi - \alpha_{2\varepsilon}\psi = \kappa_2\frac{\partial^2\psi}{\partial t^2}, \tag{12}$$

where

$$\begin{cases} \lambda_\varepsilon = \lambda + \lambda_\nu\dfrac{\partial}{\partial t}, & \mu_\varepsilon = \mu + \mu_\nu\dfrac{\partial}{\partial t}, \\ b_\varepsilon = b + b_\nu\dfrac{\partial}{\partial t}, & d_\varepsilon = d + d_\nu\dfrac{\partial}{\partial t}, \\ \alpha_{1\varepsilon} = \alpha_1 + \alpha_{1\nu}\dfrac{\partial}{\partial t}, & \alpha_{2\varepsilon} = \alpha_2 + \alpha_{2\nu}\dfrac{\partial}{\partial t}, \\ \alpha_{3\varepsilon} = \alpha_3 + \alpha_{3\nu}\dfrac{\partial}{\partial t}, & \alpha_\varepsilon = \alpha + \alpha_\nu\dfrac{\partial}{\partial t}, \\ \gamma_\varepsilon = \gamma + \gamma_\nu\dfrac{\partial}{\partial t}, & b_{1\varepsilon} = b_1 + b_{1\nu}\dfrac{\partial}{\partial t}, \end{cases} \tag{13}$$

and

$$\lambda_\nu, \mu_\nu, b_\nu, d_\nu, \alpha_{1\nu}, \alpha_{2\nu}, \alpha_{3\nu}, \alpha_\nu, \gamma_\nu, b_{1\nu}$$

are the viscoelastic constants.

3. Formulation and solution of the problem

A one dimensional problem of a homogeneous, isotropic, double porous, viscoelastic half space medium has been considered. The state variables of the problem depend only on the space variable, distance x and on time t. Therefore, the displacement components u_i, volume fractions φ and ψ are taken as

$$\{u_i, \varphi, \psi\} = \{u_i(x,t), \varphi(x,t), \psi(x,t)\}. \tag{14}$$

A normal force is considered to be acted upon the bounding surface $x = 0$. Now, to find the solution, dimensionless parameters are defined as

$$\begin{cases} x_1' = \dfrac{\omega_1}{c_1} x_1, \;\; x_3' = \dfrac{\omega_1}{c_1} x_3, \;\; u_1' = \dfrac{\omega_1}{c_1} u_1, \\ u_3' = \dfrac{\omega_1}{c_1} u_3, \;\; t_{ij}' = \dfrac{t_{ij}}{\lambda_\varepsilon}, \;\; \varphi' = \dfrac{\kappa_1 \omega_1^2}{\alpha_{1\varepsilon}} \varphi, \\ \psi' = \dfrac{\kappa_1 \omega_1^2}{\alpha_{1\varepsilon}} \psi, \;\; t' = \omega_1 t, \;\; \sigma_1' = \left(\dfrac{c_1}{\alpha_\varepsilon \omega_1} \right) \sigma_1, \\ \tau_1' = \left(\dfrac{c_1}{\alpha_\varepsilon \omega_1} \right) \tau_1, \end{cases} \tag{15}$$

where

$$\omega_1 = \frac{\lambda}{\kappa_1} \quad \text{and} \quad c_1^2 = \frac{\lambda_\varepsilon + 2\mu_\varepsilon}{\rho}\,. \tag{16}$$

Here, we consider

$$Q_i = \omega_1 \left(\frac{\lambda_\nu}{\lambda}, \frac{\mu_\nu}{\mu}, \frac{b_\nu}{b}, \frac{d_\nu}{d}, \frac{\alpha_\nu}{\alpha}, \frac{\gamma_\nu}{\gamma}, \frac{\alpha_{1\nu}}{\alpha_1}, \frac{\alpha_{2\nu}}{\alpha_2}, \frac{\alpha_{3\nu}}{\alpha_3}, \frac{b_{1\nu}}{b_1} \right), i = 1, 2, \ldots, 10\,, \tag{17}$$

and hence

$$\begin{aligned} &(\lambda_\varepsilon, \mu_\varepsilon, b_\varepsilon, d_\varepsilon, \alpha_\varepsilon, \alpha_{1\varepsilon}, \alpha_{2\varepsilon}, \alpha_{3\varepsilon}, \gamma_{1\varepsilon}, \gamma_{2\varepsilon}, b_{1\varepsilon}) \\ &\quad = (\lambda, \mu, b, d, \alpha_1, \alpha_2, \alpha_3, \gamma_1, \gamma_2, b) \left(1 + Q_i \frac{\partial}{\partial t'} \right);\; i = 1, 2 \ldots, 10. \end{aligned} \tag{18}$$

Using (15) in equations (10)-(12), and then using (14), we get

$$\frac{\partial^2 u}{\partial x^2} + \delta_1 \frac{\partial \varphi}{\partial x} + \delta_2 \frac{\partial \psi}{\partial x} = \frac{\partial^2 u}{\partial t^2}, \tag{19}$$

$$\delta_3 \frac{\partial^2 \varphi}{\partial x^2} + \delta_4 \frac{\partial^2 \psi}{\partial x^2} - \delta_5 \frac{\partial u}{\partial x} - \delta_6 \varphi - \delta_7 \psi = \frac{\partial^2 \varphi}{\partial t^2}, \tag{20}$$

$$\delta_8 \frac{\partial^2 \varphi}{\partial x^2} + \delta_9 \frac{\partial^2 \psi}{\partial x^2} - \delta_{10} \frac{\partial u}{\partial x} - \delta_{11} \varphi - \delta_{12} \psi = \frac{\partial^2 \psi}{\partial t^2}, \tag{21}$$

where

$$\begin{cases} \delta_1 = \dfrac{b_\varepsilon \alpha_{1\varepsilon}}{\rho c_1^2 k_1 \omega_1^2}, \quad \delta_2 = \dfrac{d_\varepsilon \alpha_{1\varepsilon}}{\rho c_1^2 k_1 \omega_1^2}, \quad \delta_3 = \dfrac{\alpha_\varepsilon}{c_1^2 k_1}, \delta_7 = \dfrac{\alpha_{3\varepsilon}}{k_1 \omega_1^2}, \\ \delta_8 = \dfrac{b_{1\varepsilon}}{c_1^2 k_2}, \quad \delta_9 = \dfrac{\gamma_\varepsilon}{c_1^2 k_2}, \quad \delta_4 = \dfrac{b_{1\varepsilon}}{c_1^2 k_1}, \\ \delta_5 = \dfrac{b_\varepsilon}{\alpha_{1\varepsilon}}, \quad \delta_6 = \dfrac{\alpha_{1\varepsilon}}{k_1 \omega_1^2}, \quad \delta_{10} = \dfrac{d_\varepsilon k_1}{\alpha_1 k_2}, \\ \delta_{11} = \dfrac{\alpha_{3\varepsilon}}{k_2 \omega_1^2}, \quad \delta_{12} = \dfrac{\alpha_{2\varepsilon}}{k_2 \omega_1^2}. \end{cases} \tag{22}$$

Assuming harmonic form of solution of equations (19)-(21) as

$$\{u(x,t), \varphi(x,t), \psi(x,t)\} = (\bar{u}, \bar{\varphi}, \bar{\psi}) e^{-i\omega t}\,. \tag{23}$$

Equations (19)-(21) with the help of (23) give

$$\bar{u}_{,11} = N_1 \bar{u} + N_2 \bar{\phi}_{,1} + N_3 \bar{\psi}_{,1}, \tag{24}$$

$$\bar{\phi}_{,11} = M_1 \bar{\psi}_{,11} + M_2 \bar{u}_{,1} + M_3 \bar{\phi} + M_4 \bar{\psi}, \tag{25}$$

$$\bar{\psi}_{,11} = M_5 \bar{\phi}_{,11} + M_6 \bar{u}_{,1} + M_7 \bar{\phi} + M_8 \bar{\psi}, \tag{26}$$

where

$$\begin{cases} N_1 = -\omega^2, \quad N_2 = -\delta_1, \quad N_3 = -\delta_2, \\ M_1 = \dfrac{-\delta_4}{\delta_3}, \quad M_2 = \dfrac{\delta_5}{\delta_3}, \quad M_3 = \dfrac{\delta_6 - \omega^2}{\delta_3}, \\ M_4 = \dfrac{\delta_7}{\delta_3}, \quad M_5 = \dfrac{-\delta_8}{\delta_9}, \quad M_6 = \dfrac{\delta_{10}}{\delta_9}, \\ M_7 = \dfrac{\delta_{11}}{\delta_9}, \quad M_8 = \dfrac{\delta_{12} - \omega^2}{\delta_9}\,. \end{cases} \tag{27}$$

Equations (24)-(26) can be re-written as

$$\bar{u}_{,11} = N_1 \bar{u} + N_2 \bar{\phi}_{,1} + N_3 \bar{\psi}_{,1}, \tag{28}$$

$$\bar{\phi}_{,11} = N_4 \bar{u}_{,1} + N_5 \bar{\phi} + N_6 \bar{\psi}, \tag{29}$$

$$\bar{\psi}_{,11} = N_7 \bar{u}_{,1} + N_8 \bar{\phi} + N_9 \bar{\psi}, \tag{30}$$

where

$$\begin{cases} N_4 = \dfrac{M_1 M_6 + M_2}{M_9}, \quad N_5 = \dfrac{M_1 M_7 + M_3}{M_9}, \quad N_6 = \dfrac{M_1 M_8 + M_4}{M_9}, \\ N_7 = M_5 N_4 + M_6, \quad N_8 = M_5 N_5 + M_7, \quad N_9 = M_5 N_6 + M_8. \end{cases} \tag{31}$$

4. State-space formulation

The equations (28)-(30) can be written in the matrix form as

$$\frac{dV(x,\omega)}{dx} = A(\omega)V(x,\omega), \tag{32}$$

where the values of $A(\omega)$, $V(x,\omega)$ are given in the appendix. Taking the displacement $\bar{u}$ and volume fractions $\bar{\varphi}$ and $\bar{\psi}$ as state variables, the solution of the matrix form of equation (32) can be taken as

$$V(x,\omega) = \exp[A(\omega)x]V(0,\omega), \tag{33}$$

where the value of $V(0,\omega)$ is given in the appendix. To get the matrix $\exp[A(\omega)x]$, Cayley-Hamilton theorem is used.

The characteristic equation of the matrix $A(\omega)$ is obtained as

$$\lambda^6 + D_1\lambda^4 + D_2\lambda^2 + D_3 = 0, \tag{34}$$

where

$$\begin{cases} D_1 = -N_1 - N_5 - N_9 - N_2N_4 - N_3N_7, \\ D_2 = N_1N_5 + N_1N_9 + N_5N_9 - N_6N_8 + N_2N_4N_9 \\ \qquad -N_2N_6N_7 + N_3N_5N_7 - N_3N_4N_8, \\ D_3 = N_1N_6N_8 - N_1N_5N_9. \end{cases} \tag{35}$$

Equation (34) is cubic in λ^2, yield three roots say λ_1^2, λ_2^2, λ_3^2.

The Taylor series expansion for $\exp[A(\omega)x]$ is taken as

$$\exp[A(\omega)x] = \sum_{n=0}^{\infty} \left\{ \frac{[A(\omega)x]^n}{n!} \right\}. \tag{36}$$

Truncating this infinite series and by using Cayley-Hamilton theorem, we get

$$\exp[A(\omega)x] = a_0 I + a_1 A + a_2 A^2, \tag{37}$$

where a_0, a_1, a_2 are the parameters depending on x and ω. Now, the eigen values $-\lambda_1$, $-\lambda_2$, $-\lambda_3$ of the matrix A satisfy equation (37) as per the Cayley-Hamilton theorem. Thus, we have

$$\begin{cases} \exp[-\lambda_1 x] = a_0 - a_1\lambda_1 + a_2\lambda_1^2, \\ \exp[-\lambda_2 x] = a_0 - a_1\lambda_2 + a_2\lambda_2^2, \\ \exp[-\lambda_3 x] = a_0 - a_1\lambda_3 + a_2\lambda_3^2. \end{cases} \tag{38}$$

Solving equations (38), the values of a_0, a_1 and a_2 are obtained as given in appendix. Using these values of a_0, a_1 and a_2 in (37), we obtain

$$\exp[A(\omega)x] = L(x,\omega), \tag{39}$$

and hence equation (33) can be written as

$$V(x,\omega) = L(x,\omega)V(0,\omega), \tag{40}$$

where $L(x,\omega)$ is a 6×6 matrix.

Thus, the solution in the frequency domain is obtained as,

$$\begin{bmatrix} \bar{u} \\ \bar{\varphi} \\ \bar{\psi} \end{bmatrix} = \begin{bmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}, \tag{41}$$

where

$$\begin{cases} l_{11} = a_0 + a_2N_1, \;\; l_{12} = 0, l_{13} = 0, \;\; l_{21} = 0, \;\; l_{22} = a_0 + a_2N_5, \\ l_{23} = a_2N_6, \;\; l_{31} = 0, \;\; l_{32} = a_2N_8, \;\; l_{33} = a_0 + a_2N_9\,. \end{cases} \tag{42}$$

5. Boundary conditions

As per the formulation of the problem, a normal force is considered to be acted upon the bounding surface $x = 0$. Therefore, the boundary conditions at $x = 0$ are written as,

$$\text{(i)}t_{11} = -F_1 \exp[-i\omega t], \tag{43}$$

$$\text{(ii)}\sigma_1 = -F_2 \exp[-i\omega t], \tag{44}$$

$$\text{(iii)}\tau_1 = -F_3 \exp[-i\omega t], \tag{45}$$

where F_t; $(i = 1, 2, 3)$ are the magnitudes of the force applied on the boundary.

Substituting the values of $\bar{u}$, $\bar{\varphi}$, $\bar{\psi}$ from equation (41), with the help of (14), (15) and (23), in equations (7)-(9) and then the resulting expressions in the boundary conditions (43)-(45), we obtain

$$\begin{bmatrix} Q_1 & Q_2 & Q_3 \\ Q_4 & Q_5 & Q_6 \\ Q_7 & Q_8 & Q_9 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} -F_1 \\ -F_2 \\ -F_3 \end{bmatrix}. \tag{46}$$

The values of $Q_1, Q_2, \ldots, Q_9$, are given in appendix.

Solving the above system of equations for A_1, A_2, A_3, we obtain

$$A_1 = \frac{\Gamma_1}{\Gamma}, \quad A_2 = \frac{\Gamma_2}{\Gamma}, \quad A_3 = \frac{\Gamma_3}{\Gamma}, \tag{47}$$

where

$$\begin{cases} \Gamma = \begin{vmatrix} Q_1 & Q_2 & Q_3 \\ Q_4 & Q_5 & Q_6 \\ Q_7 & Q_8 & Q_9 \end{vmatrix}, \qquad \Gamma_1 = \begin{vmatrix} -F_1 & Q_2 & Q_3 \\ -F_2 & Q_5 & Q_6 \\ -F_3 & Q_8 & Q_7 \end{vmatrix}, \\ \Gamma_2 = \begin{vmatrix} Q_1 & -F_1 & Q_3 \\ Q_4 & -F_2 & Q_6 \\ Q_7 & -F_3 & Q_9 \end{vmatrix}, \quad \Gamma_3 = \begin{vmatrix} Q_1 & Q_2 & -F_1 \\ Q_4 & Q_5 & -F_2 \\ Q_7 & Q_8 & -F_3 \end{vmatrix}. \end{cases} \tag{48}$$

Using these expressions in (41) and then using (7)-(9) with the help of (23), we get the normal stress and the components of equilibrated stress as

$$t_{11} = \left(S_1 \frac{\Gamma_1}{\Gamma} + S_2 \frac{\Gamma_2}{\Gamma} + S_3 \frac{\Gamma_3}{\Gamma} \right) e^{-i\omega t}, \tag{49}$$

$$\sigma_1 = \left(S_4 \frac{\Gamma_1}{\Gamma} + S_5 \frac{\Gamma_2}{\Gamma} + S_6 \frac{\Gamma_3}{\Gamma} \right) e^{-i\omega t}, \tag{50}$$

$$\tau_1 = \left(S_7 \frac{\Gamma_1}{\Gamma} + S_8 \frac{\Gamma_2}{\Gamma} + S_9 \frac{\Gamma_3}{\Gamma} \right) e^{-i\omega t}, \tag{51}$$

where $S_1, S_2, \ldots, S_9$ are given in appendix.

6. Particular case

Taking $F_3 = 0$ in equations (49)-(51), we get the resulting expressions for the corresponding problem for an elastic porous medium.

7. Numerical results and discussion

To illustrate the theoretical results of the problem numerically, we take the material as copper material. Sherief and Saleh [34] has given the physical parameters for this material as

$$\begin{aligned}
&\lambda = 7.76 \times 10^{10} Nm^{-2}, && C^* = 3.831 \times 10^3 m^2 s^{-2} K^{-1},\\
&\mu = 3.86 \times 10^{10} Nm^{-2}, && t = 0.1s,\\
&\alpha_2 = 2.4 \times 10^{10} Nm^{-2}, && \alpha_3 = 2.5 \times 10^{10} Nm^{-2},\\
&\rho = 8.954 \times 10^3 Kgm^{-3}, && \gamma = 1.1 \times 10^{-5} N,\\
&\alpha = 1.3 \times 10^{-5} N, && b_1 = 0.12 \times 10^{-5} N,\\
&d = 0.1 \times 10^{10} Nm^{-2}, && \kappa_1 = 0.1456 \times 10^{-12} Nm^{-2} s^2,\\
&b = 0.9 \times 10^{10} Nm^{-2}, && \alpha_1 = 2.3 \times 10^{10} Nm^{-2},\\
&\kappa_2 = 0.1546 \times 10^{-12} Nm^{-2} s^2
\end{aligned}$$

$$\begin{aligned}
&Q_1 = 0.005, \quad Q_2 = 0.003, \quad Q_3 = 0.004, \quad Q_4 = 0.005,\\
&Q_5 = 0.001, \quad Q_6 = 0.004, \quad Q_7 = 0.002, \quad Q_8 = 0.008,\\
&Q_9 = 0.005, \quad Q_{10} = 0.006
\end{aligned}$$

The variation with respect to distance x in the values of normal stress and equilibrated stresses are obtained using Matlab software and are shown in Figures 1-3, respectively. To show the effect of viscosity, in all these figures, the graphs for viscoelastic double porous material (VDP) and elastic double porous material (DP) are drawn.

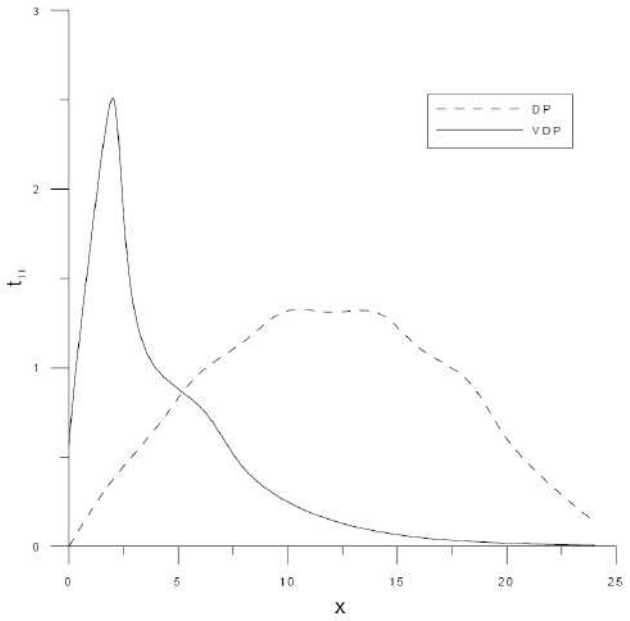


Figure 1: Variation of normal stress t_{11} with respect to distance x

Figure 1 shows the variation of t_{11} with respect to distance x. It is found that for VDP, the value of t_{11} increases for the range $0 < x \leq 3$ and then decreases for $x > 3$, whereas for DP, the value increases for the range $0 < x \leq 13$ and then decreases for the remaining range. Also, the magnitude value is more for DP in comparison to VDP except for the region $0 < x \leq 5$.

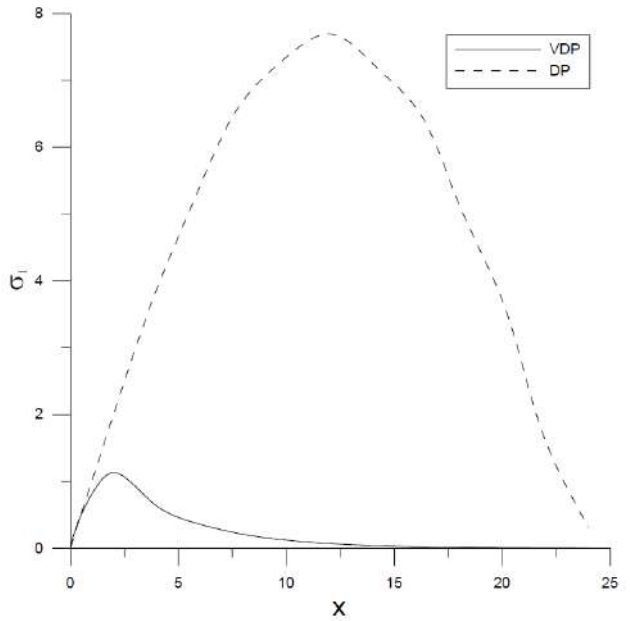


Figure 2: Variation of equilibrated stress σ_1 with respect to distance x

Figure 2 represents the variation of σ_1 with respect to distance x. It is found that for VDP, the value of σ_1 increases for the values $0 < x \leq 3$ and then decreases for the remaining region with the increase in the value of x, whereas for DP the value increases for the region $0 < x \leq 13$ and then decreases for the remaining region. Also, the magnitude value is more for DP in comparison to VDP.

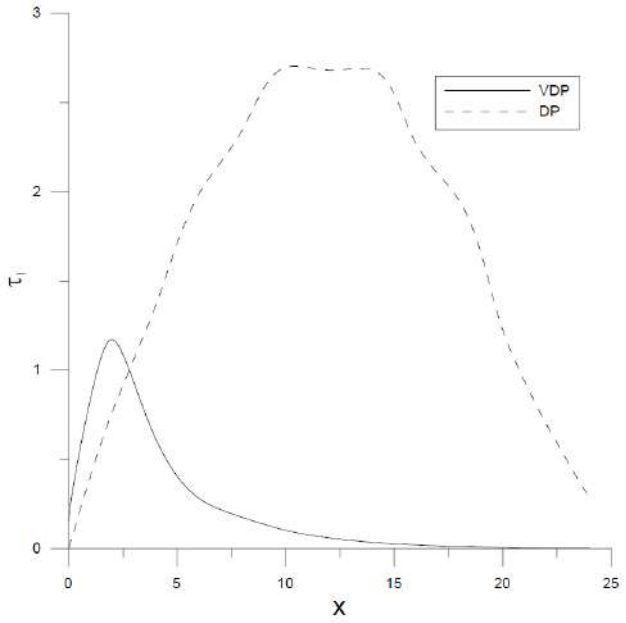


Figure 3: Variation of equilibrated stress τ_1 with respect to x

Figure 3 depicts the variation of τ_1 with respect to distance x. It is found that for VDP, the value of τ_1 increase for region $0 < x \leq 1$ and then decreases as $x > 1$ with the increase in the value of x whereas for DP, the values increase for the region $0 < x \leq 13$ and then decreases for the remaining region. Also, the magnitude value is more for DP in comparison to VDP.

8. Conclusion

State space approach has been used to solve a deformation problem for a homogeneous isotropic double porous viscoelastic medium. The components of normal stress and equilibrated stress have been found and computed numerically for a particular material, i.e., copper. The numerical results obtained for these components have been discussed graphically to show the viscous effect. A particular case has been deduced to show the generalization of the model taken for the problem.

Appendix

$$A(\omega)=\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\\ N_1 & 0 & 0 & 0 & N_2 & N_3\\ 0 & N_5 & N_6 & N_4 & 0 & 0\\ 0 & N_8 & N_9 & N_7 & 0 & 0 \end{bmatrix},$$

$$V(x,\omega)=\begin{bmatrix} \bar{u}(x,\omega)\\ \bar{\varphi}(x,\omega)\\ \bar{\psi}(x,\omega)\\ (\bar{u}(x,\omega))_{,1}\\ (\bar{\varphi}(x,\omega))_{,1}\\ (\bar{\psi}(x,\omega))_{,1} \end{bmatrix},$$

$$V(0,w)=\begin{bmatrix} \bar{u}(0,\omega)\\ \bar{\varphi}(0,\omega)\\ \bar{\psi}(0,\omega)\\ (\bar{u}(0,\omega))_{,1}\\ (\bar{\varphi}(0,\omega))_{,1}\\ (\bar{\psi}(0,\omega))_{,1} \end{bmatrix},$$

$$a_0=e^{-\lambda_1 x}\left[\frac{\lambda_2\lambda_3}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}\right]+e^{-\lambda_2 x}\left[\frac{\lambda_1\lambda_3}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}\right]+e^{\lambda_3 x}\left[\frac{\lambda_1\lambda_2}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}\right],$$

$$a_1=e^{-\lambda_1 x}\left[\frac{(\lambda_2+\lambda_3)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}\right]+e^{-\lambda_2 x}\left[\frac{(\lambda_1+\lambda_3)}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}\right]+e^{-\lambda_3 x}\left[\frac{(\lambda_1+\lambda_2)}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}\right],$$

$$a_2=e^{-\lambda_1 x}\left[\frac{1}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}\right]+e^{-\lambda_2 x}\left[\frac{1}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}\right]+e^{-\lambda_3 x}\left[\frac{1}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}\right],$$

$$Q_1=P_1(Z_1+N_1Z_3),\quad Q_2=P_2\left(a_0^0+a_2^0N_5\right)+P_3\left(a_2^0N_8\right),$$

$$Q_3=P_2\left(a_2^0N_6\right)+P_3\left(a_0^0+a_2^0N_9\right),$$

$$Q_4=0,\quad Q_5=P_4(Z_1+N_5Z_3)+P_5N_8Z_3,\quad Q_6=P_4N_6Z_3+P_5(Z_1+N_9Z_3),$$

$$Q_7=0,\quad Q_8=P_5(Z_1+N_5Z_3)+P_6N_8Z_3,\quad Q_9=P_4N_6Z_3+P_6(Z_1+N_9Z_3),$$

where

$$P_1 = \frac{\lambda + 2\mu}{\lambda}, P_2 = \frac{b\alpha_1}{\lambda k_1 \omega^2}, P_3 = \frac{d\alpha_1}{\lambda k_1 \omega^2}, P_4 = \frac{\alpha_1}{k_1 \omega^2}, P_5 = \frac{b_1 \alpha_1}{\alpha k_1 \omega^2},$$
$$P_6 = \frac{\gamma \alpha_1}{\alpha k_1 \omega}, Z_1 = -\lambda_1 D_{11} - \lambda_2 D_{12} - \lambda_3 D_{13}, Z_2 = -\lambda_1 D_{21} - \lambda_2 D_{22} - \lambda_3 D_{23},$$
$$Z_3 = -\lambda_1 D_{31} - \lambda_2 D_{32} - \lambda_3 D_{33},$$

and

$$D_{11} = \frac{\lambda_2 \lambda_3}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}, D_{12} = \frac{\lambda_1 \lambda_3}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)},$$
$$D_{13} = \frac{\lambda_1 \lambda_2}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}, D_{31} = \frac{1}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)},$$
$$D_{32} = \frac{1}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)}, D_{33} = \frac{1}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)},$$

and

$$\text{at } x = 0; a_0 = a_0^0, a_2 = a_2^0.$$

Further,

$$S_1 = P_1 (Y_1 + N_1 Y_3), S_2 = 0, S_3 = 0, S_4 = 0, S_5 = P_4 (Y_1 + N_5 Y_3) + P_5 N_8 Y_3,$$
$$S_6 = P_4 N_6 Y_3 + P_5 (Y_1 + N_9 Y_3), S_7 = 0, S_8 = P_5 (Y_1 + N_5 Y_3) + P_6 N_8 Y_3,$$
$$S_9 = P_5 N_6 Y_3 + P_6 (Y_1 + N_9 Y_3),$$

where

$$Y_1 = -\lambda_1 D_{11} e^{-\lambda_1 x} - \lambda_2 D_{12} e^{-\lambda_2 x} - \lambda_3 D_{13} e^{-\lambda_3 x},$$
$$Y_2 = -\lambda_1 D_{11} e^{-\lambda_1 x} - \lambda_2 D_{12} e^{-\lambda_2 x} - \lambda_3 D_{13} e^{-\lambda_3 x},$$
$$Y_3 = -\lambda_1 D_{21} e^{-\lambda_1 x} - \lambda_2 D_{22} e^{-\lambda_2 x} - \lambda_3 D_{23} e^{-\lambda_3 x}.$$

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Models and theories for the choice of teaching strategies in mathematics

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Abstract. This paper aims to highlight the role of the subject, not only as a school discipline but also a tool to educate students to rational thinking and a means for the development of social skills. The intention is to dispel the myth of a cold and pre-packaged mathematics, accessible to the chosen few: the idea of a mathematics for all is supported through an experience with a student with specific learning disorders, with which a personalized teaching strategy has allowed the achievement of set goals. A case study of a student with specific reading and writing disorders attended the mathematical analysis course 1 of the degree course in architecture and the personalized teaching strategy created for him are described. Finally, some mathematical models for the choice of better teaching strategies are exhibited.

Keywords: inquiry model, social skills, personalized didactic strategy, decision making, AHP.

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1. Introduction

Today society is characterized by its complexity and flexibility, markets and communication are based on globalization and internationalization. The high unpredictability of the social context in which we find ourselves urges the educational systems to a great responsibility, that of stimulating all students to an autonomy of thought and action through the construction of a personality that can respond to the demands of a reality that, now more than ever, is characterized by a general state of uncertainty. The objective of fostering the pupil freedom of choice and reaction provides the school with an orienting character where the term "orientation" indicates a continuous and personal process that involves awareness, learning and education for choice ([4]). In particular, to put orientation as the main purpose of teaching "means to develop appropriate strategies, methodologies and targeted content from the acquisition of awareness to the understanding of the complex society and the mechanisms that govern the world of studies and work" ([7]).

The current historical context requires that the school recognizes as fundamental the equality of educational opportunities and promotes the achievement of appropriate levels of knowledge and skills by all in order to "ensure to all students high levels of mastery, or at least fully appropriate, of the basic skills in the educational curricula" ([2], p. 12). Therefore, a convergent personalization (ibid.) is fundamental, with respect to which the differentiation of training paths and the programmed and opportunely diversified technical solutions allow everyone to achieve common training goals. However, the enhancement of personal differences is also essential, i.e. promoting different types of individual potential, in relation to the different kinds of talent involved. Therefore, a divergent personalization (ibid.) is necessary, that is the diversification of the training goals, according to the promotion of individual potential. The personalization of learning, then, could represent an effective strategy in the mathematics teaching and learning, in particular for students with specific learning disorders (SLD), for whom ad-hoc educational choices are required.

Therefore, the teacher has to provide, by selecting the best teaching strategy, cognitive tools necessary to start the processes of personalization, self-learning and self-orientation. From an operational point of view, teaching thus becomes a decision making problem and must therefore be addressed as such. Properly, in fact, one can speak of decision when in a situation are present: the alternatives, understood as the possibility of being able to act in several different ways; the probability, seen as the possibility that the outcomes related to each alternative will be realized; the consequences associated with the outcomes. In the school sector, the three elements mentioned (alternatives-probability-consequences) are characteristics of the environment and therefore choosing a teaching strategy is fully equivalent to making a decision as a process that goes from the representation of the problem to the selection of the options through the estimation of expectations and the evaluation of outcomes.

This research considers the teaching process as a decision-making problem. It aims to provide the mathematics teacher with a model that allows him/her to evaluate, case by case, the educational alternatives and choose the "best teaching strategies", i.e. those best suited to the context being analyzed, particularly with reference to students with SLD, for whom the teacher (not always specialized) often has to make decisions in situations of complete uncertainty.

Our research question is: "Can decision theory, applied to mathematics teaching/learning, be an effective tool for the development of personalized educational strategies for students with SLD?". In order to answer this question, we tried to develop a decision-making model that would also take into account relevant results in mathematics education research. More in detail, we started from the hierarchical analytical method of Saaty ([18], [19]), which sets a general objective and takes into account some criteria and alternatives. We identified the Specific General Objective, Specific Criteria and Specific Alternatives according to our personal beliefs and results in mathematics education research. Finally, we customized this model by applying it to a case study of a student with SLD attending the mathematical analysis course 1 of the degree course in architecture. According to the model, we assigned weights to each alternative starting from an observation of the student's behaviors and learning styles. The model provided us with a personalized teaching strategy, which we carried out in the specific case study.

In the following sections we describe the decision-making model from a mathematical standpoint (Sections 2.1 and 2.2), identify the overall objective, criteria and alternatives for the model (Section 2.3), and assign weights to the alternatives based on observation of the specific case study (Section 3). Then, we apply the educational strategy provided by the model to the case study (Section 3.1) and try to answer our research question (Section 4).

2. Teaching as a decision problem

The study of decisions can be made in terms of absolute rationality or limited rationality whose difference lies in the fact that, while the former model is based on an ideal combination of rationality and information preferring the best alternative, the latter recognizes the objective narrowness of the human mind by proposing the selection of the most satisfactory alternative (Lanciano, [9]). In this regard, it must be stressed that the consequences of a decision derive not only from the chosen course of action but also from the state of nature that represents the context in which the decision making process develops. On the basis of the knowledge that the decision maker has of the state of nature, they differ:

- decisions in a situation of certainty: the decision maker knows the state of nature;

- decisions in risk situations: the decision maker has a measure of probability for each state of nature even though he does not know it directly;
- decisions in situations of uncertainty: the decision maker has neither information regarding the state of nature nor the probabilities associated with it.

The attitude used by decision makers towards situations determines the distinction between the normative approach, which bases the choice with reference to rational decision-making ideals, and the descriptive approach, which analyses how to make a decision according to the context.

Based on the objectives and context the teacher must then consider what alternatives he can follow and for each alternative what the various consequences might be. For each pair (alternative, circumstance) the teacher obtains a result in accordance with a utility function: the decision is rational if it is based on the criterion of obtaining a maximum value for the utility function. This decision is however subjective depending on the objectives and utility functions of the teacher and therefore depends on the culture, experience and preferences. Moreover, even if the choice is rational, it is made in terms of limited rationality because, in general, there are few alternatives, but it increases as the teacher expands his/her culture and experience (Delli Rocili, Maturo, [6]; Maturo, Zappacosta, [12]).

2.1 A model for evaluating educational alternatives

Multi-Criteria Decision Analysis (MCDA) provides support to the decision maker, or a group of decision makers, when many conflicting assessments have to be considered, especially in data synthesis phase while working with complex and heterogeneous pieces of information.

Let $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ be the set of the alternatives, i. e. the possible educational strategies.

Let $\mathcal{O} = \{O_1, O_2, \ldots, O_n\}$ be the set of the objectives that we want to achieve.

Let $\mathcal{D} = \{D_1, D_2, \ldots, D_k\}$ be the set of the decision making processes.

In the first phase decision makers must establish a procedure to assign to each pair (alternative A_i, objective O_j) a score p_{ij} that measures the degree in which the alternative A_i satisfies the objective O_j.

Assume that $p_{ij} \in [0, 1]$, where:

- $p_{ij} = 0$ if the objective O_j is not at all satisfied by A_i;
- $p_{ij} = 1$ if the objective O_j is completely satisfied by A_i

At the end of the procedure we obtain a matrix $P = (p_{ij})$ of the scores which is the starting point of the elaborations that lead to the choice of the alternative, or at least to their ordering, possibly even partial (Maturo, Ventre, [10, 11]).

There may be constraints. It could be necessary to establish for each objective O_j a threshold $j > 0$, with the constraint $p_{ij} \geq j$, for each $i = 1, \ldots, m$.

Furthermore, consideration should be given to mixed strategies that is a convex linear combinations of alternatives A_i. So, a mixed strategy has the form:

$$A(h_1, h_2, \ldots, h_m) = h_1 A_1 + h_2 A_2 + \cdots + h_m A_m$$

with

- $h_1, h_2, \ldots, h_m$ non–negative real numbers;
- the h_i's are such that $h_1 + h_2 + \cdots + h_m = 1$.

The number h_i can represent the fraction of time in which the teaching strategy A_i is adopted.

If we consider also the mixed strategies, then the single alternatives A_i are called pure strategies.

The need to consider mixed strategies also arises in particular if there are "at risk" alternatives: this kind of alternatives have high scores for certain objectives and low for others (possibly below the threshold).

It is appropriate to construct a ranking of the alternative educational plans, i. e., a linear ordering of the alternatives that takes into account the objectives which contribute to the most suitable formation of the student. Such a ranking can be usefully obtained by means of the application of the Analytic Hierarchy Process (Saaty [18], [19]).

2.2 The Analytic Hierarchy Process: attributions of weights and scores

The Analytic Hierarchy Process (AHP) is both a method and a technique that allows to compare alternatives of different qualitative and quantitative nature, not easily comparable in a direct way, through the assignment of numerical values that specify their priority. The first step is the construction of the hierarchical structure that represents the elements to be considered in the decision making problem. The Analytic Hierarchy Process is based on the representation of a decision problem in terms of a directed graph $G = (V, A)$. Let us recall that (Knuth [8]):

- a *directed graph*, or *digraph*, is a pair $G = (V, A)$, where V is a non-empty set whose elements are called *vertices* and A is a set of ordered pairs of vertices, called *arcs*;
- a vertex is indicated with a latin letter; for every arc (u, v), u is called the *initial vertex* and v the *final vertex* or *end vertex*;
- an ordered n-tuple of vertices $(v_1, v_2, \ldots, v_n), n > 1$, is called a *path* with length $n - 1$, if, and only if, every pair $(v_i, v_{i+1}), i = 1, 2, \ldots, n - 1$, is an arc of G.

Furthermore, in our context, we assume the following conditions be satisfied from a directed graph:

1. the vertices are distributed in a fixed integer number $n \geq 2$ of levels, each level is indexed from 1 to n;
2. there is only one vertex of level 1, called the *root* of the directed graph;
3. for every vertex v different from the root there is at least one path having the root as the initial vertex and v as the final vertex;
4. every vertex u of level $i < n$ is the initial vertex of at least one arc and there are no arcs with the initial vertex of level n;
5. if an arc has the initial vertex of level $i < n$, then it has the end vertex in the level $i + 1$.

To this aim it is worth to describe the functional aspects of each level. If $n = 3$ we consider the digraph in which the level 1 vertex is called the *General Objective* and denoted by GO; the vertices of level 2 are called *Criteria* and vertices of level 3 are called *Alternatives*. Briefly, the general objective indicates the choice or the goal of the entire decision-making process; the Alternatives are the different ways we consider in order to reach the GO or the different options we must choose and the criteria are the parameters used to evaluate the alternatives.

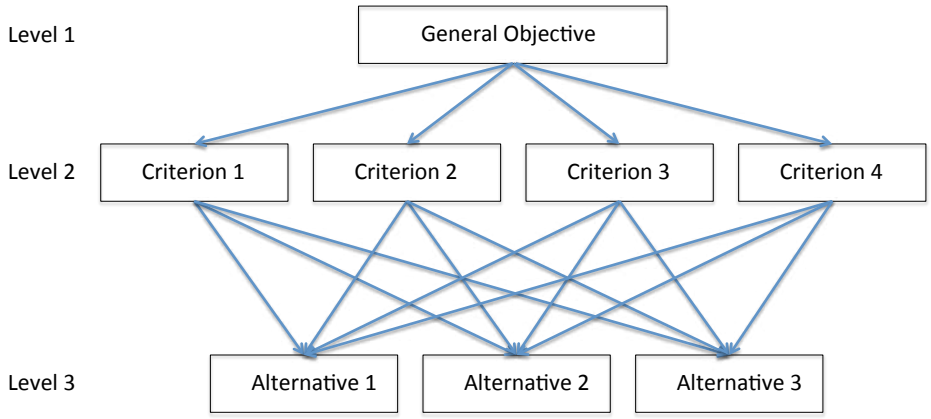


Figure 1: The digraph of the functional aspects

A decision maker assigns a score to each arc following the AHP procedure (Maturo, Ventre, [10], [11]). The second step consists into collecting data to determine the ratios of preference of the elements of a level over any of the higher, i.e. previous, level. The third step implements the estimation of local assessments, i.e. the weightings that express the relative importance of the elements of a hierarchical level over any element of the next higher level. We

therefore compare alternatives $A1, A2$ and $A3$ with respect to each criterion and the individual criteria with respect to the general objective. In order to determine the value of each comparison the following scale of evaluations is used:

Value a_{ij}	**Interpretation**
1	i and j are equally important
3	i is a slightly more important than j
5	i is quite more important than j
7	i is definitely more important than j
9	i is absolutely more important than j
1/3	i is a slightly less important than j
1/5	i is quite less important than j
1/7	i is definitely less important than j
1/9	i is absolutely less important than j

Table 1 – Scale of evaluations

Let us remark the following when assigning values to the comparisons:

1. if alternative i assumes the value x in comparison with alternative j with respect to a criterion, then the comparison of alternative j with alternative i with respect to the same criterion assumes the value $1/x$. Analogous is the procedure to assign values when comparing couples of criteria;

2. since equally important alternatives correspond to value 1, the diagonal of the matrices are composed entirely of unit values.

Matrices satisfying claims 1 and 2 above are called *pairwise comparison matrices*. Comparison matrices represent quantitative preferences between criteria or between alternatives.

After constructing the pairwise comparison matrices, we have to fix the weights of the elements in each level that assess the relevance in a scale of values ranging from 0 to 1. These weights must meet the normality condition:

$$w_1 + w_2 + \cdots + w_n = 1.$$

The way to determine the weights is based on the consideration that, if the decision maker knew all the actual weights of the elements of the pairwise comparison matrix, then it would be:

$$A = \left(\frac{w_i}{w_j}\right) = \begin{pmatrix} \frac{w_1}{w_1} & \cdots & \frac{w_1}{w_n} \\ \vdots & \ddots & \vdots \\ \frac{w_n}{w_1} & \cdots & \frac{w_n}{w_n} \end{pmatrix}.$$

In this case the weights would be obtained from any of the rows which are all multiple of the same row $\left(\frac{1}{w_1}, \frac{1}{w_2}, \ldots, \frac{1}{w_n}\right)$. It follows that the matrix A has rank 1.

Being $\boldsymbol{w} = (w_1, w_2, \ldots, w_n)^T$, we get:

$$A\boldsymbol{w} = n\boldsymbol{w}.$$

Thus, from the equation above, n is an eigenvalue of A and $\boldsymbol{w}$ is one of the eigenvectors associated with n. Since the elements on the diagonal are all 1, denoted with $\lambda_1, \lambda_2, \ldots, \lambda_n = n$ the eigenvalues of A, the value of the trace of A is:

$$tr(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n = n.$$

As n is an eigenvalue of A the other $n-1$ eigenvalues of A must be zero.

The matrix A satisfies the condition $a_{ij}a_{jk} = a_{ik}$ for every i, j, k, called *consistency condition*, that implies transitivity of the preferences, and A is said to be *consistent*.

In practice the decision maker does not know the vector $\boldsymbol{w}$: the a_{ij} values that he assigns according to his judgement may deviate from the unknown $\frac{w_i}{w_j}$. So the decision maker may produce inconsistent pairwise comparison matrices. However the closer the a_{ij} values are to $\frac{w_i}{w_j}$, the closer the maximum eigenvalue is to n and the closer the other eigenvalues are to zero.

Therefore the vector of the weights $\boldsymbol{w'}^T = (w'_1, w'_2, \ldots, w'_n)$ associated to the maximum eigenvalue (among the infinite $\boldsymbol{w'}^T$ we choose the one for which $w'_1 + w'_2 + \cdots + w'_n = 1$) will be an estimate of the vector $\boldsymbol{w}^T = (w_1, w_2, \ldots, w_n)$) the more precise the more the maximum eigenvalue λ_{max} of A is close to n what is due to the continuity of the involved operations (Ventre, [21]).

With these premises, the ratio

$$CI = \frac{\lambda_{max} - n}{n-1}$$

is defined as the *consistency index* of A and reveals how far the matrix is from consistency.

2.3 A decision-making model for mathematical education

Mathematics is one of the disciplines in which many students manifest difficulties that compromise the relationship with the subject. A student who comes out of secondary school has a long series of "failures" accompanied by the conviction that she can never do mathematics because she is not good at it. Mathematics, still today, is often taught using a transmissive model, derived from behavioral theories (e.g. Skinner, [20]). According to this model, a mathematics lesson is usually developed through the following phases: correction of the assigned tasks, resolution by the teacher or a student of more complex ones, explanation by the teacher of a part of the program, carrying out exercises on the new topic. A transmissive model leads to the assumption that, in mathematics, the absence of errors certifies the absence of difficulties and on the other hand the absence of difficulties guarantees the absence of errors (Zan, [24]). This identification leads to the didactic objective of obtaining the greatest number of correct answers

and, inevitably, to the fear of making mistakes and the conviction that mathematics is not for everyone (Zan and Di Martino, [25]). Many researchers (e.g. Piaget, [16]; Von Glasersfeld, [23]) showed that the student has to actively build mathematical concepts and techniques in order to achieve meaningful learning. This led first to the development of constructivist models and then to the socio-constructivist models (e.g. Vygotskij, [22] and Radford, [17]), which consider learning as a necessarily social process, which takes place first among people and only later within the student. Among the socio-constructivist models, the collaborative group and peer tutoring (Pesci, [14] and [15]) that take on both the disciplinary dimension and the affective and social dimension and facilitate discussion in the classroom. Another constructivist model is the inquiry model (Borasi, [5]) that proposes a positive and fundamental role of errors in mathematics teaching. This model sees knowledge as a dynamic process of investigation where cognitive conflict and doubt represent the motivations to continuously search for a more and more refined understanding. Therefore, instead of eliminating ambiguities and contradictions to avoid confusion or errors, these elements must be highlighted to stimulate and give shape to ideas and discussions. Questions such as "what would happen if this result were true?" or "under what circumstances could this error be corrected?" lead to a reformulation of the problem where error is only the starting point for a deeper understanding. Communication in the classroom plays a fundamental role and so does the conception of mathematics as a humanistic discipline: the teacher provides the necessary support for the student's autonomous search for understanding, who in turn is an active member of a research community (Pascucci, [13]).

Starting from the above premises and our personal beliefs, we set the general objective, criteria and alternatives for the decisional model.

- General objective: to guarantee a training that, through education, ensures the formation of knowledge and the development of action and communication skills;

- Criterion 1: acquisition of the objectives set out in the educational plan. This criterion considers how much a teaching strategy allows you to address all the topics set out from the course of study through good time management;

- Criterion 2: ability to measure oneself against peers in a safe and rational way while respecting the ideas of others in accordance with the Indications for the curriculum (2007) (see [27]) where mathematics "contributes to the development of ability to communicate and discuss, to argue correctly, to understand the points of view and arguments of others";

- Criterion 3: internalization of the objectives set by the didactic plan, intended as the ability to re-elaborate knowledge from a critical and personal

point of view because the most effective way to bring students closer to mathematics is therefore the image of a "method to face problems, a language, a box of tools that allow us to enhance our intuition" (Baccaglini Frank et al., [1]);

- Criterion 4: ability to cope with trials, planned or not, without negative moods, anxiety and terror of judgement; This criterion is inspired by the need to build learning teaching models that take into account the emotions: knowing how to manage situations by overcoming the fear of failure is a skill that the school must teach with the same care with which teach disciplinary skills;

- Alternative 1: new didactics, inspired by the inquiry model that is a model of teaching-learning that proposes a positive and fundamental role of errors in mathematics teaching and that puts the pupil and her emotions at the center of the context with cooperative learning experiences;

- Alternative 2: traditional didactic, that prefers frontal lessons under the assumption that, in terms of learning, the acquisition and internalization take place at the same time and that the error is the manifestation of the failure to complete one of the two processes;

- Alternative 3: distance learning, characterized by virtual interactions among teacher and students, mediated through the use of technological tools.

These alternatives were chosen because Alternative 2 is a "passive old strategy" based on the transmissive method which is the most widespread in Italian schools, while Alternatives 1-3 are "new active strategies", developed in the field of research in mathematics teaching, based on the idea that knowledge is a dynamic process of investigation. The comparison between Alternative 1 and Alternative 3 allows to understand if the teaching can be conducted to an online practice, given the recent diffusion of telematic courses.

3. A case study

In this section we describe a case study to which the model described above has been applied to build a customized teaching strategy for a student with specific learning disorders.

To this end, indeed, the Regional School Offices are committed to promote the issuance of detailed certifications that allow as much as possible, together with parents and figures who follow the student in school activities, a Personalized Didactic Plan (PDP) (MIUR, [29]).

A common trait to the different SLDs, which is particularly interesting from a didactical point of view, is the problem these students encounter with short term memory. In the field of cognitive psychology, the concept is extended with

that of "Working memory" (Baddeley and Hitch, [3]) where the term specifically indicates that set of notions necessary for written and oral productions that remain short in the student's mind. As this capacity is reduced, temporary storage and the first management/manipulation of data is compromised, leading to the difficulty of taking notes, difficulty of attention and the need for longer periods of time.

It is therefore necessary to reduce the information load, leaving more space for working time to store the fundamental concepts. For this reason, students with SLDs learn better through observation and experimentation, supported by visual aids.

From [26], some indications for the didactic strategy are:

- Use of schemes and concept maps;
- Dispense with reading aloud and mnemonic study;
- Privileging learning from experience and laboratory teaching;
- Encourage students to self-assess their own learning processes;
- Encourage peer tutoring and promote collaborative learning;
- Guarantee longer times for written tests and study;
- Take an encouraging attitude to improve self-esteem;
- Evaluate according to progress and difficulties;
- Use of calculator and digital devices.

Since learning disorders are various and create different deficits that require different compensatory tools and dispensation measures, to approach the case and to identify objectives, strategies and criteria it was essential to see how the student's specific disorders, dyslexia and dysgraphia, interfere in the learning of mathematics. Briefly:

- **Dyslexia:** it leads to difficulties in reading which is slow and incorrect, making it difficult to understand the text and to distinguish useful parts of it from those containing incidental information. In mathematics, these deficits therefore involve difficulty in recognizing the hypotheses and elements used in a problem making even the simplest ones complex. In addition, students affected by this disorder, have difficulty in studying the theory from the book and therefore, to develop the full potential of these students, psychologists suggest the constant use of mind maps that allow you to fix and rework ideas slowly until the acquisition of the argument. This type of maps is the one that best suits the way of learning of students with specific reading disorders because, through images and colors, it aims to stimulate visual memory through the inclusion of mental associations (see [28]).

- **Dysgraphia:** a deficit of a motor nature and refers to the graphic aspects of handwriting. The dysgraphers produce poorly readable texts (even by themselves) with words often misaligned and characterized by letters of different sizes. The elaboration of a written text is a difficult and long process with obvious repercussions in the study of mathematics. Mathematics, in fact, has its own language, characterized by symbols, signs and letters of the Greek and Latin alphabet. Inaccurate writing may cause errors in the resolution of algebraic expressions, for example by confusing the letter "z" and the number "2", or in the resolution of a linear system which requires many transcription steps. Another difficulty due to alignment is found in the case of powers where it confuses base and exponent with a multi-digit number (28 instead of 2^8).

The student, at the beginning of the course, presented his certificate and was immediately interested in possible recovery activities. After almost a month, when the first difficulties began to emerge, he constantly benefited from the hours of tutoring available during which a personalized teaching strategy was put in place that led the boy to the acquisition of his own method of study which proved to be successful.

3.1 The model applied to the case study

Let us specify the meanings of the GO, the criteria and the alternatives in our context.

- Specific General Objective: to promote the achievement of the GO through a personalized didactic plan.

About alternatives we have:

- Specific Alternative 1;
- Specific Alternative 2;
- Specific Alternative 3.

They are based on the educational models of Alternative 1, Alternative 2 and Alternative 3 but considering the compensatory instruments and dispensation measures.

The criteria become:

- Specific Criterion 1: acquisition of the objectives set by the educational plan that in quantitative terms are reduced compared to those considered in Criterion 1. In fact, a student with SLDs, although having an intelligence in the norm, unlike his peers, learns at a slower pace because during the study he dissipates most of his energy to compensate for his disorders;

- Specific Criterion 2: ability to measure oneself against peers in a safe and rational way while respecting the ideas of others as much as possible also increasing the level of self-esteem and personal gratification;
- Specific Criterion 3: see Criterion 3;
- Specific Criterion 4: see Criterion 4 with the difference that, in this case, in the process of evaluation of written tests the content rather than the form is taken into account, not penalizing errors when the concept expressed is clear and giving more weight to oral checks.

At this moment we can build the first matrix of the pair comparisons by assigning to each Specific Criterion a score that indicates the weight of the criterion in the achievement of the Specific General Objective.

Let us consider the following:

- SC2 is slightly more important than SC1 because working with a student with SLDs it is preferable to reduce the quantity and ensure the quality of the study topics giving more weight to the human aspect of mathematics even if the learning objectives may have flexible size;
- SC3 and SC4 are more important than SC2 because the school must guarantee the training of a citizen who is able to react in the face of trials and with the ability to put the "thinking" and the "doing" in close relationship.

SGO	SC1	SC2	SC3	SC4
SC1	1	3	1	1
SC2	1/3	1	1/3	1/5
SC3	1	3	1	1
SC4	1	5	1	1

M1: Comparison of criteria by the ratio of preference with respect to the general objective

The following matrices are obtained by comparing each specific alternative with the individual specific criteria and the number represents the extent to which the alternatives meet the criteria.

About SC1 let us consider:

- SA2 satisfies SC1 less than SA1 and SA3 because with a traditional didactics the student's understanding is not taken up by the method but is delegated to the individual act of the subject.

SC1	SA1	SA2	SA3
SA1	1	3	1
SA2	1/3	1	1/3
SA3	1	3	1

M2: Comparison of alternatives by the ratio of preference with respect to the criterion 1

About SC2 let us consider:

- traditional didactic is far from the social character of human learning so SA1 satisfies more SC2;

- sharing the same space with others is a fundamental thing for personal and social development, necessary for the student to learn to respect rules and roles. So SA2 satisfies more SC2 than SA3.

SC2	SA1	SA2	SA3
SA1	1	5	7
SA2	1/5	1	3
SA3	1/7	1/3	1

M3: Comparison of alternatives by the ratio of preference with respect to the criterion 2

About SC3 let us consider:

- distance learning offers insufficient physical interaction between student-teacher and student-student: expressions and gestures make the difference in the learning process; so SA3 is a slightly less important than SA1 to satisfy SC3;

- to overcome mechanical learning an "active" teaching can make the student able to master the knowledge and use it in other contexts. For this reason, the SA1 and SA3 are more appropriate than SA2 for building skills.

SC3	SA1	SA2	SA3
SA1	1	5	3
SA2	1/5	1	1/3
SA3	1/3	3	1

M4: Comparison of alternatives by the ratio of preference with respect to the criterion 3

About SC4 let us consider:

- exercising young people to face tests in a lucid way is a fundamental aspect for the construction of a personality that faces in a competitive way the working challenges of a competitive society. Distance learning facilitates the examination phase because the student does not have direct contact with those who have to judge him/her. For this reason, the student may not learn how to manage emotions and therefore SA1 and SA2 are more appropriate than SA3 to implement SC4;

- the Inquiry model is a teaching-learning model where error is only the starting point for a deeper understanding. So, the evaluation phase becomes a new opportunity to get involved and not a scary moment of judgment. With more serenity the student will face the challenges in the best way.

SC4	SA1	SA2	SA3
SA1	1	3	7
SA2	1/3	1	5
SA3	1/7	1/5	1

M5: Comparison of alternatives by the ratio of preference with respect to the criterion 4

In our case, with the help of MATLAB, we get:

	M1	M2	M3	M4	M5
λ_{max}	4,0328	3,0000	3,0649	3,0385	3,0649
CI	0,0109	0	0,0324	0,0193	0,0193

We can therefore write the local priorities obtained by proceeding with the last step of the AHP method which, through the aggregation of the relative weights of each level, provides a weighted ranking of the alternatives.

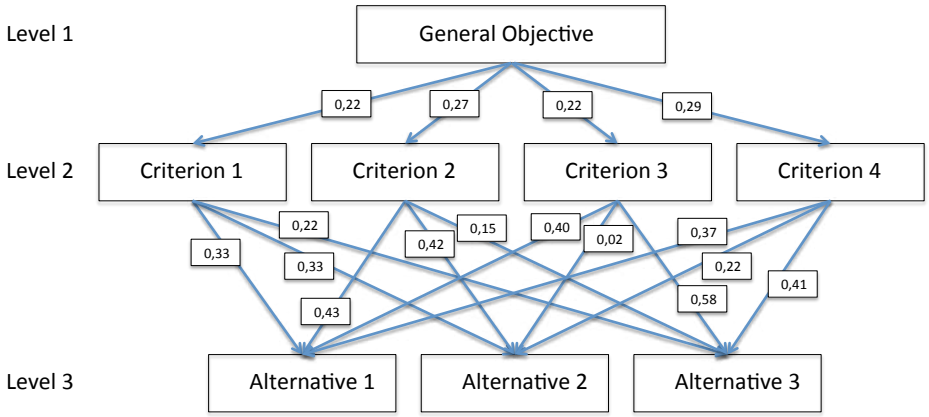


Figure 2: The digraph of the functional aspects, with weights

We observe that scores are non-negative real numbers and such that the sum of the scores of the arcs coming out of the same vertex u is equal to 1. The score assigned to an arc (u, v) indicates the extent to which the final vertex v meets the initial vertex u: the score of a path is the product of the scores of the arcs that form the path.

- For every vertex v different from GO the score $p(v)$ of v is the sum of the scores of all the paths that start from GO and arrive in v.

- For every level, the sum of the points of the vertices of level i is equal to 1.

The global properties determined will then be:

- A1=0,22*0,33+0,27*0,43+0,22*0,40+0,29*0,37=0,38;
- A2=0,22*0,33+0,27*0,42+0,22*0,02+0,29*0,22=0,26;
- A3=0,22*0,22+0,27*0,15+0,22*0,58+0,29*0,41=0,36;

We can conclude that, in order to achieve the objective, a new teaching strategy is preferred to a traditional one and direct communication and human contact are factors not to be overlooked. Alternative 2 got the lowest score considering that learning cannot be a process of transmission but of meaning construction. Alternatives 1 and 3 differ for the different weight of the emotional dimension: canceling the physical contact the level of communication decreases and the emotional and social distancing increases. The online lesson would be too impersonal to adapt it to the real needs of the student.

4. Discussion and results

We applied SA2 that emerged from the model to the student.

In our case, the meetings consisted of afternoon lessons lasting two hours, usually held shortly after the lesson held in the classroom in the morning by the teacher.

Most of the meeting was based on the study of the last topics explained by the teacher, of which the student collected notes in a bare and confusing way, characterized by erasures and large empty spaces, common characteristic of the disgraphics.

The theory was gradually joined by practice, trying as much as possible to provide constructivist learning and, once the new topic was finished, work was done on the previous ones.

To avoid the use of the book, since reading was slow and incorrect due to dyslexia, sometimes also the theory was approached through exercises or questions to answer, as the didactic strategy provided.

During the lessons we gave the student more time to carry out the activities to transmit calm and serenity for the completion of the SC4.

In order to prevent the pupil from distracting himself, we often drew his attention by questioning him and repeating the concept in different ways.

To stimulate and give shape to ideas and discussions, weight has been given to questions such as “what would happen if this result was true?” or “under what circumstances could this error be corrected?”. With this approach we built dialogues during which the student was not afraid to express his difficulties.

Difficulties began to come up from the first topics when, in the numerical set exercises, there was a lot of confusion between the parentheses used to describe the intervals. For example:

- interval from "a" to "b" not including the extremes, $]a, b[$;
- interval from "a" to "b" including the extremes, $[a, b]$.

To cope with this difficulty, with a bit of imagination, the teacher made him imagine drawing two arms "embracing" the number if the number had to be present in the interval or, otherwise, they refused to do so. In this way, together with the use of graphic representations, the pupil acquired the competence to distinguish between the two types of parentheses and rarely found confusion until the end of the course.

In order to facilitate the repetition of the topics dealt with, we have extensively used diagrams and maps, compensatory instruments.

The mind maps, which we built together, were of great help especially in solving equations and inequalities with absolute values. Initially, the student's difficulty consisted in not being able to "visualize" the writing of the systems that came out of the procedure. With the use of a map, he was able, after several lessons, to carry out the simplest exercises correctly but still presenting difficulties for the more complex ones.

The result was satisfactory, however, because we believe that these difficulties were linked not so much to a lack of understanding of the absolute value function, but rather to a lack of ability to concentrate for so long on the same procedure (as pointed out above regarding the Specific Criterion 1 so SA2 requires to "evaluate according to progress and difficulties").

Towards the end of the course, other boys who were more frequent users of teacher's service were added to the tutoring lessons in view of the exam. In this way, the teacher was able to work on the last topics of the course, the derivatives and the study of graphs of functions, experimenting a collective study during which the student with specific learning disorders discussed with his or her classmates to the point of realizing, under teacher's guidance, his or their mistakes.

The dialogue between the students, being free and spontaneous, has favored the concretization of the SC2.

Finally, during the lessons, the teacher often used the calculator to reduce the material to memorize as much as possible. For example, when studying the domain of a function, initially the student made extensive use of tables summarizing the domains of elementary functions. Subsequently, by using the calculator that reports "ERR" if the function is not applied to an element of the domain, he really understood what this procedure meant by obtaining the right conditions for the simple functions such as roots and logarithms.

At the end of the course the boy passed the exam and reached the minimum targets. In fact, he has acquired the fundamental concepts to the point of facing the final test independently. As the tutoring activity progressed, the student also managed to follow the explanations more and more with a more profitable study. The participation of peers during the tutoring activity certainly helped

the boy who, with an "equal" comparison, acquired such security that he was able to guide his companions in difficulty during the exercises.

5. Conclusions

The aim of this paper is therefore to emphasize that students with SLD can compensate for their deficits if they are properly followed. Everyone is unique from a biological, neurological and psychological point of view and it is necessary to enhance the alternative ways of learning with scenes that allow the possibility to grow by compensating for the difficulties.

Therefore the design of a learning plan must be planned so that each student (while facing learning at different levels and in different ways) can access the processes proper to the acquisition of skills and knowledge ensuring, in this way, a formative success and the enhancement of all students.

The hierarchical analytical method of Saaty has allowed the formation of a conscious environment able to respect the "different" way of learning of a student with SLDs whose cognitive abilities and physical characteristics are in the norm. The didactic plan has, in fact, allowed a learning on a par with his companions and overcoming problems such as low self-esteem or a sense of ineffectiveness by allowing social skills to grow. About our case the ultimate goal has been to maximize understanding with the means available by bringing the student to the peak of their abilities.

Another consideration is that the affective dimension, understood as everything related to relationships and emotions, both of teachers and students, is crucial for a serene and safe climate in which students find motivations and stimuli that are necessary conditions to achieve specific goals set by the educational plans.

This strategy, having been applied to only one student, cannot be generalized because each case requires different objectives and tools depending on the disturbance and the problems that this entails. However, the suggested indications are a good start to approach starting from a general scheme that must be modified for the different cases, considering new specific criteria and new specific alternatives.

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The projective character tables of the maximal subgroups of M_{22} and its automorphism group M_{22}:2

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Abstract. In this paper, a routine written in the computational algebra system GAP is presented to compute the irreducible projective characters IrrProj(G, α_i) with associated factor sets α_i for the maximal subgroups of the sporadic simple Mathieu group M_{22} and its automorphism group M_{22}:2. The said routine derived its fundamentals from the theory of finding all the irreducible projective characters of a finite group G from the ordinary irreducible characters of a so-called representation group $R = M(G).G$ of G, where $M(G)$ denotes the Schur multiplier of G.

Keywords: projective character table, factor set, Schur multiplier, representation group, GAP routine.

1. Introduction

It is a well-known fact that all the sets of irreducible projective characters IrrProj(G, α_i), i=$1, 2, \ldots, m$, of a finite group G with factor sets α_i can be obtained from the ordinary irreducible characters of a so-called representation group $R \cong M(G).G$ of G, where $M(G)$ denotes the Schur multiplier of the group G and m the number of cohomology classes $[\alpha_i]$ in $M(G)$. Using this theory, a routine written in GAP [5] is discussed in Section 3 of this paper to compute the sets IrrProj(G, α_i) for a finite group G. In particular, this said GAP routine will be used to compute the sets IrrProj(G, α_i) for each maximal subgroup G of the sporadic simple Mathieu group M_{22} and its automorphism group $Aut(M_{22}) = M_{22}$:2, except for the ones whose projective character information already appear in the ATLAS [3]. The groups M_{22} and Aut(M_{22}) have Schur multipliers of cyclic orders 12 and 2 respectively (see [3], [7], [11]). Also, their irreducible projective character tables are available in [3] or GAP library [5]. This paper will form part of a series of articles by the current author on the computation of the irreducible projective characters IrrProj(G, α_i) of the maximal subgroups of the sporadic simple Mathieu groups M_{11}, M_{12}, M_{22}, M_{23}, M_{24}

and their automorphism groups. In this regard, the sets IrrProj(G, α_i) for M_{11}, M_{12} and Aut(M_{12}) are computed in [13].

The aforementioned GAP routine has resulted from a question posed by the current author in [18] about the number $|\text{IrrProj}(G, \alpha)|$ of irreducible projective characters IrrProj(G, α) with factor set α of a finite group G. Then G. Robinson [20] provided a proof under which conditions the relation $|\text{IrrProj}(G, \alpha)| \leq |\text{Irr}(G)|$ are satisfied. From this proof, a GAP routine was written by J. Schmidt [21] to determine the number $|\text{IrrProj}(G, \alpha_i)|$ of irreducible projective characters of a finite group G associated with any factor set α_i. The results mentioned above were used in the papers [14], [15],[16] and [17] by the current author to compute irreducible projective characters of a so-called inertia factor group of an extension group $\overline{G} = N.G$. These projective characters were needed to construct the ordinary character table of $\overline{G}$ using the technique of Fischer matrices [4]. Further on, this GAP routine by [21] was modified and extended by the current author in [13] to compute all the sets IrrProj(G, α_i) for any given finite group G on condition that one can compute the ordinary character table of a representation group R (Schur cover) of G. In the current paper, the extended GAP routine in [13] (see Section 3) is slightly adjusted to compute all the sets IrrProj(G, α_i) for each maximal subgroup G of M_{22} and $Aut(M_{22})$. Especially, when the size of a maximal subgroup of M_{22} or its automorphism group $Aut(M_{22})$ becomes too large for the GAP routine in [13] to handle, then the full cover group of G is first converted to a permutation group in GAP or MAGMA [2] before its projective character tables are computed with the said GAP routine.

In Section 2, some definitions and results on projective character theory are given and in Section 4 all the results of the sets IrrProj(G, α_i) for each maximal subgroup G of M_{22} and $Aut(M_{22})$ are summarized in table format. Computations are done in GAP and MAGMA and notations in both GAP and the ATLAS [3] are followed.

2. Preliminary results on projective characters

In this section, some of the basic ideas of projective character theory will be discussed and for this purpose, the outline given in [12] is followed closely. In what will follows, it will be understood that G is a finite group, $\mathbb{C}$ the field of complex numbers, $\mathbb{C}^*$ the nonzero complex numbers, $GL(n, \mathbb{C})$ the group of non-singular $n \times n$ matrices over the complex numbers $\mathbb{C}$, $Z(G)$ the center of G, G' the derived subgroup of G, Irr(G) the set of ordinary irreducible characters of G and IrrProj(G, α) the irreducible projective characters of G with associated factor set α. Interested readers are referred to [6], [8], [9], [10] and [19] for a detailed treatment on ordinary and projective character theory.

Definition 2.1. *A projective representation of a group G of degree n over the complex numbers is a map P:$G \to GL(n, \mathbb{C})$, such that*

(*i*) $P(1) = I_n$, *and*

(*ii*) *given* $x, y \in G$, *there exists* $\alpha(x,y) \in \mathbb{C}^*$ *such that* $P(x)P(y) = \alpha(x,y)P(xy)$.

Since multiplication in G and $GL(n,\mathbb{C})$ is associative it follows that $\alpha(xy,z)\alpha(x,y) = \alpha(x,yz)\alpha(y,z)$ for all $x,y,z \in G$. In addition, a map $\alpha : G \times G \to \mathbb{C}^*$ that satisfies this condition is called a factor set (or 2-cocycle) α of G in $\mathbb{C}$. We say that P is a projective representation with factor set α. Define $\xi(g) = Trace(P(g))$ for all $g \in G$, then ξ is called a projective character of G with factor set α. We say that ξ is irreducible if P is irreducible. An irreducible projective representation P of a group G is essentially defined in a similar way then an ordinary irreducible representation of G.

Definition 2.2. *Two projective representations* P_1 *and* P_2 *of* G *of degree* n *with factor sets* α_1 *and* α_2 *respectively are said to be projectively equivalent if there exists a mapping* $\phi : G \to \mathbb{C}^*$ *and a matrix* $T \in GL(n,\mathbb{C})$ *such that*

$$P_1(x) = \phi(x)T^{-1}P_2(x)T, \ \ \forall x \in G.$$

If P_1 and P_2 are projectively equivalent, then it follows from Definition 2.2 that, $\forall x, y \in G$,

$$\alpha_2(x,y) = \phi(x)\phi(y)(\phi(xy))^{-1}\alpha_1(x,y)$$

This is an equivalence relation and the equivalence class of the factor set α_1 is denoted by $[\alpha_1]$. The set of all equivalence classes of factor sets of G has a finite abelian group structure, and is called the Schur multiplier $M(G)$ (also known as the second cohomology group $H^2(G,\mathbb{C}^*)$ of G.

Definition 2.3. *A group* $C = A.G$ *is a central extension for* G *if there exists a homomorphism* π *from* C *onto* G *such that* $A = ker(\pi) \leq Z(R) \cap R'$. *In addition, if* $A \cong M(G)$, *then we called the central extension* C *a representation group* R *of* G.

A description on how to obtain the irreducible projective representations of a group G with factor set α from the ordinary irreducible representations of a central extension $C = A.G$ of G, will follow. Let $C = A.G$ be a central extension of the group G with $A = ker(\pi)$. Let $X = \{x_g | g \in G\}$ be a set of coset representatives of A in C, such that $\pi(x_g) = g$ (one-to-one correspondence of elements of X with the elements of G). Therefore, $C = \bigcup_{g \in G} Ax_g$. Then, for all $g, h \in G$, let $a(g,h)$ be the unique element in A such that $x_g x_h = a(g,h)x_{gh}$. Since the product operation to combine two elements in C and G is associative, then it follows that $a(g,h)a(gh,k) = a(g,hk)a(h,k)$ for all $g,h,k \in G$. Now, let λ be a linear character of the abelian group A and put $\alpha(g,h) = \lambda(a(g,h))$ for all $g,h \in G$, then it follows from the relation in the previous sentence that α is a factor set of G. Now, let T be an ordinary irreducible representation of C of degree n and let $P(g) = T(x_g)$ for all $g \in G$, then P is a irreducible projective representation of G with factor set α, i.e., $P(g)P(h) = \lambda(a(g,h))P(gh)$ for all $g,h \in G$. Hence, we can formulate the following definition:

Definition 2.4. *A projective representation P of G constructed from an ordinary irreducible representation T of C in the above manner is said to be linearized by the ordinary representation T (or lifted to C). Furthermore, P is irreducible if and only if T is irreducible.*

Each irreducible projective representation of G with corresponding factor set α can be linearized by an ordinary irreducible representation of a representation group R of G. So the problem of constructing all irreducible projective characters of a finite group G reduces to that of finding the ordinary irreducible characters of a representation group R of G.

Definition 2.5. *A covering group D for G will normally be a quotient $D \cong R/B$ of a representation group $R = M(G).G$ of G by a subgroup B of $M(G)$. If $M(G)/B$ has order n we sometimes refer to the covering group as a n-fold cover of G.*

Projective representations of G are found in the representation group R for all the equivalence classes of factors sets in $M(G)$ but however in a n-fold cover D of G only the n equivalence classes which D covers will be represented [6].

Definition 2.6. *An element $x \in G$ is said to be α-regular if $\alpha(x, g) = \alpha(g, x)$ for all $g \in C_G(x)$. Furthermore, it is well known that $g \in G$ is* α-regular *if and only if $\xi(g) \neq 0$ for some $\xi \in IrrProj(G, \alpha)$ or equivalently that g is α-irregular if and only if $\xi(g) = 0$ for all $\xi \in IrrProj(G, \alpha)$.*

Now, if $x \in G$ is α-regular, then so is every conjugate of x and therefore it is meaningful to speak about α-regular classes of G. The number of IrrProj(G, α) equals the number of α-regular classes of a group G. Projective characters also satisfy the usual orthogonality relations and have analogues to ordinary characters.

3. A GAP routine to compute IrrProj(G, α_i)

In this section, the GAP routine given in [21] which has its origin in Proposition 3.1 below is extended to give a GAP routine that can computes directly all the distinct sets IrrProj(G, α_i) of G from a representation group R of G.

Proposition 3.1. *Let $R = M(G).G$ be a representation group of a finite group G, where $M(G)$ denotes the Schur multiplier of G. Then the number of irreducible characters $\chi_j \in Irr(R)$ of R which lies over a linear character λ of $M(G)$ is less or equal to $|Irr(G)|$.*

Proof. (see [14] , [15] or [20]). □

From the proof of Propositon 3.1 it is mentioned that the number of irreducible characters $\chi_j \in \mathrm{Irr}(R)$ of a representation group $R = M(G).G$ of G which lies over a linear character $\lambda \in \mathrm{Irr}(M(G))$ (in other words $\lambda \in \mathrm{Irr}(M(G)$

is an irreducible constituent of $\chi_j \downarrow_{M(G)}$, that is, $< \chi_j \downarrow_{M(G)}, \lambda > \neq 0$) is given by $\sum_{\chi \in \mathrm{Irr}(R)} \frac{<\chi\downarrow_{M(G)}, \lambda>}{\chi(1)}$. Using this fact (see the line of the GAP routine starting with "n"), we are able to compute all the sets IrrProj(G, α_i), i=1, 2, ..., |Irr($M(G)$|, of G. In addition, it is shown in the proof of Proposition 3.1 that the quantity $\sum_{\chi \in \mathrm{Irr}(R)} \frac{<\chi\downarrow_{M(G)}, \lambda>}{\chi(1)} \leq |\mathrm{Irr}(G)|$ and the inequality becomes strict if there is a non-identity element $x \in M(G) \setminus \ker(\lambda)$ which is a commutator in R. That means that the number |IrrProj(G, α_i)| of irreducible projective characters of G with factor set α_i is always less or equal to the number |Irr(G)| of ordinary irreducible characters of G. In the below GAP routine the output "N" gives |Irr($M(G)$)| blocks coming from a representation group R (denoted as "Source(f)" in the below routine), where each block will contains one of the sets IrrProj(G, α_i), $i = 1, 2, .., |\mathrm{Irr}(M(G)|$. Whereas the output "Display(ct)" will display each individual irreducible projective character table of G with factor set α_i in GAP.

Now, the below GAP routine can computes directly all the distinct sets IrrProj(G, α_i) of G from a suitable representation group R (full covering group) of G. Especially, if the finite group G has a relatively small order then the representation group (Schur cover) of G can be computed easily in GAP (as it was in the case of the groups in [13]). But if the group G becomes too large then GAP experiences difficulties to compute the Schur cover of G and then we have to employ additional techniques in computing these sets IrrProj(G, α_i). This was the case with most of the maximal subgroups of M_{22} and Aut(M_{22}). To overcome this constraint, we compute the representation group R of Perm := G using the GAP command "S:=SchurCover(Perm)" and then use the GAP commands "iso:=IsomorphismPermGroup(S)" and "x:=Image(iso)" to convert R into a permutation group x. Then to find the normal subgroup z of the group x such that x/z $\cong G$ the command "Nor:=Filtered(NormalSubgroups(x),h1$->$ Size(h1)=Size(x)/Size(Perm))" is used. Furthermore, the command "f:= NaturalHomomorphismByNormalSubgroup(x,z)" is employed and then we can follow the rest of the GAP routine below from the input line "'I1:=ImagesSource(f)" to compute the sets IrrProj(G, α_i) of G successfully. The MAGMA routine find in [1] was used to convert the Schur cover of the group $L_3(4)$:2_2 (see Table 5) into a permutation group before applying the below GAP routine.

```
gap> Perm := G;; (Permutation group with generators found in [22] or can be
generated in GAP.)
gap> f := EpimorphismSchurCover(Perm);;
gap> z := Kernel(f);;
gap> x := Source(f);;
gap> I1:=ImagesSource(f);; (Quotient group I1 ≅ G)
gap> t:=Irr(I1);;
gap> 2t:=Irr(x);;
gap> F:=FusionConjugacyClassesOp(f);
gap> map:=ProjectionMap(F);
```

```
>Cen:=SizesCentralizers(CharacterTable(I1));
>Cl:=OrdersClassRepresentatives(CharacterTable(I1));
gap> N:=[];
gap> for i in [1..Size(Irr(z))] do
> n:= Filtered(Irr(x),chi− >not IsZero(ScalarProduct
                                    (RestrictedClassFunction(chi,z),Irr(z)[i])));
> s:=List(n,x− >x{map});
> Add(N,s);
> ct:=function()local ct ;ct:=rec();
>ct.SizesCentralizers:=Cen;;
>ct.OrdersClassRepresentatives:=Cl;;
> ct.Irr:=N[i];;
>ct.UnderlyingCharacteristic:=0;ct.Id:="G";
> ConvertToLibraryCharacterTableNC(ct);return ct;end;ct:=ct();
>SetInfoLevel(InfoCharacterTable,2);
>Display(ct);
>od;
gap>N;
```

4. The sets IrrProj(G, α_i) for the maximal subgroups of M_{22} and Aut(M_{22}).

In this section, the Schur multipliers $M(G)$ and the sets IrrProj(G, α_i) for each maximal subgroup G of M_{22} and Aut(M_{22}) are computed using the method discussed in Section 3 except for the ones of the groups $L_3(4)$, A_7, M_{10}, $L_2(11)$, M_{22}, Aut(A_6) and $L_2(11)$:2 which are found in the ATLAS or GAP library. All of the above-mentioned information is summarized in the tables below.

In Table 1, a summary of all the information concerning the structures of the Schur multipliers $M(G)$ and the number $|\text{IrrProj}(G, \alpha_i)|$ of sets IrrProj(G, α_i) for all the maximal subgroups of M_{22} and Aut(M_{22}) is given. Note that in the last column of Table 1 the number $|\text{IrrProj}(G, \alpha_i)|$ of irreducible projective characters with factor set α_i for each set IrrProj(G, α_i) of a maximal subgroup G of M_{22} and Aut(M_{22}) is given even if some of the sets contain the same information. But for the groups $L_3(4)$, A_7 and M_{22} only the number of projective characters in each distinct set IrrProj(G, α_i) is listed as it appeared in the ATLAS. Note that the structure of the Schur Multiplier $M(G)$ for each group G can also be determined by the GAP command $AbelianInvariantsMultiplier(G)$. The ordinary irreducible characters Irr(G) of the maximal subgroups of M_{22} and Aut(M_{22}) are always listed in the first block of the the below tables which contain the information of the sets IrrProj(G, α_i) of each maximal subgroup G.

We will consider briefly some of the groups whose Schur multiplier has order greater or equal to 4. The maximal subgroup $G = 2^4{:}A_6$ of M_{22} has a Schur multiplier $M(G) \cong 2 \times 4 \times 3$ of order 24. Therefore, $M(G)$ will have 24 coho-

mology classes $[\alpha_i]$ which consist out of the the trivial class [1], 3 classes of order 2, 2 classes of order 3, 4 classes of order 4, 6 classes of order 6 and 8 classes of order 12. The number of disctinct sets $\mathrm{IrrProj}(G, \alpha_i)$ of G associated with these classes will be the ordinary irreducible characters of G, three sets with factor sets $\alpha_i^2 = 1$, one set with factor set $\alpha_i^3 = 1$, two sets with factor sets $\alpha_i^4 = 1$, three sets with factor sets $\alpha_i^6 = 1$ and four sets with factor sets $\alpha_i^{12} = 1$. All the sets $\mathrm{IrrProj}(G, \alpha_i)$ of $2^4{:}A_6$ with factor sets α_i of order 3, 4, 6 and 12 can be recovered by applying all possible algebraic conjugation to the classes and irreducible projective characters of G with factor sets α_i of order 3, 4, 6 and 12 which appear in Table 2. For example, there are two projective character tables with factor sets α_i of order 3, but the entries of the first table (see Table 2) will be just the complex entries of the second table. Similar situations arose for the four sets $\mathrm{IrrProj}(G, \alpha_i)$ of $2^4{:}S_5$ with factor sets α_i of order 4. Here we will have two distinct sets $\mathrm{IrrProj}(G, \alpha_i)$ with $\alpha_i^4 = 1$ (see the bottom two blocks of Table 3), where the other two are just the complex entries of these two tables. In addition, the three sets $\mathrm{IrrProj}(G, \alpha_i)$ of $2^4{:}S_5$ with factor sets $\alpha_i^2 = 1$ are all inequivalent. The Schur Multiplier $M(G)$ of $2^3{:}L_3(2)$ is the Klein four-group 2^2 and thus $2^3{:}L_3(2)$ will have 3 $\mathrm{IrrProj}(G, \alpha_i)$ sets with non-trivial factor sets α_i of order 2 (see Table 4). Notice that the two sets in the bottom of Table 4 are the same except for the entries in the last column which just differ with a minus sign. In [21] it is mentioned that the sets $\mathrm{IrrProj}(G, \alpha_i)$ of G are only defined universally up to sign and it is possible that one can obtain different signs if the sets $\mathrm{IrrProj}(G, \alpha_i)$ are re-calculated using a different representation group $R = M(G).G$. But these signs are calculated consistently with a "special factor set" (as explained in [6] or Section 2 of this paper) and so the inner product and conjugacy results of [6] apply to the sets $\mathrm{IrrProj}(G, \alpha_i)$. For this reason, the two above-mentioned sets $\mathrm{IrrProj}(G, \alpha_i)$, $\alpha_i^2 = 1$, in Table 4 just carry the same information. The maximal subgroup $(2^3{:}L_3(2)) \times 2$ of $\mathrm{Aut}(M_{22})$ is a direct product of $2^3{:}L_3(2)$ with the cyclic group of order 2 and therefore we have that the three sets $\mathrm{IrrProj}(G, \alpha_i)$ of $(2^3{:}L_3(2)) \times 2$ with nontrivial factor sets of order two will just be twice the number of irreducible projective characters of that of $2^3{:}L_3(2)$ (see Tables 4 and 8). Since the 2 sets $\mathrm{IrrProj}(G, \alpha_i)$ of $(2^3{:}L_3(2)) \times 2$ with ten irreducible projective characters each contain the same information, only one set at the bottom of Table 8 is listed. The group A_7 has a Schur multiplier of cyclic order six and hence there will be three distinct sets of irreducible projective characters with factor sets of order 2, 3 and 6 (see [3]).

The Schur multiplier of the maximal subgroup $L_3(4){:}2_2$ of $\mathrm{Aut}(M_{22})$ is cyclic of order four and will have 3 sets $\mathrm{IrrProj}(L_3(4){:}2_2, \alpha_i)$ with nontrivial factor sets α_2, α_2^2 and α_2^3 of order 4, 2 and 4, respectively. The trivial factor set $\alpha_1 = \alpha_2^4 = 1$ is associated with the ordinary irreducible characters $\mathrm{Irr}(L_3(4){:}2_2)$ of $L_3(4){:}2_2$. Since $\alpha_2^3 = \alpha_2^{-1} = \overline{\alpha_2}$, the entries of the set $\mathrm{IrrProj}(L_3(4){:}2_2, \alpha_2)$ of $L_3(4){:}2_2$ in the third block of Table 5 are just the complex conjugates of the entries of the set $\mathrm{IrrProj}(L_3(4){:}2_2, \alpha_2^3)$. Therefore, only the set $\mathrm{IrrProj}(L_3(4){:}2_2, \alpha_2)$ of

$L_3(4)$:2_2 is listed at the bottom of Table 5 since IrrProj($L_3(4)$:$2_2, \alpha_2^3$) can be easily deduced from it.

Table 1: $M(G)$ **and** $|\mathbf{IrrProj}(G, \alpha_i)|$ **of maximal subgroups** G **of** M_{22} **and** $\mathbf{Aut}(M_{22})$

Maximal subgroups of M_{22}	$\lvert G\rvert$	$[M_{22}{:}G]$	$M(G)$	$\lvert\mathrm{IrrProj}(G, \alpha_i)\rvert$
$L_3(4)$	$20160 = 2^6.3^2.5.7$	22	$4 \times 4 \times 3$	[10,8,6,7,9,7,5,6]
$2^4{:}A_6$	$5760 = 2^7.3^2.5$	77	$2 \times 4 \times 3$	[12,9,11,10,10,10,7,7,6,6,7,7, 9,9,8,8,5,5,5,5,4,4,4,4]
A_7	$2520 = 2^3.3^2.5.7$	176	6	[9,7,7,5]
A_7	$2520 = 2^3.3^2.5.7$	176	6	[9,7,7,5]
$2^4{:}S_5$	$1920 = 2^7.3.5$	231	4×2	[12,11,8,7,7,7,5,5]
$2^3{:}L_3(2)$	$1344 = 2^6.3.7$	330	2^2	[11,8,5,5]
$M_{10} \cong A_6.2_3$	$720 = 2^4.3^2.5$	616	3	[8,7,7]
$L_2(11)$	$660 = 2^2.3.5.11$	672	2	[8,7]
Maximal subgroups of $\mathrm{Aut}(M_{22})$	$\lvert G\rvert$	$[\mathrm{Aut}(M_{22}){:}G]$	$M(G)$	$\lvert\mathrm{IrrProj}(G, \alpha_i)\rvert$
M_{22}	$443520 = 2^7.3^2.5.7.11$	2	12	[12,11,8,11,10,7]
$L_3(4){:}2_2$	$40320 = 2^7.3^2.5.7$	22	4	[14,13,11,11]
$2^4{:}S_6$	$11520 = 2^8.3^2.5$	77	2^2	[21,9,16,11]
$2^5{:}S_5$	$3840 = 2^8.3.5$	231	2^3	[24,10,13,13,16,14,5,9]
$(2^3{:}L_3(2)) \times 2$	$2688 = 2^7.3.7$	330	2^2	[22,16,10,10]
$Aut(A_6) \cong A_6.2^2$	$1440 = 2^5.3^2.5$	616	2	[13,10]
$L_2(11){:}2$	$1320 = 2^3.3.5.11$	672	2	[13,11]

Table 2: **IrrProj(G,**α_i**) for** $2^4{:}A_6$**,** $\alpha_1 = 1$

$[g]_G$	1a	2a	4a	4b	2b	3a	6a	8a	4a	5a	5b	3b
$\lvert C_G(g)\rvert$	5760	384	16	32	32	36	12	8	8	5	5	9
χ_1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	5	5	1	1	1	-1	-1	-1	-1	0	0	2
χ_3	5	5	1	1	1	2	2	-1	-1	0	0	-1
χ_4	8	8	0	0	0	-1	-1	0	0	T	*T	-1
χ_5	8	8	0	0	0	-1	-1	0	0	*T	T	-1
χ_6	9	9	1	1	1	0	0	1	1	-1	-1	0
χ_7	10	10	-2	-2	-2	1	1	0	0	0	0	1
χ_8	15	-1	-1	3	-1	3	-1	1	-1	0	0	0
χ_9	15	-1	-1	-1	3	3	-1	-1	1	0	0	0
χ_{10}	30	-2	-2	2	2	-3	1	0	0	0	0	0
χ_{11}	45	-3	1	1	-3	0	0	-1	1	0	0	0
χ_{12}	45	-3	1	-3	1	0	0	1	-1	0	0	0

Table 2 (continued): **IrrProj(G,α_i) for** $2^4{:}A_6$

$\alpha_2^2 = \alpha_3^2 = \alpha_4^2 = \alpha_5^3 = \alpha_6^4 = \alpha_7^4 = 1$

$[g]_G$	1a	2a	4a	4b	2b	3a	6a	8a	4a	5a	5b	3b
$\lvert C_G(g)\rvert$	5760	384	16	32	32	36	12	8	8	5	5	9
χ_1	4	4	0	0	0	-1	-1	0	0	-1	1	-2
χ_2	4	4	0	0	0	2	2	0	0	-1	1	1
χ_3	8	8	0	0	0	1	1	0	0	T	-*T	-1
χ_4	8	8	0	0	0	1	1	0	0	*T	-T	-1
χ_5	10	10	0	0	0	-1	-1	O	O	0	0	1
χ_6	10	10	0	0	0	-1	-1	-O	-O	0	0	1
χ_7	30	-2	0	0	0	3	-1	-O	O	0	0	0
χ_8	30	-2	0	0	0	3	-1	O	-O	0	0	0
χ_9	60	-4	0	0	0	-3	1	0	0	0	0	0
χ_1	6	-2	0	-2	-2	3	-1	0	0	1	1	0
χ_2	10	2	0	2	-2	1	1	0	-2	0	0	1
χ_3	10	2	0	2	-2	1	1	0	2	0	0	1
χ_4	10	2	0	-2	2	1	1	K	0	0	0	1
χ_5	10	2	0	-2	2	1	1	-K	0	0	0	1
χ_6	18	-6	0	2	2	0	0	0	0	T	*T	0
χ_7	18	-6	0	2	2	0	0	0	0	*T	T	0
χ_8	24	-8	0	0	0	3	-1	0	0	-1	-1	0
χ_9	30	-10	0	-2	-2	-3	1	0	0	0	0	0
χ_{10}	40	8	0	0	0	-2	-2	0	0	0	0	1
χ_{11}	40	8	0	0	0	1	1	0	0	0	0	-2
χ_1	10	2	K	0	0	-1	-1	P	O	0	0	1
χ_2	10	2	K	0	0	-1	-1	-P	-O	0	0	1
χ_3	10	2	-K	0	0	-1	-1	P	-O	0	0	1
χ_4	10	2	-K	0	0	-1	-1	-P	O	0	0	1
χ_5	12	-4	0	0	0	3	-1	0	0	-*T	T	0
χ_6	12	-4	0	0	0	3	-1	0	0	-T	*T	0
χ_7	24	-8	0	0	0	-3	1	0	0	-1	1	0
χ_8	36	-12	0	0	0	0	0	0	0	1	-1	0
χ_9	40	8	0	0	0	-1	-1	0	0	0	0	-2
χ_{10}	40	8	0	0	0	2	2	0	0	0	0	1
χ_1	3	A	-1	$\overline{\text{D}}$	D	0	0	1	-D	T	U	0
χ_2	3	A	-1	$\overline{\text{D}}$	D	0	0	1	-D	*T	V	0
χ_3	6	B	2	-$\overline{\text{F}}$	-F	0	0	0	0	1	-$\overline{\text{D}}$	0
χ_4	9	C	1	-$\overline{\text{D}}$	-D	0	0	1	-D	-1	$\overline{\text{D}}$	0
χ_5	15	D	-1	$\overline{\text{A}}$	D	0	-$\overline{\text{F}}$	1	D	0	0	0
χ_6	15	E	1	$\overline{\text{D}}$	D	0	0	-1	D	0	0	0
χ_7	15	D	-1	$\overline{\text{D}}$	A	0	-$\overline{\text{F}}$	-1	-D	0	0	0
χ_8	30	F	-2	-$\overline{\text{F}}$	-F	0	$\overline{\text{F}}$	0	0	0	0	0
χ_9	45	-A	1	-$\overline{\text{D}}$	-A	0	0	-1	-D	0	0	0
χ_{10}	45	-A	1	-$\overline{\text{A}}$	-D	0	0	1	D	0	0	0
χ_1	4	0	0	K	0	2	0	Q	0	-1	1	1
χ_2	20	0	0	K	0	-2	0	-Q	0	0	0	2
χ_3	20	0	0	K	0	4	0	-Q	0	0	0	-1
χ_4	32	0	0	0	0	-2	0	0	0	-*T	T	-1
χ_5	32	0	0	0	0	-2	0	0	0	-T	*T	-1
χ_6	36	0	0	K	0	0	0	Q	0	1	-1	0
χ_7	40	0	0	L	0	2	0	0	0	0	0	1
χ_1	16	0	0	0	0	-2	0	0	0	1	1	-2
χ_2	16	0	0	0	0	4	0	0	0	1	1	1
χ_3	32	0	0	0	0	2	0	0	0	-*T	-T	-1
χ_4	32	0	0	0	0	2	0	0	0	-T	-*T	-1
χ_5	40	0	0	0	0	-2	0	R	0	0	0	1
χ_6	40	0	0	0	0	-2	0	-R	0	0	0	1

Table 2 (continued): **IrrProj(G,α_i) for** 2^4:A_6

$\alpha_8^6 = \alpha_9^6 = \alpha_{10}^6 = \alpha_{11}^{12} = \alpha_{12}^{12} = \alpha_{13}^{12} = \alpha_{14}^{12} = 1$

$[g]_G$	1a	2a	4a	4b	2b	3a	6a	8a	4a	5a	5b	3b
$\lvert C_G(g)\rvert$	5760	384	16	32	32	36	12	8	8	5	5	9
χ_1	6	B	0	0	0	0	0	-O	S	1	$\overline{\text{D}}$	0
χ_2	6	B	0	0	0	0	0	O	-S	1	$\overline{\text{D}}$	0
χ_3	12	G	0	0	0	0	0	0	0	-*T	V	0
χ_4	12	G	0	0	0	0	0	0	0	-T	U	0
χ_5	30	F	0	0	0	0	-$\overline{\text{F}}$	-O	-S	0	0	0
χ_6	30	F	0	0	0	0	-$\overline{\text{F}}$	O	S	0	0	0
χ_7	60	H	0	0	0	0	$\overline{\text{F}}$	0	0	0	0	0
χ_1	6	F	0	$\overline{\text{F}}$	F	0	-$\overline{\text{F}}$	0	0	1	-$\overline{\text{D}}$	0
χ_2	18	-B	0	-$\overline{\text{F}}$	-F	0	0	0	0	T	U	0
χ_3	18	-B	0	-$\overline{\text{F}}$	-F	0	0	0	0	*T	V	0
χ_4	24	I	0	0	0	0	-$\overline{\text{F}}$	0	0	-1	$\overline{\text{D}}$	0
χ_5	30	J	0	$\overline{\text{F}}$	F	0	$\overline{\text{F}}$	0	0	0	0	0
χ_6	30	B	0	$\overline{\text{F}}$	-F	0	0	0	F	0	0	0
χ_7	30	B	0	$\overline{\text{F}}$	-F	0	0	0	-F	0	0	0
χ_8	30	B	0	-$\overline{\text{F}}$	F	0	0	K	0	0	0	0
χ_9	30	B	0	-$\overline{\text{F}}$	F	0	0	-K	0	0	0	0
χ_1	12	$\overline{\text{H}}$	0	0	0	0	-F	0	0	-*T	$\overline{\text{V}}$	0
χ_2	12	$\overline{\text{H}}$	0	0	0	0	-F	0	0	-T	$\overline{\text{U}}$	0
χ_3	24	$\overline{\text{I}}$	0	0	0	0	F	0	0	-1	-D	0
χ_4	30	$\overline{\text{B}}$	K	0	0	0	0	P	$\overline{\text{S}}$	0	0	0
χ_5	30	$\overline{\text{B}}$	K	0	0	0	0	-P	-$\overline{\text{S}}$	0	0	0
χ_6	30	$\overline{\text{B}}$	-K	0	0	0	0	P	-$\overline{\text{S}}$	0	0	0
χ_7	30	$\overline{\text{B}}$	-K	0	0	0	0	-P	$\overline{\text{S}}$	0	0	0
χ_8	36	- $\overline{\text{G}}$	0	0	0	0	0	0	0	1	D	0
χ_1	12	0	0	M	0	0	0	Q	0	-*T	V	0
χ_2	12	0	0	M	0	0	0	Q	0	-T	U	0
χ_3	24	0	0	N	0	0	0	0	0	-1	-$\overline{\text{D}}$	0
χ_4	36	0	0	-M	0	0	0	Q	0	1	$\overline{\text{D}}$	0
χ_5	60	0	0	M	0	0	0	-Q	0	0	0	0
χ_1	12	0	0	- M	0	0	0	$\overline{\text{Q}}$	0	-*T	V	0
χ_2	12	0	0	-M	0	0	0	$\overline{\text{Q}}$	0	-T	U	0
χ_3	24	0	0	-N	0	0	0	0	0	-1	-$\overline{\text{D}}$	0
χ_4	36	0	0	M	0	0	0	$\overline{\text{Q}}$	0	1	$\overline{\text{D}}$	0
χ_5	60	0	0	-M	0	0	0	-$\overline{\text{Q}}$	0	0	0	0
χ_1	24	0	0	0	0	0	0	-R	0	-1	D	0
χ_2	24	0	0	0	0	0	0	R	0	-1	D	0
χ_3	48	0	0	0	0	0	0	0	0	T	$\overline{\text{U}}$	0
χ_4	48	0	0	0	0	0	0	0	0	*T	$\overline{\text{V}}$	0
χ_1	24	0	0	0	0	0	0	-R	0	-1	$\overline{\text{D}}$	0
χ_2	24	0	0	0	0	0	0	R	0	-1	$\overline{\text{D}}$	0
χ_3	48	0	0	0	0	0	0	0	0	T	U	0
χ_4	48	0	0	0	0	0	0	0	0	*T	V	0

where A = = $\frac{-3+3\sqrt{3}\text{i}}{2}$ = 3b3, B = $-3+3\sqrt{3}$i = 6b3,
C = $\frac{-9+9\sqrt{3}\text{i}}{2}$ = 9b3, D = $\frac{1-\sqrt{3}\text{i}}{2}$ = -b3, E = $\frac{-15+15\sqrt{3}\text{i}}{2}$ = 15b3,
F = $1-\sqrt{3}$i = 1-i3, G = $-6+6\sqrt{3}$i = 12b3, H = $2-2\sqrt{3}$i = -4b3,
I = $4-4\sqrt{3}$i = -8b3, J = $5-5\sqrt{3}$i = 5-5i3, K = -2i,
L = 4i, M = $2\text{E}(12)^{11}$, N = $-4\text{E}(12)^{11}$, O = $-\sqrt{2}$ = -r2,
P = $\sqrt{2}$i = -i2, Q = -1-i, R = $-2\text{E}(8)$, S = $\text{E}(24)^{11}$-$\text{E}(24)^{17}$,
T = $\frac{1-\sqrt{5}}{2}$ = -b5, U = -E(15)-$\text{E}(15)^4$, V = -$\text{E}(15)^7$-$\text{E}(15)^{13}$

Table 3: **IrrProj(G,α_i) for** $2^4{:}S_5$

$\alpha_1 = \alpha_2^2 = \alpha_3^2 = \alpha_4^2 = \alpha_5^4 = \alpha_6^4 = 1$

$[g]_G$	1a	2a	4a	2b	4b	4c	2c	3a	6a	4d	8a	5a
$\lvert C_G(g)\rvert$	1920	128	16	48	16	32	32	6	6	8	8	5
χ_1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	-1	-1	1	1	1	1	-1	-1	-1	1
χ_3	4	4	2	2	0	0	0	1	-1	0	0	-1
χ_4	4	4	-2	-2	0	0	0	1	1	0	0	-1
χ_5	5	5	-1	-1	1	1	1	-1	-1	1	1	0
χ_6	5	5	1	1	1	1	1	-1	1	-1	-1	0
χ_7	6	6	0	0	-2	-2	-2	0	0	0	0	1
χ_8	15	-1	-1	3	-1	-1	3	0	0	1	-1	0
χ_9	15	-1	-1	3	-1	3	-1	0	0	-1	1	0
χ_{10}	15	-1	1	-3	-1	-1	3	0	0	-1	1	0
χ_{11}	15	-1	1	-3	-1	3	-1	0	0	1	-1	0
χ_{12}	30	-2	0	0	2	-2	-2	0	0	0	0	0
χ_1	6	-2	0	0	0	-2	2	0	0	-2	0	1
χ_2	6	-2	0	0	0	-2	2	0	0	2	0	1
χ_3	6	-2	0	0	0	2	-2	0	0	0	A	1
χ_4	6	-2	0	0	0	2	-2	0	0	0	-A	1
χ_5	10	2	-2	-2	0	-2	-2	1	-1	0	0	0
χ_6	10	2	0	4	0	2	2	1	-1	0	0	0
χ_7	10	2	2	2	0	-2	-2	1	1	0	0	0
χ_8	10	2	0	-4	0	2	2	1	1	0	0	0
χ_9	20	4	2	-2	0	0	0	-1	-1	0	0	0
χ_{10}	20	4	-2	2	0	0	0	-1	1	0	0	0
χ_{11}	24	-8	0	0	0	0	0	0	0	0	0	-1
χ_1	6	-2	0	0	A	0	0	0	0	E	F	1
χ_2	6	-2	0	0	A	0	0	0	0	-E	-F	1
χ_3	6	-2	0	0	-A	0	0	0	0	-E	F	1
χ_4	6	-2	0	0	-A	0	0	0	0	E	-F	1
χ_5	20	4	0	0	0	0	0	-2	0	0	0	0
χ_6	20	4	0	0	0	0	0	1	B	0	0	0
χ_7	20	4	0	0	0	0	0	1	-B	0	0	0
χ_8	24	-8	0	0	0	0	0	0	0	0	0	-1
χ_1	4	4	0	0	0	0	0	2	0	0	0	-1
χ_2	4	4	0	0	0	0	0	-1	B	0	0	-1
χ_3	4	4	0	0	0	0	0	-1	-B	0	0	-1
χ_4	6	6	0	0	0	0	0	0	0	E	E	1
χ_5	6	6	0	0	0	0	0	0	0	-E	-E	1
χ_6	30	-2	0	0	0	0	0	0	0	E	-E	0
χ_7	30	-2	0	0	0	0	0	0	0	-E	E	0
χ_1	4	0	0	2	0	2	0	-1	C	0	G	-1
χ_2	4	0	0	-2	0	2	0	-1	-C	0	-G	-1
χ_3	16	0	0	4	0	0	0	-1	-C	0	0	1
χ_4	16	0	0	-4	0	0	0	-1	C	0	0	1
χ_5	20	0	0	-2	0	2	0	1	-C	0	G	0
χ_6	20	0	0	2	0	2	0	1	C	0	-G	0
χ_7	24	0	0	0	0	-4	0	0	0	0	0	-1
χ_1	16	0	0	0	0	0	0	-2	0	0	0	1
χ_2	16	0	0	0	0	0	0	1	D	0	0	1
χ_3	16	0	0	0	0	0	0	1	-D	0	0	1
χ_4	24	0	0	0	0	0	0	0	0	0	H	-1
χ_5	24	0	0	0	0	0	0	0	0	0	-H	-1

A = -2i, B = $-\sqrt{3}$i = -i3, C = i, D = $\sqrt{3}$ = r3,

E = $\sqrt{2}$i = i2, F = $-\sqrt{2}$ = -r2, G = 1-i, H = $2^*E(8)^3$

Table 4: **IrrProj(G,α_i) for** 2^3:$L_3(2)$

$\alpha_1 = \alpha_2^2 = \alpha_3^2 = \alpha_4^2 = 1$

$[g]_G$	1a	2a	2b	2c	4a	3a	6a	4a	4b	7a	7b
$\lvert C_G(g)\rvert$	1344	192	32	32	16	6	6	8	8	7	7
χ_1	1	1	1	1	1	1	1	1	1	1	1
χ_2	3	3	-1	-1	-1	0	0	1	1	C	$\overline{\text{C}}$
χ_3	3	3	-1	-1	-1	0	0	1	1	$\overline{\text{C}}$	C
χ_4	6	6	2	2	2	0	0	0	0	-1	-1
χ_5	7	7	-1	-1	-1	1	1	-1	-1	0	0
χ_6	7	-1	-1	3	-1	1	-1	-1	1	0	0
χ_7	7	-1	3	-1	-1	1	-1	1	-1	0	0
χ_8	8	8	0	0	0	-1	-1	0	0	1	1
χ_9	14	-2	2	2	-2	-1	1	0	0	0	0
χ_{10}	21	-3	-3	1	1	0	0	1	-1	0	0
χ_{11}	21	-3	1	-3	1	0	0	-1	1	0	0
χ_1	4	4	0	0	0	1	1	0	0	-$\overline{\text{C}}$	C
χ_2	4	4	0	0	0	1	1	0	0	-C	$\overline{\text{C}}$
χ_3	6	6	0	0	0	0	0	B	B	-1	1
χ_4	6	6	0	0	0	0	0	-B	-B	-1	1
χ_5	8	8	0	0	0	-1	-1	0	0	1	-1
χ_6	14	-2	0	0	0	-1	1	B	-B	0	0
χ_7	14	-2	0	0	0	-1	1	-B	B	0	0
χ_8	28	-4	0	0	0	1	-1	0	0	0	0
χ_1	8	0	0	0	0	2	0	0	0	1	-1
χ_2	8	0	0	0	0	-1	A	0	0	1	-1
χ_3	8	0	0	0	0	-1	-A	0	0	1	-1
χ_4	24	0	0	0	0	0	0	0	0	C	-$\overline{\text{C}}$
χ_5	24	0	0	0	0	0	0	0	0	$\overline{\text{C}}$	-C
χ_1	8	0	0	0	0	2	0	0	0	1	1
χ_2	8	0	0	0	0	-1	A	0	0	1	1
χ_3	8	0	0	0	0	-1	-A	0	0	1	1
χ_4	24	0	0	0	0	0	0	0	0	C	$\overline{\text{C}}$
χ_5	24	0	0	0	0	0	0	0	0	$\overline{\text{C}}$	C

A = -$\sqrt{3}$i = -i3, B = -$\sqrt{2}$ = -r2, C = $\frac{-1-\sqrt{7}\text{i}}{2}$ = -1-b7

Table 5: **IrrProj(G,α_i) for** $L_3(4){:}2_2$

$\alpha_1 = \alpha_2^2 = \alpha_3^4 = 1$

$[g]_G$	1a	5a	2a	7a	7b	14a	14b	2b	4a	3a	6a	8a	4b	4c
$\lvert C_G(g)\rvert$	40320	5	336	14	14	14	14	128	32	18	6	8	16	16
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	-1	1	1	-1	-1	1	1	1	-1	-1	1	-1
χ_3	20	0	-6	-1	-1	1	1	4	0	2	0	0	0	-2
χ_4	20	0	6	-1	-1	-1	-1	4	0	2	0	0	0	2
χ_5	35	0	-7	0	0	0	0	3	3	-1	-1	1	-1	1
χ_6	35	0	7	0	0	0	0	3	3	-1	1	-1	-1	-1
χ_7	45	0	-3	B	$\overline{\text{B}}$	-B	-$\overline{\text{B}}$	-3	1	0	0	-1	1	1
χ_8	45	0	-3	$\overline{\text{B}}$	B	-$\overline{\text{B}}$	-B	-3	1	0	0	-1	1	1
χ_9	45	0	3	B	$\overline{\text{B}}$	B	$\overline{\text{B}}$	-3	1	0	0	1	1	-1
χ_{10}	45	0	3	$\overline{\text{B}}$	B	$\overline{\text{B}}$	B	-3	1	0	0	1	1	-1
χ_{11}	64	-1	8	1	1	1	1	0	0	1	-1	0	0	0
χ_{12}	64	-1	-8	1	1	-1	-1	0	0	1	1	0	0	0
χ_{13}	70	0	0	0	0	0	0	6	-2	-2	0	0	2	0
χ_{14}	126	1	0	0	0	0	0	-2	-2	0	0	0	-2	0
χ_1	10	0	-4	-$\overline{\text{B}}$	-B	$\overline{\text{B}}$	-B	2	-2	-1	-1	0	0	0
χ_2	10	0	-4	-B	-$\overline{\text{B}}$	B	-$\overline{\text{B}}$	2	-2	-1	-1	0	0	0
χ_3	10	0	4	-$\overline{\text{B}}$	-B	-$\overline{\text{B}}$	B	2	-2	-1	1	0	0	0
χ_4	10	0	4	-B	-$\overline{\text{B}}$	-B	$\overline{\text{B}}$	2	-2	-1	1	0	0	0
χ_5	36	-1	-6	-1	-1	1	-1	4	0	0	0	0	0	2
χ_6	36	-1	6	-1	-1	-1	1	4	0	0	0	0	0	-2
χ_7	56	-1	0	0	0	0	0	-8	0	-2	0	0	0	0
χ_8	64	1	8	-1	-1	1	-1	0	0	-1	-1	0	0	0
χ_9	64	1	-8	-1	-1	-1	1	0	0	-1	1	0	0	0
χ_{10}	70	0	0	0	0	0	0	-2	-2	2	0	-D	0	0
χ_{11}	70	0	0	0	0	0	0	-2	-2	2	0	D	0	0
χ_{12}	90	0	-6	1	1	1	-1	2	2	0	0	0	0	-2
χ_{13}	90	0	6	1	1	-1	1	2	2	0	0	0	0	2
χ_1	20	0	6	-A	-A	-1	A	0	D	-D	0	F	0	0
χ_2	20	0	-6	-A	-A	1	-A	0	D	-D	0	-F	0	0
χ_3	36	A	6	A	A	-1	A	0	D	0	0	-F	0	0
χ_4	36	A	-6	A	A	1	-A	0	D	0	0	F	0	0
χ_5	56	A	0	0	0	0	0	0	E	-D	0	0	0	0
χ_6	64	-A	-8	A	A	-1	A	0	0	A	1	0	0	0
χ_7	64	-A	8	A	A	1	-A	0	0	A	-1	0	0	0
χ_8	80	0	-4	C	-$\overline{\text{C}}$	B	$\overline{\text{C}}$	0	0	-A	-1	0	0	0
χ_9	80	0	-4	-$\overline{\text{C}}$	C	$\overline{\text{B}}$	-C	0	0	-A	-1	0	0	0
χ_{10}	80	0	4	C	-$\overline{\text{C}}$	-B	-$\overline{\text{C}}$	0	0	-A	1	0	0	0
χ_{11}	80	0	4	-$\overline{\text{C}}$	C	-$\overline{\text{B}}$	C	0	0	-A	1	0	0	0

A = -i, B = $\frac{-1-\sqrt{7}\text{i}}{2}$ = -1-b7, C = -E(28)^3-E(28)^{19}-E(28)^{27},
D = 2i, E = -4i, F = 1-i

Table 6: **IrrProj(G,α_i) for** $2^4{:}S_6$

$\alpha_1 = \alpha_2^2 = \alpha_3^2 = \alpha_4^2 = 1$

$[g]_G$	1a	2a	2b	4a	2c	3a	6a	2d	4b	4c	6b	12a	4d	8a	2e	4d	4e	8b	5a	6c	3b
$\lvert C_G(g)\rvert$	11520	768	384	96	128	72	24	64	32	64	12	12	16	16	192	64	16	16	5	6	18
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	-1	-1	-1	1	1	1	1	1	-1	-1	1	1	-1	-1	-1	-1	1	-1	1
χ_3	5	5	3	3	3	2	2	1	1	1	0	0	-1	-1	-1	-1	1	1	0	-1	-1
χ_4	5	5	-3	-3	-3	2	2	1	1	1	0	0	-1	-1	1	1	-1	-1	0	1	-1
χ_5	5	5	-1	-1	-1	-1	-1	1	1	1	-1	-1	-1	-1	3	3	1	1	0	0	2
χ_6	5	5	1	1	1	-1	-1	1	1	1	1	1	-1	-1	-3	-3	-1	-1	0	0	2
χ_7	9	9	3	3	3	0	0	1	1	1	0	0	1	1	3	3	-1	-1	-1	0	0
χ_8	9	9	-3	-3	-3	0	0	1	1	1	0	0	1	1	-3	-3	1	1	-1	0	0
χ_9	10	10	-2	-2	-2	1	1	-2	-2	-2	1	1	0	0	2	2	0	0	0	-1	1
χ_{10}	10	10	2	2	2	1	1	-2	-2	-2	-1	-1	0	0	-2	-2	0	0	0	1	1
χ_{11}	15	-1	-7	1	1	3	-1	3	-1	-1	-1	1	1	-1	-3	1	-1	1	0	0	0
χ_{12}	15	-1	5	1	-3	3	-1	-1	-1	3	-1	1	-1	1	-3	1	1	-1	0	0	0
χ_{13}	15	-1	7	-1	-1	3	-1	3	-1	-1	1	-1	1	-1	3	-1	1	-1	0	0	0
χ_{14}	15	-1	-5	-1	3	3	-1	-1	-1	3	1	-1	-1	1	3	-1	-1	1	0	0	0
χ_{15}	16	16	0	0	0	-2	-2	0	0	0	0	0	0	0	0	0	0	0	1	0	-2
χ_{16}	30	-2	-2	2	-2	-3	1	2	-2	2	1	-1	0	0	-6	2	0	0	0	0	0
χ_{17}	30	-2	2	-2	2	-3	1	2	-2	2	-1	1	0	0	6	-2	0	0	0	0	0
χ_{18}	45	-3	9	-3	1	0	0	1	1	-3	0	0	-1	1	-3	1	-1	1	0	0	0
χ_{19}	45	-3	-9	3	-1	0	0	1	1	-3	0	0	-1	1	3	-1	1	-1	0	0	0
χ_{20}	45	-3	-3	-3	5	0	0	-3	1	1	0	0	1	-1	-3	1	1	-1	0	0	0
χ_{21}	45	-3	3	3	-5	0	0	-3	1	1	0	0	1	-1	3	-1	-1	1	0	0	0
χ_1	4	4	0	0	0	-1	-1	0	0	0	A	A	0	0	0	0	0	0	1	0	-2
χ_2	4	4	0	0	0	-1	-1	0	0	0	-A	-A	0	0	0	0	0	0	1	0	-2
χ_3	4	4	0	0	0	2	2	0	0	0	0	0	0	0	0	0	0	0	1	C	1
χ_4	4	4	0	0	0	2	2	0	0	0	0	0	0	0	0	0	0	0	1	-C	1
χ_5	16	16	0	0	0	2	2	0	0	0	0	0	0	0	0	0	0	0	-1	0	-2
χ_6	20	20	0	0	0	-2	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	2
χ_7	60	-4	0	0	0	6	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_8	60	-4	0	0	0	-3	1	0	0	0	-A	A	0	0	0	0	0	0	0	0	0
χ_9	60	-4	0	0	0	-3	1	0	0	0	A	-A	0	0	0	0	0	0	0	0	0
χ_1	6	2	4	2	0	3	-1	-2	0	-2	1	-1	0	0	0	0	-2	0	1	0	0
χ_2	6	2	-4	-2	0	3	-1	-2	0	-2	-1	1	0	0	0	0	2	0	1	0	0
χ_3	10	-2	-4	2	0	1	1	-2	0	2	-1	-1	-2	0	-4	0	0	0	0	-1	-1
χ_4	10	-2	-4	2	0	1	1	-2	0	2	-1	-1	2	0	4	0	0	0	0	1	-1
χ_5	10	-2	4	-2	0	1	1	-2	0	2	1	1	2	0	-4	0	0	0	0	-1	-1
χ_6	10	-2	4	-2	0	1	1	-2	0	2	1	1	-2	0	4	0	0	0	0	1	-1
χ_7	20	-4	0	0	0	2	2	4	0	-4	0	0	0	0	0	0	0	0	0	0	-2
χ_8	24	8	8	4	0	3	-1	0	0	0	-1	1	0	0	0	0	0	0	-1	0	0
χ_9	24	8	-8	-4	0	3	-1	0	0	0	1	-1	0	0	0	0	0	0	-1	0	0
χ_{10}	30	10	-4	-2	0	-3	1	-2	0	-2	-1	1	0	0	0	0	-2	0	0	0	0
χ_{11}	30	10	4	2	0	-3	1	-2	0	-2	1	-1	0	0	0	0	2	0	0	0	0
χ_{12}	36	12	0	0	0	0	0	4	0	4	0	0	0	0	0	0	0	0	1	0	0
χ_{13}	40	-8	8	-4	0	1	1	0	0	0	-1	-1	0	0	0	0	0	0	0	0	2
χ_{14}	40	-8	-8	4	0	1	1	0	0	0	1	1	0	0	0	0	0	0	0	0	2
χ_{15}	40	-8	0	0	0	-2	-2	0	0	0	0	0	0	0	-8	0	0	0	0	1	-1
χ_{16}	40	-8	0	0	0	-2	-2	0	0	0	0	0	0	0	8	0	0	0	0	-1	-1
χ_1	20	-4	0	0	0	-2	-2	0	0	0	0	0	0	B	0	0	0	0	0	0	-2
χ_2	20	-4	0	0	0	-2	-2	0	0	0	0	0	0	-B	0	0	0	0	0	0	-2
χ_3	24	8	0	0	0	6	-2	0	0	0	0	0	0	0	0	0	0	0	1	0	0
χ_4	24	8	0	0	0	-3	1	0	0	0	-A	A	0	0	0	0	0	0	1	0	0
χ_5	24	8	0	0	0	-3	1	0	0	0	A	-A	0	0	0	0	0	0	1	0	0
χ_6	36	12	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	B	-1	0	0
χ_7	36	12	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-B	-1	0	0
χ_8	40	-8	0	0	0	-1	-1	0	0	0	A	A	0	0	0	0	0	0	0	0	2
χ_9	40	-8	0	0	0	-1	-1	0	0	0	-A	-A	0	0	0	0	0	0	0	0	2
χ_{10}	40	-8	0	0	0	2	2	0	0	0	0	0	0	0	0	0	0	0	0	C	-1
χ_{11}	40	-8	0	0	0	2	2	0	0	0	0	0	0	0	0	0	0	0	0	-C	-1

A = -$\sqrt{3}$i = -i3, B = -2$\sqrt{2}$i = -2i2, C = $\sqrt{3}$ = r3

Table 7: **IrrProj(G,α_i) for** $2^5{:}S_5$
$\alpha_1 = \alpha_2^2 = \alpha_3^2 = \alpha_4^2 = 1$

$[g]_G$	1a	2a	2b	2c	2d	4a	4b	2e	2f	4c	4d	4e	4f	2g	6a	3a	6b	6c	4g	8a	8b	4h	10a	5a
$\lvert C_G(g)\rvert$	3840	256	384	640	96	32	32	96	64	64	32	32	64	64	12	12	12	12	16	16	16	16	10	10
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	-1	-1	1	1	-1	-1	1	1	1	-1	-1	-1	-1	1	1	-1	1	1	-1	-1	-1	1
χ_3	1	1	-1	-1	-1	-1	1	1	1	1	1	-1	-1	-1	-1	1	-1	1	-1	-1	1	1	-1	1
χ_4	1	1	1	1	-1	-1	-1	-1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	1	1
χ_5	4	4	-4	-4	2	2	-2	-2	0	0	0	0	0	0	-1	1	-1	1	0	0	0	0	1	-1
χ_6	4	4	-4	-4	-2	-2	2	2	0	0	0	0	0	0	-1	1	1	-1	0	0	0	0	1	-1
χ_7	4	4	4	4	-2	-2	-2	-2	0	0	0	0	0	0	1	1	1	1	0	0	0	0	-1	-1
χ_8	4	4	4	4	2	2	2	2	0	0	0	0	0	0	1	1	-1	-1	0	0	0	0	-1	-1
χ_9	5	5	-5	-5	1	1	-1	-1	1	1	1	-1	-1	-1	1	-1	1	-1	-1	-1	1	1	0	0
χ_{10}	5	5	-5	-5	-1	-1	1	1	1	1	1	-1	-1	-1	1	-1	-1	1	1	1	-1	-1	0	0
χ_{11}	5	5	5	5	-1	-1	-1	-1	1	1	1	1	1	1	-1	-1	-1	-1	1	1	1	1	0	0
χ_{12}	5	5	5	5	1	1	1	1	1	1	1	1	1	1	-1	-1	1	1	-1	-1	-1	-1	0	0
χ_{13}	6	6	6	6	0	0	0	0	-2	-2	-2	-2	-2	-2	0	0	0	0	0	0	0	0	1	1
χ_{14}	6	6	-6	-6	0	0	0	0	-2	-2	-2	2	2	2	0	0	0	0	0	0	0	0	-1	1
χ_{15}	15	-1	3	-5	3	-1	-1	3	3	-1	-1	-1	3	-1	0	0	0	0	1	-1	-1	1	0	0
χ_{16}	15	-1	3	-5	-3	1	1	-3	3	-1	-1	-1	3	-1	0	0	0	0	-1	1	1	-1	0	0
χ_{17}	15	-1	-3	5	-3	1	-1	3	3	-1	-1	1	-3	1	0	0	0	0	-1	1	-1	1	0	0
χ_{18}	15	-1	-3	5	3	-1	1	-3	3	-1	-1	1	-3	1	0	0	0	0	1	-1	1	-1	0	0
χ_{19}	15	-1	3	-5	3	-1	-1	3	-1	3	-1	-1	-1	3	0	0	0	0	-1	1	1	-1	0	0
χ_{20}	15	-1	3	-5	-3	1	1	-3	-1	3	-1	-1	-1	3	0	0	0	0	1	-1	-1	1	0	0
χ_{21}	15	-1	-3	5	-3	1	-1	3	-1	3	-1	1	1	-3	0	0	0	0	1	-1	1	-1	0	0
χ_{22}	15	-1	-3	5	3	-1	1	-3	-1	3	-1	1	1	-3	0	0	0	0	-1	1	-1	1	0	0
χ_{23}	30	-2	6	-10	0	0	0	0	-2	-2	2	2	-2	-2	0	0	0	0	0	0	0	0	0	0
χ_{24}	30	-2	-6	10	0	0	0	0	-2	-2	2	-2	2	2	0	0	0	0	0	0	0	0	0	0
χ_1	12	4	0	0	0	0	0	0	0	0	A	0	0	0	0	0	0	0	0	0	0	0	0	2
χ_2	12	4	0	0	0	0	0	0	0	0	-A	0	0	0	0	0	0	0	0	0	0	0	0	2
χ_3	20	-4	8	0	0	0	0	0	0	0	0	0	0	0	2	2	0	0	0	0	0	0	0	0
χ_4	20	-4	-8	0	0	0	0	0	0	0	0	0	0	0	-2	2	0	0	0	0	0	0	0	0
χ_5	20	-4	-8	0	0	0	0	0	0	0	0	0	0	0	1	-1	C	-C	0	0	0	0	0	0
χ_6	20	-4	-8	0	0	0	0	0	0	0	0	0	0	0	1	-1	-C	C	0	0	0	0	0	0
χ_7	20	-4	8	0	0	0	0	0	0	0	0	0	0	0	-1	-1	-C	-C	0	0	0	0	0	0
χ_8	20	-4	8	0	0	0	0	0	0	0	0	0	0	0	-1	-1	C	C	0	0	0	0	0	0
χ_9	24	8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	F	-1
χ_{10}	24	8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-F	-1
χ_1	6	2	0	-4	0	0	0	0	0	0	B	0	-B	-B	0	0	0	0	D	E	-D	-E	-1	1
χ_2	6	2	0	-4	0	0	0	0	0	0	B	0	-B	-B	0	0	0	0	-D	-E	D	E	-1	1
χ_3	6	2	0	-4	0	0	0	0	0	0	-B	0	B	B	0	0	0	0	D	-E	-D	E	-1	1
χ_4	6	2	0	-4	0	0	0	0	0	0	-B	0	B	B	0	0	0	0	-D	E	D	-E	-1	1
χ_5	6	2	0	4	0	0	0	0	0	0	B	0	B	B	0	0	0	0	-D	-E	-D	-E	1	1
χ_6	6	2	0	4	0	0	0	0	0	0	B	0	B	B	0	0	0	0	D	E	D	E	1	1
χ_7	6	2	0	4	0	0	0	0	0	0	-B	0	-B	-B	0	0	0	0	-D	E	-D	E	1	1
χ_8	6	2	0	4	0	0	0	0	0	0	-B	0	-B	-B	0	0	0	0	D	-E	D	-E	1	1
χ_9	20	-4	0	0	0	0	0	0	0	0	0	0	-A	A	0	2	0	0	0	0	0	0	0	0
χ_{10}	20	-4	0	0	0	0	0	0	0	0	0	0	A	-A	0	2	0	0	0	0	0	0	0	0
χ_{11}	24	8	0	-16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	-1
χ_{12}	24	8	0	16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	-1
χ_{13}	40	-8	0	0	0	0	0	0	0	0	0	0	0	0	0	-2	0	0	0	0	0	0	0	0
χ_1	6	2	0	-4	0	0	0	0	2	2	0	2	0	0	0	0	0	0	-2	0	-2	0	-1	-1
χ_2	6	2	0	-4	0	0	0	0	2	2	0	2	0	0	0	0	0	0	2	0	2	0	-1	-1
χ_3	6	2	0	4	0	0	0	0	2	2	0	-2	0	0	0	0	0	0	2	0	-2	0	1	-1
χ_4	6	2	0	4	0	0	0	0	2	2	0	-2	0	0	0	0	0	0	-2	0	2	0	1	-1
χ_5	6	2	0	-4	0	0	0	0	-2	-2	0	-2	0	0	0	0	0	0	0	B	0	B	-1	-1
χ_6	6	2	0	-4	0	0	0	0	-2	-2	0	-2	0	0	0	0	0	0	0	-B	0	-B	-1	-1
χ_7	6	2	0	4	0	0	0	0	-2	-2	0	2	0	0	0	0	0	0	0	-B	0	B	1	-1
χ_8	6	2	0	4	0	0	0	0	-2	-2	0	2	0	0	0	0	0	0	0	B	0	-B	1	-1
χ_9	20	-4	0	0	0	0	0	0	-4	4	0	0	0	0	0	2	0	0	0	0	0	0	0	0
χ_{10}	20	-4	0	0	0	0	0	0	4	-4	0	0	0	0	0	2	0	0	0	0	0	0	0	0
χ_{11}	24	8	0	-16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
χ_{12}	24	8	0	16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	1
χ_{13}	40	-8	0	0	0	0	0	0	0	0	0	0	0	0	0	-2	0	0	0	0	0	0	0	0

Table 7 (continued): **IrrProj(G,α_i) for** 2^5:S_5

$\alpha_5^2 = \alpha_6^2 = \alpha_7^2 = \alpha_8^2 = 1$

$[g]_G$	1a	2a	2b	2c	2d	4a	4b	2e	2f	4c	4d	4e	4f	2g	6a	3a	6b	6c	4g	8a	8b	4h	10a	5a
$\lvert C_G(g)\rvert$	3840	256	384	640	96	32	32	96	64	64	32	32	64	64	12	12	12	12	16	16	16	16	10	10
χ_1	10	-2	-4	0	4	0	-2	-2	2	-2	0	2	0	0	1	1	-1	-1	0	0	0	0	0	0
χ_2	10	-2	-4	0	2	-2	0	-4	-2	2	0	-2	0	0	1	1	1	1	0	0	0	0	0	0
χ_3	10	-2	-4	0	-2	2	0	4	-2	2	0	-2	0	0	1	1	-1	-1	0	0	0	0	0	0
χ_4	10	-2	-4	0	-4	0	2	2	2	-2	0	2	0	0	1	1	1	1	0	0	0	0	0	0
χ_5	10	-2	4	0	-4	0	-2	-2	2	-2	0	-2	0	0	-1	1	1	-1	0	0	0	0	0	0
χ_6	10	-2	4	0	-2	2	0	-4	-2	2	0	2	0	0	-1	1	-1	1	0	0	0	0	0	0
χ_7	10	-2	4	0	2	-2	0	4	-2	2	0	2	0	0	-1	1	1	-1	0	0	0	0	0	0
χ_8	10	-2	4	0	4	0	2	2	2	-2	0	-2	0	0	-1	1	-1	1	0	0	0	0	0	0
χ_9	12	4	0	0	0	0	0	0	-4	-4	0	0	0	0	0	0	0	0	0	0	0	0	0	-2
χ_{10}	12	4	0	0	0	0	0	0	4	4	0	0	0	0	0	0	0	0	0	0	0	0	0	-2
χ_{11}	20	-4	-8	0	2	2	-2	2	0	0	0	0	0	0	-1	-1	1	1	0	0	0	0	0	0
χ_{12}	20	-4	-8	0	-2	-2	2	-2	0	0	0	0	0	0	-1	-1	-1	-1	0	0	0	0	0	0
χ_{13}	20	-4	8	0	-2	-2	-2	2	0	0	0	0	0	0	1	-1	-1	1	0	0	0	0	0	0
χ_{14}	20	-4	8	0	2	2	2	-2	0	0	0	0	0	0	1	-1	1	-1	0	0	0	0	0	0
χ_{15}	24	8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	F	1
χ_{16}	24	8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-F	1
χ_1	4	4	4	4	0	0	0	0	0	0	0	0	0	0	2	-2	0	0	0	0	0	0	-1	1
χ_2	4	4	-4	-4	0	0	0	0	0	0	0	0	0	0	-2	-2	0	0	0	0	0	0	1	1
χ_3	4	4	-4	-4	0	0	0	0	0	0	0	0	0	0	1	1	-C	-C	0	0	0	0	1	1
χ_4	4	4	-4	-4	0	0	0	0	0	0	0	0	0	0	1	1	C	C	0	0	0	0	1	1
χ_5	4	4	4	4	0	0	0	0	0	0	0	0	0	0	-1	1	C	-C	0	0	0	0	-1	1
χ_6	4	4	4	4	0	0	0	0	0	0	0	0	0	0	-1	1	-C	C	0	0	0	0	-1	1
χ_7	6	6	-6	-6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-D	-D	-D	-D	-1	-1
χ_8	6	6	-6	-6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	D	D	D	D	-1	-1
χ_9	6	6	6	6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	D	D	-D	-D	1	-1
χ_{10}	6	6	6	6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-D	-D	D	D	1	-1
χ_{11}	30	-2	6	-10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-D	D	-D	D	0	0
χ_{12}	30	-2	6	-10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	D	-D	D	-D	0	0
χ_{13}	30	-2	-6	10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	D	-D	-D	D	0	0
χ_{14}	30	-2	-6	10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-D	D	D	-D	0	0
χ_1	4	4	0	0	0	0	0	0	0	0	0	0	0	0	0	-2	0	0	0	0	0	0	F	1
χ_2	4	4	0	0	0	0	0	0	0	0	0	0	0	0	0	-2	0	0	0	0	0	0	-F	1
χ_3	8	8	0	0	0	0	0	0	0	0	0	0	0	0	0	2	0	0	0	0	0	0	0	2
χ_4	12	12	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-2
χ_5	60	-4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_1	2	2	0	0	0	0	0	0	2	2	2	0	0	0	0	2	0	0	0	0	0	0	0	2
χ_2	6	6	0	0	0	0	0	0	-2	-2	-2	0	0	0	0	0	0	0	0	0	0	0	F	1
χ_3	6	6	0	0	0	0	0	0	-2	-2	-2	0	0	0	0	0	0	0	0	0	0	0	-F	1
χ_4	8	8	0	0	0	0	0	0	0	0	0	0	0	0	0	2	0	0	0	0	0	0	0	-2
χ_5	10	10	0	0	0	0	0	0	2	2	2	0	0	0	0	-2	0	0	0	0	0	0	0	0
χ_6	30	-2	0	0	0	0	0	0	6	-2	-2	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_7	30	-2	0	0	0	0	0	0	-2	6	-2	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_8	30	-2	0	0	0	0	0	0	-2	-2	2	-4	0	0	0	0	0	0	0	0	0	0	0	0
χ_9	30	-2	0	0	0	0	0	0	-2	-2	2	4	0	0	0	0	0	0	0	0	0	0	0	0

A = -4i, B = -2i, C = $\sqrt{3}$i = i3, D = $\sqrt{2}$i = i2, E = $\sqrt{2}$ = r2, F = -$\sqrt{5}$= -r5

Table 8: **IrrProj(G,α_i) for** $(2^3{:}L_3(2)) \times 2$

$\alpha_1 = \alpha_2^2 = \alpha_3^2 = 1$

$[g]_G$	1a	2a	2b	2c	2d	2e	2f	2g	4a	4b	4c	4d	4e	4f	3a	6a	6b	6c	7a	14a	14b	7b
$\lvert C_G(g)\rvert$	2688	2688	384	384	64	64	64	64	32	32	16	16	16	16	12	12	12	12	14	14	14	14
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	-1	-1	1	-1	1	1	-1	1	-1	-1	1	1	-1	1	-1	-1	1	1	-1	-1	1
χ_3	3	-3	-3	3	1	-1	-1	1	-1	1	-1	1	1	-1	0	0	0	0	C	-C	-$\overline{C}$	$\overline{C}$
χ_4	3	-3	-3	3	1	-1	-1	1	-1	1	-1	1	1	-1	0	0	0	0	$\overline{C}$	-$\overline{C}$	-C	C
χ_5	3	3	3	3	-1	-1	-1	-1	-1	-1	1	1	1	1	0	0	0	0	C	C	$\overline{C}$	$\overline{C}$
χ_6	3	3	3	3	-1	-1	-1	-1	-1	-1	1	1	1	1	0	0	0	0	$\overline{C}$	$\overline{C}$	C	C
χ_7	6	6	6	6	2	2	2	2	2	2	0	0	0	0	0	0	0	0	-1	-1	-1	-1
χ_8	6	-6	-6	6	-2	2	2	-2	2	-2	0	0	0	0	0	0	0	0	-1	1	1	-1
χ_9	7	7	7	7	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	1	1	1	1	0	0	0	0
χ_{10}	7	-7	-7	7	1	-1	-1	1	-1	1	1	-1	-1	1	1	-1	-1	1	0	0	0	0
χ_{11}	7	7	-1	-1	3	3	-1	-1	-1	-1	1	1	-1	-1	1	1	-1	-1	0	0	0	0
χ_{12}	7	7	-1	-1	-1	-1	3	3	-1	-1	-1	-1	1	1	1	1	-1	-1	0	0	0	0
χ_{13}	7	-7	1	-1	1	-1	3	-3	-1	1	1	-1	1	-1	1	-1	1	-1	0	0	0	0
χ_{14}	7	-7	1	-1	-3	3	-1	1	-1	1	-1	1	-1	1	1	-1	1	-1	0	0	0	0
χ_{15}	8	8	8	8	0	0	0	0	0	0	0	0	0	0	-1	-1	-1	-1	1	1	1	1
χ_{16}	8	-8	-8	8	0	0	0	0	0	0	0	0	0	0	-1	1	1	-1	1	-1	-1	1
χ_{17}	14	14	-2	-2	2	2	2	2	-2	-2	0	0	0	0	-1	-1	1	1	0	0	0	0
χ_{18}	14	-14	2	-2	-2	2	2	-2	-2	2	0	0	0	0	-1	1	-1	1	0	0	0	0
χ_{19}	21	21	-3	-3	1	1	-3	-3	1	1	-1	-1	1	1	0	0	0	0	0	0	0	0
χ_{20}	21	21	-3	-3	-3	-3	1	1	1	1	1	1	-1	-1	0	0	0	0	0	0	0	0
χ_{21}	21	-21	3	-3	3	-3	1	-1	1	-1	-1	1	-1	1	0	0	0	0	0	0	0	0
χ_{22}	21	-21	3	-3	-1	1	-3	3	1	-1	1	-1	1	-1	0	0	0	0	0	0	0	0
χ_1	4	-4	-4	-4	0	0	0	0	0	0	0	0	0	0	1	1	1	1	-$\overline{C}$	-$\overline{C}$	-C	-C
χ_2	4	-4	-4	-4	0	0	0	0	0	0	0	0	0	0	1	1	1	1	-C	-C	-$\overline{C}$	-$\overline{C}$
χ_3	4	4	4	-4	0	0	0	0	0	0	0	0	0	0	1	-1	-1	1	-$\overline{C}$	$\overline{C}$	C	-C
χ_4	4	4	4	-4	0	0	0	0	0	0	0	0	0	0	1	-1	-1	1	-C	C	$\overline{C}$	-$\overline{C}$
χ_5	6	-6	-6	-6	0	0	0	0	0	0	A	-A	-A	A	0	0	0	0	-1	-1	-1	-1
χ_6	6	-6	-6	-6	0	0	0	0	0	0	-A	A	A	-A	0	0	0	0	-1	-1	-1	-1
χ_7	6	6	6	-6	0	0	0	0	0	0	A	A	A	A	0	0	0	0	-1	1	1	-1
χ_8	6	6	6	-6	0	0	0	0	0	0	-A	-A	-A	-A	0	0	0	0	-1	1	1	-1
χ_9	8	8	8	-8	0	0	0	0	0	0	0	0	0	0	-1	1	1	-1	1	-1	-1	1
χ_{10}	8	-8	-8	-8	0	0	0	0	0	0	0	0	0	0	-1	-1	-1	-1	1	1	1	1
χ_{11}	14	14	-2	2	0	0	0	0	0	0	-A	-A	A	A	-1	1	-1	1	0	0	0	0
χ_{12}	14	14	-2	2	0	0	0	0	0	0	A	A	-A	-A	-1	1	-1	1	0	0	0	0
χ_{13}	14	-14	2	2	0	0	0	0	0	0	-A	A	-A	A	-1	-1	1	1	0	0	0	0
χ_{14}	14	-14	2	2	0	0	0	0	0	0	A	-A	A	-A	-1	-1	1	1	0	0	0	0
χ_{15}	28	28	-4	4	0	0	0	0	0	0	0	0	0	0	1	-1	1	-1	0	0	0	0
χ_{16}	28	-28	4	4	0	0	0	0	0	0	0	0	0	0	1	1	-1	-1	0	0	0	0
χ_1	8	-8	0	0	0	0	0	0	0	0	0	0	0	0	2	-2	0	0	1	-1	-1	1
χ_2	8	8	0	0	0	0	0	0	0	0	0	0	0	0	2	2	0	0	1	1	1	1
χ_3	8	-8	0	0	0	0	0	0	0	0	0	0	0	0	-1	1	B	-B	1	-1	-1	1
χ_4	8	-8	0	0	0	0	0	0	0	0	0	0	0	0	-1	1	-B	B	1	-1	-1	1
χ_5	8	8	0	0	0	0	0	0	0	0	0	0	0	0	-1	-1	B	B	1	1	1	1
χ_6	8	8	0	0	0	0	0	0	0	0	0	0	0	0	-1	-1	-B	-B	1	1	1	1
χ_7	24	-24	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	C	-C	-$\overline{C}$	$\overline{C}$
χ_8	24	-24	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$\overline{C}$	-$\overline{C}$	-C	C
χ_9	24	24	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	C	C	$\overline{C}$	$\overline{C}$
χ_{10}	24	24	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$\overline{C}$	$\overline{C}$	C	C

A = -$\sqrt{2}$ = -r2, B = -$\sqrt{3}$i = -i3, C = $\frac{-1+\sqrt{7}\mathrm{i}}{2}$ = b7

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One point paracompactification (metacompatification)

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Abstract. The notion of locally paracompact (metacompact) are studied and utilized to obtain the one point paracompactification (metacompatification). Several related results are also introduced.

Keywords: paracompact, metacompact, α-paracompact, α-metacompact.

1. Introduction

Paracompact spaces were first introduced by Dieudonné [3] in 1944 as a natural generalization of compact spaces still retaining enough structure to enjoy many properties of compact spaces. The notion of paracompactness gained stature with the proof, by A. H. Stone [7], that every metric space is paracompact and the subsequent use of the result is the solution of the general metrization problem by Bing [2], Nagata [5] and Smirnov [6].

In this paper, we define the notion of local paracompactness and study their main properties, we prove that any locally paracompact Hausdorff spaces can be embedded in a certain paracompact Hausdorff space called the one point paracompactification.

Finally, we define the notion of locally metacompactness, and by applying almost similar methods we obtain several results concerning locally metacompact spaces. For notions not defined here we refer the reader to Engelking [4]. All spaces are assumed to be T_2.

2. Locally paracompact spaces

In this section locally paracompact spaces are defined, different results concerning them and their relations with some topological spaces are studied.

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Definition 2.1. *A space X is called locally paracompact if each point of X has an open neighbourhood U such that $\overline{U}$ is paracompact.*

Clearly, every paracompact space is locally paracompact. However, the converse need no be true as the following example tells.

Example. Let Ω_o be the set of all ordinal numbers less than the first uncountable ordinal Ω equipped with the order topology whose subbase consists of sets of the form $\{x : x < \alpha\}$ or $\{x : \alpha < x\}$ for some α in Ω_o. Then Ω_o is a locally compact T_2-space. Hence, it is a locally paracompact T_2-space. However, Ω_o is not paracompact since the open cover $\{x : x < \alpha\}$ has no locally finite open refinement.

Definition 2.2. *A subset A of a topological space (X, τ) is called α-paracompact if every cover of A by members of τ has a locally finite refinement by members of τ which covers A.*

Remark. It is easy to see that if a subset A of a topological space (X, τ) is α-paracompact, then A is a paracompact subspace.

Theorem 2.1. *Let A be a closed paracompact subset of a space X and F be a closed subset of the interior of A. Then F is α-paracompact.*

Theorem 2.2. *A closed subset of an α-paracompact set is α-paracompact.*

Theorem 2.3. *An α-paracompact subset A of a T_2-space X is closed.*

Theorem 2.4. *Let (X, τ) be a locally paracompact T_2-space. Then each point of X has an open neighbourhood U such that $\overline{U}$ is α-paracompact.*

Proof. Since X is locally paracompact T_2-space, it is not difficult to see that X is a T_3-space. Since X is locally paracompact, each $x \in X$ has an open neighbourhood V such that $\overline{V}$ is paracompact. Since X is a T_3-space. there is an open set U such that $x \in U \subseteq \overline{U} \subseteq V \subseteq \overline{V}$. Thus, it follows from Theorem 2.1 that $\overline{U}$ is α-paracompact. □

Theorem 2.5. *Every closed subset of a locally paracompact space is locally paracompact.*

Proof. Let X be a locally paracompact space and F be closed in X. If $x \in F$, then there exists an open neighbourhood U of x such that $\overline{U}$ is paracompact. Now $U \cap F$ is open in F and $x \in U \cap F$. Also, $\overline{U \cap F}^F = \overline{U \cap F} \cap F$ is closed in X, and thus closed in $\overline{U}$, but $\overline{U}$ is paracompact, so, $\overline{U \cap F}^F$ is paracompact (as a closed subset of a paracompact space is paracompact). □

Theorem 2.6. *Let $X = \cup\{F_\alpha : \alpha \in \Lambda\}$, where $\{F_\alpha : \alpha \in \Lambda\}$ is a locally finite closed cover of X. Then X is locally paracompact if and only if F_α is locally paracompact.*

We recall the following known result.

Lemma 2.7. *Let $f : X \xrightarrow{onto} Y$ be a perfect function, where Y is paracompact. Then X is paracompact.*

Theorem 2.8. *Let $f : X \xrightarrow{onto} Y$ be a perfect function, where Y is a locally paracompact space. Then X is a locally paracompact space.*

Proof. Let $f : X \xrightarrow{onto} Y$ be a perfect function, where Y is a locally paracompact space. Since Y is locally paracompact, for each $x \in X$, there is an open neighbourhood K of $f(x)$ such that $\overline{K}$ is paracompact. Since f is continuous, there is an an open neighbourhood G of x such that $f(G) \subseteq K$ and $f(\overline{G}) \subseteq \overline{f(G)}$. Thus, $f(\overline{G}) \subseteq \overline{K}$, and therefore, $\overline{G} \subseteq f^{-1}(\overline{K})$. Since $\overline{K}$ is closed and $f : X \xrightarrow{onto} Y$ is perfect, so, $f : f^{-1}(\overline{K}) \xrightarrow{onto} \overline{K}$ is perfect. Since $\overline{K}$ is paracompact, it follows from Lemma 2.7 that $f^{-1}(\overline{K})$ is paracompact, but $\overline{G}$ is closed in $f^{-1}(\overline{K})$, so, $\overline{G}$ is paracompact (as a closed subset of a paracompact space is paracompact). Hence, X is locally paracompact. □

Corollary 2.8.1. *The product of a compact space X and a locally paracompact space Y is a locally paracompact space.*

Proof. Let X be compact space and Y be a locally paracompact space. Since X is compact, the projection function $P_Y : X \times Y \to Y$ is a perfect onto function. Since Y is locally paracompact, it follows from Theorem 2.8 that $X \times Y$ is locally paracompact. □

Recall that a space X is called C-scattered [8] if every nonempty closed subspace A of X has a point $x \in A$ with a compact neighbourhood in A.

Theorem 2.9. *If X is a paracompact C-scattered space, then X is locally compact.*

Proof. Let X be a paracompact C-scattered space. Let $X^{(1)} = \{y \in X : y$ does not have a compact neighbourhood in $X\}$. Suppose that, for some ordinal number $\alpha > 1$, $X^{(\alpha)}$ has been defined. If $\beta = \alpha + 1$, define $X^{(\beta)} = (X^{(\alpha)})^{(1)}$ and $X^{(\beta)} = \bigcap_{\alpha<\beta} X^{(\alpha)}$ otherwise. If X is C-scattered, it is easy to check that $X^{(\alpha)} = \phi$ for some ordinal α. For $\alpha = 1$, if $X^{(\alpha)} = \phi$, then X is locally compact. Suppose that the result holds for all $\beta < \alpha$, i.e if $X^{(\beta)} = \phi$ for $\beta < \alpha$, then X is locally compact. Now we want to prove that if $X^{(\alpha)} = \phi$, then X is locally compact.

Case 1. There is α such that $\alpha = \beta + 1$ for some $\beta < \alpha$. In this case $X^{(\alpha)} = \phi$ implies $X^{(\beta)}$ is locally compact. For each $y \in X^{(\beta)}$, there exists an open U_y, containing y such that $\overline{U_y} \cap X^{(\beta)}$ is compact. For each $x \in X - X^{(\beta)}$ there is an open set U_x containing x such that $x \in \overline{U_x} \subseteq X - X^{(\beta)}$ Hence for each $x \in X - X^{(\beta)}$, $(\overline{U_x}^{(\beta)}) = \phi$. By the induction assumption, for each $x \in X - X^{(\beta)}$, $\overline{U_x}$ is locally compact. Now $\underline{U} = \{U_x : x \in X - X^{(\beta)}\} \cup \{U_x : x \in X^{(\beta)}\}$ is

an open cover of X. Since X is paracompact, $\underline{U}$ has a locally finite closed refinement, say $\underline{V}$. Let $\underline{V'} = \{\overline{V} : V \in \underline{V}, V \subseteq U_x$, for some $x \in X - X^{(\beta)}\}$. Now $\{X^{(\beta)}\} \cup \underline{V'}$ is a closed locally finite cover of X, so X is locally compact.
Case 2. $X^{(\alpha)} = \bigcap_{\beta<\alpha} X^{(\beta)} = \phi$. Consider the open cover $\underline{U} = \{X - X^{(\beta)} : \beta < \alpha\}$ of X. By the induction assumption, it is easy to see that $X - X^{(\beta)}$ is a locally compact subset of X, thus X is locally compact. □

Definition 2.3. *A space X is called bilocally compact if each point of X has an open neighbourhood U such that $\overline{U}$ is locally compact.*

Theorem 2.10. *If X is a locally paracompact C-scattered space, then X is a bilocally compact space.*

Proof. Let X be a locally paracompact C-scattered space. Then for each $x \in X$, there is an open neighbourhood U such that $\overline{U}$ is paracompact. Since X is C-scattered, it follows from Theorem 2.9 that $\overline{U}$ is locally compact. Hence X is bilocally compact. □

Theorem 2.11. *If X is a paracompact bilocally compact space, then $X \times Y$ is paracompact for any paracompact space Y.*

Proof. Since X is a paracompact bilocally compact space, for each $x \in X$ there is an open neighbourhood U_x of x such that $\overline{U_x}$ is locally compact and paracompact. Since X is paracompact,$\{U_x : x \in X\}$ has a locally finite closed refinement $\{F_\alpha : \alpha \in \Lambda\}$, where each F_α is locally compact and paracompact. Now $\{F_\alpha \times Y : \alpha \in \Lambda\}$ is a locally finite closed cover of $X \times Y$, where $F_\alpha \times Y$ is paracompact. Hence, $X \times Y$ is paracompact. □

Definition 2.4. *A subset A of a space X is called Lindelöf if every cover of A by open subsets of X has a countable subcover of A.*

Definition 2.5. *A space X is called locally Lindelöf if each point of X has a Lindelöf neighbourhood U_x.*

Theorem 2.12. *Let X be a locally paracompact separable space. Then X is locally Lindelöf.*

Proof. Since X be a locally paracompact separable space. For each $x \in X$ there is an open neighbourhood U such that $\overline{U}$ is paracompact. Since X is separable and U is open, U is separable.too, and thus, $\overline{U}$ is also separable. Therefore, $\overline{U}$ is both paracompact and separable, but it is known that a paracompact separable space is Lindelöf, so $\overline{U}$ is Lindelöf. Hence, X is locally Lindelöf. □

3. One point paracompactification

A paracompactification of a topological space (X,τ) is a paracompact space (X^*,τ^*), that contains (X,τ) as a dense subspace. A paracompactification is called the N-point paracompactification if $X^* - X$ consists of N points. If $N = 1$, X^* is called the one point paracompactification of X.

Theorem 3.1. *Let (X,τ) be a locally paracompact T_2-space, $p \notin X$ and $X^* = X \cup \{p\}$. Let τ^* be a family of subsets of X^* of the following types:*

1. *All open subsets of X.*
2. *All subsets of X of the form $X^* - K$, where K is an α-paracompact subset of X.*
3. *The sets X^* and ϕ.*

Then each of the following holds:

(*i*) *τ^* is a topology on X^*.*

(*ii*) *(X^*,τ^*) is a T_2- space.*

(*iii*) *(X^*,τ^*) is a paracompact space.*

(*iv*) *(X,τ) is dense in (X^*,τ^*).*

Proof. (i) Let $\bigcup_{\alpha\in\Lambda} E_\alpha$ be an arbitrary union of members of τ^*. Now if $E_\alpha = X^*$ for some α then $\bigcup_{\alpha\in\Lambda} E_\alpha = X^* \in \tau^*$. Suppose that $E_\alpha \neq X^*$ for each $\alpha \in \Lambda$ then $\bigcap_{\alpha\in\Lambda} E_\alpha = L_p \cup M_q$ where L_p is a union of elements of E_α's of type 1. and M_q is a union of elements of E_α's of type 2. then L_p is of type 1. $M_q = \cup X^* - K_q = X^* - \cap K_q$, where K_q is an α-paracompact subset of X for each q. Since an α-paracompact subset of T_2-space is closed and any intersection of closed subsets of an α-paracompact set is α-paracompact, we get that $\cap K_q$ is α-paracompact, therefore M_q is of type 2. Hence it is suffices to prove that a union of a set A of type 1 and a set B of type 2 is in τ^*. Now $A \cup B = (X - (X - A)) \cup (X^* - K)$ where K is an α-paracompact closed subset of X.
$A \cup B = (X^* - (X^* - A)) \cup (X^* - K) = X^* - ((X^* - A) \cap K)$. Since $(X^* - A) \cap K)$ is an α-paracompact closed subset of X, we get that $A \cup B$ is of type 2, so $A \cup B \in \tau^*$. Let $G, H \in \tau^*$, if both G, H are of type 1, then $G \cap H$ is of type 1 also, and if both G, H are of type 2, then $G \cap H$ is of type 2 also. Let G be of type 1 and H be of type 2, then $G \cap H = G \cap (X^* - M)$ where M is $\alpha-$paracompact, now $G \cap H = (X^* - (X^* - G)) \cap (X^* - M) = X^* - ((X^* - G) \cup M)$, hence $G \cap H \in \tau^*$.

(ii) To show that (X^*,τ^*) is a T_2-space. Let $x \neq y$ are elements in X. It is suffices to assume that $y = p$ and $x \neq p$. Since X is a locally paracompact

T_2-space, there is an open neighbourhood U containing x such that $\overline{U}$ is α-paracompact. Now $V = X^* - \bar{U}$ is an open set containing y and $V \cap U = \phi$. Hence X^* is a T_2-space.

(iii) To show that (X^*, τ^*) is paracompact let $\vec{U} = \{U_\alpha : \alpha \in \Lambda\}$ be any open cover of X^*. Let $p \in U_{\alpha_o}$ for some $\alpha_o \in \Lambda$. Then $U_{\alpha_o} = X^* - R$ where R is an α-paracompact closed subset of X. Since $\vec{U}$ is an open cover of R, there is a open locally finite refinement $\vec{M}$ of $\vec{U}$ which covers R. Therefore, $\vec{H} = \vec{M} \cup \{U_{\alpha_o}\}$ is an open locally finite refinement $\vec{U}$ that covers X. Hence, X is paracompact.

(iv) (X, τ) is dense in (X^*, τ^*), since for each open set U in X^*, we have $U \cap X \neq \phi$. □

Observe that (X^*, τ^*) in the above theorem represents the one point paracompactification of (X, τ).

4. Locally metacompact spaces

In this section we introduce the notion of locally metacompact spaces and obtain several results using almost similar techniques that used in the case of locally paracompact spaces.

Definition 4.1. *A space X is called locally metacompact if each point of X has an open neighbourhood U such that $\overline{U}$ is metacompact.*

Remark. Obviously every metacompact space is locally metacompact, however, the space Ω_o of Example **??** is an example of a locally metacompact space which is not metacompact.

Remark. Since every paracompact space is metacompact, it is clear that every locally paracompact space is locally metacompact.

Definition 4.2. *A subset A of a topological space (X, τ) is called α-metacompact if every cover of A by members of τ has a point finite refinement by members of τ which covers A.*

Remark. Clearly, if a subset A of a topological space (X, τ) is α-paracompact, then A is α-metacompact.

Remark. It is easy to see that if a subset A of a topological space (X, τ) is α-metacompact, then A is a metacompact subspace.

Theorem 4.1. *Let A be a closed metacompact subset of a space X and F be a closed subset of the interior of A. Then F is α-metacompact.*

Theorem 4.2. *A closed subset of an α-metacompact set is α-metacompact.*

Theorem 4.3. *Let X be a locally metacompact T_2-space. Then each point of X has a open neighbourhood U such that $\overline{U}$ is α-metacompact.*

Proof. Similar to the proof of Theorem 2.4. □

Theorem 4.4. *Every closed subspace of a locally metacompact space is locally metacompact.*

Proof. Let X be a locally metacompact space and F be closed in X. If $x \in F$, then there exists an open neighbourhood U of x such that $\overline{U}$ is metacompact. Now $U \cap F$ is open in $F$and $x \in U \cap F$. Also, $\overline{U \cap F}^F = \overline{U \cap F} \cap F$ is closed in X, and thus closed in $\overline{U}$, but $\overline{U}$ is metacompact, so, $\overline{U \cap F}^F$ is metacompact (as a closed subset of a metacompact space is metacompact). □

Theorem 4.5. *Let $X = \cup\{F_\alpha : \alpha \in \Lambda\}$, where $\{F_\alpha : \alpha \in \Lambda\}$ is a locally finite closed cover of X. Then X is locally metacompact if and only if each F_α is locally metacompact.*

The following lemma is a (probably) known result, the straightforward proof is omitted.

Lemma 4.6. *Let $f : X \xrightarrow{onto} Y$ be a perfect function, where Y is metacompact. Then X is metacompact.*

Theorem 4.7. *Let $f : X \xrightarrow{onto} Y$ be a perfect function, where Y is a locally metacompact space. Then X is a locally metacompact space.*

Proof. Let $f : X \xrightarrow{onto} Y$ be a perfect function, where Y is locally metacompact space. Since Y is locally metacompact, for each $x \in X$, there is an open neighbourhood K of $f(x)$ such that $\overline{K}$ is metacompact. Since f is continuous, there is an an open neighbourhood G of x such that $f(G) \subseteq K$ and $f(\overline{G}) \subseteq \overline{f(G)}$. Thus, $f(\overline{G}) \subseteq \overline{K}$, and therefore, $\overline{G} \subseteq f^{-1}(\overline{K})$. Since $\overline{K}$ is closed and $f : X \xrightarrow{onto} Y$ is perfect, so, $f : f^{-1}(\overline{K}) \xrightarrow{onto} \overline{K}$ is perfect. Since $\overline{K}$ is metacompact, it follows from Lemma 4.6 that $f^{-1}(\overline{K})$ is metacompact, but $\overline{G}$ is closed in $f^{-1}(\overline{K})$, so, $\overline{G}$ is metacompact (as a closed subset of a metacompact space is metacompact). Hence, X is locally metacompact. □

Corollary 4.7.1. *The product of a compact space X and a locally metacompact space Y is a locally metacompact space.*

Proof. Let X be compact space and Y be a locally metacompact space. Since X is compact, the projection function $P_Y : X \times Y \to Y$ is a perfect onto function. Since Y is locally metacompact, it follows from Theorem 4.7 that $X \times Y$ is locally metacompact. □

Lemma 4.8. *A metacompact separable space X is Lindelöf.*

Proof. Let X be a metacompact separable space and let $\mathcal{U}$ be an open cover of X. Since X is metacompact, $\mathcal{U}$ has a point finite open refinement $\mathcal{V}$, say. We will show that $\mathcal{V}$ is countable, to do so, assume not, that is, assume that

$\mathcal{V}$ is uncountable. Since X is separable, X has a countable dense set A, say. Thus, $A \cap V \neq \emptyset$, for every $V \in \mathcal{V}$, but A is countable and $\mathcal{V}$ is uncountable, so there exists $x \in A$ such that x belongs to uncoutably many members of $\mathcal{V}$. This contradicts that X is metacompact. Thus, $\mathcal{V}$ is countable, but $\mathcal{V}$ is a refinement of $\mathcal{U}$, so $\mathcal{U}$ has a countable subcover. Hence, X is Lindelöf. □

Since every locally paracompact space is locally metacompact (Remark 4), the following theorem generalizes Theorem 2.12.

Theorem 4.9. *Let X be a locally metacompact separable space. Then X is locally Lindelöf.*

Proof. Since X is locally metacompact, each point x of X has a open neighbourhood U_x such that $\overline{U_x}$ is metacompact. Since X is separable and U_x is open, U_x is separable too, and thus $\overline{U_x}$ is also separable. By Lemma 4.8, $\overline{U_x}$ is Lindelöf. Hence, X is locally Lindelöf. □

Theorem 4.10. *Let (X, τ) be a locally metacompact T_2- space, $p \notin X$ and $X^* = X \cup \{p\}$. Let τ^* be a family of all subsets of X^* of the following types:*

1. *All open subsets of X.*
2. *All sets of the form $X^* - K$, where K is an α-metacompact closed subset of X.*
3. *The sets X^* and ϕ.*

Then each of the following holds:

(*i*) *τ^* is a topology on X^*.*

(*ii*) *(X^*, τ^*) is a metacompact T_2 space.*

(*iii*) *(X, τ) is dense in (X^*, τ^*).*

The one point metacompactification of a topological space X is a metacompact space (X^*, τ^*) that contains (X, τ) as a dense subspace and $X^* - X$ is a singleton set.

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On property $(a\omega)$ and hypercyclic/supercyclic operators

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Abstract. In this paper, we show that property $(a\omega)$ holds for the adjoint of hypercyclic/ supercyclic operator. Also, we characterize hypercyclic/ supercyclic operators satisfying the property $(a\omega)$. We establish that for a hypercyclic/ supercyclic operators, the property $(a\omega)$ holds if and only if property (ω) holds, if and only if a-Weyl's theorem holds.

Keywords: a-Weyl's theorem, a-Browder's theorem, property(ω), property$(a\omega)$, hypercyclic/ supercyclic operators, single valued extension property.

1. Introduction

An operator acting on Banach space satisfies property $(a\omega)$ if the complement of its Weyl spectrum in its spectrum is its set of eigenvalues of finite multiplicity which are isolated in the approximate spectrum. This class is a variant of Weyl's theorem (Weyl's theorem has been introduced in [8]) that has been introduced in [6] and studied in many papers e.g. [5, 6]. In [21], Rakočević was introduced two variants of Weyl's theorem, the so called a-Weyl's theorem and the property (ω) studied also in [2, 3, 12, 21]. Note that property (ω) and $(a\omega)$ are also related to Browder's, a-Browder's, Weyl's and a-Weyl's theorems [1, 2, 5, 6].

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In this work, we show that property $(a\omega)$ holds for the adjoint of hypercyclic/supercyclic operator. Also, we prove that under hypercyclicity or supercyclicity hypothesis, property $(a\omega)$, property $(a\omega)$ and a-Weyl's theorem are equivalent.

Hypercyclic/supercyclic operators satisfying property (ω) or Weyl's theorem was studied in [12, 11]. In [2], it was proved that property (ω) holds for T if and only if a-Browder's theorem hold for T and $p_{00}^a(T) = \pi_{00}(T)$. In [12], it was proved that for hypercyclic or supercyclic operator, property (ω) holds for T if and only if $\pi_{00}(T) = \pi_{00}(T^*)$. In [4], the authors characterize hypercyclic and supercyclic operators satisfying the property (gw). It is natural to ask about the relation between hypercyclic/supercyclic operators and property $(a\omega)$. In [5], characterization of property $(a\omega)$ in terms of Browder's theorem was given. This paper is a continuation of these results, more precisely, we show that for hypercyclic or supercyclic operator property $(a\omega)$ holds if and only if $\pi_{00}^a(T) = \pi_{00}(T^*)$.

2. Preliminaries

Throughout this paper, $\mathcal{X}$ denotes a generally infinite-dimensional complex Banach space and $\mathcal{B}(\mathcal{X})$ the algebra of bounded linear operators on $\mathcal{X}$. We denote the null space and range of $T \in \mathcal{B}(\mathcal{X})$ by $N(T)$ and $R(T)$, respectively. Let $\alpha(T) := \dim N(T)$, $\beta(T) := \operatorname{codim} R(T)$ be the nullity and the deficiency of T respectively. T^*, $\sigma(T)$, $\sigma_p(T)$, and $\sigma_a(T)$ denote respectively the adjoint, the spectrum, the point spectrum, and the approximate point spectrum of T.

Recall that an operator $T \in \mathcal{B}(\mathcal{X})$ is called a semi-Fredholm operator if $T(\mathcal{X})$ is closed and $\alpha(T) < \infty$ or $\beta(T) < \infty$. In the sequel, $\rho_{sF}(T)$ will denote the semi-Fredholm resolvent set of T. If both $\alpha(T)$ and $\beta(T)$ are finite, T is called Fredholm operator. The index of a Fredholm operator T is given by

$$ind(T) := \alpha(T) - \beta(T).$$

An operator $T \in \mathcal{B}(\mathcal{X})$ is called Weyl if it is Fredholm of index zero. The ascent $p(T)$ and the descent $q(T)$ are given by

$$p(T) = \inf\left\{n \in \mathbb{N} \ : \ N(T^n) = N(T^{n+1}\right\},$$
$$q(T) = \inf\left\{n \in \mathbb{N} \ : \ R(T^n) = R(T^{n+1}\right\},$$

where the infimum over the empty set is taken ∞. It is well known that if $p(T)$ and $q(T)$ are both finite then $p(T) = q(T)$, [1, 19, 17].

An operator T is called Browder if it is Fredholm of finite ascent and descent; equivalently ([13, Theorem 7.9.3]) if T is Fredholm and $T - \lambda I$ is invertible for sufficiently small $\lambda \in \mathbb{C} \setminus \{0\}$. The essential spectrum $\sigma_e(T)$, the Weyl spectrum

$\sigma_w(T)$, and the Browder spectrum $\sigma_b(T)$ of $T \in B(X)$ are defined by ([1], [13])

$$\begin{aligned}\sigma_e(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not Fredholm}\},\\ \sigma_w(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not Weyl}\},\\ \sigma_b(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not Browder}\},\end{aligned}$$

respectively. Evidently

$$\sigma_e(T) \subseteq \sigma_w(T) \subseteq \sigma_b(T) = \sigma_f(T) \cup \operatorname{acc}\sigma(T),$$

where we write acc K (respectively iso K) for the accumulation (respectively isolated) points of a subset $K \subseteq \mathbb{C}$. $T \in \mathcal{B}(\mathcal{X})$. We denote by (see [1] page 164 and 176)

$$\pi_{00}(T) := \{\lambda \in \text{iso } \sigma(T) : 0 < \alpha(\lambda I - T) < \infty\},$$

the set of isolated eigenvalues of finite multiplicity, and by

$$\pi^a_{00}(T) := \{\lambda \in \text{iso } \sigma_a(T) : 0 < \alpha(\lambda I - T) < \infty\},$$

we denote the set of eigenvalues of finite multiplicity which are isolated in $\sigma_a(T)$. We write

$$p_{00}(T) := \sigma(T) \setminus \sigma_b(T) = \{\lambda \in \sigma(T) \ : \ \lambda I - T \text{ is Browder operator } \},$$

for the set of Riesz points of T, and

$$p^a_{00}(T) := \sigma_a(T) \setminus \sigma_{ub}(T) = \{\lambda \in \sigma_a(T) \ : \ \lambda I - T \in \mathcal{B}_+(\mathcal{X})\},$$

where $\mathcal{B}_+(\mathcal{X})$ denote the class of all upper semi-Browder operators. It should be noted that

$$p_{00}(T) \subseteq p^a_{00}(T) \subseteq \pi^a_{00}(T),\ p_{00}(T) \subseteq \pi_{00}(T) \subseteq \pi^a_{00}(T) \text{ and } p_{00}(T) = p_{00}(T^*).$$

We say that Weyl's theorem holds for $T \in \mathcal{B}(\mathcal{X})$ if

$$\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T),$$

and that Browder's theorem holds for $T \in \mathcal{B}(\mathcal{X})$ if

$$\sigma(T) \setminus \sigma_w(T) = p_{00}(T).$$

To describe some of the above mentioned results, we consider the following sets [1, 2]. The class of all upper semi-Fredholm operators

$$\Phi_+(\mathcal{X}) := \{T \in \mathcal{B}(\mathcal{X}) \ : \ \alpha(T) < \infty \text{ and } T(\mathcal{X}) \text{ is closed } \},$$

the class of all lower semi-Fredholm operators

$$\Phi_-(\mathcal{X}) := \{T \in \mathcal{B}(\mathcal{X}) \ : \ \beta(T) < \infty\},$$

the class of all semi-Fredholm operators and the class of all Fredholm operators are

$$\Phi_{\pm}(\mathcal{X}) := \Phi_+(\mathcal{X}) \cup \Phi_-(\mathcal{X}) \text{ and } \Phi(\mathcal{X}) := \Phi_+(\mathcal{X}) \cap \Phi_-(\mathcal{X}),$$

respectively. The class of all upper semi-Browder operators and the class of all lower semi-Browder are defined by

$$\mathcal{B}_+(\mathcal{X}) := \{T \in \mathcal{B}(\mathcal{X}) \ : \ p(T) < \infty\} \text{ and } \mathcal{B}_-(\mathcal{X}) := \{T \in \mathcal{B}(\mathcal{X}) \ : \ q(T) < \infty\},$$

respectively. The class of all Browder operators is

$$\mathcal{B}_0(\mathcal{X}) := \mathcal{B}_+(\mathcal{X}) \cap \mathcal{B}_-(\mathcal{X}).$$

The class of all upper semi-Weyl operators and the class of all lower semi-Weyl operators will be denoted by

$$\mathcal{W}_+(\mathcal{X}) := \{T \in \Phi_+(\mathcal{X}) : ind(T) \leq 0\}, \ \mathcal{W}_-(\mathcal{X}) := \{T \in \Phi_-(\mathcal{X}) : \ ind(T) \geq 0\},$$

respectively. The class of all Weyl operators is defined by

$$\mathcal{W}(\mathcal{X}) := \mathcal{W}_+(\mathcal{X}) \cap \mathcal{W}_-(\mathcal{X}).$$

These classes of operators motivate the following spectra. The upper semi-Browder spectrum $\sigma_{ub}(T)$ and the lower semi-Browder spectrum $\sigma_{lb}(T)$ of T are defined by

$$\sigma_{ub}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin \mathcal{B}_+(\mathcal{X})\} \text{ and } \sigma_{lb}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin \mathcal{B}_-(\mathcal{X})\}.$$

The upper semi-Weyl spectrum $\sigma_{uw}(T)$ and the lower semi-Weyl spectrum $\sigma_{lw}(T)$ of T are defined by

$$\sigma_{uw}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin \mathcal{W}_+(\mathcal{X})\} \text{ and } \sigma_{lw}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin \mathcal{W}_-(\mathcal{X})\}.$$

Recall that $T \in \mathcal{B}(\mathcal{X})$ is said to be bounded below if T is injective and has closed range (and then $\sigma_a(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not bounded below }\}$).

It is well known [1, 19], that

$$\sigma_{uw}(T) \subseteq \sigma_a(T) \text{ and } \sigma_{lw}(T) \subseteq \sigma_a(T^*).$$

Recall (see [14]) that an operator $T \in \mathcal{B}(\mathcal{X})$ is said to be satisfying Browder's theorem if $\sigma_w(T) = \sigma_b(T)$, or equivalently

$$\sigma(T) \setminus \sigma_w(T) = p_{00}(T),$$

and T is said to be satisfy a-Browder's theorem if $\sigma_{uw}(T) = \sigma_{ub}(T)$, or equivalently

$$\sigma_a(T) \setminus \sigma_{uw}(T) = p_{00}^a(T).$$

We say that Weyl's theorem holds for T if

$$\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T),$$

and T is said to be satisfy a-Weyl's theorem if

$$\sigma_a(T) \setminus \sigma_{uw}(T) = \pi^a_{00}(T).$$

We say that property (ω) holds for T if

$$\sigma_a(T) \setminus \sigma_{uw}(T) = \pi_{00}(T).$$

It is well known [10, 14, 21] that for $T \in \mathcal{B}(\mathcal{X})$ we have

$$\text{a-Weyl's theorem} \implies \text{Weyl's theorem} \implies \text{Browder's theorem},$$

$$\text{a-Weyl's theorem} \implies \text{a-Browder's theorem} \implies \text{Browder's theorem},$$

$$\text{property } (\omega) \implies \text{Weyl's theorem}.$$

A variant of Weyl's theorem has been introduced by Berkani and Zariouh in [6]. The operator $T \in \mathcal{B}(\mathcal{X})$ is said to satisfy property $(a\omega)$ if

$$\sigma(T) \setminus \sigma_w(T) = \pi^a_{00}(T).$$

The following theorem [5, Theorem 2.2] establishes the precise relationship between property $(a\omega)$ and Browder's theorem.

Theorem 2.1 ([5])**.** *Let $T \in \mathcal{B}(\mathcal{X})$. Then the following statements are equivalent.*

(1) *The property $(a\omega)$ holds for T.*

(2) *Browder's theorem holds for T and $\pi^a_{00}(T) = p_{00}(T)$.*

We say that $T \in \mathcal{B}(\mathcal{X})$ has the single-valued extension property at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at λ_0) if for every open neighborhood U of λ_0, the only analytic solution $f : U \longrightarrow \mathcal{X}$ of the equation $(T - \lambda I)f(\lambda) = 0$ for all $\lambda \in U$ is the zero function, we say that T has the SVEP if T has the SVEP at every point $\lambda \in \mathbb{C}$ [1, 9, 19].

In [1], it was shown that

$$\lambda_0 \in \text{ iso } \sigma_a(T) \Longrightarrow T \text{ has SVEP at } \lambda_0. \tag{1}$$

In particular if the point spectrum $\sigma_p(T)$ is empty then T satisfied the SVEP.

$$p(\lambda_0 I - T) < \infty \Longrightarrow T \text{ has SVEP at } \lambda_0. \tag{2}$$

Dually,

$$q(\lambda_0 I - T) < \infty \Longrightarrow T^* \text{ has SVEP at } \lambda_0. \tag{3}$$

Moreover, if we assume that $\lambda_0 I - T \in \Phi_{\pm}(\mathcal{X})$, then the implications (1), (2) and (3) are equivalences [2].

Let $T \in \mathcal{B}(\mathcal{X})$ such that $\mathcal{X}$ is separable. The orbit of a vector $x \in \mathcal{X}$ under T is defined by

$$Orb(T,x) := \{T^n x \ : \ n \geq 0\}.$$

The operator T is said to be hypercyclic if there is some vector $x \in \mathcal{X}$ such that $Orb(T,x)$ is dense in $\mathcal{X}$. Such vector x is called a hypercyclic vector for T. The set of all hypercyclic vectors for T will be denoted by $\mathcal{HP}(\mathcal{X})$. The operator T is said to be supercyclic [18] if there is some vector $x \in \mathcal{X}$ such that

$$\mathbb{C}Orb(T,x) := \{\alpha T^n x \ : \ \alpha \in \mathbb{C},\ n \geq 0\},$$

is dense in $\mathcal{X}$. Such vector x is called a supercyclic vector for T , and the set of all supercyclic vectors under T will be denoted by $\mathcal{SP}(\mathcal{X})$.

3. Main results

The following Lemma was proved by second the author and Tajmouati in [12, Proof of Proposition 2.1]. We give here the proof for completeness.

Lemma 3.1. *Let $T \in \mathcal{HP}(\mathcal{X}) \cup \mathcal{SP}(\mathcal{X})$. Then Weyl's theorem holds for T^*. Moreover,*

$$p_{00}(T) = p_{00}^a(T). \tag{4}$$

Proof. We show first that T^* satisfies Weyl's theorem. Let $T \in \mathcal{HP}(\mathcal{X}) \cup \mathcal{SP}(\mathcal{X})$, then, by [15], $\sigma_p(T^*) = \emptyset$ or $\sigma_p(T^*) = \{\alpha\}$, for some $\alpha \in \mathbb{C} \setminus \{0\}$ such that $\alpha \notin \sigma_b(T^*)$, hence T^* has the SVEP. Consequently a-Browder's theorem (in particular Browder's theorem) holds for T and T^*. Suppose that $\sigma_p(T^*) = \emptyset$ then $\pi_{00}(T^*) = \emptyset$. Since $p_{00}(T^*) \subset \pi_{00}(T^*)$, thus $p_{00}(T^*) = \pi_{00}(T^*) = \emptyset$, since Browder's theorem holds for T^*, it follows, by [2, Theorem 2.16], T^* satisfies Weyl's theorem. Now, if $\sigma_p(T^*) = \{\alpha\}$, for some $\alpha \in \mathbb{C} \setminus \{0\}$ such that $\alpha \notin \sigma_b(T^*)$, then $\pi_{00}(T^*) = \{\alpha\}$. Since $p_{00}(T^*) \subset \pi_{00}(T^*)$, thus $\sigma(T^*) \setminus \sigma_b(T^*) = p_{00}(T^*) = \{\alpha\}$, consequently $p_{00}(T^*) = \pi_{00}(T^*) = \{\alpha\}$. Hence, by [2, Theorem 2.16], Weyl's theorem holds for T^*.

In the other hand, as a-Browder's theorem holds for T, thus $\sigma_{ub}(T) = \sigma_{uw}(T)$ and $\sigma_b(T) = \sigma_w(T)$. Since T^* has SVEP, then $\sigma(T) = \sigma_a(T)$ and $\sigma_{uw}(T) = \sigma_w(T)$, see [3, Theorem 1.5]. Therefore,

$$p_{00}(T) = \sigma_a(T) \setminus \sigma_{uw}(T) = p_{00}^a(T). \quad \square$$

To motivate the following theorem we start by an example which shows that the adjoint of supercyclic operator satisfies property $(a\omega)$.

Example 1. Consider the operator shift T defined on $l^2(\mathbb{N})$ by

$$T(x_1, x_2, x_3, \ldots) = \left(\frac{x_2}{2}, \frac{x_3}{3}, \ldots\right), \quad \text{for all } (x_n) \in l^2(\mathbb{N}).$$

Then T is surjective quasi-nilpotent, $\overline{\cup_{k\geq 1} N(T^k)} = \mathcal{X}$, $\sigma(T) = \{0\}$ and $\pi_{00}^a(T) = \pi_{00}(T) = \{0\}$, thus, by [20, corollary 1], the operator T is supercyclic. We know that the adjoint of T is defined by

$$T^*(x_1, x_2, x_3, \ldots) = \left(0, \frac{x_1}{2}, \frac{x_2}{3}, \ldots\right).$$

Since T^* are quasi-nilpotent, thus $\sigma(T) = \sigma(T^*) = \sigma_w(T) = \sigma_w(T^*) = \{0\}$. But T^* is injective $\pi_{00}(T^*) = \pi_{00}^a(T^*) = \emptyset$. Since, by Lemma 3.1, T^* satisfies Weyl's theorem. Hence, by [5, Theorem 2.4], T^* satisfies property $(a\omega)$.

In the following, we will give sufficient conditions for the adjoint of hypercyclic/supercyclic operator to satisfies property $(a\omega)$.

Theorem 3.1. *Let $T \in \mathcal{HP}(\mathcal{X}) \cup \mathcal{SP}(\mathcal{X})$. If*

$$\textit{iso } \sigma_a(T^*) \subset \{\lambda \in \mathbb{C} \,:\, R(\lambda I - T^*) \textit{ is closed }\},$$

then property $(a\omega)$ holds for T^*.

Proof. Let $T \in \mathcal{HP}(\mathcal{X}) \cup \mathcal{SP}(\mathcal{X})$. Then, by Lemma 3.1, T^* has the SVEP and Weyl's theorem holds for T^*.

Now, since Weyl's theorem holds for T^*, thus, in order to show that property $(a\omega)$ holds for T^* it suffices to prove that $q(\lambda I - T^*) < \infty$ for all $\lambda \in \pi_{00}^a(T^*)$ [5, Theorem 2.4]. Let $\lambda \in \pi_{00}^a(T^*)$, then λ is an isolated point of $\sigma_a(T)$ and $0 < \alpha(\lambda I - T^*) < \infty$ then $R(\lambda I - T^*)$ is closed. But T^* has the SVEP at λ, thus $0 < p(\lambda I - T^*) < \infty$. Hence, by [1, Theorem 3.4], $q(\lambda I - T^*) < \infty$. □

In the next Theorem, we will shows that a hypercyclic/supercyclic operator T satisfies property $(a\omega)$ if and only if a-Weyl's theorem holds for T.

Theorem 3.2. *Let $T \in \mathcal{HP}(\mathcal{X}) \cup \mathcal{SP}(\mathcal{X})$. Then T satisfies property $(a\omega)$ if and only if a-Weyl's theorem holds for T.*

Proof. Let $T \in \mathcal{HP}(\mathcal{X}) \cup \mathcal{SP}(\mathcal{X})$, then, by Lemma 3.1, T^* has the SVEP.

($\Longrightarrow$) Suppose that T satisfies property $(a\omega)$, thus, by [5, Theorem 2.3], Weyl's theorem holds for T. Hence, by [1, Theorem 3.108], a-Weyl's theorem holds for T.

($\Longleftarrow$) Suppose that a-Weyl's theorem holds for T. Then $\sigma_a(T) \setminus \sigma_{uw}(T) = \pi_{00}^a(T)$ and Weyl's theorem holds for T, thus $\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T)$. Since T^* has SVEP, then $\sigma(T) = \sigma_a(T)$ and $\sigma_{uw}(T) = \sigma_w(T)$, see [3, Theorem 1.5]. Hence $\pi_{00}^a(T) = \sigma(T) \setminus \sigma_w(T)$. Therefore property $(a\omega)$ holds for T. □

Example 2. Consider the operator shift T defined on $l^2(\mathbb{N})$ by

$$T(x_1, x_2, x_3, \ldots) = \left(\frac{x_2}{2}, \frac{x_3}{3}, \ldots\right) \quad \text{for all } (x_n) \in l^2(\mathbb{N}).$$

Then the operator T is supercyclic (Example 1). Since T^* is injective quasi-nilpotent, thus $\sigma(T^*) = \{0\}$ and $\pi_{00}(T^*) = \emptyset$. Hence $\pi^a_{00}(T) \neq \pi_{00}(T^*)$. On the other hand,

$$\{0\} = \pi^a_{00}(T) \neq \sigma(T) \setminus \sigma_w(T) = \emptyset.$$

Therefore, T does not satisfy property $(a\omega)$.

In the following, we give necessary and sufficient conditions for hypercyclic/supercyclic operators to have property $(a\omega)$.

Theorem 3.3. *Let $T \in \mathcal{HP}(\mathcal{X}) \cup \mathcal{SP}(\mathcal{X})$. Then the following statements are equivalent.*

(1) *The property $(a\omega)$ holds for T.*

(2) $\pi^a_{00}(T) = \pi_{00}(T^*)$.

Proof. (1) $\Longrightarrow$ (2) Suppose that property $(a\omega)$ holds for T, then by Theorem 2.1, $p_{00}(T) = \pi^a_{00}(T)$. Since Weyl's theorem holds for T^*, thus

$$\pi_{00}(T^*) = p_{00}(T^*) = p_{00}(T).$$

Therefore $\pi^a_{00}(T) = \pi_{00}(T^*)$.

(1) $\Longleftarrow$ (2) Suppose that $\pi^a_{00}(T) = \pi_{00}(T^*)$. Since T^* satisfies Weyl's theorem, thus $p_{00}(T^*) = \pi_{00}(T^*)$. Then

$$\pi_{00}(T^*) = p_{00}(T^*) = p_{00}(T).$$

Which imply that $p_{00}(T) = \pi^a_{00}(T)$. Hence

$$\sigma(T) \setminus \sigma_b(T) = \sigma(T) \setminus \sigma_w(T) = \pi^a_{00}(T).$$

Therefore, the property $(a\omega)$ holds for T. □

It was shown in [15] that if $T \in \mathcal{HP}(\mathcal{X})$ then $\sigma_p(T^*) = \emptyset$. Using this result, we get a necessary and sufficient condition for an operator $T \in \mathcal{HP}(\mathcal{X})$ to satisfies property $(a\omega)$.

Corollary 3.1. *Let $T \in \mathcal{B}(\mathcal{X})$ such that $T \in \mathcal{HP}(\mathcal{X})$. Then property $(a\omega)$ holds for T if and only if*

$$\pi^a_{00}(T) = \emptyset.$$

Proof. Assume that $p^a_{00}(T) = \emptyset$. Since $T \in \mathcal{HP}(\mathcal{X})$. Thus $\sigma_p(T^*) = \emptyset$, hence $\pi^a_{00}(T) = \pi_{00}(T^*) = \emptyset$. Therefore, by Theorem 3.3, property $(a\omega)$ holds for T. For the converse, assume that property $(a\omega)$ holds for T. Since $T \in \mathcal{HP}(\mathcal{X})$, thus $\pi_{00}(T^*) = \emptyset$. Hence $\pi^a_{00}(T) = \emptyset$. □

Corollary 3.2. *Let $T \in \mathcal{HP}(\mathcal{X}) \cup \mathcal{SP}(\mathcal{X})$ such that $\pi_{00}^a(T) = \pi_{00}(T^*)$. Then a-Weyl's theorem hols for T.*

Proof. Since $T \in \mathcal{HP}(\mathcal{X}) \cup \mathcal{SP}(\mathcal{X})$ with $\pi_{00}^a(T) = \pi_{00}(T^*)$. Thus, by Theorem 3.3, property $(a\omega)$ holds for T. Hence, by Theorem 3.2, a-Weyl's theorem hols for T. □

Given $T \in \mathcal{B}(\mathcal{X})$. Let $\rho_K(T)$ denote the Kato resolvent set of T;

$$\rho_K(T) := \left\{\lambda \in \mathbb{C} \ : \ R(\lambda I - T) \text{ is closed }, N(\lambda I - T) \subset \cap_{k\geq 1} R(\lambda I - T)^k\right\}.$$

It is well-known that $\rho_K(T)$ is open and $\rho_K(T) = \rho_K(T^*)$. Let

$$\rho_{su}(T) := \{\lambda \in \mathbb{C} \ : \ (\lambda I - T)\mathcal{X} = \mathcal{X}\}.$$

Let $\sigma_{su}(T) = \mathbb{C} \setminus \rho_{su}(T)$ be the surjectivity spectrum of T. It is well-known that $\rho_{su}(T)$ is open and $\rho_{su}(T) \subseteq \rho_K(T)$.

We denote by $\mathcal{H}_c(\sigma(T))$ the set of all complex-valued functions which are analytic and non constant in a neighborhood of the spectrum $\sigma(T)$, where the operator $f(T)$ is defined by the classical functional calculus.

Corollary 3.3. *Let $\mathcal{X}$ be a separable Banach space and suppose that $T \in \mathcal{B}(\mathcal{X})$ is such that $\lambda I - T$ is surjective and $\overline{\cup_{k\geq 1} N(\lambda I - T)^k} = \mathcal{X}$ for some $\lambda \in \rho_{su}(T)$. Then $f(T)$ satisfies property $(a\omega)$ for every $f \in \mathcal{H}_c(\sigma(T))$.*

Proof. First, we will show that $\text{iso}\, \sigma_a(f(T)) \subseteq f(\text{iso}\, \sigma_a(T))$. Let $\lambda_0 \in \text{iso}\, \sigma_a(f(T))$. By the spectral mapping theorem for the approximate point spectrum, $\lambda_0 \in \text{iso}\, f(\sigma_a(T))$. Then there exists $\mu_0 \in \sigma_a(T)$ such that $f(\mu_0) = \lambda_0$. Denote by G the component of the domain of f which contains μ_0. If $\mu_0 \in \text{acc}\, \sigma_a(T)$, then there exists a sequence $(\mu_n)_{n\geq 1} \subset G \cap \sigma_a(T)$ of distinct scalars such that $\mu_n \longrightarrow \mu_0$ as $n \longrightarrow \infty$. Let $K = \{\mu_0, \mu_1, \mu_2, \ldots\}$, then K is a compact subset of G, thus f may assume the value $\lambda_0 = f(\mu_0)$ only a finite number of points of K, hence for n sufficiently large $f(\mu_n) \neq f(\mu_0) = \lambda_0$, and since $f(\mu_n) \longrightarrow f(\mu_0) = \lambda_0$ as $n \longrightarrow \infty$, $\lambda_0 \in \text{acc}\, f(\sigma_a(T))$, and this is a contradiction. Hence $\mu_0 \in \text{iso}\, \sigma_a(T)$. Therefore

$$\text{iso}\, \sigma_a(f(T)) \subseteq f(\text{iso}\, \sigma_a(T)). \tag{5}$$

Since $\overline{\cup_{k\geq 1} N(\lambda I - T)^k} = \mathcal{X}$ for some $\lambda \in \rho_{su}(T)$. Thus, by [7, Theorem 3], $f(T)$ is supercyclic for $f \in \mathcal{H}_c(\sigma(T))$, and since T^* has de SVEP, hence $\sigma(T) = \sigma_a(T)$. Moreover, we have $f(T)^* = f(T^*)$ for every $f \in \mathcal{H}_c(\sigma(T))$. On the other hand, by [16, Proposition 3], $\sigma(T)$ is connected. But

$$\sigma(f(T)^*) = \sigma(f(T)) = f(\sigma(T)).$$

Thus, $\sigma(f(T))$ and $\sigma(f(T)^*)$ are connected. Then,

$$\pi_{00}(f(T)) = \pi_{00}(f(T)^*) = \emptyset,$$

by (5), we have $\pi_{00}^a(f(T)) = \emptyset$. Hence, $\pi_{00}^a(f(T)) = \pi_{00}(f(T)^*)$. Therefore, by Theorem 3.3, $f(T)$ satisfies property $(a\omega)$ for every $f \in \mathcal{H}_c(\sigma(T))$. □

In the following result we shows that for hypercyclic/supercyclic operator property (ω) and property $(a\omega)$ are equivalent.

Theorem 3.4. *Let $T \in \mathcal{HP}(\mathcal{X}) \cup \mathcal{SP}(\mathcal{X})$. Property $(a\omega)$ holds for the operator T if and only if T satisfies property (ω).*

Proof. Let $T \in \mathcal{HP}(\mathcal{X}) \cup \mathcal{SP}(\mathcal{X})$. Then T^* has the SVEP, thus a-Browder's theorem (in particular Browder's theorem) holds for T, hence as above,

$$p_{00}(T) = \sigma(T) \setminus \sigma_w(T) = \sigma_a(T) \setminus \sigma_{uw}(T) = p_{00}^a(T). \tag{6}$$

$(\Longrightarrow)$ Assume that T satisfies property $(a\omega)$, then, by Theorem 3.2, a-Weyl's theorem (in particular Weyl's theorem) hols for T. Thus

$$\sigma_a(T) \setminus \sigma_{uw}(T) = \pi_{00}^a(T).$$

Hence, $\sigma_a(T) \setminus \sigma_{uw}(T) = \pi_{00}(T)$. Therefore, T satisfies property (ω).

$(\Longleftarrow)$ Assume that property (ω) holds for T. Then Weyl's theorem holds for T. Thus

$$\sigma(T) \setminus \sigma_w(T) = \sigma_a(T) \setminus \sigma_{uw}(T) = \pi_{00}(T).$$

But $T \in \mathcal{HP}(\mathcal{X}) \cup \mathcal{SP}(\mathcal{X})$, thus $\pi_{00} = \pi_{00}^a(T)$, $(\sigma(T) = \sigma_a(T))$. Hence $\pi_{00}^a(T) = \sigma(T) \setminus \sigma(T)$. Therefore, property $(a\omega)$ holds for T. □

In the following, some characterizations of hypercyclic/supercyclic operators satisfying property $(a\omega)$ will be given.

Corollary 3.4. *Let $T \in \mathcal{HP}(\mathcal{X}) \cup \mathcal{SP}(\mathcal{X})$. Then the following statements are equivalent.*

(1) *The property $(a\omega)$ holds for T.*

(2) *The property (ω) holds for T.*

(3) $\pi_{00}^a(T) = \pi_{00}(T^*)$.

(4) $\pi_{00}(T) = \pi_{00}(T^*)$

Proof. By Theorem 3.4, we have $((1) \Longleftrightarrow (2))$. Using Theorem 3.3, we get $((1) \Longleftrightarrow (3))$. For $((2) \Longleftrightarrow (4))$ we use [12, Proposition 2.1]. □

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Coefficient estimates for new subclass of pseudo-type meromorphic bi-univalent functions

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Abstract. In the present article, we define a new subclass of pseudo-type meromorphic bi-univalent functions defined on $\mathbb{U}^* = \{z \mid : z \in \mathbb{C} \text{ and } 1 < |z| < \infty\}$, and investigate the initial coefficient estimates $|b_0|$ and $|b_1|$. Further we mention several new or known consequences of our results.
Keywords: analytic functions, univalent functions, meromorphic univalent functions, Bi-univalent functions, meromorphic bi-univalent functions, pseudo functions.

1. Introduction

Let A denote the class of functions $f(z)$ of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open unit open disk $\mathbb{U} = \{z : z \in \mathbb{C}, |z| < 1\}$.

Also, let $\mathbb{S}$ be class of all functions in A which are univalent and normalized by the conditions

$$f(0) = 0 = f^{'}(0) - 1$$

in $\mathbb{U}$. Some of the important and well-investigated subclasses of the univalent function class $\mathbb{S}$ includes the class $S^*(\alpha)(0 \leq \alpha < 1)$ of starlike functions of order α in $\mathbb{U}$ and the class $K(\alpha)(0 \leq \alpha < 1)$ of convex functions of order α.

Let Σ denote the class of meromorphic univalent functions g of the form

$$g(z) = z + \sum_{n=0}^{\infty} \frac{b_n}{z^n} \tag{2}$$

defined on the domain $\mathbb{U}^* = \{z : z \in \mathbb{C}, 1 < |z| < \infty\}$. It is well known that every function $g \in \Sigma$ has an inverse $g^{-1} = h$, defined by

$$g^{-1}(g(z)) = z, \quad (z \in \mathbb{U}^*),$$

and

$$g^{-1}(g(w)) = w, \quad (M < |w| < \infty, \ M > 0),$$

where

$$g^{-1}(w) = h(w) = w + \sum_{n=0}^{\infty} \frac{B_n}{w^n}$$
$$= w - b_0 - \frac{b_1}{w} - \frac{b_1 b_0 + b_2}{w^2} - \frac{b_1^2 + b_1 b_0^2 + 2b_0 b_2 + b_3}{w^3} + \cdots . \tag{3}$$

A simple computation shows that

$$w = g(h(w)) = (b_0 + B_0) + w + \frac{b_1 + B_1}{w} + \frac{B_2 - b_1 B_0 + b_2}{w^2}$$
$$+ \frac{B_3 - b_1 B_1 + b_1 B_0^2 - 2b_2 B_0 + b_3}{w^3} + \cdots . \tag{4}$$

Comparing the initial coefficients in (4), we find that

$$b_0 + B_0 = 0 \Rightarrow \quad B_0 = -b_0,$$
$$b_1 + B_1 = 0 \Rightarrow \quad B_1 = -b_1,$$
$$B_2 - b_1 B_0 + b_2 = 0 \Rightarrow B_2 = -(b_2 + b_1 b_0),$$
$$B_3 - b_1 B_1 + b_1 B_0^2 - 2b_2 B_0 + b_3 = 0 \Rightarrow B_3 = -(b_3 + 2b_0 b_1 + b_1 b_0^2 + b_1^2).$$

A function $f \in \Sigma$ is said to be meromorphic bi-univalent if $f^{-1} \in \Sigma$. The family of all meromorphic bi-univalent functions is denoted by Σ'. Estimates on the coefficient of meromorphic univalent functions were widely investigated in the literature; for example, Schiffer [1] obtained the estimate $|b_2| < \frac{3}{2}$ for meromorphic univalent functions $f \in \mathbb{S}$ with $b_0 = 0$. In 1983, Duren [2] obtained the inequality $|b_2| < \frac{2}{n+1}$ for $f \in \mathbb{S}$ with $b_k = 0, 1 \leq k \leq \frac{n}{2}$. For the coefficients of inverses of meromorphic univalent functions Springer [3] proved that

$$|B_3| < 1 \text{ and } |B_3 + \frac{1}{2} B_1^2| < \frac{1}{2},$$

and conjectured that

$$|B_{2n-1}| \leq \frac{(2n-2)!}{n!(n-1)!} \ (n = 1, 2, ...).$$

In 1977, Kubota [4] has proved that the Springer conjecture is true for n = 3;4;5; and subsequently Schober [5] obtained sharp bounds for $|B_{2n-1}|$ if $1 \leq n \leq 7$. In 2007. Kapoor and Mishra [6] found the coefficient estimates for the inverse of meromorphic starlike univalent functions of order α in $\mathbb{U}^*$.

Recently, several researchers (for example [7], [9], [10], [11], [12], [13], [14], [15], [16], [18], [19]) introduced various subclasses of bi-univalent functions and meromorphically bi-univalent functions. Estimates on the initial coefficients for functions in each of these subclasses are obtained.

In 2013, Babalola [20] defined a new subclass λ-pseudo starlike function of order $0 \leq \beta < 1$ satisfying the analytic condition

$$\Re\left\{\frac{z(f(z)')^{\lambda}}{f(z)}\right\} > \beta \ (\lambda \geq 1,\ z \in \mathbb{U}). \tag{5}$$

In particular, Babalola [20] proved that all λ-pseudo-starlike functions are Bazilevic of type $1-\frac{1}{\lambda}$ and order $\beta^{\frac{1}{\lambda}}$ and are univalent in open unit disk $\mathbb{U}$.

In this paper, motivated by the previous works, we introduce two new subclasses of pseudo-type of meromorphically bi-univalent functions and obtained the estimates for the initial coefficients $|b_0|$ and $|b_1|$ of functions in these subclasses.

In order to derive our main results, we first recall the following lemma.

Carathéodory's Lemma (see, for example, [21]). If $p \in P$, then

$$|p_j| \leq 2, \qquad (j \in \mathbb{N}),$$

where P is the family of all functions $p(z)$, analytic in $\mathbb{U}^*$, for which

$$\Re(p(z)) > 0, \qquad (z \in \mathbb{U}^*),$$

where

$$p(z) = 1 + \frac{p_1}{z} + \frac{p_2}{z_2} + \frac{p_3}{z_3} + ... \qquad (z \in \mathbb{U}^*),$$

$\mathbb{N}$ being the set of positive integers.

2. Coefficient bounds for the function class $\Sigma'_{\lambda,\beta}(\alpha)$

We begin by introducing the function class $\Sigma'_{\lambda,\beta}(\alpha)$ by means of the following definition.

Definition 2.1. *A function $g(z) \in \Sigma'$ given by (2) is said to be in the class $\Sigma_{\lambda,\beta}(\alpha)$ if the following conditions are satisfied:*

$$\left|\arg\left[(1-\beta)\left(\frac{g(z)}{z}\right)^{\lambda} + \beta\left(\frac{z(g(z)')^{\lambda}}{g(z)}\right)\right]\right| < \frac{\alpha\pi}{2}, \tag{6}$$

$(0 < \beta \leq 1, 0 < \alpha \leq 1, \lambda \geq 1, z \in \mathbb{U}^*)$, *and*

$$\left|\arg\left[(1-\beta)\left(\frac{h(w)}{w}\right)^{\lambda} + \beta\left(\frac{w(h(w)')^{\lambda}}{h(w)}\right)\right]\right| < \frac{\alpha\pi}{2}, \tag{7}$$

$(0 < \beta \leq 1, 0 < \alpha \leq 1, \lambda \geq 1, w \in \mathbb{U}^*)$, *where the function h is given by (3).*

We denote by $\Sigma'_{\lambda,\beta}(\alpha)$ the class of meromorphically strongly (λ,β)-pseudo-starlike bi-univalent functions of order $\alpha \in \mathbb{U}^*$.

We begin this section by finding the estimates on the Coefficients $|b_1|$ and $|b_2|$ for functions in the class $\Sigma'_{\lambda,\beta}(\alpha)$.

Theorem 2.1. *Let $g(z)$ be given by (2) be in the class $\Sigma'_{\lambda,\beta}(\alpha)$. Then*

$$|b_0| \leq \frac{2\alpha}{(\beta+\beta\lambda-\lambda)} \tag{8}$$

and

$$|b_1| \leq 2\alpha^2 \left(\frac{\sqrt{[\lambda(\lambda-1)(1-\beta)+2\beta]^2+(\lambda-\lambda\beta-\beta)^4}}{(\beta+2\lambda\beta-\lambda)(\lambda-\lambda\beta-\beta)^2} \right). \tag{9}$$

Proof. Let $g \in \Sigma'_{\lambda,\beta}(\alpha)$. Then, by Definition 2.1 of meromorphically bi-univalent function class $\Sigma'_{\lambda,\beta}(\alpha)$, the conditions (6) and (7) can be rewritten as follows:

$$(1-\beta)\left(\frac{g(z)}{z}\right)^{\lambda} + \beta\left(\frac{z(g(z)')^{\lambda}}{g(z)}\right) = [p(z)]^{\alpha} \tag{10}$$

and

$$(1-\beta)\left(\frac{h(w)}{w}\right)^{\lambda} + \beta\left(\frac{w(h(w)')^{\lambda}}{h(w)}\right) = [q(w)]^{\alpha}, \tag{11}$$

respectively. Here, and in what follows,the functions $p(z) \in P$ and $q(w) \in P$ have the following forms:

$$p(z) = 1 + \frac{p_1}{z} + \frac{p_2}{z^2} + \frac{p_3}{z^3} + \cdots \qquad (z \in \mathbb{U}^*) \tag{12}$$

and

$$q(w) = 1 + \frac{q_1}{w} + \frac{q_2}{w^2} + \frac{q_3}{w^3} + \cdots \qquad (w \in \mathbb{U}^*). \tag{13}$$

Clearly, we have

$$[p(z)]^{\alpha} = 1 + \frac{\alpha p_1}{z} + \frac{\frac{1}{2}\alpha(\alpha-1)p_1^2 + \alpha p_2}{z^2} + \frac{\frac{1}{6}\alpha(\alpha-1)(\alpha-2)p_1^3 + \alpha(\alpha-1)p_1p_2 + \alpha p_3}{z^3} + \cdots$$

and

$$[q(w)]^{\alpha} = 1 + \frac{\alpha q_1}{w} + \frac{\frac{1}{2}\alpha(\alpha-1)q_1^2 + \alpha q_2}{w^2} + \frac{\frac{1}{6}\alpha(\alpha-1)(\alpha-2)q_1^3 + \alpha(\alpha-1)q_1q_2 + \alpha q_3}{w^3} + \cdots .$$

We also find that

$$\begin{aligned} &(1-\beta)\left(\frac{g(z)}{z}\right)^{\lambda} + \beta\left(\frac{z(g(z)')^{\lambda}}{g(z)}\right) \\ &= 1 + \frac{(\lambda-\lambda\beta-\beta)b_0}{z} + \frac{\frac{1}{2}(\lambda(\lambda-1)(1-\beta)+2\beta)b_0^2 + (\lambda-\beta-2\lambda\beta)b_1}{z^2} \\ &\quad + \frac{\frac{1}{6}(\lambda(\lambda-1)(\lambda-2)(1-\beta)-6\beta)b_0^3 + (\lambda(\lambda-1)(1-\beta)+2\beta+\lambda\beta)b_0b_1 + (\lambda-\beta-3\lambda\beta)b_2}{z^3} + \cdots \end{aligned} \tag{14}$$

and

$$
\begin{aligned}
&(1-\beta)\left(\frac{h(w)}{w}\right)^{\lambda}+\beta\left(\frac{w(h(w)')^{\lambda}}{h(w)}\right)\\
&=1-\frac{(\lambda-\lambda\beta-\beta)b_0}{w}+\frac{\frac{1}{2}(\lambda(\lambda-1)(1-\beta)+2\beta)b_0^2+(\beta-\lambda+2\lambda\beta)b_1}{w^2}\\
&\quad+\frac{\frac{1}{6}(6\beta-\lambda(\lambda-1)(\lambda-2)(1-\beta))b_0^3+(\lambda(\lambda-1)(1-\beta)}{w^3}\\
&\quad+\frac{-\lambda(1-\beta)+3\beta+3\lambda\beta)b_0b_1+(\beta-\lambda+3\lambda\beta)b_2}{w^3}+\cdots\quad. \qquad (15)
\end{aligned}
$$

Now, equating the Coefficients in (10) and (11), we get

$$(\lambda-\lambda\beta-\beta)b_0=\alpha p_1, \qquad (16)$$

$$\frac{1}{2}(\lambda(\lambda-1)(1-\beta)+2\beta)b_0^2+(\lambda-\beta-2\lambda\beta)b_1=\frac{1}{2}\alpha(\alpha-1)p_1^2+\alpha p_2, \qquad (17)$$

$$-(\lambda-\lambda\beta-\beta)b_0=\alpha q_1, \qquad (18)$$

$$\frac{1}{2}(\lambda(\lambda-1)(1-\beta)+2\beta)b_0^2+(\beta-\lambda+2\lambda\beta)b_1=\frac{1}{2}\alpha(\alpha-1)q_1^2+\alpha q_2. \qquad (19)$$

From (16) and (18), we find that

$$p_1=-q_1 \qquad (20)$$

and

$$2(\lambda-\lambda\beta-\beta)^2b_0^2=\alpha^2(p_1^2+q_1^2) \qquad (21)$$

that is,

$$b_0^2=\frac{\alpha^2(p_1^2+q_1^2)}{2(\lambda-\lambda\beta-\beta)^2}.$$

Applying Carathéodory's Lemma for the Coefficients p_1 and q_1, we immediately have

$$|b_0|\leq\frac{2\alpha}{(\beta+\lambda\beta-\lambda)}.$$

This gives the bound on$|b_0|$ as asserted in (8).

Next, in order to find the bound on $|b_0|$, by using the equation (17) and the equation (19),we get

$$
\begin{aligned}
&\left(\frac{1}{2}(\lambda(\lambda-1)(1-\beta)+2\beta)b_0^2+(\lambda-\beta-2\lambda\beta)b_1\right)\\
&\cdot\left(\frac{1}{2}(\lambda(\lambda-1)(1-\beta)+2\beta)b_0^2+(\beta-\lambda+2\lambda\beta)b_1\right)
\end{aligned}
$$

$$= \left(\frac{1}{2}\alpha(\alpha-1)p_1^2 + \alpha p_2\right).\left(\frac{1}{2}\alpha(\alpha-1)q_1^2 + \alpha q_2\right),$$

$$\frac{1}{4}(\lambda(\lambda-1)(1-\beta)+2\beta)^2 b_0^4 - (\beta-\lambda+2\lambda\beta)^2 b_1^2$$

$$= \frac{1}{4}\alpha^2(\alpha-1)^2 p_1^2 q_1^2 + \frac{1}{2}\alpha^2(\alpha-1)\left(p_2 q_1^2 + p_1^2 q_2\right) + \alpha^2 p_2 q_2,$$

$$(\beta-\lambda+2\lambda\beta)^2 b_1^2 = \left(\frac{1}{2}(\lambda(\lambda-1)(1-\beta)+2\beta)b_0^2\right)^2$$

$$-\frac{1}{4}\alpha^2(\alpha-1)^2 p_1^2 q_1^2 - \frac{1}{2}\alpha^2(\alpha-1)\left(p_2 q_1^2 + p_1^2 q_2\right) - \alpha^2 p_2 q_2$$

and

$$(\beta-\lambda+2\lambda\beta)^2 b_1^2 = \left(\frac{1}{4}(\lambda(\lambda-1)(1-\beta)+2\beta)^2 \frac{\alpha^4(p_1^2+q_1^2)^2}{4(\lambda-\lambda\beta-\beta)^4}\right)$$

$$-\frac{1}{4}\alpha^2(\alpha-1)^2 p_1^2 q_1^2 - \frac{1}{2}\alpha^2(\alpha-1)\left(p_2 q_1^2 + p_1^2 q_2\right) - \alpha^2 p_2 q_2.$$

Applying Carathéodory's Lemma once again for the Coefficients p_1, q_1, p_2 and q_2 , we get

$$|b_1|^2 \leq \alpha^4\left(\frac{(2(\lambda(\lambda-1)(1-\beta)+2\beta))^2 + 4(\lambda-\lambda\beta-\beta)^4}{(\beta+2\lambda\beta-\lambda)^2(\lambda-\lambda\beta-\beta)^4}\right) \Rightarrow$$

$$|b_1| \leq 2\alpha^2\left(\frac{\sqrt{[\lambda(\lambda-1)(1-\beta)+2\beta]^2 + (\lambda-\lambda\beta-\beta)^4}}{(\beta+2\lambda\beta-\lambda)(\lambda-\lambda\beta-\beta)^2}\right),$$

which evidently completes the proof of Theorem 2.2. □

3. Coefficient bounds for the function class $\Sigma'_{\lambda,\beta}(\mu)$

We first introduce the function class $\Sigma'_{\lambda,\beta}(\mu)$ as follows.

Definition 3.1. *A function $g(z) \in \Sigma'$ given by (2) is said to be in the class $\Sigma'_{\lambda,\beta}(\mu)$ if the following conditions are satisfied:*

$$\Re\left((1-\beta)\left(\frac{g(z)}{z}\right)^{\lambda} + \beta\left(\frac{z(g(z)')^{\lambda}}{g(z)}\right)\right) > \mu, \tag{22}$$

$(0 < \beta \leq 1,\ 0 \leq \mu < 1,\ \lambda \geq 1,\ z \in \mathbb{U}^*)$, *and*

$$\Re\left((1-\beta)\left(\frac{h(w)}{w}\right)^{\lambda} + \beta\left(\frac{w(h(w)')^{\lambda}}{h(w)}\right)\right) > \mu, \tag{23}$$

$(0 < \beta \leq 1,\ 0 \leq \mu < 1,\ \lambda \geq 1,\ w \in \mathbb{U}^*)$ *where the function h is given by (3).*

We call $\Sigma'_{\lambda,\beta}(\mu)$ the class of meromorphically λ-pseudo-starlike bi-univalent functions of order μ.

Next, we now derive the estimates on the Coefficients $|b_0|$ and $|b_1|$ for the meromorphically bi univalent function class $\Sigma'_{\lambda,\beta}(\mu)$.

Theorem 3.1. *Let $g(z)$ be given by (2) be in the class $\Sigma'_{\lambda,\beta}(\mu)$. Then*

$$|b_0| \leq \frac{2(1-\mu)}{(\beta+\lambda\beta-\lambda)} \tag{24}$$

and

$$|b_1| \leq 2(1-\mu)\left(\frac{\sqrt{(1-\mu)^2(\lambda(\lambda-1)(1-\beta)+2\beta)^2+(\lambda-\lambda\beta-\beta)^2}}{(\beta+\lambda\beta-\lambda)(\beta+2\lambda\beta-\lambda)}\right). \tag{25}$$

Proof. Let $g \in \Sigma'_{\lambda,\beta}(\mu)$. Then, by Definition 2.1 of meromorphically bi-univalent function class $\Sigma'_{\lambda,\beta}(\mu)$, the conditions (22) and (23) can be rewritten as follows:

$$(1-\beta)\left(\frac{g(z)}{z}\right)^{\lambda}+\beta\left(\frac{z(g(z)')^{\lambda}}{g(z)}\right)=\mu+(1-\mu)p(z) \tag{26}$$

and

$$(1-\beta)\left(\frac{h(w)}{w}\right)^{\lambda}+\beta\left(\frac{w(h(w)')^{\lambda}}{h(w)}\right)=\mu+(1-\mu)q(w), \tag{27}$$

respectively. Here, just as in our proof of Theorem 2.2, with the functions $p(z) \in P$ and $q(w) \in P$ have the forms in (12) and (13). Clearly we have

$$\mu+(1-\mu)p(z)=1+\frac{(1-\mu)p_1}{z}+\frac{(1-\mu)p_2}{z^2}+\frac{(1-\mu)p_3}{z^3}+\cdots$$

and

$$\mu+(1-\mu)q(w)=1+\frac{(1-\mu)q_1}{w}+\frac{(1-\mu)q_2}{w^2}+\frac{(1-\mu)q_3}{w^3}+\cdots \quad .$$

Now, equating the Coefficients in (26) and (27) and using (14) and (15), we get

$$(\lambda-\lambda\beta-\beta)b_0=(1-\mu)p_1, \tag{28}$$

$$\frac{1}{2}(\lambda(\lambda-1)(1-\beta)+2\beta)b_0^2+2(\lambda-\beta-2\lambda\beta)b_1=(1-\mu)p_2, \tag{29}$$

$$-(\lambda-\lambda\beta-\beta)b_0=(1-\mu)q_1, \tag{30}$$

$$\frac{1}{2}(\lambda(\lambda-1)(1-\beta)+2\beta)b_0^2+2(\beta-\lambda+2\lambda\beta)b_1=(1-\mu)q_2. \tag{31}$$

From (28) and (30), we obtain

$$p_1=-q_1 \tag{32}$$

and

$$2(\lambda-\lambda\beta-\beta)^2 b_0^2=(1-\mu)^2(p_1^2+q_1^2) \tag{33}$$

that is,

$$b_0^2=\frac{(1-\mu)^2(p_1^2+q_1^2)}{2(\lambda-\lambda\beta-\beta)^2}.$$

Applying Carathéodory's Lemma for the Coefficients p_1 and q_1 , we immediately have

$$|b_0|\leq\frac{2(1-\mu)}{(\beta+\lambda\beta-\lambda)}.$$

This gives the bound on$|b_0|$ as asserted in (24).

Next, in order to find the bound on $|b_1|$, by using the equation (29) and the equation (31), we get

$$\begin{aligned}
&\left(\frac{1}{2}(\lambda(\lambda-1)(1-\beta)+2\beta)b_0^2+(\lambda-\beta-2\lambda\beta)b_1\right)\\
&\cdot\left(\frac{1}{2}(\lambda(\lambda-1)(1-\beta)+2\beta)b_0^2+(\beta-\lambda+2\lambda\beta)b_1\right)\\
&=((1-\mu)p_2)\,.\,((1-\mu)p_2)\,,\\
&\frac{1}{4}(\lambda(\lambda-1)(1-\beta)+2\beta)b_0^4-(\beta+2\lambda\beta-\lambda))^2b_1^2=(1-\mu)^2p_2q_2,\\
&(\beta+2\lambda\beta-\lambda)^2b_1^2=\frac{1}{4}\left((\lambda(\lambda-1)(1-\beta)+2\beta)^2b_0^2\right)^2-(1-\mu)^2p_2q_2
\end{aligned}$$

and

$$(\beta+2\lambda\beta-\lambda)^2b_1^2=\frac{1}{4}(\lambda(\lambda-1)(1-\beta)+2\beta)^2\frac{(1-\mu)^4(p_1^2+q_1^2)}{2(\lambda-\lambda\beta-\beta)^2}-(1-\mu)^2p_2q_2.$$

Applying Carathéodory's Lemma once again for the Coefficients $p_1,\ q_1,\ p_2$ and q_2, we get

$$(\beta+2\lambda\beta-\lambda)^2|b_1^2|\leq 4(\lambda(\lambda-1)(1-\beta)+2\beta)^2\frac{(1-\mu)^4}{(\lambda-\lambda\beta-\beta)^2}+4(1-\mu)^2,$$

that is,

$$|b_1|\leq 2(1-\mu)\left(\frac{\sqrt{(1-\mu)^2(\lambda(\lambda-1)(1-\beta)+2\beta)^2+(\lambda-\lambda\beta-\beta)^2}}{(\beta+\lambda\beta-\lambda)(\beta+2\lambda\beta-\lambda)}\right).$$

This completes the proof of Theorem 3.2. □

Remark 3.1. By taking $\beta=1$, in Theorem 2.2 and Theorem 3.2 we can deduce the corresponding coefficient estimates for several simpler meromorphically bi-univalent function classes, (see [19]).

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Transmuted shanker distribution: properties and applications

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Abstract. In this paper a new continuous distribution is proposed. This distribution will generalize Shanker distribution to the transmuted Shanker distribution using the quadratic rank transmutation map. Some properties of this distribution are studied. A numerical study is conducted to calculate the mean, standard deviation, skewness, kurtosis and the coefficient of variation of this distribution. An application to a data set is conducted. It shows that the new distribution performs better than Shanker distributions and some other distributions, like Shanker itself and Lindley in performance of the values of $-2lnL, W, A, AIC, KS$-statistic and p-value.

Keywords: Shanker distribution, transmuted Shanker distribution, moments, skewness, kurtosis, entropy, order statistics, quantile function, median, MLE.

1. Introduction

Classical families of distributions might not be adequate for modeling many practical data. Therefore, generalizing the existing distributions by adding one or more parameters allows the resulted distributions to be more appropriate to fit real life data. [23] employed the quadratic rank transmutation map to generate a general and flexible family of distributions. This family is called transmuted family of distributions. We use the quadratic rank transmutation where $(f(x))$ and $(F(x))$ are the *pdf* and CDF of the base distribution map to

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introduce a new distribution called transmuted Shanker distribution (TSHD). This proposed distribution is a generalization of Shanker distribution.

The CDF of the quadratic rank transmutation map is defined as:

$$G(x) = (1+\lambda)F(x) - \lambda F^2(x), \qquad |\lambda| \leq 1. \tag{1}$$

The *pdf* is defined as:

$$g(x) = f(x)((1+\lambda) - 2\lambda F(x)), \qquad |\lambda| \leq 1. \tag{2}$$

Many authors used these maps to generalize some distributions. [3] proposed the transmuted Janardan distribution. [7] has worked out the transmuted Mukherjee-Islam distribution: a generalization of Mukherjee-Islam distribution. Transmuted Burr type XII distribution: a generalization of the Burr type XII distribution [15]. A generalization of the new Weibull-Pareto distribution [2]. [1] worked out the transmuted two-parameter Lindley distribution. [11] generalized two distributions through her master thesis, namely: the transmuted Gamma-Gompertz and transmuted Generalized Type-II Half-Logistics Distributions. [10] worked out the transmuted reciprocal and the transmuted two-parameter weighted exponential distributions. [18] used the quadratic transmutation map to generalize the power function and Type-I half logistic distributions. [6] generalized Sujatha and Amarendra distributions using quadratic transmutation map. [5] used the same map to generalize Rani and generalized Akash distributions. [19] used the quadratic map to propose transmuted Akash distribution and the cubic map to produce the cubic Mukherjee-Islam distribution. [13] proposed the transmuted Aradhana distribution. [14] generalized Ishita distribution to transmuted Ishita with applications.

The remaining of this paper is organized as follows: Section 2, defines pdf and cdf of the proposed distribution. In Section 3, we defined the reliability, hazard rate functions. While in Section 5, the maximum likelihood estimates of the TSHD parameters are derived. The quantile function and the median of the distribution are derived in Section 6. Rényi entropy is defined in Section 7. An application to real data set is presented in Section 8.

2. Transmuted Shanker distribution

[20] has suggested a one-parameter distribution for modeling real lifetime data sets, it is called Shanker distribution. This distribution is a mixture of exponential (θ) and gamma (2,θ). [20] defined his distribution as:

Definition 2.1. *A random variable X is said to have Shanker distribution if its probability density function (pdf) and cumulative distribution function (CDF)*

are defined as [20]:

$$f(x) = \begin{cases} \frac{\theta^2}{\theta^2+1}(\theta + x)e^{-\theta x}, & x > 0, \quad \theta > 0, \\ 0, & o.w. \end{cases} \tag{3}$$

$$F(x) = \begin{cases} 1 - \frac{\theta^2+1+\theta x}{\theta^2+1}e^{-\theta x}, & x > 0, \quad \theta > 0, \\ 0, & o.w. \end{cases} \tag{4}$$

The transmuted Shanker distribution *pdf* and CDF are; respectively, defined as:

$$\begin{aligned} g(x) &= \Big(\frac{\theta^2(\theta + x)}{\theta^2 + 1}e^{-\theta x}\Big)\Big(1 + \lambda - 2\lambda\Big(1 - \frac{\theta^2 + 1 + \theta x}{\theta^2 + 1}e^{-\theta x}\Big), \\ &= \Big(\frac{\theta^2(\theta + x)}{\theta^2 + 1}e^{-\theta x}\Big)\Big(1 - \lambda + 2\lambda\frac{\theta^2 + 1 + \theta x}{\theta^2 + 1}e^{-\theta x}\Big) \\ &= \frac{(1-\lambda)\theta^3}{\theta^2+1}e^{-\theta x} + \frac{(1-\lambda)\theta^2}{\theta^2+1}xe^{-\theta x} + \frac{2\lambda\theta^3}{\theta^2+1}e^{-2\theta x} \\ &+ \frac{2\lambda\theta^2(2\theta^2+1)}{\theta^2+1}xe^{-2\theta x} + \frac{2\lambda\theta^3}{\theta^2+1}x^2e^{-2\theta x}\, x, \theta > 0, |\lambda| \leq 1, \end{aligned} \tag{5}$$

$$\begin{aligned} G(x) &= (1 + \lambda)\Big(1 - \frac{\theta^2 + 1 + \theta x}{\theta^2 + 1}e^{-\theta x}\Big) - \lambda\Big(1 - \frac{\theta^2 + 1 + \theta x}{\theta^2 + 1}e^{-\theta x}\Big)^2 \\ &= 1 - (1 - \lambda)\frac{\theta^2 + 1 + \theta x}{\theta^2 + 1}e^{-\theta x} + \lambda\Big(\frac{\theta^2 + 1 + \theta x}{\theta^2 + 1}\Big)^2 e^{-2\theta x}\, x, \theta > 0, |\lambda| \leq 1. \end{aligned} \tag{6}$$

3. Reliability and hazard rate functions

The reliability function (survival function) is defined as:

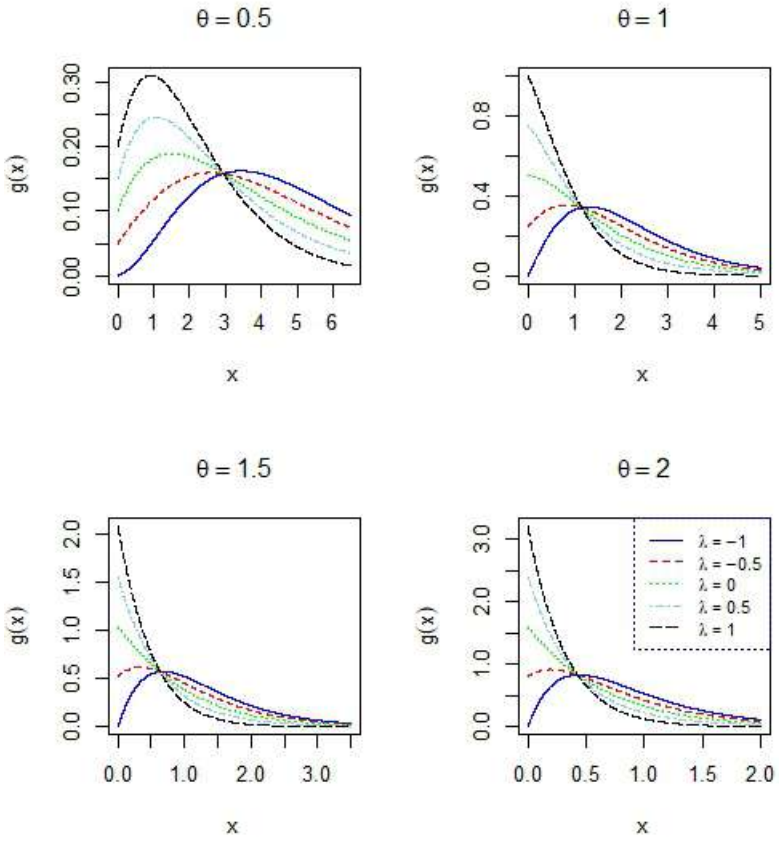
$$R(t) = p(T \geq t) = 1 - G(t).$$

Therefore, the reliability function for TSHD is given by:

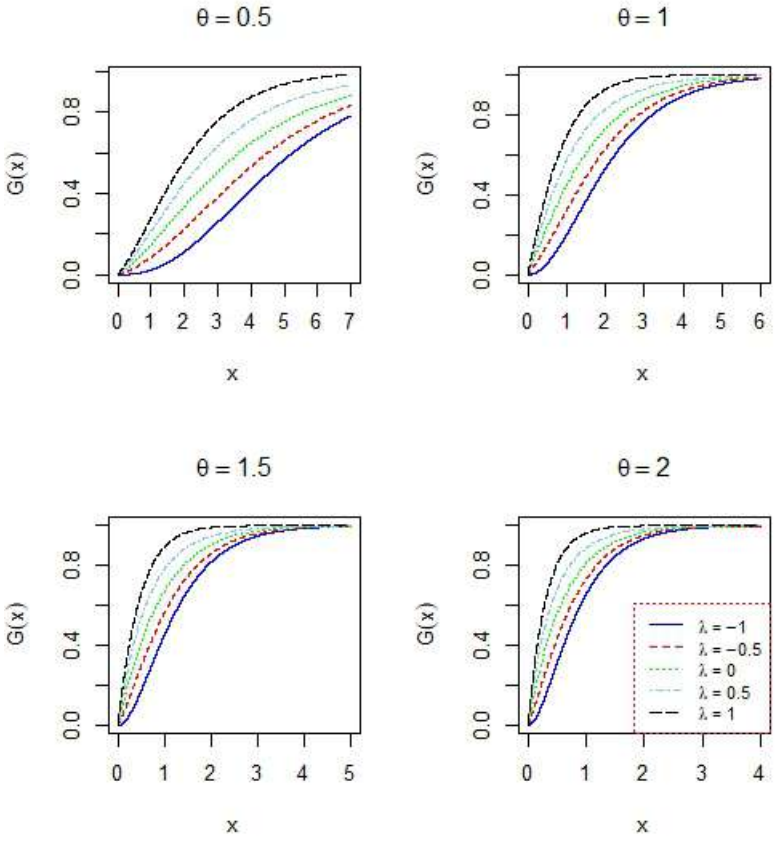
$$\begin{aligned} R(t) &= 1 - \Big[1 - (1 - \lambda)\frac{\theta^2 + 1 + \theta x}{\theta^2 + 1}e^{-\theta x} + \lambda\Big(\frac{\theta^2 + 1 + \theta x}{\theta^2 + 1}\Big)^2 e^{-2\theta x}\Big] \\ &= (1 - \lambda)\frac{\theta^2 + 1 + \theta x}{\theta^2 + 1}e^{-\theta x} - \lambda\Big(\frac{\theta^2 + 1 + \theta x}{\theta^2 + 1}\Big)^2 e^{-2\theta x} \\ &= \frac{\theta^2 + 1 + \theta x}{\theta^2 + 1}e^{-\theta x}\Big[1 - \lambda - \lambda\Big(\frac{\theta^2 + 1 + \theta x}{\theta^2 + 1}\Big)e^{-\theta x}\Big]. \end{aligned} \tag{7}$$

The hazard rate function is defined to be $h(t) = \frac{g(t)}{1-G(t)}$

$$
\begin{aligned}
h(t) &= \frac{\left(\frac{\theta^2(\theta+x)}{\theta^2+1}e^{-\theta x}\right)\left(1-\lambda+2\lambda\frac{\theta^2+1+\theta x}{\theta^2+1}e^{-\theta x}\right)}{\frac{\theta^2+1+\theta x}{\theta^2+1}e^{-\theta x}\left[1-\lambda-\lambda\left(\frac{\theta^2+1+\theta x}{\theta^2+1}\right)e^{-\theta x}\right]} \\
&= \frac{\theta^2(\theta+x)\left(1-\lambda+2\lambda\frac{\theta^2+1+\theta x}{\theta^2+1}e^{-\theta x}\right)}{(\theta^2+1+\theta x)\left[1-\lambda-\lambda\left(\frac{\theta^2+1+\theta x}{\theta^2+1}\right)e^{-\theta x}\right]} \\
&= \frac{\theta^2(\theta+x)\Big((1-\lambda)(\theta^2+1)+2\lambda(\theta^2+1+\theta x)e^{-\theta x}\Big)}{(\theta^2+1+\theta x)\Big[(1-\lambda)(\theta^2+1)-\lambda(\theta^2+1+\theta x)e^{-\theta x}\Big]}. \qquad (8)
\end{aligned}
$$

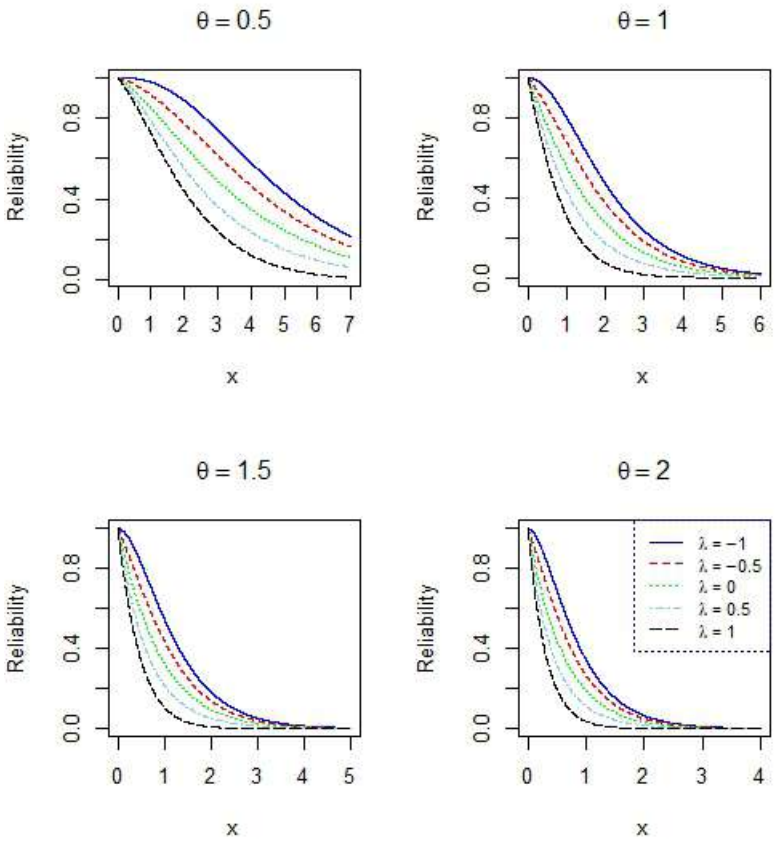


a) The pdf of the TSHD for different values of λ and θ.

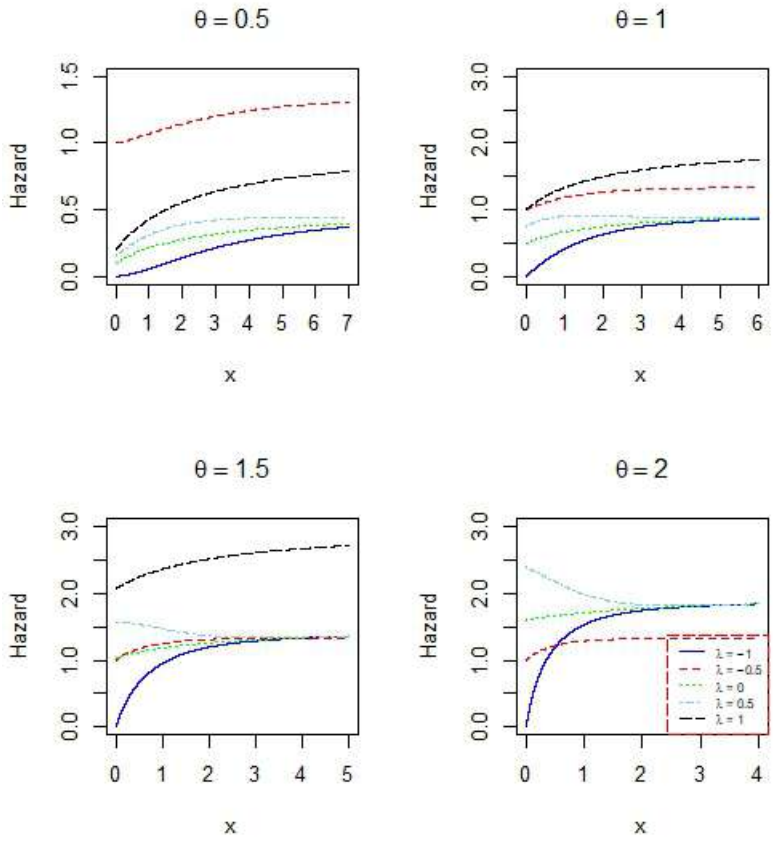


b) The cdf of the TSHD for different values of λ and θ.

Figures a and b show the plots of the *pdf* and CDF of TSHD for values of $\theta = 0.5, 1, 1.5$ and 2 and values of $\lambda = -1, -0.5, 0, 0.5$ and 1.



c) The reliability of the TSHD for different values of λ and θ.



d) The hazard rate function of the TSHD for different values of λ and θ.

Figures c and d show the plots of the reliability and hazard rate functions of TSHD for values of $\theta = 0.5, 1, 1.5$ and 2 and values of $\lambda = -1, -0.5, 0, 0.5$ and 1.

4. Properties of the TSHD

4.1 Moments

Theorem 4.1. *Let X be a TSHD random variable, then the rth moment is defined to be*

$$E(X^r) = \frac{(1-\lambda)\Gamma(r+1)}{\theta^{r-3}(\theta^2+1)} + \frac{(1-\lambda)\Gamma(r+2)}{\theta^{r-1}(\theta^2+1)} + \frac{\lambda\Gamma(r+1)}{2^{r-1}\theta^{r-2}(\theta^2+1)} + \frac{\lambda(2\theta^2+1)\Gamma(r+2)}{2^r\theta^{r-1}(\theta^2+1)} + \frac{\lambda\Gamma(r+3)}{2^{r+1}\theta^{r-1}(\theta^2+1)} \tag{9}$$

Proof.

$$\begin{aligned}
E(X^r) &= \int_0^\infty x^r g(x)dx, \\
&= \int_0^\infty x^r \left[\frac{(1-\lambda)\theta^3}{\theta^2+1}e^{-\theta x} + \frac{(1-\lambda)\theta^2}{\theta^2+1}xe^{-\theta x} + \frac{2\lambda\theta^3}{\theta^2+1}e^{-2\theta x} \right. \\
&\left. + \frac{2\lambda\theta^2(2\theta^2+1)}{\theta^2+1}xe^{-2\theta x} + \frac{2\lambda\theta^3}{\theta^2+1}x^2e^{-2\theta x} \right] dx \\
&= \frac{(1-\lambda)\theta^3}{\theta^2+1}\int_0^\infty x^r e^{-\theta x}dx \\
&+ \frac{(1-\lambda)\theta^2}{\theta^2+1}\int_0^\infty x^{r+1}e^{-\theta x}dx + \frac{2\lambda\theta^3}{\theta^2+1}\int_0^\infty e^{-2\theta x}dx \\
&+ \frac{2\lambda\theta^2(2\theta^2+1)}{\theta^2+1}\int_0^\infty x^{r+1}e^{-2\theta x}dx + \frac{2\lambda\theta^3}{\theta^2+1}\int_0^\infty x^{r+2}e^{-2\theta x}dx \\
&= \frac{(1-\lambda)\theta^3}{\theta^2+1}\frac{\Gamma(r+1)}{\theta^r} + \frac{(1-\lambda)\theta^2}{\theta^2+1}\frac{\Gamma(r+2)}{\theta^{r+1}} + \frac{2\lambda\theta^3}{\theta^2+1}\frac{\Gamma(r+1)}{(2\theta)^r} \\
&+ \frac{2\lambda\theta^2(2\theta^2+1)}{\theta^2+1}\frac{\Gamma(r+2)}{(2\theta)^{r+1}} + \frac{2\lambda\theta^3}{\theta^2+1}\frac{\Gamma(r+3)}{(2\theta)^{r+2}} \\
&= \frac{(1-\lambda)\Gamma(r+1)}{\theta^{r-3}(\theta^2+1)} + \frac{(1-\lambda)\Gamma(r+2)}{\theta^{r-1}(\theta^2+1)} + \frac{\lambda\Gamma(r+1)}{2^{r-1}\theta^{r-2}(\theta^2+1)} \\
&+ \frac{\lambda(2\theta^2+1)\Gamma(r+2)}{2^r\theta^{r-1}(\theta^2+1)} + \frac{\lambda\Gamma(r+3)}{2^{r+1}\theta^{r-1}(\theta^2+1)}.
\end{aligned}$$

□

Therefore, the first four moments can be calculated by putting $r = 1, 2, 3$ and 4 in Equation (9). Thus

$$\begin{aligned}
E(X) &= \frac{(1-\lambda)\Gamma(2)}{\theta^{-2}(\theta^2+1)} + \frac{(1-\lambda)\Gamma(3)}{(\theta^2+1)} + \frac{\lambda\Gamma(2)}{\theta^{-1}(\theta^2+1)} \\
&+ \frac{\lambda(2\theta^2+1)\Gamma(3)}{2(\theta^2+1)} + \frac{\lambda\Gamma(4)}{2^2(\theta^2+1)} \\
&= \frac{1}{(\theta^2+1)}\left[(1-\lambda)\theta^2 + 2(1-\lambda) + \lambda\theta + \lambda(2\theta^2+1) + \frac{3}{2}\lambda \right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{(\theta^2+1)}\left[(1+\lambda)\theta^2+\lambda\theta+\frac{\lambda}{2}+2\right],\\
E(X^2) &= \frac{(1-\lambda)\Gamma(3)}{\theta^{-1}(\theta^2+1)}+\frac{(1-\lambda)\Gamma(4)}{\theta(\theta^2+1)}+\frac{\lambda\Gamma(3)}{2(\theta^2+1)}\\
&+\frac{\lambda(2\theta^2+1)\Gamma(4)}{2^2\theta(\theta^2+1)}+\frac{\lambda\Gamma(5)}{2^3\theta(\theta^2+1)}\\
&= \frac{1}{(\theta^2+1)}\left[2\theta(1-\lambda)+\frac{6(1-\lambda)}{\theta}+\lambda+\frac{3\lambda(2\theta^2+1)}{2\theta}+\frac{3\lambda}{\theta}\right]\\
&= \frac{1}{(\theta^2+1)}\left[2\theta(1-\lambda)+\lambda+\frac{6\lambda\theta^2-3\lambda+12}{2\theta}\right]\\
&= \frac{1}{(\theta^2+1)}\left[(2+\lambda)\theta+\lambda+\frac{12-3\lambda}{2\theta}\right],\\
E(X^3) &= \frac{(1-\lambda)\Gamma(4)}{(\theta^2+1)}+\frac{(1-\lambda)\Gamma(5)}{\theta^2(\theta^2+1)}+\frac{\lambda\Gamma(4)}{2^2\theta(\theta^2+1)}\\
&+\frac{\lambda(2\theta^2+1)\Gamma(5)}{2^3\theta^2(\theta^2+1)}+\frac{\lambda\Gamma(6)}{2^4\theta^2(\theta^2+1)}\\
&= \frac{1}{(\theta^2+1)}\left[6(1-\lambda)+\frac{3\lambda}{2\theta}+\frac{24(1-\lambda)}{\theta^2}+\frac{3\lambda(2\theta^2+1)}{\theta^2}+\frac{15\lambda}{2\theta^2}\right],\\
E(X^4) &= \frac{(1-\lambda)\Gamma(5)}{\theta(\theta^2+1)}+\frac{120(1-\lambda)}{\theta^3(\theta^2+1)}+\frac{24\lambda}{8\theta^2(\theta^2+1)}\\
&+\frac{120\lambda(2\theta^2+1)}{16\theta^3(\theta^2+1)}+\frac{720\lambda}{32\theta^3(\theta^2+1)}\\
&= \frac{1}{(\theta^2+1)}\left[\frac{24(1-\lambda)}{\theta}+\frac{24\lambda}{\theta^2}+\frac{15\lambda\theta^2-90\lambda+120}{\theta^3}\right]\\
&= \frac{1}{\theta^3(\theta^2+1)}\left[3\theta^2(8-3\lambda)+24\lambda\theta+30(4-3\lambda)\right].
\end{aligned}
$$

Therefore, the variance of TSHD random variable is defined to be:

$$
\begin{aligned}
\sigma^2 = Var(X) &= E(X^2)-(E(X))^2\\
&= \frac{1}{(\theta^2+1)}\left[(2+\lambda)\theta+\lambda+\frac{12-3\lambda}{2\theta}\right]\\
&- \left[\frac{1}{(\theta^2+1)}\left[(1+\lambda)\theta^2+\lambda\theta+\frac{\lambda}{2}+2\right]\right]^2
\end{aligned}
$$

$$= \frac{-1}{4\theta(\theta^2+1)^2}\Bigg[4(\lambda+1)^2\theta^5 + 4(2\lambda^2+\lambda-2)\theta^4 + 8(\lambda^2+2\lambda+2)\theta^3$$

$$+ 2(2\lambda^2+9\lambda-16)\theta^2 + (\lambda^2+4\lambda+16)\theta + 8(2\lambda-3)\Bigg]. \tag{10}$$

The coefficient of variation (cv) is defined to be the ratio of standard deviation of the random variable to its expected value, that is $cv = \frac{\sqrt{var(X)}}{E(X)} = \frac{\sigma}{\mu}$. Therefore,

$$cv = \frac{\left[\frac{-1}{\theta}\left[4(\lambda+1)^2\theta^5 + 4(2\lambda^2+\lambda-2)\theta^4 + 8(\lambda^2+2\lambda+2)\theta^3 \right.\right.}{\left.\left. + 2(2\lambda^2+9\lambda-16)\theta^2 + (\lambda^2+4\lambda+16)\theta + 8(2\lambda-3)\right]\right]^{\frac{1}{2}}}{\Big[2(1+\lambda)\theta^2 + 2\lambda\theta + \lambda + 4\Big]}.$$

Skewness is a measure of symmetry, or more precisely, the lack of symmetry. A distribution, or data set, is symmetric if it looks the same to the left and right of the center point. It is defined as

$$Sk(X) = \frac{E(X^3) - 3E(X^2)E(X) + 2(E(X))^3}{\sigma^3}.$$

Thus

$$sk(X) = \frac{\left[\begin{array}{l} \frac{1}{(\theta^2+1)}\left[6(1-\lambda) + \frac{3\lambda}{2\theta} + \frac{24(1-\lambda)}{\theta^2} + \frac{3\lambda(2\theta^2+1)}{\theta^2} + \frac{15\lambda}{2\theta^2}\right] \\ \frac{-3}{(\theta^2+1)^2}\left[(2+\lambda)\theta + \lambda + \frac{12-3\lambda}{2\theta}\right]\left[(1+\lambda)\theta^2 + \lambda\theta + \frac{\lambda}{2} + 2\right] \\ + \frac{2}{(\theta^2+1)^3}\left[(1+\lambda)\theta^2 + \lambda\theta + \frac{\lambda}{2} + 2\right]^3 \end{array}\right]}{\left[\begin{array}{l} \frac{-1}{4\theta(\theta^2+1)^2}\left[4(\lambda+1)^2\theta^5 + 4(2\lambda^2+\lambda-2)\theta^4 + 8(\lambda^2+2\lambda+2)\theta^3 \right. \\ \left. + 2(2\lambda^2+9\lambda-16)\theta^2 + (\lambda^2+4\lambda+16)\theta + 8(2\lambda-3)\right] \end{array}\right]^{3/2}}.$$

Kurtosis is a measure of whether the data are heavy-tailed or not. That is, data sets with high kurtosis tend to have heavy tails, or outliers. Data sets with low kurtosis tend to have light tails, or lack of outliers. Kurtosis is defined as

$$Kur(X) = \frac{E(X^4) - 4E(X^3)(E(X)) + 6(E(X))^2(Var(X)) + 3(E(X))^4}{(Var(X))^2},$$

$$Kur(X) = \frac{\left[\begin{array}{l} \frac{1}{\theta^3(\theta^2+1)}\left[3\theta^2(8-3\lambda)+24\lambda\theta+30(4-3\lambda)\right] \\ -\frac{4}{(\theta^2+1)^2}\left[6(1-\lambda)+\frac{3\lambda}{2\theta}+\frac{24(1-\lambda)}{\theta^2}+\frac{3\lambda(2\theta^2+1)}{\theta^2}+\frac{15\lambda}{2\theta^2}\right]\left[(1+\lambda)\theta^2+\lambda\theta+\frac{\lambda}{2}+2\right] \\ -\frac{3}{2\theta(\theta^2+1)^3}\left[(2+\lambda)\theta+\lambda+\frac{12-3\lambda}{2\theta}\right]\left[4(\lambda+1)^2\theta^5+4(2\lambda^2+\lambda-2)\theta^4\right. \\ \left.+8(\lambda^2+2\lambda+2)\theta^3+2(2\lambda^2+9\lambda-16)\theta^2+(\lambda^2+4\lambda+16)\theta+8(2\lambda-3)\right] \\ \times\left[4(\lambda+1)^2\theta^5+4(2\lambda^2+\lambda-2)\theta^4+8(\lambda^2+2\lambda+2)\theta^3+2(2\lambda^2+9\lambda-16)\theta^2\right. \\ \left.+(\lambda^2+4\lambda+16)\theta+8(2\lambda-3)\right]+\frac{3}{(\theta^2+1)^4}\left[(1+\lambda)\theta^2+\lambda\theta+\frac{\lambda}{2}+2\right]^4 \end{array}\right]}{\left[\begin{array}{l} \frac{-1}{4\theta(\theta^2+1)^2}\left[4(\lambda+1)^2\theta^5+4(2\lambda^2+\lambda-2)\theta^4+8(\lambda^2+2\lambda+2)\theta^3\right. \\ \left.+2(2\lambda^2+9\lambda-16)\theta^2+(\lambda^2+4\lambda+16)\theta+8(2\lambda-3)\right] \end{array}\right]^2}$$

Table 1: The mean, standard deviation, skewness, kurtosis and the coefficient of variation of the TSHD for different values of λ when $\theta = 0.5$, 1, 1.5 and 2

λ	θ	μ	σ_x	Sk	Kur	CV (%)	θ	μ	σ_x	Sk	Kur	CV (%)
-1.0	0.5	5.080	2.945	1.221	5.384	57.974	1.5	1.292	0.901	1.425	6.125	69.760
-0.5	0.5	4.340	2.967	1.243	5.380	68.369	1.5	1.082	0.888	1.513	6.401	82.062
0.0	0.5	3.600	2.800	1.452	6.121	77.778	1.5	0.872	0.822	1.779	7.593	94.301
0.5	0.5	2.860	2.404	1.768	7.887	84.065	1.5	0.662	0.689	2.201	10.351	104.186
1.0	0.5	2.120	1.620	1.338	5.559	76.433	1.5	0.452	0.432	1.807	7.711	95.617
-1.0	1.0	2.188	1.424	1.313	5.682	65.090	2.0	0.895	0.645	1.502	6.482	72.106
-0.5	1.0	1.844	1.417	1.376	5.826	76.834	2.0	0.748	0.632	1.600	6.831	84.609
0.0	1.0	1.500	1.323	1.620	6.796	88.192	2.0	0.600	0.583	1.876	8.159	97.183
0.5	1.0	1.156	1.121	1.997	9.057	96.959	2.0	0.453	0.486	2.321	11.231	107.471
1.0	1.0	0.812	0.726	1.608	6.650	89.377	2.0	0.305	0.299	1.906	8.329	98.073

Table 1 shows that the values of the mean are decreasing as the values of the distribution parameters are increasing. It also shows that the standard deviation values are increasing in general. TSHD is positively skewed as all values of the skewness are positive. The diffusional kurtosis is a quantitative measure of the degree to which the diffusion displacement probability distribution deviates from a Gaussian form (kurtosis = 3). As all kurtosis values are greater than 3, this tells us that TSHD is a heavily tailed distribution.

4.2 Moment generating function

Theorem 4.2. *The moment generating function (MGF) of the TSHD random variable is given by*

$$M_X(t) = \frac{\theta^2}{\theta^2+1}\left[\frac{(1-\lambda)\theta}{(t-\theta)} + \frac{(1-\lambda)}{(t-\theta)^2} + \frac{2\lambda\theta}{(t-2\theta)} + \frac{2\lambda(2\theta^2+1)}{(t-2\theta)^2} + \frac{4\lambda\theta}{(t-2\theta)^3}\right]. \tag{11}$$

Proof.

$$\begin{aligned}
M_X(t) &= E(e^{tX}) \\
&= \int_0^\infty e^{tx} g(x)dx \\
&= \int_0^\infty e^{tx}\left[\frac{(1-\lambda)\theta^3}{\theta^2+1}e^{-\theta x} + \frac{(1-\lambda)\theta^2}{\theta^2+1}xe^{-\theta x} + \frac{2\lambda\theta^3}{\theta^2+1}e^{-2\theta x}\right. \\
&\left. + \frac{2\lambda\theta^2(2\theta^2+1)}{\theta^2+1}xe^{-2\theta x} + \frac{2\lambda\theta^3}{\theta^2+1}x^2e^{-2\theta x}\right]dx \\
&= \int_0^\infty \left[\frac{(1-\lambda)\theta^3}{\theta^2+1}e^{-(\theta-t)x} + \frac{(1-\lambda)\theta^2}{\theta^2+1}xe^{-(\theta-t)x} + \frac{2\lambda\theta^3}{\theta^2+1}e^{-(2\theta-t)x}\right. \\
&\left. + \frac{2\lambda\theta^2(2\theta^2+1)}{\theta^2+1}xe^{-(2\theta-t)x} + \frac{2\lambda\theta^3}{\theta^2+1}x^2e^{-(2\theta-t)x}\right]dx \\
&= \frac{\theta^2}{\theta^2+1}\left[\frac{(1-\lambda)\theta}{(t-\theta)} + \frac{(1-\lambda)}{(t-\theta)^2} + \frac{2\lambda\theta}{(t-2\theta)} + \frac{2\lambda(2\theta^2+1)}{(t-2\theta)^2} + \frac{4\lambda\theta}{(t-2\theta)^3}\right]. \square
\end{aligned}$$

5. Maximum likelihood estimates

Definition 5.1. *Let $X_1, X_2, ..., X_n$ be a random sample size n with a pdf $g(x)$. The likelihood function is defined as the joint density of the random sample, which is defined as*

$$\ell = L(\lambda, \theta | x_1, x_2, ..., x_n) = \prod_{i=1}^{n} g(x_i|\lambda, \theta).$$

Hence, the likelihood function is given by

$$L = \prod_{i=1}^{n}\left(\left(\frac{\theta^2(\theta+x_i)}{\theta^2+1}e^{-\theta x_i}\right)\left(1-\lambda+2\lambda\frac{\theta^2+1+\theta x_i}{\theta^2+1}e^{-\theta x_i}\right)\right)$$

$$= \prod_{i=1}^{n} \Big(\frac{\theta^2(\theta + x_i)}{\theta^2 + 1} e^{-\theta x_i} \Big) \prod_{i=1}^{n} \Big(1 - \lambda + 2\lambda \frac{\theta^2 + 1 + \theta x_i}{\theta^2 + 1} e^{-\theta x_i} \Big),$$

$$L = \prod_{i=1}^{n} \Big(\frac{\theta^2}{\theta^2 + 1} \Big) \prod_{i=1}^{n} (\theta + x_i)$$

$$\times e^{-\sum_{i=1}^{n} \theta x_i} \prod_{i=1}^{n} \Big(1 - \lambda + 2\lambda \frac{\theta^2 + 1 + \theta x_i}{\theta^2 + 1} e^{-\theta x_i} \Big). \tag{12}$$

Therefore, the log-likelihood function is given by:

$$\ell = lnL = ln \Bigg(\prod_{i=1}^{n} \Big(\frac{\theta^2}{\theta^2 + 1} \Big) \times \prod_{i=1}^{n} (\theta + x_i) \times e^{-\sum_{i=1}^{n} \theta x_i}$$

$$\times \prod_{i=1}^{n} \Big(1 - \lambda + 2\lambda \Big(\frac{\theta^2 + 1 + \theta x_i}{\theta^2 + 1} \Big) e^{-\theta x_i} \Big) \Bigg),$$

$$\ell = 2nln\theta - nln(\theta^2 + 1) + \sum_{i=1}^{n} ln(\theta + x_i) - \theta \sum_{i=1}^{n} x_i$$

$$+ \sum_{i=1}^{n} ln \Bigg[1 - \lambda + 2\lambda \Big(\frac{\theta^2 + 1 + \theta x_i}{\theta^2 + 1} \Big) e^{-\theta x_i} \Bigg]. \tag{13}$$

Deriving with respect to the parameters, we get

$$\frac{\partial \ell}{\partial \lambda} = \sum_{i=0}^{n} \frac{(-1 + 2(\theta^2 + 1 + \theta x_i) e^{-\theta x_i}}{(1 - \lambda)(\theta^2 + 1) + 2\lambda(\theta^2 + 1 + \theta x_i) e^{-\theta x_i}} \Bigg],$$

$$\frac{\partial \ell}{\partial \theta} = \frac{2n}{\theta(\theta^2 + 1)} + \sum_{i=0}^{n} \Bigg[\frac{1}{\theta + x_i} \Bigg] - \sum_{i=1}^{n} x_i$$

$$- \sum_{i=1}^{n} \frac{2\lambda x_i(\theta^3 + \theta^2 x_i + 3\theta + x_i)}{(\theta^2 + 1) \Big[(1 - \lambda)(\theta^2 + 1) e^{\theta x_i} + 2\lambda(\theta^2 + 1 + \theta x_i) \Big]}.$$

Estimates of distribution parameters can be found by equating the derivatives of the log-likelihood function in (13) to zero. Thus,

$$\sum_{i=0}^{n} \frac{(-1 + 2(\theta^2 + 1 + \theta x_i) e^{-\theta x_i}}{(1 - \lambda)(\theta^2 + 1) + 2\lambda(\theta^2 + 1 + \theta x_i) e^{-\theta x_i}} \Bigg] = 0,$$

$$\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \frac{2\lambda x_i(\theta^3 + \theta^2 x_i + 3\theta + x_i)}{(\theta^2 + 1) \Big[(1 - \lambda)(\theta^2 + 1) e^{\theta x_i} + 2\lambda(\theta^2 + 1 + \theta x_i) \Big]} - \sum_{i=0}^{n} \Bigg[\frac{1}{\theta + x_i} \Bigg]$$

$$= \frac{2n}{\theta(\theta^2 + 1)}.$$

There is no exact solutions to this non-linear system of equations, but we can solve it numerically.

6. Quantile function

Consider a random variable X with a continuous cumulative distribution function $G(x)$. If $G(x)$ is a strictly increasing function, then the quantile X_q is the value satisfies $q = G^{-1}(x_q)$. Therefore,

$$\begin{aligned}(1+\lambda)F(x_q) - \lambda\big(F(x_q)\big)^2 &= q,\\ \lambda\big(F(x_q)\big)^2 - (1+\lambda)F(x_q) - q &= 0.\end{aligned}$$

Solving this equation with respect to, x

$$\begin{aligned}F(x_q) &= \frac{(1+\lambda) \pm \sqrt{(1+\lambda)^2 - 4\lambda q}}{2\lambda},\\ 1 - \frac{\theta^2+1+\theta x_q}{\theta^2+1}e^{-\theta x_q} &= \frac{(1+\lambda) \pm \sqrt{(1+\lambda)^2 - 4\lambda q}}{2\lambda},\\ \frac{\theta^2+1+\theta x_q}{\theta^2+1}e^{-\theta x_q} &= 1 - \frac{(1+\lambda) \pm \sqrt{(1+\lambda)^2 - 4\lambda q}}{2\lambda},\\ (\theta^2+1+\theta x_q)e^{-\theta x_q} &= \left[1 - \frac{(1+\lambda) \pm \sqrt{(1+\lambda)^2 - 4\lambda q}}{2\lambda}\right](\theta^2+1).\end{aligned}$$

This nonlinear equation has no exact solution, but it can be solved numerically.

7. Rényi entropy

Theorem 7.1. *The Rényi entropy of $\beta \geq 0$ for TSHD is defined is*

$$E_R = \Big(\frac{1}{1-\beta}\Big) log \sum_{i=0}^{\beta}\sum_{k=0}^{\beta}\sum_{r=0}^{i}\binom{\beta}{i}\binom{\beta}{k}\binom{i}{r}\Big(\frac{(2\lambda)^i\theta^{3\beta-2k-1}(1-\lambda)^{\beta-i}\Gamma(k+r+1)}{(\theta^2+1)^{\beta+r}(\beta+i)^{k+r+1}}\Big).$$

Proof.

$$\begin{aligned}E_R &= \Big(\frac{1}{1-\beta}\Big)\log\int_0^\infty (g(x))^\beta dx\\ &= \Big(\frac{1}{1-\beta}\Big)\log\int_0^\infty \left[\Big(\frac{\theta^2(\theta+x)}{\theta^2+1}e^{-\theta x}\Big)\Big(2\lambda\frac{\theta^2+1+\theta x}{\theta^2+1}e^{-\theta x}+1-\lambda\Big)\right]^\beta dx\\ &= \Big(\frac{1}{1-\beta}\Big)\log\Big(\frac{\theta^2}{\theta^2+1}\Big)^\beta\int_0^\infty\sum_{i=0}^{\beta}\left[(\theta+x)^\beta e^{-\theta\beta x}\binom{\beta}{i}\right.\\ &\quad\left.\cdot\Big(2\lambda\frac{\theta^2+1+\theta x}{\theta^2+1}e^{-\theta x}\Big)^i(1-\lambda)^{\beta-i}\right]dx\end{aligned}$$

$$= \Big(\frac{1}{1-\beta}\Big) \log \sum_{i=0}^{\beta} \binom{\beta}{i} \Big(\frac{(2\lambda)^i (1-\lambda)^{\beta-i} \theta^{2\beta}}{(\theta^2+1)^{\beta+i}}\Big) \int_0^{\infty} (\theta+x)^{\beta} (\theta^2+1+\theta x)^i e^{-(\beta+i)\theta x} dx$$

$$= \Big(\frac{1}{1-\beta}\Big) \log \sum_{i=0}^{\beta} \sum_{k=0}^{\beta} \sum_{r=0}^{i} \binom{\beta}{i} \binom{\beta}{k} \binom{i}{r} \Big(\frac{(2\lambda)^i (1-\lambda)^{\beta-i} \theta^{2\beta}}{(\theta^2+1)^{\beta+i}}\Big) \theta^{\beta-k} (\theta)^r (\theta^2+1)^{i-r}$$

$$\times \int_0^{\infty} x^{k+r} e^{-(\beta+i)\theta x} dx$$

$$= \Big(\frac{1}{1-\beta}\Big) \log \sum_{i=0}^{\beta} \sum_{k=0}^{\beta} \sum_{r=0}^{i} \binom{\beta}{i} \binom{\beta}{k} \binom{i}{r} \Big(\frac{(2\lambda)^i \theta^{3\beta-2k-1} (1-\lambda)^{\beta-i} \Gamma(k+r+1)}{(\theta^2+1)^{\beta+r} (\beta+i)^{k+r+1}}\Big).$$

□

8. Applications

In this section, an application to real life data is presented to illustrate the usefulness of the proposed distribution in fitting such data. The data set represents the times between successive failures of air conditioning equipment in a Boeing 720 airplane and is given as: 74, 57, 48, 29, 502, 12, 70, 21, 29, 386, 59, 27, 153, 26, 326. (Data reference: [17])

The proposed distribution is used to fit this data in addition to the following distributions:

- Lindley distribution [16]:$f(x) = \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}; x > 0, \theta > 0.$
- Ishita distribution [22]: $f(x) = \frac{\theta^3}{\theta^3+2}(\theta + x^2)e^{-\theta x}; x > 0, \theta > 0.$
- Akash distribution [21]: $f(x) = \frac{\theta^3}{\theta^2+2}(1 + x^2)e^{-\theta x} \quad ; x > 0, \theta > 0.$
- Shanker distribution [20]: $f(x) = \frac{\theta^2}{\theta^2+1}(\theta + x)e^{-\theta x} \quad ; x > 0, \theta > 0.$

For all these fitted distributions, we compute the Cramer-von Mises Criterion (W), Anderson-Darling Criterion (A), Akaike Information Criterion (AIC), -2 log likelihood ($-2lnL$), Kolmogorov Smirnov (KS) statistic and its p-value and the results are reported in Table 2. It is clear from Table 2 that the proposed distribution has the smallest values of the $-2lnL$, W, A, AIC, KS-statistic and largest p-value among all fitted distributions. Consequently, the proposed distribution is more adequate to fit this real life data than other fitted distributions (Lindley, Ishita, Akash, Shanker).

Table 3 provides the maximum likelihood estimates (MLEs) of the parameters of the fitted distributions and their confidence intervals.

9. Conclusion

This paper proposed the two-parameter transmuted Shanker distribution. This distribution is a generalization of Shanker distribution. Its properties were stud-

Table 2: $-2lnL$, W, A, AIC, KS statistic and its p-value for fitted distributions.

Distribution	$-2lnL$	***W***	***A***	***AIC***	***KS-Statistic***	***p-value***
Lindley	181.3412	0.1769775	1.023697	183.3412	0.3858	0.02302
Ishita	194.3217	0.1738195	1.00636	196.3217	0.4557	0.003939
Akash	194.2997	0.1742572	1.00889	196.2997	0.4556	0.003946
Shanker	181.5784	0.1735927	1.005124	183.5784	0.388	0.02187
Transmuted Shanker	179.3349	0.1567872	0.920379	183.3349	0.3267	0.0814

Table 3: The MLEs of the parameters of the fitted distributions and their confidence intervals

Distribution	**Parameter**	**MLE**	**Standard error**	**95% Confidence Interval**	
				Lower Limit	**Upper Limit**
Lindley	θ	0.01638021	0.002979601	0.01054	0.02222
Ishita	θ	0.02473989	0.003681941	0.017523	0.031956
Akash	θ	0.02473481	0.003680836	0.01752	0.031949
Shanker	θ	0.01651167	0.003003094	0.010626	0.022398
Transmuted Shanker	θ	0.01491089	0.002985547	0.009059	0.020763
	λ	0.5779029	0.21318088	0.160068375	0.995737425

ied and investigated. Application to real data was presented and showed that the proposed TSHD outperforms Shanker distribution and some other considered distributions. For future work, it might be of interest to consider this distribution to adaptive methods suggested by [9, 8, 4].

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Harmonic meromorphic starlike functions of complex order involving Mittag-Leffler operator

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Abstract. The main goal of the present paper is to introduce the class $\breve{\mathcal{M}}\mathcal{T}_{\mathcal{H}}(\alpha,\gamma,d)$ of harmonic meromorphic starlike functions in order to define the Mittag-Leffler operator. Thus, we give some relations devoted to different classes of harmonic meromorphic functions to be a starlike function. Furthermore, we provide the fundamental and sufficient conditions for these considered functions to characterize some important subclasses of $\breve{\mathcal{M}}\mathcal{T}_{\mathcal{H}}(\alpha,\gamma,d)$.

Keywords: meromorphic univalent function, Mittag-Leffler operator, starlike function.

1. Introduction

A continuous function $f = u + iv$ is harmonic in the complex domain $F \subset \mathbb{C}$, where the real harmonic are u and v in F. Thus, we can express $f(z) = h(z) + \overline{g(z)}$, such that $h(z)$ and $g(z)$ are analytic in F. In addition, we say that $h(z)$ the analytic part and $g(z)$ the co-analytic part of f, see Clunie and Sheil-Small [5]. Then, as showing in [1]-[3] the class of harmonic functions

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$f(z) = h(z) + \overline{g(z)}$ are described by $\mathcal{H}$ in which have sense-preserving for the disk of type $U = \{z : |z| < 1\}$, where:

$$f(z) = h(z) + \overline{g(z)} = \frac{1}{z} + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^n, \ |b_1| < 1. \tag{1}$$

Assume that the family of functions $f(z) = h(z) + \overline{g(z)}$ denote by $\mathcal{MT}_{\mathcal{H}}$, that are harmonic in $\mathcal{U}$ with the normalization:

$$h(z) = \frac{1}{z} - \sum_{n=2}^{\infty} a_n z^n, g(z) = \sum_{n=1}^{\infty} b_n z^n, \ \ a_n \geq 0, b_n \geq 0, \ b_1 < 1. \tag{2}$$

Note that $\mathcal{H}$ lessens to $\mathcal{S}$, the normalized analytic functions class such that, the co-analytic part $g \equiv 0$.

Let $\breve{\mathcal{M}}\mathcal{T}_{\mathcal{H}}(\alpha, \gamma, d)$ denote the class of any functions in $\mathcal{MT}_{\mathcal{H}}$, for which:

$$\Re e \left\{ 1 + \frac{1}{d} \left(\frac{z \left(E_{\alpha,\gamma} h(z)\right)' - z \left(\overline{E_{\alpha,\gamma} g(z)}\right)'}{E_{\alpha,\gamma} h(z) + \overline{E_{\alpha,\gamma} g(z)}} - 1 \right) \right\} > 0 \tag{3}$$

with $d \in \mathbb{C} \backslash \{0\}$ (see, [9]).

Now, consider the operator $E_{\alpha,\gamma}$ of Mittag-Leffler type [15] described by $E_{\alpha,\gamma} f(z) = \frac{1}{z} + \sum_{n=2}^{\infty} D_n(\alpha, \gamma) a_n z^n$, where:

$$D_n(\alpha, \gamma) = \frac{\Gamma(\gamma)}{\Gamma(\alpha(n - 1 + \gamma)}, \ z, \alpha, \gamma \in \mathbb{C}; \ \Re e(\alpha) > 0; \ \gamma \neq 0, -1, -2, \ldots$$

and $\Gamma(z)$ is gamma function. Currently, we can define the Mittag-Leffler operator as:

$$E_{\alpha,\gamma} f(z) = E_{\alpha,\gamma} h(z) + \overline{E_{\alpha,\gamma} g(z)} \tag{4}$$

where:

$$E_{\alpha,\gamma} h(z) = \frac{1}{z} - \sum_{n=2}^{\infty} D_n(\alpha, \gamma) a_n z^n$$

and

$$E_{\alpha,\gamma} g(z) = \sum_{n=1}^{\infty} D_n(\alpha, \gamma) b_n \bar{z}^n.$$

We note that $\breve{\mathcal{M}}\mathcal{T}_{\mathcal{H}}$ is a harmonic function class in the unit disk studied such as in [8], [10].

Let $\breve{\mathcal{R}}\mathcal{T}_{\mathcal{H}}(\ \alpha, \gamma, d)$ indicates the subclass at $\mathcal{MT}_{\mathcal{H}}$ of function $f(z) = h(z) + \overline{g(z)} \in \mathcal{MT}_{\mathcal{H}}$, such that satisfied the condition:

$$\sum_{n=1}^{\infty} (2(n - 1 + |d|) D_n(\alpha, \gamma) a_n + (n + 1 + |n + 1 - 2d|) D_n(\alpha, \gamma) b_n) \leq 4|d|. \tag{5}$$

Also let $\check{\mathcal{S}}\mathcal{T}_{\mathcal{H}}(\alpha,\gamma,d)$ the subclass of $\mathcal{MT}_{\mathcal{H}}$ that verified the following:

$$\begin{aligned}&\sum_{n=1}^{\infty}\left[(n-1)\frac{\Re e(d)}{|d|}+|d|\right]D_n(\alpha,\gamma)a_n\\ &+\left[(n+1)\frac{\Re e(d)}{|d|}-|d|\right]D_n(\alpha,\gamma)b_n\leq 2|d|.\end{aligned}\tag{6}$$

Numerous researchers was investigated the class $\mathcal{H}$ of harmonic univalent functions in different situations [1], [5], [6], [10], [13], [14]. Thus, there were many related research papers are explored the class $\mathcal{H}$ associated with its subclass which is presented by Jahangiri and Noor et al. as in [6], [12] for meromorphic functions. So, Silverman and Jahangiri in 1999 are studied these results to the case of meromorphic functions through the negative coefficients [14]. More recently, Juma et al. 2021 contribute an important extension on bi-univalent functions is associated with the operator of Srivastava Attiya type, through the class of starlike or convex functions of order α [11]. Recently, it was proved that the sufficient coefficient conditions:

$$\sum_{n=2}^{\infty}n\left(|a_n|+|b_n|\right)\leq 1, b_1=0$$

is sufficient for $f(z)=h(z)+\overline{g(z)}$ to be harmonic univalent and starlike function [4]. Furthermore, the constant condition is also needed if $b_1=0$ and a_n, $b_n\leq 0$, in equation (1), and hence Jahangiri in [6] is shown that if $f(z)=h(z)+\overline{g(z)}$, then we have:

$$\sum_{n=2}^{\infty}\left(\frac{n-\lambda}{1-\lambda}|a_n|+\frac{n+\lambda}{1-\lambda}|b_n|\right)\leq 1,\tag{7}$$

where $0\leq\lambda<1$ and then f is harmonic univalent and starlike function in $\mathcal{U}$. Consequently, the condition (7) is necessary if $h(z)$ and $g(z)$ are of the form (1), for example $\lambda=0$ is and for $\lambda=b_1=0$, (see [6], [7], [14]).

2. The main results

In this section, the main important results are stated to give some relations devoted to different classes in the next outcomes.

Theorem 2.1. $\check{\mathcal{R}}\mathcal{T}_{\mathcal{H}}(\alpha,\gamma,d)\subset\check{\mathcal{M}}\mathcal{T}_{\mathcal{H}}(\alpha,\gamma,d)$.

Proof. Let $f\in\check{\mathcal{R}}\mathcal{T}_{\mathcal{H}}(\alpha,\gamma,d)$, and according to the equation (2) allows to show that condition (5) holds, and then:

$$\Re e\left\{\frac{(d-1)\left[E_{\alpha,\gamma}h(z)+\overline{E_{\alpha,\gamma}g(z)}+z\left(E_{\alpha,\gamma}h(z)\right)'-z\left(\overline{E_{\alpha,\gamma}g(z)}\right)'\right]}{d\left(E_{\alpha,\gamma}h(z)+\overline{E_{\alpha,\gamma}}g(z)\right)}\right\}>0,$$

where $d \in \mathbb{C}\backslash\{0\}$. Now by using the fact that:

$$\alpha \geq 0 \text{ and } \Re e(w) > \alpha \Leftrightarrow |1+w| > |1-w|, \ \forall w \in \mathbb{C}$$

then, we can get:

$$\begin{aligned}
&\left|(2d-1)\left[E_{\alpha,\gamma}h(z)+\overline{E_{\alpha,\gamma}g(z)}\right]+z\left(E_{\alpha,\gamma}h(z)\right)'-z\left(\overline{E_{\alpha,\gamma}g(z)}\right)'\right| \\
&-\left|E_{\alpha,\gamma}h(z)+\overline{E_{\alpha,\gamma}g(z)}-z\left(E_{\alpha,\gamma}h(z)\right)'-z\left(\overline{E_{\alpha,\gamma}g(z)}\right)'\right| \\
&=(2d-1)\left[\frac{1}{z}-\sum_{n=2}^{\infty}D_n(\alpha,\gamma)a_nz^n+\sum_{n=1}^{\infty}D_n(\alpha,\gamma)b_n\bar{z}^n\right]-\frac{1}{z} \\
&+\left|\sum_{n=2}^{\infty}D_n(\alpha,\gamma)na_nz^n+\sum_{n=1}^{\infty}D_n(\alpha,\gamma)nb_n\bar{z}^n\right| \\
&-\left|\frac{1}{z}-\sum_{n=2}^{\infty}D_n(\alpha,\gamma)na_nz^n+\sum_{n=1}^{\infty}D_n(\alpha,\gamma)nb_n\bar{z}^n\right. \\
&\left.-\frac{1}{z}+\sum_{n=2}^{\infty}D_n(\alpha,\gamma)na_n\bar{z}^n+\sum_{n=1}^{\infty}D_n(\alpha,\gamma)nb_n\bar{z}^n\right| \\
&\left|2dz^{-1}-\sum_{n=2}^{\infty}(2d-1+n)D_n(\alpha,\gamma)a_nz^n\right. \\
&\left.-\sum_{n=1}^{\infty}(n+1-2d)D_n(\alpha,\gamma)b_n\bar{z}^n\right|-\left|\sum_{n=2}^{\infty}(n-1)D_n(\alpha,\gamma)a_nz^n\right. \\
&\left.-\sum_{n=1}^{\infty}(n+1)D_n(\alpha,\gamma)b_n\bar{z}^n\right| \\
&\geq 2|d||z|^{-1}-\sum_{n=2}^{\infty}2(n-1+|d|)D_n(\alpha,\gamma)a_n|z|^n \\
&-\sum_{n=1}^{\infty}(n+1+|n+1-2d|)D_n(\alpha,\gamma)b_n|\bar{z}|^n2d \\
&-\left(\sum_{n=2}^{\infty}2(n-1+|d|)D_n(\alpha,\gamma)a_n+\right. \\
&\left.\sum_{n=1}^{\infty}(n+1+\mid n+1-2d\mid)D_n(\alpha,\gamma)b_n\right)\geq 0.
\end{aligned}$$

Consequently, the function:

$$f(z)=\frac{1}{z}-\sum_{n=2}^{\infty}\frac{|d|}{(n-1+|d|)}l_nz^n+\frac{2|d|}{n+1+\mid n+1-2d}t_n\bar{z}^n, \tag{8}$$

where $l_n, t_n \geq 0$. Then, in this case:

$$\sum_{n=2}^{\infty} l_n + \sum_{n=1}^{\infty} t_n = 1$$

and hence the functions $f(z)$ in (8) belong to $\breve{\mathcal{M}}\mathcal{T}_{\mathcal{H}}(\alpha, \gamma, d)$, since:

$$\sum_{n=2}^{\infty} 2(n-1+|d|)n(\alpha,\gamma)a_n + \sum_{n=1}^{\infty}(n+1+|n+1-2d|)D_n(\alpha,\gamma)b_n$$
$$= 2|d|\left(\sum_{n=2}^{\infty} l_n + \sum_{n=1}^{\infty} t_n\right) = 4|d|. \qquad \square$$

Next, we show that $\breve{\mathcal{M}}\mathcal{T}_{\mathcal{H}}(\alpha, \gamma, d) \subset \breve{\mathcal{S}}\mathcal{T}_{\mathcal{H}}(\alpha, \gamma, d)$.

Theorem 2.2. $\breve{\mathcal{M}}\mathcal{T}_{\mathcal{H}}(\alpha, \gamma, d) \subset \breve{\mathcal{S}}\mathcal{T}_{\mathcal{H}}(\alpha, \gamma, d)$.

Proof. Assume that $f \in \breve{\mathcal{M}}\mathcal{T}_{\mathcal{H}}(\alpha, \gamma, d)$, from (3), we get:

$$\Re e\left\{\frac{1}{d}\left(\frac{z\left(E_{\alpha,\gamma}h(z)\right)'-z\left(\overline{E_{\alpha,\gamma}g(z)}\right)'}{E_{\alpha,\gamma}h(z)+\overline{\left(E_{\alpha,\gamma}g(z)\right)}}\right)\right\} \geq -1,$$

$$\Re e\left\{\frac{1}{d}\left(\frac{\frac{-1}{z}-\sum_{n=2}^{\infty}D_n(\alpha,\gamma)na_nz^n+\sum_{n=1}^{\infty}D_n(\alpha,\gamma)nb_n\bar{z}^n}{\frac{1}{z}-\sum_{n=2}^{\infty}D_n(\alpha,\gamma)a_nz^n+\sum_{n=1}^{\infty}D_n(\alpha,\gamma)b_n\bar{z}^n}\right)\right\} > -1$$

therefore, we get:

$$\Re e\left\{\frac{1}{d}\left(\frac{-\sum_{n=2}^{\infty}(n-1)D_n(\alpha,\gamma)a_nz^n-\sum_{n=1}^{\infty}(n+1)D_n(\alpha,\gamma)b_n\bar{z}^n}{\frac{1}{z}-\sum_{n=2}^{\infty}D_n(\alpha,\gamma)a_nz^n+\sum_{n=1}^{\infty}D_n(\alpha,\gamma)b_n\bar{z}^n}-1\right)\right\} > -1.$$

Taking $z \to 1^-$ on the real axis, we get:

$$\frac{\sum_{n=2}^{\infty}(n-1)D_n(\alpha,\gamma)a_n-\sum_{n=1}^{\infty}(n+1)D_n(\alpha,\gamma)b_n}{1-\sum_{n=2}^{\infty}D_n(\alpha,\gamma)a_n+\sum_{n=1}^{\infty}D_n(\alpha,\gamma)b_n}\Re e\left(\frac{1}{d}\right) \leq 1,$$

then:

$$\sum_{n=2}^{\infty}(n-1)D_n(\alpha,\gamma)a_n - \sum_{n=1}^{\infty}(n+1)D_n(\alpha,\gamma)b_n$$
$$\leq \frac{|d|^2}{\Re e(d)}\left(1-\sum_{n=2}^{\infty}D_n(\alpha,\gamma)a_n+\sum_{n=1}^{\infty}D_n(\alpha,\gamma)b_n\right)$$

and so:

$$\sum_{n=1}^{\infty}\left[(n-1)\frac{\Re e(d)}{|d|}+|d|\right]D_n(\alpha,\gamma)a_n+\left[(n+1)\frac{\Re e(d)}{|d|}-|d|\right]D_n(\alpha,\gamma)b_n \leq 2|d|.$$

Then, by (7), we get $f \in \breve{\mathcal{S}}\mathcal{T}_{\mathcal{H}}(\alpha, \gamma, d)$. $\square$

Now, we demonstrate that the subclass $\breve{\mathcal{RT}}_{\mathcal{H}}(\alpha,\gamma,d)$ is equal to subclass $\breve{\mathcal{ST}}_{\mathcal{H}}(\alpha,\gamma,d)$.

Theorem 2.3. $\mathcal{R}\breve{\mathcal{T}}_{\mathcal{H}}(\alpha,\gamma,d) = \breve{\mathcal{MT}}_{H}(\alpha,\gamma,d) = \breve{\mathcal{ST}}_{\mathcal{H}}(\alpha,\gamma,d)$ *where*

$$(0 < d \leq 1, z, \alpha, \gamma \in \mathbb{C}, \Re e(\alpha) > 0; \gamma \neq 0, -1, -2, \ldots).$$

Proof. If $d \in (0,1]$, then the inequalities (5) and (6) are equivalent. Thus, by using Theorems 2.1 and 2.2, we getting $\breve{\mathcal{RT}}_{\mathcal{H}}(\alpha,\gamma,d) = \breve{\mathcal{ST}}_{\mathcal{H}}(\alpha,\gamma,d)$. □

The following outcome proves $\breve{\mathcal{ST}}_{\mathcal{H}}(\alpha,\gamma,d) \nsubseteq \breve{\mathcal{MT}}_{\mathcal{H}}(\alpha,\gamma,d)$.

Theorem 2.4. $\breve{\mathcal{ST}}_{\mathcal{H}}(\alpha,\gamma,d) \nsubseteq \breve{\mathcal{MT}}_{\mathcal{H}}(\alpha,\gamma,d)$ *if* $\Re e(d) \leq 0$ *and* $\Re e(d) \neq -\frac{1}{2}$ *otherwise* $d \in \left(\frac{3}{2}, \infty\right)$.

Proof. Assume that $f(z) = z - \frac{1}{\alpha+1}z^2, \alpha > -1, f \in \breve{\mathcal{ST}}_{\mathcal{H}}(\alpha,\gamma,d)$. Since:

$$\sum_{n=1}^{\infty}\left[(n-1)\frac{\Re e(d)}{|d|} + |d|\right] D_n(\alpha,\gamma)a_n + \left[(n+1)\frac{\Re e(d)}{|d|} - |d|\right] D_n(\alpha,\gamma)b_n$$
$$= |d|.1 + \frac{\Re e(d)}{|d|} + |d| = 2|d| + \frac{\Re e(d)}{|d|} \leq 2|d|,$$

when $d \in \mathbb{C}\backslash\{0\}$ and $\Re e(d) < 0$. Moreover, if we suppose:

$$r = \Re e(d) < 0, k \geq 0, \ni 1 + 2r(1-k) > 0.$$

Now, in this situation, if we choose $z = \frac{d(1-\hbar)}{1+d(1-k)}$, then $z \in U$.

Since $E_{\alpha,\gamma}f(z) = z - z^2$, we get:

$$1 + \frac{1}{d}\left(\frac{z\left(E_{\alpha,\gamma}h(z)\right)' - z\left(\overline{E_{\alpha,\gamma}g(z)}\right)'}{E_{\alpha,\gamma}h(z) + \overline{\left(E_{\alpha,\gamma}g(z)\right)}} - 1\right) = k < 0$$

and hence $f \notin \breve{\mathcal{MT}}_{\mathcal{H}}(\alpha,\gamma,d)$. Now, let:

$$f(z) = z - \frac{1}{\alpha+1}\bar{z}^2.$$

Then $f \in \breve{\mathcal{ST}}_{\mathcal{H}}(\alpha,\gamma,d)$, where $d \in \left(\frac{3}{2}, +\infty\right)$, because:

$$\sum_{n=1}^{\infty}\left[(n-1)\frac{\Re e(d)}{|d|} + |d|\right] D_n(\alpha,\gamma)a_n + \left[(n+1)\frac{\Re e(d)}{|d|} - |d|\right] D_n(\alpha,\gamma)b_n$$
$$= |d|.1 + \left(3\frac{\Re e(d)}{|d|} - |d|\right)\cdot 1$$
$$= 3\frac{\Re e(d)}{|d|} \leq 2|d|.$$

Suppose that $k \geq 0, \ni d + d(k-1) < 0$. We can take $z = -\frac{d(\mathfrak{k}-1)}{3+d(\mathfrak{k}-1)}$, then $z \in \mathcal{U}$ and through the definition of f, we have:

$$1 + \frac{1}{d}\left(\frac{z\left(E_{\alpha,\gamma}h(z)\right)' - z\left(\overline{E_{\alpha,\gamma}g(z)}\right)'}{E_{\alpha,\gamma}h(z) + \left(E_{\alpha,\gamma}g(z)\right)} - 1\right) = k < 0$$

and hence $f \notin \breve{\mathcal{M}}\mathcal{T}_{\mathcal{H}}(\alpha, \gamma, d)$. □

In the next theorem, we verify that the $\breve{\mathcal{M}}\mathcal{T}_{\mathcal{H}}(\ \alpha, \gamma, d) \nsubseteq \breve{\mathcal{R}}\mathcal{T}_{\mathcal{H}}(\ \alpha, \gamma, d)$.

Theorem 2.5. *$\breve{\mathcal{M}}\mathcal{T}_{\mathcal{H}}(\ \alpha, \gamma, d) \nsubseteq \breve{\mathcal{R}}\mathcal{T}_{\mathcal{H}}(\ \alpha, \gamma, d)$, whenever $d < -1, \gamma \geq 0$.*

Proof. Let the function:

$$f_\delta = z - \delta z^2. \tag{9}$$

Then $f \in \breve{\mathcal{M}}\mathcal{T}_{\mathcal{H}}(\alpha, \gamma, d)$ and $\delta > \frac{d}{1+d}$, we have:

$$\Re e\left\{1 + \frac{1}{d}\left(\frac{z\left(E_{\alpha,\gamma}h(z)\right)' - z\left(\overline{E_{\alpha,\gamma}g(z)}\right)'}{E_{\alpha,\gamma}^{A(z)+\left(E_{\alpha,\gamma}g(z)\right)}} - 1\right)\right\} = \left\{1 + \frac{\delta z}{d(\delta z - 1)}\right\} > 0, z \in \mathcal{U}.$$

Also:

$$\sum_{n=1}^{\infty}\left(2(n-1+|d|)D_n(\alpha,\gamma)a_n + (n+1+|n+1-2d|)D_n(\alpha,\gamma)b_n\right)$$
$$= 2|d| + 2(1 + |d|\delta > 4)$$

because $\delta > \frac{d}{1+d} > 1$, then $f \notin \breve{\mathcal{R}}\mathcal{T}_{\mathcal{H}}(\alpha, \gamma, d)$. □

3. Conclusion

We have been shown that the harmonic meromorphic functions which are starlike functions of complex order. More precisely, various interesting results concerning some relations devoted to different classes of harmonic meromorphic functions in connection to the Mittag-Leffler operator are demonstrated and analyzed.

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On time delay velocity feedback control of the composite cantilever beam model subjected to multi-excitation forces

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Abstract. The vibration rate of a structural dynamic design modelling a nonlinear composite cantilever beam at simultaneous sub-harmonic and internal resonance excitation is suppressed using delayed velocity feedback control (DVFC) in this article. This system of second order differential equations with nonlinearity due to quadratic and cubic terms, excited by parametric and external excitations, is presented. The procedure of multiple time scales perturbation is used to achieve an estimated solution for this scheme. At this approximation order, all potential resonance cases are extracted and numerically investigated. Both frequency response formulas and phase-plane trajectories are used to assess the system's stability during the worst resonance. The outcomes of various parameters on the device and controller are numerically investigated. The simulation results are achieved using Matlab and Maple programs. The time-dependent displacements are provided, and excellent consensus is reached as compared to results produced utilizing the Rung-Kutta procedure.

Keywords: Composite cantilever beam, time delay, multiple time scales method, sub-harmonic resonance, delayed feedback controller, stability analysis.

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1. Introduction

In recent times, composite beams have seen a lot of progress in mechanical science and aerospace. Conscious or smart systems are generated when active devices such as actuators and sensors are incorporated into the composite together with an appropriate control algorithm. They are used to optimize operating parameters or lower structure disruption and vibration. The coupled nonlinear vibration of several mechanical vibration systems can be simplified to nonlinear second order differential equations that can be solved analytically and numerically. The effect of modal correlations on the nonlinear behavior of harmonically disturbed structural and dynamical models was studied theoretically and experimentally by A. H. Nayfeh and B. Balachandran [1] and A. H. Nayfeh et al. [2]. They made discussions about rings, shells, pendulums, arches, ships, beam systems, and surface waves, as well as the parallels in their qualitative behavior. Quadratic nonlinear effects distinguish the structures, which can lead to 2:1 internal and combined parametric resonances. This resonances induce nonlinear periodic, quasi-periodic, and disorderly motions due to a coupling between the modes involved in the resonance. Highly low excitation levels were discovered to generate erratic motions under certain conditions.

An experimental analysis of the nonlinear behavior of an internally-resonant composite structural system to primary resonant excitation was presented by B. Balachandran and A. H. Nayfeh [3]. Nonlinear periodic, regularly modulated, and chaotically modulated responses arise from interaction between both the structure's directly and explicitly excited vibrational modes. Because of the relationship among flexural and tensional vibrational modes, too many of the described nonlinear responses are nonplanar. Weakly nonlinear analysis can be conducted to model the qualitative and quantitative reaction of an internal and external resonant structural configuration subjected to constrained primary resonant oscillations, as shown by B. Balachandran and A. H. Nayfeh [4]. The research can also be employed to forecast the Hopf bifurcation, defining the control component values at which regularly and chaotically modulated motions are most likely to appear. The effectiveness of a velocity feedback dependent nonlinear resonant controller to monitor the free and imposed self-excited vibration of a nonlinear beam was introduced by Joy Mondal and S. Chatterjee [5]. The nonlinear function of the derivative of the filter vector, which is transmitted via a second-order filter with the velocity stimulus from the sensor, is considered to evaluate the control power. Liang Li et al. [6] established a novel hierarchical methodology for turbulence analyses of rotating flexible beams with enhanced active restricted layer damping (EACLD) processing that is partly shielded for vibration surveys. Via simulating the EACLD patch's edge portion as an analogous spring with attached tip force the mass influence of the two inserted edge elements is utilized. The distinct rigid-flexible associated dynamic equations of hub-beam schemes with EACLD processing in open-loop and closed-loop circumstances was obtained through the asserted mode strategy and

Lagrange's equations. The beam is non-uniform and buckled at its left side to its disk's nucleus, where torque management operates, according to Boumediène and Smaoui [7], whereas a memory barrier control is located at the right end. The normal torque control is presented first, accompanied by the boundary control that is constructed using a particular form of memory phenomena as well as the dynamic characteristics of the input. Centered on a robust rotating beam vibration principle, an image recognition technique, and the lifting technique of data processing, L.F. Lyu and W.D. Zhu [8] introduced a new operational modal analysis (OMA) approach for a rotating structure. They created a tracking continually scanning laser Doppler vibrometer (TCSLDV) technique for controlling, scanning a rotating structure. Image processing was applied to specify the rotating structure's real-time situation allowing the TCSLDV device to detect a time-varying scanning path on the rotating structure.

To control the vibration of essentially assisted flexible plates, Sadek et al. [9] utilized piezoelectric-patch-actuators as active control. To measure actuator's voltage, an optimum control rule was inferred using the maximum principle theory. For directing polymer actuator displacement control, Wang et al. [10] adopted a reverse feed-forward controller system. The direct integration scheme [11] is used to analyze broad deformation study for a cantilever beam with a variable bending stiffness under both static and dynamic loads. Chentouf [12] investigated a model consisting of a nonhomogeneous flexible beam fastened to a rigid disc at its left end and free at its right end, where another rigid body is connected. They dealt with a variety of physical circumstances and devised feedback management rules to match. Kapuria and Yasin [13] investigated piezoelectric fiber-reinforced composite structures for successful vibration repression of metal laminated surfaces. Warminski et al. [14] investigated the experimental and theoretical motions of an autoparametric structure made up of two rectangular-sectioned beams. For irregular and harmonic oscillations, the experimental analysis of the system, configured at 1:4 internal resonance, is accomplished. Batista [15] derived the solution of large deflections of a beam within three-point bending in terms of Jacobi elliptical functions. Li et al. [16] and Peng et al. [17] implemented the effect of shear deformation and frictional end supports in the broad deformation of Timoshenko beam study. Huang et al. [18] recently used FIM to analyze static broad deformation under a dead focused vertical or horizontal force with exceptionally high precision. FIM is applied to the static/dynamic large deformation of a beam exposed to various types of loads using the Trapezoidal law and the integration matrix. Using the elliptic-function and the shooting process, Nguyen and Buntara [19] investigated broad deflections for uniform/non-uniform cantilever beams submitted to a moment at the top.

Zhang et al. [20] discussed the chaotic dynamics of a six-dimensional nonlinear structure that describes the averaged equation of a composite laminated piezoelectric rectangular plate subjected to transverse compression. The case of 1:2:4 internal resonances is considered, as well as in-plane excitations and

excitation filled by piezoelectric layers. Zhang and Li [21] investigated the resonant turbulent movements of a strictly assisted rectangular narrow plate with parametrically and externally excitations including exponential dichotomies and an averaging method. Sayed and Kamel [22, 23] reviewed the influence of various regulators on the vibrating mechanism, as well as the saturation control of a linear absorber, in order to minimize vibrations caused by rotor blade flapping motions. Both phase plane approaches and frequency response equations are used to analyze the numerical solution's stability. Kamel et al. [24] analyzed a configuration that was subjected to multiple excitation forces from the outside. The model is a two-degree-of-freedom device with a main system (machine head) and an absorber that simulates ultrasonic machining (USM). The perturbation method is used to extract the solution up to the second order approximation. Both phase-plane and frequency response functions are used to scrutinize the system's stability numerically. To minimize the vibrations of a hybrid Rayleigh–van der Pol–Duffing oscillator, the delayed velocity feedback control (DVFC) is used by Amer et al. [25]. This system is a one-degree-of freedom containing the cubic and fifth nonlinear terms and an external force. F. O. Darwesh et al. [26] investigated the effect of negative velocity feedback controller on a magnetic levitation system under External and parametric forces. They concluded that the control of linear velocity feedback is better to reduce the resulting vibration than the cubic one. Amer et al. [27] analyzed the vibration properties of a motor rotor and the effect of electro - mechanical couplers with multi-excitation forces and speed controller. They found that the velocity feedback controller is more effective for this system under the effect of external and parametric excitation forces in case of the worst resonance case. The dynamics of the dissipative system with a cubic nonlinear time-delayed of the type of the damping Duffing equation were investigated by Yusry O. El-Dib [28], they showed numerically that the time delay play dual roles in the stability behavior.

The aim of this research is to compare the effectiveness of delayed velocity feedback control (DVFC) strategies for significant vibrations of flexible composite beam structures to other previous studies that used different types of control systems. Approximate and numerical models are used to produce the results. The method of multiple scales perturbation technique is applied throughout to determine the approximation solution, whereas, the Rung-Kutta procedure of fourth order is applied for obtaining the numerical solution. The stability of the system is investigated applying both frequency response equations and phase-plane method. Also, the frequency and force-response curves are determined. The effects of all different parameters on system behavior are studied numerically. Comparisons between the numerical and analytical simulations are investigated.

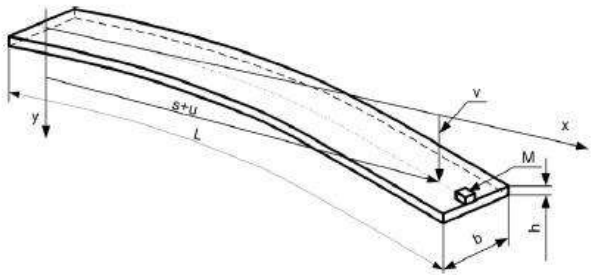


Figure 1: Semi-direct Shearer Drive Cutting Transmission Mechanism

2. System model and mathematical analysis

Consider the composite flexible beam in figure 1 given in [29] with a rectangular cross-section, the beam is assumed to be capable of significant flexural oscillations. The Hamilton's theory of last operation is used to derive the equation of motion for the current beam model. The same procedure, but for more complicated beam system, is shown in Warminske et al. [30]. The model is considered as the Euler- Bernoulli beam with added nonlinear curvature component $\rho_\nu = \nu''[1 + \frac{1}{2}(\nu')^2]$.
The dimensionless equation of motion takes form:

$$\nu + \nu^{IV}(1 + \nu'^2) + 4\nu'''\nu''\nu'^3 = \lambda\nu'', \tag{1}$$

where $\lambda = -\frac{1}{2}\int_1^s \int_0^s \nu'^2\, ds\, ds.$ is the Lagrange multiplier.

The horizontal and vertical displacements of the first vibration mode are taken into consideration in further investigations using Galerkin's approach [31] on system (1). The linear cantilever beam is supposed to be near to the function of the mode shape [29]. In the Galerkin's orthogonalization technique, the mode of the corresponding linear system is utilized As a result, we have a nonlinear ordinary differential equation for the first bending mode (based on a linear mode shape in the orthogonalization procedure):

$$\ddot{X} + 2\mu\,\omega\,\dot{X} + \omega^2 X + \beta X^3 - \delta(X\,\dot{X}^2 + X^2\ddot{X}) = 0, \tag{2}$$

where X is the generalized coordinate that corresponds to the displacement of the beam's tip, are the damping, cubic nonlinear term coefficients. The analysis is performed for an epoxy, carbon fiber reinforced beam with physical parameters [29]: $\rho = 2100\frac{kg}{m^3}$, $b = 12.8mm$, $h = 1mm$, $l = 236mm$, $M = 5 * 10^{-4}kg$, and the dimensionless coefficients $\omega^2 = 9.382, \beta = 14.4108, \delta = 0.930024, \mu = 0.01$. The oscillations of the nonlinear composite beam model are described by adapted nonlinear differential equations (with various types of controllers) in

[32, 33, 34, 35]. In this paper, the system with delayed velocity feedback control (DVFC) is given by:

$$\begin{aligned} &\ddot{p} + 2\varepsilon\ \hat{\mu}\ \omega\ \dot{p} + \omega^2 p\varepsilon\ \hat{\alpha} p^3 - \varepsilon\ \hat{\delta}(\, p\ \dot{p}^2 + p^2 \ddot{p}) \\ &= \varepsilon(\ \hat{f}_1 \cos(\Omega_1 t) + \ p\, \hat{f}_2 \cos(\Omega_2 t) - \ \hat{G}\ \dot{p}(\, t - \tau)), \end{aligned} \tag{3}$$

where $\hat{\mu} = \frac{\mu}{\varepsilon}, \hat{\alpha} = \frac{\alpha}{\varepsilon}, \hat{\delta} = \frac{\delta}{\varepsilon}, \hat{f}_1 = \frac{f_1}{\varepsilon},\ \hat{f}_2 = \frac{f_2}{\varepsilon},\ \hat{G} = \frac{G}{\varepsilon}, p$ represent the composite beam system displacement, μ is the viscous damping coefficient, ω is the natural frequency accompanied by the composite beam, $f_n, \Omega_n, n = 1, 2$ are the excitation forces and the excitation frequencies, and α, δ are the main system nonlinear stiffness coefficient which may be due to material of the beam nonhomogeneity.

2.1 Perturbation analysis

The second order approximation is given by applying the Multiple Time Scales (MTSM) procedure [36] so $p(t)$ in power series form will be:

$$p(\, T_o,\, T_1, \varepsilon) = \ p_o(\, T_o,\, T_1) + \varepsilon\, p_1(\, T_o,\, T_1) + \ O(\varepsilon^2), \tag{4}$$

where the time derivative takes the values:

$$\frac{d}{dt} = D_0 + \varepsilon D_1 + O(\varepsilon^2),\ \frac{d^2}{dt^2} = D_o^2 + 2\varepsilon D_0 D_1 + O(\varepsilon^2) \tag{5}$$

and $T_n = \varepsilon^n t,\ \ D_n = \frac{\partial}{\partial T_n},\ n = 0, 1.$

Switching equations (4), (5) into (3) and equating same power of ε coefficients to take out:

$$\begin{aligned} &O(\varepsilon^0): \\ &\qquad (D_o^2 + \omega^2)\, p_0 = 0, \end{aligned} \tag{6}$$

$$\begin{aligned} &O(\varepsilon): \\ &(D_o^2 + \omega^2)\, p_1 = -2\, D_o\, D_1\, p_o - \ 2\hat{m}u\omega\, D_o\, p_o\ + \hat{f}_1 \cos(\Omega_1\, t) \\ &+ \hat{f}_2\, p_o \cos(\Omega_2\, t) - \hat{\alpha}\, p_o^3\ + \hat{\delta}\, p_0[(\, D_o\, p_o)^2 + \ p_o(\, D_o^2\, p_o)] - \ G\, D_o\, p_{o\tau}. \end{aligned} \tag{7}$$

And the homogeneous solution of (6)is given by:

$$p_0 = A(T_1)\, e^{i\omega\, T_0} + \ cc, \tag{8}$$

where cc represents the complex conjugate of the preceding term, and A is complex function in T_1.

Let:

$$\begin{aligned} p(t - \tau) &= \ p_{o\tau}(T_o, T_1) + \varepsilon\, p_{1\tau}(T_o, T_1) + \ O(\varepsilon^2) \\ &= \ p_o(\, t - \tau, \varepsilon(\, t - \tau)) + \varepsilon\, p_1(\, t - \tau, \varepsilon(\, t - \tau)) + \ O(\varepsilon^2), \end{aligned} \tag{9}$$

where

$$
\begin{aligned}
& p_o(t-\tau, \varepsilon(t-\tau)) = p_{o\tau}(t, \varepsilon t) = p_{o\tau}(T_o, \varepsilon T_1), \\
& p_1(t-\tau, \varepsilon(t-\tau)) = p_{1\tau}(t, \varepsilon t) = p_{1\tau}(T_o, \varepsilon T_1).
\end{aligned} \tag{10}
$$

using equation (8), yields:

$$
\begin{aligned}
& p_{o\tau}(t, \varepsilon t) = p_o(t-\tau, \varepsilon(t-\tau)) \\
& = A(\varepsilon(t-\tau))\, e^{i\omega(t-\tau)} + \bar{A}(\varepsilon(t-\tau))\, e^{-i\omega(t-\tau)} \\
& = A(T_1 - \varepsilon\tau) e^{i\omega(t-\tau)} + \bar{A}(T_1 - \varepsilon\tau)\, e^{-i\omega(t-\tau)}.
\end{aligned} \tag{11}
$$

now expanding $A(T_1) - \varepsilon\tau)$ using Taylor series:

$$
A(T_1 - \varepsilon\tau) = A(T_1) - \varepsilon\tau \dot{A}(T_1) + \cdots \simeq A(T_1) + O(\varepsilon^2). \tag{12}
$$

Substituting (12) into **??** and using only the first term approximation so:

$$
p_{o\tau}(T_o, T_1) = A(T_1) e^{i\omega(To-\tau)} + \bar{A}(T_1) e^{-i\omega(To-\tau)}. \tag{13}
$$

Introducing (8), and (13) into 7, yields:

$$
\begin{aligned}
(D_o^2 + \omega^2) p_1 = & -2(i\omega D_1 A e^{i\omega T_o} - i\omega D_1 \bar{A} e^{-i\omega T_o}) \\
& - 2\omega\hat{\mu}(i\omega A e^{i\omega T_o} - i\omega \bar{A} e^{-i\omega T_o}) - \hat{\alpha}((A e^{i\omega T_o} - \bar{A} e^{-i\omega T_o})^3) \\
& + \hat{\delta}(A e^{i\omega T_o} - \bar{A} e^{-i\omega T_o})((i\omega A e^{i\omega T_o} - i\omega \bar{A} e^{-i\omega T_o})^2) \\
& - (\omega^2)\hat{\delta}((A e^{i\omega T_o} - \bar{A} e^{-i\omega T_o})^2) + \frac{\hat{f}_1}{2}(e^{i\Omega_1 T_o} - e^{-i\Omega_1 T_o}) \\
& + \frac{\hat{f}_2}{2}(A e^{i\omega T_o} - \bar{A} e^{-i\omega T_o})(e^{i\Omega_2 T_o} - e^{-i\Omega_2 T_o}) \\
& - G(i\omega A e^{i\omega(T_o-\tau)} - i\omega \bar{A} e^{-i\omega(T_o-\tau)}).
\end{aligned} \tag{14}
$$

We study now the system's severe operating approaches due to the resonance case, the worst case is the combined primary and subharmonic case i.e.:

$$
\Omega_1 = \omega + \varepsilon\,\sigma_1, \Omega_2 = 2\,\omega + \varepsilon\,\sigma_2, \tag{15}
$$

where $\sigma_i, i = 1, 2$ are called the detuning parameters. Inserting (15) into (14) and eliminating coefficients of all secular terms, yields:

$$
-2i\omega D_1 A - 2i\hat{\mu}\omega^2 A - 3\hat{\alpha}A^2\bar{A} + \frac{\hat{f}_1}{2} e^{i\sigma_1 T_1} + \frac{\hat{f}_2 \bar{A}}{2} e^{i\sigma_2 T_1} - Gi\omega A e^{i\omega\tau} = 0. \tag{16}
$$

Converting A into the polar form:

$$
A = \frac{a}{2} e^{i\gamma},\ D_1 A = \frac{e^{i\gamma}}{2}[\dot{a} + i\, a\, \dot{\gamma}], \tag{17}
$$

where a and γ are the system amplitude and phase, respectively.
Introducing (17) into (16) and comparing the real and imaginary parts in the resulting equation, then we get:

$$\dot{a} = -\mu\omega\, a + \frac{\hat{f}_1 \sin(\theta_1)}{2\omega} + \frac{\hat{f}_2 \sin(\theta_2)}{4\omega} - \frac{G\, a\, \cos(\omega\tau)}{2}, \tag{18}$$

$$a\,\dot{\theta_1} = \sigma_1\, a - \frac{3\,\hat{\alpha}\, a^3}{8\,\omega} - \frac{\omega\hat{\delta}\, a^3}{4} + \frac{\hat{f}_1 \cos(\theta_1)}{2\,\omega} + \frac{\hat{f}_2\, a\, \cos(\theta_2)}{4\,\omega} - \frac{G\, a\, \sin(\omega\tau)}{2}, \tag{19}$$

where

$$\theta_1 = \sigma_1 T_1 - \gamma, \quad \theta_2 = \sigma_2 T_1 - 2\,\gamma. \tag{20}$$

2.2 Steady state solution

The amplitude and phase steady state solution is obtained by substituting $\dot{a} = \dot{\theta_1} = 0$ into Eqs. (18), and (19), then we obtain:

$$\frac{\hat{f}_1 \sin(\theta_1)}{2\,\omega} + \frac{\hat{f}_2\, a\, \sin(\theta_2)}{4\,\omega} = \hat{mu}\omega\, a + \frac{G\, a\, \sin(\omega\tau)}{2}, \tag{21}$$

$$\frac{\hat{f}_1 \cos(\theta_1)}{2\,\omega} + \frac{\hat{f}_2\, a\, \cos(\theta_2)}{4\,\omega} = \frac{3\hat{\alpha}\, a^3}{8\omega} + \frac{\omega\hat{\delta}\, a^3}{4} + \frac{G\, a\, \sin(\omega\tau)}{2} - \sigma_1\, a\,. \tag{22}$$

Squaring and adding equations (21) and (22) to have the frequency response equation in the form:

$$\Gamma_1\, a^6 + \Gamma_2\, a^5 + \Gamma_3\, a^4 + \Gamma_4\, a^3 + \Gamma_5\, a^2 + \Gamma_6\, a + \Gamma_7 = 0, \tag{23}$$

where:

$$\begin{aligned} &\Gamma_1 = (\frac{3\,\hat{\alpha}}{8\,\omega} + \frac{\omega\hat{\delta}}{4})^2, \Gamma_2 = 0, \Gamma_3 = (\frac{G\, \sin(\omega\tau)}{2} - \sigma_1)(\frac{3\,\hat{\alpha}}{4\,\omega} + \frac{\omega\hat{\delta}}{2}), \\ &\Gamma_4 = 0, \Gamma_5 = (\hat{\mu}\omega + \frac{G\sin(\omega\tau)}{2})^2 + (\frac{G\sin(\omega\tau)}{2} - \sigma_1)^2 - \frac{\hat{f_2}^2}{16\,\omega^2}, \\ &\Gamma_6 = -\frac{\hat{f}_1\hat{f}_2}{4\omega^2}, \Gamma_7 = -\frac{\hat{f_1}^2}{4\omega^2}, \sigma_2 = 2\,\sigma_1. \end{aligned} \tag{24}$$

2.3 Stability analysis

Linearizing equations (18), and (19) according to the Lyapunov first (indirect) form [37, 38, 39] to address the stability behavior of these solutions, to give the following system:

$$\begin{bmatrix} \dot{a} \\ \dot{\theta}_1 \end{bmatrix} = J \begin{bmatrix} a \\ \theta_1 \end{bmatrix} = \begin{bmatrix} \upsilon_{11} & \upsilon_{12} \\ \upsilon_{21} & \upsilon_{22} \end{bmatrix} \begin{bmatrix} a \\ \theta_1 \end{bmatrix}, \tag{25}$$

where

$$
\begin{aligned}
\upsilon_{11} &= \frac{\partial \dot{a}}{\partial a} = -\hat{\mu}\omega + \frac{\hat{f}_2 \sin(2\,\theta_1)}{4\,\omega} - \frac{G\,\sin(\omega\tau)}{2},\\
\upsilon_{12} &= \frac{\partial \dot{a}}{\partial \theta_1} = \frac{\hat{f}_1 \cos(\theta_1)}{2\,\omega} + \frac{\hat{f}_2\, a\, \cos(2\,\theta_1)}{2\,\omega},\\
\upsilon_{21} &= \frac{\partial \dot{\theta_1}}{\partial\, a} = \frac{\sigma_1}{a} - \frac{9\hat{\alpha}\, a}{8\omega} - \frac{3\omega\hat{\delta}\, a}{4} + \frac{\hat{f}_2 \cos(2\,\theta_1)}{4\,\omega\, a} - \frac{G\,\sin(\omega\tau)}{2\,\omega},\\
\upsilon_{22} &= \frac{\partial \dot{\theta_1}}{\partial \dot{\theta_1}} = -\frac{\hat{f}_1 \sin(\theta_1)}{2\,\omega\, a} - \frac{\hat{f}_2 \sin(2\,\theta_1)}{2\,\omega}.
\end{aligned}
\tag{26}
$$

The stability of the steady-state solution therefore, depends on the Jacobian matrix's eigenvalues that can be obtained by solving the following characteristic equation:

$$
\begin{vmatrix} \upsilon_{11} - \lambda & \upsilon_{12} \\ \upsilon_{21} & \upsilon_{22} - \lambda \end{vmatrix} = 0, \tag{27}
$$

or

$$
\lambda^2 - tr(J)\lambda + \left| J \right| = 0, \tag{28}
$$

where λ denotes the Jacobian matrix $\left| J \right|$ eigenvalues, As shown by the Routh-Huriwitz criterion [40] the solution of the state is asymptotically stable if and only if

$$
tr(J) > 0, \left| J \right| > 0. \tag{29}
$$

Routh-Huriwitz conditions (29) are established numerically with the help of MATLAB software for estimating the stable and the unstable regions in the frequency response curves.

3. Results and discussion

In this section, the behavior of the system's amplitude and phase in resonance and non-resonance situations are demonstrated. The Rung-Kutta procedure of order four is utilized to evaluate the numerical solution of the given scheme, supposing the following parameters: $\mu = 0.001, f_1 = 0.01, f_2 = 0.05, \delta = 0.0001, \alpha = 0.002, \omega = 4$.

The time response is seen in Figure 1 (a) magnitudes at non-resonant situation and without control ($\Gamma = 0$), where Figure 2 (b) represents the system phase plane, the periodic solution is shown with asymptotically stable situation. Figure 3 (a) shows the time response for the system without applying control with resonance instance, $\Omega_1 = \omega + \varepsilon\,\sigma_1$, $\Omega_2 = 2\,\omega + \varepsilon\sigma_2$, furthermore, a Poincaré maps obtained in Figure 3 (b) for uncontrolled system at the same parameters. We observe that trajectory approaches a center of the graph in a very long

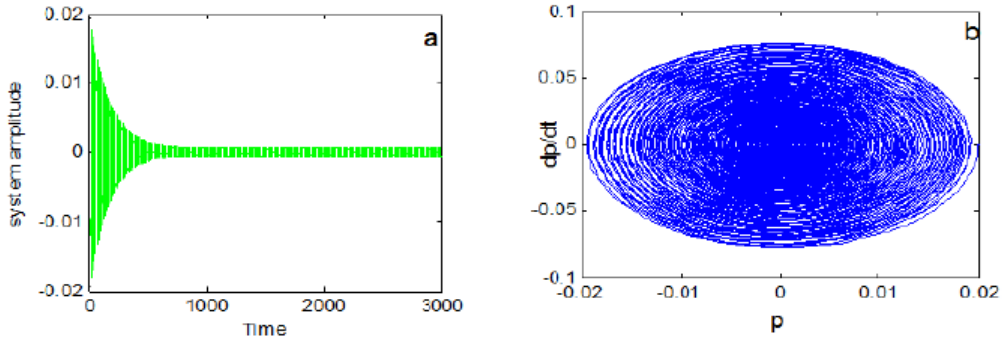


Figure 2: Time response in the normal operation (without resonance) and without applying control. (b) System phase plane.

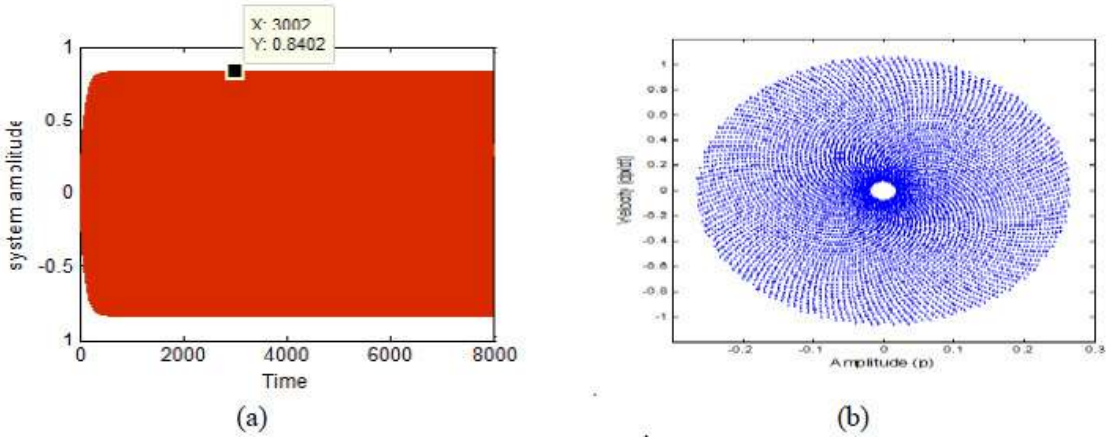


Figure 3: (a) Time response in the resonance case $\Omega_1 = \omega$, $\Omega_2 = 2\,\omega$ and without applying control ($G = 0$). (b) Poincaré mapping without control ($G = 0$).

time (stable center). The influence of applying delayed velocity feedback control (DVFC) on the system is clarified in Figure 4 (a),$G = 1.5$, $\tau = 0.274$, while Figure 4 (b) shows Poincaré maps obtained for the controlled model and trajectory approaches a center of the graph in a very short time compared to uncontrolled model. Figure 5 shows a comparison between the (DVFC) and active control introduced in [32]. From this figure, we conclude that the (DVFC) is more effective than the previous work in [32] and also more effective than the delayed control in the displacement. The effective of (DVFC) is about 99.93%where the active control introduced in [32] it is about 96.71 %. Another one important advantage of using (DVFC) is that the system reaches the steady state more

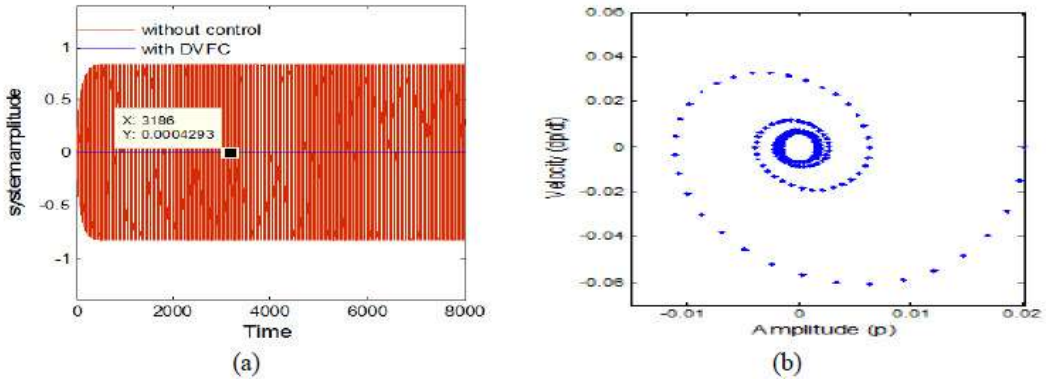


Figure 4: (a) Time response in the resonance case and with applying DVFC,($G = 1.5$) (blue curve) and without applying control ($G = 0$) (red curve). (b) Poincaré mapping with control ($G = 1.5$, $\tau = 0.274$).

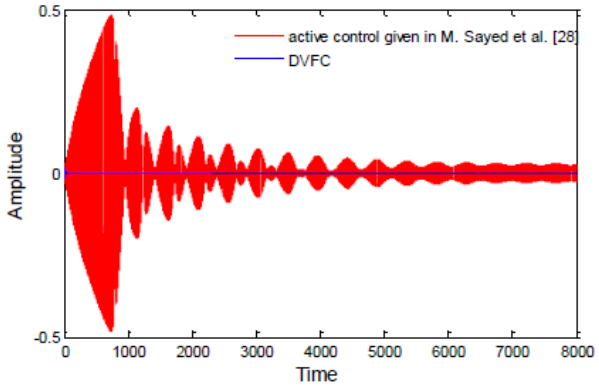


Figure 5: Comparison between (DVFC) and active control given in [32], $G = 1.5$, for both case.

rapidly than the case of active control introduced in M. Sayed et al. [32]. In Figure 6, for controlled processes with $G = 1.5$, $\tau = 0.274$, and near resonance case $\Omega_1 = \omega + \varepsilon\,\sigma_1$, $\Omega_2 = 2\,\omega + \varepsilon\,\sigma_2$, the response curve is shown for different values of the external force $f_1 = 0.01,\ 0.05,\ 0.1$. This curves show that, system peaks is proportional to f_1, and inversely proportional to the damping coefficient μ as shown in Figure 7 for. Figure 8 illustrate the frequency response by applying different values for the controller gain G, the figure clarify the inverse relation between the amplitude and the gain . Modifying the original natural frequency ω of the system modes changes the amplitudes in the response curves inversely, as clarified in Figures 9 at the values $\omega = 0.5, 1, 2$. Both solution

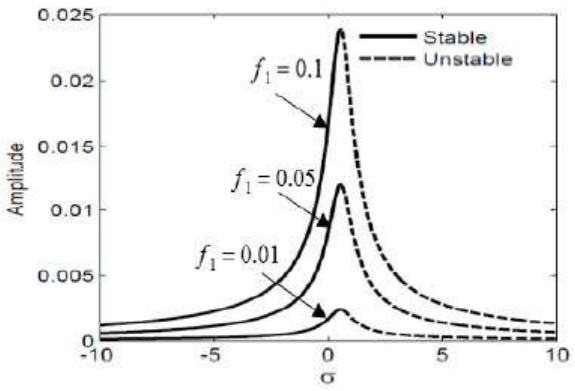


Figure 6: Effect of external excitation f_1, $f_1 = 0.01, 0.05, 0.1$, $G = 1.5$, $\tau = 0.274$).

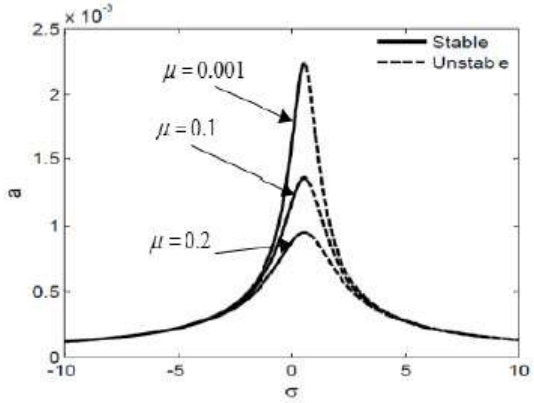


Figure 7: Effect of damping coefficient $\mu, \mu = 0.01, 0.1, 0.2$, $G = 1.5$, $\tau = 0.274$).

induced by (MTSM) and numerical solution using Rung-Kutta Method (RKM) of fourth order are matched in Figures 10-13 for time history and frequency response curves for the system. Figure 10, show good agreement between the approximate solution (blue curve) and the numerical outcome (green curve) of the resonance case without applying control scheme, while time response with applying (DVFC) is shown in Figures 11 by applying both RKM and MTSM at $G = 1.5$. Good results for the frequency response between the numerical and approximate method for frequency response at $G = 1.5, G = 0$ are shown in Figures 12, and 13 respectively. The change of amplitude range with varying of the delay time τ is shown in Figure 14, we observe that the optimum range for the time delay on the velocity feedback controller is $0 \leq \tau < 0.347$. Parameters effect on the system amplitude are illustrated on Figures 15, 16, the system

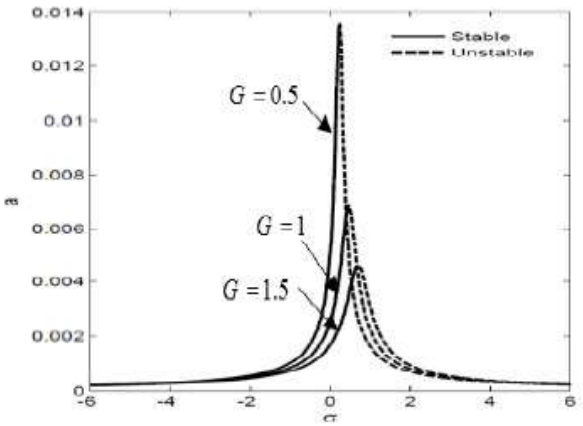


Figure 8: Effect of controller gain G, $G = 0.5, 1, 1.5$, $\tau = 0.274$).

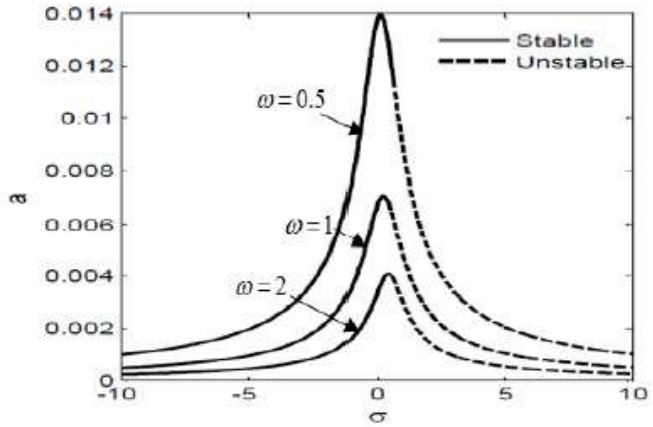


Figure 9: Effect of natural frequency $\omega, \omega = 0.5, 1, 2$, $G = 1.5$, $\tau = 0.274$).

mounts is decreased with the increase of the damping coefficient μ, we observed that from Figure 15, the optimum damping coefficient is $\mu > 0.1$, the system will has minimum amplitude and will be asymptotically stable with larger values of μ, but we cannot find a material with suitable value of damping coefficient or it can be found with highly costs, so we use a suitable material for the system and adding controller like (DVFC) for saving the system from damaging during resonance states and save system stability. A suitable value for controller gain $G > 0.5$ is illustrated in Figure 16.

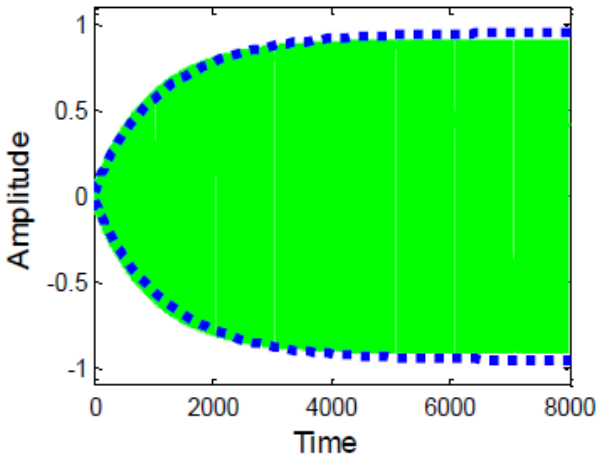


Figure 10: Comparison between numerical results (green curve) and approximate solution (blue curve), $G = 0$.

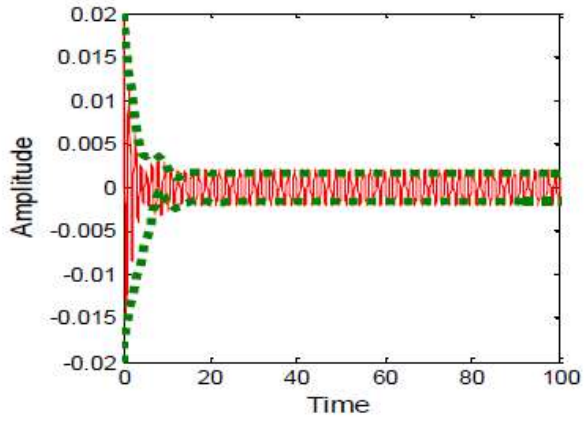


Figure 11: Comparison between numerical results (red curve) and approximate solution (green curve), $G = 1.5$, $\tau = 0.274$).

4. Conclusion

A dynamical representation of structural model architecture exhibiting a nonlinear composite cantilever beam with multi-excitation forces was investigated and solved in this analysis using multiple time scales methodology. The tragic resonance events were treated using the delayed velocity feedback control DVFC. The effective range for the delay time on the velocity feedback controller have been given by $0 \leq \tau < 0.34$. The worse situation of resonance is the simultaneous case ($\Omega_1 = \omega$, $\Omega_2 = 2\,\omega$). The amplitude of the composite beam was

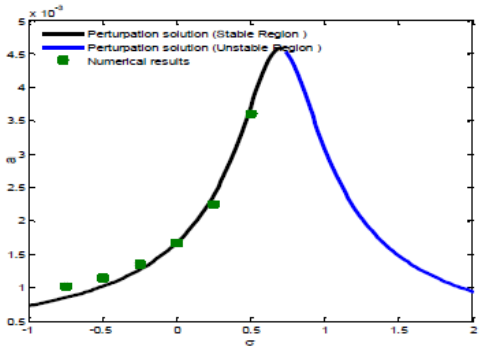


Figure 12: Comparison between numerical results and approximate solution with resonance, $G = 1.5$, $\tau = 0.274$).

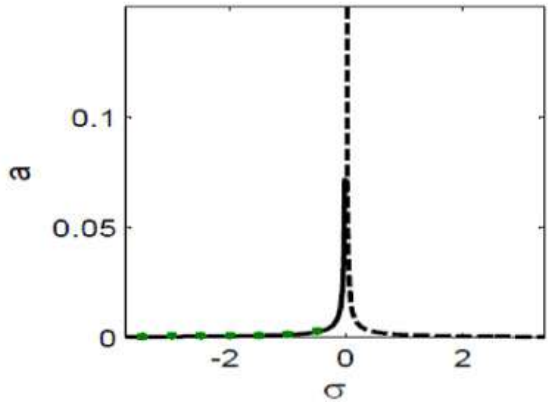


Figure 13: Comparison between numerical results (green curve) and approximate stable solution (solid black curve), with resonance, $G = 0$.

increased to about 1203 times of the excitation forces amplitude at resonance instance. The impact of DVFC was explained and debated. The control effect was found to be 99.93 ٪according to the DVFC result, and 96.71 ٪due to the effect of the active control applied in the previous work for M. Sayed et al. [32], implying that the DVFC was more efficient for this device in the resonance situation. An appropriate stability analysis was also performed and adequate feedback gains are detected in order to significantly reduce the amplitude peak and increase the stability region. Furthermore, numerical formulations achieved through using fourth-order Rung-Kutta technique were compared to approximate solutions derived via the multiple scales method. Both approximately and numerical techniques, the comparison provided strong consensus. Finally, the

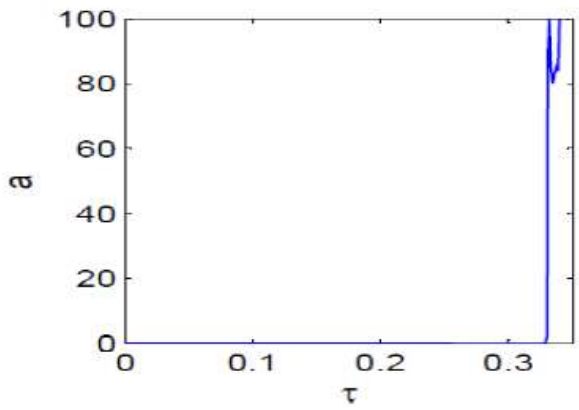


Figure 14: Effect of the time delay on the amplitude.

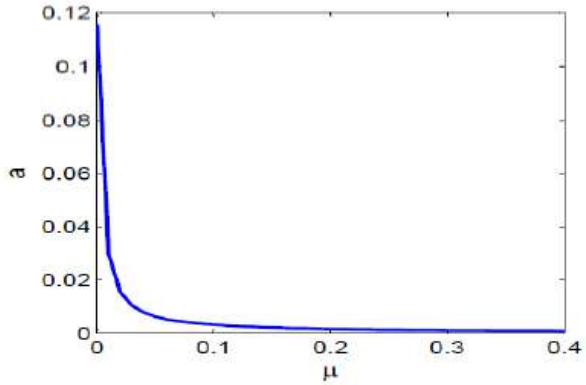


Figure 15: Effect of the damping coefficient on the amplitude.

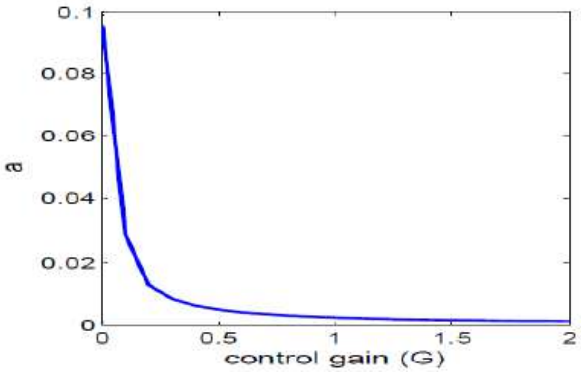


Figure 16: Effect of the controller gain on the amplitude.

influence of parameters on the scheme amplitude and the implications for the system's stability requirements were planned.

Acknowledgement

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Fekete-Szegö inequality for analytic and bi-univalent functions subordinate to Horadam polynomials

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Abstract. In the present paper, a subclass of analytic and bi-univalent functions by means of Horadam Polynomials is introduced. Certain coefficients bounds for functions belonging to this subclass are obtained. Furthermore, the Fekete-Szegö problem for this subclass is solved.

Keywords: Horadam polynomials, bi-univalent functions, analytic functions, Fekete-Szegö problem.

1. Introduction

Let $\mathcal{A}$ denote the class of all analytic functions f defined in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Thus each $f \in \mathcal{A}$ has a Taylor-Maclaurin series expansion of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in \mathbb{U}). \tag{1}$$

Further, let $\mathcal{S}$ denote the class of all functions $f \in \mathcal{A}$ which are univalent in $\mathbb{U}$ (for details, see [17]). See, also, some of the recent investigations [1, 2, 7, 9, 10, 11, 28]).

Two of the important and well-investigated subclasses of the analytic and univalent function class $\mathcal{S}$ are the class $\mathcal{S}^*(\alpha)$ of starlike functions of order α in $\mathbb{U}$ and the class $\mathcal{K}(\alpha)$ of convex functions of order α in $\mathbb{U}$. By definition, we have

$$\mathcal{S}^*(\alpha) := \left\{ f : \ f \in \mathcal{S} \ \text{ and } \ \mathrm{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha, \quad (z \in \mathbb{U}; 0 \leq \alpha < 1) \right\}, \tag{2}$$

and

$$\mathcal{K}(\alpha) := \left\{ f : \ f \in \mathcal{S} \ \text{ and } \ \mathrm{Re}\left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha, \quad (z \in \mathbb{U}; 0 \leq \alpha < 1) \right\}. \tag{3}$$

It is clear from the definitions (2) and (3) that $\mathcal{K}(\alpha) \subset \mathcal{S}^*(\alpha)$. Also we have

$$f(z) \in \mathcal{K}(\alpha) \text{ iff } zf'(z) \in \mathcal{S}^*(\alpha),$$

and

$$f(z) \in \mathcal{S}^*(\alpha) \text{ iff } \int_0^z \frac{f(t)}{t} dt = F(z) \in \mathcal{K}(\alpha).$$

It is well-known that, if $f(z)$ is an univalent analytic function from a domain $\mathbb{D}_1$ onto a domain $\mathbb{D}_2$, then the inverse function $g(z)$ defined by

$$g\left(f(z)\right) = z, \quad (z \in \mathbb{D}_1),$$

is an analytic and univalent mapping from $\mathbb{D}_2$ to $\mathbb{D}_1$. Moreover, by the familiar Koebe one-quarter theorem (for details, see [17]), we know that the image of $\mathbb{U}$ under every function $f \in \mathcal{S}$ contains a disk of radius $\frac{1}{4}$.

According to this, every function $f \in \mathcal{S}$ has an inverse map f^{-1} that satisfies the following conditions:

$$f^{-1}(f(z)) = z, \quad (z \in \mathbb{U}),$$

and

$$f\left(f^{-1}(w)\right) = w, \quad \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4}\right).$$

In fact, the inverse function is given by

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \tag{4}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1). Examples of functions in the class Σ are

$$\frac{z}{1-z}, \ -\log(1-z), \ \frac{1}{2}\log\left(\frac{1+z}{1-z}\right), \cdots .$$

It is worth noting that the familiar Koebe function is not a member of Σ, since it maps the unit disk $\mathbb{U}$ univalently onto the entire complex plane except the part of the negative real axis from $-1/4$ to $-\infty$. Thus, clearly, the image of the domain does not contain the unit disk $\mathbb{U}$. For a brief history and some intriguing examples of functions and characterization of the class Σ, see Srivastava et al. [24] and Yousef et al. [25, 26, 27, 29, 12, 13, 14, 15].

In 1967, Lewin [22] investigated the bi-univalent function class Σ and showed that $|a_2| < 1.51$. Subsequently, Brannan and Clunie [16] conjectured that $|a_2| \leq \sqrt{2}$. The coefficient estimate problem for each of the following Taylor-Maclaurin coefficients $|a_n|$ $(n \in \mathbb{N}\backslash\{1, 2\})$ for each $f \in \Sigma$ given by (1) is presumably still an open problem.

In 2009, Horzum and Kocer [20] considered the Horadam polynomials $h_n(x)$, which are given by the following recurrence relation:

$$h_n(x) = pxh_{n-1}(x) + qh_{n-2}(x), \quad (n \in \mathbb{N} \setminus \{1, 2\}), \tag{5}$$

with

$$h_1(x) = a,\ h_2(x) = bx \text{ and } h_3(x) = pbx^2 + aq \tag{6}$$

for some real constants a, b, p and q.

First off, we present some special cases of the polynomials $h_n(x)$ (see, [20] and [19]):

1. For $a = b = p = q = 1$, we get the Fibonacci polynomials $F_n(x)$.
2. For $a = 2$ and $b = p = q = 1$, we get the Lucas polynomials $L_n(x)$.
3. If $a = 1$, $b = p = 2$ and $q = 1$, then we get the Chebyshev polynomials $U_n(x)$ of the second kind.

The Fibonacci polynomials, the Lucas polynomials, the Chebyshev polynomials and the families of orthogonal polynomials and other special polynomials also their generalizations are very important in different disciplines in the mathematical, physical, statistical, and engineering sciences. These kind of polynomials have been studied in several papers from a theoretical point of view (see, [3, 4, 5, 6, 8, 18, 21, 23]).

Theorem 1.1 ([20])**.** *Let $\Omega(x, z)$ be the generating function of the Horadam polynomials $h_n(x)$. Then,*

$$\Omega(x, z) = \sum_{n=1}^{\infty} h_n(x) z^{n-1} = \frac{a + (b - ap)xz}{1 - pxz - qz^2}. \tag{7}$$

2. The class $\mathfrak{B}_{\Sigma}^{\mu}(a, \lambda, \delta)$

We begin this section by defining the class $\mathfrak{B}_{\Sigma}^{\mu}(a, \lambda, \delta)$ as follows:

Definition 2.1. *For $\lambda \geq 1$, $\mu \geq 0$ and $\delta \geq 0$, a function $f \in \Sigma$ given by (1) is said to be in the class $\mathfrak{B}_{\Sigma}^{\mu}(a, \lambda, \delta)$ if the following subordinations are satisfied:*

$$(1 - \lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} + \xi\delta z f''(z) \prec \Omega(x, z) + 1 - a$$

and

$$(1-\lambda)\left(\frac{f^{-1}(w)}{w}\right)^{\mu} + \lambda\left(f^{-1}(w)\right)'\left(\frac{f^{-1}(w)}{w}\right)^{\mu-1} + \xi\delta z\left(f^{-1}(w)\right)'' \prec \Omega(x, w) + 1 - a,$$

where f^{-1} is given by (4) and $\xi = \frac{2\lambda+\mu}{2\lambda+1}$.

3. Initial coefficient estimates

In this section, we propose to find the estimates on the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in the class $\mathfrak{B}_{\Sigma}^{\mu}(a,\lambda,\delta)$, which we introduced in Definition 2.1.

We first state Theorem 3.1.

Theorem 3.1. *For $\lambda \geq 1$, $\mu \geq 0$ and $\delta \geq 0$, let $f \in \mathcal{A}$ belongs to the class $\mathfrak{B}_{\Sigma}^{\mu}(a,\lambda,\delta)$. Then*

$$|a_2| \leq \frac{2\,|bx|\,\sqrt{|bx|}}{\sqrt{\left|(\mu+2\lambda)\left[1+\mu+\frac{12\delta}{2\lambda+1}\right](bx)^2-2\,(\mu+\lambda+2\xi\delta)^2\,(pbx^2+aq)\right|}},$$

and

$$|a_3| \leq \frac{(bx)^2}{(\mu+\lambda+2\xi\delta)^2}+\frac{|bx|}{(\mu+2\lambda)\left(1+\frac{6\delta}{2\lambda+1}\right)}.$$

Proof. Let $f \in \mathfrak{B}_{\Sigma}^{\mu}(a,\lambda,\delta)$. From Definition 2.1, for some analytic functions ϕ, ψ such that $\phi(0)=\psi(0)=0$ and $|\phi(z)|<1$, $|\psi(w)|<1$ for all $z,w\in\mathbb{U}$, then we can write

$$\begin{aligned}(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu}+\lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}+\xi\delta z f''(z)\\ =1+h_1(x)-a+h_2(x)\phi(z)+h_3(x)\phi^2(z)+\cdots\end{aligned} \tag{8}$$

and

$$\begin{aligned}(1-\lambda)\left(\frac{f^{-1}(w)}{w}\right)^{\mu}+\lambda\left(f^{-1}(w)\right)'\left(\frac{f^{-1}(w)}{w}\right)^{\mu-1}+\xi\delta z\left(f^{-1}(w)\right)''\\ =1+h_1(x)-a+h_2(x)\psi(w)+h_3(x)\psi^2(w)+\cdots.\end{aligned} \tag{9}$$

From the equalities (8) and (9), we obtain that

$$\begin{aligned}(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu}+\lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}+\xi\delta z f''(z)\\ =1+h_2(x)r_1z+\left[h_2(x)r_2+h_3(x)r_1^2\right]z^2+\cdots\end{aligned} \tag{10}$$

and

$$\begin{aligned}(1-\lambda)\left(\frac{f^{-1}(w)}{w}\right)^{\mu}+\lambda\left(f^{-1}(w)\right)'\left(\frac{f^{-1}(w)}{w}\right)^{\mu-1}+\xi\delta z\left(f^{-1}(w)\right)''\\ =1+h_2(x)s_1w+\left[h_2(x)s_2+h_3(x)s_1^2\right]w^2+\cdots.\end{aligned} \tag{11}$$

It is fairly well known that if

$$|\phi(z)|=\left|r_1z+r_2z^2+r_3z^3+\cdots\right|<1,\quad(z\in\mathbb{U})$$

and

$$|\psi(w)| = \left|s_1 w + s_2 w^2 + s_3 w^3 + \cdots\right| < 1, \quad (w \in \mathbb{U}),$$

then

$$|r_k| < 1 \text{ and } |s_k| < 1 \text{ for } k \in \mathbb{N}. \tag{12}$$

Thus, upon comparing the corresponding coefficients in (10) and (11), we have

$$(\mu + \lambda + 2\xi\delta)\, a_2 = h_2(x) r_1, \tag{13}$$

$$(\mu + 2\lambda)\left[\left(\frac{\mu - 1}{2}\right) a_2^2 + \left(1 + \frac{6\delta}{2\lambda + 1}\right) a_3\right] = h_2(x) r_2 + h_3(x) r_1^2, \tag{14}$$

$$-(\mu + \lambda + 2\xi\delta)\, a_2 = h_2(x) s_1, \tag{15}$$

and

$$(\mu + 2\lambda)\left[\left(\frac{\mu + 3}{2} + \frac{12\delta}{2\lambda + 1}\right) a_2^2 - \left(1 + \frac{6\delta}{2\lambda + 1}\right) a_3\right] = h_2(x) s_2 + h_3(x) s_1^2. \tag{16}$$

It follows from (13) and (15) that

$$r_1 = -s_1 \tag{17}$$

and

$$2(\mu + \lambda + 2\xi\delta)^2 a_2^2 = h_2^2(x)\left(r_1^2 + s_1^2\right). \tag{18}$$

If we add (14) and (16), we get

$$(\mu + 2\lambda)\left[1 + \mu + \frac{12\delta}{2\lambda + 1}\right] a_2^2 = h_2(x)(r_2 + s_2) + h_3(x)\left(r_1^2 + s_1^2\right). \tag{19}$$

Substituting the value of $\left(r_1^2 + s_1^2\right)$ from (18) in the right hand side of (19), we deduce that

$$\left[(\mu + 2\lambda)\left[1 + \mu + \frac{12\delta}{2\lambda + 1}\right] h_2^2(x) - 2(\mu + \lambda + 2\xi\delta)^2 h_3(x)\right] a_2^2 = h_2^3(x)(r_2 + s_2). \tag{20}$$

Moreover, computations using (6), (12) and (20), we find that

$$|a_2| \leq \frac{2|bx|\sqrt{|bx|}}{\sqrt{\left|(\mu + 2\lambda)\left[1 + \mu + \frac{12\delta}{2\lambda+1}\right](bx)^2 - 2(\mu + \lambda + 2\xi\delta)^2 (pbx^2 + aq)\right|}}.$$

Moreover, if we subtract (16) from (14), we obtain

$$2(\mu+2\lambda)\left(1+\frac{6\delta}{2\lambda+1}\right)\left(a_3-a_2^2\right)=h_2(x)(r_2-s_2)+h_3(x)\left(r_1^2-s_1^2\right). \tag{21}$$

Then, in view of (17) and (18), Eq. (21) becomes

$$a_3=\frac{h_2^2(x)}{2(\mu+\lambda+2\xi\delta)^2}\left(r_1^2+s_1^2\right)+\frac{h_2(x)}{2(\mu+2\lambda)\left(1+\frac{6\delta}{2\lambda+1}\right)}(r_2-s_2).$$

Thus applying (6), we conclude that

$$|a_3|\leq\frac{(bx)^2}{(\mu+\lambda+2\xi\delta)^2}+\frac{|bx|}{(\mu+2\lambda)\left(1+\frac{6\delta}{2\lambda+1}\right)}.$$

So, the proof of Theorem 3.1 is complete. □

By setting $\mu=\delta=0$ and $\lambda=1$ in Theorem 3.1, we obtain the following consequence.

Corollary 3.1. *If f belongs to the class $\mathfrak{B}_\Sigma(a,1)=\mathcal{S}_\Sigma^*$ of bi-starlike functions, then*

$$|a_2|\leq\frac{2|bx|\sqrt{|bx|}}{\sqrt{\left|2(bx)^2-2(pbx^2+aq)\right|}},$$

and

$$|a_3|\leq(bx)^2+\frac{|bx|}{2}.$$

4. Fekete-Szegö problem for the function class $\mathfrak{B}_\Sigma^\mu(a,\lambda,\delta)$

In this section, we aim to provide Fekete-Szegö inequalities for functions in the class $\mathfrak{B}_\Sigma^\mu(a,\lambda,\delta)$. These inequalities are given in the following theorem.

Theorem 4.1. *For $\lambda\geq1$, $\mu\geq0$ and $\delta\geq0$, let $f\in\mathcal{A}$ belongs to the class $\mathfrak{B}_\Sigma^\mu(a,\lambda,\delta)$. Then*

$$\left|a_3-\upsilon a_2^2\right|\leq\begin{cases}\dfrac{|bx|}{(\mu+2\lambda)\left(1+\frac{6\delta}{2\lambda+1}\right)}, & |\upsilon-1|\leq\dfrac{1}{2(\mu+2\lambda)\left(1+\frac{6\delta}{2\lambda+1}\right)}\times|\Upsilon(x)|,\\ \dfrac{2|bx||1-\upsilon|}{|\Upsilon(x)|}, & |\upsilon-1|\geq\dfrac{1}{2(\mu+2\lambda)\left(1+\frac{6\delta}{2\lambda+1}\right)}\times|\Upsilon(x)|,\end{cases}$$

where $\Upsilon(x)=[(\mu+2\lambda)[1+\mu+\frac{12\delta}{2\lambda+1}]-2(\mu+\lambda+2\xi\delta)^2\frac{(pbx^2+aq)}{(bx)^2}]$.

Proof. From (20) and (21)

$$\begin{aligned} a_3 - \upsilon a_2^2 &= (1-\upsilon)\frac{h_2^3(x)\,(r_2+s_2)}{\left[(\mu+2\lambda)\left[1+\mu+\frac{12\delta}{2\lambda+1}\right]h_2^2(x) - 2\,(\mu+\lambda+2\xi\delta)^2\,h_3(x)\right]} \\ &\quad + \frac{h_2(x)}{2\,(\mu+2\lambda)\left(1+\frac{6\delta}{2\lambda+1}\right)}\,(r_2-s_2) \\ &= h_2(x)\left[\left[\varphi(\upsilon,x) + \frac{1}{2\,(\mu+2\lambda)\left(1+\frac{6\delta}{2\lambda+1}\right)}\right]r_2\right. \\ &\quad \left. + \left[\varphi(\upsilon,x) - \frac{1}{2\,(\mu+2\lambda)\left(1+\frac{6\delta}{2\lambda+1}\right)}\right]s_2\right], \end{aligned}$$

where

$$\varphi(\upsilon,x) = \frac{h_2^2(x)\,(1-\upsilon)}{\left[(\mu+2\lambda)\left[1+\mu+\frac{12\delta}{2\lambda+1}\right]h_2^2(x) - 2\,(\mu+\lambda+2\xi\delta)^2\,h_3(x)\right]},$$

Then, in view of (6), we conclude that

$$\left|a_3 - \upsilon a_2^2\right| \leq \begin{cases} \frac{|h_2(x)|}{(\mu+2\lambda)\left(1+\frac{6\delta}{2\lambda+1}\right)} & 0 \leq |\varphi(\upsilon,x)| \leq \frac{1}{2(\mu+2\lambda)\left(1+\frac{6\delta}{2\lambda+1}\right)}, \\ 2\,|h_2(x)|\,|\varphi(\upsilon,x)| & |\varphi(\upsilon,x)| \geq \frac{1}{2(\mu+2\lambda)\left(1+\frac{6\delta}{2\lambda+1}\right)}. \end{cases}$$

Which completes the proof of Theorem 4.1. □

Putting $\mu = \delta = 0$ and $\lambda = 1$ in Theorem 4.1, we conclude the following result:

Corollary 4.1. *If f belongs to the class $\mathcal{S}_\Sigma^*$, then*

$$\left|a_3 - \upsilon a_2^2\right| \leq \begin{cases} \dfrac{|bx|}{2}, & |\upsilon - 1| \leq \left|\dfrac{(bx)^2 - \left(pbx^2+aq\right)}{2(bx)^2}\right| \\ \dfrac{2\,|bx|^3\,|1-\upsilon|}{|(bx)^2 - (pbx^2+aq)|}, & |\upsilon - 1| \geq \left|\dfrac{(bx)^2 - \left(pbx^2+aq\right)}{2(bx)^2}\right|. \end{cases}$$

Putting $\upsilon = 1$ in Theorem 4.1, we conclude the following result:

Corollary 4.2. *If f belongs to the class $\mathcal{S}_\Sigma^*$, then*

$$\left|a_3 - a_2^2\right| \leq \frac{|bx|}{(\mu+2\lambda)\left(1+\frac{6\delta}{2\lambda+1}\right)}.$$

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On the nullity of Nijenhuis torsion of a vector valued 1-form

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Abstract. On a differentiable manifold, the data of a vector valued 1-form L with a constant rank, and the nullity of Nijenhuis torsion define a foliation. We study on the scalar forms the cohomology of a derivation with degree 1 associated to L. The obtained results of this study generalize those of Willmore. The vector fields by which Lie derivative of L is null, constitute a Lie subalgebra of infinitesimal automorphisms of the foliation. We study the Chevalley-Eilenberg cohomology space associated to L when L is a product structure and this study is different from the one made by Lehmann-Lejeune who studied the same cohomology in case of L as a tangent structure. As far as Lie algebra defined by any vector valued 1-form L is concerned, the derivations are not always adjoint linear applications of vector fields.
Keywords: differentiable manifold, vector 1-form, Lie algebra, Nijenhuis torsion.

1. Introduction

We consider M a smooth manifold, paracompacte, connected with $(n+m)$ dimension. All objects are supposed to be C^∞ class. In the first place, we will study the d_L-cohomology on the scalar forms in which some of the results have been mentioned in [1]. We will use in this study the theory of Frölicher-Nijenhuis [5] and [9] which shows in particular that all derivation of degree 1 on the scalar forms that commutes (in the sense of the graduated rings) with the exterior differentiation d, is of the form d_L, where L is a vector valued 1-form determined only by derivation. If $L = I$, I being the identity vector valued 1-form, d_I equals to d. The cohomologic study of d_L naturally assumes that the square of d_L is zero ($d_L^2 = 0$). To put it differently, the torsion of Nijenhuis of L is null.

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Willmore [22] has proved that if L is regular then the d_L-cohomology is isomorphic to that of de Rham. Whereas if L is singular, Poincaré's Lemma is false. In addition, E. Ayassou demonstrated in [2] that if L is the tangent structure then all form in the kernel of L^*, which is d_L-closed, is globally d_L-exact. These properties derive essentially from the nullity of the square of d_L. The nullity of d_L^2 implies the nullity of the Nijenhuis torsion of L. It is assumed in the following section that the manifold M is provided with a vector valued 1-form L of rank m and Nijenhuis torsion zero. The nullity of the Nijenhuis torsion of L defines on M a foliation such that the image space of L corresponds to the tangent space to leaves. We prove Poincaré Lemma for the L-semi-basic forms which, if L is regular, are the forms on M. The d_L-cohomology on any leave is isomorphic to that of de Rham. For homogeneous forms, we adapt a method used by J. Klein and A. Voutier [12]. The use of a projetable vector valued 1-form makes it possible to compare the cohomologies between certain manifolds. We write, in our formalism, the classical derivations d' and d'' on foliated manifolds and on complex manifolds. We give in [10] some versions for the L-semi-basic forms, of Fröbenius' theorem and a result of H. Jacobowitz's work. In the second place, our study consists in working on the cohomology of Chevalley-Eilenberg of $\mathfrak{A}_L$ where $\mathfrak{A}_L$ denotes the set of vector fields by which the Lie derivative of L is zero. F. Takens in [21] has proved that any derivation of the Lie algebra of the vector fields $\chi(M)$ on a manifold M is inner. A. Lichnerowicz [15] considered the Lie algebra of the infinitesimal automorphisms of a foliation $\mathcal{L}_{\mathcal{F}}$ on a manifold and showed in particular that, whatever the foliation, any derivation of $\mathcal{L}_{\mathcal{F}}$ is inner. As for the considered manifold (M, L) , $\mathfrak{A}_L$ is a subalgebra of $\mathcal{L}_{\mathcal{F}}$. If L is the vector valued 1-form of the transverse bundle to a foliation ($rank(L) = n$ and $L^2 = 0$), Lehmann-Lejeune in [14] proved that the derivations of $\mathfrak{A}_L$ are adjoint linear applications of the normalizer of $\mathfrak{A}_L$ in $\chi(M)$. One can wonder the derivation behaviors of $\mathfrak{A}_L$ for any vector valued 1-form. If we denote by $\overline{\chi(M)}$ the complete bearing on the tangent bundle TM of the vector fields $\chi(M)$ on M, the first cohomology of the normalizer of $\overline{\chi(M)}$ in $\chi(TM)$ is of dimension 1, whereas that of $\overline{\chi(M)}$ is zero. If L is a tangent structure, $\overline{\chi(M)}$ is a subalgebra of $\mathfrak{A}_L$; we can demonstrate by using the result on $\overline{\chi(M)}$, that any derivation of $\mathfrak{A}_L$ comes from its normalizer - it is a result of Lehmann-Lejeune's work in [14] in this particular case - and that any derivation of the normalizer of $\mathfrak{A}_L$ is interior. If L is a connection in the sense of Grifone [8] ($L^2 = I$) then $\mathfrak{A}_L$ is isomorphic to $\chi(M) \times \chi(\mathbb{R}^n)$, therefore, any derivation of $\mathfrak{A}_L$ is interior; it is also a result of [16] by a different approach. Finally, we give an example of a vector valued 1-form of null Nijenhuis torsion admitting non-adjoint derivations of the vector fields. For Chevalley-Eilenberg cohomology space studies in other cases, the results can be found in [16], [17], [18] and [19].

2. d_L-Cohomologies

2.1 Preliminaries

Let M be a differentiable manifold, TM as the tangent bundle to M, $\mathcal{F}(M)$ the $\mathbb{R}$-algebra of differentiable functions on M, $\phi^p(M)$ the $\mathcal{F}(M)$-module of scalar p-forms on M and $\phi^0(M) = \mathcal{F}(M)$; $\psi^1(M)$ the $\mathcal{F}(M)$-module of 1-vector forms on M; $\psi^0(M)$ or $\chi(M)$ the $\mathcal{F}(M)$-module of vector fields on M, and $\phi(M)$, $\psi(M)$ or simply ϕ, ψ if there is no possible confusion on the corresponding graduated rings. The formalism of Frölicher-Nijenhuis will be the fundamental tool of this work (see [5] and [9]).

Definition 2.1 ([5], [9]). *Let's consider $L \in \psi^1$. We define the endomorphism L^* of ϕ by:*

- *a) $L^* f = f$ if $f \in \phi^0$*
- *b) $(L^*\omega)\left(X^1, X^2, \ldots, X^p\right) = \omega\left(LX^1, LX^2, \ldots, LX^p\right)$ if $\omega \in \phi^p$ and $X^1, \ldots, X^p \in \psi^0$.*

Definition 2.2 ([5], [9]). *Let's assume $\omega \in \phi^p$ and $L \in \psi^l$. We define the element $i_L\omega$ of ϕ^{p+l-1} by:*

$$(i_L\omega)\left(X^1, X^2, \ldots, X^{p+l-1}\right) = \frac{1}{(p-1)!l!}\sum_{\alpha} sgn(\alpha)\,\omega\left(L\left(X^{\alpha_1}, \ldots, X^{\alpha_l}\right), X^{\alpha_{l+1}}, \ldots, X^{\alpha_{p+l-1}}\right),$$

for all $X^1, X^2, \ldots, X^{p+l-1} \in \psi^0$; the summation above being taken from the group of permutations $\{1, 2, \ldots, p+l-1\}$. As for $\omega \in \phi^0$, $i_L\omega$ is null by definition. The application $i_L : \omega \longmapsto i_L\omega$ is a derivation of degree $l-1$ on ϕ.

Definition 2.3 ([5], [9]). *The commutator of two derivations $[i_L, d] = i_L d - (-1)^l d i_L$ defines a derivation of degree l, noted d_L. In particular, if I is the vector valued 1-form identity then $d_I = d$ and d_X is Lie derivative with respect to X, which will be noted in the following by θ_X, for $X \in \psi^0$.*

Definition 2.4 ([5], [9]). *Let's consider $K \in \psi^k$ and $L \in \psi^l$. We define $[K, L]$, element of ψ^{k+l}, by $[d_K, d_L] = d_{[K,L]}$. In particular, if $L \in \psi^1$, we have $[d_L, d_L] = 2d_L^2 = d_{[L,L]}$.*

The bracket $N_L = \frac{1}{2}[L, L]$ is called Nijenhuis torsion of L. We have the following formula:

$$\frac{1}{2}[L, L](X, Y) = [LX, LY] + L^2[X, Y] - L[LX, Y] - L[X, LY]$$

for all $X, Y \in \psi^0$.

As a result, the study of Poincaré's Lemma on the derivation d_L, L being any vector valued 1-form, supposes the nullity of Nijenhuis torsion of L that is, $[LX, LY] = L[LX, Y] + L[X, LY] - L^2[X, Y]$.

If L is of constant rank in a neighborhood U of a point z_0 of M, according to the Fröbenius theorem, the distribution which at all z of U associates L_zT_z, is completely integrable.

Let M be a paracompact, differentiable manifold of $(n+m)$-dimension with a vector valued 1-form L of rank $m \geq 1$ and Nijehnuis torsion null. Thus L defines on M a completely integrable distribution $\mathfrak{D}$ with m dimension. The structure of foliated manifold by L can be defined by an overlap of open sets $\mathfrak{U}$ of M and by the data, in each $U \in \mathfrak{U}$, of a coordinate system $\left(x^1, \ldots, x^n, y^1, \ldots, y^m\right)$ such that $\frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^m}$ form a tangent local basis to the leaves. Note $a, b, \ldots$ etc, the indexes ranging from 1 to n and $i, j, \ldots$ etc, those ranging from 1 to m. For U, U' of $\mathfrak{U}$, with $U \cap U' \neq \emptyset$, we have $x^{b'} = x^{b'}\left(x^a\right)$ and $y^{j'} = y^{j'}\left(x^a, y^i\right)$. Then, the distribution $\mathfrak{D}$ is defined by equations $dx^a = 0$.

2.2 Generalization of Willmore's theorem

Proposition 2.1. *Let's assume $X \in \psi^0$ and $L \in \psi^1$. Then $i_X L^* = L^* i_{LX}$.*

Proof. Taking into account the definitions of the inner product i_X and the endomorphism L^* in the Definition 2.1, we can write successively:

$$\begin{aligned}\left(i_X L^*\omega\right)\left(X^1, \ldots, X^{p-1}\right) &= \left(L^*\omega\right)\left(X, X^1, \ldots, X^{p-1}\right) \\ &= \omega\left(LX, LX^1, \ldots, LX^{p-1}\right) \\ &= \left(L^* i_{LX}\omega\right)\left(X^1, \ldots, X^{p-1}\right),\end{aligned}$$

where $X^1, \ldots, X^{p-1} \in \psi^0$ and $\omega \in \phi^p$. □

Proposition 2.2. *If $[L, L] = 0$ then $L^* d = d_L L^*$.*

Proof. Let's consider $\omega \in \phi^p$ and $X^0, \ldots, X^p \in \psi^0$, the symbol $\widehat{\ }$ means omission. We explain the expression $d_L L^*$:

$$\left(d_L L^*\omega\right)\left(X^0, \ldots, X^p\right) = \left(\left(i_L d - d i_L\right) L^*\omega\right)\left(X^0, \ldots, X^p\right). \tag{1}$$

By using formulas of derivations i_L and d, the equation (1) becomes

$$\begin{aligned}&\sum_{0\leq u\leq p} (-1)^u LX^u.\omega\left(LX^0, \ldots, \widehat{LX^u}, \ldots, LX^p\right) \\ &+ \sum_{0\leq u<v\leq p} (-1)^{u+v}\omega\left(L\left[LX^u, X^v\right] + L\left[X^u, LX^v\right]\right. \\ &\left.-L^2\left[X^u, X^v\right], LX^0, \ldots, \widehat{LX^u}, \ldots, \widehat{LX^v}, \ldots, LX^p\right) \\ &= \sum_{0\leq u\leq p} (-1)^u LX^u.\omega\left(LX^0, \ldots, \widehat{LX^u}, \ldots, LX^p\right) \\ &+ \sum_{0\leq u<v\leq p} (-1)^{u+v}\omega\left(\left[LX^u, LX^v\right], LX^0, \ldots, \widehat{LX^u}, \ldots, \widehat{LX^v}, \ldots, LX^p\right)\end{aligned}$$

$$-\sum_{0\leq u<v\leq p}(-1)^{u+v}\,\omega\{\frac{1}{2}\left[L,L\right]\left(X^u,X^v\right),LX^0,\ldots,\widehat{LX^u},\ldots,\widehat{LX^v},\ldots,LX^p\}.$$

This last equality is due to the formula of 2.4. If Nijenhuis torsion of L is null then $\left(d_L L^*\omega\right)\left(X^0,\ldots,X^p\right)=\left(L^*d\omega\right)\left(X^0,\ldots,X^p\right)$. □

Remark 2.1. If L is reversible then

$$L^*dL^{*-1}=d_L. \tag{2}$$

It is the result which makes it possible to obtain immediately Willmore's theorem (cf. [22]). Indeed, if $d_L\omega=0$ then $dL^{*-1}\omega=0$. By applying Poincaré's Lemma, there is $\pi\in\phi^{p-1}$ such that $L^{*-1}\omega=d\pi$. In other words, taking into account equation (2), we obtain $\omega=L^*d\pi=d_LL^*\pi$.

Proposition 2.3. *For all z of the manifold M, there is an open-set U containing z and an endomorphism K of $\phi(U)$ verifying:*

1. *For $p\geq 1$, $K\left(\phi^p(U)\right)\subset\phi^{p-1}(U)$,*
2. *For every p-form ω on U, there is a p-form ω_0 determined by ω such that $\omega=Kd\omega+dK\omega+\omega_0$ and $i_L\omega_0=0$;*
3. *If $L^*\omega=0$ then $L^*K\omega=0$.*

For the demonstration, we use the following classic result:

Lemma 2.1 ([7])**.** *We consider J_h, with $h=0,1$, the injections of $\mathbb{R}^{n+m}$ in $\mathbb{R}^{n+m}\times\mathbb{R}$ defined by $J_h(z)=(z,h)$. There is a linear application $k:\phi\left(\mathbb{R}^{n+m}\times\mathbb{R}\right)\longrightarrow\phi\left(\mathbb{R}^{n+m}\right)$ having the following properties:*

1. *$k\left(\phi^p\left(\mathbb{R}^{n+m}\times\mathbb{R}\right)\right)\subset\phi^{p-1}\left(\mathbb{R}^{n+m}\right)$, $p\geq 1$;*
2. *$dk+kd=J_1^*-J_0^*$.*

For understanding the next, we will give the definition of k. We define the application $k:\phi\left(\mathbb{R}^{n+m}\times\mathbb{R}\right)\longrightarrow\phi\left(\mathbb{R}^{n+m}\right)$ by:

- $kf=0$ if $f\in\mathcal{F}\left(\mathbb{R}^{n+m}\times\mathbb{R}\right)$,
- $k\alpha=0$ if $\alpha=\lambda_{A,I}dx^A\wedge dy^I$, A and I denote the multi-indexes;
- $k\alpha=\int_0^1\lambda_{A,I}dtdx^A\wedge dy^I$ if $\alpha=\lambda_{A,I}dt\wedge dx^A\wedge dy^I$.

Proof of the Proposition 2.3. Let U be an open set containing z, which is the compatible chart domain of M. We denote by H the differentiable application of $U\times\mathbb{R}$ on U defined by $H\left(x^a,y^i,t\right)=\left(x^a,ty^i\right)$. HJ_1 is the identical

application of U, and HJ_0 the application which has $\left(x^a, y^i\right) \longrightarrow (x^a, 0)$. Then we have $(HJ_1)^* \omega = \omega$. By setting $\omega = \lambda_{A,I} dx^A \wedge dy^I$, we obtain

$$(HJ_0)^* \omega = \begin{cases} 0, & \text{if } I \neq \emptyset \\ \lambda_{A,\emptyset}(x,0)\, dx^A, & \text{otherwise.} \end{cases}$$

So, by considering the previous Lemma, we can write $\omega = (J_1^* - J_0^*) H^*\omega + (HJ_0)^* \omega = dkH^*\omega + kdH^*\omega + (HJ_0)^* \omega$.

As $dH^* = H^*d$ we obtain $\omega = dkH^*\omega + kH^*d\omega + (HJ_0)^* \omega$. Pose $K\omega = kH^*\omega$ and $\omega_0 = (HJ_0)^* \omega$, then ω_0 thus defined is zero or of the form $\lambda_A(x,0)\, dx^A$, so $i_L\omega_0 = 0$, and we have the following expression:

$$\omega = dK\omega + Kd\omega + \omega_0.$$

If $L^*\omega = 0$, ω is then of the form $\omega = \lambda_{A,I} dx^A \wedge dy^I$ with $A \neq \emptyset$. According to the definition of K and for such ω, $L^*K\omega = 0$. □

Proposition 2.4. *Let's consider $\omega \in \phi^p(U)$ with $p \geq 1$. If $L^*\omega$ is d_L-closed, we have $\pi \in \phi^{p-1}(U)$ such that $L^*\omega = d_L(L^*\pi)$.*

Proof. According to 2. of the Proposition 2.3, ω can be written $\omega = dK\omega + Kd\omega + \omega_0$. Since $L^*\omega_0 = 0$ then $L^*\omega = L^*Kd\omega + L^*dK\omega$. By applying Proposition 2.2, namely $L^*d = d_L L^*$, we have $L^*d\omega = d_L L^*\omega = 0$. The last equality is due to the hypothesis on $L^*\omega$. And considering 3. of the Proposition 2.3, $L^*Kd\omega = 0$. Finally, according to the Proposition 2.2 we have $L^*\omega = d_L(L^*K\omega) = d_L(L^*\pi)$ by assuming $\pi = K\omega$. □

Definition 2.5. *A p-form ω is said to be L-semi-basic if for any vector fields X in the kernel of L, the interior product $i_X\omega$ is zero.*

Remark 2.2. If L is a tangent structure, an L-semi-basic form is, in the usual sense, semi-basic.

Proposition 2.5. *The following two relationships are equivalent:*

1. *ω is an L-semi-basic p-form,*
2. *There is a p-form π such that $\omega = L^*\pi$.*

Proof. 2. $\Longrightarrow$ 1. For that, we just need to see if X is into the kernel of L, according to Proposition 2.1, we have $i_X L^* = L^* i_{LX} = 0$.

1. $\Longrightarrow$ 2. With the appropriate coordinate system, $L^*dx^a = 0,\ a = 1, \ldots, n$; and $L^*dy^i = d_L y_i,\ i = 1, \ldots, m$ form a base of the orthogonal space at the kernel of L. Let's represente this kernel by N, the dual by N^*, the orthogonal by $N^\perp$; then $N_z^\perp \oplus N_z^* = \phi_z^1$ with a base of ϕ_z^1 adapted to the direct sum, the semi-basic condition with respect to L is translated by $\omega = \lambda_{\emptyset,\, I} d_L y^I = L^*\left(\lambda_{\emptyset,\, I} dy^I\right) = L^*\pi$ by assuming $\pi = \lambda_{\emptyset,\, I} dy^I$. □

Theorem 2.1. *Let $\mathcal{F}$ be the set of functions on M, $\mathcal{F}_L$ the set of constant functions on the leaves, ϕ_L^p the set of L-semi-basic p-forms. So, for every point z of M, there is an open U containing z such that the following sequence is exact:*

$$0 \longrightarrow \mathcal{F}_L(U) \longrightarrow \mathcal{F}(U) \xrightarrow{d_L} \phi_L^1(U) \xrightarrow{d_L} \ldots \xrightarrow{d_L} \phi_L^{m-1}(U) \xrightarrow{d_L} \phi_L^m(U) \xrightarrow{d_L} 0.$$

Proof of Theorem 2.1. Let U be a domain of the adapted chart to foliation and $\omega \in \phi_L^p(U)$. According to Proposition 2.5 there is $\alpha \in \phi^p(U)$ such that $\omega = L^*\alpha$. If ω is d_L-closed, by Proposition 2.4, there is $\pi \in \phi^{p-1}$ such that $\omega = d_L(L^*\pi)$. As indicated in Proposition 2.5, $L^*\pi \in \phi_L^{p-1}$. In addition, ω being L-semi-basic, $d_L\omega$ is also L-semi-basic, according to Propositions 2.1 and 2.4, $d_L\omega = d_L L^*\alpha = L^* d\alpha$. To finish the cohomological resolution, we just need to note if the dimension of $\phi_L^1(U)$ is equal to m and $d_L f = 0$ is equivalent to $f \in \mathcal{F}_L$.

Theorem 2.2. *On every leave, the cohomology of d_L is isomorphic to that of de Rham.*

Proof of Theorem 2.2. Indeed, a leaf S can be considered as a submanifold of M. The application $L^* : \phi^p(S) \longrightarrow \phi_L^p(S)$ is an isomorphism. In applying Proposition 2.2 we find the result.

2.3 Homogeneous forms

In this section, we assume that the leaves of L are the fibers of a vector fibration of M. The group of homotheties on U: $H_t\left(x^a, y^i\right) = \left(x^a, e^t y^i\right)$ with $t \in \mathbb{R}$ defines a Lie group to a parameter of transformations on M. Let's consider C as a vector field that generates this group. If S stands for a vector field such that $LS = C$, then $i_S\omega$ depends only on C, for any L-semi-basic form. In other words, $i_S\omega = i_{S'}\omega$ if $LS = LS' = C$. This derives from Propositions 2.1 and 2.5.

Definition 2.6. *A scalar form ω is said to be homogeneous of degree r, if the Lie derivative of ω with respect to C is such that $\theta_C\omega = r\omega$.*

A vector form K is said to be homogeneous of degree k if $[C, K] = kK$.

Theorem 2.3. *If L is homogeneous of degree l, any p-form ω with $p \geq 1$, L-semi-basic, homogeneous of degree $r \neq lp$ which is d_L-closed, is d_L-exacte and $(r - lp)\,\omega = d_L i_S\omega$, where S designates a vector field on M such that $LS = C$.*

Remark 2.3. This theorem gives a version of Theorem 2.1 for the forms globally defined in this particular case, and generalizes a result of J. Klein and A. Voutier, see [12], namely: If J is the vector valued 1-form defining the tangent structure of M, then $(r + p)\,\omega = d_J i_S\omega$.

To prove Theorem 2.3, we will use the following Lemma:

Lemma 2.2. $L\,[S,L] = lL$.

Proof. From (2.4) and the relation $LS = C$, we have $\frac{1}{2}\,[L,L]\,(S,X) = [C,LX] - L\,[C,X] + L^2\,[S,X] - L\,[S,LX]$ for all $X \in \psi^0$. Now, according to the formulas of [5], we have $[C,L]\,X = [C,LX] - L\,[C,X]$ and $[S,L]\,X = [S,LX] - L\,[S,X]$; then nullity of the Nijenhuis torsion implies $L\,[S,L] = [C,L]$. But L is homogeneous of degree l by hypothesis. Hence $L\,[S,L] = lL$. □

Demonstration of Theorem 2.3. We have the following formula (cf. [5] p.352)

$$i_S d_L\omega + d_L i_S\omega = \theta_C\omega - i_{[S,L]}\omega. \tag{3}$$

Since ω is homogeneous of degree r, then $\theta_C\omega = r\omega$. Moreover ω is L-semi-basic, according to Proposition 2.5 there is $\pi \in \phi$ such that $\omega = L^*\pi$. Thus, from the formula 2.2 and the Lemma, we can write

$$\begin{aligned} &i_{[S,L]}\omega\left(X^1,\ldots,X^p\right) \\ &= \sum_{u=1}^{p} \pi\left(LX^1,\ldots,L\,[S,L]\,X^u,\ldots,LX^p\right) = lpL^*\pi\left(X^1,\ldots,X^p\right), \end{aligned}$$

for all $X^1,\ldots,X^p \in \psi^0$.

Finally, the expression (3) becomes $i_S d_L\omega + d_L i_S\omega = (r-lp)\,\omega$.

If $d_L\omega = 0$ then $d_L i_S\omega = (r-lp)\,\omega$. □

Remark 2.4. The result of Theorem 2.3 is not true if $r - lp = 0$. Indeed, $\mathbb{R}^2$ the manifold and ω the following form on $T\mathbb{R}^2 \setminus \{0\}$: $\omega = \frac{y^1dx^1 - y^2dx^2}{(y^1)^2+(y^2)^2}$. It is clear that ω is C^∞ on $T\mathbb{R}^2 \setminus \{0\}$, homogeneous of degree -1 and d_j-closed, where J denotes the tangent structure. It is well known that ω does not admit a primitive on $T\mathbb{R}^2 \setminus \{0\}$.

2.4 Fröbenius' theorem on L-semi-basics Pfaff system

We present here a version of Fröbenius' theorem with the d_L derivation.

Definition 2.7. *The L-semi-basic Pfaff system of rank $s \leq m$, $(\omega^u)_{u=1,\ldots,s}$ is said to be completely integrable in an open U provided that there is on U s independent differentiable functions f^u such that $\omega^u = \lambda^u_v d_L f^v$, $v = 1,\ldots,s$; where λ^u_v are the differentiable functions on U and elements of a regular matrix.*

We note that s must be less than or equal to m in order to get such property, otherwise the system $(\omega^u)_{u=1,\ldots,s}$ would be bound.

Theorem 2.4. *If an L-semi-basic Pfaff system is fully integrable into $U \subset M^{n+m}$, it passes through each generic point $z \in U$ an integral manifold of dimension $m-s$ defined in a neighborhood of this point which is a submanifold of M^{n+m}.*

Proof of Theorem 2.3. The leaves of foliation may be considered as submanifolds of M. Let S be a submanifold passing through $z \in U$, the restriction of $L^* : \phi^1(S) \longrightarrow \phi^1_L(U)$ is an isomorphism. We consider $\tilde{L^*}$ this isomorphism, $\tilde{L^*}^{-1}\omega^u = \lambda^u_v dy^v$ is a completely integrable system. According to the classical Fröbenius theorem, it passes through z an integral variety of dimension $m-s$ which is a submanifold of the submanifold S of M. This submanifold is also an integral variety of ω^u. Indeed, if we consider (W, g) as an integral variety of $\tilde{L^*}^{-1}\omega^u$, we have: $g^*\tilde{L^*}^{-1}\omega^u = 0$ on W, that is, $\lambda^u_v \circ gd(y^v \circ g) = 0$ on W. So, we can write $\tilde{L^*}g^*\tilde{L^*}^{-1}\omega^u = \lambda^u_v \circ gd_L(y^v \circ g) = g^*\omega^u = 0$ on W.

Hence the result.

Theorem 2.5. *A necessary and sufficient condition for an L-semi-basic Pfaff system to be completely integrable in the neighborhood of a generic point, is that in a neighborhood of this point the system is d_L-closed, that is*

$$d_L\omega^u \wedge \omega^1 \wedge \cdots \wedge \omega^s = 0,\ u = 1, \ldots, s.$$

The proof is implied directly from the classical Fröbenius theorem via isomorphism $\tilde{L^*}$ previously used.

2.5 Projection of a vector valued 1-form

In this section, we will presente some results on a projectable vector valued 1-form and establish homotopic links for the d_L-cohomology; the study of projection of a vector valued 1-form is done in [6], [13]. Let μ be a differentiable application of a manifold M in N, μ_* the tangent linear application, μ^* the cotangent linear application.

Definition 2.8. *A vector l-form L on M is projectable provided that there is a vector l-form L' on N such that*

$$\mu_* L_z\left(X^1, \ldots, X^l\right) = L'_{\mu(z)}\left(\mu_* X^1, \ldots, \mu_* X^l\right),$$

for all $X^1, \ldots, X^l \in T_z(M)$ and for any $z \in M$.

Proposition 2.6 ([6])**.** *Let $i : M \longrightarrow N$ be an application of rank equals to the dimension of M, and L' a vector valued 1-form on N. There is a unique vector valued 1-form L on M which projects on L' if and only if $L'(X) \in i_*(TM)$ for all $X \in i_*(TM)$.*

Proposition 2.7 ([6])**.** *Let $\mathfrak{p} : M \longrightarrow N$ be a fibration to the connected bundles. A vector valued 1-forme L is projectable if and only if $L(Y)$ is a vertical field and $\theta_Y L$ has vertical values for all vertical field Y.*

Proposition 2.8. *If L is a projectable vector valued 1-form, then we have $\mu^* d_{\mu_* L} = d_L \mu^*$ and $\mu_*[L, L] = [\mu_* L, \mu_* L]$ where $\mu_* L$ is the projection of L by μ.*

Proof. The proof is obtained from Theorem 1, p.233 of [6]. □

Theorem 2.6. *Let L be a μ-projectable vector valued 1-form, H_L the algebra of d_L-cohomology of L-semi-basic forms, μ^* a homomorphism of algebras of $H_{\mu_* L}(N) \longrightarrow H_L(M)$.*

Proof of Theorem 2.4. The Proposition 2.8 proves that the manifold N is $\mu_* L$-foliated, that the image of $d\mu_* L$-closed form is a d_L-closed form, and that the image of $d\mu_* L$-exact form is d_L-exact form. Furthermore, by doing a simple calculation, we also have $\mu^*((\mu_* L)^* \omega) = L^*(\mu^* \omega)$, for all $\omega \in \phi(N)$. Hence the result.

2.6 Derivations having the types d' and d'' on a foliated manifold

Let M be a paracompact differentiable manifold of $n+m$ dimension, $\mathcal{D}$ an integrable distribution of m dimension on M. The variety M thus defined is foliated. By giving on M a riemannian metric, we can define a distribution $\mathcal{D}^\perp$ orthogonal to $\mathcal{D}$. The variety M is therefore provided with an almost produced structure Γ (cf. [20]). We assume $h = \frac{1}{2}(I + \Gamma)$ and $v = \frac{1}{2}(I - \Gamma)$ where I designates the identical application; h and v are the projectors of respectively ranks n and m. The distribution $\mathcal{D}^\perp$ is completely integrable if and only if Nijenhuis torsion of Γ is zero. In other words, $[h,h] = [v,v] = 0$. It is assumed in the following that Nijenhuis torsion of Γ is zero.

Proposition 2.9. *$h^* d_v = 0$ and $v^* d_h = 0$.*

Proof. This proposition derives from the properties of the projectors h and v, namely: $h \circ v = 0$, $v \circ h = 0$ and $[h,h] = [v,v] = 0$. Since they play symmetrical roles, all we need to do is to calculate $h^* d_v$. We assume ω as a scalar p-form. According to the definition of the derivation d_v, we have $h^* d_v \omega = h^*(i_v d - d i_v)\omega$. From relation $v \circ h = 0$ we obtain $h^* i_v d\omega = 0$. What is to be calculated next is $h^* d i_v \omega$. By doing a simple calculation, we get:

$$\begin{aligned}&(h^* d i_v \omega)\left(X^0, \ldots, X^p\right)\\ &= \sum_{0 \le u < t \le p} (-1)^{u+t}\, i_v \omega\left(\left[hX^u, hX^t\right], hX^0, \ldots, \widehat{hX^u}, \ldots, \widehat{hX^t}, \ldots, hX^p\right),\end{aligned}$$

for all $X^0, \ldots, X^p \in \psi^0$. Since Nijenhuis torsion of h is zero and $h^2 = h$ we have

$$\left[hX^u, hX^t\right] = h\left[hX^u, X^t\right] + h\left[X^u, hX^t\right] - h\left[X^u, X^t\right].$$

Hence, taking into account the relation $v \circ h = 0$ we get $h^* d i_v \omega = 0$. □

Proposition 2.10. *$v^* d_v = d_v v^*$ and $h^* d_h = d_h h^*$.*

Proof. From Proposition 2.9 and the relation $h = I - v$, we have

$$v^* d_{I-v} = 0 = v^* d - v^* d_v.$$

According to Proposition 2.2, we have $v^*d = d_v v^*$, that is $v^* d_v = d_v v^*$. It is the same for $h^* d_h = d_h h^*$. □

Theorem 2.7. *Let consider v a projector on M of constant rank m and of Nijenhuis torsion null. Any p-form ω with $p \geq 1$ which is d_v-closed, is locally d_v-exact if and only if $(I - v)^* \omega = 0$. Especially if $p > n$, any p-form, that is, d_v-closed is locally d_v-exact.*

Proof of Theorem 2.5. The condition is necessary. Indeed, if ω is a p-form such that $\omega = d_v \pi$, with $\pi \in \phi^{p-1}$, according to Proposition 2.9, we have $(I - v)^* \omega = h^* d_v \pi = 0$. The condition is sufficient. With the coordinates system of the section 2.1 and knowing that $dx^a = d_h x^a$, $d_v y^i = dy^i$, ω can be written: $\omega = \lambda_{A,I} d_v y^I \wedge d_h x^A$, where A, I stand for multi-indexes such that $|A| + |I| = p$. The condition $h^* \omega = 0$ means $|I| \geq 1$. Since $d_v (d_h x^a) = 0$, the condition d_v-closed ω becomes $d_v \omega = d_v \left(\lambda_{A,I} d_v y^I\right) \wedge d_h x^A = 0$. It implies that $d_v \left(\lambda_{A,I} d_v y^I\right) = 0$. According to Proposition 2.4 there is a π' form such that $d_v \pi' = \lambda_{A,I} d_v y^I$. So, the expression of ω becomes $\omega = d_v \pi' \wedge d_h x^A = d_v \left(\pi' \wedge d_h x^A\right)$. If $p > n$, it is immediate to note that $h^* \omega = 0$.

Theorem 2.8. *Let ω be a scalar p-form ($p \geq 1$) d_v-closed. There is locally a scalar $(p-1)$-form π and a p-form ω' such that*

1. *$\omega = d_v \pi + \omega'$;*
2. *ω' a basic p-form of the foliation defined by the distribution $\mathcal{D}$.*

Proof of Theorem 2.6. On a suitable open U, any p-form ω can be written $\omega = \lambda_A dx^A + \mu_{B,I} dx^B \wedge d_v y^I$ where A, B, I denote the multi-indexes such that $|A| = p, |B| + |I| = p$ and $|I| \geq 1$.

If ω is d_v-closed, we have $d_v (\lambda_A) \wedge dx^A + d_v \left(\mu_{B,I} dx^B \wedge d_v y^I\right) = 0$. That implies $d_v (\lambda_A) = 0$ and $d_v \left(\mu_{B,I} dx^B \wedge d_v y^I\right) = 0$. In other words, $\lambda_A dx^A$ is a constant p-form on the leaves defined by the distribution $\mathcal{D}$, and $\mu_{B,I} dx^B \wedge d_v y^I$ is a p-form which makes the conditions of Theorem 2.7 satisfying. Consequently, by assuming $\omega' = \lambda_A dx^A$, there is π a $(p-1)$-form such that $\omega = d_v \pi + \omega'$.

Remarks 2.1.

1. The derivations d_v, d_h are classical derivations d', d'' on foliated manifolds. The obtained results make it possible to study the scalar forms without having to introduce the notion of scalar forms of type (p, q).

2. Let M be a complex paracompact differentiable manifold and F be an almost complex structure $\left(F^2 = -I\right)$. The projectors associated with F are $P = \frac{1}{2}(I - iF)$ and $Q = \frac{1}{2}(I + iF)$ where $i = \sqrt{-1}$. The nullity of Nijenhuis torsion of F is equivalent to $[P, P] = [Q, Q] = 0$.

   As for the derivations d_P and d_Q, we get results similar to those of the derivations d_v and d_h. The classical derivations d' and d'' are found on complex manifolds.

2.7 Some types of d_L-cohomologies

Proposition 2.11. *On an L-foliated manifold M, the following conditions are supposed to be satisfactory:*

1. *There is a Pfaff system of rank n, (θ^a), $a = 1, \ldots, n$ such that, (θ^a), $a = 1, \ldots, n$ and $\left(d_L y^i\right)$, $i = 1, \ldots, m$ form a basis of $\phi^1(U)$;*
2. *the 1-forms (θ^a), $a = 1, \ldots, n$ are d_L-closed.*

Then for all p-form ω $(p \geq 1)$ on M, d_L-closed, there is locally a $(p-1)$-form π and a p-form ω' such that

i) $\omega = d_L \pi + \omega'$;

ii) ω' *is a p-form generated by* (θ^a) *on the constant functions to the leaves.*

Proof. This is a version of Theorem 2.8 in its general cases. The demonstration is done with the same principle. □

Proposition 2.12. *If the kernel of L noted N is completely integrable, then for all p-form ω $(p \geq 0)$ and for any $X^i \in N$, we have $d_L\omega\left(X^0, \ldots, X^p\right) = 0$.*

Proof. For that, we just calculate $d_L\omega\left(X^0, \ldots, X^p\right)$ for all $X^0, \ldots, X^p \in N$ and we find

$$\begin{aligned} & d_L\omega\left(X^0, \ldots, X^p\right) \\ & = -\sum_{0 \leq i < j \leq n} (-1)^{i+j}\, \omega\left(L\left[X^i, X^j\right], X^0, \ldots, \widehat{X^i}, \ldots, \widehat{X^j}, \ldots, X^p\right). \end{aligned}$$

If the kernel is fully integrable, we get $d_L\omega\left(X^0, \ldots, X^p\right) = 0$. □

Theorem 2.9. *Let M be an L-foliated manifold. If L verifies one of the following two conditions:*

1. *L is regular on the leaves;*
2. *The image of L equals to the kernel of L.*

Then all p-form ω $(p \geq 1)$ that is d_L-closed is locally d_L-exact if and only if $\omega\left(X^1, \ldots, X^p\right) = 0$ for any fields $X^1, \ldots, X^p$ in the kernel of L.

Proof of Theorem 2.7. If L confirms one of the two above conditions, the kernel of L is completely integrable according to the nullity of Nijenhuis torsion of L. In this case $d_L\pi\left(X^1, \ldots, X^p\right)$ is null for all $(p-1)$-form π and for all fields $X^1, \ldots, X^p$ in the kernel of L as stated in Proposition 2.12. In addition, there are functions $f^1, \ldots, f^n$ such that df^a, $a = 1, \ldots, n$ and $\left(d_L y^i\right)$, $i = 1, \ldots, m$ form a local basis of $\phi^1(M)$ and that $d_L df^a = 0$. Hence the result by considering Proposition 2.11.

Theorem 2.10. *Let M be an L-foliated manifold such that L is regular on the leaves. The algebra of d_L-cohomology is locally isomorphic to that of the basic forms.*

Proof of Theorem 2.8. With the coordinates system in Section 2.1, we note that $\left(dx^a, d_L y^i\right)$, $a = 1, \ldots, n$ and $i = 1, \ldots, m$ form a basis of $\phi^1(U)$. Since $d_L(dx^a) = 0$, as indicated in Proposition 2.11, we get the result we've been searching for.

Proposition 2.13. *If the leaves of L are simply connected and at the same time fibers of a fibration of M, then all L-semi-basic p-form $(p \geq 1)$ which is d_L-closed is generally d_L-exact.*

Proof. This is a consequence of Theorems 2.1 and 2.2, because the derivation of d_L acts only on the fibers, which makes it possible to paste again the pieces by means of the unit partition. □

Corollary 2.1. *Let M be a paracompact differentiable manifold, Γ a connection of Grifone $\left(\Gamma^2 = I\right)$ on M with null curvature, v the vertical projector. For all p-form ω on TM, d_v-closed, with $p \geq 1$, there is a $(p-1)$-form π and a p-form ω' such that*

1. $\omega = d_v \pi + \omega'$
2. $\omega' = \mathfrak{p}^* \omega''$ *where* $\omega'' \in \phi^p(M)$, $\mathfrak{p}$ *is the projection of* $TM \longrightarrow M$

The cohomology space $H^p_{d_v}(TM)$ is therefore isomorphic to $\phi^p(M)$.

Proof. This is a version of Theorem 2.8 for globally defined forms, because the leaves are fibers of the tangent space to M and the basic forms are the elements of $\mathfrak{p}^* \phi(M)$. □

Corollary 2.2 ([2])**.** *Let J be the vector valued 1-form defining on TM the natural structure of tangent variety. Any p-form ω on TM $(p \geq 1)$ is exact if and only if $d_J \omega = 0$ and $J^* \omega = 0$.*

Proof. This is a version of Theorem 2.9 for the tangent structure J. Here, the condition of Proposition 2.12 is translated by $J^* \omega = 0$. □

Theorem 2.11. *Let ω be a d_J-closed p-form, with $p \geq 1$. There is a $(p-1)$-form π and a p-form ω' such that*

1. $\omega = d_J \pi + \omega'$
2. $J^* \omega' = \mathfrak{p}^* \omega''$ *where* $\omega'' \in \phi^p(M)$.

The cohomology space $H^p_{d_J}(TM)$ is therefore isomorphic to $\phi^p(M)$.

Proof of Theorem 2.9. This is a consequence of Proposition 2.11 and Theorem 2.9.

2.8 Poincaré's lemma for $d_L\omega = F(z,\omega)$

In this section, we provide in d_L-derivations a version of Jacobowitz's theorem on the existence of local solutions for non-linear problems having the following types: $d\omega = F(\omega)$ (cf. [10]). Let (M,L) be an L-foliated manifold, ϕ_L^p the set of L-semi-basic p-forms on M with $p \geq 0$, F a differentiable application of $\phi_L^p \longrightarrow \phi_L^{p+1}$, $G_L^p(z)$ the set of germs of L-semi-basic p-forms at the point z, M' a submanifold of M. For all forms α and $\beta \in \phi_L^p$, we denote by $\alpha_{|_{M'}} = \beta_{|_{M'}}$ if $i^*\alpha = i^*\beta$, where i means the canonical injection of M' in M.

Definition 2.9. *We say that F satisfies the condition of compatibility $(\star)$ at point z, if $d_L F(\omega) = 0$ at z for all $\omega \in G_L^p(z)$ such that $(d_L\omega - F(\omega))(z) = 0$.*

Theorem 2.12. *We suppose that F verifies $(\star)$ for all point z of M. Let M' be a submanifold of M passing through a point z_0. If $\omega_0 \in G_L^p(z_0)$ satisfies $d_L\omega_{0|_{M'}} = F(\omega_0)_{|_{M'}}$, then there is $\omega \in G_L^p(z)$ such that $d_L\omega = F(\omega)$, and $\omega = \omega_0$ on M' ($\omega(z) = \omega_0(z)$ for all $z \in M'$).*

Corollary 2.3. *If $\omega_0 \in \phi_L^p(z_0)$ and F verifies $(\star)$ at neighborhood of z_0, so there is $\omega \in G_L^p(z)$ such that $d_L\omega = F(\omega)$ and $\omega(z_0) = \omega_0$.*

The Corollary 2.3 is inferred from Theorem 2.12 by posing $M' = \{z_0\}$.

Particular cases 2.1.

1. *If F does not depend on ω, that is, if $F(\omega) = \beta$, where β is an L-semi-basic $(p+1)$-form, the condition $(\star)$ becomes $d_L\beta = 0$. Thus by adding an initial condition, we get the Theorem 2.1.*
2. *For $p = 0$, we obtain a version of Fröbenius' theorem.*

Proof of Theorem 2.10. Let U be a compatible chart of the L-foliated manifold M, z a point of U. We have $z = (x,y)$, where y means the coordinates of z on the leaves. An element β of $G_L^p(z)$ is written: $\beta \underset{U}{=} \beta_I(x,y)\, d_L y^I$, I being a multi-index $(i_1,\ldots,i_p)$ with $1 \leq i_1,\ldots,i_p \leq m$ and $|I| = p$. Similarly, if $F(z,\beta)$ is L-semi-basic, we have $F(z,\beta) \underset{U}{=} F_J(x,y,\beta)\, d_L y^J$, with $|J| = p+1$. Each F_J is a function of coefficients of the form β and the point z of M: $F_J\left(x^1,\ldots,x^n,y^1,\ldots,y^m,\beta_{K_1},\ldots,\beta_{K_p}\right) \underset{U}{=} F_J\left(\beta_K d_L y^K\right)$, $|K| = p$. Since the derivation d_L works only on the variables $\left(y^i\right)$, $i = 1,\ldots,m$, that is, for all function $f \in \mathcal{F}(M)$, $d_L f(x,y) = \frac{\partial f}{\partial y^i}(x,y)\, d_L y^i$, the components $\left(x^a\right)$, $a = 1,\ldots,n$ play the role of parameters. Thus, our study can be reduced to the results of Jacobowitz's work with parameters, see [10].

Proposition 2.14. *Let F be a differentiable application of $\phi^p(M)$ in $\phi^{p+1}(M)$, with $p \geq 0$. If L confirms one of the following two conditions:*

1. *L is regular on the leaves;*
2. *the image of L equals to the kernel of L;*

then we have Poincaré's lemma for $d_L\omega = F(\omega)$, if F verifies the condition of compatibility $(\star)$ and $F(\beta)\left(X^0,\ldots,X^p\right) = 0$ for all p-form β and for all vector fields $X^0,\ldots,X^p$ in the kernel of L.

Proof. We have seen during the demonstration of Theorem 2.9 that for such L, there are functions $f^1,\ldots,f^n$ such that $df^a,\ a = 1,\ldots,n$ and $d_L y^i,\ i = 1,\ldots,m$ form a local basis of $\phi^1(M)$ and that $d_L(df^a) = 0$. It is possible to write $\beta = \beta_{I,A} d_L y^I \wedge df^A$ for all p-form β , where I, A designate multi-indexes $I \subset [1,m]\,,\ A \subset [1,n]$ such that $|I| + |A| = p$. Similarly, we have $F(z,\beta) = F_{J,B}(z,\beta)\, d_L y^J \wedge df^B$ with $|J|+|B| = p+1$. The condition $F(\beta)\left(X^0,\ldots,X^p\right) = 0$ for any p-form β and for all vector fields $X^0,\ldots,X^p$ in the kernel of L means that the length of J, $|J| \geq 1$. Since $d_L(df^a) = 0,\ a = 1,\ldots,n$, the relation $d_L F(z,\beta) = 0$ is equivalent to $d_L\left(F_{J,B}(z,\beta)\, d_L y^J\right) = 0$.

Similarly, the relation

$$(d_L\beta - F(\beta)) = 0$$

implies $\left(d_L\left(\beta_{I,A} d_L y^I\right) - F_{J,A} d_L y^J\right)(z) = 0$. Thus, the condition of compatibility $(\star)$ for F is translated by the condition $(\star)$ for L-semi-basic forms. Hence the result, by taking Theorem 2.12 into account. □

Remark 2.5. If L confirms the conditions of Proposition 2.11, it is obvious from the demonstration of Proposition 2.14, to have a similar result.

Particular cases 2.2. Let $F : \phi^p(M) \longrightarrow \phi^{p+1}(M)$, G_z^p be the set of germes of p-forms at point z.

1. F satisfies to the condition of compatibility at point z for a projector v, if $d_v F(\omega) = 0$ at point z for all $\omega \in G_z^p$ such that $(d_v\omega - F(\omega))(z) = 0$ and if $(I - v)^* F(\omega) = 0$ for all $\omega \in G_z^p$.

2. If L and its image space equals to the kernel of L then the latter condition becomes $L^* F(\omega) = 0$ for all $\omega \in G_z^p$.

3. If F does not depend on ω, we obtain Theorem 2.9 by adding an initial condition.

The Proposition 2.14, according to Remark 2.5 generates an extension of Jacobowitz's theorem for the classical derivations having the types d' and d''.

3. Chevalley-Eilenberg cohomologies defined by vector valued 1-form

3.1 Generalities

Definition 3.1. *Let $\mathfrak{A}$ be a Lie $\mathbb{R}$-algebra. A p-cochain C of $\mathfrak{A}$ is an alternated p-linear application of $\mathfrak{A}^p$ in $\mathfrak{A}$ such that, if $p = 0$, C is being itself identified in an element of $\mathfrak{A}$.*

The cobord operator ∂ is defined by

$$\partial C\left(X^0,\ldots,X^p\right)=\sum_{0\leq i\leq p}(-1)^i\left[X^i,C\left(X^0,\ldots,\widehat{X^i},\ldots,X^p\right)\right]$$
$$+\sum_{0\leq i<j\leq p}(-1)^{i+j}C\left(\left[X^i,X^j\right],X^0,\ldots,\widehat{X^i},\ldots,\widehat{X^j},\ldots,X^p\right),$$

where $X^i\in\mathfrak{A},\ i=0,1,\ldots,p$; and the symbol $\widehat{\ }$ designates the omission.

It implies that $\partial^2=0$. We will denote by $\mathcal{H}^p(\mathfrak{A})$ the corresponding cohomology spaces. It is called Chevalley-Eilenberg's cohomology. If $p=0$, $\partial C=-ad_X$, where ad_X is the adjoint linear application $Y\longmapsto[X,Y]$. If $p=1$, then $\partial C[X,Y]=[CX,Y]+[X,CY]-C[X,Y]$ for all $X,Y\in\mathfrak{A}$. Thus, $\partial C=0$ means that C is a derivation of Lie algebra $\mathfrak{A}$ and $\mathcal{H}^1(\mathfrak{A})=\{0\}$ is equivalent to saying that any derivation of $\mathfrak{A}$ is inner.

3.2 Lie algebra defined by a vector valued 1-form

Let M be a connected, paracompact, differentiable manifold with $m+n$ dimension; L a vector valued 1-form of rank $m\geq 1$; θ_X Lie derivation with respect to the vector field X on M. The relation $\theta_X L=0$ is also written like this: $[X,L]=0$, in other words, for all $Y\in\chi(M)$, $[X,LY]=L[X,Y]$.

We denote by $\mathfrak{A}_L$ the set of vector fields X such that $\theta_X L=0$. According to Jacobi's identity, $\mathfrak{A}_L$ is a real Lie algebra. The study of such algebra is interesting (cf. [11]).

We suppose in the following sequence that Nijenhuis torsion of L is zero. We saw in the section 2.1 that the nullity of Nijenhuis torsion of L defines a foliation with m dimension. A. Lichnerowicz considered in [15] $\mathcal{L}$ Lie algebra of tangent vector fields to a foliation, $\mathcal{L}_{\mathcal{F}}$ Lie algebra of the infinitesimal automorphisms of foliation, that is, $X\in\mathcal{L}_{\mathcal{F}}$ if and only if $\mathcal{L}\supset[X,\mathcal{L}]$. He proved in particular that whatever the considered foliation is, $\mathcal{H}^1(\mathcal{L})\approx\mathcal{L}_{\mathcal{F}}/\mathcal{L}$, and $\mathcal{H}^1(\mathcal{L}_{\mathcal{F}})=\{0\}$. $\mathfrak{A}_L$ is a subalgebra of $\mathcal{L}_{\mathcal{F}}$ of the L-foliated variety. If L is the vector valued 1-form of the fibrous transverse to a foliation ($rang(L)=n$ et $L^2=0$), J. Lehmann-Lejeune [14] proved that the derivations of $\mathfrak{A}_L$ are adjoint linear applications of the normalizer of $\mathfrak{A}_L$ in $\chi(M)$. In this section, we will give some properties of $\mathfrak{A}_L$ that will be used later.

Proposition 3.1. *If $L\mathfrak{A}_L$ denotes the space image of $\mathfrak{A}_L$ by L, $L\mathfrak{A}_L$ is an ideal of $\mathfrak{A}_L$. In particular if $L^2=0$ then $L\mathfrak{A}_L$ is a commutative ideal.*

Proof. From the definition of $\mathfrak{A}_L$, we have immediately $[X,L\mathfrak{A}_L]\subset L\mathfrak{A}_L$ for all $X\in\mathfrak{A}_L$. The nullity of Nijenhuis torsion of L, for all $X,\ Y\in\chi(M)$ can be written as follows: $[LX,LY]=L[LX,Y]+L[X,LY]-L^2[X,Y]$. If $X\in\mathfrak{A}_L$, the relation becomes for all $Y\in\chi(M)$: $[LX,LY]=L[LX,Y]$, that is, according to the definition of $\mathfrak{A}_L$, we have $L\mathfrak{A}_L\subset\mathfrak{A}_L$. Similarly, if $Y\in\mathfrak{A}_L$, we have $[LX,LY]=L^2[X,Y]$, which proves that $[L\mathfrak{A}_L,L\mathfrak{A}_L]\subset L\mathfrak{A}_L$ and that, if $L^2=0$, we get $[LX,LY]=0$. □

Proposition 3.2. *$\mathfrak{A}_L$ makes the kernel of L to remain stable.*

Proof. From the definition of $\mathfrak{A}_L$, we have $[X, LY] = L[X, Y]$, for all $X \in \mathfrak{A}_L$ and $Y \in \chi(M)$. If Y is an element of the kernel of L then we get $L[X, Y] = 0$. □

Proposition 3.3. *L is an endomorphism of modules*

- *from the centralizer of $\mathfrak{A}_L$ in $\chi(M)$,*
- *the normalizer of $\mathfrak{A}_L$ in $\chi(M)$,*
- *the algebra of derivations of $\mathfrak{A}_L$.*

Proof. If Y is a vector field commuting with $\mathfrak{A}_L$, as indicated in the definition of $\mathfrak{A}_L$, we have successively $[X, Y] = 0 = L[Y, X] = [LY, X]$ for all $X \in \mathfrak{A}_L$; this proves that LY commutes with $\mathfrak{A}_L$.

If Y is an element of the normalizer of $\mathfrak{A}_L$, for all $X \in \mathfrak{A}_L$, we have $[X, Y] \in \mathfrak{A}_L$; that is, LY is also an element of the normalizer of $\mathfrak{A}_L$.

Let D be a derivation of $\mathfrak{A}_L$. For all $X, Y \in \mathfrak{A}_L$ we get $D[X, Y] = [DX, Y] + [X, DY]$; according to the definition of $\mathfrak{A}_L$ elements, we can write $LD[X, Y] = L[DX, Y] + L[X, DY] = [LDX, Y] + [X, LDY]$; therefore, $L \circ D$ is also a derivation of $\mathfrak{A}_L$. □

3.3 Lie algebras associated with complete recovery of vector fields

Let M be a connected, paracompact, differentiable manifold with n dimension; $\chi(M)$ the module of vector fields on M; $\overline{\chi(M)}$ the complete recovery of $\chi(M)$ on the tangent bundle TM; C the canonical field generated by the group of homotheties with a parameter on TM. Uppon a domain U of a chart of M, we can write, if $X = X^i \frac{\partial}{\partial x^i}$ then $\overline{X} = X^i \frac{\partial}{\partial x^i} + y^j \frac{\partial X^i}{\partial x^j} \frac{\partial}{\partial y^i}$ and $C = y^i \frac{\partial}{\partial x^i}$, with $i, j = 1, \ldots, n$.

Proposition 3.4. *The centralizer of $\overline{\chi(M)}$ in $\chi(TM)$ is generated by C.*

Proof. The result is easily obtained by writing in local coordinates the relation

$$\left[Z, \overline{X}\right] = 0, \text{ avec } Z \in \chi(TM) \text{ et } \forall\, \overline{X} \in \overline{\chi(M)}. \quad \square$$

Proposition 3.5. *The normalizer of $\overline{\chi(M)}$ in $\chi(TM)$ is $\overline{\chi(M)} + \mathbb{R}C$.*

Proof. We consider $Z \in \chi(TM)$ such that for all $\overline{X} \in \overline{\chi(M)}$, we have $[Z, \overline{X}] \in \overline{\chi(M)}$. Since the elements of $\overline{\chi(M)}$ are homogeneous of degree 0, according to the identity of Jacobi, we get $[[C, Z], \overline{X}] = 0$ for all $\overline{X} \in \overline{\chi(M)}$. In conformity with Proposition 3.4, we obtain $[C, Z] = KC$ with $K \in \mathbb{R}$. Furthermore, Z is an element of normalizer of $\overline{\chi(M)}$, this is only possible if $Z \in \overline{\chi(M)} + \mathbb{R}C$. □

Theorem 3.1. *All derivations of* $\overline{\chi(M)} + \mathbb{R}C$ *are inner or having the* D *form such that* D *becomes null on* $\overline{\chi(M)}$ *and* $D(C) = KC$*, with* $K \in \mathbb{R}$*. The cohomology space of* $\mathcal{H}^1\left(\overline{\chi(M)} + \mathbb{R}C\right)$ *is therefore of dimension* 1*.*

Proof of Theorem 3.1. According to a result of [21], the derivative ideal of $\chi(M)$ equals to $\chi(M)$, $\overline{\chi(M)}$ and $\chi(M)$ being isomorphic, we get

$$\left[\overline{\chi(M)}, \overline{\chi(M)}\right] = \overline{\chi(M)}.$$

It implies that $\left[\overline{\chi(M)} + \mathbb{R}C, \overline{\chi(M)} + \mathbb{R}C\right] = \overline{\chi(M)}$. In other words, $\overline{\chi(M)}$ is a characteristic ideal of $\overline{\chi(M)} + \mathbb{R}C$. All derivation D de $\overline{\chi(M)} + \mathbb{R}C$ is therefore a derivation of $\overline{\chi(M)}$. Since all derivation of $\overline{\chi(M)}$ is inner, all derivation D not identically zero on $\overline{\chi(M)}$ has the form $D = ad_{\overline{Z}}$ with $\overline{Z} \in \overline{\chi(M)}$. If D is zero on $\overline{\chi(M)}$ then $D(C) = KC$ with $K \in \mathbb{R}$; because $\mathbb{R}C$ is the center of $\overline{\chi(M)} + \mathbb{R}C$ according to Proposition 3.4. Hence the result.

Remark 3.1. It is easy to find that the derivation D, which becomes zero on $\overline{\chi(M)}$ and such that $D(C) = KC$, is not an adjoint linear application of the vector fields on TM.

3.4 Case of a tangent structure

Let J be the vector valued 1-form defining on the manifold M the natural tangent structure. So, the rank of J is n which is the dimension of M, the square of J is zero so is its Nijenhuis torsion. The image space of J is the vertical space of TM. Lie algebra defined by J, according to [3] is $\mathfrak{A}_J = \overline{\chi(M)} + J\overline{\chi(M)}$.

Proposition 3.6. *The centralizer of* $\mathfrak{A}_J$ *in* $\chi(TM)$ *is zero.*

Proof. If an element Z of $\chi(TM)$ commutes with $\mathfrak{A}_J$, we have $\left[Z, \overline{\chi(M)}\right] = 0$ and $\left[Z, J\overline{\chi(M)}\right] = 0$. According to Proposition 3.4, $\left[Z, \overline{\chi(M)}\right] = 0$ if and only if $Z = KC$ for all $K \in \mathbb{R}$. Since $[C, J] = -J$, we get $\left[KC, J\overline{\chi(M)}\right] = 0$ which is equivalent to $K = 0$. □

Proposition 3.7. *The normalizer of* $\mathfrak{A}_J$ *in* $\chi(TM)$ *is* $\mathfrak{A}_J + \mathbb{R}C$*.*

Proof. The normalizer of $\mathfrak{A}_J$ can be calculated on the coordinates system and we find the result. □

Proposition 3.8. $J\mathfrak{A}_J$ *is a characteristic ideal of* $\mathfrak{A}_J$*.*

Proof. We know that $J\mathfrak{A}_J$ is an ideal of $\mathfrak{A}_J$ as stated in Proposition 3.1. What is left to be proved is that, if D is meant to be any derivation of $\mathfrak{A}_J$, D preserves

$J\mathfrak{A}_J$. In other words, if $X \in \mathfrak{A}_J$, we must have $DJX \in J\mathfrak{A}_J$. In conformity with the property of J, namely $ImJ = KerJ$, we can say

$$JDJX = 0. \tag{4}$$

We assume that $X, Y \in \mathfrak{A}_J$. From definition of $\mathfrak{A}_J$ we have

$$J[X,Y] = [JX,Y] = [X,JY]. \tag{5}$$

According to Proposition 3.3, JD is a derivation of $\mathfrak{A}_J$; by applying JD to the relation (5) and taking into account that $J\mathfrak{A}_J$ is commutative, we get

$$JDJ[X,Y] = [JDJX,Y] = [X,JDJY]. \tag{6}$$

Thus we can write $2JDJ[X,Y] = [JDJX,Y] + [X,JDJY]$. In conformity with property (5) of $\mathfrak{A}_J$, the previous expression is written like this: $2JDJ[X,Y] = [DJX,JY] + [JX,DJY]$. Since D is a derivation of $\mathfrak{A}_J$, we have $2JDJ[X,Y] = D[JX,JY] = 0$. The last equality is due to the commutativity of $J\mathfrak{A}_J$. Consequently, as stated in relation (6), we find $[JDJX,Y] = 0$ for all X and Y of $\mathfrak{A}_J$. This means that $JDJX$ is an element belonging to the center of $\mathfrak{A}_J$. However, according to the Proposition 3.6, the center of $\mathfrak{A}_J$ is reduced to 0. Hence $JDJX = 0.$ for all $X \in \mathfrak{A}_J$. In conformity with relation (4), it's the result we've been searching for. □

Proposition 3.9. *$J\mathfrak{A}_J$ is characteristic ideal of $J\mathfrak{A}_J + \mathbb{R}C$; $J\mathfrak{A}_J$, $J\mathfrak{A}_J + \mathbb{R}C$ and $\mathfrak{A}_J$ are characteristic ideals of $\mathfrak{A}_J + \mathbb{R}C$.*

Proof. It is immediate to note that $J\mathfrak{A}_J$ is a caracteristic ideal of $J\mathfrak{A}_J + \mathbb{R}C$. Indeed, as we see in the following properties: $J\mathfrak{A}_J = J\overline{\chi(M)}$ and $[C,J] = -J$, the derivative ideal of $J\mathfrak{A}_J + \mathbb{R}C$ is given by $[J\mathfrak{A}_J + \mathbb{R}C, J\mathfrak{A}_J + \mathbb{R}C] = J\mathfrak{A}_J$. similarly, $\mathfrak{A}_J$ is a characteristic ideal of $\mathfrak{A}_J + \mathbb{R}C$; because $\mathfrak{A}_J = \overline{\chi(M)} + J\overline{\chi(M)}$ and $\left[\overline{\chi(M)}, \overline{\chi(M)}\right] = \overline{\chi(M)}$; we get $[\mathfrak{A}_J + \mathbb{R}C, \mathfrak{A}_J + \mathbb{R}C] = \mathfrak{A}_J$. $J\mathfrak{A}_J$ being a characteristic ideal of $\mathfrak{A}_J$ according to Proposition 3.8, and $\mathfrak{A}_J$ is a charactéristic ideal of $\mathfrak{A}_J + \mathbb{R}C$ in coonformity with the result above; so $J\mathfrak{A}_J$ is a characteristic ideal of $\mathfrak{A}_J + \mathbb{R}C$. We then just need to demonstrate that $J\mathfrak{A}_J + \mathbb{R}C$ is a characteristic ideal of $\mathfrak{A}_J + \mathbb{R}C$. Let D be any derivation of $\mathfrak{A}_J + \mathbb{R}C$. Since $J\mathfrak{A}_J$ is a characteristic ideal of $\mathfrak{A}_J + \mathbb{R}C$, we just need to prove that the DC vector field is vertical. For all $\overline{X} \in \overline{\chi(M)}$, we have $\left[DC, \overline{X}\right] = D\left[C, \overline{X}\right] - \left[C, D\overline{X}\right]$, as we see $\left[C, \overline{X}\right] = 0$ and $\left[C, D\overline{X}\right]$ vertical, because $D\overline{X} \in \mathfrak{A}_J$, the expression $\left[DC, \overline{X}\right]$ is vertical. This implies that DC is a vertical field of $\mathfrak{A}_J + \mathbb{R}C$. □

Lemma 3.1. *Let φ be an $\mathbb{R}$-linear application $\mathbb{R}$ of $\mathfrak{A}_J$ in $J\mathfrak{A}_J$ such that $\varphi^2 = 0$ and $\varphi\left[\overline{X}, \overline{Y}\right] = \left[\varphi\overline{X}, \overline{Y}\right]$ for all $\overline{X}, \overline{Y} \in \overline{\chi(M)}$. So, $\varphi = KJ$ with $K \in \mathbb{R}$.*

The proof is obtained by a simple calculation in local coordinates.

Proposition 3.10. *Let D be a derivation of $\mathfrak{A}_J + \mathbb{R}C$ such that $D : \mathfrak{A}_J + \mathbb{R}C \longrightarrow J\mathfrak{A}_J + \mathbb{R}C$ so there is $\overline{X} \in \mathfrak{A}_J$ such that $D = ad_{J\overline{X}+KC}$ with $K \in \mathbb{R}$.*

Proof. $J\mathfrak{A}_J$ is a characteristic ideal of $\mathfrak{A}_J + \mathbb{R}C$, according to Proposition 3.9, for all $\overline{X} \in \overline{\chi(M)}$ we have $DJ\overline{X} \in J\mathfrak{A}_J$ because $J\mathfrak{A}_J = J\overline{\chi(M)}$. From relation $[C, J] = -J$, we also get $\left[C, J\overline{X}\right] = -J\overline{X}$. By applying the derivation D to the latter relation, we have $-DJ\overline{X} = \left[DC, J\overline{X}\right] + \left[C, DJ\overline{X}\right] = \left[DC, J\overline{X}\right] - DJ\overline{X}$, because $DJ\overline{X} \in J\mathfrak{A}_J$ is homogeneous of degree -1. This implies, for all $\overline{X} \in \overline{\chi(M)}$, $\left[DC, J\overline{X}\right] = 0$. We note that DC commutes with $J\mathfrak{A}_J$ and, $DC \in J\mathfrak{A}_J + \mathbb{R}C$ since $J\mathfrak{A}_J + \mathbb{R}C$ is a characteristic ideal of $\mathfrak{A}_J + \mathbb{R}C$ according to Proposition 3.9. This causes $DC \in J\mathfrak{A}_J$. So, there is $\overline{X}_C \in \mathfrak{A}_J$ such that

$$DC = J\overline{X}_C. \tag{7}$$

By hypothesis, for all $\overline{X} \in \overline{\chi(M)}$ we have $D\overline{X} \in J\mathfrak{A}_J + \mathbb{R}C$; Yet, $\overline{X} \in \mathfrak{A}_J$ and $\mathfrak{A}_J$ is a characteristic ideal of $\mathfrak{A}_J + \mathbb{R}C$, in fact, we have $D\overline{X} \in J\mathfrak{A}_J$.

From relation $\left[DC, \overline{X}\right] = D\left[C, \overline{X}\right] - \left[C, D\overline{X}\right]$, we get $\left[DC, \overline{X}\right] = D\overline{X}$. That is, according to the relation (7) we find

$$\left[J\overline{X}_C, \overline{X}\right] = D\overline{X}. \tag{8}$$

Now, we need to specify the value of D on $J\overline{\chi(M)}$.

We have $D\left[J\overline{X}, \overline{Y}\right] = DJ\left[\overline{X}, \overline{Y}\right] = \left[DJ\overline{X}, \overline{Y}\right]$ for all $\overline{X}, \overline{Y} \in \overline{\chi(M)}$, since by definition $J\left[\overline{X}, \overline{Y}\right] = \left[J\overline{X}, \overline{Y}\right]$ and by hypothesis $\left[J\overline{X}, D\overline{Y}\right] = 0$. Thus, DJ is an $\mathbb{R}$-linear application verifying the conditions of the Lemme, we get for all $\overline{X} \in \overline{\chi(M)}$,

$$DJ\overline{X} = -KJ\overline{X}, \tag{9}$$

with $K \in \mathbb{R}$. Therefore, according to relations (7), (8) and (9), we find

$$D\left(\overline{X} + J\overline{Y} + K'C\right) = ad_{J\overline{X}_C + KC}\left(\overline{X} + J\overline{Y} + K'C\right),$$

for all $\overline{X}, \overline{Y} \in \overline{\chi(M)}$ and $K' \in \mathbb{R}$.

Hence, $D = ad_{J\overline{X}_C + KC}$. □

Proposition 3.11. *Let $\mathfrak{A}$ be a Lie algebra, I a characteristic ideal of $\mathfrak{A}$, D a derivation on $\mathfrak{A}$, π the linear application: $\mathfrak{A} \xrightarrow{\pi} \mathfrak{A}/I$.*

By assuming $\overset{\circ}{D}\pi(X) = \pi(DX)$ for all $X \in \mathfrak{A}$, $\overset{\circ}{D}$ defines a derivation on $\mathfrak{A}/I$. In particular, if $D = ad_X$ then $\overset{\circ}{D} = ad_{\pi(X)}$.

Proof. The proof is classic. □

Proposition 3.12. *Any derivation of the normalizer of $\mathfrak{A}_J$ in $\chi(TM)$ is inner.*

Proof. According to Proposition 3.9, $J\mathfrak{A}_J + \mathbb{R}C$ is a characteristic ideal of $\mathfrak{A}_J + \mathbb{R}C$ which is the normalizer of $\mathfrak{A}_J$; we can therefore consider the application π: $\mathfrak{A}_J + \mathbb{R}C \longrightarrow \frac{\mathfrak{A}_J + \mathbb{R}C}{J\mathfrak{A}_J + \mathbb{R}C}$ and π verifies the properties of the Proposition 3.11.

Since $\frac{\mathfrak{A}_J + \mathbb{R}C}{J\mathfrak{A}_J + \mathbb{R}C} \approx \overline{\chi(M)}$ and all derivation of $\overline{\chi(M)}$ is inner, then all derivation of quotient algebra has the form $ad_{\pi(X)}$ with $X \in \mathfrak{A}_J + \mathbb{R}C$. Let D be a

derivation of $\mathfrak{A}_J+\mathbb{R}C$ such that $\overset{\circ}{D} = ad_{\pi(X)}$, if we pose $D' = D - ad_X$, by definition $\overset{\circ}{D'} = \overset{\circ}{D} - ad_{\pi(X)} = 0$. In other words, $\pi(D'Y) = 0$ for all $Y \in \mathfrak{A}_J+\mathbb{R}C$. So, we have $D'Y \in J\mathfrak{A}_J + \mathbb{R}C$ for all $Y \in \mathfrak{A}_J + \mathbb{R}C$. According to Proposition 3.10, there is $\overline{X'} \in \mathfrak{A}_J$ and $K \in \mathbb{R}$ such that $D' = ad_{J\overline{X'}+KC}$. Thus the expression of D is written as follows: $D = ad_{\overline{X}} + ad_{J\overline{X'}+KC} = ad_{\overline{X}+J\overline{X'}+KC}$. □

Proposition 3.13. *The derivation algebras of $\mathfrak{A}_J$ is isomorphic to that of its normalizer in $\chi(TM)$.*

Proof. $\mathfrak{A}_J$ being a characteristic ideal of $\mathfrak{A}_J + \mathbb{R}C$ according to Proposition 3.9, any derivation D' of $\mathfrak{A}_J + \mathbb{R}C$ is a derivation of $\mathfrak{A}_J$ by restriction of D' on $\mathfrak{A}_J$. Let D be any derivation of $\mathfrak{A}_J$, we will prove the extension of D on $\mathfrak{A}_J + \mathbb{R}C$. If D' is a derivation on $\mathfrak{A}_J + \mathbb{R}C$ such that the restriction of D' on $\mathfrak{A}_J$ equals to D: $D'_{|_{\mathfrak{A}_J}} = D$; then $D' - D$ is a null derivation of $\mathfrak{A}_J$. Let D'' be a derivation of $\mathfrak{A}_J + \mathbb{R}C$ that becomes zero on $\mathfrak{A}_J$. So, for all $\overline{X} \in \overline{\chi(M)}$ we have $D''\left[C, \overline{X}\right] = \left[D''C, \overline{X}\right] + \left[C, D''\overline{X}\right]$; considering $\left[C, \overline{X}\right] = 0$ and $D''\overline{X} = 0$, we get $\left[D''C, \overline{X}\right] = 0$ for all $\overline{X} \in \overline{\chi(M)}$. That is, $D"C$ commutes with $\overline{\chi(M)}$. In addition, we have

$$D"[C, JX] = [D"C, J\overline{X}] + [C, D"J\overline{X}].$$

Since $[C, J\overline{X}] = -J\overline{X}$ and $D"J\overline{X} = 0$, we get $[D"C, J\overline{X}] = 0$, for all $\overline{X} \in \chi(M)$. Yet, $\mathfrak{A}_J = \overline{\chi(M)} + J\chi(M)$, we note that $D''C$ commutes with $\mathfrak{A}_J$ and, according to Proposition 3.6, we find $D''C = 0$. By hypothesis, D'' becomes null on $\mathfrak{A}_J$, therefore D'' becomes zero on $\mathfrak{A}_J + \mathbb{R}C$. We have just proved the uniqueness of an extension of a derivation D of $\mathfrak{A}_J$ on $\mathfrak{A}_J + \mathbb{R}C$ and the extension of D must be zero on $\mathbb{R}C$. Reciprocally, it is immediate to note that by posing $D' = D$ on $\mathfrak{A}_J$ and $D'(C) = 0$, D' is a derivation of $\mathfrak{A}_J + \mathbb{R}C$, because the centralizer of $\mathfrak{A}_J$ in $\mathfrak{A}_J + \mathbb{R}C$ is zero. □

Theorem 3.2 ([14])**.** *The derivations of $\mathfrak{A}_J$ correspond to the adjoint representation of $\mathfrak{A}_J + \mathbb{R}C$. The first cohomology space of $\mathfrak{A}_J$ is thus with 1 dimension: $dim\left(\mathcal{H}^1(\mathfrak{A}_J)\right) = 1$.*

Proof of Theorem 3.2. This is an immediate consequence of Propositions 3.12 and 3.13.

Remark 3.2. According to a result seen by [3], Lie algebra $\mathfrak{A}_J$ is isomorphic to Lie algebra $\chi_D(M)$ of dual vector fields on M. In this section, we note that the derivation algebras $D(\chi_D(M))$ of $\chi_D(M)$ is such that $\mathcal{H}^1(D(\chi_D(M))) = 0$ whereas $dim\left(\mathcal{H}^1(\chi_D(M))\right) = 1$. $\mathfrak{A}_J$ admits a commutative characteristic ideal $J\mathfrak{A}_J$. It is also $J\mathfrak{A}_J$ a characteristic ideal of its vertical normalizer which is $J\mathfrak{A}_J + \mathbb{R}C$.

3.5 Case of a connection with null curvature

In this section we develop another point of view of the case given in [16]. Let M be a connected, paracompact, differentiable manifold with n dimension; J the vector valued 1-form defining on M the natural tangent structure. A connection Γ in the sense of Grifone [8] can be defined by $J\Gamma = J$ and $\Gamma J = -J$, it is an almost produced structure ($\Gamma^2 = I$). By posing $h = \frac{1}{2}(I + \Gamma)$ and $v = \frac{1}{2}(I - \Gamma)$, h means the horizontal projector, v the vertical projector and $TTM = Im(v) + Im(h)$. The curvature of Γ is defined by $R = \frac{1}{2}[h, h]$. From the definitions of Γ, h and v, we can see that Lie algebra of $\mathfrak{A}_\Gamma$ is $\mathfrak{A}_\Gamma = \mathfrak{A}_h = \mathfrak{A}_v$.

In the following section, we will use the notation $\mathfrak{A}_h$.

Proposition 3.14. *The elements of $\mathfrak{A}_h$ are projectable vector fields.*

Proof. Let's consider $X \in \mathfrak{A}_h$. From definition of $\mathfrak{A}_h$, we have $[X, hY] = h[X, Y]$ for all $Y \in \chi(TM)$. Since h is semi-basic ($hJ = 0$), in particular, we have $h[X, JY] = 0$ for all $Y \in \chi(TM)$. In other words, $J[X, JY] = 0$. This means that X is projectable (see [11]). □

In the following, we assume that the curvature R is zero, that is, Nijenhuis torsion of h is zero.

Proposition 3.15. *The horizontal vector fields of $\mathfrak{A}_h$ form a characteristic ideal of $\mathfrak{A}_h$ isomorphic with $\chi(M)$. In particular, all derivation of $h\mathfrak{A}_h$ is inner.*

Proof. We consider the application $h : \overline{\chi(M)} \longrightarrow h\mathfrak{A}_h$ where $\overline{\chi(M)}$ is the complete recovery of $\chi(M)$. Since the kernel of horizontal projector h is a vertical space, h is injective on $\overline{\chi(M)}$. Let's assume $\overline{X} \in \overline{\chi(M)}$, $h\overline{X}$ is a projectable vector fields, so we have $\left[h\overline{X}, h\right] JY = 0$ for all $Y \in \chi(TM)$. The nullity of the curvature ensures $\left[h\overline{X}, h\right] hY = 0$ for all $Y \in \chi(TM)$. Now, any vector field on TM is the direct sum of its horizontal part and its vertical part; we get $\left[h\overline{X}, h\right] = 0$ for all $\overline{X} \in \overline{\chi(M)}$. In other words, $h\left(\overline{\chi(M)}\right) = h\mathfrak{A}_h$. Hence the bijectivity of $h : \overline{\chi(M)} \longrightarrow h\mathfrak{A}_h$. On the other hand, from the nullity of curvature and relation $h^2 = h$, we can write

$$h\left[\overline{X}, \overline{Y}\right] = h\left[h\overline{X}, \overline{Y}\right] + h\left[\overline{X}, h\overline{Y}\right] - \left[h\overline{X}, h\overline{Y}\right],$$

for all $\overline{X}$, $\overline{Y} \in \overline{\chi(M)}$. We have just seen that $h\overline{X}, h\overline{Y}$ are elements of $\mathfrak{A}_h$. So, we have $h\left[h\overline{X}, \overline{Y}\right] = \left[h\overline{X}, h\overline{Y}\right]$ and $h\left[\overline{X}, h\overline{Y}\right] = \left[h\overline{X}, h\overline{Y}\right]$. Thus, the nullity of the curvature is written as follows: $h\left[\overline{X}, \overline{Y}\right] = \left[h\overline{X}, h\overline{Y}\right]$. This proves that the application $h : \overline{\chi(M)} \longrightarrow h\mathfrak{A}_{ha}$ is an isomorphism of Lie algebras. In addition, $h\mathfrak{A}_h$ is an ideal of $\mathfrak{A}_h$ according to Proposition 3.1. From the relation $\left[\overline{\chi(M)}, \overline{\chi(M)}\right] = \overline{\chi(M)}$ and by isomorphism, we have $[h\mathfrak{A}_h, h\mathfrak{A}_h] = h\mathfrak{A}_h$; that is, $h\mathfrak{A}_h$ is a characteristic ideal of $\mathfrak{A}_h$. □

Proposition 3.16. $\mathfrak{A}_h = h\mathfrak{A}_h \times v\mathfrak{A}_h$.

Proof. We notice that $v\mathfrak{A}_h$ is an ideal of $\mathfrak{A}_h$, because we have $\mathfrak{A}_h = \mathfrak{A}_v$ and $[h,h] = [v,v] = 0$; according to Proposition 3.1, $v\mathfrak{A}_v$ which is $v\mathfrak{A}_h$ is an ideal of $\mathfrak{A}_v$ which is $\mathfrak{A}_h$ too. Let's consider $X \in \mathfrak{A}_h$. From the definition of $\mathfrak{A}_h$, we have $[X, hY] = h[X,Y]$ for all $Y \in \chi(TM)$. By applying h to the above relation, we get

$$h[X, hY] = h^2[X,Y]. \tag{10}$$

The nullity of the curvature is written like this: $[hX,hY] - h[hX,Y] + h^2[X,Y] - h[X,hY] = 0$; by considering relation (10), we find $[hX,hY] - h[hX,Y] = 0$ for all $Y \in \chi(TM)$. In other words, $hX \in \mathfrak{A}_h$. By means of an analogous reasoning for v, we obtain $vX \in \mathfrak{A}_h$. We have just proved that $\mathfrak{A}_h = h\mathfrak{A}_h \oplus v\mathfrak{A}_h$. What is to be proved next is that, for all $X,\ Y \in \mathfrak{A}_h$, we must have $[hX, vY] = 0$. by taking into account the definition of $\mathfrak{A}_h$, considering that $h^2 = h$ and $I - h = v$, we can write successively $[hX,vY] = [hX,Y] - [hX,hY] = h[X,Y] - h[X,Y] = 0$ for all $X,\ Y \in \mathfrak{A}_h$. □

Proposition 3.17. *$v\mathfrak{A}_h$ is the centralizer of $h\mathfrak{A}_h$ in $\chi(TM)$.*

Proof. Let Y be an element of the centralizer of $h\mathfrak{A}_h$. This implies

$$[hX, Y] = 0 \tag{11}$$

for all $X \in \mathfrak{A}_h$. By applying h to relation (11) and taking into account $hX \in \mathfrak{A}_h$, we get $[hX,hY] = 0$; which means that hY commutes with $h\mathfrak{A}_h$. According to Proposition 3.15, $h\mathfrak{A}_h$ is the set of horizontal and projectable fields; hY is therefore null. By applying v to relation (11), we also have

$$[hX, vY] = 0 \tag{12}$$

for all $X \in \mathfrak{A}_h$. We will demonstrate that $vY \in \mathfrak{A}_h$, that is,

$$[vY, h] = 0. \tag{13}$$

Note that equation (12) is always true on vertical fields. Therefore, in order to have the relation (13), we just need to verify onto the horizontal fields. To put it differently, we must have

$$v[vY, hZ] = 0 \tag{14}$$

for all $Z \in \chi(TM)$. However, according to Proposition 3.15, $h\mathfrak{A}_h$ generates locally the horizontal space; there is therefore $h_1, \ldots, h_n \in h\mathfrak{A}_h(U)$ such that $hZ \underset{U}{=} \sum_i f_i h_i$ where f_i are functions on TU. The relation (14) becomes $\sum_i f_i v[vY, h_i] + \sum_i (vY.f_i)\, v(h_i) = 0$. This is verified, in conformity with teh relation (12) for the first term, and $v \circ h = 0$ for the second term. Therefore, we have $vY \in \mathfrak{A}_h$. The reciprocal is obtained by Proposition 3.16. □

Theorem 3.3. *Let M be a connected, paracompact, differentiable manifold with n dimension; Γ a flat connection on M; $\chi(M)$ the module of vector fields on M. Lie algebra $\mathfrak{A}_\Gamma$ is isomorphic to $\chi(M)\times\chi(\mathbb{R}^n)$. The first cohomology space of Chevalley-Eilenberg $\mathcal{H}^1(\mathfrak{A}_\Gamma)$ of $\mathfrak{A}_\Gamma$ is therefore zero.*

Proof of Theorem 3.3. The Proposition 3.15 gives the isomorphism between $\chi(M)$ and $h\mathfrak{A}_h$. The ideal $v\mathfrak{A}_h$ is on vertical space and it is the centralizer of $h\mathfrak{A}_h$ according to Proposition 3.17, since $h\mathfrak{A}_h$ is the set of horizontal and projectable fields as stated in Proposition 3.15, $v\mathfrak{A}_h$ is therefore constituted by fields that come only from the fiber which is $\mathbb{R}^n$, according to Fröbenius' theorem since the horizontal space is completely integrable. In other words, $v\mathfrak{A}_h$ is isomorphic to $\chi(\mathbb{R}^n)$. Finally, Proposition 3.16 proves that $\mathfrak{A}_h = h\mathfrak{A}_h \times v\mathfrak{A}_h$. As for the cohomology, we apply a result of F. Takens' work on $\chi(M)$ and $\chi(\mathbb{R}^n)$.

Remark 3.3. If v is a projector on M (see section 2.6), an adaptation of the method used in this section on an adapted chart to a foliation makes it possible to have Lie algebra $\mathfrak{A}_v$ such that $\mathfrak{A}_v \approx \chi_h \times \chi_v$, where χ_h denotes the set of basic vector fields, that is, the vector fields on $Im(h)$, which only depend on $x^a,\ a=1,\dots,n$, χ_v are the set of vector fields on $Im(v)$, which only depend on $y^i,\ i=1,\dots,m$. All derivation of $\mathfrak{A}_v$ is therefore interior.

Example 3.1 (**Example of a vector valued 1-form admitting non-adjoint derivations of vector fields**)**.** Let $M=\mathbb{R}^3$ be a manifold. We designate by (x,y,z) the coordinates system of $\mathbb{R}^3$ and by $\left(\overset{\bullet}{x},\overset{\bullet}{y},\overset{\bullet}{z}\right)$ that of the fiber of the tangent bundle $T\mathbb{R}^3$. The vector valued 1-form $L=dx\otimes C$ is defined on $T\mathbb{R}^3$, where C means the canonical field such that $C=\overset{\bullet}{x}\frac{\partial}{\partial x}+\overset{\bullet}{y}\frac{\partial}{\partial y}+\overset{\bullet}{z}\frac{\partial}{\partial z}$. It is easy to note that $L^2=0$ and $[L,L]=0$; Nijenhuis torsion of L is therefore zero. So, Lie algebra $\mathfrak{A}_L$ is given by the following relation: $T\in\mathfrak{A}_L \Longleftrightarrow [T,dx\otimes C]=0$. By assuming the coordinates of T by $\left(X,Y,Z,\overset{\bullet}{X},\overset{\bullet}{Y},\overset{\bullet}{Z}\right)\in T\mathbb{R}^3$, we find $X(x)$, $Y(x,y,z)$, $Z(x,y,z)$ and

$$\begin{pmatrix}\overset{\bullet}{X}\\ \overset{\bullet}{Y}\\ \overset{\bullet}{Z}\end{pmatrix}=\begin{pmatrix}\frac{\partial X}{\partial x}(x) & \alpha_2^1(x,y,z) & \alpha_3^1(x,y,z)\\ \alpha_1^2(x,y,z) & \frac{\partial X}{\partial x}(x) & \alpha_3^2(x,y,z)\\ \alpha_1^3(x,y,z) & \alpha_2^3(x,y,z) & \frac{\partial X}{\partial x}(x)\end{pmatrix}\begin{pmatrix}\overset{\bullet}{x}\\ \overset{\bullet}{y}\\ \overset{\bullet}{z}\end{pmatrix}$$

$X(x)$) is only just a function of x and Y,Z, α_j^i with $i\neq j$ are arbitrary functions of x,y,z. We see that by denoting the canonical divergence of $\mathbb{R}^3$ by φ, $D=(dx+\varphi)\otimes C$ is a non-adjoint derivation of a vector fields of $\mathfrak{A}_L$ on $T\mathbb{R}^3$.

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Orthogonality preserving maps on $B(H)$

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Abstract. In this paper, we study the linear maps on $B(H)$ preserving a class of operators in $B(H)$. We also introduce pointwise orthogonality on $B(H)$ and provide the conditions under which a linear mapping φ on $B(H)$ preserves pointwise orthogonality on $B(H)$. Finally, we prove that a Jordan $*-$homomorphism φ preserves OP operators in $B(H)$. Moreover, under additional condition of injectivity, φ preserves OP operators in both directions.

Keywords: preserving maps, orthogonality preserving operators, normal operators.

1. Introduction

In an inner product space, two elements x and y are called orthogonal and denoted by $x \perp y$ whenever the inner product $\langle x\ , y\rangle$ is zero. Now let $\|.\|$ be the induced norm by the inner product on the underlying space. It is straightforward to check that for each scalar λ

$$\|x\| \leq \|x + \lambda y\| \Leftrightarrow x \perp y.$$

This relation provided a good motivation to define orthogonality in normed linear spaces which became known as Birkhoff-James orthogonality so that two elements x and y in a normed linear space are called BJ orthogonal, denoted by

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$x \perp_B y$, whenever $\|x\| \leq \|x + \lambda y\|$, for every scalar λ. Other types of orthogonality have also been defined and studied in linear normed spaces, such as
Roberts orthogonality: $\|x - \lambda y\| = \|x + \lambda y\|$, for every scalar λ, $(x \perp_R y)$;
Isosceles orthogonality: $\|x - y\| = \|x + y\|$, $(x \perp_I y)$;
Pythagorean orthogonality: $\|x - y\|^2 = \|x\|^2 + \|y\|^2$, $(x \perp_P y)$.
Sikorska [12], Alonso and Benitez [1, 2] studied the properties of these orthogonalities and compared the relations between them. Among them, the most widely used is Birkhoff-James orthogonality which is nondegenerate ($x \perp_B x$ if and only if $x = 0$), homogenous ($x \perp_B y \Rightarrow (\lambda x \perp_B \mu y$ for all scalar $\lambda, \mu)$), non symmetric ($x \perp_B y$ does not imply $y \perp_B x$), and non additive ($x \perp_B y$ and $x \perp_B z$ does not imply $x \perp_B y + z$). Also, for every $x, y \in X$ there is an scalar λ such that $x \perp_B \lambda x + y$.

After introducing the orthogonal elements in inner product and normed spaces and studying their properties, mathematicians and researchers concentrated on the study of maps that preserve the property of the orthogonality between two spaces. Therefore, a wide field of study has been appeared under the title of "linear orthogonality preserver problems".

Let X, Y be two inner product spaces over the same field. An operator $T : X \to Y$ is said to be orthogonality preserving (OP in short), if for every $x, y \in X$, $x \perp y \Rightarrow Tx \perp Ty$. Chmieliński [5] characterized linear OP operators between inner product spaces. Actually, he proved that a linear operator $T : X \to Y$ is OP, if and only if it is a scalar multiple of an isometry(in the sense that there exists a scalar γ such that $\|Tx\| = \gamma\|x\|$, for all $x \in X$).

In the setting of real normed spaces Koldobsky [6] had already proved Chmieliński's theorem [5, Theorem 2.1] with the Birkhoff-James orthogonality, and then Blanco and Turnšek [4], generalized this theorem to complex normed spaces. Also recently, Ansari, Sanati and Kardel [3] proved that, for a Hilbert space H, a linear OP operator $T : H \to H$ is a scalar multiple of a unitary between Hilbert space H and $ran(T)$. Furthermore, different types of approximately orthogonality preserving mappings and their stability have been studied. For more details we refer to [5, 11, 14, 15, 16, 17].

In this paper, at first we study the linear maps on $B(H)$ that preserve a class of OP operators on $B(H)$ and then, we introduce the notion of pointwise orthogonality on $B(H)$ to studying pointwise orthoginality preservers.

Let H be an infinite-dimentional Hilbert space and $B(H)$ be the C^*-algebra of all bounded linear operators on H. The ideals of compact operators and finite rank operators in $B(H)$ is denoted by $K(H)$ and $F(H)$, respectively. An operator $T \in B(H)$ is said to be Fredholm if its image is closed and its kernel and co-kernel are both finite dimention. T is semi-Fredholm if its image is closed and its kernel or its co-kernel is finite dimention. The class of all Fredholm and semi-Fredholm operators are denoted by $FR(H)$ and $SF(H)$, respectively. For more details about Fredholm operators, see [9, 13]. The set of normal operators in $B(H)$ is also denoted by $N(H)$.

An operator $T \in B(H)$ is said to be generalized invertible if there is an operator $R \in B(H)$ such that $TRT = T$. The set of all generalized invertible operators in $B(H)$ is denoted by $G(H)$. Note that $T \in G(H)$ if and only if $ran(T)$ is closed [10].

We say that an operator $T \in B(H)$ is pointwise orthogonal to another operator $R \in B(H)$, denoted by $T \perp_p R$, if $\langle Tx, Rx\rangle = 0$ for all $x \in H$. Since pointwise orthogonality is based on inner product, it is nondegenerate, homogeneous, symmetric and additive on $B(H)$. This orthogonality is also continuous, for if $T_n \to T$ and $R_n \to R$ in $B(H)$ and $T_n \perp_p R_n$, for every $n \in \mathbb{N}$, then for every $x \in H$ we have,

$$\langle Tx, Rx\rangle = \lim_{n\to\infty} \langle T_n x, R_n x\rangle = 0,$$

and therefore $T \perp_p R$.

Here, we give two examples of operators which are pointwise orthogonal to each other.

Example 1.1. let M be a closed subspace of H. If $P : H \to M$ and $P^{\perp} : H \to M^{\perp}$ are the projections of H on M and $M^{\perp}$, respectively. Then $P \perp_p P^{\perp}$.

Example 1.2. If $T, R : \mathbb{R}^2 \to \mathbb{R}^2$ are defined by $T(x, y) = (x, -y)$ and $R(x, y) = (y, x)$. Then, for each $(x, y) \in \mathbb{R}^2$ we have,

$$\langle T(x, y), R(x, y)\rangle = \langle (x, -y), (y, x)\rangle = xy - yx = 0.$$

In the next section, we show that every non-zero OP operator in $B(H)$ is semi-Fredholm and every non-zero normal OP operator is Fredholm of index zero. Also, we concentrate on the linear mappings on $B(H)$ which preserve some classes of operators. For instance, we prove that a Jordan $*$-homomorphism φ on $B(H)$ preserves OP operators. Moreover, if the map φ is also one to one, then it preserves OP operators in both directions. In the last section, we compare the pointwise orthogonality on $B(H)$ with Birkhoff-James, Roberts, Isosceles, Pythagorean orthogonalities on $B(H)$ as a normed space. Finally, we provide the conditions under which a linear mapping φ preserves pointwise orthoginality on $B(H)$.

2. Orthogonality preserving maps on $B(H)$

The main theorem of this section is Theorem 2.2.

Proposition 2.1. *Let $\varphi : B(H) \to B(H)$ be a linear map which preserves the isometry maps of $B(H)$, then φ is an isometry on $OP(H)$.*

Proof. Let $T \in OP(H)$. We can write $T = \gamma U$, for some positive scalar γ and an isometry map U on H, and thus $\|T\| = \gamma$. On the other hand, since $\varphi(U)$ is an isometry we have, $\|\varphi(U)\| = 1$ so $\|\varphi(T)\| = \|\gamma\varphi(U)\| = \gamma\|\varphi(U)\| = \gamma$ and so $\|\varphi(T)\| = \|T\|$. □

Proposition 2.2. *Let T be a non-zero orthogonality preserving operator in $B(H)$, then T is semi-Fredholm. Moreover, if T is a normal operator, then it is Fredholm with zero index.*

Proof. Let $T \in OP(H)$. Since T is a multiple of an isometry, T is injective and $ran(T)$ is a closed subspace of H. Therefore, T is a semi-Fredholm operator. But we have,

$$\langle (TT^* - T^*T)x, x\rangle = \langle TT^*x, x\rangle - \langle T^*Tx, x\rangle = \|T^*x\|^2 - \|Tx\|^2.$$

Now, if T belongs to $N(H)$, then $ker(T) = ker(T^*)$ and since $ker(T^*) = (ran(T))^{\perp}$ we obtain that, T is a Fredholm operator and $ind(T) = nul(T) - def(T) = dim(ker(T)) - dim(coker(T)) = 0$. □

The following Lemma plays an important role in expressing the relationship between compact and Fredholm operators.

Lemma 2.1. *Let $T \in B(H)$. Then the following are equivalent:*

(*i*) *T is compact.*

(*ii*) *For every $R \in FR(H)$, $R + T \in FR(H)$.*

(*iii*) *For every $R \in SF(H)$, $R + T \in SF(H)$.*

Proof. This follows from [7, Lemma 2.2] and [8, Lemma 2.2]. □

Proposition 2.3. *Let $T \in B(H)$ be a non-zero normal OP operator, then for every compact operator R in $B(H)$ we have, $T + R$ is Fredholm with zero index.*

Proof. Let $T \in (OP(H) \cap N(H)) \setminus \{0\}$ and $R \in K(H)$, then Proposition 2.2 implies that $T \in FR(H)$ and $ind(T) = 0$. Now by Lemma 2.1 we have, $T + R \in FR(H)$. On the other hand, by using [9, Theorem 1.4.18], for each comapact operator R and Fredholm operator T, we obtain that $ind(T + R) = ind(T)$. Thus $T + R$ is a Fredholm operator of zero index. □

Theorem 2.1. *Let T be a non-zero orthogonality preserving operator in $B(H)$, then T^*T is invertible with inverse $\gamma^{-2}T^*T$, for some $\gamma > 0$.*

Proof of Theorem 2.1. Let x, y be two orthogonal elements of H. Since $\langle x, y\rangle = 0$ and T is OP, we have $\langle Tx, Ty\rangle = 0$ so $\langle x, T^*Ty\rangle = 0$. Therefore $\langle Tx, TT^*Ty\rangle = 0$, hence $\langle T^*Tx, T^*Ty\rangle = 0$, which implies that T^*T is OP and so it is a multiple of an isometry. Thus T^*T is injective.

Now, take $M = ran(T^*T)$. By using [3, Theorem 3.2] for T^*T, there is a unitary operator $U : H \to M$ such that $T^*T = \gamma U$ for some $\gamma > 0$ and $UU^* = id_M$, $U^*U = id_H$. Let U^+ be the adjoint of the operator U as an element of $B(H)$. Clearly, $U^+|_M = U^*$. Since $H = M \oplus M^{\perp}$, we can write for every $x \in H : x = T^*Ty + z$ for some $y \in H, z \in M^{\perp}$. Now we have

$$UU^+x = U(U^+(T^*Ty + z)) = U(U^+T^*Ty + U^+z). \tag{1}$$

We have $U^+z = 0$, because $z \in M^\perp = ran(T^*T)^\perp$ and $T^*T = \gamma U$, so $z \in ran(U)^\perp = ker(U^+)$. Now, equation (1) implies that

$$UU^+x = UU^+(T^*Ty) = UU^*(T^*Ty) = T^*Ty.$$

Hence, for every $x \in H$, we obtain that

$$UU^+x = \begin{cases} x, & x \in M, \\ 0, & x \in M^\perp. \end{cases}$$

On the other hand, $U^+Ux = U^*Ux = x$. Since $T^*T = \gamma U$ is normal, U is normal. Therefore, $UU^+x = U^+Ux$. But it is impossible unless, $M^\perp = 0$. Therefore, $ran(T^*T) = H$, i.e., T^*T is surjective.

So, T^*T is an invertible operator of $B(H)$. Moreover,

$$(\gamma^{-2}T^*T)(T^*T) = (T^*T)(\gamma^{-2}T^*T) = id_H.$$

The idea of proving the surjectivity of T^*T in Theorem 2.1, originates from [3, Theorem 3.3].

Theorem 2.2. *Let $\varphi : B(H) \to B(H)$ be a Jordan $*$-homomorphism on $B(H)$. Then:*

(i) $\varphi(OP(H)) \subseteq OP(H)$.

(ii) $\varphi(N(H) \cap OP(H)) \subseteq N(H) \cap OP(H)$ *and* $\varphi(N(H) \cap OP(H)) \nsubseteq F(H)$.

Moreover, if φ is one to one, then

(iii) φ *preserves OP operators in both directions.*

(iv) φ *preserves normal OP operators in both directions.*

Proof of Theorem 2.2. *(i)* Let $T \in OP(H)$ and $M = ran(T)$, then by [3, Theorem 3.2] there exists a positive scalar γ and a unitary map $U : H \to M$ such that $T = \gamma U$. Since $\varphi(U)^*\varphi(U) = \varphi(U^*U) = \varphi(id_H) = id_H$ we have,

$$\|\varphi(U)(x)\|^2 = \langle \varphi(U)(x), \varphi(U)(x)\rangle = \langle \varphi(U)^*\varphi(U)(x), x\rangle = \|x\|^2.$$

Therefore, $\varphi(U)$ is an isometry map of $B(H)$. On the other hand, $\varphi(T) = \gamma\varphi(U)$ which implies that $\varphi(T) \in OP(H)$.

(ii) Let $T \in N(H) \cap OP(H)$. By *(i)* we know that $\varphi(T)$ is OP. Also $\varphi(T)$ is normal, because

$$\varphi(T)^*\varphi(T) = \varphi(T^*T) = \varphi(TT^*) = \varphi(T)\varphi(T)^*.$$

Thus, $\varphi(T) \in N(H) \cap OP(H)$.

$\varphi(T)$ is surjective by a similar proof to what we saw in Theorem 2.1 but, since H is an infinite-dimentional space, $\varphi(T)$ can not be finite-rank.

(iii) From (i) we have, $\varphi(OP(H)) \subseteq OP(H)$. For the converse, suppose that $T \in B(H)$ such that $\varphi(T) \in OP(H)$. Then, there exists a positive scalar γ and a unitary map $V : H \to ran(\varphi(T))$ such that $\varphi(T) = \gamma V$. So $\varphi(T)^* = \gamma V^*$ and

$$\varphi(T^*T) = \varphi(T)^*\varphi(T) = \gamma^2 V^*V = \gamma^2 id_H = \gamma^2\varphi(id_H).$$

Hence, $T^*T - \gamma^2 id_H \in ker(\varphi)$ and, since φ is one to one we have, $T^*T = \gamma^2 id_H$. Now if $\langle x, y\rangle = 0$, then

$$\langle Tx, Ty\rangle = \langle T^*Tx, y\rangle = \gamma^2 \langle x, y\rangle = 0.$$

Therefore, $T \in OP(H)$.

(iv) Using (ii) we have, $\varphi(N(H) \cap OP(H)) \subseteq N(H) \cap OP(H)$. Conversely, let $T \in B(H)$ such that $\varphi(T) \in N(H) \cap OP(H)$. By (iii), we obtain that T is OP. Since $\varphi(T)\varphi(T)^* = \varphi(T)^*\varphi(T)$ we have, $TT^* - T^*T \in ker(\varphi)$. So T is normal, because φ is injective.

Corollary 2.1. *Suppose that φ is a Jordan $*$-homomorphism on $B(H)$. If $T \in N(H) \cap OP(H)$, then $\varphi(T)^* \in OP(H)$.*

Proof. Suppose that $x \in H$. Then, we have

$$\langle (T^*T - TT^*)x, x\rangle = \langle T^*Tx, x\rangle - \langle TT^*x, x\rangle = \|Tx\|^2 - \|T^*x\|^2.$$

Since T is normal and OP we have, $\|T^*x\| = \|Tx\| = \gamma\|x\|$, for some $\gamma > 0$. So T^* is OP and by Theorem 2.2(i), $\varphi(T)^* = \varphi(T^*) \in OP(H)$. □

Remark 2.1. Suppose that $\varphi : B(H) \to B(H)$ is a linear map. Since $K(H) \cap SF(H) = \emptyset$, Proposition 2.2 implies that $K(H) \cap OP(H) = \{0\}$ and $F(H) \cap OP(H) = \{0\}$. These imply that $\varphi(K(H) \cap OP(H)) \subseteq K(H) \cap OP(H)$ and $\varphi(F(H) \cap OP(H)) \subseteq F(H) \cap OP(H)$, respectively. Moreover, if φ is one to one, then φ preserves compact OP and finite rank OP operators in both directions.

The following example shows that for preserving OP operators under φ, Jordan's condition can not be removed from the hypothesis of Theorem 2.2.

Example 2.1. Let $K \in K(H) \setminus \{0\}$ and let $\varphi : B(H) \to B(H)$ is defined by $\varphi(T) := KTK^*$, for each $T \in B(H)$. It is clear that the linear map φ preserves the adjoint on $B(H)$, but it is not necessarily Jordan homomorphism. Note that $id \in OP(H)$, but $\varphi(id) \in K(H)$ and $OP(H) \cap K(H) = \{0\}$. Consequently, $\varphi(id) \in OP(H)$ implies $KK^* = 0$, which means that $K = 0$. This is a contradiction. Therefore, $\varphi(OP(H)) \nsubseteq OP(H)$.

To prove the fact that (iii) implies injectivity of the linear map φ in Theorem 2.2, φ does not need to be Jordan $*$-homomorphism. We provide a condition under which the linear map φ preserving OP operators in both directions should be injective.

Proposition 2.4. *Let $\varphi : B(H) \to B(H)$ be a linear map. If φ preserves OP operators in both directions and $ker(\varphi) \subseteq K(H)$, then φ is injective.*

Proof. For all $B \in ker(\varphi)$ we have, $\varphi(B) \in OP(H)$. Since φ preserves OP operators in both directions, $B \in OP(H)$. On the other hand, $B \in K(H)$. Thus, $B \in OP(H) \cap K(H)$ and Remark 2.1 implies that $B = 0$ which means that φ is injective. □

3. Pointwise orthogonality on $B(H)$

Proposition 3.1. *The pointwise orthogonality implies:*

(i) *the Birkhoff-James orthogonality on $B(H)$.*

(ii) *Roberts, Isosceles, Pythagorean orthogonalities on $B(H)$.*

Proof. Let T, R be two operators in $B(H)$ and $T \perp_p R$.

(i) Since for every $x \in H$, $\langle Tx, Rx\rangle = 0$ and H is a Hilbert space, we have $\|Tx\| \leq \|Tx+\lambda Rx\|$, for all scalar $\lambda \in \mathbb{C}$ and $x \in H$. Therefore, $\|T\| \leq \|T+\lambda R\|$ which means that $T \perp_B R$.

(ii) Since for every $x \in H$, $\langle Tx, Rx\rangle = 0$ we have,

$$\begin{aligned}\langle Tx - \lambda Rx, Tx - \lambda Rx\rangle &= \langle Tx + \lambda Rx, Tx + \lambda Rx\rangle \\ &= \langle Tx, Tx\rangle + |\lambda|^2 \langle Rx, Rx\rangle\end{aligned}$$

which implies that

$$\|Tx - \lambda Rx\|^2 = \|Tx + \lambda Rx\|^2 = \|Tx\|^2 + |\lambda|^2\|Rx\|^2.$$

Therefore, the following relations come true:

$$\|T - \lambda R\|^2 = \|T + \lambda R\|^2 = \|T\|^2 + |\lambda|^2\|R\|^2$$

and

$$\|T - R\|^2 = \|T + R\|^2 = \|T\|^2 + \|R\|^2.$$

Equivalently, $T \perp_R R$, $T \perp_I R$ and $T \perp_P R$. □

Lemma 3.1. *Let $T, R \in B(H)$. If $T \perp_p R$ and one of the following conditions:*

(i) *one of T^* or R^* belongs to $OP(H)$;*

(ii) *$TR^* = R^*T$;*

holds, then $T^* \perp_p R^*$.

Proof. If we assume that $T^* \in OP(H)$ there is a unitary map $U : H \to ran(T^*)$ such that $T^* = \gamma\, U$ for some $\gamma > 0$ so $TT^* = \gamma^2 id_H$. On the other hand, since for every $x \in H$, $\langle Tx, Rx\rangle = 0$ we have $\langle TT^*x, RT^*x\rangle = 0$. So, $\langle \gamma^2 x, RT^*x\rangle = 0$. Therefore, $\langle R^*x, T^*x\rangle = 0$ which means $T^* \perp_p R^*$.

By condition (ii), since for every $x \in H$, $\langle Tx, Rx\rangle = 0$ we obtain that $\langle R^*Tx, x\rangle = 0$. Thus, $\langle TR^*x, x\rangle = 0$ and so $\langle R^*x, T^*x\rangle = 0$ which implies that $T^* \perp_p R^*$. □

Now, we define the set A as follows:

$$A = \{(T, R) \in B(H) \times B(H) |\ T * RT* = T*\}.$$

A is non empty, because for every generalized invertible element $T \in B(H)$, there is an operator $R \in B(H)$ such that $TRT = T$. It is straightforward to check that $(T, R^*) \in A$. It is also clear that A is self adjoint.

Theorem 3.1. *Let* $T, R \in B(H)$ *and* $\varphi : B(H) \to B(H)$ *be a linear map which preserves the adjoints. Then:*

(i) *If* $T \perp_p R$ *and* $(T, R) \in A$, *then* $\varphi(T) \perp_p \varphi(R)$ *and* $(\varphi(T), \varphi(R)) \in A$.

Moreover, if φ *is one to one, then:*

(ii) $T \perp_p R$ *and* $(T, R) \in A$ *if and only if* $\varphi(T) \perp_p \varphi(R)$ *and* $(\varphi(T), \varphi(R)) \in A$.

Proof of Theorem 3.1. Take $A = \{(T, R) \in B(H) \times B(H) |\ \mathrm{T^*RT^* = T^*}\}$.

(i) Since for every $x \in H$, $\langle Tx, Rx\rangle = 0$ we obtain that $\langle TT^*x, RT^*x\rangle = 0$ so $\langle T^*x, T^*RT^*x\rangle = 0$. Therefore, for every $x \in H$

$$\|T^*x\|^2 = \langle T^*x, T^*x\rangle = \langle T^*x, T^*RT^*x\rangle = 0,$$

which implies that $\|T^*\| = 0$ and thus $T^* = 0$. Now since φ preserves the adjoint on $B(H)$ we have $\varphi(T)^* = 0$, so $(\varphi(T), \varphi(R)) \in A$. Also

$$\langle \varphi(T)x, \varphi(R)x\rangle = \langle x, \varphi(T^*)\varphi(R)x\rangle = 0.$$

Hence, $\varphi(T) \perp_p \varphi(R)$.

(ii) By (i), if $T \perp_p R$ and $(T, R) \in A$, then $\varphi(T) \perp_p \varphi(R)$ and $(\varphi(T), \varphi(R)) \in A$. Conversly, suppose that $\varphi(T) \perp_p \varphi(R)$, $(\varphi(T), \varphi(R)) \in A$. By the similar proof in (i), we obtain that $\varphi(T)^* = 0$. Since φ is one to one, $T^* = 0$. Thus, $(T, R) \in A$ and

$$\langle Tx, Rx\rangle = \langle x, T^*Rx\rangle = 0,$$

which means $T \perp_p R$.

Remark 3.1. All pointwise orthogonal members $T, R \in B(H)$ satisfy $\langle Tx, Rx\rangle = 0$ $(x \in H)$. Therefore, $T \perp_p T^*RT$, $R \perp_p R^*TR$, $T^* \perp_p T^*RT^*$, $R^* \perp_p R^*TR^*$.

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Signed product cordial labeling of corona product between paths and second power of fan graphs

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Abstract. In this work, we contribute some new results on signed product cordial labeling and investigate necessary and sufficient conditions of the corona product between paths and second power of fan graphs to be signed product cordial.
Keywords: corona operation, second power, signed product cordial, path graph, fan graph.

1. Introduction

The field of graph theory plays an important role in various areas of pure and applied sciences. One of the main problems in this field is graph labeling which is an assignment of integers to the vertices or edges, or both, subject to certain conditions. It is a very powerful tool that eventually makes things in different fields very ease to be handled in mathematical way. While the labeling of graphs is perceived to be a primarily theoretical subject in the field of graph theory and discrete mathematics, it serves as models in a wide range of application like astronomy, coding theory, x-ray crystallography, circuit design and communication networks addressing [1]. There are many contributions and different types of labeling (see, [2], [3], [4]). An excellent reference for this purpose is the survey written by Gallian [5].

In this paper, all graphs are finite, simple and un directed. The original concept of cordial graphs is due to Cahit [6] who called a graph G cordial if there is a vertex labeling $f : V(G) \to 0, 1$ such that the induced labeling $f^* : E(G) \to 0, 1$, defined by $f^*(xy) = |f(x) - f(y)|$, for all edges $x, y \in E(G)$ and with the following inequalities holding: $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$, where $v_f(i)$ (respectively $e_f(i)$) is the number of vertices (respectively, edges)

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labeled with i. Motivated through the concept of cordial labeling the product cordial labeling was introduced by Sundaram et al [7] where absolute difference of vertex labels is replaced by product of vertex labels.

Definition 1. A graph $G = (V, E)$ is called signed product cordial [8] if it is possible to label the vertex by the function $f : V \to -1, 1$ and label the edges by $f^* : E \to -1, 1$, where $f^*(uv) = f(u).f(v)$, $u, v \in V$ so that $|v_{-1} - v_1| \leq 1$ and $|e_{-1} - e_1| \leq 1$.

Definition 2. The corona $G \bigodot H$ of two graphs G (with n_1 vertices and m_1 edges) and H (with n_2 vertices and m_2 edges) is the graph denoted by $G \bigodot H$ and is obtained by taking one copy of G and n_1 copies of H, and then joining the ith vertex of G with an edge to every vertex in the ith copy of H (see, [9]). It follows from the definition of the corona that $G \bigodot H$ has $n_1 + n_1.n_2$ vertices and $m_1 + n_1.m_2 + n_1.n_2$ edges.

It is easy to see that $G \bigodot H$ is not in general isomorphic to $H \bigodot G$.

Definition 3. A second power of a fan F_m^2 is the graph obtained from the join of the second power of a path P_m^2 and a null graph N_1, i.e.$F_m^2 = P_m^2 + N_1$. So the order of F_m^2 is $m + 1$ and its size is $3m - 3$, in particular $F_1^2 \equiv P_2$,$F_2^2 \equiv C_3$ and $F_3^2 \equiv K_4$.

In this paper we proposed the corona $P_k \bigodot F_m^2$ and show that is signed product cordial for all $K \geq 1$ and $m \geq 4$.

2. Terminology and notation

A path with n vertices and $n - 1$ edges is denoted by P_n, and second power of fan graph has $n + 1$ vertices and $3n - 3$ edges is denoted by F_n^2. We let $M_r$denote the labeling $(-1)1(-1)1...(-1)1$, zero-one repeated r-times if r is even and $(-1)1(-1)1...(-1)1(-1)$ if r is odd; for example, $M_6 = (-1)1(-1)1(-1)1$ and $M_5 = (-1)1(-1)1(-1)$. We let M'_{2r} denote the labeling $1(-1)1(-1)...1(-1)$. Sometimes, we modify the labeling M_r or M'_r by adding symbols at one end or the other (or both). We let L_{4r} denote the labeling $(-1)_2 11$ $(-1)_2 11...(-1)_2 11$ (repeated r-times), Let L'_{4r} denote the labeling $(-1)11(-1)(-1)11(-1)...(-1)11$ (-1) (repeated r-times). The labeling $11(-1)_2 11(-1)_2...11(-1)_2$ (repeated r-times) and labeling $1(-1)_2 11(-1)_2 1...1(-1)_2 1$ (repeated r-times) are written S_{4r} and S'_{4r}. Let M_r denote the labeling $(-1)1$ $(-1)1...(-1)1$, zero-one repeated $r$times if r is even and $(-1)1$ $(-1)1...(-1)1(-1)$ if r is odd; for example, $M_6 = (-1)1(-1)1(-1)1$ and $M_5 = (-1)1(-1)1(-1)$. We let M'_r denote the labeling $1(-1)1(-1)...1(-1)$. Sometimes, we modify the labeling M_r or M'_r by adding symbols at one end or the other (or both). Also, L_{4r} (or L'_{4r}) with extra labeling from right or left (or both sides) [9-13]. For specific labeling L and M of $G \bigodot H$ where G is path and H is a second power of fan graph, we let $[L; M]$ denote the corona labeling. Additional notation that we use is the following. For a given labeling of the corona $G \bigodot H$, we let v_i and e_i (for $i = (-1), 1$) be the numbers

of labels that are i as before, we let x_i and a_i be the corresponding quantities for G, and we let $y_i$and b_i be those for H, which are connected to the vertices labeled (-1) of G. Likewise, let y'_i and b'_i be those for H, which are connected to the vertices labeled 1 of G. In case it increases by one more vertex, so $y"_i$ and $b"_i$ will be those for H, which are connected to the vertex labeled 1 or (-1) of G. It is easy to verify that $v_{-1} = x_{-1}+y_{-1}x_{-1}+y'_{-1}x_1, v_1 = x_1+y_1x_{-1}+y'_1x_1$ and $e_{-1} = a_{-1}+b_{-1}x_{-1}+b'_{-1}x_1+y_1x_{-1}+y'_1x_{-1}, e_1 = a_1+b_1x_{-1}+b'_1x_1+y_1x_1+y'_{-1}x_{-1}$ Thus, $v_{-1} - v_1 = (x_{-1} - x_1) + x_{-1}(y_{-1} - y_1) + x_1(y'_{-1} - y'_1)$ and $e_{-1} - e_1 = (a_{-1} - a_1) + x_{-1}(b_{-1} - b_1) + x_1(b'_{-1} - b'_1) - x_{-1}(y_{-1} - y_1) + x_1(y'_{-1} - y'_1)$. When it comes to the proof, we only need to show that, for each specified combination of labeling, $|v_{-1} - v_1| \leq 1$ and $|e_{-1} - e_1| \leq 1$.

3. The signed product cordial of corona between paths and second power of Fan graphs

In this section, we show that the corona between paths and second power of Fan graphs $P_k \bigodot F_m^2$ is signed product cordial for all $k \geq 1$, and $m \geq 4$. This target will be achieved after the following series of lemmas.

Lemma 3.1. *$P_k \bigodot F_m^2$ is signed product cordial for all $k \geq 1$ and $m \equiv 0(mod4)$.*

Proof of Lemma 3.1. We need to examine the following cases:
Case (1). $k \equiv 0(mod4)$.

Let $k = 4r$,$r \geq 1$. Then, one can choose the labeling $[L_{4r} : (-1)M'_{4s}, (-1)M'_{4s}, 1M_{4s}, 1M_{4s}, \ldots (r-times)]$ for $P_{4r} \bigodot F_{4s}^2$. Therefore $x_{-1} = x_1 = 2r, a_{-1} = 2r-1, a_1 = 2r, y_{-1} = 2s+1, y_1 = 2s, b_{-1} = 6s-1, b_1 = 6s-2, y'_{-1} = 2s, y'_1 = 2s+1, b'_{-1} = 6s-1$ *and* $b'_1 = 6s-2$. Hence, one can easily show that $v_{-1}-v1 = 0$ and $e_{-1} - e_1 = -1$. Thus $P_{4r} \bigodot F_{4s}^2$, $s \geq 1$ is signed product cordial. As an example, Figure 1 illustrates this case $P_4 \bigodot F_4^2$.

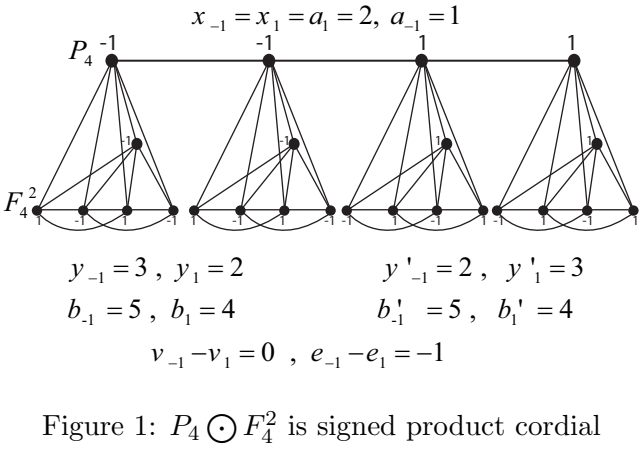


Figure 1: $P_4 \bigodot F_4^2$ is signed product cordial

Case (2). *$k \equiv 1(mod\ 4)$.*

Let $k = 4r + 1$, $r > 0$. Then, one can choose the labelling $[L_{4r}1 : (-1)M_{4s}, (-1)M_{4s}, 1M_{4s}, 1M_{4s}, \dots (r-times), (-1)M_{4s}]$ for $P_{4r+1} \bigodot F^2_{4s}$. Therefore $x_{(-1)} = 2r, x_1 = a_{(-1)} = 2r - 1, a_1 = 2r + 1, y_{-1} = 2s + 1, y_1 = 2s, b_{-1} = 6s - 2, b_1 = 6s - 1, y'_{-1} = 2s, y'_1 = 2s + 1, b'_{-1} = 6s - 2, b'_1 = 6s - 1, y''_{-1} = 2s + 1, y''_1 = 2s, b''_{-1} = 6s-2$ *and* $b''_1 = 6s-1$. Hence, one can easily show that $v_{-1}-v_1 = 0$ and $e_{-1} - e_1 = 0$. Thus $P_{4r+1} \bigodot F^2_{4s}$, $s \geq 1$ is signed product cordial.

Case (3). $k \equiv$ *2(mod 4).*

Let $k = 4r + 2$, $r > 0$. Then, one can choose the labelling $[L_{4r}1(-1) : (-1)M'_{4s}, (-1)M'_{4s}, 1M_{4s}, 1M_{4s}, \dots (r-times), 1M_{4s}, (-1)M'_{4s}]$ for $P_{4r+2} \bigodot F^2_{4s}$. Therefore $x_{-1} = x_1 = 2r + 1, a_{-1} = 2r, a_1 = 2r + 1, y_{-1} = 2s + 1, y_1 = 2s, b_{-1} = 6s - 1, b_1 = 6s - 2, y'_{-1} = 2s, y'_1 = 2s + 1, b'_{-1} = 6s - 1$ and $b'_1 = 6s - 2$. Hence, one can easily show that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = -1$. Thus $P_{4r+2} \bigodot F^2_{4s}$, $s \geq 1$ is signed product cordial.

Case (4). $k \equiv$ *3(mod 4).*

Let $k = 4r + 3, r > 0$. Then, one can choose the labelling $[L_{4r}11(-1) : (-1)M'_{4s}, (-1)M'_{4s}, 1M'_{4s}, 1M'_{4s} \dots (r - times), 1M'_{4s}, 1M_{4s}, (-1)M'_{4s}]$ for $P_{4r+3} \bigodot F^2_{4s}$. Therefore $x_{-1} = 2r + 1, x_1 = a_{-1} = 2r, a_1 = 2r + 2, y_{-1} = 2s+1, y_1 = 2s, b_{-1} = 6s-1, b_1 = 6s-2, y'_{-1} = 2s, y'_1 = 2s+1, b'_{-1} = 6s-1, b'_1 = 6s - 2, y''_{-1} = 2s + 1, y''_1 = 2s, b''_{-1} = 6s - 1$ and $b''_1 = 6s - 2$. Hence, one can easily show that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = 0$. Thus $P_{4r+3} \bigodot F^2_{4s}, s \geq 1$ is signed product cordial.

Lemma 3.2. $P_k \bigodot F^2_m$ *is signed product cordial for all* $k \geq 1$ *and* $m \equiv 1(mod4)$.

Proof of Lemma 3.2. We need to examine the following cases:

Case (1). $\boldsymbol{k \equiv 0(mod4)}$.

Let $k = 4r$, $r \geq 1$. Then, one can choose the labeling $[L_{4r} : 11_3(-1)_2M_{4s-4}, 11_3\ (-1)_2M_{4s-4}, (-1)_41_2M'_{4s-4}, (-1)_41_2M'_{4s-4}, \dots (r - times)]$ for $P_{4r} \bigodot F^2_{4s+1}$. Therefore $x_{-1} = x_1 = a_{-1} = 2r - 1, a_1 = 2r, y_{-1} = 2s, y_1 = 2s + 2, b_{-1} = 6s - 1, b_1 = 6s + 1, y'_{-1} = 2s + 2, y'_1 = 2s$ and $b'_{-1} = 6s - 1, b'_1 = 6s + 1$. Hence, one can easily show that $v_{-1}-v_1 = 0$ and $e_{-1}-e_1 = -1$.Thus $P_{4r} \bigodot F^2_{4s+1}, s \geq 1$ is signed product cordial. As an example, Figure 2 illustrates this case $P_4 \bigodot F^2_5$.

Case (2). $\boldsymbol{k \equiv 1(mod4)}$.

Let $k = 4r + 1, r > 0$. Then, one can choose the labeling $[L4r(-1) : 11_3(-1)_2M_{4s-4}, 11_3(-1)_2M_{4s-4}, (-1)_41_2M'_{4s-4}, (-1)_41_2\ M'_{4s-4}, \dots (r - times), 1(-1)_31_2M_{4s-4}]$ for $P_{4r+1} \bigodot F^2_{4s+1}$. Therefore $x_{-1} = 2r + 1, x_1 = a_{-1} = a_1 = 2r, y_{-1} = 2s + 2, y_1 = 2s, b_{-1} = 6s + 1, b_1 = 6s - 1, y'_{-1} = 2s, y'_1 = 2s + 2, b'_{-1} = 6s + 1, b'_1 = 6s - 1, y''_{-1} = y''_1 = 2s + 1$, and $''_{-1} = b''_1 = 6s$. Hence, one can easily show that $v_{-1} - v_1 = 1$ and $e_{-1} - e_1 = 0$. Thus $P_{4r+1} \bigodot F^2_{4s+1}, s \geq 1$, is signed product cordial.

Case (3). $\boldsymbol{k \equiv 2(mod4)}$.

Let $k = 4r + 2, r \geq 0$. Then, one can choose the labeling $[L_{4r}1(-1)1_4(-1)_2\ M_{4s-4}, 11_3(-1)_2M_{4s-4}, (-1)_41_2M'_{4s-4}, (-1)_41_2M'_{4s-4}, \dots (r - times), (-1)_41_2$

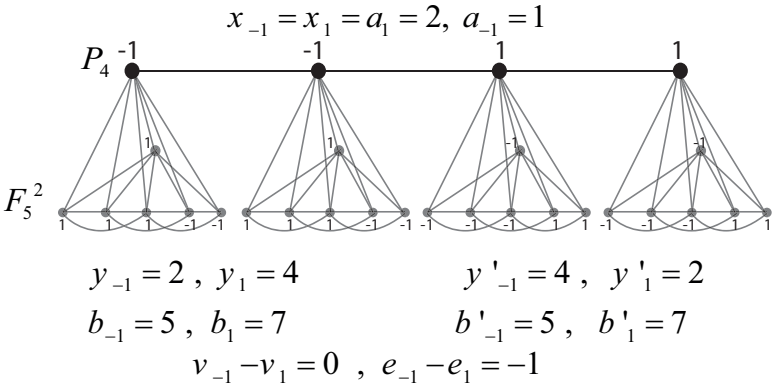


Figure 2: $P_4 \bigodot F_5^2$ is signed product cordial

$M'_{4s-4}, 11_3(-1)_2 M_{4s-4}]$ for $P_{4r+2} \bigodot F^2_{4s+1}$. Therefore $x_{-1} = x_1 = a_{-1} = 2r, a_1 = 2r+1, y_{-1} = 2s+2, y_1 = 2s, b_{-1} = 6s+1, b_1 = 6s-1, y'_{-1} = 2s, y'_1 = 2s+2$ and $b'_{-1} = 6s+1, b'_1 = 6s-1$. Hence one can easily show that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = -1$. Thus $P_{4r+2} \bigodot F^2_{4s+1}, s \geq 1$, is signed product cordial.

Case (4). $k \equiv 3(mod 4)$.

Let $k = 4r+3, r \geq 0$. Then, one can choose the labeling $[L_{4r}1(-1)1 : 1_4(-1)_2 M_{4s-4}\ \ , 1_4(-1)_2 M_{4s-4}, (-1)_4 1_2 M'_{4s-4}, (-1)_4 1_2 M'_{4s-4}, \ldots (r - times), (-1)_4\ 1_2 M'_{4s-4}, 1_4\ (-1)_2 M_{4s-4}, (-1)_3 1_3 M'_{4s-4}]$ for $P_{4r+3} \bigodot F^2_{4s+1}$. Therefore $x_{-1} = a_1 = a_{-1} = 2r+1, x_1 = 2r+2, y_{-1} = 2s, y_1 = 2s+2, b_{-1} = 6s-1, b_1 = 6s+1, y'_{-1} = 2s+2, y'_1 = 2s, b'_{-1} = 6s-1, b'_1 = 6s+1, y''_{-1} = y''_1 = 2s+1$, and $b''_{-1} = b''_1 = 6s$. Hence, one can easily show that $v_{-1} - v_1 = -1$ and $e_{-1} - e_1 = 0$. Thus $P_{4r+3} \bigodot F^2_{4s+1}, s \geq 1$, is signed product cordial.

Lemma 3.3. *$P_k \bigodot F^2_m$ is signed product cordial for all $k \geq 1$ and $\equiv 2(mod 4)$.*

Proof of Lemma 3.3. We need to study the following cases:

Case (1). $k \equiv 0(mod 4)$.

Let $k = 4r, r \geq 1$. Then, one can choose the labeling $[L_{4r} : (-1)M'_{4s+2}, (-1)M'_{4s+2}, 1M_{4s+2}, 1M_{4s+2}, \ldots (r - times)]$ for $P_{4r} \bigodot F^2_{4s+2}$. Therefore $x_{-1} = x_1 = 2r, a_{-1} = 2r-1, a_1 = 2r, y_{-1} = 2s+2, y_1 = 2s+1, b_{-1} = 6s+1, b_1 = 6s+2, y'_{-1} = 2s+1, y'_1 = 2s+2, b'_{-1} = 6s+1$ and $b'_1 = 6s+2$. Hence, one can easily show that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = 1$. Thus, $P_{4r} \bigodot F^2_{4s+2}, r \geq 1$ is signed product cordial. As an example, Figure 3 illustrates this case $P_4 \bigodot F^2_6$.

Case (2). $k \equiv 1(mod 4)$.

Let $k = 4r+1, r \geq 0$. Then, one can choose the labeling $[L_{4r}1 : (-1)M'_{4s+2}, (-1)M'_{4s+2}, 1M'_{4s+2}, 1M'_{4s+2}, \ldots, (r - times), (-1)M_{4s+2}]$ for $P_{4r+1} \bigodot F^2_{4s+2}$. Therefore $x_{-1} = 2r, x_1 = 2r+1, a_{-1} = 2r-1, a_1 = 2r+1, y_{-1} = 2s+2, y_1 = 2s+1, b_{-1} = 6s+2, b_1 = 6s+1, y'_{-1} = 2s+1, y'_1 = 2s+2, b'_{-1} = 6s+2, b'_1 = 6s+1, y"_{-1} = 2s+2, y''_1 = 2s+1, b''_{-1} = 6s+2$ and $b"_1 = 6s+1$. Hence, one

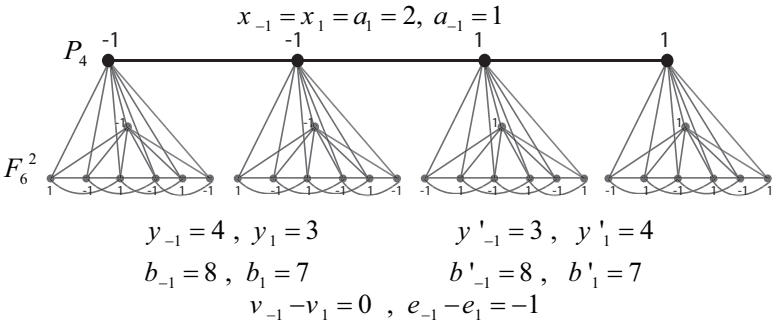


Figure 3: $P_4 \bigodot F_6^2$ is signed product cordial

can easily show that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = 0$. Thus$P_{4r+1} \bigodot F_{4s+2}^2, s \geq 1$ is signed product cordial.

Case (3). $k \equiv 2(mod4)$.

Let $k = 4r + 2, r \geq 0$. Then, one can choose the labeling $[L_{4r}1(-1) : (-1)\ M'_{4s+2}, (-1)M'_{4s+2}, 1M_{4s+2}, 1M_{4s+2}, \ldots, (r - times), 1M_{4s+2}, (-1)M'_{4s+2}]$ for $P_{4r+2} \bigodot F_{4s+2}^2$. Therefore $x_{-1} = x_1 = a_1 = 2r + 1, a_{-1} = 2r, y_{-1} = 2s+2, y_1 = 2s+1, b_{-1} = 6s+2, b_1 = 6s+1, y'_{-1} = 2s+1, y'_1 = 2s+2, b'_{-1} = 6s+2$ and $b'_1 = 6s+1$. Hence, one can easily show that $v_{-1}-v_1 = 0$ and $e_{(-1)}-e_1 = -1$. Thus $P_{4r+2} \bigodot F_{4s+2}^2, s \geq 1$ is signed product cordial.

Case (4). $k \equiv 3(mod4)$.

Let $k = 4r + 3, r \geq 0$. Then, one can choose the labeling $[L_{4r}1_2(-1) : (-1)M'_{4s+2}, (-1)M'_{4s+2}, 1M_{4s+2}, 1M_{4s+2}, \ldots, (r-times), 1M_{4s+2}, (-1)_3 1_2 M'_{4s-2}, (-1)M'_{4s+2}]$ for $P_{4r+3} \bigodot F_{4s+2}^2$. Therefore $x_{-1} = 2r+1, x_1 = a_1 = 2r+2, y_{-1} = 2s+2, a_{-1} = 2r, y_{-1} = 2s+2, y_1 = 2s+1, b_{-1} = 6s+2, b_1 = 6s+1, y'_{-1} = 2s+1, y'_1 = 2s+2, b'_{-1} = 6s+2, b'_1 = 6s+1, y''_{-1} = 2s+2, y''_1 = 2s+1, b''_{-1} = 6s+2$ and $b''_1 = 6s+1$. Hence, one can easily show that $v_{-1}-v_1 = 0$ and $e_{-1}-e_1 = 0$. Thus $P_{4r+3} \bigodot F_{4s+2}^2, s \geq 1$ is signed product cordial.

Lemma 3.4. *$P_k \bigodot F_m^2$ is signed product cordial for all $k \geq 1$ and $m \equiv 3(mod4)$.*

Proof of Lemma 3.4. Will be examined following cases:

Case (1). $k \equiv 0(mod4)$.

Let $k = 4r, r \geq 1$. Then, one can choose the labeling $[L_{4r} : 1(-1)_3 M'_{4s}, 1(-1)_3 M'_{4s}, 1(-1)1_2 M_{4s}, 1(-1)1_2 M_{4s}, \ldots, (r-times)]$ for $P_{4r} \bigodot F_{4s+3}^2$. Therefore $x_{-1} = x_1 = a_1 = 2r, a_{-1} = 2r - 1, y_{-1} = 2s + 3, y_1 = 2s + 1, b_{-1} = 6s + 4, b_1 = 6s + 2, y'_{-1} = 2s + 1, y'_1 = 2s + 3$ and $b'_{-1} = 6s + 4, b'_1 = 6s + 2$. Hence, one can easily show that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = -1$. Thus $P_{4r} \bigodot F_{4s+3}^2, s \geq 1$ is signed product cordial. As an example, Figure 4 illustrates this case $P_4 \bigodot F_7^2$.

Case (2). $k \equiv 1(mod4)$.

Let $k = 4r+1, r \geq 0$. Then, one can choose the labeling $[L_{4r}1 : 1(-1)_3 M'_{4s}, 1 (-1)_3 M'_{4s}, 1(-1)1_2 M_{4s}, 1(-1)1_2 M_{4s}, \ldots, (r - times), (-1)1_2(-1)M'_{4s}]$ for P_{4r+1}

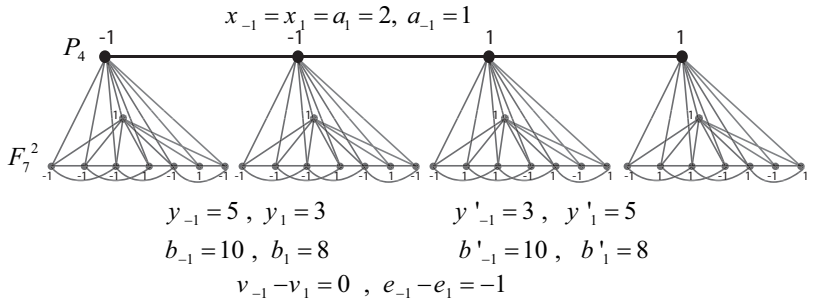


Figure 4: $P_4 \bigodot F_7^2$ is signed product cordial

$\bigodot F_{4s+3}^2$. Therefore $x_{-1} = 2r, x_1 = a_{-1} = 2r-1, a_1 = 2r+1, y_{-1} = 2s+3, y_1 = 2s+1, b_{-1} = 6s+2, b_1 = 6s+4, y'_{-1} = 2s+1, y'_1 = 2s+3, b'_{-1} = 6s+2, b'_1 = 6s+4, y''_{-1} = y''_1 = 2s+2, b''_{-1} = 6s+2$ and $b''_1 = 6s+4$. Hence, one can easily show that $v_{-1} - v_1 = -1$ and $e_{-1} - e_1 = 0$. Thus $P_{4r+1} \bigodot F_{4s+3}^2, s \geq 1$ is signed product cordial.

Case (3). $k \equiv 2(mod\ 4)$.

Let $k = 4r+2, r \geq 0$. Then, one can choose the labeling $[L_{4r}1(-1) : 1(-1)_3 M'_{4s}, 1\,(-1)_3 M'_{4s}, 1(-1)1_2 M_{4s}, 1(-1)1_2 M_{4s}, \ldots, (r-times), 1(-1)1_2\, M_{4s}, 1(-1)_3 M'_{4s}]$ for $P_{4r+2} \bigodot F_{4s+3}^2$. Therefore, $x_{-1} = x_1 = a_1 = 2r+1, a_{-1} = 2r, y_{-1} = 2s+3, y_1 = 2s+1, b_{-1} = 6s+4, b_1 = 6s+2, y'_{-1} = 2s+1, y'_1 = 2s+3, b'_{-1} = 6s+4$ and $b'_1 = 6s+2$. Hence, one can easily show that $v_{-1} - v1 = 0$ and $e_{-1} - e_1 = -1$. Thus $P_{4r+2} \bigodot F_{4s+3}^2, s \geq 1$ is signed product cordial.

Case (4). $k \equiv 3(mod 4)$.

Let $k = 4r+3, r \geq 0$. Then, one can choose the labeling $[L_{4r}1(-1)_2 : 1(-1)_3 M'_{4s}, 1(-1)_3 M'_{4s}, 1(-1)1_2 M_{4s}, 1(-1)1_2 M_{4s}, \ldots, (r-times), 1(-1)1_2 M_{4s}, 1(-1)_3 M'_{4s}, (-1)1_2(-1) M'_{4s}]$ for $P_{4r+3} \bigodot F_{4s+3}^2$. Therefore $x_{-1} = a_1 = 2r+2, x_1 = 2r+1, a_{-1} = 2r, y_{-1} = 2s+3, y_1 = 2s+1, b_{-1} = 6s+4, b_1 = 6s+2, y'_{-1} = 2s+1, y'_1 = 2s+3, b'_{-1} = 6s+4, b'_1 = 6s+2, y''_{-1} = y''_1 = 2s+2, b''_{-1} = 6s+4$ and $b''_1 = 6s+2$. Hence one can easily show that $v_{-1} - v_1 = 1$ and $e_{-1} - e_1 = 0$. Thus $P_{4r+3} \bigodot F_{4s+3}^2, s \geq 1$ is signed product cordial.

As a consequence of all previous lemmas, we can establish the following theorem.

Theorem 3.1. *The corona between path and fourth power of fan graphs $P_k \bigodot F_m^2$ is signed product cordial for all k and m.*

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A study on Pythagorean fuzzy soft topological spaces and continuous mappings

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Abstract. This paper is a study on Pythagorean fuzzy soft topological spaces and continuous mappings. Here, relevant topological notions are generalized into the new setting and proved some fundamental results. Pythagorean fuzzy soft mappings and continuous mappings are defined and illustrated with examples. Also, certain interesting properties of these mappings are obtained. Finally, we proved a couple of characterization theorems for Pythagorean fuzzy soft mappings.
Keywords: Pythagorean fuzzy soft sets, Topological spaces, Continuous mappings.

1. Introduction

Mathematical modeling of a real-life event is an incredibly difficult task. Many mathematical theories were developed to model an event with vagueness. Among these, the most important one is the invention of fuzzy sets (FS) by Zaheh [30]. FS handled vagueness with the help of the membership function. Atanassov [2] generalized FS by introducing intuitionistic fuzzy sets (IFS) and, then Yager [29] enhanced it by bringing out the Pythagorean fuzzy sets (PFS). PFS explains vagueness by the use of the membership function and non-membership function,

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where their square sum lies between 0 and 1. Molodtsov's soft set theory [8, 11] is another discovery that can describe fuzziness with the help of a parameter set. Later, hybridizing FS and soft set, fuzzy soft set (FSS) is proposed by Maji et.al. [16]. The same authors generalized FSS by introducing intuitionistic fuzzy soft sets (IFSS) [15]. Similarly, Peng [25] introduced the Pythagorean fuzzy soft set (PFSS) by combining PFS with the soft set. Since PFSS have a higher potential to capture fuzziness, many theories based on PFSS are on developing [3, 1, 4, 10, 15].

The classical topology has also been explored along with generalized sets mentioned above. Chang [8] proposed a suitable 'topological structure' on FS and since many unsolved problems in classical topology have solutions in this domain, it is a widely discussing area of research. Coker [9] introduced IFS topology, and Olgun et.al. [21] used it to define PFS topological structure. Shabir and Naz [28] studied soft topological spaces and according to them soft topological space is a parametrized family of topologies on the universal set. Later, Çağman et.al. [7] proposed a new topological structure on soft sets. Instead of considering a parametrized family of topologies, they used a collection of soft subsets to define the new topology. A nice survey of both approaches can be found in chapter 3 of the book [11]. After that many studies on soft topology are developed and that can be seen in [31, 32, 33, 34, 35]. Followed by Çağman's approach, a study on mappings and transformations on soft sets is carried out in [13, 17, 37, 38, 39]. Later, Roy and Samantha [27] considered FSS topological space and then a study on IFSS topological spaces is carried out in [6, 22, 14]. Subsequently Karata et.al. [12] defined IFSS mappings and IFSS continuous mappings. Recently Riaz et.al. [12] introduced the PFSS topological space and explained the multi-criteria decision-making problem as an application. Related studies on PFSS topology are done in [23, 24, 36]. Also, the notions of compactness, connectedness, continuous mappings and homeomorphisms are suitably defined and explored in all the above-mentioned generalized sets.

The notion of topology is used to indicate the relationship between spatial objects and their characteristics. There are real-life situations in which the boundaries of objects are vague. Using the techniques of topological structures considered on generalized sets one can analyze objects along with vagueness. Here we introduce a topological structure on PFSS. As PFSS is a parametrized family of PFSs, it is a very useful tool to model uncertainty from both parametric as well as membership points of view. Since PFSS can be considered as a single representation for a collection of certain related PFSs, it reduces the complexity of analysis and calculations, particularly in decision-making problems [10, 20]. Thus introducing a topological structure on PFSS has significant relevance and is of our interest.

In this paper, we have done a study on the topological structure of PFSS. Fundamental definitions of new notions and the corresponding examples are included. Some theorems related to interior and open sets are proved. We illustrated PFSS mappings with examples and certain properties of PFSS mapping

are shown. In analogy with the classical settings, we identified a suitable definition for PFSS continuous mappings and characterized it in terms of PFSS open sets. The notion of PFSS continuous mappings can be used to compare two PFSS topological spaces and one can easily see that the collection of all PFSS with PFSS continuous mappings forms a category. It is also worth mentioning that our definition of PFSS continuous mapping generalizes IFSS continuous mapping. Hence it can be used to handle vagueness in many fields such as machine learning, data analysis, etc. as we have developed the continuous mappings from one topological space into another. As in [20], we can improve the algorithm for decision making with the help of topological space. So if we can identify the mapping corresponding to one set of parameters to another set and the corresponding mapping among the universal set, and if the mapping identified turns out to be continuous, then decision making in the latter situation is quite easy.

This paper is divided into three sections, besides the introduction. In section 2, we recall some basic definitions which are used in the entire discussion. Section 3 deals with PFSS topological spaces and the related topological notions. In section 4, we introduce PFSS mappings, PFSS continuous mappings and proved a couple of characterization theorems.

2. Preliminaries

This section addresses some basic definitions beneficial for entire discussions. Those are the main definitions connected with Pythagorean fuzzy soft sets and Pythagorean fuzzy soft topology. Unless otherwise specified, $\mho$ be the universal set, $\mathfrak{L}$ be parameter set, and $P(\mho)$ be power set of $\mho$.

2.1 Pythagorean fuzzy soft sets

Definition 2.1 ([29]). *A Pythagorean fuzzy set $\mathcal{P}$ on $\mho$ is defined as the set $\{(u, \mu_p(u), \nu_p(u)) : u \in \mho\}$ where $\mu_p : \mho \to [0,1]$ and $\nu_p : \mho \to [0,1]$ with $0 \leq \mu_p^2 + \nu_p^2 \leq 1$.*

Definition 2.2 ([18]). *The pair $(\mathcal{S}, \mathfrak{L})$ is said to be soft set over $\mho$, if $\mathcal{S}$ is a mapping from $\mathfrak{L}$ to $P(\mho)$. i.e., for each $\mathfrak{e} \in \mathfrak{L}$, $S(\mathfrak{e})$ is considered as the set of $\mathfrak{e}$-approximate elements of the soft set.*

Definition 2.3 ([25]). *A Pythagorean fuzzy soft set(PFSS) is defined as the pair $(\mathcal{P}, \mathfrak{L})$ where, $\mathcal{P} : \mathfrak{L} \to PFS(\mho)$ and $PFS(\mho)$ is the set of all Pythagorean fuzzy subsets of $\mho$.*

We are using the notation $0_{(\mho,\mathfrak{L})}$ for the PFSS with $\mu_p(u) = 0$ and $\nu_p(u) = 1$ corresponding to each $\mathfrak{e} \in \mathfrak{L}$ and $1_{(\mho,\mathfrak{L})}$ for the PFSS with $\mu_p(u) = 1$ and $\nu_p(u) = 0$ corresponding to each $\mathfrak{e} \in \mathfrak{L}$. For $\mho = \{u_1, u_2, \ldots, u_n\}$ and $\mathfrak{L} = \{\mathfrak{e}_1, \mathfrak{e}_2, \ldots \mathfrak{e}_m\}$

the PFSS $(\mathcal{P}, \mathfrak{L})$ is represented as

$$(\mathcal{P}, \mathfrak{L}) = \begin{pmatrix} \mathcal{P}(\mathfrak{e}_1) & = & \{u_1/(a_{11}, b_{11}), u_2/(a_{12}, b_{12}), \ldots u_n/(a_{1n}, b_{1n})\} \\ \mathcal{P}(\mathfrak{e}_2) & = & \{u_1/(a_{21}, b_{21}), u_2/(a_{22}, b_{22}), \ldots u_n/(a_{2n}, b_{2n})\} \\ . & = & \\ . & = & \\ . & = & \\ \mathcal{P}(\mathfrak{e}_m) & = & \{u_1/(a_{m1}, b_{m1}), u_2/(a_{m2}, b_{m2}), \ldots u_n/(a_{mn}, b_{mn})\} \end{pmatrix},$$

where $(a_{ij}, b_{ij}) = (\mu_{\mathcal{P}_{(\mathfrak{e}_i)}}(u_j), \nu_{\mathcal{P}_{(\mathfrak{e}_i)}}(u_j))$ i.e., membership value and nonmembership value corresponding to parameter $\mathfrak{e}_i$ and universal element u_j.

The following definition is helpful to recollect various operations on PFSS.

Definition 2.4 ([20]). *Let $(\mathcal{P}_1, \mathfrak{L}_1)$ and $(\mathcal{P}_2, \mathfrak{L}_2)$ be two PFSSs on $\mho$. Then*

1. *$(\mathcal{P}_1, \mathfrak{L}_1)$ is subset of $(\mathcal{P}_2, \mathfrak{L}_2)$ i.e., $(\mathcal{P}_1, \mathfrak{L}_1) \sqsubseteq (\mathcal{P}_2, \mathfrak{L}_2)$ if:*
   (a) *$\mathfrak{L}_1 \subseteq \mathfrak{L}_2$;*
   (b) *for each $\mathfrak{e} \in \mathfrak{L}_1$, $\mu_{\mathcal{P}_1(\mathfrak{e})}(u) \leq \mu_{\mathcal{P}_2(\mathfrak{e})}(u)$ and $\nu_{\mathcal{P}_1(\mathfrak{e})}(u) \geq \nu_{\mathcal{P}_2(\mathfrak{e})}(u)$, $\forall u \in \mho$.*
2. *The complement of $(\mathcal{P}, \mathfrak{L})$ i.e., $(\mathcal{P}, \mathfrak{L})^c = (\mathcal{P}^c, \mathfrak{L})$.*
3. *The union of $(\mathcal{P}_1, \mathfrak{L}_1)$ and $(\mathcal{P}_2, \mathfrak{L}_2)$ i.e., $(\mathcal{P}_1, \mathfrak{L}_1) \sqcup (\mathcal{P}_2, \mathfrak{L}_2) = (\mathcal{P}, \mathfrak{L}_1 \cup \mathfrak{L}_2)$ where, $\mathcal{P}(e) = \{\big(u, \max\{\mu_{\mathcal{P}_1}(u), \mu_{\mathcal{P}_2}(u)\}, \min\{\nu_{\mathcal{P}_1}(u), \nu_{\mathcal{P}_2}(u)\}\big) : u \in \mho\}$ for each $\mathfrak{e} \in \mathfrak{L}_1 \cup \mathfrak{L}_2$.*
4. *The intersection of $(\mathcal{P}_1, \mathfrak{L}_1)$ and $(\mathcal{P}_2, \mathfrak{L}_2)$ i.e., $(\mathcal{P}_1, \mathfrak{L}_1) \sqcap (\mathcal{P}_2, \mathfrak{L}_2) = (\mathcal{P}, \mathfrak{L}_1 \cap \mathfrak{L}_2)$ where, $\mathcal{P}(e) = \{\big(u, \min\{\mu_{\mathcal{P}_1}(u), \mu_{\mathcal{P}_2}(u)\}, \max\{\nu_{\mathcal{P}_1}(u), \nu_{\mathcal{P}_2}(u)\}\big) : u \in \mho\}$ for each $\mathfrak{e} \in \mathfrak{L}_1 \cap \mathfrak{L}_2$.*

2.2 Pythagorean fuzzy soft topology

Definition 2.5 ([26]). *Let $\mho$ be the universal set and $\mathfrak{L}$ be the set of parameters. A collection τ of Pythagorean fuzzy soft subsets of $\mho$ over $\mathfrak{L}$ is said to be PFSS topology if*

i. $0_{(\mho, \mathfrak{L})}$, $1_{(\mho, \mathfrak{L})} \in \tau$;

ii. for any $(\mathcal{P}, \mathfrak{L}), (\mathcal{G}, \mathfrak{L}) \in \tau$, we have $(\mathcal{P}, \mathfrak{L}) \sqcap (\mathcal{G}, \mathfrak{L}) \in \tau$;

iii. for any $(\mathcal{P}_i, \mathfrak{L})_{i \in I} \in \tau$, we have $\sqcup_{i \in I}(\mathcal{P}_i, \mathfrak{L}) \in \tau$, where I is any indexed set.

Hereinafter the triptlet $(\mho, \tau, \mathfrak{L})$ refered PFSS topological space. Every element in τ is called an open PFSSs. Closed PFSSs is the complement of open PFSSs. Also, interior of a PFSS $(\mathcal{P}, \mathfrak{L})$ is the biggest open PFSS embedded in $(\mathcal{P}, \mathfrak{L})$ and closure of $(\mathcal{P}, \mathfrak{L})$ is the smallest closed PFSS containing $(\mathcal{P}, \mathfrak{L})$. Interior and closure of $(\mathcal{P}, \mathfrak{L})$ is denoted as $(\mathcal{P}, \mathfrak{L})^\circ$ and $\overline{(\mathcal{P}, \mathfrak{L})}$ respectively.

3. More results on PFSS topological space

In this section, some interesting theorems on PFSS topological space are explained. At first, the relationship between PFSS topological space and fuzzy soft bitopological space is explained. After that studies on interior and neighborhood are carried out and a characterization theorem of open sets is proved.

Now, $(\mho, \tau_1, \tau_2, \mathfrak{L})$ be known as fuzzy soft bitopological space [19] if the topologies $(\mho, \tau_1, \mathfrak{L})$ and $(\mho, \tau_2, \mathfrak{L})$ independently satisfy the axioms of fuzzy soft topological space. The following theorem says that corresponding to each PFSS topological space we get a fuzzy soft bitopological space.

Theorem 3.1. *Let $(\mho, \tau, \mathfrak{L})$ be a Pythagorean fuzzy soft topological spaces and τ is given by $\tau = \{0_{(\mho,\mathfrak{L})}, 1_{(\mho,\mathfrak{L})}, (\mathcal{P}_i, \mathfrak{L})_{i\in I}\}$ where I is arbitrary indexed set. Then $(\mho, \tau_1, \tau_2, \mathfrak{L})$ is a fuzzy soft bitopological space on $\mho$, where $\tau_1 = \{\widetilde{0}, \widetilde{1}, (\mu_{\mathcal{P}_i}, \mathfrak{L})_{i\in I}\}$ and $\tau_2 = \{\widetilde{0}, \widetilde{1}, (\nu_{\mathcal{P}_i}, \mathfrak{L})^{\complement}_{i\in I}\}$.*

Proof. We have to prove that both τ_1 and τ_2 satisfy all the three axioms of fuzzy soft topology. $0_{(\mho,\mathfrak{L})}, 1_{(\mho,\mathfrak{L})} \in \tau$ gives that $\widetilde{0}, \widetilde{1} \in \tau_1$ and $\widetilde{0}, \widetilde{1} \in \tau_2$. Thus first condition is obtained.

Second condition is obtained from the fact that $(\mathcal{P}_1, \mathfrak{L}) \sqcap (\mathcal{P}_2, \mathfrak{L}) = (\mu_{\mathcal{P}_1\wedge\mathcal{P}_2}, \nu_{\mathcal{P}_1\vee\mathcal{P}_2}, \mathfrak{L})$. So, $(\mathcal{P}_1, \mathfrak{L}) \sqcap (\mathcal{P}_2, \mathfrak{L}) \in \tau$ gives $(\mu_{\mathcal{P}_1\wedge\mathcal{P}_2}, \mathfrak{L}) \in \tau_1$ and $(\nu_{\mathcal{P}_1\vee\mathcal{P}_2}, \mathfrak{L})^{\complement} = (\nu_{\mathcal{P}_1^{\complement}\wedge\mathcal{P}_2^{\complement}}, \mathfrak{L}) \in \tau_2$.

Let $J \subseteq I$ be any indexed set. Then, we have $\sqcup_{j\in J}(\mathcal{P}_j, \mathfrak{L}) = (\mu_{\vee_{j\in J}\mathcal{P}_j}, \nu_{\wedge_{j\in J}\mathcal{P}_j}, \mathfrak{L})$. So, $\sqcup_{j\in J}(\mathcal{P}_j, \mathfrak{L}) \in \tau$ gives $(\mu_{\vee_{j\in J}\mathcal{P}_j}, \mathfrak{L}) \in \tau_1$ and $(\nu_{\wedge_{j\in J}\mathcal{P}_j}, \mathfrak{L})^{\complement} = (\nu_{\vee_{j\in J}\mathcal{P}_j^{\complement}}, \mathfrak{L}) \in \tau_2$. Thus, $(\mho, \tau_1, \tau_2, \mathfrak{L})$ is a fuzzy soft bitopological spaces. □

The effect of Theorem 3.1 is from a given PFSS topology we can generate fuzzy soft bitopological space. But one can easily identify that converse of Theorem 3.1 is not valid always. But the following example says, even if ζ be a collection of PFSSs containing $0_{(\mho,\mathfrak{L})}$ and $1_{(\mho,\mathfrak{L})}$ with the property that $(\mho, \zeta_1, \mathfrak{L})$ and $(\mho, \zeta_2, \mathfrak{L})$ are fuzzy soft topological spaces, ζ need not be a PFSS topology.

Example 3.1. Let the universal set $\mho$ be $\{u_1, u_2\}$ and parameter set $\mathfrak{L} = \{\mathfrak{e}_1, \mathfrak{e}_2\}$. The collection of PFSS ζ is given by $\zeta = \{0_{(\mho,\mathfrak{L})}, 1_{(\mho,\mathfrak{L})}, (\mathcal{P}_1, \mathfrak{L}), (\mathcal{P}_2, \mathfrak{L}), (\mathcal{P}_3, \mathfrak{L}), (\mathcal{P}_4, \mathfrak{L})\}$, where

$$(\mathcal{P}_1, \mathfrak{L}) = \left(\begin{array}{lcl} \mathcal{P}_1(\mathfrak{e}_1) & = & \{u_1/(0.7, 0.3), u_2/(0.2, 0.7)\} \\ \mathcal{P}_1(\mathfrak{e}_2) & = & \{u_1/(0.6, 0.6), u_2/(0.1, 0.7)\} \end{array}\right),$$

$$(\mathcal{P}_2, \mathfrak{L}) = \left(\begin{array}{lcl} \mathcal{P}_2(\mathfrak{e}_1) & = & \{u_1/(0.8, 0.4), u_2/(0.3, 0.6)\} \\ \mathcal{P}_2(\mathfrak{e}_2) & = & \{u_1/(0.7, 0.3), u_2/(0.4, 0.5)\} \end{array}\right),$$

$$(\mathcal{P}_3, \mathfrak{L}) = \left(\begin{array}{lcl} \mathcal{P}_3(\mathfrak{e}_1) & = & \{u_1/(0.3, 0.3), u_2/(0.2, 0.6)\} \\ \mathcal{P}_3(\mathfrak{e}_2) & = & \{u_1/(0.2, 0.3), u_2/(0.1, 0.5)\} \end{array}\right),$$

$$(\mathcal{P}_4, \mathfrak{L}) = \left(\begin{array}{lcl} \mathcal{P}_4(\mathfrak{e}_1) & = & \{u_1/(0.8, 0.4), u_2/(0.5, 0.7)\} \\ \mathcal{P}_4(\mathfrak{e}_2) & = & \{u_1/(0.8, 0.6), u_2/(0.7, 0.7)\} \end{array}\right).$$

It is clear that ζ is not a PFSS topology. Because $(\mathcal{P}_1, \mathfrak{L}) \sqcup (\mathcal{P}_2, \mathfrak{L})$ is not in ζ. But ζ_1 and ζ_2 is given by

$$\zeta_1 = \Big\{\widetilde{0},\widetilde{1}, \begin{pmatrix}\mu_{\mathcal{P}_1(\mathfrak{e}_1)} &=& \{u_1/0.7, u_2/0.2\}\\ \mu_{\mathcal{P}_1(\mathfrak{e}_2)} &=& \{u_1/0.6, u_2/0.1\}\end{pmatrix}, \begin{pmatrix}\mu_{\mathcal{P}_2(\mathfrak{e}_1)} &=& \{u_1/0.8, u_2/0.3\}\\ \mu_{\mathcal{P}_2(\mathfrak{e}_2)} &=& \{u_1/0.7, u_2/0.4\}\end{pmatrix},$$
$$\begin{pmatrix}\mu_{\mathcal{P}_3(\mathfrak{e}_1)} &=& \{u_1/0.3, u_2/0.2\}\\ \mu_{\mathcal{P}_3(\mathfrak{e}_2)} &=& \{u_1/0.2, u_2/0.1\}\end{pmatrix}, \begin{pmatrix}\mu_{\mathcal{P}_4(\mathfrak{e}_1)} &=& \{u_1/0.8, u_2/0.5\}\\ \mu_{\mathcal{P}_4(\mathfrak{e}_2)} &=& \{u_1/0.8, u_2/0.7\}\end{pmatrix}\Big\}$$
$$\zeta_2 = \Big\{\widetilde{0},\widetilde{1}, \begin{pmatrix}\mu_{\mathcal{P}_1(\mathfrak{e}_1)} &=& \{u_1/0.7, u_2/0.3\}\\ \mu_{\mathcal{P}_1(\mathfrak{e}_2)} &=& \{u_1/0.4, u_2/0.3\}\end{pmatrix}, \begin{pmatrix}\mu_{\mathcal{P}_2(\mathfrak{e}_1)} &=& \{u_1/0.6, u_2/0.4\}\\ \mu_{\mathcal{P}_2(\mathfrak{e}_2)} &=& \{u_1/0.7, u_2/0.5\}\end{pmatrix},$$
$$\begin{pmatrix}\mu_{\mathcal{P}_3(\mathfrak{e}_1)} &=& \{u_1/0.7, u_2/0.4\}\\ \mu_{\mathcal{P}_3(\mathfrak{e}_2)} &=& \{u_1/0.7, u_2/0.5\}\end{pmatrix}, \begin{pmatrix}\mu_{\mathcal{P}_4(\mathfrak{e}_1)} &=& \{u_1/0.6, u_2/0.3\}\\ \mu_{\mathcal{P}_4(\mathfrak{e}_2)} &=& \{u_1/0.4, u_2/0.3\}\end{pmatrix}\Big\}.$$

We can easily verify that ζ_1 and ζ_2 are fuzzy soft topologies.

The following is the definition of neighborhood of a PFSS which is somewhat different from the classical one. Chang [8] gave this notion for fuzzy topological space and following this idea here we introduced the definition given below.

Definition 3.1. *Let $(X, \tau, \mathfrak{L})$ be PFSS topological space and $(\mathcal{P}, \mathfrak{L})$ be a PFSS. The PFSS $(N, \mathfrak{L})$ is said to be a neighbourhood of $(\mathcal{P}, \mathfrak{L})$ if there exist a $(\mathcal{G}, \mathfrak{L}) \in \tau$ such that $(\mathcal{P}, \mathfrak{L}) \sqsubseteq (\mathcal{G}, \mathfrak{L}) \sqsubseteq (N, \mathfrak{L})$.*

Definition 3.2. *Let $(\mathcal{P}, \mathfrak{L})$ and $(\mathcal{G}, \mathfrak{L})$ be PFSS in the PFSS topological space $(X, \tau, \mathfrak{L})$ such that $(\mathcal{G}, \mathfrak{L}) \sqsubseteq (\mathcal{P}, \mathfrak{L})$. Then $(\mathcal{G}, \mathfrak{L})$ serves as interior PFSS of $(\mathcal{P}, \mathfrak{L})$ if $(\mathcal{P}, \mathfrak{L})$ is a neighbourhood of $(\mathcal{G}, \mathfrak{L})$.*

The following example illustrates the definition of the neighborhood of a PFSS and interior PFSS.

Example 3.2. Consider the PFSS topology given below.

$\tau = \{0_{(\mho,\mathfrak{L})}, 1_{(\mho,\mathfrak{L})}, (\mathcal{P}_1, \mathfrak{L}), (\mathcal{P}_2, \mathfrak{L}), (\mathcal{P}_3, \mathfrak{L}), (\mathcal{P}_4, \mathfrak{L}), (\mathcal{P}_5, \mathfrak{L})\}$ is a PFSS topology on $\mho$, where the PFSSs $(\mathcal{P}_i, \mathfrak{L})$, $i = 1, 2, 3, 4, 5$ is given by

$$(\mathcal{P}_1, \mathfrak{L}) = \begin{pmatrix}\mathcal{P}_1(\mathfrak{e}_1) &=& \{u_1/(0.8, 0.1), u_2/(0.9, 0.1), u_3/(0.3, 0.4)\}\\ \mathcal{P}_1(\mathfrak{e}_2) &=& \{u_1/(0.8, 0.1), u_2/(0.8, 0.1), u_3/(0.6, 0.5)\}\end{pmatrix},$$
$$(\mathcal{P}_2, \mathfrak{L}) = \begin{pmatrix}\mathcal{P}_2(\mathfrak{e}_1) &=& \{u_1/(0.8, 0.1), u_2/(0.6, 0.2), u_3/(0.3, 0.7)\}\\ \mathcal{P}_2(\mathfrak{e}_2) &=& \{u_1/(0.4, 0.4), u_2/(0.8, 0.1), u_3/(0.6, 0.5)\}\end{pmatrix},$$
$$(\mathcal{P}_3, \mathfrak{L}) = \begin{pmatrix}\mathcal{P}_3(\mathfrak{e}_1) &=& \{u_1/(0.6, 0.3), u_2/(0.6, 0.2), u_3/(0.2, 0.7)\}\\ \mathcal{P}_3(\mathfrak{e}_2) &=& \{u_1/(0.4, 0.4), u_2/(0.6, 0.2), u_3/(0.5, 0.6)\}\end{pmatrix},$$
$$(\mathcal{P}_4, \mathfrak{L}) = \begin{pmatrix}\mathcal{P}_4(\mathfrak{e}_1) &=& \{u_1/(0.6, 0.3), u_2/(0.9, 0.1), u_3/(0.2, 0.4)\}\\ \mathcal{P}_4(\mathfrak{e}_2) &=& \{u_1/(0.8, 0.1), u_2/(0.6, 0.2), u_3/(0.5, 0.6)\}\end{pmatrix},$$
$$(\mathcal{P}_5, \mathfrak{L}) = \begin{pmatrix}\mathcal{P}_5(\mathfrak{e}_1) &=& \{u_1/(0.9, 0.1), u_2/(0.9, 0.0), u_3/(0.5, 0.2)\}\\ \mathcal{P}_5(\mathfrak{e}_2) &=& \{u_1/(0.8, 0.0), u_2/(0.8, 0.1), u_3/(0.7, 0.2)\}\end{pmatrix}.$$

The PFSS $(\mathcal{P}, \mathfrak{L})$ and $(N, \mathfrak{L})$ is given by,

$$(\mathcal{P}, \mathfrak{L}) = \begin{pmatrix}\mathcal{P}(\mathfrak{e}_1) &=& \{u_1/(0.3, 0.5), u_2/(0.4, 0.3), u_3/(0.1, 0.8)\}\\ \mathcal{P}(\mathfrak{e}_2) &=& \{u_1/(0.2, 0.6), u_2/(0.6, 0.3), u_3/(0.2, 0.7)\}\end{pmatrix},$$
$$(N, \mathfrak{L}) = \begin{pmatrix}N(\mathfrak{e}_1) &=& \{u_1/(0.9, 0.0), u_2/(0.7, 0.1), u_3/(0.5, 0.5)\}\\ N(\mathfrak{e}_2) &=& \{u_1/(0.8, 0.3), u_2/(0.9, 0.1), u_3/(0.9, 0.3)\}\end{pmatrix}.$$

It can be seen that there exist a PFSS $(\mathcal{P}_2, \mathfrak{L}) \in \tau$ such that $(\mathcal{P}, \mathfrak{L}) \sqsubseteq (\mathcal{P}_2, \mathfrak{L}) \sqsubseteq (N, \mathfrak{L})$. So $(N, \mathfrak{L})$ is a neighbourhood of the PFSS $(\mathcal{P}, \mathfrak{L})$ and thus $(\mathcal{P}, \mathfrak{L})$ is the interior PFSS of $(N, \mathfrak{L})$.

Theorem 3.2. *In a PFSS topological space, a PFSS $(\mathcal{P}, \mathfrak{L})$ is open if and only if each subsets of $(\mathcal{P}, \mathfrak{L})$ are interior PFSSs of $(\mathcal{P}, \mathfrak{L})$.*

Proof. Let $(\mathcal{P}, \mathfrak{L})$ be open PFSS and let $(\mathcal{G}, \mathfrak{L})$ be a subset of $(\mathcal{P}, \mathfrak{L})$. So we have $(\mathcal{P}, \mathfrak{L}) \in \tau$ such that $(\mathcal{G}, \mathfrak{L}) \sqsubseteq (\mathcal{P}, \mathfrak{L}) \sqsubseteq (\mathcal{P}, \mathfrak{L})$. Thus each subsets of $(\mathcal{P}, \mathfrak{L})$ are interior PFSSs of $(\mathcal{P}, \mathfrak{L})$. Conversly, $(\mathcal{P}, \mathfrak{L})$ is a neighbourhood of each of its subsets so $(\mathcal{P}, \mathfrak{L})$ is the neighbourhood of itself. By definition there exist $(O, \mathfrak{L}) \in \tau$ with $(\mathcal{P}, \mathfrak{L}) \sqsubseteq (O, \mathfrak{L}) \sqsubseteq (\mathcal{P}, \mathfrak{L})$. Thus $(\mathcal{P}, \mathfrak{L}) = (O, \mathfrak{L})$ and so $(\mathcal{P}, \mathfrak{L})$ is open. □

Theorem 3.2 is a characterisation of PFSS open set using Definition 3.1. And so this theorem helps to identify the PFSS opens sets. The following theorem is a modification of Theorem 3.2 because it gives the relationship between the interior of a PFSS $(\mathcal{P}, \mathfrak{L})$ and interior PFSSs of $(\mathcal{P}, \mathfrak{L})$ and also says when a PFSS is open in terms of the interior of a PFSS.

Theorem 3.3. *The union of all interior PFSSs of $(\mathcal{P}, \mathfrak{L})$ gives the interior of $(\mathcal{P}, \mathfrak{L})$ and $(\mathcal{P}, \mathfrak{L})$ is open if and only if $(\mathcal{P}, \mathfrak{L}) = (\mathcal{P}, \mathfrak{L})^\circ$.*

Proof. Let $\mathscr{G} = \sqcup(\mathcal{G}, \mathfrak{L})$, where the union varies over all the interior PFSS of $(\mathcal{P}, \mathfrak{L})$. We will prove that $\mathscr{G}$ is the biggest open set embedded in $(\mathcal{P}, \mathfrak{L})$. Since $(\mathcal{P}, \mathfrak{L})$ is a neighbourhood of each interior PFSS $(\mathcal{G}, \mathfrak{L})$, $(\mathcal{P}, \mathfrak{L})$ is a neighbourhood of $\mathscr{G}$. Hence there exist $(O, \mathfrak{L}) \in \tau$ with $\mathscr{G} \sqsubseteq (O, \mathfrak{L}) \sqsubseteq (\mathcal{P}, \mathfrak{L})$. But $(O, \mathfrak{L})$ also an interior PFSS of $(\mathcal{P}, \mathfrak{L})$ so $(O, \mathfrak{L}) \sqsubseteq \mathscr{G}$. Thus $\mathscr{G}$ is open and biggest open set embedded in $(\mathcal{P}, \mathfrak{L})$. Therefore $\mathscr{G} = (\mathcal{P}, \mathfrak{L})^\circ$. For $(\mathcal{P}, \mathfrak{L})$ is open, $(\mathcal{P}, \mathfrak{L}) \sqsubseteq (\mathcal{P}, \mathfrak{L})^\circ$. Because, $(\mathcal{P}, \mathfrak{L})$ itself is interior PFSS of $(\mathcal{P}, \mathfrak{L})$. Thus $(\mathcal{P}, \mathfrak{L}) = (\mathcal{P}, \mathfrak{L})^\circ$. The converse part is straight forward from the definition of $(\mathcal{P}, \mathfrak{L})^\circ$. □

4. PFSS mappings and continuous mappings

This section includes the definition and properties of PFSS mapping and continuous mapping. From here on, the collection of all PFSS over $\mho$ under the parameter set $\mathfrak{L}$ is denoted as $PFS(\mho, \mathfrak{L})$. The following is the definition of PFSS mapping and some related features.

Definition 4.1. *Let $PFS(X, \mathfrak{L}_1), PFS(Y, \mathfrak{L}_2)$ be collection of all PFSSs over $(X, \mathfrak{L}_1)$ and $(Y, \mathfrak{L}_2)$ correspondingly. Also, let $\mathcal{J} : X \to Y$ and $\Omega : \mathfrak{L}_1 \to \mathfrak{L}_2$ be two mappings. Then*

(*i*) *the PFSS mapping $\mathcal{J}_\Omega : PFS(X, \mathfrak{L}_1) \to PFS(Y, \mathfrak{L}_2)$ is defined as, for $(\mathcal{P}, \mathfrak{L}_1) \in PFS(X, \mathfrak{L}_1)$, $\mathcal{J}_\Omega[\mathcal{P}, \mathfrak{L}_1] = (\mathcal{J}(\mathcal{P}), \Omega(\mathfrak{L}_1))$ and $(\mathcal{J}(\mathcal{P}), \Omega(\mathfrak{L}_1)) \in$*

$PFS(Y,\mathfrak{L}_2)$ and it is given by

$$\mu_{\mathcal{J}(\mathcal{P})(\beta)}(y)=\begin{cases}\sup\limits_{\substack{\alpha\in\Omega^{-1}(\beta)\cap\mathfrak{L}_1,\\ x\in\mathcal{J}^{-1}(y)}}\mu_{\mathcal{P}(\alpha)}(x), & \text{if } \mathcal{J}^{-1}(y)\neq\emptyset\\ 0, & \text{otherwise,}\end{cases}$$

$$\text{and } \nu_{\mathcal{J}(\mathcal{P})(\beta)}(y)=\begin{cases}\inf\limits_{\substack{\alpha\in\Omega^{-1}(\beta)\cap\mathfrak{L}_1,\\ x\in\mathcal{J}^{-1}(y)}}\nu_{\mathcal{P}(\alpha)}(x), & \text{if } \mathcal{J}^{-1}(y)\neq\emptyset\\ 1, & \text{otherwise,}\end{cases}\text{, for each } \beta\in$$

$\mathfrak{L}_2$ *and* $y\in Y$

(*ii*) *the inverse image under $\mathcal{J}_\Omega$, is defined as, for $(\mathcal{G},\mathfrak{L}_2)\in PFS(Y,\mathfrak{L}_2)$, $\mathcal{J}_\Omega^{-1}[\mathcal{G},\mathfrak{L}_2]=(\mathcal{J}^{-1}(\mathcal{G}),\Omega^{-1}(\mathfrak{L}_2))\in PFS(X,\mathfrak{L}_1)$ and is given by $\mu_{\mathcal{J}^{-1}(\mathcal{G})(\alpha)}(x)$ $=\mu_{G(\Omega(\alpha))}\mathcal{J}(x)$ and $\nu_{\mathcal{J}^{-1}(\mathcal{G})(\alpha)}(x)=\nu_{\mathcal{G}(\Omega(\alpha))}\mathcal{J}(x)$*

(*iii*) *$\mathcal{J}_\Omega$ is injective (surjective) mapping if $\mathcal{J}$ and Ω are injective (surjective) mappings*

(*iv*) *$\mathcal{J}_\Omega$ is identity mapping if $\mathcal{J}$ and Ω are identity mappings*

Remark 4.1. It can be easily obtained that $\mathcal{J}_\Omega[\mathcal{P},\mathfrak{L}_1]$ and $\mathcal{J}_\Omega^{-1}[\mathcal{G},\mathfrak{L}_2]$ are PFSSs. For $\mathcal{J}^{-1}(y)\neq\emptyset$,

$$\begin{aligned}\left(\mu_{\mathcal{J}(\mathcal{P})(\beta)}(y)\right)^2+\left(\nu_{\mathcal{J}(\mathcal{P})(\beta)}(y)\right)^2&=\left(\sup_{\substack{\alpha\in\Omega^{-1}(\beta)\cap\mathfrak{L}_1,\\ x\in\mathcal{J}^{-1}(y)}}\mu_{\mathcal{P}(\alpha)}(x)\right)^2+\left(\inf_{\substack{\alpha\in\Omega^{-1}(\beta)\cap\mathfrak{L}_1,\\ x\in\mathcal{J}^{-1}(y)}}\nu_{\mathcal{P}(\alpha)}(x)\right)^2\\&=\sup_{\substack{\alpha\in\Omega^{-1}(\beta)\cap\mathfrak{L}_1,\\ x\in\mathcal{J}^{-1}(y)}}\mu^2_{\mathcal{P}(\alpha)}(x)+\inf_{\substack{\alpha\in\Omega^{-1}(\beta)\cap\mathfrak{L}_1,\\ x\in\mathcal{J}^{-1}(y)}}\nu^2_{\mathcal{P}(\alpha)}(x)\\&\leq\sup_{\substack{\alpha\in\Omega^{-1}(\beta)\cap\mathfrak{L}_1,\\ x\in\mathcal{J}^{-1}(y)}}(1-\nu^2_{\mathcal{P}(\alpha)}(x))+\inf_{\substack{\alpha\in\Omega^{-1}(\beta)\cap\mathfrak{L}_1,\\ x\in\mathcal{J}^{-1}(y)}}\nu^2_{\mathcal{P}(\alpha)}(x)\leq 1.\end{aligned}$$

For $\mathcal{J}^{-1}(y)=\emptyset$ we get, $\left(\mu_{\mathcal{J}(\mathcal{P})(\beta)}(y)\right)^2+\left(\nu_{\mathcal{J}(\mathcal{P})(\beta)}(y)\right)^2=1$. This is true for each $\beta\in\mathfrak{L}_2$ and $y\in Y$ implies that $\mathcal{J}_\Omega[\mathcal{P},\mathfrak{L}_1]\in PFS(Y,\mathfrak{L}_2)$. Similarly, it is obtained that $\mathcal{J}_\Omega^{-1}[\mathcal{G},\mathfrak{L}_2]\in PFS(X,\mathfrak{L}_1)$.

Example 4.1. Let $X=\{x_1,x_2,x_3\}$ and $Y=\{y_1,y_2,y_3\}$ are universal sets and $A=\{\alpha_1,\alpha_2,\alpha_3\}$ and $B=\{\beta_1,\beta_2\}$ are the parameter sets. $\mathcal{J}:X\to Y$ is given by $\mathcal{J}(x_1)=y_1$, $\mathcal{J}(x_2)=y_2$, $\mathcal{J}(x_3)=y_2$ and $\Omega:\mathfrak{L}_1\to\mathfrak{L}_2$ is given by $\Omega(\alpha_1)=\beta_1$, $\Omega(\alpha_2)=\beta_1$, $\Omega(\alpha_3)=\beta_2$. Let $(\mathcal{P},A)\in PFS(X,A)$ and $(\mathcal{G},B)\in PFS(Y,B)$ which are given by

$$(\mathcal{P},A)=\begin{pmatrix}\mathcal{P}(\alpha_1) & = & \{x_1/(0.8,0.1),x_2/(0.6,0.5),x_3/(0.7,0.1)\}\\ \mathcal{P}(\alpha_2) & = & \{x_1/(0.9,0.0),x_2/(0.2,0.1),x_3/(0.4,0.4)\}\\ \mathcal{P}(\alpha_3) & = & \{x_1/(0.9,0.0),x_2/(0.2,0.1),x_3/(0.4,0.4)\}\end{pmatrix}$$

and

$$(\mathcal{G},B) = \left(\begin{array}{lcl} \mathcal{G}(\beta_1) & = & \{y_1/(0.1,0.4), y_2/(0.6,0.2), y_3/(0.5,0.6)\} \\ \mathcal{G}(\beta_2) & = & \{y_1/(0.9,0.2), y_2/(0.2,0.1), y_3/(0.8,0.3)\} \end{array}\right).$$

Now, the image of $(\mathcal{P},A)$ under $\mathcal{J}_\Omega : PFS(X,A) \to PFS(Y,B)$ is given by,

$$\mathcal{J}_\Omega[\mathcal{P},A] = (\mathcal{J}(\mathcal{P}),\Omega(A)) = \left(\begin{array}{lcl} \mathcal{J}(\mathcal{P})(\beta_1) & = & \{y_1/(0.9,0.0), y_2/(0.7,0.1), y_3/(0.0,1.0)\} \\ \mathcal{J}(\mathcal{P})(\beta_2) & = & \{y_1/(0.9,0.0), y_2/(0.4,0.1), y_3/(0.0,1.0)\} \end{array}\right).$$

The inverse image of $(\mathcal{G},B)$ under the PFSS mapping $\mathcal{J}_\Omega : PFS(X,A) \to PFS(Y,B)$ is given by

$$\mathcal{J}_\Omega^{-1}[\mathcal{G},B] = (\mathcal{J}^{-1}(\mathcal{G}),\Omega^{-1}(\mathfrak{L}_2)) = \left(\begin{array}{lcl} \mathcal{P}(\alpha_1) & = & \{x_1/(0.1,0.4), x_2/(0.6,0.2), x_3/(0.6,0.2)\} \\ \mathcal{P}(\alpha_2) & = & \{x_1/(0.1,0.4), x_2/(0.6,0.2), x_3/(0.6,0.2)\} \\ \mathcal{P}(\alpha_3) & = & \{x_1/(0.9,0.2), x_2/(0.2,0.1), x_3/(0.2,0.1)\} \end{array}\right).$$

The following two theorems give some properties of PFSS mappings.

Theorem 4.1. *Let $(\mathcal{P}_1,A_1),(\mathcal{P}_2,A_2) \in PFS(X,A)$ and $\mathcal{J} : X \to Y$ and $\Omega : A \to B$ be two mappings. Then,*

1. $(\mathcal{P}_1,A_1) \sqsubseteq \mathcal{J}_\Omega^{-1}[\mathcal{J}_\Omega[\mathcal{P}_1,A_1]]$, *the equality holds when $\mathcal{J}_\Omega$ is injective*
2. $(\mathcal{P}_1,A_1) \sqsubseteq (\mathcal{P}_2,A_2) \implies \mathcal{J}_\Omega[\mathcal{P}_1,A_1] \sqsubseteq \mathcal{J}_\Omega[\mathcal{P}_2,A_2]$
3. $\mathcal{J}_\Omega[(\mathcal{P}_1,A_1) \sqcup (\mathcal{P}_2,A_2)] = \mathcal{J}_\Omega[\mathcal{P}_1,A_1] \sqcup \mathcal{J}_\Omega[\mathcal{P}_2,A_2]$
4. $\mathcal{J}_\Omega[(\mathcal{P}_1,A_1) \sqcap (\mathcal{P}_2,A_2)] \sqsubseteq \mathcal{J}_\Omega[\mathcal{P}_1,A_1] \sqcap \mathcal{J}_\Omega[\mathcal{P}_2,A_2]$

Proof. We only prove 3 & 4 remaining are trivial.

3. Let $(\mathcal{P}_1,A_1) \sqcup (\mathcal{P}_2,A_2) = (I, A_1 \cup A_2)$. Then,

$$\begin{array}{rcl} \mathcal{J}_\Omega[\mathcal{P}_1,A_1] \sqcup \mathcal{J}_\Omega[\mathcal{P}_2,A_2] & = & (\mathcal{J}(\mathcal{P}_1),\Omega(A_1)) \sqcup (\mathcal{J}(\mathcal{P}_2),\Omega(A_2)) \\ & = & (J, \Omega(A_1) \cup \Omega(A_2)). \end{array}$$

Now, we will prove that $\mathcal{J}(I) = J$ for all $y \in Y$ & $\beta \in \Omega(A_1) \cup \Omega(A_2)$.
Case 1. If $\mathcal{J}^{-1}(y)=\emptyset$ then $\mu_{\mathcal{J}(I)(\beta)}(y)=\mu_{J(\beta)}(y)=0$, $\nu_{\mathcal{J}(I)(\beta)}(y) = \nu_{J(\beta)}(y)=1$.
Case 2. If $\beta \in \Omega(A_1) - \Omega(A_2)$ then it is obtained that there does not exist $\alpha \in A_2$ such that $\Omega(\alpha) = \beta$. i.e., $\Omega^{-1}(\beta) \cap (A_1 \cup A_2) = \Omega^{-1}(\beta) \cap (A_1 - A_2)$. That implies

$$\begin{array}{rcl} \mu_{\mathcal{J}(I)(\beta)}(y) & = & \sup\limits_{\substack{\alpha \in \Omega^{-1}(\beta) \cap (A_1 \cup A_2), \\ x \in \mathcal{J}^{-1}(y)}} \mu_{I(\alpha)}(x) \\ & = & \sup\limits_{\substack{\alpha \in \Omega^{-1}(\beta) \cap (A_1 - A_2), \\ x \in \mathcal{J}^{-1}(y)}} \mu_{I(\alpha)}(x) \\ & = & \sup\limits_{\substack{\alpha \in \Omega^{-1}(\beta) \cap (A_1 - A_2), \\ x \in \mathcal{J}^{-1}(y)}} \mu_{\mathcal{P}_1(\alpha)}(x) \\ & = & \mu_{J(\beta)}(y). \end{array}$$

Likewise it is able to show that $\nu_{\mathcal{J}(I)(\beta)}(y) = \nu_{J(\beta)}(y)$.

Case 3. If $\beta \in \Omega(A_2) - \Omega(A_1)$ then analogous to case 2, it is obtained that $\mu_{\mathcal{J}(I)(\beta)}(y) = \mu_{J(\beta)}(y)$ and $\nu_{\mathcal{J}(I)(\beta)}(y) = \nu_{J(\beta)}(y)$.
Case 4. If $\beta \in \Omega(A_1) \cap \Omega(A_2)$ then

$$\begin{aligned}
\mu_{\mathcal{J}(I)(\beta)}(y) &= \sup_{\substack{\alpha\in\Omega^{-1}(\beta)\cap(A_1\cup A_2),\\ x\in\mathcal{J}^{-1}(y)}} \mu_{I(\alpha)}(x)\\
&= \sup_{\substack{\alpha\in\Omega^{-1}(\beta)\cap A_1 \,\cup\, \Omega^{-1}(\beta)\cap A_2,\\ x\in\mathcal{J}^{-1}(y)}} \mu_{I(\alpha)}(x)\\
&= \sup_{\substack{\alpha\in\Omega^{-1}(\beta)\cap A_1-\Omega^{-1}(\beta)\cap A_2,\\ x\in\mathcal{J}^{-1}(y)}} \mu_{\mathcal{P}_1(\alpha)}(x)\\
&\quad \bigvee \sup_{\substack{\alpha\in\Omega^{-1}(\beta)\cap A_2-\Omega^{-1}(\beta)\cap A_1,\\ x\in\mathcal{J}^{-1}(y)}} \mu_{\mathcal{P}_2(\alpha)}(x)\\
&\quad \bigvee \sup_{\substack{\alpha\in\Omega^{-1}(\beta)\cap A_1\cap\Omega^{-1}(\beta)\cap A_2,\\ x\in\mathcal{J}^{-1}(y)}} \max\left\{\mu_{\mathcal{P}_1(\alpha)}(x), \mu_{\mathcal{P}_2(\alpha)}(x)\right\}\\
&= \max\left\{ \sup_{\substack{\alpha\in\Omega^{-1}(\beta)\cap A_1,\\ x\in\mathcal{J}^{-1}(y)}} \mu_{\mathcal{P}_1(\alpha)}(x), \sup_{\substack{\alpha\in\Omega^{-1}(\beta)\cap A_2,\\ x\in\mathcal{J}^{-1}(y)}} \mu_{\mathcal{P}_2(\alpha)}(x) \right\}\\
&= \mu_{J(\beta)}(y).
\end{aligned}$$

In similar way we can prove that $\nu_{\mathcal{J}(I)(\beta)}(y) = \nu_{J(\beta)}(y)$. Thus, from all the four possible cases it can be concluded that $\mathcal{J}(I) = J$ for all $y \in Y$ & $\beta \in \Omega(A_1) \cup \Omega(A_2)$.
4. Let $(\mathcal{P}_1, A_1) \sqcap (\mathcal{P}_2, A_2) = (I, A_1 \cap A_2)$. Then,

$$\begin{aligned}
\mathcal{J}_\Omega[\mathcal{P}_1, A_1] \sqcap \mathcal{J}_\Omega[\mathcal{P}_2, A_2] &= (\mathcal{J}(\mathcal{P}_1), \Omega(A_1)) \sqcap (\mathcal{J}(\mathcal{P}_2), \Omega(A_2))\\
&= (J, \Omega(A_1) \cap \Omega(A_2)).
\end{aligned}$$

If $\mathcal{J}^{-1}(y) = \emptyset$ then $\mu_{\mathcal{J}(I)(\beta)}(y) = \mu_{J(\beta)}(y) = 0$, $\nu_{\mathcal{J}(I)(\beta)}(y) = \nu_{J(\beta)}(y) = 1$.
Otherwise,

$$\begin{aligned}
\mu_{\mathcal{J}(I)(\beta)}(y) &= \sup_{\substack{\alpha\in\Omega^{-1}(\beta)\cap(A_1\cap A_2),\\ x\in\mathcal{J}^{-1}(y)}} \mu_{I(\alpha)}(x)\\
&= \sup_{\substack{\alpha\in\Omega^{-1}(\beta)\cap(A_1\cap A_2),\\ x\in\mathcal{J}^{-1}(y)}} \max\left\{\mu_{\mathcal{P}_1(\alpha)}(x), \mu_{\mathcal{P}_2(\alpha)}(x)\right\}\\
&\le \max\left\{ \sup_{\substack{\alpha\in\Omega^{-1}(\beta)\cap A_1,\\ x\in\mathcal{J}^{-1}(y)}} \mu_{\mathcal{P}_1(\alpha)}(x), \sup_{\substack{\alpha\in\Omega^{-1}(\beta)\cap A_2,\\ x\in\mathcal{J}^{-1}(y)}} \mu_{\mathcal{P}_2(\alpha)}(x) \right\}\\
&= \mu_{J(\beta)}(y).
\end{aligned}$$

In the similar way we can prove that $\nu_{\mathcal{J}(I)(\beta)}(y) \geq \nu_{J(\beta)}(y)$ for all $y \in Y$ & $\beta \in \Omega(A_1) \cap \Omega(A_2)$. Thus $\mathcal{J}_\Omega[(\mathcal{P}_1, A_1) \sqcap (\mathcal{P}_2, A_2)] \sqsubseteq \mathcal{J}_\Omega[\mathcal{P}_1, A_1] \sqcap \mathcal{J}_\Omega[\mathcal{P}_2, A_2]$ □

The following examples show that the equality need not always hold for 1 and 4.

Example 4.2. This example shows that $(\mathcal{P}, A)$ not always equal to $\mathcal{J}_\Omega^{-1}[\mathcal{J}_\Omega[\mathcal{P}, A]]$. From Example 4.1, we got

$$\mathcal{J}_\Omega[\mathcal{P}, A] = \begin{pmatrix} \mathcal{J}(\mathcal{P})(\beta_1) & = & \{y_1/(0.9, 0.0), y_2/(0.7, 0.1), y_3/(0.0, 1.0)\} \\ \mathcal{J}(\mathcal{P})(\beta_2) & = & \{y_1/(0.9, 0.0), y_2/(0.4, 0.1), y_3/(0.0, 1.0)\} \end{pmatrix}.$$

Now,

$$\mathcal{J}_\Omega^{-1}[\mathcal{J}_\Omega[\mathcal{P}, A]] = \begin{pmatrix} \mathcal{P}(\alpha_1) & = & \{x_1/(0.9, 0.0), x_2/(0.7, 0.1), x_3/(0.7, 0.1)\} \\ \mathcal{P}(\alpha_2) & = & \{x_1/(0.9, 0.0), x_2/(0.7, 0.1), x_3/(0.7, 0.1)\} \\ \mathcal{P}(\alpha_3) & = & \{x_1/(0.9, 0.0), x_2/(0.4, 0.1), x_3/(0.4, 0.1)\} \end{pmatrix}.$$

So, $(\mathcal{P}, A) \neq \mathcal{J}_\Omega^{-1}[\mathcal{J}_\Omega[\mathcal{P}, A]]$. But $(\mathcal{P}, A) \sqsubseteq \mathcal{J}_\Omega^{-1}[\mathcal{J}_\Omega[\mathcal{P}, A]]$.

Example 4.3. From example 4.1, consider

$$(\mathcal{P}_1, A_1) = \begin{pmatrix} \mathcal{P}_1(\alpha_1) & = & \{x_1/(0.8, 0.1), x_2/(0.6, 0.5), x_3/(0.7, 0.1)\} \\ \mathcal{P}_1(\alpha_2) & = & \{x_1/(0.9, 0.0), x_2/(0.2, 0.1), x_3/(0.4, 0.4)\} \\ \mathcal{P}_1(\alpha_3) & = & \{x_1/(0.9, 0.0), x_2/(0.2, 0.1), x_3/(0.4, 0.4)\} \end{pmatrix}, \text{ and}$$

$$(\mathcal{P}_2, A_2) = \begin{pmatrix} \mathcal{P}_2(\alpha_1) & = & \{x_1/(0.9, 0.1), x_2/(0.2, 0.5), x_3/(0.6, 0.6)\} \\ \mathcal{P}_2(\alpha_2) & = & \{x_1/(0.8, 0.1), x_2/(0.2, 0.5), x_3/(0.5, 0.4)\} \\ \mathcal{P}_3(\alpha_3) & = & \{x_1/(0.9, 0.1), x_2/(0.2, 0.2), x_3/(0.4, 0.4)\} \end{pmatrix}. \text{ Then,}$$

$$\mathcal{J}_\Omega[\mathcal{P}_1, A_1] = \begin{pmatrix} \mathcal{J}(\mathcal{P})(\beta_1) & = & \{y_1/(0.9, 0.0), y_2/(0.7, 0.1), y_3/(0.0, 1.0)\} \\ \mathcal{J}(\mathcal{P})(\beta_2) & = & \{y_1/(0.9, 0.0), y_2/(0.4, 0.1), y_3/(0.0, 1.0)\} \end{pmatrix}, \text{ and}$$

$$\mathcal{J}_\Omega[\mathcal{P}_2, A_2] = \begin{pmatrix} \mathcal{J}(\mathcal{P})(\beta_1) & = & \{y_1/(0.9, 0.1), y_2/(0.6, 0.3), y_3/(0.0, 1.0)\} \\ \mathcal{J}(\mathcal{P})(\beta_2) & = & \{y_1/(0.9, 0.1), y_2/(0.4, 0.1), y_3/(0.0, 1.0)\} \end{pmatrix}.$$

$$\mathcal{J}_\Omega[(\mathcal{P}_1, A_1) \sqcap (\mathcal{P}_2, A_2)] = \begin{pmatrix} \mathcal{J}(\mathcal{P})(\beta_1) & = & \{y_1/(0.8, 0.1), y_2/(0.6, 0.3), y_3/(0.0, 1.0)\} \\ \mathcal{J}(\mathcal{P})(\beta_2) & = & \{y_1/(0.9, 0.1), y_2/(0.4, 0.2), y_3/(0.0, 1.0)\} \end{pmatrix}$$

$$\mathcal{J}_\Omega[\mathcal{P}_1, A_1] \sqcap \mathcal{J}_\Omega[\mathcal{P}_2, A_2] = \begin{pmatrix} \mathcal{J}(\mathcal{P})(\beta_1) & = & \{y_1/(0.9, 0.0), y_2/(0.6, 0.3), y_3/(0.0, 1.0)\} \\ \mathcal{J}(\mathcal{P})(\beta_2) & = & \{y_1/(0.9, 0.1), y_2/(0.4, 0.1), y_3/(0.0, 1.0)\} \end{pmatrix}$$

Thus $\mathcal{J}_\Omega[(\mathcal{P}_1, A_1) \sqcap (\mathcal{P}_2, A_2)] \neq \mathcal{J}_\Omega[\mathcal{P}_1, A_1] \sqcap \mathcal{J}_\Omega[\mathcal{P}_2, A_2]$.

Theorem 4.2. *Let $(\mathcal{G}_1, B_1), (\mathcal{G}_2, B_2) \in PFS(Y, B)$ and $\mathcal{J} : X \to Y$ and $\Omega : A \to B$ be two mappings. Then*

1. *$\mathcal{J}_\Omega[\mathcal{J}_\Omega^{-1}[\mathcal{G}_1, B_1]] \sqsubseteq (\mathcal{G}_1, B_1)$, the equality holds when $\mathcal{J}_\Omega$ is surjective;*
2. *$(\mathcal{G}_1, B_1) \sqsubseteq (\mathcal{G}_2, B_2) \implies \mathcal{J}_\Omega^{-1}[\mathcal{G}_1, B_1] \sqsubseteq \mathcal{J}_\Omega^{-1}[\mathcal{G}_2, B_2]$;*
3. *$\mathcal{J}_\Omega^{-1}[(\mathcal{G}_1, B_1) \sqcup (\mathcal{G}_2, B_2)] = \mathcal{J}_\Omega^{-1}[\mathcal{G}_1, B_1] \sqcup \mathcal{J}_\Omega^{-1}[\mathcal{G}_2, B_2]$;*
4. *$\mathcal{J}_\Omega^{-1}[(\mathcal{G}_1, B_1) \sqcap (\mathcal{G}_2, B_2)] = \mathcal{J}_\Omega^{-1}[\mathcal{G}_1, B_1] \sqcap \mathcal{J}_\Omega^{-1}[\mathcal{G}_2, B_2]$.*

Proof. Proof is analogous to Theorem 4.1. □

Since continuity is an inevitable concept in the study of topological space, the next discussion is about the PFSS continuous mapping. Definition and two characterisation theorems of continuous mapping are given.

Definition 4.2. *Let $(X, \tau_1, \mathfrak{L}_1)$ and $(Y, \tau_2, \mathfrak{L}_2)$ be two PFSS topological spaces. A PFSS mapping $\mathcal{J}_\Omega : PFS(X, \mathfrak{L}_1) \to PFS(Y, \mathfrak{L}_2)$ is said to be PFSS continuous if for each subsets $(\mathcal{P}, \mathfrak{L}_1)$ of $(X, \mathfrak{L}_1)$ and for any neighbourhood $(V, \mathfrak{L}_2)$ of $\mathcal{J}_\Omega[\mathcal{P}, \mathfrak{L}_1]$ there exist a neighbourhood $(\mho, \mathfrak{L}_1)$ of $(\mathcal{P}, \mathfrak{L}_1)$ in a way $\mathcal{J}_\Omega[\mho, \mathfrak{L}_1] \sqsubseteq (V, \mathfrak{L}_2)$.*

The following two theorems are characterizing the PFSS continuous mappings.

Theorem 4.3. *Let $(X, \tau_1, \mathfrak{L}_1)$ and $(Y, \tau_2, \mathfrak{L}_2)$ be two PFSS topological spaces and $\mathcal{J}_\Omega : PFS(X, \mathfrak{L}_1) \to PFS(Y, \mathfrak{L}_2)$ PFSS continuous iff for each subsets $(\mathcal{P}, \mathfrak{L}_1)$ of $(X, \mathfrak{L}_1)$ and for any neighbourhood $(V, \mathfrak{L}_2)$ of $\mathcal{J}_\Omega[\mathcal{P}, \mathfrak{L}_1]$, $\mathcal{J}_\Omega^{-1}[V, \mathfrak{L}_2]$ is a neighbourhood of $(\mathcal{P}, \mathfrak{L}_1)$.*

Proof. Let $\mathcal{J}_\Omega$ is continuous. Consider a subset $(\mathcal{P}, \mathfrak{L}_1)$ of $(X, \mathfrak{L}_1)$ and a neighbourhood $(V, \mathfrak{L}_2)$ of $\mathcal{J}_\Omega[\mathcal{P}, \mathfrak{L}_1]$. By Definition 4.2 we have $(\mho, \mathfrak{L}_1) \sqsubseteq \mathcal{J}_\Omega^{-1}[V, \mathfrak{L}_2]$, where $(\mho, \mathfrak{L}_1)$ is neighbourhood of $(\mathcal{P}, \mathfrak{L}_1)$. That is, $(\mathcal{P}, \mathfrak{L}_1) \sqsubseteq (\mho, \mathfrak{L}_1) \sqsubseteq \mathcal{J}_\Omega^{-1}[V, \mathfrak{L}_2]$. Since $(\mho, \mathfrak{L}_1)$ is neighbourhood of $(\mathcal{P}, \mathfrak{L}_1)$, by Definition 3.1, $\mathcal{J}_\Omega^{-1}[V, \mathfrak{L}_2]$ is a neighbourhood of $(\mathcal{P}, \mathfrak{L}_1)$.

Conversely, let $(\mathcal{P}, \mathfrak{L}_1)$ be subset of $(X, \mathfrak{L}_1)$ and $(V, \mathfrak{L}_2)$ be neighbourhood of $\mathcal{J}_\Omega[\mathcal{P}, \mathfrak{L}_1]$. Then by assumption $\mathcal{J}_\Omega^{-1}[V, \mathfrak{L}_2]$ is a neighbourhood of $(\mathcal{P}, \mathfrak{L}_1)$. From Definition 3.1, there exist $(O, \mathfrak{L}_1) \in \tau_1$ such that $(\mathcal{P}, \mathfrak{L}_1) \sqsubseteq (O, \mathfrak{L}_1) \sqsubseteq \mathcal{J}_\Omega^{-1}[V, \mathfrak{L}_2]$. That implies $\mathcal{J}_\Omega[O, \mathfrak{L}_1] \sqsubseteq (V, \mathfrak{L}_2)$ and hence $\mathcal{J}_\Omega$ is continuous. □

Theorem 4.4. *Let $(X, \tau_1, \mathfrak{L}_1)$ and $(Y, \tau_2, \mathfrak{L}_2)$ be two PFSS topological spaces and $\mathcal{J}_\Omega : PFS(X, \mathfrak{L}_1) \to PFS(Y, \mathfrak{L}_2)$ PFSS continuous iff every open subset $(V, \mathfrak{L}_2)$ of $(Y, \mathfrak{L}_2)$, $\mathcal{J}_\Omega^{-1}[V, \mathfrak{L}_2]$ is open subset of $(X, \mathfrak{L}_1)$.*

Proof. Assume $\mathcal{J}_\Omega$ is continuous. Consider an open subset $(V, \mathfrak{L}_2)$ of $(Y, \mathfrak{L}_2)$ and let $(\mathcal{P}, \mathfrak{L}_1) \sqsubseteq \mathcal{J}_\Omega^{-1}[V, \mathfrak{L}_2]$. i.e., $\mathcal{J}_\Omega[\mathcal{P}, \mathfrak{L}_1] \sqsubseteq (V, \mathfrak{L}_2)$. Since $(V, \mathfrak{L}_2)$ is open and from Theorem 3.2, there is a neighbourhood $(N, \mathfrak{L}_2)$ such that $\mathcal{J}_\Omega[\mathcal{P}, \mathfrak{L}_1] \sqsubseteq (N, \mathfrak{L}_2) \sqsubseteq (V, \mathfrak{L}_2)$. We have $\mathcal{J}_\Omega$ is continuous so by Theorem 4.3, $\mathcal{J}_\Omega^{-1}[N, \mathfrak{L}_2]$ is a neighbourhood of $(\mathcal{P}, \mathfrak{L}_1)$ and contained in $\mathcal{J}_\Omega^{-1}[V, \mathfrak{L}_2]$. Since $(\mathcal{P}, \mathfrak{L}_1)$ is arbitrary subset of $\mathcal{J}_\Omega^{-1}[V, \mathfrak{L}_2]$, by Theorem 3.2, $\mathcal{J}_\Omega^{-1}[V, \mathfrak{L}_2]$ is open subset of $(X, \mathfrak{L}_1)$.

On the other hand, let $\mathcal{J}_\Omega$ be a PFSS mapping from $(X, \tau_1, \mathfrak{L}_1)$ to $(Y, \tau_2, \mathfrak{L}_2)$. Also let $(\mathcal{P}, \mathfrak{L}_1)$ be subset of $(X, \mathfrak{L}_1)$ and $(V, \mathfrak{L}_2)$ be a neighbourhood $\mathcal{J}_\Omega[\mathcal{P}, \mathfrak{L}_1]$. Then by Definition 3.1 we obtain an open set $(O, \mathfrak{L}_2)$ so that $\mathcal{J}_\Omega[\mathcal{P}, \mathfrak{L}_1] \sqsubseteq (O, \mathfrak{L}_2) \sqsubseteq (V, \mathfrak{L}_2)$. By assumption $\mathcal{J}_\Omega^{-1}[O, \mathfrak{L}_2]$ is open subset of $(X, \mathfrak{L}_1)$. Thus we get $(\mathcal{P}, \mathfrak{L}_1) \sqsubseteq \mathcal{J}_\Omega^{-1}[O, \mathfrak{L}_2] \sqsubseteq \mathcal{J}_\Omega^{-1}[V, \mathfrak{L}_2]$. i.e., $\mathcal{J}_\Omega^{-1}[V, \mathfrak{L}_2]$ is a neighbourhood of $(\mathcal{P}, \mathfrak{L}_1)$. Hence Theorem 4.3 implies that $\mathcal{J}_\Omega$ is continuous. □

5. Conclusion

A study on the PFSS topological structure is carried out in this paper. We introduced a definition for PFSS continuous mappings which generalizes the existing definition for IFSS continuous mappings. Finally, we obtained a couple

of characterization theorems for PFSS continuity. Subject to its real-life applications, PFSS is one of the developing areas and thus our future work will be analyzing the further topological properties of PFSS and studying the algebraic structures associated with a PFSS.

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Matrix methods for the up and down Steenrod squares

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Abstract. Let $\mathrm{P}(n)$ be the polynomial algebra which is a graded left module over the Steenrod algebra. The divided power algebra $\mathrm{DP}^d(n)$ is defined as the Hopf dual of $\mathrm{P}^d(n)$. The dual of the Steenrod square Sq^k is the linear map $Sq_k : \mathrm{DP}^{d+k}(n) \to \mathrm{DP}^d(n)$, called the down Steenrod square, defined by $Sq_k(u) = v$ for $u \in \mathrm{DP}^{d+k}(n)$, where $v(f) = \big(Sq_k(u)\big)(f) = u\big(Sq^k(f)\big)$ for $f \in \mathrm{P}^d(n)$. In this article we consider the down Steenrod squares and establish some results about. Further, we show that there is some periodic calculation on the down Steenrod squares hanging on the limited leading period. Finally, using this fact, we exhibit a matrix method for manipulating the down Steenrod squares.

Keywords: Steenrod algebra, down Steenrod squares, divided power algebra.

1. Introduction

Consider the polynomial algebra $\mathrm{P}(n) = \mathbb{F}_2[x_1, \ldots, x_n] = \bigoplus_{d\geq 0} \mathrm{P}^d(n)$, viewed as a graded left module over the Steenrod algebra $\mathcal{A}_2$ at the prime 2. The grading is by the homogeneous polynomials $\mathrm{P}^d(n)$ of degree d in the n variables $x_1, \ldots, x_n$ of grading 1. The algebra $\mathrm{P}(n)$ realizes the cohomology of the product of n copies of infinite real projective spaces.

The Steenrod algebra $\mathcal{A}_2$ is briefly defined as the graded algebra over the finite field $\mathbb{F}_2$, generated by symbols Sq^k, called the up Steenrod squares, in grading k, for $k \geq 0$, subject to Adem relations and the condition $Sq^0 = 1$ [4, 7]. In classical texts, the word 'up' is not written for the Steenrod squares Sq^k. However, we use this word since the dual of the Steenrod squares Sq^k, called the 'down' Steenrod squares Sq_k, are considered in this article. The latter word

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was first used in [7] for the 'going down' integral Steenrod square SQ_2 (see also [6]).

In Definition 2.1, the divided power algebra $\mathrm{DP}^d(n)$ is defined as the Hopf dual of $\mathrm{P}^d(n)$. By the down Steenrod square (Definition 2.2) it is meant the dual of $Sq^k : \mathrm{P}^d(n) \to \mathrm{P}^{d+k}(n)$ which is the linear map $Sq_k : \mathrm{DP}^{d+k}(n) \to \mathrm{DP}^d(n)$ defined by $Sq_k(u) = v$ for $u \in \mathrm{DP}^{d+k}(n)$, where

$$v(f) = \big(Sq_k(u)\big)(f) = u\big(Sq^k(f)\big),$$

for $f \in \mathrm{P}^d(n)$. Some properties of the Sq_k, especially the dual version of Cartan formula for these operations, are discussed in Section 2. Moreover, the Steenrod kernel $\mathrm{K}^d(n)$ is defined as the set of elements $v \in \mathrm{DP}^d(n)$ such that $Sq_k(v) = 0$ for all $k > 0$. Thus, the Steenrod kernel $\mathrm{K}^d(n)$, as a subspace of $\mathrm{DP}^d(n)$, is the transpose dual of the cohit sub-module $\mathrm{Q}^d(n)$ of $\mathrm{P}^d(n)$. The dual of the hit problem is to determine $\mathrm{K}^d(n)$. For more information about the hit and dual hit problems we cite [3, 5]. The hit and dual hit problems are not in the scope of this article.

In Section 3, we show that to manipulate the down operations Sq_k, it suffices to know only the $Sq_k(v_1^{(d_1)} \cdots v_n^{(d_n)})$ for limited values of d_i and for higher values the calculations are periodically repeated. Using this fact, in the last Section 4, certain matrices are presented and a matrix method for computing the down Steenrod squares Sq_k is exhibited. This method may be used for computer calculations of the down Steenrod squares. Also, the idea beside can be experienced for the Steenrod power operations in the mod-p Steenrod algebra $\mathcal{A}_p$ for any odd prime p.

2. Preliminaries and some properties of down squares

In this section, we review some preliminaries of the up and down Steenrod squares and establish some properties of the down squares. First, recall some fundamental concepts of the up Steenrod squares from [5, Section 1.1].

The study of Steenrod squares is often started from the total Steenrod square $\mathrm{Sq} : \mathrm{P}(n) \to \mathrm{P}(n)$ acting on the variable x_i, $1 \leq i \leq n$, by $\mathrm{Sq}(x_i) = x_i + x_i^2$ and $\mathrm{Sq}(1) = 1$. Then, for $k, d \geq 0$, the up Steenrod square Sq^k is defined to be the restriction of Sq to $\mathrm{P}^d(n)$ and projecting on $\mathrm{P}^{d+k}(n)$. Therefore, Sq is the formal sum $\mathrm{Sq} = \sum_{k\geq 0} Sq^k$ of its graded parts. The basic property of the up Steenrod square $Sq^k : \mathrm{P}^d(n) \to \mathrm{P}^{d+k}(n)$ is explained in the next result.

Proposition 2.1. *For all $x \in \mathrm{P}^1(n)$, $Sq^k(x^d) = \binom{d}{k} x^{d+k}$. In particular, $Sq^k(x^k) = x^{2k}$.*

The up Steenrod squares work out on products of cohomology classes $f, g \in \mathrm{P}(n)$ by Cartan formula [1].

$$Sq^k(fg) = \sum_{i+j=k} Sq^i(f) Sq^j(g).$$

The next fundamental result shows why Sq^k is called a 'square' operation.

Proposition 2.2. *For a homogeneous element $f \in \mathrm{P}(n)$ we have $Sq^k(f) = f^2$ if $k = \deg(f)$ and $Sq^k(f) = 0$ if $k > \deg(f)$.*

The following corollary is immediately concluded.

Corollary 2.1. *Let $k = d_1 + \cdots + d_n$. Then,*

$$Sq^k(x_1^{d_1} \cdots x_n^{d_n}) = x_1^{2d_1} \cdots x_n^{2d_n}.$$

For a definition of 'down' Steenrod operators, the concept of the divided power algebra is needed.

Definition 2.1. For $n > 0$ and $d \geq 0$, the linear dual of the $\mathbb{F}_2$-vector space $\mathrm{P}^d(n)$ is denoted by $\mathrm{DP}^d(n) = \mathrm{Hom}(\mathrm{P}^d(n), \mathbb{F}_2)$. The divided power algebra is defined as the infinite sum $\mathrm{DP}(n) = \sum_{d\geq 0} \mathrm{DP}^d(n)$ for which the grading and product is defined as follows.

Denote by $v_1, \ldots, v_n$ the basis of $\mathrm{DP}^1(n)$ dual to the basis $x_1, \ldots, x_n$ of $\mathrm{P}^1(n)$, so that $\langle v_i, x_j\rangle$ is 1 if $i = j$ and is 0 otherwise. The d-monomial $v_1^{(d_1)} \cdots v_n^{(d_n)}$ in $\mathrm{DP}(n)$ is defined as the dual of the monomial $x_1^{d_1} \cdots x_n^{d_n}$ in $\mathrm{P}(n)$, where the parenthesized exponents are called divided powers. For any $v \in \mathrm{DP}^1(n)$, put $v^{(0)} = 1$, the identity map of $\mathbb{F}_2$ which is also the identity element of $\mathrm{DP}(1)$. Let $v^{(1)} = v$ have degree 1 and define the degree of the d-monomial $v_1^{(d_1)} \cdots v_n^{(d_n)}$ as $d = d_1 + \cdots + d_n$. The degree of a d-polynomial in $\mathrm{DP}(n)$ is defined naturally.

Now, we introduce a product on $\mathrm{DP}(n)$. We start with one variable $v_1 = v$ and define the product of d-monomials in $\mathrm{DP}(1)$ by

$$v^{(r)}v^{(s)} = \binom{r+s}{r} v^{(r+s)}, \tag{1}$$

where the binomial coefficient is taken mod 2. For a field of characteristic 0 in place of $\mathbb{F}_2$, the corresponding situation is $v^{(r)} = v^r/r!$ since $v^{(r)}v = (r+1)v^{(r+1)}$ by (1). That is why $v^{(r)}$ is called the r-th divided power of v.

For $d > 0$, define $\mathrm{bin}(d) = \{2^{d_1}, \ldots, 2^{d_r}\}$, where $d = 2^{d_1} + \cdots + 2^{d_r}$ with distinct terms and $\mathrm{bin}(0) = \emptyset$. Then, the product (1) can be written as follows.

$$v^{(r)}v^{(s)} = \begin{cases} v^{(r+s)}, & \text{if } \mathrm{bin}(r) \cap \mathrm{bin}(s) = \emptyset, \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

For a nonzero d-polynomial $u = a_0 + a_1 v + a_2 v^{(2)} + \cdots + a_s v^{(s)}$ in $\mathrm{DP}(1)$, with $a_s \neq 0$, if $a_0 = 0$ then $u^2 = 0$ by the product formula (2).

Now, the product in $\mathrm{DP}(n)$ can be defined on d-monomials by using either of the equivalent product formulas (1) or (2), for each variable v_i, and is extended for d-polynomials by linearity. This product is commutative and associative and $\mathrm{DP}(n)$ is a graded algebra over $\mathbb{F}_2$ for all $n \geq 1$. In fact, $\mathrm{DP}(n)$ is the Hopf dual of $\mathrm{P}(n)$ [5, Section 9.1].

Definition 2.2. The down Steenrod square $Sq_k : \mathrm{DP}^{d+k}(n) \to \mathrm{DP}^d(n)$ is defined to be the linear dual of the up Steenrod square $Sq^k : \mathrm{P}^d(n) \to \mathrm{P}^{d+k}(n)$. More precisely, Sq_k is defined by $Sq_k(u) = v$ for $u \in \mathrm{DP}^{d+k}(n)$, where

$$v(f) = \big(Sq_k(u)\big)(f) = u\big(Sq^k(f)\big),$$

for $f \in \mathrm{P}^d(n)$. Since Sq_k lowers degree by k, it is called the 'down' square operation in opposite to the 'up' square operation Sq^k which ups the degree. In the bilinear literature, Sq_k is redefined by

$$\langle Sq_k(u), f\rangle = \langle u, Sq^k(f)\rangle.$$

The identity property of Sq_0 is now apparent. In other words, Sq_0 is the identity homomorphism. The algebra generated by the down squares Sq_k is isomorphic to the opposite algebra $\mathcal{A}_2^{\mathrm{op}}$, and not to $\mathcal{A}_2$ itself since

$$\langle Sq_a Sq_b(u), f\rangle = \langle Sq_b(u), Sq^a(f)\rangle = \langle u, Sq^b Sq^a(f)\rangle.$$

Due to the duality essence, we started the study of down Steenrod squares with the graded parts Sq_k. The total down square $\mathrm{Sq}_* : \mathrm{DP}(n) \to \mathrm{DP}(n)$, the dual of the total up square Sq, is the linear map $\mathrm{Sq}_*(u) = \sum_{k\geq 0} Sq_k(u)$, where $u \in \mathrm{DP}(n)$. Therefore, $\langle \mathrm{Sq}_*(u), f\rangle = \langle u, \mathrm{Sq}(f)\rangle$ for $u \in \mathrm{DP}(n)$ and $f \in \mathrm{P}(n)$.

In the sequel, we assume v and all its subscripts are in $\mathrm{DP}^1(n)$. The only indecomposable single monomials in $\mathrm{P}(n)$ are x_i, $1 \leq i \leq n$, and we saw that $\mathrm{Sq}(x_i) = x_i + x_i^2$. However, by the product formula (2), the indecomposable single d-monomials in $\mathrm{DP}(n)$ are $v^{(2^t)}$ for all $t \geq 0$ and if $\mathrm{bin}(d) = \{2^{t_1}, \ldots, 2^{t_s}\}$ then $v^{(d)} = v^{(2^{t_1})} \cdots v^{(2^{t_s})}$. Now, $\mathrm{Sq}_*(v^{(2^t)}) = v^{(2^t)} + \sum_{i=0}^{t-1} v^{(2^t - 2^i)}$.

The next result is the basic property of the down Steenrod operations [5, Proposition 9.3.6].

Proposition 2.3. *For all $v \in \mathrm{DP}^1(n)$,*

$$Sq_k(v^{(d)}) = \binom{d-k}{k} v^{(d-k)}.$$

In particular, $Sq_k(v^{(2k)}) = v^{(k)}$.

Proposition 2.3 is the dual version of Proposition 2.1. The particular case in Proposition 2.3 is somehow the dual of the first part of Proposition 2.2 for one variable. The dual of the last part of Proposition 2.2 is the following.

Corollary 2.2. *For $u \in \mathrm{DP}(n)$, $Sq_k(u^{(d)}) = 0$ if $k > d/2$.*

An immediate conclusion is dual to Corollary 2.1.

Corollary 2.3. *Let $k = d_1 + \cdots + d_n$. Then,*

$$Sq_k(v_1^{(2d_1)} \cdots v_n^{(2d_n)}) = v_1^{(d_1)} \cdots v_n^{(d_n)}.$$

It is proved [5, Proposition 1.4.11] that $\binom{a}{b} = 0$ if and only if $\mathrm{bin}(b) \not\subseteq \mathrm{bin}(a)$. This determines the d-monomials in the null image of Sq_k.

Corollary 2.4. *For $k > 0$, $Sq_k(v^{(d)}) = 0$ if and only if $\mathrm{bin}(k) \not\subseteq \mathrm{bin}(d-k)$.*

The following corollaries are immediately deduced from Corollary 2.4.

Corollary 2.5. *For any non-negative integers t and m,*

$$Sq_{2^t-1}\Big(v^{\big(m2^t+2(2^t-1)\big)}\Big) = v^{\big((m+1)2^t-1\big)}.$$

Corollary 2.6. *Let $1 \leq k < 2^t$. Then,*

$$Sq_k(v^{(2^t+r)}) = 0,$$

where $r = k, \ldots, 2k-1$.

Corollary 2.7. *For any $t > 0$,*

$$Sq_{2^t}(v^{(2^{t+1}+2^t+r)}) = 0,$$

where $r = 0, \ldots, 2^t - 1$.

Corollary 2.7 determines zeros of $Sq_k(v^{(d)})$ when k is a 2-power. The next corollary indicates the same determination when d is a 2-power.

Corollary 2.8. *Let $1 \leq i \leq t-2$, $t \geq 3$. Then, for $2^i < k < 2^{i+1}$,*

$$Sq_k(v^{(2^t)}) = 0.$$

Cartan formula for divided power algebras enables us to calculate the action of Sq_k on the products of elements in $\mathrm{DP}(n)$ [5, Proposition 9.3.4].

Theorem 2.1 (Cartan formula for $\mathrm{DP}(n)$)**.** *For $k \geq 0$ and any $u, w \in \mathrm{DP}(n)$, we have*

$$Sq_k(uw) = \sum_{i+j=k} Sq_i(u)Sq_j(w). \tag{3}$$

For simplicity, we refer to (3) as Cartan formula. A d-monomial of the form $v_1^{(2^{j_1}-1)} \cdots v_r^{(2^{j_r}-1)}$ for some positive powers is called a d-spike. The down operations annihilates the d-spikes.

Proposition 2.4. *Let $k > 0$. Then, $Sq_k(v^{(2^j-1)}) = 0$ for any positive integer j. In general, for any d-spike $u = v_1^{(2^{j_1}-1)} \cdots v_r^{(2^{j_r}-1)}$, $Sq_k(u) = 0$.*

Proof. We prove the one-variable case. The case $k > 2^{j-1}$ follows from Proposition 2.3 since $d = 2^j - 1 < 2^j = 2 \cdot 2^{j-1} < 2k$. For $k \leq 2^{j-1}$, we have $\mathrm{bin}(k) \not\subseteq \mathrm{bin}(2^{j-1} - k)$ and the result is established. The general statement is obtained from Cartan formula. □

$i=1$						
s	2		3		4	
The d-spike:	$u^{(1)}v^{(3)}$		$u^{(1)}v^{(7)}$		$u^{(1)}v^{(15)}$	
for p:	0	1	0	1	0	1
is the Sq_3 of:	$u^{(1)}v^{(6)}$	$u^{(2)}v^{(5)}$	$u^{(1)}v^{(10)}$	$u^{(2)}v^{(9)}$	$u^{(1)}v^{(18)}$	$u^{(2)}v^{(17)}$

$i=2$								
s	3				4			
The d-spike:	$u^{(3)}v^{(7)}$				$u^{(3)}v^{(15)}$			
for p:	0	1	2	3	0	1	2	3
is the Sq_3 of:	$u^{(3)}v^{(10)}$	$u^{(4)}v^{(9)}$	$u^{(5)}v^{(8)}$	$u^{(6)}v^{(7)}$	$u^{(3)}v^{(18)}$	$u^{(4)}v^{(17)}$	$u^{(5)}v^{(16)}$	$u^{(6)}v^{(15)}$

$i=3$				
s	4			
The d-spike:	$u^{(7)}v^{(15)}$			
for p:	0	1	2	3
is the Sq_3 of:	$u^{(7)}v^{(18)}$	$u^{(8)}v^{(17)}$	$u^{(9)}v^{(16)}$	$u^{(10)}v^{(15)}$

Table 1: Proposition 2.6 for $t = 2$.

Definition 2.3. The Steenrod kernel $\mathrm{K}(n) = \sum_{d\geq 0} \mathrm{K}^d(n)$, where $\mathrm{K}^d(n)$ is the set of elements $v \in \mathrm{DP}^d(n)$ such that $Sq_k(v) = 0$ for all $k > 0$.

Proposition 2.4 says that d-spikes are always contained in the Steenrod kernel. We know that spikes cannot appear in the image of any Sq^k for $k > 0$. However, due to the duality nature, d-spikes are always in the image of some Sq_k. In corollary 2.5 (for $m = 0$) we experienced this in one variable, which says that d-spikes may be obtained from special down operators with spike gradings. The next result, which is trivial by Corollary 2.4, shows a template for d-spikes in one variable.

Proposition 2.5. *Let $v \in \mathrm{DP}^1(n)$. Then, for any $t > 0$,*

$$Sq_i(v^{(2^t-1+i)}) = v^{(2^t-1)},\ i = 1, 2, \ldots, 2^t - 1.$$

Conversely, any d-spike is obtained in this way.

Proposition 2.5 is extended for two variables.

Proposition 2.6. *For $t > 0$ let $\ell = 2^t - 1$, $s \geq t$, and $i \geq 1$. Then, for $s > i$,*

$$u^{(2^i-1)}v^{(2^s-1)} = Sq_\ell\big(u^{(2^i-1+p)}v^{(2^s-1+\ell-p)}\big),$$

where $p = 0, 1, \ldots, 2^i - 1$ if $i < t$ and $p = 0, 1, \ldots, \ell$ otherwise.

Proof. The zero case $p = 0$ is trivial. For positive p the result comes from the one-variable case in Proposition 2.5 and Corollary 2.2 using Cartan formula. □

Example 2.1. Take $t = 2$. Then, Table 1 applies Proposition 2.6 for $i = 1, 2, 3$.

Proposition 2.6 creates the 2-variable d-spikes with distinct exponents. For equal exponents we have the following result with the proof discussion as above.

Proposition 2.7. *For $t > 0$ let $\ell = 2^t - 1$ and $s \geq t$. Then,*

$$u^{(2^s-1)}v^{(2^s-1)} = Sq_\ell\big(u^{(2^s-1+r)}v^{(2^s-1+\ell-r)}\big),$$

where $r = 0, 1, \ldots, 2^{t-1} - 1$.

Example 2.2. Again, take $t = 2$. Then, Table 2 is an example of Proposition 2.7 for some values of s.

As seen in Examples 2.1 and 2.2, the methods in Propositions 2.6 and 2.7 lead often to repeated d-spikes. Contrary to the one-variable case in Proposition 2.5, the converses of Propositions 2.6 and 2.7 are not always true. For instance, the d-spike $u^{(3)}v^{(7)}$ may be created as

$$u^{(3)}v^{(7)} = Sq_8\big(u^{(4)}v^{(14)}\big) = Sq_8\big(u^{(5)}v^{(13)}\big) = Sq_8\big(u^{(6)}v^{(12)}\big).$$

Also, we can obtain $u^{(7)}v^{(7)}$ by $Sq_6\big(u^{(7)}v^{(13)}\big)$ or $Sq_6\big(u^{(5)}v^{(11)}\big)$.

For more variables than two, however, the pre-image d-monomials should have all but two variables in power of spikes. If this happens then, due to Proposition 2.4, we will have the same forms as in Propositions 2.6 and 2.7 except for the number of spike-power variables. For instance, compare the following relations with Table 1 (middle one).

$$\begin{aligned} u^{(3)}v^{(15)}w^{(63)} &= Sq_3\big(u^{(3)}v^{(18)}w^{(63)}\big) = Sq_3\big(u^{(4)}v^{(17)}w^{(63)}\big) \\ &= Sq_3\big(u^{(5)}v^{(16)}w^{(63)}\big) = Sq_3\big(u^{(6)}v^{(15)}w^{(63)}\big). \end{aligned}$$

The existence of the above condition in more than two variables is necessary; otherwise, non-d-spike terms may occur. For example,

$$Sq_3\big(u^{(2)}v^{(2)}w^{(4)}\big) = u^{(1)}v^{(1)}w^{(3)} + u^{(1)}v^{(2)}w^{(2)}.$$

All the above computations in more than one variable are up to permutation of variables.

3. Periodic calculations

For the multisets $\boldsymbol{d} = (d_1, d_2, \ldots, d_n)$ and $\boldsymbol{m} = (m_1, m_2, \ldots, m_n)$ of non-negative integers, we shall adopt the following abbreviated notations.

$$\begin{aligned} \boldsymbol{x}^{\boldsymbol{d}} &= x_1^{d_1}x_2^{d_2}\cdots x_n^{d_n}, \\ \boldsymbol{x}^{\boldsymbol{m}2^t} &= x_1^{m_1 2^t}x_2^{m_2 2^t}\cdots x_n^{m_n 2^t}, \\ \boldsymbol{x}^{\boldsymbol{m}2^t+\boldsymbol{d}} &= x_1^{m_1 2^t+d_1}x_2^{m_2 2^t+d_2}\cdots x_n^{m_n 2^t+d_n}. \end{aligned} \tag{4}$$

where $t > 0$. To handle with the up operations Sq^k, it suffices to know only the values of the $Sq^k(x_1^{d_1}x_2^{d_2}\cdots x_n^{d_n})$ with $d_1, d_2, \ldots, d_n < 2^t$ and $1 \le k < 2^t$, for some $t > 0$ [2, Corollary 3.3].

s	2		3		4	
The d-spike:	$u^{(3)}v^{(3)}$		$u^{(7)}v^{(7)}$		$u^{(15)}v^{(15)}$	
for r:	0	1	0	1	0	1
is the Sq_3 of:	$u^{(3)}v^{(6)}$	$u^{(4)}v^{(5)}$	$u^{(7)}v^{(10)}$	$u^{(8)}v^{(9)}$	$u^{(15)}v^{(18)}$	$u^{(16)}v^{(17)}$

Table 2: Proposition 2.7 for $t = 2$.

Theorem 3.1. *Let $1 \leq d_i < 2^t$, $1 \leq i \leq n$, and let $0 \leq k < 2^t$. Then*

$$Sq^k(\boldsymbol{x}^{\boldsymbol{m}2^t+\boldsymbol{d}}) = \boldsymbol{x}^{\boldsymbol{m}2^t} Sq^k(\boldsymbol{x}^{\boldsymbol{d}}).$$

We establish the dual statements of the general Theorem 3.1. To this end, we start by one variable.

Proposition 3.1. *Let $2^{t-1} \leq k \leq 2^t - 1$, $t > 0$. Let also $d = 2k + \ell$, where $\ell = 0, 1, \ldots, 2^t - 1$. Then, for any non-negative integer m,*

$$Sq_k(v^{(m2^t+d)}) = v^{(m2^t)} Sq_k(v^{(d)}).$$

Remark 3.1. Corollary 2.5 shows the establishment of Proposition 3.1 for double of d-spikes. In fact, given any non-negative integer t,

$$Sq_{2^t-1}\Big(v^{\big(m2^t+2(2^t-1)\big)}\Big) = v^{(m2^t)} Sq_{2^t-1}\Big(v^{\big(2(2^t-1)\big)}\Big) = v^{\big((m+1)2^t-1\big)}, m \geq 0.$$

Proof of Theorem 3.1. We prove the result for $m = 1$, that is,

$$Sq_k(v^{(2^t+d)}) = v^{(2^t)} Sq_k(v^{(d)}).$$

Cartan formula, using the product formula (2), is applied.

$$Sq_k(v^{(2^t+d)}) = Sq_k(v^{(2^t)} v^{(d)}) = \sum_{i+j=k} Sq_i(v^{(2^t)}) Sq_j(v^{(d)}).$$

Appealing to Corollaries 2.2 and 2.8, the range of summation is reduced.

$$Sq_k(v^{(2^t+d)}) = v^{(2^t)} Sq_k(v^{(d)}) + \sum_{p=0}^{t-1} Sq_{2^p}(v^{(2^t)}) Sq_{k-2^p}(v^{(d)}). \tag{5}$$

Take $0 \leq p \leq t-1$. Then, $Sq_{2^p}(v^{(2^t)}) = v^{2^t-2^p}$. We have $\mathrm{bin}(2^t - 2^p) = \{2^{p+1}, \ldots, 2^t\}$. So, regarding the product formula (2), the necessary condition for the term $Sq_{2^p}(v^{(2^t)}) Sq_{k-2^p}(v^{(d)})$ to be nonzero is that $\mathrm{bin}(2^t - 2^p) \cap \mathrm{bin}(d - (k - 2^p)) = \emptyset$. This condition is true whenever $d - (k - 2^p) < 2^p$ which is impossible. Therefore, all terms of the summation in (5) are zero and the result follows for $m = 1$. For $m > 1$, the proof is similar.

Example 3.1.

$$\begin{aligned}
Sq_1(v^{(18)}) &= Sq_1(v^{(8\cdot 2+2)}) = v^{(8\cdot 2)} Sq_1(v^{(2)}) = v^{(16)} v^{(1)} = v^{(17)},\\
Sq_4(v^{(33)}) &= Sq_4(v^{(3\cdot 2^3+9)}) = v^{(3\cdot 2^3)} Sq_4(v^{(9)}) = v^{(24)} v^{(5)} = v^{(29)},\\
Sq_{22}(v^{(85)}) &= Sq_{22}(v^{(2^5+53)}) = v^{(2^5)} Sq_{22}(v^{(53)}) = v^{(32)} v^{(31)} = v^{(63)}.
\end{aligned}$$

The following main result extends Proposition 3.1 and establishes the dual of Theorem 3.1. We adopt the same notation as in (4).

Theorem 3.2. *For $t > 0$, let $2^{t-1} \leq k \leq 2^t - 1$. Then,*

$$Sq_k(\boldsymbol{v}^{(\boldsymbol{m}2^t+\boldsymbol{d})}) = \boldsymbol{v}^{(\boldsymbol{m}2^t)} Sq_k(\boldsymbol{v}^{(\boldsymbol{d})}),$$

where for each $1 \leq i \leq n$, d_i is $2k + \ell$ for $n = 1$ and is $k + \ell$ for $n > 1$, and $\ell = 0, 1, \ldots, 2^t - 1$.

Proof. The result for $n = 1$ is Proposition 3.1. We prove the result for $n = 2$; that is,

$$Sq_k(v_1^{(m_1 2^t+d_1)} v_2^{(m_2 2^t+d_2)}) = v_1^{(m_1 2^t)} v_2^{(m_2 2^t)} Sq_k(v_1^{(d_1)} v_2^{(d_2)}). \tag{6}$$

Cartan formula implies

$$Sq_k(v_1^{(m_1 2^t+d_1)} v_2^{(m_2 2^t+d_2)}) = \sum_{i+j=k} Sq_i(v_1^{(m_1 2^t+d_1)}) Sq_j(v_2^{(m_2 2^t+d_2)}).$$

For positive i, j, consider the general term

$$Sq_i(v_1^{(m_1 2^t+d_1)}) Sq_j(v_2^{(m_2 2^t+d_2)}). \tag{7}$$

Let $2^{r-1} \leq i \leq 2^r - 1$ and $2^{s-1} \leq j \leq 2^s - 1$, where r, s are positive and $\leq t$ if $k = 2^{t-1}$ and are $\leq t - 1$ if $k > 2^{t-1}$. Write (7) as

$$Sq_i(v_1^{(n_r 2^r+d_1)}) Sq_j(v_2^{(n_s 2^s+d_2)}),$$

where $n_r 2^r = m_1 2^t$ and $n_s 2^s = m_2 2^t$. We have

$$\begin{aligned} &Sq_k(v_1^{(m_1 2^t+d_1)} v_2^{(m_2 2^t+d_2)}) \\ &\qquad = v_1^{(m_1 2^t+d_1)} Sq_k(v_2^{(m_2 2^t+d_2)}) + Sq_k(v_1^{(m_1 2^t+d_1)}) v_2^{(m_2 2^t+d_2)} \\ &\qquad\quad + \sum_{\substack{i+j=k \\ i,j>0}} Sq_i(v_1^{(n_r 2^r+d_1)}) Sq_j(v_2^{(n_s 2^s+d_2)}). \end{aligned}$$

But, using Proposition 3.1, the summation equals

$$\begin{aligned} &\sum_{\substack{i+j=k \\ i,j>0}} v_1^{(n_r 2^r)} v_2^{(n_s 2^s)} Sq_i(v_1^{(d_1)}) Sq_j(v_2^{(d_2)}) \\ &= v_1^{(m_1 2^t)} v_2^{(m_2 2^t)} \sum_{\substack{i+j=k \\ i,j>0}} Sq_i(v_1^{(d_1)}) Sq_j(v_2^{(d_2)}). \end{aligned}$$

The result (6) is now followed. For $n > 2$, the proof goes the same lines. Finally, in the case where $n \geq 2$, we show that each d_i, $1 \leq i \leq n$, varies from k to $k + 2^t - 1$. The endpoint is trivial by the selection of k. We claim that $Sq_k(\boldsymbol{v}^{(\boldsymbol{d})}) = 0$ if all $d_i < k$. The claim

$$Sq_k(v_1^{(d_1)} v_2^{(d_2)}) = \sum_{i+j=k} Sq_i(v_1^{(d_1)}) Sq_j(v_2^{(d_2)}) = 0.$$

is true for $n = 2$ by Corollary 2.2. The value of $Sq_k(\boldsymbol{v}^{(\boldsymbol{d})})$ for $n > 2$ is calculated recursively from two variables. Therefore, the claim holds by Cartan formula. □

Remark 3.2. Propositions 2.6 and 2.7 applies, somehow, Theorem 3.2 for the cases where $s > i + 1$. This can be seen in Tables 1 and 2.

Example 3.2.

$$\begin{aligned} Sq_3(u^{(10)}v^{(14)}) &= Sq_3(u^{(2^2+6)}v^{(2\cdot 2^2+6)}) = u^{(2^2)}v^{(2\cdot 2^2)}Sq_3(u^{(6)}v^{(6)}) \\ &= u^{(4)}v^{(8)}\big(u^{(3)}v^{(6)} + u^{(6)}v^{(3)}\big) = u^{(7)}v^{(14)} + u^{(10)}v^{(11)}; \end{aligned}$$

$$\begin{aligned} Sq_8(u^{(32)}v^{(32)}w^{(32)}) &= Sq_8(u^{(3\cdot 2^3+8)}v^{(3\cdot 2^3+8)}w^{(3\cdot 2^3+8)}) \\ &= u^{(24)}v^{(24)}w^{(24)}Sq_8(u^{(8)}v^{(8)}w^{(8)}) \\ &= u^{(24)}v^{(24)}w^{(24)}\big(u^{(4)}v^{(4)}w^{(8)} + u^{(8)}v^{(4)}w^{(4)} + u^{(4)}v^{(8)}w^{(4)} \\ &\quad + u^{(6)}v^{(6)}w^{(4)} + u^{(4)}v^{(6)}w^{(6)} + u^{(6)}v^{(4)}w^{(6)}\big) \\ &= u^{(28)}v^{(28)}w^{(32)} + u^{(32)}v^{(28)}w^{(28)} + u^{(28)}v^{(32)}w^{(28)} \\ &\quad + u^{(30)}v^{(30)}w^{(28)} + u^{(28)}v^{(30)}w^{(30)} + u^{(30)}v^{(28)}w^{(30)}. \end{aligned}$$

4. Matrix methods

In this section, we exhibit some matrix methods to manipulate the up and down Steenrod operations. First, the up Steenrod operation is considered.

Definition 4.1. For the monomial $x_1^{r_1}x_2^{r_2}\cdots x_n^{r_n}$, the string function pow is defined by

$$\mathrm{pow}(x_1^{r_1}x_2^{r_2}\cdots x_n^{r_n}) = r_1\, r_2 \ldots r_n,$$

where $r_1\, r_2 \ldots r_n$ is supposed as a string. This definition is extended naturally for the polynomial

$$f = x_1^{r_1}x_2^{r_2}\cdots x_n^{r_n} + \cdots + x_1^{s_1}x_2^{s_2}\cdots x_n^{s_n}$$

in $\mathrm{P}(n)$ additively.

$$\mathrm{pow}(f) = r_1\, r_2 \ldots r_n + \cdots + s_1\, s_2 \ldots s_n.$$

For example, $\mathrm{pow}(x^2y^5 + x^4y^3) = 2\,5 + 4\,3$.

Note that we use only the correspondence feature of this function, not the algebraic nature. The motivation for this definition is nothing but simplification.

Definition 4.2. For a positive integer k, by the up Steenrod k-matrix it is meant the square matrix $[\mathcal{U}^k]$ of size $k + 1$ with the string entries

$$[\mathcal{U}^k]_{ij} = \mathrm{pow}\left(Sq^{i-1}(x^j)\right),$$

for $i, j = 1, 2, \ldots, k+1$. It is noteworthy that, due to the string nature, this definition is free of variable; that is,

$$[\mathcal{U}^k]_{ij} = \text{pow}\left(Sq^{i-1}(x^j)\right) = \text{pow}\left(Sq^{i-1}(y^j)\right).$$

Denote by $[\mathcal{U}^k]^t$ the transpose of $[\mathcal{U}^k]$, and $[\mathcal{U}^k]^r$ the matrix obtained from $[\mathcal{U}^k]$ by reversing the order of rows. In other words,

$$\begin{aligned}([\mathcal{U}^k]^t)_{ij} &= [\mathcal{U}^k]_{ji} = \text{pow}\left(Sq^{j-1}(x^i)\right),\\ ([\mathcal{U}^k]^r)_{ij} &= [\mathcal{U}^k]_{(k+2-i)j} = \text{pow}\left(Sq^{k+1-i}(x^j)\right).\end{aligned}$$

For instance,

$$[\mathcal{U}^3] = \begin{bmatrix}1&2&3&4\\2&0&4&0\\0&4&5&0\\0&0&6&0\end{bmatrix},\ [\mathcal{U}^3]^t = \begin{bmatrix}1&2&0&0\\2&0&4&0\\3&4&5&6\\4&0&0&0\end{bmatrix},\ [\mathcal{U}^3]^r = \begin{bmatrix}0&0&6&0\\0&4&5&0\\2&0&4&0\\1&2&3&4\end{bmatrix}$$

A large size $[\mathcal{U}^{32}]$ is depicted in Figure 1, where dots indicate zeroes. In the sequel, following a similar simplification as in Definition 4.1, the string matrices defined in Definition 4.2 are multiplied together to produce another string matrices. More precisely, instead of the matrix multiplication

$$\begin{bmatrix}x^2&x^5\\x^3&x^8\end{bmatrix}\begin{bmatrix}y^4&y^2\\y^2&y^9\end{bmatrix} = \begin{bmatrix}x^2y^4+x^5y^2&x^2y^2+x^5y^9\\x^3y^4+x^8y^2&x^3y^2+x^8y^9\end{bmatrix}, \tag{8}$$

we will apply

$$\begin{bmatrix}2&5\\3&8\end{bmatrix}\begin{bmatrix}4&2\\2&9\end{bmatrix} = \begin{bmatrix}2\,4+5\,2&2\,2+5\,9\\3\,4+8\,2&3\,2+8\,9\end{bmatrix}, \tag{9}$$

which is, in fact,

$$\begin{bmatrix}\text{pow}(x^2y^4+x^5y^2)&\text{pow}(x^2y^2+x^5y^9)\\\text{pow}(x^3y^4+x^8y^2)&\text{pow}(x^3y^2+x^8y^9)\end{bmatrix}.$$

There is no confusion in the matrix multiplication (9) since the two matrices on the left sides of the equations (8) and (9) due to different variables.

The next result indicates a matrix method to calculate of the up Steenrod operations in the two variables x and y.

Theorem 4.1. *Let k be a positive integer. Then,*

$$\text{pow}\left(Sq^k(x^iy^j)\right) = \left([\mathcal{U}^k]^t\,[\mathcal{U}^k]^r\right)_{ij}.$$

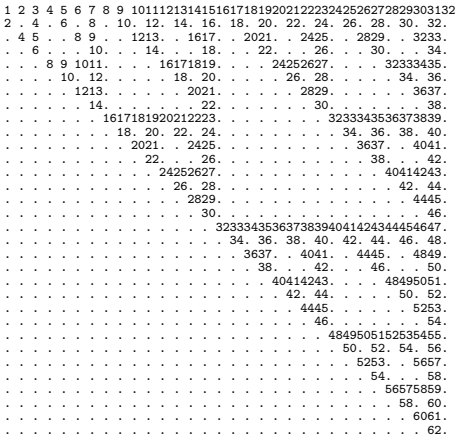


Figure 1: The up Steenrod matrix $[\mathcal{U}^{32}]$

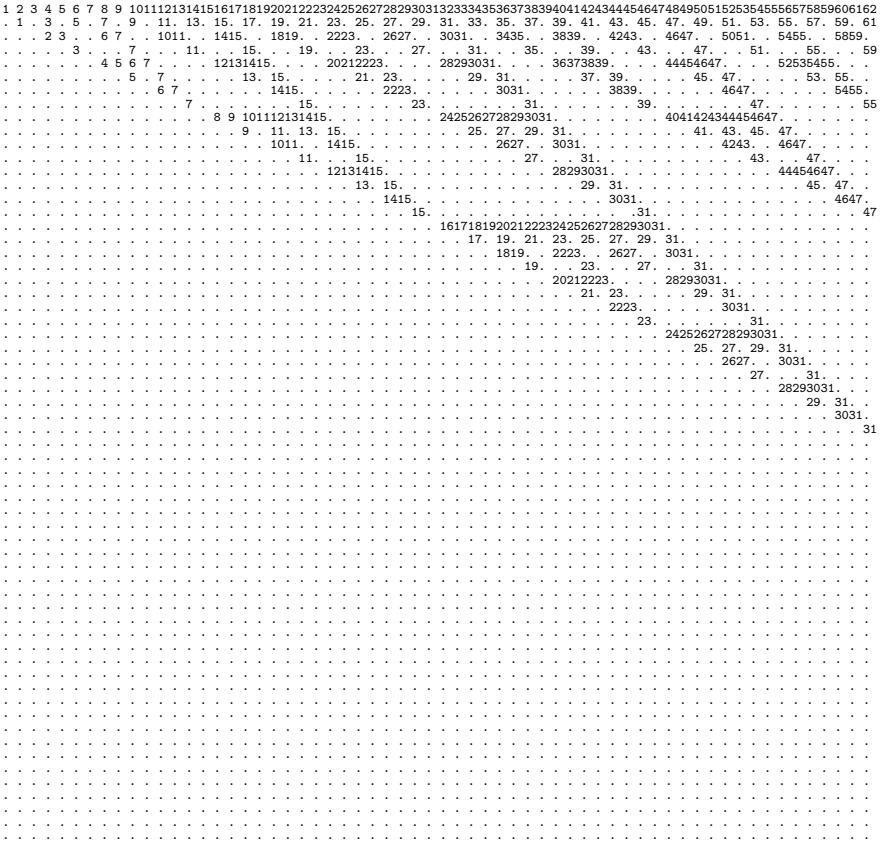


Figure 2: The down Steenrod matrix $[\mathcal{D}_{64}]$

Proof. For $i, j = 1, 2, \ldots, k+1$, by Cartan formula, we have

$$\begin{aligned} Sq^k(x^i y^j) &= \sum_{r+s=k} Sq^r(x^i) Sq^s(y^j) \\ &= \sum_{m=1}^{k+1} Sq^{m-1}(x^i) Sq^{k+1-m}(y^j). \end{aligned}$$

On the other hand, by Definition 4.2,

$$([\mathcal{U}^k]^t)_{im} = \mathrm{pow}\big(Sq^{m-1}(x^i)\big),\ ([\mathcal{U}^k]^r)_{mj} = \mathrm{pow}\big(Sq^{k+1-m}(y^j)\big).$$

The result is now follows from Definition 4.1. □

The above-mentioned matrix method may be extended inductively.

Definition 4.3. Consider the monomial $x_{i_1}^{d_1} \cdots x_{i_n}^{d_n} \in \mathrm{P}^d(n)$, where $d_1 + \cdots + d_n = d$.

For a positive integer k, the up Steenrod $k(i_1, \ldots, i_n)$-matrix $[\mathcal{U}^k_{i_1,\ldots,i_{n-1}|i_n}]$ of size $k+1$ is defined as the following square matrix with string entries.

$$[\mathcal{U}^k_{i_1,\ldots,i_{n-1}|i_n}]_{pq} = \mathrm{pow}\left(Sq^{p-1}(x_{i_1}^{d_1} \cdots x_{i_{n-1}}^{d_{n-1}} x_{i_n}^{q})\right),$$

where $p, q = 1, 2, \ldots, k+1$. For each single variable x_{i_j}, we have the free of variable up Steenrod k-matrix $[\mathcal{U}^k]$ as in Definition 4.2. The transpose $[\mathcal{U}^k_{i_1,\ldots,i_{n-1}|i_n}]^t$, and the row-reversed $[\mathcal{U}^k_{i_1,\ldots,i_{n-1}|i_n}]^r$ is defined as before.

$$\begin{aligned} ([\mathcal{U}^k_{i_1,\ldots,i_{n-1}|i_n}]^t)_{pq} &= [\mathcal{U}^k_{i_1,\ldots,i_{n-1}|i_n}]_{qp} = \mathrm{pow}\left(Sq^{q-1}(x_{i_1}^{d_1} \cdots x_{i_{n-1}}^{d_{n-1}} x_{i_n}^{p})\right), \\ ([\mathcal{U}^k_{i_1,\ldots,i_{n-1}|i_n}]^r)_{pq} &= [\mathcal{U}^k_{i_1,\ldots,i_{n-1}|i_n}]_{(k+2-p)q} = \mathrm{pow}\left(Sq^{k+1-p}(x_{i_1}^{d_1} \cdots x_{i_{n-1}}^{d_{n-1}} x_{i_n}^{q})\right). \end{aligned}$$

The next result extends Theorem 4.1. The proof is similar.

Theorem 4.2. *Let k be a positive integer and $n > 2$. Let the $Sq^k(x_{i_1}^{d_1} \cdots x_{i_n}^{d_n})$ has already been calculated by the above-mentioned matrix method for the values less than n. Take the variables $x_{i_2}, \ldots, x_{i_{n-1}}$ and fix them. Then,*

$$\mathrm{pow}\left(Sq^k(x_{i_1}^{d_1} x_{i_2}^{d_2} \cdots x_{i_{n-1}}^{d_{n-1}} x_{i_n}^{q})\right) = \left([\mathcal{U}^k_{i_1}]^t\, [\mathcal{U}^k_{i_2,\ldots,i_{n-1}|i_n}]^r\right)_{pq}.$$

Now, we concentrate on the down Steenrod operations. The next definition is analogous to Definition 4.1

Definition 4.4. For the d-monomial $v_1^{(r_1)} v_2^{(r_2)} \cdots v_n^{(r_n)}$, we correspond a string $r_1\, r_2 \ldots r_n$ written as

$$\mathrm{pow}(v_1^{(r_1)} v_2^{(r_2)} \cdots v_n^{(r_n)}) = r_1\, r_2 \ldots r_n.$$

This definition is extended naturally for sums of d-monomials.

Definition 4.5. For a positive integer k, the down Steenrod k-matrix $[\mathcal{D}_k]$ of size $k+1$ is defined as

$$[\mathcal{D}_k]_{ij} = \mathrm{pow}\left(Sq_{i-1}(v^{(j)})\right),$$

for $i, j = 1, 2, \ldots, k+1$. The matrix $[\mathcal{D}_k]^{tc}$ is obtained by reversing the order of the columns of the transpose $[\mathcal{D}_k]^t$ of $[\mathcal{D}_k]$. More precisely,

$$([\mathcal{D}_k]^{tc})_{ij} = ([\mathcal{D}_k]^t)_{i(k+2-j)} = ([\mathcal{D}_k])_{(k+2-j)i} = \mathrm{pow}\left(Sq_{k+1-j}(v^{(i)})\right).$$

For example,

$$[\mathcal{D}_3] = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ [\mathcal{D}_3]^t = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 4 & 3 & 2 & 0 \end{bmatrix}, \ [\mathcal{D}_3]^{tc} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 2 & 3 & 4 \end{bmatrix}$$

Figure 2 illustrates a large size $[\mathcal{D}_{64}]$, where dots stand for zeroes. To calculate the down Steenrod operations in two variables the next result is applied.

Theorem 4.3. *Let k be a positive integer. Then,*

$$\mathrm{pow}\left(Sq_k(u^{(i)}v^{(j)})\right) = \left([\mathcal{D}_k]^{tc}\,[\mathcal{D}_k]\right)_{ij}.$$

Proof. The proof goes along the same lines as the proof of Theorem 4.1 with a little difference. By Cartan formula, we have

$$\begin{aligned} Sq_k(u^{(i)}v^{(j)}) &= \sum_{r+s=k} Sq_r(u^{(i)})Sq_s(v^{(j)}) \\ &= \sum_{m=1}^{k+1} Sq_{k+1-m}(u^{(i)})Sq_{m-1}(v^{(j)}), \end{aligned}$$

for $i, j = 1, 2, \ldots, k+1$. By Definition 4.5 we have

$$([\mathcal{D}_k]^{tc})_{im} = \mathrm{pow}\big(Sq_{k+1-m}(u^{(i)})\big), \ ([\mathcal{D}_k])_{mj} = \mathrm{pow}\big(Sq_{m-1}(v^{(j)})\big).$$

The result is now clear by Definition 4.4. □

Definition 4.5 is extended in a similar way as Definition 4.3. In particular, the down Steenrod $k(i_1, \ldots, i_n)$-matrix $[\mathcal{D}_k^{i_1,\ldots,i_{n-1}|i_n}]$ of size $k+1$ is defined as follows.

$$[\mathcal{D}_k^{i_1,\ldots,i_{n-1}|i_n}]_{pq} = \mathrm{pow}\left(Sq_{p-1}(v_{i_1}^{(d_1)} \cdots v_{i_{n-1}}^{(d_{n-1})} v_{i_n}^{(q)})\right),$$

for $p, q = 1, 2, \ldots, k+1$. The matrix $[\mathcal{D}_k^{i_1,\ldots,i_{n-1}|i_n}]^{tc}$ is defined as before.

$$([\mathcal{D}_k^{i_1,\ldots,i_{n-1}|i_n}]^{tc})_{pq} = \mathrm{pow}\left(Sq_{k+1-q}(v_{i_1}^{(d_1)} \cdots v_{i_{n-1}}^{(d_{n-1})} v_{i_n}^{(p)})\right).$$

Now, the extended Theorem analogous to Theorem 4.2 can be stated.

Theorem 4.4. *Given the integers $k > 0$ and $n > 2$, let the $Sq_k(v_{i_1}^{(d_1)} \cdots v_{i_n}^{(d_n)})$ has already been calculated by the above-mentioned matrix method for the values less than n. Take the variables $v_{i_2}, \ldots, v_{i_n}$ and fix them. Then,*

$$\mathrm{pow}\left(Sq_k(v_{i_1}^{(p)} v_{i_2}^{(d_2)} \cdots v_{i_{n-1}}^{(d_{n-1})} v_{i_n}^{(q)})\right) = \left([\mathcal{D}_k^{i_2,\ldots,i_{n-1}|i_n}]^{tc}\,[\mathcal{D}_k^{i_1}]\right)_{pq}.$$

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The cycle-complete graph Ramsey numbers $R(C_7, K_9)$ and $R(C_8, K_9)$

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Abstract. The Ramsey number $R(G, H)$ is the smallest integer n such that any graph of order n contains the graph G or its complement contains the graph H. In this paper, we prove that $R(C_s, K_9) = 8s - 7$ for $s = 7, 8$, where C_s is a cycle of order s and K_r is the complete graph of order r.

Keywords: Ramsey number, cycle graph, complete graph.

1. Introduction

In this paper, all graphs are considered simple graphs. The Ramsey number $R(G, H)$ is the smallest integer n such that any graph of order n contains the graph G, or its complement contains the graph H. Note that if a complement of a graph Y, denoted by Y^c, contains K_r, then $\alpha(Y) \geqslant r$, where $\alpha(Y)$ is the number of independence of a graph Y. The vertex set of a graph Y is denoted by $V(Y)$ and the edge set by $E(Y)$. The order of a graph Y is the number of vertices in the graph, denoted by $\mid V(Y) \mid$ or simply by $\mid Y \mid$. A path of order s is denoted by P_s. The graph $K_1 + P_s$ is obtained by adding one vertex to the path P_s and connecting this vertex to each vertex of P_s. The neighborhood $N(u)$ of a vertex u is the set of vertices adjacent to u and $N[u] = N(u) \cup \{u\}$. The

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neighborhood of a vertex u in a graph Y is denoted by $N_Y(u)$. For $U \subseteq V(G)$, $N(U)$ is defined as $\bigcup_{u\in U} N(u)$ and $N[U] = N(U) \cup U$. The degree of a vertex u of a graph G is $\mid N(u) \mid$ and is denoted by $d(u)$. The minimum degree of a graph Y is denoted by $\delta(Y)$. The graph obtained by deleting the vertex u and all edges adjacent to u from the graph G is denoted by $G - \{u\}$ and $G - U = \cup_{i=1}^{n}(G - \{u_i\})$ where $n = |U|$ or simply by $G \setminus \{u\}$ and $G \setminus U$.

In 1976, it has been conjectured by Erdös et al. [10] that

$$R(C_m, K_n) = (n-1)(m-1) + 1; \quad \forall \ 3 \leq n \leq m,$$

except for $n = m = 3$, since $R(3, 3) = 6$. The conjecture was proved for $m \geq n^2 - 2$ by Bondy and Erdös [6]. Moreover, in the early works on Ramsey number, the conjecture was proved for $n = 3$ [11, 19]. Yang et al. [22] and Bollobás et al. [5] confirmed the conjecture for $n = 4$ and $n = 5$, respectively. Schiermeyer [20] showed that the conjecture is true for $n = 6$. Furthermore, Cheng et al. [8] confirmed the result in the conjecture for $n = 7$ and proved that $R(C_7, K_8) = 43$. Jaradat and Alzaleq [15] proved the conjecture for $n = m = 8$. Moreover, Bataineh, Jaradat and Al-Zaleq [4] proved the conjecture for $m = 9$ and $n = 8$. Greenwood and Gleason [12] and Clancy [9] showed that $R(C_3, K_4) = 9$ and $R(C_4, K_5) = 14$, respectively. Whereas, $R(C_4, K_8) = 26$ and $R(C_4, K_9) = 30$ were proved by Radziszowski and Kung-Kuen Tse[18]. Grinstead and Roberts [13] proved that $R(C_3, K_8) = 28$ and $R(C_3, K_9) = 36$. Recently, Grinstead and S. Roberts [16] showed that $R(C_5, K_8) = 29$, $R(C_6, K_9) = 41$ and $R(C_5, K_9) \leq 36$. Hendry [14] proved that $R(K_1 + P_4, K_5) = 19$. The graph $H = (n-1)K_{m-1}$ does not contain C_m and H^c doesn't contain K_n, which proves $R(C_m, K_n) \geq (n-1)(m-1) + 1$.

2. Main results

The purpose of this section is to find the Ramsey number for a cycle of order 7 and 8 versus the complete graph of order 9. Since $R(C_m, K_n) \geq (m-1)(n-1)+1$, then it is sufficient to prove that $R(C_s, K_9) \leq 8(s-1)+1 = 8s-7$, for $s = 7, 8$.

Theorem 2.1. *$R(C_s, K_9) = 8s - 7$, where $s = 7, 8$.*

To prove Theorem 2.1, we first prove a sequence of lemmas that are necessary to the proof.

Lemma 2.1. *Let G be a graph of order $n \geq 8s - 7$, $s = 7, 8$, that contains neither C_s nor a 9-element independent set, then $\delta(G) \geq s - 1$.*

Proof. Let u be a vertex of G such that $d(u) < s - 1$, then $|V(G - N[u])| \geq 7s - 6$. Since $R(C_s, K_8) = 7s - 6; s = 7, 8$, then $G - N[u]$ contains an 8-element independent set. This set together with the vertex u is a 9-element independent set, thus a contradiction. Therefore, $d(u) \geq s - 1$. The proof is done. □

Note that, with a similar arguments as in Lemma 2.1, $|N(\{v_1, \ldots, v_t\})| \geq t(s-2)+1$ for any idependent set $\{v_1, \ldots, v_t\}$, $2 \leq t \leq 8$ and $s = 7, 8$. Now, in what follows we let $s = 7, 8$, and we only consider graphs G of order $8s-7$ with the following conditions:

(1) $\delta(G) \geq s-1$.

(2) G does not contain C_s.

Lemma 2.2. *If G contains K_{s-1}, then G contains a 9-element independent set.*

Proof. Let $U = \{u_1, \ldots, u_{s-1}\}$ be the vertex set of K_{s-1}, $R = G - U$ and $U_i = N(u_i) \cap V(R)$; $1 \leq i \leq s-1$.

By (1)

$$(*) \qquad U_i \neq \phi, \quad 1 \leq i \leq s-1.$$

Note that, between any two vertices of K_{s-1} there are paths of orders $s-1, s-2, \ldots, 2$. By $(*)$ and (2), for $1 \leq i < j \leq s-1$, we have the following observations:

1. $U_i \cap U_j = \phi$.

2. $N(U_i) \cap N(U_j) \cap R = \phi$.

3. if $x \in N[U_i] \cap R$ and $y \in N[U_j] \cap R$, then x and y are independent.

If $\alpha(N[U_i] \cap R) \geq 2$ for all $i = 1, \ldots, s-1$, then each $N[U_i] \cap R$ contains two independent vertices, hence we get at least $12 \geq 9$ independent vertices in G. If $\alpha(N[U_i] \cap R) = 1$ for some i, we may assume $\alpha(N[U_1] \cap R) = 1$, then $N[U_1] \cap R$ is complete. By condition (1) $|N[U_1] \cap R| \geq s-1$. Hence, by (2) if $x, y \in (N[U_1] \cap R)$, then $N(x) \cap N(y) \cap (R \setminus N[U_1]) = \phi$ and $uv \notin E(G)$ for any $u \in N(x) \cap (R \setminus N[U_1])$ and $v \in N(y) \cap (R \setminus N[U_1])$, and hence $\alpha(R \cap N(N[U_1])) > s-2$.

Also, by (2) $N(N[U_1]) \cap R \cap U_i = \phi$; $2 \leq i \leq s-1$ and if $w \in N(N[U_1]) \cap R$ and $v \in U_i$; $2 \leq i \leq s-1$, then v and w are independent. Therefore, $\alpha(G) \geq \alpha((N(N[U_1]) \cap R) \cup (\cup_{i=2}^{s-1} U_i)) \geq s-2+s-2 \geq 9$. The proof is done. □

Lemma 2.3. *If G contains K_{s-2} and doesn't contain K_{s-1}, then G contains a 9-element independent set.*

Proof. Let $U = \{u_1, \ldots, u_{s-2}\}$ be the vertex set of K_{s-2}, $R = G - U$ and $U_i = N(u_i) \cap R$, $i = 1, \ldots, s-2$. By (1) $|U_i| \geq 2$; $i = 1, \ldots, s-2$. To complete the proof, we consider two cases:

Case 2.3.1. *$U_i \cap U_j \neq \phi$, for some $1 \leq i < j \leq s-2$.* □

Proof. Without loss of generality, assume that $U_1 \cap U_2 \neq \phi$ and let $u \in U_1 \cap U_2$.

By (1) $U_i - \{u\} \neq \phi; 1 \leq i \leq s-2$ and by (2) $xy \notin E(G)$ for any $x \in U_i$ and $y \in U_j$, $x \neq y$, $1 \leq i < j \leq s-2$. Moreover, by (2) $(U_i - \{u\}) \cap (U_j - \{u\}) = \phi$ and $N(U_i - \{u\}) \cap N(U_j - \{u\}) \cap R = \phi$; $1 \leq i < j \leq s-2$ and $\{i, j\} \neq \{1, 2\}$.

Now, let $v_i \in U_i - \{u\}$; $i = 3, \ldots, s-2$ and $V_i = N[v_i] \cap R$, then by (1) $|V_i| \geq s-1$. Since G doesn't contain K_{s-1}, then $\alpha(V_i) \geq 2$.

Also, by (2) if $x \in V_i$ and $y \in V_j$, then x and y are independent, $3 \leq i < j \leq s-2$. Thus, $\alpha(V_3 \cup \ldots \cup V_{s-2}) \geq 2(s-4)$.

Now, we consider three cases:

Case 2.3.1.1. $|U_1 \cap U_2| \geq 3$. □

Proof. Suppose that $|U_1 \cap U_2| \geq 3$. Then, by (2) $xy \notin E(G)$ for any $x \in U_1$ and $y \in U_2$ and hence $\{x, y, u\}$ is a 3-element independent set. Therefore, $\alpha(G) \geq 2(s-4) + 3 = 2s - 5 \geq 9$, because s=7,8.

Case 2.3.1.2. $|U_1 \cap U_2| = 2$. □

Proof. Suppose that $|U_1 \cap U_2| = 2$, say $\{u, a\} = U_1 \cap U_2$. By (2), if $x \in (N(u) \cap R)$ and $y \in (N(a) \cap R)$, then $\{y, x, u_1\}$ is a 3-element independent set. Therefore, $\alpha(G) \geq 9$.

Case 2.3.1.3. $|U_1 \cap U_2| = 1$. □

Proof. Suppose that $|U_1 \cap U_2| = 1$. Then by (2) u is isolated in U_1 and U_2, so $\alpha(U_1 \cup U_2) \geq 3$ and hence $\alpha(G) \geq 9$.

Case 2.3.2. $U_i \cap U_j = \phi$; $1 \leq i < j \leq s-2$. □

Proof. By (2) $N[U_i] \cap N[U_j] \cap R = \phi$; $1 \leq i < j \leq s-2$. Also, by (2), if $x \in N[U_i] \cap R$ and $y \in N[U_j] \cap R$, then x and y are independent.

Now, by (1) $|N[U_i] \cap R| \geq s-1$; $1 \leq i \leq s-2$. Since G doesn't contain K_{s-1}, then

$\alpha(N[U_i] \cap R) \geq 2$; $1 \leq i \leq s-2$. Since $s-2 \geq 5$, then $\alpha(G) \geq \alpha(\cup_{i=1}^{s-2} N[U_i] \cap R) \geq 2(s-2) \geq 10 \geq 9$. The proof is done. □

Lemma 2.4. *If G contains $K_1 + P_{s-2}$ and does not contain K_{s-2}, then G contains a 9-element independent set.*

Proof. Let $U = \{u_1, \ldots, u_{s-2}\}$ be the vertices of P_{s-2} and v be the vertex of K_1. Suppose that $R = G - (U \cup \{v\})$ and $U_i = N(u_i) \cap R$; $i = 1, \ldots s-2$. To complete the proof, we consider two cases:

Case 2.4.1. $U_i \cap U_j = \phi$; $1 \leq i < j \leq s-2$. □

Proof. By (2) if $x \in U_i$ and $y \in U_j$, then x and y are independent. Also, $N[U_i] \cap N[U_j] \cap R = \phi$ and $xy \notin E(G)$ for any $x \in N[U_i] \cap R$ and $y \in N[U_j] \cap R$, $i = 1, \ldots s-2$. Now, since G does not contain K_{s-2} and $|N[U_i] \cap R| \geq s-1$, then $\alpha(N[U_i] \cap R) \geq 2$. Therefore, $\alpha(G) \geq 2(s-2) = 2s-4 \geq 9$.

Case 2.4.2. *$U_i \cap U_j \neq \phi$ for some $1 \leq i < j \leq s-2$. By (2) $\{i, j\}$ can be $\{2, 4\}$ when s=7, and any pair of $\{2, 4\}, \{2, 5\}, \{3, 5\}$ when s=8.* □

Proof. Note that, by (2) only one pair of the $U_i's$ is allowed to intersect, so we may assume this pair to be U_2 and U_4, $U_2 \cap U_4 \neq \phi$. Let $u \in U_2 \cap U_4$. By(2), $U_i \cap U_j = \phi$; $1 \leq i < j \leq s-2$ and $\{i, j\} \neq \{2, 4\}$. If $x \in U_i$ and $y \in U_j$ and $x \neq y$, then x and y are independent, $1 \leq i < j \leq s-2$. Also, by (2) $N[U_i] \cap N[U_j] \cap R = \phi$ and $xy \notin E(G)$ for any $x \in N[U_i] \cap R$ and $y \in N[U_j] \cap R$, $1 \leq i < j \leq s-2$ and $\{i, j\} \neq \{2, 4\}$. By (1) $|N[U_i] \cap R| \geq s-2$, hence $\alpha(N[U_i] \cap R) \geq 2$; $i \in \{1, \ldots, s-2\} \setminus \{2, 4\}$. If s=8, then $\alpha(G) \geq 4(2) + |\{u\}| = 9$. For s=7, by (2) $xv \notin E(G)$ for any $x \in N[U_i] \cap R$, $i = 1, 3, 5$, and $xv \notin E(G)$ for any $x \in N(u) \cap R$. Now, if $\alpha(N(u) \cap R) \geq 2$, then $\alpha(G) \geq 2(3)+2+|\{v\}| = 9$. And, if $\alpha(N(u) \cap R) = 1$, then $N[u] \cap R$ is K_4 and $\alpha(N(N[u] \cap R) \cap (R - (N[u] \cap R))) \geq 2$. Therefore, $\alpha(G) \geq 2(3) + 2 + |\{u\}| = 9$. The proof is done. □

Lemma 2.5. *If G contains $K_1 + C_{s-3}$ and does not contain K_{s-2} and $K_1 + P_{s-2}$, then G contains a 9-element independent set.*

Proof. Let $U = \{u_1, \ldots, u_{s-3}\}$ and v be the vertex set of P_{s-3} and K_1, respectively. Also, let $R = G - (U \cup \{v\})$ and $U_i = N(u_i) \cap R$, $i = 1, \ldots s-3$. By (1) $|U_i| \geq 2$. If $U_i \cap U_j \neq \phi$ for some $i = a$ and $j = b$, then $U_i \cap U_j = \phi$ for $1 \leq i < j \leq s-3$ and $\{i, j\} \neq \{a, b\}$. Now, by (2) if $x \in U_a \cap U_b$, then x is isolated in U_a and in U_b. Since G does not contain $K_1 + P_{s-2}$, then by (1) $|(U_a \cup U_b) \setminus \{x\}| \geq 2$. Moreover, by (2) $uw \notin E(G)$ for any $u \in U_i$ and $w \in U_j$, or $u \in N(U_i) \cap R$ and $w \in N(U_j) \cap R$, $i = 1, \ldots s-3$. Note that there are at least four independent vertices, y_1, y_2, y_3, and y_4 with $y_i \in U_i$, $y_i \neq x$ and $|N(y_i) \cap R| \geq s-3$. Also, these points exist even if $U_i \cap U_j = \phi$, $1 \leq i < j \leq s-3$. Since G doesn't contain K_{s-2}, then $\alpha(N[y_i] \cap R) \geq 2$, $i = 1, 2, 3, 4$. Therefore, $\alpha(G) \geq \alpha(\cup_{i=1}^{4}(N[y_i] \cap R) \cup \{v\}) \geq 9$, where v is the vertex of K_1. If $U_i \cap U_j = \phi$, then with a similar arguments as above we have $\alpha(G) \geq 9$. The proof is done. □

Lemma 2.6. *If G contains $K_1 + P_{s-3}$ and doesn't contain $K_1 + P_{s-2}$, $K_1 + C_{s-3}$ and K_{s-2}, then G contains a 9-element independent set.*

Proof. Let $U = \{u_1, \ldots, u_{s-3}\}$ and v be the vertex set of P_{s-3} and K_1, respectively. Also, let $R = G - (U \cup \{v\})$ and $U_i = N(u_i) \cap R$, $i = 1, \ldots s-3$. By (1) $|U_i| \geq 2$. We consider two cases :

Case 2.6.1. *$x_i x_j \in E(G)$ for some $x_i, x_j \in N(v) \cap R$. Suppose that $x_1 x_2 \in E(G)$. Now, since G doesn't contain $K_1 + P_{s-2}$, then $x_i \notin U_j$ where $i = 1, 2$ and $j = 1, \ldots, s-3$ except for $j = 3$ when $s = 8$.* □

Proof. For $s = 7$, $xy \notin E(G)$ for any $x \in U_i$ and $y \in U_j$, $x \neq y$ and $1 \leq i < j \leq 4$. Thus, by (2) there exist $y_i \in U_i$, $1 \leq i < j \leq s-3$, such that $\{y_1, y_2, \ldots, y_{s-3}\}$ is an independent set. Also, by (2) $|U_i \cap U_j \cap U_k| \leq 1$ for any

$\{i \neq j \neq k\} \in \{1, \ldots, s-3\}$. Therefore, by (1) $|N(y_i) \cap R| \geq s-3$ for at least three vertices of $\{y_1, \ldots, y_{s-3}\}$. Moreover, by (2) $N(y_i) \cap N(y_j) \cap R = \phi$ and $xy \notin E(G)$ for any $x \in N(y_i)$ and $y \in N(y_j)$; $1 \leq i < j \leq 4$. Also, $x_1 \notin N(y_i)$, $N(x_1) \cap N(y_i) \cap R = \phi$ and $ab \notin E(G)$ for any $a \in N(y_i) \cap R$ and $b \in N[x_1] \cap R$, $1 \leq i \leq 4$. By (1) $|N[x_1] \cap R| \geq s-1$ and hence $\alpha(N[x_1] \cap R) \geq 2$. Now, since G doesn't contain K_{s-2} and $|N[y_i] \cap R| \geq s-2$ for at least 3 values of i, then $\alpha(N(y_i) \cap R) \geq 2$ for at least three values of i. Therefore, $\alpha(G) \geq \alpha(\cup_{i=1}^{4}(N(y_i) \cap R) \cup (N[x_1] \cap R)) \geq 3(2) + 1 + 2 = 9$.

For $s = 8$, we consider two subcases:

Case 2.6.1.1. *$x_i u_3 \in E(G)$ for some $i = 1, 2$.* □

Proof. By (2) $U_i \cap U_j = \phi$ for $i < 3$ and $j > 3$ and $U_i \cap (N[x_j] \cap R) = \phi$ for $1 \leq i \leq 5$, $i \neq 3$ and $1 \leq j \leq 2$. Also, by (2) $|U_i \cap U_j \cap U_k| \leq 1$ for any $\{i \neq j \neq k\} \in \{1, \ldots, s-3\}$ and hence $|N[U_i] \cap R| \geq 6$ for at least three values of $i \neq 3$, say $i = 1, 2, 4$. Since G does not contain K_{s-2}, then $\alpha(N(U_i) \cap R) \geq 2$; for $i = 1, 2, 4$ and $\alpha(N[\{x_1, x_2\}] \cap R) \geq 2$.

By (2) $xy \notin E(G)$ for any $x \in (N[U_i] \cap R)$ and $y \in ((N[U_j] \cup N[x_1, x_2]) \cap R)$, $i = 1, 2, 4, 5$, $j = 1, 2, 4, 5$ and $i \neq j$. Therefore, $\alpha(G) \geq \alpha(\cup_{i=1, i\neq 3}^{5}(N[U_i] \cap R) \cup (N[x_1, x_2] \cap R)) \geq 3(2) + 1 + 2 = 9$.

Case 2.6.1.2. *$x_i u_3 \notin E(G)$; $i = 1, 2$.* □

Proof. Two subsubcases are considered:

Case 2.6.1.2.1. *$U_3 \cap N(x_i) \neq \phi$ for some $i = 1, 2$.* □

Proof. By (2) $U_i \cap (N(x_j) \cap R) = \phi$, $1 \leq i \leq 5$, $i \neq 3$, $j = 1, 2$. Also, $U_i \cap U_j = \phi$ and $N(U_i) \cap N(U_j) \cap R = \phi$, $1 \leq i < j \leq 5$ and hence $|N[U_i] \cap R| \geq 7$ and $\alpha(N[U_i] \cap R) \geq 2$.

Moreover, $xy \notin E(G)$ for any $x \in N[U_i] \cap R$ and $y \in N[U_j] \cap R$, $1 \leq i < j \leq 5$. Therefore, $\alpha(G) \geq \alpha(\cup_{i=1, i\neq 3}^{5}(N[U_i] \cap R) \cup \{x_1\}) \geq 4(2) + 1 = 9$.

Case 2.6.1.2.2. *$U_3 \cap N(x_i) \cap R = \phi$,$i = 1, 2$.* □

Proof. By (2) $U_j \cap N(x_i) \cap R = \phi$ and $(N(U_j) \cap R) \cap (N(x_i) \cap R) = \phi$, $j = 1, \ldots, 5$, $i = 1, 2$. Moreover, $xy \notin E(G)$ for any $x \in N[U_i] \cap R$ and $y \in ((N[U_j] \cup N[x_1, x_2]) \cap R)$. Also, $|N(U_i) \cap R| \geq 5$ for at least 3 values of i, say i = 1,2, 3, and hence $\alpha(N(U_i) \cap R) \geq 2$, $1 \leq i \leq 3$, and $\alpha(N(x_1) \cap R) \geq 2$. Therefore, $\alpha(G) \geq \alpha(\cup_{i=1}^{3}(N[U_i] \cap R) \cup (N[x_1] \cap R) \cup \{b\})) \geq 9$, where $b \in (N(U_4) \cap R)$ such that $|N[U_4] \cap R| \leq 5$.

Case 2.6.2. *$x_i x_j \notin E(G)$ for any $x_i, x_j \in N(v) \cap R$.* □

Proof. Note that $\alpha(N(v))$ is the minimum when $d(v) = s-1$ which will be consider in the proof. Since G does not contain $K_1 + C_{s-3}$ and $K_1 + P_{s-2}$, then $k = \{x_1, x_2, u_1, u_{s-3}\}$ is an independent set. Since k is independent, then $|N(k)| \geq 4(s-2) + 1$ as, otherwise, G contains a 9-element independent set.

Therefore, $|N(k) \setminus (K_1 + P_{s-3} \setminus \{k\})| \geq 4s-7-(s-4) = 3s-3 \geq R(C_{s-3}, K_6)$. Moreover, since between any two points of k there is a path of order 3 and G doesn't contain $K_1 + C_{s-3}$, then $N(k) \cap R$ doesn't contain C_{s-3} and hence $\alpha((N(k) \cap R) \cup \{v\}) \geq 7$, say $W = \{w_1, \ldots, w_6, v\}$ is an independent set. Also, since G does not contain $K_1 + C_{s-3}$, then by (2) $N(W) \cap (G \setminus N[v])$ doesn't contain C_{s-3}. Note that, $|N(W)| \geq 7(s-2)+1$ and $|N(W) \setminus N(v)| \geq 6s-12$. Now, since $R(C_{s-3}, K_8) \leq 6s-12$, then $\alpha(N(W) \setminus N[v]| \geq 8$ and hence $\alpha(N(W) \cup \{v\}) \geq 9$.

These cases complete the proof. □

Remark 1. Suppose that G doesn't contain K_{s-2} and $K_1 + P_{s-3}$. If G contains $K_4 = \{x_1, x_2, x_3, x_4\}$, then

$$\alpha(N(x_1, x_2, x_3)) \geq \begin{cases} 3, & \text{if } s = 7 \\ 2, & \text{if } s = 8 \end{cases}$$

.

Proof. The proof is clear for $s = 8$ because G does not contain K_{s-2}. For $s = 7$, let $R = V(G) \setminus V(K_4)$. Since G doesn't contain $K_1 + P_{s-3}$, then $N(x_i) \cap N(x_j) \cap R = \phi$, $1 \leq i < j \leq 4$. By(1) $|N(x_i)| \geq 3$, $i = 1, 2, 3$. By(2), if $xy \in E(G)$ for some $x \in N(x_1) \cap R$ and $y \in N(x_2) \cap R$, then x is isolated in $N(x_1) \cap R$ and y is isolated in $N(x_2) \cap R$. Therefore, $\alpha(N(x_1) \cup N(x_2)) \geq 2$ and $\alpha(N(x_3)) \geq 1$. By(2) $xy \notin E(G)$ for any $x \in N(x_3)$ and $y \in N(x_i); i = 1, 2$, therefore, $\alpha(N(x_1, x_2, x_3) \cap R) \geq 3$. If $xy \notin E(G)$ for any $x \in N(x_i)$ and $y \in N(x_j); 1 \leq i < j \leq 3$, then $\alpha(N(x_1, x_2, x_3) \cap R) \geq 3$. The proof is done. □

Remark 2. *If G doesn't contain K_{s-2} and $K_1 + P_{s-3}$ and $\{u, v\}$ are independent vertices that are adjacent to the vertex of K_1, then $\alpha(N(u, v) \setminus K_1) \geq 3$.*

Proof. Since u and v are independent, then $|N(u, v)| \geq 2s - 3$ and hence $|N(u, v) \setminus K_1| \geq 2s - 4$. Since $R(C_{s-3}, K_3) = 2s - 7 \leq 2s - 4$, then $\alpha(N(u, v) \setminus K_1) \geq 3$. The proof is done. □

Lemma 2.7. *If G contains K_{s-3} and doesn't contain $K_1 + P_{s-3}$ and K_{s-2}, then G contains a 9-element independent set.*

Proof. Let $U = \{u_1, \ldots, u_{s-3}\}$ be the vertex set of K_{s-3}, $R = G - U$ and $U_i = R \cap N(u_i)$, $i = 1, \ldots, s - 3$. Since G doesn't contain $K_1 + P_{s-3}$, then $U_i \cap U_j = \phi; 1 \leq i < j \leq s - 3$. By (1) $\lceil U_i \geq 3$, $i = 1, \ldots, s - 3$. Also, by (2) $R \cap N(U_i) \cap N(U_j) = \phi$ and $xy \notin E(G)$ for any $x \in N(U_i) \cap R$ and $y \in N(U_j) \cap R$, $1 \leq i < j \leq s - 3$. To complete the proof we consider two cases:

Case 2.7.1. *$xy \in E(G)$ for some $x \in U_i$ and $y \in U_j$, $1 \leq i < j \leq s - 3$.* □

Proof. Without loss of generality, assume that $x \in U_1$ and $y \in U_2$. By (2) x is isolated in U_1 and y is isolated in U_2, therefore $\alpha(U_1 \cup U_2) \geq 3$. Also, by (2) $st \notin E(G)$ for any $s \in U_i$ and $t \in U_j$, $i = 3, 4$, $1 \leq j \leq 4$ and $i \neq j$. Now, by Remarks 1 and 2 , $\alpha((N[U_3] \ldots \cup N[U_{s-3}]) \cap R) \geq 6$. Therefore, $\alpha(G) \geq \alpha(U_1 \cup U_2 \cup (N[U_3 \ldots \cup U_{s-3}] \cap R)) \geq 3 + 6 = 9$.

Case 2.7.2. *$xy \notin E(G)$ for any $x \in U_i$ and $y \in U_j$, $1 \leq i < j \leq s-3$.* □

Proof. By Remarks 1 and 2, $\alpha(N[U_i] \cap R) \geq 3$ for all $1 \leq i \leq s-3$, therefore, $\alpha(G) \geq 9$.

The proof is done. □

Remark 3. *Suppose that G contains $K_1 + P_4$ and $|N(P_4) \setminus K_1| \geq 10$. If G doesn't contain $C_8, K_1 + P_5$ and K_5, then $\alpha(N(P_4) \setminus K_1) \geq 4$.*

Proof. If $N(P_4) \setminus K_1$ contains path of order 4, say L, then since G doesn't contain C_8 the vertices of L are adjacent to one vertex in $V(P_4)$. Also, since G doesn't contain $K_1 + P_5$ and C_8, then the vertices of L and $(N(P_4) \setminus K_1) \setminus L$ are disjoint. Therefore, $\alpha(N(P_4) \setminus K_1) \geq \alpha(L) + \alpha((N(P_4) \setminus K_1) \setminus L) \geq 2 + 2 = 4$. If $N(P_4 \setminus K_1)$ doesn't contain P_4, then $\alpha(N(P_4) \setminus K_1) \geq \lceil \frac{|N(P_4) \setminus K_1|}{3} \rceil \geq 4$. The proof is done. □

Remark 4. *Suppose that G contains $K_1 + P_3$ and $|N(P_3) \setminus K_1| \geq 9$. If G doesn't contain $C_7, K_1 + P_4$ and K_4, then $\alpha(N(K_1 + P_3)) \geq 4$.*

Proof. Since G doesn't contain C_7 and K_4, then $N(P_3) \setminus K_1$ doesn't contain C_3. Now, since $R(C_3, K_4) = 9$, then $\alpha(N(P_3) \setminus K_1) \geq 4$. The proof is done □

Lemma 2.8. *If G contains $K_1 + P_{s-4}$ and doesn't contain K_{s-3} and $K_1 + P_{s-3}$, then G contains a 9-element independent set.*

Proof. Let $U = \{u_1, \ldots, u_{s-4}\}$ and v be the vertex set of P_{s-4} and K_1, respectively. Also, let $R = G - (U \cup \{v\})$ and let $U_i = N(u_i) \cap R$, $i = 1, \ldots, s-4$. By (1) $|U_i| \geq 3$, $i = 1, \ldots, s-4$. To complete the proof, we consider two cases:

Case 2.8.1. *$x_i x_j \in E(G)$ for some $x_i, x_j \in N(v) \cap R$, say $x_1, x_2 \in E(G)$.* □

Proof. By (2) and since G doesn't contain $K_1 + P_{s-3}$, then $x_i \notin U_j$ for all $i = 1, 2$ and $j = 1, \ldots, s-4$. Also, by (2) $N(U_j) \cap N(x_i) \cap R = \phi$ and $xy \notin E(G)$ for any $x \in N(U_j) \cap R$, $y \notin U_j$ and $y \in (N(x_i)) \cap R)$ for $i = 1, 2$ and $j = 1, \ldots, s-4$. Now, two subcases are considered:

Case 2.8.1.1. *$U_j \cap N(x_i) \cap R = \{u\}$ for some $i = 1, 2$ where $j = 2$ for $s = 7$ and where $j = 2$ or 3 for $s = 8$.* □

Proof. Without loss of generality, assume that $u \in U_2 \cap N(x_2) \cap R$, then by (2) $U_i \cap U_j = \phi$ for $1 \leq i < j \leq s-4$, except for $\{i, j\} = \{1, 2\}$ when $s = 8$. Moreover, $U_j \cap N(x_i) \cap (R \setminus \{u\}) = \phi$ for $i = 1, 2$, $j = 1, \ldots, s-4$ and $\{i, j\} \neq \{1, 2\}$. Also, by (2) $xy \notin E(G)$ for any $x \in U_k$ and $y \in (U_j \cup N(x_i)) \cap R$ for all $i = 1, 2$, $k, j = 1, \ldots, s-4$ and $j \neq k$. Therefore, there are two disjoint independent sets $Y \subseteq N(P_{s-4}) \setminus K_1$ and $X \subseteq N(x_1) \cap (R - \{u\})$. Note that Y has at least three independent vertices, say $W = \{w_1, w_2, w_3\}$, with each vertex adjacent to different u_i, where $i \neq 2$ for $s = 8$. By (1) $|N(w_i) \cap R| \geq s-4$ and $|X| \geq s-4$, $i = 1, 2, 3$. Since G doesn't contain K_{s-3}, then $\alpha(G) \geq \alpha(\cup_{i=1}^{3} N[w_i] \cup X \cup \{u\}) \geq 9$.

Case 2.8.1.2. *$U_j \cap N(x_i) \cap R = \phi$ for all $i = 1, 2$ and $j = 1, \ldots, s-4$.* □

Proof. Since G doesn't contain K_{s-3}, then $\alpha(U) = 2$, and hence $|N(U)| \geq 2s-3$ and $|N(U) \cap R| \geq s+2$. By (2) there are two disjoint independent sets $Y \subseteq N(U) \cap R$ and $X \subseteq N(x_1, x_2) \cap R$. Since G doesn't contain K_{s-3} and $K_1 + P_{s-3}$, then $|X| \geq 2$, with two vertices in X adjacent to x_1 and x_2, and by Remarks 3 and 4, $|Y| \geq 4$. Also, by (2) $N(X) \cap (R \setminus \{x_1, x_2\})$ and $N(Y) \cap R$ are disjoint. Note that $N(Y) \cap R \geq 3s-2$ and $N(X) \cap (R \setminus \{x_1, x_2\}) \geq s+2$. With a similar argument as in Remarks 3 and 4, we get $\alpha(N(X)(R \setminus \{x_1, x_2\})) \geq 4$. Also, since $R(C_{s-3}, K_5) \leq 3s-2$ and G doesn't contain C_s and $K_1 + C_{s-3}$, then $\alpha(N(Y) \cap R) \geq 5$. Therefore, $\alpha(G) \geq 5 + 4 = 9$.

Case 2.8.2. *$x_i x_j \notin E(G)$ for all $x_i, x_j \in N(v) \cap R$.* □

Proof. In the proof, we consider the case when $|N(v)| = s-1$. Note that $\alpha(N(v)) = 5$, say $W = \{w_1, \ldots, w_5\}$ is an independent set. Therefore, $|N(N(v)) \cap R| \geq 4s-4 \geq R(C_{s-3}, K_7)$. Since G does not contain C_s and $K_1 + C_{s-3}$, then $\alpha((N(W) \cap R) \cup \{v\}) \geq 8$, say $H = \{h_1, \ldots, h_7, v\}$ is an independent set. Note that, $|N(H) \setminus (P_{s-4} \cup N(v))| \geq 7s - 14 \geq R(C_{s-3}, K_8)$. Therefore, $\alpha((N(H) \setminus N(v)) \cup \{v\}) \geq 9$.

The proof is done. □

Lemma 2.9. *If G contains K_{s-4} and doesn't contain $K_1 + P_{s-4}$ and K_{s-3}, then G contains a 9-element independent set.*

Proof. Let $U = \{u_1, \ldots, u_{s-4}\}$ be the vertex set of K_{s-4}, $R = G - U$ and $U_i = N(u_i) \cap R$. Since G doesn't contain $K_1 + P_{s-4}$, then $U_i \cap U_j = \phi$, $1 \leq i < j \leq s-4$. By (2) $\alpha(\cup_{i=1}^{s-4} U_i) \geq 6$. We consider the case when $\alpha(\cup_{i=1}^{s-4} U_i) = 6$, say $\{y_1, \ldots, y_6\}$ is an independent set. Since $R(K_4, K_3) = 9$ and $R(K_5, K_3) = 14$, then $|N(y_1, \ldots, y_6)| \geq 49 - 8$ for $s = 7$ and $|N(y_1, \ldots, y_6)| \geq 57 - 13 = 44$ for $s = 8$. Thus, $|N(y_1, \ldots, y_6) \setminus K_{s-4}| \geq 38$ for $s = 7$ and $|N(y_1, \ldots, y_6) \setminus K_{s-4}| \geq 40$ for $s = 8$. Since $R(C_4, K_9) = 30$ and $R(C_5, K_9) \leq 36$, then $\alpha(N(y_1, \ldots, y_6) \cap R) \geq 9$. The proof is done. □

Lemma 2.10. *If G doesn't contain K_{s-4} and $K_1 + P_{s-4}$, then G contains a 9-element independent set.*

Proof. Since $R(K_3, K_9) = 36$, then $\alpha(G) \geq 9$ for $s = 7$.

Whereas for $s = 8$, since $\delta(G) \geq 7$, then $\alpha(N(v)) \geq 4$. Here we consider the case when $\alpha(N(v)) = 4$, say $\{y_1, \ldots, y_4\}$ is an independent set. Since, $R(K_1 + P_4, K_5) = 19$, then $|N(y_1, \ldots, y_4) \setminus \{v\})| \geq 34$. Also, since $R(C_5, K_8) = 29$, then $\alpha(N(y_1, \ldots, y_4) \setminus \{v\}) \geq 8$, say $W = \{w_1, \ldots, w_8\}$ is an independent set. Moreover, $|N(W)| \geq 49$ and $|N(W) \setminus (\{y_1, \ldots, y_4\} \cup \{v\})| \geq 44$. Since, $R(C_5, K_9) \leq 36$, then $\alpha(G) \geq 9$. The proof is done.

□

Proof of Theorem 2.1. Let G be a graph of order $8s-7$, where $s = 7, 8$. Suppose that G does not contain C_s. If G contains K_{s-4} then by Lemma 2.9 G contains a 9-element independent set. If G does not contain K_{s-4}, then by Lemma 2.10 G contains a 9-element independent set. Therefore, G contains either C_s, $s = 7, 8$, or a 9-element independent set. Thus, $R(C_s, K_9) = 8s - 7$, $s = 7, 8$. □

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A bi-endomorphism induces a new type of derivations on B-algebras

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Abstract. The goal of this paper is to introduce the concept of an (l, r) and an (r, l)-τ-derivation on a B-algebra which is induced by a left and a right bi-endomorphism and to provide important properties. The study found that the composition of (l, r) and an (r, l)-τ-derivations is also an (l, r) and an (r, l)-τ-derivation on a 0-commutative B-algebra, respectively. In addition, the relationship among those derivations is also considered.

Keywords: B-algebra, bi-endomorphism, derivation.

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1. Introduction

The notion of B-algebras was introduced by Neggers and Kim [10] in 2002, it is a new algebraic structure, they took some properties from BCI and BCH-algebras (see, [5, 7]), called a B-algebra.

In 2010, Al-Shehrie [3] introduced the notion of a left-right and a right-left derivation of a B-algebra and investigated some related properties. Also, he studied the notion of derivation of 0-commutative B-algebra and investigated some of its properties. Ardekani and Davvaz [4] introduced a generalization of a derivation of a B-algebra, that is, the notion of an f-derivation and an (f, g)-derivation of a B-algebra and investigated some properties of an (f, g)-derivation of a commutative B-algebra in 2014. In a UP-algebra, Sawika et al. [11] introduced the concepts of an (l, r) and an (r, l)-derivation and a derivation, Iampan [6] introduced the concept of an f-derivation, and in the next year, Tippanya et al. [12] introduced the concepts of a left and a right-f-derivation of type I, and of a left and a right-f-derivation of type II. In 2021, Muangkarn et al. [9] studied some properties of an outside and an inside f_q-derivation of a B-algebra. In addition, they defined and studied some properties of a (right-left) and a (left-right) f_q-derivation on a B-algebra from the concept of Al-Kadi [2].

In this paper, we introduce the concept of an (l, r) and an (r, l)-τ-derivation on a B-algebra which is induced by a left and a right bi-endomorphism and to provide important properties. In addition, the relationship among those derivations is also considered. Also, using the concept of a derivation in past investigate some of its properties.

2. Preliminaries

First, let's review some basic definitions and theorems that are required in our work.

Definition 2.1 ([10]). *By a B-algebra we mean an algebra $B = (B, \rightsquigarrow, 0)$ of type $(2, 0)$ satisfying the followings:*

(B1) $(\forall x \in B)(x \rightsquigarrow x = 0)$,

(B2) $(\forall x \in B)(x \rightsquigarrow 0 = x)$,

(B3) $(\forall x, y, z \in B)((x \rightsquigarrow y) \rightsquigarrow z = x \rightsquigarrow (z \rightsquigarrow (0 \rightsquigarrow y)))$.

A non-empty subset S a B-algebra B is called a subalgebra of B if $x \rightsquigarrow y \in S$, for all $x, y \in S$.

After that a B-algebra $(B, \rightsquigarrow, 0)$ is denoted by B.

Example 2.1 ([9]). Let $B = \{0, 1, 2, 3\}$ with the Cayley table as follows:

$\rightsquigarrow$	0	1	2	3
0	0	2	1	3
1	1	0	3	2
2	2	3	0	1
3	3	1	2	0

Then, B is a B-algebra. Let $S = \{0, 3\}$. Then, S is a subalgebra of B.

Theorem 2.1 ([10]). *If B is a B-algebra, then*

(B4) $(\forall x, y \in B)((x \rightsquigarrow y) \rightsquigarrow (0 \rightsquigarrow y) = x)$,

(B5) $(\forall x, y, z \in B)(x \rightsquigarrow (y \rightsquigarrow z) = (x \rightsquigarrow (0 \rightsquigarrow z)) \rightsquigarrow y)$,

(B6) $(\forall x, y \in B)(x \rightsquigarrow y = 0 \Rightarrow x = y)$,

(B7) $(\forall x \in B)(0 \rightsquigarrow (0 \rightsquigarrow x) = x)$,

(B8) $(\forall x, y, z \in B)(x \rightsquigarrow z = y \rightsquigarrow z \Rightarrow x = y)$,

(B9) $(\forall x, y, z \in B)(z \rightsquigarrow x = z \rightsquigarrow y \Rightarrow x = y)$.

Theorem 2.2 ([10]). *An algebra B is a B-algebra if and only if it satisfies the followings:*

(B1) $(\forall x \in B)(x \rightsquigarrow x = 0)$,

(B7) $(\forall x \in B)(0 \rightsquigarrow (0 \rightsquigarrow x) = x)$,

(B10) $(\forall x, y, z \in B)((x \rightsquigarrow z) \rightsquigarrow (y \rightsquigarrow z) = x \rightsquigarrow y)$,

(B11) $(\forall x, y \in B)(0 \rightsquigarrow (x \rightsquigarrow y) = y \rightsquigarrow x)$.

Definition 2.2 ([8]). *A B-algebra B is said to be* 0*-commutative if*

$$(\forall x, y \in B)(x \rightsquigarrow (0 \rightsquigarrow y) = y \rightsquigarrow (0 \rightsquigarrow x)).$$

Example 2.2. In Example 2.1, we have B is 0-commutative.

Theorem 2.3 ([8]). *If B is a 0-commutative B-algebra, then*

(B12) $(\forall x, y \in B)((0 \rightsquigarrow x) \rightsquigarrow (0 \rightsquigarrow y) = y \rightsquigarrow x)$,

(B13) $(\forall x, y, z \in B)((z \rightsquigarrow y) \rightsquigarrow (z \rightsquigarrow x) = x \rightsquigarrow y)$,

(B14) $(\forall x, y, z \in B)((x \rightsquigarrow y) \rightsquigarrow z = (x \rightsquigarrow z) \rightsquigarrow y)$,

(B15) $(\forall x, y \in B)((x \rightsquigarrow (x \rightsquigarrow y)) \rightsquigarrow y = 0)$,

(B16) $(\forall x, y, z, t \in B)((x \rightsquigarrow z) \rightsquigarrow (y \rightsquigarrow t) = (t \rightsquigarrow z) \rightsquigarrow (y \rightsquigarrow x))$,

(B17) $(\forall x, y, z \in B)((x \rightsquigarrow y) \rightsquigarrow z = x \rightsquigarrow (y \rightsquigarrow z))$,

(B18) $(\forall x, y \in B)(x \rightsquigarrow (x \rightsquigarrow y) = y)$.

For a B-algebra B, we denote $x \wedge y = y \rightsquigarrow (y \rightsquigarrow x)$, for all $x, y \in B$. We can refer to (B7) and (B18) as follows:

(B7) $(\forall x, y \in B)(x \wedge 0 = x)$,

(B18) $(\forall x, y \in B)(x \wedge y = x)$ if B is a 0-commutative B-algebra.

Definition 2.3 ([3]). *A self-map d on B is called*

(1) *a (left-right)-derivation ((l, r)-derivation) of B if*

$$(\forall x, y \in B)(d(x \rightsquigarrow y) = (d(x) \rightsquigarrow y) \wedge (x \rightsquigarrow d(y))),$$

(2) *a (right-left)-derivation ((r, l)-derivation) of B if*

$$(\forall x, y \in B)(d(x \rightsquigarrow y) = (x \rightsquigarrow d(y)) \wedge (d(x) \rightsquigarrow y)),$$

(3) *a derivation of B if it is both an (l, r) and an (r, l)-derivation of B.*

Definition 2.4 ([3]). *A self-map d on B is said to be regular if $d(0) = 0$. Otherwise, d is said to be irregular.*

Example 2.3. In Example 2.1, we define a self-map d on B by

$$d(x) = \begin{cases} 3, & \text{if } x = 0 \\ 2, & \text{if } x = 1 \\ 1, & \text{if } x = 2 \\ 0, & \text{if } x = 3. \end{cases}$$

Then, d is a derivation of B, and we see that d is irregular.

Definition 2.5 ([1]). *A non-empty subset I of B is called an ideal of B if it satisfies the followings:*

(I1) $0 \in I$,

(I2) $(\forall x, y \in B)(x \rightsquigarrow y \in I, y \in I \Rightarrow x \in I)$.

We known that every subalgebra of B is an ideal.

Example 2.4. In Example 2.1, let $I = \{0, 2, 3\}$. Then, I is an ideal of B.

Proposition 2.1 ([1]). *Let I be an ideal of B. If $x \in B$ and $y \in I$ such that $x \rightsquigarrow y = 0$, then $x \in I$.*

Proposition 2.2 ([1]). *Let I be an ideal of B and $x, y \in I$. Then, $x \rightsquigarrow (0 \rightsquigarrow y) \in I$.*

Proposition 2.3 ([1]). *If I is an ideal of B such that $0 \rightsquigarrow x = x$, for all $x \in I$, then I is a subalgebra of B.*

3. Main results

In this section, we introduce bi-endomorphism on B-algebra and prove some results of some results of bi-endomorphisms of B-algebra B and its derivations as follows:

Definition 3.1. *A mapping $\tau : B \times B \to B$ is called*

(1) *a left bi-endomorphism on B if*

$$(\forall x, y, z \in B)(\tau(x \rightsquigarrow y, z) = \tau(x, z) \rightsquigarrow \tau(y, z)),$$

(2) *a right bi-endomorphism on B if*

$$(\forall x, y, z \in B)(\tau(x, y \rightsquigarrow z) = \tau(x, y) \rightsquigarrow \tau(x, z)),$$

(3) *a bi-endomorphism on B if it is a left and a right bi-endomorphism on B.*

Throughout this section, we assume that τ_l and τ_r are a left and a right bi-endomorphism of B, respectively.

Let $x \in B$. By (B2), we have

$$\tau_l(x, x) \rightsquigarrow 0 = \tau_l(x, x) = \tau_l(x \rightsquigarrow 0, x) = \tau_l(x, x) \rightsquigarrow \tau_l(0, x)$$

and

$$\tau_r(x, x) \rightsquigarrow 0 = \tau_r(x, x) = \tau_r(x, x \rightsquigarrow 0) = \tau_r(x, x) \rightsquigarrow \tau_r(x, 0).$$

By (B9), we have

(B19) $(\forall x \in B)(\tau_l(0, x) = 0)$,

(B20) $(\forall x \in B)(\tau_r(x, 0) = 0)$.

Example 3.1. In Example 2.1, we define a mapping $\tau_1 : B \times B \to B$ by

$$\tau_1(x, y) = \begin{cases} 3, & \text{if } (x, y) \in \{(1, 0), (2, 0)\} \\ 0, & \text{otherwise.} \end{cases}$$

Then, τ_1 is a left bi-endomorphism on B.

Definition 3.2. *A mapping $\tau : B \times B \to B$ is said to be symmetric if $\tau(x, y) = \tau(y, x)$, for all $x, y \in B$.*

Remark 3.1. For any B-algebra, define a mapping $0 : B \times B \to B$ by $0(x) = 0$, for all $x \in B$, which is a bi-endomorphism on B. Every symmetric left (right) bi-endomorphism on B is a bi-endomorphism.

Definition 3.3. *An ideal I of B is called a τ-ideal of B if $\tau(x, y) \in I$, for all $x, y \in I$.*

Example 3.2. From Example 2.4 and Example 3.1, we have $I = \{0, 2, 3\}$ is an ideal of B and $\tau_1(x, y) \in I$, for all $x, y \in I$. Thus I is a τ_1-ideal on B.

Next, we generalize derivations on B-algebras with a left (right) bi-endomorphism $\tau : B \times B \to B$ from the concept of the Definition 2.3.

Definition 3.4. *Let $\tau : B \times B \to B$ be a left (right) bi-endomorphism on B. A self-map d_τ on B is called*

(1) *an (l, r)-τ-derivation of B if*

$$(\forall x, y \in B)(d_\tau(x \rightsquigarrow y) = (d_\tau(x) \rightsquigarrow \tau(x, y)) \wedge (\tau(x, y) \rightsquigarrow d_\tau(y))),$$

(2) *an (r, l)-τ-derivation of B if*

$$(\forall x, y \in B)(d_\tau(x \rightsquigarrow y) = (\tau(x, y) \rightsquigarrow d_\tau(y)) \wedge (d_\tau(x) \rightsquigarrow \tau(x, y))),$$

(3) *a τ-derivation of B if it is both an (l, r)- and an (r, l)-τ-derivation of B.*

Example 3.3. In Example 2.1, we define $\tau = 0$ and $d_\tau : B \to B$ by $d_\tau(x) = 3$. It is obvious that τ is a bi-endomorphism on B. Then, d_τ is an (l, r)-τ-derivation of B.

Definition 3.5. *Let d be a self-map on B. An ideal I of B is said to be d-invariant if $d(I) \subseteq I$.*

Example 3.4. In Example 2.1 and Example 2.4, we have $I = \{0, 2, 3\}$ is an ideal of B. From Example 3.3, we have d_τ is an (l, r)-τ-derivation of B. It is obvious that $d_\tau(x) \in I$, for all $x \in I$, that is, $d_\tau(I) \subseteq I$. Hence, I is d_τ-invariant.

Theorem 3.1. *Let d_{τ_l} be an (l, r)-τ_l-derivation on B. Then, d_{τ_l} is regular if and only if $d_{\tau_l}(0 \rightsquigarrow x) = 0$, for all $x \in B$.*

Proof. Suppose that d_{τ_l} is regular. Let $x \in B$. Then

$$\begin{aligned}
d_{\tau_l}(0 \rightsquigarrow x) &= (d_{\tau_l}(0) \rightsquigarrow \tau_l(0, x)) \wedge (\tau_l(0, x) \rightsquigarrow d_{\tau_l}(x)) \\
&= (0 \rightsquigarrow 0) \wedge (0 \rightsquigarrow d_{\tau_l}(x)) && \text{(B19)} \\
&= 0 \wedge (0 \rightsquigarrow d_{\tau_l}(x)) && \text{(B1)} \\
&= (0 \rightsquigarrow d_{\tau_l}(x)) \rightsquigarrow ((0 \rightsquigarrow d_{\tau_l}(x)) \rightsquigarrow 0) \\
&= (0 \rightsquigarrow d_{\tau_l}(x)) \rightsquigarrow (0 \rightsquigarrow d_{\tau_l}(x)) && \text{(B2)} \\
&= 0. && \text{(B1)}
\end{aligned}$$

Conversely, suppose that $d_{\tau_l}(0 \rightsquigarrow x) = 0$, for all $x \in B$. Then

$$\begin{aligned}
0 &= d_{\tau_l}(0 \rightsquigarrow 0) \\
&= (d_{\tau_l}(0) \rightsquigarrow \tau_l(0, 0)) \wedge (\tau_l(0, 0) \rightsquigarrow d_{\tau_l}(0)) \\
&= (d_{\tau_l}(0) \rightsquigarrow 0) \wedge (0 \rightsquigarrow d_{\tau_l}(0)) && \text{(B19)} \\
&= d_{\tau_l}(0) \wedge (0 \rightsquigarrow d_{\tau_l}(0)) && \text{(B2)} \\
&= (0 \rightsquigarrow d_{\tau_l}(0)) \rightsquigarrow ((0 \rightsquigarrow d_{\tau_l}(0)) \rightsquigarrow d_{\tau_l}(0)).
\end{aligned}$$

By (B6) and (B2), we get $(0 \rightsquigarrow d_{\tau_l}(0)) \rightsquigarrow 0 = 0 \rightsquigarrow d_{\tau_l}(0) = (0 \rightsquigarrow d_{\tau_l}(0)) \rightsquigarrow d_{\tau_l}(0)$. Using (B9), we have $d_{\tau_l}(0) = 0$. Hence, d_{τ_l} is regular. □

Theorem 3.2. *Let d_{τ_r} be an (l,r)-τ_r-derivation on B. Then, d_{τ_r} is regular and $\tau_r(0,x) = 0$, for all $x \in B$ if and only if $d_{\tau_r}(0 \rightsquigarrow x) = 0$, for all $x \in B$.*

Proof. Suppose that d_{τ_r} is regular and $\tau_r(0,x) = 0$, for all $x \in B$. Let $x \in B$. Then

$$\begin{aligned} d_{\tau_r}(0 \rightsquigarrow x) &= (d_{\tau_r}(0) \rightsquigarrow \tau_r(0,x)) \wedge (\tau_r(0,x) \rightsquigarrow d_{\tau_r}(x)) \\ &= (0 \rightsquigarrow 0) \wedge (0 \rightsquigarrow d_{\tau_r}(x)) \\ \text{(B1)} \qquad &= 0 \wedge (0 \rightsquigarrow d_{\tau_r}(x)) \\ &= (0 \rightsquigarrow d_{\tau_r}(x)) \rightsquigarrow ((0 \rightsquigarrow d_{\tau_r}(x)) \rightsquigarrow 0) \\ \text{(B2)} \qquad &= (0 \rightsquigarrow d_{\tau_r}(x)) \rightsquigarrow (0 \rightsquigarrow d_{\tau_r}(x)) \\ \text{(B1)} \qquad &= 0. \end{aligned}$$

Conversely, suppose that $d_{\tau_r}(0 \rightsquigarrow x) = 0$, for all $x \in B$. By (B1), we have $d_{\tau_r}(0) = d_{\tau_r}(0 \rightsquigarrow 0) = 0$. Thus d_{τ_r} is regular. Let $x \in B$. Then

$$\begin{aligned} 0 &= d_{\tau_r}(0 \rightsquigarrow x) \\ &= (d_{\tau_r}(0) \rightsquigarrow \tau_r(0,x)) \wedge (\tau_r(0,x) \rightsquigarrow d_{\tau_r}(x)) \\ &= (0 \rightsquigarrow \tau_r(0,x)) \wedge (\tau_r(0,x) \rightsquigarrow d_{\tau_r}(x)) \\ &= (\tau_r(0,x) \rightsquigarrow d_{\tau_r}(x)) \rightsquigarrow ((\tau_r(0,x) \rightsquigarrow d_{\tau_r}(x)) \rightsquigarrow (0 \rightsquigarrow \tau_r(0,x))). \end{aligned}$$

By (B6) and (B2), we get $(\tau_r(0,x) \rightsquigarrow d_{\tau_r}(x)) \rightsquigarrow 0 = \tau_r(0,x) \rightsquigarrow d_{\tau_r}(x) = (\tau_r(0,x) \rightsquigarrow d_{\tau_r}(x)) \rightsquigarrow (0 \rightsquigarrow \tau_r(0,x))$. Using (B9), we have $0 \rightsquigarrow \tau_r(0,x) = 0$. Using (B6) again, we obtain $\tau_r(0,x) = 0$. □

Next, we give a necessary and sufficient condition for (r,l)-τ_l-derivation and (r,l)-τ_r-derivation to be regular with some properties.

Theorem 3.3. *Let d_{τ_l} be an (r,l)-τ_l-derivation on B. Then, d_{τ_l} is regular and $\tau_l(x,0) = 0$, for all $x \in B$ if and only if $d_{\tau_l}(x) = 0$, for all $x \in B$.*

Proof. Suppose that d_{τ_l} is regular and $\tau_l(x,0) = 0$, for all $x \in B$. Let $x \in B$. Then

$$\begin{aligned} \text{(B2)} \qquad d_{\tau_l}(x) &= d_{\tau_l}(x \rightsquigarrow 0) \\ &= (\tau_l(x,0) \rightsquigarrow d_{\tau_l}(0)) \wedge (d_{\tau_l}(x) \rightsquigarrow \tau_l(x,0)) \\ &= (0 \rightsquigarrow 0) \wedge (d_{\tau_l}(x) \rightsquigarrow 0) \\ \text{(B2)} \qquad &= 0 \wedge d_{\tau_l}(x) \\ &= d_{\tau_l}(x) \rightsquigarrow (d_{\tau_l}(x) \rightsquigarrow 0) \\ \text{(B2)} \qquad &= d_{\tau_l}(x) \rightsquigarrow d_{\tau_l}(x) \\ \text{(B1)} \qquad &= 0. \end{aligned}$$

Conversely, suppose that $d_{\tau_l}(x) = 0$, for all $x \in B$. Then, d_{τ_l} is regular. Let $x \in B$. Then

$$\begin{aligned}
0 &= d_{\tau_l}(x) \\
\text{(B2)} \quad &= d_{\tau_l}(x \rightsquigarrow 0) \\
&= (\tau_l(x,0) \rightsquigarrow d_{\tau_l}(0)) \wedge (d_{\tau_l}(x) \rightsquigarrow \tau_l(x,0)) \\
&= (\tau_l(x,0) \rightsquigarrow 0) \wedge (0 \rightsquigarrow \tau_l(x,0)) \\
\text{(B2)} \quad &= \tau_l(x,0) \wedge (0 \rightsquigarrow \tau_l(x,0)) \\
&= (0 \rightsquigarrow \tau_l(x,0)) \rightsquigarrow ((0 \rightsquigarrow \tau_l(x,0)) \rightsquigarrow \tau_l(x,0)).
\end{aligned}$$

By (B6) and (B2), we get $(0 \rightsquigarrow \tau_l(x,0)) \rightsquigarrow 0 = 0 \rightsquigarrow \tau_l(x,0) = (0 \rightsquigarrow \tau_l(x,0)) \rightsquigarrow \tau_l(x,0)$. Using (B9), we have $\tau_l(x,0) = 0$. □

Theorem 3.4. *Let d_{τ_r} be an (r,l)-τ_r-derivation on B. Then, d_{τ_r} is regular if and only if $d_{\tau_r}(x) = 0$, for all $x \in B$.*

Proof. Suppose that d_{τ_r} is regular. Let $x \in B$. Then

$$\begin{aligned}
\text{(B2)} \quad d_{\tau_r}(x) &= d_{\tau_r}(x \rightsquigarrow 0) \\
&= (\tau_r(x,0) \rightsquigarrow d_{\tau_r}(0)) \wedge (d_{\tau_r}(x) \rightsquigarrow \tau_r(x,0)) \\
\text{(B20)} \quad &= (0 \rightsquigarrow 0) \wedge (d_{\tau_r}(x) \rightsquigarrow 0) \\
\text{(B2)} \quad &= 0 \wedge d_{\tau_r}(x) \\
&= d_{\tau_r}(x) \rightsquigarrow (d_{\tau_r}(x) \rightsquigarrow 0) \\
\text{(B2)} \quad &= d_{\tau_r}(x) \rightsquigarrow d_{\tau_r}(x) \\
\text{(B1)} \quad &= 0.
\end{aligned}$$

The converse is obvious that d_{τ_r} is regular. □

By (B19), we define an (l,r)-τ_l-derivation d_{τ_l} (or (r,l)-τ_l-derivation) on B by $d_{\tau_l}(x) = \tau_l(x,0)$, for all $x \in B$, and obtain the following proposition.

Proposition 3.1. *Let d_{τ_l} be an (l,r)-τ_l-derivation on B. If $d_{\tau_l}(x) = \tau_l(x,0)$, for all $x \in B$, then d_{τ_l} is the zero self-map on B and d_{τ_l} is regular.*

Proof. Suppose that $d_{\tau_l}(x) = \tau_l(x,0)$, for all $x \in B$. By (B19), we have $d_{\tau_l}(0) = \tau_l(0,0) = 0$, that is, d_{τ_l} is regular. Let $x \in B$. Then

$$\begin{aligned}
\text{(B2)} \quad d_{\tau_l}(x) &= d_{\tau_l}(x \rightsquigarrow 0) \\
&= (d_{\tau_l}(x) \rightsquigarrow \tau_l(x,0)) \wedge (\tau_l(x,0) \rightsquigarrow d_{\tau_l}(0)) \\
&= (d_{\tau_l}(x) \rightsquigarrow d_{\tau_l}(x)) \wedge (d_{\tau_l}(x) \rightsquigarrow d_{\tau_l}(0)) \\
\text{(B1)} \quad &= 0 \wedge (d_{\tau_l}(x) \rightsquigarrow d_{\tau_l}(0)) \\
&= (d_{\tau_l}(x) \rightsquigarrow d_{\tau_l}(0)) \rightsquigarrow ((d_{\tau_l}(x) \rightsquigarrow d_{\tau_l}(0)) \rightsquigarrow 0) \\
\text{(B2)} \quad &= (d_{\tau_l}(x) \rightsquigarrow d_{\tau_l}(0)) \rightsquigarrow (d_{\tau_l}(x) \rightsquigarrow d_{\tau_l}(0)) \\
\text{(B1)} \quad &= 0.
\end{aligned}$$

□

Proposition 3.2. *Let d_{τ_l} be an (r,l)-τ_l-derivation on B. If $d_{\tau_l}(x) = \tau_l(x,0)$, for all $x \in B$, then $d_{\tau_l}(0 \rightsquigarrow x) = 0 \rightsquigarrow d_{\tau_l}(x)$, for all $x \in B$ and d_{τ_l} is regular.*

Proof. Suppose that $d_{\tau_l}(x) = \tau_l(x,0)$, for all $x \in B$. By (B19), we have $d_{\tau_l}(0) = \tau_l(0,0) = 0$, that is, d_{τ_l} is regular. Let $x \in B$. Then

$$\begin{aligned} d_{\tau_l}(0 \rightsquigarrow x) &= (\tau_l(0,x) \rightsquigarrow d_{\tau_l}(x)) \wedge (d_{\tau_l}(0) \rightsquigarrow \tau_l(0,x)) \\ \text{(B19)} \qquad &= (0 \rightsquigarrow d_{\tau_l}(x)) \wedge (0 \rightsquigarrow 0) \\ \text{(B1)} \qquad &= (0 \rightsquigarrow d_{\tau_l}(x)) \wedge 0 \\ \text{(B7)} \qquad &= 0 \rightsquigarrow d_{\tau_l}(x). \end{aligned}$$

□

By (B20), we define an (l,r)-τ_l-derivation d_{τ_l} (or (r,l)-τ_l-derivation) on B by $d_{\tau_r}(x) = \tau_r(0,x)$, for all $x \in B$, and obtain the following proposition.

Proposition 3.3. *Let d_{τ_r} be an (l,r)-τ_r-derivation on B. If $d_{\tau_r}(x) = \tau_r(0,x)$, for all $x \in B$, then $d_{\tau_r}(0 \rightsquigarrow x) = 0 \rightsquigarrow d_{\tau_r}(x)$, for all $x \in B$ and d_{τ_r} is regular.*

Proof. Suppose that $d_{\tau_r}(x) = \tau_r(0,x)$, for all $x \in B$. By (B20), we have $d_{\tau_r}(0) = \tau_r(0,0) = 0$, that is, d_{τ_r} is regular. Let $x \in B$. Then

$$\begin{aligned} d_{\tau_r}(0 \rightsquigarrow x) &= (d_{\tau_r}(0) \rightsquigarrow \tau_r(0,x)) \wedge (\tau_r(0,x) \rightsquigarrow d_{\tau_r}(x)) \\ &= (d_{\tau_r}(0) \rightsquigarrow d_{\tau_r}(x)) \wedge (d_{\tau_r}(x) \rightsquigarrow d_{\tau_r}(x)) \\ \text{(B1)} \qquad &= (d_{\tau_r}(0) \rightsquigarrow d_{\tau_r}(x)) \wedge 0 \\ \text{(B7)} \qquad &= d_{\tau_r}(0) \rightsquigarrow d_{\tau_r}(x) \\ &= 0 \rightsquigarrow d_{\tau_r}(x). \end{aligned}$$

□

Proposition 3.4. *Let d_{τ_r} be an (r,l)-τ_r-derivation on B. If $d_{\tau_r}(x) = \tau_r(0,x)$, for all $x \in B$, then d_{τ_r} is the zero self-map on B and d_{τ_r} is regular.*

Proof. Suppose that $d_{\tau_r}(x) = \tau_r(0,x)$, for all $x \in B$. By (B20), we have $d_{\tau_r}(0) = \tau_r(0,0) = 0$, that is, d_{τ_r} is regular. Let $x \in B$. Then

$$\begin{aligned} d_{\tau_r}(x) &= d_{\tau_r}(x \rightsquigarrow 0) \\ &= (\tau_r(x,0) \rightsquigarrow d_{\tau_r}(0)) \wedge (d_{\tau_r}(x) \rightsquigarrow \tau_r(x,0)) \\ \text{(B20)} \qquad &= (0 \rightsquigarrow 0) \wedge (d_{\tau_r}(x) \rightsquigarrow 0) \\ \text{(B2)} \qquad &= 0 \wedge d_{\tau_r}(x) \\ &= d_{\tau_r}(x) \rightsquigarrow (d_{\tau_r}(x) \rightsquigarrow 0) \\ \text{(B2)} \qquad &= d_{\tau_r}(x) \rightsquigarrow d_{\tau_r}(x) \\ \text{(B1)} \qquad &= 0. \end{aligned}$$

□

Theorem 3.5. *Let d_{τ_l} be a regular (r,l)-τ_l-derivation on B. Every τ_l-ideal of B is d_{τ_l}-invariant.*

Proof. Let B be an element in a τ_l-ideal I of B. Then

$$\begin{aligned}
&\text{(B2)} \quad d_{\tau_l}(x) \rightsquigarrow 0 = d_{\tau_l}(x)\\
&\text{(B2)} \quad = d_{\tau_l}(x \rightsquigarrow 0)\\
&\quad = (\tau_l(x,0) \rightsquigarrow d_{\tau_l}(0)) \wedge (d_{\tau_l}(x) \rightsquigarrow \tau_l(x,0))\\
&\quad = (\tau_l(x,0) \rightsquigarrow 0) \wedge (d_{\tau_l}(x) \rightsquigarrow \tau_l(x,0))\\
&\text{(B2)} \quad = \tau_l(x,0) \wedge (d_{\tau_l}(x) \rightsquigarrow \tau_l(x,0))\\
&\quad = (d_{\tau_l}(x) \rightsquigarrow \tau_l(x,0)) \rightsquigarrow ((d_{\tau_l}(x) \rightsquigarrow \tau_l(x,0)) \rightsquigarrow \tau_l(x,0))\\
&\text{(B10)} \quad = d_{\tau_l}(x) \rightsquigarrow (d_{\tau_l}(x) \rightsquigarrow \tau_l(x,0)).
\end{aligned}$$

By (B9), we get $0 = d_{\tau_l}(x) \rightsquigarrow \tau_l(x,0)$. Since I is a τ_l-ideal of B and by (B6), we have $d_{\tau_l}(x) = \tau_l(x,0) \in I$. Hence, I is d_{τ_l}-invariant. □

Theorem 3.6. *Let d_{τ_r} be a regular (r,l)-τ_r-derivation on B. Every ideal of B is d_{τ_r}-invariant. In particular, every τ_r-ideal of B is d_{τ_r}-invariant.*

Proof. Let B be an element in an ideal I on B. Then

$$\begin{aligned}
&\text{(B2)} \quad d_{\tau_r}(x) = d_{\tau_r}(x \rightsquigarrow 0)\\
&\quad = (\tau_r(x,0) \rightsquigarrow d_{\tau_r}(0)) \wedge (d_{\tau_r}(x) \rightsquigarrow \tau_r(x,0))\\
&\text{(B20)} \quad = (0 \rightsquigarrow 0) \wedge (d_{\tau_r}(x) \rightsquigarrow 0)\\
&\text{(B2)} \quad = 0 \wedge d_{\tau_r}(x)\\
&\quad = d_{\tau_r}(x) \rightsquigarrow (d_{\tau_r}(x) \rightsquigarrow 0)\\
&\text{(B2)} \quad = d_{\tau_r}(x) \rightsquigarrow d_{\tau_r}(x)\\
&\text{(B1)} \quad = 0 \in I.
\end{aligned}$$

Hence, I is d_{τ_r}-invariant. □

Next, we focus on composition of τ_l and τ_r-derivations.

Theorem 3.7. *Let d_{τ_r} and D_{τ_r} be (l,r)-τ_r-derivations on a 0-commutative B-algebra B. If $\tau_r(x,y) = y$, for all $x, y \in B$, then $d_{\tau_r} \circ D_{\tau_r}$ is also an (l,r)-τ_r-derivation on B.*

Proof. Suppose that $\tau_r(x,y) = y$, for all $x, y \in B$. Then

$$\begin{aligned}
&(d_{\tau_r} \circ D_{\tau_r})(x \rightsquigarrow y)\\
&= d_{\tau_r}(D_{\tau_r}(x \rightsquigarrow y))\\
&= d_{\tau_r}((D_{\tau_r}(x) \rightsquigarrow \tau_r(x,y)) \wedge (\tau_r(x,y) \rightsquigarrow D_{\tau_r}(y)))\\
&= d_{\tau_r}((D_{\tau_r}(x) \rightsquigarrow y) \wedge (y \rightsquigarrow D_{\tau_r}(y)))
\end{aligned}$$

$$
\begin{aligned}
\text{(B18)} \quad & = d_{\tau_r}(D_{\tau_r}(x) \rightsquigarrow y) \\
& = (d_{\tau_r}(D_{\tau_r}(x)) \rightsquigarrow \tau_r(D_{\tau_r}(x), y)) \wedge (\tau_r(D_{\tau_r}(x), y) \rightsquigarrow d_{\tau_r}(y)) \\
\text{(B18)} \quad & = d_{\tau_r}(D_{\tau_r}(x)) \rightsquigarrow \tau_r(D_{\tau_r}(x), y) \\
& = d_{\tau_r}(D_{\tau_r}(x)) \rightsquigarrow y \\
& = d_{\tau_r}(D_{\tau_r}(x)) \rightsquigarrow \tau_r(x, y) \\
\text{(B18)} \quad & = (d_{\tau_r}(D_{\tau_r}(x)) \rightsquigarrow \tau_r(x, y)) \wedge (\tau_r(x, y) \rightsquigarrow d_{\tau_r}(D_{\tau_r}(y)) \\
& = ((d_{\tau_r} \circ D_{\tau_r})(x) \rightsquigarrow \tau_r(x, y)) \wedge (\tau_r(x, y) \rightsquigarrow (d_{\tau_r} \circ D_{\tau_r})(y)).
\end{aligned}
$$

Hence, $d_{\tau_r} \circ D_{\tau_r}$ is an (l, r)-τ_l-derivation on B. □

Theorem 3.8. *Let d_{τ_l} and D_{τ_l} be (r, l)-τ_l-derivations on a 0-commutative B-algebra B. If $\tau_l(x, y) = x$, for all $x, y \in B$, then $d_{\tau_l} \circ D_{\tau_l}$ is also an (r, l)-τ_l-derivation on B.*

Proof. Suppose that $\tau_r(x, y) = x$, for all $x, y \in B$. Then

$$
\begin{aligned}
& (d_{\tau_l} \circ D_{\tau_l})(x \rightsquigarrow y) \\
& = d_{\tau_l}(D_{\tau_l}(x \rightsquigarrow y)) \\
& = d_{\tau_l}((\tau_l(x, y) \rightsquigarrow D_{\tau_l}(y)) \wedge (D_{\tau_l}(x) \rightsquigarrow \tau_l(x, y))) \\
& = d_{\tau_l}((x \rightsquigarrow D_{\tau_l}(y)) \wedge (D_{\tau_l}(x) \rightsquigarrow x)) \\
\text{(B18)} \quad & = d_{\tau_l}(x \rightsquigarrow D_{\tau_l}(y)) \\
& = (\tau_l(x, D_{\tau_l}(y)) \rightsquigarrow d_{\tau_l}(D_{\tau_l}(y))) \wedge (d_{\tau_l}(x) \rightsquigarrow \tau_l(x, D_{\tau_l}(y))) \\
\text{(B18)} \quad & = \tau_l(x, D_{\tau_l}(y)) \rightsquigarrow d_{\tau_l}(D_{\tau_l}(y)) \\
& = x \rightsquigarrow d_{\tau_l}(D_{\tau_l}(y)) \\
& = \tau_l(x, y) \rightsquigarrow d_{\tau_l}(D_{\tau_l}(y)) \\
\text{(B18)} \quad & = (\tau_l(x, y) \rightsquigarrow d_{\tau_l}(D_{\tau_l}(y))) \wedge (d_{\tau_l}(D_{\tau_l}(x)) \rightsquigarrow \tau_l(x, y)) \\
& = (\tau_l(x, y) \rightsquigarrow (d_{\tau_l} \circ D_{\tau_l})(y)) \wedge ((d_{\tau_l} \circ D_{\tau_l})(x) \rightsquigarrow \tau_l(x, y)).
\end{aligned}
$$

Hence, $d_{\tau_l} \circ D_{\tau_l}$ is an (r, l)-τ_l-derivation on B. □

Proposition 3.5. *Let d_{τ_r} be an (r, l)-τ_r-derivation and D_{τ_r} be an (l, r)-τ_r-derivation on a 0-commutative B-algebra B. Then*

(1) $(\forall x \in B)((d_{\tau_r} \circ D_{\tau_r})(x) = 0)$ *if d_{τ_r} is regular,*

(2) $(\forall x \in B)((D_{\tau_r} \circ d_{\tau_r})(x) = D_{\tau_r}(0) \rightsquigarrow \tau_r(0, d_{\tau_r}(0)))$,

(3) $(\forall x \in B)((d_{\tau_r} \circ D_{\tau_r})(x) = 0 = (D_{\tau_r} \circ d_{\tau_r})(x))$ *if d_{τ_r} and D_{τ_r} are regular.*

Proof. (1) Suppose that d_{τ_r} is regular. Let $x \in B$. Then

$$\begin{aligned}
(\text{B2}) \qquad (d_{\tau_r} \circ D_{\tau_r})(x) &= (d_{\tau_r} \circ D_{\tau_r})(x \rightsquigarrow 0) \\
&= d_{\tau_r}(D_{\tau_r}(x \rightsquigarrow 0)) \\
(\text{B18}) \qquad &= d_{\tau_r}(D_{\tau_r}(x) \rightsquigarrow \tau_r(x, 0)) \\
(\text{B20}) \qquad &= d_{\tau_r}(D_{\tau_r}(x) \rightsquigarrow 0) \\
(\text{B18}) \qquad &= \tau_r(D_{\tau_r}(x), 0) \rightsquigarrow d_{\tau_r}(0) \\
(\text{B20}) \qquad &= 0 \rightsquigarrow 0 \\
(\text{B1}) \qquad &= 0.
\end{aligned}$$

(2) Let $x \in B$. Then

$$\begin{aligned}
(\text{B2}) \qquad (D_{\tau_r} \circ d_{\tau_r})(x) &= (D_{\tau_r} \circ d_{\tau_r})(x \rightsquigarrow 0) \\
&= D_{\tau_r}(d_{\tau_r}(x \rightsquigarrow 0)) \\
(\text{B18}) \qquad &= D_{\tau_r}(\tau_r(x, 0) \rightsquigarrow d_{\tau_r}(0)) \\
(\text{B20}) \qquad &= D_{\tau_r}(0 \rightsquigarrow d_{\tau_r}(0)) \\
(\text{B18}) \qquad &= D_{\tau_r}(0) \rightsquigarrow \tau_r(0, d_{\tau_r}(0)).
\end{aligned}$$

(3) It is straightforward by (1) and (2). □

Let $\mathrm{Der}_\tau(B)$ be the set of all τ-derivations on a 0-commutative B-algebra B. For $d_\tau, D_\tau \in \mathrm{Der}_\tau(B)$, we define the binary operation $\curlywedge$ on $\mathrm{Der}_\tau(B)$ as follows:

$$(\forall x \in B)((d_\tau \curlywedge D_\tau)(x) = d_\tau(x) \wedge D_\tau(x)).$$

Indeed, let $x \in B$. Then

$$\begin{aligned}
(d_\tau \curlywedge D_\tau)(x) &= d_\tau(x) \wedge D_\tau(x) \\
(\text{B18}) \qquad &= d_\tau(x).
\end{aligned}$$

Hence, $d_\tau \curlywedge D_\tau = d_\tau$, for all $d_\tau, D_\tau \in \mathrm{Der}_\tau(B)$.

Therefore, we immediately get the result as the following proposition.

Proposition 3.6. *For a 0-commutative B-algebra B, $(\mathrm{Der}_\tau(B), \curlywedge)$ is the left zero semigroup.*

4. Conclusion and discussion

In this paper, we have introduced the concept of an (l, r) and an (r, l)-τ-derivation on a B-algebra which is induced by a left and a right bi-endomorphism and provided important properties. The study found that the composition of (l, r) and an (r, l)-τ-derivations is also an (l, r) and an (r, l)-τ-derivation on a 0-commutative B-algebra, respectively. In addition, we can show that the set of all τ-derivations on a 0-commutative B-algebra B is the left zero semigroup. Finally, the study of a bi-endomorphism on other algebras (d/BH/BF/BG-algebras) is an interesting open problem.

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Some results on finite soluble groups and p-supersoluble groups

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Abstract. In this paper, motivated by the conjecture of Heliel, we firstly investigated the solubility of a group by the properties of primes 3 and 5. Further, we also studied the p-supersolubility and p-nilpotency under the conditions on primary subgroups with given order.

Keywords: solubility, p-supersolubility, p-nilpotency.

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1. Introduction

All groups considered in this paper are finite. We shall use the standard notation, as [1]. In particular, let G denote a group, $|G|$ denote the order of G, $\pi(G)$ denote the set of all prime divisors of $|G|$. Let H_G be the core of H in G when $H \leq G$ and let S_n be the symmetric group of degree n. Let $M < \cdot G$ denote M is a maximal subgroup of G.

As we all know, the study of the generalized supplementation of some primary subgroups is one of the most important topic in Finite Group Theory ([2], [3], [4]). Meanwhile, the embedded properties of subgroups are a crucial approach to characterize the structure of a group. For example, in 1998, Ballester-Bolinches and Pedraza-Aguilera [5] demonstrated that if all maximal subgroup of all Sylow subgroups of G are S-quasinormally embedded in G, then G is supersoluble. Recently, Miao [6] introduced the concept of $\mathcal{M}_p$-embedded subgroups which is closely related to $\mathcal{M}_p$-supplementation and obtained some characterization of p-supersolubility and saturated formations containing all supersoluble groups.

On the other hand, many scholars concentrated on the relationship between the solubility of some primary subgroups and their generalized supplementation. For instance, in 1937, Hall [7] proved that G is soluble if and only if each Sylow subgroup of G is complemented in G. In 1982, Arad and Ward [8] showed that G is soluble if and only if every Sylow 2-subgroup and Sylow 3-subgroup of G are complemented in G. In 2000, Ballester-Bolinches, Wang and Guo [9] asserted that G is soluble if and only if each Sylow subgroup of G is c-supplemented in G. In 2008, Asaad and Ramadan [10] proved that G is soluble if and only if every subgroup of prime order of G is c-supplemented in G. In 2009, Miao and Qian [11] stated that G is soluble if every Sylow subgroup of G is $\mathcal{M}$-supplemented in G. In 2014, Heliel [12] showed that G is soluble if and only if every Sylow subgroup of odd order of G is c-supplemented in G. Moreover, Heliel stated a conjecture as follows.

Conjecture 1.1. *Let G be a finite group such that every noncyclic Sylow subgroup P of odd order of G has a subgroup D such that $1 < D \leq P$ and all subgroups H of P with $|H| = |D|$ are c-supplemented in G. Is G soluble?*

It is easy to check that $A_5 \times C_5$ is a counterexample. With the further consideration, Our goal is to investigate the influence of the embedded property of some primary subgroups on the construction of a group by considering only two prime divisors $\{3, 5\}$ and obtain some new characterization about solubility of finite groups.

Definition 1.1 ([13])**.** *Let π be a set of primes. A subgroup H of a group G is called $\mathcal{M}_\pi$-supplemented in G, if there exists a subgroup B of G such that $G = HB$ and $H_1B < G$ for every maximal subgroup H_1 of H with $\pi(|H : H_1|) \subseteq \pi$. In particular, if $\pi = \{p\}$, then H is called $\mathcal{M}_p$-supplemented in G.*

Definition 1.2 ([6], Definition 1.2). *A subgroup H of G is called $\mathcal{M}_p$-embedded in G, if there exists a p-nilpotent subgroup B of G such that $H_p \in Syl_p(B)$ and B is $\mathcal{M}_p$-supplemented in G.*

The Example 1.3 in [6] indicates that an $\mathcal{M}_p$-embedded subgroup need not to be $\mathcal{M}_p$-supplemented in G.

2. Preliminaries

For the sake of convenience, we firstly list some known results here which will be useful in the sequel.

Lemma 2.1 ([6], Lemma 2.1). *Let G be a group. Then*

(1) *Let $N \trianglelefteq G$ and $N \leq H$. If H is $\mathcal{M}_p$-embedded in G, then H/N is $\mathcal{M}_p$-embedded in G/N.*

(2) *Let π be a set of primes. Let N be a normal π'-subgroup and H be a π-subgroup of G. If H is $\mathcal{M}_p$-embedded in G, then HN/N is $\mathcal{M}_p$-embedded in G/N.*

Lemma 2.2 ([14], Lemma 4). *If P is a Sylow p-subgroup of a group G and $N \trianglelefteq G$ such that $P \cap N \leq \Phi(P)$, then N is p-nilpotent.*

Lemma 2.3 ([13], Lemma 4). *Let H be a $\mathcal{M}_\pi$-supplemented subgroup in a group G and B be a $\mathcal{M}_\pi$-supplement to H. If H_1 is a maximal subgroup in H and $\pi(H : H_1) \subseteq \pi$, then $|G : H_1B| = |H : H_1|$.*

Lemma 2.4 ([13], Theorem 1). *Let p be the smallest prime divisor of $|G|$ and H be a p-nilpotent subgroup containing a Sylow p-subgroup of G. If H is $\mathcal{M}_p$-supplemented in G, then G is p-nilpotent.*

Lemma 2.5 ([13], Theorem 2). *Let G be a group, $\pi(G)=\{p_1, p_2 = p, \ldots, p_n\}$, $p_1 < p_2 = p < \ldots < p_n$, and H be a p-nilpotent subgroup containing a Sylow p-subgroup of G. If H is $\mathcal{M}_p$-supplemented in G, then G is p-supersoluble.*

Lemma 2.6 ([15], Lemma 2.8). *Let G be a p-supersoluble group. If $O_{p'}(G) = 1$, then G is supersoluble.*

Lemma 2.7 ([16], Theorem 5.4). *If $(15, |G|) = 1$, then G is soluble.*

Lemma 2.8. *Let G be a group, $\pi(G)=\{p_1, p_2 = p, \ldots, p_n\}$, $p_1 < p_2 < \ldots < p_n$ and P be a Sylow p-subgroup of G. Suppose that $1 < d \leq |P|$. If every subgroup H of P with $|H| = d$ is $\mathcal{M}_p$-embedded in G, then G is p-soluble.*

Proof. It is easy to find that $O_{p'}(G) = 1$ by Lemma 2.1. Let H be a subgroup of P with $|H| = d$. By hypothesis, H is $\mathcal{M}_p$-embedded in G, there exists a p-nilpotent subgroup B of G such that $H \in Syl_p(B)$ and B is $\mathcal{M}_p$-supplemented in G. There exists a subgroup K of G such that $G = BK = HB_{p'}K$, where $B_{p'}$ is the normal p-complement of B, $B_i = H_iB_{p'}$ and $B_iK = H_iB_{p'}K < G$ for every

maximal subgroup H_i of H. By Lemma 2.3, $|G : B_iK| = p$. Then $G/(B_iK)_G$ is isomorphic to a subgroup of the symmetric group S_p and so $(B_iK)_G \neq 1$. We may pick a minimal normal subgroup L of G such that $L \leq (B_iK)_G$. If $|L_p| < d$, then we may choose a subgroup F with order d such that $L_p < F$. By hypothesis, F is $\mathcal{M}_p$-embedded in G, there exists a p-nilpotent subgroup B^* of G such that $F \in Syl_p(B^*)$ and B^* is $\mathcal{M}_p$-supplemented in G. There exists a subgroup K_1 of G such that $G = B^*K_1 = FB^*_{p'}K_1 = LB^*_iK_1$, where $B^*_{p'}$ is the normal p-complement of B^*, $B^*_i = F_iB^*_{p'}$ and $B^*_iK_1 = F_iB^*_{p'}K_1 < G$ for every maximal subgroup F_i of F. Next, we consider subgroup B^*L. By Lemma 2.5, B^*L is p-supersoluble and L is p-supersoluble. By Lemma 2.6, L is supersoluble and L is a p-group. Then G/L satisfies the hypothesis of the theorem and so G is p-soluble by induction.

If $|L_p| \geq d$, then we may take a subgroup E with order d such that $L_p \geq E$. By hypothesis, E is $\mathcal{M}_p$-embedded in G, there exists a p-nilpotent subgroup T of G such that $E \in Syl_p(T)$ and T is $\mathcal{M}_p$-supplemented in G. There exists a subgroup K_2 of G such that $G = TK_2 = ET_{p'}K_2 = LT_iK_2$, where $T_{p'}$ is the normal p-complement of T, $T_i = E_iT_{p'}$ and $T_iK_2 = E_iT_{p'}K_2 < G$ for every maximal subgroup E_i of E. Clearly, $L \nleq (T_iK_2)_G$ and so $L \cap (T_iK_2)_G = 1$. Then $L \cong L(T_iK_2)_G/(T_iK_2)_G$ is a subgroup of $G/(T_iK_2)_G$. Since $|G : (T_iK_2)_G| = 2^\alpha p$, L is soluble and so $|L| = p$. Hence $|L| = d = p$ and all subgroups of P with order p are $\mathcal{M}$-supplemented in G. By [[17], Corollary 3.6], G is p-soluble. □

Lemma 2.9. *Let G be a group and P a Sylow p-subgroup of G where p is the smallest prime divisor of $|G|$. Suppose that $1 < d \leq |P|$. If every subgroup H of P with $|H| = d$ is $\mathcal{M}_p$-embedded in G, then G is p-soluble.*

Proof. By [[18], Lemma 2.6], Lemma 2.4 and the proof of Lemma 2.8, it is easy to be verified. □

Lemma 2.10. *Let G be a group and P be a Sylow p-subgroup of G where p is an odd prime divisor of $|G|$. If every prime order subgroup H of P is complemented in G and $N_G(H)/C_G(H)$ is a p-group, then G is p-nilpotent.*

Proof. Assume that the assertion is false and choose G to be a counterexample of minimal order. Further, we have

(1) $|P| > p$.

Assume that $|P| = p$. Since $N_G(P)/C_G(P)$ is a p-group and $N_G(P)/C_G(P) \hookrightarrow Aut(P)$, we get $N_G(P) = C_G(P)$ and G is p-nilpotent by Burnside's Theorem, a contradiction.

(2) $G = P \rtimes Q$, where Q is a Sylow q-subgroup of G.

Since G is not p-nilpotent, G has a minimal non-p-nilpotent subgroup A of the form $A = A_p \rtimes Q$, where Q is a Sylow q-subgroup of A. Assume that $A < G$. If $|A_p| = p$, then $N_A(A_p)/C_A(A_p) \hookrightarrow Aut(A_p)$. Further, $N_A(A_p)/C_A(A_p) \cong (N_G(A_p) \cap A)C_G(A_p)/C_G(A_p) \leq N_G(A_p)/C_G(A_p)$ is a p-group. Hence $N_A(A_p) = C_A(A_p)$ and A is p-nilpotent, a contradiction. If $|A_p| > p$, then A satisfies

the hypothesis of the theorem and so A is p-nilpotent by the choice of G, a contradiction. Hence $G = A = P \rtimes Q$.

(3) $P = R_1 \times R_2 \times \cdots \times R_t$, where R_i is the minimal normal subgroup of G of order p, $i = 1, 2, \ldots, t$.

We assert that P is the product of the minimal normal subgroups of G with order p. If $P \cap \Phi(G) \neq 1$, then we pick a subgroup H of $P \cap \Phi(G)$ with order p. By hypothesis, there exists a subgroup B of G such that $G = HB$ and $H \cap B = 1$. Then $G = HB = B < G$, a contradiction. Hence $P \cap \Phi(G) = 1$. By [[11], Theorem 1.8.17], $P = R_1 \times R_2 \times \cdots \times R_t$, where R_i is the minimal normal subgroup of G, $i = 1, 2, \ldots, t$. For any $L \in \{R_1, R_2, \ldots, R_t\}$, we may choose a subgroup H of L with order p such that H is complemented in G. There exists a subgroup B of G such that $G = HB = LB$ and $H \cap B = 1$. It is easy to find that either $L \cap B = L$ or $L \cap B = 1$. If $L \cap B = L$, then $L \leq B$ and $G = LB = B < G$, a contradiction. Hence $L \cap B = 1$ and $L = |G : B| = p$.

(4) Final contradiction.

By hypothesis, $N_G(L)/C_G(L) = G/C_G(L)$ is a p-group. Since $N_G(L)/C_G(L) \hookrightarrow Aut(L)$, $G = C_G(L)$, $L \leq Z(G)$. Hence $P \leq Z(G)$ and G is p-nilpotent by [[21], Satz 5.5], a contradiction.

The final contradiction completes the proof. □

Lemma 2.11 ([19], Theorem 1.8.17)**.** *Let N be a non-trivial soluble normal subgroup of G. If $N \cap \Phi(G) = 1$, then the Fitting subgroup $F(N)$ of N is the direct product of minimal normal subgroups of G which are contained in N.*

Lemma 2.12 ([19], Theorem 1.8.19)**.** *If G is a p-soluble group, where p is a prime divisor of $|G|$, then $C_G(F_p(G)) \leq F_p(G)$.*

3. Main results

Theorem 3.1. *Let G be a group. Then G is soluble provided G satisfies the following two conditions:*

(1) *If $3 \in \pi(G)$, then every subgroup H_1 of every Sylow 3-subgroup P of G such that $|H_1| = d_1$ and $1 < d_1 \leq |P|$ is $\mathcal{M}_3$-embedded in G.*

(2) *If $5 \in \pi(G)$, then every subgroup H_2 of every Sylow 5-subgroup Q of G such that $|H_2| = d_2$ and $1 < d_2 \leq |Q|$ is $\mathcal{M}_5$-embedded in G.*

Proof of Theorem 3.1. Assume that the assertion is false and choose G to be a counterexample of minimal order. Further, we have

(1) $15 \mid |G|$ and G is 3-soluble.

If $(15, |G|) = 1$, then G is soluble by Lemma 2.7, a contradiction. If $(15, |G|) = 5$, then G is 5-soluble by Lemma 2.9. Moreover, the composition factor of G is 5-group or $5'$-group. Obviously, $5'$-group is soluble and so G is soluble, a contradiction. If $(15, |G|) = 3$, with the similar discussion as above, then G is soluble, a contradiction. Hence $15 \mid |G|$ and G is 3-soluble.

(2) $O_{\pi'}(G) = 1$, $\pi = \{3, 5\}$.

If $N = O_{\pi'}(G) \neq 1$, then we consider the factor group G/N. By Lemma 2.1 (2), G/N satisfies the condition of the theorem and so G/N is soluble by the choice of G. Hence G is soluble, a contradiction.

(3) If $O_p(G) \neq 1$ for $p = 3$ or $p = 5$, then every minimal normal subgroup L of G contained in $O_p(G)$ is order p and $L \not\leq \Phi(G)$.

Set $d = d_1$ or d_2. We choose a minimal normal subgroup L of G such that $L \leq O_p(G)$. If $|L| < d$, then G/L satisfies the hypothesis of the theorem and so G is soluble by the minimality of G, a contradiction. Hence $|L| \geq d$. We may fix a subgroup H with order d such that $H \leq L$. By hypothesis, H is $\mathcal{M}_p$-embedded in G, there exists a p-nilpotent subgroup B of G such that $H \in Syl_p(B)$ and B is $\mathcal{M}_p$-supplemented in G. There exists a subgroup K of G such that $G = BK = HB_{p'}K = LB_iK$, where $B_{p'}$ is the normal p-complement of B, $B_i = H_iB_{p'}$ and $B_iK = H_iB_{p'}K < G$ for every maximal subgroup H_i of H. Clearly, $L \not\leq \Phi(G)$ and $L \cap B_iK = 1$ by [[20], Lemma 2.8]. Then $|L| = p$ by Lemma 2.3.

(4) Final contradiction.

Since G is not soluble, $G'' \neq 1$. We may choose a minimal normal subgroup L of G such that $L \leq G''$. Since G is 3-soluble, L is 3-group or $3'$-group. If L is 3-group, then $|L| = 3$ by (3). If L is $3'$-group, then $L_5 \neq 1$ by (2).

If $|L_5| < d_2$, then we may pick a subgroup F with order d_2 such that $L_5 < F$. By hypothesis, F is $\mathcal{M}_5$-embedded in G, there exists a 5-nilpotent subgroup C of G such that $F \in Syl_5(C)$ and C is $\mathcal{M}_5$-supplemented in G. Next, we consider subgroup CL. Set $CL = M$ and $P^* \in Syl_5(M)$. Since C is $\mathcal{M}_5$-supplemented in M, there exists a subgroup T such that $M = CT$ and $|M : C_iT| = 5$, where $C_{5'}$ is the normal 5-complement of C, $C_i = P_i^*C_{5'}$ and $C_iT = P_i^*C_{5'}T < M$ for every maximal subgroup P_i^* of P^*. Then $M/(C_iT)_M \hookrightarrow S_5$ and so $M/ \cap (C_iT)_M$ is 5-soluble. Since $P^* \cap (\cap(C_iT)_M) = P^* \cap (\cap(P_i^*C_{5'}T)_M) = \cap P_i^* = \Phi(P^*)$, $\cap(C_iT)_M$ is 5-nilpotent by Lemma 2.2. Hence M is 5-soluble and L is 5-soluble. Then L is soluble and so $|L| = 5$ by (3).

If $|L_5| \geq d_2$, then we may fix a subgroup E with order d_2 such that $L_5 \geq E$. By hypothesis, E is $\mathcal{M}_5$-embedded in G, there exists a 5-nilpotent subgroup U of G such that $E \in Syl_5(U)$ and U is $\mathcal{M}_5$-supplemented in G. There exists a subgroup W such that $G = UW = EU_{5'}W = LU_iW$, where $U_{5'}$ is the normal 5-complement of U, $U_iW = E_iU_{5'}W < G$ for every maximal subgroup U_i of U. Then $L/L \cap (U_iW)_G \cong L(U_iW)_G/(U_iW)_G$ is a subgroup of $G/(U_iW)_G \hookrightarrow S_5$. If $L \cap (U_iW)_G = 1$, then L is soluble and so $|L| = 5$ by (3). If $L \cap (U_iW)_G = L$, then $G = UW = EU_{5'}W = LU_iW = U_iW < G$, a contradiction.

Hence $|L|$ is a prime and $G/C_G(L)$ is isomorphic to a subgroup of $Aut(L)$. Then $G' \leq C_G(L)$ and $L \leq Z(G')$. Moreover, $L \leq Z(G') \cap G'' \leq \Phi(G') \leq \Phi(G)$ by [[1], A.9.3(d)], which contradicts (3).

The final contradiction completes the proof.

In general, the $\mathcal{M}_p$-embedded property is not subgroup closed. Note that the generalized Fitting subgroup $F^*(G)$ play an important role in studying the

global structure of a group. Here we give the following theorem to investigate the solubility of $F^*(G)$ by considering its primary subgroups which are $\mathcal{M}_p$-embedded in a group G.

Theorem 3.2. *Let G be a group. Then $F^*(G)$ is soluble provided $F^*(G)$ satisfies the following two conditions:*

(1) *If $3 \in \pi(F^*(G))$, then every subgroup H_1 of every Sylow 3-subgroup P of $F^*(G)$ such that $|H_1| = d_1$ and $1 < d_1 \leq |P|$ is $\mathcal{M}_3$-embedded in G.*

(2) *If $5 \in \pi(F^*(G))$, then every subgroup H_2 of every Sylow 5-subgroup Q of $F^*(G)$ such that $|H_2| = d_2$ and $1 < d_2 \leq |Q|$ is $\mathcal{M}_5$-embedded in G.*

Proof of Theorem 3.2. By [[3], Lemma 2.6(2)], $F^*(G)/F(G) = R_1/F(G) \times \cdots \times R_s/F(G)$, $R_i/F(G)$ is minimal normal subgroups of $C_G(F(G))F(G)/F(G)$, $i = 1, \ldots, s$. If $(2, |R_i/F(G)|) = 1, i = 1, \ldots, s$, then $F^*(G)$ is soluble. Hence there exists $R_i/F(G)$ such that $2 \in \pi(R_i/F(G))$. We may take a normal subgroup L of G such that $R_i \leq L \leq F^*(G)$. If $(15, |L|) = 1$, then $F^*(G)$ is soluble by Lemma 2.7. If $(15, |L|) \neq 1$, without loss of generality, then we assume that $(15, |L|) = 15$.

Case one. $|L_3| \geq d_1$. We may fix a subgroup X with order d_1 such that $L_3 \geq X$. By hypothesis, X is $\mathcal{M}_3$-embedded in G, there exists a 3-nilpotent subgroup I of G such that $X \in Syl_3(I)$ and B is $\mathcal{M}_3$-supplemented in G. There exists a subgroup K of G such that $G = IK = XI_{3'}K = LI_iK$, where $I_{3'}$ is the normal p-complement of I, $I_i = X_iI_{3'}$ and $I_iK = X_iI_{3'}K < G$ for every maximal subgroup X_i of X. Then $L/L \cap (I_iK)_G \cong L(I_iK)_G/(I_iK)_G$ is a subgroup of $G/(I_iK)_G \hookrightarrow S_3$. If $L \cap (I_iK)_G = 1$, then L is soluble and R_i is soluble. Hence $F^*(G)$ is soluble. If $L \cap (I_iK)_G \neq 1$, then we set $L \cap (I_iK)_G = N_1$ and $L = N_0$. With the similar discussion of L, we have a series $1 = N_{s+1} \trianglelefteq N_s \trianglelefteq \cdots \trianglelefteq N_1 \trianglelefteq N_0 = L$ such that N_i/N_{i+1} is soluble, $i = 0, 1, \ldots, s$. Then L is soluble and R_i is soluble. Hence $F^*(G)$ is soluble.

Case two. $|L_3| < d_1$. We may choose a subgroup H with order d_1 such that $L_3 < H$. By hypothesis, H is $\mathcal{M}_3$-embedded in G, there exists a 3-nilpotent subgroup B of G such that $H \in Syl_3(B)$ and B is $\mathcal{M}_3$-supplemented in G. Next, we consider subgroup BL. By Lemma 2.9, BL is 3-soluble and so L is 3-soluble.

If $|L_5| < d_2$, then we may pick a subgroup F with order d_2 such that $L_5 < F$. By hypothesis, F is $\mathcal{M}_5$-embedded in G, there exists a 5-nilpotent subgroup C of G such that $F \in Syl_5(C)$ and C is $\mathcal{M}_5$-supplemented in G. Next, we consider subgroup $CL = M$. Set $P^* \in Syl_5(M)$. Since C is $\mathcal{M}_5$-supplemented in M, there exists a subgroup T such that $M = CT$ and $|M : C_iT| = 5$, where $C_{5'}$ is the normal 5-complement of C, $C_i = P_i^*C_{5'}$ and $C_iT = P_i^*C_{5'}T < M$ for every maximal subgroup P_i^* of P^*. Then $M/(C_iT)_M \hookrightarrow S_5$ and so $M/ \cap (C_iT)_M$ is 5-soluble. Since $P^* \cap (\cap(C_iT)_M) = P^* \cap (\cap(P_i^*C_{5'}T)_M) = \cap P_i^* = \Phi(P^*)$, $\cap(C_iT)_M$ is 5-nilpotent by Lemma 2.2. Hence M is 5-soluble and L is 5-soluble. It follows from L is soluble that $F^*(G)$ is soluble.

If $|L_5| \geq d_2$, then L is 5-soluble by the similar discussion in Case one. It follows from L is soluble that $F^*(G)$ is soluble.

Theorem 3.3. *Let G be a p-soluble group and P be a Sylow p-subgroup of G where p is a prime divisor of $|G|$. Suppose that $1 < d \leq |P|$. If every subgroup H of P with $|H| = d$ is $\mathcal{M}_p$-embedded in G, then G is p-supersoluble.*

Proof of Theorem 3.3. Assume that the assertion is false and choose G to be a counterexample of minimal order. Further, we have

(1) $O_{p'}(G)$=1.

If $N = O_{p'}(G) \neq 1$, then we consider the factor group G/N. By Lemma 2.1(2), G/N satisfies the condition of the theorem and so G/N is p-supersoluble by the choice of G. Hence G is p-supersoluble, a contradiction.

(2) $O_p(G) \neq 1$.

Since G is p-soluble and $O_{p'}(G)$=1, the minimal normal subgroup of G is a abelian p-group and $O_p(G) \neq 1$.

(3) $O_p(G) \cap \Phi(G) = 1$.

If $O_p(G) \cap \Phi(G) \neq 1$, then we may choose a minimal normal subgroup L of G such that $L \leq O_p(G) \cap \Phi(G)$. If $|L| < d$, then G/L satisfies the hypothesis of the theorem and so G is p-supersoluble by the minimality of G, a contradiction. Hence $|L| \geq d$. We may pick a subgroup H with order d such that $H \leq L$. By hypothesis, H is $\mathcal{M}_p$-embedded in G, there exists a p-nilpotent subgroup B of G such that $H \in Syl_p(B)$ and B is $\mathcal{M}_p$-supplemented in G. There exists a subgroup K of G, $G = BK = HB_{p'}K = LB_iK$, where $B_{p'}$ is the normal p-complement of B, $B_i = H_iB_{p'}$ and $B_iK = H_iB_{p'}K < G$ for every maximal subgroup H_i of H. Since $L \leq O_p(G) \cap \Phi(G)$, $G < B_iK$, a contradiction.

(4) $O_p(G) = R_1 \times R_2 \times \cdots \times R_t$, where $R_1, R_2, \ldots, R_t$ are the minimal normal subgroups of G with order p.

By [[19], Theorem 1.8.17] and (3), $O_p(G) = R_1 \times R_2 \times \cdots \times R_t$, where R_i is a minimal normal subgroup of G, $i = 1, 2, \ldots, t$. Let $L \in \{R_1, \ldots, R_t\}$. If $|L| \geq d$, then we may take a subgroup E with order d such that $E \leq L$. By hypothesis, E is $\mathcal{M}_p$-embedded in G, there exists a p-nilpotent subgroup U of G such that $E \in Syl_p(U)$ and U is $\mathcal{M}_p$-supplemented in G. There exists a subgroup W of G, $G = UW = EU_{p'}W = LU_iW$, where $U_{p'}$ is the normal p-complement of U, $U_i = E_iU_{p'}$ and $U_iK = E_iU_{p'}W < G$ for every maximal subgroup E_i of E. By [[20], Lemma 2.8], $L \cap U_iW \trianglelefteq G$. If $L \cap U_iW = L$, then $G < U_iW$, a contradiction. Hence $L \cap U_iW = 1$ and $|L| = p$. If $|L| < d$, then we may fix a subgroup F with order d such that $L < F$. By hypothesis, F is $\mathcal{M}_p$-embedded in G, there exists a p-nilpotent subgroup C of G such that $F \in Syl_p(C)$ and C is $\mathcal{M}_p$-supplemented in G. There exists a subgroup T of G, $G = CT = FC_{p'}T$, where $C_{p'}$ is the normal p-complement of C, $C_i = F_iC_{p'}$ and $C_iT = F_iC_{p'}T < G$ for every maximal subgroup F_i of F. By (3), $G = CT = FC_{p'}T = LSC_{p'}T$, where $S < \cdot F$ such that $LS = F$. By Lemma 2.3, $|L| = p$.

(5) Final contradiction.

By [[19], Theorem 1.8.19], (1) and (4), $C_G(O_p(G)) = O_p(G) = \bigcap_{i=1}^{t} C_G(R_i)$. Since $G/O_p(G) = G/C_G(O_p(G)) = G/\bigcap_{i=1}^{t} C_G(R_i) \hookrightarrow G/C_G(R_1) \times \cdots \times G/C_G(R_t)$ and $G/C_G(R_i)$ is cyclic, $G/O_p(G)$ is p-supersoluble and G is p-supersoluble.

The final contradiction completes the proof.

Theorem 3.4. *Let G be a group and P be a Sylow p-subgroup of G where p is an odd prime divisor of $|G|$. G is p-nilpotent if and only if every subgroup H of P with $|H| = d$ and $1 < d \leq |P|$ is $\mathcal{M}_p$-embedded in G and $N_G(H)/C_G(H)$ is a p-group.*

Proof of Theorem 3.4. The necessity part is obvious. We only need to prove the sufficiency party. Assume that the assertion is false and choose G to be a counterexample of minimal order. Further, we have

(1) $O_{p'}(G)$=1.

Similar to step (1) in the proof of Theorem 3.3.

(2) G is not a simple group.

Assume that G is a simple group and we take a subgroup H with order d. By hypothesis, H is $\mathcal{M}_p$-embedded in G, there exists a p-nilpotent subgroup B of G such that $H \in Syl_p(B)$ and B is $\mathcal{M}_p$-supplemented in G. There exists a subgroup K of G such that $G = BK = HB_{p'}K$, where $B_{p'}$ is the normal p-complement of B, $B_i = H_iB_{p'}$ and $B_iK = H_iB_{p'}K < G$ for every maximal subgroup H_i of H. Since $(B_iK)_G = 1$, G is isomorphic to a subgroup of the symmetric group S_p and so $|P| = p$. Hence $H = P$, $N_G(P) = C_G(P)$ and G is p-nilpotent by Burnside's Theorem, a contradiction.

(3) For every minimal normal subgroup L of G, $|L| = p$.

We may choose a minimal normal subgroup L of G. If $|L_p| \geq d$, then we may pick a subgroup X with order d such that $L_p \geq X$. By hypothesis, X is $\mathcal{M}_p$-embedded in G, there exists a p-nilpotent subgroup C of G such that $X \in Syl_p(C)$ and I is $\mathcal{M}_p$-supplemented in G. There exists a subgroup T of G, $G = CT = XC_{p'}T = LC_iT$, where $C_{p'}$ is the normal p-complement of C, $C_i = X_iC_{p'}$ and $C_iT = X_iC_{p'}T < G$ for every maximal subgroup X_i of X. We assert that $(C_iT)_G \neq 1$ for every C_i. If there exists a $(C_jT)_G$ such that $(C_jT)_G = 1$, then G is isomorphic to a subgroup of the symmetric group S_p and so $d = |P| = p$. Hence G is p-nilpotent by Burnside's Theorem, a contradiction. Clearly, $L \not\leq (C_jT)_G$ and $L \cap (C_jT)_G = 1$. Then L is isomorphic to a subgroup of the symmetric group S_p and $|L_p| = d = p$. By Lemma 2.10, G is p-nilpotent, a contradiction.

If $|L_p| < d$, then we may take a subgroup E with order d such that $L_p < E$. By hypothesis, E is $\mathcal{M}_p$-embedded in G, there exists a p-nilpotent subgroup U of G such that $E \in Syl_p(U)$ and U is $\mathcal{M}_p$-supplemented in G. There exists a subgroup W of G, $G = UW = EU_{p'}W$, where $U_{p'}$ is the normal p-complement of U, $U_i = E_iU_{p'}$ and $U_iW = E_iU_{p'}W < G$ for every maximal subgroup E_i of E. Next, we consider subgroup UL. If there exists a minimal normal subgroup L of G such that $UL < G$, then $|E| < |P|$, $E \in Syl_p(UL)$ and E is $\mathcal{M}_p$-embedded in

UL. Since $N_{UL}(E)/C_{UL}(E) \cong (N_G(E) \cap UL)C_G(E)/C_G(E) \leq N_G(E)/C_G(E)$ is a p-group, UL is p-nilpotent by the choice of G. Then L is p-nilpotent. By (1), L is a p-group and so $O_p(G) \neq 1$. If $UL = G$ for every minimal normal subgroup of G, then G/L is p-nilpotent and $|E| = |P|$. Since the class of all p-nilpotent groups is a saturated formation, L is a unique minimal normal subgroup of G. We assert that $(U_iW)_G \neq 1$ for every U_i. If not, then there exists a $(U_jW)_G$ such that $(U_jW)_G = 1$. Hence G is isomorphic to a subgroup of the symmetric group S_p and $|P| = d = p$. Hence L is a p'-group, contradicts (1). Since $P \cap (\cap(U_iW)_G) = P \cap (\cap(P_iU_{p'}W)_G) = \cap P_i = \Phi(P)$, $\cap(U_iW)_G$ is p-nilpotent by Lemma 2.2. Hence L is p-nilpotent and L is a p-group by (1). Further, $|L| = p$ by Lemma 2.3.

(4) Final contradiction.

By (3) and hypothesis, every subgroup H/L of P/L with order $d/|L|$ is $\mathcal{M}_p$-embedded in G/L. Set $C_{G/L}(H/L) = A/L$. Then $|N_{G/L}(H/L)/C_{G/L}(H/L)| = |N_G(H)/L/A/L| = |N_G(H)/C_G(H)/A/C_G(H)|$ is a p-group, and so G/L is p-nilpotent. Since the class of all p-nilpotent groups is a saturated formation, L is a unique minimal normal subgroup of G and $L = O_p(G) \not\leq \Phi(G)$ by [[19], Theorem 1.8.17].

So G is p-soluble and $C_G(L) = L$ by (1) and [[19], Lemma 1.8.19]. Since $|L| < d$ and $|L| = p$ by (3), $G/C_G(L) = G/L$ is isomorphic to a subgroup of $Aut(L)$, $L = P$, a contradiction.

The final contradiction completes the proof.

4. Applications

The results in section 3 have many applications. By Lemma 2.8, 2.9, Theorem 3.1, 3.3, 3.4, we obtained the following corollaries.

Corollary 4.1. *Let G be a group and P be a Sylow p-subgroup of G where p is the smallest prime divisor of $|G|$. G is p-nilpotent if and only if every subgroup H of P with $|H| = d$ and $1 < d \leq |P|$ is $\mathcal{M}_p$-embedded in G.*

Corollary 4.2. *Let G be a group, $\pi(G)=\{p_1, p_2 = p, \ldots, p_n\}$, $p_1 < p_2 < \ldots < p_n$ and P be a Sylow p-subgroup of G. Suppose that $1 < d \leq |P|$. If every subgroup H of P with $|H| = d$ is $\mathcal{M}_p$-embedded in G, then G is p-supersoluble.*

Corollary 4.3. *Let G be a p-soluble group and P be a Sylow p-subgroup of $F_p(G)$ where p is a prime divisor of $|G|$. Suppose that $1 < d \leq |P|$. If every subgroup H of P with $|H| = d$ is $\mathcal{M}_p$-embedded in G, then G is p-supersoluble.*

Corollary 4.4. *Let G be a group. Suppose that $1 < d \leq |P|$. If every subgroup H of every Sylow subgroup P of G with $|H| = d$ is $\mathcal{M}_p$-embedded in G, then G is supersoluble.*

Corollary 4.5. *Let G be a group and P be a Sylow p-subgroup of G where p is an odd prime divisor of $|G|$. G is p-nilpotent if and only if every subgroup*

H of P with $|H| = d$ and $1 < d \leq |P|$ is $\mathcal{M}_p$-embedded in G and $N_G(H)$ is p-nilpotent.

Corollary 4.6. *Let G be a group. Suppose that $1 < d \leq |P|$. If every subgroup H of every Sylow subgroup P of odd order of G with $|H| = d$ is $\mathcal{M}_p$-embedded in G, then G is soluble.*

Corollary 4.7 ([13], Lemma 2.12)**.** *Let G be a group and P be a Sylow p-subgroup of G where p is the smallest prime divisor of $|G|$. G is p-nilpotent if and only if P has a subgroup D such that $1 < D \leq P$ and every subgroup H of P with $|H| = |D|$ is $\mathcal{M}$-supplemented in G.*

Corollary 4.8 ([19], Corollary 3.6)**.** *Let G be a group, $\pi(G)=\{p_1, p_2 = p, \ldots, p_n\}$, $p_1 < p_2 < \ldots < p_n$ and P be a Sylow p-subgroup of G. Suppose that P has a subgroup D such that $1 < D \leq P$. If every subgroup H of P with $|H| = |D|$ is $\mathcal{M}$-supplemented in G, then G is p-supersoluble.*

Corollary 4.9 ([15], Theorem 3.1)**.** *Let G be a p-soluble group and P be a Sylow p-subgroup of G where p is a prime divisor of $|G|$. Suppose that P is $\mathcal{M}$-supplemented in G, then G is p-supersoluble.*

Corollary 4.10 ([15], Theorem 3.4)**.** *Let G be a group. Suppose that every Sylow subgroup P of G is $\mathcal{M}$-supplemented in G, then G is soluble.*

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Some separation axioms on topological B-algebra and the quotient B-topological space

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Abstract. This paper provides some separation axioms on a topological B-algebra. In particular, a characterization of a discrete topological B-algebra is presented. Furthermore, this paper introduces the quotient B-topological space and presents some properties. Moreover, this paper gives some conditions which under a topological B-algebra (resp. quotient topological B-algebra), the quotient B-topological space (resp. B-topological space) have the topological properties T_0, T_1, and Hausdorff.

Keywords: B-topological space, topological B-algebra, quotient B-topological space, quotient topological B-algebra.

1. Introduction

In 1998, D.S. Lee and D.N. Ryu [7] defined a topological BCK-algebra and found some properties of this structure. On the following year, Y.B. Jun, X.L. Xin, and D.S. Lee [5] introduced the notion of topological BCI-algebras and gave a characterization in terms of neighborhoods. In 2002, J.Neggers and H.S. Kim [8] introduced and investigated B-algebras. In 2015, N. Kouhestani and S. Mehrshad [6] studied separation axioms and connected properties on topological quotient BCK-algebras. In 2019, an initial study on topological B-algebra was conducted by N.C. Gonzaga Jr. [4] where he investigated some properties of a topological B-algebra. Moreover, he characterized a topological B-algebra with respect to neighborhoods.

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2. Preliminaries

Definition 2.1 ([8]). *A B-algebra is a triple $(X, *, 0)$ where X is a nonempty set with a constant 0 and a binary operation "$*$" satisfying the following axioms for all x, y, z in X:*

$$\textit{(B1)}\ x * x = 0; \quad \textit{(B2)}\ x * 0 = x;; \quad \textit{(B3)}\ (x * y) * z = x * [z * (0 * y)].$$

Let us denote a B-algebra $(X, *, 0)$ simply as X.

Example 2.2 ([9]). Let $X = \{0, a, b, c, d, e\}$ be a set with the following Cayley table:

$*$	0	a	b	c	d	e
0	0	b	a	c	d	e
a	a	0	b	d	e	c
b	b	a	0	e	c	d
c	c	d	e	0	b	a
d	d	e	c	a	0	b
e	e	c	d	b	a	0

Then $(X, *, 0)$ is a B-algebra.

Definition 2.3 ([4]). *Let A and B be nonempty subsets of a B-algebra X. The product of A and B, denoted by $A * B$, is given by $A * B = \{a * b | a \in A, b \in B\}$.*

Lemma 2.4 ([8]). *Let $(X, *, 0)$ be a B-algebra. Then for any $x, y \in X$,*

(i) $x * y = 0$ *implies* $x = y$;

(ii) $0 * x = 0 * y$ *implies* $x = y$;

(iii) $0 * (0 * x) = x$.

Definition 2.5 ([10]). *Let X be a B-algebra. A nonempty subset N of X is called a subalgebra of X if $x * y \in N$ for any $x, y \in N$.*

The following remark follows directly from Definition 2.5 and axiom (B1) of a B-algebra.

Remark 2.6. Suppose X is a B-algebra and S a subalgebra of X. Then $0 \in S$.

Suppose X is a B-algebra and I a normal subalgebra of X. The relation "$\cong^I$" defined by $x \cong^I y$ if and only if $x * y, y * x \in I$ is a congruence relation on X for any $x, y \in X$. That is, $\cong^I$ is an equivalence relation and for each $a, b, x, y \in X$, if $x \cong^I y$ and $a \cong^I b$, then $a * x \cong^I b * y$. Let $I_x = \{y : y \cong^I x\}$ denote the equivalence class of x and $X/I = \{I_x : x \in X\}$. Then X/I is a B-algebra called the *quotient B-algebra* under the binary operation given by $I_x * I_y = I_{x*y}$ [3]. Define a mapping $\pi_I : X \to X/I$ by $\pi_I(x) = I_x$ for any $x \in X$. Then the mappping π_I is a B-epimorphism called the *canonical epimorphism* of B-algebras [9].

Definition 2.7 ([9]). Let X be a B-algebra. A nonempty subset S of X is said to be *normal* in X if for any $x * y, a * b \in S$, $(x * a) * (y * b) \in S$.

Theorem 2.8 ([10]). *Let N be a subalgebra of a B-algebra X. Then the following statements are equivalent:*

(i) N is a normal subalgebra;

(ii) if $x \in X$ and $y \in N$, then $x * (x * y) \in N$.

Definition 2.9 ([3]). Let X and Y be B-algebras. A mapping $\phi : X \to Y$ is called a *B-homomorphism* from X into Y if $\phi(x * y) = \phi(x) * \phi(y)$ for any $x, y \in X$. A B-homomorphism ϕ is called a *B-monomorphism, B-epimorphism,* or *B-isomorphism* if ϕ is one-to-one, onto, or a bijection, respectively.

Let X be a set. A *topology* (or topological structure) in X is a family τ of subsets of X that satisfies the following:

(i) Each union of members of τ is also a member of τ;

(ii) Each finite intersection of members of τ is also a member of τ; and

(iii) $\emptyset$ and X are members of τ.

A couple (X, τ) consisting of a set X and a topology τ in X is called a *topological space*. We also say "τ is the topology of the space X". The members of τ are called *open sets* of (X, τ). Let (X, τ_X) and (Y, τ_Y) be topological spaces. A map $f : X \to Y$ is called *continuous* if the inverse image of each open set in Y is open in X (that is, if f^{-1} maps τ_Y into τ_X) and f is called *open* if the image of each open set in X is open in Y (see, [1]).

Let (X, τ) be a topological space and $A \subset X$. By a *neighborhood of an element* x in X (denoted as $U(x)$) is meant any open set (that is, member of τ) containing x. The *interior* $Int(A)$ of A is the largest open set contained in A, that is, $\text{Int}(A) = \bigcup\{U | U \in \tau, U \subset A\}$. A point a is an interior point of A if $a \in \text{Int}(A)$, that is, there exists $U(a) \in \tau$ such that $U(a) \subset A$. A is open if and only if $\text{Int}\,(A) = A$. A set $Y \subset X$ is a *closed set* in X if its complement is open. A point $x \in X$ is *adherent* to Y if each neighborhood of x contains at least one point of Y. The set $\overline{Y} = \{x \in X | \forall U(x), U(x) \cap Y \neq \emptyset\}$ of all points in X adherent to Y is called the *closure* of Y (see, [1]).

Definition 2.10 ([1]). *Let $Y = (Y, \tau)$ be a space. Y is called a T_0-space if for each pair of distinct points, at least one has a neighborhood not containing the other. Y is called a T_1-space if for each pair of distinct points, each one has a neighborhood not containing the other. Y is Hausdorff (or T_2-space) if each two distinct points have nonintersecting neighborhoods, that is, if $p \neq q$, there exists $U(p), U(q)$ such that $U(p) \cap U(q) = \emptyset$.*

The next remark follows directly from Definition 2.10.

Remark 2.11. Every Hausdorff space is a T_1-space and every T_1-space is a T_0 space.

Lemma 2.12 ([1])**.** *Let Y be a Hausdorff space. Then $\bigcap_{i\in\mathcal{A}} U_i(p) = \{p\}$ for any $p \in Y$.*

Definition 2.13 ([1])**.** In the set X, Let $\mathcal{D} = \mathcal{P}(X)$. Then $\mathcal{D}$ is called the *discrete topology* where every set is an open set having basis of the form $\{\{x\}|x \in X\}$. Hence, (X, τ) is called a *discrete space*.

Remark 2.14. We note that a discrete space is Hausdorff while the converse does not necessarily hold. Consider the Euclidean 1-space (E^1, τ). An open set in E^1 is an open interval denoted as $B(x; r)$ where $B(x; r) = \{y : |y - x| < r\}$. (E^1, τ) is a Hausdorff space but is not a discrete space.

Definition 2.15 ([1])**.** *Let $\{Y_\alpha|\alpha \in \mathcal{A}\}$ be any family of topological spaces. For each $\alpha \in \mathcal{A}$, let τ_α be the topology for Y_α. The cartersian product topology in $\prod_\alpha Y_\alpha$ is that having for subbasis all sets $< U_\beta >= \rho_\beta^{-1}(U_\beta)$, where $\rho : \prod_\alpha Y_\alpha \to Y_\alpha$, U_β ranges over all members of τ_β and β over all elements of $\mathcal{A}$.*

Definition 2.16 ([4])**.** *Let X be a B-algebra. A topology τ furnished on X is called a B-topology on X. A B-topological space (X, τ) is called a topological B-algebra if τ is a B-topology on X and the binary operation $* : X \times X \to X$ is continuous, where $X \times X$ is furnished by the Cartesian product topology.*

Theorem 2.17 ([4])**.** *Let $X = (X, *, 0)$ be a B-algebra and τ a B-topology on the set X. Then (X, τ) is a topological B-algebra if and only if for all x, y in X and for every neighborhood W of $x * y$, there are neighborhoods U and V of x and y, respectively, such that $U * V \subseteq W$.*

3. Topological B-algebra

Example 3.1. Let $X = \{0, a, b, c\}$ and $Y = \{0, 1, 2\}$ with binary operations "$*_1$ and "$*_2$, respectively, be defined on the Cayley tables provided.

$*_1$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

$*_2$	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

(i) $(X; *_1, 0)$ is a B-algebra (see [2]). Let $\tau = \big\{X, \emptyset, \{0, c\}, \{a, b\}\big\}$. Then τ is a B-topology on X and by routine calculations, (X, τ) is a topological B-algebra.

(ii) $(Y; *_2, 0)$ is a B-algebra (see [8]). Let $\tau = \big\{\{0, 2\}, \{1, 2\}, \{2\}, Y, \emptyset\big\}$. Then τ is a B-topology on Y. Note that for $0 * 1 = 2 \in \{0, 2\}$, $\{0, 2\} * \{1, 2\} = \{0, 2\} * Y = Y * \{1, 2\} = Y * Y = Y \not\subseteq \{0, 2\}$. Hence, (Y, τ) is not a topological B-algebra.

The following results involve some separation axioms on a topological B-algebra. Moreover, the succeeding theorem is a characterization of a discrete topological B-algebra.

Theorem 3.2. *Let X be a topological B-algebra. Then X is discrete if and only if $\{0\}$ is an open set.*

Proof. If X is discrete, $\{0\}$ is clearly an open set. Suppose $\{0\}$ is an open set in a topological B-algebra X. Note that by (B1), $x * x = 0 \in \{0\}$ for all $x \in X$. By Theorem 2.17, there exists $U(x), V(x) \in \tau$ such that $U(x) * V(x) \subset \{0\}$. Let $W = U(x) \cap V(x) \in \tau$. Then $W * W \subset U(x) * V(x) \subset \{0\}$. Since $x \in W$, it follows that $W * \{x\} = \{0\}$. By Lemma 2.4 (i), $W = \{x\}$. □

The next theorem is a characterization of a topological B-algebra provided any B-topological space followed by a corollary under the condition that a B-topological space is discrete.

Theorem 3.3. *Suppose X is a B-topological space such that for any $x \in X$, $\{x\} \in \tau$. Then X is a topological B-algebra.*

Proof. Suppose X is a B-topological space such that for any $x \in X$, $\{x\} \in \tau$. Note that for any $x, y \in X$ and $U(x * y) \in \tau$, there exist $\{x\}, \{y\} \in \tau$ such that $\{x\} * \{y\} = \{x * y\} \subseteq U(x * y)$. Therefore, X is a topological B-algebra. □

Corollary 3.4. *If a B-topological space X is a discrete space, then X is a topological B-algebra.*

In any topological space, every discrete space is Hausdorff while the converse is not always true as discussed in Remark 2.14. However in a topological B-algebra, the converse holds. This is stated in the next corollary which follows from Remark 2.14, Theorem 3.2, and Lemma 2.12.

Corollary 3.5. *Suppose X is a topological B-algebra such that $\bigcap_{i \in \mathcal{A}} U_i(0)$ is an open set. Then X is Hausdorff if and only if it is discrete.*

Theorem 3.6. *Let X be a topological B-algebra. Then the following are equivalent:*

(*i*) X is Hausdorff;

(*ii*) X is T_1;

(*iii*) X is T_0.

Proof. In view of Remark 2.11, it remains to show that (iii) $\implies$ (i). Suppose X is a T_0-space and let $x \neq y$ for any $x, y \in X$. Then $x * y \neq 0$ or $y * x \neq 0$. Without loss of generality, consider $x * y \neq 0$. Since X is T_0, there exists $U(x * y)$ such that $0 \notin U(x * y)$. Since X is a topological B-algebra, there exists $V(x), V(y) \in \tau$ such that $V(x) * V(y) \subset U(x * y)$.

Claim: $V(x) \cap V(y) = \emptyset$. Assume on the contrary that $V(x) \cap V(y) \neq \emptyset$. Then there exists $z \in V(x) \cap V(y)$. By (B1), $0 = z * z \in V(x) \cap V(y) \subset U(x * y)$. This implies that $0 \in U(x * y)$ which is a contradiction. Hence, the claim holds. Therefore, X is Hausdorff. □

4. Quotient B-topological space

Henceforth, we shall assume that I is a normal subalgebra of a B-algebra X.

Definition 4.1. Let X be a B-algebra, I a subalgebra of X, and $\tilde{\tau}$ a topology on X/I. Then $\tilde{\tau}$ is called a *quotient B-topology* on X/I and the couple $(X/I, \tilde{\tau})$ is called a *quotient B-topological space*. If $(X/I, \tilde{\tau})$ is a topological B-algebra, we call the couple a *quotient topological B-algebra*.

The neighborhood of $I_x \in X/I$ shall be denoted by $\tilde{U}(I_x) \in \tilde{\tau}$ and a quotient B-topological space $(X/I, \tilde{\tau})$ shall be denoted as X/I unless otherwise specified.

Example 4.2. Let $X = \{0, a, b, c\}$ be a set with the following Cayley table:

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Then X is a B-algebra [2]. Let $I = \{0, a\}$. By routine calculations, I is a subalgebra of X. By Theorem 2.8, I is a normal subalgebra of X. Moreover, $X/I = \{I, I_b\}$ where $I = I_0 = I_a, I_b = I_c = \{c, b\}$. Let $\tilde{\tau} = \{X/I, \emptyset, \{I\}, \{I_b\}\}$. Then $\tilde{\tau}$ is a quotient B-topology on X/I and $(X/I, \tilde{\tau})$ is a quotient B-topological space. By Theorem 3.3, X/I is a quotient topological B-algebra.

Example 4.3. Consider the quotient B-topological space $X/I = \{I, I_b\}$ with $\tilde{\tau} = \{X/I, \emptyset, \{I\}, \{I_b\}\}$ from Example 4.2. Then, $\tilde{U}(I_0) = \tilde{U}(I_a) \in \{X/I, \{I\}\}$ and $\tilde{U}(I_b) = \tilde{U}(I_c) \in \{X/I, \{I_b\}\}$ are neighborhoods of I_0 and I_b, respectively.

Definition 4.4. Let X be a B-topological space and X/I a quotient B-topological space. We say that X satisfies *open condition* if for any subalgebra I, the canonical epimorphism π_I is an open map.

The next example shows that there exist a topological B-algebra that does not satisfy the open condition and π_I not continuous in which X/I is not a topological B-algebra.

Example 4.5. Consider the topological B-algebra $X = \{0, a, b, c\}$ with $\tau = \{X, \emptyset, \{0, c\}, \{a, b\}\}$ in Example 3.1. Let $I = I_0 = \{0\}$. Clearly, I is a normal subalgebra. Then $I_a = \{a\}, I_b = \{b\}, I_c = \{c\}$ so that $X/I = \{I, I_a, I_b, I_c\}$. Let $\tilde{\tau} = \{X/I, \emptyset, \{I, I_a, I_b\}, \{I_c\}\}$. Then $\tilde{\tau}$ is a quotient B-topology. Note that

there exists $\{I_c\} \in \tilde{\tau}$ such that $\pi_I^{-1}(\{I_c\}) = \{c\} \notin \tau$ which implies that π_I is not continuous and there exists $\{0, c\} \in \tau$ such that $\pi_I(\{0, c\}) = \{I_0, I_c\} \notin \tilde{\tau}$ which implies that π is not open. Now, for $I_a, I_b \in X/I$ and for $I_a * I_b = I_{a*b} = I_c \in \{I_c\} = \tilde{U}(I_c)$, $\tilde{U}(I_a) * \tilde{U}(I_b) = X/I \nsubseteq \tilde{U}(I_c)$ for all $\tilde{U}(I_a), \tilde{U}(I_b) \in \tilde{U}$. This implies that X/I is not a topological B-algebra.

However, if the canonical epimorphism π_I is open and continuous on a topological B-algebra, then the quotient B-topological space is a topological B-algebra as shown in the next result.

Theorem 4.6. *If X is a topological B-algebra that satisfies open condition with π_I continuous, the quotient B-topological space X/I is a topological B-algebra. In this case, we say that X/I is a quotient topological B-algebra.*

Proof. Suppose X is a topological B-algebra that satisfies open condition with π_I continuous. Let $I_x, I_y \in X/I$, $z = x * y$, and $\tilde{W}(I_z) \in \tilde{\tau}$. Let $\tilde{W} = \tilde{W}(I_z)$. Since $I_z \in \tilde{W}$ and π_I is surjective, there exists $z \in X$ such that $\pi_I(z) = I_z \in \tilde{W}$. This implies that $x * y = z \in \pi_I^{-1}(\tilde{W})$. Since $\tilde{W} \in \tilde{\tau}$ and π_I^{-1} is continuous, $\pi_I^{-1}(\tilde{W}) \in \tau$. That is, $\pi_I^{-1}(\tilde{W})$ is an open set containing $x * y$. Since X is a topological B-algebra, there exist $U(x), U(y) \in \tau$ such that $U(x) * U(y) \subseteq \pi_I^{-1}(\tilde{W})$ by Theorem 2.17. Since $x \in U(x)$ and π_I is open, $\pi_I(x) \in \pi_I\big(U(x)\big) \in \tilde{\tau}$ or $I_x \in \pi_I\big(U(x)\big) \in \tilde{\tau}$. Similarly, $I_y \in \pi_I\big(U(y)\big) \in \tilde{\tau}$. That is, $\pi_I\big(U(x)\big)$ and $\pi_I\big(U(y)\big)$ are open sets containing I_x and I_y, respectively.
Claim: $\pi_I\big(U(x)\big) * \pi_I\big(U(y)\big) = \pi_I\big(U(x) * U(y)\big)$.

Since π_I is a homomorphism, $\pi_I\big(U(x)\big) * \pi_I\big(U(y)\big) = \{\pi_I(a) | a \in \pi_I\big(U(x)\big)\} * \{\pi_I(b) | b \in \pi_I\big(U(y)\big)\} = \{\pi_I(a) * \pi_I(b) | a \in U(x), b \in U(y)\} = \{\pi_I(a * b) | a \in U(x), b \in U(y)\} = \pi_I\big(U(x) * U(y)\big)$. This proves the claim.

By the claim and since $U(x) * U(y) \subseteq \pi_I^{-1}(\tilde{W})$, $\pi_I\big(U(x)\big) * \pi_I\big(U(y)\big) = \pi_I\big(U(x) * U(y)\big) \subseteq \pi_I\big(\pi_I^{-1}(\tilde{W})\big) = \tilde{W}$. Therefore, $(X/I, \tilde{\tau})$ is a topological B-algebra. □

The next example shows that there exists a quotient topological B-algebra X/I where X does not satisfy open condition and π_I not continuous in which X is not a topological B-algebra.

Example 4.7. Consider the quotient topological B-algebra $X/I = \{I, I_b\}$ with $\tilde{\tau} = \{X/I, \emptyset, \{I\}, \{I_b\}\}$ where $X = \{0, a, b, c\}$ from Example 4.2. Let $\tau = \{X, \emptyset, \{0, a, b\}, \{c\}\}$. Then τ is a B-topology on X. Note that there exists $\{0, a, b\} \in \tau$ such that $\pi_I(\{0, a, b\}) = \{I_a, I_b\} \notin \tilde{\tau}$ which implies that π_I is not open and there exists $\{I_b\} \in \tilde{\tau}$ such that $\pi_I^{-1}(\{I_b\}) = \{b\} \notin \tau$ which implies that π_I is not continuous. Now, for $b * c = a \in \{0, a, b\}$, $\{0, a, b\} * \{c\} = \{c, b, a\} \nsubseteq \{0, a, b\}$, $\{0, a, b, \} * X = X * \{c\} = X * X = X \nsubseteq \{0, a, b\}$. Hence, $U(b) * U(c) \nsubseteq \{0, a, b\}$ for all $U(b), U(c) \in \tau$. Therefore, X is not a topological B-algebra.

However, provided a B-topological space satisfying open condition with π_I continuous and X/I a topological B-algebra for any subalgebra I of X, then X is also a topological B-algebra which is presented in the next theorem.

Theorem 4.8. *Suppose X is a B-topological space satisfying open condition with π_I continuous. If X/I is a topological B-algebra, then X is also a topological B-algebra.*

Proof. Suppose X is a B-topological space satisfying open condition with π_I continuous and X/I a topological B-algebra. Let $x, y \in X$ and $U(x*y) \in \tau$. Since $x*y \in U(x*y)$ and π_I is open, $I_x * I_y = I_{x*y} = \pi_I(x*y) \in \pi_I\big(U(x*y)\big) \in \tilde{\tau}$. This imples that $\pi_I\big(U(x*y)\big)$ is an open set containing $I_x * I_y$. By Theorem 2.17, there exist $\tilde{U}(I_x), \tilde{U}(I_y) \in \tilde{\tau}$ such that $\tilde{U}(I_x) * \tilde{U}(I_y) \subseteq \pi_I\big(U(x*y)\big)$. Since $I_x \in \tilde{U}(I_x)$ and π_I continuous, $x = \pi_I^{-1}(I_x) \in \pi_I^{-1}\big(\tilde{U}(I_x)\big) \in \tau$. Hence, $\pi_I^{-1}\big(\tilde{U}(I_x)\big)$ is an open set containing x. Similarly, $\pi_I^{-1}\big(\tilde{U}(I_y)\big)$ is an open set containing y.
Claim: $\pi_I^{-1}\big(\tilde{U}(I_x)\big) * \pi_I^{-1}\big(\tilde{U}(I_y)\big) = \pi_I^{-1}\big(\tilde{U}(I_x) * \tilde{U}(I_y)\big)$.

$\pi_I^{-1}\big(\tilde{U}(I_x)\big) * \pi_I^{-1}\big(\tilde{U}(I_y)\big) = \{a|\pi_I(a) \in \tilde{U}(I_x)\} * \{b|\pi_I(b) \in \tilde{U}(I_y)\} = \{a * b|\pi_I(a) \in \tilde{U}(I_x), \pi_I(b) \in \tilde{U}(I_y)\} = \pi_I^{-1}\big(\tilde{U}(I_x) * \tilde{U}(I_y)\big)$. This proves the claim. Hence, $\pi_I^{-1}\big(\tilde{U}(I_x)\big) * \pi_I^{-1}\big(\tilde{U}(I_y)\big) = \pi_I^{-1}\big(\tilde{U}(I_x) * \tilde{U}(I_y)\big) \subseteq \pi_I^{-1}(\pi_I\big(U(x*y)\big)) \subseteq U(x*y)$. Therefore, X is a topological B-algebra. □

Theorem 4.9. *Suppose X is a B-topological space and I a subalgebra of X. Define $\tilde{\tau}' = \{O \subseteq X/I|\pi^{-1}(O) \in \tau\}$ where π is the canonical epimorphism. Then $\tilde{\tau}'$ is a quotient B-topology on X/I.*

Proof. Suppose X is a B-topological space and I a subalgebra of X. Clearly, X/I and $\emptyset \in \tilde{\tau}'$. Let $O_1, O_2 \in \tilde{\tau}'$. Then $\pi^{-1}(O_1), \pi^{-1}(O_2) \in \tau$. Hence, $\pi^{-1}(O_1) \cap \pi^{-1}(O_2) \in \tau$.
Claim 1: $\pi^{-1}(O_1 \cap O_2) = \pi^{-1}(O_1) \cap \pi^{-1}(O_2)$.

Suppose $x \in \pi^{-1}(O_1 \cap O_2)$. Then $\pi(x) \in O_1 \cap O_2$. That is, $\pi(x) \in O_1$ and $\pi(x) \in O_2$. Hence, $x \in \pi^{-1}(O_1)$ and $x \in \pi^{-1}(O_2)$. That is, $x \in \pi^{-1}(O_1) \cap \pi^{-1}(O_2)$ implying that $\pi^{-1}(O_1 \cap O_2) \subseteq \pi^{-1}(O_1) \cap \pi^{-1}(O_2)$. The converse follows accordingly. This proves Claim 1.

By Claim 1, $\pi^{-1}(O_1 \cap O_2) \in \tau$. Hence, $O_1 \cap O_2 \in \tilde{\tau}'$. Next, let $\{O_i\}_{i \in \mathcal{A}}$ be an arbitrary family of sets in $\tilde{\tau}'$. Then for all $i \in \mathcal{A}$, $\pi^{-1}(O_i) \in \tau$. Hence, $\bigcup_{i \in \mathcal{A}} \pi^{-1}(O_i) \in \tau$.
Claim 2: $\pi^{-1}(\bigcup_{i \in \mathcal{A}} O_i) = \bigcap_{i \in \mathcal{A}} \pi^{-1}(O_i)$.

Suppose $x \in \pi^{-1}(\bigcup_{i \in \mathcal{A}} O_i)$. Then $\pi(x) \in \bigcup_{i \in \mathcal{A}} O_i$. Hence, $\pi(x) \in O_j$ for some $j \in \mathcal{A}$. This implies that $x \in \pi^{-1}(O_j)$ for some $j \in \mathcal{A}$. That is, $x \in \bigcup_{i \in \mathcal{A}} \pi^{-1}(O_i)$ implying that $\pi^{-1}(\bigcup_{i \in \mathcal{A}} O_i) \subseteq \bigcup_{i \in \mathcal{A}} \pi^{-1}(O_i)$. The converse follows accordingly. This proves Claim 2.

By Claim 2, $\pi^{-1}(\bigcup_{i \in \mathcal{A}} O_i) \in \tau$. That is, $\bigcup_{i \in \mathcal{A}} O_i \in \tilde{\tau}'$. Therefore, $\tilde{\tau}'$ is a quotient B-topology on X/I. □

Theorem 4.10. *Suppose X/I is a quotient B-topological space. Define $\tau' = \{O \subseteq X | \pi(O) \in \tilde{\tau}\}$ where π is the canonical epimorphism. Then τ' is a B-topology on X.*

Proof. Suppose X/I is a quotient B-topological space. Clearly, X and $\emptyset \in \tau'$. Let $O_1, O_2 \in \tau'$. Then $\pi(O_1), \pi(O_2) \in \tilde{\tau}$.
Claim: $\pi(O_1 \cap O_2) = \pi(O_1) \cap \pi(O_2)$.

$\pi(O_1 \cap O_2) = \{\pi(x) | x \in O_1 \cap O_2\} = \{\pi(x) | x \in O_1 \text{ and } O_2\} = \{\pi(x) | x \in O_1\} \cap \{\pi(x) | x \in O_2\} = \pi(O_1) \cap \pi(O_2)$. This proves the claim. By the claim, $\pi(O_1 \cap O_2) = \pi(O_1) \cap \pi(O_2) \in \tilde{\tau}$. This implies that $O_1 \cap O_2 \in \tau'$. Now, suppose $O_i \in \tau'$ for all $i \in \mathcal{A}$. Then $\pi(O_i) \in \tilde{\tau}$ for each $i \in \mathcal{A}$.
Claim: $\pi(\bigcup_{i\in\mathcal{A}} O_i) = \bigcup_{i\in\mathcal{A}} \pi(O_i)$.

Suppose $z \in \pi(\bigcup_{i\in\mathcal{A}} O_i)$. Then $z = \pi(x)$ for some $x \in \bigcup_{i\in\mathcal{A}} O_i$. Then $x \in O_j$ for some $j \in \mathcal{A}$. Consequently, $z \in \pi(O_j)$ which implies that $z \in \bigcup_{i\in\mathcal{A}} \pi(O_i)$. The converse follows accordingly. This proves the claim. By the claim, $\pi(\bigcup_{i\in\mathcal{A}} O_i) = \bigcup_{i\in\mathcal{A}} \pi(O_i) \in \tilde{\tau}$. Hence, $\bigcup_{i\in\mathcal{A}} O_i \in \tau'$. Therefore, τ' is a B-topology on X. □

Note that in Examples 4.5 and 4.7, the canonical epimorphism π_I is not continuous and X does not satisfy open condition as shown in the examples. However, the next corollaries state that π_I is continuous with respect to $\tilde{\tau}'$ and X satisfies open condition with respect to τ'. These corollaries follow directly from Theorems 4.9 and 4.10.

Corollary 4.11. *Suppose X is a B-topological space and X/I a quotient B-topological space with $\tilde{\tau}'$ as the quotient B-topology. Then the canonical epimorphism π_I is a continuous map.*

Corollary 4.12. *Suppose X is a B-topological space with τ' as the B-topology and X/I is a quotient B-topological space. Then, X satisfies open condition.*

Proposition 4.13. *Let X be a B-algebra. Then for any $S \subseteq X$ and I a subalgebra of X, $\pi_I^{-1}\big(\pi_I(S)\big) = \bigcup_{x\in S} I_x$ where π_I is the canonical epimorphism.*

Proof. Suppose $y \in \pi_I^{-1}\big(\pi_I(S)\big)$. Then $I_y = \pi_I(y) \in \pi_I(S)$. Since π_I is an epimorphism, there exists an $x \in S$ such that $\pi_I(x) = I_y$. Hence, $I_x = I_y$ or $x \cong^I y$. This implies that $y \in I_x$ for some $x \in S$. It follows that $y \in \bigcup_{x\in S} I_x$ so that $\pi_I^{-1}\big(\pi_I(S)\big) \subseteq \bigcup_{x\in S} I_x$. Conversely, suppose $y \in \bigcup_{x\in S} I_x$. Then $y \in I_x$ for some $x \in S$. This implies that $y \cong^I x$ or $I_y = I_x$. Hence, $\pi_I(y) = \pi_I(x)$. Since $\pi_I(x) \in \pi_I(S)$, it follows that $\pi_I(y) \in \pi_I(S)$ implying that $y \in \pi_I^{-1}\big(\pi_I(S)\big)$. That is, $\bigcup_{x\in S} I_x \subseteq \pi_I^{-1}\big(\pi_I(S)\big)$. Therefore, $\pi_I^{-1}\big(\pi_I(S)\big) = \bigcup_{x\in S} I_x$. □

Note that in Example 4.5, the canonical epimorphism π_I is not open with respect to $\tilde{\tau}$ and is not continuous with respect to τ as indicated. However, the next propositions show that π_I is open with respect to $\tilde{\tau}'$ and π_I is continuous with respect to τ'.

Proposition 4.14. *Suppose X is a B-topological space and X/I is a quotient B-topological space with $\tilde{\tau}'$ as the quotient B-topology. If $I_x \in \tilde{\tau}'$ for any $x \in X$, then X satisfies open condition.*

Proof. Suppose X is a B-topological space and X/I is a quotient B-topological space with $\tilde{\tau}'$ as the quotient B-topology. Note that for any $O \in \tau$ and by Proposition 4.13, $\pi_I^{-1}\big(\pi_I(O)\big) = \bigcup_{x\in O} I_x \in \tau$. By definition of $\tilde{\tau}'$, it follows that $\pi_I(O) \in \tilde{\tau}'$. □

Proposition 4.15. *Suppose X is a B-topological space with τ' as the B-topology and let X/I be a quotient B-topological space. Then π_I is continuous.*

Proof. Suppose X is a B-topological space with τ' as the B-topology and let X/I be a quotient B-topological space. Let $O \in \tilde{\tau}$. Since π_I is surjective, $\pi_I\big(\pi_I^{-1}(O)\big) = O \in \tilde{\tau}$. This implies that $\pi_I^{-1}(O) \in \tau'$. Hence, π_I is continuous. □

In view of Corollaries 4.11 and 4.12, Proposition 4.14 and 4.15, the next corollaries follow from Theorems 4.6, 4.8 and 3.6, and the succeeding corollary with respect to $\tilde{\tau}'$ and τ'. Specifically, Corollary 4.16 illustrates that the relationship between a topological B-algebra and a quotient topological B-algebra is that they are equivalent carrying with them the topologies (B-topology and quotient B-topology) τ' and $\tilde{\tau}'$, respectively. Also, Corollaries 4.17 and 4.18 states that every topological B-algebra and quotient topological B-algebra is a Hausdorff (equivalently, T_1, and T_0) space.

Corollary 4.16. *Suppose (X, τ') is a B-topological space and $(X/I, \tilde{\tau}')$ is a quotient B-topological space. Then X is a topological B-algebra if and only if X/I is a topological B-algebra.*

Corollary 4.17. *Suppose (X, τ') is a topological B-algebra. Then $(X/I, \tilde{\tau}')$ is a Hausdorff (equivalently, T_1, and T_0) space.*

Corollary 4.18. *Suppose $(X/I, \tilde{\tau}')$ is a topological B-algebra. Then (X, τ') is a Hausdorff (equivalently, T_1, and T_0) space.*

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Erratum to: On some (p,q)-φ relative Gol'dberg type and (p,q)-φ relative Gol'dberg weak type based growth properties of entire functions of several complex variables

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The following corrections are needed in [1]:

1) At page 405, in line 7, read the heading "**Definition 2** ([4])**.**" instead of "**Definition 2**"

2) At page 405, in line 14-15, read ".....n-complex variables, one may see [2, 3]." instead of ".....n-complex variables, one may see [3]."

3) In consequence with Definition 2 already mentioned at page 405, one may define (p,q)-φ Gol'dberg type $\sigma_D^{(p,q)}(f,\varphi)$, (p,q)-φ Gol'dberg lower type $\overline{\sigma}_D^{(p,q)}(f,\varphi)$, (p,q)-φ Gol'dberg weak type $\tau_D^{(p,q)}(f,\varphi)$ and the growth indicator $\overline{\tau}_D^{(p,q)}(f,\varphi)$ as:

$$\begin{matrix}\sigma_D^{(p,q)}(f,\varphi)\\ \overline{\sigma}_D^{(p,q)}(f,\varphi)\end{matrix} = \lim_{R\to+\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log^{[p-1]}(M_{f,D}(R))}{\left(\log^{[q-1]}\varphi(R)\right)^{\rho^{(p,q)}(f,\varphi)}}, \text{where } 0<\rho^{(p,q)}(f,\varphi)<\infty$$

and

$$\begin{matrix}\overline{\tau}_D^{(p,q)}(f,\varphi)\\ \tau_D^{(p,q)}(f,\varphi)\end{matrix} = \lim_{R\to+\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log^{[p-1]}(M_{f,D}(R))}{(\log^{[q-1]}\varphi(R))^{\lambda^{(p,q)}(f,\varphi)}}, \text{ where } 0<\lambda^{(p,q)}(f,\varphi)<\infty.$$

In the above definitions if $\varphi(R)=R$, the above definitions reduce to the definitions of (p,q)-th Gol'dberg type $\sigma_D^{(p,q)}(f)$, (p,q)-th Gol'dberg lower type

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$\overline{\sigma}_D^{(p,q)}(f)$, (p,q)-th Gol'dberg weak type $\tau_D^{(p,q)}(f)$ and another growth indicator $\overline{\tau}_D^{(p,q)}(f)$ which are as follows:

$$\begin{matrix}\sigma_D^{(p,q)}(f) \\ \overline{\sigma}_D^{(p,q)}(f)\end{matrix} = \lim_{R\to+\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log^{[p-1]}\left(M_{f,D}(R)\right)}{\left(\log^{[q-1]} R\right)^{\rho^{(p,q)}(f)}}, \text{where } 0<\rho^{(p,q)}(f)<\infty$$

and

$$\begin{matrix}\overline{\tau}_D^{(p,q)}(f) \\ \tau_D^{(p,q)}(f)\end{matrix} = \lim_{R\to+\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log^{[p-1]}\left(M_{f,D}(R)\right)}{\left(\log^{[q-1]} R\right)^{\lambda^{(p,q)}(f)}}, \text{ where } 0<\lambda^{(p,q)}(f)<\infty.$$

4) At page 406, in line 6, read "In [3], Biswas has further introduced...." instead of "In [4], Biswas has further introduced...."

5) In Definition 6, Definition 7 and Definition 8, read ".....with index-pairs (m,q)-φ and (m,p) respectively,....." instead of ".....with index-pairs (m,q) and (m,p) respectively,....."

6) Read the last paragraph of the section "Introduction, definitions and notations" as "Various authors {cf. [1] to [13]} have been investigating growth properties of entire functions of several complex variables for a long time, using several growth indicators, such as Gol'dberg order, (p,q)-th Gol'dberg order, relative Gol'dberg order etc. as devices to understand their growth behaviour. In this paper we wish to study some growth properties based upon the notion of (p,q)-φ relative Gol'dberg type, (p,q)-φ relative Gol'dberg weak type and the other growth indicators of entire functions of several complex variables." instead of "In this paper we wish to study some growth properties based upon the notion of (p,q)-φ relative Gol'dberg type, (p,q)-φ relative Gol'dberg weak type and the other growth indicators of entire functions of several complex variables."

7) At page 408, in Lemma 1, read "....regular (m,p)-th Gol'dberg growth and with non zero finite (m,p)-th Gol'dberg order. Also let $f(z)$ be another entire function of n complex variables with index-pair (m,q)-φ." instead of "....regular (m,p)-φ growth and with non zero finite (m,p)-φ Gol'dberg order. Also let $f(z)$ be another entire function of n complex variables with index-pair (m,q) ."

References

[1] T. Biswas and R. Biswas, *On some (p,q)-φ relative Gol'dberg type and (p,q)-φ relative Gol'dberg weak type based growth properties of entire functions of several complex variables*, Ital. J. Pure Appl. Math., 44 (2020), 403-414.

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(p,q)-order and (p,q)- lower order of entire functions of several complex variables on the basis of central index

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Abstract. In this paper we discuss about the growth rates of the central indices of composition of entire functions of several complex variables with their corresponding left or right factor.
Keywords: entire function, central index, (p,q)-order and (p,q)-lower order.

1. Introduction, definitions and notations

We denote complex n-space by $\mathbb{C}^n$ and indicate its elements (points):

$$(z_1, z_2, \ldots, z_n), (|z_1|, |z_2|, \ldots, |z_n|), (r_1, r_2, \ldots, r_n), (k_1, k_2, \ldots, k_n) \text{ etc.}$$

by their corresponding symbols $z, |z|, r, k$ etc. Throughout $\Omega = \Omega_n$ stands for a nonempty open complete n-circular region in $\mathbb{C}^n$(see §3, 3 of [1]) with centre at $(0, 0, \ldots, 0)$, the zero element of $\mathbb{C}^n$.

We write

$$|\Omega| = \{r : r = |z| \text{ for some } z \in \Omega\}$$

and

$$\Omega^+ = \{r : r \in |\Omega|, \text{ no } r_j = 0\}$$

and regard these as subsets of the n-dimensional Euclidean space $\mathbb{C}^n$. For any $r, s \in |\Omega|$ we say that

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(i) $r \leq s$ or $s \geq r$, if and only if $r_j \leq s_j$ for $1 \leq j \leq n$

(ii) $r < s$ or $s > r$, if and only if $r \leq s$ but r is not equal to s and

(iii) $r << s$ or $s >> r$, if and only if $r_j << s_j$ for $1 \leq j \leq n$.

A function $f(z)$, $z \in \mathbb{C}^n$ is said to be analytic at a point $\xi \in \mathbb{C}^n$ if it can be extended in some neighborhood of ξ as an absolutely convergent power series. If we assume $\xi = (0, 0, \ldots, 0)$, then $f(z)$ has representation (see [3] and [5])

$$f(z) = \sum_{k=(0,0,\ldots,0)}^{\infty} a_{k_1,k_2,\ldots,k_n} \; z_1^{k_1} \; z_2^{k_2} \ldots z_n^{k_n} = \sum_{|k|=0}^{\infty} a_k \; z^k,$$

where $k = (k_1, k_2, \ldots, k_n)$ belongs to $\mathcal{N} = \{k : k \in C^n$, each k_j is rational integer$\}$ and $|k| = k_1 + k_2 + \ldots + k_n$.

Now, corresponding to an entire function $f(z)$, we define the functions (mappings): the maximum term

$$\mu(r) = \mu(r, f),$$

the maximum modulus

$$M(r) = M(r, f)$$

and the central index

$$\nu(r) = \nu(r, f) = (\nu_1(r, f), \nu_2(r, f), \ldots, \nu_n(r, f))$$

on $|\Omega|$ by (see, [3] and [4])

$$\mu(r) = \mu(r, f) = \max_{k \in \mathcal{N}} \{|a_k| \, r^k\},$$
$$M(r) = M(r, f) = \max_{|z|=r} \{|f(z)|\}$$

and

$$\nu_j(r) = \nu_j(r, f) = \begin{cases} \max[k_j : |a_k| r^k = \mu(r), & \text{if } \mu(r) > 0] \\ 0, & \text{if } \mu(r) = 0, \text{ for } 1 \leq j \leq n. \end{cases}$$

Also, the central index $\nu(r, f)$ for which maximum term is achieved

$$|\nu(r, f)| = \nu_1(r, f) + \nu_2(r, f) + \ldots + \nu_n(r, f).$$

We say that a mapping f with domain D in a euclidean space and also with range in a euclidean space is increasing (in D), if and only if $f(r) \leq f(s)$ for any $r, s \in D$ such that $r \leq s$. Then, in view of Hartogs theorem and maximum principal ([1], $p.21, p.51$), $M(r, f) = M(r_1, r_2, \ldots, r_n, f)$ is an increasing function of $r_1, r_2, \ldots, r_n$.

Krishna, J.G. ([3], Theorem 2.6) proved that $\mu(r)$ is increasing and continuous in $|\Omega|$. In the same paper Krishna, J.G. ([3], Corollary 2.9) has also proved that $\nu_j(r)$ is increasing and right continuous in j-th variable for $1 \leq j \leq n$. The following definitions are well known:

Definition 1 ([1], p.339)**.** *The order ρ_f and lower order λ_f of an entire function $f(z) = f(z_1, z_2, ..., z_n)$ are defined by*

$$\rho_f = \limsup_{r_1,r_2,...,r_n\to\infty} \frac{\log^{[2]} M(r_1, r_2, \ldots, r_n, f)}{\log(r_1 r_2 \ldots r_n)}$$

and

$$\lambda_f = \liminf_{r_1,r_2,...,r_n\to\infty} \frac{\log^{[2]} M(r_1, r_2, \ldots, r_n, f)}{\log(r_1 r_2 \ldots r_n)},$$

where

$$\log^{[k]} x = \log\left(\log^{[k-1]} x\right) \text{ for } k = 1, 2, 3, ... \text{and } \log^{[0]} x = x.$$

However, an entire function of n-complex variables, for which order and lower order are the same is said to be of regular growth. The function $\exp(z_1 z_2 \ldots z_n)$ is an example of regular growth of entire function of n-complex variables. Further, the functions which are not of regular growth is said to be of irregular growth.

Similarly, one can define the hyper order (hyper lower order), generalised order (generalised lower order) of entire function of n-complex variables as follows.

Definition 2. *The hyper order $\overline{\rho}_f$ and the hyper lower order $\overline{\lambda}_f$ of an entire function f of n-complex variables are defined by*

$$\overline{\rho}_f = \limsup_{r_1,r_2,...,r_n\to\infty} \frac{\log^{[3]} M(r_1, r_2, \ldots, r_n, f)}{\log(r_1 r_2 \ldots r_n)}$$

and

$$\overline{\lambda}_f = \liminf_{r_1,r_2,...,r_n\to\infty} \frac{\log^{[3]} M(r_1, r_2, \ldots, r_n, f)}{\log(r_1 r_2 \ldots r_n)}.$$

Definition 3 ([8])**.** *Let l be an integer ≥ 1. The generalised order $\rho_f^{[l]}$ and the generalised lower order $\lambda_f^{[l]}$ of an entire function f of n-complex variables are defined as follows*

$$\rho_f^{[l]} = \limsup_{r_1,r_2,...,r_n\to\infty} \frac{\log^{[l+1]} M(r_1, r_2, \ldots, r_n, f)}{\log(r_1 r_2 \ldots r_n)}$$

and

$$\lambda_f^{[l]} = \liminf_{r_1,r_2,...,r_n\to\infty} \frac{\log^{[l+1]} M(r_1, r_2, \ldots, r_n, f)}{\log(r_1 r_2 \ldots r_n)}.$$

Note that, for $l = 1$, we have the usual order (lower order) and for $l = 2$, the hyper order (hyper lower order).

Now, in the line of Juneja, Kapoor and Bajpai [2], (p,q)-order and (p,q)-lower order of an entire function f of n-complex variables are respectively defined as follows:

$$\rho_f(p,q) = \limsup_{r_1,r_2,\ldots,r_n\to\infty} \frac{\log^{[p+1]} M(r_1,r_2,\ldots,r_n,f)}{\log^{[q]}(r_1r_2\ldots r_n)} \tag{1}$$

and

$$\lambda_f(p,q) = \liminf_{r_1,r_2,\ldots,r_n\to\infty} \frac{\log^{[p+1]} M(r_1,r_2,\ldots,r_n,f)}{\log^{[q]}(r_1r_2\ldots r_n)}, \tag{2}$$

where p,q are any two positive integers with $p \geq q$. In paticular, if we consider $q=1$, then the above definition is reduced to Definition 3. Further, an entire function f of n-complex variables is said to be of regular (p,q) growth if its (p,q)-order concides with its (p,q)-lower order, otherwise f is said to be of irregular (p,q) growth.

Recently, many authors (see [5], [6] and [7]) investigated the growth of entire functions of several complex variables in terms of their central index.

In this paper we would like to establish (p,q)-order and (p,q)-lower order of an entire function of several complex variables in term of its central index. The main purpose of this paper is to investigate the growth estimates of the central indices of composite entire functions of several complex variables as compared to their left or right factor on the basis of (p,q)-order ((p,q) -lower order) , where p,q are any two positive integers with $p \geq q$.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 1 ([3]). *Let $p,r \in |\Omega|$ and let $\mu(p)$ and $\mu(r)$ be both positive. Then, the line integral,*

$$I = \int_p^r \sum_{j=1}^{n} \frac{\nu_j(x)}{x_j} dx_j$$

taken over any connected polygon in $|\Omega|$ with sides parallel to the axes and from p to r,

(i) exists,

(ii) is independent of the polygon and

(iii) is such that $\log \mu(r) = \log \mu(p) + I$.

Lemma 2 ([3]). *Let $r \in |\Omega|$. Let $p \in |\mathbb{C}^n|$ and be such that $p >> (1,1,\ldots,1)$, while $pr = (p_1r_1, p_2r_2, \ldots, p_nr_n)$ still $\in |\Omega|$.*

Let

$$N_j = \max_{r\le t\le pr} \nu_j(t), \quad \text{for } 1 \le j \le n.$$

Then

(i) $\mu(r) \le M(r) \le \mu(r) \prod_{j=1}^{n} \left[N_j + \frac{p_j}{p_j - 1}\right]$;

(ii) $\mu(r) = M(r)$, if and only if the series $\sum_{|k|=0}^{\infty} a_k r^k$ has at most one non vanishing term;

(iii) the last relation in (i) is an equality if and only if $\mu(r) = 0$.

3. Theorems

In this section we present the main results of the paper.

Theorem 1. *Let $f(z)$ be an entire function of n-complex variables with (p,q)-order $\rho_f(p,q)$ and (p,q)-lower order $\lambda_f(p,q)$, where p,q are positive integers with $p \ge q$. If $\nu(r,f) = \nu(r_1, r_2, \ldots, r_n, f)$ be the central index of f, then*

$$\limsup_{r_1,r_2,\ldots,r_n\to\infty} \frac{\log^{[p]} |\nu(r_1, r_2, \ldots, r_n, f)|}{\log^{[q]}(r_1 r_2 \ldots r_n)} = \rho_f(p,q) \tag{3}$$

and

$$\liminf_{r_1,r_2,\ldots,r_n\to\infty} \frac{\log^{[p]} |\nu(r_1, r_2, \ldots, r_n, f)|}{\log^{[q]}(r_1 r_2 \ldots r_n)} = \lambda_f(p,q). \tag{4}$$

Proof. Setting

$$f(z) = \sum_{k=(0,0,\ldots,0)}^{\infty} a_{k_1,k_2,\ldots,k_n}\, z_1^{k_1}\, z_2^{k_2} \ldots z_n^{k_n} = \sum_{|k|=0}^{\infty} a_k\, z^k. \qquad \square$$

By Lemma 1, we see that the maximum term $\mu(r)$ of f satisfies

$$\log \mu(r) = \log \mu(p) + \int_p^r \sum_{j=1}^{n} \frac{\nu_j(x)}{x_j} dx_j. \tag{5}$$

Since Krishna, J.G. ([3], Corollary 2.9) proved that $\nu_j(r)$ is increasing and right continuous in j-th variable for $1 \le j \le n$. Therefore, for any $p, r \in |\Omega|$ such that $\mu(r) > 0$ and $p >> (1,1,\ldots,1)$, we get for $1 \le j \le n$,

$$\nu_j(r) \le \frac{1}{\log p_j} \int_p^r \nu_j(r_1, \ldots, r_{j-1}, \ldots, r_n) \frac{dx_j}{x_j}. \tag{6}$$

From (5) and (6) we get

$$\log \mu(r) \ge \log \mu(p) + \sum_{j=1}^{n} \nu_j(r) \log p_j. \tag{7}$$

By Lemma 2, we have

$$\mu(r,f) \leq M(r,f). \tag{8}$$

It follows from (7) and (8) that

$$\sum_{j=1}^{n} \nu_j(r) \log p_j \leq \log M(r,f) + C_1. \tag{9}$$

As $p >> (1,1,\ldots,1)$ i.e., $p = (p_1,p_2,\ldots,p_n) >> (1,1,\ldots,1)$, choosing $p_j = 2$ for $1 \leq j \leq n$, we get

$$\begin{aligned}
&\sum_{j=1}^{n} \nu_j(r) \log 2 \leq \log M(r,f) + C_1 \\
&\Rightarrow |\nu(r,f)| \log 2 \leq \log M(r,f) + C_1 \\
&\Rightarrow \log^{[p]} |\nu(r,f)| + \log^{[p+1]} 2 \leq \log^{[p+1]} M(r,f) + C_2,
\end{aligned}$$

where $C_j(>0)$ $(j=1,2)$ are suitable constants. By this and (1), we have

$$\limsup_{r_1,r_2,\ldots,r_n \to \infty} \frac{\log^{[p]} |\nu(r_1,r_2,\ldots,r_n,f)|}{\log^{[q]}(r_1 r_2 \ldots r_n)} \leq \rho_f(p,q). \tag{10}$$

On the other hand, by choosing $p_j = 2$ for $1 \leq j \leq n$ i.e., $p = (2,2,\ldots,2)$ in (i) of Lemma 2, we have

$$\begin{aligned}
&M(r,f) \leq \mu(r,f) \prod_{j=1}^{n} [N_j + 2], \\
&\text{where } N_j = \max_{r \leq t \leq pr} \nu_j(t) \text{ for } 1 \leq j \leq n. \\
&\Rightarrow M(r,f) \leq \left|a_{\nu(r,f)}\right| r^{\nu(r,f)} \prod_{j=1}^{n} [N_j + 2]. \qquad (11)
\end{aligned}$$

Since $\{|a_k|\}$ is bounded, from (11) we get

$$\begin{aligned}
\log M(r,f) &\leq \sum_{j=1}^{n} \nu_j(r) \log r_j + \sum_{j=1}^{n} \log N_j + C_3 \\
&\leq \sum_{j=1}^{n} |\nu(r,f)| \log r_j + \sum_{j=1}^{n} \log N_j + C_3 \\
&\leq |\nu(r,f)| \log(r_1 r_2 \ldots r_n) + \log(N_1 N_2 \ldots N_n) + C_3 \\
&\Rightarrow \log^{[p+1]} M(r,f) \leq \log^{[p]} |\nu(r,f)| + \log^{[p+1]}(r_1 r_2 \ldots r_n) \\
&+ \log^{[p+1]}(N_1 N_2 \ldots N_n) + C_4,
\end{aligned}$$

where $C_j(>0)(j=3,4)$ are suitable constants. By this and (1), we get

$$\rho_f(p,q) \leq \limsup_{r_1,r_2,\ldots,r_n\to\infty} \frac{\log^{[p]}|\nu(r_1,r_2,\ldots,r_n,f)|}{\log^{[q]}(r_1r_2\ldots r_n)}. \tag{12}$$

By (10) and (12), it follows that

$$\limsup_{r_1,r_2,\ldots,r_n\to\infty} \frac{\log^{[p]}|\nu(r_1,r_2,\ldots,r_n,f)|}{\log^{[q]}(r_1r_2\ldots r_n)} = \rho_f(p,q).$$

Similarly, we can prove that

$$\liminf_{r_1,r_2,\ldots,r_n\to\infty} \frac{\log^{[p]}|\nu(r_1,r_2,\ldots,r_n,f)|}{\log^{[q]}(r_1r_2\ldots r_n)} = \lambda_f(p,q).$$

Theorem 2. *If f and g are entire functions n-complex variables such that $0<\lambda_{fog}(p,q)\leq\rho_{fog}(p,q)<\infty$ and $0<\lambda_g(m,q)\leq\rho_g(m,q)<\infty$, then*

$$\begin{aligned}\frac{\lambda_{fog}(p,q)}{\rho_g(m,q)} &\leq \liminf_{r_1,r_2,\ldots,r_n\to\infty} \frac{\log^{[p]}|\nu(r_1,r_2,\ldots,r_n,fog)|}{\log^{[m]}|\nu(r_1,r_2,\ldots,r_n,g)|} \leq \frac{\lambda_{fog}(p,q)}{\lambda_g(m,q)} \\ &\leq \limsup_{r_1,r_2,\ldots,r_n\to\infty} \frac{\log^{[p]}|\nu(r_1,r_2,\ldots,r_n,fog)|}{\log^{[m]}|\nu(r_1,r_2,\ldots,r_n,g)|} \leq \frac{\rho_{fog}(p,q)}{\lambda_g(m,q)},\end{aligned}$$

where p,q,m are positive integers with $p\geq q\geq m$.

Proof. Using Theorem 1 for the entire function fog and g respectively, we have for arbitrary positive ε and for all sufficiently large values of $r_1,r_2,\ldots,r_n$ that

$$\log^{[p]}|\nu(r_1,r_2,\ldots,r_n,fog)| \geq (\lambda_{fog}(p,q)-\varepsilon)\log^{[q]}(r_1r_2\ldots r_n) \tag{13}$$

and

$$\log^{[m]}|\nu(r_1,r_2,\ldots,r_n,g)| \leq (\rho_g(m,q)+\varepsilon)\log^{[q]}(r_1r_2\ldots r_n). \tag{14}$$ □

Now, from (13) and (14) it follows for all sufficiently large values of $r_1,r_2,\ldots,r_n$,

$$\frac{\log^{[p]}|\nu(r_1,r_2,\ldots,r_n,fog)|}{\log^{[m]}|\nu(r_1,r_2,\ldots,r_n,g)|} \geq \frac{\lambda_{fog}(p,q)-\varepsilon}{\rho_g(m,q)+\varepsilon}.$$

As $\varepsilon(>0)$ is arbitrary, we obtain that

$$\liminf_{r_1,r_2,\ldots,r_n\to\infty} \frac{\log^{[p]}|\nu(r_1,r_2,\ldots,r_n,fog)|}{\log^{[m]}|\nu(r_1,r_2,\ldots,r_n,g)|} \geq \frac{\lambda_{fog}(p,q)}{\rho_g(m,q)}. \tag{15}$$

Again, for a sequence of values of each of $r_1,r_2,\ldots,r_n$ tending to infinity

$$\log^{[p]}|\nu(r_1,r_2,\ldots,r_n,fog)| \leq (\lambda_{fog}(p,q)+\varepsilon)\log^{[q]}(r_1r_2\ldots r_n) \tag{16}$$

and for all large values of $r_1, r_2, \ldots, r_n$

$$\log^{[m]} |\nu(r_1, r_2, \ldots, r_n, g)| \geq (\lambda_g (m,q) - \varepsilon) \log^{[q]}(r_1 r_2 \ldots r_n). \tag{17}$$

So, combining (16) and (17) we get for a sequence of values of each of $r_1, r_2, \ldots, r_n$ tending to infinity,

$$\frac{\log^{[p]} |\nu(r_1, r_2, \ldots, r_n, fog)|}{\log^{[m]} |\nu(r_1, r_2, \ldots, r_n, g)|} \leq \frac{\lambda_{fog} (p,q) + \varepsilon}{\lambda_g (m,q) - \varepsilon}.$$

Since $\varepsilon(> 0)$ is arbitrary, it follows that

$$\liminf_{r_1, r_2, \ldots, r_n \to \infty} \frac{\log^{[p]} |\nu(r_1, r_2, \ldots, r_n, fog)|}{\log^{[m]} |\nu(r_1, r_2, \ldots, r_n, g)|} \leq \frac{\lambda_{fog} (p,q)}{\lambda_g (m,q)}. \tag{18}$$

Also, for a sequnce of values of each of $r_1, r_2, \ldots, r_n$ tending to infinity,

$$\log^{[m]} |\nu(r_1, r_2, \ldots, r_n, fog)| \leq (\lambda_g (m,q) + \varepsilon) \log^{[q]}(r_1 r_2 \ldots r_n). \tag{19}$$

Now, from (13) and (19) we obtain for a sequence of values of each of $r_1, r_2, \ldots, r_n$ tending to infinity,

$$\frac{\log^{[p]} |\nu(r_1, r_2, \ldots, r_n, fog)|}{\log^{[m]} |\nu(r_1, r_2, \ldots, r_n, g)|} \geq \frac{\lambda_{fog} (p,q) - \varepsilon}{\lambda_g (m,q) + \varepsilon}. \tag{20}$$

Choosing $\varepsilon \to 0$ we get that

$$\limsup_{r_1, r_2, \ldots, r_n \to \infty} \frac{\log^{[p]} |\nu(r_1, r_2, \ldots, r_n, fog)|}{\log^{[m]} |\nu(r_1, r_2, \ldots, r_n, g)|} \geq \frac{\lambda_{fog} (p,q)}{\lambda_g (m,q)}. \tag{21}$$

Also, for all large values of $r_1, r_2, \ldots, r_n$,

$$\log^{[p]} |\nu(r_1, r_2, \ldots, r_n, fog)| \leq (\rho_{fog} (p,q) + \varepsilon) \log^{[q]}(r_1 r_2 \ldots r_n). \tag{22}$$

So, from (17) and (22) it follows for all large values of $r_1, r_2, \ldots, r_n$,

$$\frac{\log^{[p]} |\nu(r_1, r_2, \ldots, r_n, fog)|}{\log^{[m]} |\nu(r_1, r_2, \ldots, r_n, g)|} \leq \frac{\rho_{fog} (p,q) + \varepsilon}{\lambda_g (m,q) - \varepsilon}.$$

As $\varepsilon(> 0)$ is arbitrary, we obtain that

$$\limsup_{r_1, r_2, \ldots, r_n \to \infty} \frac{\log^{[p]} |\nu(r_1, r_2, \ldots, r_n, fog)|}{\log^{[m]} |\nu(r_1, r_2, \ldots, r_n, g)|} \leq \frac{\rho_{fog} (p,q)}{\lambda_g (m,q)}. \tag{23}$$

Thus, the theorem follows from (15), (18), (21) and (23).

Theorem 3. *Let f and g are entire functions of n-complex variables such that $0 < \lambda_{fog}(p,q) \leq \rho_{fog}(p,q) < \infty$ and $0 < \lambda_g(m,q) \leq \rho_g(m,q) < \infty$. Then*

$$\liminf_{r_1,r_2,\ldots,r_n\to\infty} \frac{\log^{[p]} |\nu(r_1,r_2,\ldots,r_n,fog)|}{\log^{[m]} |\nu(r_1,r_2,\ldots,r_n,g)|} \leq \frac{\rho_{fog}(p,q)}{\rho_g(m,q)}$$
$$\leq \limsup_{r_1,r_2,\ldots,r_n\to\infty} \frac{\log^{[p]} |\nu(r_1,r_2,\ldots,r_n,fog)|}{\log^{[m]} |\nu(r_1,r_2,\ldots,r_n,g)|},$$

where p,q,m are positive integers with $p \geq q \geq m$.

Proof. Using Theorem 1 for the entire function g, we get for a sequence of values of each of $r_1,r_2,\ldots,r_n$ tending to infinity that

$$\log^{[m]} |\nu(r_1,r_2,\ldots,r_n,g)| \geq (\rho_g(m,q)-\varepsilon)\log^{[q]}(r_1r_2\ldots r_n). \tag{24}$$ □

Now, from (22) and (24) it follows for a sequence of values of each of $r_1,r_2,\ldots,r_n$ tending to infinity,

$$\frac{\log^{[p]} |\nu(r_1,r_2,\ldots,r_n,fog)|}{\log^{[m]} |\nu(r_1,r_2,\ldots,r_n,g)|} \leq \frac{\rho_{fog}(p,q)+\varepsilon}{\rho_g(m,q)-\varepsilon}.$$

As $\varepsilon(>0)$ is arbitrary, we obtain that

$$\liminf_{r_1,r_2,\ldots,r_n\to\infty} \frac{\log^{[p]} |\nu(r_1,r_2,\ldots,r_n,fog)|}{\log^{[m]} |\nu(r_1,r_2,\ldots,r_n,g)|} \leq \frac{\rho_{fog}(p,q)}{\rho_g(m,q)}. \tag{25}$$

Again, for a sequence of values of each of $r_1,r_2,\ldots,r_n$ tending to infinity

$$\log^{[p]} |\nu(r_1,r_2,\ldots,r_n,fog)| \geq (\rho_{fog}(p,q)-\varepsilon)\log^{[q]}(r_1r_2\ldots r_n). \tag{26}$$

So, combining (14) and (26) we get for a sequence of values of each of $r_1,r_2,\ldots,r_n$ tending to infinity,

$$\frac{\log^{[p]} |\nu(r_1,r_2,\ldots,r_n,fog)|}{\log^{[m]} |\nu(r_1,r_2,\ldots,r_n,g)|} \geq \frac{\rho_{fog}(p,q)-\varepsilon}{\rho_g(m,q)+\varepsilon}.$$

Since $\varepsilon(>0)$ is arbitrary, it follows that

$$\limsup_{r_1,r_2,\ldots,r_n\to\infty} \frac{\log^{[p]} |\nu(r_1,r_2,\ldots,r_n,fog)|}{\log^{[m]} |\nu(r_1,r_2,\ldots,r_n,g)|} \geq \frac{\rho_{fog}(p,q)}{\rho_g(m,q)}. \tag{27}$$

Thus, the theorem follows from (25) and (27) .

The following theorem is a natural consequence of Theorem 2 and Theorem 3.

Theorem 4. *If f and g are entire functions of n-complex variables such that $0 < \lambda_{fog}(p,q) \leq \rho_{fog}(p,q) < \infty$ and $0 < \lambda_g(m,q) \leq \rho_g(m,q) < \infty$, then*

$$\liminf_{r_1,r_2,\ldots,r_n\to\infty} \frac{\log^{[p]} |\nu(r_1,r_2,\ldots,r_n,fog)|}{\log^{[m]} |\nu(r_1,r_2,\ldots,r_n,g)|} \leq \min\left\{\frac{\lambda_{fog}(p,q)}{\lambda_g(m,q)}, \frac{\rho_{fog}(p,q)}{\rho_g(m,q)}\right\}$$
$$\leq \max\left\{\frac{\lambda_{fog}(p,q)}{\lambda_g(m,q)}, \frac{\rho_{fog}(p,q)}{\rho_g(m,q)}\right\} \leq \limsup_{r_1,r_2,\ldots,r_n\to\infty} \frac{\log^{[p]} |\nu(r_1,r_2,\ldots,r_n,fog)|}{\log^{[m]} |\nu(r_1,r_2,\ldots,r_n,g)|},$$

where p,q,m are positive integers with $p \geq q \geq m$.

Remark 1. If we take $0 < \lambda_f(m,q) \leq \rho_f(m,q) < \infty$ instead of $0 < \lambda_g(m,q) \leq \rho_g(m,q) < \infty$ and the other conditions remain the same, then also Theorem 2, Theorem 3 and Theorem 4 hold with g replaced by f in the denominator.

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Hypothesis test of sample mean of random intervals and comparing between methods based on Hausdorff distance and the maximum likelihood ratio

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Abstract. In this article, we will introduce some important elements of random sets theory, and we will use the tools of topology, probability theory, and statistical inference aims to develop a coherent mathematical framework to analyze random elements whose realization are intervals.

Hypothesis testing is one of the most widely used methods in statistical inference that involves asking a question, collecting data, and then examining what the data tells us about how to proceed.

The hypothesis tests that we will examine in this paper is the test of sample mean of random sets in a Euclidean space, and in particular in the case where these sets are intervals in a Euclidean space. We will use two test statistics, one is based on Hausdorff distance with using central limit theorem and the second is based on the maximum likelihood ratio.

Keywords: random set, random interval, Hausdorff distance, Minkowski average, maximum likelihood ratio and R software.

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1. Introduction

Statistical inference is a very important field in the theory of random sets, including the search for estimator of the expectation of random sets which are considered independent and identically distributed and follow the same capacity functional, in this article, we will demonstrate that the maximum likelihood estimator of this expectation is the Minkowski average of these random sets. This naturally leads to testing hypotheses and building confidence sets for expectation of random set and its subsets in a manner that is analogous to classical methods for scalar-valued random variables. Among the test statistics used for this unilateral hypothesis test which treats the equality of sample mean of independent and identically distributed random sets to a deterministic set, we find the test statistic based on the Hausdorff distance, under the assumption that the size of this sample is very large and using the central limit theorem. In that sense, we propose another method based on the maximum likelihood ratio for testing the equality of sample mean of the convex compact random sets to a deterministic compact and convex set in euclidean space R^d. Finally, we will apply those two methods of test for hypotheses testing of random intervals which are a particular case of random sets in IR , and we will study an empirical example with using the " R" Project for Statistical Computing and discuss the different results obtained from " R" code .

2. Definition of random closed set in Euclidean space IR^d and the Capacity functional

In the theory of random set (see, [1], [2] and [3]), X is called a random closed set in euclidean space IR^d, if X is a map from a probalility space (Ω, T, P) to the family of closed set F in IR^d and $X^-(K)= \{\omega : X(\omega) \cap K \neq \emptyset\}$ belongs to the σ-algebra T on Ω for each compact set K in IR^d. Each random element must be modeled by a probability distribution, in our case this random element is a set, then we use the capacity functional, which have an important role in the theory of random set.

2.1 Definition of the capacity functional

The capacity functional (see, [1], [4], [5] and [6]) of a random set X, is a map from $\mathcal{K}$ a family of compacts sets in IR^d to $[0, 1]$ defining by: $T_X(K) = P(X \cap K \neq \emptyset)$ with $K \in \mathcal{K}$. If $X = \{x\}$, then $T_X(K) = P(\{x\} \cap K \neq \emptyset) = P_x(K)$ and in this case T_X becomes the probability distribution of random variable x.

2.2 Proprieties of the capacity functional

In general case, T_X is a sub-additive capacity functional (see, [7] and [8]), and for each two compacts sets K_1 and K_2 from $\mathcal{K}$ we have:

$$T_X(K_1 \cup K_2) \leq T_X(K_1) + T_X(K_2).$$

And we have the equality, if and only if K_1 and K_2 are disjoints, it means $K_1 \cap K_2 = \emptyset$.

3. Selections of random set

3.1 Definition of selection of random set

We consider a random set X in IR^d, if x is a random element with values in IR^d, such that $x(\omega) \in X(\omega)$ almost surely, then we called x a selection of X. We denote by $Sel(X)(orL^0(X))$ a set of all selections ([2] and [3]) from X, then: $Sel(X) = \{x(\omega) : x(\omega) \in X(\omega), \omega \in \Omega\}$. In particular, if $X = \{x\}$, then $Sel(X) = \{x\}$.

3.2 Fundamental selection theorem

If $X : \Omega \to F$ is an almost surely non-empty random closed set in IR^d, then X has a measurable selection (see, [1] and [2]).

4. Expectation of random set in IR^d

It is difficult to define the expectation of a random set in euclidean space, because the space of closed sets is not linear (see, [2] and [10]), one approach relies on representing a random set using the family of its selections, and taking their expectation. For to have an integrable random closed set X in IR^d, it suffices that X possesses at least one an integrable selection, it means $\exists x(\omega) \in X(\omega)$ almost surly and $\|E(x)\| < +\infty$. Then, an integrable random closed set is non-empty, with probability one. We denote by $L^p(X)$ the family of all p-integrable selections of X, with $p \in [0, +\infty[$. Usually, we are interested in $L^1(X)$ and $L^2(X)$ which are respectively the families of all integrable and square-integrable selections of set X. And we called X respectively an integrable and square-integrable random closed set.

4.1 Definition of expectation of random set in IR^d

The selection expectation EX (see, [1], [2] and [3]) of an integrable random closed set X is the closure of the family of expectations of all its integrable selections $x \in L^1(X)$. Then, we denote $EX = cl\{Ex, x \in L^1(X)\}$.

4.2 Minkowski sum of expectation of random set in IR^d

The Minkowski sum of expectation of two random sets X and Y in IR^d is defining by the formula:

$$E(X + Y) = EX + EY = \{a + b : a \in EX, b \in EY\}.$$

5. Law of large numbers for integrably bouneded random sets

If $X_1, \ldots, X_n$ is a n-sample of independent and identically distributed integrably bounded random compact sets in IR^d, then $d_H(\frac{S_n}{n}, EX_i) \to 0$, almost surely as $n \to \infty$ (see, [2] and [3]). With $S_n = X_1 + \ldots + X_n$, $EX_i = EX_j$, $T_{X_i} = T_{X_j}$ for each$(i, j) \in \{1, \ldots, n\}$, and d_H is the Hausdorff distance of two sets, and we denote:

$$d_H(X, Y) = inf\{r : X \subset Y^r, Y \subset X^r\}$$

with

$$X^r = \{a \in IR^d, d(a, X) \leq r\}$$

and

$$Y^r = \{a \in IR^d, d(a, Y) \leq r\}.$$

According to the law of large numbers, we can sense that estimation of sample mean of random closed sets can be the Minkowski average.

6. Estimation of sample mean of random sets in Euclidean space IR^d

Theorem 6.1. *Let $X_1, \ldots, X_n$ a n-sample of compact random sets, integrable and non-empty in IR^d, Suppose that $(X_i)_{i\in\{1,\ldots,n\}}$ are independent and identically distributed. It means $T_{X_i} = T_{X_j}$ for all $i \neq j$, and $EX_i = EX_j = K$. Then the estimation of sample mean is the Minkowski average and we denote:*

$$\hat{EX} = \hat{K} = \frac{S_n}{n} = \bar{X}_n$$

with S_n is the Minkowski sum of X_i, and K a compact set in IR^d.

Proof. We use the Maximum likelihood function denoted L, we have:

$$L(X_1, \ldots, X_n, K) = P(X_1 \cap K \neq \emptyset, ..., X_n \cap K \neq \emptyset).$$

The events $\{\omega, X_i(\omega) \cap K \neq \emptyset\}$ are independents for each $i \in \{1, \ldots, n\}$, then:

$$L(X_1, \ldots, X_n, K) = \prod_{i=1}^{n} P(X_i \cap K \neq \emptyset).$$

These random sets are identically distributed, then we have:

$$L(X_1, \ldots, X_n, K) = \prod_{i=1}^{n} T_{X_i}(K) = T_{X_i}(K)^n$$

The estimator of K is denoted by $\hat{K}$, and it takes the value that maximizes L. For maximizing L it suffices to maximize $T_{X_i}(K)^n$ for each $i \in \{1, ..., n\}$.

We have $T_{X_i}(K) \in [0, 1]$, therefore : $\max_K T_{X_i}(K) = T_{X_i}(\hat{K})$, Then $P(X_i \cap K \neq \emptyset) = 1$, it means the event $\{\omega, X_i(\omega) \cap K \neq \emptyset\}$ is certain, and it suffices to take $\hat{K} = X_i$ for each $i \in \{1, ..., n\}$. For having an exhaustive estimator, we take $n\hat{K} = S_n = \sum_{i=1}^{n} X_i$, then $\hat{K} = \frac{Sn}{n} = \bar{X}_n$. □

6.1 Expectation of K

We have the expectation of Minkowski sum is the Minkowski sum of expectation of random sets $X_1, \ldots, X_n$ then:

$$E(X_1 + \ldots + X_n) = EX_1 + \ldots + EX_n.$$

Then:

$$E\hat{K} = E(\frac{X_1 + \ldots + X_n}{n}) = \frac{1}{n}\sum_{i=1}^{n} EX_i = \frac{1}{n}\sum_{i=1}^{n} K = K.$$

And we conclude that $\hat{K}$ is an unbiased estimator.

7. Central limit theorem of random sets

Let $X_1, \ldots, X_n$ n-sample of random closed sets in IR^d, such that $E\|X_i\|^2 < \infty$, then:

$$\sqrt{n} d_H(\frac{S_n}{n}, EX_i) ==> \|\xi(u)\|_\infty = \sup\{|\xi(u)|; u \in S^{d-1}\}, \text{ as } n \to \infty.$$

With $\|X_i\| = d_H(X_i, \{0\})$ and $\xi(u)$ is a centered Gaussian random field on the unit sphere S^{d-1} in IR^d, with the covariance function $E(\xi(u)\xi(v))$ (see, [1], [2] and [3]).

8. Hypothesis testing

8.1 Hypothesis testing with using central limit theorem

Let $X_1, \ldots, X_n$ n-sample of random compact sets in IR^d, independent and identically distributed such that $EX_i = K$ and $E\|X_i\|^2 < \infty$, for each $i \in 1, \ldots, n$ with $\bar{X_n}$ the Minkowski average of this sample.

Let K_0 be a deterministic non-empty convex compact set in IR^d, in order to test at a prespecified significance level $\alpha \in [0, 1]$, then we consider the following bilateral hypothesis testing $H_0 : K = K_0$, vs $H_1 : K \neq K_0$.

We use the test statistic based on Hausdorff distance (see, [2]), as $n \to \infty$

$$T_n = \sqrt{n} d_H(\bar{X_n}, EX_i) => \|\xi(u)\|_\infty = \sup\{|\xi(u)|; u \in S^{d-1}\}$$

Under H_0, we denote $T_n(K_0) = \sqrt{n} d_H(\bar{X_n}, K_0)$, and we use the following criterion:

$$\begin{cases} \text{Reject } H_0, \text{ if } T_n(K_0) > C_\alpha \\ \text{Fail reject } H_0, \text{ if } T_n(K_0) < C_\alpha \end{cases},$$

where C_α is chosen so that $P(\|\xi(u)\|_\infty > C_\alpha) = \alpha$.

8.2 Hypothesis testing with using maximum liklihood ratio

Now, we started from the above hypothesis, and using the Maximum likelihood function$L$, then we have :

$$\Lambda_n = \frac{L(X_1, \ldots, X_n, K)_{H_1}}{L(X_1, \ldots, X_n, K)_{H_0}} = \frac{L(X_1, \ldots, X_n, \hat{K})}{L(X_1, \ldots, X_n, K_0)}$$

with $\hat{K}$ is the maximum likelihood estimator of K and verifying $\sqrt{n}d_H(\hat{K}, K) => \|\xi(u)\|_\infty$with $\xi(u)$ is a centered Gaussian random field on the unit sphere S^{d-1} in IR^d. Then, the test statistic becomes:

$$T_n = 2\ln(\Lambda_n) \sim \chi^2_m.$$

Such that $m = df(H_1) - df(H_0)$, with $df(H_1)$ is the degree of freedom of parameters of K under H_1, and $df(H_0)$ is the degree of freedom of parameters of K under H_0. Then, we calculate $2\ln(\Lambda_n)$ and we will compare it with $\chi^2_{m,1-\alpha}$, such that α is the type I error.

We have $L(X_1, \ldots, X_n, K) = T_{X_i}(K)^n$, and $\bar{X}_n = \hat{K}$, then:

$$2\ln(\Lambda_n) = 2n[\ln T_{X_i}(\bar{X}_n) - \ln T_{X_i}(K_0)].$$

Then, we define the region of acceptance of hypothesis H_0 :

$$R^{acc} = \{\Lambda_n^{obs}, 2\ln(\Lambda_n) \leq \chi^2_{m,1-\alpha}\}.$$

9. Case of random intervals in IR

Random intervals are a special case of random sets on IR, let $Y = [Y^l, Y^u]$ be a random compact interval on the line IR, with square integrable end points (Y^l and Y^u are two dependent random variables). The unit sphere on the line IR is $\{-1, 1\}$ (see, [2]).

The expectation of Y denoted $EY = [EY^l, EY^u]$ is a compact set in IR, while the covariance function of this random interval can be identified as a 2x2 matrix:

$$C_Y = \begin{pmatrix} var(Y^l) & -cov(Y^l, Y^{u)} \\ -cov(Y^l, Y^u) & var(Y^u) \end{pmatrix}.$$

9.1 Hypothesis testing with using central limit theorem: case of random interval

Let $Y_1, \ldots, Y_n$ be independent and identically distributed copies of a random compact interval, $Y = [Y^l, Y^u]$ in IR which are closed and bounded, such that $E\|Y\|^2 < \infty$. Using above result in section 8.1, we consider the following bilateral hypothesis testing $H_0 : [EY^l, EY^u] = [a_0, b_0]$ vs $H_1 : [EY^l, EY^u] \neq [a_0, b_0]$.

We use the test statistic based on Hausdorff distance, as $n \to \infty$

$$T_n = \sqrt{n}d_H(\bar{Y}_n, [EY^l, EY^u]) => \|\xi(u)\|_\infty = \sup\{|\xi(u)|; u \in \{-1, 1\}\}$$

with $\bar{Y}_n$ is the Minkowski average of $Y_1, \ldots, Y_n$ denoted $\bar{Y}_n = [\bar{Y}_n^l, \bar{Y}_n^u]$, and $\bar{Y}_n^k = \frac{1}{n}\sum_{i=1}^n Y_i^k, k \in l, u$. Then:

$$T_n = \sqrt{n} d_H([\bar{Y}_n^l, \bar{Y}_n^u], [EY^l, EY^u]) => \sup\{|\xi(-1)|, |\xi(1)|\}.$$

The Hausdorff distance between two intervals equals the maximum of the absolute difference of their corresponding end-points. Then:

$$T_n = \sqrt{n} \max(|\bar{Y}_n^l - EY^l|, |\bar{Y}_n^u - EY^u|) => \sup\{|\xi(-1)|, |\xi(1)|\}$$

$\bar{Y}_n^l$ and $\bar{Y}_n^u$ are the averages of the left and right end points of independent and identically distributed random intervals $Y_1, \ldots, Y_n$. And $(\xi(-1), \xi(1))$ is the bivariate centred Gaussian vector with covariance matrix C_Y (see, [2] and [9]). We denote $\bar{y}_n^l$ and $\bar{y}_n^u$ the realizations of $\bar{Y}_n^l$ and $\bar{Y}_n^u$, respectively.

We calculate the value of $\max(|\bar{y}_n^l - EY^l|, |\bar{y}_n^u - EY^u|)$ and we conclude that:

$$\begin{cases} \text{If } \max(|\bar{y}_n^l - EY^l|, |\bar{y}_n^u - EY^u|) = |\bar{y}_n^l - EY^l|, \text{ then } T_n \sim N(0, V(Y_i^l)) \\ \text{Else if } \max(|\bar{y}_n^l - EY^l|, |\bar{y}_n^u - EY^u|) = |\bar{y}_n^u - EY^u|, \text{ then } T_n \sim N(0, V(Y_i^u)) \end{cases}.$$

The variances of Y_i^l, and Y_i^u are unknown, but as $n \to \infty$, we have:

$$V(Y_i^l) = S_{Y_i^l}^2 = \frac{1}{n-1}\sum_{i=1}^n (Y_i^l - \bar{Y}^l)^2$$

and

$$V(Y_i^u) = S_{Y_i^u}^2 = \frac{1}{n-1}\sum_{i=1}^n (Y_i^u - \bar{Y}^u)^2.$$

Under H_0, $T_n = \sqrt{n}\max(|\bar{Y}_n^l - a_0|, |\bar{Y}_n^u - b_0|)$ and we conclude the following result:

- if $\max(|\bar{y}_n^l - a_0|, |\bar{y}_n^u - b_0|) = |\bar{y}_n^l - a_0|$, then, $\frac{|\bar{Y_n}^l - a_0|}{S_{Y_i^l}} \sim N(0,1)$ and $R^{accp} = \{{Y_i^l}^{obs}, \frac{|\bar{Y}_n^l - a_0|}{S_{Y_i^l}} \leq z_{\frac{\alpha}{2}}\}$, else, if $\max(|\bar{y}_n^l - a_0|, |\bar{y}_n^u - b_0|) = |\bar{y}_n^u - b_0|$, then, $\frac{|\bar{Y}_n^u - b_0|}{S_{Y_i^u}} \sim N(0,1)$ and $R^{accp} = \{{Y_i^l}^{obs}, \frac{|\bar{Y}_n^u - a_0|}{S_{Y_i^u}} \leq z_{\frac{\alpha}{2}}\}$.

9.2 Hypothesis testing with using maximum likelihood ratio: case of random interval

We started from the above hypothesis, and using the Maximum likelihood function L, then we have :

$$\Lambda_n = \frac{L(Y_1, \ldots, Y_n, [EY^l, EY^u])_{H_1}}{L(Y_1, \ldots, Y_n, [EY^l, EY^u])_{H_0}} = \frac{L(Y_1, \ldots, Y_n, \bar{Y}_n)}{L(Y_1, \ldots, Y_n, [a_0, b_0])}$$

with $\bar{Y_n} = [\bar{Y}_n^l, \bar{Y}_n^u]$ is the maximum likelihood estimator of $[EY^l, EY^u]$. The test statistic becomes:

$$T_n = 2\ln(\Lambda_n) \sim \chi^2_m.$$

Such that $m = df(H_1) - df(H_0)$, with $df(H_1)$ is the degree of freedom of parameters of K under H_1, and $df(H_0)$ is the degree of freedom of parameters of K under H_0.

In this case $df(H_1) = 2$ and $df(H_0) = 0$, then $m = 2$ and under H_0 we have:

$$T_n = 2\ln(\Lambda_n) \sim \chi^2_2.$$

Then, we calculate $2\ln(\Lambda_n)$ and we will compare it with $\chi^2_{2,1-\alpha}$, such that $1-\alpha$ is the confidence level of test.

On the other hand we have $L(Y_1, \ldots, Y_n, [EY^l, EY^u]) = T_{Y_i}([EY^l, EY^u])^n$ that implies:

$$T_n = 2\ln(\Lambda_n) = 2n[\ln T_{Y_i}([\bar{Y}_n^l, \bar{Y}_n^u]) - \ln T_{Y_i}([a_0, b_0])].$$

We define the region of acceptance of hypothesis H_0 :

$$R^{acc} = \left\{\Lambda_n^{obs}, T_n \leq \chi^2_{2,1-\alpha}\right\}.$$

9.3 Simplification of expression of T_n

The problem now is how to calculate the test statistic T_n and to compare it with the value of $\chi^2_{2,1-\alpha}$.

On the one hand, we should to simplify the expression of T_n.

We have:

$$T_{Y_i}([\bar{Y}_n^l, \bar{Y}_n^u]) = P([\bar{Y}_n^l, \bar{Y}_n^u] \cap Y_i \neq \emptyset] = P([\bar{Y}_n^l, \bar{Y}_n^u] \cap [Y_i^l, Y_i^u] \neq \emptyset)$$

$$= P((Y_i^l < \bar{y}_n^l \leq Y_i^u) \cup (\bar{Y}_n^l \leq Y_i^l \leq \bar{Y}_n^u)).$$

The events $\{\omega \in \Omega, Y_i^l(\omega) < \bar{Y}_n^l(\omega) \leq Y_i^u(\omega)\}$ and $\{\omega \in \Omega, \bar{Y}_n^l(\omega) \leq Y_i^l(\omega) \leq \bar{Y}_n^u(\omega)\}$ are independents then:

$$T_{Y_i}([\bar{Y}_n^l, \bar{Y}_n^u]) = P(Y_i^l < \bar{Y}_n^l \leq Y_i^u) + P(\bar{Y}_n^l \leq Y_i^l \leq \bar{Y}_n^u).$$

On the other hand, we have:

$$T_{Y_i}([a_0, b_0]) = P([a_0, b_0] \cap Y_i \neq \emptyset] = P([a_0, b_0] \cap [Y_i^l, Y_i^u] \neq \emptyset)$$

$$= P[(Y_i^l < a_0 \leq Y_i^u) \cup (a_0 \leq Y_i^l \leq b_0)].$$

The events $\{\omega \in \Omega, Y_i^l(\omega) < a_0 \leq Y_i^u(\omega)\}$ and $\{\omega \in \Omega, a_0 \leq Y_i^l(\omega) \leq b_0\}$ are independents then:

$$T_{Y_i}([a_0, b_0]) = P(Y_i^l < a_0 \leq Y_i^u) + P(a_0 \leq Y_i^l \leq b_0).$$

We conclude that:

$$2\ln(\Lambda_n) = 2n(\ln T_{Y_i}([\bar{y}_n^l, \bar{y}_n^u]) - lnT_{Y_i}([a_0, b_0]))$$

$$= 2n \ln \frac{P(Y_i^l < \bar{y}_n^l \leq Y_i^u) + P(\bar{y}_n^l \leq Y_i^l \leq \bar{y}_n^u)}{P(Y_i^l < a_0 \leq Y_i^u) + P(a_0 \leq Y_i^l \leq b_0)}.$$

Y_i^l and Y_i^u are dependent variables, with $P(\ Y_i^l < Y_i^u) = 1$ then, we simplify the expression of $P(Y_i^l < \bar{Y}_n^l \leq Y_i^u)$ and $P(Y_i^l < a_0 \leq Y_i^u)$.

On the one hand, we have:

$$\begin{aligned}&\{\omega \in \Omega, Y_i^l(\omega) < \bar{Y}_n^l(\omega) \leq Y_i^u(\omega)\}\\ &= \{\omega \in \Omega, Y_i^l(\omega) < \bar{Y}_n^l(\omega)\} \cap \{\omega \in \Omega, \bar{Y}_n^l(\omega) \leq Y_i^u(\omega)\}\end{aligned}$$

and $(\{\omega \in \Omega, Y_i^l(\omega) < \bar{Y}_n^l(\omega)\} \cap \{\omega \in \Omega, , \bar{Y}_n^l(\omega) \leq Y_i^u(\omega)\}) \cap \{\omega \in \Omega, Y_i^u(\omega) < \bar{Y}_n^l(\omega)\} = \emptyset$. Then, the events $\{\omega \in \Omega, Y_i^l(\omega) < \bar{Y}_n^l(\omega)\} \cap \{\omega \in \Omega, \bar{Y}_n^l(\omega) \leq Y_i^u(\omega)\}$ and $\{\omega \in \Omega, Y_i^u(\omega) < \bar{Y}_n^l(\omega)\}$ are disjoints, and we have:

$$\begin{aligned}&P\{\omega \in \Omega, Y_i^l(\omega) < \bar{Y}_n^l(\omega) \leq Y_i^u(\omega)\} + P\{\omega \in \Omega, Y_i^u(\omega) < \bar{Y}_n^l(\omega)\}\\ &= P[(\{\omega \in \Omega, Y_i^l(\omega) < \bar{Y}_n^l(\omega)\} \cap \{\omega \in \Omega, \bar{Y}_n^l(\omega)\\ &\leq Y_i^u(\omega)\}) \cup \{\omega \in \Omega, Y_i^u(\omega) < \bar{Y}_n^l(\omega)\}]\\ &= P[(\{\omega \in \Omega, Y_i^l(\omega) < \bar{Y}_n^l(\omega)\} \cup \{\omega \in \Omega, Y_i^u(\omega) < \bar{Y}_n^l(\omega)\}) \cap \Omega]\\ &= P(\{\omega \in \Omega, Y_i^l(\omega) < \bar{Y}_n^l(\omega)\} \cup \{\omega \in \Omega, Y_i^u(\omega) < \bar{Y}_n^l(\omega)\}).\end{aligned}$$

On the other hand, we have $Y_i^l < Y_i^u$ almost surely, then $Y_i^u < \bar{Y}_n^l$ implies $Y_i^l < \bar{Y}_n^l$, then:

$$\{\omega \in \Omega, Y_i^u(\omega) < \bar{Y}_n^l(\omega)\} \subset \{\omega \in \Omega, Y_i^l(\omega) < \bar{Y}_n^l(\omega)\}.$$

And we have:

$$\begin{aligned}&P(\{\omega \in \Omega, Y_i^l(\omega) < \bar{Y}_n^l(\omega) \leq Y_i^u(\omega)\}) + P(\{\omega \in \Omega, Y_i^u(\omega) < \bar{Y}_n^l(\omega)\})\\ &= P(\{\omega \in \Omega, Y_i^l(\omega) < \bar{Y}_n^l(\omega)\}).\end{aligned}$$

Then $P(Y_i^l < \bar{Y}_n^l \leq Y_i^u) + P(Y_i^u < \bar{Y}_n^l) = P(Y_i^l < \bar{Y}_n^l)$ and we conclude that:

$$\begin{cases}P(Y_i^l < \bar{Y}_n^l \leq Y_i^u) = P(Y_i^l < \bar{Y}_n^l) - P(Y_i^u < \bar{Y}_n^l),\\ P(Y_i^l < a_0 \leq Y_i^u) = P(Y_i^l < a_0) - P(Y_i^u < a_0).\end{cases}$$

On the other hand, we have:

$$\begin{cases}P(\bar{Y}_n^l \leq Y_i^l \leq \bar{Y}_n^u) = P(Y_i^l \leq \bar{Y}_n^u) - P(Y_i^l \leq \bar{Y}_n^l),\\ P(a_0 \leq Y_i^l \leq b_0) = P(Y_i^l \leq b_0) - P(Y_i^l \leq a_0).\end{cases}$$

Then:

$$T_n = 2n \ln \frac{P(Y_i^l < \bar{Y}_n^l) - P(Y_i^u < \bar{Y}_n^l) + P(Y_i^l \leq \bar{Y}_n^u) - P(Y_i^l \leq \bar{Y}_n^l)}{P(Y_i^l < a_0) - P(Y_i^u < a_0) + P(Y_i^l \leq b_0) - P(Y_i^l \leq a_0)}.$$

And finally we conclude that:

$$T_n = 2n[\ln(P(Y_i^l \leq \bar{Y}_n^u) - P(Y_i^u < \bar{Y}_n^l)) - \ln(P(Y_i^l \leq b_0) - P(Y_i^u < a_0))].$$

We denote by $Z_i^l = Y_i^l - \bar{Y}_n^u$ and $Z_i^u = Y_i^u - \bar{Y}_n^l$ then:

$$\begin{cases} P(Y_i^l \leq \bar{Y}_n^u) = P(Z_i^l \leq 0) = F_{Z_i^l}(0), \\ P(Y_i^u < \bar{Y}_n^l) =) P(Z_i^u \leq 0) = F_{Z_i^u}(0) \end{cases}$$

and

$$\begin{cases} P(Y_i^l \leq b_0) = F_{Y_i^l}(b_0), \\ P(Y_i^u < a_0) = F_{Y_i^u}(a_0), \end{cases}$$

we replace these equations in the formula of T_n, and we have:

$$T_n = 2n[\ln(F_{Z_i^l}(0) - F_{Z_i^u}(0)) - \ln(F_{Y_i^l}(b_0) - F_{Y_i^u}(a_0))].$$

9.4 Estimation of test statistic T_n

Now, we will estimate the expression of T_n, and for do it, we must to estimate the values of the distribution functions $F_{Z_i^l}(0)$, $F_{Z_i^u}(0)$, $F_{Y_i^l}(b_0)$ and $F_{Y_i^u}(a_0)$.

For that, we use the empirical distribution function F_n denoted by: $\forall x \in IR, F_n(x) = \frac{1}{n}\sum_{i=1}^n 1_{X_i \leq x}$ for a n-sample $X_1, \ldots, X_n$ of random variables, then:

$$\begin{cases} \hat{F}_{Z_i^l}(0) = F_n^{Z_i^l}(0) = \dfrac{1}{n}\displaystyle\sum_{i=1}^n 1_{Z_i^l \leq 0}, \\ \hat{F}_{Z_i^u}(0) = F_n^{Z_i^u}(0) = \dfrac{1}{n}\displaystyle\sum_{i=1}^n 1_{Z_i^u \leq 0} \end{cases}$$

and

$$\begin{cases} \hat{F}_{Y_i^l}(b_0) = F_n^{Y_i^l}(b_0) = \dfrac{1}{n}\displaystyle\sum_{i=1}^n 1_{Y_i^l \leq b_0}, \\ \hat{F}_{Y_i^u}(a_0) = F_n^{Y_i^u}(a_0) = \dfrac{1}{n}\displaystyle\sum_{i=1}^n 1_{Y_i^u \leq a_0}. \end{cases}$$

Then:

$$\begin{aligned} \hat{T}_n &= 2n[\ln(\hat{F}_{Z_i^l}(0) - \hat{F}_{Z_i^u}(0)) - \ln(\hat{F}_{Y_i^l}(b_0) - \hat{F}_{Y_i^u}(a_0))] \\ &= 2n[\ln(F_n^{Z_i^l}(0) - F_n^{Z_i^u}(0)) - \ln(F_n^{Y^l}(b_0) - F_n^{Y^u}(a_0))]. \end{aligned}$$

And, finally, we have:

$$\hat{T}_n = 2n[\ln(\frac{1}{n}\sum_{i=1}^{n} 1_{Z_i^l \leq 0} - \frac{1}{n}\sum_{i=1}^{n} 1_{Z_i^u \leq 0}) - \ln(\frac{1}{n}\sum_{i=1}^{n} 1_{Y_i^l \leq b_0} - \frac{1}{n}\sum_{i=1}^{n} 1_{Y_i^u \leq a_0})]$$
$$= 2n[\ln(\sum_{i=1}^{n} 1_{Z_i^l \leq 0} - \sum_{i=1}^{n} 1_{Z_i^u \leq 0}) - \ln(\sum_{i=1}^{n} 1_{Y_i^l \leq b_0} - \sum_{i=1}^{n} 1_{Y_i^u \leq a_0})].$$

9.5 The Glivenko-Cantelli theorem

Let $X_1, ..., X_n$ be $i.i.d$ real-valued random variables with distribution function $F(x) = P(X_i \leq x)$. We denote the standard empirical distribution function by $F_n(x) = \frac{1}{n}\sum_{i=1}^{n} 1_{X_i<x}$ then $\sup_{x\in R} |F(x) - F_n(x)| \to 0$ almost surely as $n \to \infty$.

We conclude that $F_n(x)$ uniformly converges to its theoretical counterpart $F(x)$ (see, [11], [12], [13] and [14]). Then, $F_n^{Z_i^l}(0)$, $F_n^{Z_i^u}(0)$, $F_n^{Y^l}(b_0)$ and $F_n^{Y^u}(a_0)$ uniformly converges to $F_{Z_i^l}(0)$, $F_{Z_i^u}(0)$,$F_{Y_i^l}(b_0)$ and $F_{Y_i^u}(a_0)$ respectively as $n \to \infty$. And, we have:

$$\lim_{n\infty} \frac{\hat{T}_n}{T_n} = \lim_{n\infty} \frac{2n[\ln(F_n^{Z_i^l}(0) - F_n^{Z_i^u}(0)) - \ln(F_n^{Y_i^l}(b_0) - F_n^{Y_i^u}(a_0))]}{2n[\ln(F_{Z_i^l}(0) - F_{Z_i^u}(0)) - \ln(F_{Y_i^l}(b_0) - F_{Y_i^u}(a_0))]}$$
$$= \lim_{n\infty} \frac{\ln(F_n^{Z_i^l}(0) - F_n^{Z_i^u}(0)) - \ln(F_n^{Y_i^l}(b_0) - F_n^{Y_i^u}(a_0))}{\ln(F_{Z_i^l}(0) - F_{Z_i^u}(0)) - \ln(F_{Y_i^l}(b_0) - F_{Y_i^u}(a_0))}.$$

Such that:

$$\begin{cases} F_n^{Z_i^l}(0) - F_n^{Z_i^u}(0) \to F_{Z_i^l}(0) - F_{Z_i^u}(0) \text{ as } n \to \infty \\ F_n^{Y_i^l}(b_0) - F_n^{Y_i^u}(a_0) \to F_{Y_i^l}(b_0) - F_{Y_i^u}(a_0) \text{ as } n \to \infty \end{cases}$$

Then $\lim_{n\infty} \frac{\hat{T}_n}{T_n} = 1$, and we conclude that $\hat{T}_n \sim T_n$ as $n \to \infty$. Then, the test statistic using for hypothesis testing H_0 is $\hat{T}_n$, and finally we calculate the value of $\hat{T}_n$, and we will compare it with $\chi^2_{2,1-\alpha}$, and we conclude the region of acceptance of this hypothesis:

$$R^{acc} = \{Y_i^{u^{obs}}, Y_i^{l^{obs}}, \hat{T}_n \leq \chi^2_{2,1-\alpha}\}.$$

10. Simulation study

In this paragraph we will to compare between decisions taking for hypothesis testing of random interval with using the test statistics based on Hausdorff distance and the maximum likelihood ratio. Let's be 4000 observations of random interval in IR, our objective is to test the equality between the expectation of this sample and a deterministic interval. Using the " R" Project for Statistical Computing, we denote the first 20 observations and the last 20 observations from our simulation data denoted " simulation.csv"

Table1: The first 20 and last 20 observations of random interval from R Software

Row	Y^l	Y^u	*Row*	Y^l	Y^u
1	20	35	3981	99520	99535
2	45	60	3982	99545	99560
3	70	85	3983	99570	99585
4	95	110	3984	99595	99610
5	120	135	3985	99620	99635
6	145	160	3986	99645	99660
7	170	185	3987	99670	99685
8	195	210	3988	99695	99710
9	220	235	3989	99720	99735
10	245	260	3990	99745	99760
11	270	285	3991	99770	99785
12	295	310	3992	99795	99810
13	320	335	3993	99820	99835
14	345	360	3994	99845	99860
15	370	385	3995	99870	99885
16	395	410	3996	99895	99910
17	420	435	3997	99920	99935
18	445	460	3998	99945	99960
19	470	485	3999	99970	99985
20	495	510	4000	99995	100010

Source: Author

We consider the hypothesis testing:

$H_0 : [EY_l, EY_u] = [50000, 50020]$ vs $H_1 : [EY_l, EY_u] \neq [50000, 50020]$

10.1 Test statistic based on Hausdorff distance using the central limit theorem

We have the test statistic is denoted by:

$$T_n = \sqrt{n}\max(|\bar{Y}_n^l - EY^l|, |\bar{Y}_n^u - EY^u|) => \sup\{|\xi(-1)|, |\xi(1)|\}.$$

Under H_0, we have:

$$T_{4000} = \sqrt{4000}\max(|\bar{Y}^l - 50000|, |\bar{Y}^u - 50020|).$$

We have:

$$\max(|\bar{Y}^l - 50000|, |\bar{Y}^u - 50020|) = |\bar{Y}^l - 50000|,$$

and

$$\frac{\bar{Y}^l - 50000}{S_{Y_l}} \sim N(0, 1).$$

Taken the type 1 error $\alpha = 0.05$, we have:

$$\frac{\bar{Y}^l - 50000}{S_{Y_l}} = 0,01642962 \leq Z_{0.025} = 1,96.$$

Then, we accept the hypothesisH_0 (see, appendices).

10.2 Test statistic based on maximum likelihood ratio

Test statistic based on maximum likelihood ratio is denoted by:

$$\hat{T}_n = 2n[\ln(\sum_{i=1}^{n} 1_{Z_i^l \leq 0} - \sum_{i=1}^{n} 1_{Z_i^u \leq 0}) - \ln(\sum_{i=1}^{n} 1_{Y_i^l \leq b_0} - \sum_{i=1}^{n} 1_{Y_i^u \leq a_0})].$$

In our simulation data, we have n=4000, and under H_0 :

$$\hat{T}_{4000} = 2 * 4000[\ln(\sum_{i=1}^{4000} 1_{Z_i^l \leq 0} - \sum_{i=1}^{4000} 1_{Z_i^u \leq 0}) - \ln(\sum_{i=1}^{4000} 1_{Y_i^l \leq 50020} - \sum_{i=1}^{4000} 1_{Y_i^u \leq 50000})].$$

With $Z_i^l = Y_i^l - \bar{Y}^u$ and $Z_i^u = Y_i^u - \bar{Y}^l$.

We have $\hat{T}_{4000} \leq \chi^2_{2,0.05} = 5.991$, then we accept the hypothesis H_0(see appendices).

11. Discussion

In this part we made the comparison between two statistical test methods in the theory of random sets especially the random intervals, the first is that based on the central limit theorem, using the Hausdorff distance which is most known when the size of the sample is very large.

In this perspective, we proposed another test method based on the maximum likelihood ratio, the test statistic of which depends on unknown probability distributions, which forced us to estimate them, in order to be able to calculate this test statistic and compare it with the value of the chi-square distribution.

The decision taken by the two tests are equivalent, so we can replace the test method based on the central limit theorem by the maximum likelihood ratio test which is a very powerful test.

Appendices

Code from R software of test statistic based on Hausdorff distance

```
data = read.csv("C:/Users/abourakadi/Desktop/simulation.csv",sep=";")
summary (data)
```

Yl		Yu	
Min.	20	Min.	35
1st Qu.	25014	1st Qu.	25029
Median	50008	Median	50023
Mean	50008	Mean	50023
3rd Qu.	75001	3rd Qu.	75016
Max.	99995	Max.	100010

```
a0=50000
b0=50020
N=sqrt(nrow(data))*max(mean(data[,1])-a0,mean(data[,2]-b0))
N
[1]474.3416
T=N/sd(data[,1])
T
[1] 0.01642962
```

Source: Author

Code from R software of test statistic based on maximum likelihood ratio

$table(data[,1] \leq mean(data[,2]))$

FALSE	TRUE
1999	2001

$table(data[,2] \leq mean(data[,1]))$

FALSE	TRUE
2001	1999

$table(data[,1] \leq 50020)$

FALSE	TRUE
1999	2001

$table(data[,2] \leq 50000)$

FALSE	TRUE
2001	1999

```
T=2*4000*(log(2001-1999)-log(2001-1999))
T
[1] 0
```

Source: Author

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Decomposition of continuity via bioperation associated with new sets

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Abstract. In this paper, we introduce some new types of sets in bioperation topological space. Also we discuss some properties and characterize these new sets and finally, we find some relationships between $(\gamma,\gamma')-(\beta,\beta')-A_1$ continuous, $(\gamma,\gamma')-(\beta,\beta')-t$ continuous and $(\gamma,\gamma')-(\beta,\beta')-B_1$ continuous functions.

Keywords: (γ,γ')-open set, $(\gamma,\gamma')-A_1$-set, $(\gamma,\gamma')-A_2$-set, $(\gamma,\gamma')-t$-set, $(\gamma,\gamma')-B_1$-set.

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1. Introduction

Generalized open sets actually play an important role in General Topology and represent an interesting research topics of many topologist worldwide. The study, characterization, modified forms and separation axioms of the notions of generalized open sets, represent an important theme in General Topology and Real analysis. It is well known that, Kasahara [2], introduced and studied the concept of an operation γ on topological spaces. Ogata and Maki [5], introduced and characterized the notions of γ-open sets in a topological space (X, τ) and denote by τ_γ as the collection of all γ-open sets in a topological space (X, τ). Carpintero et al. [3], studied and investigate the concept of (γ, γ')-preopen set in a topological space and described some important properties. Again Carpintero et al. [4], using the notions of bioperations, introduced some new types of sets and obtained a new decomposition of bioperation continuity. In this paper, we define and study new types of sets in a bioperation topological space. Also, we discuss some properties and give a characterization and relationships between these new sets.

Finally, we find some relationships between $(\gamma, \gamma') - (\beta, \beta') - A_1$ continuous, $(\gamma, \gamma') - (\beta, \beta') - t$ continuous and $(\gamma, \gamma') - (\beta, \beta') - B_1$ continuous functions.

2. Preliminaries

Definition 2.1 ([2]). *Let (X, τ) be a topological space. An operation γ on the topology τ is a function from τ into the power set $\mathcal{P}(X)$ of X such that $V \subset V^\gamma$ for each $V \in \tau$, where V^γ denotes the value of γ at V. It is denoted by $\gamma : \tau \to \mathcal{P}(X)$.*

The notions of closure and the interior of a subset A of (X, τ) associated with an operation γ are denoted by $\gamma - \mathrm{Cl}(A)$ and $\gamma - \mathrm{Int}(A)$, respectively.

Definition 2.2 ([5]). *A topological space (X, τ) equipped with two operations, say, γ and γ' defined on τ is called a bioperation topological space, it is denoted by $(X, \tau, \gamma, \gamma')$.*

Definition 2.3 ([5]). *A subset A of a bioperation topological space $(X, \tau, \gamma, \gamma')$ is said to be (γ, γ')-open set if for each $x \in A$ there exist open neighborhoods U and V of x such that $U^\gamma \cup V^{\gamma'} \subset A$. The complement of a (γ, γ')-open set is called a (γ, γ')-closed set. $\tau_{(\gamma,\gamma')}$ denotes the set of all (γ, γ')-open sets in $(X, \tau, \gamma, \gamma')$.*

Definition 2.4 ([5]). *A subset A of a bioperation topological space $(X, \tau, \gamma, \gamma')$, $(\gamma, \gamma') - \mathrm{Cl}(A)$ denotes the intersection of all (γ, γ')-closed sets containing A, that is, $(\gamma, \gamma') - \mathrm{Cl}(A) = \cap\{F : A \subset F, X \backslash F \in \tau_{(\gamma,\gamma')}\}$.*

Definition 2.5. *Let A be any subset of X. The $(\gamma, \gamma') - \mathrm{Int}(A)$ is defined as $(\gamma, \gamma') - \mathrm{Int}(A) = \cup\{U : U$ is a (γ, γ')-open set and $U \subset A\}$.*

Definition 2.6. *Let (X, τ) be a topological space, γ and γ' be operations on τ. A subset A of X is said to be:*

1. *$(\gamma, \gamma')-\alpha$-open if $A \subset (\gamma, \gamma') - \mathrm{Int}((\gamma, \gamma') - \mathrm{Cl}((\gamma, \gamma') - \mathrm{Int}(A)))$*
2. *(γ, γ')-semi-preopen if $A \subset (\gamma, \gamma') - \mathrm{Cl}((\gamma, \gamma') - \mathrm{Int}((\gamma, \gamma') - \mathrm{Cl}(A)))$*
3. *(γ, γ')-regular open if $A = (\gamma, \gamma') - \mathrm{Int}((\gamma, \gamma') - \mathrm{Cl}(A))$.*
4. *$(\gamma, \gamma') - t$ set if $(\gamma, \gamma') - \mathrm{Int}((\gamma, \gamma') - \mathrm{Cl}(S)) = (\gamma, \gamma') - \mathrm{Int}(S)$,*

Definition 2.7. *A function $f : (X, \tau, \gamma, \gamma') \longrightarrow (Y, \sigma, \beta, \beta')$ is said to be:*

1. *$(\gamma, \gamma') - (\beta, \beta')$ continuous if $f^{-1}(V)$ is (γ, γ')-open in X for eve-ry $V \in \sigma_{(\beta,\beta)}$,*
2. *$(\gamma, \gamma') - (\beta, \beta') - \alpha$ continuous if $f^{-1}(V)$ is $(\gamma, \gamma') - \alpha$-open in X for every $V \in \sigma_{(\beta,\beta)}$,*
3. *$(\gamma, \gamma') - (\beta, \beta')$ semi precontinuous if $f^{-1}(V)$ is (γ, γ')-semi-preopen in X for every $V \in \sigma_{(\beta,\beta)}$.*

3. Some subsets in bioperation topological space

Given a bioperation topological space $(X, \tau, \gamma, \gamma')$, we define new sets and find its relations between them.

Definition 3.1. *A subset A of a bioperation topological space $(X, \tau, \gamma, \gamma')$ is called:*

1. *$(\gamma, \gamma') - A_1$ set if $(\gamma, \gamma') - \mathrm{Int}(A) = (\gamma, \gamma') - \mathrm{Int}((\gamma, \gamma') - \mathrm{Cl}((\gamma, \gamma') - \mathrm{Int}(A)))$*
2. *$(\gamma, \gamma') - A_2$ set if $(\gamma, \gamma') - \mathrm{Cl}(A) = (\gamma, \gamma') - \mathrm{Cl}((\gamma, \gamma') - \mathrm{Int}((\gamma, \gamma') - \mathrm{Cl}(A)))$.*

Definition 3.2. *A subset A of a bioperation topological space $(X, \tau, \gamma, \gamma')$ is called:*

1. *$(\gamma, \gamma') - B_1$ set if $A = U \cap V$, where $U \in \tau_{(\gamma,\gamma')}$ and V is a $(\gamma, \gamma') - A_1$ set.*
2. *$(\gamma, \gamma') - B_2$ set if $A = U \cap V$, where $U \in \tau_{(\gamma,\gamma')}$ and V is a $(\gamma, \gamma') - A_2$ set.*

Proposition 3.3. *Let $(X, \tau, \gamma, \gamma')$ be a bioperation topological space. The following properties hold:*

1. *Every (γ, γ')-regular open set is a $(\gamma, \gamma') - A_2$ set.*
2. *Every (γ, γ')-regular closed set is a $(\gamma, \gamma') - A_1$ set.*

Observe that in the following example, the converse of the Proposition 3.3 is not true.

Example 3.4. Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$. Define the operations $\gamma, \gamma' : \tau \longrightarrow \mathcal{P}(X)$ as follows

$$A^{\gamma} = \begin{cases} A & \text{if b} \notin \text{A}, \\ \text{Cl}(A) & \text{if b} \in \text{A}, \end{cases} \quad \text{and} \quad A^{\gamma'} = \begin{cases} \text{Cl}(A) & \text{if b} \notin \text{A}, \\ A & \text{if b} \in \text{A}. \end{cases}$$

The $\tau_{(\gamma,\gamma')} = \{\emptyset, \{b\}, \{a, b\}, \{a, c\}, X\}$. Observe that, $\{a, b\}$ is a $(\gamma, \gamma') - A_2$ set but is not a (γ, γ')-regular open and $\{c\}$ is a $(\gamma, \gamma') - A_1$ set but is not (γ, γ')-regular closed.

Theorem 3.5. *Let $(X, \tau, \gamma, \gamma')$ be a bioperation topological space and A a subset of X. The following properties hold:*

1. *A is a $(\gamma, \gamma') - A_1$ set if, and only if it is (γ, γ')-semi-preclosed.*
2. *A is a $(\gamma, \gamma') - A_2$ set if, and only if it is (γ, γ')-semi-preopen.*

Proof. (1) Let A be a $(\gamma, \gamma') - A_1$ set. Then we have the $(\gamma, \gamma') - \text{Int}(A) = (\gamma, \gamma') - \text{Int}((\gamma, \gamma') - \text{Cl}((\gamma, \gamma') - \text{Int}(A)))$. Then the $(\gamma, \gamma') - \text{Int}((\gamma, \gamma') - \text{Cl}((\gamma, \gamma') - \text{Int}(A))) \subset A$. Therefore A is (γ, γ')-semi-preclosed. Conversely, assume that A is a (γ, γ')-semi-preclosed. Then the $(\gamma, \gamma') - \text{Int}((\gamma, \gamma') - \text{Cl}((\gamma, \gamma') - \text{Int}(A))) \subset A$. Then the $(\gamma, \gamma') - \text{Int}((\gamma, \gamma') - \text{Cl}((\gamma, \gamma') - \text{Int}(A))) \subset (\gamma, \gamma') - \text{Int}(A)$. Since we also have the $(\gamma, \gamma') - \text{Int}(A) \subset (\gamma, \gamma') - \text{Int}((\gamma, \gamma') - \text{Cl}((\gamma, \gamma') - \text{Int}(A)))$. Then the $(\gamma, \gamma') - \text{Int}(A) = (\gamma, \gamma') - \text{Int}((\gamma, \gamma') - \text{Cl}((\gamma, \gamma') - \text{Int}(A)))$. Hence A is a $(\gamma, \gamma') - A_1$ set.

(2) Let A be a $(\gamma, \gamma') - A_2$ set. Then we have the $(\gamma, \gamma') - \text{Cl}(A) = (\gamma, \gamma') - \text{Cl}((\gamma, \gamma') - \text{Int}((\gamma, \gamma') - \text{Cl}(A)))$. Then $A \subset (\gamma, \gamma') - \text{Cl}((\gamma, \gamma') - \text{Int}((\gamma, \gamma') - \text{Cl}(A)))$. Therefore, A is (γ, γ')-semi-preopen.

Conversely, assume that A is (γ, γ')-semi-preopen. We have $A \subset (\gamma, \gamma') - \text{Cl}((\gamma, \gamma') - \text{Int}((\gamma, \gamma') - \text{Cl}(A)))$. Hence $(\gamma, \gamma') - \text{Cl}(A) \subset (\gamma, \gamma') - \text{Cl}((\gamma, \gamma') - \text{Int}((\gamma, \gamma') - \text{Cl}(A)))$. Since the $(\gamma, \gamma') - \text{Cl}((\gamma, \gamma') - \text{Int}((\gamma, \gamma') - \text{Cl}(A))) \subset A$, $(\gamma, \gamma') - \text{Cl}(A) = (\gamma, \gamma') - \text{Cl}((\gamma, \gamma') - \text{Int}((\gamma, \gamma') - \text{Cl}(A)))$. Hence A is $(\gamma, \gamma') - A_2$ set. □

Theorem 3.6. *Let $(X, \tau, \gamma, \gamma')$ be a bioperation topological space and A and B subsets of X. The following properties hold:*

1. *If A and B are $(\gamma, \gamma') - A_1$ sets, then $A \cap B$ is a $(\gamma, \gamma') - A_1$ set.*
2. *If A and B are $(\gamma, \gamma') - A_2$ sets, then $A \cup B$ is a $(\gamma, \gamma') - A_2$ set.*

Proof. (1) Let A and B be $(\gamma, \gamma') - A_1$ sets. Then, the $(\gamma, \gamma') - \text{Int}(A \cap B) \subset (\gamma, \gamma') - \text{Int}((\gamma, \gamma') - \text{Cl}((\gamma, \gamma') - \text{Int}(A \cap B))) \subset (\gamma, \gamma') - \text{Int}((\gamma, \gamma') - \text{Cl}((\gamma, \gamma') - \text{Int}(A)) \cap (\gamma, \gamma') - \text{Cl}((\gamma, \gamma') - \text{Int}(B))) = (\gamma, \gamma') - \text{Int}((\gamma, \gamma') - \text{Cl}((\gamma, \gamma') - \text{Int}(A))) \cap (\gamma, \gamma') - \text{Int}((\gamma, \gamma') - \text{Cl}((\gamma, \gamma') - \text{Int}(B))) = (\gamma, \gamma') - \text{Int}(A \cap B) (\gamma, \gamma') - \text{Int}(A \cap B) \subset (\gamma, \gamma') - \text{Int}((\gamma, \gamma') - \text{Cl}((\gamma, \gamma') - \text{Int}(A \cap B))) \subset (\gamma, \gamma') - \text{Int}(A \cap B) = (\gamma, \gamma') - \text{Int}((\gamma, \gamma') - \text{Cl}((\gamma, \gamma') - \text{Int}(A \cap B)))) = A \cap B$ is a $(\gamma, \gamma') - A_1$ set.

(2) Let A and B be $(\gamma,\gamma')-A_2$ sets. Then the $(\gamma,\gamma')-\mathrm{Cl}(A\cup B) = (\gamma,\gamma')-\mathrm{Cl}((\gamma,\gamma')-\mathrm{Int}((\gamma,\gamma')-\mathrm{Cl}(A)\cup(\gamma,\gamma')-\mathrm{Cl}(B))) \subset (\gamma,\gamma')-\mathrm{Cl}((\gamma,\gamma')-\mathrm{Int}((\gamma,\gamma')-\mathrm{Cl}(A))\cup(\gamma,\gamma')-\mathrm{Int}((\gamma,\gamma')-\mathrm{Cl}(B)))(\gamma,\gamma')-\mathrm{Cl}(A\cup B) \subset (\gamma,\gamma')-\mathrm{Cl}((\gamma,\gamma')-\mathrm{Int}((\gamma,\gamma')-\mathrm{Cl}(A\cup B)))$. Also the $(\gamma,\gamma')-\mathrm{Int}((\gamma,\gamma')-\mathrm{Cl}(A\cup B)) \subset (A\cup B))(\gamma,\gamma')-\mathrm{Cl}((\gamma,\gamma')-\mathrm{Int}((\gamma,\gamma')-\mathrm{Cl}(A\cup B))) \subset (\gamma,\gamma')-\mathrm{Cl}(A\cup B)(\gamma,\gamma')-\mathrm{Cl}(A\cup B) = (\gamma,\gamma')-\mathrm{Cl}((\gamma,\gamma')-\mathrm{Int}((\gamma,\gamma')-\mathrm{Cl}(A\cup B)))) = A\cup B$ is a $(\gamma,\gamma')-A_2$ set. □

Theorem 3.7. *A set A is a $(\gamma,\gamma')-A_1$ set if, and only if its complement is a $(\gamma,\gamma')-A_2$ set.*

Proof. The proof follows from Definition 3.1. □

Theorem 3.8. *Let (X,τ,γ,γ') be a bioperation topological space. A subset A of X is a (γ,γ')-open set if, and only if it is a $(\gamma,\gamma')-\alpha$-open set and $(\gamma,\gamma')-B_1$ set.*

Proof. Let A be a (γ,γ')-open, then the $(\gamma,\gamma')-\mathrm{Cl}(A) = (\gamma,\gamma')-\mathrm{Cl}((\gamma,\gamma')-\mathrm{Int}(A))$. It is clear that $A \subset (\gamma,\gamma')-\mathrm{Int}((\gamma,\gamma')-\mathrm{Cl}((\gamma,\gamma')-\mathrm{Int}(A)))$. Hence A is a $(\gamma,\gamma')-\alpha$-open set. Let $U = A \in \tau_{(\gamma,\gamma')}$ and $V = X$ be a $(\gamma,\gamma')-A_1$ set containing A. Hence $A = U\cap V$; A is a $(\gamma,\gamma')-B_1$ set. Conversely, let A be a $(\gamma,\gamma')-\alpha$-open set and a $(\gamma,\gamma')-B_1$ set. Then $A = U\cap V$, where $U \in \tau_{(\gamma,\gamma')}$ and V is a $(\gamma,\gamma')-A_1$ set. Since V is a $(\gamma,\gamma')-A_1$ set, $(\gamma,\gamma')-\mathrm{Int}(V) = (\gamma,\gamma')-\mathrm{Int}((\gamma,\gamma')-\mathrm{Cl}((\gamma,\gamma')-\mathrm{Int}(V)))$. Since A is $(\gamma,\gamma')-\alpha$-open, $A \subset (\gamma,\gamma')-\mathrm{Int}((\gamma,\gamma')-\mathrm{Cl}((\gamma,\gamma')-\mathrm{Int}(A))) = (\gamma,\gamma')-\mathrm{Int}((\gamma,\gamma')-\mathrm{Cl}((\gamma,\gamma')-\mathrm{Int}(U\cap V))) = (\gamma,\gamma')-\mathrm{Int}((\gamma,\gamma')-\mathrm{Cl}((\gamma,\gamma')-\mathrm{Int}(U)))\cap(\gamma,\gamma')-\mathrm{Int}((\gamma,\gamma')-\mathrm{Cl}((\gamma,\gamma')-\mathrm{Int}(V)))$. Hence $U\cap V = (\gamma,\gamma')-\mathrm{Int}((\gamma,\gamma')-\mathrm{Cl}(U))\cap(\gamma,\gamma')-\mathrm{Int}(V)$. Consider $U\cap V = (U\cap V)\cap U \subset U\cap(\gamma,\gamma')-\mathrm{Int}(V)$; hence $V \subset (\gamma,\gamma')-\mathrm{Int}(V)$; A is (γ,γ')-open. □

Theorem 3.9. *Let (X,τ,γ,γ') be a bioperation topological space and A a subset of X. If A is a $(\gamma,\gamma')-t$-set, then $(\gamma,\gamma')-\mathrm{Int}(A)$ is a $(\gamma,\gamma')-t$-set.*

Proof. If A is a $(\gamma,\gamma')-t$-set, then $(\gamma,\gamma')-\mathrm{Int}((\gamma,\gamma')-\mathrm{Cl}((\gamma,\gamma')-\mathrm{Int}(A))) \subset (\gamma,\gamma')-\mathrm{Int}((\gamma,\gamma')-\mathrm{Cl}(A)) = A$. Follows that $(\gamma,\gamma')-\mathrm{Int}((\gamma,\gamma')-\mathrm{Cl}((\gamma,\gamma')-\mathrm{Int}(A))) \subset (\gamma,\gamma')-\mathrm{Int}(A)$. This implies that $(\gamma,\gamma')-\mathrm{Int}(A)$ is a (γ,γ')-semi closed. Follows from [1], that $(\gamma,\gamma')-\mathrm{Int}(A)$ is a $(\gamma,\gamma')-t$-set. □

Theorem 3.10. *Let (X,τ,γ,γ') be a bioperation topological space and A a subset of X. If A is a $(\gamma,\gamma')-t$-set, then A is $(\gamma,\gamma')-A_1$ set.*

Proof. If A is a $(\gamma,\gamma')-t$-set, by Theorem 3.9 and then $(\gamma,\gamma')-\mathrm{Int}((\gamma,\gamma')-\mathrm{Cl}((\gamma,\gamma')-\mathrm{Int}(A))) \subset A$. This implies that A is a (γ,γ')-semi preclosed and then by Theorem 3.5, A is a $(\gamma,\gamma')-A_1$ set. □

The following example, shows that the converse of Theorem 3.10, is not necessarily true.

Example 3.11. Observe that in Example 3.4:

1. $(\gamma,\gamma')-A_1$ sets $=\{\emptyset,\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\},X\}$
2. $(\gamma,\gamma')-t$ sets $=\{\emptyset,\{b\},\{c\},\{a,c\},\{b,c\},X\}$

We can see that $\{a,b\}$ is a $(\gamma,\gamma')-A_1$ set but is not a $(\gamma,\gamma')-t$ set.

4. Decomposition of bioperation-continuity

In this section, we define and study the notions of: $(\gamma,\gamma')-(\beta,\beta')-A_1$ continuous, $(\gamma,\gamma')-(\beta,\beta')-A_2$ continuous, $(\gamma,\gamma')-(\beta,\beta')-t$ continuous and $(\gamma,\gamma')-(\beta,\beta')-B_1$ continuous and find some relation between them and with others forms of $(\gamma,\gamma')-(\beta,\beta')$ continuous functions

Definition 4.1. *A function $f:(X,\tau,\gamma,\gamma')\longrightarrow(Y,\sigma,\beta,\beta')$ is said to be:*

1. *$(\gamma,\gamma')-(\beta,\beta')-A_1$ continuous if $f^{-1}(V)$ is a $(\gamma,\gamma')-A_1$ set in X for each $V\in\sigma_{(\beta,\beta')}$,*
2. *$(\gamma,\gamma')-(\beta,\beta')-A_2$ continuous if $f^{-1}(V)$ is a $(\gamma,\gamma')-A_2$ set in X for each $V\in\sigma_{(\beta,\beta')}$,*
3. *$(\gamma,\gamma')-(\beta,\beta')-t$ continuous if $f^{-1}(V)$ is a $(\gamma,\gamma')-t$ set in X for each $V\in\sigma_{(\beta,\beta')}$,*
4. *$(\gamma,\gamma')-(\beta,\beta')-B_1$ continuous if $f^{-1}(V)$ is a $(\gamma,\gamma')-B_1$ set in X for each $V\in\sigma_{(\beta,\beta')}$,*

Theorem 4.2. *Let (X,τ,γ,γ') and (Y,σ,β,β') be two bioperation topological spaces. A function $f:(X,\tau,\gamma,\gamma')\longrightarrow(Y,\sigma,\beta,\beta')$ is $(\gamma,\gamma')-(\beta,\beta')-A_2$ continuous if, and only if it is $(\gamma,\gamma')-(\beta,\beta')$-semi-pre continuous.*

Proof. The proof follows from Definition 4.1(2) and Theorem 3.5(2). □

Theorem 4.3. *Let (X,τ,γ,γ') and (Y,σ,β,β') be two bioperation topological spaces.A function $f:(X,\tau,\gamma,\gamma')\longrightarrow(Y,\sigma,\beta,\beta')$ is $(\gamma,\gamma')-(\beta,\beta')$ continuous if, and only if it is $(\gamma,\gamma')-(\beta,\beta')-\alpha$ continuous and $(\gamma,\gamma')-(\beta,\beta')-B_1$ continuous.*

Proof. The proof follows from Definition 2.7 and Theorem 3.8. □

Theorem 4.4. *Let (X,τ,γ,γ') be a (γ,γ')-extremally disconnected space. If $f:(X,\tau,\gamma,\gamma')\longrightarrow(Y,\sigma,\beta,\beta')$ is $(\gamma,\gamma')-(\beta,\beta')-t$ continuous, then f is $(\gamma,\gamma')-(\beta,\beta')-A_1$ continuous.*

Proof. The proof follows from Definitions 4.1 and Theorem 3.10. □

The following theorem shows that if $f : (X, \tau, \gamma, \gamma') \longrightarrow (Y, \sigma, \beta, \beta')$ is $(\gamma, \gamma') - (\beta, \beta') - A_1$ continuous and $(\gamma, \gamma') - (\beta, \beta')$ continuous, then it is $(\gamma, \gamma') - (\beta, \beta') - B_1$ continuous.

Theorem 4.5. *Let $(X, \tau, \gamma, \gamma')$ and $(Y, \sigma, \beta, \beta')$ be two bioperation topological spaces. A function $f : (X, \tau, \gamma, \gamma') \longrightarrow (Y, \sigma, \beta, \beta')$ is $(\gamma, \gamma') - (\beta, \beta') - A_1$ continuous and $(\gamma, \gamma') - (\beta, \beta')$ continuous, then it is $(\gamma, \gamma') - (\beta, \beta') - B_1$ continuous.*

Proof. The proof follows from Definitions 2.7, 3.2 and 4.1. □

Conclusions

1. We define the new sets: $(\gamma, \gamma') - A_1$ set, $(\gamma, \gamma') - A_2$ set, $(\gamma, \gamma') - B_1$ set, $(\gamma, \gamma') - B_2$ set, $(\gamma, \gamma') - t$ set and find the relationship between them.
2. We find the relationships between $(\gamma, \gamma') - (\beta, \beta') - A_1$ continuous, $(\gamma, \gamma') - (\beta, \beta') - t$ continuous and $(\gamma, \gamma') - (\beta, \beta') - B_1$ continuous functions and with others forms of $(\gamma, \gamma') - (\beta, \beta')$ continuous functions.

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Hyper BN-algebras: hyperstructure theory applied to BN-algebras

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Abstract. In this paper, we introduce the notion of hyper BN-algebras. We prove that it is a generalization of BN-algebras. We also compare it to other existing hyper algebras. We give some routine properties of hyper BN-algebra. We also give a certain condition for when a hyper BN-algebra becomes a hyper B-algebra and a hypergroup. Finally, we define the notion of hyper subBN-algebra and obtain some preliminary properties.

Keywords: hyper BN-algebra, hyper subBN-algebra.

1. Introduction

BCK-algebras were initiated by Y. Imai and K. Iséki [7] in 1966. In the same year, K. Iséki [9] introduced BCI-algebras to generalize BCK-algebras. BCI-algebras are logical algebras which are the algebraic formulations of the set difference and the implicational functor in logical systems. The names of BCK- and BCI-algebras are originated from the combinators B, C, K, and I in combinatory logic. Another generalization of BCK-algebras was introduced by Y. Komori [12] called BCC-algebras and proved that the class of BCC-algebras were not a variety.

On the other hand, B-algebras were introduced by Neggers, J. and Kim, H.S. [14] in 2002. It is a class of algebra which coincides with the class of groups. Many papers were published regarding the theory of B-algebra. In

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2013, another class of algebra was introduced by Kim, C.B., and Kim H.S. [11] called BN-algebras.

In 1934, F. Marty [13] brought the concept of the algebraic hyperstructure theory to the world at the 8th Congress of Scandinavian Mathematicians. This is a generalization of the classical algebraic structure. Several applications of the hyperstructure theory were given by P. Corsini and V. Leoreanu [3]. Papers were also published on providing examples of hyperstructures in inheritance issues in genetics [4] and in the interaction of elementary particles in physics [6].

In 2006, Xin, X.L. [17] applied the concept of hyperstructure to BCI-algebras, thereby forming hyper BCI-algebras. Hyper BCI-algebras are the generalization of the hyperstructure counterpart of BCK-algebras called hyper BCK-algebras introduced by Jun, Y.B. et al [10] in 2000. The theory of hyper BCK-algebras is widely studied specifically on β-relations [15] and state operators [18]. The hyperstructure counterpart of the BCC-algebras was defined by R. Borzooei, W. Dudek, and N. Koohestani [1] in 2006. They have shown that the class of hyper BCC-algebras were a generalization of the class of hyper BCK-algebras. In 2016, Indangan, R. and Petalcorin Jr., G. [8] introduced the notion of hyper GR-algebra, which is based on hyper BCI-algebra by culling out one property and replacing it with another property requirement. A hyperstructure counterpart of B-algebras was formulated by Endam J. in his dissertation and was called hyper B-algebras. Some other hyper algebras include hyper quasi-MV algebras as introduced by Chen, W. and Davvaz, B. [2] in 2018.

Motivated by the introduction of various hyper algebras, we will introduce the notion of hyper BN-algebras and show that this hyper algebra is a generalization of BN-algebras. Furthermore, we will differentiate hyper BN-algebras to hyper BCI-algebras, hyper BCC-algebras, hyper B-algebras, and hyper GR-algebras. Moreover, we will investigate the structure of hyper BN-algebra with regard to its properties and related concepts.

2. Preliminaries

This section provide some preliminary concepts and results needed for this paper.

Definition 2.1 ([11])**.** A *BN-algebra* is an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the following axioms:

$$(BN_1)\ \ x * x = 0,$$

$$(BN_2)\ \ x * 0 = x, \text{ and}$$

$$(BN_3)\ \ (x * y) * z = (0 * z) * (y * x),$$

for any $x, y, z \in X$.

Definition 2.2 ([5])**.** Define $\mathcal{P}(H)$ to be the power set of H and $\mathcal{P}^*(H) = \mathcal{P}(H) \setminus \{\emptyset\}$. A *hyperoperation* on a nonempty set H is a function $\circledast : H \times H \to \mathcal{P}^*(H)$. The value $(x, y) \in H \times H$ under $\circledast$ is defined by $x \circledast y$. If $x \in H$ and $\emptyset \neq A, B \subseteq H$, then

$$(i)\ A \circledast B = \bigcup_{a \in A, b \in B} a \circledast b;$$

$$(ii)\ A \circledast x = A \circledast \{x\} \text{ and } x \circledast B = \{x\} \circledast B.$$

Definition 2.3 ([3])**.** Let H be a nonempty set and $\circledast$ be a hyperoperation. Then $(H, \circledast)$ is called a *hypergroup* if it satisfies the following axioms: for all $x, y, z \in H$,

$$(H_1)\ x \circledast (y \circledast z) = (x \circledast y) \circledast z, \text{ and}$$

$$(H_2)\ x \circledast H = H = H \circledast x.$$

A hypergroup $(H, \circledast)$ is said to be *commutative* if $x \circledast y = y \circledast x$ for all $x, y \in H$.

Definition 2.4 ([8])**.** Let $(H, \circledast, 0)$ be a hyper algebra, $x, y \in H$, and $\emptyset \neq A, B \subseteq H$. Then $x \ll y$ if and only if $0 \in x \circledast y$ and $A \ll B$ if and only if for any $a \in A$, there exists $b \in B$ such that $a \ll b$. $\ll$ is called a *hyper order* on H. Let $x \in H$ and $A \subseteq H$. $x \ll A$ implies $\{x\} \ll A$. Similarly, $A \ll x$ implies $A \ll \{x\}$.

Definition 2.5 ([17])**.** Let H be a nonempty set and $\circledast$ be a hyperoperation on H. Then $(H, \circledast, 0)$ is called a *hyper BCI-algebra*, if $0 \in H$ and the following conditions hold: for all $x, y, z \in H$,

$$(HI_1)\ (x \circledast z) \circledast (y \circledast z) \ll x \circledast y,$$

$$(HI_2)\ (x \circledast y) \circledast z = (x \circledast z) \circledast y,$$

$$(HI_3)\ x \ll x,$$

$$(HI_4)\ x \ll y \text{ and } y \ll x \text{ imply } x = y, \text{ and}$$

$$(HI_5)\ 0 \circledast (0 \circledast x) \ll x,\ x \neq 0.$$

Example 2.1 ([17])**.** Let $H = \{0, 1, 2\}$ be a set. If we define a hyperoperation "$\circledast$" on H as follows:

$\circledast$	0	1	2
0	$\{0, 1\}$	$\{0, 1\}$	$\{0, 1\}$
1	$\{1\}$	$\{0, 1\}$	$\{0, 1\}$
2	$\{2\}$	$\{1, 2\}$	$\{0, 1, 2\}$

then $(H, \circledast, 0)$ is a hyper BCI-algebra.

Definition 2.6 ([1])**.** Let H be a nonempty set and $\circledast$ be a hyperoperation on H. Then $(H, \circledast, 0)$ is called a *hyper BCC-algebra*, if $0 \in H$ and the following conditions hold: for all $x, y, z \in H$,

(HC_1) $(x \circledast z) \circledast (y \circledast z) \ll x \circledast y$,

(HC_2) $0 \circledast x = \{0\}$,

(HC_3) $x \circledast 0 = \{x\}$, and

(HC_4) $x \ll y$ and $y \ll x$ imply $x = y$.

Example 2.2 ([1])**.** Let $H = \{0, 1, 2, 3\}$ be a set. If we define a hyperoperation "$\circledast$" on H as follows:

$\circledast$	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{0\}$	$\{0\}$
2	$\{2\}$	$\{2\}$	$\{0, 1\}$	$\{2\}$
3	$\{3\}$	$\{1, 3\}$	$\{0, 1, 3\}$	$\{0, 1, 3\}$

then $(H, \circledast, 0)$ is a hyper BCC-algebra.

Definition 2.7 ([8])**.** Let H be a nonempty set and $\circledast$ be a hyperoperation on H. Then $(H, \circledast, 0)$ is called a *hyper GR-algebra*, if $0 \in H$ and the following conditions hold: for all $x, y, z \in H$,

(HGR_1) $(x \circledast z) \circledast (y \circledast z) \ll x \circledast y$,

(HGR_2) $(x \circledast y) \circledast z = (x \circledast z) \circledast y$,

(HGR_3) $x \ll x$,

(HGR_4) $0 \circledast (0 \circledast x) \ll x$, $x \neq 0$, and

(HGR_5) $(x \circledast y) \circledast z \ll y \circledast z$.

Example 2.3 ([8])**.** Let $H = \{0, 1, 2\}$ be a set. If we define a hyperoperation "$\circledast$" on H as follows:

$\circledast$	0	1	2
0	$\{0\}$	$\{0\}$	$\{1\}$
1	$\{1\}$	$\{0, 1\}$	$\{0, 1\}$
2	$\{2\}$	$\{0, 2\}$	$\{0, 1, 2\}$

then $(H, \circledast, 0)$ is a hyper GR-algebra.

Definition 2.8 ([16])**.** Let H be a nonempty set and $\circledast$ be a hyperoperation on H. Then $(H, \circledast, 0)$ is called a *hyper B-algebra*, if $0 \in H$ and the following conditions hold: for all $x, y, z \in H$,

(HB_1) $x \ll x$,

(HB_2) $x \circledast H = H = H \circledast x$, and

(HB_3) $(x \circledast y) \circledast z = x \circledast [z \circledast (0 \circledast y)]$.

Example 2.4 ([16])**.** Let $H = \{0, a, b, c\}$ be a set. If we define a hyperoperation "$\circledast$" on H as follows:

$\circledast$	0	a	b	c
0	$\{0\}$	$\{0\}$	$\{0, a, b\}$	$\{0, a, c\}$
a	$\{0\}$	$\{0\}$	$\{0, a, b\}$	$\{0, a, c\}$
b	$\{0, a, b\}$	$\{0, a, b\}$	$\{0, a, b\}$	$\{b, c\}$
b	$\{0, a, c\}$	$\{0, a, c\}$	$\{b, c\}$	$\{0, a, c\}$

then $(H, \circledast, 0)$ is a hyper B-algebra.

3. Hyper BN-algebras

We will formally define in this section what we meant by a hyper BN-algebra. We also present some of its routine properties. We will also present a characterization of a hyper BN-algebra.

Definition 3.1. Let H be a nonempty set and $\circledast$ be a hyperoperation on H. Then $(H, \circledast, 0)$ is called a *hyper BN-algebra*, if $0 \in H$ and the following conditions hold: for all $x, y, z \in H$,

(HBN_1) $x \ll x$,

(HBN_2) $x \circledast 0 = \{x\}$, and

(HBN_3) $(x \circledast y) \circledast z = (0 \circledast z) \circledast (y \circledast x)$.

The following example will show that the class of BN-algebras is just a subclass of the class of hyper BN-algebras.

Example 3.1. Let $(X, *, 0)$ be a BN-algebra. Define a hyperoperation "$\circledast$" on X by $x \circledast y = \{x * y\}$ for all $x, y \in X$. Note that, by (BN_1), $0 = x * x$. Thus, $x \circledast x = \{0\}$ which implies that $x \ll x$ holding (HBN_1). (HBN_2) and (HBN_3) follow directly from (BN_2) and (BN_3), respectively. Therefore, $(X, \circledast, 0)$ is a hyper BN-algebra.

Example 3.2. Let $H = \{0, a, b\}$ be a set. If we define a hyperoperation "$\circledast$" on H as follows:

$\circledast$	0	a	b
0	$\{0\}$	$\{a\}$	$\{b\}$
a	$\{a\}$	$\{0, a\}$	$\{b\}$
b	$\{b\}$	$\{b\}$	$\{0, b\}$

then $(H, \circledast, 0)$ is a hyper BN-algebra. However, it is not a hyper BCI-algebra nor a hyper GR-algebra because it violates the condition $(x \circledast y) \circledast z = (x \circledast z) \circledast y$ which is common to both hyper algebras. Particularly, the condition is violated when $x = b$, $y = a$, and $z = b$, that is, $(b \circledast a) \circledast b = \{0, b\}$ and $(b \circledast b) \circledast a = \{a, b\}$ which imply that $(b \circledast a) \circledast b \neq (b \circledast b) \circledast a$. Also, $(H, \circledast, 0)$ is not a hyper BCC-algebra because for $x = a$, $0 \circledast a = \{a\} \neq \{0\}$. Finally, H is not a hyper B-algebra because it violates (HB_2). Particularly, when $x = b$, we have $b \circledast H = \{0, b\} \neq H$.

Remark 3.1. In Example 2.1, the hyper BCI-algebra $(H, \circledast, 0)$ is not a hyper BN-algebra because it violates condition (HBN_3). In particular, the violation is incurred when $x = 1$, $y = 0$, and $z = 0$, that is, $(1 \circledast 0) \circledast 0 = \{1\}$ and $(0 \circledast 0) \circledast (0 \circledast 1) = \{0, 1\}$ implying that $(1 \circledast 0) \circledast 0 \neq (0 \circledast 0) \circledast (0 \circledast 1)$. Also, in Example 2.2, the hyper BCC-algebra $(H, \circledast, 0)$ is not a hyper BN-algebra because it does not satisfy (HBN_3) specifically when $x = 1$, $y = 0$, and $z = 0$, that is, $(1 \circledast 0) \circledast 0 = \{1\}$ and $(0 \circledast 0) \circledast (0 \circledast 1) = \{0\}$ which says that $(1 \circledast 0) \circledast 0 \neq (0 \circledast 0) \circledast (0 \circledast 1)$. Also, in Example 2.3, the hyper GR-algebra $(H, \circledast, 0)$ is not a hyper BN-algebra because it violates condition (HBN_3). Particularly, when $x = 1$, $y = 0$ and $z = 0$, we have $(1 \circledast 0) \circledast 0 = \{0, 1, 2\}$ and $(0 \circledast 0) \circledast (0 \circledast 1) = \{0\}$ implying that $(1 \circledast 0) \circledast 0 \neq (0 \circledast 0) \circledast (0 \circledast 1)$. Finally, the hyper B-algebra in Example 2.4 is not a hyper BN-algebra because $2 \circledast 0 = \{0, 1, 2\} \neq \{2\}$. Together with Example 3.2, we can conclude that hyper BN-algebra is different from hyper BCI-algebra, hyper BCC-algebra, hyper GR-algebra, and hyper B-algebra.

Example 3.3. Let $H = \{0, 1, 2\}$ be a set. If we define a hyperoperation "$\circledast$" on H as follows:

$\circledast$	0	1	2
0	$\{0\}$	$\{1\}$	$\{2\}$
1	$\{1\}$	$\{0, 2\}$	$\{0, 1\}$
2	$\{2\}$	$\{0, 1\}$	$\{0, 1\}$

then $(H, \circledast, 0)$ is a hyper BN-algebra. Also, it is a hyper B-algebra. However, it is not a hyper BCI-algebra because $0 \ll 1$ and $1 \ll 0$ but $0 \neq 1$. It is also not a hyper GR algebra because it violates (HGR_1): $(x \circledast z) \circledast (y \circledast z) \ll x \circledast y$. The violation is incurred when $x = y = z = 0$, that is $(0 \circledast 0) \circledast (0 \circledast 0) = \{0, 1, 2\}$ and $0 \circledast 0 = \{0, 1\}$ but $\{0, 1, 2\} \not\ll \{0, 1\}$ since $2 \not\ll 0$ and $2 \not\ll 1$.

Example 3.4. Let $H = \{0, 1, 2, 3\}$ be a set. If we define a hyperoperation "$\circledast$" on H as follows:

$\circledast$	0	1	2	3
0	$\{0\}$	$\{1\}$	$\{3\}$	$\{2\}$
1	$\{1\}$	$\{0, 1\}$	$\{0, 1, 2\}$	$\{0, 1, 3\}$
2	$\{2\}$	$\{0, 1, 3\}$	$\{0, 1, 2, 3\}$	$\{0, 2\}$
3	$\{3\}$	$\{0, 1, 2\}$	$\{0, 3\}$	$\{0, 1, 2, 3\}$

then $(H, \circledast, 0)$ is a hyper BN-algebra.

Example 3.5. Let $\mathbb{Z}$ be the set of integers. Define a hyperoperation "$\circledast$" on $\mathbb{Z}$ by:

$$x \circledast y = \begin{cases} \{x\}, & \text{if } y = 0 \\ \{y\}, & \text{if } x = 0 \\ \{x - y, y - x, x + y\}, & \text{otherwise.} \end{cases}$$

Then, we can show that $(\mathbb{Z}, \circledast, 0)$ is a hyper BN-algebra. Note that, the same holds when $\mathbb{Z}$ is replaced by $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$.

Example 3.6. Let $\mathbb{N} = \{0, 1, 2, \ldots\}$. Define a hyperoperation "$\circledast$" on $\mathbb{N}$ by:

$$x \circledast y = \begin{cases} \{x\}, & \text{if } y = 0 \\ \{y\}, & \text{if } x = 0 \\ \{0, x\}, & \text{if } x = y \neq 0 \\ \{x, y\}, & \text{otherwise.} \end{cases}$$

Then we can show that $(\mathbb{N}, \circledast, 0)$ is a hyper BN-algebra.

Observe that by (HBN_1) and (HBN_2), we have $0 \circledast 0 = \{0\}$. Thus, we have the following remark:

Remark 3.2. In any hyper BN-algebra, $0 \circledast 0 = \{0\}$.

Hereinafter, we may write the singleton set $\{x\}$ just by x as needed for the purpose of simplification.

Theorem 3.1. *In any hyper BN-algebra H, the following hold: for any $x, y, z \in H$ and $\emptyset \neq A, B, C \subseteq H$,*

(i) $x \circledast x = \{x\} \Leftrightarrow x = 0$,

(ii) $x \ll 0 \Rightarrow x = 0$,

(iii) $0 \circledast (0 \circledast x) = \{x\}$,

(iv) $0 \circledast (x \circledast y) = y \circledast x$,

(v) $x \circledast y = (0 \circledast y) \circledast (0 \circledast x)$,

(vi) $(0 \circledast x) \circledast y = (0 \circledast y) \circledast x$,

(vii) $x \ll y \Rightarrow y \ll x$,

(viii) $0 \circledast x = 0 \circledast y \Rightarrow x = y$,

(ix) $(x \circledast z) \circledast (y \circledast z) = (z \circledast y) \circledast (z \circledast x)$,

(x) $A \ll A$,

(xi) $A \subseteq B \Rightarrow A \ll B$,

(xii) $A \subseteq B$ *and* $B \ll C$ *imply* $A \ll C$,

(xiii) $A \ll \{0\} \Rightarrow A = \{0\}$,

(xiv) $A \circledast \{0\} = \{0\} \Rightarrow A = \{0\}$, *and*

(xv) $(A \circledast B) \circledast C = (0 \circledast C) \circledast (B \circledast A)$.

Proof. Let H be a hyper BN-algebra, $x, y, z \in H$ and $\emptyset \neq A, B, C \subseteq H$.

(*i*) Note that by (HBN_1), $0 \in x \circledast x = \{x\}$. Thus, $x = 0$. The converse follows from Remark 3.2.

(*ii*) $x \ll 0$ means $0 \in x \circledast 0$. But $x \circledast 0 = \{x\}$ by (HBN_2). Thus, $x = 0$.

(*iii*) Let $y = z = 0$ in (HBN_3). Then $(x \circledast 0) \circledast 0 = (0 \circledast 0) \circledast (0 \circledast x)$. By (HBN_2) and Remark 3.2, we have $0 \circledast (0 \circledast x) = \{x\}$.

(*iv*) Let $x = y$, $y = x$ and $z = 0$ in (HBN_3). Then $(y \circledast x) \circledast 0 = (0 \circledast 0) \circledast (x \circledast y)$. By (HBN_2) and Remark 3.2, we have $0 \circledast (x \circledast y) = y \circledast x$.

(*v*) By (HBN_2) and (HBN_3), $x \circledast y = (x \circledast 0) \circledast y = (0 \circledast y) \circledast (0 \circledast x)$,

(*vi*) By (HBN_3), $(0 \circledast x) \circledast y = (0 \circledast y) \circledast (x \circledast 0)$. Applying (HBN_2), we get the desired result.

(*vii*) Since $0 \in x \circledast y$, we have, by (iv) $\{0\} = 0 \circledast 0 \subseteq 0 \circledast (x \circledast y) = y \circledast x$. Thus, $0 \in y \circledast x$ and the result follows.

(*viii*) If $0 \circledast x = 0 \circledast y$, then by (iii), we have $\{x\} = 0 \circledast (0 \circledast x) = 0 \circledast (0 \circledast y) = \{y\}$. Thus, $x = y$.

(*ix*) By (HBN_3) and (iv), we have $(x \circledast z) \circledast (y \circledast z) = (0 \circledast (y \circledast z)) \circledast (z \circledast x) = (z \circledast y) \circledast (z \circledast x)$.

(*x*) From (HBN_1), $0 \in a \circledast a$ for all $a \in A$. Hence, $A \ll A$.

(*xi*) Since $A \ll A$, by (x), for all $a \in A$, $a \ll a$. Take $b = a$ since $A \subseteq B$. Then there exists $b \in B$ such that $a \ll b$. Thus, $A \ll B$.

(*xii*) Let $a \in A$. Then $a \in B$ since $A \subseteq B$ and there is an element $c \in C$ such that $a \ll c$ because $B \ll C$. Thus, $A \ll C$.

(*xiii*) Assume $A \ll \{0\}$ and let $a \in A$. Then $a \ll 0$ and so $a = 0$ by (ii). Hence, $A = \{0\}$.

(*xiv*) Suppose $A \circledast \{0\} = \{0\}$ and let $a \in A$. Then $a \circledast 0 = \{0\}$. By (HBN_2), $a \circledast 0 = \{a\}$. Thus, $a = 0$. Consequently, $A = \{0\}$.

(*xv*) Straightforward. □

The next result says that the concepts of hyper BN-algebras and hyper BCC-algebras are almost entirely different.

Theorem 3.2. *A hyper BN-algebra with at least two elements is not a hyper BCC-algebra.*

Proof. Let H be a hyper BN-algebra and let $x \in H$ such that $x \neq 0$. Suppose on the contrary that H is also a hyper BCC-algebra. By (HC_2), $0 \circledast x = \{0\}$. This means that $0 \ll x$. By Theorem 3.1(vii), $x \ll 0$. Thus, $x = 0$ by Theorem 3.1(ii). This is a contradiction to our assumption that $x \neq 0$. Therefore, H must not be a hyper BCC-algebra. □

Remark 3.3. The only intersection of the class of hyper BN-algebras and the class of hyper BCC-algebras is when $H = \{0\}$.

Definition 3.2. A hyper algebra $(H, \circledast, 0)$ is said to be 0-*commutative* if $x \circledast (0 \circledast y) = y \circledast (0 \circledast x)$ for all $x, y \in H$.

Theorem 3.3. *A hyper BN-algebra H is 0-commutative.*

Proof. Let $x, y \in H$. Then by Theorem 3.1(iii), $x \circledast (0 \circledast y) = [0 \circledast (0 \circledast x)] \circledast (0 \circledast y)$. Applying (HBN_3), we have $[0 \circledast (0 \circledast x)] \circledast (0 \circledast y) = [0 \circledast (0 \circledast y)] \circledast [(0 \circledast x) \circledast 0]$. Finally, by Theorem 3.1(iii) and (HBN_2), $[0 \circledast (0 \circledast y)] \circledast [(0 \circledast x) \circledast 0] = y \circledast (0 \circledast x)$. Therefore, $x \circledast (0 \circledast y) = y \circledast (0 \circledast x)$. □

The next result is a characterization of a hyper BN-algebra.

Theorem 3.4. $(H, \circledast, 0)$ *is a hyper BN-algebra if and only if it satisfies the following conditions: for all $x, y \in H$,*

(i) $x \ll x$,

(ii) $x \circledast 0 = \{x\}$,

(iii) H *is* 0-*commutative, and*

(iv) $0 \circledast (x \circledast y) = y \circledast x$.

Proof. ($\Rightarrow$): (i) and (ii) are just (HBN_1) and (HBN_2), respectively. (iii) follows from Theorem 3.3. Finally, (iv) follows from Theorem 3.1(iv).
($\Leftarrow$): We only need to show (HBN_3). Let $x, y, z \in H$. Now, by (iv), $(0 \circledast z) \circledast (y \circledast x) = (0 \circledast z) \circledast [0 \circledast (x \circledast y)]$. By (iii), $(0 \circledast z) \circledast [0 \circledast (x \circledast y)] = (x \circledast y) \circledast [0 \circledast (0 \circledast z)]$. Applying (iv) again, we have $(x \circledast y) \circledast [0 \circledast (0 \circledast z)] = (x \circledast y) \circledast (z \circledast 0)$. Finally by (ii), we have $(x \circledast y) \circledast (z \circledast 0) = (x \circledast y) \circledast z$. Therefore, $(x \circledast y) \circledast z = (0 \circledast z) \circledast (y \circledast x)$ holding (HBN_3). □

Definition 3.3. A hyper BN-algebra H is said to be *commutative* if for all $x, y \in H$, $x \circledast y = y \circledast x$.

Example 3.7. The hyper BN-algebras in Example 3.2 and Example 3.3 are commutative while the hyper BN-algebra in Example 3.4 is not commutative because $2 \circledast 0 = \{2\} \neq \{3\} = 0 \circledast 2$.

Theorem 3.5. *Let H be a hyper BN-algebra. Then H is commutative if and only if $0 \circledast x = \{x\}$ for all $x \in H$.*

Proof. Suppose H is a hyper BN-algebra. If H is commutative, then by (HBN_2), $\{x\} = x \circledast 0 = 0 \circledast x$ for all $x \in H$. For the converse, let $0 \circledast x = \{x\}$ for all $x \in H$ and let $y, z \in H$. Then by Theorem 3.3 and the assumption, $y \circledast z = y \circledast (0 \circledast z) = z \circledast (0 \circledast y) = z \circledast y$. Hence, H is commutative. □

Theorem 3.6. *Let $(H_1, \circledast_1, 0)$ and $(H_2, \circledast_2, 0)$ be hyper algebras such that $H_1 \cap H_2 = \{0\}$. Suppose $H = H_1 \cup H_2$. Define a hyperoperation $\circledast$ on H by*

$$x \circledast y = \begin{cases} x \circledast_1 y, & \text{if } x, y \in H_1 \\ x \circledast_2 y, & \text{if } x, y \in H_2 \\ \{x, y\}, & \text{otherwise.} \end{cases}$$

H_1 and H_2 are commutative hyper BN-algebras if and only if H is a commutative hyper BN-algebra.

Proof. Suppose H_1 and H_2 are commutative hyper BN-algebras. Let $x \in H$. If $x \in H_1$, then (HBN_1) follows because H_1 is a hyper BN-algebra. Similarly, if $x \in H_2$, then (HBN_1) holds. Thus, for all $x \in H$, (HBN_1) holds for H. Note that $0 \in H$. Further, $0 \in H_1$ and $0 \in H_2$. Let $x \in H$. If $x \in H_1$, (HBN_2) holds because H_1 is a hyper BN-algebra. Same argument when $x \in H_2$. Thus, (HBN_2) holds for H. Since either H_1 or H_2 is commutative, whether $z \in H_1$ or $z \in H_2$, we have $0 \circledast z = \{z\}$ by Theorem 3.5. And so, (HBN_3) reduces to $(x \circledast y) \circledast z = z \circledast (y \circledast x)$. Let $x, y, z \in H$. (HBN_3) holds clearly when $x, y, z \in H_1$ and $x, y, z \in H_2$ since H_1 and H_2 are hyper BN-algebras. So we are left with the following cases: $x \in H_1$ and $y, z \in H_2$, $y \in H_1$ and $x, z \in H_2$, $z \in H_1$ and $x, y \in H_2$, $x, y \in H_1$ and $z \in H_2$, $x, z \in H_1$ and $y \in H_2$, and $y, z \in H_1$ and $x \in H_2$.

Case 1. $x \in H_1$ and $y, z \in H_2$.

$$(x \circledast y) \circledast z = \{x, y\} \circledast z = (x \circledast z) \cup (y \circledast_2 z) = \{x, z\} \cup (y \circledast_2 z),$$

and

$$z \circledast (y \circledast x) = z \circledast \{x, y\} = (z \circledast x) \cup (z \circledast_2 y) = \{x, z\} \cup (z \circledast_2 y).$$

Note that $y \circledast_2 z = z \circledast_2 y$ since H_2 is commutative. Thus, (HBN_3) holds for this case.

Case 2. $y \in H_1$ and $x, z \in H_2$.

$$(x \circledast y) \circledast z = \{x, y\} \circledast z = (x \circledast_2 z) \cup (y \circledast z) = (x \circledast_2 y) \cup \{y, z\},$$

and

$$z \circledast (y \circledast x) = z \circledast \{x, y\} = (z \circledast_2 x) \cup (z \circledast y) = (z \circledast_2 x) \cup \{y, z\}.$$

Note that $x \circledast_2 z = z \circledast_2 x$ since H_2 is commutative. Thus, (HBN_3) holds for this case.

Case 3. $z \in H_1$ and $x, y \in H_2$.

$$(x \circledast y) \circledast z = (x \circledast_2 y) \circledast z = \bigcup_{a \in x \circledast_2 y} \{a, z\}.$$

Note that $x \circledast_2 y = y \circledast_2 x$ since H_2 is commutative. Thus, we have

$$z \circledast (y \circledast x) = z \circledast (y \circledast_2 x) = \bigcup_{a \in y \circledast_2 x} \{a, z\}.$$

Thus, (HBN_3) holds for this case.
The other three cases can be proven similarly. Thus, for all $x, y, z \in H$, (HBN_3) holds. Therefore, $(H, \circledast, 0)$ is a hyper BN-algebra. Commutativity follows directly from the definition of the hyperoperation of H. The converse of this theorem is clear. □

4. Hyper BN-algebra, Hyper B-algebra, and hypergroup

We have shown in Example 3.2 that the given hyper BN-algebra is not a hyper B-algebra. This means that in general, a hyper BN-algebra may not be a hyper B-algebra. In this section, we provide a certain condition for when a hyper BN-algebra becomes a hyper B-algebra and a hypergroup.

Definition 4.1. A hyper algebra $(H, \circledast, 0)$ is said to have a *condition* (D) if $(x \circledast y) \circledast z = x \circledast (z \circledast y)$ for all $x, y, z \in H$.

Example 4.1. The hyper BN-algebra in Example 3.3 satisfies condition (D). The hyper BN-algebra in Example 3.2 does not satisfy condition (D) because $(a \circledast b) \circledast b = \{0, b\} \neq \{a, b\} = a \circledast (b \circledast b)$.

Theorem 4.1. *If H is a hyper BN-algebra with the condition (D), then*

(i) $0 \circledast x = \{x\}$ for all $x \in H$, and

(ii) H is commutative.

Proof. Let H be a hyper BN-algebra that satisfies condition (D).

(*i*) Let $x = z = 0$ in (D). Then $(0 \circledast y) \circledast 0 = 0 \circledast (0 \circledast y)$. Applying (HBN_2) on the left hand side and Theorem 3.1(iv) on the right hand side of the equation, we obtain $0 \circledast y = \{y\}$.

(*ii*) Since (i) holds, we have by Theorem 3.5 that H is commutative.

□

The converse of Theorem 4.1 is not true as shown in the following example:

Example 4.2. Consider the hyper BN-algebra in Example 3.2. Notice that H is commutative and it satisfies $0 \circledast x = \{x\}$ for all $x \in H$. But it does not satisfy condition (D) as shown in Example 4.1.

Notice that the hyper BN-algebra in Example 3.2 does not satisfy condition (D) as shown in Example 4.1. Furthermore, it is not a hyper B-algebra. On the other hand, the hyper BN-algebra in Example 3.3 satisfies condition (D) and as shown it is a hyper B-algebra. Now, consider another example.

Example 4.3. Consider $H = \{0, 1, 2, 3\}$ and a hyperoperation $\circledast$ on H defined by the following Cayley table:

$\circledast$	0	1	2	3
0	$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$
1	$\{1\}$	$\{0, 1\}$	$\{0, 1, 2\}$	$\{0, 1, 3\}$
2	$\{2\}$	$\{0, 1, 2\}$	$\{0, 2\}$	$\{0, 2, 3\}$
3	$\{3\}$	$\{0, 1, 3\}$	$\{0, 2, 3\}$	$\{0, 3\}$

By routine calculations, H is a hyper BN-algebra satisfying condition (D). Furthermore, it is a hyper B-algebra.

The next result generalizes our previous observation.

Proposition 4.1. *Let $(H, \circledast, 0)$ be a hyper BN-algebra. If H satisfies condition (D), then H is a hyper B-algebra.*

Proof. Suppose H is a hyper BN-algebra satisfying condition (D). (HB_1) is just the same as (HBN_1). We will show (HB_3) first. To do that, let $x, y, z \in H$. Then by Theorem 4.1 and condition (D), $x \circledast [z \circledast (0 \circledast y)] = x \circledast (z \circledast y) = (x \circledast y) \circledast z$ holding (HB_3). We will use (HB_3) in proving (HB_2). Note that $x \circledast H \subseteq H$ and $H \circledast x \subseteq H$. We are left to show the other inclusions. Now, let $x, h \in H$. By Theorem 3.1(iv), (HBN_1), and (HB_3) we proved earlier, we have $\{h\} = 0 \circledast (0 \circledast h) \subseteq (x \circledast x) \circledast (0 \circledast h) = x \circledast [(0 \circledast h) \circledast (0 \circledast x)] \subseteq x \circledast H$. Thus, $h \in x \circledast H$ implying that $H \subseteq x \circledast H$. Hence, $x \circledast H = H$. Also, by (HBN_2), (HBN_1), Theorem 3.1(iv), and (HB_3), we have $\{h\} = h \circledast 0 \subseteq h \circledast (x \circledast x) = h \circledast [x \circledast (0 \circledast (0 \circledast x))] = [h \circledast (0 \circledast x)] \circledast x \subseteq H \circledast x$. Thus, $h \in H \circledast x$ implying that $H \subseteq H \circledast x$. Hence, $H = H \circledast x$. Thus, (HB_2) holds. Therefore, $(H, \circledast, 0)$ is a hyper B-algebra. □

In general, the converse of Proposition 4.1 is not true as shown in the next example.

Example 4.4. Consider the set $H = \{0, a, b\}$ with hyperoperation defined by the Cayley table as follows:

$\circledast$	0	a	b
0	$\{0\}$	$\{b\}$	$\{a\}$
a	$\{a\}$	$\{0, a, b\}$	$\{0, a, b\}$
b	$\{b\}$	$\{0, a, b\}$	$\{0, a, b\}$

Then H is a hyper BN-algebra. Note that H is also a hyper B-algebra but it does not satisfy condition (D) because $(0 \circledast a) \circledast 0 = \{b\}$ and $0 \circledast (0 \circledast a) = \{a\}$ implying that $(0 \circledast a) \circledast 0 \neq 0 \circledast (0 \circledast a)$.

The next result shows how a hyper BN-algebra becomes a hypergroup.

Theorem 4.2. *Let $(H, \circledast, 0)$ be a hyper BN-algebra. If H satisfies condition (D), then H is a commutative hypergroup.*

Proof. Let $(H, \circledast, 0)$ be a hyper BN-algebra satisfying condition (D). Then by Theorem 4.1(ii), H is commutative. We will now show (H_1) and (H_2) of Definition 2.3. The proof of (H_2) is similar to the proof in Proposition 4.1. By condition (D), for all $x, y, z \in H$, we have $(x \circledast y) \circledast z = x \circledast (z \circledast y) = x \circledast (y \circledast z)$ since H is commutative. Therefore, $(H, \circledast)$ is a commutative hypergroup. □

5. Hyper SubBN-algebras

We focus our attention in this section to hyper subBN-algebras. We will also introduce the concepts of reflexivity and normality leading to the concept of reflexive normal hyper subBN-algebras.

Definition 5.1. Let $(H, \circledast, 0)$ be a hyper BN-algebra and let S be a subset of H containing 0. If S is a hyper BN-algebra with respect to the hyperoperation "$\circledast$" on H, we say that S is a *hyper subBN-algebra* of H.

Example 5.1. Consider the hyper BN-algebra H in Example 3.2. Let $S = \{0, a\}$ and $T = \{0, b\}$. By routine calculations, both S and T are hyper subBN-algebra of H. If we consider the hyper BN-algebra H in Example 3.3, then the set $L = \{0, 1\}$ is not a hyper subBN-algebra of H because $1 \circledast 1 = \{0, 2\}$ but $2 \notin L$. Also, the set $M = \{0, 2\}$ is not a hyper subBN-algebra of H because $2 \circledast 2 = \{0, 1\}$ but $1 \notin M$.

Proposition 5.1. *Let S be a nonempty subset of a hyper BN-algebra H. If $x \circledast y \subseteq S$ for all $x, y \in S$, then $0 \in S$.*

Proof. Assume that $x \circledast y \subseteq S$ for all $x, y \in S \subseteq H$ and let $a \in S$. Then $a \in H$ and $a \ll a$ implying that $0 \in a \circledast a \subseteq S$. □

Theorem 5.1. *Let S be a nonempty subset of a hyper BN-algebra. Then S is a hyper subBN-algebra if and only if $x \circledast y \subseteq S$, for all $x, y \in S$.*

Proof. Let H be a hyper BN-algebra and S be a nonempty subset of H. Suppose that S is a hyper subBN-algebra. Then by Definition 5.1, S is a hyper BN-algebra. Thus, $x \circledast y \subseteq S$ for all $x, y \in S$. Conversely, suppose that $x \circledast y \subseteq S$ for all $x, y \in S$. Now, by Proposition 5.1, $0 \in S$. Also, if $x, y, z \in S$, then $x, y, z \in H$. And so, the properties (HBN_1) to (HBN_3) of a hyper BN-algebra are already satisfied since H is a hyper BN-algebra. □

The following corollary is just a consequence of Theorem 5.1 and Remark 3.2.

Corollary 5.1. *The set $\{0\}$ is a hyper subBN-algebra of any hyper BN-algebra.*

Theorem 5.2. *The intersection of family of hyper subBN-algebras of a hyper BN-algebra H is a hyper subBN-algebra of H.*

The following example will show that the union of two hyper subBN-algebras of a hyper BN-algebra is not necessary a hyper subBN-algebra.

Example 5.2. Let $H = \{0, 1, 2, 3\}$ be a set. If we define a hyperoperation "$\circledast$" on H as follows:

$\circledast$	0	1	2	3
0	$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$
1	$\{1\}$	$\{0, 1\}$	$\{1, 3\}$	$\{2, 3\}$
2	$\{2\}$	$\{1, 3\}$	$\{0, 2\}$	$\{3\}$
3	$\{3\}$	$\{2, 3\}$	$\{3\}$	$\{0, 3\}$

then H is a hyper BN-algebra. Let $S_1 = \{0, 1\}$ and $S_2 = \{0, 2\}$. By Theorem 5.1, S_1 and S_2 are hyper subBN-algebras of H. However, $S_1 \cup \S_2 = \{0, 1, 2\}$ is not a hyper subBN-algebra of H because $1, 2 \in S_1 \cup S_2$ but $1 \circledast 2 = \{1, 3\} \not\subseteq S_1 \cup S_2$.

Definition 5.2. Let N be a nonempty subset of a hyper BN-algebra. Then N is called *normal* if $(x \circledast a) \circledast (y \circledast b) \subseteq N$ whenever $x \circledast y, a \circledast b \subseteq N$.

Remark 5.1. *If N a nonempty subset of a hyper BN-algebra is normal, then $0 \in N$.*

Proof. Choose $x, y \in H$ such that $x \circledast y \subseteq N$. By normality of N, we have $0 = 0 \circledast 0 \subseteq (x \circledast x) \circledast (y \circledast y) \subseteq N$. Thus, $0 \in N$. □

Example 5.3. Consider the hyper BN-algebra $H = \{0, a, b\}$ in Example 3.2. Let $N_1 = \{0, a\}$ and $N_2 = \{0, b\}$. Then it can be shown that N_1 is normal. However, N_2 is not normal because $0 \circledast b = \{b\} \subseteq N_2$ and $a \circledast b = \{b\} \subseteq N_2$ but $(0 \circledast a) \circledast (b \circledast b) = \{a, b\} \not\subseteq N_2$.

Definition 5.3. A nonempty subset I of a hyper BN-algebra H is said to be *reflexive* if $x \circledast x \subseteq I$ for all $x \in H$.

Let H be a hyper BN-algebra and I be a reflexive subset of H. Note that for all $x \in H$, $0 \in x \circledast x \subseteq I$. Thus, we have the following remark:

Remark 5.2. *If I is a reflexive subset of a hyper BN-algebra, then $0 \in I$.*

Example 5.4. Let $H = \{0, 1, 2\}$ with hyperoperation $\circledast$ defined by the following Cayley table:

$\circledast$	0	1	2
0	$\{0\}$	$\{1\}$	$\{2\}$
1	$\{1\}$	$\{0, 1\}$	$\{2\}$
2	$\{2\}$	$\{2\}$	$\{0, 1\}$

H is a hyper BN-algebra by routine calculations. Let $I = \{0, 1\}$. Then it is reflexive because $x \circledast x \subseteq I$ for $x = 0, 1, 2$. Let $J = \{0, 2\}$. Then J is not reflexive because $1 \circledast 1 \not\subseteq J$.

The concepts of reflexivity and normality are different as shown in the following example.

Example 5.5. The normal subset N_1 as shown in Example 5.3 is not reflexive because $b \circledast b = \{0, b\} \not\subseteq N_1$. Let $H = \{0, 1, 2\}$ with hyperoperation $\circledast$ defined by the folowing Cayley table:

$\circledast$	0	1	2
0	$\{0\}$	$\{1\}$	$\{2\}$
1	$\{1\}$	$\{0, 1\}$	$\{1\}$
2	$\{2\}$	$\{1\}$	$\{0, 1\}$

By routine calculations, H is a hyper BN-algebra. Let $I = \{0, 1\}$. Then I is reflexive because $x \circledast x \subseteq I$ for all $x \in H$ but I is not normal because $1 \circledast 0 = \{1\} \subseteq I$ and $1 \circledast 2 = \{1\} \subseteq I$ but $(1 \circledast 1) \circledast (0 \circledast 2) = \{1, 2\} \not\subseteq I$.

Theorem 5.3. *Every normal subset N of a hyper BN-algebra H is a hyper subBN-algebra of H.*

Proof. Let H be a hyper BN-algebra and N a normal subset of H. Note that $0 \in N$ by Remark 5.1. Now, let $x, y \in N$. Then $x \circledast 0 = \{x\} \subseteq N$ and $y \circledast 0 = \{y\} \subseteq N$. By normality of N, $x \circledast y = (x \circledast y) \circledast 0 = (x \circledast y) \circledast (0 \circledast 0) \subseteq N$. By Theorem 5.1, N is a hyper subBN-algebra of H. □

The converse of Theorem 5.3 is not true in general. The following example will show that.

Example 5.6. Consider the hyper BN-algebra $H = \{0, a, b\}$ in Example 3.2. Let $N_2 = \{0, b\}$. It has been shown that N_2 is a hyper subBN-algebra in Example 5.1 but it is not normal as shown in Example 5.3.

Definition 5.4. A hyper subBN-algebra S of a hyper BN-algebra H is called *reflexive* (resp. *normal*) *hyper subBN-algebra* if it is reflexive (resp. normal). S is called a *reflexive normal hyper subBN-algebra* if it is both reflexive and normal.

Example 5.7. Consider the set $H = \{0, 1, 2, 3, 4\}$. Define the hyperoperation "$\circledast$" by the following Cayley table:

$\circledast$	0	1	2	3	4
0	$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$
1	$\{1\}$	$\{0, 3\}$	$\{3\}$	$\{1\}$	$\{4\}$
2	$\{2\}$	$\{3\}$	$\{0, 3\}$	$\{2\}$	$\{4\}$
3	$\{3\}$	$\{1\}$	$\{2\}$	$\{0, 3\}$	$\{4\}$
4	$\{4\}$	$\{4\}$	$\{4\}$	$\{4\}$	$\{0, 3\}$

By routine calculations, H is a hyper BN-algebra. Let $I = \{0, 3\}$. Then I is a reflexive normal hyper subBN-algebra of H.

Theorem 5.4. *The intersection of family of reflexive normal hyper subBN-algebras of a hyper BN-algebra H is a reflexive normal hyper subBN-algebra of H.*

Theorem 5.5. *Let $(H, \circledast, 0)$ be a hyper BN-algebra. Then the set $S_N := \{x \in H | x \circledast x = \{0\}\}$ is a hyper subBN-algebra of H whenever $\{0\}$ is normal in H.*

Proof. Note that $0 \in S_N$ by Remark 3.2. Thus, $S_N \neq \emptyset$. Let $x, y \in S_N$. Then $x \circledast x = \{0\}$ and $y \circledast y = \{0\}$. Let $a \in x \circledast y$. By normality of $\{0\}$, we have, $\{0\} \subseteq a \circledast a \subseteq (x \circledast y) \circledast (x \circledast y) \subseteq \{0\}$. Hence, $a \circledast a = \{0\}$. Thus, $a \in S_N$. Consequently, $x \circledast y \subseteq S_N$. Therefore, Theorem 5.1 says that S_N is a hyper subBN-algebra of H. □

6. Conclusion

We have defined a new concept of hyper algebra called hyper BN-algebras. We also successfully obtained some routine properties and gave a characterization of hyper BN-algebras. We showed some hyper BN-algebras that exhibit a certain condition to be a hyper B-algebra and a hypergroup. Finally, we developed the concept of hyper subBN-algebra and obtained some results. It is recommended to study further hyper subBN-algebras. It is also recommended to introduce the notion of ideals of a hyper BN-algebra and construct its quotient structure.

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Almost ω-continuous functions in bitopological spaces

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Abstract. In this paper, as a generalization of u-ω-continuous functions, we introduce the notion of almost ω-continuous functions in bitopological spaces and obtain several characterizations and some of its properties.
Keywords: bitopological spaces, u-ω-open sets, almost continuous function.

1. Introduction

The concept of bitopological spaces was first introduced by Kelly [12]. After the introduction of the definition of a bitopological space by Kelly, a large number of

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topologists have turned their attention to the generalization of different concepts of a single topological space in this space. Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real analysis concerns the various modified forms of continuity, separation axioms etc. by utilizing generalized open sets. Recently, as generalization of closed sets, the notion of ω-closed sets were introduced and studied by Hdeib [10]. Several characterizations and properties of ω-closed sets were provided in [4, 2, 3, 10, 11, 15]. In this paper, we introduce and study upper (lower) (i,j)-almost ω-continuous functions on bitopological space and obtain several characterizations and some of its properties.

2. Preliminaries

A triple (X,τ_1,τ_2) consisting of a nonempty set X with two topologies τ_1 and τ_2 on X is called bitopological spaces. Throughout this paper, the indices i and j take values in $\{1,2\}$ and $i \neq j$. For a A of (X,τ_1,τ_2), $i\operatorname{Int} A$ and $i\operatorname{cl} A$ means, respectively, the interior and closure of$A$ with respect to the topology τ_i. A point $x \in X$ is called a condensation point of A if for each $U \in \tau$ with $x \in U$, the set $U \cap A$ is uncountable. A is said to be ω-closed [10] if it contains all its condensation points. The complement of an ω-closed set is said to be an ω-open set. It is well known that a subset W of a space (X,τ) is ω-open if and only if for each $x \in W$, there exists $U \in \tau$ such that $x \in U$ and $U\backslash W$ is countable. The family of all ω-open subsets of a topological space (X,τ) forms a topology on X finer than τ. The intersection of all ω-closed sets containing A is called the ω-closure [10] of A and is denoted by $\omega\operatorname{cl}(A)$. The family of all ω-open sets of X is denoted by $\omega(\tau)$.

Definition 2.1. *A subset A of a bitopological space (X,τ_1,τ_2) is said to be:*

1. *(i,j)-sem-iopen [8] if $A \subset j\operatorname{cl}(i\operatorname{Int}(A))$;*
2. *(i,j)-preopen [14] if $A \subset i\operatorname{Int}(j\operatorname{cl}(A))$;*
3. *(i,j)-semi-preopen [14] if $A \subset j\operatorname{cl}(i\operatorname{Int}(j\operatorname{cl}(A)))$;*
4. *(i,j)-regular open [7] if $A = i\operatorname{Int}(j\operatorname{cl}(A))$,*

On each definition above, $i,j = 1,2$ and $i \neq j$.

The complement of an (i,j)-semi-open (resp. (i,j)-preopen, (i,j)-semi-preopen, (i,j)-regular open) set is called an (i,j)-semi-closed (resp. (i,j)-preclosed, (i,j)-semi-preclosed, (i,j)-regular closed) set. The family of all (i,j)-semi-open (resp. (i,j)-preopen, (i,j)-semi-preopen, (i,j)-regular open, (i,j)-semi-closed, (i,j)-preclosed, (i,j)-semi-preclosed, (i,j)-regular closed) sets of (X,τ_1,τ_2) is denote by (i,j)-$SO(X)$ (resp. (i,j)-$PO(X)$, (i,j)-$SPO(X)$, (i,j)-$RO(X)$, (i,j)-$SC(X)$, (i,j)-$PC(X)$, (i,j)-$SPC(X)$, (i,j)-$RC(X)$).

Definition 2.2. *A point x of X is said to be an (i,j)-δ-cluster point [13] of A if $A \cap U \neq \emptyset$ for every (i,j)-regular open set U containing x, the set of all (i,j)-δ-cluster points of A is called the (i,j)-δ-closure of A, a subset A of X is said to be (i,j)-δ-closed if the set of all (i,j)-δ-cluster points of A is a subset of A. The complement of an (i,j)-δ-closed set is an called (i,j)-δ-open set or a subset A of X is called (i,j)-δ-open if and only if there exist (i,j)-regular open sets A_k, $k \in I$ such that $A = \bigcup_{k\in I} A_k$.*

Remark 2.3. Let A be a subset of a bitopological space (X, τ_1, τ_2). Then (i,j)-$\mathrm{Int}_\delta(A)$=$\cup\{B : B \subset A,\ B$ an (i,j)-δ -open set$\}$ is called the (i,j)-δ-interior of A, and (i,j)-$\mathrm{cl}_\delta(A) = \cap\{B : B \supset A, B$ an (i,j)-δ-closed set$\}$ is called the (i,j)-δ-closure of A.

Definition 2.4 ([1])**.** *Let (X, τ_1, τ_2) be a bitopological space and let $A \subset X$. Then*

1. *A is said to be u-ω-open in (X, τ_1, τ_2) if $A \in \omega(\tau_1) \cup \omega(\tau_2)$,*
2. *A is said to be u-ω-closed in (X, τ_1, τ_2) if $X - A$ is u-ω-open in (X, τ_1, τ_2).*

The family of all u-ω-open sets in (X, τ_1, τ_2) is denoted by $\omega(\tau_1, \tau_2)$, and the family of all u-ω-closed sets in (X, τ_1, τ_2) is denoted by $\omega C(\tau_1, \tau_2)$.

Definition 2.5.
1. *The u-ω-closure of A in (X, τ_1, τ_2) is denoted by (τ_1, τ_2)-$\omega\,\mathrm{cl}(A)$ and defined as follows: (τ_1, τ_2)-$\omega\,\mathrm{cl}(A) = \omega\,\mathrm{cl}_{\tau_1}(A) \cap \omega\,\mathrm{cl}_{\tau_2}(A)$.*
2. *The u-ω-interior of A in (X, τ_1, τ_2) is denoted by (τ_1, τ_2)-$\omega\,\mathrm{Int}(A)$ and defined as follows: (τ_1, τ_2)-$\omega\,\mathrm{Int}(A) = \omega\,\mathrm{Int}_{\tau_1}(A) \cup \omega\,\mathrm{Int}_{\tau_2}(A)$.*

Lemma 2.6. *Let (X, τ_1, τ_2) be a bitopological space and A a subset of X. Then*

1. *(τ_1, τ_2)-$\omega\,\mathrm{Int}(A)$ is u-ω-open;*
2. *(τ_1, τ_2)-$\omega\,\mathrm{cl}(A)$ is u-ω-closed;*
3. *A is u-ω-open if and only if $A = (\tau_1, \tau_2)$-$\omega\,\mathrm{Int}(A)$;*
4. *A is u-ω-closed if and only if $A = (\tau_1, \tau_2)$-$\omega\,\mathrm{cl}(A)$;*
5. *(τ_1, τ_2)-$\omega\,\mathrm{Int}(X\backslash A) = X\backslash(\tau_1, \tau_2)$-$\omega\,\mathrm{cl}(A)$;*
6. *(τ_1, τ_2)-$\omega\,\mathrm{cl}(X\backslash A) = X\backslash(\tau_1, \tau_2)$-$\omega\,\mathrm{Int}(A)$.*

Lemma 2.7. *Let (X, τ_1, τ_2) be a bitopological space and $A \subset X$. A point $x \in (\tau_1, \tau_2)$-$\omega\,\mathrm{cl}(A)$ if and only if $U \cap A \neq \emptyset$ for every u-ω-open set U of (X, τ_1, τ_2).*

Definition 2.8 ([1])**.** *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be u-ω-continuous if for each $x \in X$ and each $V \in \sigma_1 \cup \sigma_2$ containing $f(x)$, there exists an u-ω-open set U containing x such that $f(U) \subset V$.*

3. Properties of (i,j)-almost ω-continuous functions

Definition 3.1. *A function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is said to be an (i,j)-almost ω-continuous at a point $x\in X$ if for each σ_i-open set V of Y containing $f(x)$, there exists an u-ω-open set U of X containing x such that $f(U)\subset i\operatorname{Int}(j\operatorname{cl}(V))$. If f is (i,j)-almost ω-continuous at every point x of X, then it is called (i,j)-almost ω-continuous on X.*

It is obvious from the definition that u-ω-continuity implies (i,j)-almost ω-continuity. However, the converse is not true in general as it is shown in the following example.

Example 3.2. Take $X=\mathbb{R}$ and two topologies $\tau_1=\tau_2=\{\emptyset,\mathbb{R},\mathbb{R}\setminus\mathbb{Q}\}$ and $Y=\{a,b,c\}$ with topologies $\sigma_1=\sigma_2=\{\emptyset,Y,\{a\},\{a,b\}\}$. Define $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ as

$$f(x)=\begin{cases}a, & \text{if } x\in\mathbb{Q},\\ b, & \text{if } x\notin\mathbb{Q}.\end{cases}$$

It is easy to see that f is (i,j)-almost ω-continuous but does not is u-ω-continuous function.

Theorem 3.3. *For a function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$, the following statements are equivalent:*

1. *f is (i,j)-almost ω-continuous.*
2. *For each $x\in X$ and each (i,j)-regular open set V of Y containing $f(x)$, there exists an u-ω-open U in X containing x such that $f(U)\subset V$.*
3. *For each $x\in X$ and each (i,j)-δ-open set V of Y containing $f(x)$, there exists an u-ω-open U in X containing x such that $f(U)\subset V$.*

Proof. (1) $\Rightarrow$ (2): Let $x\in X$ and let V be any (i,j)-regular open subset of Y containing $f(x)$. By (1), there exists an u-ω-open set U of X containing x such that $f(U)\subset i\operatorname{Int}(j\operatorname{cl}(V))$. Since V is (i,j)-regular open, $i\operatorname{Int}(j\operatorname{cl}(V))=V$. Therefore, $f(U)\subset V$.

(2) $\Rightarrow$ (3): Let $x\in X$ and let V be any (i,j)-δ-open set of Y containing $f(x)$. Then there exists a σ_i-open set G containing $f(x)$ such that $G\subset i\operatorname{Int}(j\operatorname{cl}(G))\subset V$. Since $i\operatorname{Int}(j\operatorname{cl}(G))$ is (i,j)-regular open set of Y containing $f(x)$. By (2), there exists an u-ω-open set U in X containing x such that $f(U)\subset i\operatorname{Int}(j\operatorname{cl}(G))\subset V$.

(3) $\Rightarrow$ (1): Let $x\in X$ and let V be any σ_i-open set of Y containing $f(x)$. Then $i\operatorname{Int}(j\operatorname{cl}(V)$ is (i,j)-δ-open set of Y containing $f(x)$. By (3), there exists an u-ω-open set U in X containing x such that $f(U)\subset i\operatorname{Int}(j\operatorname{cl}(V))$. Therefore, f is (i,j)-almost ω-continuous. □

Theorem 3.4. *For a function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$, the following statements are equivalent:*

1. *f is (i,j)-almost ω-continuous.*
2. *$f^{-1}(i\,\mathrm{Int}(j\,\mathrm{cl}(V)))$ is u-ω-open set in X for each σ_i-open set V in Y.*
3. *$f^{-1}(i\,\mathrm{cl}(j\,\mathrm{Int}(F)))$ is u-ω-closed set in X for each σ_i-closed set F in Y.*
4. *$f^{-1}(F)$ is u-ω-closed set in X for each (i,j)-regular closed set F of Y.*
5. *$f^{-1}(V)$ is u-ω-open set in X for each (i,j)-regular open set V of Y.*

Proof. (1) $\Rightarrow$ (2): Let V be any σ_i-open set in Y. We have to show that $f^{-1}(i\,\mathrm{Int}(j\,\mathrm{cl}(V)))$ is u-ω-open set in X. Let $x \in f^{-1}(i\,\mathrm{Int}(j\,\mathrm{cl}(V)))$. Then $f(x) \in i\,\mathrm{Int}(j\,\mathrm{cl}(V))$ and $i\,\mathrm{Int}(j\,\mathrm{cl}(V))$ is an (i,j)-regular open set in Y. Since f is (i,j)-almost ω-continuous, by Theorem 3.3, there exists an u-ω-open set U of X containing x such that $f(U) \subset i\,\mathrm{Int}(j\,\mathrm{cl}(V))$. Which implies that $x \in U \subset f^{-1}(i\,\mathrm{Int}(j\,\mathrm{cl}(V)))$. Therefore, $f^{-1}(i\,\mathrm{Int}(j\,\mathrm{cl}(V)))$ is u-ω-open set in X.

(2) $\Rightarrow$ (3): Let F be any σ_i-closed set of Y. Then $Y\backslash F$ is a σ_i-open set of Y. By (2), $f^{-1}(i\,\mathrm{Int}(j\,\mathrm{cl}(Y\backslash F)))$ is an u-ω-open set in X and $f^{-1}(i\,\mathrm{Int}(j\,\mathrm{cl}(Y\backslash F))) = f^{-1}(i\,\mathrm{Int}(Y\backslash j\,\mathrm{Int}(F))) = f^{-1}(Y\backslash i\,\mathrm{cl}(j\,\mathrm{Int}(F))) = X\backslash f^{-1}(i\,\mathrm{cl}(j\,\mathrm{Int}(F)))$ is an u-ω-open set in X and hence $f^{-1}(i\,\mathrm{cl}(j\,\mathrm{Int}(F)))$ is an u-ω-closed set in X.

(3) $\Rightarrow$ (4): Let F be any (i,j)-regular closed set of Y. Then F is a σ_i-closed set of Y. By (3), $f^{-1}(i\,\mathrm{cl}(j\,\mathrm{Int}(F)))$ is an u-ω-closed set in X, since F is (i,j)-regular closed set. Then $f^{-1}(i\,\mathrm{cl}(j\,\mathrm{Int}(F))) = f^{-1}(F)$. Therefore, $f^{-1}(F)$ is an u-ω-closed set in X.

(4) $\Rightarrow$ (5): Let V be any (i,j)-regular open set of Y. Then $Y\backslash V$ is an (i,j)-regular closed set of Y and by (4), we have $f^{-1}(Y\backslash V) = X\backslash f^{-1}(V)$ is u-ω-closed set in X and hence $f^{-1}(V)$ is u-ω-open set in X.

(5) $\Rightarrow$ (1): Let $x \in X$ and let V be any (i,j)-regular open set of Y containing $f(x)$. Then $x \in f^{-1}(V)$. By (5), we have $f^{-1}(V)$ is u-ω-open set in X. Therefore, we obtain $f(f^{-1}(V)) \subset V$. Hence by Theorem 3.3, f is (i,j)-almost ω-continuous. □

Theorem 3.5. *For a function $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$, the following statements are equivalent:*

1. *f is (i,j)-almost ω-continuous.*
2. *(τ_1,τ_2)-$\omega\,\mathrm{cl}(f^{-1}(V)) \subset f^{-1}(i\,\mathrm{cl}(V))$ for each (j,i)-semi-preopen set V of Y.*
3. *$f^{-1}(i\,\mathrm{Int}(F)) \subset (\tau_1,\tau_2)$-$\omega\,\mathrm{Int}(f^{-1}(F))$ for each (j,i)-semi-preclosed set F of Y.*
4. *$f^{-1}(i\,\mathrm{Int}(F)) \subset (\tau_1,\tau_2)$-$\omega\,\mathrm{Int}(f^{-1}(F))$ for each (j,i)-semi-closed set F of Y.*
5. *(τ_1,τ_2)-$\omega\,\mathrm{cl}(f^{-1}(V)) \subset f^{-1}(i\,\mathrm{cl}(V))$ for each (j,i)-semi-open set V of Y.*

Proof. (1) $\Rightarrow$ (2): Let V be any (j,i)-semi-preopen set of Y. Since $i\,\mathrm{cl}(V)$ is an (i,j)-regular closed set in Y and f is (i,j)-almost ω-continuous. Then by Theorem 3.4, $f^{-1}(V)$ is u-ω-closed set in X. Therefore, we obtain (τ_1,τ_2)-$\omega\,\mathrm{cl}(f^{-1}(V)) \subset f^{-1}(j\,\mathrm{cl}(V))$.

(2) $\Rightarrow$ (3): Let F be any (j,i)-semi-preclosed set of Y. Then $Y\backslash F$ is (j,i)-semi-preopen set of Y and by (2), we have

$$(\tau_1,\tau_2)-\omega\,\mathrm{cl}(f^{-1}(Y\backslash F)) \subset f^{-1}(i\,\mathrm{cl}(Y\backslash F)) \subset f^{-1}(Y\backslash i\,\mathrm{Int}(F)) \subset X\backslash f^{-1}(i\,\mathrm{Int}(F)).$$

Therefore, $f^{-1}(i\,\mathrm{Int}(F)) \subset (\tau_1,\tau_2)$-$\omega\,\mathrm{Int}(f^{-1}(F))$.

(3) $\Rightarrow$ (4): This is obvious since every (j,i)-semi-closed set is (j,i)-semi-preclosed set.

(4) $\Rightarrow$ (5): Let V be any (j,i)-semi open set of Y. Then $Y\backslash V$ is (j,i)-semi-closed set and by (4), we have $f^{-1}(i\,\mathrm{Int}(Y\backslash V)) \subset (\tau_1,\tau_2)$-$\omega\,\mathrm{Int}(f^{-1}(Y\backslash V)) \subset (\tau_1,\tau_2)$-$\omega\,\mathrm{Int}(X\backslash f^{-1}(V)) \subset X\backslash(\tau_1,\tau_2)$-$\omega\,\mathrm{cl}(f^{-1}(V))$.

Therefore, (τ_1,τ_2)-$\omega\,\mathrm{cl}(f^{-1}(V)) \subset f^{-1}(i\,\mathrm{cl}(V))$.

(5) $\Rightarrow$ (1): Let F be any (i,j)-regular closed set of Y. Then F is (j,i)-semi-open set of Y. By (5), we have (τ_1,τ_2)-$\omega\,\mathrm{cl}(f^{-1}(F)) \subset f^{-1}(i\,\mathrm{cl}(F)) = f^{-1}(F)$. This shows that $f^{-1}(F)$ is u-ω-closed set in X.

Therefore, by Theorem 3.4, f is (i,j)-almost ω-continuous. □

Theorem 3.6. *A function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is (i,j)-almost ω-continuous if and only if $f^{-1}(V) \subset (\tau_1,\tau_2)$-$\omega\,\mathrm{Int}(f^{-1}(i\,\mathrm{Int}(j\,\mathrm{cl}(V))))$ for each σ_i-open set V of Y.*

Proof. Let V be any σ_i-open set of Y. Then $V \subset i\,\mathrm{Int}(j\,\mathrm{cl}(V))$ and $i\,\mathrm{Int}(j\,\mathrm{cl}(V))$ is (i,j)-regular open set in Y. Since f is (i,j)-almost ω-continuous, by Theorem 3.4, $f^{-1}(i\,\mathrm{Int}(j\,\mathrm{cl}(V)))$ is u-ω-open set in X and hence we obtain that $f^{-1}(V) \subset f^{-1}(i\,\mathrm{Int}(j\,\mathrm{cl}(V))) = (\tau_1,\tau_2)$-$\omega\,\mathrm{Int}(f^{-1}(i\,\mathrm{Int}(j\,\mathrm{cl}(V))))$. Conversely, let V be any (i,j)-regular open set of Y. Then V is σ_i-open set of Y. By hypothesis, we have $f^{-1}(V) \subset (\tau_1,\tau_2)$-$\omega\,\mathrm{Int}(f^{-1}(i\,\mathrm{Int}(j\,\mathrm{cl}(V)))) = (\tau_1,\tau_2)$-$\omega\,\mathrm{Int}(f^{-1}(V))$. Therefore, $f^{-1}(V)$ is u-ω-open set in X and hence by Theorem 3.4, f is (i,j)-almost ω-continuous. □

Corollary 3.7. *A function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is (i,j)-almost ω-continuous if and only if (τ_1,τ_2)-$\omega\,\mathrm{cl}(f^{-1}(j\,\mathrm{cl}(i\,\mathrm{Int}(F)))) \subset f^{-1}(F)$ for each σ_i-closed set F of Y.*

Theorem 3.8. *Let $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ be an (i,j)-almost ω-continuous function and let V be any σ_i-open subset of Y.*

If $x \in (\tau_1,\tau_2)$-$\omega\,\mathrm{cl}(f^{-1}(V))\backslash f^{-1}(V)$, then $f(x) \in (\tau_1,\tau_2)$-$\omega\,\mathrm{cl}(V)$.

Proof. Let $x \in X$ be such that $x \in (\tau_1,\tau_2)$-$\omega\,\mathrm{cl}(f^{-1}(V))\backslash f^{-1}(V)$ and suppose $f(x) \notin (\tau_1,\tau_2)$-$\omega\,\mathrm{cl}(V)$. Then there exists an u-ω-open set H containing $f(x)$ such that $H \cap V = \emptyset$. Then $j\,\mathrm{cl}(H) \cap V = \emptyset$ implies $i\,\mathrm{Int}(j\,\mathrm{cl}(H)) \cap V = \emptyset$ and

$i\,\mathrm{Int}(j\,\mathrm{cl}(H))$ is (i,j)-regular open set. Since f is (i,j)-almost ω-continuous, by Theorem 3.4, there exists an u-ω-open set U in X containing x such that $f(U) \subset i\,\mathrm{Int}(j\,\mathrm{cl}(H))$. Therefore, $f(U) \cap V = \emptyset$. However, since $x \in (\tau_1,\tau_2)$-$\omega\,\mathrm{cl}(f^{-1}(V))$, $U \cap f^{-1}(V) \neq \emptyset$ for every u-ω-open set U in X containing x, so that $f(U) \cap V \neq \emptyset$. We have a contradiction. Hence $f(x) \in (\tau_1,\tau_2)$-$\omega\,\mathrm{cl}(V)$. □

Theorem 3.9. *The set of all points x of X at which $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is not (i,j)-almost ω-continuous is identical with the union of the (τ_1,τ_2)-ω-frontiers of the inverse images of (i,j)-regular open subsets of Y containing $f(x)$.*

Proof. If f is not (i,j)-almost ω-continuous at $x \in X$, then there exists an (i,j)-regular open set V containing $f(x)$ such that for every u-ω-open set U of X containing x, $f(U) \subset (Y \backslash V) \neq \emptyset$. This means that for every u-ω-open set U of X containing x, we must have $U \cap (X \backslash f^{-1}(V)) \neq \emptyset$. Hence it follows that $x \in (\tau_1,\tau_2)$-$\omega\,\mathrm{cl}(X \backslash f^{-1}(V))$. But $x \in f^{-1}(V)$ and hence $x \in (\tau_1,\tau_2)$-$\omega\,\mathrm{cl}(f^{-1}(V))$. This means that x belongs to the (τ_1,τ_2)-ω-frontier of $f^{-1}(V)$. Conversely, suppose that x belongs to the (τ_1,τ_2)-ω-frontier of $f^{-1}(V_1)$ for some (i,j)-regular open subset V_1 of Y such that $f(x) \in V_1$. Suppose that f is (i,j)-almost ω-continuous at x. Then by Theorem 3.3, there exists an u-ω-open set U of X containing x such that $f(U) \subset V_1$. Then we have $U \subset f^{-1}(V_1)$. This shows that $x \in (\tau_1,\tau_2)$-$\omega\,\mathrm{Int}(f^{-1}(V_1))$. Therefore, we have $x \in (\tau_1,\tau_2)$-$\omega\,\mathrm{cl}(X \backslash f^{-1}(V_1))$ and $x \notin (\tau_1,\tau_2)$-$\omega Fr(f^{-1}(V_1))$. But this is a contradiction. This means that f is not almost u-ω-continuous. □

Theorem 3.10. *Let $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$. Then the following statements are equivalent.*

1. *f is an (i,j)-almost ω-continuous mapping.*
2. *$f^{-1}(B)$ is an u-ω-open set of X for each (i,j)-δ-open set B of Y.*
3. *$f^{-1}(B)$ is an u-ω-closed set of X for each (i,j)-δ-closed set B of Y.*
4. *$f((\tau_1,\tau_2)$-$\omega\,\mathrm{cl}\,A) \subset (i,j)$-$\mathrm{cl}_\delta(f(A))$ for each set A of X.*
5. *(τ_1,τ_2)-$\omega\,\mathrm{cl}(f^{-1}(B)) \subset f^{-1}((i,j)$-$\mathrm{cl}_\delta(B)$ for each set B of Y.*
6. *$f^{-1}((i,j)$-$\mathrm{Int}_\delta(B)) \subset (\tau_1,\tau_2)$-$\omega\,\mathrm{Int}(f^{-1}(B))$ for each set B of Y.*

Proof. (1)⇒ (2): Let B be any (i,j)-δ-open set of Y. Then $B = \bigcup_{\alpha\in I} B_\alpha$, where B_α is an (i,j)-regular open set of Y, for each $\alpha \in I$. From $f^{-1}(B) = f^{-1}(\bigcup_{\alpha\in I} B_\alpha) = \bigcup_{\alpha\in I} f^{-1}(B_\alpha)$ it follows that $f^{-1}(B)$ is an u-ω-open set as the union of u-ω-open sets.

(2)⇒(3): Can be proved by using the complement.

(3)⇒(4): Let A be any set of X. Then (i,j)-$\mathrm{cl}_\delta(f(A))$ is an (i,j)-δ closed set of Y. By assumption $f^{-1}((i,j)$-$\mathrm{cl}_\delta(f(A)))$ is an (i,j)-preclosed set of X. Hence

(τ_1,τ_2)-$\omega\,\mathrm{cl}(A) \subset (\tau_1,\tau_2)$-$\omega\,\mathrm{cl}(f^{-1}(f(A))) \subset (\tau_1,\tau_2)$-$\omega\,\mathrm{cl}(f^{-1}((i,j)\text{-}\mathrm{cl}_\delta(f(A))) = f^{-1}((i,j)\text{-}\mathrm{cl}_\delta(f(A)))f((\tau_1,\tau_2)\text{-}\omega\,\mathrm{cl}(A)) \subset (i,j)\text{-}\mathrm{cl}_\delta(f(A))$.

(4)$\Rightarrow$ (5): Let B be any set of Y. From the assumption it follows that (τ_1,τ_2)-$\omega\,\mathrm{cl}(f^{-1}(B)) \subset f^{-1}(f(\tau_1,\tau_2)\text{-}\omega\,\mathrm{cl}(f^{-1}(B))) \subset f^{-1}((i,j)\text{-}\mathrm{cl}_\delta(B))$.

(5)$\Rightarrow$(6): Can be proved by using the complement.

(6)$\Rightarrow$(1): Let B be any (i,j)-regular open set of Y. Then $B = (i,j)$-$\mathrm{Int}_\delta(B)$. According to the assumption $f^{-1}(B) = f^{-1}((i,j)\text{-}\mathrm{Int}_\delta(B) \subset (\tau_1,\tau_2)$-$\omega\,\mathrm{Int}(f^{-1}(B)) \subset f^{-1}(B)$. Hence $f^{-1}(B) = (\tau_1,\tau_2)$-$\omega\,\mathrm{Int}(f^{-1}(B))$, so $f^{-1}(B)$ is a u-ω-open set. Thus f is an (i,j)-almost ω-continuous function. □

Theorem 3.11. *Let $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ be a bijective function. The function f is (i,j)-almost ω-continuous if and only if (i,j)-$\mathrm{Int}_\delta(f(A)) \subset f((i,j)$-$\mathrm{Int}(A))$ for each subset A of X.*

Proof. We suppose that f is (i,j)-almost ω-continuous. Then

$$\begin{aligned} f^{-1}((i,j)\text{-}\mathrm{Int}_\delta(f(A)) &= (\tau_1,\tau_2)\text{-}\omega\,\mathrm{Int}(f^{-1}f^{-1}((i,j)\text{-}\mathrm{Int}_\delta(f(A))) \\ &\subseteq (\tau_1,\tau_2)\text{-}\omega\,\mathrm{Int}(f^{-1}(f(A))) \\ &= (\tau_1,\tau_2)\text{-}\omega\,\mathrm{Int}(A). \end{aligned}$$

Again, since f is surjective, we obtain

$$\begin{aligned} (i,j)\text{-}\mathrm{Int}_\delta(f(A)) &= f(f^{-1}((i,j)\text{-}\mathrm{Int}_\delta(f(A))) \\ &\subset f((\tau_1,\tau_2)\text{-}\omega\,\mathrm{Int}(A)). \end{aligned}$$

Conversely, let B be any (i,j)-δ-open set of Y. Then (i,j)-$\mathrm{Int}_\delta(B) = B$. According to the assumption.

$$\begin{aligned} f((\tau_1,\tau_2)\text{-}\omega\,\mathrm{Int}(f^{-1}(B))) &\supset (i,j)\text{-}\mathrm{Int}_\delta(f(f^{-1}(B))) \\ &= (i,j)\text{-}\mathrm{Int}_\delta(B) \\ &= B. \end{aligned}$$

Thus, implies that $f^{-1}(f((\tau_1,\tau_2)\text{-}\omega\,\mathrm{Int}(f^{-1}(B)))) \supset f^{-1}(B)$. Since f is injective we obtain (τ_1,τ_2)-$\omega\,\mathrm{Int}(f^{-1}(B)) = f^{-1}(f((i,j)\text{-}\mathrm{Int}(f^{-1}(B)))) \supset f^{-1}(B)$. Hence, (τ_1,τ_2)-$\omega\,\mathrm{Int}(f^{-1}(B)) = f^{-1}(B)$, so $f^{-1}(B)$ is an u-ω-open set. Thus, f is (i,j)-almost ω-continuous function. □

For a function $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$, we define $D_{a\omega}(f)$ as follows: $D_{a\omega}(f) = \{x \in X : f$ is not (i,j)-almost ω-continuous at $x\}$.

Theorem 3.12. *For a function $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$, the following properties hold:*

$$\begin{aligned} D_{a\omega}(f) &= \bigcup_{V\in(j,i)-SPO(Y)} \{(\tau_1,\tau_2) - \omega\,\mathrm{cl}(f^{-1}(V))\backslash f^{-1}(i\,\mathrm{cl}(V))\} \\ &= \bigcup_{F\in(j,i)-SPC(Y)} \{f^{-1}(i\,\mathrm{Int}(F))\backslash(\tau_1,\tau_2) - \omega\,\mathrm{Int}(f^{-1}(F))\} \end{aligned}$$

$$= \bigcup_{F\in(j,i)-SC(Y)} \{f^{-1}(i\,\mathrm{Int}(F))\backslash(\tau_1,\tau_2) - \omega\,\mathrm{Int}(f^{-1}(F))\}$$
$$= \bigcup_{V\in(j,i)-SO(Y)} \{(\tau_1,\tau_2) - \omega\,\mathrm{cl}(f^{-1}(V))\backslash f^{-1}(i\,\mathrm{cl}(V))\}$$
$$= \bigcup_{V\in\sigma_i} \{f^{-1}(V)\backslash(\tau_1,\tau_2) - \omega\,\mathrm{Int}(f^{-1}(i\,\mathrm{Int}(j\,\mathrm{cl}(V))))\}.$$

Proof. The proof follows from Theorem 3.5. □

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Lower bound for the index of convergence of the singular integral in a multidimensional analogue of Tarry's problem

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Abstract. In this paper, a general lower bound is obtained for the index of convergence of the singular integral in a multidimensional analogue of Tarry's problem. In all cases when the exact estimate is known, our estimate coincides with it.

Keywords: oscillating integral, singular integral, index of convergence, Tarry's problem.

1. Introduction

Let $r \geq 1$ be a natural number, and

$$A = \{(t_1, ..., t_r) \mid t_1 \geq 0, ..., t_r \geq 0\};\ (0, ..., 0) \notin A \tag{1}$$

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be a finite set of integer vectors $(t_1, ..., t_r)$. Consider the integral

$$\theta_k = \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} \left| \int_0^1 ... \int_0^1 exp\{2\pi i F(x_1, ..., x_r)\} dx_1 ... dx_r \right|^{2k} d\alpha, \tag{2}$$

where $F(x_1, ..., x_r) = \sum_{(t_1,...,t_r)\in A} \alpha(t_1, ..., t_r)\, x_1^{t_1} ... x_r^{t_r}$. The integral (2) with r = 1 is present as a factor in the formula for the number of solutions of the system of diophantine equations obtained in 1938 by Hua Loo-Keng using the Vinogradov method of trigonometric sums [1]:

$$\begin{aligned} &x_1 + ... + x_k = y_1 + ... + y_k, \\ &x_1^2 + ... + x_k^2 = y_1^2 + ... + y_k^2, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &x_1^n + ... + x_k^n = y_1^n + ... + y_k^n, \end{aligned} \tag{3}$$

where $x_1, ..., x_k, y_1, ..., y_k$ take integer values from 1 to some sufficiently large value P. For $r > 1$, a similar formula was obtained in the works of G.I.Arkhipov, V.N.Chubarikov and A.A. Karatsuba [2].

The number γ is called the index of convergence of the integral (2) if, for any positive number ϵ, the integral (2) converges for $2k > \gamma + \epsilon$, and diverges for $2k < \gamma + \epsilon$. For $r = 1$, the Hua Loo-Keng problem of the index of convergence of the integral (2) was completely solved by G.I. Arkhipov, V.N. Chubarikov and A.A. Karatsuba [2, 3, 4, 5, 6]. In the multidimensional case, for the sets

$$A = A(n_1, ..., n_r) = \{(t_1, ..., t_r) \,|\, 0 \le t_1 \le n_1, ..., 0 \le t_r \le n_r\}, (0, ..., 0) \notin A, \tag{4}$$

$$A = A(n) = \{(t_1, ..., t_r) \,|\, 1 \le t_1 + ... + t_r \le n\}, \tag{5}$$

the same authors obtained upper estimates for the convergence index of the singular integral (2), for the sets (4) and (5) they are equal, respectively, to $\gamma = \gamma_A \le n \cdot m$ and $\gamma = \gamma_A \le r\binom{n+r}{r+1} + r$. Here $n = \max(n_1, ..., n_r)$, $m = (n_1 + 1)...(n_r + 1) - 1$. I.A. Ikromov [7] obtained lower estimates

$$\gamma \ge \frac{1}{2} n \cdot m + 1; \quad \gamma \ge \binom{n+r}{r+1} + 1, \tag{6}$$

for the sets (4) and (5), respectively. Also in [7], in one particular case of the set A having the form

$$A = A(n) = \{(t_1, ..., t_r) \mid \left[\frac{n}{2}\right] + 1 \le t_1 \le n,\, 0 \le t_2 \le t_1, ..., 0 \le t_r \le t_1\},$$

the exact value of the convergence index was obtained:

$$\gamma = \gamma_A = \sum_{r=\left[\frac{n}{2}\right]+1}^{n} k(k+1)^{r-1}.$$

More exact lower bounds than (6) were obtained in [8], where exact bounds were also obtained in one particular case and the theorem was proved:

Theorem 1. *Let the integral* (2) *converge, and let the vector* $(0, ..., 0, n_s, 0, ...)$, *where* $n_s = \max\{t_s \,|\, (t_1, ..., t_r) \,|\, \in A\}, s = 1, ..., r$, *belong to the set* A. *Then*

$$\gamma \geq \sum_{(t_1,...,t_r)\in A} t_s.$$

The aim of the article is to remove the restriction that the vector $(0, ..., 0, n_s, 0, ..., 0)$ belongs to the set A, which significantly expands the possibility of applying Theorem 1. Namely, we prove the theorem:

Theorem 2. *Let* $r \geq 2$ *be a natural number, let* $A = \{(t_1, ..., t_r) \,|\, t_1 \geq 0, ..., t_r \geq 0\}\}, (0, ..., 0) \notin A$ *be some finite set of integer vectors. If the integral (2) converges, then*

$$\gamma \geq \max_{1\leq s\leq r} \sum_{(t_1,...,t_r)\in A} t_s.$$

2. Proof of Theorem 2

To prove Theorem 2, we need a lemma from [8, 9]:

Lemma 1. *Let* $f(x) = \alpha_n x^n + ... + \alpha_1 x$ *be a polynomial of degree* n *with real coefficients. Then for* $k = 1$ *the following representation holds*

$$\begin{aligned} I &= \int_0^1 exp\{2\pi i\,(\alpha_n x^n + ... + \alpha_1 x)\}dx \\ &= \frac{c_n}{|\alpha_n|^{\frac{1}{n}}} + O\left(\frac{|\alpha_1|}{|\alpha_n|^{\frac{2}{n}}} + ... + \frac{|\alpha_{n-2}|}{|\alpha_n|^{\frac{n-1}{n}}} + \frac{|\alpha_{n-1}|}{|\alpha_n|}\,\ln|\alpha_n|\right), \\ c_n &= \int_0^{+\infty} exp\{2\,\pi\,i\,x^n sign(\alpha_n)\}dx \neq 0. \end{aligned}$$

For definiteness, let us assume that $\max_{1\leq s\leq r} n_s = n_r = n$ and that the vector $(t_1^0, t_2^0, ..., n) \in A$. We represent the integral $I = \int_0^1 ... \int_0^1 \exp(2\,\pi\,i\,F(x_1, ..., x_r))dx$ as a sum $J_1 + J_2$, where

$$\begin{aligned} J_1 &= \int_0^{\varepsilon}\int_0^1 ... \int_0^1 \exp(2\pi i\,F(x_1, ..., x_r))dx, \\ J_2 &= \int_{\varepsilon}^1\int_0^1 ... \int_0^1 \exp(2\pi i\,F(x_1, ..., x_r))dx. \end{aligned}$$

Then, $\theta_k(J_2) = \theta_k(I - J_1) \leq 2^{2k}\,\theta_k(I) + 2^{2k}\,\theta_k(J_1)$. In the second integral we make the replacement $x_1 = \varepsilon\,y_1$ and then put $\beta(t_1, ..., t_r) = \varepsilon^{t_1}\,\alpha(t_1, ..., t_r)$; $(t_1, ..., t_r) \in A$. Then, we get $\theta_k(J_1) = \varepsilon^{2k-M}\theta_k(I)$, where $M = \sum_{(t_1,...,t_r)\in A} t_1$.

This implies $\theta_k(J_2) \le 2^{2k}\,\varepsilon^{2k-M}\,\theta_k(I) + 2^{2k}\,\theta_k(I) \le 2^{2k}\,(1+\varepsilon^{2k-M})\,\theta_k(I)$. That is, for convergence of $\theta_k(I)$, the convergence of the integral $\theta_k(J_2)$ is necessary. Similarly, variables $x_2, ..., x_{r-1}$ can be separated from zero. Let $J = \int_0^1 ... \int_0^1 \int_0^\varepsilon \exp(2\,\pi\,i\,F(x_1, ..., x_r))dx$. Then, for convergence of $\theta_k(I)$, the convergence of the integral $\theta_k(J)$ is necessary. We write the polynomial F in a different form, explicitly highlighting the summation over one variable: $F(x_1, ..., x_r) = \sum_{k=0}^{n} \beta_k x^k$, where $x = x_r$; $\beta_k = \sum_{(t_1,...,t_{r-1},k)\in A} \alpha(t_1, ..., t_{r-1}, k) x_1^{t_1} ... x_{r-1}^{t_{r-1}}$. Then

$$\theta_k(J) = \int_{R^N} \left| \int_\varepsilon^1 ... \int_\varepsilon^1 \exp\{2\,\pi i \beta_0\} dx_1 ... dx_{r-1} \int_0^1 \exp\{2\,\pi i f(x)\} dx \right|^{2k} d\alpha,$$

where $f(x) = \beta_n x^n + ... + \beta_1 x$. Define the set $\Omega \subset R^N$, where $N = \sum_{(t_1,...,t_r)\in A} 1$ is the number of elements of the set A, as follows:

$$\Omega = \{\alpha_n(t_1, ..., t_r), \quad (t_1, ..., t_r) \in A \,|\, \alpha_n = \alpha(t_1^0, ..., t_{r-1}^0, n) \ge \frac{2}{\varepsilon^{t_1^0 + ... + t_{r-1}^0}},$$
$$|\alpha(t_1, ..., t_{r-1}, n)| \le \frac{\varepsilon}{N}\alpha_n; \quad (t_1, ..., t_{r-1}, n) \ne (t_1^0, ..., t_{r-1}^0, n),$$
$$|\alpha(t_1, ..., t_{r-1}, n-1)| \le \frac{\varepsilon \cdot \alpha_n^{(n-1)/n}}{N \,\ln \alpha_n},$$
$$|\alpha(t_1, ..., t_{r-1}, s)| \le \frac{\varepsilon}{N}\alpha_n^{s/n}, \quad s = 1, ..., n-2,$$
$$(7) \quad |\alpha(t_1, ..., t_{r-1}, 0)| \le \frac{\delta}{N}\}.$$

Here, ε and δ are positive numbers that we define later. It's clear that $\theta_k(J) \ge \int_\Omega |J(\alpha)|^{2k} d\alpha$, where

$$(8) \qquad J(\alpha) = \int_\varepsilon^1 ... \int_\varepsilon^1 \exp\{2\,\pi i \beta_0\}\, dx_1 ... dx_{r-1} \int_0^1 \exp\{2\,\pi i f(x)\}\, dx.$$

By the definition of the set Ω, we have:

$$|\beta_0| = \left| \sum_{(t_1,...,t_{r-1},0)\in A} \alpha(t_1, ..., t_{r-1}, 0)\, x_1^{t_1} ... x_{r-1}^{t_{r-1}} \right|$$
$$\le \sum_{(t_1,...,t_{r-1},0)\in A} |\alpha(t_1, ..., t_{r-1}, 0)| \le \sum_{(t_1,...,t_r)\in A} \frac{\delta}{N} = \delta,$$
$$|\beta_s| = \left| \sum_{(t_1,...,t_{r-1},s)\in A} \alpha(t_1, ..., t_{r-1}, s)\, x_1^{t_1} ... x_{r-1}^{t_r} \right|$$
$$\le \sum_{(t_1,...,t_r)\in A} \frac{\varepsilon}{N}\, \alpha_n^{s/n} \le \varepsilon \cdot \alpha_n^{s/n}, \quad s = 1, ..., n-2,$$
$$|\beta_{n-1}| = \left| \sum_{(t_1,...,t_{r-1},n-1)\in A} \alpha(t_1, ..., t_{r-1}, n-1)\, x_1^{t_1} ... x_{r-1}^{t_{r-1}} \right| \le \frac{\varepsilon\, \alpha_n^{(n-1)/n}}{\ln \alpha_n},$$

$$|\beta_n - \alpha_n| = \left| \sum_{(n,t_2,...,t_r)\in A} \alpha(t_1,...,t_{r-1},n)\, x_1^{t_1}...\, x_{r-1}^{t_{r-1}} - \alpha(t_1^0,...,t_{r-1}^0,n) \right|$$

$$\leq \sum_{(t_1,...,t_r)\in A} \frac{\varepsilon}{N}\, \alpha_n = \varepsilon \cdot \alpha_n. \tag{9}$$

By Lemma 1, for the integral

$$I_1 = I(x_1,...,x_{r-1},\alpha) = \int_0^1 \exp\{2\pi i f(x)\}dx = \int_0^{\int} \exp\{2\pi i(\beta_n x^n + ... + \beta_1 x)\}dx,$$

we have:

$$I_1 = \frac{c_n}{\beta_n^{1/2}} + O\left(\frac{|\beta_1|}{\beta_n^{2/n}} + ... + \frac{|\beta_{n-2}|}{\beta_n^{(n-1)/n}} + \frac{|\beta_{n-1}|}{\beta_n} \cdot \ln\, \beta_n + \frac{1}{\beta_n} \right),$$

where $c_n = \int_0^{+\infty} \exp\{2\pi i x^n\}dx \neq 0$. Given the inequalities (9) and the fact that

$$\left| \frac{1}{\beta_n^{1/n}} - \frac{1}{\alpha_n^{1/n}} \right| \leq \frac{\varepsilon}{n\alpha_n^{1/n}},$$

we obtain

$$I_1 = \frac{c_n}{\alpha_n^{1/n}} + \varepsilon \cdot \left(\frac{1}{\alpha_n^{1/n}} \right).$$

Hence, for the integral (8) we have:

$$J_\alpha = \int_\varepsilon^1 ... \int_\varepsilon^{\int} I_1 \cdot \exp\{2\pi i \beta_0\}\, dx_1...dx_{r-1} = \int_\varepsilon^{\int} ... \int_\varepsilon^1 I_1\, dx_1...dx_{r-1}$$
$$+ \int_\varepsilon^1 ... \int_\varepsilon^1 (\exp\{2\pi i\beta_0\} - 1)\; I_1\, dx_1...dx_{r-1}$$
$$= \frac{c_n}{\alpha_n^{1/n}} + \varepsilon \cdot O\left(\frac{1}{\alpha_n^{1/n}} \right) + \delta \cdot O\left(\frac{1}{\alpha_n^{1/n}} \right),$$

because $|\exp\{2\pi i\beta_0\} - 1| \leq 2\pi\, |\beta_0| \leq 2\pi\delta$. Therefore, choosing ε and δ sufficiently small, we obtain:

$$|J(\alpha)| \geq \frac{|c_n|}{2\, \alpha_n^{1/n}},$$

for all $\alpha \in \Omega$. Given the definition of the set Ω, for the singular integral $\theta_k(J)$ we obtain the estimate

$$\theta_k(J) \geq \frac{|c_n|^{2k} \cdot \varepsilon^N \cdot \delta^N}{2^{2k} \cdot N^N} \int_{\frac{2}{\varepsilon^{t_1+...+t_{r-1}}}}^{+\infty} \frac{d\alpha_n}{\alpha_n^{\frac{2k}{n} - \frac{M-n}{n}} \cdot (\ln \alpha_n)^N},$$

where $M = \sum_{(t_1,...,t_{r-1},s)\in A} s$. Therefore, for the convergence of the integral $\theta_k(J)$, it is necessary to satisfy the inequality $2k \geq M$. The theorem is proved.

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Certain efficient techniques to solve the unreported cases of 2019−nCoV epidemic model

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Abstract. One of the furthermost intimidations that the death faced after the second World War is 2019-nCoV epidemic and most crucial large-scale health disaster of this century. We devote the current work to discuss the epidemic prediction for the epidemic model created for 2019-nCoV in Wuhan, China by certain approximate analytical methods such as differential transform method and variational iteration method. Further, we recognize unreported cases in numbers and the parameters of model are due to reported case data. For the considered system demonstrating the model of coronavirus, the series solution is conventional in the structure of the differential transform method. The obtained solutions are discussed in figures which show the performance of considered model. The results show that the used schemes are definite and trouble-free to execution for the system of nonlinear ODEs. The solutions exposed that the

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both schemes are in total agreement, correct and well-organized for solving systems of nonlinear differential equations.

Keywords: mathematical models, epidemic model, corona-virus (2019-nCoV), reported and unreported cases, differential transform method, variational iteration method.

1. Introduction

The entire globe is facing tough time, since December 2019 due to 2019-nCoV, which was first identified in Wuhan, China. The detailed history of this virus, its behavior, its characteristics, how it has from animal to human and how this spreads exponentially person to person and its origin all those and what not, one can refer in the following list of works [2, 10, 12, 13, 16, 17, 18, 20, 30, 31, 32, 33, 34, 35, 36, 37, 40, 41, 42, 44, 46].

From the time when the mathematical epidemiological model was developed, there has been triggering concentration to apply its application to various biological transmittable diseases to understand the disease and its behavior. In view of the fact that these models be capable of replicate the special effects of the infection at countless levels like within-host to metapopulation. Also, these models are constructive in comprising, proposing, planning, implementing, testing theories, prevention, evaluating various detection, therapy and control programs. All the epidemic models are framed by means of ODEs may be linear or nonlinear with initial conditions. To control the disease one requires strong formation of epidemic mathematical models and effective methods to solve. Generally, the solutions will be obtained by numerical methods or exact solution methods or approximate method. But mainly these methods are rigorous in computation, need complex agent to compute. Usually, obtaining the exact solutions for such proposed epidemic models are out-of-the-way and exceptionally complex.

The differential transform method (DTM) is a numerical and analytic method in solving differential equations. The conception of DTM was first initiated by Zhou [45] in the filed of electrical circuit which showed the new path in solving both linear and nonlinear initial value and boundary value problems. This method is an iterative procedure for obtaining Taylor series solutions of differential equations and partial differential equations (see [28, 29]). The main advantage of this method is that it can be applied directly to linear and nonlinear ODEs without requiring linearization, discretization or perturbation. Another important advantage is that this method is capable of greatly reducing the size of computational work while still accurately providing the series solution with fast convergence rate.

In 2009, Arenas et al. [5] employed differential transform approach to solve seasonal diseases of epidemic model associated with non-autonomous ODEs in the sense of nonlinear and found solution in series form and compared with numerical results. In 2012, Do and Jang [14] considered the population models namely simple and Gauss type prey - predator models which involve the complex nonlinear system of ODEs and obtained solutions by the numerical and

analytical methods namely variational iteration method (VIM), RK4 and DTM and proved that DTM is efficient method to solve such a nonlinear differential equations. In 2014, Akinboro et al. [3] (see also [27]) investigated the applications of DTM and VIM methods to solve SIR epidemic nonlinear model and showed that DT method has its own advantage to solve a large set of ODEs which occur in diverse area of study. In 2017, Chakraborty et al. [9] used DTM successfully to SAEIQRS epidemic model which consists the six nonlinear ODEs with constraints and compared the solutions with RK4 and proved that the solutions obtained by DTM are efficient than RK4. In 2019, Astuti et al. [6] employed DTM to solve the SEIR model which is modeled for influenza virus with disease resistance and obtained the exact solution. Recently Harir et al. [20] introduced the four compartmental model namely SEIR coronavirus epidemiological mathematical model with the help of four differential equations which are in the sense of non-linear and studied its behavior with the help of VIM and DTM, also they showed that the solutions obtained by both methods are in good agreement.

In this paper, we applied DTM for the considered system of nonlinear differential equations describing the reported and unreported cases which results our solution to be analytical with the minimum number of orders.

2. Basics of Differential Transform Method (DTM)

The differential transform of a function is defined as follows:

$$\mathcal{V}(p) = \frac{1}{p!}\left[\nu^{(p)}(t)\right]_{t=t_0}. \tag{1}$$

In equation (1), ν is the original function and $\mathcal{V}$ is the transformed function, which is called $T-$function. Differential inverse transform for $\mathcal{V}$ is defined as:

$$\nu(t) = \sum_{p=0}^{\infty} \mathcal{V}(p)\,(t-t_0)^p. \tag{2}$$

It follows from (1) and (2), we get

$$\nu(t) = \sum_{p=0}^{\infty} \frac{(t-t_0)^p}{p!}\left[\nu^{(p)}(t)\right]_{t=t_0}. \tag{3}$$

Equation (3) shows the concept of differential transform is derived from the Taylor series expansion and relative derivatives are calculated by an iterative way which is described by the transformed equations of the original functions.

Using equations (1) and (2), the following mathematical operations can be obtained. Few of the properties listed below are applied in this work. For more properties one could refer recent work [21]:

(i) If $\nu(t) = \nu_1(t) \pm \nu_2(t)$, then $\mathcal{V}(p) = \mathcal{V}_1(p) \pm \mathcal{V}_2(p)$.

(ii) If $\nu(t) = \nu_1^{(m)}(t)$, then $\mathcal{V}(p) = (p+m)!\,\mathcal{V}_1(p+m)$.

(iii) If $\nu(t) = t^m$, then

$$\mathcal{V}(p) = \delta(p-m) = \begin{cases} 1, & \text{if } p = m \\ 0, & \text{if } p \neq m, \end{cases}$$

where δ is the Kronecker delta.

(iv) If $\nu(t) = \nu_1(t)\,\nu_2(t)$, then

$$\mathcal{V}(p) = \sum_{m=0}^{p} \mathcal{V}_1(m)\,\mathcal{V}_2(p-m).$$

If $\nu(t)$ is finite series then (2) can be written

$$\nu(t) \approx \sum_{p=0}^{N} \mathcal{V}(p)\,(t-t_0)^p \tag{4}$$

with N is a convergence of natural frequency.

2.1 Applications of DTM to 2019-nCoV epidemic model

Recently, Liu et al. [36] considered a mathematical model for 2019-nCoV to identify the unreported cases essentially inspired by the works of [15, 38]. Also, Liu et al. [36] discussed the severity of this epidemic due to unreported cases and other related issues. Actually, this model was developed by Tang et al. [41], and it was modeled without unreported cases. This current model consist of four sections β_i, where $i = 1, 2, 3, 4$ namely susceptible as individuals, asymptomatically infectious, reported symptomatically infectious and unreported symptomatically infectious respectively.

$$\beta_1'(t) = -\alpha\,\beta_1(t)\,[\beta_2(t) + \beta_4(t)], \tag{5}$$

$$\beta_2'(t) = \alpha\,\beta_1(t)\,[\beta_2(t) + \beta_4(t)] - \beta\,\beta_2(t), \tag{6}$$

$$\beta_3'(t) = \lambda\,\beta_2(t) - \gamma\,\beta_3(t), \tag{7}$$

$$\beta_4'(t) = \phi\,\beta_2(t) - \gamma\,\beta_4(t). \tag{8}$$

where $'$ represents the time derivative.

Further, we apply the differential transformation techniques discussed in previous section for the considered mathematical model (5) - (8) with initial

conditions $\mathcal{A}(0) = \mathcal{A}_0$, $\mathcal{B}(0) = \mathcal{B}_0$, $\mathcal{C}(0) = \mathcal{C}_0$, $\mathcal{D}(0) = \mathcal{D}_0$, we have

$$\mathcal{A}(p+1) = \frac{1}{(p+1)}\left[-\alpha \sum_{l=0}^{p} \mathcal{A}(l)\mathcal{B}(p-l) - \alpha \sum_{l=0}^{p} \mathcal{A}(l)\mathcal{D}(p-l)\right], \tag{9}$$

$$\mathcal{B}(p+1) = \frac{1}{(p+1)}\left[\alpha \sum_{l=0}^{p} \mathcal{A}(l)\mathcal{B}(p-l) + \alpha \sum_{l=0}^{p} \mathcal{A}(l)\mathcal{D}(p-l) - \beta\, \mathcal{B}(p)\right], \tag{10}$$

$$\mathcal{C}(p+1) = \frac{1}{(p+1)}\left[\lambda\, \mathcal{B}(p) - \gamma\, \mathcal{C}(p)\right], \tag{11}$$

$$\mathcal{D}(p+1) = \frac{1}{(p+1)}\left[\phi\, \mathcal{B}(p) - \gamma\, \mathcal{D}(p)\right]. \tag{12}$$

Next, in view of (2), we find the following iterative series from (9) - (12):

$$\beta_1(t) = \sum_{p=0}^{\infty} \mathcal{A}(p)\, t^p$$

$$= 11081000 - 3.847323200\, t - 0.7058240125\, t^2 - 0.09105854710\, t^3 + \cdots \tag{13}$$

$$\beta_2(t) = \sum_{p=0}^{\infty} \mathcal{B}(p)\, t^p$$

$$= 3.62 + 3.330180343\, t + 0.4679539880\, t^2 + 0.06877502387\, t^3 + \cdots \tag{14}$$

$$\beta_3(t) = \sum_{p=0}^{\infty} \mathcal{C}(p)\, t^p$$

$$= 0.4137142858\, t + 0.1607449992\, t^2 + 0.01017229482\, t^3 + \cdots \tag{15}$$

$$\beta_4(t) = \sum_{p=0}^{\infty} \mathcal{D}(p)\, t^p$$

$$= 4.13 - 0.4865714286\, t + 0.08232910695\, t^2 + 0.0005362709833\, t^3 + \cdots . \tag{16}$$

3. Basics of Variational Iteration Method (VIM)

In 1997, He [22] introduced an another semi-analytical method namely variational iteration method (VIM) (also see [23, 24, 25, 26]) to solve large class of non-linear differential equations effectively and it is observed that this method helps to get fast convergence. Following He [22] there are numerous works among those [1, 3, 8, 19, 20, 39, 43] have been considered and applied the VIM to solve the system of linear or non-linear ODEs and obtained solutions .

Now, consider the non-linear equation:

$$L\beta(t) + N\beta(t) = \gamma(t),$$

where L is a linear operator, N is nonlinear operator and $\gamma(t)$ is analytic function. Due to VIM, we take the correction functional as

$$\beta^{(n+1)}(t) = \beta^{(n)}(t) + \int_0^t \lambda(\tau)(L\beta(\tau) + N\beta(\tau) - \gamma(\tau))d\tau,$$

where λ is known as a general Lagrange multiplier and evaluated by variational theory, $\beta^{(0)}(t)$ is an initial approximation with possible unknowns and $\tilde{\beta}^{(n)}(t)$ is a restricted variation, that is $\delta\tilde{\beta}^{(n)}(t) = 0$.

3.1 Applications of VIM to 2019-nCOV epidemic model

Now, we apply the techniques of VIM to solve the equations (5) - (8) and write the correction functional as:

$$\begin{aligned}\beta_1^{(n+1)}(t) &= \beta_1^{(n)}(t) \\ &+ \int_0^t \lambda_1(\tau)\left[\frac{d\beta_1^{(n)}(\tau)}{d\tau} + \alpha\beta_1^{(n)}(\tau)\beta_2^{(n)}(\tau) + \alpha\beta_1^{(n)}(\tau)\beta_4^{(n)}(\tau)\right]d\tau, \end{aligned} \tag{17}$$

$$\begin{aligned}\beta_2^{(n+1)}(t) &= \beta_2^{(n)}(t) + \int_0^t \lambda_2(\tau)\left[\frac{d\beta_2^{(n)}(\tau)}{d\tau} - \alpha\beta_1^{(n)}(\tau)\beta_2^{(n)}(\tau)\right. \\ &\left. -\alpha\beta_1^{(n)}(\tau)\beta_4^{(n)}(\tau) + \beta\beta_2^{(n)}(\tau)\right]d\tau, \end{aligned} \tag{18}$$

$$\beta_3^{(n+1)}(t) = \beta_3^{(n)}(t) + \int_0^t \lambda_3(\tau)\left[\frac{d\beta_3^{(n)}(\tau)}{d\tau} - \lambda\beta_2^{(n)}(\tau) + \gamma\beta_3^{(n)}(\tau)\right]d\tau, \tag{19}$$

$$\beta_4^{(n+1)}(t) = \beta_4^{(n)}(t) + \int_0^t \lambda_4(\tau)\left[\frac{d\beta_4^{(n)}(\tau)}{d\tau} - \phi\beta_2^{(n)}(\tau) + \gamma\beta_4^{(n)}(\tau)\right]d\tau, \tag{20}$$

where λ_i, $i = 1, 2, 3, 4$. Making the above functions stationary with respect to $\beta_i^{(n)}(t)$, $i = 1, 2, 3, 4$ it can be identified as $\lambda_i(\tau) = -1$, $i = 1, 2, 3, 4$. Hence, the iteration are:

$$\begin{aligned}\beta_1^{(n+1)}(t) &= \beta_1^{(n)}(t) \\ &- \int_0^t \left[\frac{d\beta_1^{(n)}(\tau)}{d\tau} + \alpha\beta_1^{(n)}(\tau)\beta_2^{(n)}(\tau) + \alpha\beta_1^{(n)}(\tau)\beta_4^{(n)}(\tau)\right]d\tau, \end{aligned} \tag{21}$$

$$\begin{aligned}\beta_2^{(n+1)}(t) &= \beta_2^{(n)}(t) - \int_0^t \left[\frac{d\beta_2^{(n)}(\tau)}{d\tau} - \alpha\beta_1^{(n)}(\tau)\beta_2^{(n)}(\tau)\right. \\ &\left. -\alpha\beta_1^{(n)}(\tau)\beta_4^{(n)}(\tau) + \beta\beta_2^{(n)}(\tau)\right]d\tau, \end{aligned} \tag{22}$$

$$\beta_3^{(n+1)}(t) = \beta_3^{(n)}(t) - \int_0^t \left[\frac{d\beta_3^{(n)}(\tau)}{d\tau} - \lambda\beta_2^{(n)}(\tau) + \gamma\beta_3^{(n)}(\tau)\right]d\tau, \tag{23}$$

$$\beta_4^{(n+1)}(t) = \beta_4^{(n)}(t) - \int_0^t \left[\frac{d\beta_4^{(n)}(\tau)}{d\tau} - \phi\beta_2^{(n)}(\tau) + \gamma\beta_4^{(n)}(\tau)\right]d\tau. \tag{24}$$

Applying the initial approximations $\beta_i(0) = \beta_i^{(0)}$, $i = 1, 2, 3, 4$, we have

$$\begin{cases} \beta_1^{(1)}(t) = 11081000.0 - 3.847323200\,t, \\ \beta_2^{(1)}(t) = 3.62 + 3.330180343\,t, \\ \beta_3^{(1)}(t) = 0.4137142858\,t, \\ \beta_4^{(1)}(t) = 4.13 - 0.4865714286\,t. \end{cases} \tag{25}$$

For the solution after two terms:

$$\begin{cases} \beta_1^{(2)}(t) = 11081000.0 - 3.847323200\,t - 0.7058240126\,t^2, \\ \beta_2^{(2)}(t) = 3.62 + 3.330180343\,t + 0.4679539881\,t^2, \\ \beta_3^{(2)}(t) = 0.4137142858\,t + 0.1607449992\,t^2, \\ \beta_4^{(2)}(t) = 4.13 - 0.4865714286\,t + 0.08232910695\,t^2. \end{cases} \tag{26}$$

Similarly, we get the following iterative series as:

$$\begin{cases} \beta_1^{(3)}(t) = 11081000.0 - 3.847323200\,t - 0.7058240127\,t^2 \\ \qquad\qquad - 0.09105854712\,t^3, \\ \beta_2^{(3)}(t) = 3.62 + 3.330180343\,t + 0.4679539882\,t^2 + 0.06877502388\,t^3, \\ \beta_3^{(3)}(t) = 0.4137142858\,t + 0.1607449992\,t^2 + 0.01017229483\,t^3, \\ \beta_4^{(3)}(t) = 4.13 - 0.4865714286\,t + 0.08232910694\,t^2 \\ \qquad\qquad + 0.0005362709867\,t^3. \end{cases} \tag{27}$$

4. Results and discussion

The current epidemic model based on COVID-19 pandemic discussed in this work with the initial conditions $\mathcal{A}(0) = \mathcal{A}_0 = \beta_1^{(0)} = 11.081 \times 10^6$, $\mathcal{B}(0) = \mathcal{B}_0 = \beta_2^{(0)} = 3.62$, $\mathcal{C}(0) = \mathcal{C}_0 = \beta_3^{(0)} = 0$, $\mathcal{D}(0) = \mathcal{D}_0 = \beta_4^{(0)} = 4.13$ [36]. Also, we derived the series up to order three to record the procedure of the considered model. Further, we obtained the solutions $\beta_i(t)$, $i = 1, 2, 3, 4$ with respect to time (t) for the parameters $\alpha = 4.48 \times 10^{-8}$, $\beta = \dfrac{1}{7}$, $\gamma = \dfrac{1}{7}$, $\lambda = 0.8 \times \gamma$, $\phi = 0.2 \times \gamma$ which were taken from [36]. It is viewed that the projected system purely depends on the time and order. DTM plays a vital role in solving the different types of epidemic models especially nonlinear models and effectively initialized with the given conditions for the considered system which gives the interesting results that helps in better understanding of the current pandemic.

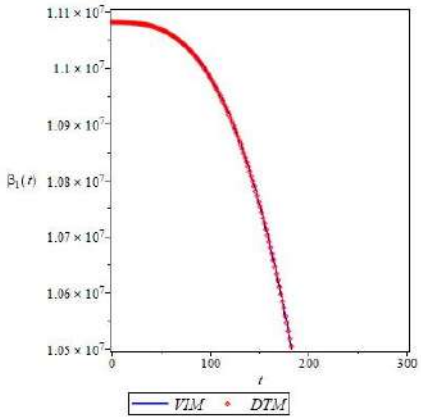


Figure 1: $\beta_1(t)$ in a time t.

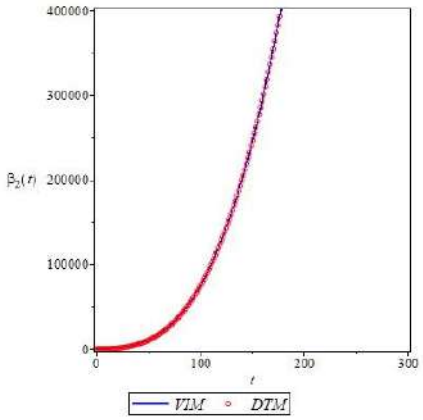


Figure 2: $\beta_2(t)$ in a time t.

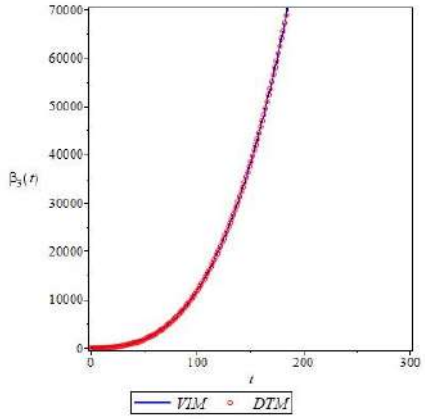


Figure 3: $\beta_3(t)$ in a time t.

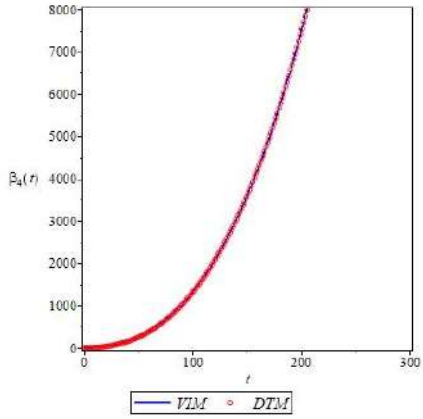


Figure 4: $\beta_4(t)$ in a time t.

5. Conclusion

The present study gives the better understanding of the proposed method and its applications to nonlinear ordinary differential equations. Here, DTM and VIM are effectively applied to the nonlinear differential equations and analyzed that the proposed methods can be implemented to various real life problems for predicting the causes and prevention. The numerical results are presented graphically. The results emphases that one can obtain the analytic form of solutions easily, efficiently by DTM and VIM. The comparison of the results obtained by the proposed methods are in good agreement.

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Smarandache pseudo-CI algebras

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Abstract. In this paper, we define the notion of Smarandache pseudo-CI algebras and we investigate their properties. We also define and study the notions of Smarandache filters, pseudo-CI Smarandache homomorphisms and modal Smarandache operators on pseudo-CI algebras. The classes of Smarandache fantastic, implicative and positive implicative filters of Smarandache pseudo-CI algebras are defined and studied by extending some results regarding Smarandache fantastic, fresh and clean ideals in Smarandache BCI-algebras and Smarandache BCH-algebras to the case of Smarandache pseudo-CI algebras. The notion of Smarandache commutative pseudo-CI algebras is defined and a characterization theorem is given. It is proved that in the case of commutative Q-Smarandache pseudo-CI algebras the notions of Smarandache filters and fantastic filters coincide.

Keywords: pseudo-CI algebra, pseudo-BE algebra, Smarandache pseudo-CI algebra, Q-Smarandache filter.

1. Introduction

Developing algebraic models for non-commutative multiple-valued logics is a central topic in the study of fuzzy systems. Pseudo-BCK algebras were introduced by G. Georgescu and A. Iorgulescu in [13] as algebras with "two differences", a left- and right-difference, and with a constant element 0 as the least element. Pseudo-BCK algebras were intensively studied in [15] (also see

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[14], [22], [21], [8]). Pseudo-BE algebras were introduced by R. A. Borzooei et al. as a generalization of BE-algebras and properties of these structures have recently been studied in [28] (also see [6]). L. C. Ciungu defined the notion of commutative pseudo-BE algebras and proved that the class of commutative pseudo-BE algebras is term equivalent to the class of commutative pseudo-BCK algebras ([9]). Recently, A. Rezaei et al. introduced the notion of pseudo-CI algebras as generalizations of CI-algebras and they provided some conditions for a pseudo-CI algebra to be a pseudo-BE algebra ([29]). The class of singular pseudo-CI algebras was defined and it was proved that any singular pseudo-CI algebra is a pseudo-BCI algebra (see [12], [29]). A. Rezaei et al. defined the dual pseudo-Q and dual pseudo-QC algebras, investigated their properties and gave characterizations of these structures ([30]). It was also proved that the class of commutative dual pseudo-QC algebras coincides with the class of commutative pseudo-BCI algebras.

Generally, a *Smarandache structure* on a set A means a weak structure W on A such that there exists a proper subset B which is embedded with a stronger structure S ([16]). Smarandache structures on multiple-valued logic algebras have been studied in [4] (also see [3], [5], [16], [17], [18], [19], [24], [25]). A. Borumand Saeid defined the notion of Smarandache (weak) BE-algebras and proved some of their properties ([2], [3]).

In this paper, we define the notion of Smarandache pseudo-CI algebras and we investigate their properties. We also define and study the notions of Smarandache filters, pseudo-CI Smarandache homomorphisms and modal Smarandache operators on pseudo-CI algebras. The classes of Smarandache fantastic, implicative and positive implicative filters of Smarandache pseudo-CI algebras are defined and studied by extending some results regarding Smarandache fantastic, fresh and clean ideals in Smarandache BCI-algebras and Smarandache BCH-algebras ([19], [18], [4]) to the case of Smarandache pseudo-CI algebras. We give a characterization of Smarandache implicative filters and we present conditions for a Smarandache filter to be a Smarandache implicative filter. For a Q-Smarandache pseudo-CI algebra we prove that any Smarandache implicative filter is a filter and any Smarandache positive implicative filter contained in Q is a Smarandache filter. We also give a characterization of Smarandache positive implicative filters. The notion of Smarandache commutative pseudo-CI algebras is defined and a characterization theorem is given. It is proved that in the case of commutative Q-Smarandache pseudo-CI algebras the notions of Smarandache filters and fantastic filters coincide. Finally, we define and investigate the notion of a Smarandache upper set in a pseudo-CI algebra and we show that every Q-Smarandache filter is a union of Q-Smarandache upper sets.

2. Preliminaries

In this section, we recall some basic notions and results regarding pseudo-CI algebras and pseudo-BE algebras. Pseudo-BE algebras were introduced in [5] as a generalization of BE-algebras (see [20]) and properties of it$'s$ have recently been studied in [30] and [6].

A *CI-algebra* ([23]) is an algebra $(X; \to, 1)$ of type $(2, 0)$ satisfying the following axioms, for all $x, y, z \in X$:
(CI_1) $x \to x = 1$;
(CI_2) $1 \to x = x$;
(CI_3) $x \to (y \to z) = y \to (x \to z)$.

We introduce a binary relation $\leq$ on X by $x \leq y$ if and only if $x \to y = 1$.

A CI-algebra $(X; \to, 1)$ is said to be a *BE-algebra* ([20]) if (BE) $x \to 1 = 1$, for all $x \in X$.

Definition 2.1. ([29]) An algebra $(X; \to, \rightsquigarrow, 1)$ of type $(2, 2, 0)$ is called a *pseudo-CI algebra if, for all* $x, y, z \in X$, *it satisfies the following axioms:*
$(psCI_1)$ $x \to x = x \rightsquigarrow x = 1$;
$(psCI_2)$ $1 \to x = 1 \rightsquigarrow x = x$;
$(psCI_3)$ $x \to (y \rightsquigarrow z) = y \rightsquigarrow (x \to z)$;
$(psCI_4)$ $x \to y = 1$ if and only if $x \rightsquigarrow y = 1$.

Remark 2.1. If $(X; \to, \rightsquigarrow, 1)$ is a pseudo-CI algebra satisfying $x \to y = x \rightsquigarrow y$, for all $x, y \in X$, then $(X; \to, 1)$ is a CI-algebra. Also, if $(X; \to, \rightsquigarrow, 1)$ is a pseudo-CI algebra, then $(X; \rightsquigarrow, \to, 1)$ is too.

Remark 2.2. Since every pseudo-BCI algebra satisfies (psCI_1)–(psCI_4), pseudo-BCI algebras are contained in the class of pseudo-CI algebras.

In the sequel, we will also refer to the pseudo-CI algebra $(X; \to, \rightsquigarrow, 1)$ by $\mathfrak{X}$.

Any pseudo-CI algebra $\mathfrak{X}$ verifying condition $(psBE)$ $x \to 1 = x \rightsquigarrow 1 = 1$, for all $x, y \in X$, is said to be a *pseudo-BE algebra* ([6]). A pseudo-CI algebra which is not a pseudo-BE algebra, pseudo-BCI algebra and pseudo-BCH algebra will be called *proper*. A pseudo-CI algebra with condition (A) or a pseudo-CI(A) algebra for short, is a pseudo-CI algebra $\mathfrak{X}$ satisfying the condition (A):

(A) for all $x, y, z \in X$, if $x \preceq y$, then $y \to z \preceq x \to z$ and $y \rightsquigarrow z \preceq x \rightsquigarrow z$.

In a pseudo-CI algebra $\mathfrak{X}$ we can introduce a binary relation $\preceq$ by:

$x \preceq y$ if and only if $x \to y = 1$ if and only if $x \rightsquigarrow y = 1$, for all $x, y \in X$.

Note that $\preceq$ is reflexive by (psCI_1).

Example 2.1. ([29]) (1) Let $X = \{1, a, b, c, d, e\}$. Define the binary operations $\to$ and $\rightsquigarrow$ on X as follows:

$\to$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	a	1	c	b	e	d
b	b	d	1	e	a	c
c	d	b	e	1	c	a
d	c	e	a	d	1	b
e	e	c	d	a	b	1

$\rightsquigarrow$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	a	1	d	e	b	c
b	b	c	1	a	e	d
c	d	e	a	1	c	b
d	c	b	e	d	1	a
e	e	d	c	b	a	1

Then, $(X; \to, \rightsquigarrow, 1)$ is a pseudo-CI algebra, but not a pseudo-BE algebra, since $a \to 1 = a \neq 1$ and $a \rightsquigarrow 1 = a \neq 1$.

(2) Let $X = \{1, a, b, c, d, e, f, g, h\}$. Define the binary operations $\to$ and $\rightsquigarrow$ on X as follows:

$\to$	1	a	b	c	d	e	f	g	h
1	1	a	b	c	d	e	f	g	h
a	1	1	1	1	d	e	f	g	h
b	1	c	1	1	d	e	f	g	h
c	1	c	b	1	d	e	f	g	h
d	d	d	d	d	1	g	h	e	f
e	e	e	e	e	h	1	g	f	d
f	f	f	f	f	g	h	1	d	e
g	h	h	h	h	e	f	d	1	g
h	g	g	g	g	f	d	e	h	1

$\rightsquigarrow$	1	a	b	c	d	e	f	g	h
1	1	a	b	c	d	e	f	g	h
a	1	1	1	1	d	e	f	g	h
b	1	c	1	1	d	e	f	g	h
c	1	c	b	1	d	e	f	g	h
d	d	d	d	d	1	h	g	f	e
e	e	e	e	e	g	1	h	d	f
f	f	f	f	f	h	g	1	e	d
g	h	h	h	h	f	d	e	1	g
h	g	g	g	g	e	f	d	h	1

Then $(X; \to, \rightsquigarrow, 1)$ is a proper pseudo-CI algebra.

Definition 2.2. ([6]) Let $\mathfrak{X}$ be a pseudo-BE algebra. A subset F of X is called a *filter* of $\mathfrak{X}$ if for all $x, y \in X$:
(F_1) $1 \in F$;
(F_2) $x \to y \in F$ and $x \in F$ imply $y \in F$.

Denote by $\mathcal{F}(X)$ set of all filters of $\mathfrak{X}$. Obviously, $\{1\}, X \in \mathcal{F}(X)$.

Definition 2.3. ([11]) Let $\mathfrak{X}$ be a pseudo-BE algebra. A mapping $f : X \longrightarrow X$ is called a *modal operator* on X if it satisfies the following conditions for all $x, y \in X$:
(M_1) $x \leq f(x)$;
(M_2) $f(f(x)) = f(x)$;
(M_3) $f(x \to y) \leq f(x) \to f(y)$ and $f(x \rightsquigarrow y) \leq f(x) \rightsquigarrow f(y)$.
The pair (X, f) is called a *modal pseudo-BE algebra*.

Denote by $\mathcal{MOD}(X)$ set of all modal operators on X.

3. Smarandache pseudo-CI algebras

In this section, we define the notion of a Smarandache pseudo-CI algebra and investigate these properties. We also define and study the notions of Smarandache filters, pseudo-CI Smarandache homomorphisms and modal Smarandache operators on pseudo-CI algebras.

Definition 3.1. A pseudo-CI algebra $\mathfrak{X}$ is said to be a *Q-Smarandache pseudo-CI algebra* if there is a proper subset Q of X such that:
(S_1) $1 \in Q$ and $|Q| \geq 2$;
(S_2) $\mathfrak{Q} = (Q; \to, \rightsquigarrow, 1)$ is a pseudo-BE algebra under the operations of $\mathfrak{X}$.
Q is called the *heart of* $\mathfrak{X}$*, if it satisfies* (S_1)*,* (S_2) *and axiom:*
(S_3) If there is $\emptyset \neq S \subseteq X$ satisfies (S_1) and (S_2), then $S \subseteq Q$
(i.e. $Q = \{x \in X : x \to 1 = 1\}$).

Remark 3.1. Using (S_3), the heart of $\mathfrak{X}$ is unique and $Q = X$ if and only if $\mathfrak{X}$ is a pseudo-BE algebra.

Example 3.1. (1) Every pseudo-BE algebra is a Smarandache pseudo-CI algebra.

(2) Consider the pseudo-CI algebra given in Example 2.1 (2), let $Q_1 = \{1, a, b, c\}$, $Q_2 = \{1, a\}$, $Q_3 = \{1, b\}$, $Q_4 = \{1, a, c\}$, and let $Q_5 = \{1, b, c\}$. Then $\mathfrak{X}$ is a Q_1, Q_2, Q_3, Q_4 and Q_5 Smarandache pseudo-CI algebra. Moreover, Q_1 satisfies (S_3), hence it is the heart of $\mathfrak{X}$.

Proposition 3.1. *In any Q-Smarandache pseudo-CI algebra* $\mathfrak{X}$ *the following hold, for all* $x, y \in X$*:*
(1) *if* $x \notin Q$*, then* $x \to 1 \notin Q$ *and* $x \rightsquigarrow 1 \notin Q$*;*
(2) $x \to 1 = 1$ *or* $x \to 1 \notin Q$*;*
(3) *if* $x \to 1 \notin Q$*, then* $x \notin Q$*;*
(4) *if* $x \to 1 = y \to 1$*, then* $x \to y \in Q$ *and* $y \to x \in Q$*;*
(5) *if* $x \rightsquigarrow 1 = y \rightsquigarrow 1$*, then* $x \rightsquigarrow y \in Q$ *and* $y \rightsquigarrow x \in Q$*;*
(6) *if* $x \in Q$ *and* $y \notin Q$*, then* $x \to y \notin Q$*,* $x \rightsquigarrow y \notin Q$ *and* $y \to x \notin Q$*,* $y \rightsquigarrow x \notin Q$*.*

Theorem 3.1. *Let* $\mathfrak{X}$ *be a proper pseudo-CI algebra. Then* $\mathfrak{X}$ *is a Q-Smarandache pseudo-CI algebra if and only if there exists* $Q \subseteq X$ *such that* $|Q| \geq 2$ *and* $x \to 1 = 1$*, for all* $x \in Q$*.*

Proof. Let $\mathfrak{X}$ be a Q-Smarandache pseudo-CI algebra. Then by definition we get there exists $Q \subseteq X$ such that $x \to 1 = 1$, for all $x \in Q$.
Conversely, consider $Q = \{x \in X \mid x \to 1 = 1\}$. It is suffice to prove that Q is a subalgebra of X. If $x, y \in Q$, then $x \to 1 = y \to 1 = 1$. By (a_4), we get

$$(x \to y) \to 1 = (x \to 1) \rightsquigarrow (y \to 1) = 1 \rightsquigarrow 1 = 1.$$

Thus $x \to y \in Q$. Similarly, $x \rightsquigarrow y \in Q$. Hence Q is a subalgebra of $\mathfrak{X}$. □

Definition 3.2. A subset F of a pseudo-CI algebra $\mathfrak{X}$ is called a *Smarandache filter* of $\mathfrak{X}$ related to $\mathfrak{Q}$ (or briefly, Q-Smarandache filter of $\mathfrak{X}$) if it satisfies, for

all $y \in Q$ and $x \in F$:
(SF_1) $1 \in F$;
(SF_2) $x \to y \in F$ implies $y \in F$;
(SF_3) $x \rightsquigarrow y \in F$ implies $y \in F$.

Denote by $\mathcal{F}_Q(X)$ set of all Q-Smarandache filters of $\mathfrak{X}$.

Example 3.2. Consider the pseudo-CI algebra given in Example 2.1 (2). We can see that $\mathfrak{X}$ is a Q-Smarandache pseudo-CI algebra where $Q = \{1, a, b, c\}$. Note that $F_1 = \{1, a, b, c, d\}$, $F_2 = \{1, h\}$ and $F_3 = \{1, g, h\}$ are Q-Smarandache filters of $\mathfrak{X}$.

The following we provide some conditions for a subalgebra to be a Q-Smarandache filter.

Theorem 3.2. *Let F be a subalgebra of $\mathfrak{X}$. Then F is a Q-Smarandache filter of $\mathfrak{X}$ if and only if for all $x, y \in X$,*

$$x \in F,\ y \in Q \setminus F \text{ imply } x \to y \in Q \setminus F \text{ and } x \rightsquigarrow y \in Q \setminus F.$$

Proof. Assume that $F \in \mathcal{F}_Q(X)$ and $x, y \in X$, such that $x \in F$ and $y \in Q \setminus F$. If $x \to y \notin Q \setminus F$, then $x \to y \in F$ (i.e. $y \in F$), which is a contradiction. Hence $x \to y \in Q \setminus F$. Now, if $x \rightsquigarrow y \notin Q \setminus F$, then $x \rightsquigarrow y \in F$ (i.e. $y \in F$), which is a contradiction. Hence $x \rightsquigarrow y \in Q \setminus F$.

Conversely, assume that the hypothesis is valid. Since F is a subalgebra, we have $1 \in F$. For every $x \in F$, let $x \to y \in F$. If $y \notin F$, then $x \to y \in Q \setminus F$ by assumption, which is a contradiction. Hence $y \in F$. Now, let $x \rightsquigarrow y \in F$. Then by hypothesis we have $y \in F$. Therefore, F is a Q-Smarandache filter of $\mathfrak{X}$. □

Theorem 3.3. *Let $\mathfrak{X}$ be a Q-Smarandache pseudo-CI algebra, and let F be a subset of X such that $Q \subseteq F$. Then F is a Smarandache filter of $\mathfrak{X}$.*

The next example shows that the converse of Theorem 3.3 is not valid in general.

Example 3.3. Let $\mathfrak{X}$ be the pseudo-CI algebra from Example 2.1 (2).
(1) If $Q = \{1, a, b, c\}$ and $F = \{1, b, c, g\}$, then F is a Q-Smarandache filter.
(2) If $Q = \{1, a, b, c\}$ we can easily see that, every filter F of $\mathfrak{X}$ containing Q is a Q-Smarandache filter of $\mathfrak{X}$. For example $F_1 = \{1, a, b, c, d, e, \}$ is a Q-Smarandache filter of $\mathfrak{X}$.

Proposition 3.2. *Any filter of a pseudo-CI algebra $\mathfrak{X}$ is a Q-Smarandache filter.*

The following example shows that the converse of above proposition is not valid in general.

Example 3.4. Consider the pseudo-CI algebra from Example 2.1 (2) and let $Q := \{1, a, b, c\}$. Then $\mathfrak{X}$ is a Q-Smarandache pseudo-CI. Also, $F = \{1, h\}$ is a Q-Smarandache filter of $\mathfrak{X}$, but it is not a filter of $\mathfrak{X}$, since $h \to g = h \rightsquigarrow g = h \in F$ and $h \in F$, but $g \notin F$.

In [7], R. A. Borzooei et al. introduced the notion of distributive pseudo-BE algebras and got some useful results. The following we define the notion of *weak distributive Q-Smarandach pseudo-CI algebras.*

Definition 3.3. A Q-Smarandache pseudo-CI algebra $\mathfrak{X}$, where Q is the heart of $\mathfrak{X}$, is said to be *weak distributive* if it satisfies only one of the following conditions, for all $x, y, z \in Q$:
(WD_1) $x \to (y \rightsquigarrow z) = (x \to y) \rightsquigarrow (x \to z)$;
(WD_2) $x \rightsquigarrow (y \to z) = (x \rightsquigarrow y) \to (x \rightsquigarrow z)$.

Remark 3.2. Take $x = y$ in (WD_1) and (WD_2) and applying (psCI_2), we get:
$x \to (x \rightsquigarrow z) = (x \to x) \rightsquigarrow (x \to z) = 1 \rightsquigarrow (x \to z) = x \to z$ and
$x \rightsquigarrow (x \to z) = (x \rightsquigarrow x) \to (x \rightsquigarrow z) = 1 \to (x \rightsquigarrow z) = x \rightsquigarrow z$.
Now, using (psCI_4), we have $x \to z = x \to (x \rightsquigarrow z) = x \rightsquigarrow (x \to z) = x \rightsquigarrow z$, for all $x, z \in Q$. Consequently, $\to = \rightsquigarrow$, and so Q is a BE-algebra.

In this paper, weak distributive pseudo-CI algebra satisfies (WD_1).

Example 3.5. (1) Let $X = \{1, a, b, c, d\}$. Define the binary operations $\to$ and $\rightsquigarrow$ on X by the following tables:

$\to$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	1	d
b	1	1	1	1	d
c	1	a	a	1	d
d	d	d	d	d	1

$\rightsquigarrow$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	1	d
b	1	1	1	1	d
c	1	a	b	1	d
d	d	d	d	d	1

Then $(X; \to, \rightsquigarrow, 1)$ is a weak distributive pseudo-CI algebra, where $Q = \{1, a, b, c\}$.

(2) Consider the Q-Smarandache pseudo-CI algebra given in Example 2.1 (2), where $Q := \{1, a, b, c\}$. Then $\mathfrak{X}$ is not a weak distributive, since $b \to (b \rightsquigarrow a) = b \to c = 1 \neq c = 1 \rightsquigarrow c = (b \to b) \rightsquigarrow (b \to a)$.

Remark 3.3. Singular pseudo-CI algebras were introduced and studied by Rezaei et al. in [29]. Now, if $\mathfrak{X}$ is a singular pseudo-CI algebra, then $Q = \{1\}$, and so $\mathfrak{X}$ is a weak distributive pseudo-CI algebra.

Proposition 3.3. *If F is a Q-Smarandache filter of weak distributive pseudo-CI algebra $\mathfrak{X}$, then for all $x, y, z \in Q$:*
(1) *$z \rightsquigarrow (y \to x) \in F$ and $z \rightsquigarrow y \in F$ imply $z \rightsquigarrow x \in F$;*
(2) *$z \to (y \rightsquigarrow x) \in F$ and $z \to y \in F$ imply $z \to x \in F$.*

Corollary 3.1. *If F is a Q-Smarandache filter of weak distributive pseudo-CI algebra $\mathfrak{X}$, then for all $x, y \in Q$:*
(1) *$y \rightsquigarrow (y \to x) \in F$ implies $y \rightsquigarrow x \in F$;*
(2) *$y \to (y \rightsquigarrow x) \in F$ implies $y \to x \in F$.*

Proposition 3.4. *Let F be a Q-Smarandache filter of a pseudo-CI algebra $\mathfrak{X}$ and $x, y \in Q$. Then*
(1) *if $x \in F$, $y \in Q$ and $x \preceq y$, then $y \in F$;*
(2) *if $\mathfrak{X}$ is weak distributive pseudo-CI algebra and $x, y \in F$, then $x \to y \in F$;*
(3) *if $\mathfrak{X}$ is weak distributive pseudo-CI algebra and $x, y \in F$, then $x \rightsquigarrow y \in F$.*

Theorem 3.4. *Any Q-Smarandache filter is a subalgebra of $\mathfrak{Q}$.*

The converse of Theorem 3.4 is not valid in general. Indeed, in Example 2.1 (1), $S = \{1, a\}$ is a subalgebra, but it is not a Q-Smarandache filter.

Theorem 3.5. *Let Q_1 and Q_2 be pseudo-BE algebras which are properly contained in a pseudo-CI algebra $\mathfrak{X}$ and $Q_1 \subseteq Q_2$. Then every Q_2-Smarandache filter is a Q_1-Smarandache filter of $\mathfrak{X}$.*

The following example shows that the converse of Theorem 3.5 is not valid in general.

Example 3.6. Let $X = \{1, a, b, c, d, e, f, g, h\}$, $Q_1 = \{1, a\}$, $Q_2 = \{1, a, b, c\}$ and $F = \{1, a, b\}$. According to Example 2.1 (2), we can see that $\mathfrak{X}$ is a Q_1-Smrandache pseudo-CI algebra and Q_2-Smrandache pseudo-CI algebra. Also, F is a Q_1-Smarandache filter of $\mathfrak{X}$, but F is not Q_2-Smrandache filter of $\mathfrak{X}$. Indeed, $b \to c = 1 \in F$, $b \in F$, $c \in Q_2$, but $c \notin F$.

Definition 3.4. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Q_X and Q_Y-Smarandache pseudo-CI algebras, respectively. A mapping $f : X \longrightarrow Y$ is called a *Smarandache pseudo-CI homomorphism* if $f_s = f_{|Q} : Q_X \longrightarrow Q_Y$ is a pseudo-BE homomorphism.

Theorem 3.6. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Q_X and Q_Y Smarandache pseudo-CI algebras and $f : X \longrightarrow Y$ be a Smarandache pseudo-CI homomorphism. Then:*
(1) *if $G \in \mathcal{F}_{Q_Y}(Y)$, then $f^{-1}(G) \in \mathcal{F}_{f^{-1}(Q_Y)}(X)$;*
(2) *if f is injective and $F \in \mathcal{F}_{Q_X}(X)$, then $f(F) \in \mathcal{F}_{f(Q_X)}(Y)$.*

Proof. (1) Assume that $G \in \mathcal{F}_{Q_Y}(Y)$ and $y \in f^{-1}(G)$. Obviously, $1_X \in f^{-1}(G)$. Let x, $x \to y \in f^{-1}(G)$ and $x \rightsquigarrow y \in f^{-1}(G)$. It follows that $f(x) \to f(y) = f(x \to y) \in G$ and $f(x) \rightsquigarrow f(y) = f(x \rightsquigarrow y) \in G$. Then $f(y) \in Q_Y$, since $f(x) \in G$ and $G \in \mathcal{F}_{Q_Y}(Y)$, we have $f(y) \in G$. Therefore, $y \in f^{-1}(G)$, and so $f^{-1}(G) \in \mathcal{F}_{f^{-1}(Q_Y)}(X)$.

(2) Assume that f is injective and $F \in \mathcal{F}_{Q_X}(X)$. Obviously, $1_Y \in f(F)$. Let a, $a \to b \in f(F)$ and $b \in f(Q_X)$. It follows that there exist x_a, $x_{a\to b} \in F$ and $x_b \in Q_X$ such that $f(x_a) = a$, $f(x_{a\to b}) = a \to b$ and $f(x_b) = b$. Now, we have

$$f(x_{a\to b}) = a \to b = f(x_a) \to f(x_b) = f(x_a \to x_b).$$

Since f is injective, we have $x_{a\to b} = x_a \to x_b \in F$, and so $x_b \in F$. Hence $b = f(x_b) \in f(F)$. Therefore, $f(F) \in \mathcal{F}_{f(Q_X)}(Y)$. □

Definition 3.5. Let $\mathfrak{X}$ be a Q-Smarandache pseudo-CI algebra. A mapping $f : X \to X$ is called a *modal Q-Smarandache operator* if $f_s = f_{|Q} : Q \to Q$ is a modal pseudo-BE algebra.

Denote by $\mathcal{SMOD}_{\mathrm{Q}}(X)$ set of all modal Q-Smarandache operators on X.

Proposition 3.5. *Let Q_1 and Q_2 be pseudo-BE algebras such that $Q_1 \subseteq Q_2 \subseteq X$. Then $\mathcal{SMOD}_{\mathrm{Q}_1}(X) \subseteq \mathcal{SMOD}_{\mathrm{Q}_2}(X)$.*

4. Commutative Smarandache pseudo-CI algebras

The commutative pseudo-BE algebras were defined and investigated in [10], while the commutative Smarandache CI-algebras have been defined and studied in [5]. In this section we introduce the notion of commutative Smarandache pseudo-CI algebras, we give characterizations of these structures and investigate some of their properties.

Let $\mathfrak{X}$ be a pseudo-CI algebra. For all $x, y \in X$, denote:

$$x \vee_1 y = (x \to y) \rightsquigarrow y \text{ and } x \vee_2 y = (x \rightsquigarrow y) \to y.$$

If $\to = \rightsquigarrow$, then the pseudo-CI algebra $\mathfrak{X}$ is a CI-algebra and

$$x \vee y = (x \to y) \to y.$$

Definition 4.1. A Q-Smarandache pseudo-CI algebra $\mathfrak{X}$ is said to be *commutative* if Q is a commutative pseudo-BE algebra, that is, it satisfies the following conditions, for all $x, y \in Q$, $x \vee_1 y = y \vee_1 x$ and $x \vee_2 y = y \vee_2 x$.

Example 4.1. Let $X = \{1, a, b, c, d, e, f, g\}$. Define the binary operations $\to$ and $\rightsquigarrow$ on X by the following tables:

$\to$	1	a	b	c	d	e	f	g
1	1	a	b	c	d	e	f	g
a	1	1	b	c	d	e	f	g
b	1	a	1	c	d	e	f	g
c	c	c	c	1	f	g	d	e
d	d	d	d	g	1	f	e	c
e	e	e	e	f	g	1	c	d
f	g	g	g	d	e	c	1	f
g	f	f	f	e	c	d	g	1

$\rightsquigarrow$	1	a	b	c	d	e	f	g
1	1	a	b	c	d	e	f	g
a	1	1	b	c	d	e	f	g
b	1	a	1	c	d	e	f	g
c	c	c	c	1	g	f	e	d
d	d	d	d	f	1	g	c	e
e	e	e	e	g	f	1	d	c
f	g	g	g	e	c	d	1	f
g	f	f	f	d	e	c	g	1

Then $(X; \to, \rightsquigarrow, 1)$ is a Q-Smarandache commutative pseudo-CI algebra, where $Q = \{1, a, b\}$.

Proposition 4.1. *Let $\mathfrak{X}$ be a Q-Smarandache commutative pseudo-CI algebra, and let $x, y \in Q$ such that $x \to y = y \to x = 1$ or $x \rightsquigarrow y = y \rightsquigarrow x = 1$. Then $x = y$.*

Proof. Consider $x, y \in Q$ such that $x \to y = y \to x = 1$. Since $\mathfrak{X}$ is commutative and applying (psCI_2), we get:
$x = 1 \rightsquigarrow x = (y \to x) \rightsquigarrow x = (x \to y) \rightsquigarrow y = 1 \rightsquigarrow y = y$. Similarly, $x \rightsquigarrow y = y \rightsquigarrow x = 1$ implies $x = y$. □

Proposition 4.2. *In any Q-Smarandache commutative pseudo-CI algebra* $\mathfrak{X}$ *the following hold, for all* $x, y \in Q$*:*
(1) $x \to y = y \vee_1 x \to y$ *and* $x \rightsquigarrow y = y \vee_2 x \rightsquigarrow y$*;*
(2) $x \vee_1 y = (x \vee_1 y) \vee_1 x$ *and* $x \vee_2 y = (x \vee_2 y) \vee_2 x$*;*
(3) $x \leq y$ *implies* $y \vee_1 x = y \vee_2 x = y$*.*

Proof. It follows by [10, Prop. 4.9]. □

Theorem 4.1. *An algebra* $\mathfrak{X}$ *of the type* $(2, 2, 0)$ *is a Q-Smarandache commutative pseudo-CI algebra if and only if the following hold, for all* $x, y, z \in Q$*:*
(P_1) $1 \to x = 1 \rightsquigarrow x = x$*;*
(P_2) $x \to 1 = x \rightsquigarrow 1 = 1$*;*
(P_3) $(x \to z) \rightsquigarrow (y \to z) = (z \to x) \rightsquigarrow (y \to x)$ *and*
$(x \rightsquigarrow z) \to (y \rightsquigarrow z) = (z \rightsquigarrow x) \to (y \rightsquigarrow x)$*;*
(P_4) $x \to (y \rightsquigarrow z) = y \rightsquigarrow (x \to z)$*;*
(P_5) $x \to y = 1$ *if and only if* $x \rightsquigarrow y = 1$*.*

Proof. It follows by [10, Th. 4.13]. □

Theorem 4.2. *An algebra* $\mathfrak{X}$ *of the type* $(2, 2, 0)$ *is a Q-Smarandache commutative pseudo-CI algebra if and only if the following hold, for all* $x, y, z \in Q$*:*
(Q_1) $(x \to 1) \rightsquigarrow y = (x \rightsquigarrow 1) \to y = y$*;*
(Q_2) $(x \to z) \rightsquigarrow (y \to z) = (z \to x) \rightsquigarrow (y \to x)$ *and*
$(x \rightsquigarrow z) \to (y \rightsquigarrow z) = (z \rightsquigarrow x) \to (y \rightsquigarrow x)$*;*
(Q_3) $x \to (y \rightsquigarrow z) = y \rightsquigarrow (x \to z)$*;*
(Q_4) $x \to y = 1$ *if and only if* $x \rightsquigarrow y = 1$*.*

Proof. It follows by [10, Th. 4.14]. □

Remark 4.1. According to [9] the following hold:
– Any pseudo BCK-algebra is a pseudo-BE algebra;
– The class of commutative pseudo-BE algebras is term equivalent to the class of commutative pseudo-BCK algebras.
It follows that in the definition of commutative Q-Smarandache pseudo-CI algebras, the pseudo-BE algebra can be replaced with a pseudo-BCK algebra.

5. Classes of Smarandache filters of Smarandache pseudo-CI algebras

Developing filter theory of multiple-valued logic algebras is a central topic in the study of fuzzy systems (see, e.g., [1, 26, 27]).

In this section we define and study the classes of Smarandache fantastic, implicative and positive implicative filters of Smarandache pseudo-CI algebras. We generalize some results regarding Smarandache fantastic, fresh and clean ideals proved in [19], [18] and [4] for Smarandache BCI-algebras and Smarandache BCH-algebras. It is proved that in the case of commutative Q-Smarandache pseudo-CI algebras the notions of Smarandache filters and fantastic filters coincide. We give a characterization of Smarandache implicative filters and we present conditions for a Smarandache filter to be a Smarandache implicative filter. For a Q-Smarandache pseudo-CI algebra we prove that any Smarandache implicative filter is a filter and any Smarandache positive implicative filter contained in Q is a Smarandache filter. Finally, we give a characterization of Smarandache positive implicative filters.

Definition 5.1. Let $\mathfrak{X}$ be a Q-Smarandache pseudo-CI algebra. A filter F of $\mathfrak{X}$ is said to be *Q-Smarandache fantastic filter* of $\mathfrak{X}$ if it satisfies the following conditions, for all $x, y \in Q$:
(FF_1) $y \to x \in F$ implies $x \vee_1 y \to x \in F$;
(FF_2) $y \rightsquigarrow x \in F$ implies $x \vee_2 y \rightsquigarrow x \in F$.

Denote by $\mathcal{F}_Q^F(X)$ set of all Q-Smarandache fantastic filters of $\mathfrak{X}$.

Theorem 5.1. *Let $\mathfrak{X}$ be a Q-Smarandache pseudo-CI algebra and let $F \subseteq X$. Then $F \in \mathcal{F}_Q^F(X)$ if and only if it satisfies the following conditions, for all $x, y \in Q$ and $z \in X$:*
(1) $1 \in F$;
(2) $z \to (y \to x) \in F$ *and* $z \in F$ *imply* $x \vee_1 y \to x \in F$;
(3) $z \to (y \rightsquigarrow x) \in F$ *and* $z \in F$ *imply* $x \vee_2 y \rightsquigarrow x \in F$.

Proof. Consider $F \in \mathcal{F}_Q(X)$. Since $1 \in F$, condition (1) is satisfied. Let $x, y \in Q$ and $z \in F$ such that $z \to (y \to x) \in F$. Obviously, $y \to x \in Q$. Since $F \in \mathcal{F}(X)$, we have $y \to x \in F$, hence $x \vee_1 y \to x \in F$, that is, condition (2). Similarly, from $z \rightsquigarrow (y \rightsquigarrow x) \in F$ and $z \in F$, we get $x \vee_2 y \rightsquigarrow x \in F$, that is, condition (3).

Conversely, let $F \subseteq X$ satisfying conditions (1), (2) and (3). Obviously, $1 \in F$. Let $x, y \in Q$ such that $x \to y$, $x \in F$. Since $x \to (1 \to y) = x \to y \in F$, using (2), we have $y = y \vee_1 1 \to y \in F$. It follows that $F \in \mathcal{F}_Q(X)$. Let $x, y \in Q$ such that $y \to x \in F$. Since $1 \to (y \to x) \in F$ and $1 \in F$, by (2), we get $x \vee_1 y \to x \in F$. Similarly, from $y \rightsquigarrow x \in F$, we get $x \vee_2 y \rightsquigarrow x \in F$. We conclude that $F \in \mathcal{F}_Q^F(X)$. □

Proposition 5.1. *Let $\mathfrak{X}$ be a pseudo-CI algebra and Q_1, Q_2 be proper subsets of $\mathfrak{X}$ such that $Q_1 \subseteq Q_2$. Then $\mathcal{F}_{Q_2}^F(X) \subseteq \mathcal{F}_{Q_1}^F(X)$.*

Proposition 5.2. *Let $\mathfrak{X}$ be a Q-Smarandache pseudo-CI(A) algebra and $F_1 \in \mathcal{F}_Q^F(X)$, $F_2 \in \mathcal{F}_Q(X)$ such that $F_1 \subseteq F_2$. Then $F_2 \in \mathcal{F}_Q^F(X)$.*

Proof. Consider $x, y \in Q$ such that $u = y \to x \in F_2$. It follows that
$$y \to (u \rightsquigarrow x) = y \to ((y \to x) \rightsquigarrow x) = 1 \in F_1.$$
Since F_1 is fantastic, we have $(u \rightsquigarrow x) \vee_1 y \to (u \rightsquigarrow x) \in F_1$.
From $F_1 \subseteq F_2$, we get $(u \rightsquigarrow x) \vee_1 y \to (u \rightsquigarrow x) \in F_2$.
Applying $(psCI_3)$, it follows that $u \rightsquigarrow ((u \rightsquigarrow x) \vee_1 y \to x) \in F_2$.
Since $u \in F_2$ and $(u \rightsquigarrow x) \vee_1 y \to x \in Q$, we get $(u \rightsquigarrow x) \vee_1 y \to x \in F_2$.
In the pseudo-BE algebra Q, $x \preceq (y \to x) \rightsquigarrow x = u \rightsquigarrow x$, hence by (A), we have $(u \rightsquigarrow x) \to y \preceq x \to y$, and $(x \to y) \rightsquigarrow y \preceq ((u \rightsquigarrow x) \to y) \rightsquigarrow y$, that is, $x \vee_1 y \preceq (u \rightsquigarrow x) \vee_1 y$.
Finally, applying again (A), $(u \rightsquigarrow x) \vee_1 y \to x \preceq x \vee_1 y \to x$.
Hence $x \vee_1 y \to x \in F_2$. Similarly, from $y \rightsquigarrow x \in F_2$, we get $x \vee_2 y \rightsquigarrow x \in F_2$.
We conclude that $F_2 \in \mathcal{F}_Q^F(X)$. □

Corollary 5.1. *Let $\mathfrak{X}$ be a Q-Smarandache pseudo-CI(A) algebra. Then $\{1\} \in \mathcal{F}_Q^F(X)$ if and only if $\mathcal{F}_Q(X) = \mathcal{F}_Q^F(X)$.*

Theorem 5.2. *If $\mathfrak{X}$ is a commutative Q-Smarandache pseudo-CI algebra, then $\mathcal{F}_Q(X) = \mathcal{F}_Q^F(X)$.*

Proof. Let $F \in \mathcal{F}_Q(X)$, and let $x, y \in Q$ such that $y \to x \in F$.
Obviously, $((y \to x) \rightsquigarrow x) \to x \in Q$ and by (a_6), $y \to x \preceq ((y \to x) \rightsquigarrow x) \to x$, hence $((y \to x) \rightsquigarrow x) \to x = y \vee_1 x \to x \in F$. Since X is commutative, we get $x \vee_1 y \to x \in X$.
Similarly, $x, y \in Q$ and $y \rightsquigarrow x \in F$ imply $x \vee_2 y \rightsquigarrow x \in F$, hence $F \in \mathcal{F}_Q^F(X)$.
We conclude that $\mathcal{F}_Q(X) \subseteq \mathcal{F}_Q^F(X)$, that is, $\mathcal{F}_Q(X) = \mathcal{F}_Q^F(X)$. □

Definition 5.2. *Let $\mathfrak{X}$ be a Q-Smarandache pseudo-CI algebra. A subset F of X is said to be a* Q-Smarandache implicative filter *of $\mathfrak{X}$ if it satisfies the following conditions, for all $x, y \in Q$ and $z \in F$:*
(IF_1) $1 \in F$;
(IF_2) $z \to ((x \to y) \rightsquigarrow x) \in F$ *implies* $x \in F$;
(IF_3) $z \rightsquigarrow ((x \rightsquigarrow y) \to x) \in F$ *implies* $x \in F$.

Denote by $\mathcal{F}_Q^i(X)$ set of all Q-Smarandache implicative filters of $\mathfrak{X}$.

Proposition 5.3. *In any Q-Smarandache pseudo-CI algebra $\mathfrak{X}$, $\mathcal{F}_Q^I(X) \subseteq \mathcal{F}_Q(X)$.*

Proof. Let $F \in \mathcal{F}_Q^I(X)$. Obviously, (SF_1) is (IF_1). Let $x \in F$ and $y \in Q$ such that $x \to y \in F$. Since $y \to ((x \to x) \rightsquigarrow x) = y \to x \in F$, by (IF_2), we get $x \in F$, that is, (SF_2) is verified. Similarly, (SF_3) follows from (IF_3), hence $F \in \mathcal{F}_Q(X)$. We conclude that $\mathcal{F}_Q^I(X) \subseteq \mathcal{F}_Q(X)$. □

Theorem 5.3. *Let $\mathfrak{X}$ be a Q-Smarandache pseudo-CI algebra, and let $F \in \mathcal{F}_Q(X)$. Then the following are equivalent; for all $x, y \in Q$,* (1) $F \in \mathcal{F}_Q^I(X)$;
(2) $(x \to y) \rightsquigarrow x \in F$ *implies* $x \in F$ *and* $(x \rightsquigarrow y) \to x \in F$ *implies* $x \in F$.

Proof. (1) $\Rightarrow$ (2) Let $F \in \mathcal{F}_Q^I(X)$, and let $x, y \in Q$ such that $(x \to y) \rightsquigarrow x \in F$. Since $1 \to ((x \to y) \rightsquigarrow x) = (x \to y) \rightsquigarrow x \in F$ and $1 \in F$, by (IF_2), we get $x \in F$. Similarly, $(x \rightsquigarrow y) \to x \in F$ implies $x \in F$.
(2) $\Rightarrow$ (1) Let $x, y \in Q$ such that $z \to ((x \to y) \rightsquigarrow x) \in F$, and let $z \in F$.
Since $F \in \mathcal{F}_Q(X)$, we get $(x \to y) \rightsquigarrow x \in F$, and applying (2), we get $x \in F$.
Similarly, $z \to ((x \rightsquigarrow y) \to x) \in F$ and $z \in F$ imply $x \in F$. Therefore, $F \in \mathcal{F}_Q^I(X)$. □

Proposition 5.4. *Let $\mathfrak{X}$ be a Q-Smarandache pseudo-CI algebra, and let $F \in \mathcal{F}_Q(X)$ such that $((x \to y) \rightsquigarrow x) \rightsquigarrow x \in F$ and $((x \rightsquigarrow y) \to x) \to x \in F$, for all $x, y \in Q$. Then $F \in \mathcal{F}_Q^I(X)$.*

Proof. Let $F \in \mathcal{F}_Q(X)$ and let $x, y \in Q$ such that $z \to ((x \to y) \rightsquigarrow x) \in F$ and $z \in F$. Since $F \in \mathcal{F}_Q(X)$, by (SF_2), we have $(x \to y) \rightsquigarrow x \in F$. Moreover, from $((x \to y) \rightsquigarrow x) \rightsquigarrow x \in F$, applying again (SF_2), we get $x \in F$. Similarly, $z \rightsquigarrow ((x \rightsquigarrow y) \to x) \in F$ and $z \in F$ imply $x \in F$. Therefore, $F \in \mathcal{F}_Q^I(X)$. □

Definition 5.3. Let $\mathfrak{X}$ be a Q-Smarandache pseudo-CI algebra. A subset F of X is said to be a *Q-Smarandache positive implicative filter* of $\mathfrak{X}$ if it satisfies the following conditions, for all $x, y, z \in Q$:
(PIF_1) $1 \in F$;
(PIF_2) $z \to (x \to y) \in F$ and $z \rightsquigarrow x \in F$ imply $z \to y \in F$;
(PIF_3) $z \rightsquigarrow (x \rightsquigarrow y) \in F$ and $z \to x \in F$ imply $z \rightsquigarrow y \in F$.

Denote by $\mathcal{F}_Q^{PI}(X)$ set of all Q-Smarandache implicative filters of $\mathfrak{X}$.

Proposition 5.5. *In any Q-Smarandache pseudo-CI algebra $\mathfrak{X}$, $\{F \in \mathcal{F}_Q^{pi}(X) \mid F \subseteq Q\} \subseteq \mathcal{F}_Q(X)$.*

Proof. Let $F \in \mathcal{F}_Q^{PI}(X)$. Obviously, (SF_1) is (PIF_1). Let $x \in F$ and $y \in Q$ such that $x \to y \in F$. Since $1 \to (x \to y) = x \to y \in F$ and $1 \rightsquigarrow x = x \in F \subseteq Q$, applying (PIF_2), we get $1 \to y = y \in F$. Thus (SF_2) is verified. Similarly, applying (PIF_3), we get (SF_3), hence $F \in \mathcal{F}_Q(X)$. We conclude that $\mathcal{F}_Q^{PI}(X) \subseteq \mathcal{F}_Q(X)$. □

Proposition 5.6. *Let $\mathfrak{X}$ be a Q-Smarandache pseudo-CI algebra, and let $F \in \mathcal{F}_Q(X)$ such that the following conditions are satisfied, for all $x, y, z \in Q$:*
(PIF_4) *$z \to (x \to y) \in F$ implies $(z \rightsquigarrow x) \to (z \to y) \in F$;*
(PIF_5) *$z \rightsquigarrow (x \rightsquigarrow y) \in F$ implies $(z \to x) \rightsquigarrow (z \rightsquigarrow y) \in F$.*
Then $F \in \mathcal{F}_Q^{PI}(X)$.

Proof. Let $F \in \mathcal{F}_Q(X)$, and let $x, y, z \in Q$ such that $z \to (x \to y) \in F$ and $z \rightsquigarrow x \in F$. By (PIF_4), we have $(z \rightsquigarrow x) \to (z \to y) \in F$ and by (SF_2), we get $z \to y \in F$. Similarly, applying (PIF_5), from $z \rightsquigarrow (x \rightsquigarrow y) \in F$ and $z \to x \in F$, we get $z \rightsquigarrow y \in F$. It follows that $F \in \mathcal{F}_Q^{PI}(X)$. □

Corollary 5.2. *Let $\mathfrak{X}$ be a Q-Smarandache pseudo-CI algebra, and let $F \in \mathcal{F}_Q(X)$ such that the following conditions are satisfied, for all $x, y \in Q$:*
$(PIF_4)'$ $x \to (x \to y) \in F$ implies $x \to y \in F$;
$(PIF_5)'$ $x \rightsquigarrow (x \rightsquigarrow y) \in F$ implies $x \rightsquigarrow y \in F$.
Then $F \in \mathcal{F}_Q^{PI}(X)$.

Lemma 5.1. *Let $\mathfrak{X}$ be a Q-Smarandache pseudo-CI algebra, and let $F \in \mathcal{F}_Q^{PI}(X)$. Then F satisfies $(PIF_4)'$ and $(PIF_5)'$, for all $x, y \in Q$.*

Proof. Let $F \in \mathcal{F}_Q(X)$, and let $x, y \in Q$ such that $x \to (x \to y) \in F$. Since $x \rightsquigarrow x = 1 \in F$, applying (PIF_2) we get $x \to y \in F$. Similarly, from $x \rightsquigarrow (x \rightsquigarrow y) \in F$, applying (PIF_3), we get $x \rightsquigarrow y \in F$. □

Theorem 5.4. *Let $\mathfrak{X}$ be a Q-Smarandache pseudo-CI algebra, and let $F \in \mathcal{F}_Q(X)$. Then $F \in \mathcal{F}_Q^{PI}(X)$ if and only if it satisfies $(PIF_4)'$ and $(PIF_5)'$.*

Proof. It follows by Lemma 5.1 and Corollary 5.2. □

Proposition 5.7. *Let $\mathfrak{X}$ be a Q-Smarandache pseudo-CI algebra, and let $F \in \mathcal{F}_Q^{PI}(X)$ such that $F \subseteq Q$. Then the following hold, for all $x, y \in Q$, $z \in F$:*
(PIF_6) $z \to (x \to (x \to y)) \in F$ implies $x \to y \in F$;
(PIF_7) $z \rightsquigarrow (x \rightsquigarrow (x \rightsquigarrow y)) \in F$ implies $x \rightsquigarrow y \in F$.

Proof. Let $F \in \mathcal{F}_Q^{PI}(X)$, $F \subseteq Q$, and let $x, y \in Q$, $z \in F$ such that $z \to (x \to (x \to y)) \in F$. Since $F \subseteq Q$ we have $z \in Q$. By Proposition 5.5, $F \in \mathcal{F}_Q(X)$ and applying (SF_2), we have $x \to (x \to y) \in F$. Hence by Lemma 5.1, we get $x \to y \in F$, thus (PIF_6) is verified. Similarly, for (PIF_7). □

Theorem 5.5. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Q_X and Q_Y-Smarandache pseudo-CI algebras and $f : X \longrightarrow Y$ be a Smarandache pseudo-CI homomorphism. Then:*
(1) *if $G \in \mathcal{F}_{Q_Y}^F(Y)$ ($\mathcal{F}_{Q_Y}^I(Y)$, $\mathcal{F}_{Q_Y}^{PI}(Y)$), then*
$f^{-1}(G) \in \mathcal{F}_{f^{-1}(Q_Y)}^F(X)$ ($\mathcal{F}_{f^{-1}(Q_Y)}^I(X)$, $\mathcal{F}_{f^{-1}(Q_Y)}^I(X)$);
(2) *if f is injective and $F \in \mathcal{F}_{Q_X}^F(X)$ ($\mathcal{F}_{Q_X}^I(X)$, $\mathcal{F}_{Q_X}^{PI}(X)$), then*
$f(F) \in \mathcal{F}_{f(Q_X)}^F(Y)$ ($\mathcal{F}_{f(Q_X)}^I(Y)$, $\mathcal{F}_{f(Q_X)}^{PI}(Y)$).

Proof. (1) Let $G \in \mathcal{F}_{Q_Y}^F(Y)$, and let $x, y \in Q_X$ such that $y \to x \in f_{Q_Y}^{-1}(G)$, that is, $f(y \to x) \in G$, so $f(y) \to f(x) \in G$. Since $G \in \mathcal{F}_{Q_Y}^F(Y)$, we have $f(x) \vee_1 f(y) \to f(x) \in G$. It follows that $f(x \vee_1 y \to x) \in G$, hence $x \vee_1 y \to x \in f^{-1}(G)$. Similarly, $y \rightsquigarrow x \in f_{Q_Y}^{-1}(G)$ implies $x \vee_2 y \rightsquigarrow x \in f^{-1}(G)$. We conclude that $f^{-1}(G) \in \mathcal{F}_{f^{-1}(Q_Y)}^F(X)$. Similarly, for $G \in \mathcal{F}_{Q_Y}^I(Y)$ and $G \in \mathcal{F}_{Q_Y}^{PI}(Y)$.
(2) Let $F \in \mathcal{F}_{Q_X}^F(X)$ and $x, y \in f(Q)$ such that $y \to x \in f(F)$. There exist $x_1, y_1, z_1 \in Q$ such that $x = f(x_1)$, $y = f(y_1)$, $y \to x = f(z_1)$. Therefore, $f(y_1) \to f(x_1) = f(z_1)$, that is, $f(y_1 \to x_1) = f(z_1)$. Since f is injective and F is fantastic, we get $y_1 \to x_1 = z_1 \in F$, hence $x_1 \vee_1 y_1 \to x_1 \in F$. It follows that $f(x_1 \vee_1 y_1 \to x_1) \in f(F)$, so $f(x_1) \vee_1 f(y_1) \to f(x_1) \in f(F)$, that is, $x \vee_1 y \to x \in f(F)$. Similarly, $y \rightsquigarrow x \in f(F)$ implies $x \vee_2 y \rightsquigarrow x \in f(F)$.
Hence $f(F) \in \mathcal{F}_{f(Q_X)}^F(Y)$. Similarly, for $F \in \mathcal{F}_{Q_X}^I(X)$ and $F \in \mathcal{F}_{Q_X}^{PI}(X)$. □

6. Q-Smarandache upper sets

In this section, we define and investigate the notion of Smarandache upper sets in a pseudo-CI algebra and we investigate some of their properties. We prove that every Q-Smarandache filter is a union of Q-Smarandache upper sets.

Let $x, y \in Q$ and $Q \subseteq X$ be a pseudo-BE algebra. Denote:

$$A(x, y) := \{z \in Q : x \to (y \rightsquigarrow z) = 1\}.$$

We call $A(x, y)$ a Q-Smarandache upper set of x and y.

Remark 6.1. It is easy to see that, $1, x, y \in A(x, y)$. The set $A(x, y)$, where $x, y \in Q$, is not a filter of $\mathfrak{X}$, in general. Also, using (psCI_3) and (psCI_4) we have

$$\begin{aligned} A(x, y) &= \{z \in Q : x \to (y \rightsquigarrow z) = 1\} \\ &= \{z \in Q : x \rightsquigarrow (y \rightsquigarrow z) = 1\} \\ &= \{z \in Q : y \rightsquigarrow (x \to z) = 1\} \\ &= \{z \in Q : y \to (x \to z) = 1\}. \end{aligned}$$

Example 6.1. (1) Consider the pseudo-CI algebras from Example 2.1 (2) and let $Q := \{1, a, c\}$. Then $A(a, c) = \{1, a, c\}$.

(2) Consider the Q-Smarandache pseudo-CI algebras from Example 4.1. Then $A(a, 1) = \{1, a, b\} \neq A(1, a) = \{1, a\}$, and so $A(x, y) \neq A(y, x)$, for some $x, y \in Q$.

Proposition 6.1. *Let $x, y \in Q$. Then*
(1) *$A(x, 1) \subseteq A(x, y)$;*
(2) *if $A(x, 1) \in F_Q(\mathfrak{X})$ and $y \in A(x, 1)$, then $A(x, y) \subseteq A(x, 1)$;*
(3) *if there is $y \in Q$, such that $y \to z = 1$ or $y \rightsquigarrow z = 1$, for all $z \in Q$, then $Q = A(x, y)$;*
(4) *$A(x, 1) = \bigcap_{y \in Q} A(x, y)$.*

Theorem 6.1. *Let $\emptyset \neq F \subseteq Q$. Then $F \in F_Q(\mathfrak{X})$ if and only if $A(x, y) \subseteq F$, for all $x, y \in F$.*

Proof. Assume that $F \in F_Q(\mathfrak{X})$ and $x, y \in F$. If $z \in A(x, y)$, then $x \to (y \rightsquigarrow z) = 1 \in F$. Since $F \in F_Q(\mathfrak{X})$ and $x, y \in F$, by (SF_2), $y \rightsquigarrow z \in F$, and so by ($\text{SF}_3$), $z \in F$. Hence $A(x, y) \subseteq F$.

Conversely, suppose $A(x, y) \subseteq F$, for all $x, y \in F$.
Since $x \to (y \rightsquigarrow 1) = x \to 1 = 1$, we get $1 \in A(x, y) \subseteq F$. Let a, $a \to b \in F$ and $a \rightsquigarrow c \in F$. Since $1 = (a \to b) \rightsquigarrow (a \to b) = a \to ((a \to b) \rightsquigarrow b)$ and
$(a \rightsquigarrow c) \to (a \rightsquigarrow c) = 1$, we have $b \in A \subseteq F$ and $c \in A \subseteq F$. Hence $b, c \in F$. Thus, $F \in F_Q(\mathfrak{X})$. □

Theorem 6.2. *Let $a \in Q$. Then the set $A(a, 1) \in F_Q(\mathfrak{X})$ if and only if the following hold, for all $x, y, z \in Q$:*
(1) *$z \preceq x \to y$ and z $\preceq$ x imply $z \preceq y$;*
(2) *$z \preceq x \rightsquigarrow y$ and z $\preceq$ x imply $z \preceq y$.*

Proof. Assume that for each $a \in Q$, $A(a,1) \in F_Q(\mathfrak{X})$. Let $x, y, z \in Q$ be such that $z \preceq x \to y$, $z \preceq x \rightsquigarrow y$, and $z \preceq x$. Then $x \to y \in A(z,1)$, $x \rightsquigarrow y \in A(z,1)$, and $x \in A(z,1)$. Since $A(z,1) \in F_Q(\mathfrak{X})$, we have $y \in A(z,1)$. Therefore, $z \preceq y$.

Conversely, consider $A(z,1)$, for $z \in Q$. Obviously, $1 \in A(z,1)$.
Let $x \to y \in A(z,1)$, and $x \rightsquigarrow b \in A(z,1)$, for all $x \in A(z,1)$ (i.e. $z \preceq x \to y$, $z \preceq x \rightsquigarrow b$ and $z \preceq x$). Then from hypothesis, $z \preceq y$ and $z \preceq b$ (i.e. $y \in A(z,1)$ and $b \in A(z,1)$). Hence $A(z,1) \in F_Q(\mathfrak{X})$, for all $z \in Q$. □

Theorem 6.3. *Let $F \in F_Q(\mathfrak{X})$ and $F \subseteq Q$, then $F = \bigcup_{x \in F} A(x,1)$.*

Proof. Assume that $F \in F_Q(\mathfrak{X})$, $F \subseteq Q$ and $z \in F$. Since $z \in A(z,1)$, we have $F \subseteq \bigcup_{z \in F} A(z,1)$. Let $z \in \bigcup_{x \in F} A(x,1)$. Then there exists $a \in F$ such that $z \in A(a,1)$, and so $a \to z = a \to (1 \rightsquigarrow z) = 1 \in F$. Since $F \in F_Q(\mathfrak{X})$ and $a \in F$, we have $z \in F$. Thus, $\bigcup_{x \in F} A(x,1) \subseteq F$. □

7. Conclusions and future work

In this paper we introduced the notion of Smarandache pseudo-CI algebras and we defined and studied some classes of Smarandache filters of Smarandache pseudo-CI algebras. This study could potentially lead to more results on Smarandache pseudo-CI algebras.

A. Borumand Saeid studied in [2] the notion of a *Smarandache weak BE-algebra*, as a BE-algebra X in which there exists a proper subset Q of X such that:

(S_1) $1 \in Q$ and $\mid Q \mid \geq 2$;
(S_2) Q is a CI-algebra under the operation of X.

Another topic of research could be to define and investigate the notion of a Smarandache weak pseudo-BE algebra.
A *Smarandache strong n-structure* on a set S means a structure W_0 on a set S such that there exists a chain of proper subsets $P_{n-1} < P_{n-2} < \cdots < P_2 < P_1 < S$, where $<$ means P_i included P_{i-1} in whose corresponding structures verify the inverse chain $W_{n-1} > W_{n-2} > \cdots > W_2 > W_1 > W_0$, where $>$ signifies strictly stronger (i.e. a structure satisfying more axioms) (see [5]).

A. Borumand Saeid and A. Rezaei introduced in [5] the notion of a Smarandache strong 3-structure of a CI-algebra X as a chain $X_1 > X_2 > X_3 > X_4$, where X_1 is a CI-algebra, X_2 is a BE-algebra, X_3 is a dual BCK-algebra, and X_4 is an implication algebra.
One could define and investigate the notion of a strong n-structure of a pseudo-CI algebra.

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Dynamical modelling of Kaziranga wildlife ecology

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Abstract. The ecology, conservation and management of wildlife is an important component in the fields of environmental science and natural resource management. Wildlife management is a very dynamic field. Kaziranga National Park in India has a good conservation history, due to its efficient management policies. The mathematical modelling of the ecosystem in Kaziranga national park located in Assam, India with the interaction of species between Indian Rhinoceros, Bengal Tiger and Swamp deer has been studied. In this system, predator-prey interaction and the effect of poaching for both Indian Rhinoceros and Bengal Tiger have been considered. Also, the indirect dependency between Indian Rhinoceros and Swamp deer accounting for dependency of both grazers on common food resource has been considered. The fixed point for three dimensional nonlinear systems at which all the species remain conserved with the stability conditions have been estimated. Using the parametric values given in census report of Kaziranga National park the future population of Indian Rhinoceros, Bengal Tiger and Swamp deer have been predicted and numerically verified with phase space and time series plots. Using simulation analysis with base value for 1999, the future value for all the three species have been estimated for the year 2016. It is observed that the forecasted values for Bengal tiger are closest to the actual value of population and the ecosystem remains in coexistence phase despite of poaching and lower growth rate of Bengal Tiger.

Keywords: nonlinear interactions, predator-prey model, stability analysis, coexistence, ecological system.

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1. Introduction

In nature, ecosystem is a functional unit where interactions between different species and the surrounding environment occur. In an ecosystem, evolution of the dynamics is governed by the various species reaction. In an ecology, every species interaction has direct and indirect dependence on each other's population growth with its own specific coefficient. The nonlinear mathematical model of inter-species interaction in the ecosystem helps to understand the evolution of population trajectories in phase space providing a projection of the possible states of stability or instability of the system of study.

Conservation of biological diversity has become a serious issue in recent times and requires immediate attention as in nature the ecological cycles are getting disturbed. Every species plays a crucial role in the food chain and if one single species becomes extinct the whole food chain gets affected which in turn disturbs the ecological cycles in nature. To save these species and to maintain the biodiversity, national parks, wildlife sanctuaries and bio-reserves has been developed across the globe. In 1972, the Indian Parliament enacted the first wild life protection act for the protection of plant and animal species with development of national parks in early 20th century A.D.

Kaziranga National park in India is one of the oldest national park developed in 1968 for the conservation of one horned Indian Rhinoceros. It is located in Assam, India 26o 40'N 93o21'E with an area of 430 square kilometre and was set up under Assam National Park Act of 1968. In this sanctuary, almost two third of world's great one horned rhinoceros are conserved and in 2006, it was declared as Tiger reserve. In Kaziranga National park Sambar Deer, Hog deer, Asian Elephant, Wild Buffalo and Wild Boar are the other major species which along with Bengal Tiger, Indian Rhinoceros and Swamp Deer constitute the overall fauna of this ecosystem.

Volterra [18] mathematically studied the mutual coexistence of species in an ecosystem with fluctuations in species abundance. Elton and Nicholas [5] derived the data based analysis of biological species, studied the Canadian Lynx population based on records of Lynx Canadensis fur collections in Canada, discussed the periodicity of fluctuations of the Canadian Lynx population with an average cycle of 9.6 years which shows dependency on Snowshoe Rabbit population cycle. Vézina and Platt [17] using inverse method analysed the food web dynamics of under sampled oceanic environment for two areas on English coast among the material fluxes in surface water coastal system and observed common patterns during late summers which indicate that how directly or through micro grazer pathway the particulate primary production flows to mesozooplankton which in turn fuel the downward particulate flux for future generation. Freedman and Ruan [7] studied the exhibition of group defence for food chain in three species prey population and taking carrying capacity as bifurcation parameter with delay in the model Hopf bifurcation was observed. Limit cycles and chaos was observed by Caswell and Neubert [2] while studying the effects of nonlinear clo-

sure terms on system dynamics of plankton food chain. It was observed that the limit cycles and chaos appearing in the dynamics of the system didn't eliminated and the food chain dynamics didn't stabilize with increase in capita mortality rate regardless of increase in density dependent or independent component.

Chauvet et al.[3] modelled the three species food chain of prey, predator and super-predator system using simulations and fixed point analysis the predator is observed to act as a conduit between prey and super-predator with possibility of extinction of predator and persistence of prey. Hsu et al. [8] discussed three and two trophic level food chain model and derived ratio dependent Michaeles-Menten type functional response conditions for extinction of three species in the form of stable steady state and limit cycle. The model had two distinct and realistic features of ratio dependence which produced extinction of prey species leading to collapse of system and dependence of system dynamics on initial population level showing feasibility of biological control in such systems. The predator-prey food chain between pest and its natural enemy in food chain with Holling type-III interaction was studied by Su et al. [12]. The pest eradication periodic solution was observed to be globally asymptotically stable when impulsive period is less than a threshold. Further, when the system is permanent it exhibits complex dynamic behaviour for the numerical simulation of impulsive perturbations on system dynamics. In study of three trophic aquatic food chain by Upadhyay [14], a new three species model based on Leslie-Gower scheme with incorporation of mutual interference in all three species is proposed. The mutual interference acts as a stabilizing and destabilizing factor for chaos. The toxic substance released by TPP population was act as a biological control which bring system to order from chaos. Ji et al.[9] discussed predator-prey model with Leslie Gower and Holling type –III schemes and stochastic perturbations. Through numerical simulation a positive solution exist with positive initial values for the long term behaviour of the system and conditions of extinction and persistence evaluated.

Etua and Rösseau [6] observed three kinds of regimes for extinction, coexistence and oscillatory phase of system dynamics in prey harvesting phenomenon with Gauss model and Holling type-III functional response for accounting interactions in the model. Application of such model in construction of real food chains for bio conservation and pest management has been highlighted. Evidence of reduction in resilience of age structured population by anthropogenic stressors for Daphnia population exposed to environmental stress was provided by Ottermanns et al. [10]. The chaos in population dynamics is observed to get repressed by enhancement of disturbance which increases the degree of synchrony. It is concluded that exposure to high concentration of chemical stressors can lead to increase in risk of extinction of Daphnia population. The above mentioned works clearly show that parameters of the ecosystem govern the direction in which system evolves. These parameters can be controlled in conservation process of bio-reserves to prevent extinction of any species for survival of other species and thus promoting coexistence of all.

Updhyay et al. [15,16] studied the emergence of spatio temporal patterns in their mathematical model for wetland Keoladeo National Park while Sahoo et al. [11] carried out their study on diseased prey model with Holling type interaction function. In this paper, non-linear interactions between the Indian Rhinoceros, Bengal Tiger and Swamp Deer for the Kaziranga ecosystem have been modeled. The species interaction type are Lotka-Voltera type and logistic growth has been considered for each species [15]. The competition between two prey species for common food resource with indirect dependency has been discussed with the mutual interaction of both the prey species with predator in the confinements of the protected environment of the national park. It is assumed that in the national park the amount of food resource is uniformly available in adequate abundance for both the prey species. With passage of time species population grow with birth of new prodigy, leading to load generation on the resource as consumption becomes higher which leads to increase in encounters between the members of species. Rhinoceros and Swamp deer compete occasionally thus both have an indirect dependency on each other's growth. As both the species are grazers and grass is the common food resource which they share. Thus growth of one species population casts a negative effect on the growth of other as rise in population leads to resource crunch for other leading to an indirect dependency. The factor of poaching has been accounted for the tiger and Rhino species national park based on the previous studies[1,4].

Using stability analysis, phase plot simulation, forecasting and validation for the system the current population level of all the three species are estimated which are observed to be mutually sustaining in the ecosystem of the Kaziranga National Park. All the three species show growth and tend to coexist mutually as per the simulation and population estimates of 2016 of Kaziranga National Park. To the best of author's knowledge, none of the authors have studied the nonlinear dynamics of Kaziranga ecosystem considering species interactions between the Indian Rhinoceros, Bengal Tiger and Swamp Deer.

2. Mathematical model

For modeling the Kaziranga national park the intermediate model is considered where one prey species acts as an intermediate between the other prey and predator species. Bengal Tiger(x) and Swamp Deer(y) are considered as predator and prey species while Indian Rhinoceros (z) is the intermediate species. The growth rate of Bengal Tiger, Swamp Deer and Indian Rhinoceros is represented by parameter a, e and i respectively. The value of a, e and i are considered to be positive for the protective ecosystem of Kaziranga where the growth of these species is encouraged for their conservation. As prey Bengal Tiger prefer hunting big ungulates like Chital, Sambar and Gaur. To a lesser extent they hunt Swamp Deer, Water Buffalo, Nilgai, Serow and Takin. As per census of 1999, the population of Sambar Deer was very low compared to that of Swamp Deer. Thus in this model of Kaziranga ecosystem Swamp deer is considered

the species on which Tiger feeds with their interaction coefficient given by parameter $b(>0)$. Tigers rarely hunt Rhinoceros but in Kaziranga and Dudhwa National parks the killings of Rhinoceros by Tigers was reported. To account this fact, the interactions between Rhinoceros and Tigers are incorporated in the model and the interaction coefficient is represented by parameter $f(>0)$. Each species is considered to follow logistic growth model as the national park has a particular carrying capacity for each species.

The interaction of Rhinoceros and Tiger is considered to affect growth rate of Tiger negatively as Rhinoceros is a large hunt for the tiger to prey on alone which leads to rise in infighting between the tigers for dominance and larger share of food. Further due to horn Rhinoceros can kill or wound the Tiger which may lead to death. During the infighting on share of food and dominance tigers can get hurt badly or chased away. In both cases due to starvation or septicemia the death can be caused to the Tiger. The chased off tigers can approach towards the borders of national park where they can be targeted easily either by poachers or by human residing there. The interaction of Tiger and Rhinoceros positively impacts the swamp deer population. Being a large hunt Rhinoceros is capable of filling the appetite of the Tiger sufficiently leaving it with no requirement for hunting the Swamp Deer frequently. Thus a hunted Rhinoceros saves many swamp deer getting hunted. The Swamp Deer similarly contributes to the Rhinoceros population by being a smaller and readily available hunt to the tigers in large numbers in groups. As both Rhinoceros and Swamp Deer are grazers by virtue of their food habits they cast indirect negative impact on each other's population because as one grown in number it grazes more to cause less availability of fodder for the other to feed upon and thus bring its growth down.

The carrying capacity of the Kaziranga ecosystem for Bengal tiger, Swamp Deer and Indian Rhino is given by $k(>0)$, $l(>0)$ and $q(>0)$ respectively. The incidents of poaching of Bengal Tiger and Indian Rhinoceros has been reported in the Kaziranga ecosystem and till date poaching is taking place despite of measures being taken to curb it. The poaching of Bengal Tiger and Indian Rhinoceros has been taken into account in the model with $c(>0)$ and $g(>0)$ representing the poaching coefficients or number of incidents of poaching per year of both the species respectively. The three dimensional non-linear model can be written as following system of equations:

$$\dot{x} = f(x,y,z) = x(a - dx + by - fz - c), \tag{1}$$

$$\dot{y} = g(x,y,z) = y(e - hy - bx) + fxz, \tag{2}$$

$$\dot{z} = h(x,y,z) = z(i - jz - g - fx) + bxy, \tag{3}$$

where $d = \frac{a}{k}$=coefficient of intra-species interaction of Bengal Tiger population, $h = \frac{e}{l}$=coefficient of of intra-species interaction of Swamp Deer population, $j = \frac{i}{q}$=coefficient of of intra-species interaction of Indian Rhinoceros population.

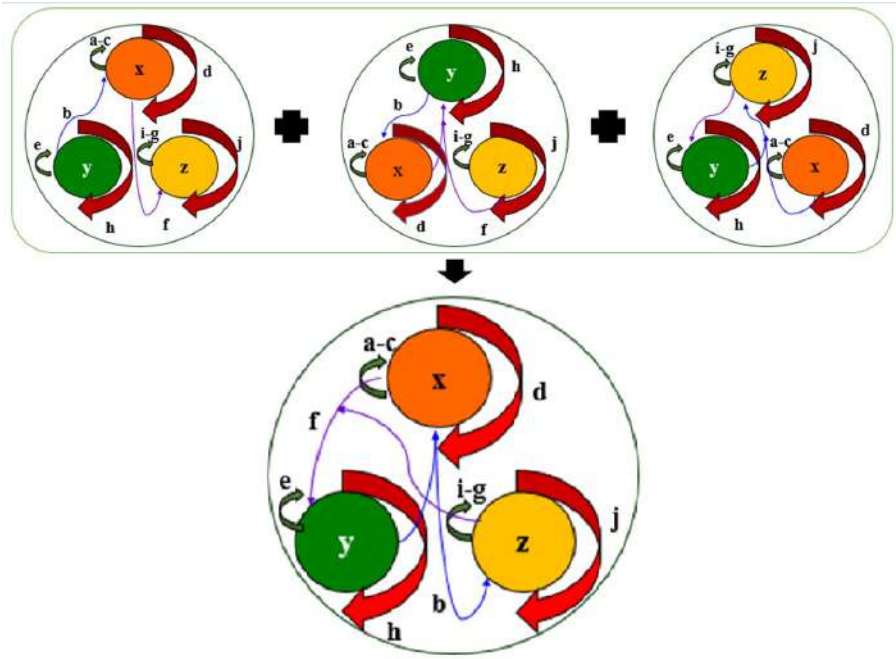


Figure 1: Schematic diagram of the Ecological Model $eq^n 1) - 3)$

3. Stability analysis

For the considered system fixed points are obtained equating $eq^n(1) - (3)$ simultaneously to zero i.e. $f(x, y, z) = g(x, y, z) = h(x, y, z) = 0$. The fixed points are mentioned as follows:

$$\begin{aligned} 1) \quad (x, y, z) &= (0, 0, 0) \\ 2) \quad (x, y, z) &= \left(0, \frac{e}{h}, \frac{(i-g)}{j}\right) \\ 3) \quad (x, y, z) &= \left(\frac{1}{d}\left[a + \frac{be}{h} - \frac{f(i-g)}{j} - c\right], \frac{e}{h}, \frac{(i-g)}{j}\right) \end{aligned}$$

Jacobian of the system is as follows:

$$J = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} & \frac{\partial h}{\partial z} \end{bmatrix} = \begin{bmatrix} m_1 & m_2 & m_3 \\ m_4 & m_5 & m_6 \\ m_7 & m_8 & m_9 \end{bmatrix}$$

where $m_1 = -2dx + a + by - fz - c$,$m_2 = bx$,$m_3 = -fx$,$m_4 = fz - by$,$m_5 = e - 2hy - bx$,$m_6 = fx$,$m_7 = by - fz$,$m_8 = bx$ and $m_9 = i - 2jz - fx - g$. The characteristic polynomial is expressed as follows:

$$\begin{vmatrix} m_1 - \lambda & m_2 & m_3 \\ m_4 & m_5 - \lambda & m_6 \\ m_7 & m_8 & m_9 - \lambda \end{vmatrix} = \lambda^3 + e_1\lambda^2 + e_2\lambda + e_3 = \quad 0$$

where $e_1 = -(m_1+m_5+m_9)$, $e_2 = m_1(m_5+m_9)+m_5m_9-m_6m_8-m_2m_4-m_3m_7$ and $e_3 = -m_1m_5m_9-m_2m_6m_7-m_3m_4m_8+m_6m_8m_1+m_2m_4m_9+m_3m_7m_5$. For a characteristic polynomial $\lambda^3+e_1\lambda^2+e_2\lambda+e_3 = 0$ as obtained from Jacobian at a particular fixed point, if roots have negative real parts with $e_1 > 0$, $e_3 > 0$ and $e_1e_2 - e_3 > 0$ then this implies stability for that fixed point by Routh-Hurwitz Criteria(RHC).

Case I: For the first fixed point FP_1= $(x, y, z) = (0, 0, 0)$, we obtain $m_2 = m_3 = m_4 = m_6 = m_7 = m_8 = 0$ and $m_1 = a - c$,$m_5 = e$, $m_9 = i - g$. Thus we get a characteristic polynomial $\lambda^3 + e_1\lambda^2 + e_2\lambda + e_3 = 0$, where $e_1 = (c + g) - (a + e + i)$;$e_2 = e(a - c) + (e + a + -c)(i - g)$;$e_3 = e(c - a)(i - g)$. It is observed that for stability of the fixed point FP_1 following conditions are required to be satisfied:

- For $e_1 > 0$, $a + e + i < c + g \rightarrow s_1 < s_2$ (condition 1).
- For $e_3 > 0$, $c > a$ and $i > g$ (condition 2).
- For $e_1e_2 - e_3 > 0$, $4(c-a)(g-i) < e^2(c-a+g-i)+(c-a)^2(g-i-e)+(g-i)^2(c-a-e) \rightarrow s_3 < s_4$ and $c > (a + e)$;$g > (i + e)$ (condition 3).

Case II: For the second fixed point FP_2= $(x, y, z) = \left(0, \frac{e}{h}, \frac{(i-g)}{j}\right)$, we obtain $m_2 = m_3 = m_6 = m_8 = 0$ and $m_1 = a+\frac{be}{h}-\frac{f(i-g)}{j}-c$,$m_4 = \frac{be}{h}-\frac{f(i-g)}{h}$,$m_5 = -e$, $m_9 = -i + g$. Thus we get a characteristic polynomial $\lambda^3 + e_1\lambda^2 + e_2\lambda + e_3 = 0$ where $e_1 = -\left(a + \frac{be}{h} - \frac{f(i-g)}{j} - c - e + g - i\right)$;$e_2 = \left(a + \frac{be}{h} - \frac{f(i-g)}{j} - c\right)(g - e - i) + f(i - g)$;$e_3 = e\left(a + \frac{be}{h} - \frac{f(i-g)}{j} - c\right)(g - j)$. It is observed that for stability of the fixed point FP_2 following conditions are required to be satisfied:

- For $e_1 > 0$, $i > g$;$\frac{f(i-g)}{j} > \frac{be}{h} \rightarrow s_5 > s_6$ and $(c + e) + (i - g) > \left(a + \frac{be}{h} - \frac{f(i-g)}{j} - c\right) \rightarrow s_7 > s_8$(condition 4)
- For $e_3 > 0$, $g < i$ and $a + \frac{be}{h} < \frac{f(i-g)}{j} + c \rightarrow s_9 < s_{10}$ (condition 5)
- For $e_1e_2 - e_3 > 0$, $\left((c+e)+(i-g)-a-\frac{be}{h}+\frac{f(i-g)}{j}\right)\left(\left(a+\frac{be}{h}-\frac{f(i-g)}{j}-c\right)U_1+f(i-g)\right) < e(i-g)\left(\frac{f(i-g)}{j}+c-a-\frac{be}{h}\right) \rightarrow s_{11} < s_{12}$, where $U_1 = (g - e - i)$ (condition 6)

Case III: For fixed point $FP_3 = (x, y, z) = \left(\frac{1}{d}\left(a + \frac{be}{h} - \frac{f(i-g)}{j} - c\right), \frac{e}{h}, \frac{(i-g)}{j}\right)$, we obtain $m_1 = \left(\frac{f(i-g)}{j} + c - a - \frac{be}{h}\right)$, $m_2 = -m_1\frac{b}{d}$,$m_3 = m_1\frac{f}{d}$,$m_4 = m_1 - c + a$,$m_5 = -e - m_2$,$m_6 = -m_3$,$m_7 = -m_4$,$m_8 = m_2$,$m_9 = g - i + m_3$. Thus we get a characteristic polynomial:

$$\lambda^3 + e_1\lambda^2 + e_2\lambda + e_3 = 0,$$

where $e_1 = -\left(m_1 + \left(-e + \frac{bm_1}{d}\right) + (g-i) + \frac{fm_1}{d}\right)$, $e_2 = m_1m_5 + m_1m_9 + m_5m_9 + m_3m_2 + (m_3 - m_2)m_4$ and $e_3 = -m_1m_5m_4 - m_2m_4(i-g) - m_3m_2m_1 + m_3m_4e$. It is observed that for stability of the fixed point FP_3 following conditions are required to be satisfied:

- For $e_1 > 0$, $i > g$ and $(i + e - g) > m_1\left(1 + \frac{(b+f)}{d}\right) \rightarrow s_{13} > s_{14}$ (condition 7)
- For $e_3 > 0$, $g < i$ and $(2m_1 + (a-c))(m_3e - m_2(i-g)) > em_1(i-g) \rightarrow s_{15} > s_{16}$ (condition 8)
- For $e_1e_2 - e_3 > 0$, $\left(i + e - g - m_1\left(1 + \frac{(b+f)}{d}\right)\right)(m_1m_5 + m_9(m_1 + m_5) + m_3m_2 + (m_3 - m_2)m_4) > ((2m_1 + a - c)(m_3e - m_2(i-g)) - em_1(i-g)) \rightarrow s_{17} > s_{18}$ (condition 9)

4. Results and discussion

As per 1999 census report for Kaziranga National Park, the average number of instants of Tigers attacking Rhinoceros were twelve from 1985-2000 [13]. Thus $f = 12$ and it is assumed that $b = 36$ as Swamp Deer with height and weight being almost half and one tenth of height and weight of a Rhinoceros respectively. Also, in 1999 the population of Bengal tigers, Indian Rhinoceros and Swamp Deer were 80, 398 and 1552 respectively which has increased to 111, 1148 and 2401 respectively as per population estimation of Kaziranga National Park for year 2016. Thus for simulation and validation, the initial population taken as $x0 = 80$, $y0 = 398$ and $z0 = 1552$. The growth rates a, e and i from 1999 to 2016 taken as $a = 1.44$, $e = 46.875$ and $i = 53.06$. As per the report submitted by Kaziranga Field Director to Guwahati High Court, the carrying capacity for rhinoceros, tiger and swamp deer are 2750, 146 and 800 respectively. Thus $d = 0.009$, $h = 0.058$ and $j = 0.01953$ respectively. Table 1, Table 2 and Table 3 gives the details about different conditions 1-9 obtained for three points and the behaviour of each point is discussed. For $FP_1 = (0, 0, 0)$ only Condition 2

Table 1: Stability of fixed point FP_1 at a =1.44, b =36, c =22, d =0.009, e =46.875, f =12, g =11, h =0.058, i =53.06, j =0.01953

Condition No.	Observation	Status
Condition 1	$s_1 > s_2$	Not Satisfied
Condition 2	$c > a$ & $i > g$	Satisfied
Condition 3	$c < a + e$, $g < i + e$ & $s_3 > s_4$	Not Satisfied

is satisfied and thus it is not stable for the considered system parameter values. For $FP_2 = (0, 808, 2153)$ none of the conditions are satisfied and thus it is not

Table 2: Stability of fixed point FP_2 at a =1.44, b =36, c =22, d =0.009, e =46.875, f =12, g =11, h =0.058, i =53.06, j =0.01953

Condition No.	Observation	Status
Condition 4	$s_5 < s_6$ & $s_7 < s_8$	Not Satisfied
Condition 5	$g < i$ & $s_9 > s_{10}$	Not Satisfied
Condition 6	$s_{11} > s_{12}$	Not Satisfied

Table 3: Stability of fixed point FP_3 at a =1.44, b =36, c =22, d =0.009, e =46.875, f =12, g =11, h =0.058, i =53.06, j =0.01953

Condition No.	Observation	Status
Condition 7	$g < i$ & $s_{13} > s_{14}$	Satisfied
Condition 8	$s_{15} > s_{16}$	Satisfied
Condition 9	$s_{17} > s_{18}$	Satisfied

stable for the considered system parameter values. For $FP_3 = (126, 808, 2153)$ all of the conditions are satisfied and thus it is stable for the considered system parameter values. Thus, from stability analysis the predicted values for tiger, rhinoceros and swamp deer population are 126, 808 and 2153 respectively for the year 2016.

From the study, it is evident that the ecosystem in Kaziranga is in stable coexistence phase and will remain in it till these parameters values are maintained. The projected trajectory that species population takes in phase space simulation is shown in Figure 2 a) which verify the phase of coexistence in Kaziranga ecosystem. Using simulation, for the year 2016 the predicted values for tiger, rhinoceros and swamp deer population are 133, 741 and 2220 respectively. The stability of fixed point is verified using Lyapunov exponent which are plotted in Figure 2 b). As the values of Lyapunov exponents are negative ($\lambda_1 = -15.888$, $\lambda_2 = -24.262$, $\lambda_3 = -43.368$) thus, system shows stable behavior. Predicted and observed values for population of Bengal Tiger, Indian Rhinoceros and Swamp Deer for the year 2016 are shown and compared in Table-4.

From Table 4, it is evident that the population tend to increase for all the three species in Kaziranga ecosystem as predicted. For Bengal Tiger and Indian Rhinoceros, the predicted values are close to the actual values but for Swamp Deer the predicted values are deviated from the actual value. The deviation between observed and predicted value exists as continuous species population data is unavailable to determine the exact parameters. However, it is observed that the predicted values for Bengal Tiger and Indian Rhinoceros are in sync with the observed values which indicates that their interactions in the ecosystem has been properly modeled. Despite the poaching phenomena the projected

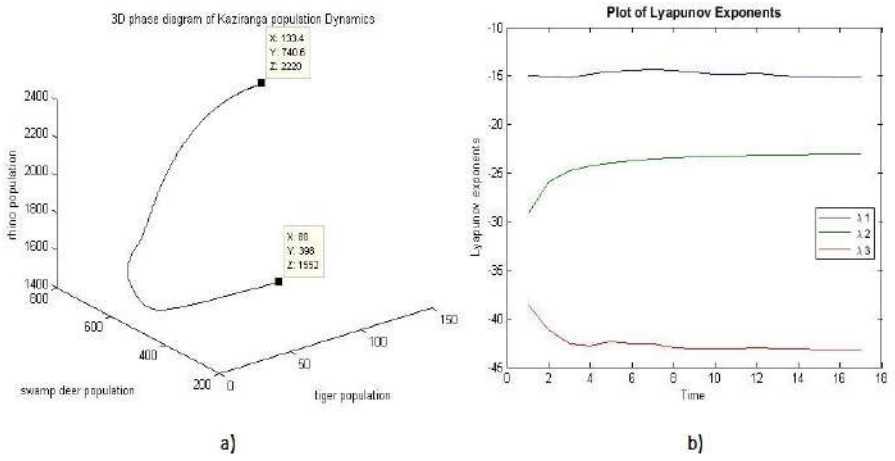


Figure 2: a) Phase Space Plot b) Plot of Lyapunov Exponents for system $eq^n 1)-3)$ at a =1.44, b =36, c =22, d =0.009, e =46.875, f =12, g =11, h =0.058, i =53.06, j =0.01953

Table 4: Predicted (P) and Actual (A) values for population of Bengal Tiger, Indian Rhinoceros and Swamp Deer using fixed point (FP) and phase plot (PP) analysis

Species	2016(P-FP)	2016(P-PP)	2016(A)
Bengal Tiger	126	133	111
Indian Rhino	2153	2200	2401
Swamp Deer	808	741	1148

trajectory of the system model does exhibit a significant rise in Rhino population as observed in previous studies[1,4]

5. Conclusion

In this paper the nonlinear interactions between tiger, rhinoceros and swamp deer in the ecosystem of the Kaziranga national park has been modeled with basic Lotka-Volterra type species interaction function. The species population was predicted using simulation for year 2016 based on the population data obtained from 1999 census report of different species in Kaziranga National Park. The predicted values of the population show an increasing trend as observed in the actual estimation of population for 2016. The predicted values of population deviate from actual values but the interaction dynamics between the rhino and tiger population are well simulated and model nearly establishes the basic species interaction dynamics which is evident from the predicted values which

are observed to be in close range of the actual values. From the stability analysis, the conditions for stable coexistence are determined. It is concluded that if the values of system parameters are maintained without much variation then despite of poaching the population of all the species can be sustained mutually.

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Solving the fuzzy assignment problems via utilizing branch and bound algorithm-practically treated

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Abstract. In this paper described of assignment cost for the numbers that are fuzzy. Here, the assignment problem, which is fuzzy or invalid, is turned into crisp values throughout robust ranking technique, and by using branch and bound algorithm we gated the optimal solution and at a less cost.

An applied example is taken from Source No. (9) to solving the assignment problem for the Cotton Industries Company in Iraq and find the optimal assignment and total cost.

Keywords: fuzzy assignment problem, branch and bound algorithm, assignment problem, robust ranking technique.

Introduction

In 1965, Zadeh presented the fuzzy set for the sake of tackling information-uncertainty. The Problem of Assignment is a specific kind of Programming Problem that is Linear. Here the crucial aim is to identify a group of jobs to identical number of individuals with cost or profit that is minimum or maximum respectively ([3]).

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Made use of in manufacturing and service systems, the Assignment Problem (AP) is to obtain the assignment schedule of the optimal jobs-total costs in which n jobs are specialized for n workers (or machines), and everyone gets a job only. The AP is a specific kind of 0–1 programming which is linear. To add more, the jobs and workers of m and n respectively can also be found, m 6 n. For every worker, there must be only one jib assigned. To add more,, the related cost C_{ij} is provided in accordance with the allocation that for the j worker the I job is specialized. Finding out such an assignment is the main problem. There will be many extensions because, in the literature, the assignment problem has been proposed. To start with, in some circumstances, the costs that are in literature are not crisp. In such a thing, some system-parameters are distinguished by information which might be uncertain such as the variables that are fuzzy and random. Thus, the AP is tackled in uncertain environments ([7]).

Literature review

Recently, a variety of papers have used the set, in other words, the technique of branch and bound , to characterize optimal solution with data-uncertainty. Liu, L., & Li, Y. (2006) presented the fuzzy quadratic assignment problem with penalty is formulated as expected value model, chance-constrained programming and dependent-opportunity programming relying on the numerous characteristics of decision, and the crisp counterparts are provided.

Strouvalis, A. M., Heckl, I., Friedler, F., & Kokossis, A. C. (2002) they proposed a methodology for implementing logic and engineering knowledge within a Branch-and-Bound algorithm and with an aim for accelerating convergence. The utility networks- assignment problem is addressed by the development emphasizing on the most favorable allocation of units for the problems of maintenance. Hahn, P., Hightower, W., Johnson, T., Guignard-Spielberg, M., & Roucairol, C. (2001), presented a new strategy for selecting nodes in a branch and bound algorithm for solving exactly the Quadratic Problem of Assignment (QAP). If it is learned, then it will be developed. This means that there is a failure on the larged-sized problems on the part of the order strategies.

The ranking method of Yager is utilized for numbers-ranking that are fuzzy, after the transformation of the problem of assignment into a crisp one via the use of linguistic variables, the (AP) has been worked out by the technique of branch and bound, presented by Gotmare, D., & Khot, P. G. (2016).

Lin, C. J., & Wen, U. P. (2004) they proposed an effective algorithm dependent upon the labeling technique for working out the programming case that is linear fractional. The algorithm starts with primary feasibility and progresses to get double feasibility while keeping complementary slackness till the primary ultimate answer is obtained. And branch-and-bound-with-underestimates, has been applied and there is the definition of the two functions which are underestimate. By Chen, G. H., & Yur, J. S. (1990, January) Their effectiveness has been shown via experiment. This is done by making a comparison between the

proposed algorithm and the Wang and Tsai's algorithm and an algorithm with $h(x) = O$. And also in our algorithm for calculating the lower bounds on solutions to the QAP. Resulting from some recently developed improvements, By Hahn, P., Grant, T., & Hall, N. (1998) the DP provides bounds that are lower and as tight as any that could be important in a branch-and-bound algorithm. These are of cheap production relatively, particularly on greater problems. The elements of the profit matrix of the problem of assignment are regarded as the triangular fuzzy numbers by Muruganandam, S., & Hema, K. (2017), and ultimate solution can be obtained through the use of branch and bound method without changing the fuzzy numbers into crisp ones. To add more, the Lone, M. A., Mir, S. A., Ismail, Y., & Majid, R. (2017) tackle the assignment problem which is Intuitionistic and fuzzy whose cost has been considered as an Intuitionistic triangular fuzzy numbers. The assignment costs are turned into crisp values via defuzzifying, and the optimum solution is got through the use of Branch and Bound method. And the Vinoliah, E. M., & Ganesan, K. (2018) have suggested a new method for getting an ultimate solution, that is fuzzy, for a fuzzy problem of assignment with the numbers that are octagonal and fuzzy. By checking its optimality via the best method of computations that is minimum, the problem is solved and considered regarded. Too reviews the Srinivas, B., & Ganesan, G. (2015) a new procedure for ranking that could be found in and is utilized for comparing the intuitionistic fuzzy numbers so that an Intuitionistic Fuzzy Branch Bound method could be used for solving the issue of assignment that is intuitionistic and fuzzy. A branch and bound algorithm is developed by Ross, G. T., & Soland, R. M. (1975) that works out the problem of assignment generalized throughout working out a binary-group of the problems of knapsack for the sake of identifying the bounds.

Definition of the problem of assignment

The Assignment problem is an NP one of hard optimization, in which n workers are assigned to n jobs, with minimum cost. The quadratic model can directly be solved as a regular model of transportation. The model of assignment is actually a certain issue of the transportation-model where the sources are represented by the workers, and the destinations are by the jobs, the amount of supply at each source and the amount of demand at each destination actually equal 1 as in table (1). The "transportation-cost" of worker i to j is C_{ij} . In effect, nevertheless, the fact that all the supply and demand amounts equal 1. The problem of assignment where n workers are assigned to n jobs, or the problem contains assignment of n facilities (or warehouses) to n locations (or sites) with the objective of reducing the costs of transportation which are associated with the material-flow between facilities and distances between locations. It did not assign many facilities to the same location or assign many locations to the same

Table 1: The general form of assignment problems

C_{11}	C_{12}	...	C_{1n}	1
C_{21}	C_{22}	...	C_{2n}	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
C_{m1}	C_{m2}	...	C_{mn}	1
1	1	...	1	

facility. The evaluation function of an assignment problem is given as ([1])

$$Minimize(z) = \sum_{i=1}^{n} C_{ij} X_{ij}, \tag{1}$$

$$\sum_{i=1}^{n} X_{ij} = 1,\ i = 1, 2, \ldots, n \tag{2}$$

$$\sum_{j=1}^{n} X_{ij} = 1,\ j = 1, 2, \ldots, n\ X_{ij} = 0 or 1 \tag{3}$$

$$X_{ij} = \begin{cases} 1, & \text{if worker } i \text{ is assigned to job } j, \\ 0, & \text{otherwise.} \end{cases}$$

Preliminaries

It has been stated earlier that the Fuzzy set was firstly presented by Zadeh in 1965. It was presented mathematically as a way for reflecting ambiguity or inaccuracy in daily life.

Definition. The $\tilde{A}$ set that is fuzzy and may be defined depending on the R, which represents real numbers- universal set, is defined to be a number that is fuzzy in case that its job of membership is having the characteristics stated below:

1. $\mu_{\tilde{A}} : R[0, 1]$ is permanent.
2. $\mu_{\tilde{A}}(x) = 0$, for all $x \in (-\infty, a] \cup [d, \infty)$.
3. $\mu_{\tilde{A}}(x)$ is increasing strictly on [a, b] and strictly decreasing on [c, d] is decreasing strictly.
4. $\mu_{\tilde{A}}(x) = 1$, for all $x \in [b, c]$, where $a < b < c < d$.

Definition. The $\tilde{A}$, which represents a fuzzy number, equals (a, b, c) is a triangular number that is fuzzy in case that its function of membership is provided

by ([3])

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x-a_1}{a_2-a_1}, & \text{if } a_1 \leqslant x \leqslant a_2 \\ \frac{a_3-x}{a_3-a_2}, & \text{if } a_2 \leqslant x \leqslant a \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

Robust ranking technique

Finding the measures of performance in terms of crisp values, it is important to clarify them within crisp ones by a ranking method of a fuzzy number. The technique of Robust ranking that satisfies and ensures compensation, linearity, and additional properties. What is more is providing the results that are consistent with the intuition of human. Give a convex fuzzy number ã, the Robust Ranking Index is defined by

$$R(\tilde{a}) = \int_0^1 (0.5)(a_\alpha^L a_\alpha^U) \tag{5}$$

$(a_\alpha^L a_\alpha^U) = \{(b-a)\alpha + a + c - (c-b)\alpha\}$, here $(a_\alpha^L a_\alpha^U)$ is the α–*level* cut of the fuzzy number a. This technique is utilized here for the fuzzy numbers- ranking. The Robust ranking index R(ã) Robust ranking index indicates the fuzzy number typical value ã. It is satisfiable for linearity and additional property ([6]).

Branch and bound algorithm

There is an m level number in the branching tree in Branch and bound technique applied to assignment problem, but is has been found zero for root node.

$$K_\gamma = [C_i j]_{i,j \in T} + \sum_{i \in x}^{n} C_{ij} (\sum_{j \in y}^{n} less C_{ij}) \tag{6}$$

such that:
K_γ; be the assignment lower bound which is partial, A up to Q_γ^m .
γ; be the assignment made in the present branching tree-node.
T; be a cells-set that is assigned up to the node from Q_γ^m the root node.
C_{ij}; is the cell-entry of the cost matrix with respect to the i row and j column.
X; be the rows-set that is not omitted up to the node Q_γ^m from the root node which is found in branching tree.
γ; be the columns-set which is not omitted up to the node Q_γ^m from the node of the root in the branching tree.
Q_γ^m; be an m level- assignment of a branching tree.

Rules of branching

- The row that is assigned as m of the problem of assignment at m level, would be designated with the most suitable column of that issue.

- If a tie exists on the lower bound, and the final node at the lower level is to be apt for additional branching.

- Stopping rule: The optimality is reached in case that the minimal lower bound is to be at a terminal node at the (n-1) level. Then the optimum solution will be shaped or formed by the assignments existed in the passage from the node of the root to that one in accordance with the row-column combination-pair which is missing ([3]).

Application

General of Company for Cotton Industries Iraq.

It is considered one of the large companies with diversified production, established in 1945, and includes four factories. A sample of production machines was selected, as each machine produces a type of cotton products, which are as follows;

- Cowboy pants production machine.
- Business suits production machine.
- Gauze and medical bandage production machine.
- Dress for men and women production machine.

The goal is to allocate four workers on the machines, and the table below shows the fuzzy times for each worker on each production machine. ([11]).

Table 2: The fuzzy number for each worker machines

	Cowboy Pants	Business Suits	Gauze and Medical Bandage	Dress for Men and Women
Worker (1)	146, 230, 424	146, 240, 434	166, 260, 454	140, 195, 489
Worker (2)	163, 263, 457	160, 200, 394	166, 260, 454	160, 210, 404
Worker (3)	146, 300, 494	146, 195, 380	140, 195, 380	158, 252, 446
Worker (4)	160, 200, 394	130, 197, 391	160, 210, 404	140, 196, 310

Using equation No. 5, convert the fuzzy numbers in Table (2) to Crisp number:
$R(\tilde{C}_{11}) = R(146, 230, 424) = \int_0^1 (0.5)\{(230-146)\alpha+146+424-(424-230)\alpha\}d\alpha = 257.5$
$R(\tilde{C}_{12}) = R(146, 240, 434) = \int_0^1 (0.5)\{(240-146)\alpha+146+434-(434-240)\alpha\}d\alpha = 265$
$R(\tilde{C}_{13}) = R(166, 260, 454) = \int_0^1 (0.5)\{(260-166)\alpha+166+454-(454-260)\alpha\}d\alpha = 285$
$R(\tilde{C}_{14}) = R(140, 195, 489) = \int_0^1 (0.5)\{(195-140)\alpha+140+489-(489-195)\alpha\}d\alpha = 254.8$
$R(\tilde{C}_{21}) = 286.5$, $R(\tilde{C}_{22}) = 238.5$, $R(\tilde{C}_{23}) = 285$, $R(\tilde{C}_{24}) = 246$
$R(\tilde{C}_{31}) = 310$, $R(\tilde{C}_{32}) = 229$, $R(\tilde{C}_{33}) = 227.5$, $R(\tilde{C}_{34}) = 277$
$R(\tilde{C}_{41}) = 238.5$, $R(\tilde{C}_{42}) = 228.8$, $R(\tilde{C}_{43}) = 246$, $R(\tilde{C}_{44}) = 210.5$
These values are replaced with their similar $R(\tilde{C}_{ij})$ and use the branch and bound method in to solve the problem of assignment.

Table 3: The crisp number for each worker machines

	Cowboy Pants	Business Suits	Gauze and Medical Bandage	Dress for Men and Women
Worker (1)	257.5	265	285	254.8
Worker (2)	286.5	238.5	285	246
Worker (3)	310	229	227.5	277
Worker (4)	238.5	228.8	246	210.5

In Fig 1, the sub-problems, which are four, under the nodes of the root are shown, lower bound the problems of solution.

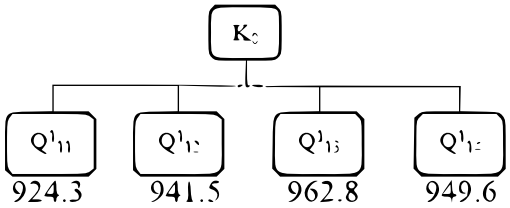


Figure 1: Further branch and bound the root nodes

Q_{11}^1 is the lower bound and by applying equation (6) we get;
$\gamma = \{(1,1)\}$, $T = \{(1,1)\}$, $x = \{2,3,4\}$, $\gamma = \{2,3,4\}$
$Q_{11}^1 = 257.5 + (228.8 + 227.5 + 210.5) = \mathbf{924.3}$
$Q_{12}^1 = 264 + (238.5 + 227.5 + 210.5) = 941.5$
$Q_{13}^1 = 285 + (238.5 + 228.8 + 210.5) = 962.8$
$Q_{14}^1 = 254.8 + (238.5 + 228.8 + 227.5) = 949.6$
Further branching is performed out of the node that is terminal and possesses the least lower bound at this stage, the terminal nodes are Q_{11}^1, Q_{12}^1, Q_{13}^1, Q_{14}^1. The node Q_{11}^1 has the least lower bound. Thus, in Fig(2), further branching from this node is shown:

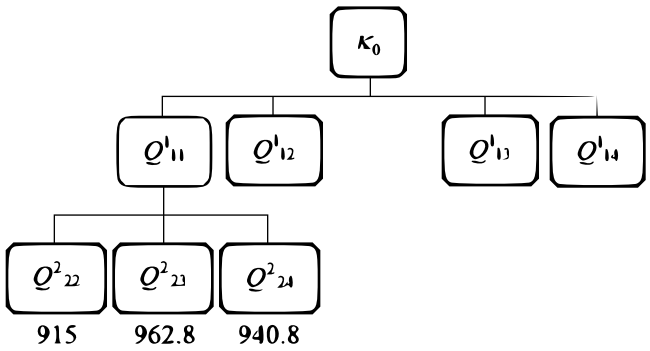


Figure 2: Further branch and bound this node

Q_{22}^2 is the lower bound and by applying equation (6) we get:

$T = \{(1,1),(2,2)\}$, $x = \{3,4\}$, $\gamma = \{3,4\}$
$Q_{22}^2 = 238.5 + (238.5 + 227.5 + 210.5) = \mathbf{915}$
$Q_{23}^2 = 285 + (238.5 + 228.8 + 210.5) = 962.8$
$Q_{24}^2 = 246 + (238.5 + 228.8 + 227.5) = 940.8$

An extra branching is performed by the final node that possesses the least lower bound at this part, the terminal nodes are Q_{22}^2, Q_{23}^2, Q_{24}^2. The node that has the minimum lower bound is Q_{22}^2. Therefore, in Fig (3) shows a further branching that is from this node:

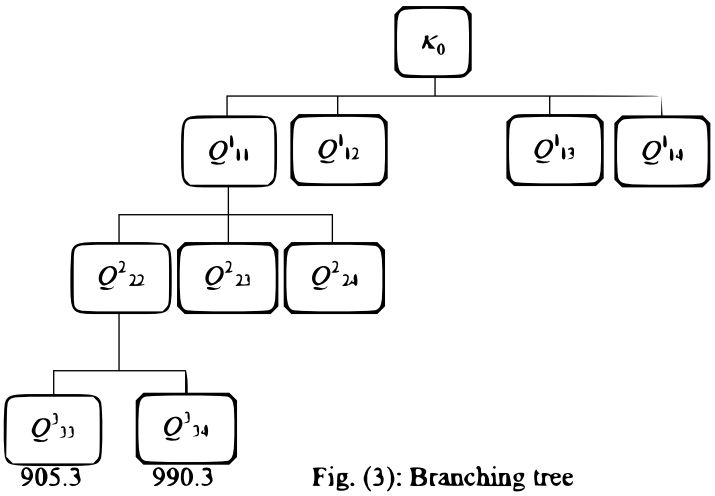


Fig. (3): Branching tree

Figure 3: Branching tree

Q_{33}^3 is the lower bound and by applying equation (6) we get;
$T = \{(1,1),(2,2),(3,3)\}$, $x = \{4\}$, $\gamma = \{4\}$
$Q_{33}^3 = 227.5 + (238.5 + 228.8 + 210.5) = \mathbf{905.3}$
$Q_{34}^3 = 277 + (238.5 + 228.8 + 246) = 990.3$
final assignment table is (1,1), (2,2), (3,3), (4,4)
Fuzzy final cost is (146, 230, 424) + (160, 200, 394) + (140, 195, 380) + (140, 196, 310). And at a cost (257.5 + 238.5 + 227.5 + 210.5 = **934**).

Conclusions

Decision making greatly relies on Assignment problem since it is of great use, e.g. problems of resource allocation, such as jobs personnel assignment, machines-tasks, etc. Not only do the managers, in real world, hope to promote each job-quality in case it is assigned to a certain individual, but also to decrease the

total cost that is used. Regardless of this, each job-cost, if quality is dependent upon here, is not a deterministic number, nor the planned total cost.

Here, the cost of assignment is regarded as Intuitionistic fuzzy numbers. By demulsifying with the preciseness function, the assignment costs are converted into crisp values on the way robust ranking technique and the optimum solution is obtained by using the Branch and Bound method.

And therefore; Q_{11}^1 be the lower bound of the first partial assignment, and the value is. 924.3 and be fuzzy assignment (146, 230, 424), And also Q_{22}^2 be the lower bound of the second assignment that is partial, and the value is. 915 and be fuzzy assignment (160, 200, 394), Q_{33}^3 be the lower bound of the third assignment that is partial, and the value is. 905.3 and be fuzzy assignment (140, 195, 380), and at a total cost (934).

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Optimal investment of DC pension plan based on a weighted utility

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Abstract. We investigate the DC pension manager's portfolio problem when he bases decisions on both absolute level of total wealth and comparisons to a certain pre-defined reference point. This setting leads to a non-concave objective utility and therefore a non-concave utility maximization problem. We apply the concavification technique to solve the non-concave optimization problem and obtain the closed-form representations of the optimal wealth process and the optimal strategies. Numerical results show that if the manager pays much more attention to the change of wealth relative to the reference point, then he will take a more conservative investment strategy due to risk aversion over gains.

Keywords: DC pension plan, weighted utility, non-concave utility, concavification.

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1. Introduction

Due to declining mortality and low interest rates, the literature on the asset allocation of pension plans has grown up significantly in recent years. There are mainly two kinds of pension fund: defined benefit (DB) pension plan and defined contribution (DC) pension plan. In a DB pension plan, the benefit is fixed in advance and therefore the plan sponsor bears the financial risks. In a DC plan, the member contributes part of the salary to the pension account and upon retirement, the member can receive regular income from the pension account. DC pension plans are becoming more popular than DB pension plans in many countries. The problem of optimal allocation during the accumulation phase for the DC pension plan has attracted a lot of attention, see Zhang et al. (2018), Wang et al. (2019), Dong and Zheng (2019, 2020).

Most earlier works on DC pension plans care about maximizing the expectation of a smooth utility of terminal wealth, see Dong et al. (2020) and Guan and Liang (2014). Under a smooth utility, people make decisions only based on absolute wealth. Prospect theory, proposed in Kahneman and Tversky (1979) and Tversky and Kahneman (1992), states that people show risk aversion over gains and risk seeking over losses and base decisions on comparisons to a certain pre-defined reference point rather than evaluating absolute level of total wealth itself. These distinguished behaviors can be characterized by an asymmetric S-shaped utility function, convex in the domain of losses and concave in the domain of gains. Recently, more and more literatures care about maximizing the expectation of an S-shaped utility of terminal wealth, see Berkelaar et al. (2004), Guan and Liang (2016), Guo (2014) and Jin et al. (2008). However, none of the above-mentioned papers consider the optimization problem based on both absolute level of total wealth and prospect theory.

As the purpose of a pension plan is to provide an adequate retirement income for a DC member, risk management based on terminal portfolio insurance (PI) constraint has attracted much attention, see Deelstra et al. (2003) and He et al. (2019). The PI constraint can well protect the members' benefits by keeping the portfolio value above a minimum guarantee at retirement. In this paper, we consider a weighted utility of a smooth utility and an S-shaped utility under PI constraint, which can include the impact of both the absolute wealth and relative to the reference point on the investment decision in our modelling.

The main contribution of this paper is that we have solved an optimization problem of a DC plan under a weighted utility and PI constraint, such that the manager can base his decisions on both absolute level of total wealth and comparisons to a certain pre-defined reference point. The rest of the paper is organized as follows. In Section 2 we formulate a DC investment problem under a weighted utility and PI constraint. In Section 3 we apply the concavification and the martingale method to solve the non-concave optimization problem and derive the explicit optimal solution. In Section 4 we numerically analysize the impact of the weight parameter on the distribution of the optimal terminal

wealth. Section 5 concludes. The appendix contains the proofs of Lemma 3.2 and Proposition 3.4.

2. The model

In a continuous-time economy, we investigate the asset allocation problem of a DC plan among a risk-free asset and a risky asset. Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtered complete probability space with the filtration $\mathbb{F} := \{\mathcal{F}_t | 0 \leq t \leq T\}$ being the natural filtration generated by a standard Brownian motion $\{W(t)\}_{0\leq t\leq T}$ and satisfying the usual conditions. The DC pension fund starts at time 0 and the retirement time is T.

The prices of the risk-free and risky assets evolve as follows:

$$dS_0(t) = rS_0(t)dt, \ S_0(0) = 1, \tag{2.1}$$

and

$$\begin{aligned} dS_1(t) &= S_1(t)(\mu dt + \sigma dW(t)) \\ &= S_1(t)(rdt + \sigma(dW(t) + \vartheta dt)), \end{aligned} \tag{2.2}$$

where $r > 0$ is a riskless interest rate, $\mu > r$ is the expected return of the risky asset, $\sigma > 0$ is the volatility of the risky asset and $\vartheta = \frac{\mu - r}{\sigma}$ is the market price of risk.

In a DC plan, the pension members contribute continuously to the pension plan before retirement time T. Let $c(t) > 0$ denote the aggregated amount of money contributed at time t of a cohort of fund participants. As explained in Dong and Zheng (2020), individual members' contribution rates may be random over the time, but the accumulated cash income of a large pension fund is in general deterministic and stable. We therefore for simplicity assume $c(t) \equiv c$ is a constant. Assume that there are no transaction costs or taxes in the financial market. The pension account is endowed with an initial endowment $x_0 \geq 0$.

Let $\pi(t)$ be the total amount allocated to the risky asset at time $t \in [0, T]$. Then, the wealth process $\{X^\pi(t)\}_{0\leq t\leq T}$ with the initial value $X^\pi(0) = x_0$ evolves as follows:

$$dX^\pi(t) = rX^\pi(t)dt + \pi(t)\sigma(dW(t) + \vartheta dt) + cdt. \tag{2.3}$$

Let $H(t)$ be the discounted value at time t of total pension contribution from t to T, where

$$H(t) = c\int_t^T e^{-r(s-t)}ds = \frac{c}{r}(1 - e^{-r(T-t)}). \tag{2.4}$$

To guarantee that a DC pension can meet the member's requirement of an elementary security supporting her living after retirement, a downside protection is necessary. Let $L > 0$ be a constant. We next define the set of admissible trading strategies.

Definition 2.1. *A portfolio strategy* $\{\pi = \pi(t) : t \in [0,T]\}$ *is said to be admissible if* $\pi(t)$ *is an* $\mathcal{F}$*-progressively measurable process, with* $\int_0^T \pi^2(t)dt < +\infty$, *a.s., for all* $t \in [0,T]$, *and* $X^\pi(t)$ *satisfies (2.3),* $X^\pi(T) \geq L$. *We denote the set of admissible portfolio strategies by* $\mathcal{A}$.

The goal of the pension fund manager is to find the optimal investment strategies within $[0,T]$. Most earlier existing works on DC pension investigate a concave utility maximization problem. Under a smooth utility, people make decisions based on absolute wealth. Kahneman and Tversky (1979) claim that people are risk averse over gains and risk seeking over losses and make decisions relative to some reference levels rather than absolute values directly. They adopt an S-shaped utility to capture these distinguished behaviors. In this paper, we consider a weighted utility of a smooth utility and an S-shaped utility, which can include the impact of both the absolute wealth and the change of wealth relative to some reference point on the investment decision in our modelling.

Firstly, we introduce an S-shaped utility, see Dong and Zheng (2020). Let $\theta > L$ be a reference point, which is chosen in advance. Assume that $U_1 : [L,\infty) \to R$ is an S-shaped utility (see, figure 1), which satisfies
1. U_1 is smooth and increasing on $[L,\infty)$, strictly convex on (L,θ) and strictly concave on $(\theta,+\infty)$.
2. $\lim_{x\to+\infty} U_1(x) = +\infty, \lim_{x\to+\infty} U_1'(x) = 0, \lim_{x\to\theta^-} U_1'(x) = \lim_{x\to\theta^+} U_1'(x) = +\infty$.
3. $\lim_{x\to\theta^-} U_1''(x) = +\infty, \lim_{x\to\theta^+} U_1''(x) = -\infty$. $\lim_{x\to+\infty} \frac{xU_1'(x)}{U_1(x)} < 1$.
4. $U_1''' > 0, x \in (L,\theta) \cup (\theta,\infty)$.

Note that U_1 is convex when x is below the reference point θ and concave when x is above θ, which demonstrates different behaviors of people in gain and loss situations relative to the reference point, that is, risk averse over gains and risk seeking over losses.

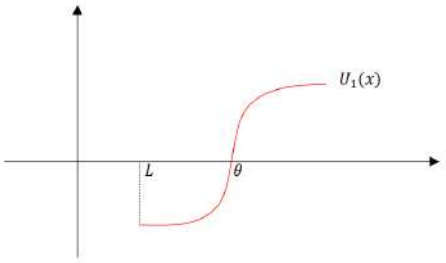


Figure 1: the utility U_1

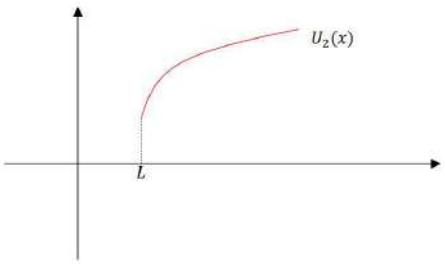


Figure 2: the utility U_2

The utility U_2 is strictly increasing, strictly concave, continuously differentiable, real-valued function defined on $[L,\infty)$ (see, figure 2) satisfying

$$\lim_{x\to+\infty} U_2(x) = +\infty, \lim_{x\to+\infty} U_2'(x) = 0, U_2'''(x) > 0,$$

and

$$\lim_{x\to+\infty}\frac{xU_2'(x)}{U_2(x)}<1.$$

We now consider a weighted utility defined by

$$U(x)=aU_1(x)+bU_2(x), x\geq L, \tag{2.5}$$

where $0\leq a,b\leq 1$ with $a+b=1$ reflect the aversion degrees of the absolute wealth and the change of wealth relative to the reference point, respectively. A larger a implies that the manager pays much more attention to the change of wealth relative to the reference point. The case $a=0$ degenerates to the optimization problem for a concave utility, while the case $a=1$ corresponds to the optimization problem for an S-shaped utility.

It is easy to conclude that the weighted utility U is strictly concave on $(\theta,+\infty)$ and the strictly decreasing function U' defined on (θ,∞) has a strictly decreasing inverse $I:(0,\infty)\to(\theta,\infty)$, that is,

$$U'(I(y))=y,\forall y>0,\ I(U'(x))=x,\forall x>\theta.$$

We consider the following optimization problem with minimum guarantee:

$$\begin{cases}\max_{\pi\in\mathcal{A}}E[U(X^\pi(T))],\\ s.t.\ X^\pi(t)\text{ satisfies (2.3) with }X^\pi(T)\geq L, a.s..\end{cases} \tag{2.6}$$

Since the market is complete, there exists the unique pricing kernel defined by

$$\xi(t)=e^{-(r+\frac{\vartheta^2}{2})t-\vartheta W(t)},\xi(0)=1. \tag{2.7}$$

Here $\xi(t)$ is the Arrow-Debreu value per probability unit of a security which pays out 1 at time t if the scenario ω happens, and 0 else. As this value is high in a recession and low in prosperous times, $\xi(t)$ directly reflects the overall state of the economy.

Following Cox and Huang (1989) and Guan and Liang (2016), we can find the optimal solution for (2.6) via the martingale method. The problem (2.6) can be rewritten as follows:

$$\begin{cases}\max_{X^\pi(T)}E[U(X^\pi(T))],\\ s.t.\ E[\xi(T)X^\pi(T)]\leq x_0+H(0), X^\pi(T)\geq L, a.s..\end{cases} \tag{2.8}$$

3. Optimal trading strategy under PI constraint

In this section, we will solve the optimization problem (2.6) and obtain the optimal wealth and optimal investment strategies.

Recall that $\lim_{x\to\theta^-}U''(x)=+\infty$ and U'' is continuous, increasing on (L,θ) and is strictly concave on (θ,∞). We can easily conclude that

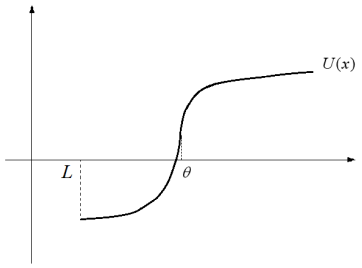


Figure 3: $U''(L+) \geq 0$

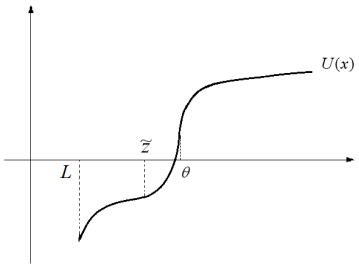


Figure 4: $U''(L+) < 0$

Case A. for $U''(L+) \geq 0$, U is strictly convex on (L, θ) and U is strictly concave on (θ, ∞) (see, figure 3).
Case B. for $U''(L+) < 0$, there exists $L < \tilde{z} < \theta$ (see, figure 4) such that

$$U''(\tilde{z}) = 0. \tag{3.1}$$

Note that, in Case B, U is strictly concave on $(L, \tilde{z}) \cup (\theta, \infty)$ and strictly convex on $(\tilde{z}, \theta)$. Furthermore, U' defined on $(L, \tilde{z})$ has a strictly decreasing inverse $\overline{I} : (U'(\tilde{z}), U'(L+)) \to (L, \tilde{z})$, that is,

$$U'(\overline{I}(y)) = y, \forall\ U'(\tilde{z}) < y < U'(L+),\ \overline{I}(U'(x)) = x, \forall\ L < x < \tilde{z}.$$

Lemma 3.1. *For $\theta > L$, there exists a unique tangent point $z > \theta$ of the straight line starting at $(L, U(L))$ to the curve $U(x), x > \theta$ satisfying the following equation*

$$U(x) - (x - L)U'(x) - U(L) = 0. \tag{3.2}$$

Proof. Denote by $f(x) = U(x) - (x - L)U'(x) - U(L)$. It is easy to check that $f(x)$ is a decreasing, continuous function in (θ, ∞) and $\lim_{x\to\theta} f(x) < 0$, $\lim_{x\to\infty} f(x) = \infty$, which yields the result. □

Lemma 3.2. *Let $\tilde{z}$ and z be given by (3.1) and (3.2), respectively. For $U'(L+) > U'(z)$, there exist $L < z_1 < \tilde{z}, z_2 > z$ satisfying*

$$\frac{U(z_2) - U(z_1)}{z_2 - z_1} = U'(z_1) = U'(z_2). \tag{3.3}$$

and

$$U(x) < U'(z_2)(x - z_2) + U(z_2) = U'(z_1)(x - z_1) + U(z_1),\ x \in [z_1, z_2]. \tag{3.4}$$

Proof. See Appendix. □

Following Carpenter(2000), we shall use the concavification technique to derive the optimal terminal wealth. We first investigate the following problem for any $y > 0$:

$$\max_{x \geq L} \{U(x) - xy\}. \tag{3.5}$$

Proposition 3.3. *Let $\tilde{z}$ and z be given by (3.1) and (3.2), respectively. For $y > 0$,*

Case A. If $U'(L+) \leq U'(z)$, then the concave envelope of U is given by

$$U^c(x) = \begin{cases} U(z) + U'(z)(x - z), & L \leq x < z, \\ U(x), & x \geq z. \end{cases} \tag{3.6}$$

Case B. If $U'(L+) > U'(z)$, then the concave envelope of U is given by

$$U^c(x) = \begin{cases} U(x), & L \leq x < z_1, \\ U(z_1) + U'(z_1)(x - z_1), & z_1 \leq x < z_2, \\ U(x), & x \geq z_2, \end{cases} \tag{3.7}$$

where $L < z_1 < \tilde{z}$ and $z_2 > z$ are determined by (3.3).

Proof. The concave envelope of U can be directly obtained from Lemma 3.2 and Lemma A.1 of Dong and Zheng (2020). □

Proposition 3.4. *Let z be given by (3.2). Assume $x_0 \geq Le^{-rT} - H(0)$. Then for the optimization problem (2.6), the optimal terminal wealth is as follows:*

For $U'(L+) \leq U'(z)$,

$$X^{\pi^*,\beta^*}(T) = \begin{cases} I(\beta^* \xi(T)), & 0 < \xi(T) < \frac{U'(z)}{\beta^*}, \\ L, & \xi(T) \geq \frac{U'(z)}{\beta^*}. \end{cases} \tag{3.8}$$

For $U'(L+) > U'(z)$,

$$X^{\pi^*,\beta^*}(T) = \begin{cases} I(\beta^* \xi(T)), & 0 < \xi(T) < \frac{U'(z_1)}{\beta^*}, \\ \overline{I}(\beta^* \xi(T)), & \frac{U'(z_1)}{\beta^*} \leq \xi(T) < \frac{U'(L+)}{\beta^*}, \\ L, & \xi(T) \geq \frac{U'(L+)}{\beta^*}. \end{cases} \tag{3.9}$$

Here the multiplier $\beta^* > 0$ is determined by

$$E[\xi(T) X^{\pi^*,\beta^*}(T)] = x_0 + H(0). \tag{3.10}$$

Proof. See Appendix. □

Remark 3.5. Note that if $a = 0$, then the optimal terminal wealth under a concave utility U_2 is given by

$$X^{\pi^*,\beta^*}(T) = \begin{cases} I_2(\beta^*\xi(T)), & 0 < \xi(T) < \frac{U_2'(L+)}{\beta^*}, \\ L, & \xi(T) \geq \frac{U_2'(L+)}{\beta^*}, \end{cases} \tag{3.11}$$

where I_2 is the inverse function of U_2.

If $a = 1$, then the optimal terminal wealth under an S-shaped utility U_1 is given by

$$X^{\pi^*,\beta^*}(T) = \begin{cases} I_1(\beta^*\xi(T)), & 0 < \xi(T) < \frac{U_1'(\hat{z})}{\beta^*}, \\ L, & \xi(T) \geq \frac{U_1'(\hat{z})}{\beta^*}, \end{cases} \tag{3.12}$$

where I_1 is the inverse function of U_1, $\hat{z}$ is the unique solution to the following equation

$$U_1(x) - (x - L)U'(x) - U(L) = 0.$$

The optimal terminal wealth $X^{\pi^*,\beta^*}(T)$ can be expressed as a function of the state price density at maturity $\xi(T)$ and the multiplier β^*. $X^{\pi^*,\beta^*}(T)$ may take a two- or three-region form according to the values of $U'(L+)$ and $U'(z)$. In all cases, the optimal terminal wealth ends up with L when the state price increases above the boundary point of the bad-states region. The PI constraint well protects the members's benefits by keeping the optimal terminal wealth always above the level L. Due to the budget constraint, we can conclude that the protection in bad economic states is at the expense of the optimal terminal wealth in good economic states.

After obtaining the optimal terminal wealth, we can derive the closed-form expression for the optimal investment strategy given in the following result.

Proposition 3.6. *Assume that $x_0 \geq Le^{-rT} - H(0)$. Then for the optimization problem (2.6), the optimal wealth process and the optimal strategy are given as follows:*

For $U'(L+) \leq U'(z)$, the optimal wealth process at time t is given by

$$X^{\pi^*,\beta^*}(t) = A_1(\xi(t), t) - H(t), \tag{3.13}$$

where

$$A_1(\xi(t), t) = F_1(t, \xi(t), d_{1,t}(U'(z))) + Le^{-r(T-t)}\Phi(d_{2,t}(U'(z))),$$

with

$$\begin{aligned} F_1(t, \xi(t), x) &= \int_x^\infty I(\beta^*\xi(t)\chi(t,u))\chi(t,u)\varphi(u)du, \\ d_{1,t}(x) &= \frac{-\ln(\frac{x}{\xi(t)\beta^*}) - (r + \frac{\vartheta^2}{2})(T-t)}{\vartheta\sqrt{T-t}}, \ d_{2,t}(x) = d_{1,t}(x) + \vartheta\sqrt{T-t}, \\ \chi(t,x) &= e^{-(r+\frac{\vartheta^2}{2})(T-t) - \vartheta\sqrt{T-t}x}, \end{aligned}$$

the multiplier $\beta^* > 0$ determined by (3.10), Φ and φ the standard normal cumulative distribution function and the standard normal density function, respectively.

The amount of wealth invested in the risky asset is given by:

$$\pi^{*,\beta^*}(t) = -\frac{\vartheta}{\sigma}(\overline{F}_1(t,\xi(t),d_{1,t}(U'(z))) + (L-z)\overline{\varphi}(U'(z))), \tag{3.14}$$

where

$$\overline{F}_1(t,\xi(t),x) = \int_x^\infty I'(\beta^*\xi(t)\chi(t,u))\beta^*\chi^2(t,u)\varphi(u)du,$$

and $\overline{\varphi}(u) = \frac{u\varphi(d_{1,t}(u))}{\vartheta\sqrt{T-t}\beta^*\xi(t)}$.

For $U'(L+) > U'(z)$, the optimal wealth process at time t is given by

$$X^{\pi^*,\beta^*}(t) = A_2(\xi(t),t) - H(t), \tag{3.15}$$

where

$$\begin{aligned} A_2(\xi(t),t) &= F_1(t,\xi(t),d_{1,t}(U'(z_1))) + F_2(t,\xi(t),d_{1,t}(U'(L+))) \\ &\quad -F_2(t,\xi(t),d_{1,t}(U'(z_1))) + Le^{-r(T-t)}\Phi(d_{2,t}(U'(L+))), \end{aligned}$$

with

$$F_2(t,\xi(t),x) = \int_x^\infty \overline{I}(\beta^*\xi(t)\chi(t,u))\chi(t,u)\varphi(u)du.$$

The amount of wealth invested in the risky asset is given by:

$$\begin{aligned} \pi^{*,\beta^*}(t) &= -\frac{\vartheta}{\sigma}(\overline{F}_1(t,\xi(t),d_{1,t}(U'(z_1))) + \overline{F}_2(t,\xi(t),d_{1,t}(U'(L+))) \\ &\quad -\overline{F}_2(t,\xi(t),d_{1,t}(U'(z_1))) + (\overline{I}(U'(z_1)) - z_2)\overline{\varphi}(U'(z_1))), \end{aligned} \tag{3.16}$$

where

$$\overline{F}_2(t,\xi(t),x) = \int_x^\infty \overline{I}'(\beta^*\xi(t)\chi(t,u))\beta^*\chi^2(t,u)\varphi(u)du.$$

Proof. The price of $X^{\pi^*,\beta^*}(T)$ at t is calculated as follows:

$$X^{\pi^*,\beta^*}(t) = \frac{1}{\xi(t)}E[\xi(T)X^{\pi^*,\beta^*}(T)|\mathcal{F}_t] - H(t). \tag{3.17}$$

Note that $\log(\frac{\xi(T)}{\xi(t)})$ is a normal distribution with a mean $-(r+\frac{\vartheta^2}{2})(T-t)$ and a variance $\vartheta^2(T-t)$. By substituting the expressions for $X^{\pi^*,\beta^*}(T)$ given in Proposition 3.4 into (3.17), we can easily obtain the formulas for $X^{\pi^*,\beta^*}(t)$ by some straightforward calculations.

Let $X^{\pi^*,\beta^*}(t) = A_i(\xi(t), t)$. Then

$$dX^{\pi^*,\beta^*}(t) = \hbar_i(t, \xi(t))dt - \frac{\partial A_i}{\partial \xi}\xi(t)\vartheta dW(t),$$

for some $\hbar_i(t, \xi(t))$, and we are only interested in the diffusion part. Comparing it with (2.3), we have

$$\pi^{*,\beta^*}(t) = -\frac{\vartheta}{\sigma}\frac{\partial A_i}{\partial \xi}\xi(t).$$

Some simple calculations yield the results. □

4. Numerical analysis

In this section, we carry out some numerical calculations to investigate the impact of the weight parameter and some other model parameters on the optimal terminal wealth. Assume that $U_1(x)$ is given by:

$$U_1(x) = \begin{cases} -B(\theta - x)^{\gamma_1}, & L \le x < \theta, \\ (x-\theta)^{\gamma_2}, & x \ge \theta, \end{cases} \tag{5.1}$$

where $\theta > 0$ is a reference point, $B > 1$ is called loss aversion degree, $0 < \gamma_1 < 1, 0 < \gamma_2 < 1$ are constants, which characterizes the degree of loss aversion and risk aversion, respectively. The concave utility $U_2(x)$ is as follows:

$$U_2(x) = x^p, x \ge L, 0 < p < 1. \tag{5.2}$$

For all numerical computations, the benchmark data used are fixed: $r = 0.03, \mu = 0.07, \sigma = 0.3, c = 1, B = 2.25, \gamma_1 = 0.2, \gamma_2 = 0.3, a = 0.5, L = 60, \theta = 100, x_0 = 20, T = 40$.

Table 1 lists expectations, standard deviations and the probabilities for different a. We can see from it that the expectation, the standard deviation and the probability $P(X^{\pi^*,\beta^*}(T) = L)$ all decrease in a. Note that the reference point can be easily obtained by investing a large proportion of wealth in the cash bond. Due to loss aversion, the manager will take a more conservative allocation strategy when he pays much more attention to the change of wealth relative to the reference point.

Table 1: expectations, standard deviations, and probabilities

a	0	0.3	0.5	0.7	1
mean	409.85	342.37	299.91	266.64	225.92
std dev	999.95	670.60	515.35	385.54	228.25
$P(X^{\pi^*,\beta^*}(T) = L)$	0.2574	0.0852	0.0384	0.0179	0.0058

Table 2 lists expectations, standard deviations and the probabilities for different (θ, L). It is observed that a higher value of L leads to a lower mean,

standard deviation and $P(X^{\pi^*,\beta^*}(T) = L)$, since the manager shall allocate less money in the risky asset to attain a higher protection level. We can also see that the mean, the standard deviation and $P(X^{\pi^*,\beta^*}(T) = L)$ also decreases in θ. As pointed out above, the reference point can be obtained by investing much money in the riskless asset for a relatively low value of θ. To attain a higher reference point, the manager will allocate more money in the riskless asset.

Table 2: expectations, standard deviations, and probabilities

(L, θ)	$(60, 80)$	$(60, 100)$	$(60, 120)$	$(40, 100)$	$(80, 100)$
means	319.87	299.91	292.15	321.53	278.27
std dev	611.11	515.35	453.85	577.47	467.11
$P(X^{\pi^*,\beta^*}(T) = L)$	0.0071	0.0384	0.0860	0.0578	0.0125

Figure 5 plots the relationship between the expectation and the parameter p for $a = 0$ and $a = 0.5$. We can observe that the expectation increases with p for a fixed a. This is due to the fact that the effect of higher risk aversion, measured by a lower p,leads to less investment in a risky asset, which brings a lower expectation.

Figure 6 plots the relationship between the standard deviation and the parameter p for $a = 0$ and $a = 0.5$. It is seen that the standard deviation increases with p for a fixed a. As explained in Figure 5, a lower p leads to a less risky portfolio, which corresponds to a lower standard deviation.

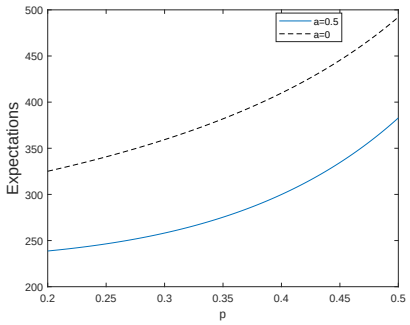


Figure 5: relationship between expectation and p

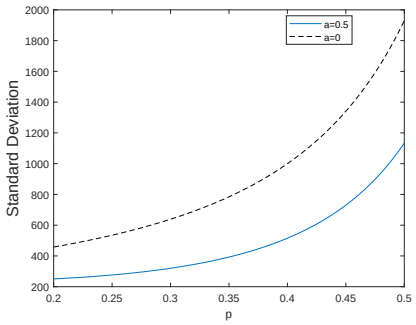


Figure 6: relationship between standard deviation and p

5. Conclusions

In this paper, we investigate the optimal portfolio selection problem of a DC pension manager when he bases decisions on both absolute level of total wealth and comparisons to a certain pre-defined reference point. By using a martingale method and a concavification technique, we derive the optimal wealth process and the optimal investment strategies. The form for the optimal terminal wealth

can take a two- or three-region form. In all cases, the optimal terminal wealth ends up with the protection level from a certain value of the price density. Numerical results show that the impact of the weight parameters which reflect the aversion degrees of the absolute wealth and the change of wealth relative to the reference point is significant. If the manager pays much more attention to the change of wealth relative to the reference point, then he will take a more conservative allocation strategy due to risk aversion over gains.

The present work might be extended in at least two directions. Firstly, we can extend a PI-constrained optimization problem to a VaR- or mutiple-VaR constrained one. Secondly, we can consider a performance fee arrangement in DC pension plan management under a weighted utility. We will leave these questions for future research.

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6. Appendix

The proof of Lemma 3.2

Proof. First, we prove the existence of z_1, z_2. Note that U is strictly concave on $(L, \tilde{z}) \cup (\theta, \infty)$ and is strictly convex on $(\tilde{z}, \theta)$ for the case $U'(L+) > U'(z)$. It is obvious that $U'(z) > U'(\tilde{z})$. Then there must exist unique $L < \eta_1 \leq \tilde{z}$ and $\eta_2 \geq \tilde{z}$, Such that

$$U'(\eta_1) = U'(z), \qquad U'(\eta_2) = U'(\tilde{z}). \tag{6.1}$$

For any $y \in [U'(\tilde{z}), U'(z)]$£¬ we define $L < c_1(y) \leq \tilde{z}, c_2(y) \geq z$ by the relation

$$U'(c_1(y)) = U'(c_2(y)) = y. \tag{6.2}$$

It is obvious that

$$\lim_{y \to U'(z)} c_1(y) = \eta_1, \lim_{y \to U'(z)} c_2(y) = z, \lim_{y \to U'(\tilde{z})} c_1(y) = \tilde{z}, \lim_{y \to U'(\tilde{z})} c_2(y) = \eta_2. \tag{6.3}$$

Furthermore, $c_1(y)$ and $c_2(y)$ are strictly decreasing and continuous on $[U'(\tilde{z}), U'(z)]$. Define $\kappa(y) = \frac{U(c_2(y)) - U(c_1(y))}{c_2(y) - c_1(y)}$. It is easy to check that $\kappa(y)$ is continuous on $[U'(\tilde{z}), U'(z)]$. It follows from concave/convex properties that

$$U(\eta_2) - U(\theta) > U'(\eta_2)(\eta_2 - \theta), \; U(\theta) - U(\tilde{z}) > U'(\tilde{z})(\theta - \tilde{z}).$$

Hence, we have

$$\kappa(U'(\tilde{z})) = \frac{U(\eta_2) - U(\tilde{z})}{\eta_2 - \tilde{z}} > U'(\tilde{z}).$$

Furthermore,

$$U(z) - U(L) = U'(z)(z - L),\ U(\eta_1) - U(L) > U'(\eta_1)(\eta_1 - L).$$

Therefore, we can obtain that

$$\kappa(U'(z)) = \frac{U(z) - U(L)}{z - L} < U'(z).$$

Then, there must exist $U'(\tilde{z}) < y_0 < U'(z)$ such that $\kappa(y_0) = y_0$. Define $L < z_1 < \tilde{z}, z_2 > z$ by the relation $U'(z_1) = U'(z_2) = y_0$, which concludes the existence of z_1, z_2. Additionally, we can conclude that the linear interpolation between $U(z_1)$ and $U(z_2)$ has to be higher than $U(.)$ on (z_1, z_2), which yields (3.4).

Now, we turn to the uniqueness of z_1, z_2. Assume there exists another pair $(\widehat{z}_1, \widehat{z}_2)$ with $L < \widehat{z}_1 < \tilde{z}, \widehat{z}_2 > z$ satisfying

$$U(\widehat{z}_2) - U(\widehat{z}_1) = U'(\widehat{z}_1)(\widehat{z}_2 - \widehat{z}_1) = U'(\widehat{z}_2)(\widehat{z}_2 - \widehat{z}_1),$$

and

$$U(x) < U(\widehat{z}_1) + U'(\widehat{z}_1)(x - \widehat{z}_1), x \in (\widehat{z}_1, \widehat{z}_2).$$

Without loss of generality, we assume that $\widehat{z}_1 > z_1$. From (3.4), we have $U(\widehat{z}_1) < U(z_2) + U'(z_2)(\widehat{z}_1 - z_2)$. So, for $z_2 \in (\widehat{z}_1, \widehat{z}_2)$,

$$U(z_2) > U(\widehat{z}_1) + U'(z_2)(z_2 - \widehat{z}_1) > U(\widehat{z}_1) + U'(\widehat{z}_1)(z_2 - \widehat{z}_1),$$

which leads to a contradiction. □

The proof of Proposition 3.2

Proof. We use the Lagrange dual theory as well as the concavification technique to solve problem (2.8). Define the Lagrangian of problem (2.8) as follows:

$$\mathcal{L}(X(T), \beta) = E[U(X(T)) - \beta\xi(T)X(T)]. \tag{6.4}$$

So, the original problem (2.8) is equivalent to the following problem:

$$\begin{cases} \inf_{\beta>0} \max_{X(T)} \mathcal{L}(X(T), \beta), \\ s.t.\ X(T) \geq L, a.s.. \end{cases}$$

To solve the above problem, we first use the concavification technique to find the optimal terminal wealth $X^{\pi^*,\beta}(T)$ for a fixed multiplier $\beta > 0$, which solves the following problem

$$\begin{cases} \max_{X(T)} \mathcal{L}(X(T), \beta), \\ s.t.\ X(T) \geq L, a.s.. \end{cases} \tag{6.5}$$

To solve (6.5), for each $y > 0$, we shall find the maximizer $x^*(y)$ solving

$$\max_{x \geq L} \{U(x) - xy\}.$$

For $U'(L+) \leq U'(z)$, it is easy to obtain the superdifferential of $U^c(x)$ given by (3.6) as follows:

$$(U^c)'(x) = \begin{cases} [U'(z), \infty), & x = L, \\ \{U'(z)\}, & L < x < z, \\ \{U'(x)\}, & x > L. \end{cases} \tag{6.6}$$

Then we can find the point $x^*(y) \in \{x | U(x) = U^c(x)\}$ solving both $\max_{x \geq L}\{U(x) - xy\}$ and $\max_{x \geq L}\{U^c(x) - xy\}$ for which 0 is in the superdifferential of $U^c(x) - xy$ given as follows:

$$x^*(y) = \begin{cases} I(y), & 0 < y < U'(z), \\ L, & y \geq U'(z). \end{cases} \tag{6.7}$$

For $U'(L+) > U'(z)$, similar to deriving (6.7), we can obtain that the maximizer is given by

$$x^*(y) = \begin{cases} I(y), & 0 < y < U'(z_1), \\ \overline{I}(y), & U'(z_1) \leq y < U'(L+), \\ L, & y \geq U'(L+). \end{cases} \tag{6.8}$$

Therefore, for each β, an optimal solution to (6.5) is

$$X^{\pi^*,\beta}(T) = x^*(\beta \xi(T)),$$

with $x^*(y)$ given by (6.7) and (6.8).

It remains to find the multiplier $\beta^* > 0$ to the following problem

$$\inf_{\beta > 0} \mathcal{L}(X^{\pi^*,\beta}(T), \beta) \tag{6.9}$$

via the complementary slackness condition (3.10).

It is easy to see that $V(\beta) = E[\xi(T) X^{\pi^*,\beta}(T)]$ is continuous and strictly decreasing in β. Furthermore, $\lim_{\beta \to 0^+} V(\beta) = +\infty$ and $\lim_{\beta \to \infty} V(\beta) = Le^{-rT}$. Thus, there exists a unique β^* satisfying (3.10), which implies that $X^{\pi^*,\beta^*}(T)$ is feasible to (2.8). Following the same arguments as used in Dong et al. (2020), we can check $X^{\pi^*,\beta^*}(T)$ is indeed the optimal terminal wealth to the problem (2.8). □

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On the multivariate extension of an increasing, form order-modular on a continuous 0-lattice

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Abstract. In the paper [17], we show a theorem, which generalizes the two methods of extensions, namely the theorem of Carathéodory and Daniell. We should be able to directly obtain an extension of modular-forms ("Valuation" under Birkhoff [2]) order-continuous increasing on 0-lattice, the two others that would flow naturally as special cases.

In this paper, we established a Multivariate case of our work intituled extension of an increasing form order-modular on a continuous 0-lattice (see, [17]).

Keywords: multivariate extension, lattice, order theory, σ-lattice, modular-form, valuation, multivariate extension of Carathéodory, extension of Daniell, measure, integral, linearity, complementary.

1. Introduction

The classic theory of integration based on extension procedures (measure or integral) is applied on sets or family of more irregular functions as those on which these concepts were originally defined.

The two most used methods are the one hand, the extension of a measure defined on a ring, which is established by Carathéodory [14]; and secondly, the extension is established by Daniell [15] as abstract integral on a vector space defined by a positive linear order-continuous form. The tools used are different views because in the first case, the Boolean structure is heavily used, and that the linearity is always true in the second case.

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2. Result

2.1 Definition and notation

A lattice as defined in [12] is a set E with an order relation satisfying, for all elements $(a_1, ..., a_k) := (a^i)_{i=1}^k$ and $(b_1, ..., b_k) := (b^i)_{i=1}^k$ of $E \subseteq \mathbb{R}^k$, there exists an upper bound and a lower bound on the $(a^i)_{i=1}^k$ and $(b^i)_{i=1}^k$.

E for providing an algebra lattice structure, denoted by $\vee$ the upper bound and $\wedge$ the lower bound defined by the following internal laws

- $(a^i)_{i=1}^k \vee (b^i)_{i=1}^k = ((a_1 \vee b_1), ..., (a_k \vee b_k)) := (\sup(a_1, b_1), ..., \sup(a_k, b_k))$.
- $(a^i)_{i=1}^k \wedge (b^i)_{i=1}^k = ((a_1 \wedge b_1), ..., (a_k \wedge b_k)) := (\inf(a_1, b_1), ..., \inf(a_k, b_k))$.

in the following, we consider $T \subseteq \mathbb{R}^k$ be a lattice with the smallest element 0_T, which will be denoted by 0-lattice [16]. We introduce the following definition.

Definition 2.1. *A sequence $((x_n^i)_{i=1}^k)_n$ is order-convergent to $(x^i)_{i=1}^k$ when there is an increasing sequence $((y_n^i)_{i=1}^k)_n$ and an decreasing sequence $((z_n^i)_{i=1}^k)_n$ such as,*

$$\vee_n^{\uparrow}(y_n^i)_{i=1}^k = (x^i)_{i=1}^k = \wedge_n^{\downarrow}(z_n^i)_{i=1}^k \quad \text{and} \quad (y_n^i)_{i=1}^k \leq (x^i)_{i=1}^k \leq (z_n^i)_{i=1}^k.$$

Where, in the following, we denote by $\uparrow$ increasing superior (respectively, by $\downarrow$ decreasing inferior).

Definition 2.2. *Let H a real function $H : T \subseteq \mathbb{R}^k \dashrightarrow \overline{\mathbb{R}}$,*

1. *We say that H is semi-continuous order inferiorly (i.e. S.C.I.) (respectively, semi-continuous order superiorly (i.e. S.C.S.), if for every sequence $((x_n^i)_{i=1}^k)_n$ converges to $(x^i)_{i=1}^k$ order, we have*

$$H((x^i)_{i=1}^k) \leq \liminf_{n \longrightarrow +\infty} H((x_n^i)_{i=1}^k) := \underline{\lim}_n H((x_n^i)_{i=1}^k)$$

(respectively, $H((x^i)_{i=1}^k) \geq \limsup_{n \longrightarrow +\infty} H((x_n^i)_{i=1}^k) := \overline{\lim}_n H((x_n^i)_{i=1}^k)$.

2. *We say that H is order-s.c.i. (respectively, order-s.c.s.) if H is verified s.c.i. (respectively, s.c.s.) for every $(x_n^i)_{i=1}^k$ in T.*

3. *We say that H is continuous order if H is order-s.c.i. and order-s.c.s..*

Remark 2.1. 1. H is increasing order-s.c.i. (respectively, order-s.c.s.), is equivalent to, $H((x_n^i)_{i=1}^k) = \sup_n^{\uparrow} H((x_n^i)_{i=1}^k)$ when $(x_n^i)_{i=1}^k = \vee_n^{\uparrow}((x_n^i)_{i=1}^k)$ (respectively, $H((x_n^i)_{i=1}^k) = \inf_n^{\downarrow} H((x_n^i)_{i=1}^k)$ when $(x_n^i)_{i=1}^k = \wedge_n^{\downarrow}((x_n^i)_{i=1}^k)$. As on [3], we introduce the following definition.

2. If H is a form modular when $H(0_T) = 0$ where $0_T := (0^i)_{i=1}^k := \underbrace{(0, ..., 0)}_{k \text{ times}}$.

Definition 2.3. *We say that H is form sub-modular (respectively, form inf-modular), if, for all $((x^i)_{i=1}^k,(y^i)_{i=1}^k)$ of T^2*

$$H((x^i)_{i=1}^k \vee (y^i)_{i=1}^k) + H((x^i)_{i=1}^k \wedge (y^i)_{i=1}^k) \leq H((x^i)_{i=1}^k) + H((y^i)_{i=1}^k)$$

(respectively, $H((x^i)_{i=1}^k \vee (y^i)_{i=1}^k) + H((x^i)_{i=1}^k \wedge (y^i)_{i=1}^k) \geq H((x^i)_{i=1}^k) + H((y^i)_{i=1}^k)$). In the following, let $T \subseteq \mathbb{R}^k$ a σ-lattice (see, [10]) (i.e. stable by countable $\wedge$ and $\vee$).

Definition 2.4. *We say that $\vee$ (respectively, $\wedge$) is order-continuous on $T \times T \subseteq \mathbb{R}^k \times \mathbb{R}^k$, if, for all increasing sequences $((x_n^i)_{i=1}^k)_n$ and $((y_n^i)_{i=1}^k)_n$ of $T \subseteq R^k$, we have*

$$(\vee_n^{\uparrow}(x_n^i)_{i=1}^k) \wedge (\vee_n^{\uparrow}(y_n^i)_{i=1}^k) = \vee_n^{\uparrow}((x_n^i)_{i=1}^k) \wedge (y_n^i)_{i=1}^k),$$

(respectively, for all decreasing sequences $((x_n^i)_{i=1}^k)_n$ and $((y_n^i)_{i=1}^k)_n$ of $T \subseteq R^k$, we have $(\wedge_n^{\downarrow}(x_n^i)_{i=1}^k) \vee (\wedge_n^{\downarrow}(y_n^i)_{i=1}^k) = \wedge_n^{\downarrow}((x_n^i)_{i=1}^k) \vee (y_n^i)_{i=1}^k)$.

In the following, we denote F, by the function defined from τ to $\mathbb{R}^+$ such that F is an increasing form order-modular and continuous, when τ is sub-lattice of $T \subseteq \mathbb{R}^k$ such that 0_τ is his smallest element.

In the following, we will gave the extension of F.

2.2 Construction of the "half σ-lattice" generated by τ

Consider the following sets

$$\tau^{\uparrow} = \{\vee_n^{\uparrow}(x_n^i)_{i=1}^k / ((x_n^i)_{i=1}^k)_n \in \tau^{\mathbb{N}}\}$$

and

$$\tau^{\downarrow} = \{\wedge_n^{\downarrow}(x_n^i)_{i=1}^k / ((x_n^i)_{i=1}^k)_n \in \tau^{\mathbb{N}}\}.$$

Remark 2.2. . Note that τ is a lattice of $T \subseteq \mathbb{R}^k$, then all $\vee$-countable (respectively, $\wedge$-countable) can be written as $\vee$-countable of an increasing sequence (respectively, $\wedge$-countable of a decreasing sequence). Therefore, it is clear that we have the following set

$$\tau^{\uparrow} = \{\vee_{(x^i)_{i=1}^k \in \mathbb{D}}(x^i)_{i=1}^k / \mathbb{D} \text{ is a countable part of } \tau\} \tag{2.1}$$

and

$$\tau^{\downarrow} = \{\wedge_{(x^i)_{i=1}^k \in \mathbb{D}}(x^i)_{i=1}^k / \mathbb{D} \text{ is a countable part of } \tau\}. \tag{2.2}$$

From the order continuity of $\wedge$ and $\vee$, it is easy to establish that $\tau^{\uparrow}$ (respectively, $\tau^{\downarrow}$) is a lattice and is stable by $\vee$-countable (respectively, $\wedge$-countable).

We have the following lemma.

Lemma 2.1. *Let $\tau^{\uparrow}$ and $\tau^{\downarrow}$ as defined in* (2.1) *and* (2.2) *then, we have*

$$\sup_n{}^{\uparrow} F((x_n^i)_{i=1}^k) = \sup_n{}^{\uparrow} F((y_n^i)_{i=1}^k) \quad \text{if} \quad \vee_n^{\uparrow}(x_n^i)_{i=1}^k = \vee_n^{\uparrow}(y_n^i)_{i=1}^k \in \tau$$

and

$$\inf_n{}^{\downarrow} F((x_n^i)_{i=1}^k) = \inf_n{}^{\downarrow} F((y_n^i)_{i=1}^k) \quad \textit{if} \quad \wedge_n^{\downarrow}(x_n^i)_{i=1}^k = \wedge_n^{\downarrow}(y_n^i)_{i=1}^k \in \tau.$$

So, we can define the extensions of F denoted by $F^{\uparrow}$ to $\tau^{\uparrow}$ and $F^{\downarrow}$ to $\tau^{\downarrow}$ by

$$F^{\uparrow}((x^i)_{i=1}^k) = \sup_n{}^{\uparrow} F((x_n^i)_{i=1}^k) \quad \textit{where} \quad (x^i)_{i=1}^k = \vee_n^{\uparrow}(x_n^i)_{i=1}^k \in \tau^{\uparrow}$$

and

$$F^{\downarrow}((x^i)_{i=1}^k) = \inf_n{}^{\downarrow} F((x_n^i)_{i=1}^k) \quad \textit{where} \quad (x^i)_{i=1}^k = \wedge_n^{\downarrow}(x_n^i)_{i=1}^k \in \tau^{\downarrow}.$$

Proof of Lemma 2.1. If $\vee_n^{\uparrow}(x_n^i)_{i=1}^k = \vee_n^{\uparrow}(y_n^i)_{i=1}^k$ where, $((x_n^i)_{i=1}^k, (y_n^i)_{i=1}^k)_n$ is a sequence of $(\tau^{\uparrow})^2$ then, we have $(x_n^i)_{i=1}^k \wedge (y_m^i)_{i=1}^k \leq (x_n^i)_{i=1}^k$, for all n, m integer and since F is increasing, we have $F((x_n^i)_{i=1}^k \wedge (y_m^i)_{i=1}^k) \leq F((x_n^i)_{i=1}^k)$, for all n, m integer.

On the other hand $\wedge$ is order-continuous on τ, then $\vee_n^{\uparrow}((x_n^i)_{i=1}^k \wedge (y_m^i)_{i=1}^k) = (y_m^i)_{i=1}^k \in \tau$, for all m integer and F is order - s.c.i. on τ, wish satisfies $\sup_n{}^{\uparrow} F((x_n^i)_{i=1}^k \wedge (y_m^i)_{i=1}^k) = F(\vee_n^{\uparrow}((x_n^i)_{i=1}^k \wedge (y_m^i)_{i=1}^k)) = F((y_m^i)_{i=1}^k)$, for all integer m. So, we have $\sup_n{}^{\uparrow} F((x_n^i)_{i=1}^k \wedge (y_m^i)_{i=1}^k) = F(\vee_n^{\uparrow}((x_n^i)_{i=1}^k \wedge (y_m^i)_{i=1}^k)) \leq \sup_n{}^{\uparrow} F((x_m^i)_{i=1}^k)$, for all integer m and

$$\sup_m{}^{\uparrow} F((y_m^i)_{i=1}^k) \leq \sup_n{}^{\uparrow} F((x_n^i)_{i=1}^k),$$

by swapping the role of the sequences, we deduce the sought equality.

In the same way, similarly, it follows the equality inf-decreasing,

$$(\text{i.e. } \inf_n{}^{\downarrow} F((x_n^i)_{i=1}^k) = \inf_n{}^{\downarrow} F((y_n^i)_{i=1}^k) \quad \text{if} \quad \wedge_n^{\downarrow}(x_n^i)_{i=1}^k = \wedge_n^{\downarrow}(y_n^i)_{i=1}^k).$$

It is concluded that the functions $F^{\uparrow}$ and $F^{\downarrow}$ are well defined.

The next propositions below gave the properties and extensions built.

Proposition 2.1. *Let τ is a 0-lattice and $F : \tau \subseteq T \subseteq \mathbb{R}^k \dashrightarrow \mathbb{R}^+$ is an increasing-order modular form and continues on $\tau^{\uparrow}$, then we have*

1. *$F^{\uparrow}$ is increasing on $\tau^{\uparrow}$;*
2. *$F^{\uparrow}$ is a modular form on $\tau^{\uparrow}$;*
3. *$F^{\uparrow}$ is order-s.c.i. on $\tau^{\uparrow}$.*

Proposition 2.2. *Let τ is a 0-lattice and $F : \tau \subseteq T \subseteq \mathbb{R}^k \dashrightarrow \mathbb{R}^+$ is an increasing-order modular form and continues on $\tau^{\downarrow}$, then we have*

1. *$F^{\downarrow}$ is increasing on $\tau^{\downarrow}$;*
2. *$F^{\downarrow}$ is a modular form on $\tau^{\downarrow}$;*

3. $F^{\downarrow}$ is order-s.c.s. on $\tau^{\downarrow}$.

Lemma 2.2. *Let $((a_n^i)_{i=1}^k)_n$ and $((b_n^i)_k^{i=1})_n$ are real positive increasing sequences (respectively, real positive decreasing sequences) of $\mathbb{R}^k$ then, we have*

$$\sup_n{}^{\uparrow}((a_n^i)_{i=1}^k + (b_n^i)_{i=1}^k) = \sup_n{}^{\uparrow}(a_n^i)_{i=1}^k + \sup_n{}^{\uparrow}(b_n^i)_k^{i=1}$$

(*respectively,* $\inf_n{}^{\downarrow}((a_n^i)_{i=1}^k + (b_n^i)_{i=1}^k) = \inf_n{}^{\downarrow}(a_n^i)_{i=1}^k + \inf_n{}^{\downarrow}(b_n^i)_k^{i=1}$).

Proof of Proposition 2.1. 1. Let $(x^i)_{i=1}^k = \vee_n^{\uparrow}(x_n^i)_{i=1}^k \leq \vee_n^{\uparrow}(y_n^i)_{i=1}^k = (y^i)_{i=1}^k$, where $((x_n^i)_{i=1}^k, (x_n^i)_{i=1}^k)_n$ is a sequence of $\tau^2 \subseteq T \times T \subset \mathbb{R}^k \times \mathbb{R}^k$ so, we have $(x^i)_{i=1}^k = (x^i)_{i=1}^k \wedge (y^i)_{i=1}^k = \vee_n^{\uparrow}((x_n^i)_{i=1}^k \wedge (y_n^i)_{i=1}^k) \in \tau^{\uparrow}$, by Lemma 2.1, we have

$$\begin{aligned} F^{\uparrow}((x^i)_{i=1}^k) = F^{\uparrow}((x^i)_{i=1}^k \wedge (y^i)_{i=1}^k) &= F^{\uparrow}(\vee_n^{\uparrow}((x_n^i)_{i=1}^k \wedge (y_n^i)_{i=1}^k)) \\ &:= \sup_n{}^{\uparrow} F((x^i)_{i=1}^k \wedge (y^i)_{i=1}^k) \\ &\leq \sup_n{}^{\uparrow} F((y^i)_{i=1}^k) := F^{\uparrow}(\vee_n^{\uparrow}(y_n^i)_{i=1}^k) \\ &= F^{\uparrow}((y^i)_{i=1}^k). \end{aligned}$$

then, we deducted that $F^{\uparrow}$ is increasing on $\tau^{\uparrow}$.

2. Let $(x^i)_{i=1}^k = \vee_n^{\uparrow}(x_n^i)_{i=1}^k$ and $\vee_n^{\uparrow}(y_n^i)_{i=1}^k = (y^i)_{i=1}^k$ are elements of $\tau^{\uparrow}$ since F is a modular-form on τ, then, for all integer n, we have

$$F(((x^i)_{i=1}^k) \vee ((y^i)_{i=1}^k)) + F((x^i)_{i=1}^k) \wedge (y^i)_{i=1}^k) = F((x^i)_{i=1}^k) + F((y^i)_{i=1}^k).$$

Moreover, F is an increasing function of $\tau \subseteq T$, and let the following real positive increasing sequences,

$$(F^{\uparrow}(x_n^i)_{i=1}^k)_n, (F^{\uparrow}(y_n^i)_{i=1}^k)_n, (F^{\uparrow}((x_n^i)_{i=1}^k) \wedge (y_n^i)_{i=1}^k)_n, \quad (F^{\uparrow}((x_n^i)_{i=1}^k) \vee (y_n^i)_{i=1}^k))_n$$

by combining the following equality's,

$$\begin{aligned} F^{\uparrow}((x^i)_{i=1}^k) \vee (y^i)_{i=1}^k) &= F^{\uparrow}(\vee_n^{\uparrow}((x_n^i)_{i=1}^k \vee (y_n^i)_{i=1}^k)) \\ &= \sup_n{}^{\uparrow} F((x_n^i)_{i=1}^k \vee (y_n^i)_{i=1}^k)) \end{aligned}$$

and

$$\begin{aligned} F^{\uparrow}((x^i)_{i=1}^k) \wedge (y^i)_{i=1}^k) &= F^{\uparrow}(\vee_n^{\uparrow}((x_n^i)_{i=1}^k \wedge (y_n^i)_{i=1}^k)) \\ &= \sup_n{}^{\uparrow} F((x_n^i)_{i=1}^k \wedge (y_n^i)_{i=1}^k)) \end{aligned}$$

and Lemma 2.2, it's obviously verified that $F^{\uparrow}$ is a modular-form of $\tau^{\uparrow}$.

3. Let $(x^i)_{i=1}^k = \vee_n^{\uparrow}(x_n^i)_{i=1}^k$ with $((x_n^i)_{i=1}^k)_n$ is a sequence of $\tau^{\uparrow}$, for all integer n, we have $(x_n^i)_{i=1}^k = \vee_s^{\uparrow}(x_n^i)_{i=1}^k)^s$ with $((x_n^i)_{i=1}^k)^s \in \tau$.

Let the following sequence defined by, $(y_m^i)_{i=1}^k = \bigvee_{\substack{s\leq m\\ n\leq m}} ((x_n^i)_{i=1}^k)^s \in \tau$, for all integer m, if $((y_m^i)_{i=1}^k)_m$ is an increasing sequence to $(x^i)_{i=1}^k$, we deducted that $\vee_m^{\uparrow}(y_m^i)_{i=1}^k \leq (x_n^i)_{i=1}^k$.

Conversely, for all integer n, s such that

$$(x_n^i)_{i=1}^k \leq (y^i_{\max(m,s)})_{i=1}^k \leq \vee_m^{\uparrow}(y_m^i)_{i=1}^k, \quad \text{for all integer } n$$

with $(y_m^i)_{i=1}^k \in \tau$, it follows that $(x^i)_{i=1}^k = \vee_m^{\uparrow}(y_m^i)_{i=1}^k$.

By Lemma 2.1, we have $F((x^i)_{i=1}^k) = \sup_m{}^{\uparrow} F((y_m^i)_{i=1}^k)$.

On the other hand, the sequence $((x^i)_{i=1}^k)^s)_s$ is increasing to $(x^i)_{i=1}^k$, for sufficiently large s, therefore we have for all integer n, such that $n \leq m$,

$$((x^i)_{i=1}^k)^s \leq (x^i)_{i=1}^k \leq (x_m^i)_{i=1}^k, \text{ for all integer } m.$$

Consequently, we have $(y_m^i)_{i=1}^k \leq (x_m^i)_{i=1}^k$, for all integer m.

Using the increasing of F and the increasing of $F^{\uparrow}$, we obtains that

$$\sup_n{}^{\uparrow} F^{\uparrow}((x_n^i)_{i=1}^k) \leq F^{\uparrow}((x^i)_{i=1}^k) = \sup_n{}^{\uparrow} F((y_n^i)_{i=1}^k) \leq \sup_n{}^{\uparrow} F^{\uparrow}((x_n^i)_{i=1}^k).$$

On the other hand, similarly, we show the Proposition 2.2. For all $(x^i)_{i=1}^k \in T$, we consider the following sets,

$$\tau^{\uparrow}_{(x^i)_{i=1}^k} = \{(t^i)_{i=1}^k \in \tau^{\uparrow}/(t^i)_{i=1}^k \geq (x^i)_{i=1}^k\} \tag{2.3}$$

and

$$\tau^{\downarrow}_{(x^i)_{i=1}^k} = \{(t^i)_{i=1}^k \in \tau^{\downarrow}/(t^i)_{i=1}^k \leq (x^i)_{i=1}^k\}. \tag{2.4}$$

From stability properties in $\vee$-countable of $\tau^{\uparrow}$ and $\wedge$-countable of $\tau^{\downarrow}$, we can easily deduce the following Propositions.

Proposition 2.3. *Let $T \subseteq R^k$ be a 0-lattice; $(x^i)_{i=1}^k, (y^i)_{i=1}^k$ in T, $((x_n^i)_{i=1}^k)_n$ is an increasing sequence of $T, \tau^{\uparrow}_{(x^i)_{i=1}^k}$ and $\tau^{\uparrow}_{(y^i)_{i=1}^k}$, are defined by (2.3), we have*

1. *if $(x^i)_{i=1}^k \leq (y^i)_{i=1}^k$, then $\tau^{\uparrow}_{(y^i)_{i=1}^k} \subseteq \tau^{\uparrow}_{(x^i)_{i=1}^k}$;*
2. *$\tau^{\uparrow}_{(x^i)_{i=1}^k}$ is a stable lattice by $\vee$-countable;*
3. *$\tau^{\uparrow}_{\vee_n^{\uparrow}(x_n^i)_{i=1}^k} = \bigcap_n^{\downarrow}(\tau^{\uparrow}_{(x_n^i)_{i=1}^k})$.*

Proposition 2.4. *Let $T \subseteq R^k$ be a 0-lattice; $(x^i)_{i=1}^k, (y^i)_{i=1}^k$ in T, $((x_n^i)_{i=1}^k)_n$ is an decreasing sequence of $T, \tau^{\downarrow}_{(x^i)_{i=1}^k}$ and $\tau^{\downarrow}_{(y^i)_{i=1}^k}$, are defined by (2.4), we have*

1. *if $(x^i)_{i=1}^k \leq (y^i)_{i=1}^k$, then $\tau^{\downarrow}_{(x^i)_{i=1}^k} \subseteq \tau^{\downarrow}_{(y^i)_{i=1}^k}$;*

2. $\tau^{\downarrow}_{(x^i)_{i=1}^k}$ *is a stable lattice by* $\wedge$*-countable;*

3. $\tau^{\downarrow}_{\vee_n^{\downarrow}(x_n^i)_{i=1}^k} = \bigcup_n^{\uparrow}(\tau^{\downarrow}_{(x_n^i)_{i=1}^k})$.

Lemma 2.3. *Let* $T \subseteq \mathbb{R}^k$ *be a 0-lattice and* $\tau^{\uparrow}_{((x_n^i)_{i=1}^k)}$ *is defined by* (2.3) *for* $(x^i)_{i=1}^k$ *in* T *then, we have*

$$\mathcal{L}_{(x^i)_{i=1}^k} = \{(x^i)_{i=1}^k \in T / \tau^{\uparrow}_{(x^i)_{i=1}^k} \neq \emptyset\} \text{ is a } \sigma - \text{lattice and } \tau \subseteq \mathcal{L}_{(x^i)_{i=1}^k}.$$

Proof of Lemma 2.3. If for all $n \in \mathbb{N}$, $(t^i)_{i=1}^k \in \tau^{\uparrow}_{(x^i)_{i=1}^k}$, then, $\vee_n(t_n^i)_{i=1}^k \in \tau^{\uparrow}_{\vee_n(x_n^i)_{i=1}^k}$ so, we have $\tau^{\uparrow}_{\vee_n(x_n^i)_{i=1}^k} \subseteq \tau^{\uparrow}_{\wedge_n(x_n^i)_{i=1}^k}$.

Remark 2.3. In the following, consider the σ-lattice $\mathcal{L}_{(x^i)_{i=1}^k}$ in place of $T \subseteq \mathbb{R}^k$ and assume that for every $(x^i)_{i=1}^k \in T, \tau^{\uparrow}_{(x^i)_{i=1}^k} \neq \emptyset$.

If $(x^i)_{i=1}^k$ of $T \subseteq \mathbb{R}^k$, let F^* and F_* the extensions of F defined on T by,

$$F^*((x^i)_{i=1}^k) := \inf_{(t^i)_{i=1}^k \in \tau^{\uparrow}_{(x^i)_{i=1}^k}} F^{\uparrow}((t^i)_{i=1}^k) \tag{2.5}$$

and

$$F_*((x^i)_{i=1}^k) := \sup_{(t^i)_{i=1}^k \in \tau^{\downarrow}_{(x^i)_{i=1}^k}} F^{\downarrow}((t^i)_{i=1}^k). \tag{2.6}$$

Note that the following inequality $F^*((x^i)_{i=1}^k) \geq F_*((x^i)_{i=1}^k)$ is always verified and the inequality's (2.5) and (2.6) are always true and we have the following propositions.

Proposition 2.5. *Let* F^* *defined as* (2.5) *then, we have*

1. F^* *is an increasing function on* T*;*

2. F^* *is a form under-modular on* T*;*

3. F^* *is order-s.c.i. on* T.

Proposition 2.6. *Let* F_* *defined as* (2.6) *and* $(x^i)_{i=1}^k = \wedge_n^{\downarrow}(x_n^i)_{i=1}^k$ *such that* $(F_*((x_n^i)_{i=1}^k))_n$ *is finite for* n *so large, then, we have*

1. F_* *is an increasing function on* T*;*

2. F_* *is a form ever-modular on* T*;*

3. *If* $(x^i)_{i=1}^k = \wedge_n^{\downarrow}(x_n^i)_{i=1}^k$ *with* $F_*((x_n^i)_{i=1}^k)$ *is finite for all large integer* n, $F_*((x^i)_{i=1}^k) = \inf_n^{\downarrow} F_*((x_n^i)_{i=1}^k)$.

Proof of Proposition 2.5. 1. Let $(x^i)_{i=1}^k$ and $(y^i)_{i=1}^k$ be two elements of $T \subseteq \mathbb{R}^k$, such that $(x^i)_{i=1}^k \leq (y^i)_{i=1}^k$; since $\tau^{\uparrow}_{(y^i)_{i=1}^k}$ is included in $\tau^{\uparrow}_{(x^i)_{i=1}^k}$ then, we have

$$F^*((x^i)_{i=1}^k) := \inf_{(t^i)_{i=1}^k \in \tau^{\uparrow}_{(x^i)_{i=1}^k}} F^{\uparrow}((t^i)_{i=1}^k) \leq \inf_{(t^i)_{i=1}^k \in \tau^{\uparrow}_{(y^i)_{i=1}^k}} F^{\uparrow}((t^i)_{i=1}^k) := F^*((y^i)_{i=1}^k)$$

so, F^* is increasing on T.

2. Let $(x^i)_{i=1}^k$ and $(y^i)_{i=1}^k$ are two elements of $T \subseteq \mathbb{R}^k$, we will show that,

$$F^*((x^i)_{i=1}^k) + F^*((y^i)_{i=1}^k) \geq F^*((x^i)_{i=1}^k) \vee (y^i)_{i=1}^k) + F^*((x^i)_{i=1}^k \wedge (y^i)_{i=1}^k).$$

If $F^*((x^i)_{i=1}^k) + F^*((y^i)_{i=1}^k) = +\infty$, we have the inequality.

So, suppose that $F^*((x^i)_{i=1}^k) + F^*((y^i)_{i=1}^k)$ is a finite real number, then for any positive real number ε, there are $((t^i)_{i=1}^k), (s^i)_{i=1}^k) \in (\tau^{\uparrow}_{(x^i)_{i=1}^k})^2$ such that $F^*((x^i)_{i=1}^k) + F^*((y^i)_{i=1}^k) + \varepsilon \geq F^{\uparrow}((t^i)_{i=1}^k) + F^{\uparrow}((s^i)_{i=1}^k)$.

On the other hand $F^{\uparrow}$ is a modular-form on $\tau^{\uparrow}$, then we have

$$F^{\uparrow}((t^i)_{i=1}^k) + F^{\uparrow}((s^i)_{i=1}^k) = F^{\uparrow}((t^i)_{i=1}^k \vee (s^i)_{i=1}^k) + F^{\uparrow}((t^i)_{i=1}^k \wedge (s^i)_{i=1}^k)$$

for $(t^i)_{i=1}^k \vee (s^i)_{i=1}^k$ is an element of $\tau^{\uparrow}_{(x^i)_{i=1}^k \vee (y^i)_{i=1}^k}$ and $(t^i)_{i=1}^k \wedge (s^i)_{i=1}^k$ is an element of $\tau^{\uparrow}_{(x^i)_{i=1}^k \wedge (y^i)_{i=1}^k}$ therefore, we have

$$\begin{aligned} F^{\uparrow}((t^i)_{i=1}^k) + F^{\uparrow}((s^i)_{i=1}^k) &\geq \inf_{(t^i)_{i=1}^k \in \tau^{\uparrow}_{(x^i)_{i=1}^k \vee (y^i)_{i=1}^k}} F^{\uparrow}((t^i)_{i=1}^k) \\ &+ \inf_{(t^i)_{i=1}^k \in \tau^{\uparrow}_{(x^i)_{i=1}^k \wedge (y^i)_{i=1}^k}} F^{\uparrow}((t^i)_{i=1}^k) \\ &\geq F^*((x^i)_{i=1}^k \vee (y^i)_{i=1}^k) + F^*((x^i)_{i=1}^k \wedge (y^i)_{i=1}^k). \end{aligned}$$

On the other hand, we have

$$F^*((x^i)_{i=1}^k) + F^*((y^i)_{i=1}^k) + \varepsilon \geq F^*((x^i)_{i=1}^k \vee (y^i)_{i=1}^k) + F^*((x^i)_{i=1}^k \wedge (y^i)_{i=1}^k).$$

for ε is arbitrary. We deducted that F^* is an over-modular form on T.

3. Now, show that F^* is order-s.c.i. on $T \subseteq \mathbb{R}^k$.

Since $(x^i)_{i=1}^k = \vee_n^{\uparrow}(x_n^i)_{i=1}^k$ and F^* is increasing, then we have the following inequality, $F^*(\vee_n^{\uparrow}(x_n^i)_{i=1}^k) \geq \sup_n{}^{\uparrow} F^*((x_n^i)_{i=1}^k)$. For the other inequality, we can assume without restriction that $\sup_n^{\uparrow} F^*((x_n^i)_{i=1}^k)$ is finite. Given a strictly positive real ε and an integer n, then by definition $F^*((x_n^i)_{i=1}^k$, and (2.5) there exist a $(t_n^i)_{i=1}^k$ of $\tau^{\uparrow}(x_n^i)_{i=1}^k$ such that

$$F^{\uparrow}((t_n^i)_{i=1}^k) \leq F^*((x_n^i)_{i=1}^k) + \frac{\varepsilon}{2^{(n+1)}}.$$

Let $(e_n^i)_{i=1}^k = \vee_{s=0}^{s=n}(t_s^i)_{i=1}^k$, it is clear that $(e_n^i)_{i=1}^k$ is still an element of $\tau^{\uparrow}_{(x_n^i)_{i=1}^k}$ and since $(x^i)_{i=1}^k = \vee_n^{\uparrow}(x_n^i)_{i=1}^k \leq \vee_n^{\uparrow}(e_n^i)_{i=1}^k$, we have $\vee_n^{\uparrow}(e_n^i)_{i=1}^k$ is a $\tau^{\uparrow}(x^i)_{i=1}^k$ element which is stable $\vee$-countable.

Show by recurrent we have the following inequality

$$(I_n) \qquad F^{\uparrow}((e_n^i)_{i=1}^k) \leq F^*((x_n^i)_{i=1}^k) + \varepsilon\left(1 - \frac{1}{2^{(n+1)}}\right).$$

For $n = 0$, $e_0 = t_0$ then inequality (I_0) is satisfied.

Suppose (I_n) is satisfied for all integer n, first note that

$$(x_n^i)_{i=1}^k = (x_n^i)_{i=1}^k \wedge (x_{n+1}^i)_{i=1}^k \leq (e_n^i)_{i=1}^k \wedge (t_{n+1}^i)_{i=1}^k.$$

Since F^* is increasing, we have

$$F^*((x_n^i)_{i=1}^k) \leq F^{\uparrow}((e_n^i)_{i=1}^k \wedge (t_{n+1}^i)_{i=1}^k). \tag{2.7}$$

As $F^{\uparrow}$ is a modular-form, we have

$$F^{\uparrow}((e_n^i)_{i=1}^k \wedge (t_{n+1}^i)_{i=1}^k) + F^{\uparrow}((e_{n+1}^i)_{i=1}^k) = F^{\uparrow}((e_n^i)_{i=1}^k) + F^{\uparrow}((t_{n+1}^i)_{i=1}^k) \tag{2.8}$$

or if we define $(t_{n+1}^i)_{i=1}^k$ by,

$$F^{\uparrow}((t_{n+1}^i)_{i=1}^k) \leq F^*((x_{n+1}^i)_{i=1}^k) + \frac{\varepsilon}{2^{(n+2)}}.$$

Therefore, according to the inequality (2.3) and (2.4) and the recurrent's hypothesis, we obtain,

$$\begin{aligned} F^{\uparrow}((e_{n+1}^i)_{i=1}^k) &= F^{\uparrow}((e_n^i)_{i=1}^k + F^{\uparrow}((t_{n+1}^i)_{i=1}^k - F^{\uparrow}((e_n^i)_{i=1}^k \wedge (t_{n+1}^i)_{i=1}^k) \\ &\leq F^*((x_n^i)_{i=1}^k) + F^*((x_{n+1}^i)_{i=1}^k - F^*((x_n^i)_{i=1}^k) \\ &\quad + \varepsilon\left(1 - \frac{1}{2^{(n+1)}}\right) + \frac{\varepsilon}{2^{(n+2)}}. \end{aligned}$$

Consequently, we have

$$F^{\uparrow}((e_{n+1}^i)_{i=1}^k) \leq F^*((x_{n+1}^i)_{i=1}^k) + \varepsilon\left(1 - \frac{1}{2^{(n+2)}}\right).$$

So, $(I_{(n+1)})$ is still satisfied.

It follows the following inequalities

$$F^*((x_n^i)_{i=1}^k) \leq \overset{\uparrow}{\sup_n} F^*((e_n^i)_{i=1}^k) \leq \overset{\uparrow}{\sup_n} F^*((x_n^i)_{i=1}^k) + \varepsilon, \qquad \text{for all integer } n,$$

when ε is arbitrary, we concluded that F^* is order-s.c.i.

On the other hand, the Proposition 6 is shown similarly.

2.3 Extension of the set $F^* = F_*$

Now, it is assumed that any element $(x^i)_{i=1}^k$ of $T \subseteq \mathbb{R}^k$, we have $\tau^{\uparrow}_{(x^i)_{i=1}^k} \neq \emptyset$ (see Lemma 3) let the following sets

$$\tau^F = \{(x^i)_{i=1}^k \in T / F^*((x^i)_{i=1}^k) = F_*((x^i)_{i=1}^k)\} \tag{2.9}$$

and

$$\tau^F_{\mathbb{R}} = \{(x^i)_{i=1}^k \in T / F^*((x^i)_{i=1}^k) = F_*((x^i)_{i=1}^k) < +\infty\}. \tag{2.10}$$

On the set (2.9), we have $F^* = F_*$. So, we denote by $\overline{F}$ the common value on τ^F.

Proposition 2.7. *Let τ is a sub-lattice and 0-lattice $F : \tau \dashrightarrow \mathbb{R}^+$ is an increasing-order continues modular-form, τ^F as (2.9) and $\tau^F_{\mathbb{R}}$ as (2.10) then, we have*

1. *τ^F is a stable set in $\vee$-countable such that, for all $((x^i)_{i=1}^k,(y^i)_{i=1}^k) \in (\tau^F)^2$,*

$$\begin{aligned} \overline{F}((x^i)_{i=1}^k) + \overline{F}((y^i)_{i=1}^k) &= \overline{F}((x^i)_{i=1}^k \vee (y^i)_{i=1}^k) + \overline{F}((x^i)_{i=1}^k \wedge (y^i)_{i=1}^k) \\ &= \overline{F}((x^i)_{i=1}^k \vee (y^i)_{i=1}^k) + F_*((x^i)_{i=1}^k \wedge (y^i)_{i=1}^k). \end{aligned}$$

2. *$\tau^F_{\mathbb{R}}$ is a stable trellis $\wedge$-countable containing τ, and $\overline{F}$ is an increasing-order continuous modular-form.*

Proof of Proposition 2.7 1. By Proposition 2.5 and Proposition 2.6, we have for all $((x^i)_{i=1}^k, (y^i)_{i=1}^k) \in T^2$,

$$\begin{aligned} F_*((x^i)_{i=1}^k) + F_*((y^i)_{i=1}^k) &\leq F_*((x^i)_{i=1}^k \vee (y^i)_{i=1}^k) + F_*((x^i)_{i=1}^k \wedge (y^i)_{i=1}^k) \\ &\leq F^*((x^i)_{i=1}^k \vee (y^i)_{i=1}^k) + F^*((x^i)_{i=1}^k \wedge (y^i)_{i=1}^k) \\ &\leq F^*((x^i)_{i=1}^k) + F^*((y^i)_{i=1}^k), \end{aligned}$$

when $((x^i)_{i=1}^k, (y^i)_{i=1}^k) \in (\tau^F)^2$ so, we have

$$F^*((x^i)_{i=1}^k \vee (y^i)_{i=1}^k) \geq F_*((x^i)_{i=1}^k \wedge (y^i)_{i=1}^k) \geq \sup(F_*((x^i)_{i=1}^k, F_*((y^i)_{i=1}^k).$$

On the other hand if $F_*((x^i)_{i=1}^k)$ or $F_*((y^i)_{i=1}^k)$ is infinite, then $(x^i)_{i=1}^k \vee (y^i)_{i=1}^k \in \tau^F$.

If $((x^i)_{i=1}^k, (y^i)_{i=1}^k) \in (\tau^F_{\mathbb{R}})^2$, then $x \vee y$ and $(x^i)_{i=1}^k \wedge (y^i)_{i=1}^k \in \tau^F_{\mathbb{R}}$. So, we have the following Lemma.

Lemma 2.4. *Let $(a^i)_{i=1}^k, (b^i)_{i=1}^k, (c^i)_{i=1}^k$ and $(d^i)_{i=1}^k$ the real numbers of $\mathbb{R}^k$ such that $(a^i)_{i=1}^k \geq (b^i)_{i=1}^k$ and $(c^i)_{i=1}^k \geq (d^i)_{i=1}^k$ when $(a^i)_{i=1}^k + (c^i)_{i=1}^k = (b^i)_{i=1}^k + (d^i)_{i=1}^k$ then $(a^i)_{i=1}^k = (b^i)_{i=1}^k$ and $(c^i)_{i=1}^k = (d^i)_{i=1}^k$.*

2. b) According to Proposition 2.5 and Proposition 2.6, F^* is order-s.c.i. and F_* increasing, if $((x_n^i)_{i=1}^k)_n$ is an increasing sequence of elements of τ^F, it is obviously that

$$F^*(\vee_n^{\uparrow}(x_n^i)_{i=1}^k) = \sup_n{}^{\uparrow} F^*((x_n^i)_{i=1}^k) = \sup_n{}^{\uparrow} F_*((x_n^i)_{i=1}^k) \leq F_*((x_n^i)_{i=1}^k),$$

and so it follows that $\vee_n^{\uparrow}(x_n^i)_{i=1}^k$ is still an element of τ^F.

Finally, F_* is order-s.c.s. in its domain, F^* increasing allows us the same way to verify that $\tau_{\mathbb{R}}^F$ was stable $\wedge$-countable, we have

$$F^*(\wedge_n^{\downarrow}(x_n^i)_{i=1}^k) \leq \inf_n{}^{\downarrow} F^*((x_n^i)_{i=1}^k) = \inf_n{}^{\downarrow} F_*((x_n^i)_{i=1}^k) = F_*(\wedge_n^{\downarrow}((x_n^i)_{i=1}^k).$$

Theorem 2.1. *Let F continuous modular form order on a trellis τ, then F is uniquely extension of the σ-lattice τ generated by an increasing modular form order-s.c.i. and order–s.c.s. in its domain.*

Proof of Theorem 2.1. Let $\tau_F = \{(x^i)_{i=1}^k \in T/\forall (y^i)_{i=1}^k \in \tau, (x^i)_{i=1}^k \wedge (y^i)_{i=1}^k \in \tau_{\mathbb{R}}^F\}$.

Step 1. We will show that τ_F is contained in τ^F, let x be an element of τ_F; since $\tau^{\uparrow}_{(x^i)_{i=1}^k} \neq \emptyset$, there exists a increasing sequence $((x_n^i)_{i=1}^k)_n$ of τ elements for which we have $(x^i)_{i=1}^k = \vee_n^{\uparrow}((x^i)_{i=1}^k \wedge ((x_n^i)_{i=1}^k)$ as $(x^i)_{i=1}^k \in \tau_F$ and $(x_n^i)_{i=1}^k \in \tau, (x^i)_{i=1}^k \wedge (x_n^i)_{i=1}^k \in \tau_{\mathbb{R}}^F \subseteq \tau^F$.

On the other hand by Proposition 2.7 and because the latter is stable $\vee$-countable, we have x is in τ^F.

Step 2. It is observably verified that τ_F is a σ-lattice containing $\tau_{\mathbb{R}}^F$, on the other hand since $\tau_{\mathbb{R}}^F$ is $\wedge$-stable and contain τ, clearly τ_F containing $\tau_{\mathbb{R}}^F$.

Let $((x_n^i)_{i=1}^k$ a sequence of τ_F and $(y_n^i)_{i=1}^k \in \tau$, then

$$\wedge_n((x_n^i)_{i=1}^k) \wedge (y^i)_{i=1}^k = \wedge_n((x_n^i)_{i=1}^k \wedge (y^i)_{i=1}^k) \in \tau_{\mathbb{R}}^F.$$

On the other hand, since, $\tau_{\mathbb{R}}^F$ is stable by $\wedge$-countable, and

$$\vee_n((x_n^i)_{i=1}^k) \wedge (y^i)_{i=1}^k = \vee_n((x_n^i)_{i=1}^k \wedge (y^i)_{i=1}^k) \in \tau_{\mathbb{R}}^F$$

then, we have τ^F is stable by $\vee$-countable, and

$$F^*((x_n^i)_{i=1}^k \wedge (y^i)_{i=1}^k) \leq F^*((y^i)_{i=1}^k) < +\infty.$$

Step 3. Based on the above, for every $((x^i)_{i=1}^k, (y^i)_{i=1}^k) \in (\tau^F)^2$ we have

$$F^*((x^i)_{i=1}^k \wedge (y^i)_{i=1}^k) = F_*((x^i)_{i=1}^k \wedge (y^i)_{i=1}^k).$$

When $(x^i)_{i=1}^k \wedge (y^i)_{i=1}^k \in \tau_F$, then we deducted from Proposition 2.7, that F is a modular-form increasing order-s.c.i. and order - s.c.s. on τ_F in its domain.

As τ_F is an σ-lattice containing τ, it contains the σ-lattice generated by τ and the restriction of F to the latter answers the problem.

In the next, we will prove the uniqueness of the extension.

Let G be a regular extension of F, G order being continuous, it coincides with $F^{\uparrow}$ on $\tau^{\uparrow}$ and $F^{\downarrow}$, on $\tau^{\downarrow}$ therefore if $(x^i)_{i=1}^k \in T, G$ implies the increasing of the following inequalities, $\forall (t^i)_{i=1}^k \in \tau^{\uparrow}_{(x^i)_{i=1}^k}$, $\forall (s^i)_{i=1}^k \in \tau^{\downarrow}_{(x^i)_{i=1}^k}$, we have

$$F^{\downarrow}((s^i)_{i=1}^k) \leq G((x^i)_{i=1}^k) \leq F^{\uparrow}((t^i)_{i=1}^k).$$

By definition of F^* and F_* , we have

$$F^*((x^i)_{i=1}^k) \leq G((x^i)_{i=1}^k) \leq F_*((x^i)_{i=1}^k).$$

The σ-lattice generated by τ being contained in the set of coincidence F^* and F_*, it is concluded that G and $\overline{F}$ are equal.

Remark 2.4. For $k = 1$, we have the Theorem of Carathéodory and Theorem of Daniell (see, [17]).

2.4 Multivariate case of the extension of Carathéodory and Daniell

In the following, we say that the σ-lattice generated by $(\mathcal{A}^i)_{i=1}^k$ is a monotone part if, for a increasing sequence $((\mathcal{A}_n^i)_{i=1}^k)_n$ of σ-lattice we have $\bigcup_n (\mathcal{A}_n^i)_{i=1}^k \in \sigma$-lattice; and for a decreasing sequence $((\mathcal{A}_n^i)_{i=1}^k)_n$ of σ-lattice we have $\bigcap_n (\mathcal{A}_n^i)_{i=1}^k$ $\in \sigma$-lattice.

From the constructions we have done before in this paper, we can deduce the multiple versions of the Theorem of Carathéodory and the Theorem of Daniell that we can call the multiple variate theorem of Carathéodory and the multiple variate theorem of Daniell.

2.4.1 Extension of Carathéodory

Corollary 2.1 (Caratheodory's theorem)**.** *Let $(\mathcal{A}^i)_{i=1}^k \subseteq \mathbb{R}^k$ a ring portions of Ω_k and $\mu_k : (\mathcal{A}^i)_{i=1}^k \dashrightarrow \mathbb{R}^+$ a measure, then there exit only one measure defined on the σ-ring generated by $(\mathcal{A}^i)_{i=1}^k$ and extending μ_k.*

Proof of Corollary 2.1. Denoted by $\mathcal{P}(\Omega_k)$ the all parts of Ω_k. Consider the order induced by the inclusion, let de following set,

$$T = \{(A^i)_{i=1}^k \in \mathcal{P}(\Omega)/(A^i)_{i=1}^k \subseteq \bigcup_n (A_n^i)_{i=1}^k \quad \text{with} \quad (A_n^i)_{i=1}^k \in (\mathcal{A}^i)_{i=1}^k\}.$$

T is a σ-lattice sequentially distributive with the smallest element is $\emptyset$, $(\mathcal{A}^i)_{i=1}^k$ is a under-lattice of T and μ is an increasing modular-form continuous order on $(\mathcal{A}^i)_{i=1}^k$.

By the Theorem 2.1, it has a only extension $\overline{\mu}_k$ on the σ-lattice generated by $(\mathcal{A}^i)_{i=1}^k$. $(\mathcal{A}^i)_{i=1}^k$ is an increasing modular form-order-s.c.i. and order-s.c.s. in its domain.

The σ-lattice generated by $(\mathcal{A}^i)_{i=1}^k$ is a monotone part (stable by increasing and decreasing) by intersection countable such that the ring $(\mathcal{A}^i)_{i=1}^k$ is included in the σ-lattice, then $((\mathcal{A}^i)_{i=1}^k)^\sigma$ is included in the σ-lattice.

We concluded that a moduler-form increasing order-s.c.i. on $((\mathcal{A}^i)_{i=1}^k)^\sigma$ is a positive measure.

2.4.2 Extension of Daniell

In this part, we suppose $T \subseteq \mathbb{R}^k$ is group (respectively, vectoriel space) of Riesz (See [8]) and τ is the set positive elements of under-group (respectively, under-vectoriel space) of Riesz of T.

Theorem 2.2. *Let $F : \tau \dashrightarrow \mathbb{R}^+$ is increasing, additive continuous order, then,*

1. *F has a only extension to σ-lattice of Riesz generated by τ, and this extention is increasing by additive, order-s.c.i. and order-s.c.s in its domain,*
2. *Under the hypothesis of* (1) *and F is positive and homogenous, then we have* (1) *and the extension is positive and homogenous in its domain.*

Remark 2.5.

1. The conditions $\lim\limits_{n\to+\infty} F((x_n^i)_{i=1}^k) = 0$ when $\wedge_n (x_n^i)_{i=1}^k = 0$ (see [11]) maked in extension of Daniell is obviously equivalent to the order continuity.
2. It's easy to see that F is a modular-form, so if $(x, y) \in \tau^2$, then,
$$(x^i)_{i=1}^k \vee (y^i)_{i=1}^k + (x^i)_{i=1}^k \wedge (y^i)_{i=1}^k = (x^i)_{i=1}^k + (y^i)_{i=1}^k$$
and F is additive.
3. The set $\mathcal{L}(x^i)_{i=1}^k = \{(x^i)_{i=1}^k \in T / \tau^\uparrow_{(x^i)_{i=1}^k} \neq \emptyset\}$ is a under-group (respectively, under-space vectoriel) of Riesz T, since $\vee_n^\uparrow (x_n^i)_{i=1}^k + \vee_n^\uparrow (y_n^i)_{i=1}^k = \vee_n^\uparrow ((x_n^i)_{i=1}^k + (y_n^i)_{i=1}^k)$ and $\alpha . \vee_n^\uparrow (x_n^i)_{i=1}^k = \vee_n^\uparrow \alpha . (x_n^i)_{i=1}^k$, for all $\alpha \geq 0$. We can assume that $\tau^\uparrow_{(x_n^i)_{i=1}^k} \neq \emptyset$ for all $(x^i)_{i=1}^k \in T$.
4. If the extension exists, it's a modular-form and by show in Theorem 2.1, this extension is exactly $\overline{F}$ on the σ-lattice generated by τ, the extension is stable by additive. In the case of the space-vectorial of the Riesz, it's obviously that the extensions $F^\uparrow, F^\downarrow, F^*$ and F_* are positively homogeneous, for this, we show the properties of F^* and F_*.

Lemma 2.5. *Let $F^\uparrow$, $F^\downarrow$, $\tau^\uparrow$ and $\tau^\downarrow$ as Lemma 2.1. and* (2.1), (2.2) *then, we have*

1. *$F^\uparrow$ is additif on $\tau^\uparrow$ and $F^\downarrow$ is additive on $\tau^\downarrow$;*
2. *For all $((x^i)_{i=1}, (y^i)_{i=1}^k) \in (\tau^F)^2$, we have*
$$\tau^\uparrow_{(x^i)_{i=1}^k} + \tau^\uparrow_{(y^i)_{i=1}^k} \subseteq \tau^\uparrow_{((x^i)_{i=1}^k + (y^i)_{i=1}^k)}, \quad \tau^\downarrow_{(x^i)_{i=1}^k} + \tau^\downarrow_{(y^i)_{i=1}^k} \subseteq \tau^\downarrow_{((x^i)_{i=1}^k + (y^i)_{i=1}^k)}.$$

3. F^* is under-additive and F_* is over-additive.

Corollary 2.2. *τ^F is stable by additive and $\overline{F}$ is additive on τ^F.*

Proof. By the Lemma 2.5, we have for all $((x^i)_{i=1}, (y^i)_{i=1}^k) \in (\tau^F)^2$

$$\begin{aligned} F^*((x^i)_{i=1}^k + (y^i)_{i=1}^k) &\leq F^*((x^i)_{i=1}^k) + F^*((y^i)_{i=1}^k) = F_*((x^i)_{i=1}^k) + F_*((y^i)_{i=1}^k) \\ &\leq F_*((x^i)_{i=1}^k + (y^i)_{i=1}^k) \end{aligned}$$

so, τ^F is stable by additive.

If $((x^i)_{i=1}^k, (y^i)_{i=1}^k) \in (\tau^F)^2$ and $t \in \tau$, we have

$$((x^i)_{i=1}^k + (y^i)_{i=1}^k) \wedge (t^i)_{i=1}^k = ((x^i)_{i=1}^k + (t^i)_{i=1}^k) \wedge ((y^i)_{i=1}^k + (t^i)_{i=1}^k) \in \tau_{\mathbb{R}}^F.$$

τ_F is stable by additive and by the next $\overline{F}$ is additive on τ_F. □

Proof of Lemma 2.5. 1. Let $(x^i)_{i=1}^k = \vee_n^{\uparrow}(x_n^i)_{i=1}^k$ and $(y^i)_{i=1}^k = \vee_n^{\uparrow}(y_n^i)_{i=1}^k$ with $((x_n^i)_{i=1}^k, (y_n^i)_{i=1}^k)_n \in (\tau^2)^{\mathbb{N}}$; then

$$(x^i)_{i=1}^k + (y^i)_{i=1}^k = \vee_n^{\uparrow}((x_n^i)_{i=1}^k + (y_n^i)_{i=1}^k)$$

by Lemma 2.1 and Proposition 2.2, we have

$$\begin{aligned} F^{\uparrow}((x^i)_{i=1}^k + (y^i)_{i=1}^k) &= \sup_n{}^{\uparrow} F((x_n^i)_{i=1}^k + (y_n^i)_{i=1}^k) \\ &= \sup_n{}^{\uparrow} F((x_n^i)_{i=1}^k) + F((y_n^i)_{i=1}^k) \\ &= \sup_n{}^{\uparrow} F((x_n^i)_{i=1}^k) + \sup_n{}^{\uparrow} F((y_n^i)_{i=1}^k) \\ &= F^{\uparrow}((x^i)_{i=1}^k) + F^{\uparrow}((y^i)_{i=1}^k). \end{aligned}$$

The procedure is same for $F^{\downarrow}$.

2. If $(t^i)_{i=1}^k \in \tau^{\uparrow}_{(x^i)_{i=1}^k}$ and $(s^i)_{i=1}^k \in \tau^{\uparrow}_{(y^i)_{i=1}^k}$ then, $(t^i)_{i=1}^k + (s^i)_{i=1}^k) \in \tau^{\uparrow}$ and

$$((t^i)_{i=1}^k + (s^i)_{i=1}^k) \geq ((x^i)_{i=1}^k + (y^i)_{i=1}^k) \quad \text{so} \quad ((t^i)_{i=1}^k + (s^i)_{i=1}^k) \in \tau^{\uparrow}_{((x^i)_{i=1}^k + (y^i)_{i=1}^k)}.$$

The procedure is the same for the following statement,

$$\tau^{\downarrow}_{(x^i)_{i=1}^k} + \tau^{\downarrow}_{(y^i)_{i=1}^k} \subseteq \tau^{\uparrow}_{((x^i)_{i=1}^k + (y^i)_{i=1}^k)}.$$

3. By (2), we have

$$\begin{aligned}
F^*(((x^i)_{i=1}^k + (y^i)_{i=1}^k)) &= \inf_{(t^i)_{i=1}^k \in \tau^{\uparrow}_{((x^i)_{i=1}^k + (y^i)_{i=1}^k)}} F^{\uparrow}((t^i)_{i=1}^k) \\
&\leq \inf_{\substack{((u^i)_{i=1}^k \in \tau^{\uparrow}_{(x^i)_{i=1}^k}, \\ (v^i)_{i=1}^k \in \tau^{\uparrow}_{(y^i)_{i=1}^k})}} F^{\uparrow}((u^i)_{i=1}^k + (v^i)_{i=1}^k) \\
&\leq \inf_{\substack{((u^i)_{i=1}^k \in \tau^{\uparrow}_{(x^i)_{i=1}^k}, \\ (v^i)_{i=1}^k \in \tau^{\uparrow}_{(y^i)_{i=1}^k})}} (F^{\uparrow}((u^i)_{i=1}^k) + F^{\uparrow}((v^i)_{i=1}^k)) \\
&\leq \inf_{((u^i)_{i=1}^k \in \tau^{\uparrow}_{(x^i)_{i=1}^k})} F^{\uparrow}((u^i)_{i=1}^k) + \inf_{((v^i)_{i=1}^k \in \tau^{\uparrow}_{(y^i)_{i=1}^k})} F^{\uparrow}((v^i)_{i=1}^k) \\
&= F^*((x^i)_{i=1}^k + F^*((y^i)_{i=1}^k).
\end{aligned}$$

By Similar proof, we have F_*.

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Computation of edge-based topological indices for zero divisor graphs of commutative rings

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Abstract. Graph constructing algorithms and their mapping with real life problems for scientific data analysis are getting popular. Standard algorithms construct the graph in a way that it can deal with all possibilities of input data to calculate the desired output. This article gives an algorithmic computational model for edge based eccentric topological indices by constructing zero divisor graph containing finite rings as $\mathbb{Z}_{p_1p_2} \times \mathbb{Z}_q$ and $\mathbb{Z}_{p^2} \times \mathbb{Z}_q$, where p, p_1, p_2 and q are primes numbers. At first algorithm classify vertices with common eccentricity and then computes first Zagreb, third Zagreb, geometric-arithmetic, atom-bond connectivity and harmonic index for zero divisor graphs containing commutative rings. Results of algorithms are verified using mathematical formulation so algorithm can be reuse or modify for applications of coding theory, algebraic cryptography, ICT, biology and chemistry easily.

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1. Introduction

Graphs structures as powerful tool of formal science is facilitating the deep and innovative research in pure sciences, particularly in the field of chemistry and biology. This strong relation of graph theory with pure sciences has evolved interdisciplinary fields like molecular graph theory and biological networks. Many other practical problems can be represented and solved using graphs by emphasizing their applications in real world. Chemical graph theory incorporate, investigation of particles and atoms where graph structures are used to model pair wise relationship between objects. In last two decades many graph invariants have been studied and used for correlation analysis in theoretical chemistry, pharmacology, toxicology, and environmental chemistry [10].

Topological index is a numerical parameter of a graph which characterize it's topology and considered as a graph invariant. In the study of chemical graph theory, topological indices are used in the development of quantitative structure-activity relationship (QSARs) in which the properties of molecules are correlated [7, 9]. A variety of topological indices are available and have application in chemical graph theory , computer networks, statistics, physics, robotics and biological networks [22].

Topological indices can be classified as "degree based topological indices" [1, 21] and "distance based topological indices" [28, 13] on the basis of their dealing with degree of vertices or distance between the vertices. Some well-known degree based topological indices are Zagreb indices [19], atom bond connectivity [15] , geometric arithmetic index [20] , Randi connectivity index [23], Harmonic index [24] and sum connectivity index. Similarly distance based topological indices are Wiener index, Hosaya index, the energy and Estrada index [28, 27].

Group and ring theory have been studied extensively for their close relationship with algebra, number theory and representation theory but also for their applications to other areas [6, 11]. Finite commutative rings is a part of group and ring theory. Now a days, finite commutative rings gain importance in many fields like algorithm analysis, engineering, combination theory, cryptography, coding theory, wireless communications, graph and iterative coding, data analysis and modeling, and finite geometry. Zero divisor graph is one of the important finite commutative rings [5, 6].

Topological indices as molecular descriptors have achieved an important place in the field of chemistry, pharmaceutical science. Moreover these indices have wide applications in nanotube structures [7, 9, 22]. Analysis of topological indices for a particular graph helps us to understand graph characteristics, their similarities and their differences with respect to the other graphs. Topological indices guide us that how the chemical structure can further grow and what

mathematical operations on the graphs with the help of topological indices can extend multidisciplinary research.

2. Definitions and notations

For a connected graph G with vertex and edge sets $V(G)$ and $E(G)$ respectively, a numerical quantity that is invariant under graph automorphisms is called topological index or topological descriptor. For a graph G the degree of a vertex v is the number of edges incident with v and is denoted by $d(v)$. The maximum degree of a graph G, denoted by $\Delta(G)$, and the minimum degrees of a graph, denoted by $\delta(G)$. The distance between any two vertices u and v is denoted by $d(u, v)$ and is defined as the number of edges in a shortest path connecting them. Mathematical definition of eccentricity is:

$$\varepsilon_u = \max\{d(u, v) : v \in V(G)\}. \tag{1}$$

Zagreb indices and their variants are useful molecular descriptors which found considerable use in QSPR and QSAR studies as summarized by Todeschini and Consonni [25]. In an analogy with the first and the second Zagreb indices, Vukičević and Graovac [26] and Ghorbani and Hosseinzadeh [19] introduced two types of Zagreb eccentricity indices that are defined as: The first Zagreb eccentricity index of G is

$$M_1^*(G) = \sum_{uv \in E(G)} (\varepsilon_u + \varepsilon_v). \tag{2}$$

The third Zagreb eccentricity index of G is

$$M_3^*(G) \sum_{uv \in E(G)} (\varepsilon_u \times \varepsilon_v). \tag{3}$$

In 2010, Ghorbani and Khaki [20], defined the geometric-arithmetic eccentric index $GA_4(G)$ as follows:

$$GA_4(G) = \sum_{uv \in E(G)} \frac{2\sqrt{\varepsilon_u \times \varepsilon_v}}{\varepsilon_u + \varepsilon_v}. \tag{4}$$

The atom-bond connectivity eccentric index $ABC_5(G)$ defined by Farahani [16] as follows:

$$ABC_5(G) = \sum_{uv \in E(G)} \sqrt{\frac{\varepsilon_u + \varepsilon_v - 2}{\varepsilon_u \times \varepsilon_v}}. \tag{5}$$

In 2017 Farahani, Ediz and Imran [17] and Gao, Ediz, Farahani and Imran [18] in 2018 defined the the fourth type of eccentric harmonic index $H_4(G)$ as:

$$H_4(G) = \sum_{uv \in E(G)} \frac{2}{\varepsilon_u + \varepsilon_v}. \tag{6}$$

Let R be a commutative ring with identity and $Z(R)$ is the set of all zero divisors of R. $G(R)$ is said to be a zero divisor graph if $x, y \in V(G(R)) = Z(R)$ and $(x, y) \in E(G(R))$ if and only if $x.y = 0$. Beck [11] introduced the notion of zero divisor graph. Anderson and Livingston [5] proved that $G(R)$ is always connected if R is commutative. For a graph G, the concept of graph parameters have always a high interest. Numerous authors briefly studied the zero-divisor and total graphs from commutative rings [3, 4, 6].

3. Method and results

While solving a problem, choosing the correct and optimum method is basic key to attain the desired solution. In mathematics one of the captivating technique for solving any problem is algorithmic approach [8]. Algorithmic solution not only compute mathematical solution rather it leads the study to apply those algorithms in future applications addressing the same problem. *Algorithm* is a well specified set of computational procedure that transforms the input to desired output. Algorithms can perform calculation, data processing, and automated reasoning tasks related to computer and mathematical operations [12]. In this article algorithm is designed to compute edge based eccentric topological indices by constructing zero divisor graph containing finite rings for two cases that are $\mathbb{Z}_{p_1p_2} \times \mathbb{Z}_q$ and $\mathbb{Z}_{p^2} \times \mathbb{Z}_q$. Algorithms only takes three prime integers as input and computes first Zagreb, third Zagreb, geometric-arithmetic, atom-bond connectivity and harmonic index for zero divisor graphs. During algorithm designing a great care is taken on accuracy of the results as well as efficiency of the algorithm.

3.1 Algorithms

Input: Three prime numbers a, b and c.
Output: Edge-based eccentric topological indices.

Algorithm 1 calculateTopologicalIndices (a, b, c)

1: createGraphSets (a,b,c)
2: $firstZagreb \leftarrow 4 \times EccEdges1 + 5 \times EccEdges2$
3: $thirdZagreb \leftarrow 4 \times EccEdges1 + 6 \times EccEdges2$
4: $GA \leftarrow 2 \times EccEdges1 + 2 \times \sqrt{6/5} \times EccEdges2$
5: $ABC5 \leftarrow \sqrt{1/2} \times EccEdges1 + \sqrt{1/2} \times EccEdges2$
6: $harmonic \leftarrow 1/2 \times EccEdges1 + 2/5 \times EccEdges2$

Input: Three prime numbers a, b and c.
Output: All possible vertex sets of zero divisor graph with eccentricity of each set.

Algorithm 2 createGraphSets (a,b,c)

```
1: for i ← 0 to a × b
2:   for j ← 0 to c
3:     if  (i ≠ 0 OR j ≠ 0)
4:      if  (a ≠ b)
5:          if (i mod a ≠ 0 AND i mod b ≠ 0 AND i ≠ 0 AND j = 0)
6:             D[di] ← AddPoint(i, j)
7:          else if (i = 0 AND j ≠ 0)
8:             A[ai] ← AddPoint(i, j)
9:          else if (i mod a = 0)
10:            if (j = 0)
11:               B[bi] ← AddPoint(i, j)
12:            else
13:               E[ei] ← AddPoint(i, j)
14:         else if (i mod b = 0)
15:            if (j ≠ 0)
16:               F[fi] ← AddPoint(i, j)
17:            else
18:               C[ci] ← AddPoint(i, j)
19:     else
20:         if (i mod a ≠ 0 AND i ≠ 0 AND j = 0)
21:            D[di] ← AddPoint(i, j)
22:         else if (i = 0 AND j ≠ 0)
23:            A[ai] ← AddPoint(i, j)
24:         else if (i mod a = 0)
25:            if (j = 0)
26:               B[bi] ← AddPoint(i, j)
27:            else
28:               C[ci] ← AddPoint(i, j)
29: if  (a ≠ b)
30:      EccEdges1 ← ai × bi + ai × ci + bi × ci
31:      EccEdges2 ← ai × di + bi × fi + ci × ei
32: else
33:      EccEdges1 ← ai × bi
34:      EccEdges2 ← ai × di + bi × ci
35: return EccEdges1, EccEdges2
```

4. Verification of algorithmic results

For any two positive integers a and b, where $G(R)$ be a zero divisor graph containing commutative ring $R = \mathbb{Z}_a \times \mathbb{Z}_b$ with vertex set $V(G(R))$ and edge set $E(G(R))$ then $V(G(R)) = \{(x, y) \in R : x|a \text{ or } y|b \text{ or } x = 0 \text{ or } y = 0\} \setminus \{(0, 0)\}$

and $E(G(R)) = \big\{\big((x_1, y_1), (x_2, y_2)\big) \in V(G(R)) \times V(G(R)) : (x_1x_2, y_1y_2) = (0, 0) \text{ in } R\big\}$.

4.1 Case 1: $\mathbb{Z}_{p^2} \times \mathbb{Z}_q$

For $a = p^2$ and $b = q$, where p, q are prime numbers. Let G_1 be a zero divisor graph containing commutative ring $R = \mathbb{Z}_{p^2} \times \mathbb{Z}_q$. From [2], we obtain $\vee_t(G_1)$ with $t = p-1, q-1, p^2-1, pq-2$ and $|\vee_{p-1}| = (p-1)(q-1)$, $|\vee_{q-1}| = p(p-1)$, $|\vee_{p^2-1}| = q-1$, $|\vee_{pq-2}| = p-1$. By hand shaking lemma $|E(G_1)| = \frac{(p-1)(4pq-3p-2)}{2}$.

Let $\Xi_{r,s} = \{uv \in E(G_1) : \varepsilon_u = r, \varepsilon_v = s\}$ be the set contain the edges with endpoints has the eccentricity r and s. From the above discussion, we have $|\Xi_{2,2}| = \frac{(p-1)(p+2q-4)}{2}$, $|\Xi_{2,3}| = (p-1)(q-1)(2p-1)$. Mathematical formulation of above edge-based eccentric topological indices can be represent for graph G_1 in the following theorem.

Theorem 4.1. *For p, q be prime numbers. Let G_1 be the zero divisor graph containing commutative ring $\mathbb{Z}_{p^2} \times \mathbb{Z}_q$, then First Zagreb eccentric index of G_1,*

$$M_1^*(G_1) = (p-1)(10pq - 8p - q - 3)$$

the third Zagreb eccentric index

$$M_3^*(G_1) = 2(p-1)(6pq - 5p - q - 1)$$

the geometric-arithmetic eccentric index

$$GA_4(G_1) = (p-1)\left(\frac{p+2q-4}{2} + \frac{2\sqrt{6}}{5}(2p-1)(q-1)\right)$$

the atom-bond connectivity eccentric index

$$ABC_5(G_1) = \frac{1}{\sqrt{2}}|E(G_1)|$$

the fourth type of eccentric harmonic index

$$H_4(G_1) = \frac{p-1}{20}\left(16pq - 11p + 2q - 12\right).$$

Proof. By putting $|\Xi_{2,2}| = \frac{(p-1)(p+2q-4)}{2}$, $|\Xi_{2,3}| = (p-1)(q-1)(2p-1)$ in equations (2), (3), (4), (5) and (6), we obtain the required results. □

For verification purpose algorithmic results are compared by substituting some instances of the values of $a = p^2$ and $c = q$ in Theorem 4.1. It is observed that both results are same that proves the accuracy of algorithm. Some of the results are are given in Table 1 for sake of computing fidelity and their use in future applications.

Table 1: Algorithmic results of edge-based topological indices for zero divisor graph with $\mathbb{Z}_{p^2} \times \mathbb{Z}_q$

$a = p^2$	$c = q$	Algorithmic Results				
		1st-Zagreb	3rd-Zagreb	GA	ABC_5	Harmonic
2	3	38	44	7.879	5.657	3
3	5	232	272	47.192	33.941	20
5	7	1176	1392	235.636	169.706	98
7	11	4140	4920	824.241	593.970	342
11	13	13080	15600	2589.086	1866.762	1068
11	7	6540	7800	1294.543	933.381	534
13	5	6192	7392	1223.755	882.469	504
17	5	10816	12928	2133.329	1538.664	876
19	3	6804	8136	1341.088	967.322	550
23	3	10076	12056	1983.996	1431.184	814

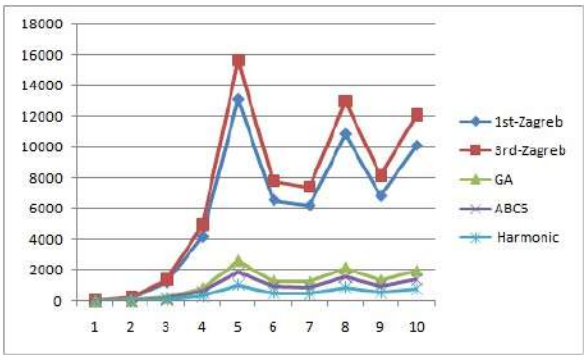


Figure 1: Pictorial representation of Table 1 results.

4.2 Case 2: $\mathbb{Z}_{p_1p_2} \times \mathbb{Z}_q$

For $a = p_1p_2$ and $b = q$, where p_1, p_2, q are prime numbers. Let G_2 be a zero divisor graph containing commutative ring $R = \mathbb{Z}_{p_1p_2} \times \mathbb{Z}_q$. From [14], we obtain $\vee_t(G_2)$ with $t = p_1p_2 - 1, p_1q - 1, p_2q - 1, q - 1, p_1 - 1, p_2 - 1$ and $|\vee_{p_1p_2-1}| = q-1$, $|\vee_{p_1q-1}| = p_2-1$, $|\vee_{p_2q-1}| = p_1-1$, $|\vee_{q-1}| = (p_1-1)(p_2-1)$, $|\vee_{p_1-1}| = (p_2-1)(q-1)$, $|\vee_{p_2-1}| = (p_1-1)(q-1)$. By hand shaking lemma $|E(G_2)| = 3p_1p_2q-2p_1p_2-2p_1q-2p_2q+p_1+p_2+q$. Let $\Xi_{r,s} = \{uv \in E(G_2) : \varepsilon_u = r, \varepsilon_v = s\}$ be the set contain the edges with endpoints has the eccentricity r and s. From the above discussion, we have $|\Xi_{2,2}| = p_1p_2+p_1q+p_2q-2p_1-2p_2-2q+3$, $|\Xi_{2,3}| = 3p_1p_2q-3p_1p_2-3p_1q-3p_2q+3p_1+3p_2+3q-3$. Mathematical formulation of above edge-based eccentric topological indices can be represent for graph G_2 in the following theorem.

Theorem 4.2. *For $p_1 < p_2$ and q be prime numbers. Let G_2 be the zero divisor graph containing commutative ring $\mathbb{Z}_{p_1p_2} \times \mathbb{Z}_q$, then First Zagreb eccentric index of G_2,*

$$M_1^*(G_2) = 15p_1p_2q - 11\,(p_1p_2 + p_1q + p_2q) + 7\,(p_1 + p_2 + q) - 3$$

the third Zagreb eccentric index

$$M_3^*(G_2) = 18p_1p_2q - 14\,(p_1p_2 + p_1q + p_2q) + 10\,(p_1 + p_2 + q) - 6$$

the geometric-arithmetic eccentric index

$$\begin{aligned} GA_4(G_1) =& \frac{6\sqrt{6}}{5}\,(p_1p_2q - 1) + \left(1 - \frac{6\sqrt{6}}{5}\right)(p_1p_2 + p_1q + p_2q) \\ &+ \left(\frac{6\sqrt{6}}{5} - 2\right)(p_1 + p_2 + q) + 3. \end{aligned}$$

the atom-bond connectivity eccentric index

$$ABC_5(G_2) = \frac{1}{\sqrt{2}}|E(G_2)|$$

the fourth type of eccentric harmonic index

$$H_4(G_2) = \frac{6}{5}\,(p_1p_2q) - \frac{7}{10}\,(p_1p_2 + p_1q + p_2q) + \frac{1}{5}\,(p_1 + p_2 + q) + \frac{3}{10}.$$

Proof. By putting $|\Xi_{2,2}| = p_1p_2 + p_1q + p_2q - 2p_1 - 2p_2 - 2q + 3$, $|\Xi_{2,3}| = 3p_1p_2q - 3p_1p_2 - 3p_1q - 3p_2q + 3p_1 + 3p_2 + 3q - 3$ in equations (2), (3), (4), (5) and (6), we obtain the required results. □

For verification purpose algorithmic results are compared by substituting some instances of the values of $a = p_1, b = p_2$ and $c = q$ in Theorem 4.1. It is observed that both results are same that proves the accuracy of algorithm. Some of the results are are given in Table 2 for sake of computing fidelity and their use in future applications.

Table 2: Algorithmic results of edge-based topological indices for zero divisor graph with $\mathbb{Z}_{p_1p_2} \times \mathbb{Z}_q$

a	b	c	Algorithmic Results 1st-Zagreb	3rd-Zagreb	GA	ABC_5	Harmonic
2	3	5	176	200	37.515	26.870	16
3	5	5	608	704	126.060	50.912	54
5	7	5	1696	1984	346.181	248.902	147
5	11	7	4096	4816	829.453	596.798	350
7	11	7	6024	7104	1214.180	873.984	510
11	13	7	11808	13968	2368.359	1705.542	990
13	17	5	12736	15040	2561.450	1844.135	1073.6
17	19	5	18976	22432	3810.175	2743.574	1594
19	23	3	13784	16160	2803.995	2016.669	1188
23	29	2	11904	13752	2476.663	1777.666	1072

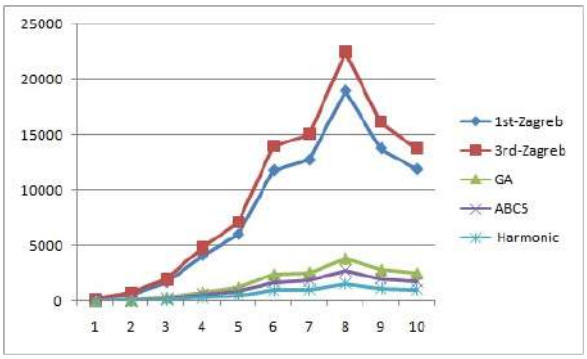


Figure 2: Pictorial representation of Table 2 results.

5. Conclusion

In this paper, we computed first Zagreb, third Zagreb, geometric-arithmetic, atom-bond connectivity and harmonic index for zero divisor graphs containing commutative rings $\mathbb{Z}_{p_1p_2} \times \mathbb{Z}_q$ and $\mathbb{Z}_{p^2} \times \mathbb{Z}_q$ algorithmically. Later on results of algorithms are verified by comparing them with mathematical calculations. Figure 1 and Figure 2 shows the trend in form of line chart about edge eccentric topological indices. In all cases, 3rd Zagreb index and Harmonic index have the maximum and minimum values, respectively. ABC_5 index is greater than Harmonic Index, GA index is greater than ABC_5 index and 1st Zagreb index is greater than GA index but lesser than 3rd Zagreb index for all zero divisor graphs. This study is further useful in understanding the attributes of various physical structures like carbohydrates, silicone structures, polymers, hexagonal chains, cylindrical fullerenes. They can likewise be useful in creating productive physical structure in mechanics as well as for different computer network problems.

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An algorithm to solve multi-objective assignment problem by using discrete PSO decomposition

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Abstract. This paper presents an algorithm to solve a multi-objective assignment problem (MOAP) using discrete particle swarm optimization method under decomposition. The mathematical modeling of the algorithm has been derived to simulate the multi-objective assignment problem that considered as a non-linear function. The approach presented in this work highlights the optimal solution of every single objective function by decomposition the objective functions. To illustrate the algorithm a numerical examples are presented.

Keywords: assignment, multi-objective decision making, discrete PSO, decomposition.

1. Introduction

One of the significant problem in the mathematical problems is the assignment problem (AP). The classical type of the assignment problems (AP), in which the

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number of machines and jobs are equal, has studied in details and introduced to solve these problems. However, this problem move to the type of uncertain assignment due to the real life uncertainty. The goal of the AP is to assign a specific number of jobs to an equal and specific number of machines in order to minimize the total time required for execution all the jobs or the assignment total cost. The objectives in the multi-objective AP are discrete PSO decomposition. The classical type of the AP refers to a specific type of the linear programming problems (LP). The LP can be defined as one of the widely used tools for making decision for solving problems in real world. In the actual situation of the decision making problem, the critical concern is the decision problems include multiple criteria (objectives or attributes). However, several cases of the decision making of the real world come to happen in a specific environment where parameters, objectives, and constrains are not accurate Liu [1]. However, two issues are addressed through application of the PSO for solving the MOPs. The first one in the multi-objective PSO is the maintenance of archive for diversity and convergence balancing. The external elitist archive is applied for storing the non- dominated solution that obtained by using an algorithm and the others solutions obtained the filtered by a specific measurement of a quality, such as density.

Some MPSO proposals where the size of archive is unconstrained [1,3], the pre-fixed maximum size of an archive is commonly used because of the non-archive number. Furthermore, physical memory has a limited power. As a result, a suitable dominated solutions will evolve very rapidly, increasing the cost of computing of upgrading archiving. Filtering these (non-dominated) solutions with poorer quality steps is thus necessary. Several method for updating archive were introduced, for example, many MPSOs [4,17,20] use the crowding distance [11] to prune the archive, whereas Kernel density was first presented in [9] and utilized in MPSO [5]. For gbest updating in MPSO, a dynamic neighborhood approach is used and proposed in [21]. To pick pbest and gbest, a decomposition method was used29 on MPSOs [19]. To lead the search process, ranking approaches [12,16] are used to classify the best solutions. A good MPSO should be able to keep the diversity of the population. The external archive technique is commonly used for this reason in order to preserve the diversity of the received solutions. An external and alternative elitist archive holds the non-dominated alternatives discovered by an algorithm and updates the archive using a specific technique to ensure diversity. However, if a MOEA (e.g., MPSO) has trouble finding any Pareto optimal solutions, and these Pareto optimal solutions show an important role in preserving the diversity of archive, the archive's diversity cannot be sustained. To preserve variety, a decomposition technique has recently been used. MOEA/D (multi-objective evolutionary algorithm developed based on decomposition) is a representative algorithm that performs well when searching for a variety of non-dominated solutions for different types of MOPs [14]. MOEA/D breaks down the process of approximating the Pareto front (PF) into

a series of single-objective optimization of sub-problems by using conventional aggregation approaches.

MOEA/D substitutes an old solution with a new solution based on their values of the aggregation function if the aggregation function values are less than the former when evolving the solution package (i.e., the latter is closer to the optimal solutions than the former). In this paper we develop such an extension of the two phase method to solve multi- objective integer linear programmers
(MOIP) $Min\{z_1(x), \ldots, z_p(x) = Cx : x \in X\}$,
where $p \geq 2$ and $C \in R^{p*n}$. X denotes the set of feasible solutions of the problem and is defined by m, $X = \{x \in Zn : Ax \leq b; x \geq 0\}$, where $A \in Rm\_n$ and $b \in Rm$. We call Rn decision space($X \subset Rn$) and R^p objective space. $Y = \{Cx : X \in X\} \subset Rp$ is called the objective space feasible set or the outcome set.

Definition 1. A feasible solution $\overset{\frown}{x} \in X$ is efficient if no other feasible solution exists for $x \in X$ such that $z(x) \leq z(\overset{\frown}{x})/.Z(x)$ (is then called a non-dominated point. If $x; x' \in X$ are such that $z(x) \leq z(x')$ we say that x dominates x' and $z(x)$ dominates $z(x')$. If $z(x) = z(x'), x$ and x' are equivalent. Y_N is the non-dominated points set of all of Y and X_E is the efficient solutions set.

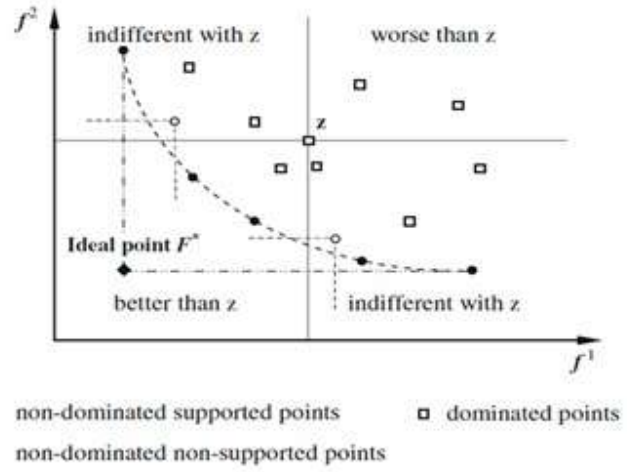


Fig. 1 Dominations in the Pareto sense in a bi-objective space

Definition 2 ([2])**.** A complete set defined by $X_E$is defined as a set of the efficient solutions at which all $x \in X \setminus X_E$ are either equivalent to or dominated by at least one $x' \in X_E$. In other words, There is at least one $x \in X_E$. for each non-dominated point $y \in Y_N$ such that $z(x) = y$.

In this work, the (AP) with the objectives is used to explain the developed methods and an algorithm to obtain the exact solution (finding a maximal or minimal complete set) of this problem is going to be proposed. In Pareto optimization (PO), the goal is to find a series of "efficient" solutions that are either

approximate or exact. Exact methods aim to solve a problem in the most efficient way possible, but their implementation on large real problems typically takes too long. Approximate approaches are used in functional applications to discover high-quality (but not necessarily optimal) alternatives in a short computational time. Approximate approaches are divided into two categories:

1. Heuristics: that are very specialized approximation approaches that take advantage of domain knowledge.

2. Meta-heuristics: Since they are not intended for a single problem, they have been effectively utilized to solve a broad variety of (combinatorial) optimization problems.

The multi-objective AP can then be solved in one of three ways: using exact methods where possible (see [3, 4], and [5]), approximating methods like those mentioned in [6,], or hybrid methods that incorporate the two preceding methods.

The below is the paper's structure: The mathematical modeling of the multi-objective AP is defined in section 2. The results and discussions are presented in Section 3. In sub-section 2.3, the optimization algorithm is presented. In sub-section 3.3, a numerical example is given to demonstrate the algorithm accuracy. Section 4 presents the conclusions and a few reflections of the suggested algorithm.

2. Mathematical model establishment of multi-objective AP

2.1 Multi-Objective Assignment Problem

This problem (MOP) is defined as follows.

$$Min(z) = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}^{k} x_{ij}, \quad k = 1, 2, 3, \ldots, n,$$

$$\sum_{i=1}^{n} x_{ij} = 1, \quad i = 1, 2, 3, \ldots, n,$$

$$\sum_{j=1}^{n} x_{ij} = 1, \quad j = 1, 2, 3, \ldots, n,$$

$$X_{ij} \geq 0, \quad \forall i, j,$$

where all coefficients of objective function c_{ij}^{k} are non-negative integers and $x = x_{11}, \ldots, x_{nn}$ are the decision variables matrix and equal 1 if the task i is assigned to agent j and 0 otherwise. The MOP can also be referred as a multi-objective perfect matching problem on a complete bipartite graph $G = (U \cup V; E)$ with $|U| = |V| = n$ so that any edge $i, j \in E$ links a vertex i in U to a vertex j in V with cost c_{ij}^{k} for any objective. A perfect matching in G is corresponding to a specific assignment in (MOAP). Let $X \subset \{0; 1\}n^2$ signify the feasible solutions set for (MOAP). To any solution $x \in X$ is associated a set $E(x)$ of n edges such that

an edge $i, j \in E$ of bipartite graph G is in $E(x)$ if and only if $xij = 1$. A solution is thus, consistently categorized by x or by $E(x)$. Finally, to any solution $x \in X$ is associated a criterion vector $z(x) = \{z_1(x), z_2(x), \ldots, z_n(x)\}$ which resembles to its image presented in the space of objective. The image set presented in the feasible solutions objective space is referred by $Z = \{z(x) : x \in X\}$.

2.2 Discrete particle swarm optimization

Position, fitness, and velocity values are all assigned to each particle. Particles benefit from historical knowledge gathered from the swarm community to travel across m-dimensional space. As a result, particles are more likely to fly into a better and stronger search field as evolution continues. Let NP stand for the swarm population size, which is given by $X_t = [X_{1t}, X_{2t}, \ldots, X_{NPt}]$. The following are the characteristics of each particle in the swarm population: The current situation was expressed by $X_i^t = [X_i^{1t}, X_i^{2t}, \ldots, X_i^{mt}]$, a current velocity represented as $V_i^t = [V_i^{1t}, V_i^{2t}, \ldots, V_i^{mt}]$ a current personal best position represented as $P_i^t = [P_i^{1t}, P_i^{2t}, \ldots, P_i^{mt}]$ and a current global best position characterized as $Gt = [g_{1t}, g_{2t}, \ldots, g_{mt}]$.

2.3 Decomposition property

Decomposition is a basic strategy in outdated multi-objective optimization. Though, it has not yet been commonly utilized in multi-objective evolutionary optimization. The decomposition idea has been used to a certain extent in several meta-heuristics for MOPs [7]–[11]. The traditional method of the mathematical programming techniques for solving MOPs used decomposition method extensively. Most MOEAs, on the other hand, regard a MOP as a whole and focus on dominance to assess solution consistency during their process of search. These algorithms could not be very successful at distributing solutions evenly around the PF.

There are some papers taking the composition approach as follow, Ref [13] took advantage of PSO as a quick, easy-to-implement, and a reliable method and specifies a smart methodology for particles updating while using decomposition to enhance and increase the solution diversity.

Since the original MOEA/D system was proposed, numerous works have been published in the literature aimed at: a) overcoming the shortcomings in components of design of the original MOEA/D, b) improving the efficiency of MOEA/D, c) presenting novel decomposition-based MOEAs, or d) adapting decomposition-based MOEAs for various types of problems. In 2016 the author [14] presents a broad survey of the decomposition- based multi-objective evolutionary algorithms anticipated in the last decade. The decomposition technique was not commonly used in the multi-objective optimization until Zhang and Li who presented the multi-objective evolutionary algorithm developed based on the principles of decomposition [15]. The proposed algorithm of Zhang and Li decompose the problem of the multi-objective optimization into a variety of

scalar sub- problems of optimization and optimizes them all at the same time using a specific evolutionary algorithm.

2.4 An introduction to PSO

In the optimization field, population-based stochastic method of the local search are a fairly recent paradigm. This family contains a variety of nature-inspired problem-solving methods that use metaphors as directions. The most well-known members are the genetic algorithms (GA), which seek solution spaces using the metaphor of genetic and evolutionary concepts of reproduction fitness selection for. Swarm Intelligence, which is described as "any attempt to design algorithms or distributed problem-solving devices inspired by the collective behavior of social insect colonies and other animal societies [19]", is motivated by the collective behavior of bird flocks, insect colonies, fish schools, and other societies of animals. The PSO approach, that used for solving optimization problems, is a relatively recent member of the Swarm Intelligence field. PSO is a recent evolutionary optimization strategy for optimizing the nonlinear continuous functions that was suggested by [20]. It takes its biological inspiration from the social contact metaphor and communication in a school of fish or flocks of birds. Among these communities, there is a leader that controls the whole swarm's activity. Every individual's movement is dependent on the leader and his own experience. Individuals (i.e. particles) in a PSO algorithm appear to obey the group leader, i.e. the one with the better results,since PSO is evolutionary in nature and population-based. In general, the model from which PSO is derived implies that each particle's action is a balance between its collective and individual memories-. The following are the rules that guide the PSO algorithm:

1. Any possible solution , a particle k in the swarm starts with a randomized position and velocity. The vectors denote the direction and velocity of particle k in the n- dimensional search space.$X_k = (x_{k1}, x_{k2}, \ldots, x_{kn})$ and $V_k = (V_{k1}, V_{k2}, \ldots, V_{kn})$, respectively, where $x_{kd}(d = 1, 2, ..., n)$ represents the location and $v_{kd}(d = 1, 2, ..., n)$ is the flying particle k velocity in the search space dth dimension.

2. Each particle k knows its position and the its objective function value. It also recalls the position at which it achieved its greatest achievement, which is $P_k^t = (P_{k1}^t, P_{k2}^t, ..., P_{kn}^t)$.

3. From every position, every particle can produce a neighborhood. It is therefore, a member of a certain neighborhood particles and recalls the particle (given by index g) which is in its best general position. The neighborhood will either be all the particles in case of the global neighborhood or a subset of particles in case of the local neighborhood.

The behavior of the particle is a compromise between the three possible options in each iteration t: following its exploration current pattern, returns to its best prior position, returns towards the all particles best historic value.

The following equations in the current iteration of the algorithm implement this compromise:

$$v_{kd}^{t+1} = wv_{kd}^{t} + c_1 r_1 (p_{kd}^{t} + x_{kd}^{t}) + c_2 r_2 (p_{gd}^{t} + x_{gd}^{t} \tag{1}$$

$$x_{kd}^{t+1} = x_{kd}^{t} + v_{kd}^{t+1} \tag{2}$$

where the symbol w is a constant value, called the inertia weight, chosen by the user for controlling the relationship between the previous velocities and the current one. The symbol c1 is the weight factor introduced to the attraction of the current particle's previous best position and the symbol c2 is the weight given to the previous best neighborhood particle location attraction. The symbols r1 and r2 in [0, 1] are random variables distributed uniformly.

2.5 The proposed algorithm (MOPSO/D) for assignment problems

In this section we are presented our new method in details as follows:

2.5.1 Multi-objective Particle Swarm Optimization (MOPSO)

In this section, the state-of-the-art PSO algorithm are examined. Two main categories of algorithms are distinguished, based on two methods: techniques that independently exploit each objective function, and schemes based on Pareto. The distinction is largely made for purposes of presentation and is not restrictive, because some algorithms combine characteristics from both methods. The exposition of the methods for each category is based on a chronological ordering.

The difficulty in determining an optimal solution of a multi-objective problem lies with the potential disagreements and contradict between the optimal solutions for the individual goals. The best solution for one purpose of a specific objective may be the worst. The resolution of these disagreements is essential to finding a multi-objective "optimal" solution. Two methods are available to resolve the multi-objective. First, the multi-objective problem would be converted into one objective problem (single problem). This was often done by weightily assembling all the objectives. However, with this technique, there is a problem of adjusting and changing the weighting factor. The second approach is mainly based on the Pareto optimality concept , which provides an ideal set of solutions rather than an optimum one. The reason for many solutions' optimalities is that with respect to all objective functions no one of them can be better than anyone else. PO is better suited to solving multi-objective problems compared to classical algorithms [12].

The (MOPSO/D) was proposed by Peng and Zhang[16]. This method utilizes the MOEA/D framework but substitutes the genetic operators (mutating and crossover) with the classical PSO inertia flight equations. MOPSO/D uses a turbulence operator (or mutation) and uses special archiving strategy (which is based on c-dominance [17]) for nondominated solutions storing initiated during

the search. In [18] they introduce a new decomposition- based multi-objective particle swarm optimizer (dMOPSO) for continuous and unconstrained MOPs. Further, the paper sought to resolve the multi-target optimum active power dispatch issue with PSO and MOPSO performances. In the IEEE Standard 30-bus system of 6 generators, the proposed technique was tested and validated. The aim is to address the multi-objective issue of the dispatch of economic power; four targets are fuel costs and effect on the environment due to emissions caused by SO2, CO2 and NOx.
MOPSO/D Procedure
Set $k := 0$ and velocity=0
Randomly initialize Point P_i for n. population;
Calculate the fitness values of initial Population: $f(P)$;
Personal best =POP;
$Zi = \min\{f(p) \setminus p \in POP(t)\}, 1 \leq i \leq m.$
Find the non-dominated solutions and initialized the archive with them (Decomposition and fitness calculation) First, by calculating the fitness value for each solution in POP(t) by crowding distance, split the POP(t) into N class solutions. Then pick some better POP(t) solutions and place them in a temporary pop population of N dimensions. Our work mostly uses the binary selection of tournament
WHILE (the termination conditions are not met)
1) PSO Steps
$p_{gd}^t = SelectLeader(POP)$
$v_{kd}^{t+1} = wv_{kd}^t + c_1r_1(p_{kd}^t + x_{kd}^t) + c_2r_2(p_{gd}^t + x_{gd}^t)$
$P_{kd}^{t+1} = P_{kd}^t + v_{kd}^{t+1}$
$P_{kd}^{new} = P_{kd}^{t+1} + 0.5 * (p_{gd}^t - p_{kd}^t)$
End
End
(Solution update) For each j = 1,..., m, if $\{Zj > (p) \setminus p \in 0\}$, then update $\{Zj > (p) \setminus p \in 0\}$. Firstly, categorize the 180 solutions of POP(k) $\cup$ O by, then pick N of the best solutions through the updated strategy of Section 3.4 and plug them into POP$(t + 1)$.
Update Global best
Update Personal best
Set $k := k + 1$;
END WHILE

3. Results and discussions

In this section, we are compares Multi-objective Partial Swarm Optimization with Decomposition (MOPSOD) practically with a stochastic meat-heuristic search algorithm,Non-Sorting Genetic Algorithm two (NSGAII), Non-Sorting Genetic Algorithm three (NSGAIII, (MOPSO), Strength Parito Evolutionary

Algorithm two (SPEA2), and (PESAII) to determine how MOPSOD is positioned against other heuristic search algorithms. In addition, through tables 1 and 2, we can say that the presented algorithm is highly efficient relative to other algorithms, and we can see this through the results that were shown or shown by statistical measures.

In this manner, standard benchmark test functions (i.e., Table 4.1) (Tarasewich and Mcmullen, 2002) presented in detail. Through the Table (4.2), i.e., we can see the performance of combine SOGS-BAT with other algorithms.

Table 1: Table (1): Comparative between algorithms by using Inverted Generational distance when (M=2)

Problem	M	D	NSGAII	NSGAIII	SPEA2	PESAII	MOPSODH
	2	5	2.7279e-2 (0.00e+0) =	2.7279e-2 (0.00e+0) =	2.7279e-2 (0.00e+0) =	2.7279e-2 (0.00e+0) =	2.7279e-2 (0.00e+0)
	2	10	1.1224e-1 (9.01e-4) =	1.1205e-1 (4.25e-4) =	1.1216e-1 (4.08e-4) =	1.1235e-1 (5.86e-4) =	1.1202e-1 (8.18e-4)
	2	15	2.4625e-1 (1.07e-3) =	2.4583e-1 (3.07e-4) =	2.4577e-1 (3.09e-4) =	2.4721e-1 (1.08e-3) =	2.4554e-1 (1.18e-3)
	2	20	4.3854e-1 (1.43e-3) =	4.3860e-1 (8.38e-4) =	4.3850e-1 (2.10e-4) =	4.3838e-1 (1.57e-3) =	4.3683e-1 (1.55e-3)
	2	25	6.7901e-1 (1.90e-3) =	6.7874e-1 (4.62e-4) =	6.7815e-1 (1.18e-3) =	6.8099e-1 (1.29e-3) =	6.7802e-1 (6.82e-4)
MOAP	2	50	2.6702e+0 (1.61e-3) =	2.6692e+0 (3.09e-3) =	2.6709e+0 (5.16e-3) =	2.6742e+0 (1.61e-3) =	2.6703e+0 (1.48e-3)
	2	100	1.0620e+1 (3.63e-3) =	1.0620e+1 (2.40e-3) =	1.0621e+1 (1.58e-3) =	1.0625e+1 (9.49e-3) =	1.0616e+1 (2.80e-3)
	2	150	2.3881e+1 (4.80e-3) =	2.3893e+1 (1.17e-2) =	2.3887e+1 (5.83e-3) =	2.3901e+1 (1.28e-2) =	2.3886e+1 (3.97e-3)
	2	200	4.2363e+1 (6.19e-3) =	4.2369e+1 (1.97e-2) =	4.2378e+1 (1.39e-2) =	4.2405e+1 (1.91e-2) =	4.2363e+1 (1.06e-2)
	2	250	6.6130e+1 (1.18e-2) =	6.6122e+1 (1.23e-2) =	6.6138e+1 (4.00e-3) =	6.6131e+1 (2.26e-2) =	6.6117e+1 (4.80e-3)
	2	500	2.6455e+2 (1.26e-2) =	2.6451e+2 (4.14e-2) =	2.6459e+2 (3.28e-2) =	2.6461e+2 (3.95e-2) =	2.6454e+2 (1.88e-2)
	2	1000	1.0578e+3 (5.05e-2) =	1.0577e+3 (6.86e-2) =	1.0577e+3 (5.07e-2) =	1.0579e+3 (1.08e-1) =	1.0577e+3 (4.78e-2)
+/-/=			0/0/12	0/0/12	0/0/12	0/0/12	

Table 2: Table (2): Comparative between algorithms by using Inverted Generational distance when (M=3)

Problem	M	D	NSGAII	NSGAIII	SPEA2	PESAII	MOPSODH
	3	5	3.3670e-2 (2.65e-4) =	3.4041e-2 (1.50e-4) =	3.3663e-2 (2.53e-4) =	3.3954e-2 (4.16e-4) =	3.3954e-2 (3.81e-4)
	3	10	1.2964e-1 (2.01e-4) =	1.3004e-1 (3.31e-4) =	1.2963e-1 (1.92e-4) =	1.2965e-1 (1.10e-3) =	1.2957e-1 (3.86e-4)
	3	15	2.9836e-1 (6.33e-4) =	2.9833e-1 (5.39e-4) =	2.9809e-1 (7.14e-4) =	2.9812e-1 (4.89e-4) =	2.9786e-1 (4.40e-4)
	3	20	5.2566e-1 (6.02e-4) =	5.2534e-1 (9.77e-4) =	5.2584e-1 (4.93e-4) =	5.2613e-1 (9.49e-4) =	5.2588e-1 (2.42e-4)
	3	25	8.1345e-1 (1.06e-3) =	8.1383e-1 (9.24e-4) =	8.1370e-1 (2.55e-4) =	8.1317e-1 (2.90e-4) =	8.1473e-1 (1.60e-3)
MOAP	3	50	3.2815e+0 (1.28e-3) =	3.2805e+0 (1.85e-3) =	3.2783e+0 (2.94e-3) =	3.2794e+0 (1.75e-3) =	3.2771e+0 (2.28e-3)
	3	100	1.2967e+1 (1.31e-3) =	1.2964e+1 (5.19e-3) =	1.2961e+1 (3.54e-3) =	1.2963e+1 (7.57e-3) =	1.2968e+1 (2.43e-3)
	3	150	2.9157e+1 (2.91e-3) =	2.9164e+1 (1.13e-2) =	2.9156e+1 (7.85e-3) =	2.9163e+1 (5.51e-3) =	2.9160e+1 (9.66e-3)
	3	200	5.1850e+1 (5.19e-3) =	5.1852e+1 (3.70e-3) =	5.1841e+1 (3.90e-3) =	5.1837e+1 (3.40e-3) =	5.1848e+1 (1.01e-2)
	3	250	8.1023e+1 (5.43e-3) =	8.1035e+1 (1.69e-2) =	8.1036e+1 (7.65e-3) =	8.1033e+1 (8.63e-3) =	8.1022e+1 (1.19e-2)
	3	500	3.2382e+2 (4.45e-2) =	3.2383e+2 (2.36e-2) =	3.2383e+2 (7.39e-3) =	3.2381e+2 (1.28e-2) =	3.2383e+2 (2.12e-2)
	3	1000	1.2954e+3 (1.23e-2) =	1.2954e+3 (4.17e-2) =	1.2954e+3 (2.37e-2) =	1.2954e+3 (1.31e-2) =	1.2954e+3 (2.75e-2)
+/-/=			0/0/12	0/0/12	0/0/12	0/0/12	

Table 3: Table (3): Comparative between algorithms by using Inverted Generational distance when (M=4)

Problem	M	D	NSGAII	NSGAIII	SPEA2	PESAII	MOPSODH
	4	5	4.0560e-2 (3.80e-5) =	4.0625e-2 (7.56e-5) =	4.0544e-2 (4.84e-5) =	4.0583e-2 (5.24e-5)	4.0639e-2 (4.05e-5) =
	4	10	1.5867e-1 (1.87e-4) =	1.5883e-1 (2.28e-4) =	1.5869e-1 (1.43e-4) =	1.5874e-1 (2.24e-4)	1.5876e-1 (4.30e-4) =
	4	15	3.3823e-1 (7.17e-4) =	3.3827e-1 (7.90e-4) =	3.3829e-1 (6.88e-4) =	3.3854e-1 (2.67e-4)	3.3873e-1 (4.10e-4) =
	4	20	6.0579e-1 (1.47e-4) =	6.0506e-1 (1.01e-3) =	6.0519e-1 (3.28e-4) =	6.0564e-1 (1.57e-4)	6.0529e-1 (4.34e-4) =
	4	25	9.3145e-1 (9.35e-4) =	9.3236e-1 (1.23e-3) =	9.3069e-1 (1.27e-3) =	9.3207e-1 (1.26e-3)	9.3255e-1 (1.04e-3) =
MOAP	4	50	3.7235e+0 (1.41e-3) =	3.7213e+0 (1.72e-3) =	3.7242e+0 (9.34e-4) =	3.7240e+0 (3.21e-3)	3.7213e+0 (8.79e-4) =
	4	100	1.5054e+1 (3.90e-3) =	1.5053e+1 (5.78e-3) =	1.5052e+1 (4.99e-3) =	1.5052e+1 (2.31e-3)	1.5049e+1 (4.86e-3) =
	4	150	3.3670e+1 (3.20e-3) =	3.3670e+1 (4.04e-3) =	3.3667e+1 (7.48e-3) =	3.3665e+1 (5.07e-3)	3.3667e+1 (1.08e-2) =
	4	200	5.9860e+1 (9.82e-3) =	5.9863e+1 (1.74e-2) =	5.9864e+1 (8.16e-3) =	5.9865e+1 (2.08e-3)	5.9871e+1 (6.53e-3) =
	4	250	9.3560e+1 (7.52e-3) =	9.3562e+1 (1.53e-2) =	9.3558e+1 (9.84e-4) =	9.3557e+1 (1.94e-2)	9.3556e+1 (5.35e-3) =
	4	500	3.7398e+2 (3.44e-3) =	3.7397e+2 (1.49e-2) =	3.7398e+2 (1.97e-2) =	3.7398e+2 (1.49e-2)	3.7397e+2 (2.50e-2) =
	4	1000	1.4949e+3 (4.76e-2) =	1.4949e+3 (2.36e-2) =	1.4948e+3 (2.71e-2) =	1.4949e+3 (2.58e-2)	1.4948e+3 (7.21e-2) =
+/-/=			0/0/12	0/0/12	0/0/12	0/0/12	

Table 4: Table (4): Comparative between algorithms by using Inverted Generational distance when (M=5)

Problem	M	D	NSGAII	NSGAIII	SPEA2	PESAII	MOPSODH
	5	5	4.3292e-2 (2.61e-4)	4.3199e-2 (4.92e-4) =	4.3210e-2 (3.35e-4) =	4.3242e-2 (3.87e-4) =	4.3541e-2 (1.05e-4) =
	5	10	1.7763e-1 (1.62e-4)	1.7790e-1 (5.55e-4) =	1.7784e-1 (1.39e-4) =	1.7774e-1 (3.83e-4) =	1.7740e-1 (2.69e-4) =
	5	15	3.8214e-1 (4.88e-4)	3.8245e-1 (3.41e-4) =	3.8196e-1 (2.80e-4) =	3.8166e-1 (6.22e-4) =	3.8201e-1 (5.16e-4) =
	5	20	6.7409e-1 (2.98e-4)	6.7531e-1 (9.57e-4) =	6.7460e-1 (1.21e-3) =	6.7356e-1 (5.86e-4) =	6.7435e-1 (6.04e-4) =
	5	25	1.0559e+0 (9.35e-4)	1.0567e+0 (1.18e-3) =	1.0558e+0 (6.62e-4) =	1.0558e+0 (5.97e-4) =	1.0548e+0 (4.12e-4) =
	5	50	4.1736e+0 (2.24e-4)	4.1759e+0 (9.58e-4) =	4.1750e+0 (1.62e-3) =	4.1743e+0 (6.88e-4) =	4.1727e+0 (2.68e-3) =
	5	100	1.6814e+1 (9.21e-4)	1.6815e+1 (1.57e-3) =	1.6813e+1 (4.65e-4) =	1.6812e+1 (2.68e-3) =	1.6808e+1 (3.70e-3) =
	5	150	3.7685e+1 (4.56e-3)	3.7691e+1 (4.36e-3) =	3.7683e+1 (6.86e-3) =	3.7689e+1 (2.83e-3) =	3.7692e+1 (2.17e-3) =
MOAP	5	200	6.6930e+1 (5.35e-3)	6.6930e+1 (5.38e-4) =	6.6927e+1 (2.82e-3) =	6.6931e+1 (1.09e-2) =	6.6924e+1 (5.36e-3) =
	5	250	1.0466e+2 (4.05e-3)	1.0465e+2 (8.90e-3) =	1.0466e+2 (1.24e-2) =	1.0465e+2 (6.49e-3) =	1.0466e+2 (1.38e-2) =
	5	500	4.1820e+2 (7.32e-3)	4.1819e+2 (1.56e-2) =	4.1820e+2 (1.13e-2) =	4.1820e+2 (3.05e-3) =	4.1819e+2 (7.92e-3) =
	5	1000	1.6714e+3 (4.46e-2)	1.6714e+3 (4.84e-2) =	1.6714e+3 (4.44e-2) =	1.6714e+3 (1.76e-2) =	1.6714e+3 (2.70e-2) =
+/-/=			0/0/12	0/0/12	0/0/12	0/0/12	

Table 5: Table (5): Comparative between algorithms by using Inverted Generational distance when (M=2)

Problem	M	D	NSGAII	NSGAIII	SPEA2	PESAII	MOPSODH
	2	5	3.9214e-3 (1.85e-6) =	3.9159e-3 (1.92e-6) =	3.9140e-3 (6.13e-19) =	3.9140e-3 (0.00e+0) =	3.9140e-3 (0.00e+0) =
	2	10	3.4271e-2 (2.71e-2) =	2.6908e-2 (1.00e-2) =	2.1268e-2 (8.52e-3) =	1.5943e-2 (7.68e-5) =	1.5943e-2 (7.68e-5) =
	2	15	5.0951e-2 (1.38e-2) =	7.8397e-2 (1.10e-2) =	8.7615e-2 (4.99e-2) =	5.0517e-2 (5.67e-3) =	5.0517e-2 (5.67e-3) =
	2	20	1.7530e-1 (3.71e-2) =	1.3909e-1 (4.43e-2) =	1.9605e-1 (1.01e-1) =	7.2658e-2 (1.97e-3) =	7.2658e-2 (1.97e-3) =
	2	25	2.3433e-1 (2.69e-2) =	2.6261e-1 (3.90e-2) =	1.9509e-1 (1.72e-2) =	1.4350e-1 (2.34e-2) =	1.4350e-1 (2.34e-2) =
MOAP	2	50	1.0685e+0 (4.15e-1) =	8.8944e-1 (1.76e-1) =	9.6106e-1 (1.25e-1) =	8.0803e-1 (1.81e-1) =	8.0803e-1 (1.81e-1) =
	2	100	3.9906e+0 (7.75e-1) =	3.7113e+0 (9.68e-1) =	3.7715e+0 (2.37e-1) =	3.4943e+0 (2.28e-1) =	3.4943e+0 (2.28e-1) =
	2	150	7.1281e+0 (4.69e-1) =	1.0267e+1 (3.11e+0) =	9.1569e+0 (2.43e+0) =	6.7541e+0 (8.92e-1) =	6.7541e+0 (8.92e-1) =
	2	200	1.4412e+1 (4.95e-1) =	1.7113e+1 (3.66e+0) =	1.4070e+1 (1.59e+0) =	1.9078e+1 (9.55e+0) =	1.9078e+1 (9.55e+0) =
	2	250	2.5003e+1 (4.02e+0) =	2.6166e+1 (6.02e+0) =	2.5468e+1 (6.81e+0) =	1.9986e+1 (9.15e-1) =	1.9986e+1 (9.15e-1) =
	2	500	1.3446e+2 (1.73e+1) =	9.2404e+1 (2.41e+1) =	1.2981e+2 (1.99e+1) =	8.9370e+1 (1.66e+1) =	8.9370e+1 (1.66e+1) =
	2	1000	4.0703e+2 (5.72e+1) =	3.7493e+2 (5.65e+1) =	3.6326e+2 (6.39e+1) =	4.1923e+2 (6.12e+1) =	4.1923e+2 (6.12e+1) =
+/-/=			0/0/12	0/0/12	0/0/12	0/0/12	

Table 6: Table (6): Comparative between algorithms by using Inverted Generational distance when (M=3)

Problem	M	D	NSGAII	NSGAIII	SPEA2	PESAII	MOPSODH
	3	5	8.7638e-3 (1.80e-3) =	9.5212e-3 (9.04e-4)	8.6174e-3 (1.51e-3) =	1.0016e-2 (1.74e-3) =	8.8364e-3 (1.60e-3) =
	3	10	3.8275e-2 (1.87e-3) =	3.7414e-2 (5.53e-3)	3.9525e-2 (2.90e-3) =	4.6431e-2 (1.67e-2) =	3.5299e-2 (3.66e-3) =
	3	15	7.9435e-2 (2.31e-2) =	6.7059e-2 (3.34e-3)	8.2517e-2 (1.36e-2) =	7.5517e-2 (7.25e-3) =	8.5704e-2 (1.69e-2) =
	3	20	1.4546e-1 (1.19e-2) =	1.3876e-1 (1.24e-2)	1.5256e-1 (6.46e-3) =	1.5567e-1 (2.76e-2) =	1.2827e-1 (1.81e-2) =
MOAP	3	25	2.2999e-1 (2.77e-2) =	2.1102e-1 (3.25e-2)	2.2502e-1 (1.82e-2) =	2.0215e-1 (9.33e-3) =	2.0230e-1 (7.35e-3) =
	3	50	9.8449e-1 (9.74e-2) =	8.3464e-1 (1.87e-1)	9.9841e-1 (1.86e-1) =	1.0157e+0 (1.58e-1) =	9.4627e-1 (1.75e-1) =
	3	100	4.8561e+0 (1.23e+0) =	4.0878e+0 (6.52e-1)	3.8836e+0 (6.12e-1) =	3.8586e+0 (6.27e-1) =	3.3895e+0 (6.80e-2) =
	3	150	6.9263e+0 (8.50e-1) =	7.7971e+0 (8.15e-1)	7.2409e+0 (1.23e+0) =	9.0822e+0 (1.11e+0) =	7.6053e+0 (1.30e+0) =
	3	200	1.4584e+1 (1.73e+0) =	1.6442e+1 (8.25e-1)	1.4504e+1 (3.55e+0) =	1.5546e+1 (1.54e+0) =	1.4802e+1 (7.23e-1) =
	3	250	2.2191e+1 (2.40e+0) =	2.1704e+1 (3.87e+0)	2.4310e+1 (2.29e+0) =	2.8338e+1 (9.28e+0) =	2.1335e+1 (2.82e+0) =
	3	500	9.3920e+1 (1.78e+1) =	8.9006e+1 (1.17e+1)	8.0480e+1 (5.38e+0) =	8.6549e+1 (9.03e+0) =	8.0611e+1 (6.14e+0) =
	3	1000	3.1176e+2 (1.35e+1) =	3.4100e+2 (4.50e+1)	3.2783e+2 (3.53e+1) =	3.6301e+2 (5.30e+1) =	3.4669e+2 (1.24e+1) =
+/-/=			0/0/12	0/0/12	0/0/12	0/0/12	

Table 7: Table (7): Comparative between algorithms by using Inverted Generational distance when (M=4)

Problem	M	D	NSGAII	NSGAIII	SPEA2	PESAII	MOPSODH
	3	5	8.7638e-3 (1.80e-3) =	9.5212e-3 (9.04e-4)	8.6174e-3 (1.51e-3) =	1.0016e-2 (1.74e-3) =	8.8364e-3 (1.60e-3) =
	3	10	3.8275e-2 (1.87e-3) =	3.7414e-2 (5.53e-3)	3.9525e-2 (2.90e-3) =	4.6431e-2 (1.67e-2) =	3.5299e-2 (3.66e-3) =
	3	15	7.9435e-2 (2.31e-2) =	6.7059e-2 (3.34e-3)	8.2517e-2 (1.36e-2) =	7.5517e-2 (7.25e-3) =	8.5704e-2 (1.69e-2) =
	3	20	1.4546e-1 (1.19e-2) =	1.3876e-1 (1.24e-2)	1.5256e-1 (6.46e-3) =	1.5567e-1 (2.76e-2) =	1.2827e-1 (1.81e-2) =
MOAP	3	25	2.2999e-1 (2.77e-2) =	2.1102e-1 (3.25e-2)	2.2502e-1 (1.82e-2) =	2.0215e-1 (9.33e-3) =	2.0230e-1 (7.35e-3) =
	3	50	9.8449e-1 (9.74e-2) =	8.3464e-1 (1.87e-1)	9.9841e-1 (1.86e-1) =	1.0157e+0 (1.58e-1) =	9.4627e-1 (1.75e-1) =
	3	100	4.8561e+0 (1.23e+0) =	4.0878e+0 (6.52e-1)	3.8836e+0 (6.12e-1) =	3.8586e+0 (6.27e-1) =	3.3895e+0 (6.80e-2) =
	3	150	6.9263e+0 (8.50e-1) =	7.7971e+0 (8.15e-1)	7.2409e+0 (1.23e+0) =	9.0822e+0 (1.11e+0) =	7.6053e+0 (1.30e+0) =
	3	200	1.4584e+1 (1.73e+0) =	1.6442e+1 (8.25e-1)	1.4504e+1 (3.55e+0) =	1.5546e+1 (1.54e+0) =	1.4802e+1 (7.23e-1) =
	3	250	2.2191e+1 (2.40e+0) =	2.1704e+1 (3.87e+0)	2.4310e+1 (2.29e+0) =	2.8338e+1 (9.28e+0) =	2.1335e+1 (2.82e+0) =
	3	500	9.3920e+1 (1.78e+1) =	8.9006e+1 (1.17e+1)	8.0480e+1 (5.38e+0) =	8.6549e+1 (9.03e+0) =	8.0611e+1 (6.14e+0) =
	3	1000	3.1176e+2 (1.35e+1) =	3.4100e+2 (4.50e+1)	3.2783e+2 (3.53e+1) =	3.6301e+2 (5.30e+1) =	3.4669e+2 (1.24e+1) =
+/-/=			0/0/12	0/0/12	0/0/12	0/0/12	

Table 8: Table (8): Comparative between algorithms by using Inverted Generational distance when (M=5)

Problem	M	D	NSGAII	NSGAIII	SPEA2	PESAII	MOPSODH
	5	5	6.2201e-3 (6.76e-6) =	7.0160e-3 (7.96e-4) =	6.2108e-3 (1.02e-5) =	6.2102e-3 (1.30e-5) =	6.5660e-3 (3.04e-4)
	5	10	2.7052e-2 (1.57e-3) =	2.9719e-2 (2.46e-4) =	2.7929e-2 (1.69e-3) =	2.7169e-2 (3.07e-3) =	3.0579e-2 (9.69e-4)
	5	15	6.2890e-2 (8.50e-3) =	5.9254e-2 (1.25e-3) =	5.7259e-2 (1.93e-3) =	5.7566e-2 (2.98e-3) =	6.6935e-2 (1.84e-3)
	5	20	1.0039e-1 (8.16e-3) =	1.1888e-1 (6.60e-3) =	1.1657e-1 (2.71e-2) =	1.0128e-1 (1.76e-3) =	1.0129e-1 (6.52e-3)
	5	25	1.6449e-1 (1.30e-2) =	1.7703e-1 (1.45e-2) =	1.6654e-1 (2.09e-2) =	1.6048e-1 (1.19e-2) =	1.9133e-1 (3.95e-2)
MOAP	5	50	6.4229e-1 (3.75e-2) =	7.9225e-1 (1.14e-1) =	6.8370e-1 (1.00e-1) =	7.2307e-1 (1.48e-1) =	6.8890e-1 (7.35e-2)
	5	100	2.8454e+0 (2.09e-1) =	2.7929e+0 (2.05e-1) =	2.3871e+0 (1.41e-2) =	2.5909e+0 (1.96e-1) =	3.0128e+0 (5.23e-1)
	5	150	6.0505e+0 (6.39e-1) =	5.8471e+0 (1.79e-1) =	6.0516e+0 (2.88e-1) =	5.8859e+0 (8.65e-1) =	5.9409e+0 (5.43e-1)
	5	200	1.0677e+1 (7.21e-1) =	1.1026e+1 (8.88e-1) =	1.0766e+1 (1.26e+0) =	1.4437e+1 (3.57e+0) =	1.0498e+1 (1.19e+0)
	5	250	1.9098e+1 (2.74e+0) =	1.8985e+1 (1.23e+0) =	1.7538e+1 (1.23e+0) =	1.7088e+1 (2.00e+0) =	1.7290e+1 (1.37e-1)
	5	500	6.6312e+1 (6.85e+0) =	6.9702e+1 (4.77e+0) =	6.9473e+1 (2.39e+0) =	6.7029e+1 (4.64e+0) =	6.4727e+1 (5.31e+0)
	5	1000	2.5046e+2 (1.83e+1) =	2.7881e+2 (7.76e+0) =	2.5666e+2 (1.37e+1) =	2.7149e+2 (2.08e+1) =	2.5650e+2 (2.66e+1)
+/-/=			0/0/12	0/0/12	0/0/12	0/0/12	

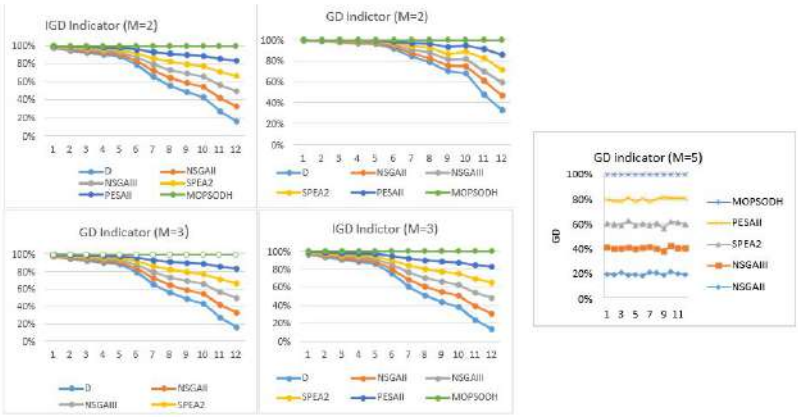


Figure 1 Shows the Priority of the Proposed Method (MOPSOD) with Others When M=2 and 3 |

4. Conclusion

This article suggests a decompose strategy-based MOPSO (MOPSO-D), in which MOPs are broken down into several subproblems of scalar optimization, and each one of them is optimized solely by using the information introduced from its several neighborhood subproblems in a single execution (run). Two performance metrics (γ and Δ), both show that MOPSO-D is highly competitive and even more effective than selected MOPSOs. The Pareto fronts also demonstrate that the MOPSO-D is able in comparison to the chosen MOPSOs to make comparatively better distributed Pareto fronts.

For further studies, it is attractive to extend the proposed algorithm and apply the technique to other evolutionary algorithms similar to Differential Evolution (DE). It is also important to expand our approach for resolution of additional performance criteria or even multi-objective parallel machines that plan problems. In addition, the presented approach can be applied for more complex machine environments to examine its validity and performance.

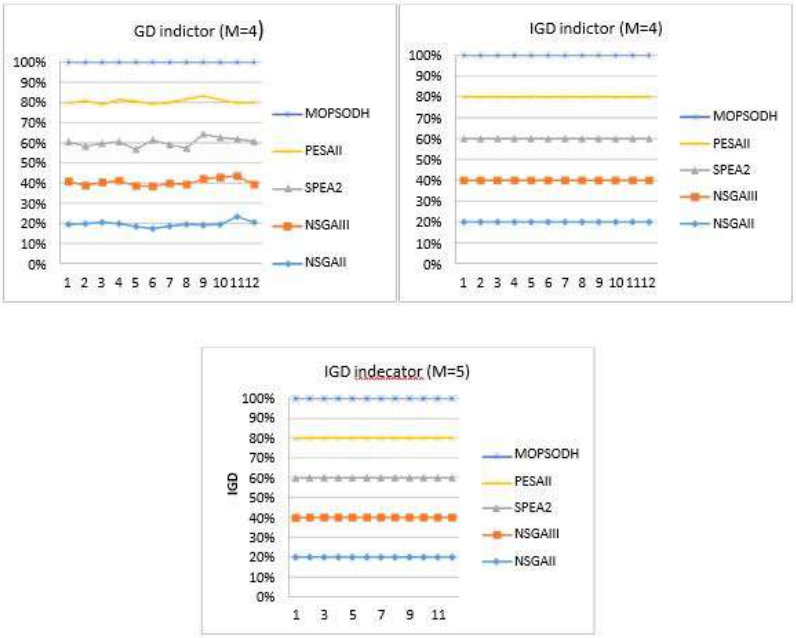


Figure 2 Shows the Priority of the Proposed Method (MOPSO/D) with Others When M=4and 5

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An extended outranking approach for multi attribute group decision making problems with intuitionistic fuzzy data

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Abstract. In group decision making problems, decision matrix depends on different attributes. The weight of each attribute indicates its relative importance comparing to others. The objective of this paper, however, is to introduce a linear model in order to determine the importance of each attribute in multi attribute group decision making problems with intuitionistic fuzzy data; so that, this model prevents the decision makers from making mistake in determining the weight and shape of standardization of their mental measurement units. Next, using the determined weights, a new method of ranking alternatives based on ELECTRE III is introduced and illustrated. In this method, group satisfaction index is used to exclude decision makers' personal opinion for selection the best alternative. Finally, a numerical example is considered to elucidate the details of the proposed method, and then the obtained results are compared with other current methods.

Keywords: intuitionistic fuzzy set, multi attribute group decision making, ranking, weight of attributes.

1. Introduction

The human is always seeking a balance between his/her requierments and goals even in making the tiniest personal decisions. For this necessity, he/she needs to use multi criteria decision making (MCDM) as a solution, which is one of the subsets of decision making. In this subset of decision making, the

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determiner selects the best choice based on some criteria from the options he/she has.

MCDM is the accepted designation for all concepts dealing with multi objective decision making (MODM) and/or multi attribute decisin making (MADM). The main difference between MODM and MADM is related to the definition of alternatives. In MODM, criteria is defined implicitly by a mathematical programming structure that results with continuous alternatives, while in MADM, the set of decision alternatives is defined explicitly by a finite list of alternative actions where discrete alternatives exit.

As lots of attributes that are sometimes antithetical involve in choosing the most appropriate alternative, this issue will be investigated as a multi attribute group decision making (MAGDM) problem. Since the decision matrix has different attributes in group decision making (GDM) models, being aware of the importance coefficient or weight for each attribute is necessary in decision making. Each attribute weight expresses its relative importance toward others. Choosing the weights knowingly and correctly is a great help in order to reach the intended target.

Imprecise and uncertainity data have considerable influence in studying and solving decision making and optimization problems. These data contain statistical or random, fuzzy, interval, rough and even a combination of the aforementioned imprecise data. One of the most important imprecise data is fuzzy set theory proposed by Zadeh [36]. He established the basics of modelling the imprecise data and approximate reasoning with mathematical equations by introducing the fuzzy sets theory which caused a massive change of its kind in mathematics and classical logic. As the distribution and density functions play the fundamental role in random or statistical environments, membership function has the main role in fuzzy data.

One of the other imprecise data is the intuitionistic fuzzy sets (IFSs) proposed by Attanassov [1] which are an extension of ordinary fuzzy sets. He believed that the non-membership degree does not always equal to one minus membership degree of ordinary fuzzy sets and there may be hesitancies in its value, too. Therefore, he introduced the IFSs that have both membership and non-membership degrees for modelling uncertainty concept in real world problems. A lot of researchers in the area of intuitionistic fuzziness, used another concept like intuitionistic fuzzy pairs [3]. Also, Attanassov [2] offered a comprehensive survey of intuitionistic fuzzy logics by reporting on both the author's research and others' findings.

Xu [28] investigated the GDM problems in which all the provided information by the decision makers (DMs) is expressed as intuitionistic fuzzy decision matrices and the information about attribute weights is partially known. A deviation-based approach to intuitionistic fuzzy MAGDM has been introduced by Xu [29] in which attributes importance is determined through an specified method. Wang et al. [25] employed projection method for MAGDM problems in intuitionistic fuzzy environment, where the weights of attributes and DMs

are predetermined. Intuitionistic fuzzy MAGDM models based on the information entropy weights for attributes have been presented by Duan et al. [8]. Yue [35] presented a TOPSIS-based fuzzy GDM methodology based on IFSs for fuzzy MAGDM where the weight of attributes is predetermined. Chen and Tan [7] proposed techniques which allow that degrees of satisfiability and non-satisfiability of each alternative with respect to a set of criteria to be presented by vague values. Furthermore, the proposed techniques allow the DM to assign a different degree of importance to each attribute. Xian et al. [27] introduced an operator for linguistic fuzzy data and evaluated its usage in GDM where the weight of each attribute is predetermined. A method for determining weights of DMs with interval numbers under GDM environment has been developed by Yue [34] . Chen et al. [6] studied the MAGDM problems with IFSs and evidential reasoning method in which weight or importance of each attribute and DMs is inductively specified. Wu et al. [26] proposed a model for GDM based on a fuzzy VIKOR approach using linguistic variables in which the attribute importance is obtained by linguistic variables. Sevastjanov and Dymova [18] studied the ranking of alternatives using hesitant fuzzy values within the Dempster-Shafer theory (DST) in which the attribute importance is predetermined.

Elimination and choice translating reality (ELECTRE) method is one of the most well-known ranking methods in decision making problems. This methodology has been proposed by Benayoun [4] and then has been developed by Van Delft and Nijkamp [22]. The dominance concept is used implicitly in this method and the alternatives are compared with each other pairwise, then dominant and weak (nondominant) alternatives are recognized and finally the weak or non-dominant alternatives are deleted [17]. Due to the complexity of new decision making problems and developing usage of MADM methods, different versions of this method have been proposed as its subsets. The primary proposed method by Benayon has been noted as ELCTRE I. Since the principle of all the new methods are the same as ELECTRE I, they have been named ELECTRE II, ELECTRE III, ELECTRE IV and ELECTRE TRI. The difference in various versions is in the mathematical operations and the kind of problems these methods are able to solve. Method I and TRI are specifically used to solve selection and allocation problems, respectively. Methods II, III and IV are used for ranking problems [23]. Shen et al. [19, 20] proposed two ranking models using ELECTRE III in MCGDM with intuitionistic fuzzy data in which the weight of criteria and DMs are previously specified. Hashemi et al. [10] evaluated the GDM problems using ELECTRE III with interval-valued intuitionistic fuzzy data in which each criterion weight is defined as interval-valued intuitionistic fuzzy values by experts however group satisfaction index is not applied.

In most of the performed studies for solving decision making problems, the importance of each attribute is either specified with ascribing a weight by the DM or with one of the defined methods, for example entropy, LINMAP, weighted least squares, eigenvector, etc. In this paper, it is tried to determine and evaluate the importance of each attribute by solving a linear programming model. Then,

an introduction to a ranking model based on ELECTRE III is carried out. Also, an MAGDM approach based on intuitionistic information is proposed. Preliminaries and the mathematical formulation is presented in Sect. 2. In Sect. 3, the ELECTRE III method and linear programming are briefly described. The proposed algorithmic method is illustrated in Sect. 4. Finally, short conclusions are given in Sect. 5.

2. Preliminaries

In the following, the definition and operations of IFSs are briefly reviewed.

Definition 2.1. *Let X be a reference set. A fuzzy set $\tilde{A}$ in X can be defined as $\tilde{A} = \{\langle x, \mu_{\tilde{A}}(x)\rangle | x \in X\}$, where the function $\mu_{\tilde{A}} : X \to [0,1]$ is the membership function of $\tilde{A}$. The function value $\mu_{\tilde{A}}(x)$ is called the grade of membership of x in $\tilde{A}$.*

Definition 2.2. *Let X be fixed. An IFS P in X is defined as $P = \{\langle x, \mu_P(x), \nu_P(x)\rangle | x \in X\}$, where the functions $\mu_P : X \to [0,1]$ and $\nu_P : X \to [0,1]$ are membership function and non-membership function of IFS P, respectively. The hesitancy degree of x in P is $\pi_P(x) = 1 - \mu_P(x) - \nu_P(x)$.*

Remark 2.1. For every $x \in X, 0 \leq \mu_P(x) \leq 1, 0 \leq \nu_P(x) \leq 1, 0 \leq \mu_P(x) + \nu_P(x) \leq 1$ and it can be seen that $\pi_P(X) \in [0,1]$. Xu [30] named $(\mu_\alpha, \nu_\alpha, \pi_\alpha)$ intuitionistic fuzzy number (IFN), denoted as α where $0 \leq \mu_\alpha \leq 1$, $0 \leq \nu_\alpha \leq 1$, $0 \leq \mu_\alpha + \nu_\alpha \leq 1$ and $\pi_\alpha = 1 - \mu_\alpha - \nu_\alpha$. If $\mu_\alpha + \nu_\alpha = 1$ then the $(\mu_\alpha, \nu_\alpha, \pi_\alpha)$ reduced to $(\mu_\alpha, 1 - \nu_\alpha)$.

Definition 2.3 ([7, 11])**.** *Let $\alpha = (\mu_\alpha, \nu_\alpha, \pi_\alpha)$ be an IFN. Score function $s(\alpha)$ which is used to measure the degree of suitability and the accuracy function $h(\alpha)$ which is used to evaluate the accuracy degree of α, as follows:*

$$s(\alpha) = \mu_\alpha - \nu_\alpha,$$
$$h(\alpha) = \mu_\alpha + \nu_\alpha.$$

In many theoretical and practical decision making problems, DMs often need to compare two IFSs or IFNs, in the same universe. Xu and Yager [31] presented some relations between any two IFNs α and β.

Definition 2.4. *Let $\alpha = (\mu_\alpha, \nu_\alpha, \pi_\alpha)$ and $\beta = (\mu_\beta, \nu_\beta, \pi_\beta)$ be two IFNs, the score and accuracy degree of α and β are defined as $s(\alpha)$ and $h(\alpha)$, respectively. Then:*

- $\alpha \prec \beta$ *if* $s(\alpha) < s(\beta)$
- $\alpha \succ \beta$ *if* $s(\alpha) > s(\beta)$

- *if* $s(\alpha) = s(\beta)$ *then*

$$\begin{cases} \alpha \prec \beta, & \text{if } h(\alpha) < h(\beta); \\ \alpha \succ \beta, & \text{if } h(\alpha) > h(\beta); \\ \alpha \approx \beta, & \text{if } h(\alpha) = h(\beta). \end{cases}$$

Definition 2.5 ([21]). *Let* $\alpha = (\mu_\alpha, \nu_\alpha, \pi_\alpha)$ *and* $\beta = (\mu_\beta, \nu_\beta, \pi_\beta)$ *be two IFNs, an intuitionistic fuzzy distance between* α *and* β *is defined as follows:*

$$d_{IFN}(\alpha, \beta) = \sqrt{\frac{1}{2}[(\mu_\alpha - \mu_\beta)^2 + (\nu_\alpha - \nu_\beta)^2 + (\pi_\alpha - \pi_\beta)^2]} \tag{1}$$

Lemma 2.1 ([30]). *Let* $\alpha = (\mu_\alpha, \nu_\alpha, \pi_\alpha)$ *and* $\beta = (\mu_\beta, \nu_\beta, \pi_\beta)$ *be two IFNs, then* $0 \leq d_{IFN}(\alpha, \beta) \leq 1$, $d_{IFN}(\alpha, \alpha) = 0$ *and* $d_{IFN}(\alpha, \beta) = d_{IFN}(\beta, \alpha)$.

3. The proposed MAGDM approach based on intuitionistic fuzzy information

In the MAGDM ranking and selection problems, the final decision is result from the preferences of a group of DMs. Uncertainty data have considerable influence in studying and solving decision making problems. For a given MAGDM problem under intuitionistic environment the decision matrix can be constructed based on IFNs for each DM. These problems consist of m alternatives, denoted by $A = \{a_1, a_2, \ldots, a_m\}$, and n attributes denoted by $C = \{c_1, c_2, \ldots, c_n\}$. Also, let $D = \{d_1, d_2, \ldots, d_q\}$ be the set of DMs. The decision matrix of DM $d_k; (k = 1, 2, .., q)$ is defined as $R_k = (r_{ijk})_{m \times n}$ in which $r_{ijk}; (i = 1, 2, \ldots, m, j = 1, 2, \ldots, n, k = 1, 2, .., q)$ is an IFN and is provided by DM d_k for the alternative a_i with respect to the attribute c_j.

3.1 The weight of attributes

In MADM techniques for evaluation of alternatives based on the attributes, the optimal alternative is selected. In almost of techniques, weight of attributes are not determined through specific methods. In most of these techniques, determining weight of attributes is based on DMs' ideas. Also in a series of the other techniques, DMs have calculated the weight by proposing a special model. In these models, there are different alternatives that are evaluated based on several attributes independently. Finally, alternatives based on the obtained values, are ranked. The importance coefficient or weight of each attribute is essential in decision making. The weight of each attribute indicates its relative importance comparing to others. A knowing and accurate selection of weights will be a great help in the way of achieving the intended goal. Therefore a linear model in order to determine the importance of each attribute in MAGDM problems with intuitionistic fuzzy data is introduced.

Let d_{ijk} be the distance of alternative $a_i; (i = 1, 2, \ldots, m)$ with respect to the attribute $c_j; (j = 1, 2, \ldots, n)$ from the ideal alternative $a^*_{jk}; (j = 1, 2, \ldots, n, k = 1, 2, \ldots, q)$, by DM $d_k; (k = 1, 2, \ldots, q)$, where d_{ijk} is defined as follows:

$$d_{ijk} = \sqrt{\tfrac{1}{2}[(\mu_{r_{ijk}} - \mu_{a^*_{jk}})^2 + (\nu_{r_{ijk}} - \nu_{a^*_{jk}})^2 + (\pi_{r_{ijk}} - \pi_{a^*_{jk}})^2]}$$

such that a^*_{jk} is calculated with Algorithms 3.1 and 3.2 in the following.

Algorithm 3.1.

Step 1. *For each attribute c_j with positive aspect in decision matrix R_k, find:*

$$\Gamma_1 := \{t \ \ | \mu_{r_{tjk}} = \max_{1 \le i \le m} (\mu_{r_{ijk}})\}$$

Step 2. *If $|\Gamma_1| = 1$, define $a^*_{jk} = (\mu_{r_{tjk}}, \nu_{r_{tjk}}, \pi_{r_{tjk}})$, $t \in \Gamma_1$. Else:*

$$a^*_{jk} = (\mu_{r_{tjk}}, \min_{t \in \Gamma_1}(\nu_{r_{tjk}}), \pi_{r_{tjk}}).$$

Algorithm 3.2.

Step 1. *For each attribute c_j with negative aspect in decision matrix R_k, find:*

$$\Gamma_2 := \{t \ \ | \nu_{r_{tjk}} = \max_{1 \le i \le m} (\nu_{r_{ijk}})\}$$

Step 2. *If $|\Gamma_2| = 1$, define $a^*_{jk} = (\mu_{r_{tjk}}, \nu_{r_{tjk}}, \pi_{r_{tjk}})$, $t \in \Gamma_2$. Else:*

$$a^*_{jk} = (\min_{t \in \Gamma_2}(\mu_{r_{tjk}}), \nu_{r_{tjk}}, \pi_{r_{tjk}}).$$

Now, consider the linear programming model (P), as follows:

$$(2) \qquad \begin{array}{lll} (P) & \gamma := & \max \frac{1}{q} \sum_{k=1}^{q} \varepsilon_k \\ s.t. & & \sum_{j=1}^{n} x_j = 1 \\ & & x_j \bar{a}_{jk} - \varepsilon_k \ge 0 \quad ; j = 1, 2, \ldots, n; k = 1, 2, \ldots, q. \end{array}$$

where x_j's are decision variables and $\bar{a}_{jk}$'s are obtained from the following relations:

$$(3) \qquad \begin{cases} \bar{a}_{jk} := \exp(\mu_{a^*_{jk}}), & \text{For attributes with positive aspect} \\ \bar{a}_{jk} := \exp(\nu_{a^*_{jk}}), & \text{For attributes with negative aspect} \end{cases}$$

it is clear that $\bar{a}_{jk} > 0$.

Lemma 3.1. *Let γ is the optimal solution of the linear programming model (2). Then $\gamma > 0$.*

Proof of Lemma 3.1. At first, consider the dual of (2), i.e. the problem (4), as follows:

$$
\begin{array}{lll}
(D) & \min & \alpha \\
s.t. & \alpha + \sum_{k=1}^{q} \beta_{jk}\bar{a}_{jk} = 0; & j = 1, 2, \ldots, n \\
& -\sum_{j=1}^{n} \beta_{jk} = \frac{1}{q}; & k = 1, 2, \ldots, q \\
& \beta_{jk} \leqslant 0; & k = 1, 2, \ldots, q, j = 1, 2, \ldots, n.
\end{array} \tag{4}
$$

Now, by contradiction assume that γ not greater than 0, i.e. $\gamma \leq 0$. If $\gamma = 0$ then according to the dual problem $\alpha = \gamma = 0$. Therefore $\sum_{k=1}^{q} \beta_{jk}\bar{a}_{jk} = 0, \forall j$. It is contradict with constraints in (4). If $\gamma < 0$ then $\alpha = \gamma < 0$, but it is also contradict with constraints in (4) and relations (3).

According to the previous explenation, let d_{ijk} is the distance of alternative a_i respect to the attribute c_j from the ideal alternative $a^*_{jk}; (j = 1, 2, \ldots, n, k = 1, 2, \ldots, q)$, in decision matrix R_k which can be obtained from the Algorithm 3.1 or Algorithm 3.2, γ and $\bar{a}_{jk}; (j = 1, 2, \ldots, n, k = 1, 2, \ldots, q)$ calculated from (2) and (3), respectively.

Now, let $W_k = \{w_{1k}, w_{2k}, \ldots, w_{nk}\}; (k = 1, 2, \ldots, q)$ be a vector of weights for n attributes corresponsive with each decision matrix $R_k; (k = 1, 2, \ldots, q)$, then this vector can be obtained by the following linear programming model:

$$
\begin{array}{lll}
\min & \sum_{i=1}^{m}\sum_{j=1}^{n} w_{jk}d_{ijk} & \\
s.t. & \sum_{j=1}^{n} w_{jk} = 1 & \\
& w_{jk}\bar{a}_{jk} \geq \gamma\ ; & j = 1, 2, \ldots, n.
\end{array} \tag{5}
$$

3.2 Ranking the alternatives

In this section ,by the weight vectors $W_k = \{w_{1k}, w_{2k}, \ldots, w_{nk}\}\ ; (k = 1, 2, \ldots, q)$, which are obtained by the stated Algorithms and models (specially (5)), a new ranking method is introduced based on ELECTRE III method with intuitionistic fuzzy information. In this method, after constructing the intuitionistic fuzzy decision matrix via DMs and the definition of the thresholds p, q and v, concordance and discordance matrixes are presented based on score function. At last, through the proposed group satisfaction degree by Goshtasby [9], the last ranking of alternatives are achieved.

Step 1 (create decision matrix). The first step is determination of the decision matrix. The characteristic of alternatives in decision matrix respect to the attributes are considered as IFN. The matrix elements related to alternative

$a_i; (i = 1, 2, \ldots, m)$, and attribute $c_j; (j = 1, 2, \ldots, n)$, by the DM $d_k; (k = 1, 2, \ldots, q)$, which is displayed as r_{ijk}.

Step 2 (define thresholds and assign weights to the attributes). In this step, in order to strengthen of the ability of identifing superior alternatives, three new thresholds are introduced as follows:

- The preference threshold p_j, which the DM shows a strongly preference of one alternative over all the others in therms of attribute c_j.
- The indifference threshold q_j, which the DM is indifferent between two alternatives outranking relationship with respect to attribute c_j.
- The veto threshold v_j, blocks the outranking relationship between alternatives for the attribute c_j.

One of the other important parameters in choosing the appropriate alternative in this step, is weight or degree of importance of each attribute. For this purpose, the weight vector W_k is obtained using the model (5) after solving (2) problem.

Step 3 (create concordance matrix for each attribute). At this level using the constructed decision matrices in the first step, the thresholds in the second step and the alternatives' relation for each attribute by any DM, a concordance matrix $\widetilde{C}_k$ is formed by any DM $d_k; (k = 1, 2, \ldots, q)$. Each of the concordance matrix's elements using the following formula are obtained:

$$\widetilde{C}_k(a_i, a_l) = \prod_{j=1}^{n} (\widetilde{c}_{jk}(a_i, a_l))^{w_{jk}}; \quad k = 1, 2, \ldots, q, \forall i \neq l \tag{6}$$

such that

$$\widetilde{c}_{jk}(a_i, a_l) = \begin{cases} 1, & \text{if } s_{ij} + q_j \geq s_{lj} \\ 0, & \text{if } s_{ij} + p_j \leq s_{lj} \\ \frac{p_j - (s_{lj} - s_{ij})}{p_j - q_j} & \text{otherwise} \end{cases} \tag{7}$$

where s_{ij} is the score function for the alternative a_i with respect to the attribute c_j i.e. $s_{ij} = \mu_{a_{ij}} - \nu_{a_{ij}}$.

Step 4 (create the discordance matrix for each attribute). As the third step, using the decision matrix and the thresholds set, a discordance matrix $\widetilde{D}$ is formed by any DM $d_k; (k = 1, 2, \ldots, q)$. Each of the discordance matrix's elements using the following formula are comparable:

$$\widetilde{D}_k(a_i, a_l) = \prod_{j=1}^{n} (\widetilde{d}_{jk}(a_i, a_l))^{w_{jk}}; \quad k = 1, 2, \ldots, q, \forall i \neq l \tag{8}$$

such that $\widetilde{d}_{jk}$ with respect to each attribute c_j by each DM using the following relations is formed:

$$\widetilde{d}_{jk}(a_i, a_l) = \begin{cases} 0, & \text{if } s_{ij} + p_j \geq s_{lj} \\ 1, & \text{if } s_{ij} + v_j \leq s_{lj} \\ \frac{s_{ij}+p_j-s_{lj}}{p_j-v_j}, & \text{otherwise} \end{cases}. \tag{9}$$

Note that, $\widetilde{d}_{jk}(a_i, a_l)$ shows the credibility of the proposition which alternative a_i is not at least as good as alternative a_l.

Remark 3.1. Note that if $s_j(a_i)$ be the score function between alternative $a_i; (i = 1, 2, \ldots, m)$ and ideal alternative with respect to the benefit or cost attribute $c_j; (j = 1, 2, \ldots, n)$, then the concept of the thresholds of preference (p_j), indifference (q_j) and veto (v_j) aims at defining the statement $a_i S a_l, (i, l = 1, 2, \ldots, m)$ for every couple a_i and a_l of the alternatives set. The statement $a_i S a_l$ means that alternative a_i outranks alternative a_l, when a_i is at least as good as a_l in the most of the attributes and never significantly worse in the rest of them. Respectively, the statement $a_i S_j a_l$ is defined for every attribute c_j. Then the attribute c_j is in agreement with $a_i S a_l$ if only $a_i S_j a_l$ and even if $s_j(a_i) > s_j(a_l) - q_j$. The ELECTRE III method is based on the definition of two matrices, the concordance and the discordance matrices, which determine if the statement $a_i S a_l$ is acceptable. The rule of concordance requests that the majority of the attributes, have to be in favour of the statement $a_i S a_l$, while the rule of discordance requests that no attribute from the minority that does not support the statement $a_i S a_l$ be strongly against it. The veto threshold (v_j) allows the complete rejection of the $a_i S a_l$ statement when the relation $s_j(a_l) > s_j(a_i) + v_j$ is valid for every attribute c_j in concordance matrices.

Step 5 (create the credibility matrix). After creating the concordance matrix $\widetilde{C}_k$ and discordance matrix $\widetilde{D}_k$, the credibility matrix $\widetilde{S}_k(a_i, a_l), \forall i \neq l$ for each DM $d_k; (k = 1, 2, \ldots, q)$ among the different alternatives and based on these two matrices is formed. Outranking matrix elements' using the following mathematical relationship can be calculated:

$$\widetilde{S}_k(a_i, a_l) = \begin{cases} \widetilde{C}_k(a_i, a_l), & \text{if } \widetilde{D}_k(a_i, a_l) \leq \widetilde{C}_k(a_i, a_l), \\ \widetilde{D}_k(a_i, a_l), & \text{if } \widetilde{D}_k(a_i, a_l) > \widetilde{C}_k(a_i, a_l). \end{cases} \tag{10}$$

Step 6 (create the net outranking flow matrix). According to the credibility matrix $\widetilde{S}_k$, the net outranking flow matrix $F_k(a_i, a_l); (k = 1, 2, \ldots, q)$ in order to represnet the net outranking alternative a_i over a_l is stated as follows:

$$F_k(a_i, a_l) = \widetilde{S}_k(a_i, a_l) - \widetilde{S}_k(a_l, a_i); \quad \forall i \neq l. \tag{11}$$

Step 7 (the net outranking flow index). In this step, the net outranking flow index $\Phi_k(a_i); (i = 1, 2, \ldots, m)$, i.e. the net outranking alternative a_i is

overall to the remaining alternatives, is calculated with the following formula:

$$\Phi_k(a_i) = \sum_{l=1, l\neq i}^{m} F_k(a_i, a_l), \quad \forall k = 1, 2, \ldots, q. \tag{12}$$

Therefore, according to the net outranking flow index $\Phi_k(a_i)$, the alternatives can be ranked.

Step 8 (the group net outranking flow index). The group net outranking flow index $\Phi_G(a_i); (i = 1, 2, \ldots, m)$, is represented as follows:

$$\Phi_G(a_i) = \sum_{k=1}^{q} \lambda_k \Phi_k(a_i). \tag{13}$$

where $\lambda_k; (k = 1, 2, \ldots, q)$, indicate the DMs weights.

Note that, $\Phi_G(a_i)$ shows the net outranking alternative a_i is overall to the other alternatives.

Step 9 (the group's satisfaction index). To gain the group's satisfaction index, the greatest deviation [9] is used between individual and group preorders. Assume $u_k(a_i)$ is the ranking order of alternative a_i and $U(a_i)$ is the group ranking order based on the obtained values using (12) and (13), respectively. Also, let:

$$d_i^k = \sum_{j=1}^{i} I\left[u_k(a_i) \leq i < U(a_j)\right]; \quad \forall k \tag{14}$$

where $I[E] = 1$ if E is true and $I[E] = 0$ if E is false. Finally, let

$$D_i^k = \sum\nolimits_{j=1}^{i} I\left[n + 1 - u_k(a_i) > U(a_j)\right]; \quad \forall k. \tag{15}$$

Then the greatest deviation between $u_k(a_i)$ and $U(a_i)$ is calculated as follows:

$$R_g^k = \frac{\max\limits_i D_i^k - \max\limits_i d_i^k}{\frac{m}{2}}; \quad \forall k. \tag{16}$$

Note that for each k, $-1 \leq R_g^k \leq 1$, R_g^k close to 1 means that there is little difference between the personal ranking order and group sorting result. Also, R_g^k close to -1 means that the group sorting result is almost the reverse of the personal order, therefore the personal disagreement is maximized.

Then the group's satisfaction index is:

$$R_G = \sum_{k=1}^{q} \lambda_k R_g^k. \tag{17}$$

Remark 3.2. If $R_G \geq \theta$, where $\theta(-1 \leq \theta \leq 1)$ is the threshold of the acceptable group satisfaction level, then R_G is an acceptable group satisfaction degree and if $R_G < \theta$, then R_G is an unacceptable group satisfaction degree, such that θ is usually determined by the DMs.

This satisfaction degree which is very useful in the interactive procedures is required to reach a consensus assignments. The DMs rank the alternatives according to the net outranking flow indices. Therefore, if the majority of preferences of the net outranking flow matrix $F_k(a_i, a_l)$ is close and a few of DMs' preferences are in disagreement, it is reasonable to change only these particular preferences.

Step 10 (the weighted deviation degree). Let $F_k(a_i, a_l) = f_{il(k)}$ and $\prod_{il} = \{f_{il(1)}, f_{il(2)}, \ldots, f_{il(q)}\}$ be the set of DMs' net outranking flow values, where $f_{il(k)}$ is the net outranking flow value of DM d_k with respect to the pair of alternatives $(a_i, a_l); (a_i, a_l \in A, i \neq l)$. Then, the weighted deviation degree ξ_{il} is calculated as follows:

$$\xi_{il} = \sqrt{\sum_{k=1}^{q} \lambda_k (f_{il(k)} - \overline{f}_{il})^2} \tag{18}$$

where $\overline{f}_{il} = \sum_{k=1}^{q} \lambda_k f_{il(k)}$ is the weighted mean of the set $\prod_{il}$.

The automatic adjustment of the group satisfaction consists of two processes (see, [19]):

- Choose the maximum element $\xi_{i^*l^*}$ from the matrix $\xi = [\xi_{il}]_{m\times m}$, then find the element $f_{i^*l^*(k^*)}$ using $\lambda_{k^*}(f_{i^*l^*(k^*)} - \overline{f}_{i^*l^*})^2 = \max\limits_{1\leq k\leq q} \lambda_k (f_{i^*l^*(k)} - \overline{f}_{i^*l^*})^2$.

- Replace $f_{i^*l^*(k^*)}$ with a numerical value $\acute{f}_{i^*l^*(k^*)} = \overline{f}_{il}$, and other elements do not change i.e., let

$$f_{il(k)}^{h+1} = \begin{cases} \overline{f}_{il}^{h}, & \text{if } i = i^*, l = l^*, and \;\; k = k^* \\ f_{il(k)}^{h}, & \text{otherwise.} \end{cases} \tag{19}$$

Let $\prod_{il}^{0} = \{f_{il(1)}^{0}, f_{il(2)}^{0}, \ldots, f_{il(q)}^{0}\}$ be an original set of the DMs' net outranking flow values and ξ_{il}^{0} be its weighted deviation degree, and $\prod_{il}^{1} = \{f_{il(1)}^{1}, f_{il(2)}^{1}, \ldots, f_{il(q)}^{1}\}$ be the corresponding first adjusted set and ξ_{il}^{1} be its corresponding weighted deviation degree, obviously then, ξ_{il}^{1} is smaller than ξ_{il}^{0}. Moreover, if one continue to change some elements of the set in this method, the weighted deviation degree approaches zero. Obviously, the recent automatic adjustment strategy is convergent. In fact, the following theorem is stated toward convergence.

Theorem 3.1. *Let* $\prod_{il}^{h} = \{f_{il(1)}^{h}, f_{il(2)}^{h}, \ldots, f_{il(q)}^{h}\}; (i, l = 1, 2, ...m, i \neq l)$ *be a set of the DMs' net outranking flow values after* h *times, and the above adjustment* ξ_{il}^{h} *be the corresponding weighted deviation degree, then*

$$\lim_{h \longrightarrow \infty} \xi_{il}^{h} = 0 \tag{20}$$

Proof of Theorem 3.1. See [19].

4. Numerical example

Due to globalization, supply chains are getting more and more risky than before. Supplier selection is one of the main concern in the context of supply chain network. In essence, supplier evaluation and selection can be taken into account as an MAGDM problem. In order to elucidate the details of the proposed method in this paper and demonstrate the method in practice, in the following a supplier evaluation problem in a high-tech company is considered.

According to the numerical example that is mentioned by Shen et al. [19], consider a high-technology company which manufactures electronic products intends to rank its five material suppliers $(a_1, a_2, a_3, a_4, a_5)$. For this purpose, the company manager decided to use the GDM to evaluate these five suppliers. Three DMs d_1, d_2 and d_3 are established; an engineering expert, a financial expert, and a quality control expert. The weight vector of experts is considered as $\lambda = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})^T$. To evaluate the five suppliers, company manager decided on four attributes; cost control (c_1), performance (c_2), technique (c_3) and service (c_4).

In order to determine the decision evaluation, the experts may not have a precise or sufficient level of knowledge of problem, and so they maybe unable to be confident about their judgments. To deal with this uncertainty, the experts' preference values are expressed using IFNs.

Three experts (DMs) give their own opinions about the performance of $a_i; (i = 1, \ldots, 5)$ with respect to each attribute $c_j; (j = 1, \ldots, 4)$, respectively. Their decision matrices with the intuitionistic fuzzy evaluation information are R_1, R_2 and R_3, as follows:

$$R_1 = \begin{pmatrix} (0.45, 0.55, 0.00) & (0.40, 0.55, 0.05) & (0.70, 0.30, 0.00) & (0.70, 0.20, 0.10) \\ (0.50, 0.40, 0.10) & (0.70, 0.30, 0.00) & (0.60, 0.40, 0.00) & (0.60, 0.35, 0.05) \\ (0.70, 0.25, 0.05) & (0.40, 0.55, 0.05) & (0.40, 0.55, 0.05) & (0.90, 0.10, 0.00) \\ (0.40, 0.55, 0.05) & (0.50, 0.50, 0.00) & (0.70, 0.25, 0.05) & (0.60, 0.40, 0.00) \\ (0.40, 0.60, 0.00) & (0.40, 0.50, 0.10) & (0.50, 0.50, 0.00) & (0.50, 0.45, 0.05) \end{pmatrix}$$

$$R_2 = \begin{pmatrix} (0.50, 0.40, 0.10) & (0.40, 0.50, 0.10) & (0.80, 0.20, 0.00) & (0.60, 0.35, 0.05) \\ (0.50, 0.50, 0.00) & (0.60, 0.40, 0.00) & (0.70, 0.30, 0.00) & (0.40, 0.25, 0.05) \\ (0.70, 0.30, 0.00) & (0.40, 0.50, 0.10) & (0.40, 0.60, 0.00) & (0.70, 0.30, 0.00) \\ (0.30, 0.60, 0.10) & (0.60, 0.35, 0.05) & (0.70, 0.25, 0.05) & (0.80, 0.10, 0.10) \\ (0.60, 0.35, 0.05) & (0.50, 0.50, 0.00) & (0.60, 0.30, 0.10) & (0.70, 0.20, 0.10) \end{pmatrix}$$

$$R_3 = \begin{pmatrix} (0.60,0.30,0.10) & (0.40,0.45,0.15) & (0.90,0.10,0.00) & (0.70,0.25,0.05) \\ (0.40,0.55,0.05) & (0.60,0.35,0.05) & (0.65,0.25,0.10) & (0.80,0.15,0.05) \\ (0.65,0.25,0.10) & (0.30,0.65,0.05) & (0.50,0.45,0.05) & (0.50,0.20,0.30) \\ (0.40,0.50,0.10) & (0.60,0.40,0.00) & (0.40,0.60,0.00) & (0.80,0.15,0.05) \\ (0.40,0.60,0.00) & (0.40,0.55,0.05) & (0.60,0.40,0.00) & (0.40,0.60,0.00) \end{pmatrix}$$

The proposed thresholds by three experts are stated in Table 1. Also, the company manager determine the acceptable group satisfaction degree $\theta = 0.6$.

Table 1: Thresholds given by three experts for attributes

DMs	q_k	p_k	ν_k
d_1	0.05	0.15	0.30
d_2	0.10	0.20	0.35
d_3	0.03	0.13	0.28

Firstly, according to the models (5) and (2), the attributes weight vector $W_k; (k = 1, 2, 3)$ for four atributes is obtained, this weights are stated in Table 2.

Table 2: Weights of attributes

k	w_{1k}	w_{2k}	w_{3k}	w_{4k}
1	0.31	0.24	0.24	0.2
2	0.27	0.29	0.22	0.22
3	0.27	0.31	0.2	0.22

Then, concordance matrices and discordance matrices are calculated using equations (6) until (9), as follows:

$$\widetilde{C}_1 = \begin{pmatrix} - & 22.38698 & 10.84675 & 22.39305 & 30.23373 \\ 23.54024 & - & 15.84675 & 34.24289 & 46.08047 \\ 27.54024 & 39.23373 & - & 39.24444 & 53.08047 \\ 15.69349 & 22.38698 & 10.84675 & - & 30.23373 \\ 14.92006 & 21.42019 & 10.26667 & 21.42596 & - \end{pmatrix}$$

$$\widetilde{C}_2 = \begin{pmatrix} - & 55.46885 & 43.02572 & 11.11707 & 38.26373 \\ 47.29283 & - & 36.62566 & 9.575696 & 32.5757 \\ 67.01011 & 67.01011 & - & 13.29286 & 46.29294 \\ 18.71713 & 18.71713 & 14.45427 & - & 12.85857 \\ 65.32744 & 65.32744 & 50.74287 & 12.97564 & - \end{pmatrix}$$

$$\widetilde{C_3} = \begin{pmatrix} - & 21.96581 & 39.99846 & 21.96581 & 49.62013 \\ 19.4719 & - & 29.8636 & 16.84187 & 37.31377 \\ 25.93184 & 22.24334 & - & 22.24334 & 50.17518 \\ 19.93184 & 17.24334 & 30.55351 & - & 38.17518 \\ 17.28098 & 14.92847 & 26.57722 & 14.92847 & - \end{pmatrix}$$

$$\widetilde{D}_1 = \begin{pmatrix} - & 65.40884 & 78.14474 & 50.96463 & 45.81636 \\ 44.05049 & - & 48.39999 & 31.69493 & 28.50063 \\ 17.31465 & 15.97031 & - & 12.43004 & 11.16959 \\ 72.05098 & 66.47623 & 79.42794 & - & 46.56289 \\ 72.05098 & 66.47623 & 79.42794 & 51.79541 & - \end{pmatrix}$$

$$\widetilde{D}_2 = \begin{pmatrix} - & 40.81855 & 72.61192 & 78.26022 & 76.24667 \\ 60.90851 & - & 79.21295 & 85.39545 & 83.19147 \\ 37.93031 & 27.73641 & - & 53.07555 & 51.71267 \\ 63.64196 & 46.35283 & 82.80566 & - & 86.97128 \\ 52.41364 & 38.28975 & 68.09317 & 73.38807 & - \end{pmatrix}$$

$$\widetilde{D}_3 = \begin{pmatrix} - & 29.62621 & 18.87567 & 29.62621 & 6.495606 \\ 56.64008 & - & 38.23261 & 61.19735 & 13.48327 \\ 26.53669 & 28.58238 & - & 28.58238 & 6.266492 \\ 55.55732 & 60.02837 & 37.49784 & - & 13.22478 \\ 56.96951 & 61.55227 & 38.45858 & 61.55227 & - \end{pmatrix}$$

In the following, the net outranking flow matrices are constructed using equation (10) and (11).

$$F_1^0 = \begin{pmatrix} - & 21.35835 & 50.6045 & -21.0863 & -26.2346 \\ -21.3583 & - & 9.16626 & -32.2333 & -20.3958 \\ -50.6045 & -9.16626 & - & -40.1835 & -26.3475 \\ 21.08634 & 32.23334 & 40.1835 & - & -5.23252 \\ 26.23462 & 20.39576 & 26.34747 & 5.23252 & - \end{pmatrix}$$

$$F_2^0 = \begin{pmatrix} - & -5.43966 & 5.601816 & 14.61826 & 10.91923 \\ 5.439662 & - & 12.20285 & 39.04262 & 17.86403 \\ -5.60182 & -12.2028 & - & -29.7301 & -16.3805 \\ -14.6183 & -39.0426 & 29.73012 & - & 13.58321 \\ -10.9192 & -17.864 & 16.38049 & -13.5832 & - \end{pmatrix}$$

$$F_3^0 = \begin{pmatrix} - & -27.0139 & 13.46178 & -25.9311 & -7.34938 \\ 27.01387 & - & 9.650228 & 1.168981 & -24.2385 \\ -13.4618 & -9.65023 & - & -8.91546 & 11.7166 \\ 25.93111 & -1.16898 & 8.915461 & - & -23.3771 \\ 7.349381 & 24.2385 & -11.7166 & 23.37708 & - \end{pmatrix}$$

Next, the net outranking flow index and the group net outranking flow index values are calculated using equation (12) and (13) which are abstracted in Table 3.

Table 3: Net outranking flow index values

	Φ_1^0	Φ_2^0	Φ_3^0	Φ_G^0
a_1	24.64188	25.69964	- 46.8326	1.1696
a_2	- 64.8212	74.54915	13.59458	7.7742
a_3	- 126.302	- 63.9153	- 20.3109	- 70.176
a_4	88.27067	- 10.3475	10.30051	29.4079
a_5	78.21038	- 25.986	43.24836	31.8243

Table 4 represents the result of the first individual and gruops' original preorders on the alternatives based on their individual net outranking flow index and the group net outranking flow index values.

Using equations (14) until (17), personal and group's original satisfaction index values are computed and abstracted in Table 5.

According to Remark 3.2, due to the relation between acceptable group satisfaction level θ and the value of R_G^0 in Table 5, i.e. $R_G^0 = 0.53 < 0.6 = \theta$, the

Table 4: The first individual and group's adjusted preorders

u_1^0	u_2^0	u_3^0	u_G^0
3	2	5	4
4	1	2	3
5	5	4	5
1	3	3	2
2	4	1	1

Table 5: The first personal and group's satisfaction index values

R_1^0	R_2^0	R_3^0	R_G^0
0.8	0	0.8	0.53

next stage is going to perform. In the second stage after calculating the value of the individual and group's adjusted preorders, the new result is $R_G^1 = 0.8$ (see Table 6 and Table 7).

Table 6: The second individual and group's adjusted preorders

u_1^1	u_2^1	u_3^1	u_G^1
3	2	5	4
4	3	2	3
5	5	4	5
1	1	3	2
2	4	1	1

Since $R_G^1 = 0.8 > 0.6 = \theta$, then the desired level of group consensus is achieved, and the final group ranking result of the alternatives are given as follows:
$a_5 \succ a_4 \succ a_2 \succ a_1 \succ a_3$.

The comparison between proposed method and ranking alternatives based on Shen et al. [19] is shown in Table 8.

The ranking order obtained by Shen et al. [19] methods are not the same as results obtained from proposed method. The main reason is that in Shen et al. [19] methods, weight of attributes is based on DMs' ideas and not through specific methods, which may lead to unreasonable decision results. By contrast, the proposed method determines weight of attributes by a linear model, which can avoid the untrue randomness and improve convincingness of decision outcomes.

Table 7: The first personal and group's satisfaction index values

R_1^1	R_2^1	R_3^1	R_G^1
0.8	0.8	0.8	0.8

Table 8: Compare the rank of proposed method with Shen et al. method

Alternatives	Proposed method	Shen et al. method
a_1	4	1
a_2	3	2
a_3	5	3
a_4	2	4
a_5	1	5

5. Conclusion

In the present paper, it has been tried to determine and evaluate importance of each attribute by a linear programming model in the MAGDM problems. Also, the application of the ELECTRE III with intuitionistic fuzzy information has been proposed to solve the MAGDM models. An automatic adjustment of group satisfaction has been utilized to overcome the convergence problems that detected in the other consensus approaches. Therefore, because the weight of attribute has been achieved via the proposed model, and it is not based on DMs' opinions, the accuracy of alternatives' ranking should be increase and the decision making errors decrease. Whiles in traditional outranking sorting methods, mainly the focuse is on DMs' opinions for determining the weight of attributes. For practical application, the proposed outranking method has been applied to solve a company supplier evaluation problem which is an important component in the supply chain management. This research could be surveyed by the other types of uncertain environments for future use. Another orientation could be the determination of the DMs' importance in MAGDM problems.

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Contragredient hom-Lie superalgebras

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Abstract. In this paper we introduce the notions of $\mathbb{Z}$-graded hom-Lie superalgebras and contragredient hom-Lie superalgebras. We introduce the invariant bilinear forms on a $\mathbb{Z}$-graded hom-Lie superalgebra and we prove that a consistent supersymmetric α-invariant form on the local part can be extended uniquely to a bilinear form with the same property on the whole $\mathbb{Z}$-graded hom-Lie superalgebra. Furthermore, we check the condition in which hom-Lie superalgebra is simple.

Keywords: hom-Lie superalgebra, contragredient hom-Lie superalgebra, $\mathbb{Z}$-graded hom-Lie superalgebra.

1. Introduction

The notion of hom-Lie algebras was introduced by J. T. Hartwig, D. Larsson and S. Silvestrov in [16] described the structures on certain deformations of Witt algebras and Virasoro algebras, which are widely utilized in the theoretical physics; such as string theory, vertex models in conformal field theory, quantum mechanics and quantum field theory [1, 10, 11, 12, 24]. The hom-Lie algebras attract more and more attention. New structures on hom-Lie algebras which have great importance and utility, have been defined and discussed by mathematicians that we express them: The hom-Lie algebras were discussed by D. Larsson and S. Silvestrov in [20, 21, 22]. The authors studied quadratic hom-Lie algebras in [9]; representation theory, cohomology and homology theory in [2, 27, 29]. In [19, 21, 23, 28], S. Silvestrov et al. introduced the general quasi-Lie algebras and including as special cases the color hom-Lie algebras

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[5, 7, 8] and in particular hom-Lie superalgebras. Recently, different features of hom-Lie superalgebras has been studied by authors in [3, 4, 6, 26, 30]. On the other hand, we have the graded structure on Lie algebras which were discussed by authors. The first basic example of graded Lie algebras was provided by Nijenhuis and then by Frolicher and Nijenhuis in [14]. In [18], Kac introduced some features of graded Lie algebras and then generalized it to Lie superalgebras in [17]. Now, we want to construct a special graded structure ($\mathbb{Z}$-graded), on hom-Lie superalgebras. In the first section of this paper, hom-Lie algebras, hom-Lie superalgebras and some of their useful related definitions are presented. In section two, we present the notion of $\mathbb{Z}$-graded hom-Lie superalgebras, local hom-Lie superalgebras and contragredient hom-Lie superalgebras. Morever we prove that we can extended uniquely a consistent supersymmetric α-invariant bilinear form on the local part to the whole $\mathbb{Z}$-graded hom-Lie superalgebra and then we prove that if contragredient hom-Lie superalgebra $\mathfrak{g}(A, \tau, \alpha)$ be finite dimensional and Cartan matrix A have some properties, then there exists a consistent supersymmetric α-invariant bilinear form $(.,.)$ on the $\mathfrak{g}(A, \tau, \alpha)$.

Throughout this paper we let $\mathbb{K}$ be an algebraically closed field of characteristic 0. All $\mathbb{Z}_2$-graded vector spaces are considered over $\mathbb{K}$ and linear maps are $\mathbb{K}$-linear maps. Each element in the hom-Lie superalgebra is supposed to be homogeneous and degree of x is denoted by $|x|$, for all $x \in \mathfrak{g}$.

We recall the definition of hom-Lie algebras and Lie superalgebras from [17, 27]. Also, we present notions of a hom-Lie superalgebra as a generalization of a Lie superalgebra [26].

Definition 1.1 ([27]). *A hom-Lie algebra is a triple $(\mathfrak{g}, [.,.], \alpha)$, where $\mathfrak{g}$ is a vector space equipped with a skew-symmetric bilinear map $[.,.] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ and a linear map $\alpha : \mathfrak{g} \to \mathfrak{g}$ such that;*

$$[\alpha(x), [y, z]] + [\alpha(y), [z, x]] + [\alpha(z), [x, y]] = 0,$$

for all x, y, z in $\mathfrak{g}$.

- *A hom-Lie algebra is called a multiplicative hom-Lie algebra, if α is an algebraic morphism, i.e. for any $x, y \in \mathfrak{g}$,*

$$\alpha([x, y]) = [\alpha(x), \alpha(y)].$$

- *We call a hom-Lie algebra regular, if α is an automorphism.*

Definition 1.2 ([17]). *A Lie superalgebra is a $\mathbb{Z}_2$-graded vector space $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$, together with a graded Lie bracket $[.,.] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ of degree zero, i.e. $[.,.]$ is a bilinear map with*

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j(mod2)},$$

such that for homogeneous elements $x, y, z \in \mathfrak{g}$, the following identities hold:

- $[x, y] = -(-1)^{|x||y|}[y, x]$,

- $[x,[y,z]] = [[x,y],z] + (-1)^{|x||y|}[y,[x,z]]$.

Now, analogous to the definition of hom-Lie algebras, a hom-Lie superalgebra is defined in a way that makes it a generalization of a Lie superalgebra.

Definition 1.3 ([26]). *A hom-Lie superalgebra is a triple* $(\mathfrak{g}, [.,.], \alpha)$ *consisting of a* $\mathbb{Z}_2$*-graded vector space* $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$*, an even linear map (bracket)* $[.,.] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ *and an even homomorphism* $\alpha : \mathfrak{g} \to \mathfrak{g}$ *satisfying the following supersymmetry and hom-Jacobi identity, i.e.*

- $[x,y] = -(-1)^{|x||y|}[y,x]$,
- $(-1)^{|x||z|}[\alpha(x),[y,z]] + (-1)^{|y||x|}[\alpha(y),[z,x]] + (-1)^{|z||y|}[\alpha(z),[x,y]] = 0$, *where* x*,* y *and* z *are homogeneous elements in* $\mathfrak{g}$.
- *A hom-Lie superalgebra is called multiplicative hom-Lie superalgebra, if* α *is an algebraic morphism, i.e. for any* $x, y \in \mathfrak{g}$ *we have*

$$\alpha([x,y]) = [\alpha(x), \alpha(y)].$$

- *A hom-Lie superalgebra is called regular hom-Lie superalgebra, if* α *is an algebraic automorphism.*

Remark 1.4 ([3]). Setting $\alpha = id$ in Definition 1.3 we obtain the definition of a Lie superalgebra. Hence hom-Lie superalgebras include Lie superalgebras as a subcategory, thereby motivating the name "hom-Lie superalgebras" as a deformation of Lie superalgebras by an endomorphism.

Example 1.5 ([3]). (Affine hom-Lie superalgebra) Let $V = V_0 \oplus V_1$ be a 3-dimensional superspace where V_0 is generated by e_1, e_2 and V_1 is generated by e_3. The triple $(V, [.,.], \alpha)$ is a hom-Lie superalgebra defined by $[e_1, e_2] = e_1$, $[e_1, e_3] = [e_2, e_3] = [e_3, 3_3] = 0$ and α is any homomorphism.

Let $(\mathfrak{g}, [.,.], \alpha)$ and $(\mathfrak{g}', [.,.]', \beta)$ be two hom-Lie superalgebras. An even homomorphism $\phi : \mathfrak{g} \to \mathfrak{g}'$ is said to be a homomorphism of hom-Lie superalgebras, if

$\phi[u,v] = [\phi(u), \phi(v))]'$,
$\phi o \alpha = \beta o \phi$.

The hom-Lie superalgebras $(\mathfrak{g}, [.,.], \alpha)$ and $(\mathfrak{g}', [.,.]', \beta)$ are isomorphic, if there is a hom-Lie superalgebra homomorphism $\phi : \mathfrak{g} \to \mathfrak{g}'$ such that ϕ be bijective [26].

A sub-vector space $I \subseteq \mathfrak{g}$ is a hom-subalgebra of $(\mathfrak{g}, [.,.], \alpha)$, if $\alpha(I) \subseteq I$ and I is closed under the bracket operation $[.,.]$, i.e. $[I, I] \subseteq I$. Also, hom-subalgebra I is called a hom-ideal of $\mathfrak{g}$, if $[I, \mathfrak{g}] \subseteq I$. Moreover, if $[I, I] = 0$, then I is Abelian [15].

Definition 1.6 ([25]). *The center of a hom-Lie superalgebra* $\mathfrak{g}$*, denoted by* $Z(\mathfrak{g})$*, is the set of elements* $x \in \mathfrak{g}$ *satisfying* $[x, \mathfrak{g}] = 0$.

Remark 1.7. When $\alpha : \mathfrak{g} \to \mathfrak{g}$ is a surjective endomorphism, then one can easily check that $(Z(\mathfrak{g}), \alpha)$ is an Abelian hom-Lie superalgebra and an hom-ideal of $(\mathfrak{g}, [.,.], \alpha)$.

Lemma 1.8. *The quotient $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is an Abelian hom-Lie superalgebra. Moreover, $[\mathfrak{g}, \mathfrak{g}]$ is the smallest hom-ideal with this property: if $\mathfrak{g}/I$ is Abelian for some hom-ideal $I \subset \mathfrak{g}$, then $[\mathfrak{g}, \mathfrak{g}] \subset I$.*

Proof. The proof is straightforward. □

We are going to need the following definition throughout the rest of the paper.

Definition 1.9 ([4])**.** *A representation of the hom-Lie superalgebra $(\mathfrak{g}, [.,.], \alpha)$ on a $\mathbb{Z}_2$-graded vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$ with respect to $\beta \in gl(V)_{\bar{0}}$ is an even linear map $\rho : \mathfrak{g} \to gl(V)$, such that for all $x, y \in \mathfrak{g}$, the following equalities are satisfied:*

$$\rho(\alpha(x)) o \beta = \beta o \rho(x);$$

$$\rho([x, y]) o \beta = \rho(\alpha(x)) o \rho(y) - (-1)^{|x||y|} \rho(\alpha(y)) o \rho(x).$$

A representation V of $\mathfrak{g}$ is called irreducible or simple, if it has no nontrivial subrepresentations. Otherwise V is called reducible.

Let $(\mathfrak{g}, [.,.], \alpha)$ be a multiplicative hom-Lie superalgebra. We consider $\mathfrak{g}$ as a representation on itself via the bracket and with respect to the morphism α.

Example 1.10 ([4])**.** The α^s-adjoint representation of the hom-Lie superalgebra $(\mathfrak{g}, [.,.], \alpha)$, which we denote by ad_s, is defined by

$$ad_s(a)(x) = [\alpha^s(a), x], \quad \text{for all} \quad a, x \in \mathfrak{g}.$$

2. Local and contragredient hom-Lie superalgebras

In [17], Kac introduced the notion of a $\mathbb{Z}$-graded Lie superalgebra and in this section we introduce concept of a $\mathbb{Z}$-graded hom-Lie superalgebra and state some results about it.

Definition 2.1. *Let $\mathfrak{g}$ be a hom-Lie superalgebra. It is called a $\mathbb{Z}$-graded hom-Lie superalgebra if it is decomposision of itself into a direct sum of finite-demensional $\mathbb{Z}_2$-graded subspaces $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$, for which $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$.*

A $\mathbb{Z}$-graded hom-Lie superalgebra is said to be consistent, if $\mathfrak{g}_{\bar{0}} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_{2i}$ and $\mathfrak{g}_{\bar{1}} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_{2i+1}$.

Remark 2.2. One can see that if $\mathfrak{g}$ is a $\mathbb{Z}$-graded hom-Lie superalgebra, then $\mathfrak{g}_0$ is a hom-Lie subalgebra and $[\mathfrak{g}_0, \mathfrak{g}_j] \subseteq \mathfrak{g}_j$; therefore the restriction of the adjoint representation to $\mathfrak{g}_0$ induces linear representation of it on the subspaces $\mathfrak{g}_j$.

Definition 2.3. *A $\mathbb{Z}$-graded hom-Lie superalgebra $\mathfrak{g}$ is called simple, if it does not have any nontrivial graded hom-ideal and $[\mathfrak{g},\mathfrak{g}] \neq \{0\}$.*

Remark 2.4. From Definition 2.3, one can consider the followings:

- A left or right graded hom-ideal of $\mathfrak{g}$ is automatically a two sided hom-ideal.
- The condition $[\mathfrak{g},\mathfrak{g}] \neq \{0\}$ serves to eliminate the zero-dimensional and two one-dimensional hom-Lie superalgebras. It follows that $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$.

Now, we consider an important notion of hom-Lie superalgebra.

Definition 2.5. *Let $\hat{\mathfrak{g}}$ be a $\mathbb{Z}_2$-graded space, decomposed into a direct sum of $\mathbb{Z}_2$-graded subspaces, $\hat{\mathfrak{g}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Suppose that whenever $|i+j| \leq 1$, a bilinear operation is defined $\mathfrak{g}_i \times \mathfrak{g}_j \to \mathfrak{g}_{i+j}$, such that $((x,y) \mapsto [x,y])$, satisfying the axiom of anticommutativity and the hom-Jacobi identity for hom-Lie superalgebras, each time the three terms of the identity are defined. Then $\hat{\mathfrak{g}}$ is called a local hom-Lie superalgebra.*

From the above definition we have the following remark.

Remark 2.6. For a $\mathbb{Z}$-graded hom-Lie superalgebra $\mathfrak{g} = \bigoplus \mathfrak{g}_i$, there corresponds a local hom-Lie superalgebra $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, which we call the local part of $\mathfrak{g}$.

Homomorphisms, transivity and bitransivity, for a local hom-Lie superalgebra are defined as for a $\mathbb{Z}$-graded hom-Lie superalgebra. In this section we consider only $\mathbb{Z}$-graded hom-Lie superalgebras $\mathfrak{g} = \bigoplus \mathfrak{g}_i$ in which the subspace $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ generates $\mathfrak{g}$.

Definition 2.7. *Let $\hat{\mathfrak{g}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a local hom-Lie superalgebra.*

- *A $\mathbb{Z}$-graded hom-Lie superalgebra $\mathfrak{g} = \bigoplus \mathfrak{g}_i$ with local part $\hat{\mathfrak{g}}$ is said to be maximal ($\mathfrak{g}_{max}(\hat{\mathfrak{g}})$) if for any other $\mathbb{Z}$-graded hom-Lie superalgebra $\mathfrak{g}'$, an isomorphism of the local part $\hat{\mathfrak{g}}$ and $\hat{\mathfrak{g}}'$ extends to an epimorphism of $\mathfrak{g}$ onto $\mathfrak{g}'$.*
- *A $\mathbb{Z}$-graded hom-Lie superalgebra $\mathfrak{g} = \bigoplus \mathfrak{g}_i$ with local part $\hat{\mathfrak{g}}$ is said to be minimal ($\mathfrak{g}_{min}(\hat{\mathfrak{g}})$) if for any other $\mathbb{Z}$-graded hom-Lie superalgebra $\mathfrak{g}'$, an isomorphism of the local part $\hat{\mathfrak{g}}$ and $\hat{\mathfrak{g}}'$ extends to an epimorphism of $\mathfrak{g}'$ onto $\mathfrak{g}$.*

Theorem 2.8 ([13])**.** *Let $\hat{\mathfrak{g}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a multiplicative local hom-Lie superalgebra such that $\alpha^2 = \alpha$. Then there exists a maximal and a minimal $\mathbb{Z}$-graded hom-Lie superalgebra whose local parts are isomorphic to $\hat{\mathfrak{g}}$.*

Corollary 2.9. *Let $\hat{\mathfrak{g}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a local hom-Lie superalgebra.*

i. Any $\mathbb{Z}$-graded hom-Lie superalgebra whose local part is isomorphic to $\hat{\mathfrak{g}}$, is a quotient of maximal $\mathbb{Z}$-graded hom-Lie superalgebra $\mathfrak{g}_{max}(\hat{\mathfrak{g}})$.

ii. Minimal $\mathbb{Z}$-graded hom-Lie superalgebra $\mathfrak{g}_{min}(\hat{\mathfrak{g}})$ is a quotient of any $\mathbb{Z}$-graded hom-Lie superalgebra whose local part is isomorphic to $\hat{\mathfrak{g}}$.

iii. $\mathfrak{g}_{max}(\hat{\mathfrak{g}})$ has a unique maximal graded hom-ideal J_{max} such that

$$J_{max} \cap \hat{\mathfrak{g}} = \{0\} \quad \text{and} \quad \mathfrak{g}_{max}(\hat{\mathfrak{g}})/J_{max} = \mathfrak{g}_{min}(\hat{\mathfrak{g}}).$$

Proof. The proof is obvious by Theorem 2.8. □

Proposition 2.10 ([13])**.** *We have the following properties*

i. A bitransitive $\mathbb{Z}$-graded hom-Lie superalgebra is minimal.

ii. A minimal $\mathbb{Z}$-graded hom-Lie superalgebra with bitransitive local part is bitransitive.

iii. Two bitransitive $\mathbb{Z}$-graded hom-Lie superalgebra are isomorphic, if and only if their local parts are isomorphic.

Definition 2.11 ([9])**.** *Let $(\mathfrak{g}, [.,.], \alpha)$ be a hom-Lie superalgebras. Let f be a bilinear form on $\mathfrak{g}$.*

The form f is said to be consistent, if

$$f(x, y) = 0 \quad \text{for all} \quad x \in g_0, y \in \mathfrak{g}_1;$$

f is said to be nondegenerate, if

$$\mathfrak{g}^{\perp} = \{x \in \mathfrak{g} | f(x, y) = 0, \quad \text{for all} \quad y \in \mathfrak{g}\};$$

f is said to be invariant, if

$$f([x, y], z) = f(x, [y, z]) \quad \text{for all} \quad x, y, z \in \mathfrak{g};$$

f is said to be supersymmetric, if

$$f(x, y) = -(-1)^{|x||y|} f(y, x) \quad \text{for all} \quad x, y, z \in \mathfrak{g}.$$

An invariant bilinear form f is said to be α-invariant, if

$$f(\alpha(x), y) = f(x, \alpha(y)) \quad \text{for all} \quad x, y, z \in \mathfrak{g}.$$

It is obvious that the kernel of an invariant form f on $\mathfrak{g}$ (that is, the set of $x \in \mathfrak{g}$ for which $f(x, \mathfrak{g}) = 0$) is an hom-ideal in $\mathfrak{g}$.

Hence, we have the following proposition:

Proposition 2.12. *Let $\mathfrak{g}$ be a simple hom-Lie superalgebra. Then every nonzero invariant form on $\mathfrak{g}$ is nondegenerate and any two invariant forms on $\mathfrak{g}$ are proportional.*

Proposition 2.13. *Let $(.,.)$ be a consistent supersymmetric α-invariant bilinear form on the local part of a $\mathbb{Z}$-graded hom-Lie superalgebra $\mathfrak{g} = \bigoplus \mathfrak{g}_i$ for which $(\mathfrak{g}_i, \mathfrak{g}_j) = 0$, where $i + j \neq 0$. If $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ generates $\mathfrak{g}$, then the form can be extended uniquely to a consistent supersymmetric α-invariant bilinear form with the same property on the whole $\mathfrak{g}$.*

Proof. Let $(\mathfrak{g}_i, \mathfrak{g}_j) = 0$, where $i + j \neq 0$. We prove by induction on k such that for $x \in \mathfrak{g}_i$, $y \in \mathfrak{g}_{-i}$, $|i| \leq k$, we can define (x, y) so that the bilinear form $(.,.)$ be α-invariant so long as all the elements in this equation lie in the space $\bigoplus_{i=-k}^{k} \mathfrak{g}_i$. When $k = 1$, the assertion holds by hypothesis, since $(.,.)$ is defined on $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$. For $i > 0$, let x_i and y_i be elements in $\mathfrak{g}_i$ and $\mathfrak{g}_{-i}$ respectively. Suppose that

$$A = (-1)^{|s-k||s|}(-1)^{|s-k||k|}([[x_{k-s}, x_s], \alpha(y_{k-s})], y_s).$$

By the induction hypothesis we have the following equality $(0 < s < k)$:

$$\begin{aligned}
A &= (-1)^{|s-k||s|}(-1)^{|s-k||k|}([[x_{k-s}, x_s], \alpha(y_{k-s})], y_s) \\
&= (-1)^{|k-s||s-k|}([\alpha(x_{k-s}), [x_s, y_{k-s}]], y_s) + (-1)^{|k-s||s|}([\alpha(x_s), [y_{k-s}, x_{k-s}]], y_s) \\
&= (-1)^{|k-s||s-k|}([\alpha(x_{k-s}), [x_s, y_{k-s}]], y_s) - (-1)^{|k-s||s|}([[y_{k-s}, x_{k-s}], \alpha(x_s)], y_s) \\
&= -(-1)^{|k-s||s-k|}(-1)^{|k-s||2s-k|}([[x_s, y_{k-s}], \alpha(x_{k-s})], y_s) \\
&\quad + (-1)^{|k-s||s|}(-1)^{|k-s||s-k|}([[x_{k-s}, y_{k-s}], \alpha(x_s)], y_s) \\
&= -(-1)^{|k-s||s-k|}(-1)^{|k-s||2s-k|}([x_s, y_{k-s}], [\alpha(x_{k-s}), y_s]) \\
&\quad + (-1)^{|k-s||s|}(-1)^{|k-s||k-s|}([x_{k-s}, y_{k-s}], [\alpha(x_s), y_s]) \\
&= (-1)^{|k-s||s-k|}(-1)^{|k-s||2s-k|}(-1)^{|k-s||s|}([x_s, y_{k-s}], [y_s, \alpha(x_{k-s})]) \\
&\quad + (-1)^{|k-s||s|}(-1)^{|k-s||k-s|}([x_{k-s}, y_{k-s}], [x_s, \alpha(y_s)]) \\
&= (-1)^{|k-s||s-k|}(-1)^{|k-s||2s-k|}(-1)^{|k-s||s|}([x_s, y_{k-s}], [y_s, \alpha(x_{k-s})]) \\
&\quad + (-1)^{|k-s||s|}(-1)^{|k-s||k-s|}([x_{k-s}, y_{k-s}], [x_s, \alpha(y_s)]) \\
&= (-1)^{|k-s||s-k|}(-1)^{|k-s||2s-k|}(-1)^{|k-s||s|}(x_s, [y_{k-s}, [y_s, \alpha(x_{k-s})]]) \\
&\quad - (-1)^{|k-s||s|}(-1)^{|k-s||k-s|}([x_s, \alpha(y_s)], [x_{k-s}, y_{k-s}]) \\
&= (-1)^{|k-s||s-k|}(-1)^{|k-s||2s-k|}(-1)^{|k-s||s|}(x_s, [y_{k-s}, [y_s, \alpha(x_{k-s})]]) \\
&\quad - (-1)^{|k-s||s|}(-1)^{|k-s||k-s|}(x_s, [\alpha(y_s), [x_{k-s}, y_{k-s}]]),
\end{aligned}$$

therefore, we have

$$\begin{aligned}
&(-1)^{|s-k||k|}([[x_{k-s}, x_s], \alpha(y_{k-s})], y_s) \\
&\qquad = (-1)^{|k-s||s-k|}(-1)^{|k-s||2s-k|}(x_s, [y_{k-s}, [y_s, \alpha(x_{k-s})]]) \\
&\qquad\quad - (-1)^{|k-s||k-s|}(x_s, [\alpha(y_s), [x_{k-s}, y_{k-s}]],
\end{aligned}$$

finally

$$([[x_{k-s}, x_s], \alpha(y_{k-s})], y_s) = (x_s, [[\alpha(y_{k-s}), y_s], x_{k-s}]).$$

If we set

$$(\alpha[x_{k-s}, x_s], [y_{k-s}, y_s]) = ([\alpha[x_{k-s}, x_s], y_{k-s}], y_s) = ([[x_{k-s}, x_s], \alpha(y_{k-s})], y_s),$$

then this definition will be well-defined and the form will satisfy the induction hypothesis. □

Definition 2.14. *Let $A = (a_{ij})$ be an $(n \times n)$-matrix with elements from a field $\mathbb{K}$. Let τ be a subset of $I = \{1, 2, ..., n\}$ that tells us which of the generators is odd. Let $\mathfrak{g}_{-1}$, $\mathfrak{g}_0$ and $\mathfrak{g}_1$ be vector spaces with bases $\{f_i\}$, $\{h_i\}$ and $\{e_i\}$, $i \in I$, respectively. It is east to see that the following relations together with linear map α such that is closed on $\mathfrak{g}_i$ determine the structure of a local hom-Lie superalgebra $\hat{\mathfrak{g}}(A, \tau, \alpha)$ on the spaces $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$:*

$$\begin{aligned}
&[e_i, f_j] = \delta_{ij} h_i, \qquad [h_i, h_j] = 0,\\
&[h_i, e_j] = a_{ij} e_j, \qquad [h_i, f_j] = -a_{ij} f_j,\\
&deg(h_i) = \bar{0},\\
&deg(e_i) = deg(f_i) = \bar{0}, \qquad i \notin \tau,\\
&deg(e_i) = deg(f_i) = \bar{1}, \qquad i \in \tau.
\end{aligned} \tag{1}$$

According to Theorem 2.8, there exists a minimal $\mathbb{Z}$-graded hom-Lie superalgebra $\mathfrak{g}(A, \tau, \alpha)$ with local part $\hat{\mathfrak{g}}(A, \tau, \alpha)$. We call $\mathfrak{g}(A, \tau, \alpha)$ contragredient hom-Lie superalgebra, A its Cartan matrix and n its rank.

Remark 2.15. By setting $\alpha = id$, we have contragredient Lie superalgebra, also if setting $\tau = \emptyset$, then we have contragredient Lie algebra.

Proposition 2.16. *Let Z be center of $\mathbb{Z}$-graded hom-Lie superalgebra $\mathfrak{g}(A, \tau, \alpha)$.*

i. Z consist of elements of the form $\sum_{i=1}^{n} b_i h_i$, where $\sum_{i=1}^{n} a_{ij} b_i = 0$.

ii. The factor $\mathfrak{g}'(A, \tau, \alpha) = \frac{\mathfrak{g}(A,\tau,\alpha)}{Z}$, with the induced gradation, is transitive.

Proof. i. By relations (1) we have

1. If $\gamma_j \in \mathfrak{g}_0$, then

$$[\sum_{i=1}^{n} b_i h_i, \gamma_j] = \sum_{i=1}^{n} b_i [h_i, \gamma_j] = 0.$$

2. If $\gamma_j \in \mathfrak{g}_{-1}$, i.e, $\gamma_j = f_j$, then

$$[\sum_{i=1}^{n} b_i h_i, \gamma_j] = \sum_{i=1}^{n} b_i [h_i, \gamma_j] = -(\sum_{i=1}^{n} b_i a_{ij}) f_j = -(\sum_{i=1}^{n} a_{ij} b_i) f_j = 0.$$

3. If $\gamma_j \in \mathfrak{g}_1$, i.e, $\gamma_j = e_j$, then

$$[\sum_{i=1}^{n} b_i h_i, \gamma_j] = \sum_{i=1}^{n} b_i [h_i, \gamma_j] = (\sum_{i=1}^{n} b_i a_{ij}) e_j = (\sum_{i=1}^{n} a_{ij} b_i) e_j = 0.$$

ii. The proof is obvious by Proposition 2.10.

□

If $c \in \mathbb{K}$ and replacing h_i by ch_i and f_i by cf_i, then the $i-th$ row of A is multiplied by c. Therefore, we can assume that a_{ii} is either 0 or 2, for any i.

If the matrix A can be obtained from the matrix $\tilde{A}$ by multiplying any rows by nonzero numbers and by renumbering the indices, then the matrix A and $\tilde{A}$ will be called equivalent. Contragredient hom-Lie superalgebras with equivalent Cartan matrices are isomorphic.

If the Cartan matrix A can be decomposed, that is, it takes the block diagonal form under a suitable permutation of indices

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},$$

then $\mathfrak{g}(A, \tau, \alpha)$ is direct sum of the contragredient hom-Lie superalgebras with the Cartan matrices A_1 and A_2.

In this paper, in many cases we will need some assumptions for the matrix A. For convenience we present them in one place:

(M1) $a_{ii} = 2$ for all i;

(M2) $a_{ij} = 0$ implies that $a_{ji} = 0$;

(M3) for any set of natural numbers $i_1, i_2, ..., i_k \in I$, such that $k \leq n$; we have

$$a_{i_1 i_2} a_{i_2 i_3} ... a_{i_k i_1} = a_{i_2 i_1} a_{i_3 i_2} ... a_{i_1 i_k};$$

(M4) for any $i, j \in I$, there exist $i_1, i_2, ...i_k \in I$ for which

$$a_{i i_1} a_{i_1 i_2} ... a_{i_k j} \neq 0.$$

Proposition 2.17. *Let $\mathfrak{g}(A, \tau, \alpha) = \bigoplus_{i=-m}^{m} \mathfrak{g}_i$ be finite dimensional and Z be its center. Then $\mathfrak{g}'(A, \tau, \alpha) = \frac{\mathfrak{g}(A,\tau,\alpha)}{Z}$ is simple if and only if condition (M4) is satisfied for the Cartan matrix A.*

Proof. Let J be a nonzero hom-ideal of $\mathfrak{g}'(A, \tau, \alpha)$ and $g = \sum_{i \geq k} g_i$ be decomposition of a nonzero element in J and k be the largest integer for all nonzero elements in J. Then $[g_k, \mathfrak{g}_1] = 0$. By using definition of α^s-adjoint representation,

$$\bigoplus_{i,j \geq 0} (ad_s(\mathfrak{g}_{-1}))^i (ad_s(\mathfrak{g}_0))^j (g_k)$$

is a nonzero hom-ideal of $\mathfrak{g}'(A,\tau,\alpha)$. Thus $k=m$ and g is a homogeneous element in J and hom-ideal generated by g_k is contained in J and is homogeneous. Therefore $J\cap(\mathfrak{g}_{-1}\oplus\mathfrak{g}_1)\neq 0$. So, for some $i\in I$, e_i or f_i is in J. It would be easily seen that condition (M4) implies that for all $i\in I$, $e_i,f_i\in J$. Thus $J=\mathfrak{g}'(A,\tau,\alpha)$. Therefore $\mathfrak{g}'(A,\tau,\alpha)$ is simple. Conversly, let condition (M4) is not satisfied for some $i,j\in I$. So it would be easily seen that hom-ideal generated by e_i does not contain e_j; therefore it is contradicted that $\mathfrak{g}'(A,\tau,\alpha)$ is simple. □

Proposition 2.18. *Let $\mathfrak{g}(A,\tau,\alpha)$ is finite dimensional and Cartan matrix A have properties (M2) and (M3). Then there exists a consistent supersymmetric α-invariant bilinear form (.,.) on the $\mathfrak{g}(A,\tau,\alpha)$.*

Proof. Let $(\mathfrak{g}_i,\mathfrak{g}_j)=0$, where $i\neq j$. According to Proposition 2.13, it is sufficient to define bilinear form (.,.) on the $\hat{\mathfrak{g}}(A,\tau,\alpha)$ of $\mathfrak{g}(A,\tau,\alpha)$. Let

$$\begin{aligned}&(e_i,f_j)=0,\quad i\neq j,\\&(e_1,f_1)=1.\end{aligned}\tag{2}$$

The form (.,.) will be invariant on $\hat{\mathfrak{g}}(A,\tau,\alpha)$ if $([e_i,f_i],h_j)=(e_i,[f_i,h_j])$ or from 1 $(h_i,h_j)=(e_i,a_{ji}f_i)$ or

$$(h_i,h_j)=a_{ji}(e_i,f_i).\tag{3}$$

Now, it is possible to define the form (.,.) on $\hat{\mathfrak{g}}(A,\tau,\alpha)$ with (M2), (M3), (2) and (3). □

The α-invariant bilinear form (.,.) constructed in the Proposition 2.18 will be called the canonical α-invariant bilinear form.

Using the above Proposition, the following corollary can be obtained.

Corollary 2.19. *For any canonical α-invariant bilinear form with property (M1), we have*

$$a_{ji}=\frac{2(h_i,h_j)}{(h_i,h_i)}.$$

Proof. If we set $i=j$ in (3), then $(h_i,h_i)=a_{ii}(e_i,f_i)$. Since $a_{ii}=2$, $(h_i,h_i)=2(e_i,f_i)\Rightarrow(e_i,f_i)=\frac{(h_i,h_i)}{2}$. Therefore, $a_{ii}=\frac{(h_i,h_j)}{(e_i,f_i)}=\frac{2(h_i,h_j)}{(h_i,h_i)}$. □

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Geometric study on modified Koebe functions imposed by confluent hypergeometric functions

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Abstract. The current endeavor, employing confluent hypergeometric functions and convolution tool, introduces a new normalized modified Koebe function in the complex open unit disk. Certain sufficient stipulations on parameters of the modified Koebe function to be starlike, convex and close-to-convex are discussed and presented. Moreover, close-to-convexity of modified Koebe functions with respect to specific functions is discussed. In addition, the stipulations on the modified Koebe function to be included in the Hardy space are also studied.

Keywords: regular function, Koebe functions, hypergeometric functions, convolution product.

1. Introduction

The theory of special functions (SFT) is a classical field in mathematics consisting of regular functions defined via series. It originated in the 19th century and evolved in the 20th century. Among the important researchers include Gauss, Jacobi, Klein and others. Recently, SFT is a crucial part of an ongoing investigation in numerous areas of researches in engineering, physical, and mathematical sciences. In complex analysis, in 1985, De Branges [7] is a mathematician at Purdue who employed a class of special functions called hypergeometric functions to solve the major problem in the geometric theory of regular functions (GTR), namely, Bieberbach conjecture. He has renewed attention in studying this class of functions. Prior to proving conjecture, there were only a few papers

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dealing with the relation between GTR and hypergeometric functions. Indeed, GTR studies the relationship between the regular features of a given function and the geometric features of its image domain. This relevance between geometrical and regularity is a pivotal feature in GTR. Accordingly, it has sparked interest in studying several geometric features of various special functions in a complex domain. In other words, diverse authors have discussed sufficient stipulations of univalence, starlikeness, convexity and close-to-convexity on the parameters for various special functions. In 1959 and 1960, Kreyszig and Todd examined the univalence feature of the error function [17], $\exp(z^2)$ function [16], and Bessel function [18]. Later, in 1961, Merkes and Scott [19] discussed the starlikeness feature of Gaussian hypergeometric functions. Beside that, in 1984, Carlson and Shaffer [8] introduced a linear operator including an incomplete beta function. The first sentence of this paper is remarkable in view of de Branges' proof a year later, as it reads "Connections between GTR and SFT have not got much attention". Afterward, in 1986, Ruscheweyh and Singh [33] considered the order of the starlikeness feature of hypergeometric functions. In 1990, Miller and Mocanu [20] derived stipulations for univalence, convexity and starlikeness of Gaussian and confluent hypergeometric functions. In 1996 and 1998, Ponnusamy studied the Hardy spaces of the above interesting functions [28], Close-to-convexity features of confluent hypergeometric functions [26]. Following this, in 2006, Baricz [36] utilized the Ponnusamy's idea and acquired the Hardy spaces of Bessel-type functions. Pursuing this line of study, a number of geometric results have recently been introduced to the literature concerning the Bessel function ([5], [6]), Struve functions ([38]), Wright functions ([29], [31]), Mittag-Leffler functions [3], Lommel functions ([37], [21]).

On the other hand, the convolution product is a significant tool introduced by Hadamard [11] in 1899. It remains one of the most effective and efficient tools to investigate various normalized functions ([21], [1]) and complex operators, ([1], [2], [13] and [14]). Consequently, this study imposes a new normalized modified Koebe function based on the confluent hypergeometric function and convolution technique. Certain sufficient stipulations are examined such that modified Koebe function are starlike, convex and close-to-convex. Furthermore, the Hardy space of modified Koebe function are also acquired.

2. Preliminaries

Let $\mathfrak{D} = \{z \in \mathbb{C} : |z| < 1\}$ identify the open unit disc on the complex domain $\mathbb{C}$. The class of regular functions in $\mathfrak{D}$ is indicated by Ω. Consider the subclass Λ of Ω consisting of regular functions ϑ of the formula:

$$\vartheta(z) = z + \sum_{\mathfrak{n}=2}^{\infty} \varkappa_{\mathfrak{n}}\, z^{\mathfrak{n}}, \qquad (z \in \mathfrak{D}), \tag{1}$$

which are normalized by $\vartheta(0) = \vartheta'(0) - 1 = 0$. Denoted by Ξ the subclass of Λ including univalent functions in $\mathfrak{D}$. The Koebe function κ defined by

$$\kappa(z) = \frac{z}{(1-z)^2} = \sum_{\mathfrak{n}=1}^{\infty} \mathfrak{n}\ z^{\mathfrak{n}}, \qquad (z \in \mathfrak{D}), \tag{2}$$

is a remarkable instance of a univalent function in Ξ. Actually, Koebe function and its rotations $e^{\upsilon}\kappa\left(e^{i\upsilon}z\right)$ are the only extremal functions for numerous problems in GTR. The known subclasses of Ω are classes of convex, starlike, close-to-convex functions, which are determined by regular and geometric stipulations. Study [34] first introduced in 1913, the class $\mathcal{C}_{\mathcal{O}}$ of convex functions which is formulated by regular stipulation: $\mathcal{C}_{\mathcal{O}} = \{\vartheta \in \Lambda :\ 0 < \Re(1 + \frac{z\vartheta''(z)}{\vartheta'(z)}),\ z \in \mathfrak{D}\}$. Geometrically, a function $\vartheta \in \Lambda$ is called convex if the image $\vartheta(\mathfrak{D})$ is a convex domain. That is $\varrho z_1 + (1-\varrho)z_2 \in \vartheta(\mathfrak{D})$ whenever $z_1, z_2 \in \vartheta(\mathfrak{D})$ and $\varrho \in [0,1]$. Nevanlinna [22] presented in 1921, the class $\mathcal{S}^*$ of starlike functions which is defined by regular stipulation: $\mathcal{S}^* = \{\vartheta \in \Lambda :\ 0 < \Re(\frac{z\vartheta'(z)}{\vartheta(z)}),\ z \in \mathfrak{D}\}$. Geometrically, a function $\vartheta \in \Lambda$ is called starlike if the image $\vartheta(\mathfrak{D})$ is a starlike domain with respect to the origin. That is, if $\varrho z \in \vartheta(\mathfrak{D})$ whenever $z \in \vartheta(\mathfrak{D})$ and $\varrho \in [0,1]$. Robertson [32] imposed in 1936, the classes $\mathcal{C}_{\mathcal{O}}(\zeta)$ and $\mathcal{S}^*(\zeta)$ of convex and starlike functions of order $\zeta \in [0,1)$ which are given by

$$\mathcal{C}_{\mathcal{O}}(\zeta) = \left\{\vartheta \in \Lambda :\ \zeta < \Re\left(1 + \frac{z\vartheta''(z)}{\vartheta'(z)}\right),\ z \in \mathfrak{D}\right\}, \tag{3}$$

and

$$\mathcal{S}^*(\zeta) = \left\{\vartheta \in \Lambda :\ \zeta < \Re\left(\frac{z\vartheta'(z)}{\vartheta(z)}\right),\ z \in \mathfrak{D}\right\}. \tag{4}$$

Clearly, $\mathcal{C}_{\mathcal{O}}(0) = \mathcal{C}_{\mathcal{O}}$ and $\mathcal{S}^*(0) = \mathcal{S}^*$. Kaplan [15] considered in 1952, the class $\mathcal{C}_{\mathcal{V}}$ of close-to convex functions which are formed by $\mathcal{C}_{\mathcal{V}} = \{\vartheta \in \Lambda :\ 0 < \Re(\frac{z\vartheta'(z)}{\mathcal{G}(z)}),\ \mathcal{G} \in \mathcal{S}^*,\ z \in \mathfrak{D}\}$. In other words, a function $\vartheta \in \Lambda$ is called close-to convex if the image $\vartheta(\mathfrak{D})$ is a close-to convex domain, meaning that, the complement of $\vartheta(\mathfrak{D})$ can be expressed as the union of non-intersecting half-lines. More generally, Noonan [23] investigated in 1973, the class $\mathcal{C}_{\mathcal{V}}(\zeta)$ of close-to convex functions of order $\zeta \in [0,1)$ which are given by

$$\mathcal{C}_{\mathcal{V}}(\zeta) = \left\{\vartheta \in \Lambda :\ \zeta < \Re\left(\frac{z\vartheta'(z)}{\mathcal{G}(z)}\right),\ \mathcal{G} \in \mathcal{S}^*(\zeta),\ z \in \mathfrak{D}\right\}. \tag{5}$$

Evidently, $\mathcal{C}_{\mathcal{V}}(0) = \mathcal{C}_{\mathcal{V}}$. Baricz [4] proposed in 2006, two interesting classes of order $\zeta \in [0,1)$ which are defined by

$$\mathcal{P}_{\mu}(\zeta) = \left\{\mathsf{p} \in \Omega : \mathsf{p}(0) = 1,\ \ \zeta < \Re\left(e^{i\mu}\mathsf{p}(z)\right),\ \mu \in \mathbb{R},\ z \in \mathfrak{D}\right\}, \tag{6}$$

and

$$\mathcal{R}_{\mu}(\zeta) = \left\{\vartheta \in \Lambda : \zeta < \Re\left(e^{i\mu}\vartheta'(z)\right),\ \mu \in \mathbb{R},\ z \in \mathfrak{D}\right\}. \tag{7}$$

Note that for $\mu = 0$, the classes (6) and (7) reduces to $\mathcal{P}(\zeta)$ and $\mathcal{R}(\zeta)$, respectively. Further, for $\mu = 0$ and $\zeta = 0$, the classes $\mathcal{P}$ and $\mathcal{R}$ are obtained.

Hadamard [11] presented in 1899, the important mathematical operation on any two regular functions, called convolution product and denoted by $*$, to yield a third regular function. It is formulated as: for $\vartheta_1,\ \vartheta_2 \in \Lambda$ given by $\vartheta_1(z) = z + \sum_{\mathfrak{n}=2}^{\infty} \varkappa_{1\mathfrak{n}}\ z^{\mathfrak{n}}$ and $\vartheta_2(z) = z + \sum_{\mathfrak{n}=2}^{\infty} \varkappa_{2\mathfrak{n}}\ z^{\mathfrak{n}}$, then

$$\vartheta_1(z) * \vartheta_2(z) = z + \sum_{\mathfrak{n}=2}^{\infty} \varkappa_{1\mathfrak{n}}\ \varkappa_{2\mathfrak{n}}\ z^{\mathfrak{n}}, \qquad (z \in \mathfrak{D}). \tag{8}$$

The scientist Wallis first utilized the term "hypergeometric series" in 1655. Then, systematic studies were conducted by Gauss (1813), Kummer (1836), and Riemann (1857). The hypergeometric function ${}_0\mathcal{F}_1(\tau, z)$ is stated as the following:

$${}_0\mathcal{F}_1(\tau, z) = \sum_{\mathfrak{n}=0}^{\infty} \frac{1}{(\tau)_{\mathfrak{n}}} \frac{z^{\mathfrak{n}}}{\mathfrak{n}!}, \qquad (\tau \in \mathbb{C} \setminus \{0, -1, \ldots\},\ z \in \mathbb{C}), \tag{9}$$

where $(\tau)_{\mathfrak{n}} = \frac{\Gamma(\tau+\mathfrak{n})}{\Gamma(\tau)} = \tau(\tau+1)...(\tau+\mathfrak{n}-1)$, denotes the (Pochhammer) symbol. This function, called a confluent hypergeometric limit function, achieves the differential equation $z\vartheta''(z) + \tau\vartheta'(z) = \vartheta(z)$, [9].

Moreover, consider the space Ω^{∞} of bounded functions ϑ on Ω. Let $\vartheta \in \Omega$, set

$$\mathfrak{M}_{\mathsf{p}}(\mathsf{r}, \vartheta) = \begin{cases} \left(\dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} \left|\vartheta\left(\mathsf{r}e^{i\tau}\right)\right|^{\mathsf{p}} d\tau \right)^{1/\mathsf{p}}, & 0 < \mathsf{p} < \infty, \\ \sup\{|\vartheta(z)| : |z| \leq \mathsf{r}\}, & \mathsf{p} = \infty. \end{cases} \tag{10}$$

The function $\vartheta \in \Omega$ belongs to the hardy space $\Omega^p (0 < p \leq \infty)$. If $\mathfrak{M}_{\mathsf{p}}(\mathsf{r}, \vartheta)$ is bounded for $\mathsf{r} \in [0, 1)$, then $\vartheta \in \Omega^{\mathsf{p}}$. Notice that $\Omega^{\infty} \subset \Omega^{\mathsf{q}} \subset \Omega^{\mathsf{p}},\ 0 < \mathsf{q} < \mathsf{p} < \infty$, [10]. Also for, in [30], $\Re\left(\vartheta'(z)\right) > 0$ in $\mathfrak{D}$, then

$$\begin{cases} \vartheta' \in \Omega^{\mathsf{q}}, & \mathsf{q} < 1, \\ \vartheta \in \Omega^{\mathsf{q}/(1-\mathsf{q})}, & 0 < \mathsf{q} < 1. \end{cases} \tag{11}$$

The following is a description of some results that will be utilized to gain the main results.

Lemma 2.1 ([24]). *If $\vartheta \in \Lambda$ achieves the inequality*

$$\left|z\vartheta''(z)\right| < \frac{1-\zeta}{4}, \qquad (z \in \mathfrak{D},\ 0 \leq \zeta < 1), \tag{12}$$

then

$$\Re z\vartheta'(z) > \frac{1+\zeta}{4}, \qquad (z \in \mathfrak{D},\ 0 \leq \zeta < 1). \tag{13}$$

Lemma 2.2 ([25]). *Let $\vartheta \in \Lambda$. If*

$$\begin{aligned} &1 \geq 2\varkappa_2 \geq ... \geq \mathfrak{n}\varkappa_{\mathfrak{n}} \geq (\mathfrak{n}+1)\varkappa_{\mathfrak{n}} \geq ... \geq 0, \\ \text{or } &1 \leq 2\varkappa_2 \leq ... \leq \mathfrak{n}\varkappa_{\mathfrak{n}} \geq (\mathfrak{n}+1)\varkappa_{\mathfrak{n}} \leq ... \leq 2, \end{aligned} \tag{14}$$

then ϑ is close-to-convex with respect $-\log(1-z)$. Moreover, if the odd function $\psi(z) = z + \ell_3 z^3 + ... + \ell_{2\mathfrak{n}+1} z^{2\mathfrak{n}+1} + ...$ is regular in $\mathfrak{D}$ and if

$$\begin{aligned} &1 \geq 3\ell_3 \geq ... \geq (2\mathfrak{n}+1)\ell_{2\mathfrak{n}+1} \geq (\mathfrak{n}+1)\varkappa_{\mathfrak{n}} \geq ... \geq 0, \\ \text{or } &1 \leq 3\ell_3 \leq ... \leq \mathfrak{n}\varkappa_{\mathfrak{n}} \geq (2\mathfrak{n}+1)\ell_{2\mathfrak{n}+1} \leq ... \leq 2, \end{aligned} \tag{15}$$

then ψ is univalent in $\mathfrak{D}$.

Lemma 2.3 ([35]). *$\mathcal{P}_0(\zeta) * \mathcal{P}_0(\xi) \subset \mathcal{P}_0(\varpi)$, where $\varpi = 1 - 2(1-\zeta)(1-\xi)$ with $\zeta, \xi < 1$ and the value of ϖ is best possible.*

Lemma 2.4 ([27]). *For $\zeta < 1, \xi < 1$ and $\varpi = 1 - 2(1-\zeta)(1-\xi)$, we have $\mathcal{R}_0(\zeta) * \mathcal{R}_0(\xi) \subset \mathcal{R}_0(\varpi)$ or equivalently $\mathcal{P}_0(\zeta) * \mathcal{P}_0(\xi) \subset \mathcal{P}_0(\varpi)$.*

Lemma 2.5 ([12]). *If $\vartheta \in \mathcal{C}_{\mathcal{V}}(\zeta)$, where $\zeta \in [0,1)$ is not of the formula*

$$\vartheta(z) = \begin{cases} \eta + \delta z \left(1 - ze^{i\rho}\right)^{2\zeta - 1}, & \zeta \neq \frac{1}{2}, \\ \eta + \delta \log\left(1 - ze^{i\rho}\right), & \zeta = \frac{1}{2}. \end{cases} \tag{16}$$

for η, $\delta \in \mathcal{C}$ and $\rho \in \mathbb{R}$, then the following statements are true:

1. *There is $\sigma = \sigma(\vartheta) > 0$ such that $\vartheta' \in \Omega^{\sigma + 1/[2(1-\zeta)]}$.*
2. *If $\zeta \in [0, 1/2)$, then there is $\omega = \omega(\vartheta) > 0$ such that $\vartheta \in \Omega^{\omega + 1/(1-2\zeta)}$.*
3. *If $\zeta \leq 1/2$, then $\vartheta \in \Omega^{\infty}$.*

3. Modified Koebe functions

This section introduces, for $\varrho \in \Omega_{\mathfrak{X}}$, a new modified Koebe function based on confluent hypergeometric limit function.

Corresponding to (2), we impose the following normalized function:

$$\begin{aligned} \mathcal{E}_{\varrho}(z) &= \frac{z}{1-z} + z\ \varrho\left[\kappa'(z) - 1\right] \qquad\qquad (0 \leq \varrho) \\ &= z + \sum_{\mathfrak{n}=1}^{\infty} z^{\mathfrak{n}} + \sum_{\mathfrak{n}=1}^{\infty} \varrho\ \mathfrak{n}^2\ z^{\mathfrak{n}} \\ &= z + \sum_{\mathfrak{n}=2}^{\infty} \left(1 + \varrho\ \mathfrak{n}^2\right) z^{\mathfrak{n}}. \end{aligned} \tag{17}$$

In view of the confluent hypergeometric function ${}_0\mathcal{F}_1(\tau, z)$ given by (7) and normalized function $\mathcal{E}_{\varrho}(z)$ defined by (15), we introduce a new modified Koebe function by using convolution product.

For $\vartheta \in \Lambda$, $z \in \mathbb{C}$, $0 \leq \varrho$, $\tau \in \mathbb{C} \setminus \{0, -1, \ldots\}$, we consider that

$$\mathcal{E}_{\varrho}(z) * \mathcal{K}_{\varrho,\tau}(z) = z\,{}_0\mathcal{F}_1(\tau, z) = z + \sum_{\mathfrak{n}=2}^{\infty} \frac{z^{\mathfrak{n}}}{(\tau)_{\mathfrak{n}-1}\ (\mathfrak{n}-1)!}. \tag{18}$$

Therefore, the calculation gives the following modified Koebe function:

$$\begin{aligned} \mathcal{K}_{\varrho,\tau}(z) &= z + \sum_{\mathfrak{n}=2}^{\infty} \frac{z^{\mathfrak{n}}}{(1+\varrho\ \mathfrak{n}^2)\ (\tau)_{\mathfrak{n}-1}\,(\mathfrak{n}-1)!} \\ &= z + \sum_{\mathfrak{n}=1}^{\infty} \frac{z^{\mathfrak{n}+1}}{(1+\varrho\ (\mathfrak{n}+1)^2)\ (\tau)_{\mathfrak{n}}\,\mathfrak{n}!}. \end{aligned} \tag{19}$$

4. Geometric features of modified Koebe functions

This section investigates certain appropriate stipulations on parameters of the modified Koebe function to order be starlike, convex and close-to-convex in $\mathfrak{D}$.

Theorem 4.1. *Let $\varrho \in [0, \infty)$ and $\zeta \in [0, 1)$. If*

$$\frac{(2-\zeta) + \sqrt{5\zeta^2 - 16\zeta + 12 - 48\zeta\varrho + 16\zeta^2\varrho + 32\varrho}}{2(1-\zeta)(1+4\varrho)} < \tau, \tag{20}$$

then $\mathcal{K}_{\varrho,\tau} \in \mathcal{S}^*(\zeta)$ in $\mathfrak{D}$.

Proof. To demonstrate the starlikeness of order ζ for the function $\mathcal{K}_{\varrho,\tau}(z)$, it is adequate to verify that

$$\left| \frac{z\mathcal{K}'_{\varrho,\tau}(z)}{\mathcal{K}_{\varrho,\tau}(z)} - 1 \right| < 1 - \zeta. \tag{21}$$

Consider

$$\mathcal{K}'_{\varrho,\tau}(z) = 1 + \sum_{\mathfrak{n}=1}^{\infty} \frac{\mathfrak{n}+1}{(1+\varrho\ (\mathfrak{n}+1)^2)\ (\tau)_{\mathfrak{n}}\,\mathfrak{n}!}\ z^{\mathfrak{n}}\ , \tag{22}$$

and

$$\frac{\mathcal{K}_{\varrho,\tau}(z)}{z} = 1 + \sum_{\mathfrak{n}=1}^{\infty} \frac{z^{\mathfrak{n}}}{(1+\varrho\ (\mathfrak{n}+1)^2)\ (\tau)_{\mathfrak{n}}\,\mathfrak{n}!}. \tag{23}$$

Furthermore, for $\mathfrak{n} \in \mathbb{N}$ leads to

$$\mathfrak{n} \leq \mathfrak{n}!,\ \ 2 \leq \mathfrak{n}+1,\ \ \text{and} \quad \tau(\tau+1)^{\mathfrak{n}-1} \leq \tau(\tau+1)_{\mathfrak{n}-1} = (\tau)_{\mathfrak{n}}\,. \tag{24}$$

In light of the famed triangle inequality and (22), (23), (24), it acquires

$$\begin{aligned}\left|\mathcal{K}'_{\varrho,\tau}(z)-\frac{\mathcal{K}_{\varrho,\tau}(z)}{z}\right| &\leq \sum_{\mathfrak{n}=1}^{\infty}\frac{\mathfrak{n}}{(1+\varrho\ (\mathfrak{n}+1)^2)\ \ (\tau)_{\mathfrak{n}}\,\mathfrak{n}!}\\ &\leq \sum_{\mathfrak{n}=1}^{\infty}\frac{1}{(1+\varrho\ (\mathfrak{n}+1)^2)\ \ \tau\ \ (\tau+1)^{\mathfrak{n}-1}} \qquad (25)\\ &\leq \frac{1}{\tau\,(1+4\varrho)}\sum_{\mathfrak{n}=1}^{\infty}\left(\frac{1}{\tau+1}\right)^{\mathfrak{n}-1}=\frac{\tau+1}{\tau^2\,(1+4\varrho)},\quad(\tau>0).\end{aligned}$$

Using (23), (24) and the reverse triangle inequality implies that

$$\begin{aligned}\left|\frac{\mathcal{K}_{\varrho,\tau}(z)}{z}\right| &\geq 1-\sum_{\mathfrak{n}=1}^{\infty}\frac{1}{(1+\varrho\ (\mathfrak{n}+1)^2)\ \ (\tau)_{\mathfrak{n}}\,\mathfrak{n}!}\\ &\geq 1-\frac{1}{\tau\,(1+4\varrho)}\sum_{\mathfrak{n}=1}^{\infty}\left(\frac{1}{\tau+1}\right)^{\mathfrak{n}-1} \qquad (26)\\ &=1-\frac{\tau+1}{\tau^2\,(1+4\varrho)}=\frac{\tau^2\,(1+4\varrho)-\tau-1}{\tau^2\,(1+4\varrho)},\quad(\tau>0).\end{aligned}$$

Hence, by combining (25) and (26), then

$$\left|\frac{z\mathcal{K}'_{\varrho,\tau}(z)}{\mathcal{K}_{\varrho,\tau}(z)}-1\right|\leq\frac{\tau+1}{\tau^2\,(1+4\varrho)-\tau-1}<1-\zeta. \qquad (27)$$

Therefore, $\mathcal{K}_{\varrho,\tau}$ is starlike function of order ζ, where $0\leq\zeta<1-\frac{\tau+1}{\tau^2(1+4\varrho)-\tau-1}$.

Theorem 4.2. *Let $\varrho\in[0,\infty)$ and $\zeta\in[0,1)$. If*

$$\frac{(5-3\zeta)+4(1-\zeta)+\sqrt{(5-3\zeta)^2+8(1-\zeta)\left[2-\zeta+13\varrho-7\zeta\varrho+2\varrho^2(1-\zeta)\right]}}{2(1-\zeta)(1+4\varrho)}<\tau, \qquad (28)$$

then $\mathcal{K}_{\varrho,\tau}\in\mathcal{C}_{\mathcal{O}}(\zeta)$ in $\mathfrak{D}$.

Proof. In order to examine the convexity of order ζ for the function $\mathcal{K}_{\varrho,\tau}(z)$, it is sufficient to prove that

$$\left|\frac{z\mathcal{K}''_{\varrho,\tau}(z)}{\mathcal{K}'_{\varrho,\tau}(z)}\right|<1-\zeta. \qquad (29)$$

Consider

$$\mathcal{K}''_{\varrho,\tau}(z)=\sum_{\mathfrak{n}=1}^{\infty}\frac{\mathfrak{n}(\mathfrak{n}+1)}{(1+\varrho\ (\mathfrak{n}+1)^2)\ \ (\tau)_{\mathfrak{n}}\,\mathfrak{n}!}\ z^{\mathfrak{n}}. \qquad (30)$$

By utilizing (30), (24) and the following inequality: for all $\mathfrak{n} \in \mathbb{N}$

$$\frac{\mathfrak{n}(\mathfrak{n}+1)}{2^{\mathfrak{n}}} \leq \mathfrak{n}! \,, \tag{31}$$

we attain

$$\begin{aligned} \left|z\mathcal{K}''_{\varrho,\tau}(z)\right| &= \left|\sum_{\mathfrak{n}=1}^{\infty} \frac{\mathfrak{n}(\mathfrak{n}+1)}{(1+\varrho\ (\mathfrak{n}+1)^2)\ \ (\tau)_{\mathfrak{n}}\, \mathfrak{n}!}\ z^{\mathfrak{n}}\right| \leq \frac{1}{(1+4\varrho)\,\tau} \sum_{\mathfrak{n}=1}^{\infty} \frac{\mathfrak{n}(\mathfrak{n}+1)}{(\tau+1)^{\mathfrak{n}-1}\ \mathfrak{n}!} \\ &\leq \frac{2}{(1+4\varrho)\,\tau} \sum_{\mathfrak{n}=1}^{\infty} \left(\frac{2}{\tau+1}\right)^{\mathfrak{n}-1} = \frac{2\,(\tau+1)}{(1+4\varrho)\,\tau\,(\tau-1)}, \quad (\tau>1). \end{aligned} \tag{32}$$

Moreover, by using (22) and the following inequalities: for $\mathfrak{n} \in \mathbb{N}$

$$2(\mathfrak{n}+1) \leq 4^{\mathfrak{n}}, \quad \text{and} \quad \mathfrak{n}! \geq 2^{\mathfrak{n}-1}, \tag{33}$$

we achieve

$$\begin{aligned} \left|\mathcal{K}'_{\varrho,\tau}(z)\right| &= \left|1+\sum_{\mathfrak{n}=1}^{\infty} \frac{\mathfrak{n}+1}{(1+\varrho\ (\mathfrak{n}+1)^2)\ \ (\tau)_{\mathfrak{n}}\, \mathfrak{n}!}\ z^{\mathfrak{n}}\right| \\ &\geq 1 - \frac{1}{(1+4\varrho)\,\tau} \sum_{\mathfrak{n}=1}^{\infty} \frac{\mathfrak{n}+1}{(\tau+1)^{\mathfrak{n}-1}\ \mathfrak{n}!} \\ &\geq 1 - \frac{2}{(1+4\varrho)\,\tau} \sum_{\mathfrak{n}=1}^{\infty} \left(\frac{2}{\tau+1}\right)^{\mathfrak{n}-1} \\ &= \frac{(1+4\varrho)\,\tau^2 - (3+4\varrho)\,\tau - 2}{\tau(\tau-1)\,(1+4\varrho)}, \quad (\tau>1). \end{aligned} \tag{34}$$

By combining (32) and (34), we yield

$$\left|\frac{z\mathcal{K}''_{\varrho,\tau}(z)}{\mathcal{K}'_{\varrho,\tau}(z)}\right| \leq \frac{2\,(\tau+1)}{(1+4\varrho)\,\tau^2 - (3+4\varrho)\,\tau - 2} < 1-\zeta. \tag{35}$$

Thus, $\mathcal{K}_{\varrho,\tau}$ is convex function of order ζ, where $0 \leq \zeta < 1 - \frac{2(\tau+1)}{(1+4\varrho)\tau^2-(3+4\varrho)\tau-2}$.

Theorem 4.3. *Let $\varrho \in [0,\infty)$ and $\zeta \in [0,1)$. If*

$$\frac{\begin{array}{l} 8+(1-\zeta)(1+4\varrho) \\ \quad +\sqrt{\zeta^2-50\zeta+113+4\varrho(1-\zeta)(4\varrho(1-\zeta)+50-2\zeta)} \end{array}}{2(1-\zeta)(1+4\varrho)} < \tau. \tag{36}$$

then $\mathcal{K}_{\varrho,\tau} \in \mathcal{C}_{\mathcal{V}}(\frac{1+\zeta}{2})$ in $\mathfrak{D}$.

Proof. Utilizing inequality (32) and Lemma 2.1, we gain

$$\left|z\mathcal{K}''_{\varrho,\tau}(z)\right| \ \leq \frac{2(\tau+1)}{(1+4\varrho)\tau(\tau-1)} \leq \frac{1-\zeta}{4}. \tag{37}$$

where $0 \leq \zeta < 1 - \frac{8(\tau+1)}{(1+4\varrho)\tau(\tau-1)}$ and

$$\frac{8+(1-\zeta)(1+4\varrho)+\sqrt{\zeta^2-50\zeta+113+4\varrho(1-\zeta)(4\varrho(1-\zeta)+50-2\zeta)}}{2(1-\zeta)(1+4\varrho)} < \tau.$$

This shows that $\mathcal{K}_{\varrho,\tau} \in \mathcal{C}_{\mathcal{V}}(\frac{1+\zeta}{2})$. Hence, $\Re\left(\mathcal{K}'_{\varrho,\tau}(z)\right) > \frac{1+\zeta}{2}$.

Theorem 4.4. *Let $\varrho \in [0,\infty)$ and $\zeta \in [0,1)$. If*

$$\frac{1+\sqrt{1+4(1-\zeta)(1+4\varrho)}}{2(1-\zeta)(1+4\varrho)} < \tau. \tag{38}$$

then $\frac{\mathcal{K}_{\varrho,\tau}}{z} \in \mathcal{P}(\zeta)$ in $\mathfrak{D}$.

Proof. To show that $\frac{\mathcal{K}_{\varrho,\tau}}{z} \in \mathcal{P}(\zeta)$, it is sufficient to verify that $|\mathfrak{A}(z)-1| < 1$ where $\frac{(\mathfrak{A}(z)/z)-\zeta}{1-\zeta}$. By utilizing the inequalities $\tau(\tau+1)^{\mathfrak{n}-1} \leq \tau(\tau+1)_{\mathfrak{n}-1} = (\tau)_{\mathfrak{n}}$ and $1 \leq \mathfrak{n}!$, for $\mathfrak{n} \in \mathbb{N}$. we gain

$$\begin{aligned} |\mathfrak{A}(z)-1| &= \left| \frac{1}{1-\zeta} \sum_{\mathfrak{n}=1}^{\infty} \frac{z^{\mathfrak{n}}}{(1+\varrho\ (\mathfrak{n}+1)^2)\ (\tau)_{\mathfrak{n}}\ \mathfrak{n}!} \right| \\ &\leq \frac{1}{(1-\zeta)(1+4\varrho)\ \tau} \sum_{\mathfrak{n}=1}^{\infty} \frac{1}{(\tau+1)_{\mathfrak{n}-1}} \\ &\leq \frac{1}{(1-\zeta)(1+4\varrho)\ \tau} \sum_{\mathfrak{n}=1}^{\infty} \left(\frac{1}{\tau+1}\right)^{\mathfrak{n}-1} = \frac{\tau+1}{\tau^2(1-\zeta)(1+4\varrho)}. \end{aligned} \tag{39}$$

Therefore, $\frac{\mathcal{K}_{\varrho,\tau}}{z} \in \mathcal{P}(\zeta)$ for $0 < \zeta < 1 - \frac{\tau+1}{\tau^2(1+4\varrho)}$.

Setting $\zeta = 0$ in the above Theorems 4.1, 4.2, 4.3 and 4.4 , respectively, we yield the required outcome.

Corollary 4.1. *Let $\varrho \in [0,\infty)$ and $z \in \mathfrak{D}$, then*

1. *If $\frac{1+\sqrt{3+8\varrho}}{1+4\varrho} < \tau$, then $\mathcal{K}_{\varrho,\tau} \in \mathcal{S}^*$.*
2. *If $\frac{9+\sqrt{41+\varrho(13+2\varrho)}}{2(1+4\varrho)} < \tau$, then $\mathcal{K}_{\varrho,\tau} \in \mathcal{C}_{\mathcal{O}}$.*
3. *If $\frac{9+4\varrho+\sqrt{113+4\varrho(4\varrho+50)}}{2(1+4\varrho)} < \tau$, then $\mathcal{K}_{\varrho,\tau} \in \mathcal{C}_{\mathcal{V}}\left(\frac{1}{2}\right)$.*
4. *If $\frac{1+\sqrt{1+4(1+4\varrho)}}{2(1+4\varrho)} < \tau$, then $\frac{\mathcal{K}_{\varrho,\tau}}{z} \in \mathcal{P}$.*

5. Close-to-convexity of modified Koebe functions with respect to specific functions

This sections discusses some stipulations on the parameters ϱ and τ under which the modified Koebe functions are assured to be close-to-convex with respect to the functions $-\log(1-z)$ and $\frac{1}{2}\log\frac{1+z}{1-z}$, respectively.

Theorem 5.1. *Let $1 \leq \varrho$ and $1 \leq \tau$, then $z \to \mathcal{K}_{\varrho,\tau}(z)$ is in $\mathcal{C}_{\mathcal{V}}$ with respect to the function $-\log(1-z)$.*

Proof. Set

$$\Xi(z) = \mathcal{K}_{\varrho,\tau}(z) = z + \sum_{\mathfrak{n}=2}^{\infty} \wp_{\mathfrak{n}-1} z^{\mathfrak{n}}. \tag{40}$$

We yield $0 < \wp_{\mathfrak{n}-1}$ for all $2 \leq \mathfrak{n}$ and $\wp_1 = \frac{1}{(1+4\varrho)\tau} \leq 1$. To show this outcome, we utilize Lemma 2.2. Thus, we have to show that $\{\mathfrak{n}\wp_{\mathfrak{n}-1}\}_{2\leq\mathfrak{n}}$ is decreasing. Now

$$\begin{aligned} &\mathfrak{n}\,\wp_{\mathfrak{n}-1} - (\mathfrak{n}+1)\,\wp_{\mathfrak{n}} \\ &= \frac{\mathfrak{n}}{(1+\varrho\ \mathfrak{n}^2)\ (\tau)_{\mathfrak{n}-1}\,(\mathfrak{n}-1)!} - \frac{\mathfrak{n}+1}{(1+\varrho\ (\mathfrak{n}+1)^2)\ (\tau)_{\mathfrak{n}}\,(\mathfrak{n})!} \\ &= \frac{1}{(\mathfrak{n}-1)!}\left[\frac{\mathfrak{n}^2\ (\tau+\mathfrak{n}-1)\ \left(1+\varrho(\mathfrak{n}+1)^2\right) - (\mathfrak{n}+1)\left(1+\varrho\ \mathfrak{n}^2\right)}{\mathfrak{n}\ (\tau)_{\mathfrak{n}}\ (1+\varrho\ \mathfrak{n}^2)\ (1+\varrho\ (\mathfrak{n}+1)^2)}\right] > 0. \end{aligned} \tag{41}$$

By utilizing the stipulations on parameters, we notice that $\mathfrak{n}\,\wp_{\mathfrak{n}-1} - (\mathfrak{n}+1)\,\wp_{\mathfrak{n}} > 0$ for all $2 \leq \mathfrak{n}$. Therefore $\{\mathfrak{n}\wp_{\mathfrak{n}-1}\}_{2\leq\mathfrak{n}}$ is a decreasing sequence. Hence the outcome.

Theorem 5.2. *Let $1 \leq \varrho$ and $1 \leq \tau$, then $z \to z\mathcal{K}_{\varrho,\tau}(z^2)$ is in $\mathcal{C}_{\mathcal{V}}$ with respect to the function $\frac{1}{2}\log\frac{1+z}{1-z}$.*

Proof. Set

$$\Xi(z) = z\mathcal{K}_{\varrho,\tau}(z^2) = z + \sum_{\mathfrak{n}=2}^{\infty} \Lambda_{2\mathfrak{n}-1}\, z^{2\mathfrak{n}-1}. \tag{42}$$

Here, $\Lambda_{2\mathfrak{n}-1} = \wp_{\mathfrak{n}-1} = \frac{1}{(1+\varrho\ \mathfrak{n}^2)\,(\tau)_{\mathfrak{n}-1}\,(\mathfrak{n}-1)!}$ for all $2 \leq \mathfrak{n}$ and hence we gain $\wp_1 = \frac{1}{\tau\,(1+4\varrho)}$ and $\Lambda_{2\mathfrak{n}-1} > 0$ for all $2 \leq \mathfrak{n}$.

To show this outcome, we apply Lemma 2.2. Thus, we have to show that $\{(2\mathfrak{n}-1)\wp_{\mathfrak{n}-1}\}_{2\leq\mathfrak{n}}$ is decreasing. Now,

$$\begin{aligned} &(2\mathfrak{n}-1)\,\wp_{\mathfrak{n}-1} - (2\mathfrak{n}+1)\,\wp_{\mathfrak{n}} \\ &= \frac{2\mathfrak{n}-1}{(1+\varrho\ \mathfrak{n}^2)\ (\tau)_{\mathfrak{n}-1}\,(\mathfrak{n}-1)!} - \frac{2\mathfrak{n}+1}{(1+\varrho\ (\mathfrak{n}+1)^2)\ (\tau)_{\mathfrak{n}}\,(\mathfrak{n})!} \\ &= \frac{1}{(\mathfrak{n}-1)!}\left[\frac{(2\mathfrak{n}^2-\mathfrak{n})(\tau+\mathfrak{n}-1)(1+\varrho(\mathfrak{n}+1)^2) - (2\mathfrak{n}+1)(1+\varrho\ \mathfrak{n}^2)}{\mathfrak{n}(\tau)_{\mathfrak{n}}(1+\varrho\ \mathfrak{n}^2)(1+\varrho\ (\mathfrak{n}+1)^2)}\right] > 0. \end{aligned} \tag{43}$$

By employing the stipulations on parameters, we notice that $(2\mathfrak{n}-1)\,\wp_{\mathfrak{n}-1} - (2\mathfrak{n}+1)\,\wp_{\mathfrak{n}} > 0$ for all $2 \leq \mathfrak{n}$. Hence $\{(2\mathfrak{n}-1)\wp_{\mathfrak{n}-1}\}_{2\leq\mathfrak{n}}$ is a decreasing sequence. Therefore the outcome.

6. Hardy spaces of modified Koebe functions

This section examines stipulations for modified Koebe functions to belong to the Hardy space.

Theorem 6.1. *Let* $\zeta \in [0,1)$ *and*

$$\frac{(5-3\zeta)+4(1-\zeta) + \sqrt{(5-3\zeta)^2+8(1-\zeta)\left[2-\zeta+13\varrho-7\zeta\varrho+2\varrho^2(1-\zeta)\right]}}{2(1-\zeta)(1+4\varrho)} < \tau.$$

Then

1. $\mathcal{K}_{\varrho,\tau} \in \Omega^{1/(1-2\zeta)}$ *for* $\zeta \in \left[0, \frac{1}{2}\right)$.
2. $\mathcal{K}_{\varrho,\tau} \in \Omega^{\infty}$ *for* $\zeta \geq \frac{1}{2}$.

Proof. By applying the following concept of hypergeometric function:

$$ {}_2\mathcal{F}_1\left(\mathfrak{a},\mathfrak{b},\mathfrak{c};z\right) = \sum_{\mathfrak{n}=0}^{\infty} \frac{\left(\mathfrak{a}\right)_{\mathfrak{n}}\left(\mathfrak{b}\right)_{\mathfrak{n}}}{\left(\mathfrak{c}\right)_{\mathfrak{n}}} \frac{z^{\mathfrak{n}}}{\mathfrak{n}!}, \tag{44}$$

we yield

$$\begin{aligned} \eta + \frac{dz}{z\left(1-ze^{i\rho}\right)^{1-2\zeta}} &= \eta + dz\ {}_2\mathcal{F}_1\left(1,1-2\zeta,1;ze^{i\rho}\right) \\ &= \eta + d\sum_{\mathfrak{n}=0}^{\infty} \frac{(1-2\zeta)_{\mathfrak{n}+1}}{\mathfrak{n}!}\, e^{i\rho\mathfrak{n}} z^{\mathfrak{n}}, \end{aligned} \tag{45}$$

for $\eta,\, \delta \in \mathcal{C}$, $\zeta \neq \frac{1}{2}$ and $\rho \in \mathbb{R}$. In addition, we obtain

$$\begin{aligned} \eta + d\log\left(1-ze^{i\rho}\right) &= \eta - dz\ {}_2\mathcal{F}_1\left(1,1,z;ze^{i\rho}\right) \\ &= \eta - d\sum_{\mathfrak{n}=0}^{\infty} \frac{1}{\mathfrak{n}+1}\, e^{i\rho\mathfrak{n}} z^{\mathfrak{n}}. \end{aligned} \tag{46}$$

Consequently, $\mathcal{K}_{\varrho,\tau}$ is not of the forms $\eta + dz\left(1-z^{i\rho}\right)^{2\zeta-1}$ for $\zeta \neq \frac{1}{2}$ and $\eta + d\log\left(1-ze^{i\rho}\right)$ for $\zeta = \frac{1}{2}$, respectively. Furthermore, by Theorem 4.2, $\mathcal{K}_{\varrho,\tau} \in \mathcal{C}_{\mathcal{O}}(\zeta)$. Thus, by utilizing Lemma 2.5, we achieve the requited outcome.

Theorem 6.2. *Let* $1 \leq \varrho$, $\frac{1+\sqrt{1+4(1+4\varrho)}}{2(1+4\varrho)} < \tau$ *and* $\vartheta \in \mathcal{R}$. *Then the convolution* $\mathcal{K}_{\varrho,\tau} * \vartheta$ *is in* $\Omega^{\infty} \cap \mathbb{R}$.

Proof. Let $\digamma(z) = \mathcal{K}_{\varrho,\tau}(z) * \vartheta(z)$. Then, $\digamma'(z) = \frac{\mathcal{K}_{\varrho,\tau}(z)}{z} * \vartheta'(z)$. Utilizing Corollary 4.1(4), we yield $\frac{\mathcal{K}_{\varrho,\tau}(z)}{z} \in \mathcal{P}$. Since $\vartheta \in \mathcal{R}$, thus by utilizing Lemma 2.3, we attain $\digamma \in \mathcal{R}$. It is obvious that $\frac{\mathcal{K}_{\varrho,\tau}(z)}{z}$ is an entire function and hence ϑ is entire. This leads to ϑ is bounded. Therefore, we have the required outcome.

Theorem 6.3. *Let $\varrho \in [0,\infty)$, $\zeta \in [0,1)$, $\frac{1+\sqrt{1+4(1-\zeta)(1+4\varrho)}}{2(1-\zeta)(1+4\varrho)} < \tau$ and $z \in \mathfrak{D}$. If $\vartheta \in \mathcal{R}(\xi)$ with $\xi < 1$, then $\mathcal{K}_{\varrho,\tau} * \vartheta \in \mathcal{R}(\varpi)$, where $\varpi = 1 - 2(1-\zeta)(1-\xi)$.*

Proof. Let $\digamma(z) = \mathcal{K}_{\varrho,\tau}(z) * \vartheta(z)$. Then, $\digamma'(z) = \frac{\mathcal{K}_{\varrho,\tau}(z)}{z} * \vartheta'(z)$. By employing Theorem 4.4, we acquire $\frac{\mathcal{K}_{\varrho,\tau}(z)}{z} \in \mathcal{P}(\zeta)$. By utilizing Lemma 2.4 and considering $\digamma'(z) \in \mathcal{P}(\xi)$, we achieve $\digamma'(z) \in \mathcal{P}(\varpi)$, where $\varpi = 1-2(1-\zeta)(1-\xi)$. Therefore, we obtain $\digamma \in \Re(\varpi)$.

Corollary 6.1. *Let $\varrho \in [0,\infty)$, $\zeta \in [0,1)$, $\frac{1+\sqrt{1+4(1-\zeta)(1+4\varrho)}}{2(1-\zeta)(1+4\varrho)} < \tau$. If $\vartheta \in \mathcal{R}(\xi)$, $\xi = (1-2\zeta)/(2-2\zeta)$, then $\mathcal{K}_{\varrho,\tau} * \vartheta \in \mathcal{R}(0)$.*

Corollary 6.2. *Let $\varrho \in [0,\infty)$, $\frac{1+\sqrt{1+4(1+4\varrho)}}{2(1+4\varrho)} < \tau$. If $\vartheta \in \mathcal{R}(1/2)$, then $\mathcal{K}_{\varrho,\tau} * \vartheta \in \mathcal{R}(0)$.*

7. Conclusions

Conducting the study of the confluent hypergeometric functions in a complex domain, we have defined a new normalized modified Koebe-type function. This Koebe-type function may be utilized to yield new classes of normalized regular functions in the open unit disk. Moreover, we have imposed sufficient stipulations on parameters of this function to be starlike, convex and close-to-convex. We have additionally investigated the Hardy spaces of the modified Koebe function. Normalized regular functions are suggested in this work by employing the confluent hypergeometric functions. These functions have been done to reveal some of the connections of the special functions.

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Interval valued vague ideals in Γ-nearrings

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Abstract. In the present paper we propose interval valued (I-V)vague ideals in Γ-nearrings, explain some of their properties and provide some applications related to this work. Also, look at the direct product of I-V vague ideals, as well as the normal I-V vague ideals Γ-nearrings.

Keywords: vague ideals, Γ-nearrings, vague ideal in Γ-nearrings, I-V ideals, direct product, normal vague ideals.

1. Introduction

Zadeh introduced fuzzy set in 1965 [5]. In this he talked about the true membership value of a function. Gau and Buehrer introduced the concept of vague theory in [13]. In vague set they talk about the membership value of a function as well as non-membership value. To increase the study of vague sets, many authors have considered several extension works of fuzzy sets. In a vague set, the universal set $\mathfrak{X}$ can be represented by a pair of functions $(\tau_{\mathfrak{X}}, \sigma_{\mathfrak{X}})$. In this $\tau_{\mathfrak{X}}$ and $\sigma_{\mathfrak{X}}$ are the functions from $\mathfrak{X}$ to [0,1] such as $\tau_{\mathfrak{X}}(\mathrm{i}) + \sigma_{\mathfrak{X}}(\mathrm{i}) \leq 1$. Here, $\tau_{\mathfrak{X}}$ gives membership value and $\sigma_{\mathfrak{X}}$ is non-membership value. In the domain of fuzzy sets, a vague set is referred to as an interval membership function, as opposed to point membership. This notion is used in a variety of fields including fuzzy control systems, decision making, fault diagnostics, knowledge discovery and many more. Vague in group theory introduced by R. Biswas [11]. A similar concept of vague set is intuitionistic set is given by many researchers in [14],[8].

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Vague ideals in Γ-nearrings introduced by Y.Bhargav and S.Ragamayi in [12]. L. Bhaskar et. al sum, vague ideal and a nearring [7].

Vague set is interval membership function but it is not I-V function because in vague set first value represent membership value and second value represent non-membership value, but in I-V function whole interval represent either membership value or non-membership values.

Here, we introduced I-V vague ideals in Γ-nearring and show that vague interval and interval valued vague ideal is different and give some examples related to this Also, talk about normal interval valued vague ideal and direct product of vague ideals. For standard definitions and results we refer to Book and article [3], [6], [1]. A vague collection $\mathfrak{A}$ in the discourse universe $\mathfrak{X}$ is a pair $(\tau_{\mathfrak{A}}, \sigma_{\mathfrak{A}})$ where $\tau_{\mathfrak{A}}$: $\mathfrak{X} \to [0, 1]$ and $\sigma_{\mathfrak{A}}$: $\mathfrak{X} \to [0, 1]$ are mappings such as $\tau_{\mathfrak{A}}$(i) + $\sigma_{\mathfrak{A}}$(i) $\leq 1 \ \forall \ i \in \mathfrak{X}$. The membership function and non-membership function in $[0,1]$ is $\tau_{\mathfrak{A}}$ and $\sigma_{\mathfrak{A}}$ correspondingly. And $[\tau_{\mathfrak{A}}, 1 - \sigma_{\mathfrak{A}}]$ is called interval vague value of i in $\mathfrak{A}$ and it is indicated by $\mathfrak{V}_{\mathfrak{A}}(i)$ that is $\mathfrak{V}_{\mathfrak{A}}(i) = [\tau_{\mathfrak{A}}(i), 1 - \sigma_{\mathfrak{A}}(i)]$.

2. Interval valued vague ideals in Γ-nearrings

Definition 2.1. *A vague collection $\mathfrak{A}$ in Γ-nearring N is I-V vague ideals if it is satisfy following properties for all i, j, $k \in N$ and $\rho \in \Gamma$:*

(*i*) $\overline{\mathfrak{V}_{\mathfrak{A}}}(i-j) \geq \min\{\overline{\mathfrak{V}_{\mathfrak{A}}}(i), \overline{\mathfrak{V}_{\mathfrak{A}}}(j)\}$;

(*ii*) $\overline{\mathfrak{V}_{\mathfrak{A}}}(j+i-j) \geq \overline{\mathfrak{V}_{\mathfrak{A}}}(i)$;

(*iii*) $\overline{\mathfrak{V}_{\mathfrak{A}}}(i\rho(j+k) - i\rho k) \geq \overline{\mathfrak{V}_{\mathfrak{A}}}(j)$;

(*iv*) $\overline{\tau_{\mathfrak{A}}}(i-j) \geq \min\{\overline{\tau_{\mathfrak{A}}}(i), \overline{\tau_{\mathfrak{A}}}(j)\}$;

(*v*) $\overline{\tau_{\mathfrak{A}}}(j+i-j) \geq \overline{\tau_{\mathfrak{A}}}(i)$;

(*vi*) $\overline{\tau_{\mathfrak{A}}}(i\rho(j+k) - i\rho k) \geq \overline{\tau_{\mathfrak{A}}}(j)$;

(*vii*) $1-\overline{\sigma_{\mathfrak{A}}}(i-j) \geq \min\{1-\overline{\sigma_{\mathfrak{A}}}(i), 1-\overline{\sigma_{\mathfrak{A}}}(j)\}$, *or* $\overline{\sigma_{\mathfrak{A}}}(i-j) \leq \max\{\overline{\sigma_{\mathfrak{A}}}(i), \overline{\sigma_{\mathfrak{A}}}(j)\}$;

(*viii*) $1 - \overline{\sigma_{\mathfrak{A}}}(j+i-j) \geq 1 - \overline{\sigma_{\mathfrak{A}}}(i)$, *or* $\overline{\sigma_{\mathfrak{A}}}(j+i-j) \leq \overline{\sigma_{\mathfrak{A}}}(i)$;

(*ix*) $1 - \overline{\sigma_{\mathfrak{A}}}(i\rho(j+k) - i\rho k) \geq 1 - \overline{\sigma_{\mathfrak{A}}}(j)$, *or* $\overline{\sigma_{\mathfrak{A}}}(i\rho(j+k) - i\rho k) \leq \overline{\sigma_{\mathfrak{A}}}(j)$.

Example 2.1. Consider integers modulo 4, $N = \{0^*, i^*, j^*, k^*\}$ be a collection with two binary operations + and ρ as described below:

+	0^*	i^*	j^*	k^*
0^*	0^*	i^*	j^*	k^*
i^*	i^*	j^*	k^*	0^*
j^*	j^*	k^*	0^*	i^*
k^*	k^*	0^*	i^*	j^*

ρ	0^*	i^*	j^*	k^*
0^*	0^*	0^*	0^*	0^*
i^*	0^*	j^*	i^*	i^*
j^*	0^*	j^*	0^*	k^*
k^*	0^*	0^*	0^*	k^*

Clearly, N is a Γ-nearring. Define as $\overline{\mathfrak{V}_{\mathfrak{A}}}(0^*) = [[0.8, 0.9], [0, 0.1]]$, $\overline{\mathfrak{V}_{\mathfrak{A}}}(i^*) = [[0.5, 0.5], [0.4, 0.5]]$, $\overline{\mathfrak{V}_{\mathfrak{A}}}(j^*) = [[0.1, 0.3], [0.6 ,0.7]]$, $\overline{\mathfrak{V}_{\mathfrak{A}}}(k^*) = [[0.6, 0.7], [0.2, 0.3]]$. Now, by routine verification we can show that $\overline{\mathfrak{V}_{\mathfrak{A}}} = [\overline{\tau_{\mathfrak{A}}}, \overline{\sigma_{\mathfrak{A}}}]$ is an I-V vague ideal in Γ-nearring.

Example 2.2. Write $N = \mathbb{Z}$ (set of integers) under operations "+" and $\Gamma = \mathbb{Z}$. Thus $(N, +, \Gamma)$ is a Γ-nearring and with the mapping, $\overline{\mathfrak{V}_{\mathfrak{A}}}$: $N \to [0, 1]$ define by

$$\overline{\tau_{\mathfrak{A}}}(i) = \begin{cases} [0.5, 0.7], & \text{if } i = 2\text{n for some n} \in \mathbb{Z}, \\ 0, & \text{if } i = 2\text{n} + 1,\ \text{n} \in \mathbb{Z}\ , \\ 1, & \text{if } i = 0 \end{cases}$$

$$\overline{\sigma_{\mathfrak{A}}}(i) = \begin{cases} [0.1, 0.3], & \text{if } i = 2\text{n for some n} \in \mathbb{Z}, \\ 1, & \text{if } i = 2\text{n} + 1,\ \text{n} \in \mathbb{Z}\ , \\ 0, & \text{if } i = 0 \end{cases} .$$

By routine verification, we can show that $\overline{\mathfrak{V}_{\mathfrak{A}}}$ is an I-V vague ideal.

Example 2.3. Write $N = \mathbb{Z}_7$ (set of integers modulo 7) and $\Gamma = \mathbb{Z}$, thus $(N, +, \Gamma)$ is a Γ-nearring. Then, the vague collection $\overline{\mathfrak{V}_{\mathfrak{A}}} = (\overline{\tau_{\mathfrak{A}}}, \overline{\sigma_{\mathfrak{A}}})$ becomes I-V vague ideal in N with the following mappings:

$$\overline{\tau_{\mathfrak{A}}}(i) = \begin{cases} [0.6, 0.9], & \text{if } i \in \mathbb{Z}_7, \\ [0, 0.1], & \text{if } i \notin \mathbb{Z}_7 \end{cases}$$

$$\overline{\sigma_{\mathfrak{A}}}(i) = \begin{cases} [0.1, 0.3], & \text{if } i \in \mathbb{Z}_7, \\ [0.5, 0.6], & \text{if } i \notin \mathbb{Z}_7 \end{cases}$$

Now, by routine calculation we can show that $\overline{\mathfrak{V}_{\mathfrak{A}}} = [\overline{\tau_{\mathfrak{A}}}, \overline{\sigma_{\mathfrak{A}}}]$ is an I-V vague ideals in Γ-nearring.

Proposition 2.1. *If $\overline{\mathfrak{V}_{\mathfrak{A}}}$ is an I-V vague ideals Γ-nearring N and if satisfies the condition $\overline{\mathfrak{V}_{\mathfrak{A}}}(j + i - j) \geq \overline{\mathfrak{V}_{\mathfrak{A}}}(i)$, then $\overline{\mathfrak{V}_{\mathfrak{A}}}(i + j) = \overline{\mathfrak{V}_{\mathfrak{A}}}(j + i)$.*

Proposition 2.2. *Let $\mathfrak{A}$ be a I-V vague ideal of Γ-nearring N and if $\overline{\mathfrak{V}_{\mathfrak{A}}}(i-j) = \overline{\mathfrak{V}_{\mathfrak{A}}}(0)$, then $\overline{\mathfrak{V}_{\mathfrak{A}}}(i) = \overline{\mathfrak{V}_{\mathfrak{A}}}(j)$.*

Lemma 2.1. *Let $\overline{\mathfrak{V}_{\mathfrak{A}}}$ be an I-V vague ideals in N and if $\overline{\mathfrak{V}_{\mathfrak{A}}}(i) < \overline{\mathfrak{V}_{\mathfrak{A}}}(j)$, then $\overline{\mathfrak{V}_{\mathfrak{A}}}(i - j) = \overline{\mathfrak{V}_{\mathfrak{A}}}(i) = \overline{\mathfrak{V}_{\mathfrak{A}}}(j - i)$.*

Lemma 2.2. *Let $\Re$ is an I-V vague ideal in N than for any $\overline{e} \leq \overline{f} \subset (0, 1]$, where $\overline{e} + \overline{f} \leq 1$, then there exists an I-V vague ideal $\overline{\mathfrak{V}_{\mathfrak{A}}}$ in N, such that $\overline{\mathfrak{V}_{\mathfrak{A}}}_{\overline{e},\overline{f}}$ $= \Re$. Here, $\overline{e}$ and $\overline{f}$ also an intervals.*

Proof. Consider an I-V vague ideal $\Re$ of N. Define a function $\overline{\mathfrak{V}_{\mathfrak{A}}}$: $N \to [0, 1]$ by

$$\overline{\mathfrak{V}_{\mathfrak{A}}}(i) = \begin{cases} [\overline{e}, \overline{f}] & \text{if } i \in \Re \\ [0, 0] & \text{if } i \notin \Re \end{cases}$$

where $\overline{e}$, $\overline{f} \subset (0, 1]$. Clearly $\overline{\mathfrak{V}_{\mathfrak{A}_{\overline{e},\overline{f}}}} = \Re$.

(i) Let $i, j \in \Re$ now $\overline{\mathfrak{V}_{\mathfrak{A}}}(i - j) = [\overline{e}, \overline{f}] \geq \min\{\overline{\mathfrak{V}_{\mathfrak{A}}}(i), \overline{\mathfrak{V}_{\mathfrak{A}}}(j)\}$. If there exists at least one of i and j is not in $\Re$, then $i - j \notin \Re$, thus $\overline{\mathfrak{V}_{\mathfrak{A}}}(i - j) = [\overline{0}, \overline{0}] \geq \min\{\overline{\mathfrak{V}_{\mathfrak{A}_1}}(i), \overline{\mathfrak{V}_{\mathfrak{A}_1}}(j)\}$.

(ii) Let $i \in \Re$ now $\overline{\mathfrak{V}_{\mathfrak{A}}}(j + i - j) = [\overline{e}, \overline{f}] \geq \overline{\mathfrak{V}_{\mathfrak{A}}}(i)$. If there $i \notin \Re$, then $j + i - j \notin \Re$ thus $\overline{\mathfrak{V}_{\mathfrak{A}}}(j + i - j) = [\overline{0}, \overline{0}] \geq \overline{\mathfrak{V}_{\mathfrak{A}}}(i)$.

(iii) Let $j \in \Re$ now $\overline{\mathfrak{V}_{\mathfrak{A}}}(i\rho(j + k) - i\rho k) = [\overline{e}, \overline{f}] \geq \overline{\mathfrak{V}_{\mathfrak{A}}}(j)$. If there $j \notin \Re$, then $i\rho(j + k) - i\rho k \notin \Re$ thus $\overline{\mathfrak{V}_{\mathfrak{A}}}(i\rho(j + k) - i\rho k) = [\overline{0}, \overline{0}] \geq \overline{\mathfrak{V}_{\mathfrak{A}_1}}(j)$. Thus, $\overline{\mathfrak{V}_{\mathfrak{A}}}$ is an I-V vague ideal of N. □

Corollary 2.1. *If $\overline{\mathfrak{V}_{\mathfrak{A}}}$ is an I-V vague ideal of N, then the collection $N_{\overline{\mathfrak{V}_{\mathfrak{A}}}}(i) = \{i \in N \mid \overline{\mathfrak{V}_{\mathfrak{A}}}(i) = \overline{\mathfrak{V}_{\mathfrak{A}}}(0)\}$ is a ideal of N.*

Proposition 2.3. *If $\overline{\mathcal{P}} \subseteq \overline{Q}$ are I-V vague ideals in Γ-nearring N, then $\overline{\tau_{\mathcal{P}}} \subseteq \overline{\tau_{\mathcal{Q}}}$ and $\overline{\sigma_{\mathcal{Q}}} \subseteq \overline{\sigma_{\mathcal{P}}}$.*

Theorem 2.1. *Let $\overline{\mathfrak{V}_{\mathfrak{A}}}$ is an I-V vague ideal of a Γ-nearring N iff each level sub collection $\overline{\mathfrak{V}_{\mathfrak{A}_{\overline{\mathfrak{b}}}}}$, $\overline{\mathfrak{b}} \in Im(\overline{\mathfrak{V}_{\mathfrak{A}}})$ is an ideal of N.*

Proof. Suppose $\overline{\mathfrak{V}_{\mathfrak{A}}}$ is an I-V vague ideal of N. Let $\overline{\mathfrak{b}} \in \mathrm{Im}(\overline{\mathfrak{V}_{\mathfrak{A}}})$ and $i, j, k \in N$.

(i) For any $i, j \in \overline{\mathfrak{V}_{\mathfrak{A}_{\overline{\mathfrak{b}}}}}$ and $\rho \in \Gamma$, we have $\overline{\mathfrak{V}_{\mathfrak{A}}}(i-j) \geq \min\{\overline{\mathfrak{V}_{\mathfrak{A}}}(i), \overline{\mathfrak{V}_{\mathfrak{A}}}(j)\} = \overline{\mathfrak{b}}$ thus, $i - j \in \overline{\mathfrak{V}_{\mathfrak{A}_{\overline{\mathfrak{b}}}}}$.

(ii) For any $i \in \overline{\mathfrak{V}_{\mathfrak{A}_{\overline{\mathfrak{b}}}}}$ we have $\overline{\mathfrak{V}_{\mathfrak{A}}}(j+i-j) \geq \overline{\mathfrak{V}_{\mathfrak{A}}}(i) = \overline{\mathfrak{b}}$ thus, $j+i-j \in \overline{\mathfrak{V}_{\mathfrak{A}_{\overline{\mathfrak{b}}}}}$.

(iii) For any $j \in \overline{\mathfrak{V}_{\mathfrak{A}_{\overline{\mathfrak{b}}}}}$, we have $\overline{\mathfrak{V}_{\mathfrak{A}}}(i\rho(j+k)-i\rho k) \geq \overline{\mathfrak{V}_{\mathfrak{A}}}(j) = \overline{\mathfrak{b}}$ thus $i\rho(j+k)-i\rho k \in \overline{\mathfrak{V}_{\mathfrak{A}_{\overline{\mathfrak{b}}}}}$. Hence, $\overline{\mathfrak{V}_{\mathfrak{A}_{\overline{\mathfrak{b}}}}}$ is an ideal of N. Conversely, assume that $\overline{\mathfrak{V}_{\mathfrak{A}_{\overline{\mathfrak{b}}}}}$ is an ideal of N for every $\overline{\mathfrak{b}} \in \mathrm{Im}(\overline{\mathfrak{V}_{\mathfrak{A}}})$ suppose that $\overline{\mathfrak{V}_{\mathfrak{A}}}(i_\circ - j_\circ) < \min\{\overline{\mathfrak{V}_{\mathfrak{A}}}(i_\circ), \overline{\mathfrak{V}_{\mathfrak{A}}}(j_\circ)\}$, for some $i_\circ, j_\circ \in N$ then, by taking $\overline{\mathfrak{b}}_\circ = \frac{1}{2}[\overline{\mathfrak{V}_{\mathfrak{A}}}(i_\circ - j_\circ) + \min\{\overline{\mathfrak{V}_{\mathfrak{A}}}(i_\circ), \overline{\mathfrak{V}_{\mathfrak{A}}}(j_\circ)\}]$, we have $\overline{\mathfrak{V}_{\mathfrak{A}}}(i_\circ - j_\circ) < \overline{\mathfrak{b}}_\circ, \overline{\mathfrak{V}_{\mathfrak{A}}}(i_\circ) > \overline{\mathfrak{b}}_\circ$ and $\overline{\mathfrak{V}_{\mathfrak{A}}}(j_\circ) > \overline{\mathfrak{b}}_\circ$. Hence, $(i_\circ - j_\circ) \notin \overline{\mathfrak{V}_{\mathfrak{A}_{\overline{\mathfrak{b}}_\circ}}}, i_\circ \in \overline{\mathfrak{V}_{\mathfrak{A}_{\overline{\mathfrak{b}}_\circ}}}$ and $j_\circ \in \overline{\mathfrak{V}_{\mathfrak{A}_{\overline{\mathfrak{b}}_\circ}}}$ this is a contradiction.

(ii) Suppose that $\overline{\mathfrak{V}_{\mathfrak{A}}}(j_\circ+i_\circ-j_\circ) < \overline{\mathfrak{V}_{\mathfrak{A}}}(i_\circ)$ for some $i_\circ, j_\circ \in N$, then by taking $\overline{\mathfrak{b}}_\circ = \frac{1}{2}[\overline{\mathfrak{V}_{\mathfrak{A}}}(j_\circ+i_\circ-j_\circ)+\overline{\mathfrak{V}_{\mathfrak{A}}}(i_\circ)]$, we have $\overline{\mathfrak{V}_{\mathfrak{A}}}(j_\circ+i_\circ-j_\circ) < \overline{\mathfrak{b}}_\circ and \overline{\mathfrak{V}_{\mathfrak{A}}}(i_\circ) > \overline{\mathfrak{b}}_\circ$. Hence, $(j_\circ + i_\circ - j_\circ) \notin \overline{\mathfrak{V}_{\mathfrak{A}_{\overline{\mathfrak{b}}_\circ}}}$ and $i_\circ \in \overline{\mathfrak{V}_{\mathfrak{A}_{\overline{\mathfrak{b}}_\circ}}}$ this is an contradiction.

(iii) Suppose that $\overline{\mathfrak{V}_{\mathfrak{A}}}(i_\circ\rho_0(j_\circ+k_\circ)-i_\circ\rho_0k_\circ) < \overline{\mathfrak{V}_{\mathfrak{A}}}(v_\circ)$ for some $i_\circ, j_\circ, k_\circ \in N$ and $\rho_0 \in \Gamma$ then, by taking $\overline{\mathfrak{b}}_\circ = \frac{1}{2}[\overline{\mathfrak{V}_{\mathfrak{A}}}(i_\circ\rho_0(j_\circ+k_\circ)-i_\circ\rho_0k_\circ)+\overline{\mathfrak{V}_{\mathfrak{A}}}(j_\circ)]$, we have $\overline{\mathfrak{V}_{\mathfrak{A}}}(i_\circ\rho_0(j_\circ+k_\circ)-i_\circ\rho_0k_\circ) < \overline{\mathfrak{b}}_\circ$ and $\overline{\mathfrak{V}_{\mathfrak{A}}}(j_\circ) > \mathfrak{b}_\circ$. Hence $i_\circ\rho_0(j_\circ+k_\circ)-i_\circ\rho_0k_\circ \notin \overline{\mathfrak{V}_{\mathfrak{A}_{\overline{\mathfrak{b}}_\circ}}}$ and $j_\circ \in \overline{\mathfrak{V}_{\mathfrak{V}_{\overline{\mathfrak{b}}_\circ}}}$ a contradiction. Thus, $\overline{\mathfrak{V}_{\mathfrak{A}}}$ is a I-V vague ideal of N. □

Theorem 2.2. *Let $\overline{\mathfrak{V}_{\mathfrak{A}}}$ be an I-V vague sub-collection of N then $\overline{\mathfrak{V}_{\mathfrak{A}}} = [\overline{\mathfrak{V}_{\mathfrak{A}}}^{-}, \overline{\mathfrak{V}_{\mathfrak{A}}}^{+}]$ is an I-V vague ideal of N if and only if $\overline{\mathfrak{V}_{\mathfrak{A}}}^{-}$, $\overline{\mathfrak{V}_{\mathfrak{A}}}^{+}$ are I-V vague ideals of N.*

Proof. Assume that $\overline{\mathfrak{V}_{\mathfrak{A}}}$ is an I-V vague ideal of N. For any $i, j \in N$, we have

$$\begin{aligned}\overline{\mathfrak{V}_{\mathfrak{A}}}(i–j) &\geq \min\{\overline{\mathfrak{V}_{\mathfrak{A}}}(i), \overline{\mathfrak{V}_{\mathfrak{A}}}(j)\} = \min\{[\overline{\mathfrak{V}_{\mathfrak{A}}}^{-}(i), \overline{\mathfrak{V}_{\mathfrak{A}}}^{+}(i)], [\overline{\mathfrak{V}_{\mathfrak{A}}}^{-}(j), \overline{\mathfrak{V}_{\mathfrak{A}}}^{+}(j)]\} \\ &= \min\{[\overline{\mathfrak{V}_{\mathfrak{A}}}^{-}(i), \overline{\mathfrak{V}_{\mathfrak{A}}}^{-}(j)], [\overline{\mathfrak{V}_{\mathfrak{A}}}^{+}(i), \overline{\mathfrak{V}_{\mathfrak{A}}}^{+}(j)]\} \\ &= [\min\{\overline{\mathfrak{V}_{\mathfrak{A}}}^{-}(i), \overline{\mathfrak{V}_{\mathfrak{A}}}^{-}(j)\}], [\min\{\overline{\mathfrak{V}_{\mathfrak{A}}}^{+}(i), \overline{\mathfrak{V}_{\mathfrak{A}}}^{+}(j)\}].\end{aligned}$$

It follow that $\overline{\mathfrak{V}_{\mathfrak{A}}}^{-}(i-j) \geq \min\{\overline{\mathfrak{V}_{\mathfrak{A}}}^{-}(i), \overline{\mathfrak{V}_{\mathfrak{A}}}^{-}(j)\}$ and

$$\overline{\mathfrak{V}_{\mathfrak{A}}}^{+}(i-j) \geq \min\{\overline{\mathfrak{V}_{\mathfrak{A}}}^{+}(i), \overline{\mathfrak{V}_{\mathfrak{A}}}^{+}(j)\}.$$

Similarly, we can prove all the conditions.

Now, conversely assume that $\overline{\mathfrak{V}_{\mathfrak{A}}}^{-}$ and $\overline{\mathfrak{V}_{\mathfrak{A}}}^{+}$ is a vague ideal of N.

Let $i, j \in N$, then $\overline{\mathfrak{V}_{\mathfrak{A}}}(i-j) = [\overline{\mathfrak{V}_{\mathfrak{A}}}^{-}(i-j), \overline{\mathfrak{V}_{\mathfrak{A}}}^{+}(i–j)] \geq [\min\{\overline{\mathfrak{V}_{\mathfrak{A}}}^{-}(i), \overline{\mathfrak{V}_{\mathfrak{A}}}^{-}(j)\}, \min\{\overline{\mathfrak{V}_{\mathfrak{A}}}^{+}(i), \overline{\mathfrak{V}_{\mathfrak{A}}}^{+}(j)\}] = \min\{[\overline{\mathfrak{V}_{\mathfrak{A}}}^{-}(i), \overline{\mathfrak{V}_{\mathfrak{A}}}^{+}(i)], [\overline{\mathfrak{V}_{\mathfrak{A}}}^{-}(j), \overline{\mathfrak{V}_{\mathfrak{A}}}^{+}(j)]\}$ $= \min\{\overline{\mathfrak{V}_{\mathfrak{A}}}(i), \overline{\mathfrak{V}_{\mathfrak{A}}}(j)\}$. Similarly, we can prove all other conditions of I-V vague ideals of N. □

Theorem 2.3. *If $\{\overline{\mathfrak{V}_{\mathfrak{A}}} \mid \dot{a} \in \Omega\}$ is a family of I-V vague ideals of N, then $\bigcap_{\dot{a}\in\Omega}\overline{\mathfrak{V}_{\mathfrak{A}}}$ is also an I-V vague ideals of N, where Ω is any index set.*

Proof. Let $i, j \in N$ and $\rho \in \Gamma$. Then, $\bigcap_{\dot{a}\in\Omega}\overline{\mathfrak{V}_{\mathfrak{A}}}(i-j) = \inf_{\dot{a}\in\Omega}\overline{\mathfrak{V}_{\mathfrak{A}}}(i-j) : \dot{a} \in \Omega\} \geq \inf_{\dot{a}\in\Omega}\min\{\overline{\mathfrak{V}_{\mathfrak{A}}}(i), \overline{\mathfrak{V}_{\mathfrak{A}}}(j)\} = \min\{\inf_{\dot{a}\in\Omega}\overline{\mathfrak{V}_{\mathfrak{A}}}(i), \inf_{\dot{a}\in\Omega}\overline{\mathfrak{V}_{\mathfrak{A}}}(i)\} = \min\{\bigcap_{\dot{a}\in\Omega}\overline{\mathfrak{V}_{\mathfrak{A}}}(i), \bigcap_{\dot{a}\in\Omega}\overline{\mathfrak{V}_{\mathfrak{A}}}(j)\}$.

$\bigcap_{\dot{a}\in\Omega}\overline{\mathfrak{V}_{\mathfrak{A}}}(j+i-j) = \inf_{\dot{a}\in\Omega}\overline{\mathfrak{V}_{\mathfrak{A}}}(j+i-j) \geq \inf_{\dot{a}\in\Omega}\overline{\mathfrak{V}_{\mathfrak{A}}}(i) = \bigcap_{\dot{a}\in\Omega}\overline{\mathfrak{V}_{\mathfrak{A}}}(i)$. Now, $\bigcap_{\dot{a}\in\Omega}\overline{\mathfrak{V}_{\mathfrak{A}}}(i\rho(j+k)-i\rho k) = \inf_{\dot{a}\in\Omega}\overline{\mathfrak{V}_{\mathfrak{A}}}(i\rho(j+k)-i\rho k) \geq \inf_{\dot{a}\in\Omega}\overline{\mathfrak{V}_{\mathfrak{A}}}(j) = \bigcap_{\dot{a}\in\Omega}\overline{\mathfrak{V}_{\mathfrak{A}}}(j)$. Therefore, $\bigcap_{\dot{a}\in\Omega}\overline{\mathfrak{V}_{\mathfrak{A}}}$ is an I-V vague ideal of N. □

Theorem 2.4. *If $\{\overline{\mathfrak{V}_{\mathfrak{A}}} \mid \dot{a} \in \Omega\}$ be a family of an is an I-V vague ideal of N, then $\bigvee_{\dot{a}\in\Omega}\overline{\mathfrak{V}_{\mathfrak{A}\dot{a}}}$ is also I-V vague ideal of N.*

Proof. Proof is similar. □

Definition 2.2. *Consider two Γ-nearrings N and M. A mapping ß: $N \to M$ is called homomorphism it satisfies the following conditions $ß(i+j) = ß(i)+ß(j)$ and $ß(i\rho j) = ß(i)\rho ß(j)$, for all $i, j \in N$ and $\rho \in \Gamma$.*

Theorem 2.5. *Let N and M be Γ-nearrings and ß: $N \to M$ is an onto homomorphism. If $\overline{\mathfrak{V}_{\mathfrak{A}}}$ is a ß invariant I-V vague ideal of N, then $ß(\overline{\mathfrak{V}_{\mathfrak{A}}})$ is an I-V vague ideal of M.*

Proof. Suppose that N and M be Γ-nearrings, ß: $N \to M$ is an onto homomorphism, $\overline{\mathfrak{V}_{\mathfrak{A}}}$ be an ß invariant I-V vague ideal of N and $i \in M$. Suppose $p \in M, t \in ß^{-1}(i), p = ß(t)$. Then $p \in ß^{-1}(i)$ that implies $ß(t) = i = ß(p)$, since $\overline{\mathfrak{V}_{\mathfrak{A}}}$ is a ß invariant, $\overline{\mathfrak{V}_{\mathfrak{A}}}(t) = \overline{\mathfrak{V}_{\mathfrak{A}}}(p) \Rightarrow ß(\overline{\mathfrak{V}_{\mathfrak{A}}})(i) = \inf_{t \in ß^{-1}(i)} \overline{\mathfrak{V}_{\mathfrak{A}}}(t) = \overline{\mathfrak{V}_{\mathfrak{A}}}(p)$. Hence, $ß(\overline{\mathfrak{V}_{\mathfrak{A}}})(i) = \overline{\mathfrak{V}_{\mathfrak{A}}}(p)$.

(i) Let $i, j \in M$. Then, there exists $p, q \in N$such that $ß(p) = i, ß(q) = j$ implies $ß(p - q) = i - j \Rightarrow ß(\overline{\mathfrak{V}_{\mathfrak{A}}})(i - j) = \overline{\mathfrak{V}_{\mathfrak{A}}}(p - q) \geq \min\{\overline{\mathfrak{V}_{\mathfrak{A}}}(p), \overline{\mathfrak{V}_{\mathfrak{A}}}(q)\} = \min\{ß(\overline{\mathfrak{V}_{\mathfrak{A}}})(i), ß(\overline{\mathfrak{V}_{\mathfrak{A}}})(j)\}$.

(ii) Let $i, j \in M$. Then, there exists $p, q \in N$ such that $ß(p) = i, ß(q) = j \Rightarrow ß(q+p-q) = j+i-j$ implies $ß(\overline{\mathfrak{V}_{\mathfrak{A}}})(j+i-j) = \overline{\mathfrak{V}_{\mathfrak{A}}}(q+p-q) \geq \overline{\mathfrak{V}_{\mathfrak{A}}}(p) = ß(\overline{\mathfrak{V}_{\mathfrak{A}}})(i)$.

(iii) Let $i, j, k \in M$ and $\rho \in \Gamma$. Then, there exists $p, q, r \in N$ such that $ß(p) = i, ß(q) = j, ß(r) = k$ implies $ß(p\rho(q + r) - q\rho r) = i\rho(j + k) - i\rho k \Rightarrow ß(\overline{\mathfrak{V}_{\mathfrak{A}}})(i\rho(j + k) - i\rho k) = \overline{\mathfrak{V}_{\mathfrak{A}}}(p\rho(q + r) - q\rho r) \geq \overline{\mathfrak{V}_{\mathfrak{A}}}(q) = ß(\overline{\mathfrak{V}_{\mathfrak{A}}})(j)$. Hence, $ß(\overline{\mathfrak{V}_{\mathfrak{A}}})$ is an I-V vague ideal of M. □

Theorem 2.6. *Let N and M be two Γ-nearrings and ß: $N \to M$ a homomorphism mapping. If $\overline{\mathfrak{V}_{\mathfrak{A}}}$ is a I-V vague ideal of M then $ß^{-1}(\overline{\mathfrak{V}_{\mathfrak{A}}})$ is an I-V vague ideal of N.*

Proof. Proof is straight forward. □

3. Normal interval valued vague ideal

Definition 3.1. *An vague set $\overline{\mathfrak{V}_{\mathfrak{A}}}$ is said to be a normal I-V vague ideal if $\overline{\mathfrak{V}_{\mathfrak{A}}}(0) = [1, 1]$ it means that $\overline{\tau_{\mathfrak{A}}}(0) = 1$ and $1 - \overline{\sigma_{\mathfrak{A}}}(0) = 1$.*

Note. Example(3) is an example of normal interval valued vague ideal.

Theorem 3.1. *Let $t \in (0, 1]$ be a real number. If $\overline{\mathfrak{V}_{\mathfrak{A}}}$ is a normal I-V vague ideal of N, then $\overline{\mathfrak{V}^t_{\mathfrak{A}}}$ is also a normal vague ideal of N and $N_{\overline{\mathfrak{V}^t_{\mathfrak{A}}}} = N_{\overline{\mathfrak{V}_{\mathfrak{A}}}}$. Here, $\overline{\mathfrak{V}^t_{\mathfrak{A}}}(i) = [\overline{\mathfrak{V}_{\mathfrak{A}}}(i)]^t$.*

Proof. Let i, j, $k \in N$ and $\rho \in \Gamma$ then, (i) $\overline{\mathfrak{V}^t_{\mathfrak{A}}}(i - j) = [\overline{\mathfrak{V}_{\mathfrak{A}}}(i - j)]^t \geq [\min\{\overline{\mathfrak{V}_{\mathfrak{A}}}(i), \overline{\mathfrak{V}_{\mathfrak{A}}}(j)\}]^t = \min\{(\overline{\mathfrak{V}_{\mathfrak{A}}}(i))^t, (\overline{\mathfrak{V}_{\mathfrak{A}}}(j))^t\} = \min\{\overline{\mathfrak{V}^t_{\mathfrak{A}}}(i), \overline{\mathfrak{V}^t_{\mathfrak{A}}}(j)\}$.

(ii) $\overline{\mathfrak{V}^t_{\mathfrak{A}}}(j + i - j) = [\overline{\mathfrak{V}_{\mathfrak{A}}}(j + i - j)]^t \geq [\overline{\mathfrak{V}_{\mathfrak{A}}}(i)]^t = \overline{\mathfrak{V}^t_{\mathfrak{A}}}(i)$.

(iii) $\overline{\mathfrak{V}^t_{\mathfrak{A}}}(i\rho(j + k) - i\rho k) = [\overline{\mathfrak{V}_{\mathfrak{A}}}(i\rho(j + k) - i\rho k)]^t \geq [\overline{\mathfrak{V}_{\mathfrak{A}}}(j)]^t = \overline{\mathfrak{V}^t_{\mathfrak{A}}}(j)$

Hence, $\overline{\mathfrak{V}^t_{\mathfrak{A}}}$ is I-V vague ideal of the N.

Now, $\mathfrak{V}^t_{\mathfrak{A}}(0) = [\overline{\mathfrak{V}_{\mathfrak{A}}}(0)]^t = [1, 0]^t\} = [1, 0]$. Therefore, $\overline{\mathfrak{V}^t_{\mathfrak{A}}}$ is a normal I-V vague ideal of the N. Now, $N_{\overline{\mathfrak{V}^t_{\mathfrak{A}}}} = \{i \in N \mid \overline{\mathfrak{V}^t_{\mathfrak{A}}}(i) = \overline{\mathfrak{V}^t_{\mathfrak{A}}}(0)\} = \{i \in N \mid \{\overline{\mathfrak{V}_{\mathfrak{A}}}(i)\}^t = [\overline{\mathfrak{V}_{\mathfrak{A}}}(0)]^t\} = \{i \in N \mid \{\overline{\mathfrak{V}_{\mathfrak{A}}}(i)\}^t = [1, 0]\} = \{i \in N \mid \overline{\mathfrak{V}_{\mathfrak{A}}}(i) = [1, 0]\} = \{i \in N \mid \overline{\mathfrak{V}_{\mathfrak{A}}}(i) = \overline{\mathfrak{V}_{\mathfrak{A}}}(0)\} = N_{\overline{\mathfrak{V}_{\mathfrak{A}}}}$. □

Theorem 3.2. *Let $\overline{\mathfrak{V}_{\mathfrak{A}}} = [\overline{\tau_{\mathfrak{A}}}, \overline{\sigma_{\mathfrak{A}}}]$ be an I-V vague ideal of N such that if $\overline{\tau_{\mathfrak{A}}}(i) + \overline{\sigma_{\mathfrak{A}}}(i) \leq \overline{\tau_{\mathfrak{A}}}(0) + \overline{\sigma_{\mathfrak{A}}}(0) \forall i \in N$, where $\overline{\mathfrak{V}_{\mathfrak{A}+}} = [\overline{\tau_{\mathfrak{A}+}}, \overline{\sigma_{\mathfrak{A}+}}]$ defined as*

$\overline{\tau_{\mathfrak{A}+}}(i) = \overline{\tau_{\mathfrak{A}}}(i) + \overline{\tau_{\mathfrak{A}}}(0) - 1$ *and* $\overline{\sigma_{\mathfrak{A}+}}(i) = \overline{\sigma_{\mathfrak{A}}}(i) - \overline{\sigma_{\mathfrak{A}}}(0)$, *then* $\overline{\mathfrak{V}_{\mathfrak{A}+}}$ *is normal I-V vague ideal of* N.

Proof. For any $i \in N$, $\overline{\tau_{\mathfrak{A}+}}(i)+\overline{\sigma_{\mathfrak{A}+}}(i) = \overline{\tau_{\mathfrak{A}}}(i)+\overline{\tau_{\mathfrak{A}}}(0)-1+\overline{\sigma_{\mathfrak{A}}}(i)-\overline{\sigma_{\mathfrak{A}}}(0) \leq 1$, thus $\overline{\tau_{\mathfrak{A}+}}, \overline{\sigma_{\mathfrak{A}+}}$ is I-V vague ideal with $\overline{\tau_{\mathfrak{A}+}}(0) = 1, \overline{\sigma_{\mathfrak{A}+}}(0) = 0$, therefore $\overline{\mathfrak{V}_{\mathfrak{A}+}}$ is normal I-V vague ideal of N. □

Corollary 3.1. *If* $\overline{\mathfrak{V}_{\mathfrak{A}}}$ *is a I-V vague ideal of* N *and if* $\overline{\mathfrak{V}_{\mathfrak{A}+}}(i) = 0$, *for some* $i \in N$ *then* $\overline{\mathfrak{V}_{\mathfrak{A}}}(i) = 0$, *for all* $i \in N$.

Theorem 3.3. *An I-V vague ideal of* $\overline{\mathfrak{V}_{\mathfrak{A}}}$ *is normal if and only if* $\overline{\mathfrak{V}_{\mathfrak{A}}} = \overline{\mathfrak{V}_{\mathfrak{A}+}}$.

Proof. Suppose that $\overline{\mathfrak{V}_{\mathfrak{A}}}$ is normal I-V vague ideal of N and let $i \in N$. Then, $\overline{\tau_{\mathfrak{A}+}}(i) = \overline{\tau_{\mathfrak{A}}}(i)+\overline{\tau_{\mathfrak{A}}}(0)-1 = \overline{\tau_{\mathfrak{A}}}(i)$ and $\overline{\sigma_{\mathfrak{A}+}}(i) = \overline{\sigma_{\mathfrak{A}}}(i)+\overline{\sigma_{\mathfrak{A}}}(0) = \overline{\sigma_{\mathfrak{A}}}(i)$. Converse is obvious. □

4. Direct product of interval valued vague ideal of Γ-nearrings

Definition 4.1. *Let* $\overline{\mathfrak{V}_{\mathfrak{A}_l}}$ *be an I-V vague ideal of* N_l *for* $l = 1, 2 \ldots n$, *then direct product of* $\overline{\mathfrak{V}_{\mathfrak{A}_l}}$ *is a function* $\overline{\mathfrak{V}_{\mathfrak{A}_1}} \times \overline{\mathfrak{V}_{\mathfrak{A}_2}} \times \ldots \times \overline{\mathfrak{V}_{\mathfrak{A}_n}}$: $N_1 \times N_2 \times \ldots \times N_n \to [0,1]$ *defined by* $(\overline{\mathfrak{V}_{\mathfrak{A}_1}} \times \overline{\mathfrak{V}_{\mathfrak{A}_2}} \times \ldots \times \overline{\mathfrak{V}_{\mathfrak{A}_n}})(i_1, i_2 \ldots, i_n) = \min\{\overline{\mathfrak{V}_{\mathfrak{A}_1}}(i_1), \overline{\mathfrak{V}_{\mathfrak{A}_2}}(i_2) \ldots \overline{\mathfrak{V}_{\mathfrak{A}_n}}(i_n)\}$.

Theorem 4.1. *If* $\overline{\mathfrak{V}_{\mathfrak{A}_1}}$ *and* $\overline{\mathfrak{V}_{\mathfrak{A}_2}}$ *be two I-V vague ideals in* Γ*-nearrings* M *and* N *respectively then* $\overline{\mathfrak{V}_{\mathfrak{A}_1}} \times \overline{\mathfrak{V}_{\mathfrak{A}_2}}$ *is an I-V vague ideal of* $M \times N$.

Corollary 4.1. *The direct product of I-V vague ideals of a* Γ*-nearrings* N *is also an I-V vague ideal of* N.

Proof. Let $\overline{\mathfrak{V}_{\mathfrak{A}_i}}$ be I-V vague ideal of Γ-nearrings N_i, for $i = 1, 2 \ldots n$. Now, let $N = N_1 \times N_2 \times \ldots \times N_n$ be the direct product of Γ-nearrings and $\rho_1, \rho_2 \ldots \rho_n \in \Gamma$ be an direct product of Γ-nearrings.

Let $i = (i_1, i_2 \ldots i_n), j = (j_1, j_2 \ldots j_n), k = (k_1, k_2 \ldots k_n) \in N$. Take $\overline{\mathfrak{V}_{\mathfrak{A}_1}} \times \overline{\mathfrak{V}_{\mathfrak{A}_2}} \times \ldots \times \overline{\mathfrak{V}_{\mathfrak{A}_n}} = \overline{\mathfrak{V}_{\mathfrak{A}}}$. Now, (i) $\overline{\mathfrak{V}_{\mathfrak{A}}}(i-j) = \overline{\mathfrak{V}_{\mathfrak{A}}}((i_1, i_2 \ldots i_n) - (j_1, j_2 \ldots j_n)) = \overline{\mathfrak{V}_{\mathfrak{A}}}(i_1-j_1, i_2-j_2 \ldots i_n-j_n) = \min\{\overline{\mathfrak{V}_{\mathfrak{A}_1}}(i_1-j_1), \overline{\mathfrak{V}_{\mathfrak{A}_2}}(i_2-j_2) \ldots \overline{\mathfrak{V}_{\mathfrak{A}_n}}(i_n-j_n)\} \geq \min\{\min\{\overline{\mathfrak{V}_{\mathfrak{A}_1}}(i_1), \overline{\mathfrak{V}_{\mathfrak{A}_1}}(j_1)\}, \min\{\overline{\mathfrak{V}_{\mathfrak{A}_2}}(i_2), \overline{\mathfrak{V}_{\mathfrak{A}_2}}(j_2)\} \ldots \min\{\overline{\mathfrak{V}_{\mathfrak{A}_n}}(i_n), \overline{\mathfrak{V}_{\mathfrak{A}_n}}(j_n)\} = \min\{(\overline{\mathfrak{V}_{\mathfrak{A}_1}} \times \overline{\mathfrak{V}_{\mathfrak{A}_2}} \times \ldots \times \overline{\mathfrak{V}_{\mathfrak{A}_n}})(i_1, i_2 \ldots i_n), (\overline{\mathfrak{V}_{\mathfrak{A}_1}} \times \overline{\mathfrak{V}_{\mathfrak{A}_2}} \times \ldots \times \overline{\mathfrak{V}_{\mathfrak{A}_n}})(j_1, j_2 \ldots j_n)\} = \min\{\overline{\mathfrak{V}_{\mathfrak{A}}}(i), \overline{\mathfrak{V}_{\mathfrak{A}}}(j)\}$.

(ii) $\overline{\mathfrak{V}_{\mathfrak{A}}}(j+i-j) = \overline{\mathfrak{V}_{\mathfrak{A}}}((j_1, j_2 \ldots j_n)+(i_1, i_2 \ldots i_n)-(j_1, j_2 \ldots j_n)) = \overline{\mathfrak{V}_{\mathfrak{A}}}\{j_1 + i_1 - j_1, j_2 + i_2 - j_2 \ldots j_n + i_n - j_n\} = \min\{\overline{\mathfrak{V}_{\mathfrak{A}_1}}(j_1 + i_1 - j_1), \overline{\mathfrak{V}_{\mathfrak{A}_2}}(j_2 + i_2 - j_2) \ldots \overline{\mathfrak{V}_{\mathfrak{A}_n}}(j_n + i_n - j_n)\} \geq \min\{\overline{\mathfrak{V}_{\mathfrak{A}_1}}(i_1), \overline{\mathfrak{V}_{\mathfrak{A}_2}}(i_2) \ldots \overline{\mathfrak{V}_{\mathfrak{A}_n}}(i_n)\} = (\overline{\mathfrak{V}_{\mathfrak{A}_1}} \times \overline{\mathfrak{V}_{\mathfrak{A}_2}} \times \ldots \times \overline{\mathfrak{V}_{\mathfrak{A}_n}})(i_1, i_2 \ldots i_n) = \overline{\mathfrak{V}_{\mathfrak{A}}}(i)$.

(iii) $\overline{\mathfrak{V}_{\mathfrak{A}}}(i\rho(j+k)-i\rho k) = \overline{\mathfrak{V}_{\mathfrak{A}}}((i_1, i_2 \ldots i_n)(\rho_1, \rho_2 \ldots \rho_n)(j_1+k_1, j_2+k_2 \ldots j_n+k_n) -(i_1, i_2 \ldots i_n)(\rho_1, \rho_2 \ldots \rho_n)(k_1, k_2 \ldots k_n)) = \overline{\mathfrak{V}_{\mathfrak{A}}}(i_1\rho_1(j_1+k_1)-i_1\rho_1k_1, i_2\rho_2(j_2 + j_2) -i_2\rho_2j_2 \ldots i_n\rho_n(j_n+k_n)-i_n\rho_nk_n) = \min\{\overline{\mathfrak{V}_{\mathfrak{A}_1}}(i_1\rho_1(j_1+k_1)-i_1\rho_1k_1), \overline{\mathfrak{V}_{\mathfrak{A}_2}}\rho_2(i_2(j_2+ k_2)-i_2\rho_2k_2) \ldots \overline{\mathfrak{V}_{\mathfrak{A}_n}}(i_n\rho_n(j_n+k_n)-i_n\rho_nk_n)\} \geq \min\{\overline{\mathfrak{V}_{\mathfrak{A}_1}}(j_1), \overline{\mathfrak{V}_{\mathfrak{A}_2}}(j_2) \ldots \overline{\mathfrak{V}_{\mathfrak{A}_n}}(j_n)\} = (\overline{\mathfrak{V}_{\mathfrak{A}_1}} \times \overline{\mathfrak{V}_{\mathfrak{A}_2}} \times \ldots \times \overline{\mathfrak{V}_{\mathfrak{A}_n}})(j_1, j_2 \ldots, j_n) = \overline{\mathfrak{V}_{\mathfrak{A}}}(j)$. Hence, $\overline{\mathfrak{V}_{\mathfrak{A}}}$ is I-V vague ideal of N. □

Lemma 4.1. *Let $\overline{\mathfrak{V}_{\mathfrak{A}_1}}$ and $\overline{\mathfrak{V}_{\mathfrak{A}_2}}$ be a two I-V vague subcollections of Γ-nearrings M and N respectively, if $\overline{\mathfrak{V}_{\mathfrak{A}_1}} \times \overline{\mathfrak{V}_{\mathfrak{A}_2}}$ is a I-V vague ideal of $M \times N$, then:*

(*i*) $(\overline{\mathfrak{V}_{\mathfrak{A}_1}} \times \overline{\mathfrak{V}_{\mathfrak{A}_2}})(0, 0^{'}) \geq (\overline{\mathfrak{V}_{\mathfrak{A}_1}} \times \overline{\mathfrak{V}_{\mathfrak{A}_2}})(i, j)$;

(*ii*) $(\overline{\mathfrak{V}_{\mathfrak{A}_1}} \times \overline{\mathfrak{V}_{\mathfrak{A}_2}} \ldots \times \overline{\mathfrak{V}_{\mathfrak{A}_n}})(0_1, 0_2 \ldots 0_n) \geq (\overline{\mathfrak{V}_{\mathfrak{A}_1}} \times \overline{\mathfrak{V}_{\mathfrak{A}_2}} \ldots \times \overline{\mathfrak{V}_{\mathfrak{A}_n}})(i_1, i_2 \ldots i_n)$.

Theorem 4.2. *Let $\overline{\mathfrak{V}_{\mathfrak{A}_1}}$ and $\overline{\mathfrak{V}_{\mathfrak{A}_2}}$ be a two vague subsets of Γ-nearrings M and N respectively. If $\overline{\mathfrak{V}_{\mathfrak{A}_1}} \times \overline{\mathfrak{V}_{\mathfrak{A}_2}}$ is a I-V vague ideal of Γ-nearring $M \times N$, then atleast one of the following must be true*

(*i*) $\overline{\mathfrak{V}_{\mathfrak{A}_1}}(0) \geq \overline{\mathfrak{V}_{\mathfrak{A}_2}}(j) \forall j \in N$;

(*ii*) $\overline{\mathfrak{V}_{\mathfrak{A}_2}}(0^{'}) \geq \overline{\mathfrak{V}_{\mathfrak{A}_1}}(i) \forall i \in M$.

Proof. Let $\overline{\mathfrak{V}_{\mathfrak{A}_1}} \times \overline{\mathfrak{V}_{\mathfrak{A}_2}}$ is the I-V vague ideal of $M \times N$. By contradiction suppose that: there exists $i \in M$ and $j \in N$ such that $\overline{\mathfrak{V}_{\mathfrak{A}_2}}(0') < \overline{\mathfrak{V}_{\mathfrak{A}_1}}(i)$ and $\overline{\mathfrak{V}_{\mathfrak{A}_1}}(0) < \overline{\mathfrak{V}_{\mathfrak{A}_2}}(j)$.

Now, $(\overline{\mathfrak{V}_{\mathfrak{A}_1}} \times \overline{\mathfrak{V}_{\mathfrak{A}_2}})(i, j) = \overline{\mathfrak{V}_{\mathfrak{A}_1}}(i) \wedge \overline{\mathfrak{V}_{\mathfrak{A}_2}}(j) > \overline{\mathfrak{V}_{\mathfrak{A}_1}}(0) \wedge \overline{\mathfrak{V}_{\mathfrak{A}_2}}(0') = (\overline{\mathfrak{V}_{\mathfrak{A}_1}} \times \overline{\mathfrak{V}_{\mathfrak{A}_2}})(0, 0')$. Thus, $\overline{\mathfrak{V}_{\mathfrak{A}_1}} \times \overline{\mathfrak{V}_{\mathfrak{A}_2}}$ is a I-V vague ideals of the $M \times N$ satisfying $(\overline{\mathfrak{V}_{\mathfrak{A}_1}} \times \overline{\mathfrak{V}_{\mathfrak{A}_2}})(i, j) > (\overline{\mathfrak{V}_{\mathfrak{A}_1}} \times \overline{\mathfrak{V}_{\mathfrak{A}_2}})(0, 0')$. This is a contradict to Lemma (26). □

Theorem 4.3. *Let $\overline{\mathfrak{V}_{\mathfrak{A}_1}}$ and $\overline{\mathfrak{V}_{\mathfrak{A}_2}}$ be an two vague subset of Γ-nearrings M and N respectively, if $\overline{\mathfrak{V}_{\mathfrak{A}_1}} \times \overline{\mathfrak{V}_{\mathfrak{A}_2}}$ is a I-V vague ideal of Γ-nearring $M \times N$, then either $\overline{\mathfrak{V}_{\mathfrak{A}_1}}$ is a I-V vague ideal of M or $\overline{\mathfrak{V}_{\mathfrak{A}_2}}$ is a I-V vague ideal of N.*

Proof. Suppose $\overline{\mathfrak{V}_{\mathfrak{A}_1}} \times \overline{\mathfrak{V}_{\mathfrak{A}_2}}$ is a I-V vague ideal $M \times N$. Then, by Theorem 27 $\overline{\mathfrak{V}_{\mathfrak{A}_2}}(0^{'}) \geq \overline{\mathfrak{V}_{\mathfrak{A}_1}}(i)$.

Now, (i) for $i, j \in M$, $\overline{\mathfrak{V}_{\mathfrak{A}_1}}(i-j) \geq \overline{\mathfrak{V}_{\mathfrak{A}_1}}(i-j) \wedge \overline{\mathfrak{V}_{\mathfrak{A}_2}}(0^{'}) = \overline{\mathfrak{V}_{\mathfrak{A}_1}}(i-j) \wedge \overline{\mathfrak{V}_{\mathfrak{A}_2}}(0^{'} - 0^{'}) = (\overline{\mathfrak{V}_{\mathfrak{A}_1}} \times \overline{\mathfrak{V}_{\mathfrak{A}_2}})((i, 0^{'}) - (j, 0^{'})) \geq \overline{\mathfrak{V}_{\mathfrak{A}_1}}(i) \wedge \overline{\mathfrak{V}_{\mathfrak{A}_2}}(0^{'}) \wedge \overline{\mathfrak{V}_{\mathfrak{A}_1}}(j) = \overline{\mathfrak{V}_{\mathfrak{A}_1}}(i) \wedge \overline{\mathfrak{V}_{\mathfrak{A}_1}}(j)$.

(ii) $\overline{\mathfrak{V}_{\mathfrak{A}_1}}(j+i-j) \geq \overline{\mathfrak{V}_{\mathfrak{A}_1}}(j+i-j) \wedge \overline{\mathfrak{V}_{\mathfrak{A}_2}}(0^{'}) = (\overline{\mathfrak{V}_{\mathfrak{A}_1}} \times \overline{\mathfrak{V}_{\mathfrak{A}_2}})((j, 0^{'}) + (i, 0^{'}) - (j, 0^{'})) \geq \overline{\mathfrak{V}_{\mathfrak{A}_1}}(i) \wedge \overline{\mathfrak{V}_{\mathfrak{A}_2}}(0^{'}) = \overline{\mathfrak{V}_{\mathfrak{A}_1}}(i)$.

(iii) For i, j, $k \in M$ and $\rho \in \Gamma$ so, $\overline{\mathfrak{V}_{\mathfrak{A}_1}}(j\rho(i+k) - j\rho k) \geq \overline{\mathfrak{V}_{\mathfrak{A}_1}}(k\rho(i+k) - j\rho k) \wedge \overline{\mathfrak{V}_{\mathfrak{A}_2}}(0^{'}) = (\overline{\mathfrak{V}_{\mathfrak{A}_1}} \times \overline{\mathfrak{V}_{\mathfrak{A}_2}})((j, 0^{'})(\rho, 0^{'})((i, 0^{'}) + (k, 0^{'})) - (j, 0^{'})(\rho, 0^{'})(k, 0^{'})) \geq \overline{\mathfrak{V}_{\mathfrak{A}_1}}(i) \wedge \overline{\mathfrak{V}_{\mathfrak{A}_2}}(0^{'}) = \overline{\mathfrak{V}_{\mathfrak{A}_1}}(i)$.

Hence, in this case $\overline{\mathfrak{V}_{\mathfrak{A}_1}}$ is a I-V vague ideal of M. Similarly, we can show $\overline{\mathfrak{V}_{\mathfrak{A}_2}}$ is a I-V vague ideal of N. □

Theorem 4.4. *Let $\overline{\mathfrak{V}_{\mathfrak{A}_1}}$ and $\overline{\mathfrak{V}_{\mathfrak{A}_2}}$ be vague subsets of Γ-nearrings M and N respectively, such that $\overline{\mathfrak{V}_{\mathfrak{A}_1}}(0) \geq \overline{\mathfrak{V}_{\mathfrak{A}_1}}(j)$, for all $j \in N$ and $\overline{\mathfrak{V}_{\mathfrak{A}_1}} \times \overline{\mathfrak{V}_{\mathfrak{A}_2}}$ is a I-V vague ideal of $M \times N$, then $\overline{\mathfrak{V}_{\mathfrak{A}_2}}$ is a I-V vague ideal of N .*

Corollary 4.2. *Let $\overline{\mathfrak{V}_{\mathfrak{A}_1}}$, $\overline{\mathfrak{V}_{\mathfrak{A}_2}} \ldots \overline{\mathfrak{V}_{\mathfrak{A}_n}}$ be a vague subset of Γ-nearrings N_1, $N_2 \ldots N_n$ respectively and if $\overline{\mathfrak{V}_{\mathfrak{A}_1}} \times \overline{\mathfrak{V}_{\mathfrak{A}_2}} \times \ldots \times \overline{\mathfrak{V}_{\mathfrak{A}_n}}$ is a I-V vague ideal of Γ-nearring $N_1 \times N_2 \times \ldots \times N_n$, then at least for one l, $\overline{\mathfrak{V}_{\mathfrak{A}_l}}(0_l) \geq \overline{\mathfrak{V}_{\mathfrak{A}_m}}(i_m)$ for all $i \in N_m$, for m = 1, 2... n where 0_l is the identity in N_l, must be true.*

Proof. We know that if N_1, $N_2 \ldots N_n$ are Γ-nearrings, then their product $N_1 \times N_2 \times \ldots \times N_n$ is also a Γ-nearring. Let $\overline{\mathfrak{V}_{\mathfrak{A}_1}} \times \overline{\mathfrak{V}_{\mathfrak{A}_2}} \times \ldots \times \overline{\mathfrak{V}_{\mathfrak{A}_n}}$ is a I-V vague ideal of$N_1 \times N_2 \times \ldots \times N_n$. By contradiction, suppose there is $i_m \in N_m$ such that $\overline{\mathfrak{V}_{\mathfrak{A}_l}}(i_l) > \overline{\mathfrak{V}_{\mathfrak{A}_m}}(0_m)$, for all l. Now, $[(\overline{\mathfrak{V}_{\mathfrak{A}_1}} \times \overline{\mathfrak{V}_{\mathfrak{A}_2}} \times \ldots \times \overline{\mathfrak{V}_{\mathfrak{A}_n}})(i_1, i_2, \ldots, i_n)] = [\overline{\mathfrak{V}_{\mathfrak{A}_1}}(i_1) \wedge \overline{\mathfrak{V}_{\mathfrak{A}_2}}(i_2) \wedge \ldots \overline{\mathfrak{V}_{\mathfrak{A}_n}}(i_n)] > \overline{\mathfrak{V}_{\mathfrak{A}_n}}(0_n) \wedge \overline{\mathfrak{V}_{\mathfrak{A}_{n-1}}}(0_{n-1}) \wedge \ldots \wedge \overline{\mathfrak{V}_{\mathfrak{A}_1}}(0_1) = (\overline{\mathfrak{V}_{\mathfrak{A}_n}} \times \overline{\mathfrak{V}_{\mathfrak{A}_{n-1}}} \times \ldots \times \overline{\mathfrak{V}_{\mathfrak{A}_1}})(0_n, 0_{n-1}, \ldots, 0_1)$. Thus, $\overline{\mathfrak{V}_{\mathfrak{A}_1}} \times \overline{\mathfrak{V}_{\mathfrak{A}_2}} \times \ldots \times \overline{\mathfrak{V}_{\mathfrak{A}_n}}$ is vague with identity $(0_1, 0_2, \ldots, 0_n) of N_1 \times N_2 \times \ldots \times N_n$ satisfying $(\overline{\mathfrak{V}_{\mathfrak{A}_1}} \times \overline{\mathfrak{V}_{\mathfrak{A}_2}} \times \ldots \times \overline{\mathfrak{V}_{\mathfrak{A}_n}})(i_1, i_2 \ldots i_n) > (\overline{\mathfrak{V}_{\mathfrak{A}_1}} \times \overline{\mathfrak{V}_{\mathfrak{A}_2}} \times \ldots \times \overline{\mathfrak{V}_{\mathfrak{A}_n}})(0_n, 0_{n-1} \ldots 0_1)$, this is a contradiction. □

Corollary 4.3. *Let* $\overline{\mathfrak{V}_{\mathfrak{A}_1}}, \overline{\mathfrak{V}_{\mathfrak{A}_2}} \ldots \overline{\mathfrak{V}_{\mathfrak{A}_n}}$ *are I-V vague ideals of* Γ*-nearrings* $N_1, N_2, \ldots, N_n$*, respectively and if* $\overline{\mathfrak{V}_{\mathfrak{A}_1}} \times \overline{\mathfrak{V}_{\mathfrak{A}_2}} \times \ldots \times \overline{\mathfrak{V}_{\mathfrak{A}_n}}$ *is a I-V vague ideal of* $N_1 \times N_2 \times \ldots \times N_n$*, then at least for one of* l, $\overline{\mathfrak{V}_{\mathfrak{A}_l}}$ *is a I-V vague ideal of* N_l.

Conclusion

In this paper we give about interval vague ideals in Γ-nearrings. After that, we give definition, properties, theorems, examples of vague ideals in Γ-nearrings. We also study normal interval valued vague ideals and the direct product of interval value vague ideals in Γ-nearrings.

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A graph modelling to measure the frustration index in signed networks

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Abstract. Computing the frustration index of a signed graph is a key step towards solving problems in many fields including social networks, political science, physics, chemistry, and biology. In social networks the frustration index determines network distance from a state of structural balance. The focus of this paper is to provide insight into computing the frustration index and show that exact values of the frustration index can be computed using optimisation models namely, 0/1 Binary Linear Model.

Keywords: frustration index, line index of balance, signed graph, integer programming, optimisation, balance theory.

1. Introduction

Heider published his important article on the psychology of cognitive organization in 1946. Since then, theories of cognitive consistency or psychological balance have proliferated remarkably. The rise of 'balance theories' to such popularity is explainable in part by their theoretical elegance and simplicity, but also by their apparent applicability to a wide range of problems in sociology and social psychology. Reference group relations, strain toward consistency in culture, persuasive communication, socialization of self-attitudes, friendship formation, the formation of group norms and value consensus, group or clique formation, group reactions to deviant behavior, the effect of initiation rites on group solidarity, reactions to status inconsistency, even some aspects of international relations, all can be given psychologically based interpretations with the aid of some form of balance theory.

Psychological principles with such broad explanatory potential clearly warrant the critical attention of sociologists. Social relations are complex and subtle (see, [16, 12]), reflecting interactions that could be either positive such as trust, agreement, approval; or negative such as distrust, disagreement, opposition. Yet until recently a number of authors began to investigate signed relations by examining the links in emerging signed social networks. Tom Coleman et.al. [16] examined the applicability of social psychology theories at multiple signed networks. F. Harrary et. al [3] further applied classification approaches to pre-

dict the sign of social relationships. Many researchers focus on discovering the interplay between relations and behavior. The nontrivial relationship between behavioral correlation and the sign of social relations motivated researchers to leverage both social and behavioral evidences when predicting signed social ties. Hence, study has been done to design models to capture the interplay between users- behavior of decision making and social interaction in the context of social networks. The problem of predicting signed social ties, such as trust and distrust, based on the acquaintance relationships in social networks thereby to determine, whether a link corresponds to a trustworthy friend or rather an enemy. The frustration index is one of the key measure for analysing signed networks.

A better understanding of balance theory and computation of frustration index can be done using signed graph models. This paper present a graph model that infer signed ties by capturing the interplay between social relations and users- behavior of decision making, and thereby compute whether the given system is balanced or not. For that we need the terms and concepts in graphs and signed graphs.

2. Graph theoretic concepts

All graphs in this paper are simple and finite. Let $G = (V, E)$ be a simple graph of order n and size m with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = \{e_1, e_2, \ldots, e_m\}$. For the terms and definitions not mentioned in the paper one may refer [13].

Signed graphs (also called sigraphs) are much studied in literature because of their extensive use in modeling a variety of socio-psychological processes and also because of their interesting connections with many classical mathematical systems (see, [2, 8, 18, 19, 20]).

Formally, a signed graph is defined as

Definition 2.1. *A signed graph is an ordered pair $\Sigma = (G, \sigma)$ where $G = (V, E)$ is a graph called the underlying graph of Σ and $\sigma : E \to \{+1, -1\}$, is a function called a* signature or signing . *$E^+(\Sigma)$ denotes the set of all edges of G that are mapped by σ to the element $+1$ and $E^-(\Sigma) = E - E^+(\Sigma)$. The elements of $E^+(\Sigma)$ are called positive edges and those of $E^-(\Sigma)$ are called negative edges of Σ.*

A signed graph is *all-positive* (respectively, *all-negative*) if all of its edges are positive (negative); further, it is said to be *homogeneous* if it is either all-positive or all-negative and *heterogeneous* otherwise. The sign of a cycle in a signed graph is the product of the signs of its edges. Thus, a cycle is positive if and only if it contains even number of negative edges. A signed graph Σ is said to be *balanced* or *cycle balanced* if all of its cycles are positive. Two signed graphs, $\Sigma_1 = (G_1, \sigma_1)$ and $\Sigma_2 = (G_2, \sigma_2)$, are *isomorphic* if there is a graph isomorphism $f : G_1 \to G_2$ that preserves signs of the edges. If $\theta : V \to \{+1, -1\}$ is *switching*

function, then *switching* of the signed graph $\Sigma = (G, \sigma)$ by θ means changing σ to σ^θ defined by:

$$\sigma^\theta(uv) = \theta(u)\sigma(uv)\theta(v).$$

The switched graph denoted by Σ^θ, is the signed graph $\Sigma^\theta = (G, \sigma^\theta)$. We call two signed graphs $\Sigma_1 = (G, \sigma_1)$ and $\Sigma_2 = (G, \sigma_2)$ to be *switching equivalent* and write $\Sigma_1 \sim \Sigma_2$, if there exists a switching function $\theta : V \to \{+1, -1\}$ such that $\Sigma_1 = \Sigma_2^\theta$. It can be seen that switching preserves many features of the two signed graphs including their eigenvalues [14] and balance.

In the classic friend-enemy interpretation of balance theory, a by-product of computing the frustration index is the partitioning of nodes into two internally solidary but mutually hostile groups.

Theorem 2.1 ([3]). *A signed graph is balanced if and only if, the vertex set V can be partitioned into two subgraphs, such that vertices within a subgraph are connected by a positive edge and vertices in separate subgraphs are connected by a negative edge.*

More generally, a signed graph Σ is said to be **clusterable** or **partitionable** if its vertex set can be partitioned into subsets (**clusters**), so that every positive edge joins the vertices within the same cluster and every negative edge joins the vertices in the different clusters. The question to be addressed is that, is it possible to partition vertices of a signed graph, so that every line that connects vertices that belong to the same cluster is positive and every line that connects two vertices that belong to different clusters is negative? If it is possible to partition vertices in this way, we call the signed graph partitionable or clusterable. However, it is a general rule that positive and negative relations among people tend towards balance. Hence, important are balanced signed graphs where vertices can be partitioned into 2 clusters.

2.1 Computation of frustration index in a signed network

Computing the frustration index of a signed graph is a key step towards solving problems in many fields including social networks, physics, material science, and biology. The frustration index determines the distance of a network from a state of total structural balance. Local ties between entities lead to global structures in networks. Ties can be formed as a result of interactions and individual preferences of the entities in the network. The dual nature of interactions in various contexts means the ties may form in two opposite types, namely positive ties and negative ties. In a social context, this is interpreted as friendship versus enmity or trust versus distrust between people. The term signed network embodies a multitude of concepts involving relationships characterizable by ties with plus and minus signs.

Signed graphs are used to model such networks where edges have positive and negative signs. Structural balance in signed graphs is a macro-scale structural property that has become a focus in network science. Structural balance theory

was the first attempt to understand the sources of tensions and conflicts in groups of people with signed ties (see, [15]). According to balance theory, some structural configurations of people with signed ties lead to social tension and therefore are not balanced. Using graph-theoretic concepts, Cartwright and Harary identified cycles of the graph as the origins of tension, in particular cycles containing an odd number of negative edges [3]. By definition, signed graphs in which no such negative cycles are present satisfy the property of structural balance. The vertex set of a balanced signed network can be partitioned into $k \geq 2$ subsets such that each negative edge joins vertices belonging to different subsets [8]. For graphs that are not totally balanced, a distance from total balance called the *measure of partial balance*, can be computed. Among various measures is the *frustration index* that indicates the minimum number of edges whose removal or equivalently, negation, results in balance (see, [18, 19, 20]). Measures based on cycles [7], triangles, and closed walks [17] are not generally consistent and do not satisfy key axiomatic properties [12]. Among all the measures, a normalised version of the frustration index is shown to satisfy many basic axioms [12]. This measure provides a clear understanding of the transition to balance in terms of the number of edges to be modified to reduce the tension, as opposed to graph cycles that were first suggested as origins of tension in unbalanced networks.

2.2 Complexity of computing frustration index

Computing the frustration index is related to the well-known unsigned graph optimization problem edge-bipartization, which requires minimization of the number of edges whose deletion makes the graph bipartite. Given an instance of the latter problem, by declaring each edge to be negative we convert it to the problem of computing the frustration index. Since edge-bipartization is known to be NP-hard, so is computing the frustration index. In the converse direction there is a reduction of the frustration index problem to edge-bipartization which increases the number of edges by a factor of at most 2. Hence, the frustration index can be computed in polynomial time for planar graphs (see, [9]), which is equivalent to the ground state calculation of a two-dimensional spin glass model with no periodic boundary conditions and no magnetic field [16]. The classic graph optimization problem MAXCUT is also a special case of the frustration index problem, as can be seen by assigning all edges to be negative.

3. Frustration count

The number of edges incident to the node $i \in V$ represents the degree of node i and is denoted $d(i)$. By net degree of a vertex v in signed graph we mean the difference of number of positive and negative edges incident at the vertex v, denoted respectively as $d^+(v)$ (or $d^-(v)$), if the net degree is positive (or negative) at the the vertex v. We apply BFS Algorithm with the initial condition

in choosing the root vertex in the given network graph as follows. Choose the root vertex say u such that net degree $d^-(u)$ is maximum. Visit the vertices that are in the first neighbourhood of the vertex u. Then visit the neighbous of all the vertices which are already visited, in the order in which they are visited. That is, represent the m undirected edges in G as ordered pairs of vertices $E = \{e_1, e_2, \ldots, e_m\} \subseteq \{(i,j), i < j, i, j \in V\}$ where a single edge e_k between nodes i and $j, i < j$, is denoted by $e_k = (i,j), i < j$. Let the number of positive and negative vertices in the nth level be respectively l_n^+ and l_n^-. BFS algorithm separates the edges into two distinct sets namely the set of tree edges and set of cycle edges (or back edges). Let us denote the set consisting of tree edges by T and the set consisting of cycle edges by C.

Let $V_i, 0 \leq i \leq n$ be the vertex set in level $l_i, 0 \leq i \leq n$. Then, $n+1$ is the number of levels fixing the root vertex u at the 0th level. Now consider the partition of the vertex set V, as $V_o = \bigcup_i V_i, i$ *odd* and $V_e = \bigcup_i V_i, i$ *even*, where i denote the level number.

To compute the frustration index of the signed graph, we only need to concentrate on cycle edges, (that are either positive or negative forming cycles), as the sign of the cycles decide the balance of the signed graph. We prove the following Lemma first.

Lemma 3.1. *Consider the partition of the vertex set as described in BFS algorithm, $V_o = \bigcup_i V_i, i$ odd and $V_e = \bigcup_i V_i, i$ even. Then, the number of negative cycle edges appear in the same level and the positive cycle edges in two consecutive levels, of the given signed graph decides the frustration index of the given signed graph.*

Proof. Consider the partition of the vertex set obtained by applying BFS algorithm, $V_o = \bigcup_i V_i, i$ *odd* and $V_e = \bigcup_i V_i, i$ *even*.
Case 1. Negative cycle edges in C.

That is, negative cycle edges are in the same partitioned set V_o (or V_e), say at the ith level. We claim that these negative edges are the 'bad edges' or 'frustrated edges' in the signed graph. Now, if the negative edge in the same level l_i is such that the end vertices of the edges are incident to negative tree edges in level l_{i-1}, which again are incident with any father negative edges in level $l_{i-j}, 1 \leq j \leq i$, then we get a negative cycle $C_n, n \geq 3$. On the other hand, if the negative cycles edges appear in the different level, then the negative edges are such that one of the end vertices of the each of the negative edges are in V_e and other in V_o. Hence, by Harray's bipartition theorem, none of these edges are 'bad edges'. Let us call such edges as 'good' or 'satisfied' edges.
Case 2. Consider positive cycle edges in C.

If, the positive cycle edges appear in the different level, then the positive edges are such that one of the end vertices of the each of the positive edges are in V_e and other in V_o. Hence, by Harray's bipartition theorem, all of these edges are 'bad edges'. On the other hand, if the positive cycle edges with both their

end vertices in the same partitioned set V_e or V_o, then by Harray's bipartition theorem, none of these positive edges are 'bad edges'.

Hence, deletion of negative cycle edges appear in the same level and the positive cycle edges in two consecutive levels, of the given signed graph, the given signed graph becomes balanced. □

Remark 3.1. Given a signed network, respectively a signed graph on n vertices, choosing a proper vertex as root vertex, and applying BFS Algorithm with this specified condition of choosing the root vertex, we get n^{n-2} spanning trees, with two sets of edges namely set with trees edges T and set C consisting of cycle edges. However, only cycle edges contribute to the frustration index of the signed graph. Hence, the problem of minimizing the frustration index depends on the choice of the spanning tree from the set of n^{n-2} spanning trees. Choose the spanning tree in such a way that it follows the following rules.
Case (i). Choose the spanning tree in which the number of negative edges with one end in V_0 and other end in V_e is maximum, and the spanning tree in which the number of negative cycle edges in same set V_o (and in V_e) are minimum.
Case (ii). Choose the spanning tree in which the number of positive edges with one end in V_0 and other end in V_e is minimum, and, the spanning tree in which the number of positive edges in same set V_o (and in V_e) is maximum.

Let us define these 'Good' and 'bad' edges based on colourings of the nodes. Colouring each node in V_o with black and the each node in V_e with white, a bad (satisfied) edge (i, j) is either a positive (negative) edge with different colours on the endpoints i, j or a negative (positive) edge with the same colours on the endpoints i, j. On the other hand, a 'good' edge (i, j) is either a negative edge with different colours on the endpoints i, j or a positive edge with the same colours on the endpoints i, j. Hence, the problem of computing the frustration index reduces to optimizing the number of negative edges with same colours and the number of positive edges with different colours. That is, the problem of computing the frustration index of a given signed graph reduces to a linear programming problem.

4. Binary linear programming and frustration count

Now, $V_o \cup V_e = V$ defines a partition of V. By Colouring each node in V_0 with black and each node in V_e with white. One can view this partition in terms of coloring, namely, $V = X \cup V - X$, where $X \subseteq V$ a subset of vertices, with colors black or white. We call X a colouring set. Now, define the binary variable x_i to denote the colour of node $i \in V$ under this colouring set X as $x_i = 1$, if $i \in X$ (black node) and $x_i = 0$ if $i \in V - X$ (white node).

Hence, the frustration count of signed graph G under colouring X is $f_G(X) := \sum_{(i,j)\in E} f_{ij}(X)$ where $f_{ij}(X)$ is the frustration state of edge (i, j), given by

$$f_{ij}(X) = \begin{cases} 0, & \text{if } x_i = x_j \text{ and } (i,j) \in E^+ \\ 1, & \text{if } x_i = x_j \text{ and } (i,j) \in E^- \\ 0, & \text{if } x_i \neq x_j \text{ and } (i,j) \in E^- \\ 1, & \text{if } x_i \neq x_j \text{ and } (i,j) \in E^+ \end{cases}. \tag{1}$$

The frustration index of a signed graph Σ depends on the choice of the spanning tree satisfying the conditions Case(i) and Case (ii) of Remark 3.1. That is, the frustration index $L(G)$ of a graph G can be found by finding the spanning tree satisfying the conditions Case (i) and Case (ii) of Remark 3.1, such that the choice of spanning tree that minimizes the frustration count $f_G(X)$, which is obtained by solving the Equation (1).

Hence, the objective function Z of the Binary Linear Programming Model is to Minimize the frustration count. That is , the binary linear programming model is as follows.

Fix the objective function to minimize the frustration count. There arises two cases.

Case (i). For counting the frustration index for the positive cycle edges:

The frustration of a positive edge (i,j) be represented by $f_{ij} = x_i + x_j - 2(x_i \text{ and } x_j), \forall (i,j) \in E^+$, using the two binary variables x_i, x_j for the endpoint colours.

Case (ii). For counting the frustration index for the negative cycle edge:

For a negative edge $f_{ij} = 1 - (x_i + x_j - 2(x_i \text{ and } x_j)), \forall (i,j) \in E^-$.

The term x_i and x_j can be replaced by binary variables $x_{ij} = x_i$ and x_j for each edge (i,j) that take value 1 whenever $x_i = x_j = 1$ (both endpoints are coloured black) and 0 otherwise.

Hence, this model is a 0/1 linear programming model which computes the frustration index in the minimisation objective function as

$Minimize_{x_i \in V;(i,j)\in E}\ \ Z = \sum_{(i,j)\in E^+}(x_i + x_j - 2x_{ij}) + \sum_{(i,j)\in E^-} 1 - ((x_i + x_j) - 2x_{ij})$

Subject to

$x_{ij} \leq x_i$, for all $(i,j) \in E^+$

$x_{ij} \leq x_j$, for all $(i,j) \in E^+$

$x_{ij} \geq x_i + x_j - 1$, for all $(i,j) \in E^-$

$x_i \in \{0,1\}$, for all $i \in V$

$x_{ij} \in \{0,1\}$, for all $(i,j) \in E$.

Remark 4.1. One may note that the dependencies between the x_{ij} and x_i, x_j values are taken into account in this constraints. This model has $n+m$ variables and $2m^+ + m^-$ constraints. The dependent variables are x_{ij}. Because of the constraints and the minimisation objective function, one may consider them as continuous variables in the unit interval, $x_{ij} \in [0,1]$.

5. Conclusion

Solving the 0/1 linear model provides us with the minimum number of frustrated edges in the given signed graph. This number determines how many edges should be removed to make the network balanced. By applying BFS algorithm and selecting the spanning tree among the set of n^{n-2} spanning trees may be done by counting the number of negative edges in consecutive levels or by identifying and counting the vertices with maximum negative net degree. This process itself give a way to minimize the bad cycle edges in the signed graph so that the optimization process may converge faster.

One may be interested in generalizing the 2-colour minimum frustration count optimization problem to k- colour ($k \geq 2$). That is, the vertex set of the signed graph is partitioned onto k subsets ($k \geq 2$) such that each negative edges joins vertices belonging to disjoint subsets. The minimum number of frustrated edges for general k ($k \geq 3$) is referred as correlation clustering problem. Another generalization of 2 colour minimum frustration count optimization problem by allowing more than two colours, which is called multi-colour minimum frustration count optimization problem. Developing a new optimization model based on optimized multi-colour minimum frustration count is an interesting problem for further investigation.

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A note on rigid rings

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Abstract. Let R be an associative ring with identity element and let σ be an endomorphism of R. σ is called rigid if whenever $a \in R$ with $a\sigma(a) = 0$, then $a = 0$. R is called a σ-rigid ring if it has a rigid endomorphism σ. In this article we study some of the elementary properties of rigid rings which are generalizations of some properties of reduced rings. We show, among other things, that the set of all rigid automorphisms of a ring R is a normal subgroup of all automorphisms of R.

Keywords: reduced rings, rigid rings, zero commutative rings, zero insertive rings.

1. Introduction

Throughout this article all rings are associative with multiplicative identity element. The ring of integers is denoted by $\mathbb{Z}$ and the ring of integers modulo a positive integer n is denoted by $\mathbb{Z}_n$. Given a ring R, $R[x]$ denotes the polynomial ring with indeterminate x and with coefficients in R. $E = End(R)$ denotes the ring of all endomorphisms of the ring R. If σ is an endomorphism of R, $R[x, \sigma]$ denotes the skew polynomial ring which is the collection of all polynomials in the indeterminate x and with coefficients in R under usual addition of polynomials while multiplication of polynomials is defined by setting $xr = \sigma(r)x$. The set of all nilpotent elements in R (i.e., the nilradical of R) is denoted by $N(R)$. A ring R is called a reduced ring if $N(R) = (0)$, i.e., if whenever $a^2 = 0$, for $a \in R$, then $a = 0$. A ring R is called a zero commutative ring (zc) if whenever $ab = 0$, then $ba = 0$, and R is called a zero insertive ring (zi) if whenever $ab = 0$, then $arb = 0$ for all $r \in R$. Reduced rings are zc and every zc ring is zi (see, for example, [4] and [9]).

As a sort of generalization of reduced rings Krempa in 1996 [8] introduced what he called rigid rings as we will see the definition in the next section. These rings where studied by many authors (see, for example, [2], [10] and [11]). In this article we will study some properties of rigid rings and see its relations with the other mentioned rings.

2. Rigid rings

Definition 2.1 ([8]). *Let R be a ring and let $\sigma \in End(R)$. σ is said to be a rigid endomorphism of the ring R if whenever $a\sigma(a) = 0$, for $a \in R$, then $a = 0$.*

A ring R is called σ-rigid (or rigid) if it has a rigid endomorphism σ.

Clearly, if σ is a rigid endomorphism of a ring R, then $ker\sigma = 0$, and if a ring R is reduced, then the identity endomorphism, id of R is rigid. So reduced rings are id-rigid rings. So rigid rings are in some sense a generalization of reduced rings. However it is not true that every endomorphism of a reduced ring is rigid as the following example shows.

Example 2.1. Let D be an integral domain and let $R = D \oplus D$. Then R is a reduced ring under the usual addition and multiplication. Now the map $\sigma : R \longrightarrow R$ given by $\sigma(a, b) = (b, a)$ is an automorphism of R and $(1, 0)\sigma(1, 0) = (1, 0)(0, 1) = (0, 0)$. So σ is not a rigid endomorphism of R.

However, an inner automorphism of a reduced ring R is rigid. For, let $\sigma : R \longrightarrow R$ be given by $\sigma(a) = uau^{-1}$ for some unit $u \in R$. Suppose that $a\sigma(a) = 0$, then $auau^{-1} = 0$. Hence $aua = 0$. Now $(au)^2 = auau = 0$, so $au = 0$ since R is reduced and hence $a = 0$. This implies that σ is rigid.

On the other hand and as was observed by A. A. Kamal in [5], if R is a σ-rigid ring, then R is a reduced ring. For, let R be a σ-rigid ring and assume $a^2 = 0$ for $a \in R$, then $a\sigma(a)\sigma(a\sigma(a)) = a\sigma(a^2)\sigma^2(a) = 0$. Thus, since σ is rigid, we have $a\sigma(a) = 0$ and so $a = 0$.

Example 2.2. Let $R = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} | a, b \in \mathbb{Z}_3 \right\}$. Then R is a reduced ring with identity. Let σ be the endomorphism of R given by $\sigma(\begin{bmatrix} a & b \\ -b & a \end{bmatrix}) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. It is easy to show that σ is a non-identity rigid automorphism.

However the ring $T = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} | a, b \in \mathbb{Z}_5 \right\}$ is a reduced ring with identity but the automorphism α of T given by $\alpha(\begin{bmatrix} a & b \\ -b & a \end{bmatrix}) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ is not a rigid automorphism.

The following example shows that for certain rings, all its ring endomorphisms, except the identity endomorphism, are not rigid.

Example 2.3. Let $\mathbb{Z}_6$ be the ring of integers modulo 6 and let σ be an endomorphism of $\mathbb{Z}_6$. Since $\mathbb{Z}_6$ is a cyclic group generated by 1 under addition modulo 6, σ, when regarded as a group homomorphism, is completely determine by $\sigma(1)$. Let $\sigma(1) = a$, then $\sigma(x) = xa$ for all $x \in \mathbb{Z}_6$. By Lagrange's Theorem for finite groups the order of a, $|a|$, must be $1, 2, 3,$ or 6. Thus $a = 0, 3, 2, 4, 1,$ or 5. But σ is a ring homomorphism, so we have $a = \sigma(1) = \sigma(1.1) = \sigma(1)^2$. i.e., a is an

idempotent element in $\mathbb{Z}_6$. This reduces the values of a to $a = 0, 1, 3, 4$. Therefore $\mathrm{End}(\mathbb{Z}_6)$ has 4 elements: $\sigma_0(x) = 0, \sigma_1(x) = x, \sigma_3(x) = 3x$, and $\sigma_4(x) = 4x$, for all $x \in \mathbb{Z}_6$. Now σ_0, σ_3, and σ_4 are not rigid because $1\sigma_0(1) = 0, 2\sigma_3(2) = 0$, and $3\sigma_4(3) = 0$, while none of the elements, of course, 1, 2 or 3 is zero in $\mathbb{Z}_6$. However σ_1 is rigid since $x\sigma_1(x) = x^2$ for all $x \in \mathbb{Z}_6$. Therefore if $x\sigma_1(x) = 0$, then $x^2 = 0$ in $\mathbb{Z}_6$, hence $x = 0$ since $\mathbb{Z}_6$ is a reduced ring.

Definition 2.2 ([4]). *A ring R is called a zero commutative (abbreviated zc) ring if whenever $ab = 0$ for $a, b \in R$, then $ba = 0$. R is called zero insertive (abbreviated zi) ring if whenever $ab = 0$ for $a, b \in R$, then $arb = 0$ for all $r \in R$.*

In literature the property zero commutative (zero-insertive) is, sometimes, called reversible (semicommutative) (See, for example [10]).

Of course every commutative ring is zc and hence zi. But the ring $\mathbb{Z}_{n^2}$ of integers modulo n^2 is zc but not reduced.

The ring S of all upper triangular matrices of the form $\begin{bmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{bmatrix}$, where $a, b, c, d \in \mathbb{Z}_3$ is zi but not zc (see, for example, [9]).

The following Lemma duo to C. Y. Hong, N. K. Kim, and T. K. Kwak, [5]. We restate it here for completeness.

Lemma 2.1 ([5]). *Let R be a σ-rigid ring. (1) If $a, b \in R$ such that $ab = 0$, then $a\sigma^n(b) = 0$ and $b\sigma^n(a) = 0$ for every positive integer n. In particular we have $b\sigma(a) = 0$ and $a\sigma(b) = 0$. (2) If $a\sigma^k(b) = 0$ for some positive integer k, then $ab = 0$.*

Proof. (1) Let $a, b \in R$ be such that $ab = 0$. Then $\sigma(ab) = \sigma(a)\sigma(b) = 0$. Now $(a\sigma^n(b))\sigma(a\sigma^n(b)) = (a\sigma^n(b))\sigma(a)\sigma^{n+1}(b) = a(\sigma^n(b)\sigma(a))\sigma(b)\sigma^n(b) = 0$. Since σ is rigid, we have $a\sigma^n(b) = 0$ for every positive integer n. Similarly we can show that $b\sigma^n(a) = 0$.

(2) Assume $a\sigma^k(b) = 0$ for some positive integer k, by (1) we have $\sigma^k(a)\sigma^k(b) = 0$. Now $\sigma^k(ab) = \sigma^k(a)\sigma^k(b) = 0$. Since σ is a monomorphism, we have $ab = 0$. □

Theorem 2.1. *Let R be a ring.*
(1) *If R is σ-rigid, then R is reduced.*
(2) *If R is a reduced ring, then R is zc.*
(3) *If R is zc, then R is zi.*
(4) *If R is zi, then every idempotent $e = e^2$ in R is central, i.e., $er = re$ for all $r \in R$*

Proof. (1) Let $a \in R$ be such that $a^2 = 0$. Then $(\sigma(a))^2 = \sigma(a^2) = 0$. Consider $(a\sigma(a))(\sigma(a\sigma(a)) = a\sigma(a)(\sigma(a)(\sigma(\sigma(a)) = a(\sigma(a))^2\sigma(\sigma(a)) = 0$. Now since σ is rigid we have $a\sigma(a) = 0$ and hence $a = 0$.

(2) Let $a, b \in R$ be such that $ab = 0$. Then $(ba)^2 = (ba)(ba) = b(ab)b = 0$. Since R is reduced, we have $ba = 0$.
(3) Let $a, b \in R$ be such that $ab = 0$. Since R is zc we have $ba = 0$. Then $bax = 0$ for all $x \in R$. Again since R is zc, we have $axb = 0$ for all $x \in R$.
(4) Since $e \in R$ is an idempotent we $e(1 - e) = 0$. Since R is zi, we have $ex(1 - e) = 0$. Hence $ex = exe$. Similarly $exe = xe$. Therefore $ex = xe$ for all $x \in R$. □

Lemma 2.2. *Let σ and τ be rigid endomorphisms of a ring R. Then $\sigma\tau$ is a rigid endomorphism of R.*

Proof. Let σ and τ be rigid endomorphisms of R and let $a \in R$ be such that

$$a((\sigma\tau)(a)) = 0. \qquad (*)$$

Now $(a\tau(a))\sigma(a\tau(a)) = (a\tau(a))\sigma(a)\sigma(\tau(a)) = a.[\tau(a)\sigma(a)].(\sigma\tau)(a) = 0$. This last equation is true because R is zi by Theorem 2.1 and by applying the zero insertivity of R to $(*)$ above. So we have $(a\tau(a))\sigma(a\tau(a)) = 0$. Now since σ is rigid we have $a\tau(a) = 0$ and since τ is rigid we have $a = 0$. This shows $\sigma\tau$ is a rigid endomorphism of R. □

Lemma 2.3. *Let R be a $\sigma -$ rigid ring. Then the identity endomorphism of R is a rigid endomorphism of R.*

Proof. Let R be a σ-rigid ring and let $id : R \longrightarrow R$ be the identity map. Suppose $a(id(a)) = 0$. Then $a^2 = 0$. But since R is a reduced ring by Theorem 2.1 we have $a = 0$. □

Remark 2.1. Let S be the set of all rigid endomorphisms of a ring R. Then by Lemma 2.2 and Lemma 2.3 above, S is a monoid under composition of maps.

Lemma 2.4. *Let R be ring. If σ is a rigid automorphism of R, then σ^{-1} is a rigid automorphism of R.*

Proof. Let σ be a rigid automorphism of R and let $a \in R$ be such that $a\sigma^{-1}(a) = 0$. Then $\sigma(a\sigma^{-1}(a)) = \sigma(0) = 0$. So $\sigma(a)(\sigma\sigma^{-1}(a)) = \sigma(a)a = 0$. Since R is zc by Theorem 2.1 we have $a\sigma(a) = 0$ and by the rigidity of σ we have $a = 0$. □

Theorem 2.2. *Let R be a ring and let N be the set of all rigid automorphisms of R. Then N is a group under composition of maps.*

Proof. Clear from the above two lemmas. □

Now, let $G = Aut(R)$ be the group of all automorphisms of R. We have the following

Theorem 2.3. *N is a normal subgroup of G.*

Proof. Clearly N is a subgroup of G. Now let $\sigma \in N, \alpha \in G$. We show that $\alpha\sigma\alpha^{-1} \in N$. For, let $a \in R$ be such that $a(\alpha\sigma\alpha^{-1}(a)) = 0$. Apply α^{-1} on this equation we get $\alpha^{-1}(a)\alpha^{-1}(\alpha\sigma\alpha^{-1}(a)) = \alpha^{-1}(0) = 0$. Hence we have $(\alpha^{-1}(a))\sigma(\alpha^{-1}(a)) = 0$. By the rigidity of σ we have $\alpha^{-1}(a) = 0$, which implies that $a = 0$. Hence $\alpha\sigma\alpha^{-1}$ is a rigid automorphism of R and then N is a normal subgroup of G. □

Example 2.4. The group $End(\mathbb{Z} \oplus \mathbb{Z})$ has nine elements each is of the form $\sigma(x, y) = (ax + by, cx + dy)$, where $(a, b) = (0, 0), (1, 0)$ and $(0, 1)$; and $(c, d) = (0, 0), (1, 0)$ and $(0, 1)$. The group $G = Aut(\mathbb{Z} \oplus \mathbb{Z})$ consists of two elements α and β, where $\alpha(x, y) = (x, y)$, the identity endomorphism and $\beta(x, y) = (y, x)$. Clearly α is a rigid automorphism and β is not a rigid automorphism. Hence the group N of rigid automorphisms of $\mathbb{Z} \oplus \mathbb{Z}$ is the trivial group.

Let σ be an endomorphism of a ring R and let S be a σ-invariant subring of R, in the sense that $\sigma(s) \in S$ for all $s \in S$. So the restriction of σ to S gives an endomorphism of S. Clearly if R is σ-rigid, then S is σ-rigid.

Let R be a ring and let σ be an endomorphism of R. σ can be extended to an endomorphism $\bar{\sigma}$ of $R[x]$ in a usual way as follows (see, [1]):

$$\bar{\sigma}(a_0 + a_1x + \cdots + a_nx^n) = \sigma(a_0) + \sigma(a_1)x + \cdots + \sigma(a_n)x^n.$$

The following Theorem states that the notion of rigidness can be extended from the ring R to the ring of polynomials $R[x]$

Theorem 2.4 ([1])**.** *Let R be a ring and let σ be an endomorphism of R. Then $\bar{\sigma}$ is a rigid endomorphism of $R[x]$ if and only if σ is a rigid endomorphism of R.*

Remark 2.2. Let R be a rigid ring, then $R[x]$ is rigid by the above Lemma and hence $R[x_1, x_2, \cdots, x_n]$ is rigid for any commuting indeterminates $x_1, x_2, \cdots, x_n$. In particular if R is a reduced ring, then $R[x_1, x_2, \cdots, x_n]$ is a reduced ring.

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Investigation the order elements 3 in certain twisted groups of lie type

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Abstract. Suppose G is a finite group and that X is a subgroup of G. The commuting graph is denoted by $C(G, X)$ which if $x \neq y$ and $xy = yx$, has a set of vertices $Xx, y \in X$ joining. Assume that X for elements of order 3 is a G-conjugacy class. In this article, to evaluate X, the commuting graphs for unique Twisted Lie form groups of function two over X are used.

Keywords: exceptional groups, commuting graph, diameter, cliques.

1. Introduction

One of the modern methods of analyzing the structure of the group which has been recently proven to be as effective is studying the action of a group on a graph. Similar approaches are also distinctively used to study the algebraic properties of rings [1]. In addition to the central function of involution in the finite group, the elements of order 3, as well as the elements of order 3, are considered significant in the analysis of finite group algebraic properties. For example, a class of elements of order 3 can be generated by the Conway group, which is the group of auto morphisms of the 24-dimensiona Leech lattice, so that both of them either commute or produce the alternating groups A_4, A_5 or $SL_2(3), SL_2(5)$. In John Thompson's Quadratic pairs of prime 3, such a class also has an essential case. See several of the studies collected in this sense [2, 3, 4, 5]. Suppose G is a finite group and X is a subset of $G; C(G, X)$ denotes the graph for commuting as X as its vertex set with two vertices $x, y \times X$ is adjacent if $x \neq y$ andx, y are commuting. In the seminal paper [6], For a given type of isomorphism of an involution centralizer, where there are finitely many non-abelian groups able to contain it up to isomorphism, Fowler and Brauer first noticed the commuting graph they were outstanding for showing. For the works of the Margulis - Platanov conjecture, these graphs are of considerable importance (see, [7] as they appeared in [6] had $X = G \{1\}$). The commuting graph is called the commuting involution graph, since G is a finite group, and X is a conjugate involution class. In the commuting graph, Rowley, Hart (number Perkins), Bates, and Bundy have some similar results

in finite groups, including, for example, diameter estimation and disc structure analysis (see, [6, 8, 9, 11]). The commuting graph $C(G, X)$ was studied in [12] by Nawawi and Rowley when G is either a sporadic simple group McL or a symmetric group Sn and X is aG-conjugacy class of elements order 3. The commuting graphs for G-classes of elements of order 3 in Mathieu groups is analyzed by Azeez and Aubad [13]. Everett and Rowley point out the various assorted commuting graphs examined in [14]. X is considered to be a G-conjugacy class of order 3 elements from now on. This analysis aims to analyze some features of the commuting graph $C(G, X)$ when G is one of the two characteristic Twisted classes of Lie form. In truth, we assume that G is one of the basic groups, ${}^3D_4(2)$ or ${}^2F_4(2)$'. The research includes studying the configurations of the disks and measuring the diameters, girths, and number of cliques for these diagrams. We will assume, from now on, that G is one of the groups described above. In addition, we can assume that t is an element of order 3 in G and $X = t^G$. For $x \in X$, the ith disc of x, which denote by $\Delta_i(x), (i \in N)$ define to be

$$\Delta_i(x) = \{y \in X | d(x, y) = i\},$$

where $d(,)$ is defined as the standard distance metric on a graph $C(G, X)$. One can immediately see that G, operating by conjugation on X, immerses G in the group of graph auto morphisms of $C(G, X)$. In addition, it is transitive to Cleary, G on the vertices of $C(G, X)$. Let $t-X$ now be a fixed class X variable. The main objective is to research the disc structure of vertex t in $C(G, X)$. The diameter of the commuting graph $C(G, X)$. is denoted by $DiamC(G, X)$ and is defined as:

$$DiamC(G, X) = \max_{x \in X} \{i | \Delta_i(x) \neq \phi \text{ and } \Delta_{i+1}(x) = \phi\}.$$

A graph's girth is defined as the length of the shortest cycle in the graph. Sometimes called a clique, a complete graph is called. The clique number of G is called the size of the largest clique that can be made up of edges and vertices of G. Finally, we can focus on the Atlas [15] for F or the names of the G conjugacy groups. Our primary outcome is as follows.

Theorem 1.1. *Let G be isomorphic to one of ${}^3D_4(2)$ or ${}^2F_4(2)$ then:*

1. *The sizes of the discs $\Delta_i(t)$ are listed in Table 1 and the G-conjugacy classes of tx for $x \in \Delta_i(t); i \in N$ are given in Table 2.*
2. *If $(G, X) = {}^3D_4(2), 3A)$ then $DimC(G, X) = 5$;*
3. *If $(G, X) = {}^3D_4(2), 3B)$ then $DimC(G, X) = 4$;*
4. *If $(G, X) = {}^2F_4(2), 3A)$ then $DimC(G, X) = 7$.*

With the assistance of Magma [16], Gap [17] and the Online Atlas [17], analytical methods have been used to achieve the above results. We allocated

G	$X = t^G$	$\lvert\Delta_1(t)\rvert$	$\lvert\Delta_2(t)\rvert$	$\lvert\Delta_3(t)\rvert$	$\lvert\Delta_4(t)\rvert$	$\lvert\Delta_5(t)\rvert$	$\lvert\Delta_6(t)\rvert$	$\lvert\Delta_7(t)\rvert$
${}^3D_4(2)$	3A	113	6048	75276	57834	504		
	3B	145	9936	261792	54270			
${}^2F_4(2)$	3A	25	216	1836	15552	76464	71442	864

Table 1: The Discs for $C(G, X)$., G ${}^3D_4(2)$, ${}^2F_4(2)$‘.

the equations to the $C_G(t)$-orbits on X (where $C_G(t)$ is acting by conjugation). In Section 2, we also provide details on the action of $C_G(t)$ on X. Specially , we collate the $C_G(t)$-orbit sizes on each $X_C = \phi$, where X_C being defined as below. For a G-Class C define the set:

$$X_C = \{x \in X | tx \in X\}.$$

One can see that if $X_C \neq \phi$ then it is equal to a union of certain $C_G(t)$orbits of X. The way of X_C breaks into $C_G(t)$orbits it will essential to determinate which discs of t contain the vertices in X_C. Also knowing the is the size of X_C can be beneficial which leads us to class structure constants. Class structure constants are the sizes of sets:

$$\{(g_1, g_2) \in C_1 \times C_2 \ | g_1 g_2 = g\},$$

where C_1, C_2, C_3 are Gconjugacy classes and g is a fixed element of C_3. Now these constants can be calculated directly from the complex character table of G which are recorded in the Atlas and are available electronically in the standard libraries of the computer algebra package Gap. If we take $C_1 = C, C_2 = X = C_3$ and $g = t$, then in this case

$$|X_C| = \frac{|G|}{|C_G(t)|\,|C_G(h)|} \sum_{i=1}^{s} \frac{\chi_i(h)\chi_i(t)\overline{\chi_i(t)}}{\chi_i(i)}.$$

Where h is a representative from C and $\chi_1, \chi_2, ..., \chi_s$ the complex irreducible characters of G. Here, we include some detail on how to get the tables, although we have made some changes, as mentioned, we use the class names in Atlas. We suppress the notation "slave" first. So for example, for the classes we write $7A, 7B7B^{*2}, 7C^{*4}$ of ${}^3D_4(2)$, respectively. Secondly, to clarify, as we mean to unite these classes and their characters are in alphabetical order, we compact the letter component of the class name.

As an example, in Table 2, for $G \cong {}^3D_4(2)$ and $X = 3B, 9AC$ is short-hand for $9A \cup 9B \cup 9C$.

G	$X = t^G$	$\Delta_1(t)$	$\Delta_2(t)$	$\Delta_3(t)$	$\Delta_4(t)$	$\Delta_5(t)$	$\Delta_6(t)$	$\Delta_7(t)$
${}^3D_4(2)$	3A	1A, 3A, 3B	7AC, 9AC (504^2), 21AC (504)	3A(378), 4A(216), 4B, 7D(216, 1512^8 8B (1512^6), 9AC 1512^2 12A 1512^2 13AC (1512^3) 21AC (1512^4) 28AC (1512^2)	2A, $3A(27, 216^2, 378)$, 216^2, 378 4A(216 ,378) 4C,6A 6B 7AC 7D (216^2) 8B (216^2), , 9AC 1512 12A (1512^2) 13AC (1512^6) ,28AC (1512^2)	3A 3A		
${}^3D_4(2)$	3B	1A, 3B (72^2)	3A,3B (216^2), 4B (216^6), 9AC 216 12A (216^{12}), 18AC (216^6)	4C,6A (648^6), 6B,7AC (324^2) 7D,8A 9AC $(324^8, 648^2)$, 12A (648^{12}), 13AC (648^{36}), 14AC $(324^8, 648^8)$, 18AC $(324^6, (324^8, (648^{14},$ 21AC (648^{24}), 28AC $(324^4, (648^{12})$	2B,3B($81^3), 4A$, 4B(81^3) 6A 4B(81^4), 7AC (108), 8B (648^{12}), 9AC 4B(108, 324), 13AC $648^{13})$, 14AC (324) 13AC 18AC (648^4), 21AC (648), 28AC (648^4)			
${}^2F_4(2)$‘	3A	1A, 3A(12^2)	3A (36^2)	$4C(27^2, 108^8)$, $6A(27^2, (108^8)$,	3A (108^4), 4C (108^4), 6A (108^{16}), 8C (108^{16}), 12AB (108^6), 13AB (108^{26}) 16AD (108^6)	$2B, 3A(54^2, 108^4)$, $4AB(108^4)$, $(108^2), 6A$ $(108^{52}), 8AB$ $(108^{38}), 8CD$ $(108^{42}), 10A$ $(108^{64}), 12AB$ $(108^{56}), 13AB$ $(108^{68}), 16AD$ (108^{40}),	2A,3A(27),4AB $(108^4), 4C(27^2,$ $54^6, 108^{18}), 5A, 6A$ $(27^2, 546, 108^{52})$, $8AB(108^{20}), 8CD$ $(108^{38}), 10A$ (108^{56}), $12AB(108^{58})$, $13AB(108^{27})$, $16AD(108^{50})$	8AB (108^2), 13AB (108^2),	

Table 2: G- Class of Goods tx for $x \in \Delta_i(t)$.

The difference between t and x in $C(G, X)$ cannot always be defined by the G-class to which tx belongs, a very remarkable finding of the present analysis. The cases where tx is included in the interval between t and x defined by the G-class are $G \cong {}^3D_4(2)$; $X = 3A$, $tx \in \{1A, 2A, 3A, 4B, 4C, 6A, 6B, 7A, 7B, 7C\}, G \cong {}^3D_4(2)$; $X = 3B$, $tx \in \{1A, 2B, 3A, 4A, 4C, 6B, 7D, 8A\}$ and $G \cong {}^2F_4(2)$; $X =$

$3A,\ tx \in \{1A, 2A, 2B, 5A\}$. The following additional cases where the distance between t and x in $C(G, X)$ is not defined by the G-class comprising tx. For instance,, when $G \cong {}^3D_4(2);\ X = 3A,\ tx \in \{9A, 9B, 9C\}$ each of $9A,\ 9B$ and $9C$ breaks into five $C_G(t)$-orbits, 2 of size 504 being in $2(t)$,2 of size 1512 being in $\Delta_3(t)$ and 1 of size 1512 being in $3(t)$. For more details, see Table 2.

2. $C_G(t)$-orbits on X

The sizes of the $C_G(t)$-orbits are given in their action on X_C in this section, as we described above., when X_C for C is a G-conjugacy class. In the succeeding table's exponential notation are used to point the multiplicity of a specific size. For instance, in the $C_G(t)$-orbits table for $C({}^3D_4(2), 3B)$ the entry $648^6,\ 81^4$ next to 6A is show that X_{6A} It is a union of ten $C_G(t)$-orbits, six of which are 648 in size and two of which are 81 in size. In the $C_G(t)$-orbits table, we will see another example of this. for $C({}^2F_4(2), 3A)$ the entry 10896 next to $16AD$ is points that each of X_{16A} , X_{16B}, X_{16C} and X_{16D} is the union of ninety six $C_G(t)$-orbits, all of them have size 108. In the next table, we include class information, including the class size, structure of the $C_G(t)$-orbits and permutation rank. For the repressions for ${}^3D_4(2)$ and ${}^2F_4(2)$ẁe applied the

Group	$X = t^G$	$\|X\|$	Permutation Rank	$C_G(t) structures$
${}^3D_4(2)$	3A	139776	118	$Z_3 x PSL(2, 8)$
	3B	326144	600	$(((Z_3 x Z_3) : Z_3) : Q_8) : Z_3$
${}^2F_4(2)$	3A	166400	1564	$((Z_3 x Z_3) : Z_3) : Z_4$

Table 3: The Classes Information

permutation representations with 819 and 1600 points respectively (see, [18]).

2.1 $C_G(t)$-orbits of commuting graph C (3D4(2),3A)

In the following table, we provide complete details on the scale of the $C_G(t)$-orbits in the case of $G\ {}^3D_4(2);\ X = 3A$. Also, the way of X_C breaks into $C_G(t)$-orbits are given: The C_G(t)-orbits of the commuting graph $C({}^3D_4(2), 3A)$ in this case distribute in five discs each with different size.

2.2 $C_G(t)$-orbits of commuting graph C (3D4(2),3B)

In the next table we give a full details about the $C_G(t)$-orbits sizes in case $G \cong {}^3D_4(2);\ X = 3B$. Also, the way of X_C breaks into $C_G(t)$-orbits are presented:

The $C_G(t)$-orbits of the commuting graph $C({}^3D_4(2), 3B)$ in this case disseminate in four discs each with different size.

Class name	$C_G(t)$-Orbit size
1A	1
2A	27^2
3A	$27, 56, 216^2, 378^2, 504$
3B	56
4A	$216^2, 378$
4B	378
4C	756^2
6A	1512^4
6B	756^2
7AC	504
7D	$216^3, 1512^8$
8B	1512^8
9AC	504^5
12A	1512^4
13AC	1512^9
21AC	1512^5
28AC	1512^4

Table 4: $C_G(t)$-Orbits sizes of $C(^3D_4(2), 3A)$

2.3 $C_G(t)$-orbits of commuting graph $C(^2F_4(2)', 3B)$

In the following table we give a full information on the $C_G(t)$-orbits sizes in case $G\ {}^2F_4(2)'$; $X = 3A$. Also, the form of X_C breaks into $C_G(t)$-orbits are given: The $C_G(t)$-orbits of the commuting graph $C(^2F_4(2)', 3B)$ in this case disseminate in seven discs each with different size.

3. Girths and cliques number

The strategy we going to use to calculate the cliques number for the commuting graph $C(G, X)$, mainly depend on the fact that G, acting by conjugation, induces graph auto morphisms of $C(G, X)$, and is transitive on its vertices. Also easy computational check inside $\Delta_1(t)$ one can find $x, y \in \Delta_1(t)$ such that x and y are commute. Thus, the girth of the commuting graph is 3.The following table the cliques number for the commuting graph $C(G, X)$, The above table telling us that the girth $C(G, X)., G \cong {}^3D_4(2),\ {}^2F_4(2)$ìs 3 while clique number for $C(G, X)., G \cong {}^3D_4(2), {}^2F_4(2)$ and $X = 3A$, equal 3 and $C(G, X)., G \cong {}^3D_4(2),\ X = 3B$ is 8. Also the maximum elementary abelian subgroup of the elements of X is of the form $Z_3 \times Z_3$.

Class name	$C_G(t)$-Orbit size
1A	1
2B	81^3
3A	216^5
3B	$72^2, 81^3, 216^2$
4A	81
4B	$81^3, 216^6$
4C	648^4
6A	$81^4, 648^6$
6B	756^2
6B	648^{10}
7ABC	$108, 324^2$
7D	648^9
8A	648^{12}
8B	648^{12}
9ABC	$108, 216, 324^9, 648^9$
12A	$216^{12}, 648^{12}$
13ABC	648^{49}
14ABC	$324^9, 648^8$
18ABC	$216^6, 324^6, 648^{18}$
21ABC	648^{25}
28ABC	$324^4, 648^{16}$

Table 5: $C_G(t)$-Orbits sizes of $C({}^2F_4(2)', 3A)$

Group	$X = t^G$	Clique number	Girth
${}^3D_4(2)$	3A	8	3
	3B	6	3
${}^2F_4(2)'$	3A	8	3

Table 6: Girth and Clique Number for $C(G, X)$., G ${}^3D_4(2)$, ${}^2F_4(2)'$.

Conclusions

This paper aims to study the algebraic structure of G-classes of elements of order 3 in exceptional groups of Lie type ${}^3D_4(2)$, ${}^2F_4(2)$. For that purpose we employed the commuting graph on the G-classes of elements of order 3 of such groups. A very important results have been gained for instance, $C_G(t)$-orbits sizes, girth, clique number of the graph. Also the maximum elementary abelian subgroup of the elements of G-classes of elements of order 3.

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Bayes estimation of Lomax parameters under different loss functions using Lindley's approximation

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Abstract. This paper discusses Maximum Likelihood and Base estimation of two parameters shape parameter β and scale parameter Θ of the Lomax distribution based on three kinds of Loss Functions; Squared Error Loss Function (SELF), Quadratic Loss Function (QLF), and Linex Loss Function (LLF). Lindley approximation was used to obtain the values for the Lomax distribution parameters and to find the best estimation of the parameters. Many sample sizes (10,20,30,50,100) were used to compare the best estimator using the Monte Carlo simulation and the statistical scale Mean Squared Error (MSE). The results show that the Bayes estimator with both the Linex loss function and the Lindley approximation is the best estimator, and when increasing the values of shape parameter due to decreasing the values of mean square error with various vales of scale parameter.

Keywords: maximum likelihood, base estimation, Lindley approximation, Squared Error Loss Function (SELF), Quadratic Loss Function (QLF), Linex Loss Function (LLF), Monte Carlo, Mean Squared Error (MSE).

1. Introduction

A number of authors have studied Lomax distribution Inference by several authors. Nasiri and Hosseini [9] were used (MLE) with an appropriate prior distribution to obtain an estimate of Bayes based on the use of the result (Quadratic

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Loss and Squared Error Loss) functions. Okasha [10] obtained the Bayes estimator for the Parameter under (the Balanced Squared Error Loss) function. A Comparison was made using the Monte Carlo simulation between the new approach and Maximum Likelihood and Bayes methods. Afaq and others [1] used Jeffery's prior and Jeffery's extension prior to the derivation of the Maximum likelihood and Bayes estimators under use (Squared error loss, Al-Bayyati, and Precautionary loss) functions. Pak and Mahmoudi [11] determined Lomax parameters by using Bayes estimators under the (SEL) function and Maximum likelihood estimator and reached Bayesian with informative priors giving smaller biases and MSE compared to the Maximum likelihood method. Fitrilia, Fithriani and Nurrohmah [3] set the shape parameter based on the assumption of the scale parameters and used Gamma as a conjugate prior distribution with a balanced squared error loss function. The three parameters generalized Lomax distribution were studied by Maurya, Tripathi, Lodhi and Rastogi [7]. Several structural and statistical properties have been investigated, including moments, quantiles, stochastic ordering, and order statistics. They used the Tierney and Kadane method to compute Bayes estimators.

2. Lomax distribution

The distribution of Lomax was introduced by Lomax world in [6], which is appropriate for many issues such as: theory of queuing, business, economics, and internet traffic modeling. The Probability Density Function of the Lomax distribution [1] is defined as:

$$f(x) = \frac{\beta}{\Theta}\left(1 + \frac{x}{\Theta}\right)^{-(\beta+1)}, \quad x > 0, \Theta, \beta > 0 \tag{1}$$

with the (shape parameter) β and the (scale parameter) Θ. The Corresponding Cumulative Distribution Function is given as [12]:

$$F(x) = 1 - \left(1 + \frac{x}{\Theta}\right)^{-\beta)}, \quad x > 0, \Theta, \beta > 0. \tag{2}$$

Figure (1) shows probability and cumulative density function curve of the Lomax distribution with several different parameters of (β, Θ). The Reliability and hazard functions are given as follows:

$$R(x) = \left(1 + \frac{x}{\Theta}\right)^{-\beta}, \tag{3}$$

$$h(x) = \frac{\beta}{\Theta}\left(1 + \frac{x}{\Theta}\right)^{-1}. \tag{4}$$

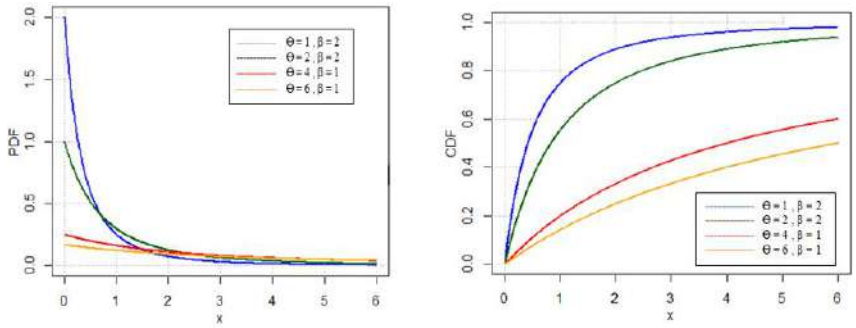


Fig 1: Probability *and* Cumulative Density Function Curve of Lomax Distribution for Various chioce for β, Θ [14]

3. Maximum Likelihood Estimator method (MLE)

Let $x = (x_1, x_2, \ldots, x_n)$ be a random sample (r.s.) of size (n) of the Lomax dist., the likelihood function is: $L(\beta, \Theta; x) = \prod_{i=1}^{n} f(x_i, \beta, \Theta)$

$$L(\beta, \Theta; x) = \frac{\beta^n}{\Theta^n} \prod_{i=1}^{n} (1 + \frac{x_i}{\Theta})^{-(\beta+1)}. \tag{5}$$

Subsequently, the partial derivative for Log Likelihood function is performed with respect to β and equating to zero, and the estimation of MLE of β denoted by $\widehat{\beta}_{MLE}$ is as follows:

$$\widehat{\beta}_{MLE} = \frac{n}{\sum_{i=1}^{n} \ln(1 + \frac{x_i}{\Theta})}. \tag{6}$$

Suppose β is known. We propose to solve Θ by using the Newton-Raphson method. Thus Θ is obtained by taking the initial value for Θ_i and iterating the process till it converges as:

$$\Theta_{i+1} = \Theta_i - \frac{-n + \frac{n}{\sum_{i=1}^{n} \ln(1+\frac{x_i}{\Theta})} \sum_{i=1}^{n} \frac{x_i}{\Theta+x_i} + \sum_{i=1}^{n} \frac{x_i}{\Theta+x_i}}{\frac{n}{\Theta}[\frac{\sum_{i=1}^{n} \frac{x_i}{\Theta+x_i}}{\sum_{i=1}^{n} \ln(1+\frac{x_i}{\Theta})}]^2 + \frac{n}{\sum_{i=1}^{n} \ln(1+\frac{x_i}{\Theta})} \sum_{i=1}^{n} \frac{x_i}{[\Theta+x_i]^2} - \sum_{i=1}^{n} \frac{x_i}{[\Theta+x_i]^2}}. \tag{7}$$

When we determine the value of $\widehat{\Theta}; (\widehat{\widehat{\Theta}}_{MLE})$, we get the value of $\widehat{\beta}; (\widehat{\widehat{\beta}}_{MLE})$.

4. Bayes estimation

In this section, we studied Bayes estimators of parameters β and Θ and reliability function R(t) based on informative prior with different loss functions [4], [13]. Prior distributions for finding Bayes estimators of the Lomax distribution parameters are taken as follows:

$$g_1(\beta) = \frac{b^a \beta^{a-1} e^{-b\beta}}{\Gamma(a)}, \quad \beta, a, b > 0, \tag{8}$$

$$g_2(\Theta) = \frac{1}{\Theta}, \quad \Theta > 0, \tag{9}$$

where $\Gamma(a)$ is the function of gamma. Thus, the joint prior distribution for β and Θ is:

$$g(\beta,\Theta)=\frac{b^a\beta^{a-1}e^{-b\beta}}{\Theta\Gamma(a)},\quad \beta,\Theta,a,b>0, \tag{10}$$

$$\pi(\beta,\Theta\setminus x_1,x_2,...,x_n)=\frac{L(x_1,x_2,...,x_n;\beta,\Theta)g(\beta,\Theta)}{\int_0^\infty\int_0^\infty L(x_1,x_2,...,x_n;\beta,\Theta)g(\beta,\Theta)d\beta d\Theta}. \tag{11}$$

Substituting L(β,Θ;x) and g(β,Θ) of (5) and (10) correspondingly, the joint posterior P(β,Θ;x) as follows:

$$\pi(\beta,\Theta\setminus x_1,x_2,...,x_n)=\frac{\frac{\beta^{n+a-1}}{\Theta^{n+1}}\prod_{i=1}^n(1+\frac{x_i}{\Theta})^{-(\beta+1)}e^{-b\beta}}{\int_0^\infty\int_0^\infty\frac{\beta^{n+a-1}}{\Theta^{n+1}}\prod_{i=1}^n(1+\frac{x_i}{\Theta})^{-(\beta+1)}e^{-b\beta}d\beta d\Theta}. \tag{12}$$

5. Bayes estimation using Lindly approximation

Lindley approximation method proposed by Lindley [5]. The procedure was designed and applied to get an approximate Bayes estimator and compute the ratio of two integrals [2], [3] as follows:

$$I=E[u(\beta,\Theta)]=\frac{\int_{(\beta,\Theta)}u(\beta,\Theta)e^{l(\beta,\Theta)+g(\beta,\Theta)}d(\beta,\Theta)}{\int_{(\beta,\Theta)}e^{l(\beta,\Theta)+g(\beta,\Theta)}d(\beta,\Theta)}, \tag{13}$$

where u(β,Θ)=function of β and Θ,l(β,Θ)=Log Likelihood and g(β,Θ)=joint prior of β and Θ. The Bayesian estimator for β and Θ by using the square error loss function is: R=E[R(β,Θ)], where R(β,Θ) is any function for β and Θ:

$$R=E[R(\beta,\Theta)]=\int_{(\beta,\Theta)}R(\beta,\Theta)\pi(\beta,\Theta\setminus x_1,x_2,...,x_n)d(\beta,\Theta). \tag{14}$$

By using Lindley's approximate method, which accession the ratio of two integrals in order to obtain the Bayes estimators β,Θ , it is possible to considering $\widehat{R}$ as follows:

$$\begin{aligned}\widehat{R}&=u(\widehat{\beta},\widehat{\Theta})+[u_1\alpha_1+u_2\alpha_2+\alpha_3+\alpha_4]\\&\quad+\frac{1}{2}[A(u_1\sigma_{11}+u_2\sigma_{12})+B(u_1\sigma_{21}+u_2\sigma_{22})],\end{aligned} \tag{15}$$

where $\widehat{\beta}$ and $\widehat{\Theta}$ are the MLE of β and Θ respectively

$$\begin{aligned}&\alpha_1=\rho_1\sigma_{11}+\rho_2\sigma_{12},\quad \alpha_2=\rho_1\sigma_{21}+\rho_2\sigma_{22},\\&\alpha_3=\frac{1}{2}[u_{12}\sigma_{12}+u_{21}\sigma_{21}],\quad \alpha_4=\frac{1}{2}[u_{11}\sigma_{11}+u_{22}\sigma_{22}],\\&B=\sigma_{11}L_{112}+2\sigma_{12}L_{122}+\sigma_{22}L_{222},\end{aligned}$$

$$u_1 = \frac{\partial u(\beta,\Theta)}{\partial \beta}, \quad u_2 = \frac{\partial u(\beta,\Theta)}{\partial \Theta}$$

$$u_{11} = \frac{\partial^2 u(\beta,\Theta)}{\partial \beta^2}, \quad u_{12} = \frac{\partial^2 u(\beta,\Theta)}{\partial \beta \partial \Theta},$$

$$u_{21} = \frac{\partial^2 u(\beta,\Theta)}{\partial \beta \partial \Theta}, \quad u_{22} = \frac{\partial^2 u(\beta,\Theta)}{\partial \Theta^2},$$

$$\rho = \ln g(\beta,\Theta) = a\ln b + (a-1)\ln\beta - b\beta - \ln\Theta - \ln\Gamma(a),$$

$$\rho_1 = \frac{\partial \rho}{\partial \beta} = \frac{a-1}{\beta} - b, \quad \rho_2 = \frac{\partial \rho}{\partial \Theta} = -\frac{1}{\Theta},$$

$$L_{11} = \frac{\partial^2 L(\beta,\Theta)}{\partial \beta^2} = \frac{-n^2}{\beta}, \quad L_{12} = \frac{\partial^2 u(\beta,\Theta)}{\partial \beta \partial \Theta} = \sum_{i=1}^{n} \frac{x_i}{(\Theta^2 + \Theta x_i)},$$

$$L_{21} = \frac{\partial^2 u(\beta,\Theta)}{\partial \Theta \partial \beta} = L_{12}, \quad L_{22} = \frac{\partial^2 u(\beta,\Theta)}{\partial \beta^2} = \frac{n}{\Theta^2} - (\beta+1)\sum_{i=1}^{n} \frac{2\Theta x_i + x_i^2}{(\Theta^2 + \Theta x_i)^2},$$

$$L_{211} = \frac{\partial^3 u(\beta,\Theta)}{\partial \Theta \partial \beta^2} = L_{121} = \frac{\partial^3 u(\beta,\Theta)}{\partial \beta^2 \partial \Theta} = L_{112} = \frac{\partial^3 u(\beta,\Theta)}{\partial \beta^2 \partial \Theta} = 0,$$

$$L_{212} = \frac{\partial^3 u(\beta,\Theta)}{\partial \beta \partial \Theta^2} = -\sum_{i=1}^{n} \frac{2\Theta + x_i}{(\Theta^2 + \Theta x_i)^2}$$

$$L_{221} = \frac{\partial^3 u(\beta,\Theta)}{\partial \Theta^2 \partial \beta} = L_{122} = \frac{\partial^3 u(\beta,\Theta)}{\partial \Theta^2 \partial \beta} = -\sum_{i=1}^{n} \frac{2\Theta x_i + x_i^2}{(\Theta^2 + \Theta x_i)^2},$$

$$L_{222} = \frac{\partial^3 u(\beta,\Theta)}{\partial \beta^3} = \frac{-2n}{\Theta^3} - \frac{(\beta+1)\sum_{i=1}^{n} 2x_i(\Theta^2 + \Theta x_i) - 2(2\Theta x_i + x_i^2)(2\Theta + x_i)}{\Theta^2 + \Theta x_i}$$

β and Θ are independent then:

$$\sigma_{12} = \sigma_{\beta\Theta} = \sigma_{\Theta\beta} = \sigma_{21} = 0, \quad \sigma_{11} = -\frac{1}{L_{11}}, \quad \sigma_{22} = -\frac{1}{L_{22}}$$

6. Estimator of Bayes with Squared Error Loss (SEL) function

The squared error loss function is given as: [1]

$$L_{BS}(\widehat{\Theta},\Theta)\alpha(\widehat{\Theta} - \Theta)^2. \tag{16}$$

Then, the Risk function under the Squared Error Loss Function is denoted by $R_{SL}(\widehat{\Theta},\Theta)$ is:

$$R_{SEL}(\widehat{\Theta},\Theta) = \int_0^{\infty} (\widehat{\Theta} - \Theta)\pi(\Theta\backslash t)d\Theta. \tag{17}$$

Using a partial derivative for $R_{SEL}(\widehat{\Theta},\Theta)$ with regard to $\widehat{\Theta}$ and fixing it equals zero, we get:

$$\widehat{\widehat{\Theta}} = E(\Theta/t) \tag{18}$$

i) If $u(\beta,\Theta)=\widehat{\beta}$ then: $u_1=1, u_2=u_{11}=u_{22}=u_{12}=u_{21}=0$. Then, the estimator β is written as follows:

$$\widehat{\beta}_{BSEL}=\beta-\frac{\rho_1}{L_{11}}+\frac{L_{111}L_{222}+L_{221}}{2L_{11}L_{22}}+\frac{L_{111}}{2L_{11}^2}+\frac{L_{221}L_{222}}{2L_{22}^2}. \tag{19}$$

ii) If $u(\beta,\Theta)=\widehat{\Theta}$ then: $u_2=1, u_2=u_{11}=u_{12}=u_{21}=u_{22}=0$. Then, the estimator Θ is written as follows:

$$\widehat{\widehat{\Theta}}_{BSEL}=\widehat{\Theta}-\frac{\rho_2}{L_{22}}-\frac{L_{111}L_{222}}{2L_{11}L_{22}^2}+\frac{L_{221}L_{222}}{2L_{22}^3}. \tag{20}$$

7. Estimator of Bayes with Quadratic Loss (QL) function

The (QL) function is given as [1]

$$L_{QL}(\widehat{\Theta},\Theta)=\left(\frac{\widehat{\Theta}-\Theta}{\Theta}\right)^2. \tag{21}$$

Then, the function of Risk with (QL) function is denoted as $R_{QL}(\widehat{\Theta}-\Theta)$:

$$R_{QL}(\widehat{\Theta},\Theta)=\int_0^{\infty}\left(\frac{\widehat{\Theta}-\Theta}{\Theta}\right)^2\pi(\Theta\backslash t)d\Theta. \tag{22}$$

Going to take partial derivative to $R_{QL}(\widehat{\Theta},\Theta)$ with regard to $\widehat{\Theta}$ and to place it equal to zero, we get:

$$\widehat{\Theta}=\left(\frac{E(\Theta^{-1}\backslash t)}{E(\Theta^{-2}\backslash t)}\right) \tag{23}$$

i) If $u(\beta,\Theta)=\frac{1}{\beta}$ then: $u_1=u_{11}=u_{12}=u_{21}=0, u_2=\frac{-1}{\beta^2}, u_{22}=\frac{2}{\beta^3}$

$$E(\beta^{-1}\backslash t)=\frac{1}{\beta}+\frac{\rho_1}{\beta^2L_{22}}-\frac{1}{\beta^3L11}-\frac{L_{221}}{2\beta^2L_{11}^2}-\frac{L_{111}}{2\beta^2L_{11}L_{22}}. \tag{24}$$

ii) If $u(\beta,\Theta)=\frac{1}{\beta^2}$ then: $u_1=\frac{-2}{\beta^3}, u_{11}=\frac{6}{\beta^4}, u_2=u_{22}=u_{12}=u_{21}=0$

$$E(\beta^{-2}\backslash t)=\frac{1}{\beta^2}+\frac{2\rho_1}{\beta^3L_{22}}-\frac{3}{\beta^4L_{11}}-\frac{L_{221}}{\beta^3L_{11}^2}-\frac{L_{111}}{\beta^3L_{11}L_{22}}. \tag{25}$$

Then, the estimator β is written as follows:

$$\widehat{\widehat{\beta}}_{BQL}=\frac{E(\beta^{-1}\backslash t)}{E(\beta^{-2}\backslash t)}=\frac{1+\frac{\rho_1}{\beta L_{22}}-\frac{1}{\beta^2L_{11}}-\frac{L_{221}}{2\beta L_{11}^2}-\frac{L_{111}}{2\beta L_{11}L_{22}}}{\frac{1}{\beta}+\frac{2\rho_1}{\beta^2L_{22}}-\frac{3}{\beta^3L_{11}}-\frac{L_{221}}{\beta^2L_{11}^2}-\frac{L_{111}}{\beta^2L_{11}L_{22}}}. \tag{26}$$

iii) If $u(\beta,\Theta)=\frac{1}{\Theta}$ then: $u_1=\frac{-1}{\Theta^2}, u_{11}=\frac{2}{\Theta^3}, u_2=u_{22}=u_{12}=u_{21}=0$

$$E(\Theta^{-1}\backslash t)=\frac{1}{\Theta}+\frac{\rho_1}{\Theta^2L_{22}}-\frac{1}{\Theta^3L11}-\frac{L_{221}}{2\Theta^2L_{22}^2}-\frac{L_{111}}{2\Theta^2L_{11}L_{22}}. \tag{27}$$

iv) If $u(\beta,\Theta)=\frac{1}{\Theta^2}$ then: $u_1=\frac{-2}{\Theta^3}, u_{11}=\frac{6}{\Theta^4}, u_2=u_{21}=u_{12}=u_{22}=0$

$$E(\Theta^{-2}\backslash t)=\frac{1}{\Theta^2}+\frac{2\rho_1}{\Theta^3L_{22}}-\frac{3}{\Theta^4L_{11}}-\frac{L_{221}}{\Theta^3L_{11}^2}-\frac{L_{111}}{\Theta^3L_{11}L_{22}}. \tag{28}$$

Then, the estimator Θ is written as follows:

$$\widehat{\widehat{\Theta}}_{BQL}=\frac{E(\Theta^{-1}\backslash t)}{E(\Theta^{-2}\backslash t)}=\frac{1+\frac{\rho_1}{\Theta L_{22}}-\frac{1}{\Theta^2L_{11}}-\frac{L_{221}}{2\Theta L_{11}^2}-\frac{L_{111}}{2\Theta L_{11}L_{22}}}{\frac{1}{\Theta}+\frac{2\rho_1}{\Theta^2L_{22}}-\frac{3}{\Theta^3L_{11}}-\frac{L_{221}}{\Theta^2L_{11}^2}-\frac{L_{111}}{\Theta^2L_{11}L_{22}}}. \tag{29}$$

8. Bayes Estimator with Linex Loss (LL) function

The (LLF) is indicated as [8]

$$L(\widehat{\Theta},\Theta)=e^{c(\widehat{\Theta}-\Theta)}-c(\widehat{\Theta}-\Theta)-1,\quad c\neq 0. \tag{30}$$

Additionally, $R_{LS}(\widehat{\Theta},\Theta)$ defines the Risk function with Linex loss function as follows:

$$R_{LS}(\widehat{\Theta},\Theta)=\int_0^\infty [e^{c(\widehat{\Theta}-\Theta)}-c(\widehat{\Theta}-\Theta)-1]\pi(\Theta\backslash t)d\Theta. \tag{31}$$

Take a partial derivative for $R_{LS}(\widehat{\Theta},\Theta)$ with due regard to $\widehat{\Theta}$ and making it equal to zero, we obtain:

$$\widehat{\widehat{\beta}}_{BLL}=-\frac{1}{c}\ln E_\beta[e^{-c\beta}], \tag{32}$$

where E_β is posterior expectation. i) If $u(\widehat{\beta},\widehat{\Theta})=e^{-c\beta}$ then: $u_1=-ce^{-c\beta}, u_{11}=c^2e^{-c\beta}, u_2=u_{21}=u_{12}=u_{22}=0$

$$E_\beta[e^{-c\beta}]=e^{-c\beta}(1+\frac{c\rho_1}{L_{11}}-\frac{c^2}{2L_{11}}-\frac{c}{2L_{11}^2}+\frac{cL_{221}}{2L_{11}L_{22}}). \tag{33}$$

The estimator β then can be written as follows:

$$\widehat{\widehat{\beta}}_{BLL}=\widehat{\beta}-\frac{1}{c}\ln(1+\frac{c\rho_1}{L_{11}}-\frac{c^2}{2L_{11}}-\frac{c}{2L_{11}^2}+\frac{cL_{221}}{2L_{11}L_{22}}). \tag{34}$$

ii) If $u(\widehat{\beta},\widehat{\Theta})=e^{-c\Theta}$ then: $u_2=-ce^{-c\Theta}, u_{22}=c^2e^{-c\Theta}, u_1=u_{21}=u_{12}=u_{11}=0$

$$\widehat{\widehat{\Theta}}_{BLL}=\frac{-1}{c}\ln E_\Theta[e^{-c\Theta}],$$

$$E_\Theta[e^{-c\Theta}]=e^{-c\Theta}(1+\frac{c\rho_2}{L_{22}}-\frac{c^2}{2L_{22}}+\frac{c}{2L_{11}^2}+\frac{cL_{111}L_{222}}{L_{11}L_{22}^2}+\frac{cL_{221}L_{222}}{2L_{22}^3}). \tag{35}$$

Then, the estimator Θ written as follows:

$$\widehat{\widehat{\Theta}}_{BLL}=\widehat{\Theta}-\frac{1}{c}\ln(1+\frac{c\rho_2}{L_{22}}-\frac{c^2}{2L_{22}}+\frac{c}{2L_{11}^2}+\frac{cL_{111}L_{222}}{L_{11}L_{22}^2}+\frac{cL_{221}L_{222}}{2L_{22}^3}). \tag{36}$$

9. Simulation study and results

The simulation of (Monte Carlo) was used in this paper to compare the various methods used to estimate the Lomax distribution parameters. This approach is outlined in the following steps:

1) Select many assumed values for the parameters β and Θ for the Lomax distribution: $\beta = 0.5, 0.75, 1.5$; $\Theta = 0.5, 0.75, 1.5$.

2) Choose a number of different sample sizes: n=10, 20, 30, 50, 100.

3) Assumed that the values of a, b and c in the Lindley approximation method are as follows: a=1.5 ,b=0.75 ,c=2.

4) Random data was generated from the distribution of the uniform to the distribution of Lomax using the inverse transformation process, which was based on the determination of the inverse of the cumulative distribution function as follows: $x_i = \Theta[(1-u_i)^{\frac{-1}{\beta}} - 1]$.

5) Repeat each experiment (L=1000) times.

6) Compare all estimation methods using (MSE) as follows:

$$MSE(\widehat{\beta}) = \frac{1}{L}\sum_{i=1}^{L}(\beta_i - \widehat{\beta}_i)^2, \quad MSE(\widehat{\Theta}) = \frac{1}{L}\sum_{i=1}^{L}(\Theta_i - \widehat{\Theta}_i)^2. \tag{37}$$

The following tables include sample of the numerical results for the empirical works.

TABLE 1: MLE and Bayes estimation for β *and* Θ by using Lindly when $\beta = 0.5$; $\Theta = 0.5, 0.75, 1.5$

Θ	n	MLE		Bayes under Square Error Loss		Bayes under Quadratic Loss		*Bayes under Linex* Loss		Best
		$\hat{\beta}_{MLE}$	$\hat{\Theta}_{MLE}$	$\hat{\beta}_{BSEL}$	$\hat{\Theta}_{BSEL}$	$\hat{\beta}_{BQL}$	$\hat{\Theta}_{BQL}$	$\hat{\beta}_{BLL}$	$\hat{\Theta}_{BLL}$	
0.5	10	0.55953	0.45823	0.33287	0.41155	0.42884	0.44991	0.11967	0.23189	BLL
	20	0.55996	0.45872	0.33891	0.41865	0.42858	0.44980	0.11861	0.23587	BLL
	30	0.55784	0.45256	0.33875	0.41557	0.42817	0.44875	0.11665	0.23147	BLL
	50	0.55875	0.45158	0.33887	0.41987	0.42789	0.44489	0.11584	0.23178	BLL
	100	0.55389	0.45145	0.33881	0.41148	0.42675	0.44451	0.10457	0.22154	BLL
0.75	10	0.42125	0.43994	0.31665	0.50520	0.48665	0.51896	0.20689	0.30884	BLL
	20	0.42147	0.43897	0.31564	0.50456	0.48663	0.51887	0.20654	0.30845	BLL
	30	0.42410	0.43911	0.31478	0.50412	0.48538	0.51874	0.20568	0.30745	BLL
	50	0.42243	0.43742	0.31458	0.50408	0.48416	0.51754	0.20548	0.30581	BLL
	100	0.42145	0.43117	0.31354	0.50388	0.48314	0.51521	0.20487	0.30511	BLL
1.5	10	0.40789	0.45554	0.31897	0.50589	0.48897	0.51296	0.21884	0.30412	BLL
	20	0.40445	0.45078	0.31489	0.50568	0.48889	0.51285	0.20954	0.30989	BLL
	30	0.40325	0.45088	0.31258	0.50498	0.48875	0.51654	0.20852	0.30895	BLL
	50	0.40158	0.45245	0.31018	0.50477	0.48787	0.51491	0.20754	0.30845	BLL
	100	0.40125	0.45014	0.31011	0.50244	0.48584	0.51284	0.20451	0.30821	BLL

10. Conclusion and recommendation

- Table 1 shows that when (β=0.5), the output of Bayes estimator with many (loss functions) of β and Θ is better than the MLE.

TABLE 2: MLE and Bayes estimation for β *and* Θ by using Lindly when $\beta = 0.75$; $\Theta = 0.5, 0.75, 1.5$

Θ	n	MLE		Bayes under Square Error Loss		Bayes under Quadratic Loss		*Bayes under Linex* Loss		Best
		$\hat{\beta}_{MLE}$	$\hat{\Theta}_{MLE}$	$\hat{\beta}_{BSEL}$	$\hat{\Theta}_{BSEL}$	$\hat{\beta}_{BQL}$	$\hat{\Theta}_{BQL}$	$\hat{\beta}_{BLL}$	$\hat{\Theta}_{BLL}$	
0.5	10	0.21989	0.22911	0.30104	032817	0.10169	0.11199	0.01888	0.02220	BLL
	20	0.21852	0.22921	0.30101	032816	0.10155	0.11182	0.01824	0.02123	BLL
	30	0.21842	0.22846	0.30092	032801	0.10114	0.11145	0.01812	0.02112	BLL
	50	0.21753	0.22834	0.30088	032781	0.10111	0.11124	0.01801	0.02012	BLL
	100	0.21701	0.22784	0.30014	032711	0.10089	0.11119	0.01789	0.02010	BLL
0.75	10	0.21424	0.22412	0.30545	032894	0.10074	0.11985	0.01987	0.02625	BLL
	20	0.21421	0.22410	0.30254	032892	0.10064	0.11978	0.01963	0.02600	BLL
	30	0.21434	0.22385	0.30224	032887	0.10832	0.11894	0.01898	0.02875	BLL
	50	0.21415	0.22289	0.30214	032784	0.10823	0.11852	0.01874	0.02852	BLL
	100	0.21401	0.22265	0.30112	032745	0.10811	0.11724	0.01785	0.02812	BLL
1.5	10	0.24211	0.22321	0.30214	032721	0.10728	0.11700	0.01545	0.02942	BLL
	20	0.21987	0.22301	0.30005	032657	0.10721	0.11664	0.01527	0.02910	BLL
	30	0.21878	0.22102	0.30001	032584	0.10700	0.11657	0.01501	0.02828	BLL
	50	0.21258	0.22044	0.26541	032541	0.10681	0.11611	0.01456	0.02712	BLL
	100	0.21189	0.22001	0.25841	032441	0.10521	0.11421	0.01412	0.02621	BLL

TABLE 3: MLE and Bayes estimation for β *and* Θ by using Lindly when $\beta = 1.5$; $\Theta = 0.5, 0.75, 1.5$

Θ	n	MLE		Bayes under Square Error Loss		Bayes under Quadratic Loss		*Bayes under Linex* Loss		Best
		$\hat{\beta}_{MLE}$	$\hat{\Theta}_{MLE}$	$\hat{\beta}_{BSEL}$	$\hat{\Theta}_{BSEL}$	$\hat{\beta}_{BQL}$	$\hat{\Theta}_{BQL}$	$\hat{\beta}_{BLL}$	$\hat{\Theta}_{BLL}$	
0.5	10	0.15524	0.17878	0.29948	030753	0.12246	0.14019	0.00885	0.00727	BLL
	20	0.15515	0.17521	0.29936	030754	0.12237	0.14015	0.00881	0.00711	BLL
	30	0.15098	0.17510	0.29912	030742	0.12244	0.14014	0.00875	0.00621	BLL
	50	0.15085	0.17489	0.29901	030722	0.12213	0.14008	0.00871	0.00601	BLL
	100	0.15009	0.17467	0.29900	030712	0.12155	0.14004	0.00865	0.00561	BLL
0.75	10	0.15847	0.17457	0.29741	030697	0.12456	0.14028	0.00842	0.00552	BLL
	20	0.15879	0.17444	0.29714	030665	0.12424	0.14026	0.00833	0.00542	BLL
	30	0.15789	0.17432	0.29706	030642	0.12412	0.14014	0.00811	0.00532	BLL
	50	0.15741	0.17414	0.29712	030621	0.12368	0.14011	0.00784	0.00456	BLL
	100	0.15701	0.17211	0.29710	030612	0.12332	0.13001	0.00745	0.00444	BLL
1.5	10	0.15856	0.17212	0.29985	030822	0.12621	0.14901	0.00739	0.00424	BLL
	20	0.15846	0.17211	0.29963	030812	0.12611	0.14822	0.00718	0.00396	BLL
	30	0.15723	0.17200	0.29921	030811	0.12601	0.14802	0.00684	0.00352	BLL
	50	0.15710	0.17166	0.29901	030711	0.12564	0.14722	0.00665	0.00331	BLL
	100	0.15612	0.17122	0.29861	030701	0.12552	0.14692	0.00656	0.00153	BLL

- According to tables (2,3), when β=0.75,1.5 the performance of (MLE) of β and Θ is better than the estimator of Bayes under (SELF) for sample sizes.

- Tables (1,2,3) display that the Bayes estimator with both the Linex loss function and the Lindley approximation is the better estimator when compared to the other estimators and for all sample sizes.

- From both tables (2 and 3), as $\beta = 0.75, 1.5$, the performance of Bayes estimator over quadratic Loss function is better than MLE and the Bayes estimator with (SEL) function of all samples.

- The estimation values of (MLE) and Bayes estimator under different loss functions are very close to each other, as shown in the tables for estimate values of α and β.

- It should be noted that increasing the values of β due to decreases the values of (MSE) in both the different values of Θ and for all sample sizes.

- The study was carried out to determine the best estimator for the parameters of the Lomax distribution. We recommend using the Bayes estimator with Lindley approximation under the Linex loss function. Experiment with new loss functions as well.

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Cusa-Huygens type inequality and a correction for generalized trigonometric functions

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Abstract. In this paper, we mainly give several Cusa-Huygens and Kober type inequalities for generalized trigonometric functions by using classical analysis. Finally, we also show that the proof of Theorem 3.9 given by Huang et.al. in the paper "Some Wilker and Cusa-Huygens type inequality for generalized trigonometric functions" is incorrect. In addition, we present a weak result by using l'Hôspital monotone rule.
Keywords: generalized trigonometric functions, Cusa-Huygens type inequality.

1. Introduction

It is well known from calculus that

$$\arcsin x = \int_0^x \frac{1}{(1-t^2)^{1/2}}dt$$

for $0 \leq x \leq 1$ and

$$\frac{\pi}{2} = \arcsin 1 = \int_0^1 \frac{1}{(1-t^2)^{1/2}}dt.$$

For $1 < p < \infty$ and $0 \leq x \leq 1$, the arcsine may be generalized as

$$\arcsin_p x = \int_0^x \frac{1}{(1-t^p)^{1/p}}dt \tag{1}$$

and

$$\frac{\pi_p}{2} = \arcsin_p 1 = \int_0^1 \frac{1}{(1-t^p)^{1/p}}dt. \tag{2}$$

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The inverse of $\arcsin_p$ on $[0, \frac{\pi_p}{2}]$ is called the generalized sine function, denoted by $\sin_p$ and may be extended to $(-\infty, \infty)$. See [5] and closely related references therein.

For $x \in [0, \frac{\pi_p}{2}]$, the generalized cosine function $\cos_p x$ is defined by

$$\cos_p x = \frac{\mathrm{d} \sin_p x}{\mathrm{d}x}. \tag{3}$$

It is easy to see that

$$\cos_p x = (1 - \sin_p^p x)^{1/p} \tag{4}$$

and

$$\frac{\mathrm{d} \cos_p x}{\mathrm{d}x} = -\cos_p^{2-p} x \sin_p^{p-1} x. \tag{5}$$

The generalized tangent function $\tan_p x$ is defined as

$$\tan_p x = \frac{\sin_p x}{\cos_p x}, \quad x \in \mathbb{R} \backslash \left\{ k\pi_p + \frac{\pi_p}{2} : k \in \mathbb{Z} \right\}. \tag{6}$$

From (1.6), it follows that

$$\frac{\mathrm{d} \tan_p x}{\mathrm{d}x} = 1 + |\tan_p x|^p, \quad x \in \left(-\frac{\pi_p}{2}, \frac{\pi_p}{2} \right). \tag{7}$$

In present, there has been a vivid interest on the generalized trigonometric and hyperbolic functions, numerous papers have been published on the studies of generalized trigonometric functions and their inequalities. Interested readers can refer to the recent review paper [12] or references [1, 2, 3, 6, 11].

The classical Cusa-Huygens inequality for trigonometric functions asserts that

$$\frac{\sin(x)}{x} < \frac{\cos(x) + 2}{3}, x \in (0, \frac{\pi}{2}), \tag{8}$$

while the Kober inequality states that

$$1 - \frac{2x}{\pi} < \cos x < 1 - \frac{x^2}{\pi}, x \in (0, \frac{\pi}{2}). \tag{9}$$

for many references related to both inequalities (8) and (9), see e.g.[9]. Due to Cusa-Huygens inequality, Klén, Vuorinen and Zhang[6] proved the Cusa-Huygens type inequalities for generalized trigonometric and hyperbolic functions in 2012. Later, Nantomah et.al. have done some meaningful work. The reader may refer to references [7, 8]. For the generalization of Kober inequality to generalized trigonometric and hyperbolic functions, the results in this aspect are rare limited to the author's knowledge.

In this note, we mainly give several new Cusa-Huygens type inequalities and Kober type inequalities for generalized trigonometric functions. Finally, we point out a mistake in proof of [4]. In addition, we present a weak result by using l'Hôspital monotone rule.

2. Cusa-Huygens type inequalities

Lemma 2.1 ([6, Theorem 3.22])**.** *For $p \in (1,2]$, the following inequality*

$$\frac{\sin_p x}{x} < \frac{\cos_p x + p}{1+p} \leq \frac{\cos_p x + 2}{3} \tag{10}$$

holds for all $x \in \left(0, \frac{\pi_p}{2}\right)$.

Next, two theorems show the Cusa-Huygens type inequalities for the generalized trigonometric functions.

Theorem 2.1. *For $p \in (1,2]$, the function $\delta(x) = \frac{x - \sin_p x}{x^{p+1}}$ is strictly decreasing from $(0, \frac{\pi_p}{2})$ onto $\left(\frac{2^p(\pi_p - 2)}{\pi_p^{p+1}}, \frac{1}{p(p+1)}\right)$. In particular, for $x \in \left(0, \frac{\pi_p}{2}\right)$, we have*

$$1 - \alpha x^p < \frac{\sin_p x}{x} < 1 - \beta x^p, \tag{11}$$

where the constants $\alpha = \frac{1}{p(p+1)}$ and $\beta = \frac{2^p(\pi_p - 2)}{\pi_p^{p+1}}$ are the best possible.

Proof of Theorem 2.1. Simple computation yields

$$\delta'(x) = \frac{(p+1)\sin_p x - x\cos_p x - px}{x^{p+1}},$$

which is negative by Lemma 2.1. Hence, $\delta(x)$ is strictly decreasing in $x \in \left(0, \frac{\pi_p}{2}\right)$. The limits follows by applying l'Hospital rule.

Lemma 2.2 ([6, Theorem 3.6])**.** *For all $p \in (1, \infty)$ and $x \in \left(0, \frac{\pi_p}{2}\right)$,*

$$\cos_p^{\alpha} x < \frac{\sin_p x}{x} < 1 \tag{12}$$

with the best constant $\alpha = \frac{1}{p+1}$.

Lemma 2.3. *For $p \geqslant 2$, the function $\beta(x) = \frac{x^p}{p+1} + \cos_p^{\frac{p}{p+1}} x$ is strictly decreasing from $(0, \frac{\pi_p}{2})$ onto $\left(\frac{\pi_p^2}{4(p+1)}, 1\right)$. In particular, for $x \in \left(0, \frac{\pi_p}{2}\right)$, we have*

$$\left[\frac{\pi_p^2 - 4x^p}{4(p+1)}\right]^{\frac{p+1}{p}} < \cos_p x < \left(1 - \frac{x^p}{p+1}\right)^{\frac{p+1}{p}}. \tag{13}$$

Proof of Lemma 2.3. By direct calculation, we get

$$\begin{aligned} \beta'(x) &= \frac{p}{p+1}\left(x^{p-1} - \cos_p^{2-p-\frac{1}{p+1}} x \sin_p^{p-1} x\right) \\ &= \frac{px^{p-1}}{p+1}\left[1 - \frac{(\frac{\sin_p x}{x})^{p-1}}{\left(\cos_p^{\frac{p^2-p-1}{p^2-1}} x\right)^{p-1}}\right]. \end{aligned}$$

Considering to $p \geqslant 2$, we easily obtain $\frac{p^2-p-1}{p^2-1} \geqslant \frac{1}{p+1}$. (In fact, this is equivalent to $p(p-2) \geqslant 0$.) Therefore, we have

$$\frac{\sin_p x}{x} > (\cos_p x)^{\frac{1}{p+1}} \geqslant (\cos_p x)^{\frac{p^2-p+1}{p^2-1}},$$

where we apply the function $\cos_p x$ is decreasing on $\big(0, \frac{\pi_p}{2}\big)$ and Lemma 2.2. This implies that $\beta'(x) \leqslant 0$.

So, $\beta(x)$ is decreasing on $\big(0, \frac{\pi_p}{2}\big)$ and the limiting values are trivial. □

By Lemma 2.3, we easily obtain the following corollary.

Corollary 2.1. *For $p \geqslant 2$ and $x \in \big(0, \frac{\pi_p}{2}\big)$, we have*

$$\frac{x^p \cos_p^{\frac{1}{p+1}} x + (p+1)\cos_p x}{p+1} < \cos_p^{\frac{1}{p+1}} x < \frac{4}{\pi_p^2}\Big(\frac{x^p \cos_p^{\frac{1}{p+1}} x + (p+1)\cos_p x}{p+1}\Big).$$

Completely similar to the proof of Lemma 2.3, we can get the following proposition.

Proposition 2.1. *For $1 < p \leqslant 2$, the function $\beta(x)$ is strictly increasing from $\big(0, \frac{\pi_p}{2}\big)$ onto $\big(1, \frac{\pi_p^2}{4(p+1)}\big)$. As a result, we have*

$$\Big(1 - \frac{x^p}{p+1}\Big)^{\frac{p+1}{p}} < \cos_p x < \Big(\frac{\pi_p^2 - 4x^p}{4(p+1)}\Big)^{\frac{p+1}{p}}$$

for $x \in \big(0, \frac{\pi_p}{2}\big)$.

Remark. Lemma 2.3, Corollary 2.1 and Proposition 2.1 show the Kober type inequality for the generalized trigonometric functions.

Theorem 2.2. *For $p > 1$, the function $r(x) = p - 1 + \cos_p^{\frac{p^2}{p+1}} x - \frac{p \sin_p^p x}{x^p}$ is strictly increasing from $(0, \frac{\pi_p}{2})$ onto $\big(0, p-1-\frac{p\cdot 2^p}{\pi_p^p}\big)$. As a result, for $x \in \big(0, \frac{\pi_p}{2}\big)$, we have*

$$\frac{\frac{p\cdot 2^p}{\pi_p^p} + \cos_p^{\frac{p^2}{p+1}} x}{p} < \frac{\sin_p^p x}{x^p} < \frac{p - 1 + \cos_p^{\frac{p^2}{p+1}} x}{p}. \tag{14}$$

Proof of Theorem 2.2. Differiating the function $r(x)$, we get

$$h'(x) = -\frac{p^2 \sin_p^{p-1} x}{x^p}\Big[\frac{x^p \cos_p^{\frac{1}{p+1}} x}{p+1} + \cos_p x - \frac{\sin_p x}{x}\Big].$$

Using Corollary 2.1, we have

$$\frac{x^p \cos_p^{\frac{1}{p+1}} x}{p+1} + \cos_p x < \cos_p^{\frac{1}{p+1}} x.$$

This implies that

$$h'(x) > -\frac{p^2 \sin_p^{p-1} x}{x^p}\Big(\cos_p^{\frac{1}{p+1}} x - \frac{\sin_p x}{x}\Big) > 0,$$

which is strictly increasing for any $p > 1$ by Lemma 2.2. The limiting values follow by easy computation.

Similar to the proof of Theorem 2.2, the following theorem holds true.

Theorem 2.3. *For $2 \leqslant p \leqslant \sqrt{2}+1$, the function $\Delta(x) = p-1+\cos_p^{\frac{2p}{p+1}} x - \frac{2\sin_p^p x}{x^p}$ is strictly increasing from $\big(0, \frac{\pi_p}{2}\big)$ onto $\Big(p-2, p-1-\frac{2^{p+1}}{\pi_p^p}\Big)$. In particular, for $x \in \big(0, \frac{\pi_p}{2}\big)$, we have*

$$\frac{\frac{2^{p+1}}{\pi_p^p} + \cos_p^{\frac{2p}{p+1}} x}{2} < \frac{\sin_p^p x}{x^p} < \frac{1+\cos_p^{\frac{2p}{p+1}} x}{2} \tag{15}$$

or

$$\frac{2\sin_p^p x}{x^p} - 1 < \cos_p^{\frac{2p}{p+1}} x < \frac{2\sin_p^p x}{x^p} - \frac{2^{p+1}}{\pi_p^p}. \tag{16}$$

Proof of Theorem 2.3. By easy calculation, we get

$$\Delta'(x) = \frac{-2p\sin_p^{p-1} x}{x^{p+1}}\Big(\frac{x^{p+1}\cos_p^{\frac{p-1}{p+1}+2-p} x}{(p+1)x} + x\cos_p x - \sin_p x\Big). \tag{17}$$

For $p \geqslant 2$ and $x \in \big(0, \frac{\pi_p}{2}\big)$, Corollary 2.1 states

$$\frac{x^p \cos_p^{\frac{p-1}{p+1}+2-p} x + (p+1)\cos_p x}{p+1} < \cos_p^{\frac{1}{p+1}} x.$$

Therefore, we get

$$\Delta'(x) > -\frac{2p\sin_p^{p-1} x}{x^p}\Big(\cos_p^{\frac{1}{p+1}} x - \frac{\sin_p x}{x}\Big) > 0, \tag{18}$$

where we applied Lemma 2.2. Thus, the function $\Delta(x)$ is strictly increasing on $\big(0, \frac{\pi_p}{2}\big)$ and the limiting values follow.

Finally, we give a conjecture about Kober type inequality.

Conjecture. For $x \in (0, \frac{\pi}{2})$ and $p \geqslant 2$, the function $f(x) = \frac{x^p(2p+1+\cos_p x)}{1-\cos_p x}$ is strictly increasing from $\big(0, \frac{\pi_p}{2}\big)$ onto $\Big(2p(p+1), \frac{(2p+1)\pi_p^p}{2^p}\Big)$? In particular, for $p \geqslant 2$ and $x \in \big(0, \frac{\pi_p}{2}\big)$, we have the following inequalities

$$\frac{\alpha - (2p+1)x^p}{\alpha + x^p} < \cos_p x < \frac{\beta - (2p+1)x^p}{\beta + x^p} \tag{19}$$

or

$$\frac{2p+2}{1+\frac{x^p}{\alpha}} - (2p+1) < \cos_p x < \frac{2p+2}{1+\frac{x^p}{\beta}} - (2p+1), \tag{20}$$

where the constants $\alpha = 2p(p+1)$ and $\beta = \frac{(2p+1)\pi_p^p}{2^p}$ are the best possible?

3. A correction

Lemma 2.1 shows the Cusa-Huygens type inequality for generalized trigonometric functions. Later, Huang et al.[4] extended and sharpened inequality (10). Their result is as stated below:

Theorem 3.1. *For $1 < p \leq 2$ and $x \in (0, \frac{\pi_p}{2})$, we have*

$$\left(\frac{p+\cos_p x}{p+1}\right)^{\alpha} < \frac{\sin_p x}{x} < \left(\frac{p+\cos_p x}{p+1}\right)^{\beta} \tag{21}$$

with the best constants $\alpha = \frac{\ln\left(\frac{2}{\pi_p}\right)}{\ln\left(\frac{p}{p+1}\right)}$ and $\beta = 1$.

In their proof, it obtained that

$$\begin{aligned} &\tfrac{x\cos_p x - \sin_p x}{x\sin_p x} \ln\left(\tfrac{p+\cos_p x}{p+1}\right) + \tfrac{\cos_p x \tan_p^{p-1} x}{p+\cos_p x} \ln\left(\tfrac{\sin_p x}{x}\right) \\ &> \ln\left(\tfrac{\sin_p x}{x}\right)\left(\tfrac{x\cos_p x - \sin_p x}{x\sin_p x} + \tfrac{\cos_p x \tan_p^{p-1} x}{p+\cos_p x}\right). \end{aligned} \tag{22}$$

Next, we explained how intermediate the above inequality (21) is logically incorrect. In fact, using inequality (10), we easily get

$$\ln\left(\frac{\sin_p x}{x}\right) < \ln\left(\frac{p+\cos_p x}{p+1}\right)$$

which gives

$$\frac{x\cos_p x - \sin_p x}{x\sin_p x} \ln\left(\frac{\sin_p x}{x}\right) > \frac{x\cos_p x - \sin_p x}{x\sin_p x} \ln\left(\frac{p+\cos_p x}{p+1}\right)$$

since $x\cos_p x - \sin_p x < 0$. This is a mathematical mistake. So their proof as they claimed can not be considered as a correct proof of inequality (21).

Lemma 3.1 ([6])**.** *For $p > 1$, the function $g_1(x) = \frac{\sin_p x}{x}$ is strictly decreasing from $(0, \frac{\pi_p}{2})$ onto $(\frac{2}{\pi_p}, 1)$.*

Lemma 3.2. *For $p > 1$, the function $g_2(x) = (\cos_p x)^{1-p}$ is strictly increasing on $(0, \frac{\pi_p}{2})$.*

Proof of Lemma 3.2. Direct calculation yields

$$(g_2(x))' = (p-1)(\cos_p x)^{2-2p}(\sin_p x)^{p-1} > 0.$$

This implies the above result. □

Lemma 3.3. *For $1 < p \leq 2$, the function $g_3(x) = \cos_p x + (\cos_p x)^{1-p}$ is strictly increasing on $(x_p^*, \frac{\pi_p}{2})$ where x_p^* is the unique root of the equation $(\cos_p x)^p + (1-p) = 0$.*

Proof of Lemma 3.3. Simple calculation yields

$$(g_3(x))' = -(\cos_p x)^{2-2p}(\sin_p x)^{p-1}[(\cos_p x)^p + (1-p)] > 0.$$

We complete the proof. □

Lemma 3.4. *For $1 < p \leq 2$, the function $g_4(x) = p\cos_p x + (p-2)(\cos_p x)^{1-p}$ is strictly decreasing on $(0, \frac{\pi_p}{2})$.*

Proof of Lemma 3.4. Using Lemma 3.2, we easily obtain the function $g_4(x)$ is strictly decreasing on $(0, \frac{\pi_p}{2})$. □

Lemma 3.5 (l'Hôspital monotone rule[6])**.** *For $-\infty < a < b < \infty$, let $f, g : [a,b] \to \mathbb{R}$ be continuous and differentiable on $[a,b]$. Let $g'(x) \neq 0$ on $[a,b]$. If $f'(x)/g'(x)$ is increasing(decreasing) on $[a,b]$, then so are*

$$\frac{f(x)-f(a)}{g(x)-g(a)} \text{ and } \frac{f(x)-f(b)}{g(x)-g(b)}.$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

Theorem 3.2. *For $1 < p \leq 2$ and $x \in (x_p^*, \frac{\pi_p}{2})$, the inequality (21) holds true with the best constants $\alpha = \frac{\ln\left(\frac{2}{\pi_p}\right)}{\ln\left(\frac{p}{p+1}\right)}$ and $\beta = 1$.*

Proof of Theorem 3.2. Let

$$f(x) = \frac{\ln\left(\frac{\sin_p x}{x}\right)}{\ln\left(\frac{p+\cos_p x}{p+1}\right)} = \frac{f_1(x)}{f_2(x)}$$

where $f_1(x) = \ln\left(\frac{\sin_p x}{x}\right)$ and $f_2(x) = \ln\left(\frac{p+\cos_p x}{p+1}\right)$ with $f_1(0^+) = 0$ and $f_2(0^+) = 0$.

Direct computation results in

$$\frac{f_1'(x)}{f_2'(x)} = \frac{(\sin_p x - x\cos_p x)(p+\cos_p x)}{x(\cos_p x)^{2-p}(\sin_p x)^p} = \frac{f_3(x)}{f_4(x)}$$

with $f_3(0^+) = 0$ and $f_4(0^+) = 0$.

Calculating again, we obtain

$$f_3'(x) = (\cos_p x)^{2-p}(\sin_p x)^{p-1}[px - \sin_p x + 2x\cos_p x]$$

and

$$f_4'(x) = (\cos_p x)^{2-p}(\sin_p x)^{p-1}\left[\sin_p x - (2-p)x(\cos_p x)^{1-p} + px\cos_p x\right].$$

Therefore, we get

$$\begin{aligned}\frac{f_3'(x)}{f_4'(x)} &= \frac{px-\sin_p x+2x\cos_p x}{\sin_p x-(2-p)x\left(\cos_p x\right)^{1-p}+px\cos_p x}\\ &= 1+\frac{p-\frac{2\sin_p x}{x}+(2-p)\left[\cos_p x+\left(\cos_p x\right)^{1-p}\right]}{\frac{2\sin_p x}{x}+p\cos_p x+(p-2)\left(\cos_p x\right)^{1-p}}\\ &= 1+\frac{\alpha_p(x)}{\beta_p(x)}\end{aligned}$$

where

$$\alpha_p(x)=p-\frac{2\sin_p x}{x}+(2-p)\left[\cos_p x+\left(\cos_p x\right)^{1-p}\right]$$

and

$$\beta_p(x)=\frac{2\sin_p x}{x}+p\cos_p x+(p-2)\left(\cos_p x\right)^{1-p}.$$

Applying to Lemma 3.1-3.4, we easily obtain the function $\alpha_p(x)$ is positive and strictly increasing and $\beta_p(x)$ is positive and strictly decreasing on $(x_p^*, \frac{\pi_p}{2})$. So we get the function $\frac{f_3'(x)}{f_4'(x)}$ is strictly increasing on $(x_p^*, \frac{\pi_p}{2})$. Using Lemma 2.5, we know the function $f(x)$ is strictly increasing on $(x_p^*, \frac{\pi_p}{2})$.

Using Hôspital rule, we easily obtain

$$f(\frac{\pi_p}{2})=\frac{\ln\left(\frac{2}{\pi_p}\right)}{\ln\left(\frac{p}{p+1}\right)}.$$

So, the left of inequality (21) is proved. On the other hand, the right of inequality (21) is trivial by applying (10). This completes the proof. □

If $p=2$ in Theorem 3.2, we easily obtain the following corollary.

Corollary 3.1 (Theorem 1.1, [10]). *For $x\in(0,\frac{\pi}{2})$, we have*

$$\left(\frac{2+\cos x}{3}\right)^{\alpha}<\frac{\sin x}{x}<\left(\frac{2+\cos x}{3}\right)^{\beta} \tag{23}$$

with the best constants $\alpha=\frac{\ln\left(\frac{2}{\pi}\right)}{\ln\left(\frac{2}{3}\right)}$ and $\beta=1$.

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Numerical integration by using polynomial $P_5(x)$ of Newton-Gregory forward interpolation formula

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Abstract. In this paper, we will appear that the classical quadrature method for the numerical solution by using polynomial $P_5(x)$ of Newton-Gregory forward interpolation formula in numerical integration.

Keywords: Numerical integration, classical quadrature formula, trapezoidal rule, Simpson's $\frac{1}{3}$ rule, Boole's rule, Weddle's rule.

1. Introduction

Throughout this paper, let $f(x)$ be continuous on the interval $[a, b]$. Then evaluation of the definite integral maybe practically impossible or at least difficult, in this two cases we tried to solve it by numerical integration (NI) is that the study of how the numerical value of an integral are often found. It is also called as quadrature which refers to find a square whose area is the same as the area

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under the curve. The inverse process to differentiation in calculus is represented by:

$$I = \int_a^b f(x)dx, \tag{1.1}$$

which means that the integral of a function $f(x)$ with respect to the independent variable x evaluated between the initial values $x = a$ and $x = b$ (Equation (1.1)). In other word, it is mean evaluate the area under the curve of the function $f(x)$ between the two vertical lines $x = a$ to $x = b$ and the x-axis. We will show that graphically in fig. 1.

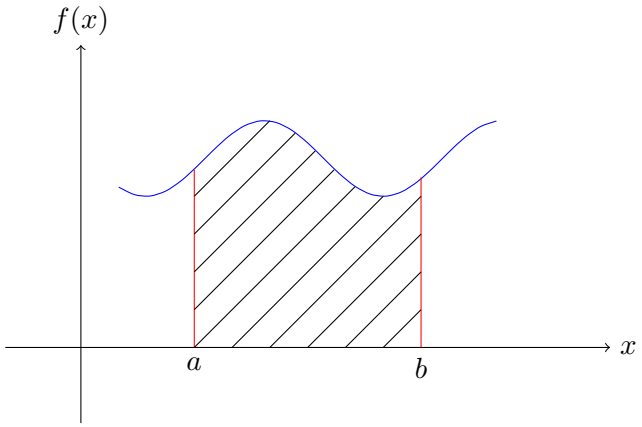


Figure 1: Graphical representation of integral of a function $f(x)$

In the integration, the problem is how can we find the shaded area. Might be the better alternative approach by using simple arithmetic operations to evaluate the area.

The problem of integration is just reduced to the matter of finding shaded area. A better alternative approach might be to use a way that uses simple arithmetic operations to compute the area. This approach is named as NI or numerical quadrature. NI method uses an interpolating polynomial (IP) $P_n(x)$ in the place of $f(x)$ which can be integrated analytically. We have seen several ways of numerical integration which replace the function $f(x)$ by straight line or polynomial $P_n(x)$. In this paper, we interest by the polynomial $P_5(x)$ of order 5. Thus:

$$I = \int_a^b f(x)dx = \int_a^b P_5(x)dx. \tag{1.2}$$

After this brief introduction, we include a section of preliminaries and notation regarding Newton-Gregory forward interpolation formula. We also, review some rules which have been deriving from this formula like Trapezoidal, Simpson's $\frac{1}{3}$, Simpson's $\frac{3}{8}$, Boole and Weddle. In Section 3, we derive the **5A** rule from this formula and we generalize it in Section 4. Finally, in section 5 we discuss and comparing some problems.

2. Preliminaries

In this section, we use the general formula for solving NI, it is also called general quadrature formula.

Choosing interval size $h = \frac{b-a}{n}$, divide the interval $[a, b]$ in to n intervals by means $(n+1)$ equally spaced point $x_0 = a$, $x_n = b$, $x_i = x_0 + ih$, $i = 1, 2, ..., n-1$ and let $y_i = f(x_i)$ for $i = 0, 1, 2, ..., n$, the basic idea as we have seen in numerical integration is to replace the unknown tabulated function $y = f(x)$ by an nth degree polynomial $P_n(x)$ say **Newton-Gregory** forward interpolation formula and carry on the integration. Thus

$$\begin{aligned} I &= \int_{a=x_0}^{b=x_n} f(x)dx \\ &\simeq \int_{x_0}^{x_n} P_n(x)dx \\ &= h\int_{q=0}^{n} [y_0 + q\Delta y_0 + \frac{q(q-1)}{2!}\Delta^2 y_0 + \frac{q(q-1)(q-2)}{3!}\Delta^3 y_0 + ... \\ &+ \frac{q(q-1)(q-2)...(q-n+1)}{n!}\Delta^n y_0]dq. \end{aligned} \tag{2.1}$$

Here, the new variable is $q = \frac{x-x_0}{h}$, so $dq = \frac{dx}{h}$ where $h = \frac{b-a}{n}$.
By changing n to different values various formulae is used to solve NI, they are:

(i) If $n = 1$, then derivation of Trapezoidal rule formula.

(ii) If $n = 2$, then derivation of Simpson's $\frac{1}{3}$rd rule formula.

(iii) If $n = 3$, then derivation of Simpson's $\frac{3}{8}$th rule formula.

(iv) If $n = 4$, then derivation of Boole's rule formula.

(v) If $n = 6$, then derivation of Weddle's rule formula.

In this paper, we mainly focus on to demonstrating of the rule when $n = 5$ in NI.

3. Main results

We start by using a function $y = f(x)$ on the interval $[x_0, x_5]$, and put the equation (1.2) in the equation (2.1) by taking $n = 5$, this means we have six

points and five intervals, so the equation (2.1) becomes:

$$
\begin{aligned}
&\int_{x_0}^{x_5} y dx = h\int_0^5 [y_0 + q\Delta y_0 + \frac{q(q-1)}{2!}\Delta^2 y_0 + \frac{q(q-1)(q-2)}{3!}\Delta^3 y_0 \\
&+ \frac{q(q-1)(q-2)(q-3)}{4!}\Delta^4 y_0 + \frac{q(q-1)(q-2)(q-3)(q-4)}{5!}\Delta^5 y_0]dq \\
&= h\int_0^5 [y_0 + q\Delta y_0 + (\frac{q^2}{2} - \frac{q}{2})\Delta^2 y_0 + (\frac{q^3}{6} - \frac{q^2}{2} + \frac{q}{3})\Delta^3 y_0 \\
&+ (\frac{q^4}{24} - \frac{q^3}{4} + \frac{11q^2}{24} - \frac{q}{4})\Delta^4 y_0 + (\frac{q^5}{120} - \frac{q^4}{12} + \frac{7q^3}{24} - \frac{5q^2}{12} + \frac{q}{5})\Delta^5 y_0]dq \\
&= h[y_0 q + \frac{q^2}{2}\Delta y_0 + (\frac{q^3}{6} - \frac{q^2}{4})\Delta^2 y_0 + (\frac{q^4}{24} - \frac{q^3}{6} + \frac{q^2}{6})\Delta^3 y_0 \\
&+ (\frac{q^5}{120} - \frac{q^4}{16} + \frac{11q^3}{72} - \frac{q^2}{8})\Delta^4 y_0 + (\frac{q^6}{720} - \frac{q^5}{60} + \frac{7q^4}{96} - \frac{5q^3}{36} + \frac{q^2}{10})\Delta^5 y_0]_0^5 \qquad (3.1)\\
&= h[5y_0 + \frac{25}{2}\Delta y_0 + \frac{175}{12}\Delta^2 y_0 + \frac{75}{8}\Delta^3 y_0 + \frac{425}{144}\Delta^4 y_0 + \frac{95}{288}\Delta^5 y_0] \\
&= \frac{h}{288}[1440y_0 + 3600(y_1 - y_0) \\
&+ 4200(y_2 - 2y_1 + y_0) + 2700(y_3 - 3y_2 + 3y_1 - y_0) \\
&+ 850(y_4 - 4y_3 + 6y_2 - 4y_1 + y_0) \\
&+ 95(y_5 - 5y_4 + 10y_3 - 10y_2 + 5y_1 - y_0)] \\
&= \frac{h}{288}[95y_0 + 375y_1 + 250y_2 + 250y_3 + 375y_4 + 95y_5] \\
&= \frac{5h}{288}[19y_0 + 75y_1 + 50y_2 + 50y_3 + 75y_4 + 19y_5].
\end{aligned}
$$

Put $A = \frac{1}{288}$ in above equation. That is

$$\int_{x_0}^{x_5} y dx = 5Ah[19y_0 + 75y_1 + 50y_2 + 50y_3 + 75y_4 + 19y_5]. \qquad (3.2)$$

So, we name equation (3.2) as $5A$ rule formula.

4. Composite formula

Now, we generalize this result by using the composite integration methods. We divide the interval $[a, b]$ into a number of sub-intervals and evaluate the integral in each sub-interval by the $5A$ rule formula.

We can derive composite formulae from average rule. If the range of integration is from a to $a + nh = b$, then this rule can be improved by dividing the interval $[a, b]$ into sub-intervals of width $5h$ and apply 3.2 for each of the sub-interval. The sum of areas of all sub-intervals is the integral of the interval

$[a, b]$. Thus, equation 3.2 becomes:

$$\begin{aligned}
&\int_{x_0}^{x_n} ydx = \int_{x_0}^{x_5} ydx + \int_{x_5}^{x_{10}} ydx + ... + \int_{x_{n-5}}^{x_n} ydx \\
&= 5Ah[19y_0 + 75y_1 + 50y_2 + 50y_3 + 75y_4 + 19y_5] \\
&+ 5Ah[19y_5 + 75y_6 + 50y_7 + 50y_8 + 75y_9 + 19y_{10}] + ... \\
&+ 5Ah[19y_{n-5} + 75y_{n-4} + 50y_{n-3} + 50y_{n-2} + 75y_{n-1} + 19y_n] \\
&= 5Ah[19(y_0 + y_n) + 38(y_5 + y_{10} + ... + y_{n-5}) \\
&+ 75(y_1 + y_4 + y_6 + y_9 + ... + y_{n-4} + y_{n-1}) \\
&+ 50(y_2 + y_3 + y_7 + y_8 + ... + y_{n-3} + y_{n-2})].
\end{aligned} \tag{4.1}$$

5. Problems

Problem 5.1. In this problem, we will find the approximate value of $\int_1^{11} \frac{dx}{x}$, using Table 1. Also, we find the exact solution and find the solution by using

Table 1: value of x and $y = f(x) = \frac{1}{x}$

x	1	2	3	4	5	6	7	8	9	10	11
y	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{1}{8}$	$\frac{1}{9}$	$\frac{1}{10}$	$\frac{1}{11}$

other method.

Finally, we comparing $5A$ rule with other method by finding the absolute errors.

Solution. We have $a = 1$, $b = 11$ and 10 intervals, then $h = \frac{b-a}{n} = \frac{11-1}{10} = 1$. Thus, we use (4.1) to find the Solution:

$$\begin{aligned}
&\int_{x_0=1}^{x_{10}=11} \frac{dx}{x} = 5Ah[19(y_0 + y_n) + 38(y_5 + y_{10} + ... + y_{n-5}) \\
&+ 75(y_1 + y_4 + y_6 + y_9 + ... + y_{n-4} + y_{n-1}) \\
&+ 50(y_2 + y_3 + y_7 + y_8 + ... + y_{n-3} + y_{n-2})] \\
&= \frac{5}{288}[19(y_0 + y_{10}) + 38(y_5) + 75(y_1 + y_4 + y_6 + y_9) \\
&+ 50(y_2 + y_3 + y_7 + y_8)] \\
&= \frac{5}{288}[19(1 + \frac{1}{11}) + 38(\frac{1}{6}) + 75(\frac{1}{2} + \frac{1}{5} + \frac{1}{7} + \frac{1}{10}) \\
&+ 50(\frac{1}{3} + \frac{1}{4} + \frac{1}{8} + \frac{1}{9})] = \frac{5}{288}(138.747114) \\
&= 2.408804.
\end{aligned} \tag{5.1}$$

Now, we find the exact solution

$$\int_1^{11} \frac{dx}{x} = ln(x)|_1^{11} = ln(11) - ln(1) = 2.397895. \tag{5.2}$$

So, the absolute error of the above solution is: $e = |2.408804 - 2.397895| = 0.010909$. The solution by using Trapezoidal rule is:

$$\begin{aligned}\int_{x_0=1}^{x_{10}=11} \frac{dx}{x} &= \frac{h}{2}[(y_0+y_n)+2(y_1+y_2+...+y_{n-1})] \\ &= \frac{1}{2}[1+\frac{1}{11}+2(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10})] \\ &= 2.474423.\end{aligned} \tag{5.3}$$

Here, the absolute error of Trapezoidal rule is: $e = |2.474423 - 2.397895| = 0.076528$. We observe that the solution which is in (5.1) gives more accuracy than the solution which is in (5.3). Hence the $5A$ rule is the better.

Problem 5.2. Find the approximate value of $\int_1^3 f(x)dx$, using Table2 below.

Table 2: value of x and $y = f(x)$

x	1	1.2	1.4	1.6	1.8	2
y	2.287355	3.094479	3.996196	4.950921	5.891435	6.718850
x	2.2	2.4	2.6	2.8	3	
y	7.296691	7.445750	6.940575	5.508762	2.834471	

Solution. Since the number of intervals is 10, we can not use Simpson$\frac{1}{3}$ rule, Simpson$\frac{3}{8}$ rule, Boole's rule and Weddle's rule. We have $a = 1$, $b = 3$ and 10 intervals, then $h = \frac{b-a}{n} = \frac{3-1}{10} = \frac{1}{5}$. Thus we use $5A$ rule to find the Solution:

$$\begin{aligned}&\int_{x_0=1}^{x_{10}=3} f(x)dx = \frac{5h}{288}[19(y_0+y_n)+38(y_5+y_{10}+...+y_{n-5}) \\ &+75(y_1+y_4+y_6+y_9+...+y_{n-4}+y_{n-1}) \\ &+50(y_2+y_3+y_7+y_8+...+y_{n-3}+y_{n-2})] \\ &= \frac{5\times\frac{1}{5}}{288}[19(y_0+y_{10})+38(y_5) \\ &+75(y_1+y_4+y_6+y_9)+50(y_2+y_3+y_7+y_8)] \\ &= \frac{1}{288}[19(2.287355+2.834471)+38(6.718850) \\ &+75(3.094479+5.891435+7.296691+5.508762) \\ &+50(3.996196+4.950921+7.445750+6.940575)] \\ &= 10.950193.\end{aligned} \tag{5.4}$$

Problem 5.3. In this problem we will find the approximate value of $\int_0^2 e^{x^2}dx$, using $5A$ rule two times. First one with $h = 0.2$ and the second with $h = 0.1$. Furthermore, we comparing the result with other methods which are already calculated from state-of-the-art studies like [7].

Solution. (i) Using $5A$ rule with $h = 0.2$, the estimate is:

$$\begin{aligned}
&\int_0^2 e^{x^2}dx = \frac{5h}{288}[19(y_0+y_n)+38(y_5+y_{10}+...+y_{n-5})\\
&+75(y_1+y_4+y_6+y_9+...+y_{n-4}+y_{n-1})\\
&+50(y_2+y_3+y_7+y_8+...+y_{n-3}+y_{n-2})]\\
&=\frac{5\times\frac{2}{10}}{288}[19(y_0+y_{10})+38(y_5)\\
&+75(y_1+y_4+y_6+y_9)+50(y_2+y_3+y_7+y_8)]\\
&=\frac{1}{288}[19(1+54.598150)+38(2.718282)\\
&+75(1.040811+1.896481+4.220696+25.533722)\\
&+50(1.173511+1.433329+7.099327+12.935817)]=16.470962.
\end{aligned} \tag{5.5}$$

(ii) Using $5A$ rule with $h = 0.1$, the estimate is:

$$\begin{aligned}
&\int_0^2 e^{x^2}dx = \frac{5h}{288}[19(y_0+y_n)+38(y_5+y_{10}+...+y_{n-5})\\
&+75(y_1+y_4+y_6+y_9+...+y_{n-4}+y_{n-1})\\
&+50(y_2+y_3+y_7+y_8+...+y_{n-3}+y_{n-2})]\\
&=\frac{5\times\frac{1}{10}}{288}[19(y_0+y_{20})+38(y_5+y_{10}+y_{15})+75(y_1+y_4+y_6+y_9+y_{11}\\
&+y_{14}+y_{16}+y_{19})+50(y_2+y_3+y_7+y_8+y_{12}+y_{13}+y_{17}+y_{18})]\\
&=\frac{1}{576}[19(1+54.598150)+38(1.284025+2.718282+9.487736)\\
&+75(1.01005+1.173511+1.433329+2.247908+3.353485+7.099327\\
&+12.935817+36.966053)+50(1.040811+1.094174+1.632316\\
&+1.896481+4.220696+5.419481+17.993310+25.533722)]\\
&=16.453120.
\end{aligned} \tag{5.6}$$

(iii) We have already the Exact Integration of $\int_0^2 e^{x^2}dx$ which is equal to 16.45263. In addition, we have been seen some methods like Trapezoidal rule (TR), Simpson's $\frac{1}{3}$ rule ($SR_{1/3}$), Simpson's $\frac{3}{8}$ rule ($SR_{3/8}$), Boole's rule (BR), Weddle's rule (WR) and proposed NI method of $\int_0^2 e^{x^2}dx$ (for reference see, [7]), which are given below:

(1) The estimate of the given integral (I) by using TR is equal to 16.95311.

(2) The estimate of the given I by using $SR_{1/3}$ is equal to 16.47144.

(3) The estimate of the given I by using $SR_{3/8}$ is equal to 16.49168.

(4) The estimate of the given I by using BR is equal to 16.45674.

(5) The estimate of the given I by using WR is equal to 16.45524.

(6) The estimate of the given I by using the proposed method when $k = 20$ is equal to 16.453890.

Table 3: Evaluating $\int_0^2 e^{x^2} dx$ using different numerical integration methods, n is the arrangement of methods that has small error.

Integration method	Area	Absolute error	n
Exact (Numerical integration)	16.45263	0	0
Trapezoidal rule	16.95311	0.50048	8
Simpson's $\frac{1}{3}$ rule	16.47144	0.01881	6
Simpson's $\frac{3}{8}$ rule	16.49168	0.03905	7
Boole's rule	16.45674	0.00411	4
Weddle's rule	16.45524	0.00261	3
Proposed method when $k = 20$	16.45389	0.00126	2
proposed method when $h = 0.2$	16.470962	0.018332	5
proposed method when $h = 0.1$	16.45312	0.00049	1

6. Conclusion

Comparing our proposed method ($5A$ rule) to the various numerical integration formulas from Table3 using the absolute errors, when $h = 0.2$, the estimation of the area under the curve are better than the $SR_{1/3}$, $SR_{3/8}$ and TR. Farther, when $h = 0.1$, the estimates of the area under the curve are better than the $SR_{1/3}$, $SR_{3/8}$, TR, BR, WR and the method which proposed when $k = 20$.

Hence, $5A$ rule with $h = 0.1$ gives a better estimate with lesser errors as compared to the $SR_{1/3}$, $SR_{3/8}$, TR, BR, WR, and the method which proposed when $k = 20$.

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Every planar graph without 4-cycles and 6-cycles is (2,9)-colorable

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Abstract. Let $d_1, d_2, \ldots, d_k$ be k nonnegative integers. A graph G is $(d_1, d_2, \ldots, d_k)$-colorable if the vertex set of G can be partitioned into k sets $V_1, V_2, \ldots, V_k$ such that the subgraph $G[V_i]$ has maximum degree at most d_i for $i = 1, \ldots, k$. Sittitrai and Nakprasit proved that every planar graph without 4-cycles and 5-cycles is (2,9)-colorable [Discrete Math., 341 (2018) 2142-2150]. In this paper, we prove that every planar graph without 4-cycles and 6-cycles is (2,9)-colorable.

Keywords: defective coloring, planar graph, cycle.

1. Introduction

Let $d_1, d_2, \ldots, d_k$ be k nonnegative integers. A graph G is $(d_1, d_2, \ldots, d_k)$-colorable if the vertex set can be partitioned into k sets $V_1, V_2, \ldots, V_k$ such that the subgraph $G[V_i]$ has maximum degree at most d_i for $i = 1, \ldots, k$. Any such coloring is called a $(d_1, d_2, \ldots, d_k)$-coloring of G. Thus, Grötzsch's Theorem [12] can be stated as every planar graph without 3-cycles is (0,0,0)-colorable. Improper colorability of graphs has been extensively studied. Cowen, Cowen, and Woodall [6] proved that every planar graph is (2,2,2)-colorable. Eaton and

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Hull [10] showed that this is sharp by exhibiting non-$(1, k, k)$-colorable planar graphs for each k.

Recently, many authors are devoted to investigating (d_1, d_2)-colorability of planar graphs. For planar graphs without one short cycle, Montassier and Ochem [15] constructed planar graphs without 3-cycles are not (d_1, d_2)-colorable for given any d_1 and d_2. Choi, Liu and Oum [9] proved that every planar graph without 4-cycles is (5, 5)-colorable and posed the following questions.

Problem 1.1. *What is the minimum d such that every planar graph without 4-cycles is (d, d)-colorable?*

Problem 1.2. *What is the minimum d such that every planar graph without 3-, 4-, 6-cycles is $(0, d)$-colorable?*

For planar graphs without two short cycles, every planar graph without 3-cycles and 4-cycles is $(2, 6)$-, $(4, 4)$-colorable proved by Havet and Sereni [11], $(3, 5)$-colorable proved by Choi and Raspaud [8], and $(1, 10)$-colorable proved by Choi, Choi, Jeong and Suh [7]. Sittitrai and Nakprasit [16] proved that every planar graph without 4-cycles and 5-cycles is $(4, 4)-, (3, 5)-, (2, 9)$-colorable. As Problems 1.1 and 1.2, one naturally asks the following question.

Problem 1.3. *For a pair of integers (i, j) where $3 \leq i < j$, what is the minimum d_1 for a given d_2 such that every planar graph without i-cycles and j-cycles is (d_1, d_2)-colorable?*

For planar graphs without more than two short cycles, more results can be found in [2, 3, 4, 5, 13, 14, 15, 17, 18, 20]. On the other hand, Xu and Wang [19] posed the following conjecture.

Conjecture 1.4. *Every planar graph without 4-cycles and 6-cycles is $(0, 0, 0)$-colorable.*

Motivated by Problem 1.3 and Conjecture 1.4, we present the following result.

Theorem 1.5. *Every planar graph without 4-cycles and 6-cycles is (2,9)-colorable.*

At the end of this section, we give some notation and terminology. For a plane graph G, we use $V(G)$, $E(G)$ and $F(G)$ to denote its vertex set, edge set and face set, respectively. A vertex v is called a *k-vertex* (*k^+-vertex* or *k^--vertex*) if $d(v) = k$ ($d(v) \geq k$ or $d(v) \leq k$). A similar notation will be used for faces. For a face $f \in F(G)$, we use $b(f)$ to denote the boundary walk of f and write $f = [u_1 u_2 \dots u_m]$ if $u_1, u_2, \dots, u_m$ are the boundary vertices of f in a cyclic order. A k-face $f = [u_1 u_2 \dots u_k]$ is called an $(a_1, a_2, \dots, a_k)$-face if $d(u_i) = a_i$ for $i = 1, 2, \dots, k$. A 2-vertex is *bad* if it is incident with a 3-face and good otherwise. A good 2-vertex v is called a *type-I* good 2-vertex if v is incident to

a 5-face and *type-II* good 2-vertex otherwise. A 3-face is *bad* if it is incident to a bad 2-vertex and good otherwise. A *pendent* 3-face of a vertex v is a 3-face which does not contain v but contains a 3-vertex u adjacent to v, and v is called a *pendent neighbor* of u. A 5_2*-face* is a 5-face incident with two 2-vertices. A vertex h is the *head* of a 5_2-face $f = [hv_1v_2v_3v_4]$ if $d(v_1) = d(v_4) = 2$. If a vertex v is incident to two 5-faces g_1 and g_2 such that g_1 and g_2 have a common edge vu, then we say v is incident with a *double 5-hole*. If v is incident with k 5-faces $f_1, f_2, \ldots, f_k$ such that f_j and f_{j+1} is a double 5-hole of v for $j = 1, \ldots k-1$, then we say v is incident with consecutive k 5-holes. In particular, let f be a 5-face incident with v, we say v is incident with a lonely 5-hole f if v is not incident with any 5-face g such that f and g have a common edge incident with v, where $f \neq g$. A 4-vertex v is *semi-poor* if v is incident with a consecutive three 5-holes and two type-I good 2-vertices. A 4-vertex v is *poor* if v is incident with a double 5-hole with $f_1 = [vv_1v_2v_3v_4], f_2 = [vv_4v_5v_6v_7]$ such that $d(v_1) = d(v_7) = 2$, and v is incident with a type-II good 2-vertex or a pendent 3-neighbor. A 4-vertex is *rich* if it is neither poor nor semi-poor.

We organize this paper as follows. In Section 2, we show the reducible configurations useful in our proof. In Section 3, we complete the proof of Theorem 1.5 by discharging process.

2. Reducible configurations

Suppose that Theorem 1.5 is not true. Let G be a counterexample with $|V(G)|+|E(G)|$ minimized, then G is connected. A face is called *simple* if its boundary forms a cycle. Otherwise, f is called *non-simple*. Since there are neither 4-cycles nor 6-cycles in G, we have the following lemma.

Lemma 2.1 ([1])**.** (1) *There are no adjacent 3-faces.*
(2) *There are no 4-faces and no simple 6-faces.*
(3) *There are no non-simple 5-faces.*
(4) *Any 5-face is not adjacent to 3-faces.*
(5) *If f is a non-simple 6-face, then its boundary consists of two edge-disjoint triangles and the two triangles have a common vertex.*
(6) *There are no non-simple 7-faces.*

Lemma 2.2. (1) *(Lemmas 1 and 2, [8]) If v is a 3^--vertex, then v is adjacent to at least two 4^+-vertices, one of which is a 11^+-vertex.*

(2) *(Lemma 3, [8]) If v is a 12^--vertex, then v is adjacent to at least one 4^+-vertex.*

Lemma 2.3. (1) *If a 2-vertex v is incident with a bad 3-face f_1, then the other face f_2 incident with v is a 8^+-face.*

(2) *If a 2-vertex v is incident with a 5-face, then the other face incident with v is a 7^+-face.*

Proof. (1) Let $f_1 = [uvw]$ be the bad 3-face. Suppose to the contrary that f_2 is a 7^--face. By Lemma 2.2(1), no two 2-vertices is adjacent, then there are no non-simple 6-faces by Lemma 2.1(5). By Lemma 2.1, we only need to consider the cases that f_2 is a simple 7-face. Let f_2 be a simple 7-face and $f_2 = [v_1v_2v_3v_4uvw]$, then $v_1v_2v_3v_4uw$ is a 6-cycle in G, a contradiction.

(2) It follows from that there are neither 4-cycles nor 6-cycles in G. □

Lemma 2.4. *If f is a k-face with $k \geq 8$, then f is incident to at most $k-7$ bad 2-vertices.*

Proof. Similar to the proof of Lemma 2.3, a cycle $C_{m-1}(m \geq 4)$ can be obtained from C_m incident to a bad 2-vertex. It follows that we can obtain C_6 from a k-face incident to $k-6$ bad 2-vertices. This completes the proof. □

Lemma 2.5. *Let $f = [v_1vv_2]$ be a bad 3-face with $d(v) = 2$. Then one of the following statements is true.*
(1) One of v_1, v_2 is a 5^+-vertex, the other is a 11^+vertex; or
(2) One of v_1, v_2 is a 4-vertex, the other is a 12^+-vertex.

Proof. By the minimality of G, $G-v$ admits a (2,9)-coloring φ. If $\varphi(v_1) = \varphi(v_2)$, then we can color v with the color $\{1,2\} \setminus \varphi(v_1)$, a contradiction. By symmetry, we assume that $\varphi(v_1) = 1$ and $\varphi(v_2) = 2$. If v_1 has at most one neighbor colored 1, then we can color v with 1 and obtain a $(2,9)$-coloring of G, a contradiction. Thus, v_1 has two neighbors colored 1. If v_2 has at most eight neighbors colored 2, then we can color v with 2 and obtain a $(2,9)$-coloring of G, a contradiction. Thus, v_2 has nine neighbors colored 2. Hence, $d(v_1) \geq 4$ and $d(v_2) \geq 11$. If $d(v_1) = 4$ and $d(v_2) = 11$, we can recolor v_1 with 2, recolor v_2 with 1 and color v with 1 or 2, then we obtain a $(2,9)$-coloring of G, a contradiction. This implies that $d(v_1) \geq 5$ or $d(v_2) \geq 12$. The lemma is proved. □

3. Proof of Theorem 1.5

To prove Theorem 1.5, we are going to derive a contradiction by a discharging procedure.

We assume that G is a counterexample to Theorem 1.5 with $|V(G)|+|E(G)|$ minimized. The initial charge function ch is defined as: $ch(v) = 2d(v) - 6$ for $v \in V(G)$ and $ch(f) = d(f) - 6$ for $f \in F(G)$. By the Euler's formula $|V(G)| - |E(G)| + |F(G)| = 2$ and the Handshaking lemma,

$$\sum_{v \in V(G)} (2d(v) - 6) + \sum_{f \in F(G)} (d(f) - 6) = -12.$$

We then define suitable discharging rules and redistribute charge accordingly. Note that any discharging procedure preserves the total charge sum of G. If the

final charge function on G such that $ch'(x) \geq 0$ for all $x \in V(G) \cup F(G)$, then we get an obvious contradiction.

$$0 \leq \sum_{x \in V(G) \cup F(G)} ch'(x) = \sum_{x \in V(G) \cup F(G)} ch(x) = -12.$$

Our discharging rules are defined as follows:

(R1) Every 4-vertex sends 1 to each incident 3-face; $\frac{1}{2}$ to each adjacent good 2-vertex; $\frac{1}{2}$ to each pendent 3-face.

(R2) Every k-vertex for $5 \leq k \leq 10$ sends $\frac{3}{2}$ to each incident 3-face; $\frac{1}{2}$ to each adjacent good 2-vertex; $\frac{1}{2}$ to each pendent 3-face.

(R3) Every 11-vertex sends $\frac{5}{2}$ to each incident 3-face; $\frac{3}{2}$ to each adjacent good 2-vertex; $\frac{3}{2}$ to each pendent 3-face.

(R4) Every 12^+-vertex sends 3 to each incident 3-face; $\frac{3}{2}$ to each adjacent good 2-vertex; $\frac{3}{2}$ to each pendent 3-face.

(R5) Every rich 4-vertex v sends $\frac{1}{2}$ to each incident 5-face f such that v has at most one 2-neighbor on $b(f)$; Every poor 4-vertex sends $\frac{1}{4}$ to each incident 5-face; Every semi-poor 4-vertex sends $\frac{1}{3}$ to each incident 5-face.

(R6) Every 5^+-vertex v sends $\frac{1}{2}$ to each incident 5-face f such that v has at most one 2-neighbor on $b(f)$

(R7) Every 8^+-face sends 1 to each incident bad 2-vertex. Every 7^+-face adjacent to a 5-face such that these two faces are both incident with a good 2-vertex sends $\frac{1}{3}$ to the 5-face.

(R8) Every bad 3-face sends 1 to each incident 2-vertex.

We shall show that each $x \in F(G) \cup V(G)$ has final charge $ch'(x) \geq 0$. First consider the vertices. Let v be a k-vertex and let t be the number of 3-faces incident with v. Let v be incident with k_1 lonely 5-holes such that v has at most one 2-neighbor on every 5-face, k_2 incident 5_2-faces such that v is the head of every 5_2-face. Note that each 5_2-face has exactly one head by Lemma 2.2(1).

(1) $k = 2$. Initially $ch(v) = -2$. By Lemma 2.2(1), v is adjacent to two 4^+-vertices, one of which is a 11^+-vertex. If v is a good 2-vertex, then $ch'(v) \geq ch(v) + \frac{1}{2} + \frac{3}{2} = 0$ by (R1)-(R4). If v is a bad 2-vertex, then v is incident to an 8^+-face by Lemma 2.3(1). Hence $ch'(v) \geq ch(v) + 1 + 1 = 0$ by (R7) and (R8).

(2) $k = 3$. Since each 3-vertex is not involved in redistributing discharge procedure, $ch'(v) = ch(v) = 0$.

(3) $k = 4$. Assume that v is incident with consecutive four 5-holes, then $ch'(v) \geq ch(v) - 4 \times \frac{1}{2} = 0$ by (R5). Assume that v is incident with consecutive three 5-holes. If v is a rich 4-vertex, then $ch'(v) \geq ch(v) - \frac{1}{2} - 3 \times \frac{1}{2} = 0$ by (R1) and (R5). If v is a semi-poor 4-vertex, then $ch'(v) \geq ch(v) - 2 \times \frac{1}{2} - 3 \times \frac{1}{3} = 0$ by (R1) and (R5). Assume that v is incident with a double 5-hole. If v is a rich 4-vertex, then $ch'(v) \geq ch(v) - 2 \times \frac{1}{2} - 2 \times \frac{1}{2} = 0$. If v is a poor 4-vertex, then $ch'(v) \geq ch(v) - \frac{1}{2} - 2 \times \frac{1}{2} - 2 \times \frac{1}{4} = 0$. Thus v is not incident with the consecutive 5-holes. By (R1) and (R5), $ch'(v) \geq ch(v) - [t + (\frac{1}{2} + \frac{1}{2})k_1 + 2 \times \frac{1}{2}k_2 + \frac{1}{2}(4 - 2t - 2k_1 - 2k_2)] = 0$.

(4) $5 \leq k \leq 10$. If v is incident with consecutive k 5-holes, then $ch'(v) \geq ch(v) - k \times \frac{1}{2} = \frac{3}{2}k - 6 > 0$ by (R6). If v is incident with the consecutive i 5-holes with $2 \leq i \leq k-1$, then v sends at most $\frac{1}{2}i + 2 \times \frac{1}{2}$ to the incident consecutive i 5-holes together with the 2-neighbors on the consecutive i 5-holes. We can think that v sends charge equality $\frac{\frac{i}{2}+2\times\frac{1}{2}}{i+1}$ to every neighbor incident to the consecutive i 5-holes. Define a real value function $f(i) = \frac{\frac{i}{2}+2\times\frac{1}{2}}{i+1}$ where $2 \leq i \leq k-1$. It is easy to check that $f_{max}(i) = f(2)$. Therefore, in order to obtain the maximal charge v sends, we only consider the case that v is incident with the double 5-hole about the consecutive 5-holes. Let v be incident with q_2 double 5-holes. Since $2t + 2q_2 \leq d(v) = k$, hence, by (R2) and (R6),

$$\begin{aligned} ch'(v) &\geq ch(v) - [\frac{3}{2}t + (\frac{1}{2} + \frac{1}{2})k_1 + 2 \times \frac{1}{2}k_2 + (2 \times \frac{1}{2} + 2 \times \frac{1}{2})q_2 \\ &\quad + \frac{1}{2}(k - 2t - 2k_1 - 2k_2 - 3q_2)] \\ &= 2k - 6 - (\frac{1}{2}t + \frac{1}{2}q_2 + \frac{1}{2}k) \\ &= \frac{3}{2}k - 6 - \frac{1}{2}(t + q_2) \\ &\geq \frac{3}{2}k - \frac{1}{2} \times \frac{k}{2} - 6 \\ &= \frac{5}{4}k - 6 > 0. \end{aligned}$$

(5) $k = 11$. If v is incident with consecutive k 5-holes, then $ch'(v) \geq ch(v) - k \times \frac{1}{2} = \frac{3}{2}k - 6 > 0$ by (R6). If v is incident with the consecutive i 5-holes with $2 \leq i \leq k-1$, then v sends at most $\frac{1}{2}i + 2 \times \frac{3}{2}$ to the incident consecutive i 5-holes together with the 2-neighbors on the consecutive i 5-holes. We can think that v sends charge equality $\frac{\frac{i}{2}+2\times\frac{3}{2}}{i+1}$ to every neighbor incident to the consecutive i 5-holes. Define a real value function $f(i) = \frac{\frac{i}{2}+2\times\frac{3}{2}}{i+1}$ where $2 \leq i \leq k-1$. It is easy to check that $f_{max}(i) = f(2)$. Therefore, in order to obtain the maximal charge v sends, we only consider the case that v is incident with the double 5-hole about the consecutive 5-holes. Let v be

incident with q_2 double 5-holes. If $t \geq 1$ or $k_1 \geq 1$ or $q_2 \geq 1$, then by (R3) and (R6),

$$\begin{aligned} ch'(v) &\geq ch(v) - [\frac{5}{2}t + (\frac{1}{2}+\frac{3}{2})k_1 + 2\times\frac{3}{2}k_2 + (2\times\frac{3}{2}+2\times\frac{1}{2})q_2 \\ &\quad +\frac{3}{2}(11-2t-2k_1-2k_2-3q_2)] \\ &= \frac{t}{2}+k_1+\frac{1}{2}q_2-\frac{1}{2}\geq 0. \end{aligned}$$

If $t = k_1 = q_2 = 0$, then v is adjacent to at least one 4^+-vertex by Lemma 2.2(2). Thus, v is adjacent to at most ten 3^--vertices. Hence, $ch'(v) \geq ch(v) - (2\times\frac{3}{2})k_2 - \frac{3}{2}\times(10-2k_2) = 1 > 0$.

(6) $k \geq 12$. Let v be adjacent to s type-II good 2-vertices, p pendent 3-faces, and let v be incident with q_i precisely consecutive i 5-holes, for $2 \leq i \leq k-1$. If v is incident with k 5-faces, then $ch'(v) \geq ch(v) - k\times\frac{1}{2} = \frac{3}{2}k - 6 > 0$ by (R6). Thus, assume that v is incident with at most $(k-1)$ 5-faces. In this case, $2(k_1+k_2+t)+p+s+\sum_{i=2}^{k-1}(i+1)q_i \leq d(v)$. Since $i \geq 2$, $\frac{3}{2}(i+1) \geq \frac{i}{2}+3$. Hence,

$$\begin{aligned} ch'(v) &\geq ch(v) - [(\frac{3}{2}+\frac{1}{2})k_1 + (2\times\frac{3}{2})k_2 \\ &\quad + \sum_{i=2}^{k-1}(i\times\frac{1}{2}+2\times\frac{3}{2})q_i + \frac{3}{2}s + 3t + \frac{3}{2}p] \\ &\geq 2d(v) - 6 - \frac{3}{2}[2k_1+2k_2+2t+p+s+\sum_{i=2}^{k-1}(i+1)q_i] \\ &\geq 2d(v) - 6 - \frac{3}{2}d(v) = \frac{1}{2}d(v) - 6 \geq 0. \end{aligned}$$

We next analyze the final charge for $f \in F(G)$.

(1) $d(f) = 6$. Since each 6-face is not involved in redistributing discharge procedure, $ch'(f) = ch(f) = 0$.

(2) $d(f) = 7$. By Lemma 2.2(1), f is incident with at most three 2-vertices. Thus, $ch'(f) \geq ch(f) - 3\times\frac{1}{3} = 0$ by (R7).

(3) $d(f) = k = 8^+$. By Lemma 2.4, every k-face with $k \geq 8$ is incident with at most $k-7$ bad 2-vertices. By Lemma 2.2(1), f is incident with at most $\lfloor\frac{k}{2}\rfloor$ 2-vertices. Let f be incident with n_1 bad 2-vertices. Then $n_1 \leq k-7$ and f is incident with at most $\lfloor\frac{k}{2}\rfloor - n_1$ good 2-vertices. Thus, $ch'(f) \geq ch(f) - n_1 - \frac{1}{3}\times(\lfloor\frac{k}{2}\rfloor - n_1) \geq \frac{5}{6}k - 6 - \frac{2}{3}n_1 \geq \frac{5}{6}k - 6 - \frac{2}{3}\times(k-7) = \frac{k-8}{6} \geq 0$ by (R7).

(4) $d(f) = 3$. If f is a bad 3-face, then f is a $(2, 5^+, 11^+)$-face or f is a $(2, 4, 12^+)$-face by Lemma 2.5. In the former case, $ch'(f) \geq ch(f) + \frac{3}{2} + \frac{5}{2} - 1 = 0$ by (R2)-(R4) and (R8). In the latter case, $ch'(f) \geq ch(f) + 1 + 3 - 1 = 0$ by (R1),(R4) and (R8).

We now consider that f is a good 3-face. If f is incident to a 3-vertex, then f is incident to a 4^+-vertex by Lemma 2.2(1). Thus f is a $(3, 3, k)-$, $(3, 3, 11^+)-$, $(3, 4^+, k)-$, $(3, 4^+, 11^+)-$ or a $(4^+, 4^+, 4^+)-$face where $4 \leq k \leq 10$. Let $f = [v_1v_2v_3]$ with $d(v_1) \leq d(v_2) \leq d(v_3)$. If f is a $(3, 3, k)$-face with $4 \leq k \leq 10$, then the pendent neighbor of a 3-vertex is a 11^+-vertex by Lemma 2.2(1). By (R1)-(R4), f receives $2 \times \frac{3}{2}$ from the pendent neighbors of v_1, v_2 and gets 1 from v_3. Hence $ch'(f) \geq ch(f) + 2 \times \frac{3}{2} + 1 > 0$. If f is a $(3, 3, 11^+)$-face, then the pendent neighbor of a 3-vertex is a 4^+-vertex by Lemma 2.2(1). By (R1)-(R4), f receives $2 \times \frac{1}{2}$ from the pendent neighbors of v_1, v_2 and gets $\frac{5}{2}$ from v_3. Hence $ch'(f) \geq ch(f) + 2 \times \frac{1}{2} + \frac{5}{2} > 0$. If f is a $(3, 4^+, k)$-face with $4 \leq k \leq 10$, then the pendent neighbor of v_1 is a 11^+-vertex by Lemma 2.2(1). By (R1)-(R4), f receives $\frac{3}{2}$ from the pendent neighbors of v_1 and gets 1×2 from v_2, v_3. Hence $ch'(f) \geq ch(f) + \frac{3}{2} + 1 \times 2 > 0$. If f is a $(3, 4^+, 11^+)$-face, then f gets 1 from v_2 and gets $\frac{5}{2}$ from v_3. Hence $ch'(f) \geq ch(f) + \frac{5}{2} + 1 > 0$. Finally, if f is a $(4^+, 4^+, 4^+)$-face, then f gets at least 1 from each of v_1, v_2, v_3 by (R1)-(R4). Hence $ch'(f) \geq ch(f) + 3 \times 1 = 0$.

(5) $d(f) = 5$. First assume that there exists neither poor 4-vertices nor semi-poor 4-vertices on $b(f)$. If there exists exactly two 2-vertices on $b(f)$, then f is a 5_2-face. By Lemma 2.2(1), a 2-vertex is adjacent to at least two 4^+-vertices, one of which is a 11^+-vertex. If the head of f is a k-vertex with $4 \leq k \leq 10$, then f is a $(2, k, 2, 11^+, 11^+)$-face. By (R6), each of the two 11^+-vertices sends at least $\frac{1}{2}$ to f. Thus, $ch^{'}(f) \geq ch(f) + 2 \times \frac{1}{2} = 0$. If the head of f is a 11^+-vertex, then f is a $(2, 11^+, 2, 4^+, 4^+)$-face. By (R5) and (R6), each of the two 4^+-vertices sends $\frac{1}{2}$ to f. Thus, $ch'(f) \geq ch(f) + 2 \times \frac{1}{2} = 0$. If there exists exactly one 2-vertex v on $b(f)$, then v is adjacent to at least two 4^+-vertices, one of which is a 11^+-vertex by Lemma 2.2(1). By (R5) and (R6), the 4^+-vertex and 11^+-vertex send $\frac{1}{2} + \frac{1}{2} = 1$ to f. Hence $ch'(f) \geq ch(f) + \frac{1}{2} + \frac{1}{2} = 0$. Thus, we assume that there is no 2-vertex on $b(f)$. If there exists one 3-vertex v on $b(f)$, then v is adjacent to at least two 4^+-vertices, one of which is a 11^+-vertex by Lemma 2.2(1). Let $f = [vv_1v_2v_3v_4]$. Assume first that $d(v_1) \geq 4$ and $d(v_4) \geq 4$. By (R5) and (R6), each of v_1 and v_4 sends $\frac{1}{2}$ to f and thus $ch'(f) \geq ch(f) + \frac{1}{2} + \frac{1}{2} = 0$. Next, assume that $d(v_1) \geq 4$ and $d(v_4) = 3$ by symmetry. Then $d(v_3) \geq 4$ by Lemma 2.2(1). By (R5) and (R6), each of v_1 and v_3 sends $\frac{1}{2}$ to f and hence $ch'(f) \geq ch(f) + \frac{1}{2} + \frac{1}{2} = 0$. If there exists no 3^--vertex v on $b(f)$, then $ch'(f) \geq ch(f) + 2 \times \frac{1}{2} = 0$ by (R5) and (R6).

Next, assume that there exists a poor 4-vertex v on $b(f)$. We first assume that f is a 5_2-face. By the definition of the poor 4-vertex, v is not the head of f. Let $f = [vv_1v_2v_3v_4]$ and $d(v_2) = d(v_4) = 2$. Then $d(v_1) \geq 4$ by

Lemma 2.2(1). Let f_1 be the other face rather than f incident with v_2 and let f_2 be the other face rather than f incident with v_4. By Lemma 2.3(2), both f_1 and f_2 are 7^+-faces. By (R7), each of f_1 and f_2 sends $\frac{1}{3}$ to f. By (R5), poor 4-vertex v sends $\frac{1}{4}$ to f and 4^+-vertex v_1 sends at least $\frac{1}{4}$ to f . Hence, $ch'(f) \geq ch(f)+2\times\frac{1}{3}+\frac{1}{4}+\frac{1}{4}=\frac{1}{6}>0$. We then assume that f is not a 5_2-face. Let $f=[vv_1v_2v_3v_4]$ and $d(v_4)=2$. By Lemma 2.2(1), $d(v_3)\geq 11$. Let f_2 be the other face rather than f incident with v_4. By Lemma 2.3(2), f_2 is a 7^+-face. By (R7), f_2 sends $\frac{1}{3}$ to f. By (R5), poor 4-vertex v sends $\frac{1}{4}$ to f. By (R6), v_3 sends $\frac{1}{2}$ to f. Hence $ch'(f)\geq ch(f)+\frac{1}{3}+\frac{1}{4}+\frac{1}{2}=\frac{1}{12}>0$.

Finally, assume that there exists a semi-poor 4-vertex v on $b(f)$. Let three consecutive 5-faces incident to v be $f_1=[vv_1v_2v_3v_4]$, $f_2=[vv_4v_5v_6v_7]$ and $f_3=[vv_7v_8v_9v_{10}]$. By symmetry, we only need to consider 5-face f_1 and f_2. By the definition of semi-poor 4-vertex, $d(v_1)=d(v_{10})=2$. We first assume that f_1 is not a 5_2-face. By Lemma 2.2(1), $d(v_2)\geq 11$. Let $f_1^{'}$ be the other face rather than f_1 incident with v_1. By Lemma 2.3(2), $f_1^{'}$ is a 7^+-face. By (R7), $f_1^{'}$ sends $\frac{1}{3}$ to f. By (R5), semi-poor 4-vertex v sends $\frac{1}{3}$ to f_1. By (R6), 11^+-vertex v_2 sends $\frac{1}{2}$ to f_1. Hence, $ch'(f_1)\geq ch(f_1)+2\times\frac{1}{3}+\frac{1}{2}=\frac{1}{6}>0$.

Next we calculate the final charge of f_2. Suppose firstly $d(v_4)=3$. Assume that $d(v_5)=3$, then $d(v_6)\geq 4$ by Lemma 2.2(1). By the definition of poor 4-vertex, neither v_6 nor v_7 is a poor 4-vertex. By (R5) and (R6), each of v_6 and v_7 sends at least $\frac{1}{3}$ to f_2, v sends $\frac{1}{3}$ to f_2. Hence $ch'(f_2)\geq ch(f_2)+3\times\frac{1}{3}=0$. Assume that $d(v_5)\geq 4$. If v_5 is a poor 4-vertex, then $d(v_6)=2$ by the definition of poor 4-vertex. Hence $d(v_7)\geq 11$ by Lemma 2.2(1). Let $f_6^{'}$ be the other face incident with v_6. By Lemma 2.3(2), $f_6^{'}$ is a 7^+-face. By (R7), $f_6^{'}$ sends $\frac{1}{3}$ to f_2. By (R6), v_7 sends $\frac{1}{2}$ to f_2. By (R5), semi-poor 4-vertex v sends $\frac{1}{3}$ to f_1. Hence $ch'(f_2)\geq ch(f_2)+\frac{1}{3}+\frac{1}{3}+\frac{1}{2}>0$. Next suppose that v_5 is not a poor 4-vertex. By (R5) and (R6), v_5 sends at least $\frac{1}{3}$ to f_2. If $d(v_6)=2$, let $f_6^{'}$ be the other face incident with v_6. By Lemma 2.3(2), $f_6^{'}$ is a 7^+-face. By (R7), $f_6^{'}$ sends $\frac{1}{3}$ to f_2. Thus $ch'(f_2)\geq ch(f_2)+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}=0$. If $d(v_6)\geq 3$, by Lemma 2.2(2), v is adjacent to at least one 4^+-vertex, then v_7 is a 4^+-vertex and v_7 is not a poor 4-vertex. By (R5) and (R6), v_7 sends at least $\frac{1}{3}$ to f_2. Thus $ch'(f_2)\geq ch(f_2)+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}=0$.

Suppose $d(v_4)\geq 4$. It is clear that v_4 is not a poor 4-vertex. If v_4 is a semi-poor 4-vertex, then $d(v_5)=2$. let $f_5^{'}$ be the other face incident with v_5. By Lemma 2.3(2), $f_5^{'}$ is a 7^+-face. By (R7), $f_5^{'}$ sends $\frac{1}{3}$ to f_2. By (R5), each of v and v_4 sends $\frac{1}{3}$ to f_2. Thus $ch'(f_2)\geq ch(f_2)+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}=0$. If v_4 is not a semi-poor 4-vertex, then v_4 sends $\frac{1}{2}$ to f_2 by (R6). Assume that $d(v_5)=2$, then the other face incident with v_5 other than f_2 sends $\frac{1}{3}$ to f_2 by Lemma 2.3(2) and (R7). Hence $ch'(f_2)\geq ch(f_2)+\frac{1}{2}+\frac{1}{3}+\frac{1}{3}=\frac{1}{6}>0$. Assume that $d(v_5)=3$. If $d(v_6)=2$, then the other face incident with v_6 other than f_2 sends $\frac{1}{3}$ to f_2 by Lemma 2.3(2) and (R7). Hence $ch'(f_2)\geq ch(f_2)+\frac{1}{2}+\frac{1}{3}+\frac{1}{3}=\frac{1}{6}>0$. If $d(v_6)=3$, then $d(v_7)\geq 4$ by Lemma 2.2(1). And v_7 is not a poor 4-vertex by the definition of the poor 4-vertex. By (R5)

and (R6), v_7 sends at least $\frac{1}{3}$ to f_2. Thus $ch'(f_2) \geq ch(f_2)+\frac{1}{3}+\frac{1}{3}+\frac{1}{2} = \frac{1}{6} > 0$. If $d(v_6) \geq 4$, then v_6 is not a poor 4-vertex. By (R5) and (R6), v_6 sends at least $\frac{1}{3}$ to f_2. Thus $ch'(f_2) \geq ch(f_2) + \frac{1}{3} + \frac{1}{3} + \frac{1}{2} = \frac{1}{6} > 0$. Assume that $d(v_5) \geq 4$, then v_5 sends at least $\frac{1}{4}$ to f_2 by (R5) and (R6). Hence $ch'(f_2) \geq ch(f_2) + \frac{1}{3} + \frac{1}{2} + \frac{1}{4} = \frac{1}{12} > 0$.

Next, assume that f_1 is a 5_2-face, then $d(v_3) = 2$. Let $f_1^{'}$ be the other face incident with v_1 and $f_3^{'}$ be the other face incident with v_3. By Lemma 2.3(2), $f_1^{'}$ and $f_3^{'}$ are 7^+-faces. By (R7), each of $f_1^{'}$ and $f_3^{'}$ sends $\frac{1}{3}$ to f_1. Hence $ch'(f_1) \geq ch(f_1) + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 0$.

Next we calculate the final charge of f_2. By Lemma 2.2(1), $d(v_4) \geq 4$. If v_4 is a poor 4-vertex, then $d(v_5) = 2$. By Lemma 2.2(1), $d(v_6) \geq 11$. By (R5), poor 4-vertex v_4 sends $\frac{1}{4}$ to f_2. By (R6), 11^+-vertex v_6 sends $\frac{1}{2}$ to f_2. Hence $ch'(f_2) \geq ch(f_2) + \frac{1}{3} + \frac{1}{2} + \frac{1}{4} = \frac{1}{12} > 0$. If v_4 is a semi-poor 4-vertex, then v_4 sends $\frac{1}{3}$ to f_2 by (R5). Assume that $d(v_5) = 3$. If $d(v_6) = 3$, then $d(v_7) \geq 4$ by Lemma 2.2(1) and v_7 is not a poor 4-vertex. Thus v_7 sends at least $\frac{1}{3}$ to f_2 by (R5) and (R6). Hence $ch'(f_2) \geq ch(f_2) + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 0$. If $d(v_6) \geq 4$, then v_6 is not a poor 4-vertex. Thus v_6 sends at least $\frac{1}{3}$ to f_2. Hence $ch'(f_2) \geq ch(f_2) + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 0$. Assume that $d(v_5) \geq 4$. If v_5 is not a poor 4-vertex, then v_5 sends at least $\frac{1}{3}$ to f_2 by (R5) and (R6). Hence $ch'(f_2) \geq ch(f_2) + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 0$. If v_5 is a poor 4-vertex, then $d(v_6) = 2$. Thus the other face incident with v_6 is a 7^+-face and sends $\frac{1}{3}$ to f_2 by Lemma 2.3(2) and (R7). Hence $ch'(f_2) \geq ch(f_2) + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 0$. If v_4 is neither a poor 4-vertex nor a semi-poor 4-vertex, then v_4 sends $\frac{1}{2}$ to f_2 by (R5) and (R6). Assume that $d(v_5) = 2$, then the other face incident with v_5 is a 7^+-face and sends $\frac{1}{3}$ to f_2 by Lemma 2.3(2) and (R7). Hence $ch'(f_2) \geq ch(f_2) + \frac{1}{3} + \frac{1}{2} + \frac{1}{3} = \frac{1}{6} > 0$. Assume that $d(v_5) = 3$. If $d(v_6) = 3$, then $d(v_7) \geq 4$. By (R5) and (R6), v_7 sends at least $\frac{1}{4}$ to f_2. Hence $ch'(f_2) \geq ch(f_2) + \frac{1}{3} + \frac{1}{2} + \frac{1}{4} = \frac{1}{12} > 0$. If $d(v_6) \geq 4$, then v_6 sends at least $\frac{1}{4}$ to f_2. Hence $ch'(f_2) \geq ch(f_2) + \frac{1}{3} + \frac{1}{2} + \frac{1}{4} = \frac{1}{12} > 0$. Assume that $d(v_5) \geq 4$. Then v_5 sends at least $\frac{1}{4}$ to f_2. Hence $ch'(f_2) \geq ch(f_2) + \frac{1}{3} + \frac{1}{2} + \frac{1}{4} = \frac{1}{12} > 0$.

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Study of generalised solutions for piezoelectric material with long memory and damage

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Abstract. We consider a dynamic problem that describes a frictional contact with damage between piezoelectric body and a conductive foundation. The constitutive law used is the electro viscoelastic with long term memory. The contact is supposed bilateral and frictional, it is modeled with Tresca's law. The damage is described by a function, its evolution is given by an inclusion of parabolic type. We derive a variational formulation for the model which is in a form of a coupled system for the displacement, the electric potential and damage function. The existence of a unique weak solution for a contact problem is established. The proof is based on the evolutionary variational inequalities, evolution inclusion and Banach's fixed point theorem.

Keywords: evolutionary variational inequality, fixed point, frictional contact, piezoelectric material.

1. Introduction

The piezoelectric materials are characterized by the combination of mechanical and electrical proprieties. The mechanical stress is generated when the electrical potential is applied and conversely the electric potential is created when the mechanical stress is present. Certain crystals, such as quartz, tourmaline, Rochelle salt, when subjected to a stress, become electrically polarized [7]. A general models for electro elastic problem can found in [13, 18, 19]. We consider material electro viscoelastic with long term memory. For problems involving such constutive law we can see for instance [21, 6]. We are interested to Models taking into account the influence of interne state variable which describe a damage of system. The subject of damage is extremely important in design engineering since it affects directly the useful life of the designed structure or component. There exists a very large engineering literature on it. General novel models for damage were derived from the virtual power principle. A mathematical analysis has been investigated for example in [9, 10]. A considerable interest in contact problems involving a damage see for instance [11, 12]. The process of damage introduce a function $\beta \in [0,1]$, There is no damage when $\beta = 1$, and it is completely damaged when $\beta = 0$.

In this paper we study a contact between electro-viscoelastic body with long term memory and a deformable conductive foundation. Our interest is to describe the evolution of the deformation of the body, the function of damage and of the electric potential on the time interval $[0, T]$. The frictional contact is bilateral and it is modeled with Tresca's law, we suppose that the acceleration of system is not negligible so that the process is dynamic. Dynamic contact problems for elastic and viscoelastic body were considered in [2, 14, 15] and in the references therein. A few mathematical results arising in the study of a dynamic frictional contact of piezoelectric bodies may be found in [1, 8, 17].

Our study serves two purposes, the first one is to obtain variational formulation of the problem and second one to prove existence and uniqueness of weak solutions. The novelly in this work is to study the case of dynamic frictional contact for piezoelectric body taking into account the influence of the internal damage of the material on the process.

So in section 2 the piezoelectric problem is stated together with three variational formulations, in section 3 we state the existence and uniqueness result of the problem $\mathcal{P}_V$ (Theorem 3.1). The proof is based on the theory of evolution equations with monotone operators, on the evolutionary inclusions related to the variational inequalities and Banach's fixed point theorem.

2. Problem statement and notations

We assume that the body occupies the bounded domain Ω , and assume that the boundary Γ of Ω is Lipschitz continuous and partitioned into three disjoint measurable open parts Γ_1, Γ_2 and Γ_3, and a partition $\Gamma_1 \sqcup \Gamma_2$ into open parts Γ_a and Γ_b. We assume that $meas\Gamma_2 > 0$ and $meas\Gamma_a > 0$. The body is clamped on Γ_1, therefore the displacement field vanishes there. A volume force of density f_0 acts in $\Omega \times (0, T)$ and surface traction of density f_2 act in $\Gamma_2 \times (0, T)$. The body may arrive in contact on $\Gamma_3 \times (0, T)$ with an obstacle, we assume that the contact is bilateral and frictional. The electric effects leads to the appearance of charges of density q_0. The process is be assumed electrically static.

We denote by $\mathbb{S}^d$ the space of second order of symétric tensors on $\mathbb{R}^d$ ($d = 1, 2, 3$) and by $(\cdot)$ et $|\cdot|$ respectively the scalar product and the Euclidean norm in $\mathbb{S}^d$ (resp in $\mathbb{R}^d$).

$$\mathbf{u}.\boldsymbol{v} = \mathbf{u}_i v_i \qquad |\mathbf{u}| = (\mathbf{u}.\mathbf{u})^{\frac{1}{2}} \quad \forall \mathbf{u}, \boldsymbol{v} \in \mathbb{R}^d, i = 1, \cdots d,$$
$$\boldsymbol{\sigma}.\boldsymbol{\tau} = \sigma_{ij}\tau_{ij} \qquad |\tau| = (\tau.\tau)^{\frac{1}{2}} \quad \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}^d, i = 1, \cdots d, j = 1, \cdots d.$$

Here and bellow the indices i, j run between 1 and d and the summation convention over repeated indices is adopted Let $\Omega \subset \mathbb{R}^d$. we shall use the notation

$$\begin{aligned}
H &= \left\{\mathbf{u} = (u_i) \mid u_i \in L^2(\Omega)\right\} = \left(L^2(\Omega)\right)^d, \\
\mathcal{W} &= \left\{\boldsymbol{D} \in H \mid div\boldsymbol{D} \in L^2(\Omega)\right\}, \\
\mathcal{H} &= \left\{\boldsymbol{\sigma} = (\sigma)_{ij} \mid \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\right\}, \\
H_1 &= \left\{\mathbf{u} = (u_i) \mid \boldsymbol{\varepsilon}(\mathbf{u}) \in \mathcal{H}\right\}, \\
\mathcal{H}_1 &= \left\{\boldsymbol{\sigma} \in \mathcal{H} \mid Div\boldsymbol{\sigma} \in H\right\},
\end{aligned}$$

with $\boldsymbol{\varepsilon} : H \to \mathcal{H}$ and $Div : \mathcal{H} \to H$ are respectively operators of deformation and divergence defined by :

$$\boldsymbol{\varepsilon}(\mathbf{u}) = \left(\boldsymbol{\varepsilon}_{ij}(\mathbf{u})\right), \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \text{ and } Div\boldsymbol{\sigma} = \left(\sigma_{ij,j}\right).$$

The tensors $\mathcal{E} = (e_{ijk})$ and its transpose $\mathcal{E}^* = (e_{kij})$ satisfy the equality

$$\mathcal{E}\boldsymbol{\sigma} \cdot \boldsymbol{v} = \boldsymbol{\sigma} \cdot \mathcal{E}^* \boldsymbol{v},$$

where the index that follows a comma indicates a partial derivative with respect to the corresponding component of the independent variable. We assume that the mass density ρ satisfy

(1) $\rho \in L_\infty(\Omega)$ and there exist $\rho_* > 0$ such that $\rho(\boldsymbol{x}) \geq \rho_*$ a.e. in Ω,

then the space H is a Hilbert space endowed with a new inner product

$$(\mathbf{u}, \boldsymbol{v})_H = \int_\Omega \rho u_i v_i \, dx.$$

The space H, $\mathcal{H}$, H^1 and $\mathcal{H}^1$ are Hilbert spaces endowed with the inner products given by

$$\begin{aligned}
(\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} &= \int_\Omega \sigma_{ij}\tau_{ij} \, dx, \\
(\mathbf{u}, \boldsymbol{v})_{H_1} &= (\mathbf{u}, \boldsymbol{v})_H + (\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}}, \\
(\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}_1} &= (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} + (Div\boldsymbol{\sigma}, Div\boldsymbol{\tau})_H.
\end{aligned}$$

The associated norms on those spaces respectively are $|\cdot|_H$, $|\cdot|_{\mathcal{H}}$, $|\cdot|_{H^1}$ and $|\cdot|_{\mathcal{H}^1}$. Since Γ is assumed be Lipschitz continuous then the unit outward normal vector ν is defined a.e., for every vector $\boldsymbol{v} \in H_1$, we use the notation $\boldsymbol{v}$ for the trace of $\boldsymbol{v}$ on Γ and we denote by $\boldsymbol{v}_\nu$ and $\boldsymbol{v}_\tau$ the normal and tangential components of $\boldsymbol{v}$ on Γ, given by $v_\nu = \boldsymbol{v} \cdot \nu$ and $\boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu \nu$. For regular stress field $\boldsymbol{\sigma}$ (say C^1), the application of its trace to ν is the Cauchy stress vector $\boldsymbol{\sigma}\nu$. We define the normal and tangential components of $\boldsymbol{\sigma}$ by $\sigma_\nu = (\boldsymbol{\sigma}\nu) \cdot \nu$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\nu - \boldsymbol{\sigma}_\nu \nu$, and recall that the Green's formula holds

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} + (Div\boldsymbol{\sigma}, \boldsymbol{v})_H = \int_\Gamma \boldsymbol{\sigma}\nu \cdot \boldsymbol{v}, \quad \forall \boldsymbol{v} \in H_1. \tag{2}$$

Let the Banach spaces $L^p(0,T;H)$ and $W^{1,p}(0,T;H)$ $1 \leq p \leq +\infty$, defined by

$$L^p(0,T;H) = \left\{ \mathbf{u} \mid \mathbf{u} : (0,T) \to H \ , \|\mathbf{u}\|_{L^p(0,T;H)} = \left(\int_0^T \|\mathbf{u}(t)\|_H^p \, dt \right)^{1/p} \right\},$$

$$W^{1,p}(0,T;H) = \Big\{ \mathbf{u} \in L^p(0,T;H), \ \dot{\mathbf{u}} = \tfrac{d\boldsymbol{u}(t)}{dt} \in L^p(0,T;H) \text{ such that}$$
$$\|\mathbf{u}\|_{W^{1,p}} = \left(\int_0^T \|\dot{\mathbf{u}}(t)\|_H^p \, dt \right)^{1/p} + \left(\int_0^T \|\mathbf{u}(t)\|_H^p \, dt \right)^{1/p} \Big\}.$$

Here and every where in this paper the dot above $\mathbf{u}$ is the derivative with respect to the time variable.

The physical model for the process is as follows :

Problem $\mathcal{P}$. Find a displacement field $\boldsymbol{u} : \Omega \times [0,T] \to \mathbb{R}^d$, an electric potential $\varphi : \Omega \times [0,T] \to \mathbb{R}$, an electric displacement field and a damage function $\beta : \Omega \times [0,T] \to \mathbb{R}$

$$\boldsymbol{\sigma} = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \mathcal{B}(\boldsymbol{\varepsilon}(\boldsymbol{u}), \beta) + \int_0^t \mathcal{C}(t-s, \boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \beta(s)) ds \tag{3}$$

$$+\mathcal{E}^* \nabla \varphi \text{ in } \Omega \times (0,T), \tag{4}$$

$$\boldsymbol{D} = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}) - \ \gamma \nabla \varphi \text{ in } \Omega \times (0,T), \tag{5}$$

$$\dot{\beta} - \hat{k} \triangle \beta + \partial \varphi_K(\beta) \ni \phi(\boldsymbol{\varepsilon}(\boldsymbol{u}), \beta), \tag{6}$$

$$\rho \ddot{\boldsymbol{u}} = Div \ \boldsymbol{\sigma} + \mathbf{f}_0 \quad \text{ in } \Omega \times (0,T), \tag{7}$$

$$div \ \boldsymbol{D} = \boldsymbol{q}_0 \text{ in } \Omega \times (0,T), \tag{8}$$

$$\boldsymbol{u} \ = 0 \quad \text{ on } \Gamma_1 \times (0,T), \tag{9}$$

$$\boldsymbol{\sigma}\nu \ = \boldsymbol{f}_2 \quad \text{ on } \Gamma_2 \times (0,T), \tag{10}$$

$$\sigma_\nu = 0 \quad \text{on } \Gamma_3 \times (0,T), \tag{11}$$
$$\|\boldsymbol{\sigma}_\tau\| \leq g,$$

$$\dot{\mathbf{u}}_\tau \neq \mathbf{0} \Rightarrow \boldsymbol{\sigma}_\tau = -g \frac{\dot{\mathbf{u}}_\tau}{\|\dot{\mathbf{u}}_{\boldsymbol{\tau}}\|} \quad \text{on } \Gamma_3 \times (0,T), \tag{12}$$

$$\varphi = 0 \quad \text{on } \Gamma_a \times (0,T), \tag{13}$$

$$\boldsymbol{D}\cdot\nu = q_b \quad \text{on}\,\Gamma_b \times (0,T)\,, \tag{14}$$

$$\boldsymbol{D}\cdot\nu = (\varphi - \varphi_0) \ \text{ on}\ \Gamma_3 \times (0,T)\,, \tag{15}$$

$$\frac{\partial\beta}{\partial\nu} = 0, \ \text{on}\ \Gamma \times (0,T), \tag{16}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \dot{\boldsymbol{u}}(0) = \boldsymbol{v}_0 \quad \beta(0) = \beta_0 \quad \text{in}\ \Omega. \tag{17}$$

During the process of contact the normal component of the electric displacement field or the free charge is assumed to be proportional to the difference between the potential of the foundation and the body's surface potential, with k as the proportionality factor.

The equations (3)-(8) and below, in order to simplify we do not indicate explicitly the dependence of various functions on the variables $x \in \Omega \cup \Gamma$ and $t \in [0,T]$. The general viscoelastic constitute law with damage and with electric effects is given by (3), where $\mathcal{A}$ is nonlinear viscosity function, $\mathcal{B}$ is nonlinear elasticity, depends on strain tensor $\boldsymbol{\varepsilon}(\boldsymbol{u})$ and the damage function β . The stress is depends on electric field $-\nabla\varphi$. The relation (5) is the electric displacement it is a linear function of strain and electric field. The expression (6) describes the evolution of the damage function, K denotes the set of admissible damage functions defined by $K = \left\{\xi \in H^1(\Omega) \mid 0 \le \xi \le 1 \text{ a.e. on } \Omega\right\}$, $H^1(\Omega)$ is the classical Sobolev space. The set $\partial\varphi_K$ is the subdifferential of the indicator function χ_K of the convex K. The equations (7) and (8) are the equilibrium equations, in equation (7) we suppose the process is dynamic with a masse density ρ. Here the conditions (9) and (10) are the displacement and traction boundary conditions, respectively conditions (11) and (12) represents the bilateral frictional contact condition, it is modeled with Tresca's law of dry friction with prescribed friction bound g (hence $g \ge 0$ and the nonseparation condition $u_\nu = 0$. The boundary is electricly conductive (13) and (14) are boundary conditions on electric potential φ and displacement field D on Γ_a and Γ_b. On a part of boundary Γ_3, and during the process of contact the normal of electric displacement field is assumed be proportional to the difference between the potential of foundation φ_0 and the body's surface potential, thus the condition (15). The equations given in (17) are the initial condition on displacement, the velocity field and the damage function. To present variational formulation of the above problem we need additional notations. Let us consider the subspaces of H_1 and H^1defined by

$$V = \{\boldsymbol{v} \in H_1 \mid \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_1\}\,,$$
$$W = \left\{\xi \in H^1 \mid \xi = 0 \text{ on } \Gamma_a\right\}$$

We recall since $meas\Gamma_1 > 0$ and $meas\Gamma_a > 0$, Korn's and Friederichs-Poincaré inequalities hold, thus there exist respectively a constant $C_K > 0$ and $c_F > 0$ which depends respectively only on Γ_1, Γ_a and Ω such that

$$|\boldsymbol{\varepsilon}(\boldsymbol{v})|_{\mathcal{H}_1} \geq c_K |\boldsymbol{v}|_{H_1}, \quad |\nabla\xi|_H \geq c_F |\xi|_{H^1}, \quad \forall \boldsymbol{v} \in V, \forall \xi \in W.$$

For $\boldsymbol{u}$, $\boldsymbol{v} \in V$ and $\forall \varphi, \xi \in W$, we have

$$(\boldsymbol{u}, \boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}_1}, \qquad (\varphi, \xi)_W = (\nabla\varphi, \nabla\xi)_H,$$

and we have $|\boldsymbol{u}|_V = (\boldsymbol{u},\boldsymbol{u})_V^{1/2}$, $|\varphi|_V = (\varphi, \varphi)_W^{1/2}$, therefore $(V, |\cdot|_V)$ and $(W, |\cdot|_W)$ are real Hilbert spaces. Moreover, by Sobolev trace theorem, there exist a constants c_0, $\tilde{c}_0$ depending only on Ω and Γ_1, Γ_a such that

$$|\xi|_{L^2(\Gamma_a)} \leq c_0 |\xi|_W, \qquad \forall \xi \in W, \tag{18}$$

$$|\boldsymbol{v}|_{L^2(\Gamma_1)^d} \leq \tilde{c}_0 |\boldsymbol{v}|_V, \qquad \forall \boldsymbol{v} \in V. \tag{19}$$

Let note V' and W' the dual spaces of V and W, so we have continuous and dense imbedding $V \subset H \subset V'$ rep ($W \subset L^2(\Omega) \subset W'$). To study problem $\mathcal{P}$ we must make some assumptions.The viscosity operator $\mathcal{A}$ and the elasticity one $\mathcal{B}$ satisfy conditions

$$\begin{cases} \text{(a) } \mathcal{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d \\ \text{(b) there exist } L_{\mathcal{A}} > 0 \text{ such that} \\ |\mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)| \leq L_{\mathcal{A}} |\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2| \ \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega, \\ \text{(c) there exist } m_{\mathcal{A}} > 0 \text{ such that} \\ (\mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \geq m_{\mathcal{A}} |\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2|^2, \\ \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega, \\ \text{(d) the mapping } \boldsymbol{x} \longmapsto \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}) \text{ is Lebesgue measurable on } \Omega, \\ \text{(e) the mapping } \boldsymbol{x} \longmapsto \mathcal{A}(\boldsymbol{x}, \boldsymbol{0}) \text{ belongs to } \mathcal{H}. \end{cases} \tag{20}$$

$$\begin{cases} \text{(a) } \mathcal{B} : \Omega \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{S}^d, \\ \text{(b) } \mathcal{B}(\mathbf{x}, \boldsymbol{\tau}, \zeta) = b(\mathbf{x}, \zeta)\boldsymbol{\tau}, \ \forall \boldsymbol{\tau} \in \mathbb{S}^d, \text{ a.e. } \mathbf{x} \in \Omega. \\ \text{(c) } b(\mathbf{x}, \zeta) = (b_{ijkl}), \ b_{ijkl} = b_{klij} \text{ belongs to } L^\infty(\Omega \times \mathbb{R}). \forall i, j, k, l. \\ \text{(d) There exists } m_{\mathcal{B}} > 0 \text{ such that} \\ \qquad \mathcal{B}(\mathbf{x}, \boldsymbol{\xi}, \zeta) \ \boldsymbol{\xi} \geq m_{\mathcal{B}} \|\boldsymbol{\xi}\|^2, \text{ for all } \mathbf{x} \in \Omega, \boldsymbol{\xi} \in \mathbb{S}^d, \zeta \in \mathbb{R}. \end{cases} \tag{21}$$

The relaxation function $\mathcal{C} : \Omega \times (0, T) \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{S}^d$, satisfies

$$\begin{cases} |\mathcal{C}(t, \boldsymbol{x}, \boldsymbol{\varepsilon}_1, \beta_1) - \mathcal{C}(t, \boldsymbol{x}, \boldsymbol{\varepsilon}_2, \beta_2)| \leq L_{\mathcal{C}} |\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2| + |\beta_1 - \beta_2|, \\ \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \ \forall \beta_1, \beta_2 \in \mathbb{R} \text{ a.e. } \boldsymbol{x} \in \Omega, \forall t \in (0, T). \\ \text{(c) the mapping } \boldsymbol{x} \longmapsto \mathcal{C}(\boldsymbol{x}, \boldsymbol{\varepsilon}, \beta) \text{ is Lebesgue measurable on } \Omega, \\ \qquad \forall \boldsymbol{\varepsilon} \in \mathbb{S}^d, \ \forall \beta \in \mathbb{R}. \\ \text{(c) the mapping } t \longmapsto \mathcal{C}(t, \boldsymbol{x}, \boldsymbol{\varepsilon}, \beta) \text{ is continious on } (0, T), \\ \qquad \forall \boldsymbol{\varepsilon} \in \mathbb{S}^d, \ \forall \beta \in \mathbb{R}. \\ \text{(d) the mapping } \boldsymbol{x} \longmapsto \mathcal{C}(t, \boldsymbol{x}, \boldsymbol{0}, 0) \text{ belongs to } \mathcal{H}. \end{cases} \tag{22}$$

The damage source function satisfies

$$(23)\quad \begin{cases} \text{(a) } \phi : \Omega \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{R} \\ |\phi(\boldsymbol{x},\ \boldsymbol{\varepsilon}_1, \beta_1) - \phi(\boldsymbol{x}, \boldsymbol{\varepsilon}_2, \beta_2)| \leq L_\phi |\boldsymbol{\varepsilon}_1 - \ \boldsymbol{\varepsilon}_2| + |\beta_1 - \beta_2|, \\ \ \forall \boldsymbol{\varepsilon}_1,\ \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d,\ \forall \beta_1, \beta_2 \in \mathbb{R} \text{ a.e. } \boldsymbol{x} \in \Omega, \\ \text{(c) the mapping } \boldsymbol{x} \longmapsto \phi(\boldsymbol{x}, \boldsymbol{\varepsilon}, \beta) \text{ is Lebesgue measurable on } \Omega, \\ \forall \boldsymbol{\varepsilon} \in \mathbb{S}^d,\ \forall \beta \in \mathbb{R} \\ \text{(d) the mapping } \boldsymbol{x} \longmapsto \phi(\boldsymbol{x}, \mathbf{0}, 0) \text{ belongs to } \mathcal{H}. \end{cases}$$

The piezoelectic tensor $\mathcal{E}$ and the permeability tensor γ satisfy

$$(24)\quad \begin{cases} \text{(a) } \mathcal{E} : \Omega \times \mathbb{S}^d \to \mathbb{R}^d \\ \text{(b) } \mathcal{E}(\boldsymbol{x}, \zeta) = (e_{ijk}(\boldsymbol{x})\zeta_{jk}), \quad \forall \zeta = (\zeta_{ij}) \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega \\ \text{(c) } e_{ijk} = e_{ikj} \in L^\infty(\Omega) \end{cases}$$

$$(25)\quad \begin{cases} \text{(a) } \gamma : \Omega \times \mathbb{R}^d \to \mathbb{R}^d, \\ \text{(b) } \gamma(\boldsymbol{x}, \mathbf{E}) = (\gamma_{ij}(\boldsymbol{x})E_j), \quad \forall \mathbf{E} = (E_i) \in \mathbb{R}^d, \text{ a.e. } \boldsymbol{x} \in \Omega, \\ \text{(c) } \gamma_{ik} = \gamma_{ji} \in L^\infty(\Omega), \\ \text{(d) there exist a constant } m_\gamma > 0 \text{ such that} \\ \gamma_{ij}(\boldsymbol{x})E_iE_j \geq m_\gamma \|\mathbf{E}\|^2, \ \forall \mathbf{E} = (E_i) \in \mathbb{R}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \end{cases}$$

The body forces and surfaces traction and free charges densities satisfy

$$(26)\quad \mathbf{f}_0 \in W^{1;1}(0, T; H),$$

$$(27)\quad \boldsymbol{f}_2 \in L^2\left(0, T; L^2(\Gamma_2)^d\right),$$

$$(28)\quad q_0 \in W^{1;2}\left(0, T; L^2(\Omega)\right),$$

$$(29)\quad q_b \in W^{1;2}\left(0, T; L^2(\Gamma_b)\right).$$

We assume $\mathbf{v}_0$ belongs to ∂j the subdifferential of j in convex sens and satisfies

$$(30)\quad \begin{aligned} &(\mathcal{A}\boldsymbol{\varepsilon}(\mathbf{v}_0),\ \boldsymbol{\varepsilon}(\mathbf{v}) - \varepsilon(\mathbf{v}_0))_{\mathcal{H}} + (\mathcal{B}\boldsymbol{\varepsilon}(\mathbf{u}_0), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{v}_0))_{\mathcal{H}} + \\ &j(\mathbf{v}) - j(\mathbf{v}_0) \geq \langle \mathbf{f}_0(0), \mathbf{v} - \mathbf{v}_0 \rangle_H \ \forall v \in V. \end{aligned}$$

We also suppose that

$$(31)\quad \varphi_0 \in L^2(\Gamma_3),$$

$$(32)\quad \hat{k} > 0, \quad \beta_0 \in K,$$

$$(33)\quad \boldsymbol{u}_0 \in V, \quad \boldsymbol{v}_0 \in V.$$

By using Riesz's representation theorem we define un element $q(t) \in W$ by

$$(34)\quad (q(t),\xi)_W = -\left(q_0(t),\xi\right)_{L^2(\Omega)} - \left(q_b(t),\xi\right)_{L^2(\Gamma_b)},\ \ \forall \xi \in W, \text{ a.e. } t \in (0,T),$$

Let $j : V \to \mathbb{R}$ be the functional such that

$$(35)\qquad j(\mathbf{v}) = \int_{\Gamma_3} g\|\mathbf{v}_\tau\|\, da + \int_{\Gamma_2} f_2 \mathbf{v} da,$$

$h : W \times W \to \mathbb{R}$ by

$$h\left(\varphi,\xi\right) \ = \ \int_{\Gamma_3} \left(\varphi - \varphi_0\right)\xi\ da,\ ,\ \forall \xi,\ \varphi \in W.$$

and a bilinear form $a : W \times W \to \mathbb{R}$ such that

$$(36)\qquad a\left(\zeta,\eta\right) = \hat{k}\int_\Omega \nabla\zeta\nabla\eta\ dx, \quad \forall \zeta,\eta \in W.$$

Note that the integrals in (35) and in (36) are well defined. If $\{\boldsymbol{u},\varphi,\beta\}$ are regular functions satisfying (3)-(17), this imply that $\boldsymbol{u}(t) \in V$, $\varphi(t) \in W$, and $\beta(t) \in H^1$, and keeping in mind the relations (2), (35), (36), we deduce the variational formulation of problem $\mathcal{P}$, noted $\mathcal{P}_V$.

Problem $\mathcal{P}_V$. Find a displacement field $\boldsymbol{u} : \Omega\times[0,T] \to \mathbb{R}^d$, an electric potential $\varphi : \Omega \times [0,T] \to \mathbb{R}$ and a damage function $\beta : \Omega \times [0,T] \to \mathbb{R}$ such that

$$\begin{aligned}(37)\qquad &\left(\ddot{\mathbf{u}}(t),\mathbf{v}-\dot{\mathbf{u}}\right)_{V'} + \left(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)),\boldsymbol{\varepsilon}(\mathbf{v}) - \varepsilon(\dot{\mathbf{u}}(t))\right)_{\mathcal{H}}\\ &+ \left(\mathcal{B}\left(\boldsymbol{\varepsilon}(\mathbf{u}(t)),\beta\left(t\right)\right) + \left(\int_0^t \mathcal{C}(t-s,\boldsymbol{\varepsilon}(\boldsymbol{u}(s)),\beta(s))ds,\boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t))\right)_{\mathcal{H}}\right.\\ &+ \left(\mathcal{E}^*\nabla\varphi(t),\boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\ \dot{\mathbf{u}}(t))\right)_{\mathcal{H}}\ + j(\mathbf{v}) - j(\dot{\mathbf{u}}(t)) \geq \left(\mathbf{f}_0(t),\mathbf{v}-\dot{\mathbf{u}}(t)\right)_H,\end{aligned}$$

$$(38)\qquad \begin{array}{c}\left(\gamma\nabla\varphi(t),\nabla\xi\right)_{\mathcal{H}} - \left(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t),\nabla\xi\right)_{\mathcal{H}} + h\left(\varphi(t),\xi\right) = \left(q(t),\xi\right)_W\\ \forall \xi \in W, \text{ a.e. on } (0,T),\end{array}$$

$$(39)\qquad \begin{array}{c}\beta(t) \in K, \quad \left(\dot{\beta}(t),\zeta-\beta(t)\right)_{L^2(\Omega)} + a\left(\beta(t),\zeta-\beta(t)\right)\\ \geq \left(\phi\left(\ \boldsymbol{\varepsilon}\left(\boldsymbol{u}(t)\right),\beta(t)\right),\zeta-\beta(t)\right)_{L^2(\Omega)},\\ \forall \zeta \in K, \text{ a.e. on } (0,T),\end{array}$$

$$(40)\qquad \boldsymbol{u}(0) = \boldsymbol{u}_0,\ \dot{\boldsymbol{u}}(0) = \boldsymbol{v}_0,\ \beta(0) = \beta_0.$$

3. An existence and uniqueness result for the problem $\mathcal{P}_V$

Theorem 3.1. *Assume that the conditions (20)-(33) hold. Then, there exists a unique solution of the problem $\mathcal{P}_V$. Moreover the solution satisfies*

$$\begin{aligned} \boldsymbol{u} &\in W^{1;\infty}(0,T;V)\cap W^{2;\infty}\left(0,T;V'\right). \\ \varphi &\in W^{1;2}(0,T;W), \\ \beta &\in K, \quad \beta\in W^{1;2}\left(0;T,L^2(\Omega)\right)\cap L^2\left(0,T;H^1(\Omega)\right). \end{aligned} \tag{41}$$

The proof of Theorem 3.1 will be carried out in several steps. It is based on results of evolution equations with monotone operators and Banach's fixed point theorem and classical results on parabolic equations bellow, see ([3]).

Theorem 3.2. *Let V and H be real Hilbert spaces satisfying $V\subset H\subset V'$, with continuous and dense injection, and let $B:V\to V'$ be a linear continuous and symmetric operator satisfying the coerciveness condition*

$$\exists\alpha_0>0,\ \alpha_1\in\mathbb{R}\ \ \textit{such that}\ \ (Bu,u)_{V',V}+\alpha_1|u|_H^2\geq\alpha_0|u|_V^2\ ,\forall u\in V.$$

Let $M:V\to 2^{V'}$ a multivalued maximal monotone operator, f a given function in $W^{1,1}(0,T;V')$ and u_0, v_0 two datas satisfying

$$u_0\in V,\ v_0\in D(M)\ \textit{and}\ (Bu_0+Mv_0)\cap H\neq\emptyset. \tag{42}$$

Then, there exist a unique function $u\in W^{1;\infty}(0,T;V)\cap W^{2;\infty}(0,T;V')$ solution to the problem

$$\begin{cases} \ddot{u}(t)+Bu(t)+M\dot{u}(t)\ni f(t),\ \textit{a.e.}\ t\in(0,T), \\ u(0)=0,\quad \dot{u}(0)=v_0. \end{cases} \tag{43}$$

Theorem 3.3. *Let $V\subset H\subset V'$ be a Gelfand triple. Let K a be nonempty and closed set of V. Let define a bilinear and symmetric and continuous form $a(.,.):V\times V\to R$ which satisfy*

$$\exists\alpha_0>0,\ \alpha_1\in\mathbb{R}\quad \textit{such that}\ a(u,u)_{V',V}\geq\alpha_0|u|_V^2+\alpha_1|u|_H^2\ ,\ \forall u\in V, \tag{44}$$

Then, for a given $u_0\in K$ and $f\in L^2(0,T;H)$, there exist a unique function u which satisfies

$$u\in L^2(0,T;V)\cap H^1([0,T];H)$$

for all $t\in[0,T]$, $u(t)\in K$, and for almost $t\in(0,T)$

$$(\dot{u}(t),v-u(t))+a(u(t),v-u(t))\geq(f(t),v-u(t))_H\ ,\qquad \forall u\in K, \tag{45}$$

$$u(0)=u_0. \tag{46}$$

Now, assume that assumptions (1) and (20)-(29) hold and let $\eta = \left(\eta^1, \eta^2\right) \in W^{1;1}(0,T;V') \times L^2(0,T;L^2(\Omega))$. We consider now the following problems.

Problem $\mathcal{P}_1^\eta$. Find a displacement field $\boldsymbol{u}_\eta : \Omega \times [0,T] \to \mathbb{R}^d$, such that

$$\begin{gathered}(\ddot{\mathbf{u}}_\eta(t), \mathbf{v} - \dot{\mathbf{u}}_\eta(t))_{V'} + (\mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_\eta(t)), \boldsymbol{\varepsilon}(\mathbf{v}) - \varepsilon(\dot{\mathbf{u}}_\eta(t)))_{\mathcal{H}} + \\ (\mathcal{B}(\boldsymbol{\varepsilon}(\mathbf{u}_\eta(t)), \beta(t)), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}_\eta(t)))_{\mathcal{H}} + (\eta^1(t), \mathbf{v} - \dot{\mathbf{u}}_\eta(t))_{V'} + \\ j(\mathbf{v}) - j(\dot{\mathbf{u}}_\eta(t)) \geq (\mathbf{f}_0(t), \mathbf{v} - \dot{\mathbf{u}}_\eta(t))_{V'}, \\ \forall \boldsymbol{v} \in V, \text{ a.e. } t \in (0,T),\end{gathered} \tag{47}$$

$$\boldsymbol{u}_\eta(0) = \boldsymbol{u}_0, \ \dot{\boldsymbol{u}}_\eta(0) = \boldsymbol{v}_0, \tag{48}$$

Problem $\mathcal{P}^\eta$. Find an electric potential $\varphi : \Omega \times [0,T] \to \mathbb{R}$, such that

$$\begin{gathered}(\gamma \nabla \varphi_\eta(t), \nabla \xi)_{\mathcal{H}} - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_\eta(t), \nabla \xi)_{\mathcal{H}} + (h(\boldsymbol{u}_\eta(t), \varphi_\eta(t)), \xi)_W \\ = (q(t), \xi)_W \, \forall \xi \in W, \text{ a.e. on } (0,T),\end{gathered} \tag{49}$$

Problem $\mathcal{P}_2^\eta$. Find a damage function $\beta_\eta : \Omega \times [0,T] \to \mathbb{R}$, such that

$$\begin{gathered}\beta_\eta(t) \in K, \ \left(\dot{\beta}_\eta(t), \zeta - \beta_\eta(t)\right)_{L^2(\Omega)} + a(\beta_\eta(t), \zeta - \beta_\eta(t)) \\ \geq \left(\eta^2(t), \zeta - \beta_\eta(t)\right)_{L^2(\Omega)}, \ \forall \zeta \in K, \text{ a.e. on } (0,T).\end{gathered} \tag{50}$$

$$\beta_\eta(0) = \beta.$$

Note that last problem is decoupled, so we can established separately existence and uniqueness solution for each one. We denote by c all different constants given in the estimates below.

Lemma 3.1. *There exists a unique solution to the problem* $\mathcal{P}_1^\eta$. *Moreover it satisfies*

$$\boldsymbol{u}_\eta \in W^{1;\infty}(0,T;V) \cap W^{2;\infty}\left(0,T;V'\right). \tag{51}$$

If $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ *are two solutions of the problem* $\mathcal{P}_1^\eta$ *corresponding to the data* η_1^1 η_2^1 *of* η^1, *then there exist a constant* $c > 0$ *such that*

$$|\mathbf{u}_1(t) - \mathbf{u}_2(t)|_V \leq c\left|\eta_2^1 - \eta_1^1\right|_{L^2(0,T;V')}, \tag{52}$$

Proof. First let rewrite the equation (47) in the operator form equation like in (43) of Theorem 3.2 where

$$M\dot{\mathbf{u}}(t) = A\dot{\mathbf{u}}(t) + \partial j \ \dot{\mathbf{u}}(t),$$

where $A : V \to V'$

$$(A\dot{\mathbf{u}}(t), \mathbf{v})_V = (\mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_\eta(t)), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}},$$

Using (20(a),(d)), (20(c)) A is bounded and monotone on V,

$$(A\mathbf{u}_1 - A\mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2)_V \geq m_A \|\mathbf{u}_1 - \mathbf{u}_2\|_V^2 \quad \forall\, \mathbf{u}_1, \mathbf{u}_2 \in V.$$

The operator A is hemicontinuous, $\forall u, v \in V$, indeed let $(s_n)_n \in [0,1]$ such that $s_n \to s$ when $n \to \infty$, we have

$$\begin{aligned} &|\langle A(\mathbf{u} + s_n\mathbf{v}) - A(\mathbf{u} + s\mathbf{v}), \mathbf{v}\rangle_V| \\ &= |(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}} + s_n\dot{\mathbf{v}}) - \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}} + s\dot{\mathbf{v}}),\ \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}}| \\ &\leq\ C_{\mathcal{A}}\,|s_n - s|\,|\dot{\mathbf{v}}|_V\,|\mathbf{v}|_V \underset{n\to\infty}{\to} 0. \end{aligned} \tag{53}$$

In the second see that the functional j defined by (35) is convex lower semi continuous. Its subdifferentiel in the convex sens ∂j is a maximal monotone operator. The multivalued $M : V \to 2^{V'}$

$$M\dot{\mathbf{u}}\,(t) = A\dot{\mathbf{u}}(t) + \partial j\ \dot{\mathbf{u}}\,(t)\,,$$

is maximal monotone with respect to $\dot{\mathbf{u}}$, Indeed it is the sum of two operators, the first A which is assumed to be monotone and continuous (20 (a)-(e)), therefore A is hemicontinuous and bounded, and the second ∂j the subdiffrential of a proper convex lower semi-continuous function, so it is maximal monotone. According to the functional analysis results on the maximal monotone operators we conclude that the sum $M\big(\dot{u}(t)\big)$ is maximal monotone. Let define $B\,(t) : V \to V$, $t \in (0,T)$ where

$$(B\,(t)\,\mathbf{u}(t), \mathbf{v})_V = (\mathcal{B}\,(\boldsymbol{\varepsilon}(\mathbf{u}_\eta(t)), \beta\,(t))\,, \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}}.$$

It is clear that by (21(a)-(d)) $B\,(t)$ is a linear operator continuous, symmetric and coercive, for almost $t \in (0,T)$. We recall that $F_\eta = \mathbf{f}_0 - \ \eta^1 \in W^{1;1}\,(0,T;V')$ and $\boldsymbol{v}_0 \in V$, (see the conditions (26), (48)).

Since we have $u_0 \in V$, $v_0 \in D(\partial j)$, the condition (33) and $F_\eta(0) \in H$, this implies that there exists an element both in $(Bu_0 + Mv_0)$ and in H. Thus, the condition $(Bu_0 + Mv_0) \cap H \neq \emptyset$ is satisfied. All conditions of the Theorem 3.2 are satisfied, then there exists a unique solution $\mathbf{u}_\eta$ of problem $\mathcal{P}_1^\eta$ with $\mathbf{u}_\eta \in W^{1;\infty}\,(0,T;V) \cap W^{2;\infty}\,(0,T;H)$. Let now $\eta_1^1\ \eta_2^1 \in W^{1;2}\,(0,T;V')$ and $\mathbf{u}_{\eta_1^1}\,(t)$, $\mathbf{u}_{\eta_2^1}\,(t)$ noted respectively $\mathbf{u}_1$ and $\mathbf{u}_2$ two solutions of problems $\mathcal{P}_I^{\eta_1^1}$ and respectively $\mathcal{P}_I^{\eta_2^1}$. Recall that $\mathcal{P}_1^{\eta_i^1}$, $i = 1,2$ can also be written as

$$\begin{aligned} &\langle \ddot{\mathbf{u}}_i, v - \dot{\mathbf{u}}_i\rangle_{V'} + \langle A\mathbf{u}_i, v - \dot{\mathbf{u}}_i\rangle_V + \langle B\mathbf{u}_i, v - \dot{\mathbf{u}}_i\rangle_V + \\ &+ j(v) - j(\dot{\mathbf{u}}_i) \geq (F_{\eta_i}, v - \dot{\mathbf{u}}_i)_{V'},\ \forall v \in V. \end{aligned} \tag{54}$$

Replacing v respectively in (54) by $\dot{\mathbf{u}}_2$ then by $\dot{\mathbf{u}}_1$, and then by adding the obtained inequalities, we find

$$\begin{aligned} &\langle \ddot{\mathbf{u}}_2 - \ddot{\mathbf{u}}_1, \dot{\mathbf{u}}_2 - \dot{\mathbf{u}}_1\rangle_{V'} + \langle A\ \dot{\mathbf{u}}_2 - A\dot{\mathbf{u}}_1, \dot{\mathbf{u}}_2 - \dot{\mathbf{u}}_1\rangle_V + \\ &\langle B\,(\mathbf{u}_2 - \mathbf{u}_1)\,, \dot{\mathbf{u}}_2 - \dot{\mathbf{u}}_1\rangle_V \leq (\eta_1^1 - \eta_2^1, \dot{\mathbf{u}}_2 - \dot{\mathbf{u}}_1)_{V'}, \end{aligned}$$

by using the property of strong monotonicity of A, the Lipshitz condition on B, we find that

$$\frac{d}{dt}\left|\dot{\mathbf{u}}_2-\dot{\mathbf{u}}_1\right|_H^2+m_A\left|\dot{\mathbf{u}}_1-\dot{\mathbf{u}}_2\right|_V^2\leq\left|\eta_2^1-\eta_1^1\right|_{V'}\left|\dot{\mathbf{u}}_2-\dot{\mathbf{u}}_1\right|_V+$$
$$C_B\left|\mathbf{u}_2-\mathbf{u}_1\right|_V\left|\dot{\mathbf{u}}_2-\dot{\mathbf{u}}_1\right|_V,$$

using the inequality $|ab|\leq\frac{\alpha}{2}|a|^2+\frac{2}{\alpha}|b|^2$ where $\alpha\in\mathbb{R}_+^*$ is fixed, we have

$$\begin{aligned}&\frac{d}{dt}\left|\dot{\mathbf{u}}_2-\dot{\mathbf{u}}_1\right|_H^2+m_A\left|\dot{\mathbf{u}}_1-\dot{\mathbf{u}}_2\right|_V^2\\ \leq\ & c\left\{\alpha\left|\dot{\mathbf{u}}_1-\dot{\mathbf{u}}_2\right|_V^2+\frac{2}{\alpha}\left|\eta_2^1-\eta_1^1\right|_{V'}^2+\frac{2}{\alpha}\left|\mathbf{u}_1-\mathbf{u}_2\right|_V^2\right\},\end{aligned}$$

$$\frac{d}{dt}\left|\dot{\mathbf{u}}_2-\dot{\mathbf{u}}_1\right|_H^2+(m_A-\alpha c)\left|\dot{\mathbf{u}}_1-\dot{\mathbf{u}}_2\right|_V^2\leq c\left\{\left|\eta_2^1-\eta_1^1\right|_H^2+\left|\mathbf{u}_1-\mathbf{u}_2\right|_V^2\right\}.$$

by the choice of α such that $m_A-\alpha c>0$, and by integration from 0 to T, we get

$$\begin{aligned}&\left|\dot{\mathbf{u}}_2(t)-\dot{\mathbf{u}}_1(t)\right|_H^2+\left(m_A-\alpha\tfrac{c}{2}\right)\left|\dot{\mathbf{u}}_1-\dot{\mathbf{u}}_2\right|_{L^2(0,t;V)}^2\\ &\quad\leq c\left|\eta_2^1-\eta_1^1\right|_{L^2(0,t;V')}^2+\left|\mathbf{u}_1-\mathbf{u}_2\right|_{L^2(0,t;V)}^2.\end{aligned}$$

$$\left|\dot{\mathbf{u}}_2(t)-\dot{\mathbf{u}}_1(t)\right|_H^2\leq c\left|\eta_2^1-\eta_1^1\right|_{L^2(0,t;V')}^2+\left|\mathbf{u}_1-\mathbf{u}_2\right|_{L^2(0,t;V)}^2$$

hence

$$\left|\dot{\mathbf{u}}_1-\dot{\mathbf{u}}_2\right|_{L^2(0,t;V)}^2\leq c\left\{\left|\ \eta_2^1-\eta_1^1\right|_{L^2(0,t;V')}^2+\left|\mathbf{u}_1-\mathbf{u}_2\right|_{L^2(0,t;V)}^2\right\}.$$

Recall that $\mathbf{u}_1(t)=\int_0^t\dot{\mathbf{u}}_1(s)\,ds+\mathbf{u}_0$, $s<t$, then

$$\begin{aligned}\left|\mathbf{u}_1(t)-\mathbf{u}_2(t)\right|_V^2&\leq\int_0^t\left|\dot{\mathbf{u}}_1(s')-\dot{\mathbf{u}}_2(s')\right|_V^2ds'\\ &\leq c\left|\eta_2^1-\eta_1^1\right|_{L^2(0,t;V')}^2+c\int_0^t\left|\mathbf{u}_1(s)-\mathbf{u}_2(s)\right|_V^2ds,\end{aligned}$$

by Gronwall's Lemma we deduce that

$$\left|\mathbf{u}_1(t)-\mathbf{u}_2(t)\right|_V^2\leq c\left|\eta_2^1-\ \eta_1^1\right|_{L^2(0,T;V')}^2,$$

The Lemma is now proved. □

Lemma 3.2. *There exists a unique solution*

$$\varphi_\eta\in W^{1,2}(0,T;W), \tag{55}$$

of problem $\mathcal{P}^\eta$. If φ_1, φ_2 are two solutions of the problem $\mathcal{P}^\eta$ corresponding to the data η_1 and η_2 of η, then there exist a constant $c>0$ such that

$$\left|\varphi_1(t)-\varphi_2(t)\right|_W\leq c\left|\mathbf{u}_1(t)-\mathbf{u}_2(t)\right|_V. \tag{56}$$

Proof. Let define the operator $A_\eta(t) : W \to W$, for $t \in [0,T]$. by

$$(A_\eta(t)\varphi_\eta(t),\xi)_W = (\gamma\nabla\varphi_\eta(t),\nabla\xi)_{\mathcal{H}} - (\mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}_\eta(t),\nabla\xi)_{\mathcal{H}} + \int_{\Gamma_3}(\varphi_\eta-\varphi_0)\,\xi, \qquad \forall\xi\in W.$$

Let φ_1, $\varphi_2 \in W$, will γ is coercive (see (25)), thus imply

$$(A_\eta(t)\varphi_1 - A_\eta(t)\varphi_2, \varphi_1-\varphi_2)_W \geq m_\gamma\,|\varphi_1-\varphi_2|_W^2\,.$$

The operator $A_\eta(t)$ is strongly monotone. Now, by Conditions (25) and (18), we have

$$(A_\eta(t)\varphi_1 - A_\eta(t)\varphi_2, \xi)_W \leq c\,|\varphi_1-\varphi_2|_W\,|\xi|_W\,,$$

thus imply that $A_\eta(t)$ is Lipshitz continuous. The equation $A_\eta(t)\varphi(t) = q(t)$, have a unique solution $\varphi_\eta(t) \in W$, for $q(t) \in W$. The function $\varphi_\eta(t)$ is then the unique solution of the problem $\mathcal{P}^\eta$. It is a classical result of evolutionary elliptic problems see for example [5]

Let now prove that $\varphi_\eta \in W^{1,2}(0,T;W)$, let t_1, $t_2 \in [0,T]$ and $\varphi_\eta(t_1) = \varphi_1$, $\varphi_\eta(t_2) = \varphi_2$, $\mathbf{u}_\eta(t_1) = \mathbf{u}_1$, $\mathbf{u}_\eta(t_2) = \mathbf{u}_2$, $q(t_1) = q_1$, $q(t_2) = q_2$. Then, we deduce from conditions (25), (18), (19), (24) that

$$m_\gamma\,|\varphi_1-\varphi_2|_W^2 + k\int_{\Gamma_3}(\varphi_1-\varphi_2)\,(\varphi_1-\varphi_2)\,da \leq \left[c_{\mathcal{E}}\,|\mathbf{u}_1-\mathbf{u}_2|_V + |q_1-q_2|_W\right]|\varphi_1-\varphi_2|_W$$

thus imply that

$$|\varphi_1-\varphi_2|_W \leq c\,\left[|\mathbf{u}_1-\mathbf{u}_2|_V + |q_1-q_2|_W\right]. \tag{57}$$

Since $q_i \in W^{1,2}(0,T;W)$ and $\mathbf{u}_i \in W^{1,\infty}(0,T;V)$, $i=1,2$, then we have $\varphi_\eta \in W^{1,2}(0,T;W)$. We show next that we have the estimation (56).

Let η_1, $\eta_2 \in W^{1;2}(0,T;H)$ and φ_1, φ_2, are respectively solutions of problems $\mathcal{P}^{\eta_i}$, $i=1,2$, and $\mathbf{u}_1$, $\mathbf{u}_2$, are solutions of problems $\mathcal{P}_1^{\eta_i}$, $i=1,2$. With similar arguments we deduce that $|\varphi_1-\varphi_2|_W \leq \frac{c_{\mathcal{E}}}{m_\gamma}|\mathbf{u}_1-\mathbf{u}_2|_V$. □

For the existence and uniqueness of solution β_η for the problem $\mathcal{P}_2^\eta$, we have

Lemma 3.3. *There exist a unique solution β_η for the problem $\mathcal{P}_2^\eta$ such that*

$$\beta_\eta \in L^2\left(0,T;L^2(\Omega)\right) \cap L^2\left(0,T;H^1(\Omega)\right). \tag{58}$$

If β_1, β_2 are two solutions of the problem $\mathcal{P}_\eta^2$ corresponding to data $\eta^2 = \eta_i^2$, $i=1,2$. Then, we have

$$|\beta_1(t)-\beta_2(t)|_{L^2(\Omega)}^2 \leq \int_0^t \left|\eta_1^2(s)-\eta_2^2(s)\right|_{L^2(\Omega)}^2\,ds. \tag{59}$$

Proof. Apply the Theorem 3.3, let $a : W \times W \to \mathbb{R}$ such that

$$a(\zeta, \xi) = \hat{k} \int_\Omega \nabla\zeta \nabla\xi \, dx, \quad \forall \zeta, \, \xi \in H^1(\Omega), \text{ a.e. } t \in (0,T). \tag{60}$$

We have $\hat{k} > 0$, then we can see that a satisfy the conditions of a Theorem 3.3 and then for a given $\beta_0 \in K$ and $, \eta^2 \in L^2(0,T;L^2(\Omega))$, there exist a unique function β_η which satisfies

$$\beta_\eta \in K, \, \beta_\eta \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2\left(0,T;H^1(\Omega)\right), \tag{61}$$

$$\left(\dot{\beta}_\eta(t), \zeta - \beta_\eta\right) + a(\beta_\eta(t), \zeta - \beta_\eta) \geq \left(\eta^2(t), \zeta - \beta_\eta\right), \quad \forall \zeta \in K \quad \text{a.e. } t) \tag{62}$$

$$\beta_\eta(0) = \beta_0. \tag{63}$$

Let $t \in [0,T]$ and β_1, β_2 the two solutions of problem $\mathcal{P}^{\eta_i}$, corresponding to data $\eta^2 = \eta_i^2$, $i = 1, 2$, then we have

$$|\beta_1(t) - \beta_2(t)|^2_{L^2(\Omega)} + \hat{k} \int_0^t |\nabla(\beta_1(s) - \beta_2(s))|^2_{L^2(\Omega)} \tag{64}$$

$$\leq \int_0^t \left|\eta_1^2 - \eta_1^2\right|^2_{L^2(\Omega)} ds + \int_0^t |\beta_1(t) - \beta_2(t)|^2_{L^2(\Omega)} ds. \tag{65}$$

Will τ_i, $i = 1, 2$, belongs to $L^2(0,T;L^2(\Omega))$, applying Gronwall's lemma to $|\beta_1(t) - \beta_2(t)|^2_{L^2(\Omega)}$, we have

$$|\beta_1(t) - \beta_2(t)|^2_{L^2(\Omega)} \leq \int_0^t \left|\eta_1^2(s) - \eta_1^2(s)\right|^2_{L^2(\Omega)} ds. \tag{66}$$

□

Let now define the operator

$$\Lambda : Q \to Q \text{ by}$$

where $Q = L^2\left(0,T;V' \times L^2(\Omega)\right), Q$ is a Banach space endowed with the norm

$$\left|(\eta^1, \eta^2)\right|_Q = \left|\eta^1\right|_{L^2(0,T;V')} + \left|\eta^2\right|_{L^2(0,T;L^2(\Omega))}$$

$$\Lambda(\eta) = \left(\mathcal{E}^* \nabla\varphi_\eta + \int_0^t \mathcal{C}(t-s, \boldsymbol{\varepsilon}(\boldsymbol{u}_\eta(s)), \beta_\eta(s)) ds \; ; \; \phi(\boldsymbol{\varepsilon}(\mathbf{u}_\eta), \beta_\eta)\right), \tag{67}$$

where $\mathbf{u}_\eta$, φ_η, β_η are respectively the unique solutions of problems $\mathcal{P}_1^\eta$, $\mathcal{P}^\eta$ and $\mathcal{P}_2^\eta$. We have this result.

Theorem 3.4. *The operator Λ has a unique fixed point*

$$\eta^* \in L^2\left(0,T;V' \times L^2(\Omega)\right).$$

Proof. Let $t \in [0,T]$, $\eta_i \in Q$, $i=1,2$, $\eta_1 = \left(\eta_1^1,\eta_1^2\right), \eta_2 = \left(\eta_2^1,\eta_2^2\right)$,and use the notation $\mathbf{u}_{\eta_i} = \mathbf{u}_i$, $\varphi_{\eta_i} = \varphi_i$ and $\beta_{\eta_i} = \beta_i$, $i=1,2$, then by applying proprieties (23 (b)), (24),

$$|\Lambda\eta_1 - \Lambda\eta_2|_{V'\times L^2(\Omega)} \leq C_{\mathcal{E}^*}\,|\varphi_1-\varphi_2|_W + |\phi\left(\boldsymbol{\varepsilon}\left(\mathbf{u}_1\right),\beta_1\right) - \phi\left(\boldsymbol{\varepsilon}\left(\mathbf{u}_2\right),\beta_2\right)|_{L^2(\Omega)}$$
$$+ \int_0^t |\mathcal{C}(t-s,\boldsymbol{\varepsilon}(\boldsymbol{u}_1(s)),\beta_1(s)) - \mathcal{C}\left(t,x,\boldsymbol{\varepsilon}\left(\mathbf{u}_2(s)\right),\beta_2(s)\right)|_{\mathcal{H}}\,ds,$$

and

$$|\Lambda\eta_1 - \Lambda\eta_2|_{V'\times L^2(\Omega)} \leq c\left\{|\varphi_1-\varphi_2|_W + |\mathbf{u}_1-\mathbf{u}_2|_V + |\beta_1-\beta_2|_{L^2(\Omega)}\right.$$
$$\left. + |\mathbf{u}_1-\mathbf{u}_2|_{L^2(0,t,V)} + |\beta_1-\beta_2|_{L^2(0,t;L^2(\Omega))}\right\}.$$

By the estimation (56) we have

$$|\Lambda\eta_1 - \Lambda\eta_2|_{V'\times L^2(\Omega)} \leq c\left\{|\mathbf{u}_1-\mathbf{u}_2|_V + |\beta_1-\beta_2|_{L^2(\Omega)}\right.$$
$$\left. + |\mathbf{u}_1-\mathbf{u}_2|_{L^2(0,t,V)} + |\beta_1-\beta_2|_{L^2(0,t;L^2(\Omega))}\right\}.$$

Apply now the inequalities (52) and (59), this leads to

$$\begin{aligned}|\Lambda\eta_1 - \Lambda\eta_2|_{V'\times L^2(\Omega)} &\leq c\left\{\left|\eta_1^1-\eta_2^1\right|_{L^2(0,t,V')} + \left|\eta_1^2-\eta_2^2\right|_{L^2(0,t;L^2(\Omega))}\right\}\\ &= c\,|\eta_1-\eta_2|_Q.\end{aligned}$$

Reiterating the estimation n times yields

$$|\Lambda^n\eta_1 - \Lambda^n\eta_2|_Q \leq \frac{(Tc)^n}{n!}\,|\eta_1-\eta_2|_Q.$$

Thus imply that the operator Λ is a contraction on Q. By Banach's fixed point theorem, Λ has a unique fixed point $\eta^* \in L^2(0,T;V'\times L^2(\Omega))$. □

Proof. *Proof of Theorem* 3.1

Existence. Let $\eta^* \in Q$ a unique fixed point of Λ and $(\mathbf{u}_{\eta^*};\varphi_{\eta^*};\beta_{\tau^*})$ be the solution of the three problems $\mathcal{P}_1^{\eta^*}$, $\mathcal{P}_2^{\eta^*}$and $\mathcal{P}^{\eta^*}$. Then, using (37), (38), (39) and (67). Keeping in mind that $\Lambda\left(\eta^*\right) = \eta^*$, we deduce that $(\mathbf{u}_{\eta^*};\varphi_{\eta^*};\beta_{\tau^*})$ is the solution of the problem $\mathcal{P}_V$. The regularity (41) provide from Lemma 3.1, Lemma 3.2 and Lemma 3.3.

Uniqueness. The uniqueness of solutions in Theorem 3.1 is the consequence of the uniqueness of the fixed point of the operator Λ given by (67). □

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Upshots of cataclysmic chemical reaction and thermophoresis on hydromagnetic flow past an inclined plate

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Abstract. A probe is made into free convection flow of electrically conducting incompressible viscous fluid passing over a plate inclined at some angle to the vertical direction. The fluid is considered to be flown through a porous medium with invariable permeability with non-uniform temperature and concentration. The very essence of this study is to scrutinize the possible effect of thermophoresis and angle of inclination on the event of fluid flow in presence of cataclysmic chemical reaction and external heat absorption. Governing equations after being transformed into suitable non-dimensional forms have been solved numerically using Matlab solver bvp4c. The effects of pertinent flow parameters are assessed. This particular analysis delineates that externally applied magnetic field, chemical reaction, thermophoresis and angle of inclination contribute in reversal of fluid velocity but permeability of porous medium prevents retardation of the same. Presence of heat sink brings down the warmth of the fluid flow. It is reported that cataclysmic chemical reaction and thermophoresis dilute the fluid concentration. Influence of different flow parameters on skin friction and Nusselt number are also examined. Graphical depictions of all these results are put forth for visual substantiation.

Keywords: porous medium, hydromagnetic, chemical reaction, heat absorption, thermophoresis.

1. Introduction

Hydromagnetic flow of a viscous fluid past an inclined plate with heat absorption and chemical reaction under free convection has allured attentions form diverse directions considering the fact that it has fundamental applications in

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various problems of engineering field, for instance in chemical and metallurgical industries, in nuclear reactors for cooling purpose, etc.

Pop et al. [1] have discussed free convection boundary layer over a flat plate with a uniform surface with heat flux which is inclined at a small angle to the horizontal. Keller box numerical method has been employed to obtain the solution. An extensive formulation natural convection over an inclined heated plate is put forwarded by Umemura and Law [2]. Unsteady natural convection hydromagnetic flow past a plate with inclination with heat and mass fluctuation has been inspected by Ganesan and Palani [3]. They have employed finite difference method for analysis of the problem. Chen [4] has carried out the study of heat and mass transfer in magnetohydrodynamic flow past a permeable inclined plate taking Ohmic heating and viscous dissipation into consideration. Kandasamy and Devi [5] have ascertained the aftermath of chemical reaction, heat and mass transfer on non-linear hydrodynamic flow over a wedge with suction and injection. The problem of radiative heat and mass transfer over an inclined plate in presence of chemical reaction has been discussed by Kumar [6].

Free convection MHD heat and mass transfer flow past an inclined plate with heat generation has been studied by Alam et al. [7]. For solving the problem a particular shooting iteration procedure have been applied by them. Uddin and Kumar [8] have examined unsteady fluid flow with viscous dissipative heat past an inclined plate imbedded in a perforated medium. Fluid flow past a plate inclined to horizontal with fluctuating surface temperature under convection effects has been scrutinized by Palani [9]. Their studies bring forth the observation that angle of inclination of the plate retards the tempo of fluid flow and coefficient of heat transfer. Singh and Makinde [10] have inspected computational dynamic of free convection electrically conducting fluid flow along an inclined plate in presence of Newtonian heating and volumetric heat source. Pattnaik et al. [11] have elucidated radiation and mass transfer aftermath on free convection electrically conducting fluid through porous medium. Influence of heat radiation on unsteady flow past an inclined plate with cataclysmic chemical reaction has been assessed by Barik et al. [12].

Navier slip and Newtonian heating condition on MHD flow using computational modelling has been evaluuated by Makinde [13]. Mishra et al. [14] have examined the influence of spatial and thermal transfer on a viscoelastic fluid with suction and heat generation. Conjugated impact of variable thermal and concentration shift on free convective electromagnetic flow past an impulsively started plate has been realised by Ali et al. [15]. Similarity transformation of heat as well as mass transfer consequence on MHD dissipative fluid passing over an inclined porous surface with chemical reaction has been investigated by Reddy et al. [17]. They utilised numerical technique to obtain the solution. Raju et al. [18] have inspected heat transfer effects on a dissipative viscous fluid past a vertical plate in presence of induced magnetic field. They have assumed the plate to be electrically non-conducting.

Reddy et al. [19] have scrutinized the influence of chemical reaction, radiation and rotation on electrically conducting nanofluid flow past a permeable plate in porous medium. Analytical solution for heat and mass transfer flow past a semi-infinite vertical porous plate with huge suction has been discussed by Khan et al. [20]. Selvarani and Govindarajan [21] have examined radiative fluid flow over an inclined plate. The duo have considered a nanofluid flow and the plate to be in nonuniform surface temperature. Free convection viscous dissipative flow past an inclined plate with heat transfer effects has been studied by Palani and Arutchelvi [22].

Thermophoresis is the movement of minuscule particles suspended in a non-isothermal fluid. The suspended particles due to presence of thermal gradient acquire a velocity relative to the fluid in the direction of retarding temperature. Such type of phenomenon has useful applicability in inventory drug mechanism, aerosol reactors, heat exchange fouling etc. Selim et al. [23] have discussed the effect of surface mass transfer on mixed convection flow past a heated vertical flat permeable plate with thermophoresis. Postelnicu [24] has examined the consequences of thermophoresis particle deposition in free convection boundary layer from a horizontal flat plate embedded in a porous medium. Effects of thermophoresis and radiation on laminar flow along a semi-infinite vertical plate has been analysed by Bakier and Gorla [25].

Noor et al. [26] have explored influence of heat and mass transfer of thermophoretic MHD flow over an inclined isothermal permeable surface in the presence of radiation and heat source/sink. Kundu et al. [27] have investigated combined effects of thermophoresis and chemical reaction on magnetohydrodynamic convection flow. Thermophoretic hydromagnetic dissipative heat and mass transfer with lateral mass flux, heat source, Ohmic heating and thermal conductivity effects have been explored by Zeuco et al. [28].

The essence of this probe is to scrutinize the possible effect of thermophoresis and angle of inclination on the event of fluid flow in presence of cataclysmic chemical reaction and external heat absorption in view of some physical situations,in particular drainage mechanism where the geometry of flow gets situated in situations other than two extreme positions viz.vertical and horizontal plate.

2. Formulation of the problem

A free convective flow of a viscous fluid past a semi-infinite plate is considered. The fluid is taken to be viscous and incompressible. The plate is inclined at an angle ψ to the vertical and is considered to be at constant concentration gradient. The flow is along the plate. $\overline{X}$ axis is considered along the plate and $\overline{Y}$ axis perpendicular to it. A uniform magnetic field of strength B_0 is applied parallel to $\overline{Y}$ axis. Induced magnetic field is neglected under the criteria that the magnetic Reynolds number is very small. Under Boussinesq's approximation, the boundary layer equations of mass, momentum, energy and concentration

past an inclined plate are given by

$$\frac{\partial V}{\partial \overline{Y}} = 0 \Rightarrow V = \text{constant} = -V_W, V_W > 0, \tag{1}$$

$$V\frac{\partial U}{\partial \overline{Y}} = \nu\frac{\partial^2 U}{\partial \overline{Y}^2} - \frac{\sigma B_0^2}{\rho}U - \frac{\nu}{K_P}U + g\beta\left(T - T_\infty\right)\cos\psi + g\bar{\beta}\left(C - C_\infty\right)\cos\psi, \tag{2}$$

$$V\frac{\partial T}{\partial \overline{Y}} = \frac{\kappa}{\rho C_P}\frac{\partial^2 T}{\partial \overline{Y}^2} + \frac{\nu}{C_P}\left(\frac{\partial U}{\partial \overline{Y}}\right)^2 - \frac{Q}{\rho C_P}\left(T - T_\infty\right), \tag{3}$$

$$V\frac{\partial C}{\partial \overline{Y}} = D\frac{\partial^2 C}{\partial \overline{Y}^2} - K_C\left(C - C_\infty\right) - \frac{\partial}{\partial \overline{Y}}\left[V_T\left(C - C_\infty\right)\right], \tag{4}$$

where U and V are velocity components along $\overline{X}$ and $\overline{Y}$ respectively, ν is kinematic viscosity, β, $\bar{\beta}$ are coefficients of thermal and spatial expansion, respectively, T, C are fluid temperature and concentration, respectively, T_∞,C_∞ are respective fluid temperature and concentration far away from the plate, σ is electrical conductivity, ρ is density of the fluid, K_P is permeability coefficient, κ is thermal conductivity, Q is heat absorption coefficient, C_P is specific heat at constant pressure, D is mass diffusion coefficient, K_C is rate of chemical reaction and $V_T(= -\frac{\nu K_T}{T_r}\frac{\partial T}{\partial \overline{Y}})$ is thermophoretic velocity with K_T and T_r being thermophoretic coefficient and reference temperature respectively.

The boundary conditions are:

$$\begin{aligned} &\overline{Y} = 0, U = U_W, T = T_W, -D\frac{\partial C}{\partial \overline{Y}} = C_W, \\ &\overline{Y} \to \infty, U \to 0, T \to T_\infty, C \to C_\infty, \end{aligned} \tag{5}$$

where U_W, T_W are velocity and temperature at the plate and C_W is concentration gradient at the plate.

For convenience, the following dimensionless variables and physical quantities are introduced:

$$\begin{aligned} &\eta = \frac{\overline{Y}}{V_W}, f = \frac{U}{U_W}, \theta = \frac{T - T_\infty}{T_W - T_\infty}, \\ &\phi = \frac{(C - C_\infty)V_W D}{C_W \nu}, G_r = \frac{g\beta(T_W - T_\infty)\nu}{U_W V_W^2}, G_m = \frac{g\bar{\beta}C_W\nu^2}{U_W V_W^3 D}, \\ &M = \frac{B_0^2\nu\sigma}{V_W^2\rho}, P_r = \frac{\rho\nu C_P}{\kappa}, K = \frac{K_P V_W^2}{\nu^2}, \\ &S_C = \frac{\nu}{D}, E = \frac{U_W^2}{(T_W - T_\infty)C_P}, S = \frac{Q\nu}{\rho C_P V_W^2}, \\ &\gamma = \frac{\nu K_C}{V_W^2}, \delta = -\frac{K_T(T_W - T_\infty)}{T_r}, \end{aligned} \tag{6}$$

where η is dimensionless ordinate, f is dimensionless velocity in the direction of

the plate, θ and ϕ are non-dimensional temperature and species concentration,G_r and G_m are Grashof numbers for heat transfer and mass transfer respectively, M is magnetic parameter, P_r is Prandtl number,$K$is permeability parameter, S_C is Schmidt number, E is Eckert number, S is heat absorption parameter, γ is chemical reaction parameter and δ is thermophretic parameter.

Using introduced surrogates from (6) in the Equations (2) to (4), the following demensionless equations are obtained:

$$f'' = -f' + \left(M + \frac{1}{K}\right) f - G_r\theta\cos\psi - G_m\phi\cos\psi, \tag{7}$$

$$\theta'' = -P_r\theta' + P_rS\theta - P_rE\left(f'\right)^2, \tag{8}$$

$$\phi'' = -S_C\phi' + S_C\gamma\phi + S_C\delta\left(\phi\theta'' + \phi'\theta'\right). \tag{9}$$

Boundary conditions represented by (5) transmute into the following shape:

$$\begin{aligned} &\eta = 0, f = 1, \theta = 1, \phi' = -1 \\ &\eta \to \infty, f \to 0, \theta \to 0, \phi \to 0. \end{aligned} \tag{10}$$

3. Method of solution

The Equations (7) to (9) with the boundary conditions (10) have been solved numerically using Matlab's built in solver bvp4c. The dimensionless velocity, skin friction, temperature, Sherwood number and concentration are represented in pictorial form with the help of graphs obtained through the computing environment supported by Matlab.

4. Results and discussion

The graphical delineations of the fluid locomotion and its thermal and spatial transfer affairs bring out the impact of various parameters that dictate flow of the fluid. Amongst the visible influences of different pertinent flow parameters, thermophoresis along with chemical reaction have been kept around the nucleus of the observation. Moreover, the effect of inclined plate on the fluid mobility and other flow characteristics has been observed. It has been emphasised that the angle of inclination is acute with exclusion of both the extreme values ($\psi = 0$ and $\psi = \pi/2$). This study is carried out with numeric values of parameters as $G_r = 5, G_m = 6, P_r = 2.4, M = 2, K = 0.5, E = 0.01, S = 2, S_C = 0.4, \psi = \pi/4, \gamma = 0.5$ and $\delta = 0.7$ unless otherwise stated.

Figures 1-6 evince the velocity profile for some key flow parameters. The profiles depict the behaviour of velocity in the region of fluid flow. It is observed that velocity drops suddenly in the vicinity of the plate and then maintains slow and steady diminution to the boundary condition away from the plate.

Figure 1 displays the impact of permeability of porous medium on velocity. The locomotive strength of the fluid gets encouraged with the rise of permeability parameter (K). Influence of magnetic parameter (M) on velocity is shown

in Figure 2. It disposes the fact that growth in the intensity of magnetic field inflicts a retarding effect on fluid velocity. Figure 3 presents the role of angle of inclination of the plate in mobility of fluid. While excluding the extreme values of angle of inclination, ψ $(0 < \psi < \pi/2)$, it is spotted that an increment in angle of inclination dwindles the velocity. Effect of chemical reaction parameter(γ), thermophoretic parameter(δ) and heat absorption parameter(S) have been represented by Figure 4, Figure 5 and Figure 6 respectively. In all the cases, an increment in the parameters results into a significant depletion in the velocity of the fluid.

Figures 7-11 illustrate the impact of permeability parameter, magnetic parameter, angle of inclination, chemical reaction parameter and thermophoretic parameter on skin friction (S_F). In view of maintaining uniformity, the impact of fore-said parameters are measured against different values of thermal Grashof number (G_r).

Figure 7 elucidates that with the hike in numerical values of permeability , the skin friction gets enhanced. Influence of magnetic parameter on skin friction has been exhibited in Figure 8. It proclaims the fact that with a hoist in magnetic parameter the skin friction gets elevated. Figure 9 assists in visualising the effect of angle of inclination the plate on skin friction. It helps in comprehending that angle of inclination induces a hindering effect on skin friction. However, the amount of hindrance inflicted for a specific position of inclination shows some fluctuation with reference to thermal buoyancy(G_r). Figure 10 and Figure 11 illustrate that chemical reaction and thermpophoresis prompt escalation in skin friction.

Figure 12 to Figure 14 elucidate the sleek variation of temperature all over the region of fluid flow. Figure 12 exhibits that with declination in thermal diffusion (i.e. rise in Prandtl number, P_r) prompts drop in fluid temperature while Figure 13 confirms that increment in kinetic energy relative to enthalpy (i.e. increase in Eckert number,E) induces rise in temperature of the fluid. Further, intensification of strength of heat absorption (S) cast significant reduction in temperature as demonstrated by Figure 14.

Figure 15 to Figure 17 demonstrate the impact of chemical reaction, thermophoresis and inclination of the plate on Nusselt number(Nu). Figure 15 and Figure 16 illustrate that inflation in chemical reaction parameter (γ) and thermophoresis parameter(δ) have impeding influence on the Nusselt number. This suggests that more there are cataclysmic chemical reaction and thermophoresis near the plate,less is the rate of heat transfer. However, increment in angle of inclination instigates an escalation in Nusselt number which is shown by Figure 17.

Figure 18 to Figure 20 represent the effect of relevant flow parameters on species concentration. Figure 18 depicts the effect of Schmidt number(S_C) on species concentration. It is noticed that a surge in Schmidt number reduces the concentration. Similar impact on species concentration is inflicted by chemical reaction parameter and thermophoresis parameter as indicated by Figure

19 and Figure 20, respectively. Thus, denser fluid species with low diffusion (larger S_C values) exposed to high rates of cataclysmic chemical reaction and thermophoresis results into diminution of the concentration throughout the flow region.

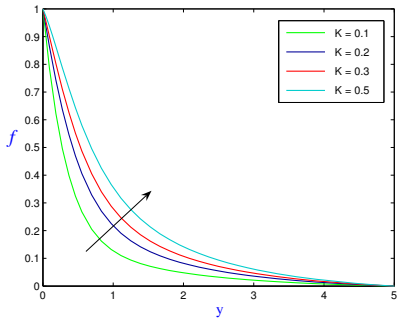


Figure 1: Velocity for K

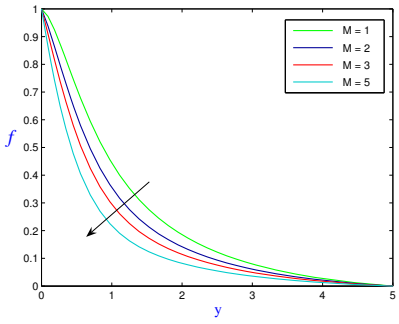


Figure 2: Velocity for M

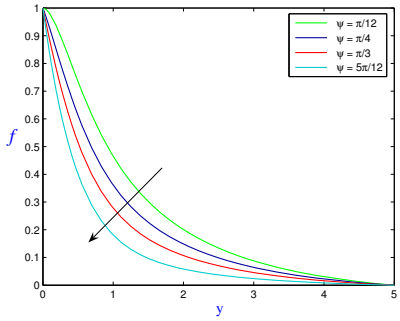


Figure 3: Velocity for ψ

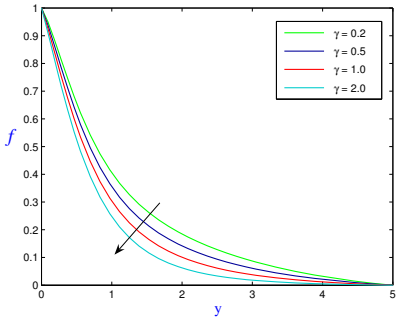


Figure 4: Velocity for γ

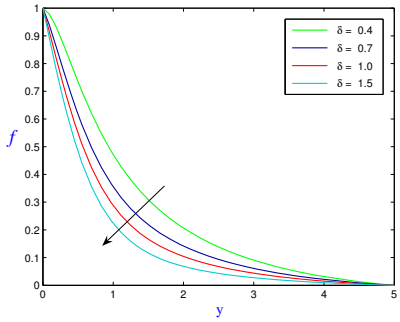


Figure 5: Velocity for δ

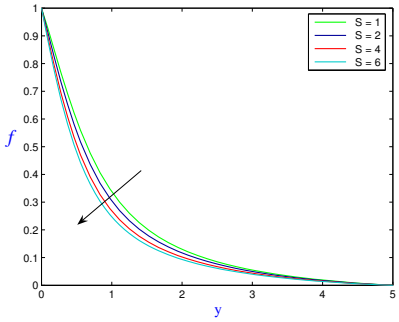


Figure 6: Velocity for S

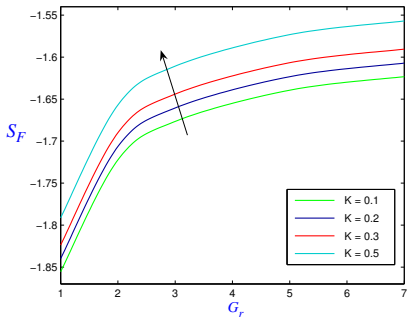


Figure 7: Skin friction for K

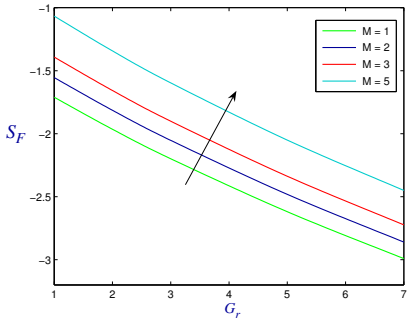


Figure 8: Skin friction for M

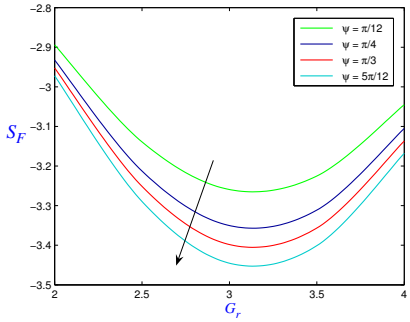


Figure 9: Skin friction for ψ

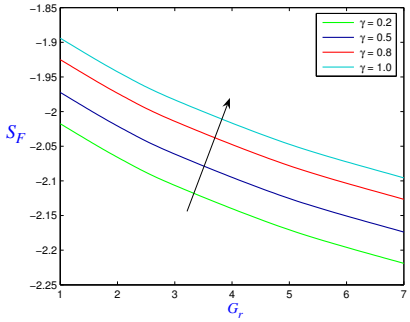


Figure 10: Skin friction for γ

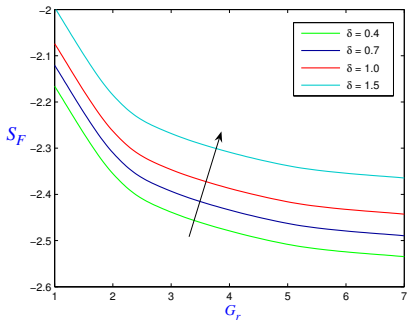


Figure 11: Skin friction for δ

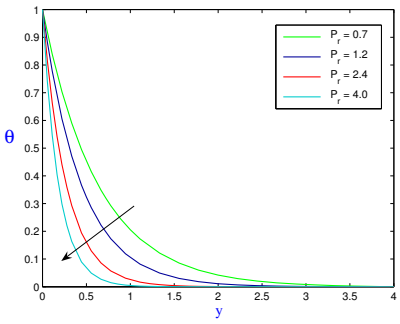


Figure 12: Temperature for P_r

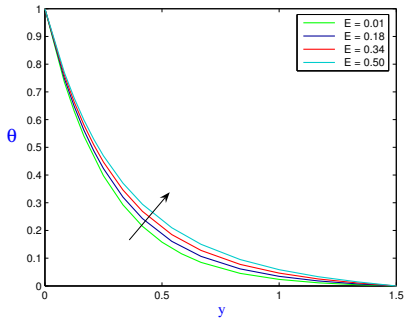


Figure 13: Temperature for E

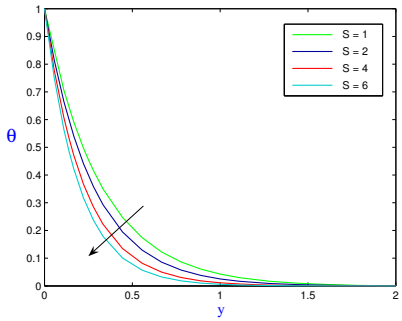


Figure 14: Temperature for S

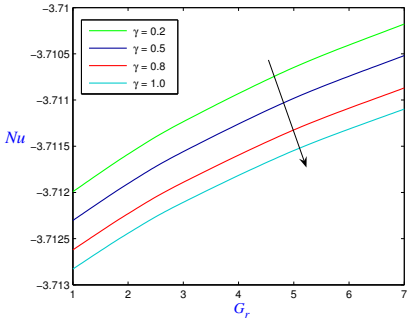


Figure 15: Nusselt number for γ

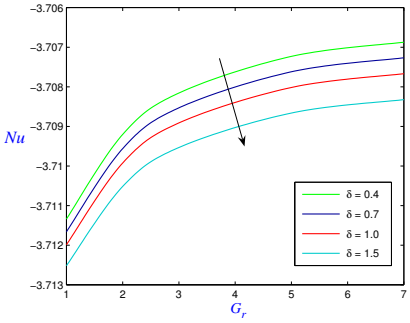


Figure 16: Nusselt number for δ

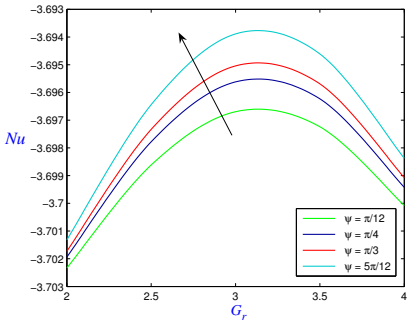


Figure 17: Nusselt number for ψ

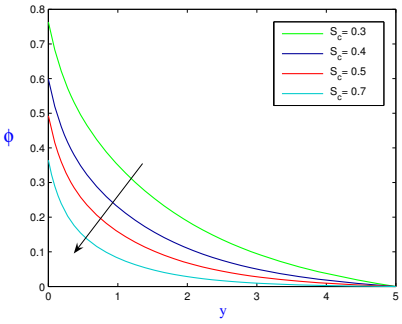


Figure 18: Concentration for S_C

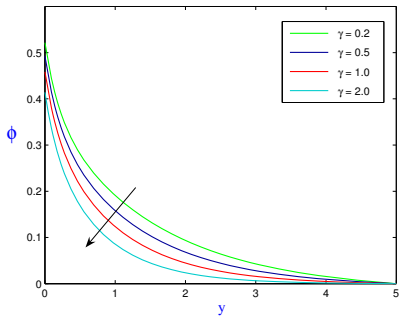


Figure 19: Concentration for γ

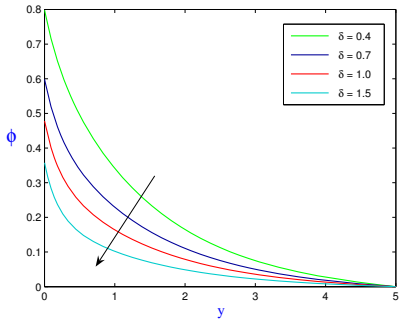


Figure 20: Concentration for δ

5. Conclusion

The present exploration regarding free convection flow of electrically conducting incompressible viscous fluid past a plate inclined at some angle under the effect of thermophoresis, cataclysmic chemical reaction and external heat absorption unveils some interesting inferences.

It is observed that intensification of magnetic parameter inflicts hindrance to fluid velocity while the same action on permeability of porous medium offers assistance to growth of velocity. Angle of inclination of the plate and thermophoresis impose impeding effect in forward locomotion of the fluid. Further, inflation in cataclysmic chemical reaction and heat absorbing source have dwindling effect on flow velocity.

An increment in the angle of inclination of the plate intrudes a reduction in skin friction while rise in magnetic parameter and permeability parameter encourages increase in skin friction. Thermophoresis and chemical reaction, when escalated, bring about growth in values of skin friction.

The temperature of the fluid flow gets lowered with the elevation in values of Prandtl number and heat absorption parameter while an opposite scenario is observed with the increment of Eckert number.

An increment in the angle of inclination encourages the growth of Nusselt number. However, increment in rate of chemical reaction and thermophoresis inflict suppressing effect on Nusselt number.

The concentration of the fluid flow erodes with the inflation in Schmidt number, chemical reaction parameter and thermophoreis parameter.

Acknowledgement

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L^p-boundedness of fractional maximal operator

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Abstract. In this paper, we are introducing a new generalised one-sided fractional maximal operator $M^+_{g,\alpha}$ and our aim is to study its boundedness between weighted Lebesgue spaces L^p_w.

Keywords: maximal operator, fractional maximal operator, weighted Lebesgue spaces, $S^+_{p,g,\alpha}$ class.

1. Introduction

The Hardy-Littlewood maximal function for a locally integrable function f on R^n is defined as

$$Mf(x) = \sup_{x\in Q} \frac{1}{|Q|} \left(\int_Q |f(t)|dt \right),$$

where the sup is taken over all the cubes Q in R^n and $|Q|$ denotes the Lebesgue measure of Q.

A weight w is measurable, locally integrable function which is finite and positive almost everywhere on $(0, \infty)$. The space L^p_w consists of all measurable functions f for which the quantity

$$\|f\|_{L^p_w} = \left(\int |f|^p w \right)^{\frac{1}{p}}$$

is finite.

∗. Corresponding author

B. Muckenhoupt [7] in 1972, proved that for $1 < p < \infty$,

$$\int_{-\infty}^{+\infty} (Mf(x))^p \, w(x)dx \leq C \int_{-\infty}^{+\infty} |f(x)|^p w(x)dx,$$

for all f holds if and only if $w \in A_p$, i.e.,

$$\left(\frac{1}{h}\int_a^{a+h} w(x)dx\right)\left(\frac{1}{h}\int_a^{a+h} w(x)^{\frac{-1}{p-1}}dx\right)^{p-1} \leq C.$$

E.T. Sawyer [9] in 1982 proved that if w and v are non-negative functions on R^n and $1 < p \leq q < \infty$, then

$$\left(\int (Mf(x))^q w(x)dx\right)^{\frac{1}{q}} \leq C\left(\int f^p(x)v(x)dx\right)^{\frac{1}{p}},$$

for all f in $L^p(v)$ if and only if

$$\int \left((M(\chi_Q v^{1-p'})(x))^q w(x)dx\right)^{\frac{1}{q}} \leq C\left(\int_Q v^{1-p'}(x)dx\right)^{\frac{1}{p}} < \infty,$$

for all cubes Q.

In [2], the authors studied the operator M_α^+ defined by

$$M_\alpha^+ f(x) = \sup_{h>0} h^{\alpha-1}\left(\int_x^{x+h} |f(t)|dt\right)$$

and gave a new characterisation for the pairs of weights in which M_α^+ is bounded.

Consider now the maximal operator defined as

$$M_g^+ f(x) = \sup_{h>0}\left(\int_x^{x+h} |f(t)|g(t)dt\right)\left(\int_x^{x+h} g(t)dt\right)^{-1},$$

where g is a positive and locally integrable function on R.

In [5], F.J.Martin-Reyes et al characterised the pairs of weights such that the operator M_g^+ where g is a locally integrable and positive function in R is of strong and weak type with different sets of weights.

In this paper, we are introducing a new one-sided fractional maximal operator

$$M_{g,\alpha}^+ f(x) = \sup_{h>0}\left(\int_x^{x+h} |f(t)|g(t)dt\right)\left(\int_x^{x+h} g(t)dt\right)^{\alpha-1}, 0 \leq \alpha < 1.$$

Remark 1.1. For $\alpha = 0$, $M_{g,\alpha}^+$ becomes the operator M_g^+ studied in the paper [5].

For $g = 1$, $M_{g,\alpha}^+$ reduces to the operator M_α^+ studied in the paper [2].

For $\alpha = 0$ and $g = 1$, $M_{g,\alpha}^+$ gives the operator M^+ studied in [8].

For the further development of these topics, one may refer to [1], [3], [4], [10] and [6].

Our primal aim here is to characterise the pairs of non-negative functions (u, v) for which the operator $M^+_{g,\alpha}$ is bounded from L^p_v to L^p_u.

Throughout the paper, $|Q|$ and χ_Q denote the Lebesgue measure and the characteristic function of the set Q, respectively, and $0.\infty$ is assumed to be 0. The letter C denotes a positive constant which is not necessarily the same at each occurence. The conjugate exponent of p is p' where $\frac{1}{p} + \frac{1}{p'} = 1$ and $p \geq 1$.

Firstly, we introduce the following definitions for $0 \leq \alpha < 1$.

Definition 1.1. *Let $p > 1$ and u, v be nonnegative functions. We say that a pair (u, v) satisfies condition $S^+_{p,g,\alpha}$ if there exists a constant $C > 0$ such that for every $I = (a, b)$ with $\int_{(-\infty,a)} u > 0$,*

$$\int_a^b \left(M^+_{g,\alpha}\left(\chi_I g^{\frac{1}{p-1}}\sigma\right)\right)^p u \leq C \int_a^b g^{p'}\sigma < \infty,$$

where $\sigma = v^{\frac{-1}{p-1}}$.

Definition 1.2. *Let $p > 1$ and u, v be nonnegative functions. We say that a pair (u, v) satisfies condition $S^-_{p,g,\alpha}$ if there exists a constant $C > 0$ such that for every $I = (a, b)$ with $\int_{(-\infty,a)} u > 0$,*

$$\int_a^b \left(M^-_{g,\alpha}\left(\chi_I g^{\frac{1}{p-1}}\sigma\right)\right)^p u \leq C \int_a^b g^{p'}\sigma < \infty,$$

where $\sigma = v^{\frac{-1}{p-1}}$.

Definition 1.3. *Let $p > 1$ and u, v be nonnegative functions. We say that a pair (u, v) satisfies condition $S_{p,g,\alpha}$ if there exists a constant $C > 0$ such that for every $I = (a, b)$ with $\int_{(-\infty,a)} u > 0$,*

$$\int_a^b \left(M_{g,\alpha}\left(\chi_I g^{\frac{1}{p-1}}\sigma\right)\right)^p u \leq C \int_a^b g^{p'}\sigma < \infty,$$

where $\sigma = v^{\frac{-1}{p-1}}$.

2. Strong boundedness

We prove the following:

Theorem 2.1. *Let u and v be weight functions and $p > 1$. The operator $M^+_{g,\alpha}$ is bounded from L^p_v to L^p_u if and only if $(u, v) \in S^+_{p,g,\alpha}$.*

Proof. Let $I = (a, b)$. The necessity follows by taking $f = \chi_I g^{\frac{1}{p-1}} v^{\frac{-1}{p-1}}$.

For the sufficiency, without loss of generality, we assume the function f in L^p_v to be non-negative bounded function having compact support. Let $O_k = \{x \in R : M^+_{g,\alpha} f(x) > 2^k\} \cap (-N, +\infty)$, where N is any positive integer and k is any positive real number. We note that O_k is open for each k, therefore, there exists a sequence $\{I_{jk}\}$ of open pairwise disjoint intervals with the following properties: $O_k = \cup_j I_{jk}$ and

$$\int_x^{b_{jk}} gf \geq 2^k \left(\int_x^{b_{jk}} g \right)^{(1-\alpha)} \tag{1}$$

holds for all $x \in I_{jk} = (a_{jk}, b_{jk})$.

Let $A_{jk} = \{x \in I_{jk} : \int_x^{b_{jk}} g^{p'} \sigma = \infty\}$, for every j and k.

If $A_{jk} \neq \varnothing$, let $x_{jk} = \sup A$; if $A_{jk} = \varnothing$, let $x_{jk} = a_{jk}$. It is clear that $\int_x^{b_{jk}} g^{p'} \sigma < \infty$ if $x > x_{jk}$ and $u = 0$ a.e. x in (a_{jk}, x_{jk}) by $S^+_{p,g,\alpha}$. Let $E_{jk} = I_{jk} \cap \left\{x : M^+_{g,\alpha} f(x) \leq 2^{k+1}\right\}$ and $F_{jk} = (x_{jk}, b_{jk}) \cap E_{jk}$.

Clearly, the sets E_{jk} are pairwise disjoint and I_{jk}, $\cup_j E_{jk} = \{x : 2^k < M^+_{g,\alpha} f(x) \leq 2^{(k+1)}\} \cap (-N, \infty)$ for every k. Then

$$\int_{-N}^{+\infty} \left(M^+_{g,\alpha} f(x)\right)^p u(x) dx = \sum_k \int_{\cup_j E_{jk}} \left(M^+_{g,\alpha} f(x)\right)^p u(x) dx$$

$$= \sum_{k,j} \int_{E_{jk}} \left(M^+_{g,\alpha} f(x)\right)^p u(x) dx = \sum_{k,j} \int_{F_{jk}} \left(M^+_{g,\alpha} f(x)\right)^p u(x) dx.$$

Using (2.1) and the definition of F_{jk}, we get

$$\left(M^+_{g,\alpha} f\right)^p u \leq 2^p \left(\int_x^{b_{jk}} gf \right)^p \left(\int_x^{b_{jk}} g \right)^{(\alpha-1)p}.$$

Now, by the last argument, we have

$$\int_{-N}^{\infty} \left(M^+_{g,\alpha} f\right)^p u(x) dx \leq 2^p \sum_{k,j} \int_{F_{jk}} \left(\int_x^{b_{jk}} gf \right)^p \left(\int_x^{b_{jk}} g \right)^{(\alpha-1)p} u(x) dx$$

$$= 2^p \sum_{k,j} \int_{F_{jk}} \left(\int_x^{b_{jk}} gf \right)^p \left(\int_x^{b_{jk}} g^{p'} \sigma \right)^{-p} \left(\int_x^{b_{jk}} g^{p'} \sigma \right)^p \left(\int_x^{b_{jk}} g \right)^{(\alpha-1)p} u(x) dx. \tag{2}$$

Let m and ν denote the Lebesgue measure on R and the counting measure on Z respectively.

Define a function ϕ on X as

$$\phi(j, k, x) = \chi_{F_{jk}(x)} \left(\int_x^{b_{jk}} g^{p'} \sigma \right)^p \left(\int_x^{b_{jk}} g \right)^{(\alpha-1)p} u(x)$$

and

$$Th(j,k,x) = \int_x^{b_{jk}} hg^{p'}\sigma \left(\int_x^{b_{jk}} g^{p'}\sigma\right)^{-1},$$

where $X = Z \times Z \times R$ and $w = \nu \times \nu \times m$ is the product measure. The inequality (2.2) transfers to

$$\int_{-N}^{\infty} \left(M_{g,\alpha}^{+} f\right)^p u \leq 2^p \int_X \left(T(f(g^{-1}v)^{\frac{1}{p-1}})\right)^p \phi dw.$$

Next, we establish that the operator T is bounded. According to Marcinkiewicz interpolation theorem, it is sufficient to show that

$$\int_{\{(j,k,x)\in X: Th(j,k,x)>\lambda\}} \phi dw \leq \frac{C}{\lambda} \int_{-\infty}^{+\infty} hg^{p'}\sigma dx.$$

Let

$$A_{jk}(\lambda) = F_{jk} \cap \{x : Th(j,k,x) > \lambda\},$$

$$s_{jk}(\lambda) = \inf A_{jk}(\lambda)$$

and

$$J_{jk} = J_{jk}(\lambda) = [s_{jk}(\lambda), b_{jk}).$$

We observe that $A_{jk}(\lambda)$ are pairwise disjoint and any two intervals J_{lm} and J_{jk} are either disjoint or one of them is contained in the other. In addition

$$\int_{J_{jk}} hg^{p'}\sigma \geq \lambda \int_{J_{jk}} g^{p'}\sigma. \tag{3}$$

The intervals J_{jk} have bounded sizes which allows us to extract a maximal subcollection $\{J_i\}$ in such a way that each J_i verify (2.3). Then by using the facts that $(u,v) \in S_{p,g,\alpha}^{+}$ and J_i are maximal elements and (2.3), we have

$$\begin{aligned}
&\int_{\{(j,k,x)\in X: Th(j,k,x)>\lambda\}} \phi dw \\
&= \sum_{(k,j)} \int_{A_{jk(\lambda)}} \left(\int_x^{b_{jk}} g^{p'}\sigma\right)^p \left(\int_x^{b_{jk}} g\right)^{(\alpha-1)p} u(x)dx \\
&\leq \sum_i \sum_{\{(k,j): J_{jk}\subset J_j\}} \int_{A_{jk(\lambda)}} \left(\int_x^{b_{jk}} g^{p'}\sigma\right)^p \left(\int_x^{b_{jk}} g\right)^{(\alpha-1)p} u(x)dx \\
&\leq \sum_i \left(M_{g,\alpha}^{+}(\chi_{J_i} g^{\frac{1}{p-1}}\sigma)(x)\right)^p u(x)dx \\
&\leq C \sum_i \int_{J_i} g^{p'}(x)\sigma(x)dx \leq \frac{C}{\lambda} \sum_i \int_{J_i} h(x)g^{p'}(x)\sigma(x)dx \\
&\leq \frac{C}{\lambda} \int_{-\infty}^{+\infty} h(x)g^{p'}(x)\sigma(x)dx
\end{aligned}$$

which proves T is bounded from $L^p(g^{p'}\sigma dx)$ to $L^p(X, \phi dw)$. As N approaches to infinity,

$$\int_{-N}^{+\infty} \left(M^+_{g,\alpha} f\right)^p u \leq C.2^p \int_{-\infty}^{+\infty} \left(f(g^{-1}v)^{\frac{1}{p-1}}\right)^p g^{p'}\sigma dx = C.2^p \int_{-\infty}^{+\infty} f^p v$$

we have the desired inequality. □

Remark 2.1. Analogous result hold changing $M^+_{g,\alpha}$ by $M^-_{g,\alpha}$ and $S^+_{p,g,\alpha}$ by $S^-_{p,g,\alpha}$ where

$$M^-_{g,\alpha} f(x) = \sup_{h>0} \left(\int_{x-h}^{x} |f(t)|g(t)dt\right) \left(\int_{x-h}^{x} g(t)dt\right)^{\alpha-1}, 0 \leq \alpha < 1.$$

Proposition 2.1. *For* $1 < p < \infty$,

$$S_{p,g,\alpha} = S^+_{p,g,\alpha} \cap S^-_{p,g,\alpha}.$$

Proposition 2.2.

$$\frac{1}{2}\left(M^+_{g,\alpha} + M^-_{g,\alpha}\right) \leq M_{g,\alpha} \leq M^+_{g,\alpha} + M^-_{g,\alpha}.$$

Remark 2.2. Let u and v be two weights on R. If (u, v) satisfies strong condition then (u, v) also satisfies the weak condition from the fact that strong convergence implies weak convergence.

3. Conclusion

Theorem 3.1. *If* $p > 1$ *and* u *and* v *are weight functions, then the operator* $M_{g,\alpha}$ *is bounded from* L^p_v *to* L^p_u *if and only if* $(u, v) \in S_{p,g,\alpha}$.

Proof. The proof follows from the Propositions 2.1 and 2.2. □

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A-contractions relative to a weak distance

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Abstract. Among self maps of a metric space, Akram et. al have considered a class of contractions called A-contractions. On a metric space the concept of a weak distance was defined by Kada et.al.

In this paper we introduce *A-contraction relative to a weak distance* among the self maps of a metric space and prove three fixed point theorems for such maps.
Keywords: weak distance, A-contraction relative to weak distance.

1. Introduction

In [3], Kada, Suzuki and Takahashi have introduced the notion of a weak distance on a metric space as follows

Definition 1.1. Let (X, d) be a metric space. A function $p : X^2 \to [0, \infty)$ is called a *weak distance* on X if it satisfies the following conditions:

(a) $p(x, y) \leqslant p(x, z) + p(z, y)$, for all $x, y, z \in X$;

(b) to each $x \in X$, $p(x, \cdot) : X \to [0, \infty)$ is lower semi continuous. That is, for each sequence $\{y_n\}$ in X with $y_n \to y$ as $n \to \infty$ we have

$$p(x, y) \leq \liminf_{n\to\infty} p(x, y_n);$$

(c) to each $\varepsilon > 0$ there is a $\delta > 0$ such that $p(z, x) < \delta$ and $p(z, y) < \delta$ imply $d(x, y) < \varepsilon$.

Example 1.2. (i) Let (X, d) be a metric space. If a function $p : X^2 \to [0, \infty)$ is defined by $p(x, y) = d(x, y)$, for all $x, y \in X$, then p is a weak distance on X. That is every metric on a set is a weak distance on it.

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(ii) Let $X_0 = [0, 1)$ with $d(x, y) = |x - y|$ for $x, y \in X_0$. Then, (X_0, d) is a metric space. If we define $p : X_0^2 \to [0, \infty)$ by $p(x, y) = y$ for $x, y \in X_0$, it can be verified that p is a weak distance on X_0.

Remark 1.3. If p is a weak distance on a metric space then neither $p(x, y) = 0$ implies $x = y$ nor $x = y$ implies $p(x, y) = 0$. Also, $p(x, y)$ and $p(y, x)$ need not be equal.

Note that $p(x, 0) = 0$ for all $x \in X_0$ in Example 1.2 (ii).

For other examples we refer to [3].

Definition 1.4. Let $\mathbb{R}_+ = [0, \infty)$ and A be the class of all functions $\alpha : \mathbb{R}_+^3 \to \mathbb{R}_+$ such that:

(i) α is continuous on $\mathbb{R}_+^3$ (with respect to Euclidean metric on $\mathbb{R}_+^3$);

(ii) $a \leq \alpha(a, b, b)$ or $a \leq \alpha(b, a, b)$ or $a \leq \alpha(b, b, a)$, for all $a, b \in \mathbb{R}_+$ implies $a \leq k.b$ for some $k \in [0, 1)$.

For example, it is proved in [1] (Proof of Theorem 3) that the function α defined by

$$\alpha(u, v, w) = k. \max\{v, w\}, \text{ for } u, v, w \in \mathbb{R}_+, \tag{1}$$

where $k \in [0, 1)$, is in the class A.

Now, we define a new class of contractions on a metric space with a weak distance on it.

Definition 1.5. Suppose (X, d) is a metric space with a weak distance p on it. A self map f of X is said to be a $A - \textit{contraction relative to } p$ if there is an $\alpha \in A$, such that $p(fx, fy) \leq \alpha(p(x, y), p(fx, x), p(fy, y))$, for all $x, y \in X$ holds.

Remark 1.6. Any A-contraction relative to d (which is a weak distance, by Example 1.2(i)) on the metric space (X, d) is an *A-contraction* introduced by Akram, Zafar and Siddiqui [1] who observed that contractions due to Kannan [4], Khan [5], Bianchini [2] and Reich [6] are all A-contractions for suitably chosen $\alpha \in A$. Also, they have proved that the class of A-contractions is a wider class which includes these contractions properly. For instance, the A-contraction with $\alpha \in A$ defined in (1) is due to Bianchini [2] which satisfies the condition:

$$d(fx, fy) \leq k. \max\{d(fx, x), d(fy, y)\}, \forall x, y \in X \text{ and for some } k \in [0, 1). \tag{2}$$

Now, we give an example of a metric space (X, d) with a weak distance p on it and a A-contraction relative to p but not relative to d.

Example 1.7. Let (X_0, d) be the metric space with weak distance p given in Example 1.2(ii). Define $f : X_0 \to X_0$ by$f(x) = \frac{2}{3}x^2$ for $x \in X_0$. First note that there is a $k \in [0, 1)$ such that

$$p(fx, fy) \leq k.\max\{p(fx, x), p(fy, y)\}, \quad \text{for all } x, y \in X_0. \tag{3}$$

In fact, (3) holds if and only if $fy \leq k.\max\{x, y\}$ for all $x, y \in X_0$ for some $k \in [0, 1)$ (in view of definition of p here); and this holds with $k = \frac{2}{3}$. Therefore, f is a A-contraction relative to p with $\alpha \in A$ as defined in (1).

Observe that, in (3) if we replace p by d we get (2), the condition for f to be a contraction due to Bianchini [2]. Now, we remark that (2) can not hold for f since $d(f(\frac{3}{4}), f(0))=|f(\frac{3}{4}) - f(0)| = \frac{3}{8}$ and $\max\{d(f(\frac{3}{4}), \frac{3}{4}),\ d(f0, 0)\} = \max\{\frac{3}{8}, 0\} = \frac{3}{8}$ gives that $d(f(\frac{3}{4}), f(0)) \not\leq k.\max\{d(f(\frac{3}{4}), \frac{3}{4}),d(f0, 0)\}$, for any $k \in [0, 1)$. That is (2) fails if $x = \frac{3}{4}$ and $y = 0$.

Thus f is not a Bianchini contraction on X_0 but it is a Bianchini contraction relative to p.

The purpose of this paper is to prove three fixed point theorems for A-contractions relative to a weak distance on a complete metric space. Also, we note that Theorem 5,Theorem 6 and Theorem 7 of [1] are special cases of our results.

2. A fixed point theorem

The following lemma proved in [3] is needed:

Lemma 2.1. *Suppose (X, d) is a complete metric space with a weak distance p on it and suppose $\{x_n\}$ is a sequence in X; $\{a_n\}$ and $\{b_n\}$ are sequences in $\mathbb{R}_+$ such that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$ and $x, y, z \in X$, Then*

(i) $p(x_n, x) \leq a_n$ and $p(x_n, y) \leq b_n$ for all $n \geq 1$ imply $x = y$. In particular, $p(z, x)=p(z, y)=0$, for some $z \in X$ implies $x = y$;

(ii) $p(x_m, x_n) \leq a_n$ for all $m > n \geq 1$ implies $\{x_n\}$ is a Cauchy sequence in (X, d).

Theorem 2.2. *Suppose (X, d) is a complete metric space with a weak distance p on it such that $p(x, .)$ is continuous for each $x \in X$. If $f : X \to X$ is a A-contraction relative to p, then:*

(i) for any $x_0 \in X$ the sequence $\{x_n\}$ defined by $x_n = fx_{n-1}$ for $n \geq 1$ converges to some $z \in X$;

(ii) any such $z \in X$ with $p(z, fz) = p(fz, z)$ is the unique fixed point of f if and only if $p(z, z) = 0$.

Proof. Take any $x_0 \in X$ and $x_n = fx_{n-1}$, for $n \geq 1$. Write $P_n = p(x_{n+1}, x_n)$ and $Q_n^r = p(x_{n+r}, x_n)$, for $n \geq 0$, $r \geq 0$. Since f is a A- contraction relative to p there is an $\alpha \in A$ such that

$$p(fx, fy) \leq \alpha(p(x, y), p(fx, x), p(fy, y)), \text{ for all } x, y \in X. \tag{4}$$

Now, for $n \geq 1$ we have by (4), that

$$\begin{aligned} P_n = p(fx_n, fx_{n-1}) &\leq \alpha(p(x_n, x_{n-1}), p(fx_n, x_n), p(fx_{n-1}, x_{n-1})) \\ &= \alpha(p(x_n, x_{n-1}), p(x_{n+1}, x_n), p(x_n, x_{n-1})) \\ &= \alpha(P_{n-1}, P_n, P_{n-1}) \end{aligned}$$

so, that by (ii) of Definition 1.4, we get $P_n \leq kP_{n-1}$ for some $k \in [0, 1)$ which on repeated use gives

$$P_n \leq kP_{n-1} \leq k^2 P_{n-2} \leq .... \leq k^n P_0 \tag{5}$$

and therefore

$$\lim_{n\to\infty} P_n = 0. \tag{6}$$

By (a) of Definition 1.1 and (5), we have

$$\begin{aligned} Q_n^r &\leq p(x_{n+r}, x_{n+r-1}) + p(x_{n+r-1}, x_{n+r-2}) + ... + p(x_{n+1}, x_n) \\ &= P_{n+r-1} + P_{n+r-2} + ... + P_n \\ &\leq k^n(1 + k + ... + k^{r-1})P_0 < \frac{k^n P_0}{1-k} = a_n. \end{aligned} \tag{7}$$

That is, if $m = n + r > n \geq 1$ then $p(x_m, x_n) = Q_n^r < a_n$, where $a_n \to 0$ as $n \to \infty$; and hence by (ii) of Lemma 2.1, the sequence $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is complete there is a $z \in X$ such that $\lim_{n\to\infty} x_n = z$, proving part (i) of the theorem.

To prove part (ii), first suppose $z \in X$ is such that $p(z, fz) = p(fz, z)$ and $p(z, z) = 0$ and we now prove that z is the unique fixed point of f.

For any $n \geq 1$, we have by (4), that

$$\begin{aligned} p(fz, x_{n+1}) = p(fz, fx_n) &\leq \alpha(p(z, x_n), p(fz, z), p(fx_n, x_n)) \\ &= \alpha(p(z, x_n), p(fz, z), P_n) \end{aligned}$$

in which letting $n \to \infty$, using the continuity of α on $\mathbb{R}_+^3$ as well as the lower semi continuity of $p(z, .)$ and (6) we get

$$p(fz, z) \leq \alpha(p(z, z), p(fz, z), 0) = \alpha(0, p(fz, z), 0)$$

so that,by (ii) of Definition 1.4, we find $p(fz, z) \leq k.0 = 0$ showing $p(fz, z) = 0$. Then, $p(fz, z) = p(z, fz) = 0$. Thus $p(z, fz) = p(z, z) = 0$ so, that by (i) of Lemma 2.1, we get $fz = z$ showing z is a fixed point of f.

If $u \in X$ is another fixed point of f, then, by (4),

$$p(u,u) = p(fu,fu) \leq \alpha(p(u,u), p(fu,u), p(fu,u)) = \alpha(p(u,u), p(u,u), p(u,u))$$

which gives $p(u,u) \leq k.p(u,u)$, so that $p(u,u) = 0$. Also,

$$\begin{aligned} p(u,z) = p(fu,fz) &\leq \alpha(p(u,z), p(fu,u), p(fz,z)) \\ &= \alpha(p(u,z), p(u,u), p(z,z)) \\ &= \alpha(p(u,z), 0, 0) \text{ gives } p(u,z) \leq k.0 = 0 \end{aligned}$$

and hence $p(u,z) = 0$. Thus, $p(u,u) = p(u,z) = 0$ showing $u = z$ by (i) of Lemma 2.1. Thus z is the unique fixed point of f.

Conversely, if $z \in X$ with $p(z,fz) = p(fz,z)$ is the unique fixed point of f then $p(z,z) = p(fz,fz) \leq \alpha(p(z,z), p(fz,z), p(fz,z))$ gives $p(z,z) \leq k.p(z,z)$ and hence $p(z,z) = 0$. □

Remark 2.3. In the case $p = d$, the metric on the set X then the condition (ii) of Theorem 2.2 holds and hence every A-contraction on a complete metric space has a unique fixed point in X which is Theorem 5 of [1].

3. Common fixed point theorems

In this section we prove common fixed point theorems for A-contractions relative to a weak distance on complete metric spaces. We introduce a notation.

Definition 3.1. Suppose (X,d) is a metric space with weak distance p on it.

If an ordered pair (F,G) of self maps of X is such that there is an $\alpha \in A$ for which $p(Fx,Gy) \leq \alpha(p(x,y), p(Fx,x), p(Gy,y))$, for all $x,y \in X$, holds then, we write $(F,G) \in \mathscr{C}_\alpha^p$.

Observe that $f : X \to X$ is a A-contraction relative to p if $(f,f) \in \mathscr{C}_\alpha^p$.

Remark 3.2. It is easy to show that $(F,G) \in \mathscr{C}_\alpha^p$ if and only if $(G,F) \in \mathscr{C}_\alpha^p$ in case p is such that $p(x,y) = p(y,x)$ for all $x,y \in X$. In particular if δ is a metric on X then $(F,G) \in \mathscr{C}_\alpha^\delta \Leftrightarrow (G,F) \in \mathscr{C}_\alpha^\delta$.

Theorem 3.3. *Suppose (X,d) is a complete metric space with weak distance p on it for which*

(i) $d(x,y) \leq p(x,y)$ *for all* $x,y \in X$.

Suppose f and g are self maps of X; of which g is continuous on (X,d), and

(ii) $(f,g) \in \mathscr{C}_\alpha^p$ *and* $(g,f) \in \mathscr{C}_\alpha^p$, *for some* $\alpha \in A$

then, for any $x_0 \in X$ the sequence $\{x_n\}$ defined for $n \geq 1$ by $x_n = fx_{n-1}$ or gx_{n-1} according as n is even or odd,converges to a point $z \in X$. If $p(z,z) = 0$ then z is the unique common fixed point of f and g.

Proof. Take any $x_0 \in X$ and for any $n \geq 1$, let

$$x_n = \begin{cases} fx_{n-1}, & \text{if } n \text{ is even,} \\ gx_{n-1}, & \text{if } n \text{ is odd.} \end{cases}$$

Write

$$p(x_{n+1}, x_n) = \begin{cases} P_n, & \text{if } n \text{ is odd,} \\ Q_n, & \text{if } n \text{ is even.} \end{cases}$$

Now, if n is odd then since $(f, g) \in \mathscr{C}_\alpha^p$

$$\begin{aligned} P_n &= p(x_{n+1}, x_n) = p(fx_n, gx_{n-1}) \\ &\leq \alpha(p(x_n, x_{n-1}), p(fx_n, x_n), p(gx_{n-1}, x_{n-1})) \\ &= \alpha(Q_{n-1}, P_n, Q_{n-1}) \leq k.Q_{n-1}, \end{aligned} \tag{8}$$

for some $k \in [0, 1)$, and if n is even then since $(g, f) \in \mathscr{C}_\alpha^p$ we get

$$\begin{aligned} Q_n &= p(x_{n+1}, x_n) = p(gx_n, fx_{n-1}) \\ &\leq \alpha(p(x_n, x_{n-1}), p(gx_n, x_n), p(fx_{n-1}, x_{n-1})) \\ &= \alpha(P_{n-1}, Q_n, P_{n-1}) \leq k.P_{n-1} \end{aligned} \tag{9}$$

so that (8) and (9) give

$$P_n \leq k^2.P_{n-2} \text{ and } Q_n \leq k^2.Q_{n-2}, \text{ for each } n \geq 1. \tag{10}$$

That is for each $n \geq 1$

$$p(x_{n+1}, x_n) \leq k^2.p(x_{n-1}, x_{n-2}) \tag{11}$$

which on repeated use gives

$$p(x_{n+1}, x_n) \leq k^n.p(x_1, x_0) \text{ or } p(x_{n+1}, x_n) \leq k^n.p(x_0, x_1)$$

according as n is even or odd so that $p(x_{n+1}, x_n) \to 0$ as $n \to \infty$, since $0 \leq k < 1$. Therefore, if $b_n = p(x_{n+r}, x_{n+r-1}) + p(x_{n+r-1}, x_{n+r-2}) + ... + p(x_{n+1}, x_n)$ for any $n \geq 1$ and $r \geq 1$ then

$$\lim_{n\to\infty} b_n = 0. \tag{12}$$

Now, for any $m = n + r > n \geq 1$, by (a) of Definition 1.1, we have that $p(x_m, x_n) = p(x_{n+r}, x_n) \leq p(x_{n+r}, x_{n+r-1}) + p(x_{n+r-1}, x_{n+r-2}) + ... + p(x_{n+1}, x_n) = b_n$ and therefore, in view of (12), we get by (ii) of Lemma 2.1 that $\{x_n\}$ is a Cauchy sequence in the complete metric space(X, d) so that there is a $z \in X$ with $\lim_{n\to\infty} x_n = z$. That is,

$$\lim_{n\to\infty} d(x_n, z) = 0. \tag{13}$$

Therefore, the continuity of $d(x,y)$ in x; the continuity of g on X (by the hypothesis) and (13) give

$$\begin{aligned} 0 &= \lim_{n\to\infty} d(x_{2n+1}, z) = \lim_{n\to\infty} d(gx_{2n}, z) \\ &= d(\lim_{n\to\infty} gx_{2n}, z) = d(g(\lim_{n\to\infty} x_n), z) = d(gz, z) \end{aligned}$$

which shows $gz = z$. That is, z is a fixed point of g.

Also, $p(fz,z) = p(fz,gz) \leq \alpha(p(z,z), p(fz,z), p(gz,z)) = \alpha(p(z,z), p(fz,z), p(z,z))$ gives $p(fz,z) \leq k.p(z,z)$ so that $p(fz,z) = 0$ if $p(z,z) = 0$. Hence, by (i) of the Theorem 3.3, $d(fz,z) = 0$ giving that $fz = z$. Thus, z is a common fixed point of f and g if $p(z,z) = 0$.

To prove the uniqueness,we first observe that $p(u,u) = 0$, for any common fixed point of f and g. In fact, $p(u,u) = p(fu,gu) \leq \alpha(p(u,u), p(fu,u), p(gu,u))$ $= \alpha(p(u,u), p(u,u), p(u,u))$ gives $p(u,u) \leq k.p(u,u)$, so that $p(u,u) = 0$.

Now, if u and z are two common fixed points of f and g then $p(u,z) = p(fu,gz) \leq \alpha(p(u,z), p(fu,u), p(gz,z)) = \alpha(p(u,z), 0, 0)$, so that $p(u,z) \leq k.0 = 0$ giving $p(u,z) = 0$. Thus, $p(u,z) = p(u,u) = 0$ shows $u = z$ completing the proof of the theorem. □

Remark 3.4. In case $p = \delta$, another metric on (X,d) then in view of Remark 3.2, we get Theorem 7 of [1].

Theorem 3.5. *Suppose (X,d) is a complete metric space with weak distance p on it such that $p(x,y) = p(y,x)$ for all $x, y \in X$ and $p(x,.)$ is continuous for each $x \in X$. Suppose $\{f_n\}$ is a sequence of self maps of X such that $(f_i, f_j) \in \mathscr{C}^p_\alpha$, for some $\alpha \in A$ and for any $i \neq j$. Then*

(i) for any $x_0 \in X$ the sequence $\{x_n\}$ defined by $x_n = f_n x_{n-1}$ for $n \geq 1$ converges to some $z \in X$;

(ii) any such $z \in X$ with $p(z,z) = 0$ is the unique common fixed point of $\{f_n\}$.

Proof. By hypothesis there is an $\alpha \in A$ such that

$$p(f_i x, f_i y) \leq \alpha(p(x,y), p(f_i x, x), p(f_j y, y)), \tag{14}$$

for all $x, y \in X$ and for $i \neq j$. Take any $x_0 \in X$ and let $x_n = f_n x_{n-1}$ for $n \geq 1$. Now, for any $n \geq 1$ we have by (14),

$$\begin{aligned} p(x_{n+1}, x_n) &= p(f_{n+1}x_n, f_n x_{n-1}) \\ &\leq \alpha(p(x_n, x_{n-1}), p(f_{n+1}x_n, x_n), p(f_n x_{n-1}, x_{n-1})) \\ &= \alpha(p(x_n, x_{n-1}), p(x_{n+1}, x_n), p(x_n, x_{n-1})) \end{aligned}$$

which shows that $p(x_{n+1}, x_n) \leq k.p(x_n, x_{n-1})$, for some $k \in [0,1)$, which on repeated use gives

$$p(x_{n+1}, x_n) \leq k.p(x_n, x_{n-1}) \leq ..... \leq k^n.p(x_1, x_0). \tag{15}$$

Now, (15) implies

$$\lim_{n\to\infty} p(x_{n+1}, x_n) = 0 \tag{16}$$

and that $\{x_n\}$ is a Cauchy sequence in the complete metric space (X, d). Hence there is a $z \in X$ such that $\lim_{n\to\infty} x_n = z$, proving part (i) of the theorem.

(ii) Suppose such $z \in X$ is with $p(z, z) = 0$. To prove z is a common fixed point of$\{f_n\}$, fix n and consider

$$\begin{aligned} p(z, f_n z) &\le p(z, x_{m+1}) + p(x_{m+1}, f_n z) = p(z, x_{m+1}) + p(f_{m+1}x_m, f_n z) \\ &\le p(z, x_{m+1}) + \alpha(p(x_m, z), p(x_{m+1}, x_m), p(f_n z, z)) \\ &= p(z, x_{m+1}) + \alpha(p(z, x_m), p(x_{m+1}, x_m), p(z, f_n z)), \end{aligned} \tag{17}$$

where we used the hypothesis that $p(x, y) = p(y, x)$, for all $x, y \in X$.

In (17) letting $m \to \infty$,using the continuity of α as well as the continuity of $p(z, .)$ and (16) we get $p(z, f_n z) \le p(z, z) + \alpha(p(z, z), 0, p(z, f_n z)) = 0 + \alpha(0, 0, p(z, f_n z))$ showing $p(z, f_n z) \le k.0$ which implies $p(z, f_n z) = 0$. Thus, $p(z, f_n z) = p(z, z) = 0$, so that $f_n z = z$. That is z is a common fixed point of $\{f_n\}$.

The uniqueness can be proved on the lines similar to earlier theorem. □

Remark 3.6. In case $p = d$, the metric on the space X, our Theorem 3.5 gives Theorem 6 of [1].

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Isomorphic properties of neighborly irregular cubic graph

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Abstract. Cubic graph can manage the uncertainty relevant to the inconsistent and indeterminate information of all real-world problems, in which fuzzy graphs possibly will not succeed into bringing about satisfactory results. Hence, in this research, we describe weak isomorphism, co-weak isomorphism and isomorphism of neighborly irregular cubic graphs. Some results on order, size and degree of nodes in isomorphic neighborly irregular and isomorphic highly irregular cubic graphs are discussed. Isomorphism between neighborly irregular and highly irregular cubic graphs are proved to be an equivalence relation. Likewise, density and balanced irregular cubic graphs are introduced.

Keywords: cubic graph, weak isomorphism, neighborly irregular, highly irregular.

1. Introduction

Fuzzy sets theory and its related fuzzy logic has been proposed by Zadeh [37]. A fuzzy set of a universe X is a function from X into the unit closed interval $[0, 1]$ of real number. In [38] Zadeh made an extension of the concept of a fuzzy set by an interval-valued fuzzy set, i.e., a fuzzy set with an interval-valued membership function. The first definition of fuzzy graphs was proposed by Kaufmann [12] in 1993. Interval-valued fuzzy sets have been actively used in real-life applications. For example, Sambuc [27] in medical diagnosis in thyroiditis pathology, Kohout

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[14] also in medicine, Turksen in preferences modeling [36], etc. These works and others show the importance of these sets. Jun et al. [11] introduced cubic sets. The fuzzy graph theory as a generalization of Euler's graph theory was first introduced by Rosenfeld [26] in 1975. Later, Bhattacharya [2] gave some remarks on fuzzy graphs and some operations on fuzzy graphs were introduced by Mordeson and Peng [15]. The complement of a fuzzy graph was dened by Mordeson [16] and further studied by Sunitha and Vijayakumar [28]. Hongmei and Lianhua gave the definition of interval-valued fuzzy graphs [10]. Akram and Dudek defined some operations on interval-valued fuzzy graphs [1]. Rashmanlou et al. [17, 18, 19] introduced some properties of highly irregular interval-valued fuzzy graphs, and new concepts of bipolar fuzzy graphs. Karunambigai et al. [13] introduced edge regular intuitionistic fuzzy graph. Samanta and Pal [29, 30, 31, 32, 33] defined fuzzy tolerance graph, fuzzy threshold graph, fuzzy k-competition graph and p-competition fuzzy graph and new concepts of fuzzy planar graph. Shao et al. [34, 35] investigated new concepts in intuitionistic fuzzy graphs and vague graphs. Borzooei et al. [3, 4, 5, 6] described several domination sets in vague graphs. Recently, some research works have been done by the authors in continuation of previous works related to interval valued intuitionistic fuzzy graphs, vague graphs, bipolar fuzzy graphs, and intuitionistic fuzzy graphs which are mentioned in [7, 8, 9, 21, 22, 23, 24, 25].

In this paper some properties of an edge regular cubic graph are given. Particularly, strongly regular, edge regular and biregular cubic graphs are defined and the necessary and sufficient condition for a cubic graph to be strongly regular is studied. Also, we have introduced a partially edge regular cubic graph and fully edge regular cubic graph with suitable illustrations.

2. Preliminaries

A graph is an ordered pair $G = (V, E)$, where V is the set of vertices of G and E is the set of edges of G. A subgraph of a graph $G = (V, E)$ is a graph $H = (W, F)$, where $W \subseteq V$ and $F \subseteq E$. A fuzzy graph $G = (\sigma, \mu)$ is a pair of functions $\sigma : V \to [0, 1]$ and $\mu : V \times V \to [0, 1]$ with $\mu(u, v) \leq \sigma(u) \wedge \sigma(v)$, for all $u, v \in V$, where V is a finite non-empty set and $\wedge$ denote minimum. We introduce below necessary notions and present a few auxiliary results that will be used throughout the paper.

A map $\lambda : X \to [0, 1]$ is called a fuzzy subset of X. For any two fuzzy subsets λ and μ of X, $\lambda \leq \mu$ means that, for all $x \in X, \lambda(x) \leq \mu(x)$. The symbol $\lambda \wedge \mu$ and $\lambda \vee \mu$ will mean the following fuzzy subsets of X.

$(\lambda \wedge \mu)(x) = \lambda(x) \wedge \mu(x)$ and $(\lambda \vee \mu)(x) = \lambda(x) \vee \mu(x)$, for all $x \in X$.

Let X be a non-empty set. A function $A : X \to [I]$ is called an interval-valued fuzzy set(shortly, an IVF set) in X. Let $[I]^X$ stands for the set of all IVF sets in X. For every $A \in [I]^X$ and $x \in X$, $A(x) = [A^-(x), A^+(x)]$ is called the degree of membership of an element x to A, where $A^- : X \to I$ and $A^+ : X \to I$ are fuzzy sets in X which are called a lower fuzzy set and an upper fuzzy set in

X, respectively. For simplicity, we denote $A = [A^-, A^+]$. For every $A, B \in [I]^X$, we define $A \subseteq B$ if and only if $A(x) \leq B(x)$, for all $x \in X$.

Definition 2.1 ([10]). *Let $A = [A^-, A^+]$, and $B = [B^-, B^+]$ be two interval-valued fuzzy set in X. Then, we define $r\min\{A(x), B(x)\} = [\min\{A^-(x), B^-(x)\}, \min\{A^+(x), B^+(x)\}]$, $r\max\{A(x), B(x)\} = [\max\{A^-(x), B^-(x)\}, \max\{A^+(x), B^+(x)\}]$.*

Definition 2.2 ([11]). *Let X be a non-empty set. By a cubic set in X, we mean a structure $A = \{< x, A(x), \lambda(x) : x \in X\}$ in which A is an interval-valued fuzzy sets in X and λ is a fuzzy set in X. A cubic set $A = \{< x, A(x), \lambda(x) : x \in X >\}$ is simply denoted by $A =< A, \lambda >$. The collection of all cubic sets in X is denoted by $CP(X)$.*

Definition 2.3 ([20]). *A cubic graph is a triple $G = (G^*, P, Q)$ where $G^* = (V, E)$ is a graph, $P = (\widetilde{\mu_P}, \lambda_P)$ is a cubic set on V and $Q = (\widetilde{\mu_Q}, \widetilde{\lambda_Q})$ is a cubic set on $V \times V$ such that $\widetilde{\mu_Q}(xy) \leq r\min\{\widetilde{\mu_P}(x), \widetilde{\mu_P}(y)\}$ and $\widetilde{\lambda_Q}(xy) \geq \max\{\widetilde{\lambda_P}(x), \widetilde{\lambda_P}(y)\}$.*

The underlying crisp graph of a cubic graph $G = (A, B)$, is the graph $G = (V, E)$, where $V = \{v : \widetilde{\mu_P}>0 \text{ and } \lambda_P>0\}$ and $E = \big\{\{u, v\} : \widetilde{\mu_Q}(\{u, v\})>0, \widetilde{\mu_Q}(\{u, v\})>0\big\}$. V is called the vertex set and E is called the edge set. A cubic graph maybe also denoted as $G = (V, E)$.

Definition 2.4 ([20]). *A cubic graph $G = (G^*, P, Q)$ is called complete if $\widetilde{\mu_Q}(xy) = r\min\{\mu_P(x), \mu_P(y)\}$ and $\widetilde{\lambda_Q}(xy) = \max\{\widetilde{\lambda_P}(x), \widetilde{\lambda_P}(y)\}$ for all $x, y \in V$.*

Definition 2.5 ([20]). *A cubic graph $G = (G^*, P, Q)$ is called strong if $\widetilde{\mu_Q}(xy) = r\min\{\mu_P(x), \mu_P(y)\}$ and $\widetilde{\lambda_Q}(xy) = \max\{\widetilde{\lambda_P}(x), \widetilde{\lambda_P}(y)\}$, for all $xy \in E$.*

Definition 2.6 ([20]). *The complement of a cubic graph $G = (A, B)$ is a cubic graph $\overline{G} = (\overline{G^*}, \overline{P}, \overline{Q})$, where $\overline{P} = (\overline{\widetilde{\mu_P}}, \overline{\lambda_P})$ and $\overline{Q} = (\overline{\widetilde{\mu_Q}}, \overline{\widetilde{\lambda_Q}})$ is defined by: $\overline{\widetilde{\mu_Q}}(xy) = r\min\{\mu_P(x), \mu_P(y)\} - \widetilde{\mu_Q}(xy)$ and $\overline{\widetilde{\lambda_Q}}(xy) = \widetilde{\lambda_Q}(xy) - \max\{\widetilde{\lambda_P}(x), \widetilde{\lambda_P}(y)\}$.*

Definition 2.7. *Let $G = (V, E)$ be a cubic graph.*

(i) *The neighborhood degree of a vertex v is defined as $d_N(v) = (d_{N_{\widetilde{\mu_P}}(v)}, d_{N_{\lambda_Q}}(v))$, where $d_{N_{\mu_P}}(v) = (\sum_{w \in N(\widetilde{\mu_P^-})} \widetilde{\mu_P^-}(w), \sum_{w \in N(\mu_P^+)} \mu_P^+(w))$ and $d_{N_{\lambda_Q}}(v) = \sum_{w \in N(\lambda_Q)} \lambda_Q(w)$.*

(ii) *The degree of a vertex v_i is defined by $d_G(v_i) = (d_{\widetilde{\mu_P}}(v_i), d_{\lambda_Q}(v_i)) = (k_1, k_2)$, where $k_1 = d_{\widetilde{\mu_P}}(v_i) = (\sum_{v_i \neq v_j} \widetilde{\mu_Q^-}(v_i v_j), \sum_{v_i \neq v_j} \widetilde{\mu_Q^+}(v_i v_j))$ and $k_2 = d_{\lambda_Q}(v_i) = \sum_{v_i \neq v_j} \lambda_Q(v_i v_j)$.*

Definition 2.8. *A cubic graph $G = (V, E)$ is said to be*

(i) $(k_1, k_2)-$*regular if* $d_G(v_i) = (k_1, k_2)$*, for all* $v_i \in V$ *and also* G *is said to be a regular cubic graph of degree* (k_1, k_2)*.*

(ii) bipartite if the vertex set V *can be partitioned into two non-empty sets* V_1 *and* V_2 *such that:*

(a) $\widetilde{\mu_Q}(v_iv_j) = 0$ *and* $\lambda_Q(v_iv_j) = 0$*, if* $(v_i, v_j) \in V_1$ *or* $(v_i, v_j) \in V_2$*;*

(b) $\widetilde{\mu_Q}(v_iv_j) = 0$*,* $\lambda_Q(v_iv_j) > 0$*, if* $v_i \in V_1$ *or* $v_j \in V_2$*;*

(c) $\widetilde{\mu_Q}(v_iv_j) > 0$*,* $\lambda_Q(v_iv_j) = 0$*, if* $v_i \in V_1$ *or* $v_j \in V_2$*, for some* i *and* j*.*

Definition 2.9. *Let* $G^* = (V, E)$ *be a crisp graph and let* $e = v_iv_j$ *be an edge in* G^**. Then, the degree of an edge* $e = v_iv_j \in E$ *is defined as* $d_{G^*}(v_iv_j) = d_{G^*}(v_i) + d_{G^*}(v_j) - 2$*.*

Definition 2.10. *(i) The order of* G *is defined to be* $O(G) = (O_{\widetilde{\mu_P}}, O_{\lambda_P})$*, where* $O_{\widetilde{\mu_P}} = \sum_{u\in V} \widetilde{\mu_P(u)}$ *and* $O_{\lambda_P} = \sum_{u\in V} \lambda_P(u)$*.*

(ii) The size of G *is defined to be* $S(G) = (S_{\widetilde{\mu_Q}}(G), S_{\lambda_Q}(G))$*, where* $S_{\widetilde{\mu_Q}}(G) = \sum_{u\neq v} \widetilde{\mu_Q(uv)}$ *and* $S_{\lambda_Q}(G) = \sum_{u\neq v} \lambda_Q(uv)$*.*

3. Isomorphic properties of neighborly irregular and highly irregular cubic graphs

In this section, we describe weak isomorphism, co-weak isomorphism, and isomorphism of neighborly irregular cubic graphs and prove that isomorphisms between neighborly irregular and highly irregular cubic graphs are equivalence relations.

Definition 3.1. *Let* $G = (A, B)$ *be a connected cubic graph.* G *is said to be a neighborly irregular cubic graph if every two adjacent vertices of* G *have distinct degree.*

Definition 3.2. *Let* $G = (A, B)$ *be a connected cubic graph.* G *is said to be a highly irregular cubic graph if every vertex of* G *is adjacent to vertices with distinct degrees.*

Definition 3.3. *A homomorphism* h *of neighborly irregular cubic graphs (highly irregular cubic graphs)* G_1 *and* G_2 *is a mapping* $h : V_1 \to V_2$ *which satisfies the following conditions:*

(a) $\widetilde{\mu}_{A_1}(u_1) \leq \widetilde{\mu}_{A_2}(h(u_1))$*,* $\lambda_{A_1}(u_1) \geq \lambda_{A_2}(h(u_1))$*, for all* $u_1 \in V_1$*,*

(b) $\widetilde{\mu}_{B_1}(u_1v_1) \leq \widetilde{\mu}_{B_2}(h(u_1)h(v_1))$*,* $\lambda_{B_1}(u_1v_1) \geq \lambda_{B_2}(h(u_1)h(v_1))$*, for all* $u_1v_1 \in E_1$*.*

Example 3.1. Let $V_1 = \{a, b, c, d\}$ and $V_2 = \{u, v, x, w\}$. Consider two neighborly irregular cubic graphs $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$. The below graph in Figure 1 shows the homomorphism of neighborly irregular cubic graphs G_1 and G_2. There is a homomorphism $h : V_1 \to V_2$ such that $h(a) = u$, $h(b) = v$, $h(c) = x$, $h(d) = w$.

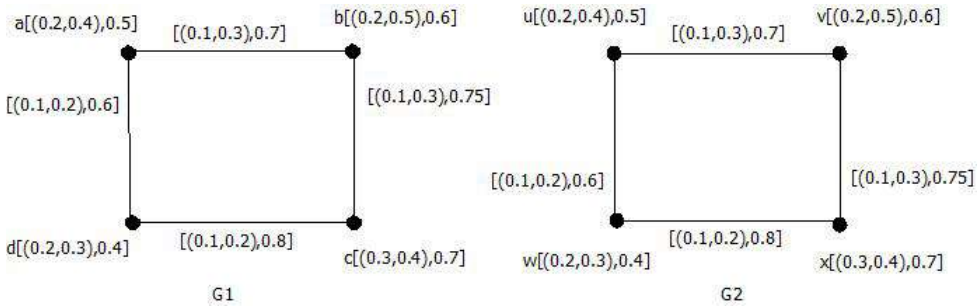


Figure 1: Homomorphism of neighborly irregular cubic graphs G_1 and G_2

Definition 3.4. *A weak isomorphism h of neighborly irregular cubic graphs (highly irregular cubic graphs) G_1 and G_2 is a bijective mapping $h : V_1 \to V_2$ which satisfies the following conditions:*

(c) h is homomorphism,

(d) $\widetilde{\mu}_{A_1}(u_1) = \widetilde{\mu}_{A_2}(h(u_1))$, $\lambda_{A_1}(u_1) = \lambda_{A_2}(h(u_1))$, for all $u_1 \in V_1$.

Example 3.2. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two highly irregular cubic graphs defined as follows.

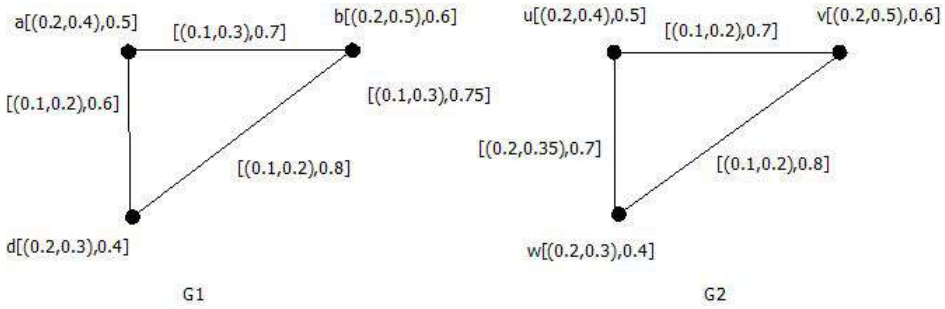


Figure 2: Weak isomorphism of highly irregular cubic graphs G_1 and G_2

There is a weak isomorphism $h : V_1 \to V_2$ such that $h(a) = u$, $h(b) = v$, $h(c) = w$.

Definition 3.5. *A co-weak isomorphism h of neighborly irregular cubic graphs (highly irregular cubic graphs) G_1 and G_2 is a bijective mapping $h : V_1 \to V_2$ which satisfies the following conditions:*

(e) h is homomorphism,

(f) $\widetilde{\mu}_{B_1}(u_1v_1) = \widetilde{\mu}_{B_2}(h(u_1)h(v_1)), \lambda_{B_1}(u_1v_1) = \lambda_{B_2}(h(u_1)h(v_1))$, for all $u_1v_1 \in E_1$.

Example 3.3. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two neighborly irregular cubic graphs given in Figure 3 as follows.

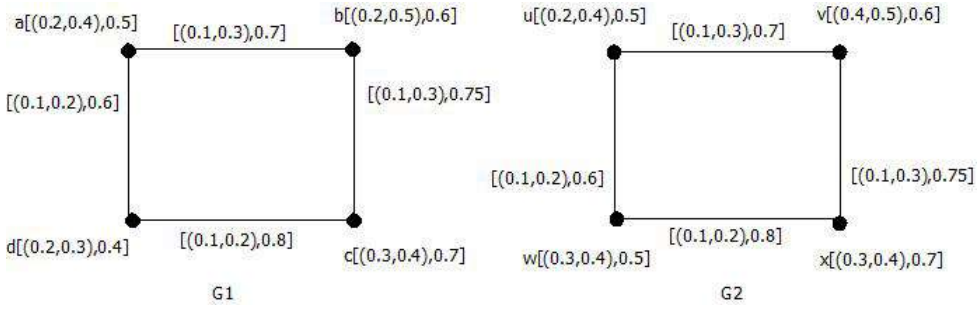


Figure 3: Co-weak isomorphism of highly irregular cubic graphs G_1 and G_2

There is a co-weak isomorphism $h : V_1 \to V_2$ such that $h(a) = u, h(b) = v, h(c) = w, h(d) = x$.

Definition 3.6. *An isomorphism h of neighborly irregular cubic graphs (highly irregular cubic graphs) G_1 and G_2 is a bijective mapping $h : V_1 \to V_2$ which satisfies the following conditions:*

(g) $\widetilde{\mu}_{A_1}(u_1) = \widetilde{\mu}_{A_2}(h(u_1)),\ \lambda_{A_1}(u_1) = \lambda_{A_2}(h(u_1)),$

(h) $\widetilde{\mu}_{B_1}(u_1v_1) = \widetilde{\mu}_{B_2}(h(u_1)h(v_1)),\ \lambda_{B_1}(u_1v_1) = \lambda_{B_2}(h(u_1)h(v_1))$, *for all* $u_1 \in V_1$ *and* $u_1v_1 \in E_1$.

Theorem 3.1. *For any two isomorphic neighborly irregular cubic graphs, their order and size are the same.*

Proof. Let h from G_1 to G_2 be an isomorphism between the neighborly irregular cubic graphs G_1 and G_2 with the underlying sets V_1 and V_2 respectively. Then $\widetilde{\mu}_{A_1}(u) = \widetilde{\mu}_{A_2}(h(u)),\ \lambda_{A_1}(u) = \lambda_{A_2}(h(u))$, for all $u \in V_1$, $\widetilde{\mu}_{B_1}(uv) = \widetilde{\mu}_{B_2}(h(u)h(v)),\ \lambda_{B_1}(uv) = \lambda_{B_2}(h(u)h(v))$, for all $u, v \in V$. So, we have

$$\begin{aligned} O(G_1) &= (\sum_{u_1 \in V_1} \widetilde{\mu}_{A_1}(u_1), \sum_{u_1 \in V_1} \lambda_{A_1}(u_1)) \\ &= (\sum_{u_1 \in V_1} \widetilde{\mu}_{A_2}(h(u_1)), \sum_{u_1 \in V_1} \lambda_{A_2}(h(u_1))) \\ &= (\sum_{u_2 \in V_2} \widetilde{\mu}_{A_2}(u_2), \sum_{u_2 \in V_2} \lambda_{A_2}(u_2)) = O(G_2) \end{aligned}$$

$$\begin{aligned} S(G_1) &= (\sum_{u_1v_1 \in E_1} \widetilde{\mu}_{B_1}(u_1v_1), \sum_{u_1v_1 \in E_1} \lambda_{B_1}(u_1v_1)) \\ &= (\sum_{u_1,v_1 \in V_1} \widetilde{\mu}_{B_2}(h(u_1)h(v_1)), \\ &\qquad \sum_{u_1,v_1 \in V_1} \lambda_{B_2}(h(u_1)h(v_1))) \\ &= (\sum_{u_2v_2 \in E_2} \widetilde{\mu}_{B_2}(u_2v_2), \sum_{u_2v_2 \in E_2} \lambda_{B_2}(u_2v_2)) \\ &= S(G_2). \quad \square \end{aligned}$$

Remark 3.1. The above theorem is also true for highly irregular cubic graphs.

Corollary 3.1. *Converse of Theorem 3.10 need not be true for both neighborly irregular and highly irregular cubic graphs. It can be seen from the following example.*

Example 3.4. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two highly irregular cubic graphs given in Fig. 4.

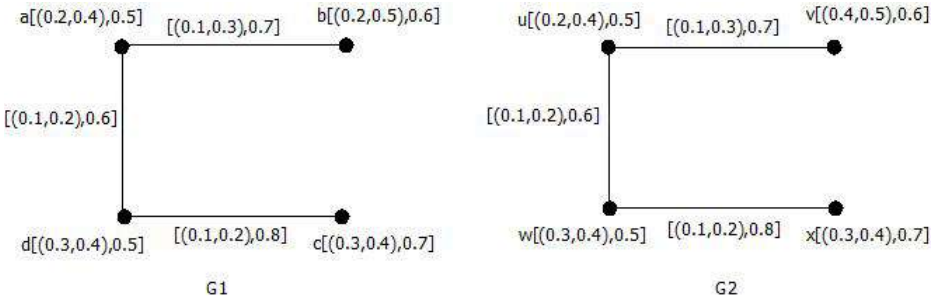


Figure 4: Highly irregular cubic graphs G_1 and G_2

In both the graphs, $O(G_1) = O(G_2) = [(1.0, 1.8), 2.3]$ and $S(G_1) = S(G_2) = [(0.3, 0.7), 2.1]$. But G_1 is not isomorphic to G_2

Proposition 3.1. *If two neighborly irregular cubic graphs (highly irregular cubic graphs) are weak isomorphic, then their orders are same. But neighborly irregular cubic graphs (highly irregular cubic graphs) of same order need not be weak isomorphic.*

Example 3.5. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two neighborly irregular cubic graphs defined as follows.

In both graphs, $O(G_1) = O(G_2) = [(1.0, 1.7), 2.3]$, but they are not weak isomorphic.

Proposition 3.2. *If two neighborly irregular cubic graphs (highly irregular cubic graphs) are co-weak isomorphic, their sizes are the same. But neighborly*

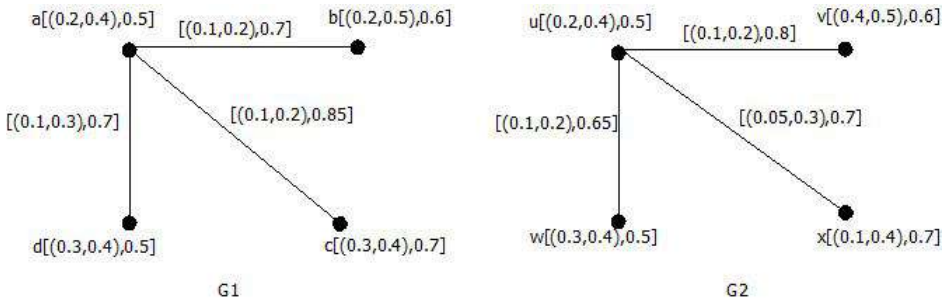


Figure 5: Neighborly irregular cubic graphs G_1 and G_2

irregular cubic graphs (highly irregular cubic graphs) of same size need not be co-weak isomorphic.

Example 3.6. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two highly irregular cubic graphs given as follows.

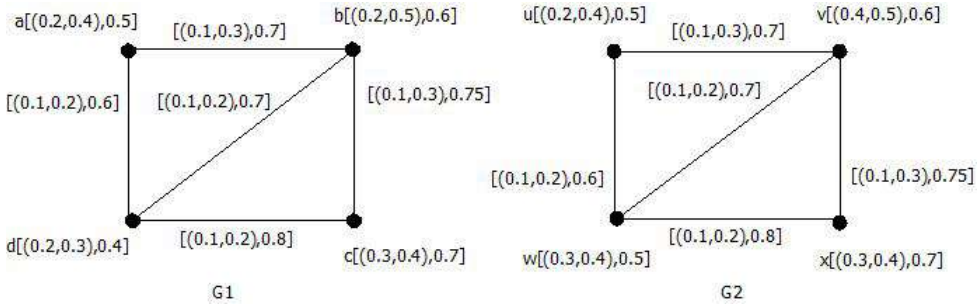


Figure 6: Highly irregular cubic graphs G_1 and G_2

The size of the above two highly irregular cubic graphs are same ($S(G_1) = S(G_2) = [(0.5, 1.2), 3.55]$. But they are not co-weak isomorphic.

Theorem 3.2. *If G_1 and G_2 are isomorphic neighborly irregular cubic graphs, then the degrees of the corresponding vertices u and $h(u)$ are preserved.*

Proof. Let $h : G_1 \to G_2$ be an isomorphism between the neighborly irregular cubic graphs G_1 and G_2 with underlying sets V_1 and V_2 respectively. Then $\widetilde{\mu}_{B_1}(u_1v_1) = \widetilde{\mu}_{B_2}(h(u_1)h(v_1))$, $\lambda_{B_1}(u_1v_1) = \lambda_{B_2}(h(u_1)h(v_1))$, for all $u_1, v_1 \in V_1$.

Therefore,

$$\begin{aligned} d_{\widetilde{\mu}}(u_1) &= \sum_{u_1,v_1 \in V_1} \widetilde{\mu}_{B_1}(u_1v_1) \\ &= \sum_{u_1,v_1 \in V_1} \widetilde{\mu}_{B_2}(h(u_1)h(v_1)) = d_{\widetilde{\mu}}(h(u_1)), \\ d_{\lambda(u_1)} &= \sum_{u_1,v_1 \in V_1} \lambda_{B_1}(u_1v_1) \\ &= \sum_{u_1,v_1 \in V_1} \lambda_{B_2}(h(u_1)h(v_1)) = d_{\lambda(h(u_1))}, \end{aligned}$$

for all $u_1 \in V_1$. That is, the degrees of the corresponding vertices of G_1 and G_2 are the same. □

Remark 3.2. The above theorem is true for highly irregular cubic graphs also.

Corollary 3.2. *Converse of the Theorem 3.18 and Remark 3.19 need not be true. This can be seen from the following example.*

Example 3.7. Consider two neighborly irregular cubic graphs $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ given in Figure 7.

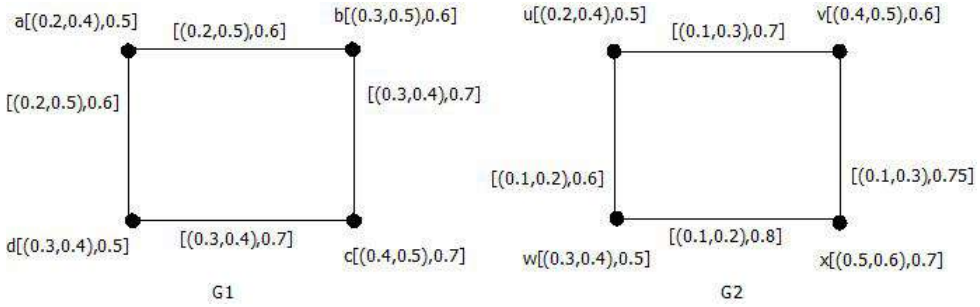


Figure 7: Neighborly irregular cubic graphs G_1 and G_2 with same degrees

$d(a) = d(u) = [(0.4, 1), 1.2]$, $d(b) = d(v) = [(0.5, 0.9), 1.3]$, $d(c) = d(w) = [(0.6, 1), 1.4]$, $d(d) = d(x) = [(0.5, 0.9), 1.3]$. In both the graphs the degrees of the corresponding vertices are the same, but G_1 and G_2 are only co-weak isomorphic but not isomorphic.

Remark 3.3. Isomorphism between neighborly irregular cubic graphs is an equivalence relation.

Theorem 3.3. *Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two highly irregular cubic graphs. G_1 and G_2 are isomorphic if and only if their complements are isomorphic. But the complement need not be highly irregular.*

Proof. Assume that G_1 and G_2 are isomorphic, with their isomorphism map $h : V_1 \to V_2$. By the definition of complement we have,

$$\begin{aligned}
\overline{\widetilde{\mu}_{B_1}}(xy) &= \min(\widetilde{\mu}_{A_1}(x), t_{A_1}(y)) - \widetilde{\mu}_{B_1}(xy) \\
&= \min(\widetilde{\mu}_{A_2}(h(x)), \widetilde{\mu}_{A_2}(h(y))) \\
&- t_{B_2}(h(x)h(y)) \\
\overline{\lambda_{B_1}}(xy) &= \lambda_{B_1}(xy) - \max(\lambda_{A_1}(x), \lambda_{A_1}(y)) \\
&= \lambda_{B_2}(h(x)h(y)) \\
&- \max(\lambda_{A_2}(h(x)), \lambda_{A_2}(h(y))),
\end{aligned}$$

for all $xy \in E_1$. Hence $\overline{G}_1 \cong \overline{G}_2$. The proof of the converse part is straight forward. □

Theorem 3.4. *Let G_1 and G_2 be two highly irregular cubic graphs. If G_1 is weak isomorphic with G_2, then $\overline{G}_1$ is weak isomorphic with $\overline{G}_2$.*

Proof. Assume that G_1 and G_2 are weak isomorphic, with their weak isomorphism map $h : V_1 \to V_2$. As $h^{-1} : V_2 \to V_1$ is bijective also, for every $x_2 \in V_2$ there is an $x_1 \in V_1$ such that $h^{-1}(x_2) = x_1$. By the definition of complement we have,

$$\begin{aligned}
\overline{\widetilde{\mu}_{B_1}}(x_1y_1) &= \min(\widetilde{\mu}_{A_1}(x_1), \widetilde{\mu}_{A_1}(y_1)) \\
&- \widetilde{\mu}_{B_1}(x_1y_1), \ \textit{for all } x_1, y_1 \in V_1.
\end{aligned}$$

$$\begin{aligned}
\overline{\widetilde{\mu}_{B_1}}(h^{-1}(x_2)h^{-1}(y_2)) &\geq \min(\widetilde{\mu}_{A_2}(h(x_1)), \\
&\widetilde{\mu}_{A_2}(h(y_1))) - \widetilde{\mu}_{B_2}(h(x_1)h(y_1)) \\
&= \min(\widetilde{\mu}_{A_2}(x_2), \widetilde{\mu}_{A_2}(y_2)) - \widetilde{\mu}_{B_2}(x_2y_2) \\
&= \overline{\widetilde{\mu}_{B_2}}(x_2y_2), \text{for all } x_2, y_2 \in V_2.
\end{aligned}$$

i.e. $\overline{\widetilde{\mu}_{B_2}}(x_2y_2) \leq \overline{\widetilde{\mu}_{B_1}}(h^{-1}(x_2)h^{-1}(y_2))$, for all $x_2, y_2 \in V_2$. Also, $\overline{\lambda_{B_1}}(x_1y_1) = \lambda_{B_1}(x_1y_1) - \max(\lambda_{A_1}(x_1), \lambda_{A_1}(y_1))$, for all $x_1, y_1 \in V_1$. So,

$$\begin{aligned}
\overline{\lambda_{B_1}}(h^{-1}(x_2)h^{-1}(y_2)) &\leq \lambda_{B_2}(h(x_1)h(y_1)) \\
&- \max(\lambda_{A_2}(h(x_1)), \lambda_{A_2}(h(y_1))) \\
&= \lambda_{B_2}(x_2y_2) - \max(\lambda_{A_2}(x_2), \lambda_{A_2}(y_2)) \\
&= \overline{\lambda_{B_2}}(x_2y_2), \quad \text{for all } x_2, y_2 \in V_2
\end{aligned}$$

i.e. $\overline{\lambda_{B_2}}(x_2y_2) \geq \overline{\lambda_{B_1}}(h^{-1}(x_2)h^{-1}(y_2))$, for all $x_2, y_2 \in V_2$. Therefore, $h^{-1} : V_2 \to V_1$ is a weak isomorphism between $\overline{G}_1$ and $\overline{G}_2$. □

Remark 3.4. The above theorem is true for neighborly irregular cubic graphs.

Theorem 3.5. *Let G_1 and G_2 be two highly irregular cubic graphs (neighborly irregular cubic graphs). If G_1 is a co-weak isomorphic with G_2, then there exists a homomorphism between $\overline{G}_1$ and $\overline{G}_2$.*

Definition 3.7. *A cubic graph G is said to be a self weak complementary if G is weak isomorphic with $\overline{G}$.*

Theorem 3.6. *Let G be a self weak complementary highly irregular cubic graph, then*

$$\sum_{x\neq y}\widetilde{\mu}_B(xy) \leq \frac{1}{2}\sum_{x\neq y}\min(\widetilde{\mu}_A(x),\widetilde{\mu}_A(y)) \quad \textit{and}$$
$$\sum_{x\neq y}\lambda_B(xy) \geq \frac{1}{2}\sum_{x\neq y}\max(\widetilde{\mu}_A(x),\widetilde{\mu}_A(y)).$$

Proof. Let $G = (A, B)$ be a self weak complementary highly irregular cubic graph of $G^* = (V, E)$. Using the definition of complement for all $x, y \in V$ we have,

$$\begin{array}{rcl}
\widetilde{\mu}_B(xy) & \leq & \overline{\widetilde{\mu}_B}(h(x)h(y)) \\
 & = & \min(\widetilde{\mu}_A(h(x)),\widetilde{\mu}_A(h(y))) - \widetilde{\mu}_B(h(x)h(y)), \\
f_B(xy) & \geq & \overline{\lambda_B}(h(x)h(y)) = \lambda_B(h(x)h(y)) \\
 & - & \max(\lambda_A(h(x)),\lambda_A(h(y))), \\
\widetilde{\mu}_B(xy) & + & \widetilde{\mu}_B(h(x)h(y)) \leq \min(\widetilde{\mu}_A(h(x)),\widetilde{\mu}_A(h(y))) \\
\lambda_B(xy) & + & \max(\lambda_A(h(x)),\lambda_A(h(y))) \geq \lambda_B(h(x)h(y)).
\end{array}$$

So,

$$\begin{array}{rcl}
\displaystyle\sum_{x\neq y}\widetilde{\mu}_B(xy) & + & \displaystyle\sum_{x\neq y}\widetilde{\mu}_B(h(x)h(y)) \\
 & \leq & \displaystyle\sum_{x\neq y}\min(\widetilde{\mu}_A(h(x)),\widetilde{\mu}_A(h(y)))
\end{array}$$

and

$$\begin{array}{rcl}
\displaystyle\sum_{x\neq y}\lambda_B(xy) & + & \displaystyle\sum_{x\neq y}\lambda_B(h(x)h(y)) \\
 & \geq & \displaystyle\sum_{x\neq y}\max(\lambda_A(h(x)),\lambda_A(h(y))).
\end{array}$$

Hence, $2\sum_{x\neq y}\widetilde{\mu}_B(xy) \leq \sum_{x\neq y}\min(\widetilde{\mu}_A(x),\widetilde{\mu}_A(y))$ and $2\sum_{x\neq y}\lambda_B(xy) \geq \sum_{x\neq y}\max(\lambda_A(x),\lambda_A(y))$. Therefore, $\sum_{x\neq y}\widetilde{\mu}_B(xy) \leq \frac{1}{2}\sum_{x\neq y}\min(\widetilde{\mu}_A(x), \widetilde{\mu}_A(y))$ and $\sum_{x\neq y}\lambda_B(xy) \geq \frac{1}{2}\sum_{x\neq y}\max(\lambda_A(x),\lambda_A(y))$. □

Definition 3.8. *Let $G = (A, B)$ be an irregular cubic graph then, the irregularity of G is defined as $Irreg(G) = (Irreg_{\widetilde{\mu}}(G), Irreg_{\lambda(G)})$ where $Irreg_{\widetilde{\mu}}(G) =$*

$\sum_{xy\in E}|d_{\widetilde{\mu}}(x)-d_{\widetilde{\mu}}(y)|$ *and* $Irreg_{\lambda}(G)=\sum_{xy\in E}|d_{\lambda}(x)-d_{\lambda}(y)|$, *for all* $x,y\in V$. *The total irregularity of the irregular cubic graph is defined as*

$$Irreg_{total}(G)=\frac{1}{2}\sum_{x,y\in V}|d(x)-d(y)|.$$

Example 3.8. Consider an irregular cubic graph G as in Figure 8.

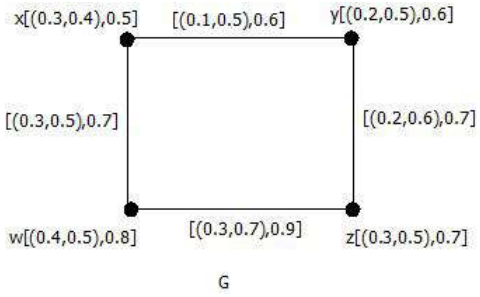


Figure 8: Irregular cubic graph G

Here, $d(x)$=$[(0.4,1),1.2]$, $d(y)$=$[(0.3,1.1),1.3]$, $d(z)$=$[(0.5,1.3),1.5]$, $d(w)=[(0.6,1.2),1.5]$. It is easy to show that $Irreg(G)=[(0.1,0.1),0.1]+[(0.2,0.2),0.2]+[(0.1,0.1),0.1]+[(0.2,0.2),0.4]=[(0.6,0.6),0.8]$. But the total irregularity is $Irreg_{total}(G)=\frac{1}{2}\{[(0.1,0.1),0.1]+[(0.1,0.3),0.3]+[(0.2,0.2),0.4]+[(0.2,0.2),0.2]+[(0.3,0.1),0.3]+[(0.1,0.1),0.3]\}=[(0.5,0.5),0.8]$.

Remark 3.5. (i) For any irregular cubic graph G, $Irreg_{total}(G)\leq Irreg(G)$.

(ii) Let $G=(A,B)$ be the irregular cubic graph which is complete then, $Irreg(G)=2[Irreg_{total}(G)]$.

Definition 3.9. *Let* $G=(A,B)$ *be an irregular cubic graph. The density of* G *is defined as* $D(G)=(D_{\widetilde{\mu}}(G),D_{\lambda}(G))$ *where* $D_{\widetilde{\mu}}(G)=\frac{2\sum_{x,y\in V}\widetilde{\mu}_B(xy)}{\sum_{xy\in E}\widetilde{\mu}_A(x)\wedge\widetilde{\mu}_A(y)}$, *for all* $x,y\in V$ *and* $D_{\lambda}(G)=\frac{2\sum_{x,y\in V}\lambda_B(xy)}{\sum_{xy\in E}\lambda_A(x)\vee\lambda_A(y)}$, *for all* $x,y\in V$.

Example 3.9. In Example 3.30 we have:

$$\begin{aligned}&D(G)=(D_{\widetilde{\mu}}(G),D_{\lambda}(G))\\&=\left(\frac{2[0.1+0.2+0.3+0.3]}{0.2+0.2+0.3+0.3},\frac{2[0.5+0.6+0.7+0.5]}{0.5+0.5+0.5+0.5},\frac{2[0.6+0.7+0.7+0.7]}{0.6+0.7+0.8+0.8}\right)\\&=[(1.8,2.42),1.86].\end{aligned}$$

Theorem 3.7. *Let* $G=(A,B)$ *be the irregular cubic graph with* $\widetilde{\mu}_B(xy)=\frac{1}{2}(\widetilde{\mu}_A(x)\wedge\widetilde{\mu}_A(y))$ *and* $\lambda_B(xy)=\frac{1}{2}(\lambda_A(x)\vee\lambda_A(y))$ *then,* $D(G)=(1,1)$.

Proof.

$$\begin{aligned} D(G) &= \Big(\frac{2\sum_{x,y\in V}\widetilde{\mu}_B(xy)}{\sum_{xy\in E}(\widetilde{\mu}_A(x)\wedge\widetilde{\mu}_A(y))}, \\ &= \Big(\frac{2\sum_{x,y\in V}\frac{1}{2}(\widetilde{\mu}_A(x)\wedge\widetilde{\mu}_A(y))}{\sum_{xy\in E}(\widetilde{\mu}_A(x)\wedge\widetilde{\mu}_A(y))}, \\ &\qquad \frac{2\sum_{x,y\in V}\frac{1}{2}(\lambda_A(x)\vee\lambda_A(y))}{\sum_{xy\in E}(\lambda_A(x)\vee\lambda_A(y))}\Big) \\ &= \Big(\frac{2(\frac{1}{2})\sum_{x,y\in V}(\widetilde{\mu}_A(x)\wedge\widetilde{\mu}_A(y))}{\sum_{xy\in E}(\widetilde{\mu}_A(x)\wedge\widetilde{\mu}_A(y))}, \\ &\qquad \frac{2(\frac{1}{2})\sum_{x,y\in V}(\lambda_A(x)\vee\lambda_A(y))}{\sum_{xy\in E}(\lambda_A(x)\vee\lambda_A(y))}\Big) = (1,1). \quad \square \end{aligned}$$

Theorem 3.8. *Let G be an irregular and complete cubic graph then, $D(G) = (2,2)$.*

Proof. Since G is complete and irregular we have $\widetilde{\mu}_B(xy) = \widetilde{\mu}_A(x)\wedge\widetilde{\mu}_A(y)$ and $\lambda_B(xy) = \lambda_A(x)\vee\lambda_A(y)$, for all $x, y \in V$. Now, $\sum_{x,y\in V}\widetilde{\mu}_B(xy) = \sum_{xy\in E}(\widetilde{\mu}_A(x)\wedge\widetilde{\mu}_A(y))$, $\sum_{x,y\in V}\lambda_B(xy) = \sum_{xy\in E}(\lambda_A(x)\vee\lambda_A(y))$. Therefore,

$$\begin{aligned} D(G) &= \Big(\frac{2\sum_{x,y\in V}\widetilde{\mu}_B(xy)}{\sum_{xy\in E}(\widetilde{\mu}_A(x)\wedge\widetilde{\mu}_A(y))}, \\ &= \Big(\frac{2\sum_{xy\in E}(\widetilde{\mu}_A(x)\wedge\widetilde{\mu}_A(y))}{\sum_{xy\in E}(\widetilde{\mu}_A(x)\wedge\widetilde{\mu}_A(y))}, \\ &\qquad \frac{2\sum_{xy\in E}(\lambda_A(x)\vee\lambda_A(y))}{\sum_{xy\in E}(\lambda_A(x)\vee\lambda_A(y))}\Big) = (2,2). \quad \square \end{aligned}$$

Definition 3.10. *An irregular cubic graph G is said to be balanced if $D_{\widetilde{\mu}}(H) \leq D_{\widetilde{\mu}}(G)$ and $D_\lambda(H) \leq D_\lambda(G)$, for all subgraph H of G.*

Example 3.10. Consider an irregular cubic graph G as in Figure 9.

Here, $D(G) = (D_{\widetilde{\mu}}(G), D_\lambda(G)) = [(1.3, 1.5), 2]$, and subgraph of G are $H_1 = \{x, y\}$, $H_2 = \{x, z\}$, $H_3 = \{x, w\}$, $H_4 = \{y, z\}$, $H_5 = \{y, w\}$, $H_6 = \{z, w\}$, $H_7 = \{x, y, z\}$, $H_8 = \{x, z, w\}, H_9 = \{x, y, w\}$, $H_{10} = \{y, z, w\}$, $H_{11} = \{x, y, z, w\}$.

Now, density $(D_{\widetilde{\mu}}(H), D_\lambda(H))$ is $D(H_1) = [(1.3, 1.5), 2]$, $D(H_2) = [(0, 0), 0]$, $D(H_3) = [(1.3, 1.5), 2]$, $D(H_4) = [(1.25, 1.5), 2]$, $D(H_5) = [(0, 0), 0]$, $D(H_6) = [(1.25, 1.5), 2]$, $D(H_7) = [(1.3, 1.5), 2]$, $D(H_8) = [(0.92, 1.5), 2]$, $D(H_9) = [(1.3, 1.5), 2]$, $D(H_{10}) = [(1.25, 1.5), 2]$, $D(H_{11}) = [(1.3, 1.5), 2]$. So, $D(H) \leq D(G)$, for all subgraphs H of G. Hence, G is balanced irregular cubic graph.

Theorem 3.9. *Let $G = (A, B)$ be an irregular cubic graph and all the edges of G are strong then, G is balanced.*

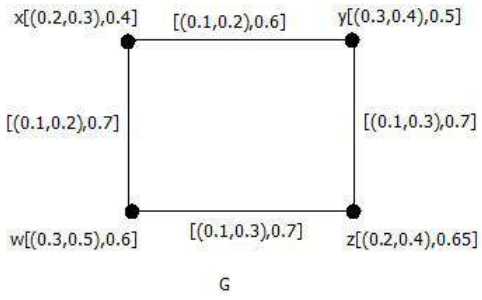


Figure 9: Balanced irregular cubic graph G

Proof. Since all the edges are strong then $\widetilde{\mu}_B(xy) = \widetilde{\mu}_A(x) \wedge \widetilde{\mu}_A(y)$ and $f_B(xy) = f_A(x) \vee f_A(y)$. Now, by Theorem 3.8, $D(G) = (2, 2)$. For the subgraphs H, $(D_{\widetilde{\mu}}(H), D_\lambda(H)) = (2, 2)$ if the vertices of H having edges, otherwise $(D_{\widetilde{\mu}}(H), D_\lambda(H)) = (0, 0)$ i.e. $D_{\widetilde{\mu}}(H) \leq D_{\widetilde{\mu}}(G)$ and $D_\lambda(H) \leq D_\lambda(G)$ which implies G is balanced. □

4. Conclusion

Fuzzy graph theory has numerous applications in modern science and technology, especially in the fields of operations research, neural networks, and decision making. Since the cubic models give more precision, flexibility and compatibility to the system as compared to the classical and fuzzy models, hence, in this paper, the definitions of partial edge regular and fully edge regular cubic graph are given and several properties of edge regular cubic graph are studied. Also, we have introduced the condition under which edge regular cubic graph and totally edge regular cubic graph are equivalent. In our future work, we will introduce cubic incidence graphs and study the concepts of connected perfect dominating set, regular perfect dominating set, inverse perfect dominating set, and independent perfect dominating set on cubic incidence graph.

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Fixed point theorem for non-self mappings satisfying generalized contraction condition of integral type in metrically convex spaces

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Abstract. A common fixed point theorem for single valued non-self mapping from nonempty closed subset K of a metrically convex metric spaces X is proved satisfying the contraction condition of integral type on the subset K. Result generalizing the previous results due to Banach [3], Branciari [4], Ćirić [6], Rhoades [11], Khan [9] and others.

Keywords: fixed point, integral type condition, metric convexity.

1. Introduction

In this paper we consider generalized contraction condition of integral type in metrically convex spaces and prove a fixed point theorem for nonself single valued maps. Here, let us point out that there do exist many situations when the mapping on a metric space or Banach space is not always a self map. Therefore, it is of great interest to find the sufficient conditions on non-self mapping which will ensure the existence of a fixed point.

In 1972, Assad and Kirk [1] introduced the definition of convexity in metric space and established a wonderful result. Since then there have been many theorems dealing with non-self mappings satisfying various types of contractive inequalities. The recent literature witness various extensions and generalizations of this theorem which includes Assad[2], Imdad et al. [7], Khan and Imdad [10], Khan [8] and others.

Here, let us recall the following results for the convenience of the reader.

Theorem 1.1 ([4]). *Let (X, d) be a complete metric space, $c \in [0, 1)$, $f : X \to X$ be a mapping such that, for each $x, y \in X$,*

$$\int_0^{d(fx,fy)} \phi(t)dt \leq \ c \int_0^{d(x,y)} \phi(t)dt,$$

where $\phi : R^+ \to R^+$ is a Lebesgue-integrable mapping which is summable,non-negative and such that, for each $\epsilon > 0$, $\int_0^{\epsilon} \phi(t)dt > 0$. Then f has a unique fixed point $z \in X$ such that, for each $x \in X, \lim_{n\to\infty} f^n x = z$.

Theorem 1.2 ([11]). *Let (X,d) be a complete metric space, $k \in [0,1)$, $f : X \to X$ be a mapping such that, for each $x, y \in X$,*

$$\int_0^{d(fx,fy)} \phi(t)dt \leq k \int_0^{m(x,y)} \phi(t)dt,$$

where $m(x,y) = \max\{d(x,y), d(x,fx), d(y,fy), \frac{d(x,fy)+d(y,fx)}{2}\}$ and $\phi : R^+ \to R^+$ is a Lebesgue-integrable mapping which is summable,non-negative and such that, for each $\epsilon > 0$, $\int_0^\epsilon \phi(t)dt > 0$. Then f has a unique fixed point $z \in X$ such that, for each $x \in X, \lim_{n\to\infty} f^n x = z$.

In the present paper we prove a fixed point theorem for a single valued non-self mapping which generalize the earlier results due to Banach [3], Branciari [4], Ćirić [6], Rhoades [11], Khan [9] and others.

Before proving the results, we collect the following definitions for future discussion.

Definition 1.3. Let (X,d) be a metric space and K be a nonempty subset of a metric space X. Let a mapping $T : K \to X$ is said to be generalized contraction condition on K, if for each $x, y \in K$,

$$\int_0^{d(Tx,Ty)} \phi(t)dt \leq c \int_0^{m(x,y)} \phi(t)dt, c \in [0,1), \tag{1.1}$$

where $m(x,y) = h \max\left\{\frac{1}{2}d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{q}\right\}, 0 < h(1+h) \leq k < 1$ and any real number q satisfying $q \geq 1+2k$, and $\phi : R^+ \to R^+$ is a Lebesgue-integrable mapping which is summable, non-negative and such that

$$\int_0^\epsilon \phi(t)dt > 0 \text{ for each } \epsilon > 0. \tag{1.2}$$

Definition 1.4 ([1]). A metric space (X,d) is said to be metrically convex if for any $x, y \in X$ with $x \neq y$ there exists a point $z \in X, x \neq z \neq y$ such that

$$d(x,z) + d(z,y) = d(x,y).$$

2. Results

The result of this paper runs as follows.

Theorem 2.1. *Let (X,d) be a complete metrically convex metric space and K be a nonempty closed subset of X. Let $T : K \to X$ be a mapping satisfying generalized contraction condtion and for each $x \in \delta K, Tx \in K$. Then T has a unique fixed point $x \in K$ such that, for each $x \in K, \lim_{n\to\infty} T^n x = x$.*

Proof. Firstly, we proceed to construct two sequences $\{x_n\}$ and $\{y_n\}$ in the following way. Let $x_0 \in K$. Define $y_1 = Tx_0$. If $y_1 \in K$ set $y_1 = x_1$. If $y_1 \notin K$, then choose $x_1 \in \delta K$ so that

$$d(x_0, x_1) + d(x_1, y_1) = d(x_0, y_1).$$

If $y_2 \in K$ then set $y_2 = x_2$. If $y_2 \notin K$, then choose $x_2 \in \delta K$ so that

$$d(x_1, x_2) + d(x_2, y_2) = d(x_1, y_2).$$

Thus, repeating the foregoing arguments, one obtains two sequences $\{x_n\}$ and $\{y_n\}$ such that

(i) $y_{n+1} = Tx_n$,

(ii) $y_n = x_n$ if $y_n \in K$,

(iii) if $x_n \in \delta K$ then $d(x_{n-1}, x_n) + d(x_n, y_n) = d(x_{n-1}, y_n)$, where $y_n \notin K$.

Here, one obtains two types of sets we denote as follows:

$$P = \{x_i \in \{x_n\} : x_i = y_i\} \text{ and } Q = \{x_i \in \{x_n\} : x_i \neq y_i\}.$$

One can note that if $x_n \in Q$ then x_{n-1} and $x_{n+1} \in P$. We, wish to estimate $d(x_n, x_{n+1})$. Now, we distinguish the following three cases.

Case 1. If x_n and $x_{n+1} \in P$, then

$$\int_0^{d(x_n, x_{n+1})} \phi(t)dt = \int_0^{d(Tx_{n-1}, Tx_n)} \phi(t)dt \leq c \int_0^{m(x_{n-1}, x_n)} \phi(t)dt.$$

Since

$$\begin{aligned} & d(Tx_{n-1}, Tx_n) \\ & \leq h \max \left\{ \frac{d(x_{n-1}, x_n)}{2}, d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), \right. \\ & \qquad \left. \frac{d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})}{q} \right\}, \end{aligned}$$

$$d(Tx_{n-1}, Tx_n) \leq h \max \left\{ d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1})}{q} \right\}. \tag{1.3}$$

Here, if $d(x_{n-1}, x_n)$ is maximum, then we have

$$d(Tx_{n-1}, Tx_n) \leq h\ d(x_{n-1}, x_n) < d(x_{n-1}, x_n).$$

Otherwise, if we suppose that $d(x_{n-1}, x_n) < d(x_n, x_{n+1})$, then we obtain

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq d(x_n, x_{n+1}),$$

which is a contradiction.

Now, if $\frac{d(x_{n-1},x_{n+1})}{q}$ is maximum then from equation (1.3) we have

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq \frac{h}{q}\{d(x_{n-1}, x_n) + d(x_n, x_{n+1})\}$$
$$\leq \frac{h}{q-h} d(x_{n-1}, x_n) < hd(x_{n-1}, x_n) < kd(x_{n-1}, x_n) < d(x_{n-1}, x_n).$$

Therefore,

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq d(x_{n-1}, x_n).$$

Hence

$$\int_0^{d(x_n,x_{n+1})} \phi(t)dt \leq \ c\int_0^{m(x_{n-1},x_n)} \phi(t)dt.$$

Case 2. If $x_n \in P$ and $x_{n+1} \in Q$, then

$$d(x_n, x_{n+1}) + d(x_{n+1}, y_{n+1}) = d(x_n, y_{n+1})$$

which in turn yields

$$d(x_n, x_{n+1}) \leq d(x_n, y_{n+1}).$$

Now, proceeding as in case 1, we have

$$\int_0^{d(x_n,x_{n+1})} \phi(t)dt \leq \ c\int_0^{m(x_{n-1},x_n)} \phi(t)dt.$$

Case 3. If $x_n \in Q$ and $x_{n+1} \in P$. Since $x_n \in Q$ and is a convex linear combination of x_{n-1} and y_n, it follows that

$$d(x_n, x_{n+1}) \leq \max\{d(x_{n-1}, x_{n+1}), d(y_n, x_{n+1})\}.$$

If $d(x_{n-1}, x_{n+1}) \leq d(x_{n+1}, y_n)$, then proceeding as in case 1, we have

$$\int_0^{d(x_n,x_{n+1})} \phi(t)dt \leq \ c\int_0^{m(x_{n-1},x_n)} \phi(t)dt.$$

Otherwise, if $d(x_{n+1}, y_n) \leq d(x_{n-1}, x_{n+1})$ then, we have

$$\int_0^{d(x_n,x_{n+1})} \phi(t)dt \leq \int_0^{d(x_{n-1},x_{n+1})} \phi(t)dt$$
$$= \int_0^{d(Tx_{n-2},Tx_n)} \phi(t)dt \leq \ c\int_0^{m(x_{n-2},x_n)} \phi(t)dt.$$

Here

$$\begin{aligned} & d(x_n, x_{n+1}) \leq \\ & h \max \left\{ \frac{d(x_{n-2}, x_n)}{2}, d(x_{n-2}, Tx_{n-2}), d(x_n, Tx_n), \right. \\ & \qquad \left. \frac{d(x_{n-2}, Tx_n) + d(x_n, Tx_{n-2})}{q} \right\} \\ & \leq h \max \left\{ \frac{d(x_{n-2}, x_n)}{2}, d(x_{n-2}, x_{n-1}), d(x_n, x_{n+1}), \right. \\ & \qquad \left. \frac{d(x_{n-2}, x_{n+1}) + d(x_n, x_{n-1})}{q} \right\}. \end{aligned}$$

Notice that

$$\begin{aligned} \frac{d(x_{n-2}, x_n)}{2} & \leq \frac{d(x_{n-2}, x_{n-1}) \ + \ d(x_{n-1}, x_n)}{2} \\ & \leq \max \{d(x_{n-2}, x_{n-1}), d(x_{n-1}, x_n)\}. \end{aligned}$$

Here, if

$$d(x_{n-2}, x_{n-1}) \leq d(x_{n-1}, x_n), \text{then } d(x_{n-2}, x_n) \leq d(x_{n-1}, x_n).$$

Otherwise, if

$$d(x_{n-1}, x_n) \leq d(x_{n-2}, x_{n-1}), \text{ then } d(x_{n-2}, x_n) \leq d(x_{n-2}, x_{n-1}).$$

Therefore, we obtain

$$\begin{aligned} d(x_n, x_{n+1}) \leq h \max \Big\{ & d(x_{n-2}, x_{n-1}), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \\ & \frac{d(x_{n-2}, x_{n+1}) + d(x_n, x_{n-1})}{q} \Big\} \end{aligned}$$

which in turn yields

$$d(x_n, x_{n+1}) \leq \begin{cases} c \ d(x_{n-1}, x_n) & \text{if } \ d(x_{n-1}, x_n) \geq d(x_{n-2}, x_{n-1}) \text{ or} \\ c \ d(x_{n-2}, x_{n-1}) & \text{if } \ d(x_{n-1}, x_n) \leq d(x_{n-2}, x_{n-1}). \end{cases}$$

Thus, in all the cases, we have

$$d(x_n, x_{n+1}) \leq c \max\{d(x_{n-1}, x_n), d(x_{n-2}, x_{n-1})\}.$$

It can be easily shown by induction that for $n > 1$, we have

$$d(x_n, x_{n+1}) \leq c \max\{d(x_0, x_1), d(x_1, x_2)\}.$$

Thus,

$$\int_0^{d(x_n,x_{n+1})} \phi(t)dt \leq c \int_0^{\max\{d(x_0,x_1),d(x_1,x_2)\}} \phi(t)dt$$

which implies that

$$\int_0^{d(x_n,x_{n+1})} \phi(t)dt \leq c \max\left\{ \int_0^{d(x_0,x_1)} \phi(t)dt, \int_0^{d(x_1,x_2)} \phi(t)dt \right\}.$$

It follows that the sequence $\{d(x_n, x_{n+1})\}$ is monotonically decreasing. Hence $\int_0^{d(x_n,x_{n+1})} \phi(t)dt \to 0$ as $n \to \infty$. From equation (1.2) it implies that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

Now, we prove that the sequence $\{x_n\}$ is a Cauchy sequence. Let on contrary that the sequence $\{x_n\}$ is not Cauchy. Then, there exists $\epsilon > 0$ for which we can find subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ such that $d(x_{m_k}, x_{n_k}) \geq \epsilon$.
Here, we proceed on the lines of Rhoades [11] it can be shown that the sequence $\{x_n\}$ is Cauchy and converges to a point say x.
From equation (1.1) we have

$$\begin{aligned}
&\int_0^{d(Tx,x_{n+1})} \phi(t)dt \leq c \int_0^{m(x,x_n)} \phi(t)dt \\
&\leq c \max\left\{ \int_0^{d(x,x_n)} \phi(t)dt, \int_0^{d(x,Tx)} \phi(t)dt, \int_0^{d(x_n,x_{n+1})} \phi(t)dt, \right. \\
&\qquad \left. \int_0^{d(x,x_{n+1})} \phi(t)dt, \int_0^{d(x_n,Tx)} \phi(t)dt, \right\}. \qquad (1.4)
\end{aligned}$$

On letting $k \to \infty$, in equation (1.4) then, we have

$$\int_0^{d(Tx,x)} \phi(t)dt \leq c \int_0^{d(Tx,x)} \phi(t)dt$$

which implies that

$$\int_0^{d(Tx,x)} \phi(t)dt = 0, \qquad (1.5)$$

which from equation (1.5), implies that $d(Tx, x) = 0 \Rightarrow Tx = x$. This shows that x is a fixed point of T.

To prove that the uniqueness of fixed points. Let us suppose that x_1 and x_2 are two fixed points of T. Then

$$\begin{aligned}
&\int_0^{d(x_1,x_2)} \phi(t)dt = \int_0^{d(Tx_1,Tx_2)} \phi(t)dt \leq c \int_0^{m(x_1,x_2)} \phi(t)dt \\
&= c \max\left\{ \int_0^{d(x_1,x_2)} \phi(t)dt, 0 \right\} = c \int_0^{d(x_1,x_2)} \phi(t)dt,
\end{aligned}$$

which implies that $\int_0^{d(x_1,x_2)} \phi(t)dt = 0$. Also imply that $d(x_1, x_2) = 0$ or $x_1 = x_2$. This shows the uniqueness of fixed point. This completes the proof. □

Remark 2.2. By setting $K = X$ and $\phi(t) = 1$ for each $t \geq 0$ in the Theorem 2.1, then we deduce a partial generalization of the result due to Banach [3].

Remark 2.3. By setting $K = X$ in the Theorem 2.1, then we deduce a sharpened version of the result due to Branciari [4].

Remark 2.4. By setting $K = X$ in the Theorem 2.1, then we deduce a partial generalization of the result due to Rhoades [11].

Remark 2.5. By setting some minor changes in the Theorem 2.1, then we deduce the result due to Khan [9].

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Finite groups whose character degrees are products of at most three prime numbers

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Abstract. Huppert and Manz determined the structure of nonsolvable groups whose character degrees are products of at most two primes. In this paper, we classify nonsolvable groups satisfying that their character degrees are products of at most three primes.

Keywords: simple group, nonsolvable group, character degree.

1. Introduction

Suppose that all groups considered are finite. Let $\mathrm{Irr}(G)$ be the set of all complex irreducible characters of G. Set $\mathrm{cd}(G) = \{\chi(1) : \chi \in \mathrm{Irr}(G)\}$, the set of degrees of all irreducible characters of G. It is well known that if G is a finite group, and p is a prime, then $p \nmid \chi(1)$ for all $\chi \in \mathrm{Irr}(G)$, if and only if G has an abelian normal Sylow p-subgroup; see Theorem 12.34 of [14]. Gow-Huphreys proved that if G is a finite group, p is a prime, and P is a Sylow p-subgroup of G, then $p\big|\chi(1)$ for all nonlinear character $\chi \in \mathrm{Irr}(G)$ if and only if G has a normal p-complement K and $C_{K'}(P) = 1$; see Theorem 7.6 of [14]. Recently, some authors considered the relation between character degrees of certain simple groups and the structure of finite groups; see [11, 13, 19] for instance.

Let $\mathbb{N}$ be the set of positive integers. If $n \in \mathbb{N}$ can be written by

$$n = \prod_{i=1}^{k} p_i^{a_i},$$

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where $p_1, p_2, \cdots, p_s$ are different primes and $a_1, a_2, \cdots, a_k$ are positive integers, then define

$$\omega(n) = \sum_{i=1}^{k} a_i$$

and $\omega(G) = \max_{\chi \in \mathrm{Irr}(G)} \omega(\chi(1))$.

If $\omega(G) = 1$, then all non-linear characters have prime-number-degrees and so by [10, pp. 243 Corollary 14.4], G is solvable. Huppert and Manz in [8] proved that the groups whose character degrees are square-free are solvable. Lewis and White in [12] classified the nonsolvable groups whose degrees of all irreducible characters are odd-square-free. Huppert and Manz in [9] have classified nonsolvable groups whose character degrees are products of at most two prime numbers. From Theorem of [9], we can get the following result.

Theorem 1.1. *Let G be a nonabelian simple group with $\omega(G) \leq 2$. Then G is isomorphic to A_5 of A_7.*

Huppert in [7] showed that the derived length of a group G is at most 4 if $\omega(G) = 2$ and G is solvable.

Let G be a finite group and put

$$\pi(G) = \{p : p \text{ is a prime divisor of } |G|\}.$$

The other notation is standard; see [4].

We first give the structure of a non-abelian simple group G with $\omega(G) \leq 3$.

Theorem 1.2. *Assume that G is a non-abelian simple group with $\omega(G) \leq 3$.*

If $\omega(G) = 2$, then G is isomorphic to A_5 or A_7.

If $\omega(G) = 3$, then the following hold.

(1) *If $|\pi(G)| = 3$, then G is isomorphic to $L_2(7)$, $L_2(8)$ or $L_2(9)$;*

(2) *If $|\pi(G)| = 4$, then G is isomorphic to $L_2(11)$, $L_2(19)$ or $L_2(27)$;*

(3) *If $|\pi(G)| = 5$, then G is isomorphic to $L_2(p)$ with either $p - 1 = 2^2 \cdot p_1$, $p + 1 = 2 \cdot 3 \cdot p_2$ or $p - 1 = 2 \cdot 3 \cdot p_1$, $p + 1 = 2^2 \cdot p_2$ where p, p_1, p_2 are different primes and $p_1, p_2 \geq 5$.*

In this paper, we will also give the structure of a nonsolvable group G with $\omega(G) = 3$.

Theorem 1.3. *Assume that G is a nonsolvable group. If $\omega(G) = 3$, then G has a normal subgroup N such that $G/N \cong L_2(q)$ for some prime power q or A_7. Furthermore, each $\theta \in \mathrm{Irr}(N)$ is invariant in G and one of the following statements holds.*

(1) *G/N is isomorphic to S where $\mathrm{cd}(N) = \{1, p\}$ for some prime p, and $S \in \{A_5, S_5, A_7\}$;*

(2) *G/N is isomorphic to S where N is abelian, $S \in \{L_2(8), S_7, L_2(8).3, L_2(7), L_2(7).2, L_2(9), L_2(11), L_2(11).2, L_2(19), L_2(19).2, L_2(27), L_2(27).2,$*

$L_2(9).2_2$, $L_2(q)$, $L_2(q).2$} with q a prime satisfying either $q-1=2^2\cdot p_1, q+1=2\cdot 3\cdot p_2$ or $q-1=2\cdot 3\cdot p_1, q+1=2^2\cdot p_2$ where p_1, p_2 are different primes and $p_1, p_2 \geq 5$.

2. Some needed results

Lemma 2.1. *Let N be a subgroup of G. Then $\omega(N) \leq \omega(G)$. If $N \trianglelefteq G$, then $\omega(G/N) \leq \omega(G)$.*

Proof. It is easy to get from the definition of $\omega(G)$. □

Lemma 2.2 ([3])**.** *If x or y is prime, then the equation $x^m - y^n = 1$, $x > 1$, $y > 1$, $m > 1$, $n > 1$, $x, y, m, n \in \mathbb{N}$, has only solution $(x, y, m, n) = (3, 2, 2, 3)$.*

Recall that a group G is called a ***simple K_n-group*** if $|\pi(G)| = n \geq 3$ and G is simple.

Lemma 2.3 ([5])**.** *If G is a simple K_3-group. Then G is isomorphic to one of the groups: A_5, A_6, $L_2(7)$, $L_2(8)$, $L_2(17)$, $L_3(3)$, $U_3(3)$ and $U_4(2)$.*

Lemma 2.4 (Lemma 1 of [18])**.** *Let G be an insolvable group. Then G has a normal series $1 \trianglelefteq K \trianglelefteq L \trianglelefteq G$, such that L/K is a direct product of isomorphic non-abelian simple groups and $|G/L| \mid |\mathrm{Out}(L/K)|$.*

Lemma 2.5. *Let L be a sporadic simple group or ${}^2F_4(2)'$ except M_{11}. Then $\omega(L) \geq 5$. In particular, $\omega(M_{11}) = 4$.*

Proof. By [4], we have Table 1, and so get the desired result. □

Table 1. Sporadic simple groups and a Tits group

L	Character	Degree	L	Character	Degree
M_{11}	χ_6	2^4	M_{12}	χ_{13}	$2^3\cdot 3\cdot 5$
J_1	χ_9	$2^2\cdot 3\cdot 5$	M_{22}	χ_{10}	$2^3\cdot 5\cdot 7$
J_2	χ_{16}	$2^4\cdot 3^2$	M_{23}	χ_{12}	$2^7\cdot 7$
${}^2F_4(2)'$	χ_{22}	2^{11}	HS	χ_{24}	$2^5 7\cdot 5^2$
J_3	χ_{21}	$2\cdot 3^4\cdot 19$	M_{24}	χ_{24}	$2^3\cdot 3^2\cdot 7\cdot 11$
M^cL	χ_{11}	$2^6\cdot 5\cdot 11$	He	χ_{30}	$2^{10}\cdot 3\cdot 7$
Ru	χ_{11}	$2^3\cdot 3^3\cdot 5^3$	Suz	χ_{43}	$2^{10}\cdot 3^5$
ON	χ_{25}	$2^9\cdot 3^4\cdot 5$	Co_3	χ_{39}	$2^7\cdot 5^2\cdot 7\cdot 11$
Co_2	χ_{60}	$3^6\cdot 5^3\cdot 23$	Fi_{22}	χ_{44}	$2^{16}\cdot 3\cdot 13$
HN	χ_{53}	$2^3\cdot 3^6\cdot 5^3\cdot 7$	Ly	χ_{55}	$2^2\cdot 3\cdot 7\cdot 11\cdot 31\cdot 37\cdot 67$
Th	χ_4	$2^3\cdot 3^3\cdot 5^3$	Fi_{23}	χ_3	$2^2\cdot 3\cdot 13\cdot 23$
Co_1	χ_6	$2\cdot 3\cdot 5^2\cdot 23$	J_4	χ_4	$3^2\cdot 29\cdot 31\cdot 37$
Fi_{24}	χ_4	$2\cdot 11\cdot 17\cdot 23\cdot 29$	B	χ_5	$2\cdot 5^4\cdot 7\cdot 23\cdot 47$
M	χ_3	$2^2\cdot 31\cdot 41\cdot 59\cdot 71$			

3. The proof of theorems

The structure of the simple groups whose degrees of the irreducible complex characters are products of at most three primes, are determined first, and then the structure of a nonsolvable group G with $\omega(G) = 3$ is given.

3.1 The proof of Theorem 1.2

Proof. If G is a non-abelian simple group, then G is isomorphic to one of the following simple groups: alternating groups A_n with $n \geq 5$, sporadic simple groups or simple groups of Lie type.

If $\omega(G) = 2$, then by [9], we conclude that G is isomorphic to A_5 or A_7.

So, we consider $\omega(G) = 3$ in the following.

(1) Let G be an alternating group A_n of degree $n \geq 5$. If $n = 5$ or $n = 7$, then by [4], $\omega(G) = 2$.

If $n = 6$, then by [4], $\mathrm{cd}(A_6) = \{1, 5, 8, 9, 10\}$ and so $\omega(A_6) = 3$, the desired result.

If $13 \geq n \geq 8$, then $\omega(A_n) \geq 4$.

Some degree for A_n with $13 \geq n \geq 5$[1]

A_n	Character χ_i	Degree $\chi_i(1)$
A_8	χ_{12}	$2^3 \cdot 7$
A_9	χ_{11}	$2^3 \cdot 7$
A_{10}	χ_4	$2^2 \cdot 3^2$
A_{11}	χ_6	$2^3 \cdot 3 \cdot 5$
A_{12}	χ_3	$2 \cdot 3^3$
A_{13}	χ_6	$2^2 \cdot 5 \cdot 11$

Note that the irreducible characters χ^α of A_n are the same as that of the symmetric group S_n if the partitions α of n are non-self-conjugate. If $n \geq 14$, then by hook formula, we have

$$\chi^\alpha(1) = \frac{(n-1)(n-2)(n-3)(n-4)(n-5)}{5!}, \quad \alpha = (n-5, 1, 1, 1, 1, 1),$$

where α is a partition of n for S_n and non-self-conjugate. So, for $n \geq 8$, we get $\omega(A_n) \geq 4$, a contradiction.

(2) Let G be a simple group of Lie type. Let G be a finite group of Lie type in characteristic p, then the Steinberg character is obviously a p-Steinberg character; see [16, pp. 304]. By Corollary 3 of [16], if a finite simple group G of has p-Steinberg character χ with $p \big| |G|$, then G is a finite simple group of Lie type in characteristic p, and χ is then the Steinberg character St of G. Note that if G is a simple group of Lie type of characteristic p, p a prime, then $\mathrm{St}(1) = |G|_p$. Then, we can construct Table 3 from [4].

Table 2. Simple groups of Lie type

L	St(1)	Condition	L	St(1)	Condition
$A_n(q)$	$q^{\frac{n(n+1)}{2}}$	$n \geq 1$	${}^2A_n(q)$	$q^{\frac{n(n+1)}{2}}$	$n \geq 2$
$B_n(q)$	q^{n^2}	$n \geq 2$	$C_n(q)$	q^{n^2}	$n \geq 3$
$D_n(q)$	$q^{n(n-1)}$	$n \geq 4$	${}^2D_n(q)$	$q^{n(n-1)}$	$n \geq 4$
${}^2G_2(q)$	q^6		$F_4(q)$	q^{24}	
$E_6(q)$	q^{36}		${}^2E_2(q)$	q^{36}	
$E_7(q)$	q^{63}		$E_8(q)$	q^{120}	
${}^2B_2(q)$	q^2	$q = 2^{2m+1} \geq 8$	${}^3D_4(q)$	q^{12}	
${}^2G_2(q)$	q^3	$q = 3^{2m+1} \geq 27$	${}^2F_4(q)$	q^{12}	$q = 2^{2m+1} \geq 2^5$

By Table 2, we have that the possible groups G satisfying $\omega(G) \leq 3$ are $A_n(q)$ with $n \geq 1$ and ${}^2A_n(q)$ with $n \geq 2$.

We consider the following cases:

Case 2.1. G is of Type A_l: $L_{l+1}(q)$.

As in this case $l(l+1)/2 \leq 3$, so $l = 1$, or 2. So, two cases are considered: $l = 1$ and $l = 2$.

Let $l = 1$. First, we assume that q is even. Then, $q = 2^n$ for some integer $n \geq 2$. By hypothesis, $2 \leq n \leq 3$ since $L_2(2^n)$ has a Steinberg character of degree 2^n. If $n = 2$, then G is isomorphic to $L_2(4) \cong A_5$ as $\mathrm{cd}(A_5) = \{1, 3, 4, 5\}$. If $n = 3$, then $G \cong L_2(8)$ as $\mathrm{cd}(L_2(8)) = \{1, 7, 8, 9\}$.

Now, we suppose that q is odd and $q \geq 7$ since if $q = 3$, $L_2(3)$ is solvable and if $q = 5$, then $L_2(5) \cong L_2(4)$ is considered. Note that

$$\mathrm{cd}(L_2(q)) = \{1, q-1, q, q+1, \frac{q+(-1)^{(q-1)/2}}{2}\}.$$

Since $q \geq 7$ is odd, then one of the numbers $q+1$ and $q-1$ has the form $4r$ for some integer $r \geq 2$. Thus, we only need to consider q. Since $\omega(G) = 3$, then $q \big| p^3$ for some odd prime p. So, we consider the following three cases: $q = p$, $q = p^2$ and $q = p^3$.

If $q = p$, then since for some $k \in \mathbb{N}$, $p = 4k+1$ or $4k-1$, one has that

$$p + 1 = 4k + 2 = 2 \cdot 3 \cdot p_1 \text{ and } p - 1 = 4k = 2^2 \cdot p_2$$

or

$$p + 1 = 2^2 \cdot p_1 \text{ and } p - 1 = 4k - 2 = 2 \cdot 3 \cdot p_2.$$

It follows that G is a simple K_n-group for $n \leq 5$.

If G is a simple K_3-group, then by Lemma 2.3, G is isomorphic to $L_2(7)$.

If G is a simple K_4-group, then by Theorem 2 of [2], G is isomorphic to $L_2(11)$ or $L_2(19)$.

If G is a K_5-group, then G is isomorphic to $L_2(p)$ satisfying one of the two conditions:

$$\begin{cases} p - 1 = 2^2 \cdot p_1 \\ p + 1 = 2 \cdot 3 \cdot p_2 \end{cases} \text{ with } p, p_1, p_2 \text{ different primes and } p_1, p_2 \geq 5$$

and

$$\begin{cases} p - 1 = 2 \cdot 3 \cdot p_1 \\ p + 1 = 2^2 \cdot p_2 \end{cases} \text{ with } p, p_1, p_2 \text{ different primes and } p_1, p_2 \geq 5.$$

If $q = p^2$, then $p^2 - 1 = (p-1)(p+1) = 8 \cdot m$ for some positive integer m and so $m = 1$, $q = 9$. Thus, G is isomorphic to $L_2(9)$.

If $q = p^3$, then we get that $p^3 + 1 = (p+1)(p^2 - p + 1) = 2m(p^2 - p + 1)$ and $p^3 - 1 = (p-1)(p^2 + p + 1) = 4n(p^2 + p + 1)$ for some $m, n \in \mathbb{N}$ or $p^3 + 1 = (p+1)(p^2 - p + 1) = 4m(p^2 - p + 1)$ and $p^3 - 1 = (p-1)(p^2 + p + 1) = 2n(p^2 + p + 1)$ for some $m, n \in \mathbb{N}$.

If the former, then $\omega(G) = 3$ implies $n = 1$. It follows that $q^2 + q + 1$ is a prime, so $p^3 - 1 = 4(p^2 + p + 1)$. Thus, $p = 5$ and $q + 1 = p^3 + 1 = 125 + 1 = 126 = 2 \cdot 3^2 \cdot 7$. Therefore $\omega(G) \geq 4$, a contradiction to the hypothesis.

If the latter, then $m = 1$, $p^2 - p + 1$ is a prime and so $p^3 + 1 = 4(p^2 - p + 1)$. Thus, $p = 3$. It follows that $p^3 - 1 = 3^3 - 1 = 26 = 2 \cdot 13$ and so $G \cong L_2(27)$.

Let $l = 2$. In this case, the Steinberg character St of $L_3(q)$ has the degree q^3, so by hypothesis, q is a prime.

If $q = 2$, then $\omega(L_3(2)) = 3$ as $L_3(2) \cong L_2(7)$ and $\mathrm{cd}(L_2(7)) = \{1, 3, 6, 7, 8\}$.

If q is an odd prime, then by [15] there is a character $\chi \in \mathrm{Irr}(L_3(q))$ such that $\chi(1) = (q-1)^2(q+1)$ is divisible by 16 since $(q-1)(q+1)$ is divisible by 8. By Lemma 2.1, $\omega(G) \geq 4$, a contradiction.

Case 2.2. G is of Type ${}^2A_l$: $U_{l+1}(q)$ $(l \geq 2)$.

In this case, $l(l+1)/2 \leq 3$, so $l = 2$ or $l = 1$. If $l = 1$, then $U_2(q) \cong L_2(q)$ is considered in Case 2.1. Now let $l = 2$. If $q = 2$, then $U_3(2)$ is solvable, so $q \geq 3$. Then by [15], there is a character $\chi \in \mathrm{Irr}(U_3(q^2))$ such that if q is odd, then $\chi(1) = (q-1)(q+1)^2 = 16k$ for some positive integer k and that if q is even, then $q \geq 4$ and $q^3 \in \mathrm{cd}(G)$. It follows that $\omega(G) \geq 4$ in both cases, a contradiction to the hypothesis $\omega(G) = 3$.

(3) Let G be a sporadic simple group or a Tits group. By Lemma 2.5, we rule out this case.

This completes the proof. □

3.2 The Proof of Theorem 1.3

Lemma 3.1. *If G is a nonsolvable group with $\omega(G) = 3$, then there is a normal series $1 \leq K \leq L \leq G$ such that L/K is isomorphic to one of the groups as listed in Main Theorem 1.2 and $|G/L| \mid |\mathrm{Out}(L/K)|$.*

Proof. Since G is nonsolvable, by Lemma 2.4 there is a normal series $1 \leq K \leq L \leq G$ such that L/K is a direct product of isomorphic non-abelian simple groups and $|G/L||\mathrm{Out}(L/K)|$. Since $\omega(G) = 3$ and G is nonsolvable, then L/K is isomorphic to $B_1 \times B_2 \times \cdots \times B_n$, where B_i is the group listed in Main Theorem 1.2. If $n \geq 2$, then $\omega(B_1 \times B_2 \times \cdots \times B_n) \geq 4 > \omega(G) = 3$, a contradiction to Lemma 2.1. Therefore $n = 1$. □

Lemma 3.2. *Let N be a normal subgroup of a group G such that G/N is isomorphic to $L_2(q)$ with $q \geq 7$ an odd prime power. If $\omega(G) = 3$, then for each $\theta \in \mathrm{Irr}(N)$, θ is G-invariant.*

Proof. Let $\theta \in \mathrm{Irr}(N)$, and let $I := I_G(\theta)$, be the inertia group of θ in G. Note that $N \trianglelefteq I \leq G$.

If $I = N$, then Problem 6.1 of [10] gives $\theta^G \in \mathrm{Irr}(G)$ for $\theta \in \mathrm{Irr}(N)$, so $\theta^G(1) = |L_2(q)|\theta(1)$ is the product of at least four primes, a contradiction.

If $N < I < G$, then by [6, Chap III, Theorem 8.27], I is isomorphic to $N : F$ where F is a Frobenius group of order $q(q-1)/2$, $N : D_{(q+1)}$, or $N : D_{(q-1)}$.

If $I = N : F$, then $|G : I| = q + 1$. Let $\eta \in \mathrm{Irr}(I/N)$ with $\eta(1) = \frac{q-1}{2}$ as I/N is a Frobenius group and $\frac{q-1}{2} \in \mathrm{cd}(I/N)$. Now by Corollary 6.17 of [10], $\eta\chi \in \mathrm{Irr}(I)$ for each irreducible constituent χ of θ^G, and $(\eta\chi)_N = \theta$. So, by Theorem 6.11 of [10], $(\eta\chi)^G \in \mathrm{Irr}(G)$, so $(\eta\chi)^G(1) = (q+1)(q-1)/2 = 8r$ or $4rq$ for some numbers $r \geq 2$ and $q \geq 2$, a contradiction.

If $I = N : D_{q-1}$, then let $\eta \in \mathrm{Irr}(I)$ with $\eta(1) = 2$ since $I/N \cong D_{q-1}$ is a dihedral group, one has $\eta^G(1) = |G : I|\eta(1) = 2q(q+1)$, also a contradiction. Similarly we may exclude $I = D_{q-1}$.

Thus, $I = G$, θ is G-invariant. □

Now, we can prove Main theorem 1.3

Proof of Main Theorem 1.3

Proof. Let $H \trianglelefteq G$. By Lemma 2.1, $\omega(H) \leq \omega(G)$ and $\omega(G/H) \leq \omega(G) = 3$. By hypothesis, G is nonsolvable.

By Lemma 3.1 we can assume that G has a normal series $1 \trianglelefteq K \trianglelefteq L \trianglelefteq G$ such that L/K is isomorphic to a group listed in Main Theorem 1.2, and that p is a prime divisor of $|G|$.

We consider the groups listed in Main Theorem 1.2.

Case 1. $L/K \cong A_5$. By Lemma 3.1, $A_5 \leq G/K \leq S_5$.

Let $G/K \cong A_5$.

Let $\theta \in \mathrm{Irr}(K)$, and let $I = I_G(\theta)$ be the inertia group of θ in G. Hypothesis and the fact $\mathrm{cd}(A_5) = \{1, 3, 4, 5\}$ show that $\mathrm{cd}(K) = \{1, p\}$, for some prime p.

Let $G/K \cong S_5$.

Let $\theta \in \mathrm{Irr}(K)$, and let $I = I_G(\theta)$, be the inertia of θ in G. Then by Lemma 3.2, $I = G$, so by Corollary 6.17 of [10], for each $\eta \in \mathrm{Irr}(G/K)$, $\chi\eta \in \mathrm{Irr}(G)$ with $\chi_K = \theta$. Hypothesis and $\mathrm{cd}(S_5) = \{1, 4, 5, 6\}$ show that $\mathrm{cd}(K) = \{1, p\}$, for some prime p.

Case 2. $L/K \cong A_7$. We know that $\mathrm{cd}(A_7) = \{1,\ 2 \cdot 3,\ 2 \cdot 5,\ 2 \cdot 7,\ 3 \cdot 5,\ 3 \cdot 7,\ 5 \cdot 7\}$ and that $\mathrm{cd}(S_7) = \{1,\ 2 \cdot 3,\ 2 \cdot 7,\ 3 \cdot 5,\ 2^2 \cdot 5,\ 3 \cdot 7,\ 5 \cdot 7\}$, so $\omega(A_7) = 2$ and $\omega(S_7) = 3$.

By Lemma 3.1, $A_7 \leq G/K \leq S_7$.

Let $G/K \cong A_7$.

Similarly as Case 1, we can get that $G/K \cong A_7$ such that for every $\theta \in \mathrm{Irr}(K)$, θ is G-invariant and $\mathrm{cd}(K) = \{1, p\}$ for some prime p.

Let $G/K \cong S_7$.

Now $\omega(S_7) = 3$. Also we can get that $G/K \cong S_7$ such that for every $\theta \in \mathrm{Irr}(K)$, θ is G-invariant and K is abelian.

Case 3. $L/K \cong L_2(7)$. Note that $\mathrm{cd}(L_2(7)) = \{1, 3, 2 \cdot 3, 7, 2^3\}$ and that $\mathrm{cd}(L_2(7).2) = \{1, 3, 2^2, 3 \cdot 3, 7, 2^3\}$, so $\omega(L_2(7)) = \omega(L_2(7).2) = 3$.

By Lemma 3.1, $L_2(7) \leq G/K \leq L_2(7).2$.

Similarly as Case 1, we can get that G/K is isomorphic to $L_2(7)$ or $L_2(7).2$ where K is abelian and for every $\theta \in \mathrm{Irr}(K)$, θ is G-invariant.

Case 4. $L/K \cong L_2(8)$. Note that $\omega(L_2(8)) = \omega(L_2(8).3) = 3$. We can also get that G/K is isomorphic to $L_2(8)$ or $L_2(8).3$, where K is abelian and for every $\theta \in \mathrm{Irr}(K)$, θ is G-invariant.

Case 5. $L/K \cong L_2(9)$. By [1],

$$\begin{aligned} \mathrm{cd}(A_6) &= \{1,5,8,9,10\}, \\ \mathrm{cd}(A_6.2_1) &= \{1,5,9,10,16\}, \\ \mathrm{cd}(A_6.2_2) &= \{1,8,9,10\}, \\ \mathrm{cd}(A_6.2_3) &= \{1,9,10,16\}, \\ \mathrm{cd}(A_6.2^2) &= \{1,9,10,16,20\}. \end{aligned}$$

By Lemma 3.1, $L_2(9) \leq G/K \leq \mathrm{Aut}(L_2(9))$, so hypothesis gives that G/K is isomorphic to A_6, or $A_6.2_2$ and that K is abelian and, for every $\theta \in \mathrm{Irr}(K)$, θ is G-invariant.

Case 6. $L/K \cong L_2(p)$ where p is a prime with either $p-1 = 2^2 \cdot p_1, p+1 = 2 \cdot 3 \cdot p_2$ or $p-1 = 2 \cdot 3 \cdot p_1, p+1 = 2^2 \cdot p_2$ where p, p_1, p_2 are different primes and $p_1, p_2 \geq 5$.

Now, we have that $\omega(L_2(p)) = 3$, and $|\mathrm{Out}(L_2(p))| = 2$.

We can get from [17, pp. 192], that $\omega(L_2(p)) = \omega(L_2(p).2) = 3$. By Lemma 3.1, $L_2(p) \leq G/K \leq L_2(p).2$, so G/K is isomorphic to $L_2(p)$ or $L_2(p).2$ where K is abelian and for every $\theta \in \mathrm{Irr}(K)$, θ is G-invariant.

Case 7. $L/K \cong L_2(11)$ or $L_2(19)$.

Note that $\mathrm{cd}(L_2(11)) = \{1,\ 5,\ 2 \cdot 5,\ 11,\ 2^2 \cdot 3\}$ and $\mathrm{cd}(L_2(11).2) = \{1,\ 2 \cdot 5,\ 11,\ 2^2 \cdot 3\}$. Then $G/K \cong L_2(11)$ or $L_2(11).2$. Corollary 6.17 of [10] means that K is abelian and $\theta \in \mathrm{Irr}(K)$ is G-invariant by Lemma 3.2. Similarly we can get that $G/K \cong L_2(19)$ or $L_2(19).2$ with K abelian and $\theta \in \mathrm{Irr}(K)$ G-invariant.

Case 8. $L/K \cong L_2(27)$.

We know from [4] that

$$\begin{aligned} \mathrm{cd}(L_2(27)) &= \{1, 13, 2 \cdot 13, 3^3, 2^2 \cdot 7\}, \\ \mathrm{cd}(L_2(27).2) &= \{1, 2 \cdot 13, 3^3, 2^2 \cdot 7\}, \\ \mathrm{cd}(L_2(27).3) &= \{1, 13, 2 \cdot 3 \cdot 13, 2^2 \cdot 3 \cdot 7\}, \\ \mathrm{cd}(L_2(27).6) &= \{1, 2 \cdot 13, 2 \cdot 3 \cdot 13, 3^3, 2^2 \cdot 3 \cdot 7\}. \end{aligned}$$

So, as the proof of Case 5, we get similarly that $G/K \cong L_2(27)$, or $L_2(27).2$ where K is abelian by Corollary 6.17 of [10], and so by Lemma 3.2, $\theta \in \mathrm{Irr}(K)$ is G-invariant.

This completes the proof. □

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Some characterization of rarely ω-continuous functions

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Abstract. The notion of rare continuity was introduced by Popa [15]. In this present paper, we introduce a new class of functions, called rarely ω-continuous function via ω-open sets and we investigate several properties of this function.
Keywords: rare set, ω-open, rarely continuous, rarely ω-continuous.

1. Introduction

Throughout the present paper, (X, τ) or X will denote a topological space and $Cl(A)$ (resp. $Int(A)$) will denote the closure (resp. interior) of a subset A of X. A subset A of X is said to be regular open (resp. regular closed) if $A = Int(Cl(A))$ (resp. $A = Cl(Int(A))$). We denote by $Ro(X)$ (resp. $Rc(X)$) the family of all regular open (resp. regular closed) sets in a space (X, τ). Let (X, τ) be a space and A a subset of X. A point $x \in X$ is called a condensation point of A if for each $U \in \tau$ with $x \in U$, the set $U \cap A$ is uncountable. A set A is said to be ω-closed [9] if it contains all its condensation points [8]. The complement of an ω-closed set is said to be ω-open. It is well known that a subset W of a space (X, τ) is ω-open if and only if for each $x \in W$, there exists $U \in \tau$ such that $x \in U$ and $U - W$ is countable. The family of all ω-open sets of a space (X, τ), denoted by τ_ω or $\omega O(X)$, forms a topology on X finer than τ. The ω-closure and ω-interior, that can be defined in the same way as $Cl(A)$ and $Int(A)$, respectively, will be denoted by $Cl_\omega(A)$ and $Int_\omega(A)$, respectively. Recently papers [1–7] have introduced some new classes of ω-open set. A subset A of X is a called a dense set if and only if the intersection of every non-empty

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open set with A is non-empty, that is, $A \cap U \neq \emptyset$ for all non-empty open $U \in \tau$. A subset A of X is called a rare set in X if and only if $Int(A) = \emptyset$ that is, if $X - A$ is a dense set.

Example 1.1. Let $(\mathbb{R}, \tau)$ be the real numbers with the left-ray topology, i.e. $\tau = \{(-\infty, a) : a \in \mathbb{R}\} \cup \{\mathbb{R}, \emptyset\}$. Let $A = [0, 1]$, since $Int(A) = \emptyset$, then A is a rare set in $\mathbb{R}$.

For a point $x \in X$, the family $\{U : x \in U \text{ and } U \in \tau\}$ is denoted by $\tau(x)$. For a point $x \in X$, the family $\{U : x \in U \text{ and } U \in \tau_\omega\}$ is denoted by $\tau_\omega(x)$. For a point $x \in X$, the family $\{U : x \in U \text{ and } U \in Ro(X)\}$ is denoted by $Ro(x)$.

Popa [15] introduced the notion of rare continuity. Recently papers [4–6] have introduced several properties of this function.

Definition 1.1. *A function $f : (X, \tau) \to (Y, \sigma)$ is called rarely continuous if for each $x \in X$ and each open set W in Y containing $f(x)$, there exist a rare set R_W with $W \cap Cl(R_W) = \emptyset$ and an open set U in X containing x such that $f(U) \subset W \cup R_W$.*

Definition 1.2. *A subset $\mathcal{N}$ of a topological space (X, τ) is called an ω-neighbourhood of a point $x \in X$ if there exists an ω-open set B such that $x \in B \subseteq \mathcal{N}$.*

Definition 1.3 ([10])**.** *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be:*

1. *ω-continuous if $f^{-1}(V) \in \tau_\omega$ for every $V \in \sigma$.*
2. *ω-open if $f(U) \in \sigma_\omega$ for every $U \in \tau_\omega$.*
3. *ω-closed if $f(U) \in (\sigma_\omega)^c$ for every $U \in (\tau_\omega)^c$.*

2. Rarely ω-continuous functions

Definition 2.1. *A function $f : (X, \tau) \to (Y, \sigma)$: is called rarely ω-continuous if for each $x \in X$ and each $W \in \sigma(f(x))$, there exist a rare set R_w with $W \cap Cl(R_w) = \emptyset$ and an ω-open set U containing x such that $f(U) \subset W \cup R_w$.*

Example 2.1. Let $\mathbb{R}$ be the set of real numbers, τ_u be the usual topology and τ_l be the left-ray topology, i.e. $\tau_l = \{(-\infty, a) : a \in \mathbb{R}\} \cup \{\mathbb{R}, \emptyset\}$. We define $f : (\mathbb{R}, \tau_u) \to (\mathbb{R}, \tau_l)$ be the identity function $f(x) = x$. Let $a \in \mathbb{R}$ then for each open set $W = (-\infty, a + \delta) \in \tau_l$ containing $f(a) = a$ there exists $U = I \cup \{a\} \in \tau_\omega(a)$ where I is the set of irrational numbers. Then, $Int[f(U) \cap (Y - W)] = Int[(I \cup \{a\}) \cap [a + \delta, \infty)] = \emptyset$. Hence, if $a \in \mathbb{R}$ then for each open set W containing $f(a) = a$ there exists $U \in \tau_\omega(a)$ such that $Int[f(U) \cap (Y - W)] = \emptyset$. Hence, f is rarely continuous function and rarely ω-continuous function.

Theorem 2.1. *Let (X, τ) and (Y, σ) be topological spaces. Then, the following statements are equivalent for a function $f : (X, \tau) \to (Y, \sigma)$:*

1. *f is rarely ω-continuous at $x \in X$.*

2. *For each $W \in \sigma(f(x))$, there exists $U \in \tau_\omega(x)$ such that $Int[f(U) \cap (Y - W)] = \emptyset$.*

3. *For each $W \in \sigma(f(x))$, there exists $U \in \tau_\omega(x)$ such that $Int[f(U)] \subset Cl(W)$.*

4. *For each $W \in \sigma(f(x)))$, there exists a rare set R_w with $W \cap Cl(R_w) = \emptyset$ such that $x \in Int_w(f^{-1}(W \cup R_w))$.*

5. *For each $W \in \sigma(f(x))$, there exists a rare set R_w with $Cl(W) \cap R_w = \emptyset$ such that $x \in Int_w(f^{-1}(Cl(W) \cup R_w))$.*

6. *For each $W \in Ro(f(x))$, there exists a rare set R_w with $W \cap Cl(R_w) = \emptyset$ such that $x \in Int_w(f^{-1}(W \cup R_w))$.*

Proof. (1) $\Rightarrow$ (2): Let $W \in \sigma(f(x))$. By $f(x) \in W \subset Int(Cl(W))$ and the fact that $Int(Cl(W)) \in \sigma(f(x))$, then there exist a rare set R_W with $Int(Cl(W)) \cap Cl(R_W) = \emptyset$ and an ω-open set $U \subset X$ containing x such that $f(U) \subset Int(Cl(W)) \cup R_W$. We have $Int[f(U) \cap (Y - W)] = Int[f(U)] \cap Int(Y - W) \subset Int[Int(Cl(W)) \cup R_W] \cap (Y - Cl(W)) \subset (Cl(W) \cup Int(R_W)) \cap (Y - Cl(W)) = \emptyset$. Hence, $Int[f(U) \cap (Y - W)] = \emptyset$.

(2) $\Rightarrow$ (3): Let $W \in \sigma(f(x))$, then by (2), there exists $U \in \tau_\omega(x)$ such that $Int[f(U) \cap (Y - W)] = \emptyset$. We have $Int[f(U) \cap (Y - W)] = Int(f(U)) \cap Int((Y - W)) = Int(f(U)) \cap (Y - Cl(W)) = \emptyset$. Then $Int[f(U)] \subset Cl(W)$.

(3) $\Rightarrow$ (1): Let $W \in \sigma(f(x))$. Then, by (3), there exists $U \in \tau_\omega(x)$ such that $Int[f(U)] \subset Cl(W)$. We have $f(U) = [f(U) - Int(f(U))] \cup Int(f(U)) \subset [f(U) - Int(f(U))] \cup Cl(W) = [f(U) - Int(f(U))] \cup W \cup (Cl(W) - W) = [f(U) - Int(f(U))] \cap (Y - W) \cup W \cup (Cl(W) - W)$. Set $R_1 = [f(U) - Int(f(U))] \cap (Y - W)$ and $R_2 = (Cl(W) - W)$. Then R_1 and R_2 are rare sets. More $R_w = R_1 \cup R_2$ is a rare set such that $Cl(R_w) \cap W = \emptyset$ and $f(U) \subset W \cup R_w$. This shows that f is rarely w-continuous function.

(1) $\Rightarrow$ (4): Suppose that $W \in \sigma(f(x))$. Then, there exists a rare set R_w with $W \cap cl(R_w) = \emptyset$ and a ω-open set U in X containing x, such that $f(U) \subset W \cup R_w$. It follows that $x \in U \subset f^{-1}(W \cup R_w)$. This implies that $x \in Int_\omega(f^{-1}(W \cup R_w))$.

(4) $\Rightarrow$ (5): Suppose that $W \in \sigma(f(x))$. Then, there exists a rare set R_w with $W \cap Cl(R_w) = \emptyset$ such that $x \in Int_w(f^{-1}(W \cup R_w))$. Since $W \cap Cl(R_w) = \emptyset$, $R_w \subset Y - W$, where $Y - W = (Y - Cl(W)) \cup (Cl(W) - W)$. Now, we have $R_w \subset (R_w \cup (Y - cl(W)) \cup (Cl(W) - W))$. Set $R_1 = R_w \cap (Y - Cl(W))$. It follows that R_1 is a rare set with $cl(W) \cap R_1 = \emptyset$. Therefore, $x \in Int_w[f^{-1}(W \cup R_w)] \subset Int_w[f^{-1}(Cl(W) \cup R_1)]$.

(5) $\Rightarrow$ (6): Assume that $W \in Ro(f(x))$. Then, there exists a rare set R_w with $Cl(W) \cap R_w = \emptyset$ such that $x \in Int_w[f^{-1}(Cl(W) \cup R_w)]$. Set $R_1 = R_w \cup (Cl(W) - W)$. It follows that R_1 is a rare set and $W \cap Cl(R_1) = \emptyset$.

Hence, $x \in Int_w[f^{-1}(cl(W) \cup R_w)] = Int_w[f^{-1}(W \cup (Cl(W) - W) \cup R_w)] = Int_w[f^{-1}(W \cup R_1)]$.

(6) $\Rightarrow$ (2): Let $W \in \sigma(f(x))$. By $f(x) \in W \subset Int(Cl(W))$ and the fact that $Int(Cl(W)) \in Ro(f(x))$, there exists a rare set R_w with $Int(Cl(W)) \cap Cl(R_w) = \emptyset$ such that $x \in Int_w[f^{-1}(Int(Cl(W)) \cup R_w)]$. Let

$U = Int_w[f^{-1}(Int(Cl(W)) \cup R_w)]$. Hence, $x \in U \in \tau_\omega(x)$ and, therefore $f(U) \subset Int(Cl(W)) \cup R_w$. We have $Int[f(U) \cap (Y - W)] = Int[f(U)] \cap Int(Y - W) \subset Int[Int(Cl(W)) \cup R_W] \cap (Y - Cl(W)) \subset (Cl(W) \cup Int(R_W)) \cap (Y - Cl(W)) = \emptyset$. □

Theorem 2.2. *Let $f : (X, \tau) \to (Y, \sigma)$. Then, every rarely continuous function is rarely ω-continuous.*

Proof. Let $x \in X$ and W be an open set in Y containing $f(x)$. Since f is rarely continuous, by there exists an open set U in X containing x such that $Int(f(U)) \subset Cl(W)$. Since $\tau \subseteq \tau_\omega$, U is an ω-open set containing x. It then f is rarely ω-continuous. □

The converse of Theorem 2.2 is not true in general as shown in the following example.

Example 2.2. Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a, b\}, \{c\}\}$ and $\sigma = \{\emptyset, Y, \{a, b\}, \{a\}, \{b\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity function. Since $\tau_\omega = P(X)$, then f is rarely ω-continuous but not rarely continuous function because take $a \in Y$ and $W = \{a\}$, the for any open set $U = \{a, b\}$ or $U = X$ in X containing a we have $Int[f(U) \cap (Y - W)] = Int[U \cap \{b, c\}] \neq \emptyset$.

Theorem 2.3. *A function $f : (X, \tau) \to (Y, \sigma)$ is rarely ω-continuous if and only if for each an open set $W \subset Y$, there exists a rare set R_w with $W \cap Cl(R_w) = \emptyset$ such that $f^{-1}(W) \subset Int_w[f^{-1}(W \cup R_w)]$.*

Proof. Let $W \in \sigma(f(x))$, then there exists a rare set R_w with $W \cap cl(R_w) = \emptyset$ and an ω-open set U in X containing x, such that $f(U) \subset W \cup R_w$. Let $y \in W$ Since $W \subset W \cup R_w$, then $y \in W \cup R_w$. It follows that $x \in f^{-1}(W) \subset f^{-1}(W \cup R_w)$. This implies that $x \in Int_w(f^{-1}(W \cup R_w))$.

Conversely, let $W \in \sigma(f(x))$, then there exists a rare set R_w with $W \cap Cl(R_w) = \emptyset$ such that $f^{-1}(W) \subset Int_w[f^{-1}(W \cup R_w)]$. Let $x \in f^{-1}(W) \subset f^{-1}(W \cup R_w)$. This implies that $x \in Int_w(f^{-1}(W \cup R_w))$. Then, by Theorem 2.1 f is rarely ω-continuous function. □

Definition 2.2. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be ω^*-continuous at $x \in X$ if for each set $W \in \sigma(f(x))$, there exists an ω-open set U in X containing x such that $Int[f(U)] \subset W$. If f has this property at each point $x \in X$, then we say that f is ω^*-continuous on X.*

Theorem 2.4. *Let Y be a regular space a function $f : (X, \tau) \to (Y, \sigma)$ is ω^*-continuous on X if and only if f is rarely ω-continuous on X.*

Proof. Let f be rarely ω^*-continuous on X and $x \in X$. Suppose that $W \in \sigma(f(x))$. Then, there exists a ω-open set U in X containing x such that $Int[f(U)] \subset W$, then $Int[f(U)] \subset Cl(W)$. We have $f(U) = [f(U) - Int(f(U))] \cup Int(f(U)) \subset [f(U) - Int(f(U))] \cup Cl(W) = [f(U) - Int(f(U))] \cup W \cup (Cl(W) - W) = [f(U) - Int(f(U))] \cap (Y - W) \cup W \cup (Cl(W) - W)$.

Set $R_1 = [f(U) - Int(f(U))] \cap (Y - W)$ and $R_2 = (Cl(W) - W)$.

Then R_1 and R_2 are rare sets. Also $R_w = R_1 \cup R_2$ is a rare set such that $Cl(R_w) \cap W = \emptyset$ and $f(U) \subset W \cup R_w$. This shows that f is rarely ω-continuous function.

Conversely, let f be rarely ω-continuous on X and $x \in X$. Suppose that $f(x) \in W$, where W is an open set in Y. By the regularity of Y, there exists an open set $W_1 \in \sigma(f(x))$ such that $Cl(W_1) \subset W$. Since f is rarely ω-continuous, there exists an ω-open set U in X containing x such that $Int[f(U)] \subset Cl(W_1)$. This implies that $Int[f(U)] \subset W$ and so f is ω^*-continuous function on X. □

Definition 2.3. *A function $f : (X, \tau) \to (Y, \sigma)$ is called to be almost weakly ω-continuous if for each open set W in Y containing $f(x)$ there exists an ω-open set U in X containing x such that $f(U) \subset Cl(W)$.*

Theorem 2.5. *Let $f : (X, \tau) \to (Y, \sigma)$ be an ω-open and rarely ω-continuous function, then f is almost weakly ω-continuous.*

Proof. Let $x \in X$ and $W \in \sigma(f(x))$. Since f is rarely ω-continuous, there exists an ω-open set U in X such that $Int(f(U)) \subset Cl(W)$. Since f is ω-open $f(U)$ is ω-open and hence $f(U) = Int_\omega(f(U)) \subseteq Int(f(U)) \subset Cl(W)$. This shows that f is almost weakly ω-continuous. □

Theorem 2.6. *Let $f : X \to Y$ be a rarely ω-continuous function, then the graph function $g : X \to X \times Y$, defined by $g(x) = (x, f(x))$ for every x in X is rarely ω-continuous.*

Proof. Let $x \in X$ and A is any open set containing $g(x)$. It follows that there exist open sets U and W in X and Y, respectively, such that $(x, f(x)) \in U \times W \subseteq A$. Since f is rarely ω-continuous, then there exists $G \in \tau_\omega(x)$ such that $Int[f(G)] \subset Cl(W)$. Let $B = U \cap G$. It follows that $B \in \tau_\omega(x)$ and we have $Int[g(B)] \subseteq Int(U \times f(G)) \subseteq U \times Cl(W) \subseteq Cl(A)$. Hence, g is a rarely ω-continuous function. □

Definition 2.4. *Let $A = \{W_i\}$ be a class of subsets of a topological space (X, τ). The rarely union sets [14] of A we mean $\{W_i \cup R_{W_i}\}$, where each R_{W_i} is a rare set such that each of $\{W_i \cap Cl(R_{W_i})\}$ is empty.*

Definition 2.5. *A subset K of a topological space (X, τ) is said to be*

1. *ω -compact relative to X if every cover of K by ω-open sets in X has a finite subcover. A space X is said to be ω-compact if X is ω-compact relative to X.*

2. *rarely almost compact relative to X if for every cover of K by open sets of X, there exists a finite subfamily whose rarely union sets cover K. A topological space X is said to be rarely almost compact if the set X is rarely almost compact relative to X.*

Theorem 2.7. *Let $f : (X, \tau) \to (Y, \sigma)$ be rarely ω-continuous and K be ω-compact relative to X. Then $f(K)$ is rarely almost compact relative to Y.*

Proof. Let $\Omega = \{W_k : k \in K\}$ be an open cover of $f(K)$. Let $\mathcal{A}$ be the set of all W in Ω such that $W \cap f(K) \neq \emptyset$. Then $\mathcal{A}$ is an open cover of $f(K)$. Hence, for each $k \in K$, there is some $W_k \in \mathcal{A}$ such that $f(k) \in W_k$. Since f is rarely ω-continuous, there exists a rare set R_{W_k} with $W_k \cap Cl(R_{V_k}) = \emptyset$ and an ω-open set U_k containing k such that $f(U_k) \subset W_k \cup R_{W_k}$. Hence, there is a finite subfamily $\{U_k : k \in \Delta\}$ which covers K, where Δ is a finite subset of K. The subfamily $\{W_k \cup R_{W_k} : k \in \Delta\}$ also covers $f(K)$. □

Lemma 2.1. *If $f : (X, \tau) \to (Y, \sigma)$ is continuous and one-to-one, then f preserves rare sets.*

Theorem 2.8. *If $f : (X, \tau) \to (Y, \sigma)$ is rarely ω-continuous and $g : (Y, \sigma) \to (Z, \rho)$ is a continuous injection, then $g \circ f : (X, \tau) \to (Z, \rho)$ is rarely ω-continuous.*

Proof. Let $x \in X$ and $(g \circ f)(x) \in V$, where V is an open set in Z. By hypothesis, g is continuous, therefore $W = g^{-1}(V)$ is an open set in Y containing $f(x)$ such that $g(W) \subset V$. Since f is rarely w-continuous, there exists a rare set R_W with $W \cap Cl(R_W) = \emptyset$ and an ω-open set U containing x such that $f(U) \subset W \cup R_W$. It follows from Lemma 2.1 that $g(R_W)$ is a rare set in Z. Since R_W is a subset of $Y - W$ and g is injective, we have $Cl(g(R_W)) \cap V = \emptyset$. This implies that $(g \circ f)(U) \subset V \cup g(R_W)$. Hence, $g \circ f$ is rarely ω-continuous. □

Definition 2.6. *A topological space (X, τ) is called rarely separated [14] if for every pair of distinct points x and y in X, there exist open sets U_x and U_y containing x and y, respectively, and rare sets R_{U_x}, R_{U_y} with $U_x \cap cl(R_{U_x}) = \emptyset$ and $U_y \cap cl(R_{U_y}) = \emptyset$ such that $(U_x \cup R_{U_x}) \cap (U_y \cup R_{U_y}) = \emptyset$.*

Definition 2.7. *A topological space (X, τ) is said to be ω-T_2 [10] if for any distinct pair of points x and y in X, there exist disjoint ω-open sets U and V in X containing x and y, respectively.*

Theorem 2.9. *Let $f : (X, \tau) \to (Y, \sigma)$ be rarely ω-continuous injection such that (Y, σ) is rarely separated, then (X, τ) is ω-T_2 space.*

Proof. Let x and y be any distinct points in X. Then $f(x) \neq f(y)$ (as f is injective). Thus there exist open sets G_x and U_y in Y containing $f(x)$ and $f(y)$, respectively, and rare sets R_{U_x} and R_{U_y} with $U_x \cap Cl(R_{U_x}) = \emptyset$ and $U_y \cap Cl(R_{U_y}) = \emptyset$ such that $(U_x \cup R_{U_x}) \cap (U_y \cap R_{U_y}) = \emptyset$. Therefore,

$Int_w[f^{-1}(U_x \cup R_{U_x})] \cap Int_w[f^{-1}(U_y \cup R_{U_y})] = \emptyset$. By Theorem 2.3 we have $x \in f^{-1}(U_x) \subset Int_w[f^{-1}(U_x \cup R_{U_x})]$ and $y \in f^{-1}(U_y) \subset Int_w[f^{-1}(U_y \cup R_{U_y})]$. Since $Int_w[f^{-1}(U_x \cap R_{U_x})]$ and $Int_w[f^{-1}(U_y \cap R_{U_y})]$ are two disjoint ω-open sets, hence (X, τ) is an ω-T_2 space. □

Theorem 2.10. *Let (X, τ) be topological space let $A \subset X$ and $f : (X, \tau) \to (A, \tau_A)$ be a rarely ω-continuous retraction of X onto A. If X is Hausdorff, then A is a closed set.*

Proof. Suppose A is not closed. Then, there exists a point $a \in Cl(A) - A$. Since f is a retraction, $f(a) \neq a$. Moreover, X is Hausdorff, there exist disjoint open sets H and W containing a and $f(a)$, respectively. Now, for the open W, there exists a rare set R_W in the subspace A and an ω-open U containing a such that $Cl(R_W) \cap W = \emptyset$, $U \subset H$ and $f(U) \subset W \cup R_W$. Since $U \cap A$ is ω-open in A, there is a point $z \in U \cap A$ such that $z \notin R_W$. So, $f(z) = z \notin W \cup R_W$, then f is not rarely ω-continuous, a contradiction. Therefore, A is closed. □

Definition 2.8 ([13])**.** *Let (X, τ) be Hausdorff space, then X is H-closed if every open cover $\Omega = \{U_\alpha : \alpha \in \Delta\}$ of X contains a finite subcollection $\Omega_0 = \{Cl(U_{\alpha_i}) : i = 1, 2, 3, ..., n\}$ covers X.*

Theorem 2.11. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function with an ω-closed graph where Y is H-closed. Then f is rarely ω-continuous function.*

Proof. Let $x \in X$ and W be a regular open set containing $f(x)$. For each $y \in Y - W$ there exist an ω-open set $U(x_y)$ containing x and open set $G(y)$ containing y such that $f(U(x_y)) \cap G(y) = \emptyset$ because f has an ω-closed graph. Since Y is Hausdorff, we may also choose $G(y)$ so that $f(x) \notin Cl(G(y))$ for each $y \in Y - W$. Thus the collection $\{G(y) : y \in Y - W\}$ forms an open cover of the regular closed set $Y - W$. Since Y is H-closed spaces, then the regular closed sets are themselves H-closed which implies the open cover $\{G(y) : y \in Y - U\}$ has a finite subcollection $\{G(y_i) : i = 1, 2....n)\}$, such that $Y - W \subset \bigcup_{i=1}^{n}(Cl(G(y_i)) = Cl(\bigcup_{i=1}^{n} G(y_i))$. Now, define $U = \bigcap_{i=1}^{n} U(x_{y_i})$ is an ω-open set. Then, by construction $f(U) \cap (\bigcup_{i=1}^{n} G(y_i)) = \emptyset$. So, if we let $R_W = (Y - W) \cap (Cl(\bigcup_{i=1}^{n} G(y_i)) - \bigcup_{i=1}^{n} G(y_i))$ then R_W is a rare set and $f(U) \subset W \cup R_W$. By Theorem 2.1, f is rarely ω-continuous function. □

Lemma 2.2. *Let $A_1, A_2, \ldots, A_n$ be a finite family of pairwise disjoint sets having the property that when $i \neq j$, either $Cl(A_i) \cap A_j = \emptyset$ or $A_i \cap Cl(A_j) = \emptyset$. If, in addition, $Int(A_i) = \emptyset$ for all $1 < i < n$, then $Int(\bigcup_{i=1}^{n} A_i) = \emptyset$.*

Theorem 2.12. *Let (X, τ) and (Y_a, σ_a) be topological spaces. Let $f_a : (X, \tau) \to (Y_a, \sigma_a)$ be a rarely ω-continuous function for each $a \in \Delta$ and define $f : (X, \tau) \to (\prod_{a\in\Delta} Y_a, \prod_{a\in\Delta} \sigma_a)$ by $f(x) = \{f_a(x)\}$. Then f is rarely ω-continuous.*

Proof. Let $x \in X$ and W be an open set containing $f(x)$. Now, let $H = W_{a_1} \times W_{a_2} \times \ldots.. \times W_{a_n} \times \Pi\{Y_\beta : \beta \neq a_1, a_2, \ldots, a_n\}$ be a basic open set

containing $f(x)$ such that $f(x) \in H \subset W$. Then, for each W_{a_1} there exists a rare set $R_{W_{a_i}}$ with $Cl(R_{W_{a_i}}) \cap W_{a_i} = \emptyset$ and some an ω-open U_{a_l} containing x such that $f_{a_i}\ (U_{a_i}) \subset W_{a_i} \cup R_{W_{a_i}}$. Define $U = \bigcap_{i=1}^{n} U_{a_i}$ which is an ω-open. Then $J(U) \subset \Pi\{Y_\beta : \beta \neq a_1, a_2, \ldots, a_n\} \times\ (W_{a1} \cup R_{W_{a1}}) \times\ (W_{a_2} \cup R_{W_{a_2}}) \times \ldots \times\ (W_{a_n} \cup R_{W_{a_n}}) = \Pi\{Y_\beta : \beta \neq a_1, a_2, \ldots, a_n\} \times\ W_{a_1} \times\ W_{a_2} \times\ \ldots \times W_{a_n} \cup (\cup\{\Pi\{Y_\beta : \beta \neq a_1, a_2, \ldots, a_n\} \times \prod_{i=1}^{n}\{A_i : A_i = W_{a_i}\ or\ Ai = R_{W_{a_i}}$ and $\prod_{i=1}^{n} A_i \neq \prod_{i=1}^{n} W_{a_i}\})$. Observe that the collection of sets $\{\Pi\{Y_\beta : \beta \neq a_1, a_2, \ldots, a_n\} \times\ \prod_{i=1}^{n}\{A_i : A_i = W_{a_i}\ or\ A_i = R_{W_{a_i}}$ and $\prod_{i=1}^{n} A_i \neq \prod_{i=1}^{n} W_{a_i}\}$ is a finite family of pairwise disjoint sets that satisfies the hypothesis of Lemma 2.2. Thus the interior of the union of this finite family is empty so if we define R_W is rare set, $Cl(R_W) \cap W = \emptyset$ and $f(U) \subset W \cup R_W$. Thus f is rarely ω-continuous function. □

Conclusion

We defined and studied a new class of functions, called rarely ω-continuous function via ω-open sets. In the future, we intend to provide more characterizations and prove soft mapping, as well as define new soft topological concepts using soft ω-open sets.

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Fuzzy structures in hyper GR-algebras

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Abstract. In this paper, we consider hyper GR-ideals of a hyper GR-algebra applied to fuzzy sets such as fuzzy hyper GR-ideal of type 2 and fuzzy distributive hyper GR-ideal of type 2. These concepts are introduced including the concept of distributive hyper GR-ideal. Several properties and characterizations of distributive hyper GR-ideals, fuzzy hyper GR-ideals of type 2 and fuzzy distributive hyper GR-ideals of type 2 are obtained. Moreover, a relationship between fuzzy hyper GR-ideals of type 2 and hyper GR-ideals and fuzzy distributive hyper GR-ideals of type 2 and distributive hyper GR-ideals with respect to its level subset are presented. Furthermore, it is obtained that the family of fuzzy distributive hyper GR-ideals of type 2 of a hyper GR-algebra is a complete distributive lattice with respect to join and meet.

Keywords: hyper GR-algebras, hyper GR-ideals, distributive hyper GR-ideal, fuzzy hyper GR-ideals of type 2, fuzzy distributive hyper GR-ideal of type.

1. Introduction

Hyperstructure theory was born in 1934 at the 8th Congress of Scandinavian Mathematicians, where Marty [7] introduced the hypergroup notion as a generalization of groups and proved its utility in solving some problems of groups, algebraic functions and rational fractions. Surveys of the theory can be found in the books of Corsini [18], Vougiouklis [22], Corsini and Leoreanu [19]. Hypergroups are studied from the theoretical point of view and for their applications to many areas of pure and applied mathematics, such as: geometry, topology, cryptography and coding theory, graphs and hypergraphs, probability theory, theory of fuzzy and rough sets, automata theory, and artificial intelligence. Moreover, hyperstructures can be associated with elementary particles in physical theory [2]. It was also used in inheritance issues in genetics [3].

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In the following years, several reaserchers worked on this area. One of the well developed hyperstructure is the hyper BCI-algebra [23]. By following this hyperstructure, Indangan and Petalcorin [11, 1] introduced a new hyperstructure which is called a hyper GR-algebra. They established some results on hyper GR-ideals of a hyper GR-algebra and hyper homomorphic properties on hyper GR-algebras. Recently, a pseudo hyper GR-algebra was established by Manzano and Petalcorin [10]. Pseudo hyper GR-ideals were also defined and classified to determine their relationship to each other.

In the real world, the classes of objects encountered may not have precisely defined criteria of membership. For example, the class of blue colors clearly includes sky blue and navy blue as its members and clearly excludes colors such as red and orange. However, colors such as blue violet and cyan have an ambiguous status with respect to the class of blue colors. With this ambiguity, the theory of fuzzy set arises. When Zadeh [15] first introduced the fuzzy set, unlike sets where the objects are precise, he defined it as a class of objects where objects are given grades of membership. In other words, a fuzzy set A is characterized by a membership function μ where each object is assigned a grade of membership between 0 to 1. The closer the value of the function at an object x to 1, the higher the membership of x in A. When A is a set in the ordinary sense, the membership function is simply the characteristic function. This concept of fuzzy sets are extremely useful for many people involved in research and development including engineers, mathematicians, computer software developers and researchers, natural scientists, medical researchers, social scientists, public policy analysts, business analysts, and jurists [4]. Moreover, it provides a natural framework for generating many concepts of general mathematics. Specifically, it had been applied to some algebras and hyperstructure algebras. Some of these studies are fuzzy hyper BCK-ideals of hyper BCK-algebras[14], fuzzy implicative hyper BCK-ideals of hyper BCK-algebras [13], and fuzzy ideals in hyper BCI-algebras [8], hyper K-subalgebras based on fuzzy points [16], on fuzzy hyper B-ideals of hyper B-algebras [12], some results on fuzzy implicative hyper GR-ideals [6], and on intuitionistic fuzzy hyper GR-ideals in hyper GR-algebras [5].

In this study, the fuzzification of a hyper GR-algebras are considered. In particular, this paper seeks to introduce the concepts of distributive hyper GR-ideal, fuzzy hyper GR-ideal of type 2 and fuzzy distributive hyper GR-ideal of type 2, and aims to establish some of their characterizations and relationships.

2. Preliminaries

In this section, we gather some basic notions and properties relevant to this study.

Let H be a nonempty set, $x \in H$ and A, B a nonempty subsets of H. Consider a hyperoperation $\circledast$. We define $A \circledast B = \bigcup_{a \in A, b \in B} a \circledast b$, $A \circledast x = A \circledast \{x\}$, and $x \circledast B = \{x\} \circledast B$. Moreover, $x \ll y$ is defined by $0 \in x \circledast y$ and $A \ll B$ is

defined by for any $a \in A$, there exists $b \in B$ such that $a \ll b$. We call " $\ll$ " a *hyperorder* on H.

Definition 2.1 ([11]). *Let H be a nonempty set and $\circledast$ be a hyperoperation on H. Then $(H; \circledast, 0)$ is called a hyper GR-algebra if it contains a constant $0 \in H$ and it satisfies the following conditions, for all $x, y, z \in H$:*

(HGR1) $(x \circledast z) \circledast (y \circledast z) \ll x \circledast y$;

(HGR2) $(x \circledast y) \circledast z = (x \circledast z) \circledast y$;

(HGR3) $x \ll x$;

(HGR4) $0 \circledast (0 \circledast x) \ll x, x \neq 0$; *and*

(HGR5) $(x \circledast y) \circledast z \ll y \circledast z$.

For the sake of simplicity, we also call H a hyper GR-algebra.

Example 2.1. Let $H = [0, 1]$ such that for any $a, b \in [0, 1]$,

$$a \circledast b = \begin{cases} [0, 0.3], & \text{if } b \neq 0 \text{ or } a = 0 = b; \\ \{a\}, & \text{if } a \neq 0 \text{ and } b = 0. \end{cases}$$

It can be seen that H is a hyper GR-algebra.

Definition 2.2 ([11]). *Let S be a subset of a hyper GR-algebra H containing 0. We say that S is a hyper subGR-algebra on H if it is a hyper GR-algebra with respect to the hyperoperation $\circledast$ on H.*

Definition 2.3 ([11]). *A subset I of a hyper GR-algebra H is said to be a hyper GR-ideal of H if*

(I1) $0 \in I$, *and*

(I2) *for all $x, y \in H$, $x \circledast y \subseteq I$ and $y \in I$ imply that $x \in I$.*

Definition 2.4 ([6]). *A nonempty subset I of a hyper GR-algebra H is called an implicative hyper GR-ideal of H if for any $x, y, z \in H$,*

(IH1) $0 \in I$, *and*

(IH2) $(x \circledast z) \circledast (y \circledast x) \subseteq I$ *and* $z \in I$ *imply* $x \in I$.

Theorem 2.1 ([6]). *Let H be a hyper GR-algebra such that $x \in x \circledast y$ for any $x, y \in H$. Then any subset I of H containing 0 is an implicative hyper GR-ideal of H.*

Definition 2.5 ([11]). *Suppose that H and K are hyper GR-algebras. Define a hyperoperation*

$$\circledast : (H \times K) \times (H \times K) \to P^*(H \times K) \text{ by}$$
$$(a,b) \circledast (c,d) = (a \circledast_H c) \times (b \circledast_K d)$$

and $(a,b) \ll (c,d)$ if and only if $a \ll_H c$ and $b \ll_K d$, for all $a,c \in H$ and $b,d \in K$ with $\ll_H$ and $\ll_K$ as hyper orders of $(H, \circledast_H, 0_H)$ and $(K, \circledast, 0_k)$, respectively. If A, B are nonempty subsets of H, then we define $(A,B) = A \times B$.

Lemma 2.1 ([11]). *Let H and K be hyper GR-algebras, $A, C \subseteq H$, anf $B, D \subseteq K$. Then $(A,B) \circledast (C,D) = (A \circledast_H C, B \circledast_K D)$.*

Definition 2.6 ([15]). *A fuzzy set μ of a nonempty set M is a function $\mu : M \to [0,1]$.*

Definition 2.7 ([20]). *Let μ be a fuzzy set of M. For a fixed $t \in [0,1]$, the set $\mu_t = \{x \in M | \mu(x) \geq t\}$ is called a level subset of μ.*

Definition 2.8 ([5]). *A fuzzy set μ in a hyper GR-algebra H is a fuzzy hyper GR-ideal of type 1 if for all $x, y \in H$,*

(F1) $\mu(0) \geq \mu(x) \geq \min\{\inf_{u \in x \circledast y} \mu(u), \mu(y)\}$.

Definition 2.9 ([17]). *A fuzzy relation on any set H is a fuzzy set*

$$\lambda : H \times H \to [0,1].$$

Definition 2.10 ([17]). *If λ is a fuzzy relation on a set H and μ is a fuzzy set in H, then λ is a fuzzy relation on μ if $\lambda(x,y) \leq \min\{\mu(x), \mu(y)\}$ for all $x, y \in H$.*

Definition 2.11 ([17]). *If μ is a fuzzy set in a set H, then a strongest fuzzy relation on H, denoted by λ_μ, is a fuzzy relation on μ given for all $x, y \in H$ by $\lambda_\mu(x,y) = \min\{\mu(x), \mu(y)\}$.*

Definition 2.12 ([17]). *Let μ_1 and μ_2 be fuzzy sets in a set H. The Cartesian product of μ_1 and μ_2 is defined for all $x, y \in H$ by*

$$(\mu_1 \times \mu_2)(x,y) = \min\{\mu_1(x), \mu_2(y)\}.$$

Definition 2.13 ([21]). *For a family $\{\mu_i : i \in I\}$ of fuzzy sets in H, define the join $\vee_{i \in I} \mu_i$ and the meet $\wedge_{i \in I} \mu_i$ as follows:*

$$(\vee_{i \in I} \mu_i)(x) = \sup_{i \in I} \mu_i(x) \text{ and } (\wedge_{i \in I} \mu_i)(x) = \inf_{i \in I} \mu_i(x),$$

for all $x \in H$, where I is any indexing set.

Definition 2.14 ([9]). *A partially ordered set P is a set in which a binary relation $x \leq y$ is defined, which satisfies for all $x, y, z \in P$ the following conditions:*

(P1) *For all x, $x \geq x$.*

(P2) *If $x \geq y$ and $y \geq x$, then $x = y$.*

(P3) *If $x \geq y$ and $y \geq z$, then $x \geq z$.*

Definition 2.15 ([9])**.** *A partially ordered set P is finite if the number of its elements is finite.*

Definition 2.16 ([9])**.** *A totally ordered set P is a partially ordered set which satisfies the following condition:*

(P4) *Given x and y in P, either $x \geq y$ or $y \geq x$.*

A totally ordered set is also called a chain.

Theorem 2.2 ([9])**.** *Any subset of a chain is a chain.*

Definition 2.17 ([9])**.** *Let X be a subset of a partially ordered set P. An upperbound of X is an element $u \in P$ such that $u \geq x$ for all $x \in X$. The least upperbound or supremum of X, denoted by $\sup X$, is an upperbound u_0 such that $u_0 \leq u$ for any upperbound u of X. A lowerbound of X is an element $l \in P$ such that $l \leq x$ for all $x \in X$. The greatest lowerbound or infimum of X, denoted by $\inf X$, is a lowerbound l_0 such that $l_0 \geq l$ for any lowerbound l of X.*

Definition 2.18 ([9])**.** *The length of a finite chain is defined to be $n-1$ where n is the number of elements of the chain. The length $l(P)$ of a partially ordered set P is defined as the least upper bound of the lengths of the chains in P. When $l(P)$ is finite, P is said to be of finite length.*

Definition 2.19 ([9])**.** *A lattice L is a partially ordered set where any two of whose elements x and y have an infimum (or "meet") denoted by $x \wedge y$, and a supremum (or "join") denoted by $x \vee y$. A lattice L is complete when each of its subsets has an infimum and supremum in L.*

Remark 2.1 ([9])**.** Any finite lattice or lattice of finite length is complete.

Definition 2.20 ([9])**.** *A sublattice of a lattice L is a subset X of L such that $a \in X$, $b \in X$ imply $a \wedge b \in X$ and $a \vee b \in X$.*

Definition 2.21 ([9])**.** *A lattice L is distributive if either $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ or $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ holds for all $x, y, z \in L$.*

Lemma 2.2 ([9])**.** *Any totally ordered lattice is a distributive lattice.*

3. Distributive hyper GR-ideal

Definition 3.1. *A nonempty subset A of a hyper GR-algebra H is called a distributive hyper GR-ideal if it satisfies the following:*

(DH1) $0 \in A$, *and*

(DH2) *for all* $x, y, z \in H$, $[(x \circledast z) \circledast z] \circledast (y \circledast z) \subseteq A$ *and* $y \in A$ *imply* $x \in A$.

Example 3.1. Let $H = \{0, 1, 2, 3\}$ with the Cayley table below.

$\circledast$	0	1	2	3
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
2	$\{0,2\}$	$\{0,2\}$	$\{0,1,2\}$	$\{0,1,2\}$
3	$\{0,3\}$	$\{0,3\}$	$\{0,3\}$	$\{0,1,3\}$

It can be verified that H is a hyper GR-algebra. By routine calculations, we see that $\{0,1\}, \{0,1,2\}$ and $\{0,1,3\}$ are both hyper GR-ideals and distributive hyper GR-ideals of H.

In Example 3.1, $[(x \circledast z) \circledast z] \circledast (y \circledast z) \subseteq \{0,1,2\}$ for any $x \neq 3$, $z \neq 0$ (or 1 or 2 or 3) and $y \neq 0$ (or 1 or 2 or 3). Will $\{0,1,2\}$ remain a distributive hyper GR-ideal of H if $[(x \circledast z) \circledast z] \circledast (y \circledast z) \subseteq \{0,1,2\}$, for all $x, y, z \in H$? The next theorem will answer this question.

Theorem 3.1. *Let A be a proper subset of a hyper GR-algebra H. If $[(x \circledast z) \circledast z] \circledast (y \circledast z) \subseteq A$, for all $x, y, z \in H$, then A is not a distributive hyper GR-ideal of H.*

Proof of Theorem 3.1. Since A is a proper subset of H, there exists $w \in H$ such that $w \notin A$. Suppose A is a distributive hyper GR-ideal of H. Let $v \in A$. By the hypothesis, $[(w \circledast z) \circledast z] \circledast (v \circledast z) \subseteq A$. Then $w \in A$. This is a contradiction. Hence, A is not a distributive hyper GR-ideal of H.

Example 3.2. Consider the set $H = \{0, 1, 2\}$ with the Cayley table below.

$\circledast$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{0,1\}$	$\{0,2\}$	$\{0\}$
2	$\{0,2\}$	$\{2\}$	$\{0\}$

It can be verified that H is a hyper GR-algebra and $\{0\}$ is a hyper GR-ideal of H. Observe that $\{0\}$ and $\{0,1\}$ are hyper GR-ideals of H. Since $[(2 \circledast 2) \circledast 2] \circledast (0 \circledast 2) = \{0\}$ and $0 \in \{0\}$ but $2 \notin \{0\}$, $\{0\}$ is not a distributive hyper GR-ideal of H. Moreover, since $[(2 \circledast 2) \circledast 2] \circledast (1 \circledast 2) = \{0\} \subset \{0,1\}$ and $1 \in \{0,1\}$ but $2 \notin \{0,1\}$, $\{0,1\}$ is also not a distributive hyper GR-ideal of H.

Remark 3.1. Example 3.2 shows that not every hyper GR-ideal of a hyper GR-algebra H is a distributive hyper GR-ideal.

In Example 3.2, we can see that $2 \in 2 \circledast 2$ and $\{0\}$ is not a distributive hyper GR-ideal of H.

Example 3.3. Consider the hyper GR-algebra $H = \mathbb{Z}$ such that $x \circledast y = \{0, x, y\}$ for $x, y \in H$. Clearly, $x \in x \circledast y$, for all $x, y \in H$. Let $a, b, c \in H$ such that $[(a \circledast c) \circledast c] \circledast (b \circledast c) \subseteq \{0, x, y\}$ and $b \in \{0, x, y\}$. Since $a \in [(a \circledast c) \circledast c] \circledast (b \circledast c)$, $a \in \{0, x, y\}$. Hence, $\{0, x, y\}$ is a distributive hyper GR-ideal of H.

The next theorem gives a generalization when $x \in x \circledast y$, for all $x, y \in H$.

Theorem 3.2. *Let H be a hyper GR-algebra, $A \subset H$ and $0 \in A$. If $x \in x \circledast y$, for all $x, y \in H$, then A is both hyper GR-ideal and distributive hyper GR-ideal of H.*

Proof of Theorem 3.2. Let $x, y \in H$ such that $x \circledast y \subseteq A$ and $y \in A$. By the hypothesis, $x \in x \circledast y \subseteq A$. Hence, A is a hyper GR-ideal of H. Moreover, let $x, y, z \in H$ such that $[(x \circledast z) \circledast z] \circledast (y \circledast z) \subseteq A$ and $y \in A$. By the hypothesis, $x \in x \circledast z$ and so $x \in (x \circledast z) \circledast z$. Since$[(x \circledast y) \circledast z] \circledast (y \circledast z) \subseteq A$, $x \circledast v \subseteq A$ for $v \in y \circledast z$. By the hypothesis $x \in x \circledast v$ and so $x \in A$. Hence, A is a distributive hyper GR-ideal of H.

By Theorems 2.1 and 3.2, we obtained a theorem that makes a hyper GR-ideal both a distributive hyper GR-ideal and an implicative hyper GR-ideal of a hyper GR-algebra.

Theorem 3.3. *If $x \in x \circledast y$ for any x and y in a hyper GR-algebra H, then a subset A containing 0 of H is a hyper GR-ideal, distributive hyper GR-ideal and implicative hyper GR-ideal of H.*

Example 3.4. Consider the hyper GR-algebra $H = \{0, 1, 2, 3\}$ with the Cayley table below.

$\circledast$	0	1	2	3
0	$\{0,1,3\}$	$\{0,1,3\}$	$\{0,1,3\}$	$\{0,1,3\}$
1	$\{0,1,3\}$	$\{0,1,3\}$	$\{0,1,3\}$	$\{0,1,3\}$
2	$\{0,1,3\}$	$\{0,1\}$	$\{0,1\}$	$\{3\}$
3	$\{0,1,3\}$	$\{0,1,3\}$	$\{0,1,3\}$	$\{0\}$

We can see that H, $\{0\}, \{0, 1, 2\}$ and $\{0, 2\}$ are the only hyper GR-ideals of H. Note that $[(x \circledast z) \circledast z] \circledast (y \circledast z) = \{0, 1, 3\}$, for all $x, y, z \in H$. This implies that all subsets of H containing 0 which are not equal to $\{0, 1, 3\}$ are distributive hyper GR-ideals of H. Particularly, $\{0, 1\}$ is a distributive hyper GR-ideal of H but not a hyper GR-ideal of H because $2 \circledast 1 = \{0, 1\}$, $1 \in \{0, 1\}$ and $2 \notin \{0, 1\}$.

Remark 3.2. Example 3.4 shows that not every distributive hyper GR-ideal is a hyper GR-ideal.

Theorem 3.4. *A distributive hyper GR-ideal A such that $A \circledast A \subseteq A$ is a hyper GR-ideal.*

Proof of Theorem 3.4. Let $x \circledast y \subseteq A$ where $y \in A$. Since $A \circledast A \subseteq A, (x \circledast y) \circledast y \subseteq A$ and $y \circledast y \subseteq A$. It implies that $[(x \circledast y) \circledast y)] \circledast (y \circledast y) \subseteq A$. Since A is a distributive hyper GR-ideal, $x \in A$. Thus, A is a hyper GR-ideal.

Theorem 3.5. *Let H be a hyper GR-algebra such that $[(x \circledast 0) \circledast (y \circledast 0)] \circledast 0 = x \circledast y$, for all $x, y \in H$. Then a distributive hyper GR-ideal A is a hyper GR-ideal.*

Proof of Theorem 3.5. Let A be a distributive hyper GR-ideal of H. Let $a, b \in H$ such that $a \circledast b \subseteq A$ and $b \in A$. Then, by HGR2 and the hypothesis, we have

$$[(a \circledast 0) \circledast 0] \circledast (b \circledast 0) = [(a \circledast 0) \circledast (b \circledast 0)] \circledast 0 = a \circledast b \subseteq A.$$

Since A is a distributive hyper GR-ideal of H, we have $a \in A$. Hence, A is a hyper GR-ideal of H.

4. Fuzzy hyper GR-ideal of type 2

Definition 4.1. *A fuzzy set μ in a hyper GR-algebra H is a fuzzy hyper GR-ideal of type 2 if for all $x, y \in H$*

(FF1) *$x \ll y$ implies $\mu(y) \leq \mu(x)$; and*

(FF2) $\mu(x) \geq \min\{\inf_{u \in x \circledast y} \mu(u), \mu(y)\}$.

Example 4.1. Consider the hyper GR-algebra H in Example 2.1. Define a fuzzy set μ in H by

$$\mu(a) = \begin{cases} l, & \text{if } a = 0 \\ k, & \text{if } a \neq 0 \end{cases}$$

where $k, l \in [0, 1]$ and $k < l$. By routine calculations we see that μ is a fuzzy hyper GR-ideal of type 2 in H.

Remark 4.1. By FF1, $x \ll y$ and $y \ll x$ imply $\mu(x) = \mu(y)$.

Example 4.2. Let $H = \{0, 1, 2, 3\}$ with the Cayley table below.

$\circledast$	0	1	2	3
0	{0,1,2}	{0,1,2}	{0,1,2}	{0,1,2}
1	{0,1}	{0,1,2}	{0,1,2}	{0,1,2}
2	{0,2}	{0,1,2}	{0,1,2}	{0,1,2}
3	{0,1,3}	{1,2,3}	{1,2,3}	{0,1,2}

It can be shown that H is a hyper GR-algebra. Define a fuzzy set μ in H by $\mu(1) = 0.75$, $\mu(2) = 0.6$, $\mu(3) = 0.5$, and $\mu(0) = 0.8$. By routine calculations, we can see that μ is a fuzzy hyper GR-ideal of type 1 in H. On the contrary, μ is not a fuzzy hyper GR-ideal of type 2 in H since $2 \ll 1$ but $\mu(2) = 0.6 < 0.75 = \mu(1)$.

Remark 4.2. Example 4.2 shows that not all fuzzy hyper GR-ideal of type 1 are fuzzy hyper GR-ideal of type 2.

In Example 3.1, we see that $2 \ll x$ and $x \ll 2$, for all $x \in H$. So, if a fuzzy set μ on H satisfies FF1, then $\mu(x) = \mu(2)$, for all $x \in H$. Hence, μ is constant. The next theorem is a generalization for this case.

Theorem 4.1. *Let H be a hyper GR-algebra and μ a fuzzy set such that $x \ll y$ implies $\mu(y) \leq \mu(x)$. If there exists $\hat{x} \in H$ such that $\hat{x} \ll x$ and $x \ll \hat{x}$, for all $x \in H$, then the fuzzy set μ is constant.*

Proof of Theorem 4.1. Let $x \in H$ and μ a fuzzy set in H such that $x \ll y$ implies $\mu(y) \leq \mu(x)$. Then by the hypothesis, $\mu(x) \leq \mu(\hat{x})$ and $\mu(\hat{x}) \leq \mu(x)$. It follows that $\mu(x) = \mu(\hat{x})$. Since x is an arbitrary element in H, μ is constant.

Theorem 4.2. *Let H be a hyper GR-algebra such that $0 \ll x$, for all $x \in H$. Then a fuzzy set μ is a fuzzy hyper GR-ideal of type 2 in H if and only if μ_d is a hyper GR-ideal of H whenever $\mu_d \neq \emptyset$, $d = \min\{\inf_{u \in x \circledast y} \mu(u), \mu(y)\}$ for $x, y \in H$ and for $x \neq y$ such that $x \ll y$ implies $d = \mu(y)$.*

Proof of Theorem 4.2. Let μ be a fuzzy hyper GR-ideal of type 2 in H and $d = \min\{\inf_{u \in x \circledast y} \mu(u), \mu(y)\}$ for $x, y \in H$. Since $0 \ll x$, for all $x \in H$, $\mu(0) \geq \mu(x)$, for all $x \in H$. This implies that $\mu(0) \geq \mu(u)$, for all $u \in x \circledast y$. Then $\mu(0) \geq \inf_{u \in 0 \circledast y} \mu(u) \geq \min\{\inf_{u \in x \circledast y} \mu(u), \mu(y)\} = d$. It follows that $0 \in \mu_d$. Let $x', y' \in H$ such that $x' \circledast y' \subseteq \mu_d$ and $y' \in \mu_d$. Then, for any $v \in x' \circledast y'$, $d \leq \mu(v)$. Since μ is a fuzzy hyper GR-ideal of type 2, $\mu(x') \geq \min\{\inf_{v \in x' \circledast y'} \mu(v), \mu(y')\}$. Note that $\mu(v) \geq d$ and $\mu(y') \geq d$. This implies that $\mu(x') \geq \min\{\inf_{v \in x' \circledast y'} \mu(v), \mu(\ y')\} \geq d$. Thus, $x' \in \mu_d$ and so μ_d is a hyper GR-ideal of H.

Conversely, let $x, y \in H$ and $d = \min\{\inf_{u \in x \circledast y} \mu(u), \mu(y)\}$. Since $\mu(y) \geq \min\{\inf_{u \in x \circledast y} \mu(u), \mu(y)\} = d$, $y \in \mu_d$. Let $w \in x \circledast y$. Then we have $\mu(w) \geq \inf_{u \in x \circledast y} \mu(u) \geq \min\{\inf_{u \in x \circledast y} \mu(u), \mu(y)\} = d$. It implies that $w \in \mu_d$ and so $x \circledast y \subseteq \mu_d$. Since μ_d is a hyper GR-ideal, $x \in \mu_d$. Then, $\mu(x) \geq d = \min\{\inf_{u \in x \circledast y} \mu(u), \mu(y)\}$. Moreover, suppose that $x \ll y$ where $x \neq y$. By hypothesis, $d = \mu(y)$. Since $x \in \mu_d$, $\mu(x) \geq d = \mu(y)$. Suppose $x = y$. Clearly, $\mu(x) = \mu(y)$. Therefore, μ is a fuzzy hyper GR-ideal of type 2 in H.

Theorem 4.3. *Let μ be a fuzzy hyper GR-ideal of type 2 in H. If $x \circledast y \ll z$, then $\mu(x) \geq \min\{\mu(z), \mu(y)\}$.*

Proof of Theorem 4.3. Let $x, y, z \in H$ such that $x \circledast y \ll z$. Then by FF1, $\mu(z) \leq \mu(u)$, for all $u \in x \circledast y$. Since μ is a fuzzy hyper GR-ideal of type 2 in H,

$$\mu(x) \geq \min\left\{\inf_{u \in x \circledast y} \mu(u), \mu(y)\right\} \geq \min\{\mu(z), \mu(y)\}.$$

5. Fuzzy distributive hyper GR-ideal of type 2

Definition 5.1. *A fuzzy set μ in a hyper GR-algebra H is a fuzzy distributive hyper GR-ideal of type 2 if it satisfies FF1 and for all $x, y, z \in H$,*

$$\text{(FFD) } \mu(x) \geq \min\left\{\inf_{v\in[(x\circledast z)\circledast z]\circledast(y\circledast z)} \mu(v), \mu(y)\right\}.$$

Example 5.1. The fuzzy set μ in Example 2.1 is also a fuzzy distributive hyper GR-ideal of type 2.

Example 5.2. Let $H = \langle a_n\rangle$ be a sequence of numbers where $n \in \mathbb{N}\cup\{0\}$ and $a_0 = 0$ such that

$$a_n \circledast a_m = \begin{cases} \{0, a_1\}, & \text{if } m \neq 0 \text{ or } n = 0 = m, \\ \{a_n\}, & \text{if } n \neq 0 \text{ and } m = 0. \end{cases}$$

It can be verified that H is a hyper GR-algebra. Define a fuzzy set μ in H by $\mu(0) = l$ and $\mu(a_n) = k$ $(n \neq 0)$ where $k, l \in [0, 1]$ and $k < l$. By routine calculations, μ is both a fuzzy hyper GR-ideal of type 2 and a fuzzy distributive hyper GR-ideal of type 2.

Theorem 5.1. *Let H be a hyper GR-algebra such that $[(x\circledast 0)\circledast(y\circledast 0)]\circledast 0 \subseteq x\circledast y$, for all $x, y \in H$. If μ is a fuzzy distributive hyper GR-ideal of type 2, then μ is a fuzzy hyper GR-ideal of type 2.*

Proof of Theorem 5.1. Let $x \in H$. By Definition 5.1, HGR2 and infimum property,

$$\begin{aligned} \mu(x) &\geq \min\left\{\inf_{u\in[(x\circledast 0)\circledast 0]\circledast(y\circledast 0)} \mu(u), \mu(y)\right\} \\ &= \min\left\{\inf_{u\in[(x\circledast 0)\circledast(y\circledast 0)]\circledast 0} \mu(u), \mu(y)\right\} \\ &\geq \min\left\{\inf_{u\in x\circledast y} \mu(u), \mu(y)\right\}. \end{aligned}$$

Hence, μ is a fuzzy hyper GR-ideal of type 2.

Theorem 5.2. *Let H be a hyper GR-algebra such that $0 \ll x$, for all $x \in H$ and $[(x\circledast 0)\circledast(y\circledast 0)]\circledast 0 = x\circledast y$, for all $x, y \in H$. Then a fuzzy set μ of a hyper GR-algebra H is a fuzzy distributive hyper GR-ideal of type 2 in H if and only if for all $x, y \in H$, μ_d is a distributive hyper GR-ideal of H whenever $\mu_d \neq \emptyset$, $d = \min\{\inf_{u\in x\circledast y} \mu(u), \mu(y)\}$ and for $x \neq y$ such that $x \ll y$ implies $d = \mu(y)$.*

Proof of Theorem 5.2. Let $d = \min\{\inf_{u\in x\circledast y} \mu(u), \mu(y)\}$ for $x, y \in H$. Suppose μ is a fuzzy distributive hyper GR-ideal of type 2 in H, $\mu_d \neq \emptyset$ and $[(x\circledast 0)\circledast(y\circledast 0)]\circledast 0 = x\circledast y$. By Theorem 5.1, μ is a fuzzy hyper GR-ideal of type 2. It follows from Theorem 4.2 that μ_d is a hyper GR-ideal of H. Hence, $0 \in \mu_d$.

Moreover, let $x, y, z \in H$ such that $[(x \circledast z) \circledast z] \circledast (y \circledast z) \subseteq \mu_d$ and $y \in \mu_d$. For any $u \in [(x \circledast z) \circledast z] \circledast (y \circledast z)$, $u \in \mu_d$. Thus, $d \leq \mu(u)$. It implies that d is a lower-bound for $\{\mu(u) : u \in [(x \circledast z) \circledast z] \circledast (y \circledast z)\}$ and hence $\inf_{u \in [(x \circledast z) \circledast z] \circledast (y \circledast z)} \mu(u) \geq d$. Since μ is a fuzzy distributive hyper GR-ideal of type 2 in H, for all $x, y, z \in H$ we have $\mu(x) \geq \min\left\{\inf_{u \in [(x \circledast z) \circledast z] \circledast (y \circledast z)} \mu(u), \mu(y)\right\} \geq \min\{d, d\} = d$. Thus, $x \in \mu_d$ and so μ_d is a distributive hyper GR-ideal of H.

Conversely, for all $x, y \in H$ such that $d = \min\{\inf_{u \in x \circledast y} \mu(u), \mu(y)\}$, μ_d is a distributive hyper GR-ideal of H, whenever $\mu_d \neq \emptyset$ and $[(x \circledast 0) \circledast (y \circledast 0)] \circledast 0 = x \circledast y$. By Theorem 3.5, μ_d is a hyper GR-ideal of H. Hence, from Theorem 4.2, μ is a fuzzy hyper GR-ideal of type 2 in H. Hence, FF1 holds. Let $x, y, z \in H$ and let $l \in [0, 1]$ such that $l = \min\left\{\inf_{u \in [(x \circledast z) \circledast z] \circledast (y \circledast z)} \mu(u), \mu(y)\right\}$. Let $v \in [(x \circledast z) \circledast z] \circledast (y \circledast z)$. Then, $\mu(v) \geq \inf_{u \in [(x \circledast z) \circledast z] \circledast (y \circledast z)} \mu(u) \geq \min\{\inf_{u \in [(x \circledast z) \circledast z] \circledast (y \circledast z)} \mu(u), \mu(y)\} = l$. It implies that $v \in \mu_l$ and hence $[(x \circledast z) \circledast z] \circledast (y \circledast z) \subseteq \mu_l$. Note that $\mu(y) \geq \min\left\{\inf_{u \in [(x \circledast z) \circledast z] \circledast (y \circledast z)} \mu(u), \mu(y)\right\} = l$. Then, $y \in \mu_l$. Since μ_l is a distributive hyper GR-ideal of H, $x \in \mu_l$. It follows that $\mu(x) \geq l = \min\left\{\inf_{u \in [(x \circledast z) \circledast z] \circledast (y \circledast z)} \mu(u), \mu(y)\right\}$. Therefore, μ is a fuzzy distributive hyper GR-ideal of type 2.

Corollary 5.1. *Let H be a hyper GR-algebra such that $0 \ll x$, for all $x \in H$ and $[(x \circledast 0) \circledast (y \circledast 0)] \circledast 0 = x \circledast y$, for all $x, y \in H$. For any subset A of H, let μ_A be a fuzzy set in H defined by*

$$\mu_A(x) = \begin{cases} d, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases},$$

for all $x \in H$ where $d = \min\{\inf_{u \in x \circledast y} \mu(u), \mu(y)\}$ and $x \ll y$ implies $d = \mu(y)$. Then A is a distributive hyper GR-ideal if and only if μ_A is a fuzzy distributive hyper GR-ideal of type 2 in H.

Proof of Corollary 5.1. Let $d = \min\{\inf_{u \in x \circledast y} \mu(u), \mu(y)\}$ for $x \in H$. Note that

$$(\mu_A)_t = \begin{cases} \emptyset, & \text{if } d < t \leq 1, \\ A, & \text{if } 0 < t \leq d \\ H, & \text{if } t \geq 0 \end{cases} \tag{1}$$

are all possible level subsets of μ_A. Let $t = d$. It follows from Equation 1 that $(\mu_A)_d = A$. By Theorem 5.2, A is a distributive hyper GR-ideal if and only if μ_A is a fuzzy distributive hyper GR-ideal of type 2 in H.

Lemma 5.1. *Let H be a hyper GR-algebra, $I \subseteq \mathbb{N}$, and for $x, y, z \in H$,*

$$A = \sup_{i \in I}\left\{\min\left\{\inf_{u \in [(x \circledast z) \circledast z] \circledast (y \circledast z)} \mu_i(u), \mu_i(y)\right\}\right\},$$

$$B = \min\left\{\sup_{i \in I}\left(\inf_{u \in [(x \circledast z) \circledast z] \circledast (y \circledast z)} \mu_i(u)\right), \sup_{i \in I} \mu_i(y)\right\},$$

$$C = \min\left\{\inf_{u \in [(x \circledast z) \circledast z] \circledast (y \circledast z)} (\sup_{i \in I} \mu_i(u)), \sup_{i \in I} \mu_i(y)\right\},$$

$$D = \inf_{i\in I}\left\{\min\left\{\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}\mu_i(u),\mu_i(y)\right\}\right\},$$

$$E = \min\left\{\inf_{i\in I}\left(\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}\mu_i(u)\right),\inf_{i\in I}\mu_i(y)\right\},\ \textit{and}$$

$$F = \min\left\{\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}\left(\inf_{i\in I}\mu_i(u)\right),\inf_{i\in I}\mu_i(y)\right\}$$

where μ_i is a fuzzy distributive hyper GR-ideal of type 2, for all $i \in I$. Then, for $x, y, z \in H$, (i) $A \geq B$; (ii) $B \geq C$; (iii) $D \geq E$; (iv) $E \geq F$; *and* (v) $\mu_i(y) \leq \mu_i(x)$ *for $i \in I$ and $x, y \in H$ imply* $\sup_{i\in I}\mu_i(y) \leq \sup_{i\in I}\mu_i(x)$ *and* $\inf_{i\in I}\mu_i(y) \leq \inf_{i\in I}\mu_i(x)$

Proof of Lemma 5.1. (*i*) Case 1: Suppose
$\min\left\{\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}\mu_i(u),\mu_i(y)\right\} = \inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}\mu_i(u)$. Then,

$$\sup_{i\in I}\left\{\min\left\{\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}\mu_i(u),\mu_i(y)\right\}\right\} = \sup_{i\in I}\left(\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}\mu_i(u)\right)$$
$$\geq \min\left\{\sup_{i\in I}\left(\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}\mu_i(u)\right),\sup_{i\in I}\mu_i(y)\right\}.$$

Case 2: Suppose $\min\left\{\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}\mu_i(u),\mu_i(y)\right\} = \mu_i(y)$. Then,

$$\sup_{i\in I}\left\{\min\left\{\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}\mu_i(u),\mu_i(y)\right\}\right\} = \sup_{i\in I}\mu_i(y)$$
$$\geq \min\left\{\sup_{i\in I}\left(\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}\mu_i(u)\right),\sup_{i\in I}\mu_i(y)\right\}.$$

So, $A \geq B$.
(*ii*) For $u \in [(x \circledast z) \circledast z] \circledast (y \circledast z)$, let $\mu_1(u), \mu_2(u) \in \{\mu_i(u) : i \in I\}$. Then

$$\mu_1(u) \wedge \mu_2(u) = \inf\{\mu_1(u), \mu_2(u)\}$$
$$= \min\{\mu_1(u), \mu_2(u)\} \in \{\mu_i(u) : i \in I\}$$

and

$$\mu_1(u) \vee \mu_2(u) = \sup\{\mu_1(u), \mu_2(u)\}$$
$$= \max\{\mu_1(u), \mu_2(u)\} \in \{\mu_i(u) : i \in I\}.$$

Hence, $\{\mu_i(u) : i \in I\}$ for $u \in [(x \circledast z) \circledast z] \circledast (y \circledast z)$ is a sublattice of [0,1]. Since $\{\mu_i(u) : i \in I\} \subseteq [0, 1]$, $\{\mu_i(u) : i \in I\}$ is of finite length. By Remark 2.1, $\{\mu_i(u) : i \in I\}$ is complete and so $\sup_{i\in I}\mu_i(u)$ and $\inf_{i\in I}\mu_i(u)$ are in $\{\mu_i(u) : i \in I\}$. This implies that $\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}(\sup_{i\in I}\mu_i(u)) \in \left\{\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}\mu_i(u) : i \in I\right\}$. Then $\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}\left(\sup_{i\in I}\mu_i(u)\right) \leq \sup_{i\in I}\left(\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}\mu_i(u)\right)$. Hence, $B \geq C$.
(*iii*) Case 1: Suppose
$\min\left\{\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}\mu_i(u),\mu_i(y)\right\} = \inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}\mu_i(u)$. Then,

$$\inf_{i\in I}\left\{\min\left\{\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}\mu_i(u),\mu_i(y)\right\}\right\} = \inf_{i\in I}\left(\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}\mu_i(u)\right)$$

$$\geq \min\left\{\inf_{i\in I}\left(\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}\mu_i(u)\right), \inf_{i\in I}\mu_i(y)\right\}.$$

Case 2: Suppose $\min\left\{\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}\mu_i(u), \mu_i(y)\right\} = \mu_i(y)$. Then,

$$\inf_{i\in I}\left\{\min\left\{\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}\mu_i(u), \mu_i(y)\right\}\right\} = \inf_{i\in I}\mu_i(y)$$
$$\geq \min\left\{\inf_{i\in I}\left(\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}\mu_i(u)\right), \inf_{i\in I}\mu_i(y)\right\}.$$

Thus, $D \geq E$.
(iv) Since $\mu_i(u) \geq \inf_{i\in I}\mu_i(u)$ for each $i \in I$,

$$\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}\mu_i(u) \geq \inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}\inf_{i\in I}\mu_i(u).$$

Hence, $E \geq F$.
(v) Since $\mu_i(y) \leq \mu_i(x)$ for $i \in I$ and $x, y \in H$,

$$\begin{aligned}
sup_{i\in I}\mu_i(y) &= \sup\{\mu_1(y), \mu_2(y), ..., \mu_n(y), ...\}\\
&\leq \sup\{\mu_1(x), \mu_2(y), ..., \mu_n(y), ...\}\\
&\leq \sup\{\mu_1(x), \mu_2(x), ..., \mu_n(y), ...\}\\
&\vdots \qquad\qquad \vdots\\
&\leq \sup\{\mu_1(x), \mu_2(x), ..., \mu_n(x), ...\} = \sup_{i\in I}\mu_i(x) \text{ and}\\
\inf_{i\in I}\mu_i(y) &= \inf\{\mu_1(y), \mu_2(y), ..., \mu_n(y), ...\}\\
&\leq \inf\{\mu_1(x), \mu_2(y), ..., \mu_n(y), ...\}\\
&\leq \inf\{\mu_1(x), \mu_2(x), ..., \mu_n(y), ...\}\\
&\vdots \qquad\qquad \vdots\\
&\leq \inf\{\mu_1(x), \mu_2(x), ..., \mu_n(x), ...\} = \inf_{i\in I}\mu_i(x).
\end{aligned}$$

Theorem 5.3. *The family M of fuzzy distributive hyper GR-ideals of type 2 of a hyper GR-algebra H is a complete distributive lattice with respect to join and meet.*

Proof of Theorem 5.3. Let M be a family of fuzzy distributive hyper GR-ideals of type 2 in H and $\mu_1, \mu_2, \mu_3 \in M$. Let a binary relation $\leq$ be defined by $\mu \leq \mu'$ if and only if $\mu(x) \leq \mu'(x)$, for all $x \in H$ and $\mu, \mu' \in M$. Clearly, $\mu_1 = \mu_1$. Suppose $\mu_1 \leq \mu_2$ and $\mu_2 \leq \mu_1$. Then $\mu_1(x) \leq \mu_2(x)$ and $\mu_2(x) \leq \mu_1(x)$, for all $x \in H$. Since $\mu_1(x), \mu_2(x) \in [0, 1]$ for all $x \in H$, $\mu_1(x) = \mu_2(x)$, for all $x \in H$ and hence $\mu_1 = \mu_2$. Let $\mu_1 \leq \mu_2$ and $\mu_2 \leq \mu_3$. Then $\mu_1(x) \leq \mu_2(x)$ and $\mu_2(x) \leq \mu_3(x)$, for all $x \in H$. Since $\mu_1(x), \mu_2(x), \mu_3(x) \in [0, 1]$, for all $x \in H$, $\mu_1(x) \leq \mu_3(x)$, for all $x \in H$ and so $\mu_1 \leq \mu_3$. Since $\mu_1(x), \mu_2(x) \in [0, 1]$, for all

$x \in H$, either $\mu_1(x) \leq \mu_2(x)$ or $\mu_2(x) \leq \mu_1(x)$, for all $x \in H$. That is, either $\mu_1 \leq \mu_2$ or $\mu_2 \leq \mu_1$. Thus, M is a totally ordered set.

Moreover, let $\mu_1, \mu_2 \in M$ and let $x, y \in H$ such that $x \ll y$. Note that $(\mu_1 \vee \mu_2)(x) = sup\{\mu_1(x), \mu_2(x)\} = \max\{\mu_1(x), \mu_2(x)\}$ and $(\mu_1 \wedge \mu_2)(x) = inf\{\mu_1(x), \mu_2(x)\} = \min\{\mu_1(x), \mu_2(x)\}$ for any $x \in H$. Since μ_1 and μ_2 are fuzzy distributive hyper GR-ideals of type 2 in H, $\mu_1(y) \leq \mu_1(x)$ and $\mu_2(y) \leq \mu_2(x)$. Then $\mu_1(y) \leq \mu_1(x) \leq \max\{\mu_1(x), \mu_2(x)\}$, $\mu_2(y) \leq \mu_2(x) \leq \max\{\mu_1(x), \mu_2(x)\}$, $\mu_1(x) \geq \mu_1(y) \geq \min\{\mu_1(y), \mu_2(y)\}$, and $\mu_2(x) \geq \mu_2(y) \geq \min\{\mu_1(y), \mu_2(y)\}$. It follows that

$$(\mu_1 \vee \mu_2)(y) = \max\{\mu_1(y), \mu_2(y)\} \leq \max\{\mu_1(x), \mu_2(x)\} = (\mu_1 \vee \mu_2)(x) \text{ and}$$
$$(\mu_1 \wedge \mu_2)(y) = \min\{\mu_1(y), \mu_2(y)\} \leq \min\{\mu_1(x), \mu_2(x)\} = (\mu_1 \wedge \mu_2)(x).$$

Let $x, y, z \in H$ and $I = \{1, 2\}$. Since μ_1, μ_2 are fuzzy distributive hyper GR-ideals of type 2 in H,

$$\mu_1(x) \geq \min\left\{\inf_{u \in [(x \circledast z) \circledast z] \circledast (y \circledast z)} \mu_1(u), \mu_1(y)\right\} \text{ and}$$
$$\mu_2(x) \geq \min\left\{\inf_{u \in [(x \circledast z) \circledast z] \circledast (y \circledast z)} \mu_2(u), \mu_2(y)\right\}.$$

Thus

$$(\mu_1 \vee \mu_2)(x) = \max_{i \in I} \mu_i(x) \geq \max_{i \in I} \left\{\min\left\{\inf_{u \in [(x \circledast z) \circledast z] \circledast (y \circledast z)} \mu_i(u), \mu_i(y)\right\}\right\}.$$

By Lemma 5.1(i) and (ii), we have

$$\begin{aligned}(\mu_1 \vee \mu_2)(x) = \max_{i \in I} \mu_i(x) &\geq \max_{i \in I} \left\{\min\left\{\inf_{u \in [(x \circledast z) \circledast z] \circledast (y \circledast z)} \mu_i(u), \mu_i(y)\right\}\right\} \\ &\geq \min\left\{\max_{i \in I}\left(\inf_{u \in [(x \circledast z) \circledast z] \circledast (y \circledast z)} \mu_i(u)\right), \max_{i \in I} \mu_i(y)\right\} \\ &\geq \min\left\{\inf_{u \in [(x \circledast z) \circledast z] \circledast (y \circledast z)} (\max_{i \in I} \mu_i(u)), \max_{i \in I} \mu_i(y)\right\} \\ &= \min\left\{\left\{\inf_{u \in [(x \circledast z) \circledast z] \circledast (y \circledast z)} (\mu_1 \vee \mu_2)(u), (\mu_1 \vee \mu_2)(y)\right\}\right\}.\end{aligned}$$

Also, by Lemma 5.1(iii) and (iv), we have

$$\begin{aligned}(\mu_1 \wedge \mu_2)(x) = \min_{i \in I} \mu_i(x) &\geq \min_{i \in I} \left\{\min\left\{\inf_{u \in [(x \circledast z) \circledast z] \circledast (y \circledast z)} \mu_i(u), \mu_i(y)\right\}\right\} \\ &\geq \min\left\{\min_{i \in I}\left(\inf_{u \in [(x \circledast z) \circledast z] \circledast (y \circledast z)} \mu_i(u)\right), \min_{i \in I} \mu_i(y)\right\} \\ &\geq \min\left\{\inf_{u \in [(x \circledast z) \circledast z] \circledast (y \circledast z)} \left(\min_{i \in I} \mu_i(u)\right), \min_{i \in I} \mu_i(y)\right\} \\ &= \min\left\{\left\{\inf_{u \in [(x \circledast z) \circledast z] \circledast (y \circledast z)} (\mu_1 \wedge \mu_2)(u), (\mu_1 \wedge \mu_2)(y)\right\}\right\}.\end{aligned}$$

It implies that $\mu_1 \vee \mu_2$ and $\mu_1 \wedge \mu_2$ are fuzzy distributive hyper GR-ideals of type 2 in H. Hence, $\mu_1 \vee \mu_2$ and $\mu_1 \wedge \mu_2$ exist in M and so M is a lattice. Since M is totally ordered set, M is distributive lattice by Lemma 2.2.

Furthermore, let $R = \{\mu_i : i \in I\}$ be a subset of M and let $x, y \in H$ such that $x \ll y$. Since μ_i is a fuzzy distributive hyper GR-ideal of type 2 in H, for all $i \in I$, $\mu_i(y) \leq \mu_i(x)$. By Lemma 5.1(v),

$$\begin{aligned}(\vee_{i\in I}\mu_i)(y) &= \sup_{i\in I}\mu_i(y) \leq \sup_{i\in I}\mu_i(x) = (\vee_{i\in I}\mu_i)(x) \text{ and}\\ (\wedge_{i\in I}\mu_i)(y) &= \inf_{i\in I}\mu_i(y) \leq \inf_{i\in I}\mu_i(x) = (\wedge_{i\in I}\mu_i)(x).\end{aligned}$$

Let $x, y, z \in H$. Since μ_i is a fuzzy distributive hyper GR-ideal of type 2 in H, for all $i \in I$, $\mu_i(x) \geq \min\left\{\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)} \mu_i(u), \mu_i(y)\right\}$.

By Lemma 5.1(i) and (ii), we get

$$\begin{aligned}(\vee_{i\in I}\mu_i)(x) = \sup_{i\in I}\mu_i(x) &\geq \sup_{i\in I}\left\{\min\left\{\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}\mu_i(u), \mu_i(y)\right\}\right\}\\ &\geq \min\left\{\sup_{i\in I}\left(\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}\mu_i(u)\right), \sup_{i\in I}\mu_i(y)\right\}\\ &\geq \min\left\{\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}\left(\sup_{i\in I}\mu_i(u)\right), \sup_{i\in I}\mu_i(y)\right\}\\ &= \min\left\{\left\{\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}(\vee_{i\in I}\mu_i)(u), (\vee_{i\in I}\mu_i)(y)\right\}\right\}.\end{aligned}$$

Also, by Lemma 5.1(iii) and (iv), we have

$$\begin{aligned}(\wedge_{i\in I}\mu_i)(x) = \inf_{i\in I}\mu_i(x) &\geq \inf_{i\in I}\left\{\min\left\{\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}\mu_i(u), \mu_i(y)\right\}\right\}\\ &\geq \min\left\{\inf_{i\in I}\left(\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}\mu_i(u)\right), \inf_{i\in I}\mu_i(y)\right\}\\ &\geq \min\left\{\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}\left(\inf_{i\in I}\mu_i(u)\right), \inf_{i\in I}\mu_i(y)\right\}\\ &= \min\left\{\left\{\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}(\wedge_{i\in I}\mu_i)(u), (\wedge_{i\in I}\mu_i)(y)\right\}\right\}.\end{aligned}$$

Hence, $\vee_{i\in I}\mu_i$ and $\wedge_{i\in I}\mu_i$ are fuzzy distributive hyper GR-ideals of type 2 in H. Thus, $\vee_{i\in I}\mu_i$ and $\wedge_{i\in I}\mu_i$ exist in M. Therefore, M is a complete distributive lattice.

Theorem 5.4. *Let μ_1 and μ_2 be fuzzy distributive hyper GR-ideals of type 2 in H. Then $\mu_1 \times \mu_2$ is a fuzzy distributive hyper GR-ideal of type 2 in $H \times H$.*

Proof of Theorem 5.4. Let $(x_1, y_1), (x_2, y_2) \in H \times H$ such that $(x_1, y_1) \ll (x_2, y_2)$. Then $x_1 \ll x_2$ and $y_1 \ll y_2$. Since μ_1 and μ_2 are fuzzy distributive

hyper GR-ideals of type 2 in H, $\mu_1(x_2) \ll \mu_1(x_1)$ and $\mu_2(y_2) \ll \mu_2(y_1)$. Now,

$$\begin{aligned}(\mu_1 \times \mu_2)(x_2, y_2) &= \min\{\mu_1(x_2), \mu_2(y_2)\} \\ &\leq \min\{\mu_1(x_1), \mu_2(y_1)\} = (\mu_1 \times \mu_2)(x_1, y_1).\end{aligned}$$

Moreover, let $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in H \times H$. Since μ_1 and μ_2 are fuzzy distributive hyper GR-ideals of type 2 in H,
$\mu_1(x_1) \geq \min\left\{\inf_{u\in[(x_1\circledast x_3)\circledast x_3]\circledast(x_2\circledast x_3)} \mu_1(u), \mu_1(x_2)\right\}$
and $\mu_2(y_1) \geq \min\left\{\inf_{v\in[(y_1\circledast y_3)\circledast y_3]\circledast(y_2\circledast y_3)} \mu_2(v), \mu_2(y_2)\right\}$. Then

$$\begin{aligned}(\mu_1 \times \mu_2)(x_1,y_1) &= \min\{\mu_1(x_1), \mu_2(y_1)\} \\ &\geq \min\left\{\min\left\{\inf_{u\in[(x_1\circledast x_3)\circledast x_3]\circledast(x_2\circledast x_3)} \mu_1(u), \mu_1(x_2)\right\},\right. \\ &\qquad \left.\min\left\{\inf_{v\in[(y_1\circledast y_3)\circledast y_3]\circledast(y_2\circledast y_3)} \mu_2(v), \mu_2(y_2)\right\}\right\} \\ &= \min\left\{\min\left\{\inf_{u\in[(x_1\circledast x_3)\circledast x_3]\circledast(x_2\circledast x_3)} \mu_1(u), \inf_{v\in[(y_1\circledast y_3)\circledast y_3]\circledast(y_2\circledast y_3)} \mu_2(v)\right\},\right. \\ &\qquad \left.\min\{\mu_1(x_2), \mu_2(y_2)\}\right\}.\end{aligned}$$

It follows that

$$\begin{aligned}(\mu_1 \times \mu_2)(x_1, y_1) &= \min\{\min\{\inf\{\mu_1(u), \mu_2(v)\}\}, (\mu_1 \times \mu_2)(x_2, y_2)\} \\ &= \min\{\inf\{\min\{\mu_1(u), \mu_2(v)\}\}, (\mu_1 \times \mu_2)(x_2, y_2)\} \\ &= \min\{\inf\{(\mu_1 \times \mu_2)(u, v)\}, (\mu_1 \times \mu_2)(x_2, y_2)\},\end{aligned}$$

for $u \in [(x_1 \circledast x_3) \circledast x_3] \circledast (x_2 \circledast x_3)$ and $v \in [(y_1 \circledast y_3) \circledast y_3] \circledast (y_2 \circledast y_3)$. By Definition 2.5 and Lemma 2.1,

$$\begin{aligned}(u, v) \in &\{[(x_1 \circledast x_3) \circledast x_3] \circledast (x_2 \circledast x_3)\} \times \{[(y_1 \circledast y_3) \circledast y_3] \circledast (y_2 \circledast y_3)\} \\ =&((x_1 \circledast x_3) \circledast x_3, (y_1 \circledast y_3) \circledast y_3) \circledast (x_2 \circledast x_3, y_2 \circledast y_3) \\ =&[(x_1 \circledast x_3, y_1 \circledast y_3) \circledast (x_3, y_3)] \circledast [(x_2, y_2) \circledast (x_3, y_3)] \\ =&\{[(x_1, y_1) \circledast (x_3, y_3)] \circledast (x_3, y_3)\} \circledast [(x_2, y_2) \circledast (x_3, y_3)].\end{aligned}$$

Hence, $(\mu_1\times\mu_2)(x_1, y_1) \geq \min\left\{\inf_{(u,v)\in\Omega}(\mu_1 \times \mu_2)(u, v), (\mu_1 \times \mu_2)(x_2, y_2)\right\}$ where $\Omega = \{[(x_1, y_1) \circledast (x_3, y_3)] \circledast (x_3, y_3)\} \circledast [(x_2, y_2) \circledast (x_3, y_3)]$. Therefore, $\mu_1 \times \mu_2$ is a fuzzy distributive hyper GR-ideal of type 2 in $H \times H$.

Theorem 5.5. *Let μ be a fuzzy set in a hyper GR-algebra H and λ_μ the strongest fuzzy relation on H. Then μ is a fuzzy distributive hyper GR-ideal of type 2 in H if and only if λ_μ is a fuzzy distributive hyper GR-ideal of type 2 in $H \times H$.*

Proof of Theorem 5.5. Let μ be a fuzzy distributive hyper GR-ideal of type 2 in H. Suppose $(x_1, x_2) \ll (y_1, y_2)$. Then $x_1 \ll y_1$ and $x_2 \ll y_2$. Since μ is a fuzzy

distributive hyper GR-ideal of type 2 in H, $\mu(y_1) \leq \mu(x_1)$ and $\mu(y_2) \leq \mu(x_2)$. Then $\lambda_\mu(y_1, y_2) = \min\{\mu(y_1), \mu(y_2)\} \leq \min\{\mu(x_1), \mu(x_2)\} = \lambda_\mu(x_1, x_2)$. Let $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in H \times H$. Then,

$$\begin{aligned}
\lambda_\mu(x_1, x_2) &= \min\{\mu(x_1), \mu(x_2)\} \\
&\geq \min\left\{ \min\left\{ \inf_{u \in [(x_1 \circledast z_1) \circledast z_1] \circledast (y_1 \circledast z_1)} \mu(u), \mu(y_1) \right\}, \right. \\
&\qquad \left. \min\left\{ \inf_{v \in [(x_2 \circledast z_2) \circledast z_2] \circledast (y_2 \circledast z_2)} \mu(v), \mu(y_2) \right\} \right\} \\
&= \min\left\{ \min\left\{ \inf_{u \in [(x_1 \circledast z_1) \circledast z_1] \circledast (y_1 \circledast z_1)} \mu(u), \inf_{v \in [(x_2 \circledast z_2) \circledast z_2] \circledast (y_2 \circledast z_2)} \mu(v) \right\}, \right. \\
&\qquad \left. \min\{\mu(y_1), \mu(y_2)\} \right\}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\lambda_\mu(x_1, x_2) &= \min\{\min\{\inf\{\mu(u), \mu(v)\}\}, \lambda_\mu(y_1, y_2)\} \\
&= \min\{\inf\{\min\{\mu(u), \mu(v)\}\}, \lambda_\mu(y_1, y_2)\} \\
&= \min\{\inf \lambda_\mu(u, v), \lambda_\mu(y_1, y_2)\}
\end{aligned}$$

for $u \in [(x_1 \circledast z_1) \circledast z_1] \circledast (y_1 \circledast z_1)$, and $v \in [(x_2 \circledast z_2) \circledast z_2] \circledast (y_2 \circledast z_2)$. By Definition 2.5 and Lemma 2.1,

$$\begin{aligned}
(u, v) \in &\{[(x_1 \circledast z_1) \circledast z_1] \circledast (y_1 \circledast z_1)\} \times \{[(x_2 \circledast z_2) \circledast z_2] \circledast (y_2 \circledast z_2)\} \\
=&([(x_1 \circledast z_1) \circledast z_1], [(x_2 \circledast z_2) \circledast z_2]) \circledast (y_1 \circledast z_1, y_2 \circledast z_2) \\
=&[(x_1 \circledast z_1, x_2 \circledast z_2) \circledast (z_1, z_2)] \circledast [(y_1, y_2) \circledast (z_1, z_2)] \\
=&\{[(x_1, x_2) \circledast (z_1, z_2)] \circledast (z_1, z_2)\} \circledast ((y_1, y_2) \circledast (z_1, z_2)).
\end{aligned}$$

Thus, $\lambda_\mu(x_1, x_2) \geq \min\left\{\inf_{(u,v) \in \omega} \lambda_\mu(u, v), \lambda_\mu(y_1, y_2)\right\}$ where $\omega = \{[(x_1, x_2) \circledast (z_1, z_2)] \circledast (z_1, z_2)\} \circledast ((y_1, y_2) \circledast (z_1, z_2))$. Therefore, λ_μ is a fuzzy distributive hyper GR-ideal of type 2 in $H \times H$.

Conversely, let λ_μ be a fuzzy distributive hyper GR-ideal of type 2 in $H \times H$. Let $x, y \in H$ such that $x \ll y$. Then $(x, x) \ll (y, y)$. It implies that

$$\mu(x) = \min\{\mu(x), \mu(x)\} = \lambda_\mu(x, x) \geq \lambda_\mu(y, y) = \min\{\mu(y), \mu(y)\} = \mu(y).$$

Moreover, let $x, y, z \in H$. Then

$$\begin{aligned}
\min\{\mu(x), \mu(0)\} &= \lambda_\mu(x, 0) \\
&\geq \min\left\{ \inf_{(u,v) \in [((x,0) \circledast (z,0)) \circledast (z,0)] \circledast ((y,0) \circledast (z,0))} \lambda_\mu(u, v), \lambda_\mu(y, 0) \right\} \\
&= \min\left\{ \inf_{(u,v) \in ([(x \circledast z) \circledast z] \circledast (y \circledast z), [(0 \circledast 0) \circledast 0] \circledast (0 \circledast 0))} \lambda_\mu(u, v), \lambda_\mu(y, 0) \right\}
\end{aligned}$$

$$
\begin{aligned}
&= \min\{\inf \lambda_\mu(u,v), \lambda_\mu(y,0)\} \\
&= \min\{\inf\{\min\{\mu(u),\mu(v)\}\}, \min\{\mu(y),\mu(0)\}\} \\
&= \min\{\min\{\inf\{\mu(u),\mu(v)\}\}, \min\{\mu(y),\mu(0)\}\} \\
&= \min\{\min\{\inf \mu(u), inf\mu(v)\}, \min\{\mu(y),\mu(0)\}\} \\
&= \min\left\{ \min\left\{ \inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)} \mu(u), \mu(y) \right\}, \right. \\
&\qquad \left. \min\left\{ \inf_{v\in[(0\circledast 0)\circledast 0]\circledast(0\circledast 0))} \mu(v), \mu(0) \right\} \right\}.
\end{aligned}
$$

Thus, $\mu(x) \geq \min\left\{\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)} \mu(u), \mu(y)\right\}$ and so μ is a fuzzy distributive hyper GR-ideal of type 2 in H.

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Product of TBCH-algebras and TBCH-algebras involving ideals

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Abstract. A BCH-algebra $(H, *, 0)$ equipped with a topology τ on H (also called a BCH-topology on H) is called a topological BCH-algebra (or TBCH-algebra) if the operation $* : H \times H \to H$, defined by $*((x, y)) = x * y$ for any $x, y \in H$, is continuous, where the Cartesian product topology on $H \times H$ is furnished by τ. In this paper, we show that given two BCH-algebras $(H_1, *_1, 0_1)$ and $(H_2, *_2, 0_2)$, an operation $*$ can be defined on the product $H = H_1 \times H_2$ so that $(H, *, 0)$, where $0 = (0_1, 0_2)$, is a BCH-algebra. Moreover, if (H_1, τ_1) and (H_2, τ_2) are TBCH-algebras, then (H, τ) is a TBCH-algebra, where τ is the product topology. We also consider in this paper TBCH-algebras involving ideals.

Keywords: BCH-algebra, topology, TBCH-algebra, product, ideals.

1. Introduction

The study of BCH-algebra, an algebraic structure that generalizes the BCI-algebra, was initiated by Hu and Li [7, 8]. In their study, the concepts of ideal and closed ideal in a BCH-algebra were defined. In 2000, Hwan, R. E. and Yu, K. S. [6] defined a new type of ideal in a BCH-algebra called a translation ideal. In 2013, Borzooei, R. A., and Zahiri, O. [5] showed that a congruence relation on

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a BCH-algebra using a translation ideal can be defined. Moreover, they defined the notion of a radical of an ideal and used it together with the concept of a closed translation ideal to construct a quotient BCH-algebra.

Jun, Y. B. et al [9] defined the concept of topological BCI-algebra (or TBCI-algebra) and investigated some of its properties which include ideals in a TBCI-algebra. In 2017, Jansi, M., and Thiruveni, V. [4] introduced the concept of topological BCH-algebra (or TBCH-algebra) and investigated some of its algebraic and topological properties. The aim of this paper is to give other structural properties of topological BCH-algebras involving product of two TBCH-algebras and ideals in a TBCH-algebra.

2. Preliminary notions and known results

Definition 2.1 ([2])**.** Let (X, τ) be a topological space and let $x \in X$. Any set $U \in \tau$ containing x is called a *neighborhood* (sometimes written as *nbhd* or τ-*nbhd*) of x.

Definition 2.2 ([7])**.** A *BCH-algebra* is a nonempty set H endowed with a operation "$*$" and constant 0 satisfying the following axioms: For all $x, y, z \in H$,

(B1) $x * x = 0$,

(B2) $x * y = 0$ and $y * x = 0$ imply $x = y$,

(B3) $(x * y) * z = (x * z) * y$.

Remark 2.1 ([3, 4])**.** In any BCH-algebra $(X, *, 0)$, the following hold:

(i) $x * 0 = x$;

(ii) $x * 0 = 0$ implies $x = 0$;

(iii) $0 * (x * y) = (0 * x) * (0 * y)$; and

(iv) $(x * (x * y)) * y = 0$.

Definition 2.3 ([4])**.** Let $(X, *, 0)$ be a BCH-algebra and U, V be any nonempty subsets of X. We define a subset $U * V$ of X by $U * V = \{x * y : x \in U, y \in V\}$.

Definition 2.4 ([3, 4])**.** A nonempty subset I of a *BCH*-algebra $(X, *, 0)$ is called an *ideal* if

(i) $0 \in I$; and

(ii) $x * y \in I$ and $y \in I$ imply $x \in I$, for all $x, y \in X$.

Definition 2.5 ([6])**.** Let X be a BCH-algebra and I an ideal of X. I is a *translation ideal* if $\forall x, y, z \in X$,

$$x * y, y * x \in I \Longrightarrow (x * z) * (y * z), (z * x) * (z * y) \in I.$$

Definition 2.6 ([4])**.** Let $(H, *, 0)$ be a BCH-algebra. A topology τ furnished on H is called a *BCH-topology* on H. In addition, (H, τ) is called a *topological BCH-algebra* (or *TBCH-algebra*) if τ is a BCH-topology on H and the function $* : H \times H \to H$ defined as $*((x, y)) = x * y$ is continuous, where the Cartesian product topology on $H \times H$ is furnished by τ.

Example 2.1. Let $X = \{0, 1, 2, 3, 4\}$ and define $*$ as follows:

$*$	0	1	2	3	4
0	0	0	0	0	4
1	1	0	0	1	4
2	2	2	0	0	4
3	3	3	3	0	4
4	4	4	4	4	0

Then, $(X, *, 0)$ is a BCH-algebra [3]. Let $\tau = \{X, \varnothing, \{4\}, \{0, 1, 2, 3\}\}$. Then τ is a BCH-topology on X. Moreover,

$$\begin{aligned} *^{-1}(X) &= X \times X \\ *^{-1}(\varnothing) &= \varnothing \\ *^{-1}(\{4\}) &= (\{0, 1, 2, 3\} \times \{4\}) \cup (\{4\} \times \{0, 1, 2, 3\}) \\ *^{-1}(\{0, 1, 2, 3\}) &= (\{0, 1, 2, 3\} \times \{0, 1, 2, 3\}) \cup (\{4\} \times \{4\}). \end{aligned}$$

This implies that $*$ is continuous. Thus, (X, τ) is a TBCH-algebra.

By continuity of $*$ in a TBCH-algebra (H, τ), the following result in [1] is easy:

Theorem 2.1. *Let τ be a BCH-topology on X. Then, (X, τ) is a TBCH-algebra if and only if for each $x, y \in X$ and each nbhd W of $x * y$, there exist nbhds U and V of x and y, respectively, such that $U * V \subseteq W$.*

3. Results

3.1 Product of two TBCH-algebras

Theorem 3.1. *Let $(H_1, *_1, 0_1)$ and $(H_2, *_2, 0_2)$ be BCH-algebras and $H = H_1 \times H_2$. Then $(H, *, 0)$ is a BCH-algebra, where $0 = (0_1, 0_2)$ and "$*$" is defined by $(x, y) * (a, b) = (x *_1 a, y *_2 b)$ for all $(x, y), (a, b) \in H$.*

Proof. Let $x = (x_1, x_2) \in H$. Then by (B1),

$$\begin{aligned} x * x &= (x_1, x_2) * (x_1, x_2) \\ &= (x_1 *_1 x_1, x_2 *_2 x_2) \\ &= (0_1, 0_2) \\ &= 0. \end{aligned}$$

Next, let $x = (x_1, x_2)$, $y = (y_1, y_2) \in H$. Suppose that $x * y = 0$ and $y * x = 0$. Then $0 = x * y = (x_1, x_2) * (y_1, y_2) = (x_1 *_1 y_1, x_2 *_2 y_2)$. Hence, $x_1 *_1 y_1 = 0_1$ and $x_2 *_2 y_2 = 0_2$. Similarly, $y_1 *_1 x_1 = 0_1$ and $y_2 *_2 x_2 = 0_2$. Since $(H_1, *_1, 0_1)$ and $(H_2, *_2, 0_2)$ are BCH-algebras, $x_1 = y_1$ and $x_2 = y_2$ by (B2). Thus, $x = (x_1, x_2) = (y_1, y_2) = y$.

Finally, let $x = (x_1, x_2)$, $y = (y_1, y_2)$, $z = (z_1, z_2) \in H$. Then by (B3),

$$\begin{aligned}
(x * y) * z &= [(x_1, x_2) * (y_1, y_2)] * (z_1, z_2) \\
&= (x_1 *_1 y_1, x_2 *_2 y_2) * (z_1, z_2) \\
&= ((x_1 *_1 y_1) *_1 z_1, (x_2 *_2 y_2) *_2 z_2) \\
&= ((x_1 *_1 z_1) *_1 y_1, (x_2 *_2 z_2) *_2 y_2) \\
&= (x_1 *_1 z_1, x_2 *_2 z_2) * (y_1, y_2) \\
&= [(x_1, x_2) * (z_1, z_2)] * (y_1, y_2) \\
&= (x * z) * y.
\end{aligned}$$

Therefore, $(H, *, 0)$ is a BCH-algebra by Definition 2.2. □

Lemma 3.1. *Let $(H_1, *_1)$ and $(H_2, *_2)$ be BCH-algebras and let $(H, *)$ be the product of H_1 and H_2 defined in Theorem* 3.1. *If $V_1, U_1 \subseteq H_1$, $V_2, U_2 \subseteq H_2$, $W_1 = V_1 \times V_2$ and $W_2 = U_1 \times U_2$, then $W_1 * W_2 = (V_1 *_1 U_1) \times (V_2 *_2 U_2)$.*

Proof. Let $(x, y) \in W_1$ and $(a, b) \in W_2$. Then $x *_1 a \in V_1 *_1 U_1$ and $y *_2 b \in V_2 *_2 U_2$. Hence, $(x, y) * (a, b) = (x *_1 a, y *_2 b) \in (V_1 *_1 U_1) \times (V_2 *_2 U_2)$ by the definition of $*$ in Theorem 3.1. Hence, $W_1 * W_2 \subseteq (V_1 *_1 U_1) \times (V_2 *_2 U_2)$.

Next, let $a_1 \in V_1$, $a_2 \in U_1$, $b_1 \in V_2$, and $b_2 \in U_2$. Then $(a_1, b_1) \in W_1$ and $(a_2, b_2) \in W_2$. Thus, by the definition of $*$ in Theorem 3.1, $(a_1 *_1 a_2, b_1 *_2 b_2) = (a_1, b_1) * (a_2, b_2) \in W_1 * W_2$. Therefore, $(V_1 *_1 U_1) \times (V_2 *_2 U_2) \subseteq W_1 * W_2$. This proves the desired equality. □

Theorem 3.2. *Let (H_1, τ_1) and (H_2, τ_2) be TBCH-algebras. Then (H, τ), where $H = H_1 \times H_2$ and τ is the product topology, is a TBCH-algebra.*

Proof. Let $(x, y), (a, b) \in H$ and let W be a τ-nbhd of $(x, y) * (a, b) = (x *_1 a, y *_2 b)$. Let $U \in \tau_1$ and $V \in \tau_2$ such that $(x *_1 a, y *_2 b) \in U \times V \subseteq W$. Since (H_1, τ_1) is a TBCH-algebra, there exist τ_1-nbhds U_x and U_a of x and a, respectively, such that $U_x *_1 U_a \subseteq U$ by Theorem 2.1. Similarly, since (H_2, τ_2) is a TBCH-algebra, there exist τ_2-nbhds V_y and V_b of y and b, respectively, such that $V_y *_2 V_b \subseteq V$. Clearly, $U_x \times V_y$ and $U_a \times V_b$ are τ-nbhds of (x, y) and (a, b), respectively. Moreover, by Lemma 3.1,

$$(U_x \times V_y) * (U_a \times V_b) = (U_x *_1 U_a) \times (V_y *_1 V_b) \subseteq U \times V \subseteq W.$$

Therefore, by Theorem 2.1, (H, τ) is a TBCH-algebra. □

3.2 Ideals in a TBCH-algebra

We denote a TBCH-algebra (X, τ) by X, unless otherwise specified.

Remark 3.1. An ideal of a TBCH-algebra may neither be an open set nor a closed set.

Example 3.1. Consider the TBCH-algebra $X = \{0, 1, 2, 3, 4\}$ with topology $\tau = \{X, \varnothing, \{4\}, \{0, 1, 2, 3\}\}$ in Example 2.1. Consider the trivial ideal $I = \{0\}$ of X. Observe that $I \notin \tau$ and $X \setminus I = \{1, 2, 3, 4\} \notin \tau$, that is, I is neither an open nor closed set in (X, τ).

The next theorem asserts that if an ideal is an open set in a TBCH-algebra, then it is also a closed set.

Theorem 3.3. *Let X be a TBCH-algebra and let A be an ideal of X. If A is an open set in X, then A is also a closed set.*

Proof. Suppose A is an open set and let $x \in X \setminus A$. Since A is an ideal of X, $x * x = 0 \in A$ by (B1) and Definition 2.4(i). By Theorem 2.1, there exist nbhds U and V of x such that $U * V \subseteq A$. Suppose $V \cap A \neq \varnothing$, say $y \in V \cap A$. Then $x * y \in U * V \subseteq A$. Since $y \in A$ and A is an ideal, $x \in A$, contrary to our assumption. Thus $V \subseteq X \setminus A$, showing that $X \setminus A$ is open. Therefore, A is a closed set in X. □

Theorem 3.4. *Let X be a TBCH-algebra and A an ideal of X. Then A is open in X if and only if 0 is an interior point of A.*

Proof. Suppose 0 is an interior point of A. Then, there exists nbhd U of 0 such that $U \subseteq A$. Let $x \in A$. By (B1), $x * x = 0 \in U$. Then, there exist nbhds G and H of x such that $G * H \subseteq U \subseteq A$ by Theorem 2.1. Let $y \in G$. Then, $y * x \in G * H \subseteq A$. Since A is an ideal and $x \in A$, $y \in A$. Hence, $x \in G \subseteq A$. Thus, A is an open set in X.
Conversely, suppose A is an open set in X. Then, $A = int(A)$. Since A is an ideal, $0 \in A = int(A)$. Therefore, 0 is an interior point of A. □

Example 3.2. Consider the *TBCH*-algebra $X = \{0, 1, 2, 3, 4\}$ with topology $\tau = \{X, \varnothing, \{4\}, \{0, 1, 2, 3\}\}$ in Example 2.1. Clearly, $\{4\}$ is an open set which is not an ideal of X. Moreover, $0 \notin int(\{4\}) = \{4\}$.

Remark 3.2. Not every open set in a *TBCH*-algebra X is an ideal of X.

The next theorem says that if every nonempty open set in a TBCH-algebra contains 0, then every nonempty open set is an ideal.

Theorem 3.5. *Let X be a TBCH-algebra such that $0 \in \bigcap_{U \in \tau \setminus \{\varnothing\}} U$. Then every nonempty open set I in X is an ideal.*

Proof. Let I be a nonempty open set in X. Then $0 \in I$ by hypothesis. Let $x, y \in X$ such that $y \in I$ and $x * y \in I$. Since I is open, there exist nbhds U and V of x and y, respectively, such that $U * V \subseteq I$ by Theorem 2.1. By the hypothesis and Remark 2.1(i), $x = x * 0 \in U * V \subseteq I$. Therefore, I is an ideal of X. □

Lemma 3.2. *Let X be a TBCH-algebra. Suppose there exists $\varnothing \neq I \subseteq X$ such that $I \subseteq U$ for each $U \in \tau$ with $0 \in U$. Then $I \subseteq V$ for any nbhd V of a point in I.*

Proof. Let V be a nbhd of some point $x \in I$. By Remark 2.1(i), $x * 0 = x \in V$. By Theorem 2.1, there exist nbhds U and W of x and 0, respectively, such that $U * W \subseteq V$. By (B1) and the hypothesis, $0 = x * x \in U * I \subseteq U * W \subseteq V$. It follows that V is an open set containing 0. Therefore, $I \subseteq V$. □

Theorem 3.6. *Let (X, τ) be a TBCH-algebra. If $I \in \tau$ and $0 \in I \subseteq U$ for each $U \in \tau$ with $0 \in U$, then I is an ideal of X.*

Proof. Let $x, y \in X$. Suppose $x * y \in I$ and $y \in I$. By Theorem 2.1, there exist nbhds U and V of x and y, respectively, such that $U * V \subseteq I$. By Lemma 3.2, $I \subseteq V$. By Remark 2.1 (i), $x = x * 0 \in U * I \subseteq U * V \subseteq I$. Therefore, I is an ideal of X. □

The following example shows that the converse of Theorem 3.6 is not necessarily true.

Example 3.3. Consider the *TBCH*-algebra $X = \{0, 1, 2, 3, 4\}$ with topology $\tau = \{X, \varnothing, \{4\}, \{0, 1, 2, 3\}\}$ in Example 2.1. Clearly, the trivial ideal $\{0\}$ is not an open set in X.

With some additional assumption, the converse of Theorem 3.3 holds.

Theorem 3.7. *Let X be a TBCH-algebra and let A be an ideal of X. Suppose there exists a nonempty open set I such that $0 \in I \subseteq U$ for each $U \in \tau$ with $0 \in U$. If A is a closed set, then it is also an open set.*

Proof. Suppose A is a closed set in X. Note that since A is an ideal of X, $0 \in A$. Assume that A is not an open set in X. By Theorem 3.4, 0 is not an interior point of A. This implies that for all nbhd U of 0, $U \nsubseteq A$. In particular, $I \nsubseteq A$. Hence, $(X \setminus A) \cap I \neq \varnothing$. Let $x \in (X \setminus A) \cap I$. Observe that $(X \setminus A) \cap I$ is an open set containing x. By Lemma 3.2, $I \subseteq (X \setminus A) \cap I \subseteq X \setminus A$. It follows that $0 \in X \setminus A$, a contradiction. Therefore, A is an open set in X. □

Corollary 3.1. *Let X be a finite TBCH-algebra and let A be an ideal of X. Then A is open if and only if it is closed.*

Proof. Suppose that A is open. Then A is closed by Theorem 3.3.

For the converse, suppose that A is closed. Let I be the intersection of all the nbhds of 0. Then I satisfies the property given in Theorem 3.7. Thus, by Theorem 3.7, A is open. □

Theorem 3.8. *Let X be a BCH-algebra and let I be a translation ideal of X. Then the relation $\cong_I$, defined on X by, $x \cong_I y$ if and only if $x*y, y*x \in I$, is a congruence relation on X. Moreover, if $x, y, z, t \in X$ and $x \cong_I y$ and $z \cong_I t$, then $z*x \cong_I t*y$.*

Proof. Let $x \in X$. By (B1) and the fact that I is an ideal of X, $x*x = 0 \in I$. Hence, $x \cong_I x$. Let $x, y \in X$ and suppose that $x \cong_I y$. Then $x*y$, $y*x \in I$. Thus, $y \cong_I x$. Now, if $x, y, z \in X$ and $x \cong_I y$ and $y \cong_I z$, then $x*y$, $y*x$, $y*z$, $z*y \in I$. Since I is a translation ideal, $(x*z)*(y*z)$, $(z*x)*(z*y) \in I$. It follows that $x*z$, $z*x \in I$ because I is an ideal of X. Hence, $x \cong_I z$. Therefore, $\cong_I$ is an equivalence relation on X.

Next, let $x, y, z, t \in X$ and suppose that $x \cong_I y$ and $z \cong_I t$. Then $x*y$, $y*x$, $z*t$, $t*z \in I$. Since I is a translation ideal, $(x*z)*(y*z), (z*x)*(z*y), (y*z)*(x*z), (z*y)*(z*x), (z*y)*(t*y), (y*z)*(y*t), (t*y)*(z*y), (y*t)*(y*z) \in I$. Using $(y*t)$ and the fact that I is a translation ideal and $(x*z)*(y*z), (y*z)*(x*z) \in I$, we have $((x*z)*(y*t))*((y*z)*(y*t)) \in I$. Hence, $(x*z)*(y*t) \in I$ because $(y*z)*(y*t) \in I$ and I is an ideal of X. Similarly, $(y*t)*(x*z) \in I$. Thus, $x*z \cong_I y*t$. Using a similar argument, it can be shown that $z*x \cong_I t*y$. Accordingly, $\cong_I$ is a congruence relation on X. □

Let X be a BCH-algebra and I a translation ideal of X. Consider the congruence relation $\cong_I$ on X defined in Theorem 3.8. The equivalence class of $x \in X$ with respect to $\cong_I$ will be denoted by I_x, that is $I_x = \{y \in X : x \cong_I y\} = \{y \in X : x*y, y*x \in I\}$.

Theorem 3.9. *Let $\mathcal{I}$ be the collection of all translation ideals in a BCH-algebra X. Then (X, τ) is a TBCH-algebra, where*

$$\tau = \{U \subseteq X : \forall x \in U, \exists I \in \mathcal{I} \text{ such that } I_x \subseteq U\}.$$

Proof. Note that for each $x \in X$ and for each $I \in \mathcal{I}$, $I_x \subseteq X$. Hence, $X \in \tau$. By definition of τ, $\varnothing \in \tau$. Let $U_1, U_2 \in \tau$. Suppose $y \in U_1 \cap U_2$. Then $y \in U_1$ and $y \in U_2$. Thus, exist $A, B \in \mathcal{I}$ such that $A_y \subseteq U_1$ and $B_y \subseteq U_2$. Let $I = A \cap B$. Let $x, y, z \in X$ and suppose that $x*y, y*x \in I$. Then $x*y, y*x \in A$ and $x*y, y*x \in B$. Since A and B are translation ideals, $(x*z)*(y*z), (z*x)*(z*y) \in A$ and $(x*z)*(y*z), (z*x)*(z*y) \in B$. Hence, $(x*z)*(y*z), (z*x)*(z*y) \in A \cap B = I$. It follows that $I \in \mathcal{I}$.

Let $x \in I_y$. Then $y*x, x*y \in I \subseteq A$. Hence, $y \cong_A x$, that is, $x \in A_y$. Similarly, $x \in B_y$. It follows that $I_y \subseteq A_y \cap B_y \subseteq U_1 \cap U_2$. This shows that $U_1 \cap U_2 \in \tau$.

Let $\{U_\alpha : \alpha \in \mathcal{A}\} \subseteq \tau$ and let $y \in \bigcup_{\alpha \in \mathcal{A}} U_\alpha$. Then $y \in U_\beta$ for some $\beta \in \mathcal{A}$. This implies that there exists $I \in \mathcal{I}$ such that $I_y \subseteq U_\beta \subseteq \bigcup_{\alpha \in \mathcal{A}} U_\alpha$. Hence, $\bigcup_{\alpha \in \mathcal{A}} U_\alpha \in \tau$. Therefore, τ is a *BCH*-topology on X.

Next, let $I \in \mathcal{I}$ and let $x \in X$. Then clearly, $I_x \in \tau$. Now, let $x, y \in X$ and let U be a nbhd of $x*y$. Then there exists $I \in \mathcal{I}$ such that $I_{x*y} \subseteq U$. By

Theorem 3.8, $\cong_I$ is a congruence relation on X and so

$$\begin{aligned} I_x * I_y &= \{a * b \in X : a \in I_x \text{ and } b \in I_y\} \\ &= \{a * b \in X : x \cong_I a \text{ and } y \cong_I b\} \\ &= \{a * b \in X : x * y \cong_I a * b\} \\ &\subseteq \{z \in X : x * y \cong_I z\} \\ &= I_{x*y}. \end{aligned}$$

Thus, I_x and I_y are τ-nbhds of x and y, respectively and $I_x * I_y \subseteq I_{x*y} \subseteq U$. By Theorem 2.1, $(X, *, \tau)$ is a TBCH-algebra. □

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Weakly semi ω-continuous functions

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Abstract. In this paper we will use the semi ω-open sets to introduce a new type of weaker form of continuous function, namely weakly semi ω-continuous. Then we discuss various properties of this form of continuous functions. Finally, we study some separation axioms by using the semi ω-open subsets and we obtain some theorems linking the concepts of $s\omega$-strongly closed graph, $s\omega$-compact spaces and $s\omega$-connected with weakly semi ω-continuous functions.

Keywords: semi-open, ω-open, semi ω-open, weakly semi ω-continuous.

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1. Introduction

Throughout this work a space will always mean a topological space in which no separation axioms is assumed unless explicitly stated. If E is a subset of a space (S,τ) then the closure of E, the interior of E and the relative topology on E in (S,τ) will be denoted by $Cl(E)$, $Int(E)$ and τ_E, respectively. A space (S,τ) is called anti-locally countable [6] if each non-empty open subset of (S,τ) is uncountable.

The point $s \in S$ is called a condensation point of E [8] if for each $G \in \tau$ with $s \in G$, the set $G \cap E$ is uncountable. The set E is called ω-closed if it contains all its condensation points [9]. The complement of an ω-closed set is called ω-open or equivalently E is ω-open [6] if for each $s \in E$, there exists an open set G containing s such that $G - E$ is countable. The family of all ω-open subsets of a space (S,τ), denoted by τ_ω, forms a topology on S finer that τ. The ω-closure of a subset E of a space (S,τ) is the closure of E in the space (S,τ_ω) and it is denoted by $Cl_\omega(E)$. Many researchers have invested this type of open sets to present many works as [1, 2, 12].

A subset E of a space (S,τ) is called semi open if there exists an open set G such that $G \subseteq E \subseteq Cl(G)$. In [5], Al-Zoubi and Al-Nashef introduced the class of semi ω-open subsets of a topological space (S,τ) by generalizing ω-open subsets in the same way that the semi-open sets has generalized open sets, where a subset E of a space (S,τ) is called semi ω-open if there exists an ω-open set G such that $G \subseteq E \subseteq Cl(G)$. The family of all semi ω-open subsets of $(S,\tau)(resp.$, the family of all semi ω-open subsets of (S,τ) containing $s)$ will be denoted by $S\omega O(S,\tau)(resp., S\omega O(S,s))$. The semi ω-closure of a subset E of a space (S,τ) is the intersection of all semi ω-closed subsets of (S,τ) containing E and denoted by $Cl_{s\omega}(E)$. The union of all semi ω-open subsets of (S,τ) which is subset of E is the semi ω-interior of E and denoted by $Int_{s\omega}(E)$.

In this paper, we use the class of semi ω-open subsets to introduce and study a generalization of continuity which called weakly semi ω-continuous. In sections 2 and 3 of this work, we present the definition of weakly semi ω-continuous and investigate some characterizations of this type of functions and study some of there basic properties. Finally in section 4, also we present some separation axioms by using the semi ω-open subsets then we introduce the concept of $s\omega$-strongly closed graph, $s\omega$-compact and $s\omega$-connected and some theorems linking these concepts with weakly semi ω-continuous will be shown.

In this paper $\mathbb{R}, \mathbb{Q}$ and $\mathbb{N}$ denote, respectively the set of real numbers, the set of rational numbers and the set of natural numbers.

Now, we begin with some notations, definitions, and result will be used in this work.

Theorem 1.1 ([5]). *A subset E of a space (S,τ) is semi ω-open iff $E \subseteq Cl(Int_\omega(E))$.*

Lemma 1.1 ([5]). *If H is an open subset of a space (S,τ) and $E \in S\omega O(S,\tau)$, then $H \cap E \in S\omega O(S,\tau)$.*

Theorem 1.2 ([5]). *Let E be a subset of a space (S,τ). Then $Int_{s\omega}(E) = E \cap Cl(Int_{\omega}(E))$.*

Lemma 1.2 ([5]). *Let (Z,τ_Z) be a subspace of a space (S,τ) and $E \subseteq Z$. Then:*
(i) $E \in S\omega O(Z,\tau_Z)$ if $E \in S\omega O(S,\tau)$.
(ii) $E \in S\omega O(S,\tau)$ if $Z \in \omega O(S,\tau)$ and $E \in S\omega O(Z,\tau_Z)$.

Corollary 1.1 ([5]). *Let (S,τ) be a space and let $E \subseteq S$. Then:*
(i) $Cl_{s\omega}(S-E) = S - Int_{s\omega}(E)$.
(ii) $Int_{s\omega}(S-E) = S - Cl_{s\omega}(E)$.

Definition 1.1 ([10]). *A function $\psi : (S,\tau) \to (Z,\sigma)$ is said to be weakly continuous if for each point $s \in S$ and open set H in (Z,σ) containing $\psi(s)$, there is an open set G in (S,τ) with $s \in G$ and $\psi(G) \subseteq Cl(H)$.*

Definition 1.2 ([3]). *A function $\psi : (S,\tau) \to (Z,\sigma)$ is called semi ω-continuous, if for every open subset H of Z, $\psi^{-1}(H) \in S\omega O(S)$.*

Lemma 1.3 ([3]). *Let $\psi : (S,\tau) \to (Z,\sigma)$ be an open continuous surjection. If $G \in S\omega O(S,\tau)$, then $\psi(G) \in S\omega O(Z,\sigma)$.*

Definition 1.3 ([13]). *A space (S,τ) is called a Urysohn space iff whenever $s \neq t$ in S, there are two non-empty open set G and H in (S,τ) with $s \in G$, $t \in H$ and $Cl(G) \cap Cl(H) = \phi$.*

Definition 1.4 ([7]). *A topological space (S,τ) is said to be almost compact if whenever $\mathcal{G} = \{G_\alpha : \alpha \in I\}$ is open cover of (S,τ) there is a finite subsets $I_\circ$ of I with $S = \bigcup_{\alpha \in I_\circ} Cl(G_\alpha)$.*

2. Weakly semi ω-continuous

In this section we will define the class of weakly semi ω-continuous functions and establish some properties of this class.

Definition 2.1. *A function $\psi : (S,\tau) \to (Z,\sigma)$ is said to be weakly semi ω-continuous if for each $s \in S$ and for each open set H in (Z,σ) containing $\psi(s)$ there is a set $G \in S\omega O(S,s)$ with $\psi(G) \subseteq Cl(H)$.*

Observe that if S is a countable set then every function $\psi : (S,\tau) \to (Z,\sigma)$ is weakly semi ω-continuous. The following diagram follows immediately from the definitions and the fact that every open set is semi ω-open.

$$\begin{array}{ccc} \text{Continuous} & \to & \text{Weakly continuous} \\ \downarrow & & \downarrow \\ \text{Semi } \omega\text{-continuous} & \to & \text{Weakly Semi } \omega\text{-continuous} \end{array}$$

The following examples show that the converses in the above diagram need not be true in general.

Example 2.1. Let $S = \mathbb{R}$ with the topology $\tau = \tau_u$ and let $Z = \{0, 1\}$ with the topology $\sigma = \{\phi, Z, \{1\}\}$. Let $\psi : (S, \tau) \to (Z, \sigma)$ be defined by $\psi(s) = \begin{cases} 1 & , s \in \mathbb{Q} \\ 0 & , s \in \mathbb{R} - \mathbb{Q} \end{cases}$. It is clear that ψ is weakly semi ω-continuous. However ψ is not semi ω-continuous since $\psi^{-1}(\{1\}) = \mathbb{Q} \notin S\omega O(\mathbb{R}, \tau_u)$.

Example 2.2. Let $S = \{0, 1, 2\}$ with the topology $\tau = \{\phi, S, \{0\}, \{1, 2\}\}$. Let $\psi : (S, \tau) \to (S, \tau)$ be the function defined by $\psi(s) = \begin{cases} 0 & , s = 0, 1 \\ 2 & , s = 2 \end{cases}$. Then ψ is weakly semi ω-continuous but it is not weakly continuous, since $\psi(1) = 0 \in \{0\}$ which is an open set in (S, τ) with $Cl(\{0\}) = \{0\}$ and there is no open set G in (S, τ) contains 1 with $\psi(G) \subseteq Cl(\{0\}) = \{0\}$.

In the following two theorems we give some characterizations of weakly semi ω-continuous functions.

Theorem 2.1. *The following are equivalent for a function $\psi : (S, \tau) \to (Z, \sigma)$:*

(i) ψ is weakly semi ω-continuous.

(ii) for each $s \in S$ and open set H in (Z, σ) containing $\psi(s)$, we have $s \in Cl(Int_\omega(\psi^{-1}(Cl(H))))$.

(iii) $\psi^{-1}(H) \subseteq Int_{s\omega}(\psi^{-1}(Cl(H)))$ for each open set H in (Z, σ).

Proof. *(i $\to$ ii)* Let $s \in S$ and H be an open set in (Z, σ) containing $\psi(s)$. Then there is $G \in S\omega O(S, s)$ with $G \subseteq \psi^{-1}(Cl(H))$. Since $G \in S\omega O(S, s)$ then $s \in G \subseteq Cl(Int_\omega(G)) \subseteq Cl(Int_\omega(\psi^{-1}(Cl(H))))$.

(ii $\to$ iii) Let H be an open set in (Z,σ) containing $\psi(s)$. Since $s \in Cl(Int_\omega(\psi^{-1}(Cl(H))))\cap\psi^{-1}(Cl(H))$ and by Theorem 1.2, $s \in Int_{s\omega}(\psi^{-1}(Cl(H))$.

(iii $\to$ i) Let $s \in S$ and H be open set in (Z, σ) containing $\psi(s)$. Put $G = Int_{s\omega}(\psi^{-1}(Cl(H))$. Then $G \in S\omega O(S, s)$ with $\psi(G) \subseteq Cl(H)$. □

Theorem 2.2. *The following are equivalent for a function $\psi : (S, \tau) \to (Z, \sigma)$:*

(i) ψ is weakly semi ω-continuous.

(ii) $Cl_{s\omega}(\psi^{-1}(Int(Cl(E)))) \subseteq \psi^{-1}(Cl(E))$ for any subset E of Z.

(iii) $Cl_{s\omega}(\psi^{-1}(Int(F))) \subseteq \psi^{-1}(F)$ for every closed subset F of Z.

(iv) $Cl_{s\omega}(\psi^{-1}(H)) \subseteq \psi^{-1}(Cl(H))$ for every open subset H of Z.

Proof. *(i $\to$ ii)* Let E be a subset of Z. Suppose that $s \in S - \psi^{-1}(Cl(E))$. Then there is an open set H in (Z, σ) with $\psi(s) \in H$ and $H \cap E = \phi$, so $Cl(H) \cap Int(Cl(E)) = \phi$. Since ψ is weakly semi ω-continuous, then there is $G \in S\omega O(S, s)$ with $\psi(G) \subseteq Cl(H)$ hence $G \cap \psi^{-1}(Int(Cl(E))) = \phi$. Thus $Cl_{s\omega}(\psi^{-1}(Int(Cl(E)))) \subseteq S - G$ and so $s \in S - Cl_{s\omega}(\psi^{-1}(Int(Cl(E))))$.

(ii $\to$ iii) The proof is clear since $Cl(F) = F$.

(iii $\to$ iv) Let H be an open in Z. Since $H \subseteq Int(Cl(H))$, then $Cl_{s\omega}(\psi^{-1}(H)) \subseteq Cl_{s\omega}(\psi^{-1}(Int(Cl(H)))) \subseteq \psi^{-1}(Cl(H))$.

(iv $\to$ i) Let H be an open set in Z containing $\psi(s)$. Then, by Corollary 1.1, $S - Int_{s\omega}(\psi^{-1}(Cl(H))) = Cl_{s\omega}(S - \psi^{-1}(Cl(H))) = Cl_{s\omega}(\psi^{-1}(Z - Cl(H))) \subseteq$

$\psi^{-1}(Cl(Z - Cl(H))) \subseteq S - \psi^{-1}(H)$. Thus $\psi^{-1}(H) \subseteq Int_{s\omega}(\psi^{-1}(Cl(H)))$ for each H open set in Z containing $\psi(s)$. Hence, by Theorem 2.1, ψ is weakly semi ω-continuous. □

3. Some properties of weakly semi ω-continuous

In this section we obtain some properties of weakly semi ω-continuous functions. The composition $\varphi \circ \psi : (S, \tau) \to (W, \theta)$ of a continuous function $\psi : (S, \tau) \to (Z, \sigma)$ and a weakly semi ω-continuous function $\varphi : (Z, \sigma) \to (W, \theta)$ is not necessarily weakly semi ω-continuous as the following example shows. Thus, the composition of weakly semi ω-continuous functions need not be weakly semi ω-continuous.

Example 3.1. Let $S = \mathbb{R}$ with the topology $\tau = \tau_u$ and let $Z = \{0, 1, 2\}$ with the topologies $\sigma = \tau_{ind}$ and $\theta = \{\phi, Z, \{1\}, \{0, 2\}\}$. Let $\psi(s) : (\mathbb{R}, \tau_u) \to (Z, \tau_{ind})$ be the function defined by $\psi(s) = \begin{cases} 1 & , s \in \mathbb{Q} \\ 0 & , s \in \mathbb{R} - \mathbb{Q} \end{cases}$ and $\varphi(s) : (Z, \tau_{ind}) \to (W, \theta)$ be the identity function. Then ψ is continuous and φ is weakly semi ω-continuous but $(\varphi \circ \psi)(s)$ is not weakly semi ω-continuous. Note that if $s \in \mathbb{Q}$, then $(\varphi \circ \psi)(s) = 1 \in H = \{1\}$. Suppose that there is $G \in S\omega O(\mathbb{R}, \tau_u)$ with $s \in G$ and $(\varphi \circ \psi)(G) \subseteq Cl(H) = \{1\}$. Then $G \subseteq \mathbb{Q}$, i.e. G is countable, a contradiction, since whenever G is a countable subset of $(\mathbb{R}, \tau_u)$, $Int_\omega(G) = \phi$. Therefore, $\varphi \circ \psi$ is not weakly semi ω-continuous.

The proof of the following result follows immediately from the definitions.

Theorem 3.1. *Let $\psi : (S, \tau) \to (Z, \sigma)$ and $\varphi : (Z, \sigma) \to (W, \theta)$ be two functions. Then:*

(i) $\varphi \circ \psi$ is weakly semi ω-continuous if φ is weakly continuous and ψ is semi ω-continuous.

(ii) $\varphi \circ \psi$ is weakly semi ω-continuous if φ is continuous and ψ is weakly semi ω-continuous.

Note that Theorem 3.1 is not true if φ is assumed to be only semi ω-continuous as it is shown in the following example.

Example 3.2. Let $Z = \{0, 1, 2\}$ with the topology $\sigma = \{\phi, Z, \{0\}, \{0, 1\}\}$, and let $W = \{a, b\}$ with the topology $\theta = \{\phi, W, \{a\}, \{b\}\}$. Let $\psi : (\mathbb{R}, \tau_u) \to (Z, \sigma)$ be the function defied by $\psi(s) = \begin{cases} 2 & , s \in \mathbb{Q} \\ 1 & , s \in \mathbb{R} - \mathbb{Q} \end{cases}$ and $\varphi : (Z, \sigma) \to (W, \theta)$ be defined as follows $\varphi(0) = \varphi(2) = a$ and $\varphi(1) = b$. Then ψ is weakly semi ω-continuous function and φ is ω-continuous function but $\varphi \circ \psi$ is not weakly semi ω-continuous at $s \in \mathbb{Q}$.

Corollary 3.1. *If $\psi : (S, \tau) \to \prod_{\alpha \in \Delta} S_\alpha$ is a weakly semi ω-continuous function then for each $\alpha \in \Delta$, $p_\alpha \circ \psi$ is weakly semi ω-continuous function, where p_α is the projection function from the product space $\prod_{\alpha \in \Delta} S_\alpha$ onto the space S_α.*

Theorem 3.2. *Let $\psi : (S, \tau) \to (Z, \sigma)$ be an open continuous surjection function. If $\varphi : (Z, \sigma) \to (W, \theta)$ is a function with $\varphi \circ \psi : (S, \tau) \to (W, \theta)$ is weakly semi ω-continuous, then φ is weakly semi ω-continuous.*

Proof. Let $t \in Z$ and $H \in \theta$ with $\varphi(t) \in H$. Choose $s \in S$ with $\psi(s) = t$. Since $\varphi \circ \psi$ is weakly semi ω-continuous, there is $G \in S\omega O(S, \tau)$ with $s \in G$ and $\varphi(\psi(G)) \subseteq Cl(H)$. But ψ is an open continuous surjection, therefore by Lemma 1.3, $\psi(G) \in S\omega O(Z, \sigma)$ with $\psi(s) \in \psi(G)$ and the result follows. □

Corollary 3.2. *Let $\psi_\alpha : (S_\alpha, \tau_\alpha) \to (Z_\alpha, \sigma_\alpha)$ be a function for each $\alpha \in \Delta$. If the product function $\psi = \prod_{\alpha\in\Delta} \psi_\alpha : \prod_{\alpha\in\Delta} S_\alpha \to \prod_{\alpha\in\Delta} Z_\alpha$ is weakly semi ω-continuous, then ψ_α is weakly semi ω-continuous for each $\alpha \in \Delta$.*

Proof. For each $\beta \in \Delta$, let $p_\beta : \prod_{\alpha\in\Delta} S_\alpha \to S_\beta$ and $q_\beta : \prod_{\alpha\in\Delta} Z_\alpha \to Z_\beta$ be the projections. Then, we have $q_\beta \circ \psi = \psi_\beta \circ p_\beta$ for each $\beta \in \Delta$. Then, by Theorems 3.2 ψ_β, is weakly semi ω-continuous. □

The following example shows that the converse is not true.

Example 3.3. Let $S = \mathbb{R}$ with the topology $\tau = \{G : G \subseteq \mathbb{Q}\} \cup \{\mathbb{R}\}$ and $Z = \{0, 1\}$ with τ_{dis}. Let $\psi : (S, \tau) \to (Z, \tau_{dis})$ be the function defined by $\psi(s) = \begin{cases} 0 & , s \in \mathbb{Q} \\ 1 & , s \in \mathbb{R} - \mathbb{Q} \end{cases}$. Note that $\mathbb{Q}$ and $\mathbb{R} - \mathbb{Q}$ are semi ω-open sets in (S, τ) and so ψ is weakly semi ω- continuous function. However, the product function $h = \psi \times \psi : (\mathbb{R} \times \mathbb{R}) \to (Z \times Z)$ defined by $h(s, t) = (\psi(s), \psi(t))$ for all $(s, t) \in (\mathbb{R} \times \mathbb{R})$ is not weakly semi ω-continuous. Let $(s, t) \in (\mathbb{R} - \mathbb{Q}) \times (\mathbb{R} - \mathbb{Q})$. Then $h(s, t) = (1, 1)$. Put $H = \{(1, 1)\}$. Then H is an open set in $((Z \times Z), (\tau_{dis} \times \tau_{dis}))$ with $h(s, t) \in H$. Now, suppose there is a semi ω-open set E in $(S \times S)$ with $h(E) \subseteq Cl(H) = H$. There is $M \in (\tau \times \tau)_\omega$ with $M \subseteq E \subseteq Cl(M)$. Then $h(M) \subseteq H$ and so $M \subseteq (\mathbb{R} - \mathbb{Q}) \times (\mathbb{R} - \mathbb{Q})$. Since $M \neq \phi$, choose $(r, t) \in M$. Note that the only open set containing (r, t) is $(\mathbb{R} \times \mathbb{R})$ therefore $(\mathbb{R} \times \mathbb{R}) - M$ is countable. Thus $(\mathbb{R} \times \mathbb{Q}) \cup (\mathbb{R} \times \mathbb{Q}) = (\mathbb{R} \times \mathbb{R}) - ((\mathbb{R} - \mathbb{Q}) \times (\mathbb{R} - \mathbb{Q})) \subseteq (\mathbb{R} \times \mathbb{R}) - M$, a contradiction.

Theorem 3.3. *Let $\psi : (S, \tau) \to (Z, \sigma)$ be a mapping and let $\varphi : (S, \tau) \to ((S \times Z), (\tau \times \sigma))$ be the graph mapping of ψ given by $\varphi(s) = (s, \psi(s))$ for every point $s \in S$. Then $\varphi : S \to (S \times Z)$ is weakly semi ω-continuous iff ψ is weakly semi ω-continuous.*

Proof. Note that $\psi = p_Z \circ \varphi$, where $p_Z : S \times Z \to Z$ is the projection function. Now, assume that φ is weakly semi ω-continuous. Then, by Theorem 3.1, ψ is weakly semi ω-continuous.

Conversely, assume that ψ is semi ω-continuous. Let $s \in S$ and H be any open set in $(S \times Z)$ containing $\varphi(s)$. Then there are open sets $F \subseteq S$ and $B \subseteq Z$ with $\varphi(s) = (s, \psi(s)) \in F \times B \subseteq H$. Since ψ is weakly semi ω-continuous, there is $M \in S\omega O(S, \tau)$ with $s \in M$ and $\psi(M) \subseteq Cl(B)$. Put $G = F \cap M$. Then

$G \in S\omega O(S,\tau)$, by Lemma 1.1, such that $s \in G$ and $\psi(G) \subseteq Cl(B)$. Therefore, we have $\varphi(G) \subseteq G \times \psi(G) \subseteq F \times Cl(B) \subseteq Cl(F) \times Cl(B) \subseteq Cl(H)$. □

Theorem 3.4. *(i) Let $\psi : (S,\tau) \to (Z,\sigma)$ be any function. If ψ is weakly semi ω-continuous and E is an open subset of S, then the restriction $\psi|_E : (E,\tau_E) \to (Z,\sigma)$ is weakly semi ω-continuous.*

(ii) Let $s \in S$. If there is an ω-open subset E of S containing s with $\psi|_E : (S,\tau) \to (Z,\sigma)$ is weakly semi ω-continuous at s, then ψ is weakly semi ω-continuous at s.

Proof. (i) Let $s \in E$ and H be any open set in (Z,σ) containing $(\psi|_E)(s) = \psi(s)$. Since ψ is weakly semi ω-continuous, there is $G \in S\omega O(S,\tau)$ containing s and $\psi(G) \subseteq Cl(H)$. Put $G_E = G \cap E$. Then by Lemma 1.1, $G_E \in S\omega O(S,\tau)$ and by Lemma 1.2, $G_E \in S\omega O(E,\tau_E)$ with $s \in G_E$ and $(\psi|_E)(G_E) = \psi(G_E) = \psi(G \cap E) \subseteq \psi(G) \cap \psi(E) \subseteq Cl(H)$. This indicates that $\psi|_E$ is weakly semi ω-continuous at s.

(ii) Let H be any open set in (Z,σ) with $\psi(s) \in H$. Since $\psi|_E$ is weakly semi ω-continuous, there is $M \in S\omega O(E,\tau_E)$ with $s \in M$ and $\psi(M) = \psi|_E(M) \subseteq Cl(H)$. Since E is an ω-open in (S,τ), then by Lemma 1.2(ii), M is a semi ω-open in (S,τ) and the result follows. □

Corollary 3.3. *If $\mathcal{G} = \{G_\alpha : \alpha \in \Delta\}$ is an open cover of S, then ψ is weakly semi ω-continuous iff $\psi|_{G_\alpha}$ is weakly semi ω-continuous for every $\alpha \in \Delta$.*

Proof. Follows from Theorem 3.4. □

The following example shows that Theorem 3.4 part (i) is not true if E is ω-open subset of (S,τ).

Example 3.4. Let $S = \mathbb{R}$ with the topology $\tau = \{G \subseteq \mathbb{R} : 1 \notin G$ and $\mathbb{R} - G$ is finite$\}$. The function $\psi : (S,\tau) \to (\{0,1\},\tau_{dis})$ defined by $\psi(s) = \left\{ \begin{array}{ll} 1, & s \in \mathbb{R} - \mathbb{Q} \\ 0, & s \in \mathbb{Q} \end{array} \right\}$ is semi ω-continuous (Example 2.2 of [3]) and so it is weakly semi ω-continuous. Put $E = (\mathbb{R} - \mathbb{Q}) \cup \{1\}$. Then E is ω-open in (S,τ) but it is not open. Now, the resection function $\psi|_E : (E,\tau_E) \to (\{0,1\},\tau_{dis})$ is not weakly semi ω-continuous at $s = 1$ since $\{1\} \notin S\omega O(S,\tau)$.

4. Some separation axioms

In this section we study some separation axioms by using the semi ω-open subsets then we introduce the concept of $s\omega$-strongly closed graph, $s\omega$-compact spaces and $s\omega$-connected and then we obtain some theorems linking these concepts with weakly semi ω-continuous functions.

Definition 4.1. *A topological space (S,τ) is said to be:*

(i) $s\omega - T_1$ if for each pair of distinct points s and t there are $G \in S\omega O(S,s)$, $O \in S\omega O(S,t)$ with $s \notin O$ and $t \notin G$.

(ii) $s\omega-T_2$ *if for each pair of distinct points* s *and* t *there are* $G \in S\omega O(S,s)$, $O \in S\omega O(S,t)$ *with* $G \cap O = \phi$.

In the following definition we will present the definition of $s\omega$-strongly closed graph, where the graph of the function $\psi : (S,\tau) \to (Z,\sigma)$ is the subset $L_\psi = \{(s,\psi(s)) : s \in S\}$ of the product space $(S \times Z)$.

Definition 4.2. *A function* $\psi : (S,\tau) \to (Z,\sigma)$ *is said to have a* $s\omega$*-strongly closed graph if for each* $(s,t) \in (S \times Z) - L_\psi$, *there are* $G \in S\omega O(S)$ *and open set* $H \in \sigma$ *containing* t *with* $(s,t) \in (G \times H)$ *and* $(G \times Cl(H)) \cap L_\psi = \phi$.

The following lemma follows immediately from above definition

Lemma 4.1. *A function* $\psi : (S,\tau) \to (Z,\sigma)$ *has a* $s\omega$*-strongly closed graph iff for each point* $(s,t) \in (S \times Z) - L_\psi$ *there are* $G \in S\omega O(S,s)$ *and open set* $H \in \sigma$ *containing* t *with* $\psi(G) \cap Cl(H) = \phi$.

Theorem 4.1. *Let* $\psi : (S,\tau) \to (Z,\sigma)$ *be a function having a* $s\omega$*-strongly closed graph. Then:*

(i) (S,τ) *is* $s\omega - T_1$ *if* ψ *is injective.*

(ii) (Z,σ) *is* T_2 *if* ψ *is surjective.*

Proof. (i) Let $s, m \in S$ be two distinct points. Since ψ has a $s\omega$-strongly closed graph there is $G \in S\omega O(S,s)$ and open set $H \in \sigma$ containing $\psi(m)$ with $\psi(G) \cap Cl(H) = \phi$. Therefore, $m \notin G$ and hence (S,τ) is $s\omega - T_1$.

(ii) Let $t, z \in Z$ be two distinct points. Since ψ is surjective there is $s \in S$ with $\psi(s) = z$. Since ψ has a $s\omega$-strongly closed graph then there are $G \in S\omega O(S,s)$ and open set $H \in \sigma$ containing t with $\psi(G) \cap Cl(H) = \phi$. Therefore, $z \in \psi(G) \subseteq Z - Cl(H)$. □

Theorem 4.2. *Let* $\psi : (S,\tau) \to (Z,\sigma)$ *be a weakly semi* ω*-continuous injective function. Then:*

(i) (S,τ) *is* $s\omega - T_2$ *if* (Z,σ) *is Urysohn.*

(ii) (S,τ) *is* $s\omega - T_1$ *if* (Z,σ) *is* T_2.

Proof. (i) Let $s, m \in S$ be two distinct points. Since $\psi(s) \neq \psi(m)$ then there are open sets $H, M \in \sigma$ with $\psi(s) \in H$, $\psi(m) \in M$ and $Cl(H) \cap Cl(M) = \phi$. Since ψ is weakly semi ω-continuous, there are $G \in S\omega O(S,s), O \in S\omega O(S,m)$ with $\psi(G) \subseteq Cl(H)$ and $\psi(O) \subseteq Cl(M)$. As $\psi^{-1}(Cl(H)) \cap \psi^{-1}(Cl(M)) = \phi$, then G and O are the requested sets. Therefore, (S,τ) is $s\omega - T_2$.

(ii) Let $s, m \in S$ be two distinct points. Since $\psi(s) \neq \psi(m)$ then there are open sets $H, M \in \sigma$ with $\psi(s) \in H$, $\psi(m) \in M$ and $\psi(m) \notin Cl(H)$, $\psi(s) \notin Cl(M)$. Since ψ is weakly semi ω-continuous, there are $G \in S\omega O(S,s), O \in S\omega O(S,m)$ with $\psi(G) \subseteq Cl(H)$ and $\psi(O) \subseteq Cl(M)$. Since $m \notin G$ and $s \notin O$ so (S,τ) is $s\omega - T_1$. □

Theorem 4.3. *Let* $\psi : (S,\tau) \to (Z,\sigma)$ *be a weakly semi* ω*-continuous. If* (Z,σ) *is a Urysohn space, then* $\psi : (S,\tau) \to (Z,\sigma)$ *has a* $s\omega$*-strongly closed graph.*

Proof. Let $(s,t) \in (S \times Z) - L_\psi$. Then there are open sets M_1,$M_2 \in \sigma$ with $t \in M_1$, $\psi(s) \in M_2$ and $Cl(M_1) \cap Cl(M_2) = \phi$. Since ψ is weakly semi ω-continuous, there is $G \in S\omega O(S,s)$ with $\psi(G) \subseteq Cl(M_1)$. It follows $\psi(G) \cap Cl(M_2) = \phi$ and so $(G \times Cl(M_2)) \cap L_\psi = \phi$. □

Theorem 4.4. *If $\psi : (S,\tau) \to (Z,\sigma)$ is weakly semi ω-continuous, $\varphi : (S,\tau) \to (Z,\sigma)$ is weakly continuous and Z is Urysohn, then the set $\{s \in S : \psi(s) = \varphi(s)\}$ is semi ω-closed in S.*

Proof. Let $E = \{s \in S : \psi(s) = \varphi(s)\}$ and $s \in S - E$. Then $\psi(s) \neq \varphi(s)$. Since Z is Urysohn, there are open sets H,$M \in \sigma$ with $\psi(s) \in H$, $\varphi(s) \in M$ and $Cl(H) \cap Cl(M) = \phi$. Since ψ is weakly semi ω-continuous there is $G \in S\omega O(S,\tau)$ containing s with $\psi(G) \subseteq Cl(H)$. Since φ is weakly continuous there is an open set $O \in \tau$ containing s with $\varphi(O) \subseteq Cl(M)$. Put $U = G \cap O$. Then $U \in S\omega O(S,\tau)$ containing s and $\psi(U) \cap \varphi(U) \subseteq Cl(H) \cap Cl(M) = \phi$. We obtain $U \cap E = \phi$ and hence E is semi ω-closed in S. □

Theorem 4.5. *Let $\psi : (S,\tau) \to (Z,\sigma)$ be weakly semi ω-continuous. If ψ is injective and has $s\omega$-strongly closed graph, then (S,τ) is $s\omega - T_2$.*

Proof. Let $s,m \in S$ be two distinct points. Then $\psi(s) \neq \psi(m)$ and $(s,\psi(m)) \notin L_\psi$. Since ψ has $s\omega$-strongly closed graph, there are $G \in S\omega O(S,s)$ and $H \in \sigma$ containing $\psi(m)$ with $\psi(G) \cap Cl(H) = \phi$. On other hand, ψ is weakly semi ω-continuous and so there is $M \in S\omega O(S,m)$ with $\psi(M) \subseteq Cl(H)$. Thus $\psi(G) \cap \psi(M) = \phi$ and hence $G \cap M = \phi$. Thus (S,τ) is $s\omega - T_2$. □

In the following definition we will present the concept of $s\omega$-compact.

Definition 4.3. *A topological space (S,τ) is said to be $s\omega$-compact if every $s\omega$-open cover of S has a finite subcover.*

It is note that every $s\omega$-compact is compact space but the converse is not true as the following example shows.

Example 4.1. The space $(\mathbb{N},\tau_{cof})$ is compact but it is not $s\omega$-compact since $\tau_\omega = \tau_{dis} \subseteq S\omega O(S)$.

Theorem 4.6. *Let $\psi : (S,\tau) \to (Z,\sigma)$ be a weakly semi ω-continuous surjective function. If (S,τ) is $s\omega$-compact, then (Z,σ) is almost compact.*

Proof. Let $\{H_\alpha : \alpha \in \Delta\}$ be an open cover of Z. For each $s \in S$ there is $\alpha(s) \in \Delta$ with $\psi(s) \in H_{\alpha(s)}$. Since ψ is weakly semi ω-continuous, there is $G_s \in S\omega O(S,s)$ with $\psi(G_s) \subseteq Cl(H_{\alpha(s)})$. Therefore, $\{G_s : s \in S\}$ is a semi ω-open cover of (S,τ) and so there is $\{s_1, s_2, ...s_n\} \subseteq S$ with $S = \bigcup_{i=1}^n G_{s_i}$. Thus $Z = \psi(S) = \psi(\bigcup_{i=1}^n G_{s_i}) \subseteq \bigcup_{i=1}^n Cl(H_{\alpha(s_i)})$ and hence (Z,σ) is almost compact. □

Definition 4.4. *A topological space (S,τ) is called to be $s\omega$-connected iff S can not be written as a union of two non-empty disjoint semi ω-open sets.*

Theorem 4.7. *If (S,τ) is a sω-connected space and $\psi : (S,\tau) \to (Z,\sigma)$ is a weakly semi ω-continuous surjection function, then (Z,σ) is connected.*

Proof. At first we show that if H is a clopen subset in (Z,σ) then $\psi^{-1}(H)$ is semi ω-clopen in (S,τ). Let H be clopen subset in (Z,σ). Then by Theorem 2.1, $\psi^{-1}(H) \subseteq Int_{s\omega}(\psi^{-1}(Cl_\sigma(H))) = Int_{s\omega}(\psi^{-1}(H))$, thus $\psi^{-1}(H) \in S\omega O(S,\tau)$.

Now, we show that $\psi^{-1}(H)$ is semi ω-closed in (S,τ). Suppose by contrary that there is $s \in Cl_{s\omega}(\psi^{-1}(H)) - \psi^{-1}(H)$. Since ψ is weakly semi ω-continuous and $(Z-H) \in \sigma$ containing $\psi(s)$, there is $G \in S\omega O(S,\tau)$ with $s \in G$ and $\psi(G) \subseteq Cl(Z-H) = Z-H$. But $s \in Cl_{s\omega}(\psi^{-1}(H))$ and so $G \cap \psi^{-1}(H) \neq \phi$. Therefore, $\phi \neq \psi(G) \cap H \subseteq (Z-H) \cap H$ a contradiction. Thus $\psi^{-1}(H)$ is semi ω-closed in (S,τ). Now, suppose that (Z,σ) is not connected. Then there are non-empty open sets H_1,$H_2 \in \sigma$ with $H_1 \cap H_2 = \phi$ and $H_1 \cup H_2 = Z$. Hence we have $\psi^{-1}(H_1) \cap \psi^{-1}(H_2) = \phi$ and $\psi^{-1}(H_1) \cup \psi^{-1}(H_2) = S$. Now, H_1 and H_2 are clopen in (Z,σ), then $\psi^{-1}(H_1)$ and $\psi^{-1}(H_2)$ are semi ω-clopen in (S,τ). This implies that (S,τ) is semi ω-disconnected, a contradiction. Therefore, (Z,σ) is connected. □

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C^*-convex sets generated by C^*-convex maps in $*$-rings and their C^*-faces

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Abstract. In this paper, we identify some C^*-convex sets generated by C^*-affine maps and C^*-convex maps on the unital $*$-rings, and we obtain the C^*-faces of such C^*-convex sets. Specially, we show that the set $Fix(f)$ of all fixed points of the C^*-affine map f on the unital $*$-ring $\mathcal{R}$, is a C^*-convex set. Moreover, we prove that for a C^*-convex map f on some $*$-rings, the set $Fix(f)$ is a C^*-face of some C^*-convex sets.

Keywords: C^*-affine map, C^*-convex set, C^*-face, $*$-ring.

1. Introduction

One of the form of non-commutative convexity is C^*-convexity. Formal study of C^*-convexity was initiated by Loebl and Paulsen in [8]. Hopenwasser, Moore, and Paulsen obtained the set of all extreme points and C^*-extreme points of some C^*-convex sets, and the distinction of these sets in [7]. Farenick and Morenz proved that each irreducible element of the C^*-algebra M_n of complex $n \times n$ matrices, is a C^*-extreme point, the relative extreme point, of the C^*-convex set that it generates [5]. Also they studied the C^*-extreme points of the C^*-convex space $S_H(A)$ of all unital completely positive linear maps from a C^*-algebra A to the algebra $B(H)$ of continuous linear operators on a complex Hilbert space H [6]. Moreover, in [9] Morenz discovered a right analogue of linear extreme points, called structural elements to prove a generalised Krein Milman theorem for C^*-convex subsets of M_n. Also he extend the notion of face from convexity to C^*-face in C^*-convexity. The author and G. H. Esslamzadeh generalised the notion of C^*-convexity and the relative extreme points, C^*-extreme points, to $*$-rings [4]. Recently, the author has investigated the C^*-extreme points of the graph and epigraph of C^*-affine maps in [2]. Also it has shawn in [2] that for a C^*-convex map f defined on a unital $*$-ring $\mathcal{R}$ with some conditions the graph of f is a C^*-face of the epigraph of f, and some other results about the C^*-faces of C^*-convex sets in $*$-rings. Moreover, the structure of the set of all C^*-convex maps and C^*-affine maps in $*$-rings has investigated in [3].

Throughout this paper, $\mathcal{R}$ is a unital $*$-ring, that is, a ring with an involution which has an identity element. An element x in $\mathcal{R}$ is called positive, written $x \geq 0$, if $x = y_1^* y_1 + y_2^* y_2 + \cdots + y_n^* y_n$ for some $y_1, y_2, \cdots, y_n$ in $\mathcal{R}$. The self-adjoint elements of $\mathcal{R}$ can be ordered by writing $x \leq y$ in case $y - x \geq 0$. Also, the interval $[x, y]$denotes the set of all elements z in $\mathcal{R}$ such that $x \leq z \leq y$. The involution of $\mathcal{R}$ is called proper if $x^* x = 0$ implies that $x = 0$ for every $x \in \mathcal{R}$. $x \in \mathcal{R}$ is called a central element of $\mathcal{R}$, if $xa = ax$ for each $a \in \mathcal{R}$. The set of all central elements of $\mathcal{R}$ is denoted by $Z(\mathcal{R})$. The reference [1] is essential for studying the $*$-rings. We denote the graph and epigraph of a map f on $\mathcal{R}$ by $graph(f)$ and $epi(f)$ respectively, which are defined as following;

$$graph(f) = \{(x, y) : x \in \mathcal{R}, y = f(x)\} \subseteq \mathcal{R} \oplus \mathcal{R},$$

$$epi(f) = \{(x, y) : x \in \mathcal{R}, f(x) \leq y\} \subseteq \mathcal{R} \oplus \mathcal{R}.$$

In the next section we state the definitions of C^*-convexity, C^*-convex map, C^*-affine map, and C^*-face in $*$-rings, and the last section of this paper is devoted to the main results of this paper.

2. Definitions and preliminaries

Definition 2.1. *A subset K of a unital $*$-ring $\mathcal{R}$ is called C^*-convex, if*

$$\sum_{i=1}^{n} a_i^* x_i a_i \in K,$$

whenever $x_i \in K$, $a_i \in \mathcal{R}$ for all i and $\sum_{i=1}^{n} a_i^* a_i = 1_{\mathcal{R}}$.

In this case the summation $\sum_{i=1}^{n} a_i^* x_i a_i$ is called a C^*-convex combination of elements $x_i \in K$.

Definition 2.2. *Let K be a C^*-convex subset of $\mathcal{R}$. An element $x \in K$ is called a C^*-extreme point of K if the condition*

$$x = \sum_{i=1}^{n} a_i^* x_i a_i, \ \sum_{i=1}^{n} a_i^* a_i = 1_{\mathcal{R}}, x_i \in K, a_i \text{ is invertible in } \mathcal{R}, n \in \mathbb{N}, \tag{1}$$

implies that all x_i are unitarily equivalent to x in $\mathcal{R}$, that is, there exist unitaries $u_i \in \mathcal{R}$ such that $x_i = u_i^* x u_i$ for all i.

The set of all C^*-extreme points of K is denoted by C^*-$ext(K)$.

In addition, if condition (1) holds, then we say that x is a proper C^*-convex combination of $x_1, \ldots, x_n$.

In [4] we defined the notion of C^*-convex maps as the following;

Definition 2.3. *Let K be a C^*-convex subset of $\mathcal{R}$. We say that a map f on K is C^*-convex if*

$$f(\sum_{i=1}^{n} a_i^* x_i a_i) \leq \sum_{i=1}^{n} a_i^* f(x_i) a_i,$$

where $n \in \mathbb{N}$, $x_i \in K$, $a_i \in \mathcal{R}$, *and* $\sum_{i=1}^{n} a_i^* a_i = 1_{\mathcal{R}}$.

If we replace the equality instead of the less or equality in this definition, we say that f *is a* C^**-affine map on* K. *Also, if* $-f$ *is* C^**-convex, we say that* f *is* C^**-concave.*

As an example $f(x) = x^*$ is a C^*-affine map on a unital $*$-ring $\mathcal{R}$. In continuation, we give some examples of C^*-convex subsets of $\mathcal{R}$.

Example 2.1. The following sets are C^*-convex sets in any unital $*$-ring $\mathcal{R}$.

(1) The set of all self adjoint elements in $\mathcal{R}$.

(2) The set of all positive elements in $\mathcal{R}$.

(3) $[0, 1_{\mathcal{R}}]$.

(4) $U(x)$ for each $x \in \mathcal{R}$, where $U(x)$ denotes the set of all elements which are unitarily equivalent to x in $\mathcal{R}$.

In the next example, we give another family of C^*-convex subsets of the unital $*$-ring $\mathcal{R}$.

Example 2.2. The following sets are C^*-convex in the unital $*$- ring $\mathcal{R}$.

(1) Every two sided ideal in $\mathcal{R}$.

(2) Every one sided $*$-ideal in $\mathcal{R}$ is a two sided ideal, and hence is a C^*-convex set.

To see this, let I be a left $*$-ideal in $\mathcal{R}$. Then $\mathcal{R}I \subseteq I$, and hence

$$I\mathcal{R} = (\mathcal{R}I)^* \subseteq I^* \subseteq I.$$

The notion of face from linear convexity has extended to C^*-face for C^*-convex subsets of a C^*-algebra in [9] by P. B. Morenz. The author and G. H. Esslamzadeh generalised this notion to $*$-rings [4].

Definition 2.4. *A nonempty subset* F *of a* C^**-convex set* $K \subseteq \mathcal{R}$ *is called a* C^**-face of* K, *if the condition* $x \in F$ *and* $x = \sum_{i=1}^{n} a_i^* x_i a_i$ *as a proper* C^**-convex combination of elements* $x_i \in K$, *implies that* $x_i \in F$ *for all* i.

Example 2.3. Let K be a C^*-convex subset of $\mathcal{R}$. Then the following sets are C^*-faces of K.

(1) The set K itself.

(2) The set $C^* - ext(K)$ of all C^*-extreme points of K, provided that it is nonempty.

3. Main results

In this section, we obtain some C^*-convex sets generated by C^*-affine maps and C^*-convex maps on the unital $*$-rings, and also we obtain C^*-faces of such C^*-convex sets.

First of all, we show that the set of all fixed points of a C^*-affine map f on a unital $*$-ring $\mathcal{R}$ is a C^*-convex subset of $\mathcal{R}$. An element $x \in \mathcal{R}$ is called a fixed point of the map f on $\mathcal{R}$ if $f(x) = x$. We denote the set of all fixed points of f by $Fix(f)$.

Proposition 3.1. *Let f be a C^*-affine map on a unital $*$-ring $\mathcal{R}$. Then, the set $Fix(f)$ of all fixed points of f is a C^*-convex subset of $\mathcal{R}$. Moreover, $Fix(f)$ is closed under the unitary orbit of its elements.*

Proof. Suppose that $x_i \in \mathcal{R}$ and $f(x_i) = x_i$ for each i $(1 \leq i \leq n)$, and $a_i \in \mathcal{R}$ such that $\sum_{i=1}^n a_i^* a_i = 1_{\mathcal{R}}$, then

$$f(\sum_{i=1}^n a_i^* x_i a_i) = \sum_{i=1}^n a_i^* f(x_i) a_i = \sum_{i=1}^n a_i^* x_i a_i.$$

So, $\sum_{i=1}^n a_i^* x_i a_i$ is a fixed point of f, and hence $Fix(f)$ is a C^*-convex subset of $\mathcal{R}$. Since, every C^*-convex set is closed under the unitary orbit of its elements, so for every $x \in Fix(f)$, we have $U(x) \subset Fix(f)$, i.e. every element in the unitary orbit of x, is a fixed point of f. □

Note that, if we consider the C^*-affine map $f(x) = x^*$ on $\mathcal{R}$, then $Fix(f)$ is the set $\mathcal{R}_{sa}$ of all self-adjoint elements in $\mathcal{R}$, which is a C^*-convex subset of $\mathcal{R}$. It is shown in [4] that for every C^*-convex map f on the unital $*$-algebra $\mathcal{A}$ the sets $\{x : f(x) \leq x\}$, and $\{x : f(x) \leq \alpha 1_{\mathcal{A}}\}$ are C^*-convex sets. In this section we extend this result as follows;

Theorem 3.1. *Suppose that f is a C^*-convex map on a unital $*$-ring $\mathcal{R}$, and $y \in \mathcal{R}$ such that $C^* - co(\{y\}) = \{y\}$ (specially $y \in Z(\mathcal{R})$). Then the set*

$$K = \{x \in \mathcal{R} \mid f(x) \leq y\},$$

is a C^*-convex set in $\mathcal{R}$, provided that it is nonempty.

Proof. Let $x_i \in K$ for each i, $(1 \leq i \leq n)$, and $\sum_{i=1}^n a_i^* x_i a_i$ be a C^*-convex combination of elements $x_i \in K$. Then $f(x_i) \leq y$, and hence $a_i^* f(x_i) a_i \leq a_i^* y a_i$ for each i, $(1 \leq i \leq n)$. Thus, by the C^*-convexity of f and using the assumption $C^* - co(\{y\}) = \{y\}$, we conclude that

$$f(\sum_{i=1}^n a_i^* x_i a_i) \leq \sum_{i=1}^n a_i^* f(x_i) a_i \leq \sum_{i=1}^n a_i^* y a_i = y.$$

Hence, $\sum_{i=1}^n a_i^* x_i a_i \in K$. Therefore, K is a C^*-convex subset of $\mathcal{R}$. □

In the next theorems we will use the following notation for every C^*-affine map $f : \mathcal{R} \longrightarrow \mathcal{R}$ and every nonempty subset B of $\mathcal{R}$.

$$f^{-1}(B) = \{x \in \mathcal{R} \mid f(x) \in B\}.$$

Theorem 3.2. *Let $f : \mathcal{R} \longrightarrow \mathcal{R}$ be a C^*-affine map on a unital $*$-ring $\mathcal{R}$. Then:*

1) *$f(K)$ is a C^*-convex subset of $\mathcal{R}$, for every C^*-convex set K in $\mathcal{R}$.*

2) *$f^{-1}(B)$ is a C^*-convex subset of $\mathcal{R}$, for every C^*-convex set B in $\mathcal{R}$.*

3) *If f be a bijective C^*-affine map, then K is a C^*-convex subset of $\mathcal{R}$ iff $f(K)$ is a C^*-convex set.*

Proof. We only prove the first part. The proof of the second part is similar, and 3) is a consequence of the first two parts.

Let K be a C^*-convex subset of $\mathcal{R}$, $y_i \in f(K)$ $(1 \leq i \leq n)$, and $\sum_{i=1}^n a_i^* y_i a_i$ be a C^*-convex combination of elements $y_i \in f(K)$. Then, there exists $x_i \in K$ such that $f(x_i) = y_i$ for each i $(1 \leq i \leq n)$. K is a C^*-convex set, so $\sum_{i=1}^n a_i^* x_i a_i \in K$. Since f is a C^*-affine map on $\mathcal{R}$, we have

$$\sum_{i=1}^n a_i^* y_i a_i = \sum_{i=1}^n a_i^* f(x_i) a_i = f(\sum_{i=1}^n a_i^* x_i a_i) \in f(K). \qquad \square$$

Theorem 3.3. *Suppose that f is a C^*-affine map on a unital $*$-ring $\mathcal{R}$, and $x, y \in \mathcal{R}$ such that $C^* - co(\{x\}) = \{x\}$, and $C^* - co(\{y\}) = \{y\}$ (specially $x, y \in Z(\mathcal{R})$). Then the following sets are C^*-convex sets in $\mathcal{R}$, provided that they are nonempty.*

1) $[0, y]$ *where* $y \geq 0$.
2) $[x, y]$ *where* $x \leq y$.
3) $f^{-1}(\{y\})$ *, (specially* $f^{-1}(\{0\})$*, the kernel of* f*).*
4) $f^{-1}([0, y])$ *where* $y \geq 0$.
5) $f^{-1}([x, y])$ *where* $x \leq y$.

Proof. 2) Let $\sum_{i=1}^n a_i^* x_i a_i$ be a C^*-convex combination of elements $x_i \in [x, y]$. Then $x \leq x_i \leq y$ for each i $(1 \leq i \leq n)$. Hence,

$$a_i^* x a_i \leq a_i^* x_i a_i \leq a_i^* y a_i.$$

So,

$$\sum_{i=1}^n a_i^* x a_i \leq \sum_{i=1}^n a_i^* x_i a_i \leq \sum_{i=1}^n a_i^* y a_i.$$

By the assumptions $C^* - co(\{y\}) = \{y\}$, and $C^* - co(\{x\}) = \{x\}$, we have

$$x \leq \sum_{i=1}^n a_i^* x_i a_i \leq y,$$

and hence $\sum_{i=1}^n a_i^* x_i a_i \in [x, y]$. Therefore, $[x, y]$ is a C^*-convex set.

The proof of 1) is similar to the proof of 2). The other parts are the conclusions of the first two parts by applying the above theorem. $\square$

Corollary 3.1. *Let f be a C^*-affine map on a unital $*$-ring $\mathcal{R}$, and $m, n \in \mathbb{N}$ such that $m \leq n$. Then the following sets are C^*-convex sets in $\mathcal{R}$, provided that they are nonempty.*

1) $f^{-1}(n1_{\mathcal{R}})$.
2) $f^{-1}([0, 1_{\mathcal{R}}])$.
3) $[m1_{\mathcal{R}}, n1_{\mathcal{R}}]$.
4) $f^{-1}([m1_{\mathcal{R}}, n1_{\mathcal{R}}])$.

Recently, in [2] the author has proved that the graph of a C^*-convex map on a unital $*$-ring $\mathcal{R}$ is a C^*-face of its epigraph ([2], Theorem 3.4) as following:

Theorem 3.4. *Let $\mathcal{R}$ be a unital $*$-ring satisfying the positive square root axiom, and $x_1^*x_1 + x_2^*x_2 + \cdots + x_n^*x_n = 0$ implies that $x_1 = x_2 = \cdots = x_n = 0$ for every $x_i \in \mathcal{R}$ and $n \in \mathbb{N}$ and f be a C^*-convex map on $\mathcal{R}$. Then, $graph(f)$ is a C^*-face of $epi(f)$.*

In the next theorem, we show that the set of all fixed points of the C^*-convex map f, is a C^*-face of the set $\{x \in \mathcal{R} | f(x) \leq x\}$.

Theorem 3.5. *Let f be a C^*-convex map on a unital $*$-ring $\mathcal{R}$ such that $x_1^*x_1 + x_2^*x_2 + \cdots + x_n^*x_n = 0$ implies that $x_1 = x_2 = \cdots = x_n = 0$ for every $x_i \in \mathcal{R}$ and $n \in \mathbb{N}$. Then, the set $Fix(f) = \{x \in \mathcal{R} | f(x) = x\}$ is a C^*-face of the set $K = \{x \in \mathcal{R} | f(x) \leq x\}$, provided that it is nonempty.*

Proof. Let $x \in Fix(f)$ and $x = \sum_{i=1}^n a_i^* x_i a_i$ be a proper C^*-convex combination of elements $x_i \in K$. We must show that $x_i \in Fix(f)$ for each i $(1 \leq i \leq n)$. Since

$$x = f(x) = f(\sum_{i=1}^n a_i^* x_i a_i) \leq \sum_{i=1}^n a_i^* f(x_i) a_i \leq \sum_{i=1}^n a_i^* x_i a_i = x,$$

so $\sum_{i=1}^n a_i^* x_i a_i = \sum_{i=1}^n a_i^* f(x_i) a_i$, and hence, $\sum_{i=1}^n a_i^*(x_i - f(x_i))a_i = 0$.

Now, since $x_i - f(x_i) \geq 0$ and using the assumption

$$x_1^*x_1 + x_2^*x_2 + \cdots + x_n^*x_n = 0 \implies x_1 = x_2 = \cdots = x_n = 0,$$

for every $x_i \in \mathcal{R}$ and $n \in \mathbb{N}$, we conclude that

$$a_i^*(x_i - f(x_i))a_i = 0,$$

for each i $(1 \leq i \leq n)$. Thus, $x_i - f(x_i) = 0$ for each i by the invertibility of a_i. Therefore, $x_i = f(x_i)$ and $x_i \in Fix(f)$ for each i $(1 \leq i \leq n)$. Hence, $Fix(f)$ is a C^*-face of the C^*-convex set K. □

Corollary 3.2. *Let $\mathcal{R}$ be a unital $*$-ring such that $x_1^*x_1 + x_2^*x_2 + \cdots + x_n^*x_n = 0$ implies that $x_1 = x_2 = \cdots = x_n = 0$, for every $x_i \in \mathcal{R}$ and $n \in \mathbb{N}$. Then:*

1) similarly, the above theorem holds for every C^*-concave map f, and the set $K = \{x \in \mathcal{R} | f(x) \geq x\}$.

2) if f be a C^*-affine map on $\mathcal{R}$, then $Fix(f)$ is a C^*-face of the sets $K_1 = \{x \in \mathcal{R} | f(x) \leq x\}$, and $K_2 = \{x \in \mathcal{R} | f(x) \geq x\}$, provided that they are nonempty.

Theorem 3.6. *Let f be a C^*-convex map on a unital $*$-ring $\mathcal{R}$ such that $x_1^*x_1 + x_2^*x_2 + \cdots + x_n^*x_n = 0$ implies that $x_1 = x_2 = \cdots = x_n = 0$ for every $x_i \in \mathcal{R}$, $n \in \mathbb{N}$, and $y \in \mathcal{R}$ such that $C^* - co(\{y\}) = \{y\}$ (specially $y \in Z(\mathcal{R})$, the center of $\mathcal{R}$). Then $F = \{x \in \mathcal{R} | f(x) = y\}$ is a C^*-face of the C^*-convex set $K = \{x \in \mathcal{R} | f(x) \leq y\}$, provided that it is nonempty.*

Proof. The proof is similar to the proof of the above theorem. □

Corollary 3.3. *If f be a C^*-convex map on a unital $*$-ring $\mathcal{R}$ such that $x_1^*x_1 + x_2^*x_2 + \cdots + x_n^*x_n = 0$ implies that $x_1 = x_2 = \cdots = x_n = 0$ for every $x_i \in \mathcal{R}$ and $n \in \mathbb{N}$. Then $F = \{x \in \mathcal{R} | f(x) = n1_{\mathcal{R}}\}$ is a C^*-face of the C^*-convex set $K = \{x \in \mathcal{R} | f(x) \leq n1_{\mathcal{R}}\}$, provided that F be nonempty. Also, the same conclusion holds for C^*-concave map f and the C^*-convex set $K = \{x \in \mathcal{R} | f(x) \geq n1_{\mathcal{R}}\}$.*

Corollary 3.4. *If f be a C^*-convex map on a unital $*$-algebra $\mathcal{A}$ such that $x_1^*x_1 + x_2^*x_2 + \cdots + x_n^*x_n = 0$ implies that $x_1 = x_2 = \cdots = x_n = 0$, for every $x_i \in \mathcal{A}$, $n \in \mathbb{N}$, and $\alpha \in \mathbb{C}$. Then, $F = \{x \in \mathcal{A} | f(x) = \alpha 1_{\mathcal{A}}\}$ is a C^*-face of the C^*-convex set $K = \{x \in \mathcal{A} | f(x) \leq \alpha 1_{\mathcal{A}}\}$, provided that F be nonempty. Also, the same conclusion holds for C^*-concave map f and the C^*-convex set $K = \{x \in \mathcal{A} | f(x) \geq \alpha 1_{\mathcal{A}}\}$.*

Note that, since every C^*-affine map is also a C^*-convex map, so the above theorem and its corollaries hold for every C^*-affine maps.

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Cubic fuzzy $\beta-$ideals of $\beta-$algebras

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Abstract. This paper extends fuzzy $\beta-$ideal into cubic fuzzy $\beta-$ideal of a $\beta-$algebra. Further, some related results using Cartesian product and homomorphism are also studied.

Keywords: $\beta-$algebra, $\beta-$ideals, fuzzy $\beta-$ideal, cubic ideals, cubic $\beta-$ideal.

1. Introduction

Fuzzy sets have been introduced by Zadeh [12, 13] in 1965 to manipulate data and information possessing non statistical uncertainties. He also discussed about the concept of a linguistic variable which presents a means of approximate characterization of phenomena which are too complicated in conventional quantitative terms. Atanassov [2] initiated Intuitionistic fuzzy sets as an extension of fuzzy set whose components have degrees of membership and non-membership. Neggers and Kim [5] presented the thought of $\beta-$algebras as a generalization of $BCK-$algebras and $BCI-$algebras. Abu Ayub Ansari and

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chandramouleeswaran [1] proposed the notion of fuzzy $\beta-$ideals of $\beta-$algebras. In [3, 4], Hemavathi et al. described the concept of interval valued fuzzy $\beta-$subalgebras also they have extended the thought of $\beta-$ideals in interval valued fuzzy set. In 2012, the thought of cubic sets has been initiated by Jun et al. [10]. The notion of cubic subalgebras and ideals of $BCK/BCI-$ algebras was described by Jun et al. [8, 9]. Also they have applied the cubic structures in to ideals of $BCI-$algebras. Senapathi et al. [7] originated the perception of cubic implicative ideal of $BCK-$algebras. The notion of cubic $BRK-$ideal of $BRK-$algebra was initiated by Osama [6] in 2015. Recently Muralikrishna et al. [11] described some aspects on cubic fuzzy $\beta-$subalgebras of $\beta-$algebra. With all these inspiration, this paper provides the study of cubic fuzzy $\beta-$ideals of $\beta-$algebras and discusses some fascinating results.

2. Preliminaries

This section reveals the necessary definitions required for the work.

Definition 2.1. *A $\beta-$ algebra is a non-empty set X with a constant 0 and two binary operations $+$ and $-$ are satisfying the following axioms:*

(i) $x - 0 = x$;

(ii) $(0 - x) + x = 0$;

(iii) $(x - y) - z = x - (z + y)$, $\forall\ x, y, z \in X$.

Example 2.1. The following Cayley table shows $(X = \{0, 1, 2, 3\}, +, -, 0)$ is a $\beta-$algebra.

$+$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	1	0
3	3	2	0	1

$-$	0	1	2	3
0	0	1	3	2
1	1	0	2	3
2	2	3	0	1
3	3	2	1	0

Example 2.2. Consider Set of all integers Z. $(Z, +, -, 0)$ is an infinite $\beta-$algebra where 0, $+$ and $-$ have usual meanings.

Definition 2.2. *A non-empty subset I of a $\beta-$algebra $(X, +, -, 0)$ is called a $\beta-$ideal of X, if*

(i) $0 \in I$;

(ii) $x + y \in I$;

(iii) $x - y,\ y \in I$ *then* $x \in I, \forall x, y \in X$.

Example 2.3. Consider the $\beta-$algebra $(X, +, -, 0)$ in the following Cayley's table

$+$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

$-$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

The subset $I_1 = \{0, 1\}$ is a $\beta-$ideal of X.

Definition 2.3. *An interval valued fuzzy set (briefly i-v fuzzy set) A defined on a non empty set X is given by A=$\{(x, [\zeta_A^L(x), \zeta_A^U(x)])\}, \forall\ x \in X$ (briefly denoted by $A = [\zeta_A^L, \zeta_A^U]$), where ζ_A^L and ζ_A^U are two fuzzy sets in X such that $\zeta_A^L(x) \leq \zeta_A^U(x), \forall\ x \in X$. Let $\overline{\zeta}_A(x) = [\zeta_A^L(x), \zeta_A^U(x)], \forall\ x \in X$ and let $D[0, 1]$ denotes the family of all closed sub intervals of $[0, 1]$. If $\zeta_A^L(x) = \zeta_A^U(x) = c$, say, where $0 \leq c \leq 1$, then we have which we also assume, for the sake of convenience, to belong to $D[0, 1]$. Thus, $\overline{\zeta}_A(x) \in D[0, 1], \forall x \in X$, and $\overline{\zeta}_A(x) \in D[0, 1], \forall\ x \in X$, and therefore the i-v fuzzy set A is given by $A = \{(x, \overline{\zeta}_A(x))\}, \forall\ x \in X$, where $\overline{\zeta}_A : X \to D[0, 1]$. Now, let us define what is known as $refined$ $mimimum$ (briefly $r\min$) of two elements in $D[0, 1]$. We also define the symbols " $\geq$ ", " $\leq$ ", and " $=$ " in case of two elements in $D[0, 1]$. Consider two elements $D_1 := [a_1, b_1]$ and $D_2 := [a_2, b_2] \in D[0, 1]$. Then, we have $r\min(D_1, D_2) = [\min\{a_1, a_2\}, \min\{b_1, b_2\}]$; $D_1 \geq D_2$ if and only if $a_1 \geq a_2,\ b_1 \geq b_2$; Similarly we may have $D_1 \leq D_2$ and $D_1 = D_2$.*

Definition 2.4. *Let ζ be a fuzzy set in a $\beta-$ algebra. Then ζ is called a fuzzy $\beta-$ideal of X, if:*

(*i*) $\zeta(0) \geq \zeta(x)$;

(*ii*) $\zeta(x + y) \geq \min\{\zeta(x), \zeta(y)\}$;

(*iii*) $\zeta(x) \geq \min\{\zeta(x - y), \zeta(y)\}, \forall x, y \in X$.

Definition 2.5. *Let $A = \{\langle x, \overline{\zeta}_A(x)\rangle : x \in X\}$ be an i-v fuzzy set in a $\beta-$algebra of X. Then A is called an interval valued fuzzy $\beta-$ ideal of X if:*

(*i*) $\overline{\zeta}_A(0) \geq \overline{\zeta}_A(x), \forall x \in X$;

(*ii*) $\overline{\zeta}_A(x + y) \geq r\min\{\overline{\zeta}_A(x), \overline{\zeta}_A(y)\}, \forall x, y \in X$;

(*iii*) $\overline{\zeta}_A(x) \geq r\min\{\overline{\zeta}_A(x - y), \overline{\zeta}_A(y)\}, \forall x, y \in X$.

Definition 2.6. *Let X be a non empty set. By a cubic set in X we mean a structure $C = \{\langle x, \overline{\zeta}_C(x), \eta_C(x)\rangle :\ x \in\ X\}$ in which $\overline{\zeta}_C$ is an interval valued fuzzy set in X and η_C is a fuzzy set in X.*

Definition 2.7. *Let $C = \{\langle x, \overline{\zeta}_C(x), \eta_C(x)\rangle : x \in X\}$ be a cubic fuzzy (CF) set in a non empty set X. Then Then, the set C is a cubic fuzzy $\beta-$ subalgebra if it satisfies the following conditions*

(*i*) $\overline{\zeta}_C(x+y) \geq r\min\{\overline{\zeta}_C(x), \overline{\zeta}_C(y)\}$ & $\overline{\zeta}_C(x-y) \geq r\min\{\overline{\zeta}_C(x), \overline{\zeta}_C(y)\}$;

(*ii*) $\eta_C(x+y) \leq \max\{\eta_C(x), \eta_C(y)\}$ & $\eta_C(x-y) \leq \max\{\eta_C(x), \eta_C(y)\}$, $\forall\ x, y \in X$.

3. Cubic fuzzy $\beta-$ ideals

This section starts with the definitions of cubic fuzzy $\beta-$ideals of $\beta-$algebras. Also throughout the paper, X is a $\beta-$algebra unless and otherwise specified.

Definition 3.1. *Let $C = \{x, \overline{\zeta}_C(x), \eta_C(x) : x \in X\}$ be a cubic fuzzy set in a $\beta-$algebra of X. C is called a cubic fuzzy $\beta-$ideal of X, if $\forall x, y \in X$:*

(*i*) $\overline{\zeta}_C(0) \geq \overline{\zeta}_C(x)$ & $\eta_C(0) \leq \eta_C(x)$;

(*ii*) $\overline{\zeta}_C(x+y) \geq r\min\{\overline{\zeta}_C(x), \overline{\zeta}_C(y)\} \& \eta_C(x+y) \leq \max\{\eta_C(x), \eta_C(y)\}$;

(*iii*) $\overline{\zeta}_C(x) \geq r\min\{\overline{\zeta}_C(x-y), \overline{\zeta}_C(y)\}$ & $\eta_C(x) \leq \max\{\eta_C(x-y), \eta_C(y)\}$.

Example 3.1. Let $X = \{0, a, b, c\}$ be a $\beta-$algebra with constant 0 and binary operations $+$ and $-$ are defined on X as in the following Cayley's table

$+$	0	a	b	c
0	0	a	b	c
a	a	b	c	0
b	b	c	0	a
c	c	0	a	b

$-$	0	a	b	c
0	0	c	b	a
a	a	0	c	b
b	b	a	0	c
c	c	b	a	0

The Cubic fuzzy set C in X defined as

$$\overline{\zeta}_C = \begin{cases} [0.4, 0.5] : & x = 0 \\ [0.3, 0.4] : & x = b \\ [0.2, 0.3] : & x = a, c \end{cases} \qquad \eta_C = \begin{cases} 0.5 : & x = a, b \\ 0.4 : & x = c \\ 0.2 : & x = 0 \end{cases}$$

Then C is a Cubic fuzzy $\beta-$ideal of X .

Theorem 3.1. *Let C be a Cubic fuzzy $\beta-$ ideal of a $\beta-$algebra X. If $x \leq y$ then $\overline{\zeta}_C(x) \geq \overline{\zeta}_C(y)$ and $\eta_C(x) \leq \eta_C(y)$.*

Proof. For $x, y \in X, x \leq y \Rightarrow x - y = 0$ then

$$\overline{\zeta}_C(x) \geq r\min\{\overline{\zeta}_C(x-y), \overline{\zeta}_C(y)\} = r\min\{\overline{\zeta}_C(0), \overline{\zeta}_C(y)\} = \overline{\zeta}_C(y),$$
$$\eta_C(x) \leq \max\{\eta_C(x-y), \eta_C(y)\} = \max\{\eta_C(0), \eta_C(y)\} = \eta_C(y). \qquad \square$$

Theorem 3.2. *Let $C = \{x, \overline{\zeta}_C(x), \eta_C(x) : x \in X\}$ be a cubic fuzzy $\beta-$ ideals of a $\beta-$algebra X. If $x \leq z + y$ then $\overline{\zeta}(x) \geq r\min\{\overline{\zeta}(z), \overline{\zeta}(y)\}$ & $\eta(x) \leq \max\{\eta(z), \eta(y)\}, \forall\ x, y, z \in X$.*

Proof. For $x, y, z \in X$

$$\begin{aligned}
\overline{\zeta}(x) &\geq r\min\{\overline{\zeta}(x-y), \overline{\zeta}(y)\} \\
&= r\min\{r\min\{\overline{\zeta}((x-y)-z), \overline{\zeta}(z)\}, \overline{\zeta}(y)\} \\
&= r\min\{r\min\{\overline{\zeta}((x-(z+y)), \overline{\zeta}(z)\}, \overline{\zeta}(y)\} \\
&= r\min\{r\min\{\overline{\zeta}(0), \overline{\zeta}(z)\}, \overline{\zeta}(y)\} \\
&= r\min\{\overline{\zeta}(z), \overline{\zeta}(y)\}, \\
\eta(x) &\leq \max\{\eta(x-y), \eta(y)\} \\
&= \max\{\max\{\eta((x-y)-z), \eta(z)\}, \eta(y)\} \\
&= \max\{\max\{\eta((x-(z+y)), \eta(z)\}, \eta(y)\} \\
&= \max\{\max\{\eta(0), \eta(z)\}, \eta(y)\} \\
&= \max\{\eta(z), \eta(y)\}.
\end{aligned}$$

□

The definitions of product, homomorphic image and inverse image of cubic fuzzy sets are defined in the following and some related results are discussed.

Definition 3.2. *Consider the cubic fuzzy sets* $A = \{\langle x, \overline{\zeta}_A(x), \eta_A(x)\rangle : x \in X\}$ *of* X *and* $B = \{\langle y, \overline{\zeta}_B(y), \eta_B(y)\rangle : y \in Y\}$ *of* Y*. Then, cubic Cartesian product of* A *and* B *is defined as* $A \times B = \{\overline{\zeta}_{A\times B}(x, y)\ \&\ \eta_{A\times B}(x, y) : x, y \in X \times Y\}$*, where* $\overline{\zeta}_{A\times B} : X \times Y \to D[0, 1]$ *is given by* $\overline{\zeta}_{A\times B}(x, y) = r\min\{\overline{\zeta}_A(x), \overline{\zeta}_B(y)\}$ *and* $\eta_{A\times B} : X \times Y \to [0, 1]$ *is given by* $\eta_{A\times B}(x, y) = \max\{\eta_A(x), \eta_B(y)\}$*.*

Theorem 3.3. *Let* A *and* B *be two cubic fuzzy sets of* X *and* Y *such that* $A \times B$ *is also a cubic fuzzy* $\beta-$ *ideal of* $X \times Y$*. Then either* A *is a cubic fuzzy* $\beta-$*ideal of* X *or* B *is a cubic fuzzy* $\beta-$*ideal of* Y*.*

Proof. We know that

$$(1) \qquad \overline{\zeta}_A(0) \geq \overline{\zeta}_A(x), \overline{\zeta}_B(0) \geq \overline{\zeta}_B(y)\ \&\ \eta_A(0) \leq \eta_A(x), \eta_B(0) \leq \eta_B(y),$$

then $\overline{\zeta}_{A\times B}(0, y) = r\min\{\overline{\zeta}_A(0), \overline{\zeta}_B(y)\}$, $\eta_{A\times B}(0, y) = \max\{\eta_A(0), \eta_B(y)\}$. Since $A \times B$ is a cubic fuzzy $\beta-$ideals of $X \times Y$,

$$\overline{\zeta}_{A\times B}((x_1, y_1), (x_2, y_2)) \geq r\min\{\overline{\zeta}_{A\times B}((x_1, y_1) - (x_2, y_2)), \overline{\zeta}_{A\times B}(x_2, y_2)\}$$

and since

$$\begin{aligned}
&\overline{\zeta}_{A\times B}((x_1, y_1), (x_2, y_2)) \geq r\min\{\overline{\zeta}_{A\times B}((x_1, y_1) - (x_2, y_2)), \overline{\zeta}_{A\times B}((x_2, y_2)\}, \\
(2)\ &\overline{\zeta}_{A\times B}((x_1, y_1), (x_2, y_2)) \geq r\min\{\overline{\zeta}_{A\times B}((x_1 - x_2), (y_1 - y_2)), \overline{\zeta}_{A\times B}(x_2, y_2)\}, \\
&\overline{\zeta}_{A\times B}((x_1 - x_2), (y_1 - y_2)) \geq r\min\{\overline{\zeta}_{A\times B}((x_1, y_1), \overline{\zeta}_{A\times B}(x_2, y_2)\}.
\end{aligned}$$

Putting $x_1 = x_2 = 0$ in (6). Then

$$\overline{\zeta}_{A\times B}((0, y_1) \geq r\min\{\overline{\zeta}_{A\times B}(0, (y_1 - y_2)), \overline{\zeta}_{A\times B}(0, y_2)\}$$

and

$$\overline{\zeta}_{A\times B}(0,(y_1-y_2)) \geq r\min\{\overline{\zeta}_{A\times B}(0,y_1),\overline{\zeta}_{A\times B}(0,y_2)\}. \tag{3}$$

Using equations (1) in (3) which yields $\overline{\zeta}_B(y_1) \geq r\min\{\{\overline{\zeta}_B(y_1-y_2),\overline{\zeta}_B(y_2)\}$ and $\overline{\zeta}_B(y_1-y_2) \geq r\min\{\overline{\zeta}_B(y_1),\overline{\zeta}_B(y_2)\}$. Similarly, $\eta_B(y_1) \leq \max\{\{\eta_B(y_1-y_2),\eta_B(y_2)\}$ and $\eta_B(y_1-y_2) \leq \max\{\eta_B(y_1),\eta_B(y_2)\}$. Hence, B is a cubic fuzzy $\beta-$ideal of Y. □

Using the definition of product of cubic fuzzy sets and the above theorem, one can prove the following theorem.

Theorem 3.4. *If A and B are two cubic fuzzy $\beta-$ideals of X and Y respectively then $A\times B$ is also a cubic fuzzy $\beta-$ ideal of $X\times Y$.*

Definition 3.3. *Let $C=\{\langle x,\overline{\zeta}_C(x),\eta_C(x)\rangle : x\in X\}$ be a cubic fuzzy set with the degree membership $\overline{\zeta}_C(x): X\to D[0,1]$ and the degree of non-membership function $\eta_C(x): X\to[0,1]$ is said to have rsup-inf property if for any subset T of X, there exists $x_0\in T$, such that $\overline{\zeta}_C(x_0)=r\sup_{x\in T}\overline{\zeta}_C(x)$ and $\eta_C(x_0)=\inf_{x\in T}\ \eta_C(x)$*

Definition 3.4. *Let f be a mapping from a set X into a set Y. Let $A=\{\langle x,\overline{\zeta}_A(x),\eta_A(x)\rangle : x\in X\}$ be a cubic fuzzy set in X, then the cubic set $B=\{\langle x,\overline{\zeta}_B(x),\eta_B(x)\rangle : x\in X\}$ of Y defined by*

$$\overline{\zeta}_A(f^-1(y))=\overline{\zeta}_B(y)=\begin{cases} r\sup_{x\in\ f^{-1}(y)}\overline{\zeta}_A(x), & \text{if } f^{-1}=\{x\in X, f(x)=y\}\neq\emptyset \\ [0,0], & \text{if otherwise}\end{cases}$$

$$\eta_A(f^{-1}(y))=\eta_B(y)=\begin{cases}\inf_{x\in\ f^{-1}(y)}\eta_A(x), & \text{if } f^{-1}=\{x\in X, f(x)=y\}\neq\emptyset \\ 1 & \text{if otherwise}\end{cases}$$

$\forall y\in Y$ is called the image of A under f.

Similarly, if B is a cubic set of Y, then the cubic set $A=B\circ f$ in X defined by $\overline{\zeta}_B(f(x))=\overline{\zeta}_A(x),\eta_B(f(x))=\eta_A(x),\forall\ x\in X$, also to be the inverse image of B under under f.

Theorem 3.5. *Let $f: X\to X$ be an endomorphism on X and $C=\{x,\overline{\zeta}_C(x),\eta_C(x) : x\in X\}$ be a cubic fuzzy $\beta-$ideal of X. Then, $C_f=\{f(x),\overline{\zeta}_f(x),\eta_f(x) : x\in X\}$, where $\overline{\zeta}_f: X\to D[0,1]$ and $\eta_f: X\to[0,1]$ are defined by $\overline{\zeta}_f(x)=\overline{\zeta}(f(x))$ and $\eta_f(x)=\eta(f(x)),\forall x\in X$, is a cubic fuzzy $\beta-$ideal of X.*

Proof. Let C be a cubic fuzzy $\beta-$ideal of X. For $x\in X$, $\overline{\zeta}_f(0)=\overline{\zeta}(f(0))=\overline{\zeta}(0)\leq\overline{\zeta}(x)$, $\eta_f(0)=\eta(f(0))=\eta(0)\geq\eta(x),\forall x\in X$. Then

$$\begin{aligned}\overline{\zeta}_f(x+y)&=\overline{\zeta}(f(x+y))\\ &\geq\overline{\zeta}(f(x)+f(y))\\ &=r\min\{\overline{\zeta}(f(x)),\overline{\zeta}(f(y))\}\\ &=r\min\{\overline{\zeta}_f(x),\overline{\zeta}_f(y)\}.\end{aligned}$$

and

$$\begin{aligned}\eta_f(x+y) &= \eta(f(x+y)) \\ &\leq \eta(f(x)+f(y)) \\ &= \max\{\eta(f(x)), \eta(f(y))\} \\ &= \max\{\eta_f(x), \eta_f(y)\}.\end{aligned}$$

Also,

$$\begin{aligned}\overline{\zeta}_f(x) &= \overline{\zeta}(f(x)) \\ &\geq r\min\{\overline{\zeta}(f(x)-f(y)), \overline{\zeta}(f(y))\} \\ &= r\min\{\overline{\zeta}(f(x-y)), \overline{\zeta}(f(y))\} \\ &= r\min\{\overline{\zeta}_f(x-y), \overline{\zeta}_f(y)\}.\end{aligned}$$

Similarly,

$$\begin{aligned}\eta_f(x) &= \eta(f(x)) \\ &\leq \max\{\eta(f(x)-f(y)), \eta(f(y))\} \\ &= \max\{\eta(f(x-y)), \eta(f(y))\} \\ &= \max\{\eta_f(x-y), \eta_f(y)\}.\end{aligned}$$

Hence, C_f is a cubic fuzzy $\beta-$ideal of X. □

Using the definition of homomorphic image and the above theorem, one can prove the following theorem.

Theorem 3.6. *Let $f : X \to Y$ be an onto homomorphism of $\beta-$algebras. If C is a cubic fuzzy $\beta-$ideal of Y, then the inverse image $f^{-1}(C)$ is a cubic fuzzy $\beta-$ideal of X.*

Conclusion

This paper introduces cubic fuzzy $\beta-$ideal of a $\beta-$algebra. In depth, the study analysed the cubic fuzzy $\beta-$ideal using homomorphic image, cartesion product. This can be extended to the other substructures like $H-$ideals and filters of a $\beta-$algebra.

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Ulam stability of linear difference equations with initial conditions

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Abstract. In this paper, we prove the Hyers-Ulam stability, Hyers-Ulam-Rassias stability, Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam-Rassias stability of the linear difference equations with initial conditions by applying Newton's Theorem. Also, the Ulam stability constants are obtained.

Keywords: Hyers-Ulam Stability, Hyers-Ulam-Rassias stability, Mittag-Leffler-Hyers-Ulam stability, Mittag-Leffler-Hyers-Ulam-Rassias stability, linear difference equations, initial conditions and Newton's theorem.

1. Introduction

In [44], Ulam proposed the universal Ulam stability problem: When is it true that by slightly changing the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true? In [12], Hyers

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gave the first confirmatory answer to the question of Ulam for additive functional equations on Banach spaces. Hyers result has since then seen many significant generalizations, both in terms of the control condition used to define the concept of approximate solution [2, 3, 4, 10, 35, 36, 37, 38, 39].

A generalization of Ulam's problem was recently proposed by replacing functional equations with differential equations: The differential equation

$$\phi\left(f, x, x', x'', ...x^{(n)}\right) = 0$$

has the Hyers-Ulam stability if for a given $\epsilon > 0$ and a function x such that

$$\left|\phi\left(f, x, x', x'', ...x^{(n)}\right)\right| \leq \epsilon,$$

there exists a solution x_a of the differential equation such that $|x(t) - x_a(t)| \leq K(\epsilon)$ and $\lim_{\epsilon \to 0} K(\epsilon) = 0$. If the preceding statement is also true when we replace ϵ and $K(\epsilon)$ by $\phi(t)$ and $\varphi(t)$, where ϕ, φ are appropriate functions not depending on x and x_a explicitly, then we say that the corresponding differential equation has the generalized Hyers-Ulam stability or Hyers-Ulam-Rassias stability.

Obloza seems to be the first author who has investigated the Hyers-Ulam stability of linear differential equations [31, 32]. Thereafter, In 1998, C. Alsina and R. Ger [1] were the first authors who investigated the Hyers-Ulam stability of differential equations. This result of C. Alsina and R. Ger [1] has been generalized by Takahasi [41]. They proved in [41] that the Hyers-Ulam stability holds true for the Banach Space valued differential equation $y'(t) = \lambda y(t)$. Those previous results were extended to the Hyers-Ulam stability of linear differential equation second order and higher order in [11, 16, 17, 20, 21, 22, 23, 24, 25] respectively.

Now a days, the Hyers-Ulam stability of difference equations has been given attention. In 2005, D. Popa [34] proved the Hyers-Ulam stability of the linear recurrence of first order with constant coefficient and nth order linear recurrence with constant coefficients in Banach spaces. Also the Hyers-Ulam-Rassias stability of a linear recurrences of higher order equations are established in [6, 33]. In 2007, J.Brzdek et.al [7] investigated the Hyers-Ulam stability of the nonlinear recurrences in an Abelian group with invariant metric.

In 2018, R. Murali et. al [30] proved the Hyers-Ulam stability of the forward and backward difference equations. Very recently, they proved [27, 29] the Hyers-Ulam stability and Hyers-Ulam-Rassias stability of the first and second order difference equation by using Z-transform method. These days, the Hyers-Ulam stability of difference equations has been investigated [5, 8, 13, 14, 15, 18, 27, 28, 29, 42, 43] and the investigation is going on.

In 2011, P. Gavruta, S.M. Jung and Y. Li [9] are studied the Hyers-Ulam stability for second order linear differential equations with initial and boundary conditions using Taylor formula. As well known, many different methods for solving differential equations have been used to study the Hyers-Ulam stability

problem for various differential equation. But using initial conditions are have more significant advantage for solving differential equations. As a discrete case also initial conditions are have a similar advantage in solving difference equations. Motivated by the results reported in [9], our main goal is to study the Hyers-Ulam stability, Hyers-Ulam-Rassias stability, Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam-Rassias stability linear difference equation of first order

$$z(n+1) + p(n)\ z(n) = 0 \tag{1}$$

with initial condition

$$z(a) = 0 \tag{2}$$

the linear difference equation of second order

$$z(n+2) + p(n)\ z(n) = 0 \tag{3}$$

with initial conditions

$$z(a) = z(a+1) = 0 \tag{4}$$

and the linear difference equation of kth order

$$z(n+k) + p(n)\ z(n) = 0 \tag{5}$$

with initial conditions

$$z(a) = z(a+1) = z(a+2) = \cdots = z(a+k-1) = 0 \tag{6}$$

by using Newton's Formula. Where $n \in (a, b+1)$, $a, b \in \mathbb{N}_0$.

2. Preliminaries

The following Definitions and theorems are very useful to prove our main results.

Definition 2.1 ([40]). *Let $t \in N_0$, $\lambda \in (-1, 1)$ and $\alpha, \beta \in R^+$. The one and two parameter discrete Mittag-Leffler functions are defined by*

$$F_\alpha(\lambda, t) = \sum_{k=0}^{\infty} \lambda^k \frac{t^{\overline{\alpha k}}}{\Gamma(\alpha k + 1)}, \tag{7}$$

$$F_{\alpha,\beta}(\lambda, t^{\overline{\alpha}}) = \sum_{k=0}^{\infty} \lambda^k \frac{t^{\overline{\alpha k}}}{\Gamma(\alpha k + \beta)}. \tag{8}$$

Theorem 2.1. *(Newton's Theorem). If $f(n)$ is a polynomial of degree k then*

$$f(n) = f(0) + \frac{n^{(1)}}{1!}\Delta f(0) + \frac{n^{(2)}}{2!}\Delta^{(2)} f(0) + \frac{n^{(3)}}{3!}\Delta^{(3)} f(0) + \cdots + \frac{n^{(k)}}{k!}\Delta^{(k)} f(0),$$

where $n^{(k)} = \frac{n!}{(n-k)!}$.

Now, we give the definitions of Hyers-Ulam stability, Hyers-Ulam-Rassias stability, Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam-Rassias stability of the linear difference equation with initial conditions.

Definition 2.2. *We say that the linear difference equation* (5) *has the Hyers-Ulam stability, if there exists a constant* $K > 0$ *with the following conditions: For every* $\epsilon > 0$ *and every* $z(n)$ *be a sequence of function of kth order difference equation satisfying the inequality*

$$|z(n+k) + p(n)\ z(n)| \leq \epsilon,$$

with initial conditions (6)*. Then, there exists some* $w(n)$ *satisfying the difference equation* $w(n+k) + p(n)\ w(n) = 0$ *with* $w(a) = w(a+1) = w(a+2) = \cdots = w(a+k-1) = 0$ *such that* $|z(n) - w(n)| \leq K\ \epsilon$*. We call such* K *as a Hyers-Ulam stability constant for the linear difference equation* (5) *with* (6)*.*

Definition 2.3. *We say that the kth order linear difference equation* (5) *has Hyers-Ulam-Rassias stability, if there exists a constant* $K > 0$ *with the following property: For every* $\epsilon > 0$*,* $z(n)$ *be a sequence of function of kth order difference equation and there exists* $\phi(n)$ *be a positive and bounded sequence satisfying the inequality* $|z(n+k) + p(n)\ z(n)| \leq \phi(n)\ \epsilon$*, with initial conditions* (6)*. Then, there exists some* $w(n)$ *satisfying the difference equation* $w(n+k)+p(n)\ w(n) = 0$ *with*

$$w(a) = w(a+1) = w(a+2) = \cdots = w(a+k-1) = 0$$

such that $|z(n) - w(n)| \leq K\ \phi(n)\ \epsilon$*. We call such* K *as a Hyers-Ulam-Rassias stability constant for the linear difference equation* (5) *with* (6)*.*

Definition 2.4. *We say that the linear difference equation* (5) *has the Mittag-Leffler-Hyers-Ulam stability, if there exists a constant* $K > 0$ *with the following conditions: For every* $\epsilon > 0$ *and every* $z(n)$ *be a sequence of function of kth order difference equation satisfying the inequality*

$$|z(n+k) + p(n)\ z(n)| \leq \epsilon F_\alpha(\lambda, n),$$

with initial conditions (6)*, where* F_α *is Mittag-Leffler function. Then, there exists some* $w(n)$ *satisfying the difference equation* $w(n+k) + p(n)\ w(n) = 0$ *with* $w(a) = w(a+1) = w(a+2) = \cdots = w(a+k-1) = 0$ *such that* $|z(n) - w(n)| \leq K\ \epsilon F_\alpha(\lambda, n)$*. We call such* K *as a Mittag-Leffler-Hyers-Ulam stability constant for the linear difference equation* (5) *with* (6)*.*

Definition 2.5. *We say that the kth order linear difference equation* (5) *has Mittag-Leffler-Hyers-Ulam-Rassias stability, if there exists a constant* $K > 0$ *with the following property: For every* $\epsilon > 0$*,* $z(n)$ *be a sequence of function of kth order difference equation and there exists* $\phi(n)$ *be a positive and bounded sequence satisfying the inequality*

$$|z(n+k) + p(n)\ z(n)| \leq \phi(n)\ \epsilon F_\alpha(\lambda, n),$$

with initial conditions (6)*, where* F_α *is Mittag-Leffler function. Then, there exists some* $w(n)$ *satisfying the difference equation* $w(n+k)+p(n)\ w(n)=0$ *with* $w(a)=w(a+1)=w(a+2)=\cdots=w(a+k-1)=0$ *such that* $|z(n)-w(n)|\le K\ \phi(n)\ \epsilon F_\alpha(\lambda,n)$*. We call such* K *as a Mittag-Leffler-Hyers-Ulam-Rassias stability constant for the linear difference equation* (5) *with* (6)*.*

3. Hyers-Ulam stability

In this section, we prove the Hyers-Ulam stability and Hyers-Ulam-Rassias stability of the linear difference equations (1), (3) and (5) with initial conditions (2), (4) and (6) respectively. Firstly, we prove the Hyers-Ulam stability and Hyers-Ulam-Rassias stability of the linear difference equation (1) with (2).

Theorem 3.1. *Let* $\max|p(n)|<\frac{\gamma}{n}$ *with* $0<\gamma<1$ *and for every* $\epsilon>0$*, there exists* $z(n)$*,* $n\in(a,b+1)$ *satisfies the inequality*

$$|z(n+1)+p(n)z(n)|\le\epsilon, \tag{9}$$

with initial condition $z(a)=0$*. Then, there exists a solution* $w(n)$ *satisfies the difference equation* $w(n+1)+p(n)w(n)=0$ *with* $w(a)=0$ *such that* $|z(n)-w(n)|\le K\epsilon$*.*

Proof. Assume that for every $\epsilon>0$, if $z(n)$ be a sequence of function satisfying the inequality (9) with $z(a)=0$. Then, by using Newton's Theorem, we have

$$z(n)=z(a)+\frac{n^{(1)}}{1!}\Delta z(a)=z(a)+\frac{n^{(1)}}{1!}[z(a+1)-z(a)].$$

By using initial condition (2), we obtain that

$$z(n)=\frac{n!}{(n-1)!\ 1!}\ z(a+1).$$

Now, taking modulus on both sides and using the inequality (9), we get

$$\begin{aligned}
|z(n)| &= \left|\frac{n!}{(n-1)!\ 1!}z(a+1)\right| \\
&\le \frac{n}{1!}\ \max|z(n+1)| \\
\max|z(n)| &\le \frac{n}{1!}\ \max|z(n+1)+p(n)z(n)-p(n)z(n)| \\
&\le \frac{n}{1!}\max|z(n+1)+p(n)z(n)|+\frac{n}{1!}\max|p(n)|\max|z(n)| \\
&\le \frac{b}{1!}\ \epsilon+\gamma\ \max|z(n)|\,.
\end{aligned}$$

Let $K=\frac{b}{(1-\gamma)}$, then we have $\max|z(n)|\le K\epsilon$. Obviously, $w(n)\equiv 0$ is a solution of the difference equation $z(n+1)+p(n)z(n)=0$ with $w(a)=0$ such that $|z(n)-w(n)|\le K\epsilon$. Hence the linear difference equation (1) has the Hyers-Ulam stability with initial condition (2). □

The following corollary shows that the Hyers-Ulam-Rassias stability of the linear difference equation (1) with (2).

Corollary 3.1. *Let* $\max|p(n)| < \frac{\gamma}{n}$ *with* $0 < \gamma < 1$ *and for every* $\epsilon > 0$ *there exists* $z(n), n \in (a, b+1)$ *and* $\phi(n)$ *be a positive and bounded sequence satisfies the inequality,*

$$|z(n+1) + p(n)z(n)| \leq \phi(n)\epsilon,$$

with initial condition $z(a) = 0$. *Then, there exists a solution* $w(n)$ *satisfies the difference equation* $w(n+1) + p(n)w(n) = 0$ *with* $w(a) = 0$ *such that* $|z(n) - w(n)| \leq K\phi(n)\epsilon$ *for all* $n \in (a, b+1)$.

Proof. Assume that given $\epsilon > 0$, there exists $z(n), n \in (a, b+1)$ and $\phi(n)$ be a positive and bounded sequence satisfies the inequality,

$$|z(n+1) + p(n)z(n)| \leq \phi(n)\epsilon, \tag{10}$$

with initial condition $z(a) = 0$. By the similar way of the Theorem 3.1, we can get the result. □

Now, we investigate the Hyers-Ulam stability and Hyers-Ulam-Rassias stability of the linear difference equation (3) with initial conditions (4).

Theorem 3.2. *Let* $\max|p(n)| < \frac{2\gamma}{n(n-1)}$ *with* $0 < \gamma < 1$ *and for every* $\epsilon > 0$, $z(n)$ *be a sequence satisfies the inequality* $|z(n+2) + p(n)z(n)| \leq \epsilon$ *with initial condition* $z(a) = z(a+1) = 0$. *Then, there exists a solution* $w(n)$ *satisfies the difference equation* $w(n+2) + p(n)w(n) = 0$ *with* $w(a) = w(a+1) = 0$ *such that* $|z(n) - w(n)| \leq K\epsilon$.

Proof. Assume that for every $\epsilon > 0$, $z(n)$ be a function of second order difference equation on $(a, b+1)$ satisfying

$$|z(n+2) + p(n)z(n)| \leq \epsilon, \tag{11}$$

with $z(a) = z(a+1) = 0$. Then, by using Newton's Theorem, we have

$$\begin{aligned} z(n) &= z(a) + \frac{n^{(1)}}{1!}\Delta^{(1)}z(a) + \frac{n^{(2)}}{2!}\Delta^{(2)}z(a) \\ &= z(a) + \frac{n^{(1)}}{1!}\left[z(a+1) - z(a)\right] + \frac{n^{(2)}}{2!}\left[z(a+2) - 2z(a+1) + z(a)\right]. \end{aligned}$$

By using initial condition (4), we obtain that $z(n) = \frac{n!}{(n-2)!\ 2!}\ z(a+2)$. Now, taking modulus on both sides and using the inequality (11), we get that

$$|z(n)| = \left|\frac{n!}{(n-2)!\ 2!}\ z(a+2)\right| \leq \frac{n(n-1)}{2!}\ \max|z(n+2)|.$$

$$\begin{aligned} \max|z(n)| &\leq \frac{n(n-1)}{2!}\ \max|z(n+2) + p(n)z(n) - p(n)z(n)| \\ &\leq \frac{b(b-1)}{2}\ \epsilon + \gamma\ \max|z(n)|. \end{aligned}$$

Let $K = \frac{b(b-1)}{2(1-\gamma)}$, then we have $\max|z(n)| \leq K\epsilon$. Obviously, $w(n) \equiv 0$ is a solution of the difference equation

$$z(n+2) + p(n)z(n) = 0$$

with $w(a) = w(a+1) = 0$ such that $|z(n) - w(n)| \leq K\epsilon$. Hence the linear difference equation (3) has the Hyers-Ulam stability with (4). □

Corollary 3.2. *Let* $\max|p(n)| < \frac{2\gamma}{n(n-1)}$ *with* $0 < \gamma < 1$ *and for every* $\epsilon > 0$, *there exists a function* $z(n)$, $n \in (a, b+1)$ *and* $\phi(n)$ *be a positive and bounded sequence satisfies the inequality*

$$|z(n+2) + p(n)z(n)| \leq \phi(n)\epsilon, \tag{12}$$

with initial condition $z(a) = z(a+1) = 0$. *Then, there exists a solution* $w(n)$ *satisfies the difference equation* $w(n+2)+p(n)w(n) = 0$ *with* $w(a) = w(a+1) = 0$ *such that*

$$|z(n) - w(n)| \leq K\phi(n)\epsilon, \qquad \forall\ n \in (a, b+1).$$

Proof. Assume for every $\epsilon > 0$, there exists $z(n), n \in (a, b+1)$ and $\phi(n)$ be a positive and bounded sequence satisfies the inequality (18) with initial condition $z(a) = z(a+1) = 0$. Then, by using above Theorem 3.2, similarly we can obtain the Hyers-Ulam-Rassias stability of (3) with (4). □

Finally, in this section we are going to establish the Hyers-Ulam stability and Hyers-Ulam-Rassias stability of the kth order linear difference equation (5) with initial conditions (6).

Theorem 3.3. *Let* $\max|p(n)| < \frac{\gamma(n-k)!\ k!}{n!}$ *with* $0 < \gamma < 1$*and for every* $\epsilon > 0$, *there exists* $z(n)$, $n \in (a, b+1)$ *satisfies the inequality*

$$|z(n+k) + p(n)z(n)| \leq \epsilon$$

with the initial conditions (6). *Then, there exists a solution* $w(n)$ *satisfies the difference equation* $w(n+k)+p(n)w(n) = 0$ *with* $w(a) = w(a+1) = w(a+2) = \cdots = w(a+k-1) = 0$ *such that* $|z(n) - w(n)| \leq K\epsilon$.

Proof. Assume that for every $\epsilon > 0$, $z(n)$ be a sequence of function of kth order difference equation on $(a, b+1)$ satisfying the inequality

$$|z(n+k) + p(n)z(n)| \leq \epsilon, \tag{13}$$

with initial conditions $z(a) = z(a+1) = z(a+2) = \cdots = z(a+k-1) = 0$. Then, by using Newton's Theorem, we have

$$z(n) = z(a) + \frac{n^{(1)}}{1!}\Delta^{(1)}z(a) + \frac{n^{(2)}}{2!}\Delta^{(2)}z(a) + \frac{n^{(3)}}{3!}\Delta^{(3)}z(a) + \cdots + \frac{n^{(k)}}{k!}\Delta^{(k)}z(a).$$

By using initial condition (6), we obtain that

$$z(n) = \frac{n!}{(n-k)!\ k!}\ x(a+k).$$

Taking modulus on both sides and using the inequality (13), we get that

$$|z(n)| = \left|\frac{n!}{(n-k)!\ k!}\ x(a+k)\right| \leq \frac{n!}{(n-k)!\ k!}\ \max|z(n+k)|,$$
$$\max|z(n)| \leq \frac{n!}{(n-k)!\ k!}\ \max|z(n+k)+p(n)z(n)-p(n)z(n)|$$

choosing $K = \frac{b!}{(1-\gamma)\ (b-k)!\ k!}$. Thus we have $\max|z(n)| \leq K\epsilon$. Obviously, $w(n) \equiv 0$ is a solution of the difference equation $z(n+k)+p(n)z(n) = 0$ with $w(a) = w(a+1) = w(a+2) = \cdots = y(n+k-1) = 0$ such that $|z(n)-w(n)| \leq K\epsilon$. Hence by the virtue of Definition 2.2, the linear difference equation (5) has the Hyers-Ulam stability with (6). □

The next corollary says that the Hyers-Ulam-Rassias stability of the linear difference equation (5) with (6).

Corollary 3.3. *Let* $\max|p(n)| < \frac{\gamma(n-k)!\ k!}{n!}$ *with* $0 < \gamma < 1$ *for every* $\epsilon > 0$, *there exists a function* $z(n)$, $n \in (a, b+1)$ *and* $\phi(n)$ *be a positive and bounded sequence satisfies the inequality*

$$|z(n+k)+p(n)z(n)| \leq \phi(n)\epsilon,$$

with initial condition (6). *Then, there exists a solution* $w(n)$ *satisfies the difference equation* $w(n+k)+p(n)w(n) = 0$ *with* $w(a) = w(a+1) = w(a+2) = \cdots = w(a+k-1) = 0$ *such that* $|z(n)-w(n)| \leq K\phi(n)\epsilon$, *for all* $n \in (a, b+1)$.

Proof. If for every $\epsilon > 0$, there exists $z(n), n \in (a, b+1)$ and $\phi(n)$ be a positive and bounded sequence satisfies the inequality,

$$|z(n+k)+p(n)z(n)| \leq \phi(n)\epsilon, \tag{14}$$

with initial condition $z(a) = z(a+1) = z(a+2) = \cdots = z(a+k-1) = 0$. Then, by above Theorem 3.3, we can easily obtain the result. □

4. Mittag-Leffler-Hyers-Ulam stability

In this section, we study the Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam-Rassias stability of the linear difference equations (1), (3) and (5) with initial conditions (2), (4) and (6) respectively.

Theorem 4.1. *Let* $\max|p(n)| < \frac{\gamma}{n}$ *with* $0 < \gamma < 1$ *and for every* $\epsilon > 0$*, there exists* $z(n)$*,* $n \in (a, b+1)$ *satisfies the inequality*

$$|z(n+1) + p(n)z(n)| \leq \epsilon F_\alpha(\lambda, n), \tag{15}$$

with initial condition $z(a) = 0$*. Then, there exists a solution* $w(n)$ *satisfies the difference equation* $w(n+1) + p(n)w(n) = 0$ *with* $w(a) = 0$ *such that* $|z(n) - w(n)| \leq K\epsilon F_\alpha(\lambda, n)$*.*

Proof. Assume that for every $\epsilon > 0$, if $z(n)$ be a sequence of function satisfying the inequality (15) with $z(a) = 0$. Then, by using Newton's Theorem, we have

$$z(n) = z(a) + \frac{n^{(1)}}{1!}\Delta z(a) = z(a) + \frac{n^{(1)}}{1!}[z(a+1) - z(a)].$$

By using initial condition (2), we obtain that

$$z(n) = \frac{n!}{(n-1)!\ 1!}\ z(a+1).$$

Now, taking modulus on both sides and using the inequality (15), we get

$$\begin{aligned}
|z(n)| &= \left|\frac{n!}{(n-1)!\ 1!}z(a+1)\right| \leq \frac{n}{1!}\ \max|z(n+1)|\,, \\
\max|z(n)| &\leq \frac{n}{1!}\ \max|z(n+1) + p(n)z(n) - p(n)z(n)| \\
&\leq \frac{n}{1!}\ \max|z(n+1) + p(n)z(n)| + \frac{n}{1!}\ \max|p(n)|\max|z(n)| \\
&\leq \frac{b}{1!}\ \epsilon F_\alpha(\lambda, n) + \gamma\ \max|z(n)|\,.
\end{aligned}$$

Let $K = \frac{b}{(1-\gamma)}$, then we have $\max|z(n)| \leq K\epsilon F_\alpha(\lambda, n)$. Obviously, $w(n) \equiv 0$ is a solution of the difference equation $z(n+1) + p(n)z(n) = 0$ with $w(a) = 0$ such that $|z(n) - w(n)| \leq K\epsilon F_\alpha(\lambda, n)$. Hence the difference equation (1) has the Mittag-Leffler-Hyers-Ulam stability with initial condition (2). □

The following corollary shows that the Mittag-Leffler-Hyers-Ulam-Rassias stability of the linear difference equation (1) with (2).

Corollary 4.1. *Let* $\max|p(n)| < \frac{\gamma}{n}$ *with* $0 < \gamma < 1$ *and for every* $\epsilon > 0$ *there exists* $z(n), n \in (a, b+1)$ *and* $\phi(n)$ *be a positive and bounded sequence satisfies the inequality* $|z(n+1) + p(n)z(n)| \leq \phi(n)\epsilon F_\alpha(\lambda, n)$*, with initial condition* $z(a) = 0$*. Then, there exists a solution* $w(n)$ *satisfies the difference equation* $w(n+1) + p(n)w(n) = 0$ *with* $w(a) = 0$ *such that*

$$|z(n) - w(n)| \leq K\phi(n)\epsilon F_\alpha(\lambda, n),$$

for all $n \in (a, b+1)$*.*

Proof. Assume that given $\epsilon > 0$ and $z(n), n \in (a, b+1)$, there exists $\phi(n)$ be a positive and bounded sequence satisfying the inequality,

$$|z(n+1) + p(n)z(n)| \leq \phi(n)\epsilon F_\alpha(\lambda, n), \tag{16}$$

with initial condition $z(a) = 0$. Then, by the similar approach of the Theorem 4.1, we can obtain the Mittag-Leffler-Hyers-Ulam-Rassias stability of the linear difference equation (1) with initial condition (2). □

Now, we investigate the Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam-Rassias stability of the linear difference equation (3) with initial conditions (4).

Theorem 4.2. *Let* $\max|p(n)| < \frac{2\gamma}{n(n-1)}$ *with* $0 < \gamma < 1$ *and for every* $\epsilon > 0$, $z(n)$ *be a sequence satisfies the inequality*

$$|z(n+2) + p(n)z(n)| \leq \epsilon F_\alpha(\lambda, n)$$

with initial condition $z(a) = z(a+1) = 0$. *Then, there exists a solution* $w(n)$ *satisfies the difference equation* $w(n+2)+p(n)w(n) = 0$ *with* $w(a) = w(a+1) = 0$ *such that* $|z(n) - w(n)| \leq K\epsilon F_\alpha(\lambda, n)$.

Proof. Assume that for every $\epsilon > 0$, $z(n)$ be a function of second order difference equation on $(a, b+1)$ satisfying

$$|z(n+2) + p(n)z(n)| \leq \epsilon F_\alpha(\lambda, n), \tag{17}$$

with $z(a) = z(a+1) = 0$. Then, by using Newton's Theorem, we have

$$\begin{aligned} z(n) &= z(a) + \frac{n^{(1)}}{1!}\Delta^{(1)}z(a) + \frac{n^{(2)}}{2!}\Delta^{(2)}z(a) \\ &= z(a) + \frac{n^{(1)}}{1!}\,[z(a+1) - z(a)] + \frac{n^{(2)}}{2!}\,[z(a+2) - 2z(a+1) + z(a)]. \end{aligned}$$

By using initial condition (4), we obtain that $z(n) = \frac{n!}{(n-2)!\ 2!}\ z(a+2)$. Now, taking modulus on both sides and using the inequality (17), we get that

$$\begin{aligned} |z(n)| &= \left|\frac{n!}{(n-2)!\ 2!}\ z(a+2)\right| \leq \frac{n(n-1)}{2!}\ \max|z(n+2)| \\ \max|z(n)| &\leq \frac{n(n-1)}{2!}\ \max|z(n+2) + p(n)z(n) - p(n)z(n)| \\ &\leq \frac{n(n-1)}{2!}\max|z(n+2)+p(n)z(n)| + \frac{n(n-1)}{2!}\max|p(n)|\max|z(n)| \\ &\leq \frac{b(b-1)}{2}\ \epsilon F_\alpha(\lambda, n) + \gamma\ \max|z(n)|\,. \end{aligned}$$

Let $K = \frac{b(b-1)}{2(1-\gamma)}$, then we have $\max|z(n)| \leq K\epsilon F_\alpha(\lambda, n)$. Obviously, $w(n) \equiv 0$ is a solution of the difference equation

$$z(n+2) + p(n)z(n) = 0$$

with $w(a) = w(a+1) = 0$ such that $|z(n) - w(n)| \leq K\epsilon F_\alpha(\lambda, n)$. Hence the linear difference equation (3) has the Mittag-Leffler-Hyers-Ulam stability with (4). □

Corollary 4.2. *Let* $\max|p(n)| < \frac{2\gamma}{n(n-1)}$ *with* $0 < \gamma < 1$ *and for every* $\epsilon > 0$, *there exists a function* $z(n)$, $n \in (a, b+1)$ *and* $\phi(n)$ *be a positive and bounded sequence satisfies the inequality*

$$|z(n+2) + p(n)z(n)| \leq \phi(n)\epsilon F_\alpha(\lambda, n), \tag{18}$$

with initial condition $z(a) = z(a+1) = 0$. *Then, there exists a solution* $w(n)$ *satisfies the difference equation* $w(n+2)+p(n)w(n) = 0$ *with* $w(a) = w(a+1) = 0$ *such that*

$$|z(n) - w(n)| \leq K\phi(n)\epsilon F_\alpha(\lambda, n), \qquad \forall\ n \in (a, b+1).$$

Proof. Assume for every $\epsilon > 0$, there exists $z(n), n \in (a, b+1)$ and $\phi(n)$ be a positive and bounded sequence satisfies the inequality (18) with initial condition $z(a) = z(a+1) = 0$. Then, by using above Theorem 4.2, one can easily prove the Mittag-Leffler-Hyers-Ulam-Rassias stability of (3) with initial conditions (4). □

Finally, in this section we are going to establish the Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam-Rassias stability of the kth order linear difference equation (5) with initial conditions (6).

Theorem 4.3. *Let* $\max|p(n)| < \frac{\gamma(n-k)!\ k!}{n!}$ *with* $0 < \gamma < 1$*and for every* $\epsilon > 0$, *and every* $z(n)$, $n \in (a, b+1)$ *satisfies the inequality* $|z(n+k) + p(n)z(n)| \leq \epsilon F_\alpha(\lambda, n)$ *with the initial conditions* (6). *Then, there exists a solution* $w(n)$ *satisfies the difference equation*

$$w(n+k) + p(n)w(n) = 0$$

with $w(a) = w(a+1) = w(a+2) = \cdots = w(a+k-1) = 0$ *such that*

$$|z(n) - w(n)| \leq K\epsilon F_\alpha(\lambda, n).$$

Proof. Assume that for every $\epsilon > 0$, $z(n)$ be a sequence of function of kth order difference equation on $(a, b+1)$ satisfying the inequality

$$|z(n+k) + p(n)z(n)| \leq \epsilon F_\alpha(\lambda, n), \tag{19}$$

with initial conditions $z(a) = z(a+1) = z(a+2) = \cdots = z(a+k-1) = 0$. Then, by using Newton's Theorem, we have

$$z(n) = z(a) + \frac{n^{(1)}}{1!}\Delta^{(1)}z(a) + \frac{n^{(2)}}{2!}\Delta^{(2)}z(a) + \cdots + \frac{n^{(k)}}{k!}\Delta^{(k)}z(a).$$

By using initial condition (6), we obtain that

$$z(n) = \frac{n!}{(n-k)!\ k!}\ x(a+k).$$

Taking modulus on both sides and using the inequality (19), we get that

$$\begin{aligned}
|z(n)| = \left|\frac{n!}{(n-k)!\ k!}\ x(a+k)\right| &\leq \frac{n!}{(n-k)!\ k!}\ \max|z(n+k)|\,, \\
\max|z(n)| &\leq \frac{n!}{(n-k)!\ k!}\ \max|z(n+k)+p(n)z(n)-p(n)z(n)| \\
&\leq \frac{n!}{(n-k)!\ k!}\ \max|z(n+k)+p(n)z(n)| \\
&\quad + \frac{n!}{(n-k)!\ k!}\ \max|p(n)|\max|z(n)|
\end{aligned}$$

Let $K = \frac{b!}{(1-\gamma)\ (b-k)!\ k!}$. Thus we have $\max|z(n)| \leq K\epsilon F_\alpha(\lambda, n)$.

Obviously, $w(n) \equiv 0$ is a solution of the difference equation $z(n+k) + p(n)z(n) = 0$ with $w(a) = w(a+1) = w(a+2) = \cdots = y(n+k-1) = 0$ such that $|z(n) - w(n)| \leq K\epsilon F_\alpha(\lambda, n)$. Hence by the virtue of Definition 2.4, the linear difference equation (5) has the Mittag-Leffler-Hyers-Ulam stability with initial conditions (6). □

The next corollary says that the Mittag-Leffler-Hyers-Ulam-Rassias stability of the linear difference equation (5) with (6).

Corollary 4.3. *Let* $\max|p(n)| < \frac{\gamma(n-k)!\ k!}{n!}$ *with* $0 < \gamma < 1$ *for every* $\epsilon > 0$, $z(n)$ *be a sequence and there exists a function* $\phi(n)$ *be a positive and bounded sequence satisfies the inequality*

$$|z(n+k) + p(n)z(n)| \leq \phi(n)\epsilon F_\alpha(\lambda, n),$$

with initial condition (6). *Then, there exists a solution* $w(n)$ *satisfies the difference equation* $w(n+k) + p(n)w(n) = 0$ *with* $w(a) = w(a+1) = w(a+2) = \cdots = w(a+k-1) = 0$ *such that*

$$|z(n) - w(n)| \leq K\phi(n)\epsilon F_\alpha(\lambda, n), \qquad \forall\ n \in (a, b+1).$$

Proof. If for every $\epsilon > 0$, there exists $z(n), n \in (a, b+1)$ and $\phi(n)$ be a positive and bounded sequence satisfies the inequality,

$$|z(n+k) + p(n)z(n)| \leq \phi(n)\epsilon F_\alpha(\lambda, n), \tag{20}$$

with initial condition $z(a) = z(a+1) = z(a+2) = \cdots = z(a+k-1) = 0$. Then, by above Theorem 4.3, we can easily obtain the result. □

Conclusion: In this paper, we studied the Hyers-Ulam stability, Hyers-Ulam-Rassias stability, Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam-Rassias stability of the linear difference equations of first order, second order and kth order using initial conditions with the help of Newton's Theorem. In fact the results obtained in this paper can be regarded as a discrete analogue of the stability results for linear differential equation in [9]. Additionally, this paper also provides another method to study the Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam stability of the difference equations.

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Some differential subordinations and fuzzy differential subordinations using generalized integral operator

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Abstract. S.S. Miller and P.T. Mocanu introduced differential subordination and derived some properties associated with it. Motivated by this studies the aim of this paper is to establish some properties of differential subordination and fuzzy differential subordination associated with generalized integral operator which defined in the open unit disk.

Keywords: differential subordination, fuzzy subordination, fuzzy set, fuzzy best dominant.

1. Introduction

The theory of differential subordination was introduced by S.S. Miller and P.T.Mocanu in [7], then developed by many authors see also [9, 10]. In [11]

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authors have extended the notion of subordination from the geometric theory of analytic functions of one complex variable to the fuzzy set theory. In [12] the authors have defined the notion of fuzzy differential subordination. In this paper we will study differential subordination and fuzzy differential subordination for certain classes of holomorphic functions.

Let $U = \{z \in \mathbb{C} : |z| < 1\}$ and $H(U)$ denote the class of holomorphic functions in U. For $a \in \mathbb{C}$ and $n \in \mathbb{N}$, we denote by $H[a, n] = \{f \in H(U) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + ..., z \in U\}$ and $\mathcal{A}_n$=$\{f \in H(U) : f(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + ..., z \in U\}$ with $\mathcal{A}_1$=$\mathcal{A}$.

Definition 1.1 ([7]). *For the functions f and g analytic in U, we say that the function f is subordinate to g in U and written as $f \prec g$, if there exist a Schwartz function w analytic in U with $w(0) = 0, |w(z)| < 1$ $(z \in U)$ such that $f(z) = g(w(z))$ $(z \in U)$. In particular, if the function g is univalent in U, the above subordination is equivalent to $f(0) = g(0),\ f(U) \subset g(U)$.*

Definition 1.2 ([19]). *Let X be a non-empty set. An application $F : X \to$ [0, 1] is called fuzzy subset. An alternate definition, more precise would be the following:*
A pair (S, F_S), where F_S: $X \to$ [0, 1] and $supp(S, F_S) = \{x \in X : 0 < F_S(x) \leq 1\}$ is called fuzzy subset. The function F_S is called membership function of the fuzzy subset (S, F_S).

Definition 1.3 ([11]). *Let two fuzzy subsets of X be (M, F_M) and (N, F_N). We say that the fuzzy subsets M and N are equal if and only if $F_M(x) = F_N(x),\ x \in X$ and we denote this by $(M, F_M) = (N, F_N)$. The fuzzy subset (M, F_M) is contained in the fuzzy subset (N, F_N) if and only if $F_M(x) \leq F_N(x),\ x \in X$ and we denote the inclusion relation by $(M, F_M) \subseteq (N, F_N)$.*
Assume that D is a set in $\mathbb{C}$ and f, g are holomorphic functions. We indicate by

$$f(D) = supp(f(D), F_{f(D)}) = \{f(z) : 0 < F_{f(D)}(f(z)) \leq 1, z \in D\}$$

and

$$g(D) = supp(g(D), F_{g(D)}) = \{g(z) : 0 < F_{g(D)}(g(z)) \leq 1, z \in D\}.$$

Definition 1.4 ([11]). *Suppose that D is a set in $\mathbb{C}$, $z_0 \in D$ is a fixed point and let the functions $f, g \in H(D)$. The function f is named a fuzzy subordinate to g and write $f \prec_F g$ or $f(z) \prec_F g(z)$ if*
(1) $f(z_0) = g(z_0)$,
(2) $F_{f(D)}(f(z)) \leq F_{g(D)}(g(z)),\ z \in D$.

Definition 1.5 ([12]). *Let h be univalent in U and $\Psi : \mathbb{C}^3 \times U \to \mathbb{C}$. If $\mathcal{P}$ holomorphic in U satisfies the fuzzy differential subordination*

$$F_{\psi(\mathbb{C}^3.U)}(\psi(\mathcal{P}(z), z\mathcal{P}'(z), z^2\mathcal{P}''(z); z)) \leq F_{h(U)}(h(z)), \tag{1}$$

i.e.,

$$\psi(\mathcal{P}(z), z\mathcal{P}'(z), z^2\mathcal{P}''(z); z) \prec_F h(z),\ z \in U$$

then $\mathcal{P}$ is called a fuzzy solution of the fuzzy differential subordination. The univalent function q is called a fuzzy dominant of the fuzzy solutions of the fuzzy differential subordination, or more simple a fuzzy dominant, if $\mathcal{P}(z) \prec_F q(z),\ z \in U$ for all $\mathcal{P}$ satisfying (1). A fuzzy dominant $\tilde{q}$ that satisfies $\tilde{q}(z) \prec_F q(z),\ z \in U$ for all fuzzy dominant q of (1)is said to be the fuzzy best dominant of (1).

Definition 1.6. *Raina and Sharma [14] (see, also, [1], [2]) defined the integral operator for $\mu > 0$ and $a, c \in \mathbb{C}$ are such that $\Re(c-a) \geq 0$, $\mathcal{J}_\mu^{a,c} : T \to T$ as*
i) For $\Re(c-a) \geq 0$ and $\Re(a) > -\mu$ by

$$\mathcal{J}_\mu^{a,c} f(z) = \frac{\Gamma(c+\mu)}{\Gamma(a+\mu)\Gamma(c-a)} \int_0^1 (1-t)^{c-a-1} t^{a-1} f(zt^\mu) dt. \tag{2}$$

ii) For a=c by

$$\mathcal{J}_\mu^{a,a} f(z) = f(z). \tag{3}$$

Where Γ stands for Euler's Gamma function (which is valid for all complex number except the non-positive integer) for $f(z)$, it follows easily from (2) and (3) that

$$\mathcal{J}_\mu^{a,c} f(z) = z + \frac{\Gamma(c+\mu)}{\Gamma(a+\mu)} \sum_{k=2}^{\infty} \frac{\Gamma(a+k\mu)}{\Gamma(c+k\mu)} a_k z^k, (\mu > 0, \Re(c) \geq \Re(a) > -\mu). \tag{4}$$

We need the following lemmas in investigating our main results.

Lemma 1.1 ([7])**.** *Let h be a convex function with $h(0) = a$, and let $\gamma \in \mathbb{C}^*$ be a complex number with $\Re(\gamma) \geq 0$. If $\mathcal{P} \in H[a, n]$ and*

$$\mathcal{P}(z) + \frac{1}{\gamma} z\mathcal{P}'(z) \prec h(z),\ z \in U$$

then

$$\mathcal{P}(z) \prec g(z) \prec h(z),\ z \in U$$

where

$$g(z) = \frac{\gamma}{n z^{\frac{\gamma}{n}}} \int_0^z h(t) t^{\frac{\gamma}{n}-1} dt,\ z \in U.$$

Lemma 1.2 ([7])**.** *Let g be a convex function in U and let $h(z) = g(z) + n\alpha z g'(z)$, for $z \in U$, where $\alpha > 0$ and n is positive integer. If $\mathcal{P}(z) = g(0) + \mathcal{P}_n z^n + \mathcal{P}_{n+1} z^{n+1} + ...,\ z \in U$ is holomorphic in U and*

$$\mathcal{P}(z) + \alpha z\mathcal{P}'(z) \prec h(z),\ z \in U$$

then

$$\mathcal{P}(z) \prec g(z),\ z \in U,$$

and the result is sharp.

Lemma 1.3 ([13]). *Let h be a convex function with $h(0) = a$, and let $\gamma \in \mathbb{C}^*$ be a complex number with $\Re(\gamma) \geq 0$. If $\mathcal{P} \in H[a, n]$ with $\mathcal{P}(0) = a$ and $\Psi : \mathbb{C}^2.U \to \mathbb{C}, \psi(\mathcal{P}(z), z\mathcal{P}'(z)) = \mathcal{P}(z) + \frac{1}{\gamma} z\mathcal{P}'(z)$ is holomorphic in U, then*

$$F_{\psi(\mathbb{C}^2.U)}\left[\mathcal{P}(z) + \frac{1}{\gamma} z\mathcal{P}'(z))\right] \leq F_{h(U)}(h(z)),$$

implies

$$F_{\mathcal{P}(U)}(\mathcal{P}(z)) \leq F_{q(U)}(q(z)) \leq F_{h(U)}(h(z)), z \in U,$$

i.e.,

$$\mathcal{P}(z) \prec_F q(z) \prec_F h(z),$$

where

$$q(z) = \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z h(t) t^{\frac{\gamma}{n}-1} dt$$

the function q is convex and is the fuzzy best (a,n)-dominant.

Lemma 1.4 ([13]). *Suppose that q is convex function in U, let $h(z) = g(z) + n\gamma z g'(z)$, $\gamma > 0$ and $n \in \mathbb{N}$. If $\mathcal{P} \in H[q(0), n]$ and $\Psi : \mathbb{C}^2 \times U \to \mathbb{C}, \psi(\mathcal{P}(z), z\mathcal{P}'(z)) = \mathcal{P}(z) + \gamma z\mathcal{P}'(z)$ is holomorphic in U, then*

$$F_{\psi(\mathbb{C}^2.U)}\left[\mathcal{P}(z) + \gamma z\mathcal{P}'(z)\right] \leq F_{h(U)}(h(z)),$$

implies

$$F_{\mathcal{P}(U)}(\mathcal{P}(z)) \leq F_{q(U)}(q(z)), z \in U,$$

i.e.,

$$\mathcal{P}(z) \prec_F q(z)$$

and q is the fuzzy best dominant.

Recently Oros and Oros [12, 13], Lupas[4, 5, 6], Hyder [8], Wanas [17], Wanas and Bulut [16], Altınkaya and Wanas [3] and Wanas and Majeed [15, 18] have obtained fuzzy differential subordination results for certain classes of holomorphic functions.Using the integral operator defined in (4), we study differential subordinations and fuzzy differential subordinations.

2. Results based on subordination

Theorem 2.1. *Let g be a convex function, $g(0) = 1$ and let h be the function $h(z) = g(z) + \frac{z}{\lambda} g'(z)$, $z \in U$, if $a, \lambda > 0, n \in \mathbb{N}, f \in \mathcal{A}$ and satisfies the differential subordination*

$$\left(\frac{\mathcal{J}_\mu^{a,c} f(z)}{z}\right)^{\lambda-1} \left(\mathcal{J}_\mu^{a,c} f(z)\right)' \prec h(z), z \in U, \tag{5}$$

then

$$\left(\frac{\mathcal{J}_\mu^{a,c} f(z)}{z}\right)^{\lambda} \prec g(z), z \in U,$$

and this result is sharp.

Proof. Let

$$\mathcal{J}_{\mu}^{a,c}f(z) = z + \frac{\Gamma(c+\mu)}{\Gamma(a+\mu)}\sum_{k=2}^{\infty}\frac{\Gamma(a+k\mu)}{\Gamma(c+k\mu)}a_k z^k, (\mu > 0, \Re(c) \geq \Re(a) > -\mu).$$

Consider

$$\mathcal{P}(z) = \left(\frac{\mathcal{J}_{\mu}^{a,c}f(z)}{z}\right)^{\lambda} = 1 + \mathcal{P}_{\lambda}z^{\lambda} + \mathcal{P}_{\lambda+1}z^{\lambda+1} + ..., \ z \in U.$$

We deduce that $\mathcal{P} \in H[1, \lambda]$. Differentiating above equation, we get

$$\left(\frac{\mathcal{J}_{\mu}^{a,c}f(z)}{z}\right)^{\lambda-1}\left(\mathcal{J}_{\mu}^{a,c}f(z)\right)' = \mathcal{P}(z) + \frac{1}{\lambda}z\mathcal{P}'(z), \ z \in U,$$

then (5) becomes

$$\mathcal{P}(z) + \frac{1}{\lambda}z\mathcal{P}'(z) \prec h(z) = g(z) + \frac{z}{\lambda}g'(z), z \in U.$$

By using Lemma 1.2, we have

$$\mathcal{P}(z) \prec g(z), z \in U,$$

i.e.,

$$\left(\frac{\mathcal{J}_{\mu}^{a,c}f(z)}{z}\right)^{\lambda} \prec g(z), \ z \in U.$$

It can be observed that this result is sharp. □

Theorem 2.2. *Let h be a holomorphic function which satisfies the inequality $Re(1 + \frac{zh''(z)}{h'(z)}) > -\frac{1}{2}, z \in U$, and $h(0) = 1$. if $a, \lambda > 0, n \in \mathbb{N}, f \in \mathcal{A}$ and satisfies the differential subordination*

$$\left(\frac{\mathcal{J}_{\mu}^{a,c}f(z)}{z}\right)^{\lambda-1}\left(\mathcal{J}_{\mu}^{a,c}f(z)\right)' \prec h(z), z \in U, \tag{6}$$

then

$$\left(\frac{\mathcal{J}_{\mu}^{a,c}f(z)}{z}\right)^{\lambda} \prec q(z), z \in U,$$

where

$$q(z) = \frac{\lambda}{z^{\lambda}}\int_0^z h(t)t^{\lambda-1}dt.$$

The function q is convex and it is the best dominant.

Proof. Let

$$\begin{aligned}\mathcal{P}(z) &= \left(\frac{\mathcal{J}_{\mu}^{a,c}f(z)}{z}\right)^{\lambda} \\ &= \left(\frac{z+\frac{\Gamma(c+\mu)}{\Gamma(a+\mu)}\sum_{k=2}^{\infty}\frac{\Gamma(a+k\mu)}{\Gamma(c+k\mu)}a_k z^k}{z}\right)^{\lambda} \\ &= \left(1+\frac{\Gamma(c+\mu)}{\Gamma(a+\mu)}\sum_{k=2}^{\infty}\frac{\Gamma(a+k\mu)}{\Gamma(c+k\mu)}a_k z^{k-1}\right)^{\lambda} \\ &=1+\sum_{j=\lambda+1}^{\infty}\mathcal{P}_j z^{j-1} \\ &=1+\lambda\left(\frac{\Gamma(c+\mu)}{\Gamma(a+\mu)}\sum_{k=2}^{\infty}\frac{\Gamma(a+k\mu)}{\Gamma(c+k\mu)}a_k z^{k-1}\right)+ \\ &+\frac{\lambda(\lambda-1)}{2!}\left(\frac{\Gamma(c+\mu)}{\Gamma(a+\mu)}\sum_{k=2}^{\infty}\frac{\Gamma(a+k\mu)}{\Gamma(c+k\mu)}a_k z^{k-1}\right)^2+...,\end{aligned}$$

for $z \in U,\ \mathcal{P} \in H[1,\lambda]$. Differentiating $\mathcal{P}$, we obtain

$$\left(\frac{\mathcal{J}_{\mu}^{a,c}f(z)}{z}\right)^{\lambda-1}\left(\mathcal{J}_{\mu}^{a,c}f(z)\right)' = \mathcal{P}(z)+\frac{1}{\lambda}z\mathcal{P}'(z), z \in U,$$

and (6) becomes

$$\mathcal{P}(z)+\frac{1}{\lambda}z\mathcal{P}'(z) \prec h(z), z \in U.$$

Using Lemma 1.1, we have

$$\mathcal{P}(z) \prec q(z), z \in U,$$

i.e.,

$$\left(\frac{\mathcal{J}_{\mu}^{a,c}f(z)}{z}\right)^{\lambda} \prec q(z) = \frac{\lambda}{z^{\lambda}}\int_0^z h(t)t^{\lambda-1}dt,\ z \in U,$$

and q is the best fuzzy dominant. □

Corollary 2.1. *Let $h(z) = \frac{1+(2\beta-1)z}{1+z}$ be a convex function in U, where $0 \leq \beta < 1$. If $a, \lambda \geq 0, n \in \mathbb{N}, f \in \mathcal{A}$ and satisfies the differential subordination*

$$\left(\frac{\mathcal{J}_{\mu}^{a,c}f(z)}{z}\right)^{\lambda-1}\left(\mathcal{J}_{\mu}^{a,c}f(z)\right)' \prec h(z), z \in U, \tag{7}$$

then

$$\left(\frac{\mathcal{J}_{\mu}^{a,c}f(z)}{z}\right)^{\lambda} \prec q(z), z \in U,$$

where q is given by

$$q(z) = (2\beta - 1) + \frac{2(1-\beta)\lambda}{z^{\lambda}} \int_0^z \frac{t^{\lambda-1}}{1+t} dt, z \in U.$$

The function q is convex and it is the best dominant.

Proof. Following the same steps as in the proof of the Theorem 2.2 and considering $\mathcal{P}(z) = \left(\frac{\mathcal{J}_{\mu}^{a,c} f(z)}{z}\right)^{\lambda}$, the differential subordination (7) becomes

$$\mathcal{P}(z) + \frac{1}{\lambda} z\mathcal{P}'(z) \prec h(z) = \frac{1 + (2\beta - 1)z}{1+z}, z \in U.$$

By using Lemma 1.2, we have $\mathcal{P}(z) \prec q(z)$ i.e.,

$$\begin{aligned}\left(\frac{\mathcal{J}_{\mu}^{a,c} f(z)}{z}\right)^{\lambda} \prec q(z) =& \frac{\lambda}{z^{\lambda}} \int_0^z h(t) t^{\lambda-1} dt \\ =& \frac{\lambda}{z^{\lambda}} \int_0^z t^{\lambda-1} \frac{(1 + (2\beta-1)t)}{1+t} dt \\ =& \frac{\lambda}{z^{\lambda}} \int_0^z \left[(2\beta-1)t^{\lambda-1} + 2(1-\beta)\frac{t^{\lambda-1}}{1+t}\right] dt.\end{aligned}$$

Therefore,

$$\left(\frac{\mathcal{J}_{\mu}^{a,c} f(z)}{z}\right)^{\lambda} \prec q(z) = (2\beta - 1) + \frac{2(1-\beta)\lambda}{z^{\lambda}} \int_0^z \frac{t^{\lambda-1}}{1+t} dt, z \in U. \qquad \square$$

3. Results based on fuzzy subordination

Theorem 3.1. *Suppose that the convex function h satisfies $h(0) = 1$. Let $f \in \mathcal{A}$ and*

$$\frac{1}{z}\mathcal{J}_{\mu}^{a,c} f(z) + \frac{\Gamma(c+\mu)}{\Gamma(a+\mu)} \sum_{k=2}^{\infty} \frac{\Gamma(a+k\mu)}{\Gamma(c+k\mu)} a_k z^{k-1} + z(\mathcal{J}_{\mu}^{a,c} f(z))''$$

is holomorphic in U. If

$$F_{\psi(\mathbb{C}^2.U)} \left[\frac{1}{z}\mathcal{J}_{\mu}^{a,c} f(z) + \frac{\Gamma(c+\mu)}{\Gamma(a+\mu)} \sum_{k=2}^{\infty} \frac{\Gamma(a+k\mu)}{\Gamma(c+k\mu)} a_k z^{k-1} + z(\mathcal{J}_{\mu}^{a,c} f(z))''\right]$$

$$\leq F_{h(U)}(h(z)), \tag{8}$$

then

$$F_{(\mathcal{J}_{\mu}^{a,c} f)'(U)}(\mathcal{J}_{\mu}^{a,c} f(z))' \leq F_{q(U)} q(z) \leq F_{h(U)} h(z),$$

i.e.,

$$(\mathcal{J}_{\mu}^{a,c} f(z))' \prec_F q(z) \prec_F h(z),$$

where $q(z) = \frac{1}{z}\int_0^z h(t)dt$ is convex and is the fuzzy best dominant.

Proof. Assume that

$$\mathcal{P}(z) = \left(\mathcal{J}_{\mu}^{a,c} f(z)\right)'. \tag{9}$$

Then $\mathcal{P} \in H[1,1]$ and $\mathcal{P}(0) = 1$. therefore, we have

(10)

$$\begin{aligned}
\mathcal{P}(z) + z\mathcal{P}'(z) &= \left(\mathcal{J}_{\mu}^{a,c} f(z)\right)' + z\left(\mathcal{J}_{\mu}^{a,c} f(z)\right)'' \\
&= 1 + \frac{\Gamma(c+\mu)}{\Gamma(a+\mu)} \sum_{k=2}^{\infty} \frac{\Gamma(a+k\mu)}{\Gamma(c+k\mu)} a_k k z^{k-1} + \\
&+ z\left(\frac{\Gamma(c+\mu)}{\Gamma(a+\mu)} \sum_{k=2}^{\infty} \frac{\Gamma(a+k\mu)}{\Gamma(c+k\mu)} a_k k(k-1) z^{k-2}\right) \\
&= 1 + \frac{\Gamma(c+\mu)}{\Gamma(a+\mu)} \sum_{k=2}^{\infty} k^2 \frac{\Gamma(a+k\mu)}{\Gamma(c+k\mu)} a_k z^{k-1} \\
&= \frac{1}{z}\mathcal{J}_{\mu}^{a,c} f(z) + \frac{\Gamma(c+\mu)}{\Gamma(a+\mu)} \sum_{k=2}^{\infty} \frac{\Gamma(a+k\mu)}{\Gamma(c+k\mu)} a_k z^{k-1} + z(\mathcal{J}_{\mu}^{a,c} f(z))''.
\end{aligned}$$

According to (8)and (10), we deduce that

$$F_{\psi(\mathbb{C}^2.U)}\left[\mathcal{P}(z) + z\mathcal{P}'(z)\right] \leq F_{h(U)}(h(z)).$$

Thus, by applying Lemma 1.3 with $\gamma = 1$, we obtain

$$F_{\mathcal{P}(U)}\mathcal{P}(z) \leq F_{q(U)}q(z) \leq F_{h(U)}h(z), z \in U.$$

From (9), we find that

$$F_{(\mathcal{J}_{\mu}^{a,c} f)'U}(\mathcal{J}_{\mu}^{a,c} f(z))' \leq F_{q(U)}q(z) \leq F_{h(U)}h(z),$$

i.e.,

$$(\mathcal{J}_{\mu}^{a,c} f(z))' \prec_F q(z) \prec_F h(z),$$

where $q(z) = \frac{1}{z}\int_0^z h(t)dt$ is convex and is the fuzzy best dominant. □

For $a = c$ and $h(z) = \frac{1+(2p-1)z}{1+z} (0 \leq p < 1)$ in Theorem 1.3, we obtain the following corollary:

Corollary 3.1. *Let $f \in \mathcal{A}$ and $zf''(z) + f'(z)$ is holomorphic in U. If*

$$zf''(z) + f'(z) \prec_F \frac{1+(2p-1)z}{1+z},$$

then

$$f'(z) \prec_F q(z) \prec_F \frac{1+(2p-1)z}{1+z},$$

where $q(z) = 2p - 1 + \frac{2(1-p)}{z} ln(1+z)$ is convex and the fuzzy best dominant.

Theorem 3.2. *Suppose that the convex function h satisfies $h(0) = 1$. Let $f \in \mathcal{A}$ and $(\mathcal{J}_{\mu}^{a,c} f(z))'$ is holomorphic in U. If*

$$F_{\psi(\mathbb{C}^2.U)} \left[(\mathcal{J}_{\mu}^{a,c} f(z))' \right] \leq F_{h(U)} h(z), \tag{11}$$

then

$$F_{(\mathcal{J}_{\mu}^{a,c} f)(U)} \left(\frac{\mathcal{J}_{\mu}^{a,c} f(z)}{z} \right) \leq F_{q(U)} q(z) \leq F_{h(U)} h(z),$$

i.e.,

$$\left(\frac{\mathcal{J}_{\mu}^{a,c} f(z)}{z} \right) \prec_F q(z) \prec_F h(z),$$

where $q(z) = \frac{1}{z} \int_0^z h(t)dt$ is convex and is the fuzzy best dominant.

Proof. Assume that

$$\mathcal{P}(z) = \frac{\mathcal{J}_{\mu}^{a,c} f(z)}{z}. \tag{12}$$

It is clear that $\mathcal{P} \in H[1, 1]$ and $\mathcal{P}(0) = 1$, we find that

$$\mathcal{P}(z) + z\mathcal{P}'(z) = (\mathcal{J}_{\mu}^{a,c} f(z))'. \tag{13}$$

In view of (13), the fuzzy differential subordination (11) becomes

$$F_{\psi(\mathbb{C}^2.U)} \left[\mathcal{P}(z) + z\mathcal{P}'(z) \right] \leq F_{h(U)}(h(z)).$$

Thus, by applying Lemma 1.3 with $\gamma = 1$, we obtain

$$F_{\mathcal{P}(U)} \mathcal{P}(z) \leq F_{q(U)} q(z) \leq F_{h(U)} h(z), z \in U.$$

From (12), we get

$$F_{(\mathcal{J}_{\mu}^{a,c} f)(U)} \left(\frac{\mathcal{J}_{\mu}^{a,c} f(z)}{z} \right) \leq F_{q(U)} q(z) \leq F_{h(U)} h(z),$$

i.e.,

$$\left(\frac{\mathcal{J}_{\mu}^{a,c} f(z)}{z} \right) \prec_F q(z) \prec_F h(z),$$

where $q(z) = \frac{1}{z} \int_0^z h(t)dt$ is convex and is the fuzzy best dominant. □

For $a = c$ and $h(z) = e^{bz}, |b| \leq 1$ in Theorem 3.2, we obtain the following

Corollary 3.2. *Let $f \in \mathcal{A}$, $f'(z)$ is holomorphic in U. If $f'(z) \prec_F e^{bz}$, then*

$$\frac{f(z)}{z} \prec_F q(z) \prec_F e^{bz},$$

where $q(z) = \frac{e^{bz}-1}{bz}$ is convex and the fuzzy best dominant.

Theorem 3.3. *Let g be a convex function, $g(0)=1$ and let h be the function $h(z)= g(z)+\frac{z}{\lambda}g'(z), z \in U$. If $a, \lambda > 0, n \in \mathbb{N}, f \in \mathcal{A}$ and $\left(\frac{\mathcal{J}_{\mu}^{a,c}f(z)}{z}\right)^{\lambda-1}\left(\mathcal{J}_{\mu}^{a,c}f(z)\right)'$ is holomorphic in U. If*

$$F_{\psi(\mathbb{C}^2.U)}\left[\left(\frac{\mathcal{J}_{\mu}^{a,c}f(z)}{z}\right)^{\lambda-1}\left(\mathcal{J}_{\mu}^{a,c}f(z)\right)'\right] \leq F_{h(U)}h(z), \tag{14}$$

then

$$F_{(\mathcal{J}_{\mu}^{a,c}f)^{\lambda}(U)}\left(\frac{\mathcal{J}_{\mu}^{a,c}f(z)}{z}\right)^{\lambda} \leq F_{g(U)}g(z),$$

i.e.,

$$\left(\frac{\mathcal{J}_{\mu}^{a,c}f(z)}{z}\right)^{\lambda} \prec_F g(z),$$

and this result is sharp.

Proof. Assume that

$$\mathcal{P}(z) = \left(\frac{\mathcal{J}_{\mu}^{a,c}f(z)}{z}\right)^{\lambda}. \tag{15}$$

Then $\mathcal{P} \in H[1,1]$ and $\mathcal{P}(0) = 1$, therefore in view of (4) and (15), we have

$$\mathcal{P}(z) + \frac{1}{\lambda}z\mathcal{P}'(z) = \left(\frac{\mathcal{J}_{\mu}^{a,c}f(z)}{z}\right)^{\lambda-1}\left(\mathcal{J}_{\mu}^{a,c}f(z)\right)'. \tag{16}$$

According to (14) and (16), We obtained

$$F_{\psi(\mathbb{C}^2.U)}\left[\mathcal{P}(z) + \frac{1}{\lambda}z\mathcal{P}'(z)\right] \leq F_{h(U)}h(z),$$

then by applying Lemma 4 with $\gamma = \lambda$ we have

$$F_{\mathcal{P}(U)}(\mathcal{P}(z)) \leq F_{q(U)}(q(z)) \leq F_{h(U)}(h(z)), z \in U.$$

From (15) we obtain

$$F_{(\mathcal{J}_{\mu}^{a,c}f)^{\lambda}(U)}\left(\frac{\mathcal{J}_{\mu}^{a,c}f(z)}{z}\right)^{\lambda} \leq F_{g(U)}g(z),$$

i.e.,

$$\left(\frac{\mathcal{J}_{\mu}^{a,c}f(z)}{z}\right)^{\lambda} \prec_F g(z),\ z \in U,$$

and this result is sharp. □

Theorem 3.4. *Let h be a holomorphic function which satisfies the inequality $Re\left(1+\frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2}$, $z \in U$ and $h(0) = 1$ if $a, \lambda > 0$, $n \in \mathbb{N}, f \in \mathcal{A}$ and $\left(\frac{\mathcal{J}_{\mu}^{a,c} f(z)}{z}\right)^{\lambda-1} (\mathcal{J}_{\mu}^{a,c} f(z))'$ is holomorphic in U. If*

$$F_{\psi(\mathbb{C}^2.U)}\left[\left(\frac{\mathcal{J}_{\mu}^{a,c} f(z)}{z}\right)^{\lambda-1} \left(\mathcal{J}_{\mu}^{a,c} f(z)\right)'\right] \leq F_{h(U)} h(z), \tag{17}$$

then

$$F_{(\mathcal{J}_{\mu}^{a,c} f)^{\lambda}(U)}\left(\frac{\mathcal{J}_{\mu}^{a,c} f(z)}{z}\right)^{\lambda} \prec F_{q(U)} q(z), z \in U,$$

i.e.,

$$\left(\frac{\mathcal{J}_{\mu}^{a,c} f(z)}{z}\right)^{\lambda} \prec_F q(z), z \in U,$$

where $q(z) = \frac{\lambda}{z^{\lambda}} \int_0^z h(t) t^{\lambda-1} dt$ the function q is convex and it is the best dominant.

Proof. Assume that

$$\mathcal{P}(z) = \left(\frac{\mathcal{J}_{\mu}^{a,c} f(z)}{z}\right)^{\lambda}. \tag{18}$$

It is clear that $\mathcal{P} \in H[1,1]$ and $\mathcal{P}(0) = 1$, we find that

$$\mathcal{P}(z) + \frac{1}{\lambda} z\mathcal{P}'(z) = \left(\frac{\mathcal{J}_{\mu}^{a,c} f(z)}{z}\right)^{\lambda-1} \left(\mathcal{J}_{\mu}^{a,c} f(z)\right)'. \tag{19}$$

According to (17) and (19), we obtain

$$F_{\psi(\mathbb{C}^2.U)}\left[\mathcal{P}(z) + \frac{1}{\lambda} z\mathcal{P}'(z)\right] \leq F_{h(U)} h(z).$$

Then by applying Lemma 3 with $\gamma = \lambda$, we have

$$F_{\mathcal{P}(U)} \mathcal{P}(z) \leq F_{q(U)} q(z) \leq F_{h(U)} h(z), z \in U.$$

From (18), we get

$$F_{\psi(\mathbb{C}^2.U)}\left[\left(\frac{\mathcal{J}_{\mu}^{a,c} f(z)}{z}\right)^{\lambda-1} \left(\mathcal{J}_{\mu}^{a,c} f(z)\right)'\right] \leq F_{q(U)} q(z) \leq F_{h(U)} h(z),$$

and

$$\left(\frac{\mathcal{J}_{\mu}^{a,c} f(z)}{z}\right)^{\lambda} \prec_F q(z), \ z \in U,$$

where $q(z) = \frac{1}{z} \int_0^z h(t) dt$ is convex and is the fuzzy best dominant. □

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Extensions of e-reversible rings

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Abstract. Kose, et al. in [12] defined and studied the right (left) e -reversible rings. We introduce a strong condition on the Kose's notion and we call it e-strongly reversible rings, we define as follows: if $ab = 0$ implies $bea = 0$ for any $a, b \in R$, and e is an idempotent element in R. We show that e-reversible ring need not be e-strongly reversible. Also, we study some ring extensions over right (left) e-reversible rings e.g., Morita context and the Jordan construction.

Keywords: reversible ring, idempotent element, ring extensions and matrix ring.

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1. Introduction

Throughout this paper all rings R are associative with identity. Let $T_2(R)$, $E(R)$, $N(R)$, and $C(R)$ denote the 2×2 upper triangular matrix ring over R, the set of all idempotent elements, the set of nilpotent elements, and the center of R, respectively. An idempotent e of a ring R is called left (right) semicentral if $ae = eae$ ($ea = eae$) for each a in R. A ring R is called *reduced* if it has no nonzero nilpotent elements. According to Lambek [14], a ring R is *symmetric* if $abc = 0$ implies $acb = 0$ for $a, b, c \in R$. Anderson and Camillo [2] showed that a ring R is *symmetric* if and only if $r_1 r_2 ... r_n = 0$ implies $r_{\sigma(1)} r_{\sigma(2)} ... r_{\sigma(n)} = 0$, for any permutation of the set $\{1, 2, ..., n\}$ and $r_i \in R$. According to [6] a ring R is said to be *reversible* if for any $a, b \in R$, $ab = 0$ implies $ba = 0$. Anderson and Camillo [14], called zero symmetric rings and denote it by ZC_2 for what is called reversible. A ring R is called *semicommutative* if for any $a, b \in R$ such that $ab = 0$ implies $aRb = 0$. Moreover, a ring R is said to be *central reversible* if for every $a, b \in R$ such that $ab = 0$ implies ba is central in R. The study of reversible and central reversible rings can be found in [6] and [10]**.** *Central reversible* rings generalize *reversible* rings as shown in [10]. Finally a ring R is called *reflexive* if $aRb = 0$ implies $bRa = 0$ for $a, b \in R$, see [13]. A ring R is called *Abelian* if all of its idempotent elements are central. Another direction for generalizing above rings can be found in [17]. A ring R is said to be *right (left) e-reduced* if $N(R)e = 0$ ($eN(R) = 0$), and R is *e-symmetric* if whenever $abc = 0$, then $acbe = 0$ for $a, b, c \in R$, e is an idempotent element in R. Also according to [11], *any ring* R is said to be right (left) *e-semicommutative* if for any $a, b \in R$ such that $ab = 0$ implies $aRbe = 0$ ($eaRb = 0$). In [8] a ring R is called right *idempotent reflexive* if $aRe = 0$ implies $eRa = 0$ for $a \in R$, left *idempotent reflexive* can be defined similarly. If a ring R is both left and right idempotent reflexive then the ring is called an *idempotent reflexive* ring. Later reversible rings extended to e-reversible rings for an idempotent $e \in R$. A ring R is said to be right (left) *e-reversible* if $ab = 0$ implies $bae = 0$ ($eba = 0$), for $a, b \in R$ and $e \in E(R)$. See [12].

Motivated by the above, we define *e-strong reversible* ring for an idempotent $e \in R$, that is if for any $a, b \in R$, $ab = 0$ implies $bea = 0$, and study its properties. The properties of e-reversible and e-strongly reversible are independent. Suitable examples are given. We examined the transfer of *e-reversibility* over some various ring extensions. Namely, we study *e-reversibility* over the ring of polynomials, the classical ring of quotient, Jordan construction and finally the Morita context is discussed.

2. e-reversible rings

In this section we discuss the property of e–reversibility and related rings.

Definition 2.1 ([12])**.** A ring R is called right (left) e-reversible if $ab = 0$ implies $bae = 0$ ($eba = 0$), for $a, b \in R$ and $e \in E(R)$.

It is clear that for an idempotent $e \in R$, if R is reversible then R is e-reversible, since $ab = 0$ implies $bae = 0$ and $(eba) = 0$, for any $a, b \in R$.

Proposition 2.1. *Let R be a reduced ring. Then*

$$T(R) = \left\{ \begin{bmatrix} a & b & c \\ 0 & d & g \\ 0 & 0 & f \end{bmatrix} \quad \textit{such that} \quad a, b, c, d, g, f \in R \right\}$$

is right E-reversible, where $E = \begin{bmatrix} e & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ *and e in $E(R)$.*

Proof. Let $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ 0 & d_1 & g_1 \\ 0 & 0 & f_1 \end{bmatrix}$, $B = \begin{bmatrix} a_2 & b_2 & c_2 \\ 0 & d_2 & g_2 \\ 0 & 0 & f_2 \end{bmatrix}$, in R such that $AB = 0$. Then $a_1 a_2 = 0$, since R is a reduced ring, we get $(a_2 a_1)^2 = 0$, $a_2 a_1 = 0$. This leads to $a_2 a_1 e = 0$. For $E = \begin{bmatrix} e & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, then $BAE = 0$. Therefore, $T(R)$ is right E-reversible. □

We use $S_l(R)$ (resp., $S_r(R)$) for the set of right (resp., left) semicentral idempotents and $Z(R)$ for the set of central idempotents of R. Note that $S_l(R) \cap S_r(R) = Z(R)$ and if R is a semiprime ring, then $S_l(R) = S_r(R) = Z(R)$.

Theorem 2.2 ([11])**.** *Let R be a ring and $e = e^2 \in R$. Then R is right (left) e-reversible iff e is left (right) semicentral and eRe is reversible.*

Proposition 2.3. *For any ring R, if R is a right e-reversible, then R is left idempotent reflexive.*

Proof. Assume R is right *e-reversible* and $eRa = 0$, then $era = 0$. Since R is right e-reversible, $aere = 0$ and as e is left semicentral, then $are = 0$ for all $r \in R$. Concluding, $aRe = 0$, thus R is *left idempotent reflexive.* □

The implication in Proposition 2.3 is irreversible as shown in the upcoming example.

Example 2.1. Consider the ring

$$R = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b, c \in \mathbb{Z}, a + c \equiv 0 \ (mod 2) \right\}.$$

R is idempotent reflexive [5]. For $a = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$, $b = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$, we have $ab = 0$, but $ba \neq 0$. Hence R is not reversible ring. Thus R is not e-reversible ring for any non zero idempotent element in R.

Proposition 2.4. *For any ring R, if eR is right e-reversible and $(1-e)R$ is right $(1-e)$-reversible, then R is reversible.*

Proof. Let $ab = 0$ then $eab = 0$. Since eR is a right e-reversible and e is a left semicentral, $beae = bae = 0$. By the assumption that $(1-e)R$ is right $(1-e)$-reversible, we have $(1-e)ab = 0$ which leads to $b(1-e)a(1-e) = 0$, and $(1-e)$ is a left semicentral implies that $(1-e)$ is a left semicentral implying that, $ba(1-e) = 0$ and hence $ba - bae = 0$. Now, since $bae = 0$, we get $ba = 0$. Therefore, R is a reversible ring. □

In the next result we study some properties of subsets and ideals of e-reversible rings.

Lemma 2.5. *For any ring R, consider the following conditions:*

(1) *R is right e-reversible.*

(2) *$AB = 0$ implies $BAe = 0$ for any nonempty subsets A and B of R.*

(3) *$IJ = 0$ implies $JI = 0$ for all left ideals I, J of R where I is a left ideal generated by an idempotent $e \in E(R)$.*

(4) *$IJ = 0$ implies $JI = 0$ for all ideals I, J of R where I is an ideal generated by an idempotent $e \in E(R)$. Then we have the following implication* (1) $\Leftrightarrow$ (2), (2) $\Leftrightarrow$ (3), (3) $\Rightarrow$ (4).

Proof. (1) $\Leftrightarrow$ (2) Let $AB = 0$ where A, B are subsets of R. For any $a \in A$, $b \in B$ with $ab = 0$, since R is right e-reversible we have $bae = 0$, and $BAe = 0$.

(2) $\Rightarrow$ (3) Let I and J be left ideals of R such that I is generated by $e \in E(R)$. Suppose $IJ = 0$. Since I is left ideal, $RI \subset I$, then $RIJ \subset IJ = 0$ and $RIJ = 0$. Since $e \in I$ we have $ReJ = 0$. By e-reversible property of R we have $JRe = 0$. Since I is generated by e we conclude that $I = Re$. Therefore, $JI = 0$.

(3) $\Rightarrow$ (2) Clearly.

(3) $\Rightarrow$ (4) Let I, J be ideals of R where I is the ideal generated by e and $IJ = 0$. Since $IRJ \subset IJ = 0$ we get $eRJ = 0$. By e-reversibility of R, we get $JeRe = 0$ and then $JRe = 0$, $JReR = 0$. Since I is generated by e, $I = ReR$ and so $JI = 0$. □

It is shown in [9] that for a reversible ring R the homomorphic image of R need not be reversible. We now study the homomorphic image property of e-reversible rings.

The authors in [12, Theorem 2.27] show that if R/I is right $\bar{e}$-reversible, then R is right e-reversible where I is a reduced ideal of R. Moreover in [12, Proposition 2.28] they show that if R is e-symmetric ring and I an ideal of R with $I = r_R(J)$ for some subset J of R, then R/I is right $\bar{e}$-reversible. While we have the following obvious results.

Lemma 2.6. *Let R be right e-reversible ring. Then R/I is right $\bar{e}$-reversible, for any ideal I in R.*

Proposition 2.7. *For any ring R, if R/I is right $\overline{e}$-reversible with I is right e-reduced ideal in R, then R is right e-reversible.*

Proof. Let $ab = 0$ and $\overline{a}, \overline{b} \in R/I$ with $\overline{a} = a + I$, $\overline{b} = b + I$. Suppose $I = \overline{a}\overline{b} = \overline{0}$. Since R/I is right $\overline{e}$-reversible, $(b+I)(a+I)(e+I) = I$. Hence $bae \in I$. But ba is a nilpotent element since $(ba)^2 = baba = 0$, e is an idempotent and I is e-reduced, then $bae = 0$. Therefore R is right e-reversible. □

Recall that an ideal P of a ring R is called strongly prime if $a^2 \in P$ implies $a \in P$.

Proposition 2.8. *For a ring R, the quotient ring R/I by a strongly prime ideal I is a right $\overline{e}$-reversible ring for every idempotent $e \in R$.*

Proof. Let $\overline{a}\overline{b} = 0$, $ab \in I$, $baba = (ba)^2 \in I$. Since I is strongly prime, then $ba \in I$ and hence $bae \in I$. Finally $\overline{b}\overline{a}\overline{e} = 0$ and it follows that R/I is right $\overline{e}$-reversible. □

Lemma 2.9. *Let R be a right e-reversible ring and $f \in E(R)$. If R satisfies one of the following conditions, then R is right f-reversible;*

(1) $eR + (1 - f)R = R$;
(2) $ea + 1 - f \in U(R)$ for some $a \in R$;
(3) $Re + R(1 - f) = R$;
(4) $ae + 1 - f \in U(R)$ for some $a \in R$.

Proof. (1) Let $ab = 0$, R is right e-reversible,therefore e is left semicentral. Noting that $eR + (1 - f)R = R$, then $fR = feR = efeR \subseteq eR$. Since R has unity, it follows that $f = ef$. Since R is right e-reversible, then $bae = 0 = baef$, $baf = 0$. Hence, R is a right f-reversible ring.

(2) Let $ea+1-f = u \in U(R)$. Then $fu = fea$ and one obtains $f = feau^{-1}$. Noting that e is left semicentral, then $f = ef$, and this gives that R is right f-reversible ring.

(3) If $Re + R(1 - f) = R$, then $Rf = Ref$. Set $f = cef$ for some $c \in R$. Then $f = ecef = ef$ since e is left semicentral, therefore R is right f-reversible.

(4) Set $ae + 1 - f = v \in U(R)$. Then $fv = fae$ and one obtains $f = faev^{-1}$. Noting that e is left semicentral, then $f = ef$, therefore R is right f-reversible. □

3. e-strong reversible rings

In this section, we introduce the definition of e-strong reversible rings and show that e-reversible ring need not be e-strongly reversible.

Definition 3.1. Let R be a ring. Then R is called an e-strong reversible ring if for any $a, b \in R$, $ab = 0$ implies $bea = 0$.

It is clear that, R is reversible if and only if R is strongly 1-reversible. Also if R is e-strong reversible ring, $e \in Z(R)$ then R is e-reversible.

The following Example shows that e-reversible ring need not to be e-strong reversible.

Example 3.1. [[15]] Let $S = \{a, b\}$ be the semigroup with multiplication $a^2 = ab = a$ and $b^2 = ba = b$. Consider the semigroup ring $R = \mathbb{Z}_2[S] = \{0, a, b, a + b\}$, and $E(R) = \{0, a, b\}$. It is easy to check that, R is left b-reversible and R is not b-strong reversible.

Example 3.2. Consider the ring $T_2(R)$. $T_2(R)$ is $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$-strong reversible but not right $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$-reversible. Since, for $x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, y = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $e = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, we have $xy = 0$, and $yex = 0$ but $yxe \neq 0$.

We discuss the relation between central reversible and e-strong reversible properties through the following Lemma.

Lemma 3.1. *Let R be a semiprime ring. Then every central reversible is e-strong reversible.*

Proof. Let $ab = 0$, then $abe = 0$ for any $e \in R$. Since R is central reversible, $bea \in C(R)$. On the other hand $beasbea = beabeas = 0$ for any $s \in R$. Since R is semiprime, we have $bea = 0$. Therefore, R is e-strong reversible. □

Theorem 3.2. *Any ring R is an e-strong reversible ring if and only if $e \in Z(R)$ and eRe is a reversible ring.*

Proof. Assume $e \in C(R)$ and eRe is a reversible ring. By Theorem 2.2, R is a right e-reversible ring, then R is e-strong reversible. Conversely, assume that R is an e-strong reversible ring. For each $a \in R$, set $g = e + ea(1 - e)$ and $f = e + (1 - e)ae \in E(R)$. Then $eg = g$, $ge = e$, $fe = f$ and $ef = e$. Since R is an e-strong reversible we have $(1 - g)g = 0$ then $ge(1 - g) = 0$ hence $e(1 - g) = 0$ therefore $e = eg = g$. Also $e(1 - f) = 0$ then $(1 - f)e = 0$ therefore $e = fe = f$. Hence $g = e = f$. One obtains $ea(1 - e) = 0 = (1 - e)ae$ for each $a \in R$. Thus $e \in C(R)$. Further, R is a right e-reversible ring. By Theorem 2.2, eRe is a reversible ring. □

4. Some extensions of e-reversible rings

In this section we observe the e-reversibility of some important kinds of ring extensions.

For a monomorphism α of a ring R, we consider the relation between a ring R and the Jordan construction of the ring $A(R, \alpha)$. Let $A(R, \alpha) = \{x^{-i} r x^i : r \in R, i \geq 0\}$ be a subset of the skew laurent polynomial ring $R[x, x^{-1}; \alpha]$, with respect to the relations $xr = \alpha(r)\, x$ and $rx^{-1} = x^{-1}\alpha(r)$ for all $r \in$

R. Then for each $j \geq 0$, we can write $x^{-i}rx^i = x^{-(i+j)}\alpha^j(r)x^{(i+j)}$. It follows that $A(R,\alpha)$ forms a subring of $R[x,x^{-1};\alpha]$ with the following operations: $(x^{-i}rx^i)+(x^{-j}sx^j) = x^{-(i+j)}(\alpha^j(r)+\alpha^i(s))x^{(i+j)}$, $(x^{-i}rx^i)(x^{-j}sx^j) = x^{-(i+j)}(\alpha^j(r)\alpha^i(s))x^{(i+j)}$, for $r,s \in R$ and $i,j \geq 0$ (see, [7]).

Lemma 4.1 ([7], Lemma 2.1)**.** *Let α be a monomorphism of a ring R. Then $E(A) = \{x^{-i}ex^i : e \in E(R)$ and $i \geq 0\}$.*

Lemma 4.2. *Let α be a monomorphism of a ring R with $\alpha(e) = e$. Then A is right $(x^{-k}ex^k)$-reversible if and only if R is right e-reversible.*

Proof. Let R be right e-reversible and $(x^{-i}rx^i)(x^{-j}sx^j) = 0$, for some $r,s \in R$ and nonnegative integers i,j. Then $x^{-(i+j)}\alpha^j(r)\alpha^i(s)x^{(i+j)} = 0$ and so $\alpha^j(r)\alpha^i(s) = 0$. For each $k \geq 0$ we have $\alpha^{j+k}(r)\alpha^{i+k}(s) = 0$. Since R is right e-reversible, we have $\alpha^{i+k}(s)\alpha^{j+k}(r)\alpha^{i+j}(e) = 0$, for each $k \geq 0$. Therefore,

$$x^{-(i+j+k)}\alpha^{i+k}(s)\alpha^{j+k}(r)\alpha^{i+j}(e)x^{i+j+k} = 0$$

and then $(x^{-j}sx^j)(x^{-i}rx^i)(x^{-k}ex^k) = 0$. Hence A is right $(x^{-k}ex^k)$-reversible. The other direction follows easy. □

An element u of a ring R is called right regular if $ur = 0$ implies $r = 0$ for $r \in R$, similarly left regular elements can be defined. An element is regular if it is both left and right regular(and hence not a zero divisor). The multiplicative closed subsets Δ of a ring R consisting of central regular elements. The elements of $\Delta^{-1}R$ is of the form $u^{-1}r$ such that $u \in \Delta$, $r \in R\}$. We suppose that every idempotent in Δ^{-1} R is of the form $u^{-1}e$ with $e \in E(R)$ and $u \in \Delta$.

Lemma 4.3. *Let Δ be a multiplicative closed subsets of a ring R consisting of central regular elements.*

(i) If Δ^{-1} R is right $u^{-1}e$-reversible, then R is right e-reversible.

(ii) If R is right e-reversible, then Δ^{-1} R is right $u^{-1}e$-reversible.

Proof. (i) Let $\Delta^{-1}R$ be a right e-reversible ring, and $ab = 0$ where $a,b \in R$. Then $u^{-1}w^{-1}ab = 0$, and $(u^{-1}a)(w^{-1}b) = 0$. Since $\Delta^{-1}R$ is right $u^{-1}e$-reversible, we have $(w^{-1}b)(u^{-1}a)(v^{-1}e) = 0$, and then $w^{-1}u^{-1}v^{-1}bae = 0$. Finally, we have $bae = 0$. Therefor R is right e-reversible.

(ii) Conversely, let R be right e-reversible and $ab = 0$, then $bae = 0$. Multiply by $v^{-1}w^{-1}u^{-1}$ then we have $(v^{-1}b)(w^{-1}a)(u^{-1}e) = 0$. Hence $\Delta^{-1}R$ is right $w^{-1}e$-reversible. □

In [1] a ring R is called reversible Armendariz if $f(x)\,g(x) = 0$, then $b_ja_i = 0$, $\forall i$, j, whenever $f(x) = \sum_{i=0}^{m} a_ix^i, g(x) = \sum_{j=0}^{n} b_jx^j$ are polynomials in the polynomial ring $R[x]$. In [9, Example 2.1] shows that a ring R being reversible need not imply $R[x]$ being reversible. While we have the following Theorem.

Theorem 4.4. *Let R be reversible Armendariz. Then $R[x]$ is $e(x)$-reversible, where $e(x)$ is an idempotent element in $R[x]$.*

Proof. Assume that $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j$, $e(x) = \sum_{k=0}^{l} e_k x^k \in R[x]$, where e_k are idempotent elements in R. Let $f(x)g(x) = 0$. Since R is reversible Armendariz, then $b_j a_i = 0 \ \forall i, j$. Then $b_j a_i e_k = 0 \quad \forall i, j, k$. Then we have $g(x)f(x)e(x) = 0$. There for $R[x]$ is right $e(x)$-reversible Armendariz. □

Corollary 4.5. *For a ring R, $R[x]$ is a right $e(x)$-reversible ring if and only if $R[x, x^{-1}]$ is right $e(x)x^n$-reversible, where $e(x)$ is an idempotent element in $R[x]$ and $e(x)x^n$ is an idempotent element in $R[x, x^{-1}]$.*

Proof. Let $\Delta = \{1, x, x^2, ...\}$. Then Δ is a multiplicative closed central subset of the ring $R[x]$ since $R[x, x^{-1}] = \Delta^{-1}R[x]$. Thus $R[x, x^{-1}]$ is $e(x)$-reversible. The sufficient condition is straight forward since a subring of e-reversible ring is also e-reversible ring and $R[x] \subseteq R[x, x^{-1}]$. □

5. Matrix ring extensions

We devote this section to study the property of right (left) e-reversible ring over some types of matrix rings.

Given a ring R and an R-bimodule ${}_RM_R$, the trivial extension of R by M is the ring $T(R, M) = R \oplus M$ with the usual addition and the multiplication

$$(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + m_1 r_2), \text{ where } r_1, r_2 \in R \text{ and } m_1, m_2 \in M.$$

This is isomorphic to the ring of all matrices $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$, where $r \in R$ and $m \in M$ with the usual matrix operations. Therefore, we have

Theorem 5.1. (1) *If $T(R, R)$ is a right $E = \begin{bmatrix} e & 0 \\ 0 & e \end{bmatrix}$-reversible then R is a right e-reversible for each $e = e^2 \in R$.*

(2) *Suppose that R is a reduced ring. If R is right e-reversible, then $T(R, R)$ is right $E = \begin{bmatrix} e & 0 \\ 0 & e \end{bmatrix}$-reversible.*

Proof. Suppose $T(R, R)$ is right E-reversible and let $ab = 0$. Let $A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$, $B = \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix}$ in $T(R, R)$, hence $AB = 0$. Since $T(R, R)$ is E-reversible $BAE = \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} e & 0 \\ 0 & e \end{bmatrix} = \begin{bmatrix} bae & 0 \\ 0 & bae \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, this leads to $bae = 0$. Thus R is right e-reversible.

Conversely, let R be right e-reversible and $AB = 0$ for any $A, B \in T(R, R)$. Then $AB = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \begin{bmatrix} c & d \\ 0 & c \end{bmatrix} = \begin{bmatrix} ac & ad + bc \\ 0 & ac \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Therefore, $ac = 0 = ad + bc$, and since R is reduced we get $(ca)^2 = caca = 0$, $ca = 0$ and then

$cae = 0$. Since $ad + bc = 0$, then $ada + bca = 0$ which implies $ada = 0$, then $adad = 0 = (ad)^2$. Since R is a reduced ring and then $ad = 0$, also R is right e-reversible hence $dae = 0$. Therefore, $bc = 0$. Moreover, since R is a right e-reversible, $cbe = 0$. Hence $BAE = \begin{bmatrix} c & d \\ 0 & c \end{bmatrix} \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \begin{bmatrix} e & 0 \\ 0 & e \end{bmatrix} = \begin{bmatrix} cae & cbe + dae \\ 0 & cae \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. So $T(R, R)$ is a right $\begin{bmatrix} e & 0 \\ 0 & e \end{bmatrix}$-reversible ring. □

Proposition 5.2. *Let R be a ring and $e \in E(R)$, $r \in R$. Then we have the following results:*

(1) *$T_3(R)$ is right $\begin{bmatrix} 1 & r & r \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$-reversible if and only if R is reversible.*

(2) *$T_3(R)$ is right $\begin{bmatrix} e & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$-reversible if and only if R is right e-reversible.*

(3) *$T_3(R)$ is right $\begin{bmatrix} e & e & e \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$-reversible ring if and only if R is right e-reversible.*

Proof. (1) Let $a, b \in T_3(R)$ and $ab = 0$. Then we have $\begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} b & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Since $T_3(R)$ is right $\begin{bmatrix} 1 & r & r \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$-reversible, then $\begin{bmatrix} b & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & r & r \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} ba & bar & bar \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. This leads to $ba = 0$, therefor R is a reversible ring. Conversely, let $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ 0 & d_1 & e_1 \\ 0 & 0 & f_1 \end{bmatrix}$, $B = \begin{bmatrix} a_2 & b_2 & c_2 \\ 0 & d_2 & e_2 \\ 0 & 0 & f_2 \end{bmatrix} \in T_3(R)$. $AB = \begin{bmatrix} a_1a_2 & a_1b_2 + b_1d_2 & a_1c_2 + b_1e_2 + c_1f_2 \\ 0 & d_1d_2 & d_1e_2 + e_1f_2 \\ 0 & 0 & f_1f_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so $a_1a_2 = d_1d_2 = f_1f_2 = 0$. Since R is a reversible ring, then $a_2a_1 = d_2d_1 = 0 = f_2f_1$. $BA \begin{bmatrix} 1 & r & r \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} a_2a_1 & a_2a_1r & a_2a_1r \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. So $T_2(R)$ is right $\begin{bmatrix} 1 & r \\ 0 & 0 \end{bmatrix}$-reversible. Similarly, we can prove (2) and (3). □

Recall that a Morita context is a 4-tuble $T = (A, W, V, B)$ and two bimodule homomorphisms ϕ, θ where A, B are rings, ${}_AW_B$ and ${}_BV_A$ are bimodules, and there exists a context product $W \times V \to A$ and $V \times W \to B$ written multiplicatively as $(w, v) \longmapsto wv$, $(v, w) \longmapsto vw$, such that $\begin{bmatrix} A & W \\ V & B \end{bmatrix}$ is an associative ring with matrix operations. Morita context were introduced in [18]. A Morita context $\begin{bmatrix} A & W \\ V & B \end{bmatrix}$ is called trivial if the context products are trivial, i.e $WV = 0$ and $VW = 0$.

According to [4], the matrix $\begin{bmatrix} r & w \\ v & s \end{bmatrix}$ is an idempotent element in the trivial Morita context if and only if $r^2 = r$, $s^2 = s$, $rw + ws = w$ and $vr + sv = v$. Therefore, the trivial idempotents are $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Other idempotents of T are of the form $\begin{bmatrix} 1 & w \\ v & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & w \\ v & 1 \end{bmatrix}$.

We study now the e-reversibility over the trivial Morita context.

Theorem 5.3. *Let $T = \begin{bmatrix} R & W \\ V & S \end{bmatrix}$ be a trivial Morita context ring, if $Wf = W$, $Ve = V$ for any $e^2 = e \in R$ and $f^2 = f \in S$. Then T is right $\begin{bmatrix} e & w \\ v & f \end{bmatrix}$-reversible if and only if R is right e-reversible and S is right f-reversible, and $V = 0, W = 0$.*

Proof. Assume that T is $\begin{bmatrix} e & w \\ v & f \end{bmatrix}$-reversible. If we take $e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $f = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, then clearly e and f are idempotents of T. Since $eTe \cong R$ and $fTf \cong S$, using Theorem [2.2], R is right e-reversible and S is right f-reversible. Conversely, Let R be right e-reversible and S be right f-reversible rings and $\begin{bmatrix} r_1 & w_1 \\ v_1 & s_1 \end{bmatrix} \begin{bmatrix} r_2 & w_2 \\ v_2 & s_2 \end{bmatrix} = 0$, then $\begin{bmatrix} r_1r_2 & w_1w_2 + w_1s_2 \\ v_1r_2 + s_1v_2 & s_1s_2 \end{bmatrix} = 0$. Since R is right e-reversible and S is right f-reversible, then $r_2r_1e = 0$ and $s_2s_1f = 0$. Hence $\begin{bmatrix} r_2 & w_2 \\ v_2 & s_2 \end{bmatrix} \begin{bmatrix} r_1 & w_1 \\ v_1 & s_1 \end{bmatrix} \begin{bmatrix} e & w \\ v & f \end{bmatrix} = \begin{bmatrix} r_2r_1e & w^* \\ v^* & s_2s_1f \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, and consequently T is right $\begin{bmatrix} e & w \\ v & f \end{bmatrix}$-reversible. □

Theorem 5.4. *Let $T = \begin{bmatrix} r & w \\ v & s \end{bmatrix}$ be right $\begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix}$-reversible. Then R is right e-reversible.*

Proof. Let T be right $\begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix}$-reversible, $r_1r_2 = 0$ and $s_1s_2 = 0$. Since $0 = \begin{bmatrix} r_1 & 0 \\ 0 & s_1 \end{bmatrix}\begin{bmatrix} r_2 & 0 \\ 0 & s_2 \end{bmatrix} = \begin{bmatrix} r_1r_2 & 0 \\ 0 & s_1s_2 \end{bmatrix}$ then $\begin{bmatrix} r_2 & 0 \\ 0 & s_2 \end{bmatrix}\begin{bmatrix} r_1 & 0 \\ 0 & s_1 \end{bmatrix}\begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} r_2r_1e & 0 \\ 0 & 0 \end{bmatrix}$. Therefore, R is right e-reversible. □

Theorem 5.5. *Let $T = \begin{bmatrix} r & w \\ v & s \end{bmatrix}$ be left $\begin{bmatrix} 0 & 0 \\ 0 & f \end{bmatrix}$-reversible. Then S is left e-reversible.*

Proof. As in Theorem [5.4]. □

Let S and T be any rings, ${}_SM_T$ an S-T-bimodule. Then $R = \begin{bmatrix} S & M \\ 0 & T \end{bmatrix}$ is the formal triangular matrix ring. Now, we give the following Corollary as a direct consequence of Theorem 5.3.

Corollary 5.6. *Let $T = \begin{bmatrix} R & W \\ 0 & S \end{bmatrix}$ be the formal triangular matrix ring. Then the following conditions are equivalent*

(1) *T is right $\begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix}$-reversible.*

(2) *R is right e-reversible, S is right f-reversible and $f \in r_S(W)$.*

Proof. (1) $\rightarrow$ (2) Let $r_1r_2 = 0$ and $s_1s_2 = 0$. Let $\begin{bmatrix} r_1 & 0 \\ 0 & s_1 \end{bmatrix}, \begin{bmatrix} r_2 & 0 \\ 0 & s_2 \end{bmatrix} \in T$. Then $0 = \begin{bmatrix} r_1 & 0 \\ 0 & s_1 \end{bmatrix}\begin{bmatrix} r_2 & 0 \\ 0 & s_2 \end{bmatrix} = \begin{bmatrix} r_1r_2 & 0 \\ 0 & s_{1s_2} \end{bmatrix}$. Since T is right $\begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix}$-reversible, $0 = \begin{bmatrix} r_2 & 0 \\ 0 & s_2 \end{bmatrix}\begin{bmatrix} r_1 & 0 \\ 0 & s_1 \end{bmatrix}\begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix} = \begin{bmatrix} r_2r_1e & 0 \\ 0 & s_2s_1f \end{bmatrix}$. Therefore, $r_2r_1e = 0$ and $s_2s_1f = 0$. We conclude that R is right e-reversible and S is right f-reversible.

(2) $\rightarrow$ (1) Let $r_1r_2 = 0$ and $s_1s_2 = 0$. Assume $\begin{bmatrix} r_1 & w_1 \\ 0 & s_1 \end{bmatrix}, \begin{bmatrix} r_2 & w_2 \\ 0 & s_2 \end{bmatrix} \in T$, such that $0 = \begin{bmatrix} r_1 & w_1 \\ 0 & s_1 \end{bmatrix}\begin{bmatrix} r_2 & w_2 \\ 0 & s_2 \end{bmatrix} = \begin{bmatrix} r_1r_2 & r_1w_2 + w_1s_2 \\ 0 & s_1s_2 \end{bmatrix}$. Since R is right e-reversible and S is right f-reversible. We have $r_2r_1e = 0$ and $s_2s_1f = 0$. Therefore, $\begin{bmatrix} r_2 & w_2 \\ 0 & s_2 \end{bmatrix}\begin{bmatrix} r_1 & w_1 \\ 0 & s_1 \end{bmatrix}\begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix} = \begin{bmatrix} t_2t_1e & t_2w_1f + w_2s_1f \\ 0 & s_2s_1f \end{bmatrix}$. Since $f \in r_S(W)$, then $t_2w_1f + w_2s_1f = 0$. Hence, T is right $\begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix}$-reversible. □

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Partial domination in prisms of graphs

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Abstract. For any graph $G = (V, E)$ and proportion $p \in (0, 1]$, a set $S \subseteq V$ is a p-dominating set if $\frac{|N[S]|}{|V|} \geq p$. The p-domination number $\gamma_p(G)$ equals the minimum cardinality of a p-dominating set in G. For a permutation π of the vertex set of G, the graph πG is obtained from two disjoint copies G_1 and G_2 of G by joining each v in G_1 to $\pi(v)$ in G_2. i.e., $V(\pi G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{(v, \pi(v)) : v \in V(G_1), \pi(v) \in V(G_2)\}$. The graph πG is called the prism of G with respect to π. In this paper, we find some relations between the domination and the p-domination numbers in the context of graph and its prism graph for particular values of p.

Keywords: permutation graph, algebraic graph theory, prism graph.

1. Introduction

The concept of prisms of graphs was first introduced by Chartrand and Harary [1] in 1967. They used the term *permutation graphs* to define such graphs; but their definition was different from the one we have for permutation graphs as defined in [2]. Later those graphs were named as prisms of graphs with respect to a permutation. Prisms of graphs play a great role in designing computer networks.

Partial domination [3, 4] in graphs is a variation of domination introduced in 2017. In [5], we see some algebraic properties of the partial dominating sets of a graph. Here, in this paper we study prism graphs in the context of partial domination.

2. Basic terminologies

Let $G = (V(G), E(G))$ be a finite, simple and undirected graph with $V(G)$ as its vertex set and $E(G)$ as its edge set. For any $v \in V(G)$, $N_G(v) = \{u \in$



(a) Domination (b) $\frac{2}{3}$-domination

Figure 1: Domination and Partial Domination

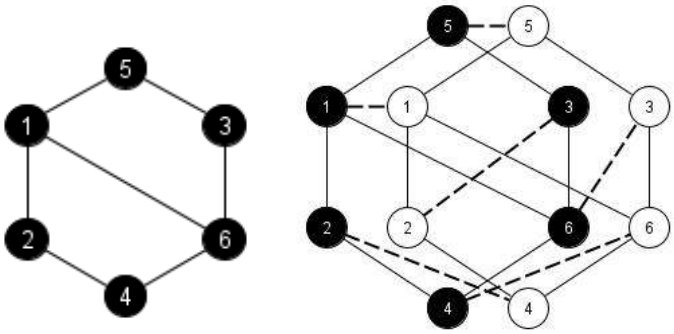


Figure 2: Graph G and its prism graph based on π=(2463)

$V(G) : uv \in E(G)\}$ and $N_G[v] = N_G(v) \cup \{v\}$ denote the open and the closed neighborhoods of v respectively. A set $S \subseteq V(G)$ is called a dominating set if every vertex in $V - S$ is adjacent to at least one vertex in S. The minimum cardinality of a dominating set is called the domination number and is denoted by $\gamma(G)$.

A subset $S \subseteq V$ is called a p-dominating set for $p \in (0, 1]$ if $\frac{|N[S]|}{|V|} \geq p$. The p-domination number, denoted by $\gamma_p(G)$ is the cardinality of the minimum p-dominating set.

In Figure 1 (a), the white vertices dominate all the vertices of the graph. In Figure 1 (b), the white vertices do not dominate the vertices of the graph. However, they dominate exactly 10 vertices of the graph. Hence, we say, the white vertices $\frac{2}{3}$-dominate the graph.

Let π be any permutation on $V(G)$. The prism πG of G with respect to π is obtained by taking two disjoint copies of G and joining each vertex v in one copy of G with $\pi(v)$ in the other copy by means of an edge. The study of domination in prisms started in 2004 [6]. Since then many works related to various domination parameters were studied [7, 8].

In [9], it has been proved that for any graph G, $\gamma(G) \leq \gamma(\pi G) \leq 2\gamma(G)$. Also, it has been defined in [6] that, G is called universal γ-fixer, if $\gamma(G) = \gamma(\pi G)$ for all permutations π of V(G) and G is called universal doubler if $\gamma(\pi G) = 2\gamma(G)$

for all permutations π of V(G). Analogous to this, in the context of partial domination we give the following definition:

Definition 2.1. *Let $p \in [0,1]$. G is called universal γ_p-fixer, if $\gamma_p(G) = \gamma_p(\pi G)$ for all permutations π of V(G) and G is called universal γ_p-doubler if $\gamma_p(\pi G) = 2\gamma_p(G)$ for all permutations π of V(G).*

Figure 2 is an example of a graph G and its prism graph with π=(2463).

3. Results

Proposition 3.1. *Let G be any n-vertex connected graph without isolated vertices and with γ=1. Then, for any permutation π of V(G) and for any $p \in (0,1]$,*

$$\gamma_p(\pi G) = \begin{cases} 1, & \text{for } p \in (0, \frac{n+1}{2n}], \\ 2, & \text{for } p \in (\frac{n+1}{2n}, 1]. \end{cases}$$

Proof. Let π be any permutation of $V(G)$. Consider πG.

Since γ(G)=1, $\exists v \in V(G)$ such that deg(v)=n-1. This v dominates n+1 vertices in πG. Hence $\frac{|N_{\pi G}[v]|}{|V(\pi G)|} = \frac{n+1}{2n}$. Thus, $\gamma_p(\pi G) = 1$, for $p \in (0, \frac{n+1}{2n}]$.

Now, consider $S = \{v, v'\}$ in πG. Let v' be the mirror image of v in the second copy of G in πG. This S dominates πG and is minimum. Thus $\gamma_p(\pi G) = 2$, for $p \in (\frac{n+1}{2n}, 1]$. □

Corollary 3.1. *Let G be any n-vertex connected graph without isolated vertices and with γ=1. Then, G is a universal γ_p-fixer for $p \in (0, \frac{n+1}{2n}]$ and is a universal γ_p-doubler for $p \in (\frac{n+1}{2n}, 1]$.*

Proposition 3.2. *Let G be any n-vertex graph. Then for any permutation π of V(G) and for any $p \in (0, \frac{n+\gamma(G)}{2n}]$, $\gamma_p(\pi G) \leq \gamma(G)$.*

Proof. Let π be any permutation of V(G). Consider πG.

Let S be a γ-set of G. Then by the definition of πG, S is a $n+\gamma(G)$ dominating set in πG. Also, if $p \leq q$ then $\gamma_p \leq \gamma_q$. Hence, for any $p \in (0, \frac{n+\gamma(G)}{2n}]$, $\gamma_p(\pi G) \leq \gamma(G)$. □

The following result shows that the above bound is sharp.

Proposition 3.3. *Let G be P_n or C_n for $n \geq 2$. Then for any permutation π on V(G), $\gamma_{\frac{n+\gamma(G)}{2n}}(\pi G) = \gamma(G)$.*

Proof. Let π be any permutation on V(G). Then by the previous proposition $\gamma_{\frac{n+\gamma(G)}{2n}}(\pi G) \leq \gamma(G)$. Hence it is enough if we prove that $\gamma_{\frac{n+\gamma(G)}{2n}}(\pi G) \geq \gamma(G)$ for any permutation π on V(G).

Let us assume the contradiction that $\gamma_{\frac{n+\gamma(G)}{2n}}(\pi G) < \gamma(G)$ for some permutation π.

WLG let $S \subseteq V(G)$ be a $\gamma_{\frac{n+\gamma(G)}{2n}}(\pi G)$-set with $\gamma(G)-1$ vertices. Let G_1 and G_2 denote the two copies of G in πG. Then two cases may arise.

Case (i). All the vertices of S are from either $V(G_1)$ or $V(G_2)$.

In this case S dominates almost $(n-1)+(\gamma(G)-1)$ vertices in πG.

$$\implies \frac{|N[S]|}{2n} \leq \frac{n-1+\gamma(G)-1}{2n} = \frac{n+\gamma(G)-2}{2n} < \frac{n+\gamma(G)}{2n}.$$

This is a contradiction to our assumption that $S \subseteq V(G)$ is a $\gamma_{\frac{n+\gamma(G)}{2n}}(\pi G)$-set. Hence the proof in this case.

Case (ii). Vertices of S are from both G_1 and G_2. WLG let there be l and m vertices from G_1 and G_2 respectively, where $l+m = \gamma(G)-1$ by our assumption. Then, we have the following:

$$\frac{|N[S]|}{2n} \leq \frac{4(l+m)}{2n} = \frac{4(\gamma(G)-1)}{2n}. \tag{1}$$

Now, $\gamma(G) = \lceil \frac{n}{3} \rceil$

$$< \frac{n}{3}+1 \implies 3\gamma(G) < n+3 < n+4. \tag{2}$$

Hence, from (1) and (2) we will get a contradiction to our assumption. Hence the proof. □

Remark 3.1. For any graph G, $\gamma_{\frac{n+\gamma(G)}{2n}}(\pi G) = 1$ if and only if $\gamma(G) = 1$.

Proposition 3.4. *Let G be any graph with u_Δ as a vertex having the maximum degree $\Delta(G)$ and u'_Δ as its mirror image in the second copy of G in πG. Then $\gamma_{\frac{n+\gamma(G)}{2n}}(\pi G) = 2$ if and only if one of the following two conditions holds for G:*

(i) $\gamma(G) = 2$;

(ii) $\gamma(G) \geq 3$ and $\Delta(G) \geq \frac{n+\gamma(G)-4+i}{2}$ where $|N[u_\Delta] \cap N[u'_\Delta]| = i$ for $0 \leq i \leq 2$.

Proof. Let us assume that $\gamma_{\frac{n+\gamma(G)}{2n}}(\pi G) = 2$. Then by the above remark $\gamma(G) \geq 2$. If $\gamma(G) = 2$,then the result is true. Hence we assume that $\gamma(G) \geq 3$. Let $|N[u_\Delta] \cap N[u'_\Delta]| = i$ for $0 \leq i \leq 2$.

We prove by the method of contradiction. Suppose $\Delta(G) < \frac{n+\gamma(G)-4+i}{2}$. Then, $|N_{\pi G}[u_\Delta] \cup N_{\pi G}[u'_\Delta]| < n+\gamma(G)$ which implies that $\gamma_{\frac{n+\gamma(G)}{2n}}(\pi G) > 2$ which is a contradiction. Hence the condition is necessary. For the proof of the sufficient part, let us assume that $\gamma(G) = 2$. Let S be a $\gamma(G)$-set of G. Then S is also a $\gamma_{\frac{n+\gamma(G)}{2n}}(\pi G)$-set of πG. Hence $\gamma_{\frac{n+\gamma(G)}{2n}}(\pi G) = 2$.

Now, let us assume the condition(ii). Consider $T = \{u_\Delta, u'_\Delta\}$. Then

$$N_{\pi G}[T] \geq 2\left[\frac{n+\gamma(G)-4+i+4-i}{2}\right] = n+\gamma(G).$$

Also, T is minimum in this context. Hence $\gamma_{\frac{n+\gamma(G)}{2n}}(\pi G) = 2$. □

Proposition 3.5. *Suppose $p \in (0,1]$ and π is any permutation on V(G), where G is any n- vertex graph. Then $\gamma_p(G) \leq \gamma_p(\pi G) \leq 2\gamma_p(G)$.*

Proof. Let us first prove the lower bound part. Let $p \in (0,1]$ and S be a γ_p-set in πG.

$$\implies \frac{|N[S]|}{2n} \geq p \implies |N[S]| \geq 2np. \tag{3}$$

Let G_1 and G_2 denote the two copies of G in πG.

Now, two cases may arise.

Case (i). All the vertices of S are from either $V(G_1)$ or $V(G_2)$ Then from 3, $|N[S]| \geq 2np > np$. Hence, S is a p-dominating set in G_1 or G_2. Thus, $\gamma_p(G) \leq \gamma_p(\pi G)$ in this case.

Case (ii): Vertices of S are from both G_1 and G_2.

Let $S = X \cup Y$ where $X \subseteq V(G_1)$ and $Y \subseteq V(G_2)$. By our assumption,

$$\frac{|N[X \cup Y]|}{2n} \geq p \implies |N[X]| + |N[Y]| \geq 2np. \tag{4}$$

Now, two cases may arise:

Subcase (i). $|N[X]| \geq np$ and $|N[Y]| \geq np$.

Let $X^* = \{w \in V(G_2)/\pi(v) = w \forall v \in X\}$ and $Y^* = \{w \in V(G_1)/\pi(w) = v \forall v \in Y\}$. Then, $X \cup Y^*$ and $X^* \cup Y$ are p-dominating sets of G. Thus,

$$\gamma_p(G) \leq |X \cup Y^*| = |X \cup Y| = \gamma_p(\pi G).$$

Subcase (ii). WLG let $|N[X]| < np$. Then $|N[Y]| \geq np + (np - |N[X]|)$ by our assumption in (4). In this case $|N[Y]| > np$. Hence, $X^* \cup Y$ is a p-dominating set of G, where X^* is defined as in the above subcase. Thus

$$\gamma_p(G) \leq |X^* \cup Y| = |X \cup Y| = \gamma_p(\pi G).$$

Hence, $\gamma_p(G) \leq \gamma_p(\pi G)$ in both the cases. Now, let us prove the upper bound part. Let S be a γ_p set of G. Let $S^* = \{v^* \in G_2/\pi(v) = v^* \forall v \in S\}$. Then, $\frac{|N[S \cup S^*]|}{2n} \geq p$. Thus, $S \cup S^*$ is a p-dominating set of πG and $|S \cup S^*| = 2\gamma_p(G)$. Hence, $\gamma_p(\pi G) \leq 2\gamma_p(G)$. □

Proposition 3.6. *Let G be a graph having an independent set $M = \{v_1, v_2, ..., v_k\}$ of k-vertices $(k \geq 1)$ each having the maximum degree $\Delta(G)$. If $N(v_i) \cap N(v_j) = \phi \forall v_i, v_j \in M$ then for $i = 1, 2, ..., k$,*

$$\gamma_p(\pi G) = i, \text{ for } p \in \left(\frac{(i-1)(\Delta(G)+2)}{2n}, \frac{i(\Delta(G)+2)}{2n} \right]$$

for any permutation π on V(G).

Proof. Let $N(v_i) \cap N(v_j) = \phi \forall v_i, v_j \in M$.

In this case, for $1 \leq i \leq k$, $S = \{v_1, v_2, ..., v_i\} \subseteq M$ dominates $i(\Delta(G) + 2)$ vertices in πG. Also since each vertex in M is of maximum degree, 'i' is the minimum number of vertices that are required to dominate $i(\Delta(G)+2)$ vertices in πG.

Hence, $\gamma_{\frac{i(\Delta(G)+2)}{2n}} = i$. By the same argument we can say that $\gamma_{\frac{(i-1)(\Delta(G)+2)}{2n}} = i-1$. Also, $\frac{(i-1)(\Delta(G)+2)}{2n}$ is the best proportion of domination possible with i-1 vertices. Hence the result is true in this case. □

Proposition 3.7. *Let G be a n-vertex graph having a set of k-mutually non-adjacent vertices($k \geq 1$) say $M = \{v_1, v_2, ..., v_k\}$ each having the maximum degree $\Delta(G)$. Let G and G' be the two copies of G in πG for any permutation π on V(G). Let $M' = \{v'_1, v'_2, ..., v'_k\}$ be the copy of M in G'. If for each $v_r \in M$, there exists exactly one $v_s \in M$ there exists exactly one such that $|N(v_r) \cap N(v_s)| = 1$, then for $i = 1, 2, ..., k$,*

$$\gamma_p(\pi G) = i, \text{ for } p \in \left(\frac{(i-1)(\Delta(G)+2)}{2n}, \frac{i(\Delta(G)+2)}{2n} \right]$$

for the following permutations π on V(G):

(i) π*=1*

(ii) $\pi(v_r) = v'_s$ *and* $\pi(v_s) = v'_r$ *for each* v_r *and* v_s *as defined above.*

(iii) $\pi \neq 1$ *and* $\pi(v_r) \neq v'_s$ *or* $\pi(v_s) \neq v'_r$ *or both and* $\pi(v_i) \in M'$ $\forall v_i \in M$.

Proof. We shall prove the theorem in three cases. For $1 \leq i \leq k$, WLG assume that i is an even number. Let $S = \{v_1, v_2, ..., v_i\} \subseteq M$ be such that there exists $\frac{i}{2}$pairs of v_r, v_s as defined above.

Case (i). $\pi = 1$.

In this case $\{v_r, v'_s\}$ for each pair of v_r, v_s will dominate $\frac{2(\Delta(G)+2)}{2n}$ vertices in πG. Thus there exists $\frac{i}{2}$ such pairs which dominate $\frac{i(\Delta(G)+2)}{2n}$ vertices in πG. Hence by an argument similar to the previous proposition, the result is proved in this case.

Case (ii). $\pi(v_r) = v'_s$ and $\pi(v_s) = v'_r$ for each v_r and v_s as defined above.

In this case $\{v_r, v'_r\}$ or $\{v_s, v'_s\}$ for each pair of v_r, v_s will serve the purpose for the required result.

Case (iii): $\pi \neq 1$ and $\pi(v_r) \neq v'_s$ or $\pi(v_s) \neq v'_r$ or both and $\pi(v_i) \in M'$ $\forall v_i \in M$. For this case we give an algorithm which returns a set T of k-vertices from $M \cup M'$ whose members are mutually non-adjacent to each other. Let $\pi(v_i) = v_i^*$ $\forall v_i \in M$ under the above permutations.

Algorithm 1 Algorithm to find T

Require: M, M', $\pi(v_x)$ $\forall v_x \in M$, $N(v_x)$ $\forall v_x$ in M and M'
Ensure: T

```
T={}, k = |M|
For v_x ∈ M
v_x^* = π(v_x)
if {v_x, v_x^*} ∩ T = φ then
    T = T ∪ {v_x}
else
    if |T| = k then
        return T
    else
        go to 2
    end if
end if
if |N(v_x) ∩ N(v_y)| = 1 then
    T = T ∪ v_y^*
end if
if |N(v_y^*) ∩ N(v_z^*)| = 1 then
    v_x = v_z and go to 4
end if
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Each vertex in T thus got from the algorithm will dominate $\frac{\Delta(G)+2}{2n}$ vertices in πG. And for $1 \leq i \leq k$, i vertices from T will dominate $\frac{i(\Delta(G)+2)}{2n}$ vertices in πG. □

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Harmonization of some fuzzy subgroups

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Abstract. This paper studies a new concept, Harmonized Fuzzy Groups (HFGs), from which different types of improvements on Rosenfeld's fuzzy groups (such as Intuitionistic Fuzzy Groups (IFGs), Pythagorean Fuzzy Groups (PFGs) and a new kind of fuzzy group presented in this paper, which is called Fermatean Fuzzy Groups (FFGs)) can be realized. Also, some properties of HFGs are studied.

Keywords: harmonized fuzzy sets, harmonized fuzzy groups, harmonized fuzzy level subgroup.

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1. Introduction

Due to the failure of the classical set to address vagueness and uncertainties, Zadeh [17] introduced the notion of fuzzy sets. These sets have become acceptable and very useful among many researchers in Mathematics, Computer Science, Engineering and other areas. Fuzzy set is able to accommodate the element in a set to some degree values between $[0, 1]$. Many works have been done in this area by many scientists. In order to put algebraic structure on these sets as we have in the case of classical group, Rosenfeld [12] developed the idea of fuzzy subgroups and fuzzy ideals. This became a very interesting area to many mathematicians. Mukherjee and Bhattacharya [6] introduced the notion of fuzzy normal subgroups and fuzzy cosets. Some results on fuzzy subgroups were also obtained by Onasanya and Ilori [7, 8, 9, 10].

However, Atanassov [1] demonstrated that fuzzy sets do not completely capture the imprecision in vagueness. A situation where the degree of indeterminacy is involved cannot be fully represented by Zadeh's fuzzy sets. In fuzzy sets, for a non empty set X, the degree of existence of an element $u \in X$ is represented by $\vartheta(u) \in [0, 1]$, its complement, the degree of nonmembership, can be $\nu(u) = 1 - \vartheta(u)$. In this case, $\vartheta(u) + \nu(u) = 1$. This has assumed no possibility of some hesitations towards either membership or nonmembership, which usually is not the case in real life. But, Atanassov demonstrated the possibility of having $\vartheta(u) + \nu(u) \leq 1$. In that case, $\pi(u) = 1 - (\vartheta(u) + \nu(u))$ is referred to as the degree of indeterminacy of $u \in X$. Hence, the triple $(u, \vartheta(u), \nu(u))$ is referred to as Intuitionistic Fuzzy Sets (IFSs).

Furthermore, Biswas [2], like Rosenfeld, developed some group structures on the IFSs of Atanassov. He also studied some of the properties. Meanwhile, Yager [14, 15] has demonstrated the possibility of $\vartheta(u) + \nu(u) \geq 1$, in which case, the inequality was replaced with $\vartheta^2(u)+\nu^2(u) \leq 1$. In this case, if $\vartheta(u) = a$, $\vartheta^2(u) = a^2$. The triple $(u, \vartheta(u), \nu(u))$ is referred to as Pythagorean Fuzzy Sets (PFSs). The measure of indeterminacy of $u \in X$ is $\pi(u) = \sqrt{1 - (\vartheta^2(u) + \nu^2(u))}$. It turned out that every IFS is a PFS but it is not true conversely. Besides, Yager [16] has extended this set to a case of $\vartheta^n(u) + \nu^n(u) \leq 1$ and $\pi(u) = \sqrt[n]{1 - (\vartheta^n(u) + \nu^n(u))}$, where $n \geq 1$.

Recently, Bhunia and Ghoria [3] put group structure on the PFSs and developed Pythagorean Fuzzy Subgroups. They studied some other algebraic properties of these new structures as in the case of fuzzy subgroups.

As always the case, some limitations of PFSs were found and demonstrated by Senapati and Yager [13]. The case of $\vartheta^2(u) + \nu^2(u) \geq 1$ was brought to bear. Thus, the condition $\vartheta^2(u) + \nu^2(u) \leq 1$ was replaced by $\vartheta^3(u) + \nu^3(u) \leq 1$, in which case, the extent of indeterminacy is $\pi(u) = \sqrt[3]{1 - (\vartheta^3(u) + \nu^3(u))}$. Hence, triple $(u, \vartheta(u), \nu(u))$ is referred to as Fermatean Fuzzy Sets (FFSs). This is a special case of the work of Yager [16].

In this paper, $\vartheta^n(u) + \nu^n(u) \leq 1$ was considered and some group structures were put on the set. The fuzzy subgroup so obtained is called Harmonized Fuzzy Group (HFG). Finally, some of the properties of HFG were studied.

The remaining part of this paper is as follows: Section 2 contains the Preliminaries; Section 3 has the Harmonized fuzzy structure. Section 4 contains the level subgroup of the Harmonized fuzzy structure and then the Conclusion in Section 5.

2. Preliminaries

Definition 2.1 ([17]). *A fuzzy set A in X is a set of ordered pairs $A = \{(u, \vartheta_A(u)) : u \in X\}$, where $\vartheta_A(u)$ is the grade of membership of $u \in A$ and $\vartheta_A : X \longrightarrow [0, 1]$ is the membership function.*

Definition 2.2 ([17]). *Let $A = \{(u, \vartheta_A(u)) : u \in X\}$ be a fuzzy set. The complement of A is defined as $A' = \{(u, \nu_A(u)) : u \in X\} = \{u \in X : \nu_A(u) = 1 - \vartheta_A(u)\}$.*

Definition 2.3 ([12]). *Let X be a group and ϑ a fuzzy subset of X. Then ϑ is called a fuzzy subgroup of X if, for any $u, v \in X$,*

$$\vartheta(uv) \geq \min\{\vartheta(u), \vartheta(v)\}$$

and

$$\vartheta(u^{-1}) \geq \vartheta(u).$$

Remark 2.1. Let ϑ and ν be fuzzy subgroups of a group X. Their intersection $\vartheta \cap \nu = \min\{\vartheta, \nu\}$ is a fuzzy subgroup [12]. But their union $\vartheta \cup \nu = \max\{\vartheta, \nu\}$ is not usually a fuzzy subgroup [4, 5]. Furthermore, let ϑ and ν be two antifuzzy subgroups of a group X. It can easily be shown that their intersection $\vartheta \cap \nu = \max\{\vartheta, \nu\}$ is an antifuzzy subgroup. Also, their union $\vartheta \cup \nu = \min\{\vartheta, \nu\}$ is an antifuzzy subgroup [7].

Theorem 2.1 ([12]). *Let X be a group and ϑ a fuzzy subset of X. Then ϑ is a fuzzy subgroup of X if and only if $\vartheta(uv^{-1}) \geq \min\{\vartheta(u), \vartheta(v)\}$ for any $u, v \in X$.*

Definition 2.4 ([1]). *An intuitionistic fuzzy set A in X is defined as*

$$A = \{\langle u, \vartheta_A(u), \nu_A(u)\rangle | u \in X\},$$

where

$$\vartheta_A : X \longrightarrow [0, 1] \text{ and } \nu_A : X \longrightarrow [0, 1]$$

are respectively degree of membership and degree of non-membership for every $u \in X$ with

$$0 \leq \vartheta_A(u) + \nu_A(u) \leq 1$$

and

$$\pi(u) = 1 - (\vartheta(u) + \nu(u))$$

is the degree of indeterminacy of $u \in X$.

Definition 2.5 ([2]). *Let X be a group and $A = \{\langle u, \vartheta_A(u), \nu_A(u)\rangle | u \in X\}$ an IFS. Then, A is an intuitionistic fuzzy group (IFG) if for any $u, v \in X$, the following are satisfied:*

(*i*) $\vartheta_A(uv) \geq \min\{\vartheta_A(u), \vartheta_A(v)\}$

(*ii*) $\vartheta_A(u^{-1}) \geq \vartheta_A(u)$

(*iii*) $\nu_A(uv) \leq \max\{\nu_A(u), \nu_A(v)\}$

(*iv*) $\nu_A(u^{-1}) \leq \nu_A(u)$

Proposition 2.1 ([2]). *Let X be a group and $A = \{\langle u, \vartheta_A(u), \nu_A(u)\rangle | u \in X\}$ an IFS. Then, A is an intuitionistic fuzzy group (IFG) if and only if for any $u, v \in X$, the following are satisfied:*

(*i*) $\vartheta_A(uv^{-1}) \geq \min\{\vartheta_A(u), \vartheta_A(v)\}$

(*ii*) $\nu_A(uv^{-1}) \leq \max\{\nu_A(u), \nu_A(v)\}$

Definition 2.6 ([14]). *A Pythagorean fuzzy set A in X is defined as*

$$A = \{\langle u, \vartheta_A(u), \nu_A(u)\rangle | u \in X\},$$

where

$$\vartheta_A : X \longrightarrow [0, 1] \text{ and } \nu_A : X \longrightarrow [0, 1]$$

are respectively degree of membership and degree of non-membership for every $u \in X$ with

$$0 \leq \vartheta_A^2(u) + \nu_A^2(u) \leq 1$$

and degree of indeterminacy of $u \in X$ is $\pi(u) = \sqrt{1 - (\vartheta^2(u) + \nu^2(u))}$.

Definition 2.7 ([3]). *Let X be a group and $A = \{\langle u, \vartheta_A(u), \nu_A(u)\rangle | u \in X\}$ a PFS. Then, A is a Pythagorean fuzzy group (PFG) if for any $u, v \in X$, the following are satisfied:*

(*i*) $\vartheta_A^2(uv) \geq \min\{\vartheta_A^2(u), \vartheta_A^2(v)\}$

(*ii*) $\vartheta_A^2(u^{-1}) \geq \vartheta_A^2(u)$

(*iii*) $\nu_A^2(uv) \leq \max\{\nu_A^2(u), \nu_A^2(v)\}$

(*iv*) $\nu_A^2(u^{-1}) \leq \nu_A^2(u)$

Proposition 2.2 ([3]). *Let X be a group and $A = \{\langle u, \vartheta_A(u), \nu_A(u)\rangle | u \in X\}$ a PFS. Then, A is a Pythagorean fuzzy group (PFG) if and only if for any $u, v \in X$,*

(*i*) $\vartheta_A^2(uv^{-1}) \geq \min\{\vartheta_A^2(u), \vartheta_A^2(v)\}$

(*ii*) $\nu_A^2(uv^{-1}) \leq \max\{\nu_A^2(u), \nu_A^2(v)\}$.

Definition 2.8 ([16]). *A generalized orthopair fuzzy set A in X is defined as*

$$A = \{\langle u, \vartheta_A(u), \nu_A(u)\rangle | u \in X\},$$

where

$$\vartheta_A : X \longrightarrow [0,1] \text{ and } \nu_A : X \longrightarrow [0,1]$$

are respectively degree of membership and degree of non-membership for every $u \in X$ with

$$0 \leq \vartheta_A^n(u) + \nu_A^n(u) \leq 1$$

and degree of indeterminacy of $u \in X$ is $\pi(u) = \sqrt[n]{1 - (\vartheta^n(u) + \nu^n(u))}$.

Remark 2.2. A special case of generalized orthopair fuzzy set for $n = 3$ was considered by [13]. In this paper, the Generalized Orthopair Fuzzy Sets in a non-empty set X are referred to as Harmonized Fuzzy Sets (HFSs) in X.

3. Harmonized fuzzy subgroups

Definition 3.1. *Let X be a group and $A = \{\langle u, \vartheta_A(u), \nu_A(u)\rangle | u \in X\}$ a HFS. Then, A is a harmonized fuzzy group (HFG), if for any $u, v \in X$ the following are satisfied:*

(*i*) $\vartheta_A^n(uv) \geq \min\{\vartheta_A^n(u), \vartheta_A^n(v)\}$

(*ii*) $\vartheta_A^n(u^{-1}) \geq \vartheta_A^n(u)$

(*iii*) $\nu_A^n(uv) \leq \max\{\nu_A^n(u), \nu_A^n(v)\}$

(*iv*) $\nu_A^n(u^{-1}) \leq \nu_A^n(u)$

Remark 3.1. For convenience we write ϑ^n and ν^n respectively for ϑ_A^n and ν_A^n, where $\vartheta^n = \underbrace{\vartheta \cdot \vartheta \cdots \vartheta}_{n \text{ times}}$, $\nu^n = \underbrace{\nu \cdot \nu \cdots \nu}_{n \text{ times}}$, if no confusion will arise. Note that for

$$\vartheta : X \longrightarrow [0,1] \text{ and } \nu : X \longrightarrow [0,1],$$
$$0 \leq \vartheta^n + \nu^n \leq \cdots \leq \vartheta^3 + \nu^3 \leq \vartheta^2 + \nu^2 \leq \vartheta + \nu \leq 1$$

when ϑ and ν are respectively membership and non-membership functions in HFS. Some operations on the HFS $A = \langle u, \vartheta, \nu\rangle$ can be obtained from [16]. These also could apply to the HFG $A = \langle u, \vartheta, \nu\rangle$.

The HFG

$$A = \{\langle u, \vartheta_A(u), \nu_A(u)\rangle | u \in X\}$$

is an IFG when $n = 1$ in which case ϑ_A is a fuzzy subgroup of X and ν_A is antifuzzy subgroup. It is a PFG when $n = 2$. Also, it is a Fermatean Fuzzy Group when $n = 3$ since the underlying fuzzy set is Fermatean Fuzzy Set.

Proposition 3.1. *Let X be a group and $A = \{\langle u, \vartheta(u), \nu(u)\rangle | u \in X\}$ a HFG. Then, for every $u \in X$*

(*i*) $\vartheta^n(u^{-1}) = \vartheta^n(u)$

(*ii*) $\nu^n(u^{-1}) = \nu^n(u)$

(*iii*) $\vartheta^n(e) \geq \vartheta^n(u)$

(*iv*) $\nu^n(e) \leq \nu^n(u)$

Proof. We begin the proof.

(*i*) To prove $\vartheta^n(u^{-1}) = \vartheta^n(u)$:
$\vartheta^n(u) = \vartheta^n((u^{-1})^{-1}) \geq \vartheta^n(u^{-1}) \geq \vartheta^n(u)$.

(*ii*) To prove $\nu^n_A(u^{-1}) = \nu^n_A(u)$:
$\nu^n(u) = \nu^n((u^{-1})^{-1}) \leq \nu^n(u^{-1}) \leq \nu^n(u)$.

(*iii*) To prove $\vartheta^n(e) \geq \vartheta^n(u)$:
$\vartheta^n(e) = \vartheta^n(uu^{-1}) \geq \min\{\vartheta^n(u), \vartheta^n(u^{-1})\} = \vartheta^n(u)$.

(*iv*) To prove $\nu^n(e) \leq \nu^n(u)$:
$\nu^n(e) = \nu^n(uu^{-1}) \leq \max\{\nu^n(u), \nu^n(u^{-1})\} = \nu^n(u)$. □

Proposition 3.2. *Let X be a group and $A = \{\langle u, \vartheta_A(u), \nu_A(u)\rangle | u \in X\}$ a HFS. Then, A is a harmonized fuzzy group (HFG) if and only if, for any $u, v \in X$*

(*i*) $\vartheta^n(uv^{-1}) \geq \min\{\vartheta^n(u), \vartheta^n(v)\}$

(*ii*) $\nu^n(uv^{-1}) \leq \max\{\nu^n(u), \nu^n(v)\}$

Proof. Let A be a harmonized fuzzy subgroup.

(*i*) By Definition 3.1 and Proposition 3.1,

$$\vartheta^n(uv^{-1}) \geq \min\{\vartheta^n(u), \vartheta^n(v^{-1})\} = \min\{\vartheta^n(u), \vartheta^n(v)\}.$$

(*ii*) By Definition 3.1 and Proposition 3.1,

$$\nu^n(uv^{-1}) \leq \max\{\nu^n(u), \nu^n(v^{-1})\} = \max\{\nu^n(u), \nu^n(v)\}.$$

Conversely,

(*i*) assume that

$$\vartheta^n(uv^{-1}) \geq \min\{\vartheta^n(u), \vartheta^n(v)\}.$$

Then,

$$\vartheta^n(e) = \vartheta^n(uu^{-1}) \geq \min\{\vartheta^n(u), \vartheta^n(u)\} = \vartheta^n(u).$$

This implies that $\vartheta^n(e) \geq \vartheta^n(u)$, $\forall u \in X$. Also note that, $\vartheta^n(u^{-1}) = \vartheta^n(eu^{-1}) \geq \min\{\vartheta^n(e), \vartheta^n(u)\} = \vartheta^n(u)$. This implies that $\vartheta^n(u^{-1}) \geq \vartheta^n(u)$, $\forall u \in X$. Furthermore, let

$$\vartheta^n(uv) = \vartheta^n(u(v^{-1})^{-1}) \geq \min\{\vartheta^n(u), \vartheta^n(v^{-1})\}.$$

There are two possible cases, namely: $\vartheta^n(u) \geq \vartheta^n(v^{-1})$ and $\vartheta^n(u) < \vartheta^n(v^{-1})$.

Case 1. If $\vartheta^n(u) \geq \vartheta^n(v^{-1})$, then $\vartheta^n(u) \geq \min\{\vartheta^n(u), \vartheta^n(v^{-1})\} = \vartheta^n(v^{-1}) \geq \vartheta^n(v)$. In this case, $\vartheta^n(u) \geq \vartheta^n(v)$. Then,

$$\begin{aligned} \vartheta^n(uv) &= \vartheta^n(u(v^{-1})^{-1}) \\ &\geq \min\{\vartheta^n(u), \vartheta^n(v^{-1})\} \\ &= \vartheta^n(v^{-1}) \\ &\geq \vartheta^n(v) \\ &= \min\{\vartheta^n(u), \vartheta^n(v)\}. \end{aligned}$$

Hence, $\vartheta^n(uv) \geq \min\{\vartheta^n(u), \vartheta^n(v)\}$.

Case 2. On the other hand, let $\vartheta^n(u) < \vartheta^n(v^{-1}) \Rightarrow \vartheta^n(v^{-1}) > \vartheta^n(u)$. We know that $\vartheta^n(v^{-1}) \geq \vartheta^n(v)$. Hence, $\vartheta^n(v^{-1}) \geq \min\{\vartheta^n(u), \vartheta^n(v)\}$. Then,

$$\begin{aligned} \vartheta^n(uv) &= \vartheta^n(u(v^{-1})^{-1}) \\ &\geq \min\{\vartheta^n(u), \vartheta^n(v^{-1})\} \\ &\geq \min\{\vartheta^n(u), \vartheta^n(v)\}. \end{aligned}$$

So, in both cases, $\vartheta^n(uv) \geq \min\{\vartheta^n(u), \vartheta^n(v)\}$.

(*ii*) assume that $\nu^n(uv^{-1}) \leq \max\{\nu^n(u), \nu^n(v)\}$. But $\nu^n(e) = \vartheta^n(uu^{-1}) \leq \max\{\nu^n(u), \nu^n(u)\} = \nu^n(u)$. This implies that $\nu^n(e) \leq \nu^n(u)$, $\forall u \in X$. Also note that, $\nu^n(u^{-1}) = \nu^n(eu^{-1}) \leq \max\{\nu^n(e), \nu^n(u)\} = \nu^n(u)$. This implies that $\nu^n(u^{-1}) \leq \nu^n(u), \forall u \in X$.

Furthermore, if $\nu^n(uv) = \nu^n(u(v^{-1})^{-1}) \leq \max\{\nu^n(u), \nu^n(v^{-1})\}$, there are two possible cases, namely: $\nu^n(u) \geq \nu^n(v^{-1})$ and $\nu^n(u) < \nu^n(v^{-1})$.

Case 1. Let $\nu^n(v^{-1}) \leq \nu^n(u)$. We know that $\nu^n(v^{-1}) \leq \nu^n(v)$. Hence, $\nu^n(v^{-1}) \leq \max\{\nu^n(u), \nu^n(v)\}$. Then,

$$\begin{aligned} \nu^n(uv) &= \nu^n(u(v^{-1})^{-1}) \\ &\leq \max\{\nu^n(u), \nu^n(v^{-1})\} \\ &\leq \max\{\nu^n(u), \nu^n(v)\}. \end{aligned}$$

Hence, $\nu^n(uv) \leq \max\{\vartheta^n(u), \vartheta^n(v)\}$.

Case 2. If $\nu^n(u) < \nu^n(v^{-1}) \Rightarrow \nu^n(v^{-1}) > \nu^n(u)$, then $\nu^n(u) < \nu^n(v^{-1}) \leq \nu^n(v)$, which implies that $\nu^n(v) = \max\{\nu^n(u), \nu^n(v)\}$. Then,

$$\begin{aligned} \nu^n(uv) &= \nu^n(u(v^{-1})^{-1}) \\ &\leq \max\{\nu^n(u), \nu^n(v^{-1})\} \\ &= \nu^n(v^{-1}) \\ &\leq \nu^n(v) \\ &= \max\{\nu^n(u), \nu^n(v)\}. \end{aligned}$$

Hence, $A = \{\langle u, \vartheta_A(u), \nu_A(u)\rangle | u \in X\}$ is a HFG.

□

Proposition 3.3. *Let X be a group and $A = \{\langle u, \vartheta(u), \nu(u)\rangle | u \in X\}$ a HFG. Then, for all $u \in X$*

(*i*) $\{\langle u, \vartheta^n(u)\rangle | u \in X\}$ *is a fuzzy subgroup of X*

(*ii*) $\{\langle u, \nu^n(u)\rangle | u \in X\}$ *is an antifuzzy subgroup of X*

Proof. The proof is straightforward by Definition 3.1. □

Example 3.1. Let the group $X = \{1, -1, i, -i\}$ such that $\vartheta(1) = 0.70$, $\vartheta(-1) = 0.60$, $\vartheta(i) = 0.40 = \vartheta(-i)$ and $\nu(1) = 0.10$, $\nu(-1) = 0.25$, $\nu(i) = 0.5 = \vartheta(-i)$. Obviously, $\langle u, \vartheta(u), \nu(u)\rangle$ is a HFG of X.

Theorem 3.1. *Let X be a group and $A = \{\langle u, \vartheta(u)\rangle | u \in X\}$ a fuzzy group (FG) of X. Then, FG $\Rightarrow$ IFG $\Rightarrow$ PFG $\Rightarrow$ FFG $\Rightarrow$ HFG.*

Proof. It has been shown in [2] that a fuzzy subgroup of a group is an IFG. Also, in [3], it was shown that an IFG is a PFG. It is only necessary to show that a PFG is a FFG and that FFG is a HFG. However, while the proof that a PFG is a FFG is left as an easy exercise, it will be shown that a FFG is a HFG.

To show that a FFG $\Rightarrow$ HFG,

(*i*) assume that $\vartheta^3(uv^{-1}) \geq \min\{\vartheta^3(u), \vartheta^3(v)\}$ and show that $\vartheta^n(uv^{-1}) \geq min\{\vartheta^n(u), \vartheta^n(v)\}$;

(*ii*) and also assume that $\nu^3(uv^{-1}) \leq \max\{\nu^3(u), \nu^3(v)\}$ and show that $\nu^n(uv^{-1}) \leq \max\{\nu^n(u), \nu^n(v)\}$.

Now,

(*i*) assume that $\vartheta^3(uv^{-1}) \geq \min\{\vartheta^3(u), \vartheta^3(v)\}$. There are two possible cases, namely: $\vartheta^3(u) \geq \vartheta^3(v)$ and $\vartheta^3(u) < \vartheta^3(v)$.

Case 1. If $\vartheta^3(u) \geq \vartheta^3(v)$, then $\vartheta^n(u) \geq \vartheta^n(v)$ which implies that $\vartheta^n(v) = \min\{\vartheta^n(u), \vartheta^n(v)\}$. Also, $\vartheta^3(uv^{-1}) \geq \min\{\vartheta^3(u), \vartheta^3(v)\} = \vartheta^3(v)$. This

implies that $\vartheta^n(uv^{-1}) \geq \vartheta^n(v) = \min\{\vartheta^n(u), \vartheta^n(v)\}$. Hence, $\vartheta^n(uv^{-1}) \geq \min\{\vartheta^n(u), \vartheta^n(v)\}$.

Case 2. If $\vartheta^3(v) > \vartheta^3(u)$, then $\vartheta^n(v) > \vartheta^n(u)$. Hence, $\vartheta^n(u) = \min\{\vartheta^n(u), \vartheta^n(v)\}$. Also, $\vartheta^3(uv^{-1}) \geq \min\{\vartheta^3(u), \vartheta^3(v)\} = \vartheta^3(u)$. This implies that $\vartheta^n(uv^{-1}) \geq \vartheta^n(u) = \min\{\vartheta^n(u), \vartheta^n(v)\}$. Hence, $\vartheta^n(uv^{-1}) \geq \min\{\vartheta^n(u), \vartheta^n(v)\}$.

(*ii*) Furthermore, assume that $\nu^3(uv^{-1}) \leq \max\{\nu^3(u), \nu^3(v)\}$. There are also two possible cases, namely: $\nu^3(u) \leq \nu^3(v)$ and $\nu^3(u) > \nu^3(v)$.

Case 1. If $\nu^3(u) \leq \nu^3(v)$, then $\nu^n(u) \leq \nu^n(v)$ which implies that $\nu^n(v) = \max\{\nu^n(u), \nu^n(v)\}$. Also, $\nu^3(uv^{-1}) \leq \max\{\nu^3(u), \nu^3(v)\} = \nu^3(v)$. This implies that $\nu^n(uv^{-1}) \leq \nu^n(v) = \max\{\nu^n(u), \nu^n(v)\}$. Hence, $\nu^n(uv^{-1}) \leq \max\{\nu^n(u), \nu^n(v)\}$.

Case 2. If $\nu^3(v) < \nu^3(u)$, then $\nu^n(v) < \nu^n(u)$. Also, $\nu^n(u) = \max\{\nu^n(u), \nu^n(v)\}$. Also, $\nu^3(uv^{-1}) \leq \max\{\nu^3(u), \nu^3(v)\} = \nu^3(u)$. This implies that $\nu^n(uv^{-1}) \leq \nu^n(u) = \max\{\nu^n(u), \nu^n(v)\}$. Hence, $\nu^n(uv^{-1}) \leq \max\{\nu^n(u), \nu^n(v)\}$. □

Remark 3.2. In [2], it was stated that an IFS set A is an IFG if and only if ϑ and ν are respectively fuzzy and antifuzzy subgroups. But this is not so, considering Example 3.1. Besides, [3] has shown that IFG is a PFG but the converse is not always true. Hence, in general, the converse of Theorem 3.1 is not always true.

Definition 3.2. *Let X be a group such that $A_1 = \langle u, \vartheta_1, \nu_1\rangle$ and $A_2 = \langle u, \vartheta_2, \nu_1\rangle$ are harmonized fuzzy groups from X. For all $u \in X$, the intersection of A_1 and A_2 is $A = \langle u, \vartheta, \nu\rangle$, where $\vartheta^n(u) = \min\{\vartheta_1^n(u), \vartheta_2^n(u)\}$ and $\nu^n(u) = \max\{\nu_1^n(u), \nu_2^n(u)\}$.*

Definition 3.3. *Let X be a group such that $A_1 = \langle u, \vartheta_1, \nu_1\rangle$ and $A_2 = \langle u, \vartheta_2, \nu_1\rangle$ are harmonized fuzzy groups from X. For all $u \in X$, the union of A_1 and A_2 is $A = \langle u, \vartheta, \nu\rangle$, where $\vartheta^n(u) = \max\{\vartheta_1^n(u), \vartheta_2^n(u)\}$ and $\nu^n(u) = \min\{\nu_1^n(u), \nu_2^n(u)\}$.*

Theorem 3.2. *Let X be a group such that $A_1 = \langle u, \vartheta_1, \nu_1\rangle$ and $A_2 = \langle u, \vartheta_2, \nu_1\rangle$ are harmonized fuzzy groups from X. For all $u \in X$, the intersection of A_1 and A_2 is $A = \langle u, \vartheta, \nu\rangle$ is also a HFG.*

Proof.

$$\begin{aligned}
\vartheta^n(uv^{-1}) &= \min\{\vartheta_1^n(uv^{-1}), \vartheta_2^n(uv^{-1})\} \\
&\geq \min\{\min\{\vartheta_1^n(u), \vartheta_1^n(v)\}, \min\{\vartheta_2^n(u), \vartheta_2^n(v)\}\} \\
&= \min\{\min\{\vartheta_1^n(u), \vartheta_2^n(u)\}, \min\{\vartheta_1^n(v), \vartheta_2^n(v)\}\} \\
&= \min\{\vartheta^n(u), \vartheta^n(v)\}.
\end{aligned}$$

$$\begin{aligned}\nu^n(uv^{-1}) &= \max\{\nu_1^n(uv^{-1}), \nu_2^n(uv^{-1})\} \\ &\leq \max\{\max\{\nu_1^n(u), \nu_1^n(v)\}, \max\{\nu_2^n(u), \nu_2^n(v)\}\} \\ &= \max\{\max\{\nu_1^n(u), \nu_2^n(u)\}, \max\{\nu_1^n(v), \nu_2^n(v)\}\} \\ &= \max\{\nu^n(u), \nu^n(v)\}. \quad \square\end{aligned}$$

Corollary 3.1. *Let X be a group such that $\{A_i = \langle u, \vartheta_i, \nu_i\rangle\}$ is an arbitrary collection of harmonized fuzzy groups from X. For all $u \in X$, the arbitrary intersection of A_i is also a HFG.*

Proof. The same step as in Theorem 3.2 will proof the corollary. $\square$

Remark 3.3. The following example shows that the union of two HFGs is not a HFG.

Example 3.2. Consider the group $(\mathbb{Z}, +)$. Let $A_1 = \langle u, \vartheta_1, \nu_1\rangle$ and $A_2 = \langle u, \vartheta_2, \nu_2\rangle$ be two HFGs such that $A = A_1 \cup A_2 = \langle u, \vartheta, \nu\rangle$ and ϑ, ν, ϑ_1, ν_1, ϑ_2, ν_2 are as defined below:

$$\vartheta(u) = \begin{cases} 0.4, & u \in 5\mathbb{Z} \\ 0.2, & u \in (2\mathbb{Z} - 5\mathbb{Z}) \\ 0.1, & elsewhere \end{cases} \qquad \nu(u) = \begin{cases} 0.1, & u \in 5\mathbb{Z} \\ 0.3, & u \in (2\mathbb{Z} - 5\mathbb{Z}) \\ 0.5, & elsewhere \end{cases}$$

$$\vartheta_1(u) = \begin{cases} 0.4, & u \in 5\mathbb{Z} \\ 0.1, & elsewhere \end{cases} \qquad \nu_1(u) = \begin{cases} 0.1, & u \in 5\mathbb{Z} \\ 0.6, & elsewhere \end{cases}$$

$$\vartheta_2(u) = \begin{cases} 0.2, & u \in 2\mathbb{Z} \\ 0.1, & elsewhere \end{cases} \qquad \nu_2(u) = \begin{cases} 0.3, & u \in 5\mathbb{Z} \\ 0.5, & elsewhere \end{cases}$$

it will be shown that from the above that $\vartheta^n(15 + (-2)) = \vartheta^n(13) = 0.1^n$ and $\min\{\vartheta^n(15), \vartheta^n(-2)\} = \min\{0.4^n, 0.2^n\} = 0.2^n$. But $0.1^n \not\geq 0.2^n$. Hence, $\vartheta^n(uv) \not\geq \min\{\vartheta^n(u), \vartheta^n(v)\}$. Also, $\nu^n(15 + (-2)) = \nu^n(13) = 0.5^n$ and $\max\{\nu^n(15), \nu^n(-2)\} = \max\{0.1^n, 0.3^n\} = 0.3^n$. But $0.5^n \not\leq 0.3^n$. Hence, $\nu^n(uv) \not\leq \max\{\nu^n(u), \nu^n(v)\}$.

Remark 3.4. By definition, if ϑ is a fuzzy subgroup and ν an antifuzzy subgroup of a group X, $\vartheta(u^s) \geq \vartheta(u)$ and $\nu(u^s) \leq \nu(u)$ for any integer s. Hence, for HFG $A = \langle u, \vartheta, \nu\rangle$, since ϑ^n is a fuzzy subgroup by Proposition 3.3, $\vartheta^n(u^s) \geq \vartheta^n(u)$. Similarly, since ν^n is an antifuzzy subgroup by Proposition 3.3, $\nu^n(u^s) \leq \nu^n(u)$.

Proposition 3.4. *Let $A = \langle u, \vartheta, \nu\rangle$ be a HFG of a group X. Let $e, u \in X$, where e is the identity. If $\vartheta^n(u) = \vartheta^n(e)$, then $\vartheta^n(uv) = \vartheta^n(v)$, $\forall v \in X$ and if $\nu^n(u) = \nu^n(e)$, then $\nu^n(uv) = \nu^n(v)$, $\forall v \in X$.*

Proof.

$$\begin{aligned}\vartheta^n(v) &= \vartheta^n(u^{-1}uv)\\ &\geq \min\{\vartheta^n(u), \vartheta^n(uv)\}\\ &= \min\{\vartheta^n(e), \vartheta^n(uv)\}\\ &= \vartheta^n(uv)\\ &\geq \min\{\vartheta^n(u), \vartheta^n(v)\}\\ &= \min\{\vartheta^n(e), \vartheta^n(v)\}\\ &= \vartheta^n(v).\end{aligned}$$

Hence, $\vartheta^n(uv) = \vartheta^n(v)$. On the other hand,

$$\begin{aligned}\nu^n(v) &= \nu^n(u^{-1}uv)\}\\ &\leq \max\{\nu^n(u), \nu^n(uv)\}\\ &= \max\{\nu^n(e), \nu^n(uv)\}\\ &= \nu^n(uv)\\ &\leq \max\{\nu^n(u), \nu^n(v)\}\\ &= \max\{\nu^n(e), \nu^n(v)\}\\ &= \nu^n(v).\end{aligned}$$

Hence, $\nu^n(uv) = \nu^n(v)$. □

Proposition 3.5. *Let $A = \langle u, \vartheta, \nu\rangle$ be a HFG of a group X. The set $\mathbb{G} = \{u \in X : \vartheta^n(u) = \vartheta^n(e)$ and $\nu^n(u) = \nu^n(e)\}$ is a subgroup of X.*

Proof. Let $u, v^{-1} \in \mathbb{G}$. By Proposition 3.4 and since A is a HFG, $\vartheta^n(uv^{-1}) = \vartheta^n(v^{-1}) = \vartheta^n(v) = \vartheta^n(e)$. Also, by the same reasons, $\nu^n(uv^{-1}) = \nu^n(v^{-1}) = \nu^n(v) = \nu^n(e)$. Thus, $uv^{-1} \in \mathbb{G}$. Hence, $\mathbb{G}$ is a subgroup of X. □

4. (α, β)-level of a harmonized fuzzy subgroup

Bhunia *et al* [3] has discussed the Pythagorean fuzzy level subgroup but the Harmonized fuzzy level subgroup (HFLG) called (α, β)-Level of a Harmonized Fuzzy Subgroup is now being discussed.

Definition 4.1. *Let X be a set and $A = \langle u, \vartheta, \nu\rangle$ a HFS of X. Then, the set*

$$A_{(\alpha,\beta)} = \{u \in X : \vartheta(u) \geq \alpha \text{ and } \nu(u) \leq \beta, \quad \alpha, \beta \in [0, 1]\},$$

where $0 \leq \vartheta^n(u) + \nu^n(u) \leq 1$, is called Harmonized fuzzy level subset of the set X.

Definition 4.2. *Let X be a group and $A = \langle u, \vartheta, \nu\rangle$ a HFG of X. Then, the set*

$$A_{(\alpha,\beta)} = \{u \in X : \vartheta(u) \geq \alpha \text{ and } \nu(u) \leq \beta, \quad \alpha, \beta \in [0, 1]\},$$

where $0 \leq \vartheta^n(u) + \nu^n(u) \leq 1$, *is called Harmonized fuzzy level subgroup of the group* X.

Proposition 4.1. *Let* $A = \langle u, \vartheta_a, \nu_a \rangle$ *and* $B = \langle u, \vartheta_b, \nu_b \rangle$ *be HFGs of a group* X. *Then, the following hold:*

(*i*) $A_{(\theta,\beta)} \subseteq B_{(\epsilon,\alpha)}$ *if* $\epsilon \leq \theta$ *and* $\alpha \geq \beta$, $\forall \alpha, \beta, \epsilon, \theta \in [0,1]$

(*ii*) $A \subseteq B \Rightarrow A_{(\theta,\beta)} \subseteq B_{(\theta,\beta)}$, $\forall \beta, \theta \in [0,1]$.

Proof. Note that $\alpha, \beta, \epsilon, \theta \in [0,1]$.

(i) Let $u \in A_{(\theta,\beta)}$. Then, $\vartheta_a^n(u) \geq \theta \geq \epsilon$ and $\nu_a^n(u) \leq \beta \leq \alpha$. This implies that $u \in A_{(\epsilon,\alpha)}$ and thus $A_{(\theta,\beta)} \subseteq A_{(\epsilon,\alpha)}$

(ii) Assume that $A \subseteq B$, then $\vartheta_a(u) \leq \vartheta_b(u)$ and $\nu_a(u) \geq \nu_b(u)$, $\forall\ u \in X$. This implies that $\vartheta_a^n(u) \leq \vartheta_b^n(u)$ and $\nu_a^n(u) \geq \nu_b^n(u)$, $\forall\ u \in X$.

Let $u \in A_{(\theta,\beta)}$, $\vartheta_a^n(u) \geq \theta$ and $\nu_a^n(u) \leq \beta$. This implies that $\vartheta_b^n(u) \geq \vartheta_a^n(u) \geq \theta$ and $\nu_b^n(u) \leq \nu_a^n(u) \leq \beta$. Hence, $\vartheta_b^n(u) \geq \theta$ and $\nu_b^n(u) \leq \beta$, which implies that $u \in B_{(\theta,\beta)}$ and $A_{(\theta,\beta)} \subseteq B_{(\theta,\beta)}$.

□

Example 4.1. Let $X = S_3$. Then, $A = \langle u, \vartheta_a, \nu_a \rangle$ and $B = \langle u, \vartheta_b, \nu_b \rangle$ are HFGs by the definitions below:

$$\vartheta_a(u) = \begin{cases} 0.9,\ u = e \\ 0.7,\ u \in \{(123),(132)\} \\ 0.5,\ elsewhere \end{cases} \qquad \nu_a(u) = \begin{cases} 0.2,\ u = e \\ 0.5,\ u \in \{(123),(132)\} \\ 0.7,\ elsewhere \end{cases}$$

$$\vartheta_b(u) = \begin{cases} 0.8,\ u = e \\ 0.5,\ elsewhere \end{cases} \qquad \nu_b(u) = \begin{cases} 0.3,\ u = e \\ 0.8,\ elsewhere \end{cases}$$

$A_{(0.7,0.5)} = \{e, (123), (132)\}$ and $A_{(0.5,0.8)} = \{e, (12), (13), (23), (123), (132)\}$, so $A_{(0.7,0.5)} \subseteq A_{(0.5,0.8)}$. Note that $B \subseteq A$ and $B_{(0.7,0.5)} = \{e\} \subseteq \{e, (123), (132)\} = A_{(0.7,0.5)}$.

Proposition 4.2. *Let* $A = \langle u, \vartheta, \nu \rangle$ *be a HFG of a group* X. *Let the set* $A_{(\theta,\beta)}$ *be a HFLG of* A *for which* $\vartheta^n(e) \geq \theta$ *and* $\nu^n(e) \leq \beta$, *where* e *is the identity of* X. *Then,* $A_{(\theta,\beta)}$ *is a subgroup of* X.

Proof. Note that $A_{(\theta,\beta)} \neq \emptyset$ since $\vartheta^n(e) \geq \theta$ and $\nu^n(e) \leq \beta$ implies that $e \in A_{(\theta,\beta)}$. If no other element is in $A_{(\theta,\beta)}$, there is no need to prove anything since $\{e\} \subseteq A_{(\theta,\beta)}$. But suppose $u, v \in A_{(\theta,\beta)}$, then

$$\begin{aligned} \vartheta^n(uv^{-1}) &\geq \min\{\vartheta^n(u), \vartheta^n(v)\} \\ &= \theta \end{aligned}$$

and

$$\begin{aligned} \nu^n(uv^{-1}) &\leq \max\{\nu^n(u), \nu^n(v)\} \\ &= \beta. \end{aligned}$$

Hence, $uv^{-1} \in A_{(\theta,\beta)}$. Thus, $A_{(\theta,\beta)}$ is a subgroup of X. □

5. Conclusion

In this paper, some existing improvements on fuzzy sets and their algebraic structures have been studied, namely: Intuitionistic Fuzzy Sets (IFSs) and Intuitionistic Fuzzy Groups (IFGs); Pythagorean Fuzzy Sets (PFSs) and Pythagorean Fuzzy Groups (PFGs); Fermatean Fuzzy Sets (FFSs) on which algebraic structures are now built, Fermatean Fuzzy Groups (FFGs). All these algebraic structures are now harmonized in one structure, Harmonized Fuzzy Group (HFG). Some of the properties of these structures were studied in detail. In subsequent studies, it will be of interest to consider further properties such as Harmonized Fuzzy Cosets and Harmonized Normal Fuzzy Subgroup and some related properties.

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Analysis of the dynamics of rabies in North Shewa, Ethiopia

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Abstract. In this paper, authors proposed and developed a deterministic model that explains the dynamics of rabies infection in human and dog populations in North Shewa, Ethiopia. The rabies model was analysed by determining a feasible region, positivity of the solution set, basic reproductive number, equilibrium points and their stability. Next Generation Matrix approach was employed in computing the basic reproduction number and its entirely dependent on the parameters of dog population. Its the threshold value that determines the dynamics of rabies transmission in North Shewa, Ethiopia. It was established that whenever $R_0 < 1$, rabies will die out from the population and whenever $R_0 > 1$, rabies persists in the population. Contribution each parameter was conducted using sensitivity analysis of the parameter values of the basic reproductive number. Some parameters were more sensitive and responsible for the spread of rabies in North Shewa region of Ethiopia. Moreover, stability analysis approach was employed to determine the disease free and endemic equilibrium of the rabies model. The results of the numerical simulation showed that as the contact rate and infecton rate increases, the disease spread faster. Moreover, as the removal rate and treatment rate increases, the disease spread decreases.

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Keywords: rabies model, contact rate, stability, model validation, sensitivity.

1. Introduction

Rabies is an acute and fatal zoontic disease remains one of the most feared and important threats public health around the world [1]. Rabies virus also circulates among domesticated animals and among various bat species. Despite geographic and ecological overlap in the ranges of many of these species, the implications of multispecies host susceptibility and the community ecology of rabies has rarely been examined [2].

The Ethiopian Health and Nutrition Research Institute, the current Ethiopian Public Health Institute, indicated that human rabies has been reported in Ethiopia in 1903 for the first time. Ethiopia is among high burden African countries in regard to human rabies virus exposure since ancient times [3]. Among the first symptom of the rabies pain, burning or numbness at the site of the infections, the production of large quantities of saliva and tear coupled with an inability to speak or swallow are typical during the later stage of the diseases; this can be result in hydro phobia in which the animal difficulty swallowing because the throat and jaw become slowly paralyzed, show panic when presented liquid to drink and cannot quench it thirst [4].

The rabies is manly transmitted from animal to human through close contact with infected saliva via bite scratch and invariably results in death [5]. Domestic and wild animals can potentially transmit the disease to humans mainly through biting [5]. The transmission of rabies virus usually begins when infected saliva of a host is passed to an uninfected animal [6].The most common mode of rabies virus transmission is through the bite and virus containing saliva of an infected host.

China and India have the most and second most reported cases, respectively [7]. Despite, the preventable nature of the disease and existence of effective and economical control strategy [8], rabies remain as one of the major public health problem resulting in an estimate loss of approximate 60,000 lives in the worldwide each year which almost all of the case belong to Africa and Asia [5].

Dogs were the reasons for 97% of rabies related human death in Ethiopia [9]. Despite, the availability of vaccine both to human and dogs in the country, only 3.9% dogs were found vaccinated as the study done from 2009 - 2012 in Addis Ababa [9]. The widespread use of traditional medications and religious approach to treat rabies cases as evil spirit are also the challenge for prevention and control of rabies as shown in Figure 1.

[10] formulated a mathematical model to investigate rabies epidemics and found that vaccination rate is one of the key components determining rabies transmission. The work by [11] further extended the existing models on rabies transmission between dogs to include dog to human transmission and concluded that human post exposure prophylaxis (PEP) with a dog vaccination campaign was the more cost effective in controlling the disease in the long run.

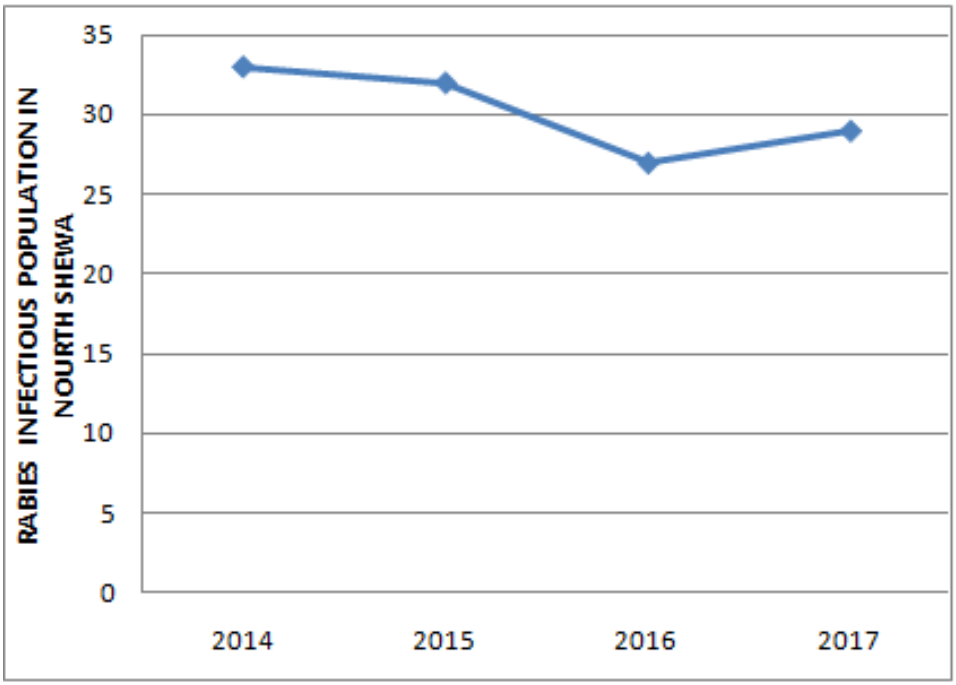


Figure 1: Rabies case reported in Nourth Shewa from 2014 to 2017

Generally, biological or epidemiological models explain the transmission dynamics of diseases and determine the persistence or die out of such diseases with time. They also determine the contribution of each parameter to the basic reproductive number and the best optimal control measure through sensitivity analysis and optimal control theory respectively [12, 13, 14].

1.1 Model description and formulation

1.1.1 Model formation

In this model we consider population of dogs and human. Each population is subdivided in to four compartments; susceptible, exposed, infected and recovers. Susceptible humans are recruited at a rate μ_h either by birth or immigration, and their number increase from individuals that come from sub-classes of rabies recovered by losing their temporary immunity with rate of σ and decrease by individuals that move to exposed compartments at a rate of β_h and natural death rate with rate θ_h. Some exposed human population move to infected compartment at the rate of δ_h and the remaining exposed human population who get the drug(vaccination) compartment join the recover compartment at a rate of τ. The infected human compartment decrease both by natural and rabbies induced death rate θ_h and ω_h respectively.

Susceptible dogs are recruited at a rate μ_d either by birth or immigration, and their number increases from individuals that come from sub-classes of rabies recover by losing their temporary immunity with rate of γ and their number decrease by individuals that move to exposed compartments with rate of β_d and natural death rate with rate θ_d. The exposed population of dog which get drug go to the recovery by rate of α and the remaining which not get the drug with time go to the infected class. The infected population of dog reduces by natural death (θ_d), diseased induce rate (ω_d) and the removing parameter(ω_2). Total human population is given by; $N(t) = S(t) + E(t) + I(t) + R(t)$

The dynamics of rabbies transmission in human and dogs population is represented in the schematic diagram as shown in Fig: 2

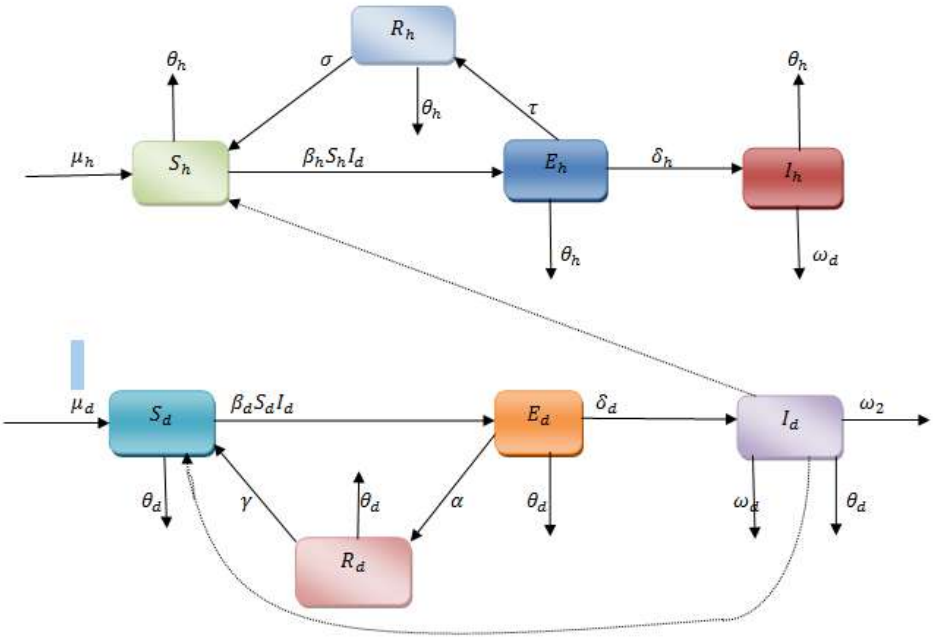


Figure 2: Schematic diagram of the model

1.1.2 Model equations

Based on the assumptions and interrelation between the variables and parameters in figure 2, the following system of ordinary differential equation generated.

$$(1)\qquad \begin{cases} \frac{dS_h}{dt} = \mu_h + \sigma R_h - (\beta_h I_d + \theta_h) S_h, \\ \frac{dE_h}{dt} = \beta_h S_h I_d - (\delta_h + \tau + \theta_h) E_h, \\ \frac{dI_h}{dt} = \delta_h E_h - (\theta_h + \omega_h) I_h, \\ \frac{dR_h}{dt} = \tau E_h - (\sigma + \theta_h) R_h, \\ \frac{dS_d}{dt} = \mu_d + \gamma R_d - (\beta_d I_d + \theta_d) S_d, \\ \frac{dE_d}{dt} = \beta_d S_d I_d - (\delta_d + \theta_d + \alpha) E_d, \\ \frac{dI_d}{dt} = \delta_d E_d - (\theta_d + \omega_2 + \omega_d) I_d, \\ \frac{dR_d}{dt} = \alpha E_d - (\theta_d + \gamma) R_d. \end{cases}$$

1.2 Model analysis

1.2.1 Invariant region

The region in which solutions of rabbies model system is uniformly bounded is the proper subset $\Omega \in \mathbb{R}^8$ and $\Omega = N_H \cup N_D \in \mathbb{R}^4 \times \mathbb{R}^4$.

Considering the human population at any time t; $N_H = S_h + E_h + I_h + R_h$.

The feasible solution of human population of model system in equation (1);

$$(2)\qquad D_h = \left\{ (S_h,\, E_h,\, I_h,\, R_h) \in \mathbb{R}^4,\, 0 \le N_H \le \frac{\mu_h}{\theta_h} \right\}.$$

Moreover, considering dogs population, denoted by N_D; $N_D = S_d + E_d + I_d + R_d$.

The feasible solution of the human population of model system in equation (1) the region $D_d = \left\{ (S_d,\, E_d,\, I_d,\, R_d) \in \mathbb{R}^4 : 0 \le N_D \le \frac{\mu_d}{\theta_d} \right\}$

Total dog and human population is given by; $N_M = N_D + N_H$

By considering;

$$\begin{aligned} N_H &= S_h + E_h + I_h + R_h, \\ N_D &= S_d + E_d + I_d + R_d, \\ N_M &= S_h + E_h + I_h + R_h + S_d + E_d + I_d + R_d. \end{aligned}$$

Feasible solution of human population of model system in equation (1);

$$(3)\quad D_d = \left\{ (S_d, E_d, I_d, R_d, S_h, E_h, I_h, R_h) \in \mathbb{R}^4 \times \mathbb{R}^4 = \mathbb{R}^8 : 0 \le N_Z \le \frac{\mu_N}{\theta_Y} \right\}.$$

1.2.2 Positivity of solutions

Theorem 1. *Let $S_h(0) > 0$, $E_h(0) \ge 0$, $I_h(0) \ge 0$, $R_h(0) \ge 0$, $S_d(0) \ge 0$, $E_d(0) \ge 0$, $I_d(0) \ge 0$ and $R_d(0) \ge 0$ then the solution set S_h, E_h, I_h, R_h, S_d, E_d, I_d and R_d of the system of the equation (1) are positive for all $t > 0$.*

Proof.

$$(4)\quad \frac{dS_h}{dt} = \mu_h + \sigma R_h - \beta_h S_h I_h - \theta_{,h}\, S_h, \frac{dS_h}{dt} + (\beta_h I_h + \theta_h) S_h = \mu_h + \sigma . R_h.$$

Let $A(t) = e^{-\int(\beta_h I_h + \theta_h)dsh}$ be the integrating factor

$$A(t) = e^{-\int(\beta_h I_h+\theta_h)dsh}(\int(\mu_h + \sigma R_h)dsh + C) \geq 0,$$

$$\frac{dE_h}{dt} = \beta_h S_h I_h - \delta_h E_h - \tau E_h - \theta_h E_h, \tag{5}$$

$$\frac{dE_h}{dt} + (\delta_h + \tau + \theta_h)E_h = \beta_h S_h I_h. \tag{6}$$

Let $B(t) = e^{-\int(\delta_h+\tau+\theta_h)deh}$ be the integrating factor

$$\frac{dB(t)}{dt} = B(t)\beta_h S_h I_h, \tag{7}$$

$$\int \frac{dB(t)}{dt} = \int (B(t)\beta_h S_h I_h)deh, \tag{8}$$

$$B(t) = e^{-\int(\delta_h+\tau+\theta_h)de}(\int \beta_h S_h I_h deh + c) \geq 0. \tag{9}$$

By apply in same approach for; $I_h(t), R_h(t), S_d(t), E_d(t), I_d(t), R_d(t)$.

Hence, $S_h(t), E_h(t), I_h(t), R_h(t), S_d(t), E_d(t), I_d(t), R_d(t)$ are positive from equation (1). □

1.2.3 Diseases free equilibrium point

Disease free equilibrium point of the system in equation (1) in the absence of rabbies infections is determined. $E_h = I_d = E_d = I_h = 0$. From equation (1)

$$\frac{dS_d}{dt} = \mu_d - \beta_d S_d I_d - \theta_d S_d + \gamma R_d = 0, \tag{10}$$

$$S_d^* = \frac{\mu_d}{\theta_d}, \tag{11}$$

$$\frac{dS_h}{dt} = \mu_h + \sigma R_h - \beta_h S_h I_d - \theta_{,h}\, S_h = 0 \tag{12}$$

$$S_h^* = \frac{\mu_h}{\theta_h}, \tag{13}$$

$$DFE = \left(\frac{\mu_h}{\theta_h}, 0, 0, 0, \frac{\mu_d}{\theta_d}, 0, 0, 0\right). \tag{14}$$

1.2.4 Basic reproductive number

This is a threshold value that governs the dynamics of rabbies. By employing the "Next Generation Matrix" [15, 16]. Considering;

$$\frac{dE_d}{dt} = \beta_d S_d I_d - \delta_d E_d - \theta_d E_d - \alpha E_d, \tag{15}$$

$$\frac{dI_d}{dt} = \delta_d E_d - \theta_d I_d - \omega_2 I_d - \omega_d I_d, \tag{16}$$

$$\frac{dE_h}{dt} = \beta_h S_h I_d - \delta_h E_h - \tau E_h - \theta_h E_h, \tag{17}$$

$$\frac{dI_h}{dt} = \delta_h E_h - \theta_h I_h - \omega_h I_h. \tag{18}$$

This can be written as;

$$F = \begin{bmatrix} \beta_d S_d I_d \\ 0 \\ \beta_h S_h I_d \\ 0 \end{bmatrix} \quad V = \begin{bmatrix} -\delta_d E_d - \theta_d E_d - \alpha E_d \\ \delta_d E_d - \theta_d I_d - \omega_2 I_d - \omega_d I_d \\ -\delta_h E_h - \tau E_h - \theta_h E_h \\ \delta_h E_h - \theta_h I_h - \omega_h I_h \end{bmatrix}.$$

Differentiating F and V with respect E_d, I_d, E_h and I_h.

$$F = \begin{bmatrix} 0 & \beta_d S_d & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \beta_h S_h & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$V = \begin{bmatrix} \delta_d + \theta_d + \alpha & 0 & 0 & 0 \\ -\delta_d & \theta_d + \omega_2 + \omega_d & 0 & 0 \\ 0 & 0 & \delta_d + \tau + \theta_h & 0 \\ 0 & 0 & -\delta_h & \theta_h + \omega_h \end{bmatrix},$$

where V^{-1} is given by

$$V^{-1} = \begin{bmatrix} \frac{1}{(\delta_d+\alpha+\theta_d)} & 0 & 0 & 0 \\ \frac{\delta_d}{(\delta_d+\theta_d+\alpha)(\theta_d+\omega_2+\omega_d)} & \frac{1}{(\theta_d+\omega_2+\omega_d)} & 0 & 0 \\ 0 & 0 & \frac{1}{(\delta_d+\tau+\theta_h)} & 0 \\ 0 & 0 & \frac{\delta_h}{(\delta_d+\theta_d+\alpha)(\theta_d+\omega_2+\omega_d)} & \frac{1}{(\theta_h+\omega_h)} \end{bmatrix}. \tag{19}$$

Determining the product of F and V^{-1};

$$FV^{-1} = \begin{bmatrix} \frac{\beta_d\mu_d\delta_d}{\theta_d(\delta_d+\theta_d+\alpha)(\theta_d+\omega_2+\omega_d)} & \frac{\beta_d\mu_d}{\theta_d(\theta_d+\omega_2+\omega_d)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\beta_h\mu_h\delta_d}{\theta_h(\delta_d+\theta_d+\alpha)(\theta_d+\omega_2+\omega_d} & \frac{\beta_h\mu_h}{\theta_h(\theta_d+\omega_2+\omega_d)} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{20}$$

Let $K = FV^{-1}$, the eigenvalues of FV^{-1} can be obtained

$$|K| = \begin{vmatrix} \frac{\beta_d\mu_d\delta_d}{\theta_d(\delta_d+\theta_d+\alpha)(\theta_d+\omega_2+\omega_d)} - \lambda & \frac{\beta_d\mu_d}{\theta_d(\theta_d+\omega_2+\omega_d)} & 0 & 0 \\ 0 & 0-\lambda & 0 & 0 \\ \frac{\beta_h\mu_h\delta_d}{\theta_h(\delta_d+\theta_d+\alpha)(\theta_d+\omega_2+\omega_d} & \frac{\beta_h\mu_h}{\theta_h(\theta_d+\omega_2+\omega_d)} & 0-\lambda & 0 \\ 0 & 0 & 0 & 0-\lambda \end{vmatrix} = 0,$$

$$\left(\frac{\beta_d\mu_d\delta_d}{\theta_d(\delta_d+\theta_d+\alpha)(\theta_d+\omega_2+\omega_d)}-\lambda\right)(\lambda)(\lambda)(\lambda)=0, \tag{21}$$

$$\frac{\beta_d\mu_d\delta_d}{\theta_d(\delta_d+\theta_d+\alpha)(\theta_d+\omega_2+\omega_d)}=0 \quad \text{or} \quad \lambda=0. \tag{22}$$

Hence, eigenvalues,

$$\lambda_1=0 \quad \text{or} \quad \lambda_2=\frac{\beta_d\mu_d\delta_d}{\theta_d(\delta_d+\theta_d+\alpha)(\theta_d+\omega_2+\omega_d)}.$$

The dorminant eigenvalue is the spectral radius (basic reproductive number).

$$R_O=\frac{\beta_d\mu_d\delta_d}{\theta_d(\delta_d+\theta_d+\alpha)(\theta_d+\omega_2+\omega_d)}. \tag{23}$$

1.2.5 Endemic equilibrium point

Endemic equilibrium points are steady state situations where the disease persists in the population. To determine the endemic equilibrium point we put the right side of equation (1) equal to zero.

The endemic equilibrium point of the model is written below;

$$E^*=(S_h^*,\,E_h^*,\,I_h^*,\,R_h^*,\,S_d^*,\,E_d^*,\,I_d^*,\,R_d^*)\,, \tag{24}$$

where

$$\begin{aligned}
S_h^*&=\frac{(\delta_h+\tau+\theta_h)\mu_h\beta_h(\sigma+\theta_h)}{\beta_h[(\delta_h+\tau+\theta_h)(\sigma+\theta_h)(\beta_hI_d^*+\theta_h)-\sigma\tau\beta_hI_d^*]},\\
E_h^*&=\frac{\mu_h\beta_hI_d^*(\sigma+\theta_h)}{[(\delta_h+\tau+\theta_h)(\sigma+\theta_h)(\beta_hI_d^*+\theta_h)-\sigma\tau\beta_hI_d^*]},\\
I_h^*&=\frac{\delta_h\mu_h\beta_hI_d^*(\sigma+\theta_h)}{(\theta_h+\omega_h)[(\delta_h+\tau+\theta_h)(\sigma+\theta_h)(\beta_hI_d^*+\theta_h)-\sigma\tau\beta_hI_d^*]},\\
R_h^*&=\frac{\tau\mu_h\beta_hI_d^*}{[(\delta_h+\tau+\theta_h)(\sigma+\theta_h)(\beta_hI_d^*+\theta_h)-\sigma\tau\beta_hI_d^*]},\\
S_d^*&=\frac{(\delta_d+\theta_d+\alpha)(\theta_d+\omega_2+\omega_d)}{\beta_d\delta_d},\\
E_d^*&=\frac{(\theta_d+\gamma)[\theta_d(\delta_d+\theta_d+\alpha)(\theta_d+\omega_2+\omega_d)-\mu_d\beta_d\delta_d]}{(\theta_d+\omega_2+\omega_d)\beta_d(\gamma\alpha-(\delta_d+\theta_d+\alpha)E_d)(\theta_d+\gamma)},\\
I_d^*&=\frac{(\theta_d+\gamma)[\theta_d(\delta_d+\theta_d+\alpha)(\theta_d+\omega_2+\omega_d)-\mu_d\beta_d\delta_d]}{(\theta_d+\omega_2+\omega_d)\beta_d(\gamma\alpha-(\delta_d+\theta_d+\alpha)E_d)(\theta_d+\gamma)},\\
R_d^*&=\frac{\alpha[\theta_d(\delta_d+\theta_d+\alpha)(\theta_d+\omega_2+\omega_d)-\mu_d\beta_d\delta_d]}{\beta_d\delta_d(\gamma\alpha-(\delta_d+\theta_d+\alpha)E_d)(\theta_d+\gamma)}.
\end{aligned}$$

1.3 Stability analysis of disease free equilibrium

1.3.1 Local stability of disease free equilibrium

Local stability of an equilibrium point means that if you put the system somewhere nearby the point then it will move itself to the equilibrium point in some time.

Theorem 2. *The disease free equilibrium point is locally asymptotically stable if $R_0 < 1$ otherwise it is unstable.*

Proof. To prove local stability of disease free equilibrium, we obtained the Jacobean's matrix of the system (1) at the disease free equilibrium (DFE).

Then, the Jacobean matrix become

(25)
$$\begin{pmatrix} -\theta_h & 0 & 0 & \sigma & 0 & 0 & -\beta_h\frac{\mu_h}{\theta_h} & 0 \\ 0 & -(\delta_h+\tau+\theta_h) & 0 & 0 & 0 & 0 & \beta_h\frac{\mu_h}{\theta_h} & 0 \\ 0 & \delta_h & -(\theta_h+\omega_h) & 0 & 0 & 0 & 0 & 0 \\ 0 & \tau & 0 & -(\sigma+\theta_h) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\theta_d & 0 & -\beta_d\frac{\mu_d}{\theta_d} & \gamma \\ 0 & 0 & 0 & 0 & 0 & -(\delta_d+\theta_d+\alpha) & \beta_d\frac{\mu_d}{\theta_d} & 0 \\ 0 & 0 & 0 & 0 & 0 & \delta_d & -(\theta_d+\omega_2+\omega_d) & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha & 0 & -(\theta_d+\gamma) \end{pmatrix}$$

(26)
$$J(DFE) = \begin{pmatrix} C1 & 0 & 0 & \sigma & 0 & 0 & -b3 & 0 \\ 0 & C2 & 0 & 0 & 0 & 0 & b3 & 0 \\ 0 & \delta_h & c3 & 0 & 0 & 0 & 0 & 0 \\ 0 & \tau & 0 & C4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c5 & 0 & -b4 & \gamma \\ 0 & 0 & 0 & 0 & 0 & C6 & b4 & 0 \\ 0 & 0 & 0 & 0 & 0 & \delta_d & b1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha & 0 & b2 \end{pmatrix}$$

where

(27) $C1 = -\theta_h$, $C6 = -(\delta_d + \theta_d + \alpha)$,

(28) $C2 = -(\delta_h + \tau + \theta_h)$, $b1 = -(\theta_d + \omega_2 + \omega_d)$,

(29) $C3 = -(\theta_h + \omega_h)$, $b2 = -(\theta_d + \gamma)$,

(30) $C4 = -(\sigma + \theta_h)$, $b3 = \beta_h\dfrac{\mu_h}{\theta_h}$,

(31) $C5 = -\theta_d$, $b4 = \beta_d\dfrac{\mu_d}{\theta_d}$

(32)
$$|J(DFE) - \lambda| = \begin{vmatrix} C1-\lambda & 0 & 0 & \sigma & 0 & 0 & -b3 & 0 \\ 0 & C2-\lambda & 0 & 0 & 0 & 0 & b3 & 0 \\ 0 & \delta_h & C3-\lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & \tau & 0 & C4-\lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & C5-\lambda & 0 & -b4 & \gamma \\ 0 & 0 & 0 & 0 & 0 & C6-\lambda & b4 & 0 \\ 0 & 0 & 0 & 0 & 0 & \delta_d & b1-\lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha & 0 & b2-\lambda \end{vmatrix} = 0$$

$$(C1-\lambda)\begin{vmatrix} C2-\lambda & 0 & 0 & 0 & 0 & b3 & 0 \\ \delta_h & C3-\lambda & 0 & 0 & 0 & 0 & 0 \\ \tau & 0 & C4-\lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C5-\lambda & 0 & -b4 & \gamma \\ 0 & 0 & 0 & 0 & C6-\lambda & b4 & 0 \\ 0 & 0 & 0 & 0 & \delta_d & b1-\lambda & 0 \\ 0 & 0 & 0 & 0 & \alpha & 0 & b2-\lambda \end{vmatrix} = 0, \tag{33}$$

$$\begin{aligned} &(C1-\lambda)(C2-\lambda)(C3-\lambda)(C4-\lambda) \\ &\cdot \begin{vmatrix} C5-\lambda & 0 & -b_4 & \gamma \\ 0 & C6-\lambda & b_4 & 0 \\ 0 & \delta_d & b_1-\lambda & 0 \\ 0 & \alpha & 0 & b_2-\lambda \end{vmatrix} = 0 \end{aligned} \tag{34}$$

$$\begin{aligned} &(C1-\lambda)(C2-\lambda)(C3-\lambda)(C4-\lambda)(C5-\lambda) \\ &\cdot \begin{vmatrix} C6-\lambda & b_4 & 0 \\ \delta_d & b_1-\lambda & 0 \\ \alpha & 0 & b_2-\lambda \end{vmatrix} = 0, \\ &(C1-\lambda)(C2-\lambda)(C3-\lambda)(C4-\lambda)(C5-\lambda)(b_2-\lambda) \\ &\cdot [(C6-\lambda)(b_1-\lambda)-b_4\delta_d] = 0, \\ &(C1-\lambda)(C2-\lambda)(C3-\lambda)(C4-\lambda)(C5-\lambda)(b_2-\lambda) = 0, \end{aligned}$$

$$\begin{aligned} \lambda_1 &= C1 & \lambda_4 &= C4 \\ \lambda_2 &= C2 & \lambda_5 &= C5 \\ \lambda_3 &= C3 & \lambda_6 &= b_2 \end{aligned}$$

or

$$\begin{aligned} &(C6-\lambda)(b_1-\lambda)-b_4\delta_d = 0, \\ &\lambda^2-(C6+b_1)\lambda+C6b_1-b_4\delta_d. \end{aligned} \tag{35}$$

Therefore, from the Routh-Hurwitz criterion of order two, it implies that the conditions, $-(c_6+b_1)>0$ and $C6\ b_1-b_4\delta_d>0$. From $C6\ b_1-b_4\delta_d>0$

$$-(\delta_d+\theta_d+\alpha)(-(\theta_d+\omega_2+\omega_d))-\beta_d\frac{\mu_d}{\theta_d}\delta_d>0, \tag{36}$$

$$(\delta_d+\theta_d+\alpha)(\theta_d+\omega_2+\omega_d)-\beta_d\frac{\mu_d}{\theta_d}\delta_d>0, \tag{37}$$

$$(\delta_d+\theta_d+\alpha)(\theta_d+\omega_2+\omega_d)>\beta_d\frac{\mu_d}{\theta_d}\delta_d, \tag{38}$$

$$(\delta_d+\theta_d+\alpha)(\theta_d+\omega_2+\omega_d)>\frac{\beta_d\mu_d\delta_d}{\theta_d}, \tag{39}$$

$$1>\frac{\beta_d\mu_d\delta_d}{\theta_d(\delta_d+\theta_d+\alpha)(\theta_d+\omega_2+\omega_d)}. \tag{40}$$

$R_0<1$ therefore DFE is locally asymptotically stable. The prove is completed. □

1.3.2 Global stability of disease free equilibrium

According to Livingstone.et.al. 2015 to get the global stability of disease free equilibrium point of system (1) we write our system as follows:

$$\begin{cases} \frac{dX_1}{dt} = B(X_1 - X_1(E_0)) + B_1 X_2 \\ \frac{dX_2}{dt} = B_2 X_2 \end{cases} \tag{41}$$

$$X_1 = \text{ transmitting compartments}, \tag{42}$$

$$X_2 = \text{ non - transmitting compartments}, \tag{43}$$

$$X_1 = (S_h,\ R_h,\ S_d,\ R_d), \tag{44}$$

$$X_2 = (E_h,\ I_h,\ E_d,\ I_d). \tag{45}$$

The DFE is denoted by $DFE = \left\{ \frac{\mu_h}{\theta_h}, 0, \frac{\mu_d}{\theta_d}, 0 \right\}$

$$X_1 - X_1(DFE) = \begin{cases} S_h - \frac{\mu_h}{\theta_h} \\ R_h \\ S_d - \frac{\mu_d}{\theta_d} \\ R_d \end{cases} \tag{46}$$

For the global stability of DFE we need to prove the following.

1. B should be a matrix with real negative Eigen values.
2. B_2 Should be a Metzler matrix

Using system (1) together with the representation in (41) the two equations can be written as follows:

$$\begin{bmatrix} \mu_h + \sigma R_h - \beta_h S_h I_d - \theta_h S_h \\ \tau E_h - \sigma R_h - \theta_h R_h \\ \mu_d - \beta_d S_d I_d - \theta_d S_d + \gamma R_d \\ \alpha E_d - \theta_d R_d - \gamma R_d \end{bmatrix} = B \begin{bmatrix} S_h - \frac{\mu_h}{\theta_h} \\ R_h \\ S_d - \frac{\mu_d}{\theta_d} \\ R_d \end{bmatrix} + B_1 \begin{bmatrix} E_h \\ I_h \\ E_d \\ I_d \end{bmatrix}$$

and

$$\begin{bmatrix} \beta_h S_h I_d - \delta_h E_h - \tau E_h - \theta_h E_h \\ \delta_h E_h - \theta_h I_h - \omega_h I_h \\ \beta_d S_d I_d - \delta_d E_d - \theta_d E_d - \alpha E_d \\ \delta_d E_d - \theta_d I_d - \omega_2 I_d - \omega_d I_d \end{bmatrix} = B_2 \begin{bmatrix} E_h \\ I_h \\ E_d \\ I_d \end{bmatrix}.$$

Matrices B, B_1 and B_2 are order 4×4 matrix Using non - transmitting elements of the Jacobean matrix of system (1) and representation in (**??**) we

get

$$B = \begin{bmatrix} -\theta_h & \sigma & 0 & 0 \\ 0 & -(\sigma+\theta_h) & 0 & 0 \\ 0 & 0 & -\theta_d & \gamma \\ 0 & 0 & 0 & -(\theta_d+\gamma) \end{bmatrix}, \tag{47}$$

$$B_1 = \begin{bmatrix} 0 & 0 & 0 & -\beta_h S_h \\ \tau & 0 & 0 & 0 \\ 0 & 0 & 0 & -\beta_d S_d \\ 0 & 0 & \alpha & 0 \end{bmatrix}, \tag{48}$$

$$B_2 = \begin{bmatrix} -\delta_d-\theta_d-\alpha & \beta_d S_d & 0 & 0 \\ \delta_d & -\theta_d-\omega_2-\omega_d & 0 & 0 \\ 0 & \beta_h S_h & -\delta_d-\tau-\theta_h & 0 \\ 0 & 0 & \delta_h & -\theta_h-\omega_h \end{bmatrix} \tag{49}$$

$$\begin{vmatrix} -\theta_h-\lambda & \sigma & 0 & 0 \\ 0 & -(\sigma+\theta_h)-\lambda & 0 & 0 \\ 0 & 0 & -\theta_d-\lambda & \gamma \\ 0 & 0 & 0 & -(\theta_d+\gamma)-\lambda \end{vmatrix} = 0,$$

$$(-\theta_h-\lambda)(-(\sigma+\theta_h)-\lambda)(-\theta_d-\lambda)(-(\theta_d+\gamma)-\lambda) = 0, \tag{50}$$

$$\lambda_1 = -\theta_h, \tag{51}$$

$$\lambda_2 = -(\sigma+\theta_h), \tag{52}$$

$$\lambda_3 = -\theta_d, \tag{53}$$

$$\lambda_4 = -(\theta_d+\gamma). \tag{54}$$

The Eigen value of matrix B is negative and the off diagonal elements of matrix B_2 are non - negative which is Metzler matrix. This proves that the DFE point of system (1) globally asymptotically stable in the region $\mathbb{R}^8$ and $R_0 < 1$.

1.4 Stability analysis of endemic equilibrium point

In this section we focus on the stability of the endemic equilibrium point. The stability of endemic equilibrium point is divided into two local and global.

1.4.1 Local stability analysis of endemic equilibrium point

Theorem 3. *The endemic equilibrium E^* of model (1) is globally asymptotically stable whenever $R_0 > 1$.*

Proof. To determine the local stability of endemic equilibrium point from the differential equation (1) first we determine the Jacobean matrix at E^*.

$$J(E^*) = \begin{pmatrix} C1 & 0 & 0 & \sigma & 0 & 0 & -b3 & 0 \\ b5 & C2 & 0 & 0 & 0 & 0 & b3 & 0 \\ 0 & \delta_h & C3 & 0 & 0 & 0 & 0 & 0 \\ 0 & \tau & 0 & C4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & C5 & 0 & -b4 & \gamma \\ 0 & 0 & 0 & 0 & b6 & C6 & b4 & 0 \\ 0 & 0 & 0 & 0 & 0 & \delta_d & b1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha & 0 & b2 \end{pmatrix} \tag{55}$$

where

$$\begin{aligned} C1 &= -\beta_h I_d^* - \theta_h & C6 &= -(\delta_d + \theta_d + \alpha) \\ C2 &= -(\delta_h + \tau + \theta_h) & b1 &= -(\theta_d + \omega_2 + \omega_d) \\ C3 &= -(\theta_h + \omega_h) & b2 &= -(\theta_d + \gamma) \\ C4 &= -(\sigma + \theta_h) & b3 &= \beta_h S_h^* \\ C5 &= -\beta_d I_d^* - \theta_d & b4 &= \beta_d S_d^* \\ b6 &= -\beta_d I_d^* & b5 &= -\beta_h I_d^* \end{aligned}$$

$$|J(E^*) - \lambda| = \begin{vmatrix} (C1-\lambda) & 0 & 0 & \sigma & 0 & 0 & -b3 & 0 \\ b5 & C2-\lambda & 0 & 0 & 0 & 0 & b3 & 0 \\ 0 & \delta_h & C3-\lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & \tau & 0 & C4-\lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & C5-\lambda & 0 & -b4 & \gamma \\ 0 & 0 & 0 & 0 & b6 & C6-\lambda & b4 & 0 \\ 0 & 0 & 0 & 0 & 0 & \delta_d & b1-\lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha & 0 & b2-\lambda \end{vmatrix} = 0 \tag{56}$$

From the above, we get

$$\lambda^7 + m_1\lambda^6 + m_2\lambda^5 + m_3\lambda^4 + m_4\lambda^3 + m_5\lambda^2 + m_6\lambda + m_7 = 0. \tag{57}$$

From the above $\lambda_1 = C3$ it is negative and to see the remains value of λ we use Routh Hurwitz criterion. To obtain the precise number of roots with nonnegative real part, proceed as follow arrange the coefficient of polynomial and values subsequently calculated form them as below:

$$m_0\lambda^n + m_1\lambda^{n-1} + \cdots + m_{n-1}\lambda + m_n = 0, \tag{58}$$

$$m_0\lambda^n + m_1\lambda^{n-1} + \cdots + m_{n-1}\lambda + m_n = 0$$

λ^7	1	m_2	m_4	m_6
λ^6	m_1	m_3	m_5	m_7
λ^5	d_1	d_2	d_3	0
λ^4	e_1	e_2	e_3	
λ^3	f_1	f_2	0	
λ^2	h_1	h_2		
λ^1	n_1	0		
λ^0	x_1			

$$d_1 = -\frac{1}{m_1}\begin{vmatrix} -1 & m_2 \\ m_1 & m_3 \end{vmatrix} = -\frac{1}{m_1}(-m_3 + m_1m_2) = \frac{1}{m_1}(-m_3 + m_1m_2), \tag{59}$$

$$d_2 = -\frac{1}{m_1}\begin{vmatrix} -1 & m_4 \\ m_1 & m_5 \end{vmatrix} = -\frac{1}{m_1}(-m_5 + m_1m_4) = \frac{1}{m_1}(-m_5 + m_1m_4), \tag{60}$$

$$d_3 = -\frac{1}{m_1}\begin{vmatrix} -1 & m_6 \\ m_1 & m_7 \end{vmatrix} = -\frac{1}{m_1}(-m_7 + m_1m_6) = \frac{1}{m_1}(-m_7 + m_1m_6) \tag{61}$$

$$e_1 = -\frac{1}{d_1}\begin{vmatrix} m_1 & m_3 \\ d_1 & d_2 \end{vmatrix} = -\frac{1}{d_1}(m_1d_2 - m_3d_1) \tag{62}$$

$$= -\frac{1}{d_1m_1}(m_1(-m_5 + m_1m_4) - m_3(-m_3 + m_1m_2)) \tag{63}$$

$$= \frac{1}{d_1m_1}[m_3^2 + m_1^2m_4 - m_1m_5 + m_3m_1m_2] \tag{64}$$

$$= \frac{1}{m_5 - m_1m_2}[-m_3^2 - m_1^2m_4 + m_1m_5 + m_3m_1m_2], \tag{65}$$

$$e_2 = -\frac{1}{d_1}\begin{vmatrix} m_1 & m_5 \\ d_1 & d_3 \end{vmatrix} = -\frac{1}{d_1}(m_1d_3 - m_5d_1) \tag{66}$$

$$= -\frac{1}{d_1m_1}(m_1(-m_7 + m_1m_6) - m_5(-m_3 + m_1m_2)) \tag{67}$$

$$= \frac{1}{d_1m_1}[-m_1^2m_6 + m_1m_7 + m_5m_1m_2 - m_5m_3] \tag{68}$$

$$= \frac{1}{m_3 - m_1m_2}[-m_1^2m_6 + m_1m_7 + m_5m_1m_2 - m_5m_3], \tag{69}$$

$$e_3 = -\frac{1}{d_1}\begin{vmatrix} m_1 & m_7 \\ d_1 & 0 \end{vmatrix} = -\frac{1}{d_1}(-d_1m_7) = m_7, \tag{70}$$

$$f_1 = -\frac{1}{e_1}\begin{vmatrix} d_1 & d_2 \\ e_1 & e_2 \end{vmatrix}, \tag{71}$$

$$f_2 = -\frac{1}{e_1}\begin{vmatrix} d_1 & d_3 \\ e_1 & e_3 \end{vmatrix} = -\frac{1}{e_1}(d_1e_3 - e_1d_3), \tag{72}$$

$$h_1 = -\frac{1}{f_1}\begin{vmatrix} e_1 & e_2 \\ f_1 & f_2 \end{vmatrix} = -\frac{1}{f_1}(e_1f_2 - f_1e_2), \tag{73}$$

$$h_2 = -\frac{1}{f_1}\begin{vmatrix} e_1 & e_3 \\ f_1 & 0 \end{vmatrix}, \tag{74}$$

$$h_2 = -\frac{1}{f_1}(0 - e_3f_1) = e_3 = m_7, \tag{75}$$

$$n_1 = -\frac{1}{f_1}\begin{vmatrix} f_1 & f_2 \\ h_1 & h_2 \end{vmatrix}, \tag{76}$$

$$n_1 = -\frac{1}{f_1}(f_1h_2 - h_1f_2), \tag{77}$$

$$n_1 = \frac{1}{f_1}(h_1f_2 - f_1h_2). \tag{78}$$

Let $h_1 = z$, , $f_1 = r$, $f_2 = y$ then, $m_1{=}\frac{1}{f_1}(zy - rm_7)$

$$n_1 = -\frac{1}{n_1}\begin{vmatrix} h_1 & h_2 \\ n_1 & 0 \end{vmatrix} = -\frac{1}{n_1}(-n_1h_2) = h_2 = e_3 = m_7.$$

By Using the Routh Hurwitz criterion it can be seen that all the Eigen values of the characteristic equation have negative real part if and only if $m_1 > 0$, $m_2 > 0$, $m_3 > 0$, $m_4 > 0$, $m_5 > 0$, $m_6 > 0$, $m_7 > 0$, $d_1 > 0$, $d_2 > 0$, $d_3 > 0$, $e_1 > 0$, $e_2 > 0$, $e_3 > 0$, $f_1 > 0$, $f_2 > 0$, $h_1 > 0$, $h_2 > 0$, $n_1 > 0$.

Since $R_0 = \frac{\mu_d\beta_d\delta_d}{\theta_d(\delta_d+\theta_d+\alpha)(\theta_d+\omega_2+\omega_d)}$. Hence, $R_0 > 1$. □

This implies that the endemic equilibrium point (E^*) is local asymptotical stable.

1.4.2 Global stability analysis of endemic equilibrium

Global stability means that the system will come to the equilibrium point from any possible starting point.

Theorem 4. *If $R_0 > 1$, E^* of the model (1) is globally asymptotically stable.*

Proof. Using the Lyapunov approach in [17], global asymptotic stability of E^*: Define

$$
\begin{aligned}
&V(S_h^*, E_h^*, I_h^*, R_h^*, S_d^*, I_d^*, I_d^*, R_d^*)\\
&= (S_h - S_h^* - S_h^* \ln \frac{S_h^*}{S_h}) + (E_h - E_h^* - E_h^* \ln \frac{E_h^*}{E_h})\\
&+ (I_h - I_h^* - I_h^* \ln \frac{I_h^*}{I_h}) + (R_h - R_h^* - R_h^* \ln \frac{R_h^*}{R_h}) + (S_d - S_d^* - S_d^* \ln \frac{S_d^*}{S_d})\\
&+ (E_d - E_d^* - E_d^* \ln \frac{E_d^*}{E_d}) + (I_d - I_d^* - I_d^* \ln \frac{I_d^*}{I_d}) + (R_d - R_d^* - R_d^* \ln \frac{R^*}{R_d}).
\end{aligned} \tag{79}
$$

By direct calculating the derivative of V along the solution of (1), we have;

$$
\begin{aligned}
&\frac{dV}{dt} = \left(1 - \frac{S_h^*}{S_h}\right) \frac{dS_h}{dt} + \left(1 - \frac{E_h^*}{E_h}\right) \frac{dE_h}{dt} + \left(1 - \frac{I_h^*}{I_h}\right) \frac{dI_h}{dt} + \left(1 - \frac{R_h^*}{R_h}\right),\\
&\frac{dR_h}{dt} + \left(1 - \frac{S_d^*}{S_d}\right) \frac{dS_d}{dt} + \left(1 - \frac{E_d^*}{E_d}\right) \frac{dE_d}{dt} + \left(1 - \frac{I_d^*}{I_d}\right) \frac{dI_d}{dt} + \left(1 - \frac{R_d^*}{R_d}\right) \frac{dR_d}{dt}
\end{aligned} \tag{80}
$$

$$
\begin{aligned}
\frac{dV}{dt} &= \mu_h + \sigma R_h + \beta_h S_h^* I_d + \theta_h S_h^* + \beta_h S_h I_d\\
&+ \delta_h E + \tau E^* + \theta_h E^* + \delta_h E_h\\
&+ \theta_h I_h^* + \omega_h I_h^* + \tau E_h + \sigma R_h^* + \theta_h R_h^*\\
&+ \mu_d + \gamma R_d + \beta_d S_d^* I_d + \theta_d S_d^* + \beta_d S_d I_d\\
&+ \delta_d E_d^* + \theta_d E_d^* + \alpha E_d^* + \delta_d E_d + \theta_d I_d^* + \omega_2 I_d^*\\
&+ \omega_d I_d^* + \alpha E_d + \theta_d R_d^* + \gamma R_d^*\\
&- \Big(- \beta_h S_h I_d - \theta_h S_h - \frac{S_h^*}{S_h} \mu_h - \frac{S_h^*}{S_h} \sigma R_h - \delta_h E_h\\
&- \tau E_h - \theta_h E_h - \frac{E^*}{E_h} \beta_h S_h I_d - \theta_h I_h - \omega_h I_h\\
&- \frac{I_h^*}{I_h} \delta_h E_h - \sigma R_h - \theta_h R_h - \frac{R_h^*}{R_h} \tau E_h - \beta_d S_d I_d - \theta_d S_d\\
&- \frac{S_d^*}{S_d} \mu_d - \frac{S_d^*}{S_d} R_d - \delta_d E_d,
\end{aligned} \tag{81}
$$

$$
\begin{aligned}
&\theta_d E_d - \alpha E_d - \frac{E_d^*}{E_d} \beta_d S_d I_d - \theta_d I_d - \omega_2 I_d - \omega_d I_d\\
&- \frac{I_d^*}{I_d} \delta_d E_d - \theta_d R_d - \gamma R_d - \frac{R_d^*}{R_d} \alpha E_d \Big),
\end{aligned} \tag{82}
$$

$$
\frac{dV}{dt} = A - B, \tag{83}
$$

where

$$
\begin{aligned}
A = {} & \mu_h + \sigma R_h + \beta_h S_h^* I_d + \theta_h S_h^* + \beta_h S_h I_d + \delta_h E^* + \tau E^* + \theta_h E^* + \delta_h E_h \\
& + \theta_h I_h^* + \omega_h I_h^* + \tau E_h + \sigma R_h^* + \theta_h R_h^* + \mu_d + \gamma R_d + \beta_d S_d^* I_d \\
& + \theta_d S_d^* + \beta_d S_d I_d + \delta_d E_d^* + \theta_d E_d^* + \alpha E_d^* + \delta_d E_d \\
& + \theta_d I_d^* + \omega_2 I_d^* + \omega_d I_d^* + \alpha E_d + \theta_d R_d^* + \gamma R_d^*,
\end{aligned} \tag{84}
$$

$$
\begin{aligned}
B = {} & -\beta_h S_h I_d - \theta_h S_h - \frac{S_h^*}{S_h}\mu_h - \frac{S_h^*}{S_h}\sigma R_h \\
& - \delta_h E_h - \tau E_h - \theta_h E_h - \frac{E^*}{E_h}\beta_h S_h I_d \\
& - \theta_h I_h - \omega_h I_h - \frac{I_h^*}{I_h}\delta_h E_h - \sigma R_h - \theta_h R_h - \frac{R_h^*}{R_h}\tau E_h - \beta_d S_d I_d \theta_d S_d - \frac{S_d^*}{S_d}\mu_d \\
& - \frac{S_d^*}{S_d} R_d - \delta_d E_d - \theta_d E_d - \alpha E_d - \frac{E_d^*}{E_d}\beta_d S_d I_d - \theta_d I_d \omega_2 I_d - \omega_d I_d \\
& - \frac{I_d^*}{I_d}\delta_d E_d - \theta_d R_d - \gamma R_d - \frac{R_d^*}{R_d}\alpha E_d.
\end{aligned} \tag{85}
$$

Thus, if $A < B$, then $\frac{dV}{dt} \leq 0$; Noting that $\frac{dV}{dt} = 0$. If and only if $S_d = S_d^*$, $E_d = E_d^*$, $I_d = I_d^*$, $R_d = R_d^*$, $S_h = S_h^*$, $E_h = E_h^*$, $I_h = I_h^*$, $R_h = R_h^*$. Therefore, the largest compact invariant set in $\{(S_h^*, E_h^*, I_h^*, R_h^*, S_d^*, I_d^*, I_d^*, R_d^*) \in \mathbb{R}^8 : \frac{dV}{dt} = 0\}$ is the singleton E^*, where E^* is the endemic equilibrium of the system (1) by invariance principle (is criterion for the asymptotic stability of autonomous (possibly nonlinear) dynamical system), it implies that E^* is globally asymptotically stable in $\mathbb{R}^8$ if $A < B$. This implies that $R_0 > 1$ [18, 19, 20]. □

1.5 Sensitivity analysis

The sensitivity analysis of the model parameters was carried out in order to determine parameters in the model that have a high transmission influence on the disease [21, 22, 23, 24]. We analysed the reproduction number to determine whether or not treatment of infective and mortality can lead to the effective elimination or control the rabies disease in the population.

We determine the most sensitive parameter by using the relation;
$P_{mi} = \frac{mi}{R_0} \times \frac{\partial R_0}{\partial mi}$, where mi are parameter, and R_0 is the reproductive number.
If $P_{mi} < 0$, then mi have an effect of controlling the disease
If $P_{mi} > 0$, then mi have an effect on expanding the disease

$$P_{\beta_d} = \frac{\beta_d}{R_0} \times \frac{\partial R_0}{\partial \beta_d}$$

The results of the sensitivity analysis as shown in Table 1 indicates that some parameters are more sensitive to the reproduction number than others. The following parameters; $\theta_d, \alpha, \omega_2, \omega_d$ help reduce the spread of the infection. However, the following parameters μ_d, β_d, δ_d increases the spread of the infection whenever their values increases.

Table 1: Sensitivity indices of model parameters to R_0

Sensitivity indices of R_0		
No	Parameter	Sensitivity index
1.	μ_d	$+ve$
2.	β_d	$+ve$
3.	δ_d	$+ve$
4.	θ_d	$-ve$
5.	α	$-ve$
6.	ω_2	$-ve$
7.	ω_d	$-ve$

1.6 Model validation and validity

Model validation is an important and yet controversial aspect of any model-based methodology in general, and system dynamics in particular. Validity of the results in a model-based study is crucially dependent on the validity of the model. The validation process is undertaken in order to ensure that the model developed is sufficiently accurate for the purpose at hand.

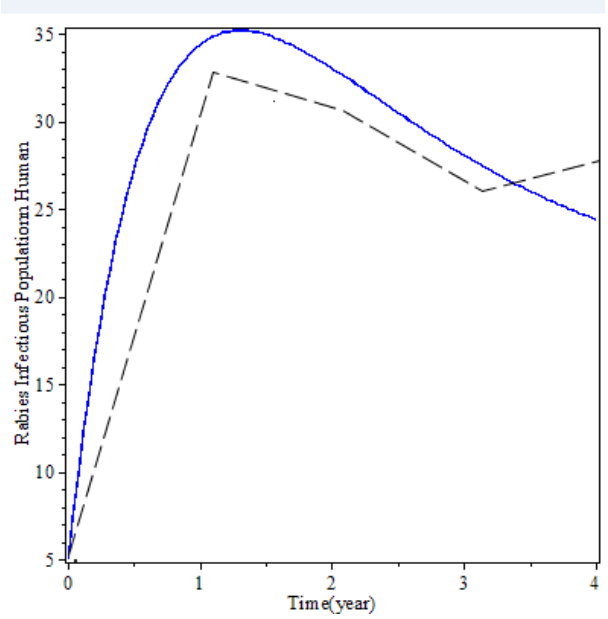


Figure 3: The comparison between the reported human rabies cases in North Shewa zone of Health Bearaua from 2014 to 2017 and the simulation of $I_h(t)$ from the model

The dashed curve represents the data reported by the North Shewa Ministry of Health while the solid curve is simulated by using our model. The initial values

used in the stimulations were $s[h](0) = 50$,$e[h](0) = 25$, $i[h](0) = 5$, $r[h](0) = 5$, $s[d](0) = 250$, $e[d](0) = 500$, $i[d](0) = 1000$, $r[d](0) = 250$.

The numerical simulation of human rabies cases in North Shewa from 2014 to 2017 is shown in Fig. 4, indicating that our model provides a good match to the reported data. The awareness of rabies for people in recent years has been enhanced gradually.

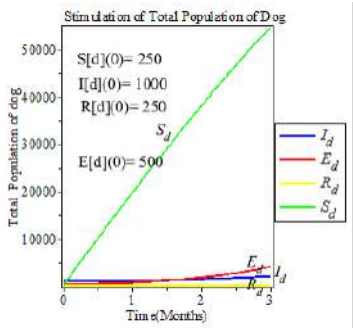


Figure 4: Simulation of total human population with initial value

2. Numerical simulation

Numerical simulations are required to study the behaviour of a systems whose mathematical model is too complex to provide analytical solution as in most non linear systems.

Table 2 shows the values of the parameters used in the various simulations.

Table 2: parameter value

Parameter	**Value**	**Source**
μ_h	200	Assumption
θ_h	0.0008	Assumption
δ_h	0.17	[25]
ω_h	1	[26]
β_h	2.29×10^{-2}	[25]
τ	0.1	[27]
σ	1	[27]
μ_d	2×10^{4}	[26]
θ_d	0.083	[26]
δ_d	2.571×10^{-2}	[4]
ω_d	0.005	Assumption
β_d	3.776×10^{-6}	Assumption
α	0.005	Assumption
γ	0.5	[26]
ω_2	0.001	Assumption

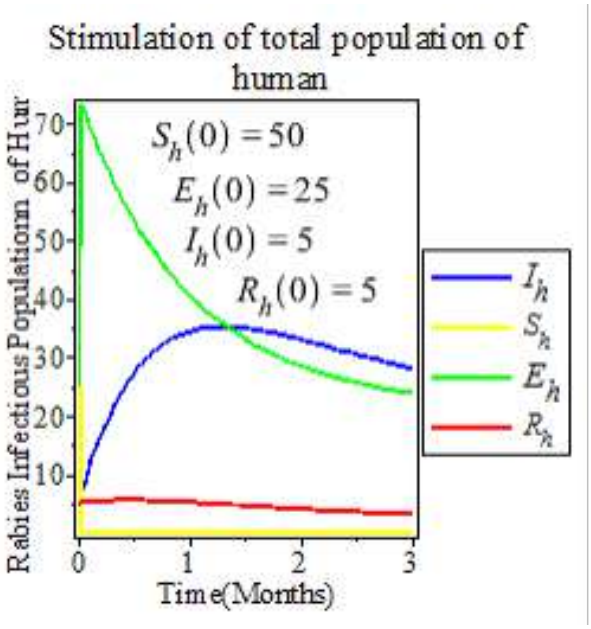


Figure 5: The stimulation of total population dogs with initial value

2.1 Effect of removing rate on rabies infectious population

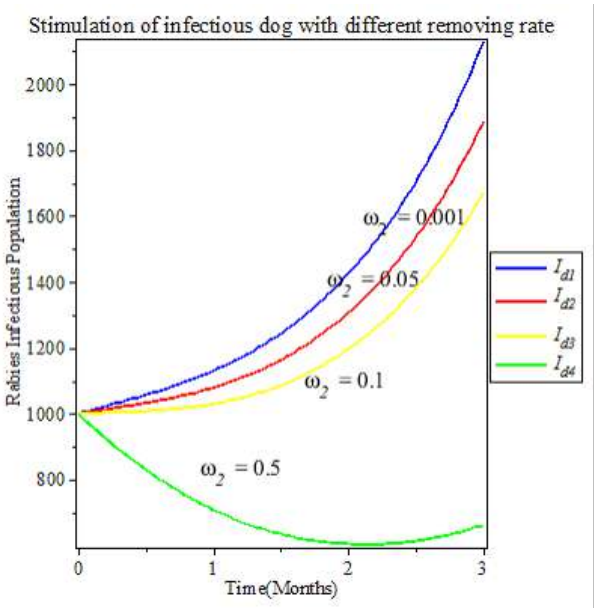


Figure 6: Simulation of infectious dogs with different removing rates.

In Fig. 6, we have experimented on the effect of ω_2 in decreasing the number of rabies infectious population. The figure shows that when the values of ω_2 increase, the number of rabies infectious population is going down (decreasing).

2.2 Effect of contact rate on exposed dogs population

In this section, as we see in Fig. 7, below the figure reflects that as the value of contact rate of rabies is increased, the exposed population increases.

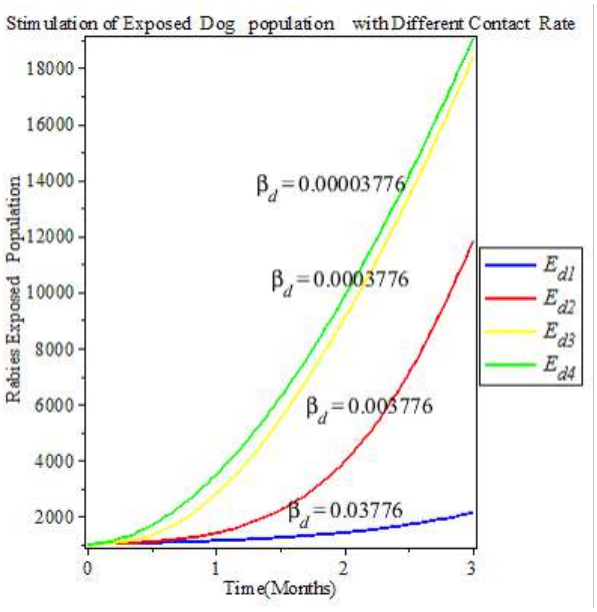


Figure 7: Simulation of infectious dogs with different removing rates.

2.3 Effects of treatment rate on recovered dogs population

We simulated the effects of treatment on recovered dogs population as shown in Fig: 8.

2.4 Effects of rate of infection on infectious dogs population

In this section, we simulated the effects of rate of infection on infectious dogs population as shown in Fig 9.

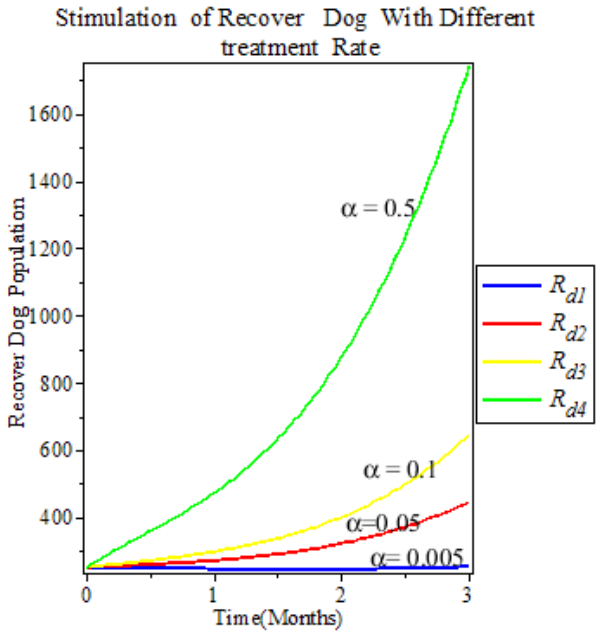


Figure 8: Simulation of infectious dogs with different removing rates.

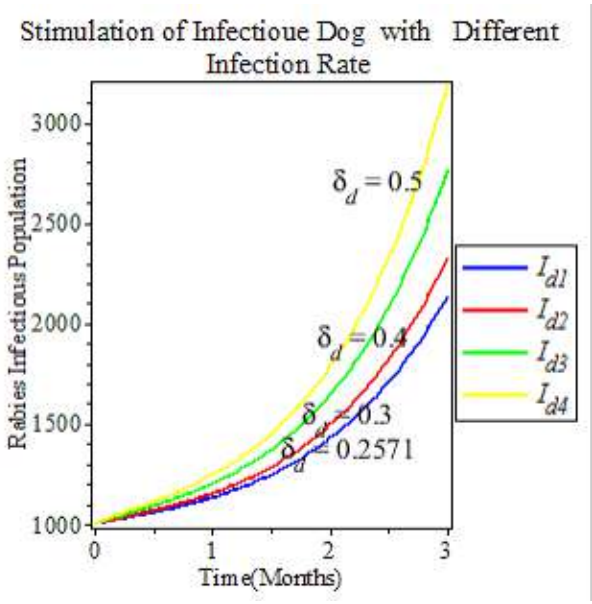


Figure 9: Simulation of infectious dogs with different infectious rates.

3. Conclusion

In this study we have formulated mathematical model of rabies. The model contains dog and human population. We defined the reproduction number in

terms of the parameters and computed it by using next generation operator. The results are depending only on the parameter of dog population. It was also established that for the basic reproduction number, $R_0 < 1$, the disease free equilibrium point is asymptotically stable so that the disease dies out after some period of time and if $R_0 > 1$, the disease free equilibrium is unstable and the disease prissiest. We also established that when $R_0 > 1$ then the endemic equilibrium is locally asymptotically stable, and unstable if $R_0 < 1$. The local stability theorems of disease free equilibrium and endemic equilibrium points of the model are proved by using Jacobean matrix and Routh-Hurwitz criterion. Further more global stability analysis of endemic equilibrium point was computed by using invariance principle. Sensitivity analysis of basic parameters and interpretation of the sensitivity index is also computed. Depending on the value of the sensitivity analysis of parameter the natural death rate, diseases induce rate, removing parameter and vaccination rate (the rate of exposed population of dog join to the recovered population) have an effects on controlling the rabies disease in the community and natural birth rate, the rate susceptible population infection by infected animal and the rate of exposed dog infected (incubation period) have an effect on expansion the rabies disease in the community. Moreover, numerical stimulation is performed in order to check the effect of each parameter in the expansion as well as in the controlling of rabies. Depending on numerical stimulation, if the removing rate and treatment rate are reduced (decrease) the disease spread increases and if the other rate like contact rate and infective rate are increase the disease spread increases.

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Data Availability Statement

The data supporting this Tuberculosis model analysis is from the Ethiopia National Institute of Health, a division of the National Health. Some of the parameter values are assumed and others are taken from published articles and are cited in this paper. These published articles are also cited at relevant places within the text as references.

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Stability analysis and modeling the dynamics of Hepatitis B with vaccination compartment

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Abstract. Resources hinder control and prevention of diseases in a number of communities in Ghana and Africa in general. Some measures have been put in place in an attempt to combat infectious diseases. A model that explains the dynamics of hepatitis B was formulated and analysed. The local and global stability of the disease as well as the basic reproduction number were determined. It was established that the hepatitis B infection is locally asymptotically stable whenever the basic reproduction number is less than unity and unstable otherwise. The population dynamics of the susceptible, infectious and recovered were determined numerically. The effects of contact rate between the infectious and susceptible population was established numerically. It was evidence that as contact rate increases, the population infected increase with time. Moreover, as the contact rate between the treated and recovered populations increase, the population of recovery increases with time.

Keywords: Hepatitis B, vaccination, treatment, equilibrium points, stability analysis.

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1. Introduction

Hepatitis B is among the world's leading health complication [1]. It can cause chronic infection, and if not well managed, can result in death from cirrhosis and hepatocellular carcinoma [2]. Approximately, hepatitis B deaths in a year ranges between 500 to $1,200,000$ people annually, and it's among the top 10 causes of death worldwide [2]. An individual is infected when the virus gets into bloodstream either through vertical transmission or blood, semen and other bodily fluids of infected persons [3].

There are safe and effective vaccines since 1982 for the prevention of hepatitis B infection through vaccination [4]. The vaccines work by activating the body to produce antibodies that ensure protection against contracting the virus. Susceptible individuals who are predisposed to the infection or are at high risk because of their particular circumstances are candidates for vaccination. Therefore, it is necessary to screen people to ascertain whether they are candidates for vaccination or not.

Research is still in progress to find a cure for chronic hepatitis B infection. Reports estimate that about one percent of the chronically infected break free from the virus each year [5]. One goal of available antiviral therapies is to help bring down the viral load and thus reduce the chance of disease progression towards liver scarring and liver cancer that are both extremely life-threatening [2].

Contact with an infectious person leading to infection can occur through sex, sharing needles, syringes and contaminated equipment, sharing toothbrushes and razors. Direct contact with blood or open sore [6, 7].The virus, after it has left its host for some days, can still infect susceptible individuals [8]. Infected individuals who are not able to fight off the hepatitis B virus become chronically infected after about 180 days of infection [9]. This stage of contagiousness in the transmission model of the diseases is known as the chronic class. If left untreated, the chronically infected is very likely to die from liver cancer or hepatocellular carcinoma [10].

Disease models generally describe the transmission dynamics of diseases, determine the best optimal control mechanism and the most effective cost to be employed in fighting the infections [11, 12].

A real phenomenon is clearly translated into a disease model for optimal cost control and employ sensitivity analysis to determine the best control measure [13, 14, 15].

2. Model description and formulation

The model divides population under study into six compartments with respect to their disease status at any point in time t. Susceptible Humans $S(t)$, Expose Population $E(t)$, Infected Population $I(t)$, Treated Population $T(t)$, Vaccinated Population $V(t)$ and Recovered Population $R(t)$. The nutural birth and death

rates are given by μ_1 and μ_2 respectively. Vaccination rate of the susceptible is given by σ_1 and rate of unsuccessful vaccination is given by η. β_1 and β_2 are transmission rates for infectious and treated populations respectively. α is the rate at which the exposed are infected. p_1 is the probability that an infected person clears the virus and the probability of failing to clear the virus is $(1-p_1)$. p_2 is the probability of an infected mother gives birth to an infected baby. σ_3 is the rate at which the recovered loss immunity and return to the susceptible class. The rate at which the treated move to the recovered class is given by σ_4.

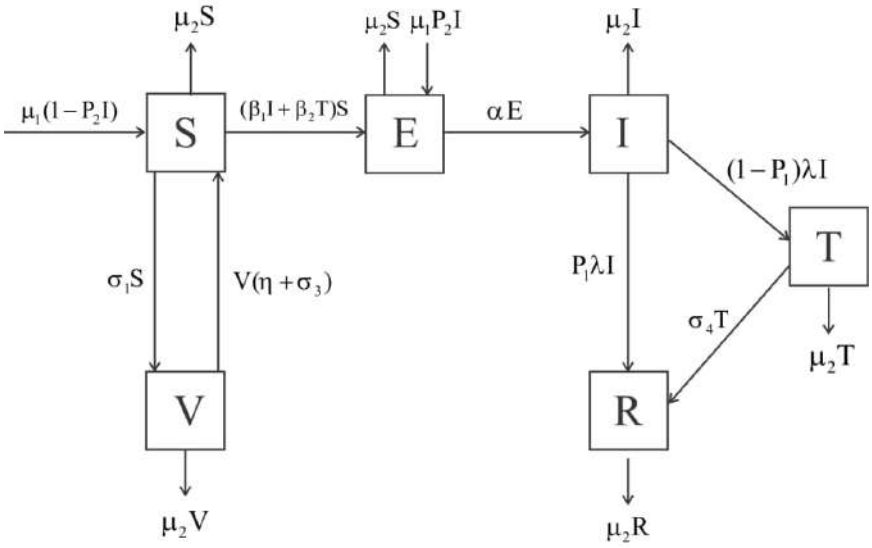


Figure 1: Hepatitis B Model flow chart

Total population, $N(t)$;

$$N = S(t) + E(t) + I(t) + T(t) + V(t) + R(t). \tag{1}$$

System of differential equation obtained from the model flow diagram in Figure (1)

$$\begin{cases} \frac{dS}{dt} &= \mu_1(1-p_2I) + \eta V + \sigma_3 R - [(\beta_1 I + \beta_2 T) + \mu_2 + \sigma_1]S, \\ \frac{dE}{dt} &= (\beta_1 I + \beta_2 T)S + \mu_1 p_2 I - [\mu_2 + \alpha]E, \\ \frac{dI}{dt} &= \alpha E - [p_1\lambda + (1-p_1)\lambda + \mu_2]I, \\ \frac{dT}{dt} &= (1-p_1)\lambda I - [\sigma_4 + \mu_2]T, \\ \frac{dV}{dt} &= \sigma_1 S - [\eta + \mu_2]V, \\ \frac{dR}{dt} &= p_1\lambda I + \sigma_4 T - [\sigma_3 + \mu_2]R. \end{cases} \tag{2}$$

3. Model analysis

We provide evidence of well-posedness of hepatitis model by proving the boundedness and positivity of the model solution.

3.1 Feasible region

The invariant region is given by, $\Upsilon = \{(S(t)+E(t)+I(t)+T(t)+V(t)+R(t))) \in \mathbb{R}^6_+; S(t)+E(t)+I(t)+T(t)+V(t)+R(t) = N \leq \frac{\mu_1}{\mu_2}\}$. From model equation, the total population N is given by

$$\begin{cases} \frac{dN}{dt} & = \frac{dS}{dt} + \frac{dE}{dt} + \frac{dI}{dt} + \frac{dT}{dt} + \frac{dV}{dt} + \frac{dR}{dt} \\ & = [\mu_1 - (S+E+I+T+V+R)\mu_2] \\ & = \mu_1 - N\mu_2, \end{cases} \tag{3}$$

$$\frac{dN}{dt} = \mu_1 - N\mu_2. \tag{4}$$

By solving

$$\begin{aligned} &\frac{dN}{\mu_1 - N\mu_2} = dt, \\ &\int \frac{dN}{\mu_1 - N\mu_2} = \int dt, \\ &\frac{-\ln|\mu_1 - N\mu_2|}{\mu_2} = t + c, \\ &-\ln|\mu_1 - N\mu_2| = \mu_2 t + c_1, \\ &\mu_1 - N\mu_2 = e^{-\mu_2 t + c_1}, \\ &\mu_1 - N\mu_2 = c_2 e^{-\mu_2 t}, \\ &N(t) = \frac{c_2 e^{-\mu_2 t} - \mu_1}{-\mu_2}, \\ &N(t) = \frac{\mu_1}{\mu_2} - c_2 e^{-\mu_2 t} \end{aligned}$$

as $t = 0, N(t) = N(0)$, $N(0) = \frac{\mu_1}{\mu_2} - c_2$, $c_2 = \frac{\mu_1}{\mu_2} - N(0)$. Therefore,

$$N(t) = \frac{\mu_1}{\mu_2} - \left(\frac{\mu_1}{\mu_2} - N(0)\right)e^{-\mu_2 t}. \tag{5}$$

Now, as t $\to \infty$, $N(t) \leq \frac{\mu_1}{\mu_2}$. Hence, Υ is positively invariant.

3.2 Positivity of model solutions

An epidemic model must epidemiologically meaningful. The positivity of the model solution must be established.

Let the initial values of the parameters be $\{S(t) \geq 0, E(t) \geq 0, I(t) \geq 0, T(t) \geq 0, V(t) \geq 0, R(t) \geq 0\} \in \Upsilon$ then the solution set $\{S(t), E(t), I(t), T(t), V(t), R(t)\} \geq 0$, $\forall\ t \geq 0$.

Proof. From

$$\frac{dS}{dt} = \mu_1(1 - p_2 I) + \eta V + \sigma_3 R - [(\beta_1 I + \beta_2 T) + \mu_2 + \sigma_1]S, \tag{6}$$

we have

$$\frac{dS}{dt} \geq -[(\beta_1 I + \beta_2 T) + \mu_2 + \sigma_1]S(t),$$
$$\frac{dS}{S(t)} \geq -[(\beta_1 I + \beta_2 T) + \mu_2 + \sigma_1]dt.$$

Applying anti-derivate on both sides, we have

$$\int \frac{dS}{S(t)} \geq \int -[(\beta_1 I + \beta_2 T) + \mu_2 + \sigma_1]dt,$$
$$\ln|S(t)| \geq -[(\beta_1 I + \beta_2 T) + \mu_2 + \sigma_1]t + c,$$
$$S(t) \geq e^{-[(\beta_1 I + \beta_2 T) + \mu_2 + \sigma_1]t + c},$$

for $t \geq 0$

$$S(t) \geq S(0)e^{-[(\beta_1 I + \beta_2 T) + \mu_2 + \sigma_1]t} \geq 0.$$

Therefore $S(t) \geq 0$. Next,consider the second equation in the model

$$\frac{dE}{dt} = (\beta_1 I + \beta_2 T)S + \mu_1 p_2 I - [\mu_2 + \alpha]E. \tag{7}$$

Now, we have

$$\frac{dE}{dt} \geq -[\mu_2 + \alpha]E(t),$$
$$\frac{dE}{E(t)} \geq -[\mu_2 + \alpha]dt.$$

Taking anti derivative of both sides of the equation, we have

$$\int \frac{dE}{E(t)} \geq \int -[\mu_2 + \alpha]dt,$$
$$\ln|E(t)| \geq -[\mu_2 + \alpha]t + c,$$
$$E(t) \geq e^{-[\mu_2 + \alpha]t + c},$$

for non-negative values of t

$$E(t) \geq E(0)e^{-[\mu_2 + \alpha]t} \geq 0.$$

Hence, $E(t) \geq 0$ similarly, it can be proved that $I(t), T(t), V(t)$ and $R(t)$ are all positively invariant for all non-negative values of t. □

3.3 Disease free equilibrium

At disease free equilibrium, there are no infections and recovery, hence;

$$\frac{dS}{dt} = \frac{dE}{dt} = \frac{dI}{dt} = \frac{dT}{dt} = \frac{dV}{dt} = \frac{dR}{dt} = 0. \tag{8}$$

The DFE of the model is obtained as;

$$\Big(A_1, 0, 0, 0, A_2, 0\Big), \tag{9}$$

where

$$A_1 = \frac{\mu_1[\eta + \mu_2]}{\big([\mu_2 + \sigma_1][\eta + \mu_2]\big) - \eta\sigma_1},$$
$$A_2 = \frac{\sigma_1\mu_1}{\big([\eta + \mu_2][\mu_2 + \sigma_1]\big) - \sigma_1\eta}.$$

3.4 Basic reproduction number, R_0

Using the next generation matrix approach as outlined in [16, 17]; Considering;

$$\begin{cases} \dfrac{dE}{dt} &= (\beta_1 I + \beta_2 T)S + \mu_1 p_2 I - [\mu_2 + \alpha]E, \\ \dfrac{dI}{dt} &= \alpha E - [\lambda + \mu_2]I, \\ \dfrac{dT}{dt} &= (1 - p_1)\lambda I - [\sigma_4 + \mu_2]T. \end{cases} \tag{10}$$

Let

$$F_i(x) = \begin{bmatrix} (\beta_1 I + \beta_2 T)S(t) \\ 0 \\ 0 \end{bmatrix}, \quad V_i(x) = \begin{bmatrix} -\mu_1 p_2 I + [\mu_2 + \alpha]E \\ -\alpha E + [\lambda + \mu_2]I \\ -(1 - p_1)\lambda I + [\sigma_4 + \mu_2]T \end{bmatrix}.$$

Taking the partial derivative of $F_i(x)$ and $V_i(x)$ at DFE;

$$F = \begin{bmatrix} 0 & S^0\beta_1 & S^0\beta_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{11}$$

and

$$V = \begin{pmatrix} [\mu_2 + \alpha] & -\mu_1 P_2 & 0 \\ -\alpha & [\lambda + \mu_2] & 0 \\ 0 & -\lambda(1 - p_1) & [\sigma_4 + \mu_2] \end{pmatrix} \tag{12}$$

finding the inverse of V

(13)
$$V^{-1} = \begin{pmatrix} \dfrac{\lambda+\mu_2}{[\alpha+\mu_2][\lambda+\mu_2]-\alpha\mu_1P_2} & \dfrac{\mu_1P_2}{[\alpha+\mu_2][\lambda+\mu_2]-\alpha\mu_1P_2} & 0 \\ \dfrac{\alpha}{[\alpha+\mu_2][\lambda+\mu_2]-\alpha\mu_1P_2} & \dfrac{\alpha+\mu_2}{[\alpha+\mu_2][\lambda+\mu_2]-\alpha\mu_1P_2} & 0 \\ \dfrac{(1-P_1)\alpha\lambda}{A_3} & \dfrac{(1-P_1)\lambda(\alpha+\mu_2)}{A_3} & \dfrac{1}{\sigma_4+\mu_2} \end{pmatrix},$$

where $A_3 = (\mu_2+\sigma_4)\big([\alpha+\mu_2][\lambda+\mu_2]-\alpha\mu_1P_2\big)$

(14)
$$FV^{-1}\begin{pmatrix} D_1 & D_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where

$$D_1 = \frac{S^0\beta_1\alpha(\sigma_4+\mu_2)+S^0\beta_2\alpha(1-P_1)\lambda}{(\mu_2+\sigma_4)\big([\alpha+\mu_2][\lambda+\mu_2]-\alpha\mu_1P_2\big)}$$
$$D_2 = \frac{S^0\beta_1(\sigma_4+\mu_2)(\alpha+\mu_2)+S^0\beta_2\alpha(1-P_1)\lambda(\alpha+\mu_2)}{(\mu_2+\sigma_4)\big([\alpha+\mu_2][\lambda+\mu_2]-\alpha\mu_1P_2\big)}.$$

The spectral radius (largest eigen value) of the matrix FV^{-1} is the basic reproductive number [18, 19, 20]. Since FV^{-1} is a triangular matrix the eigen values are $D_1, 0$ and 0 therefore the spectral radius is D_1 hence $R_0 = D_1$

(15)
$$R_0 = \frac{S^0\beta_1\alpha(\sigma_4+\mu_2)+S^0\beta_2\alpha(1-P_1)\lambda}{(\mu_2+\sigma_4)\big([\alpha+\mu_2][\lambda+\mu_2]-\alpha\mu_1P_2\big)}.$$

Our model is epidemiologically meaningful if

(16)
$$[\alpha+\mu_2][\lambda+\mu_2]-\alpha\mu_1P_2 > 0.$$

3.5 Local stability of disease-free equilibrium

Theorem 3.1. *The disease-free equilibrium point is locally asymptotically stable if $R_0 \leq 1$ and unstable if $R_0 > 1$.*

Proof. The Jacobian matrix is given by

(17)
$$J_{DFE} = \begin{bmatrix} -H_1 & 0 & -H_7 & -H_8 & \eta & \sigma_3 \\ 0 & -H_2 & H_7 & H_8 & 0 & 0 \\ 0 & \alpha & -H_3 & 0 & 0 & 0 \\ 0 & 0 & H_9 & -H_4 & 0 & 0 \\ \sigma_1 & 0 & 0 & 0 & -H_5 & 0 \\ 0 & 0 & H_{10} & \sigma_4 & 0 & -H_6 \end{bmatrix}$$

where $H_1 = [\sigma_1+\mu_2]$, $H_2 = [\mu_2+\alpha]$, $H_3 = [\lambda+\mu_2]$, $H_4 = [\sigma_4+\mu_2]$, $H_5 = [\eta+\mu_2]$, $H_6 = [\sigma_3+\mu_2]$, $H_7 = \mu_1 P_2 + \beta_1 S^0$, $H_8 = \beta_2 S^0$, $H_9 = \lambda(1-p_1)$, $H_{10} = \lambda p_1$, $S = S^0$.

The characteristics equation $P(\bar{\lambda}) = |\bar{\lambda} I - J_{DEF}| = 0$ becomes

$$P(\bar{\lambda}) = \begin{vmatrix} \bar{\lambda}+H_1 & 0 & H_7 & H_8 & -\eta & -\sigma_3 \\ 0 & \bar{\lambda}+H_2 & -H_7 & -H_8 & 0 & 0 \\ 0 & -\alpha & \bar{\lambda}+H_3 & 0 & 0 & 0 \\ 0 & 0 & -H_9 & \bar{\lambda}+H_4 & 0 & 0 \\ -\sigma_1 & 0 & 0 & 0 & \bar{\lambda}+H_5 & 0 \\ 0 & 0 & -H_{10} & -\sigma_4 & 0 & \bar{\lambda}+H_6 \end{vmatrix} = 0. \tag{18}$$

Let

$$\begin{aligned} P(\bar{\lambda}) &= P(\bar{\lambda_1}) \times P(\bar{\lambda_2}) = 0, \\ P(\bar{\lambda_1}) &= (\bar{\lambda}+H_6)[(\bar{\lambda}+H_1)(\bar{\lambda}+H_5) - \sigma_1\eta] - \sigma_1\sigma_3 = 0 \\ &= \bar{\lambda}^3 + (H_1+H_5+H_6)\bar{\lambda}^2 + (H_1H_5 + H_1H_6 + H_5H_6 - \sigma_1\eta)\bar{\lambda} \\ &\quad + H_1H_5H_6 - \sigma_1\eta H_6 - \sigma_1\sigma_3 = 0. \end{aligned} \tag{19}$$

For the other factor

$$\begin{aligned} P(\bar{\lambda_2}) &= (\bar{\lambda}+H_4)[(\bar{\lambda}+H_2)(\bar{\lambda}+H_3) - \alpha H_7] - \alpha H_8 H_9 = 0 \\ &= \bar{\lambda}^3 + (H_2+H_3+H_4)\bar{\lambda}^2 + (H_2H_3 + H_2H_4 + H_3H_4 - \alpha H_7)\bar{\lambda} \\ &\quad + H_2H_3H_4 - \alpha H_4H_7 - \alpha H_8H_9 = 0. \end{aligned}$$

Now, we analyze $P(\bar{\lambda_1}) = 0$ and $P(\bar{\lambda_2}) = 0$ separately for the nature of their roots by using the Routh-Hurwitz criteria. Let's begin with $P(\bar{\lambda_1}) = 0$. $P(\bar{\lambda_1})$ is in the form

$$P(\bar{\lambda_1}) = \bar{\lambda}^3 + m_1\bar{\lambda}^2 + m_2\bar{\lambda} + m_3 = 0 \tag{20}$$

and

$$\begin{aligned} m_1 &= H_1 + H_5 + H_6 \\ &= \sigma_1 + \mu_2 + \eta + \mu_2 + \sigma_3 + \mu_2 \\ &= \sigma_1 + \sigma_3 + \eta + 3\mu_2, \\ m_2 &= H_1H_5 + H_1H_6 + H_5H_6 - \sigma_1\eta \\ &= (\sigma_1+\mu_2)(\eta+\mu_2) + (\sigma_1+\mu_2) \\ &\quad (\sigma_3+\mu_2) + (\eta+\mu_2)(\sigma_3+\mu_2) - \sigma_1\eta \\ &= \mu_2\eta + (\sigma_1+\mu_2)(\mu_2) + (\sigma_1+\mu_2)(\sigma_3+\mu_2) \\ &\quad + (\eta+\mu_2)(\sigma_3+\mu_2), \end{aligned}$$

$$\begin{aligned} m_3 &= H_1H_5H_6 - \sigma_1\eta H_6 - \sigma_1\sigma_3 \\ &= (\sigma_1+\mu_2)(\eta+\mu_2)(\sigma_3+\mu_2) \\ &\quad - \sigma_1\eta(\sigma_3+\mu_2) - \sigma_1\sigma_3 \\ &= \sigma_1\mu_2 + \mu_2(\sigma_1+\eta+\mu_2)(\sigma_3+\mu_2). \end{aligned}$$

According to Routh-Hurwitz criteria, roots of $P(\bar{\lambda_1})$ will have negative real parts if the conditions below are satisfied

(i) $m_1 > 0$;

(ii) $m_2 > 0$;

(iii) $m_3 > 0$;

(v) $m_1m_2 > m_3$;

$$\begin{aligned} m_1 &= \sigma_1+\sigma_3+\eta+3\mu_2 > 0, \\ m_3 &= \sigma_1\mu_2 + \mu_2(\sigma_1+\eta+\mu_2)(\sigma_3+\mu_2) > 0. \end{aligned}$$

Now,

$$\begin{aligned} m_1m_2 &= (\sigma_1+\sigma_3+\eta+3\mu_2)\big[\mu_2\eta + (\sigma_1+\mu_2)(\mu_2) + (\sigma_1+\mu_2)(\sigma_3+\mu_2) \\ &\quad + (\eta+\mu_2)(\sigma_3+\mu_2)\big], \\ m_3 &= \sigma_1\mu_2 + \mu_2(\sigma_1+\eta+\mu_2)(\sigma_3+\mu_2). \end{aligned}$$

Notice that in expanded form , m_1m_2 contains all the terms in m_3 and will still be left with additional terms. Therefore $m_1m_2 > m_3$ is satisfied. By the Routh-Hurwitz criteria all the roots of $P(\bar{\lambda_1}) = 0$ are negative or have negative real parts.

Let's now analyze the second factor $P(\bar{\lambda_2})$. $P(\bar{\lambda_2})$ is in the form

$$P(\bar{\lambda_2}) = \bar{\lambda}^3 + c_1\bar{\lambda}^2 + c_2\bar{\lambda} + c_3 = 0, \tag{21}$$

where

$$\begin{aligned} c_1 &= H_2+H_3+H_4 = \alpha+\lambda+\sigma_4+3\mu_2, \\ c_2 &= H_2H_3 + H_2H_4 + H_3H_4 - \alpha H_7 = (\alpha+\mu_2)(\lambda+\mu_2) + (\alpha+\mu_2)(\sigma_4+\mu_2) \\ &\quad + (\lambda+\mu_2)(\sigma_4+\mu_2) - \alpha(\mu_1P_2+\beta_1S), \\ c_3 &= H_2H_3H_4 - \alpha H_4H_7 - \alpha H_8H_9 = (\alpha+\mu_2)(\lambda+\mu_2)(\sigma_4+\mu_2) - \alpha\mu_1P_2(\sigma_4+\mu_2) \\ &\quad - \alpha\beta_1S(\sigma_4+\mu_2) - \alpha\lambda\beta_2S(1-p_1). \end{aligned}$$

According to the Routh-Hurwitz criterion for $P(\bar{\lambda_2}) = 0$ to have negative real parts, $c_1 > 0, c_3 > 0$ and $c_1c_2 > c_3$, $c_1 = \alpha+\lambda+\sigma_4+3\mu_2 > 0, c_2 > 0 \implies$

$(\alpha+\mu_2)(\lambda+\mu_2)+(\alpha+\mu_2)(\sigma_4+\mu_2)+(\lambda+\mu_2)(\sigma_4+\mu_2)-\alpha\mu_1P_2>\alpha\beta_1S$ but from $R_0<1$ $(\alpha+\mu_2)(\lambda+\mu_2)-\mu_1P_2>\alpha\beta_1S$, $c_2>0$ holds

$$
\begin{aligned}
&c_3=(\alpha+\mu_2)(\lambda+\mu_2)(\sigma_4+\mu_2)-\alpha\mu_1P_2(\sigma_4+\mu_2)-\alpha\beta_1S(\sigma_4+\mu_2)\\
&-\alpha\lambda\beta_2S(1-p_1)>0,\\
&(\alpha+\mu_2)(\lambda+\mu_2)(\sigma_4+\mu_2)-\alpha\mu_1P_2(\sigma_4+\mu_2)>\alpha\beta_1S(\sigma_4+\mu_2)+\alpha\lambda\beta_2S(1-p_1)\\
&\alpha\beta_1S(\sigma_4+\mu_2)+\alpha\lambda\beta_2S(1-p_1)<(\alpha+\mu_2)(\lambda+\mu_2)(\sigma_4+\mu_2)-\alpha\mu_1P_2(\sigma_4+\mu_2),
\end{aligned}
$$

$$
\frac{\alpha\beta_1S(\sigma_4+\mu_2)+\alpha\lambda\beta_2S(1-p_1)}{(\alpha+\mu_2)(\lambda+\mu_2)(\sigma_4+\mu_2)-\alpha\mu_1P_2(\sigma_4+\mu_2)}<1. \tag{22}
$$

Since $R_0<1$, $m_3>0$ holds.

We now analyze $c_1c_2>c_3$ below

$$
\begin{aligned}
&(H_2+H_3+H_4)(H_2H_3+H_2H_4+H_3H_4-\alpha H_7)>H_2H_3H_4\\
&-\alpha H_4H_7-\alpha H_8H_9,\\
&(H_2+H_3)(H_2H_3+H_2H_4+H_3H_4-\alpha H_7)+H_2H_3H_4+H_2H_4H_4\\
&+H_3H_4H_4-\alpha H_4H_7>H_2H_3H_4-\alpha H_4H_7-\alpha H_8H_9,\\
&(H_2+H_3)(H_2H_3+H_2H_4+H_3H_4)-\alpha H_7(H_2+H_3)+H_4H_4(H_2+H_3)>-\alpha H_8H_9,\\
&(H_2+H_3)(H_2H_3+H_2H_4+H_3H_4+H_4H_4)>\alpha H_7(H_2+H_3)-\alpha H_8H_9,\\
&(H_2+H_3)(H_2+H_4)(H_3+H_4)>\alpha H_7(H_2+H_3)-\alpha H_8H_9,\\
&\alpha H_7(H_2+H_3)-\alpha H_8H_9<(H_2+H_3)(H_2+H_4)(H_3+H_4),
\end{aligned}
$$

substituting the model parameters

$$
\begin{aligned}
&\alpha(\mu_1P_2+\beta_1S)(\alpha+\lambda+2\mu_2)-\alpha\beta_2S\lambda(1-p_1)<\\
&(\alpha+\lambda+2\mu_2)(\alpha+\sigma_4+2\mu_2)(\lambda+\sigma_4+2\mu_2),\\
&(\alpha+\lambda+2\mu_2)\beta_1S-\alpha\beta_2S\lambda(1-p_1)<(\alpha+\lambda+2\mu_2)(\alpha+\sigma_4+2\mu_2)(\lambda+\sigma_4+2\mu_2)\\
&-\alpha(\alpha+\lambda+2\mu_2)\mu_1P_2\alpha\beta_1S-\frac{\alpha\beta_2S\lambda(1-p_1)}{\alpha+\lambda+2\mu_2}\\
&<(\alpha+\sigma_4+2\mu_2)(\lambda+\sigma_4+2\mu_2)-\alpha\mu_1P_2,
\end{aligned}
$$

$$
\alpha\beta_1S<(\alpha+\sigma_4+2\mu_2)(\lambda+\sigma_4+2\mu_2)-\alpha\mu_1P_2+\frac{\alpha\beta_2S\lambda(1-p_1)}{\alpha+\lambda+2\mu_2}. \tag{23}
$$

From $R_0<1$

$$
\begin{aligned}
&\alpha\beta_1S(\sigma_4+\mu_2)+\alpha\beta_2\lambda S(1-p_1)<(\sigma_4+\mu_2)\big[(\alpha+\mu_2)(\lambda+\mu_2)-\alpha\mu_1P_2\big],\\
&\alpha\beta_1S(\sigma_4+\mu_2)<(\sigma_4+\mu_2)\big[(\alpha+\mu_2)(\lambda+\mu_2)-\alpha\mu_1P_2\big],\\
&\alpha\beta_1S<(\alpha+\mu_2)(\lambda+\mu_2)-\alpha\mu_1P_2.
\end{aligned} \tag{24}
$$

Equation (24) is a true inequality when $R_0 < 1$, therefore equation (23) is also true because the right hand side of equation (23) is greater than right hand side of equation (24).

This proves that $c_1c_2 > c_3$. It follows that $P(\bar{\lambda}_2) = 0$ has its roots negative or negative real parts.

In conclusion, the roots of $P(\bar{\lambda}) = 0$ are all negatives or have negative real parts so the disease free-equilibrium is asymptotically stable when $R_0 < 1$ and unstable when $R_0 > 1$ □

3.6 Global stability of the disease free equilibrium point (DFE)

Theorem 3.2. *If $R_0 > 1$, the disease free equilibrium Q^0 of the model is globally asymptotically stable*

Let F_i, V_i, F and V be defined as

$$f(x,y) = (F-V)x - F_i(x,y) + V_i(x,y), \tag{25}$$

where $x = (E, I, T)^T$,

$$\begin{aligned} f(x,y) &= \begin{bmatrix} -(\alpha+\mu_2) & \beta_1 S + \mu_1 P_2 & \beta_2 S \\ \alpha & -(\mu_2+\lambda) & 0 \\ 0 & \lambda(1-p_1) & -(\mu_2+\sigma_4) \end{bmatrix} \begin{bmatrix} E \\ I \\ T \end{bmatrix} \\ &+ \begin{bmatrix} -(\beta_1 I + \beta_2 T)S - \mu_1 P_2 I + (\alpha+\mu_2)E \\ (\lambda+\mu_2)I - \alpha E \\ (\sigma_4+\mu_2)I - \lambda(1-p_1)I \end{bmatrix} \\ &= \begin{bmatrix} -(\alpha+\mu_2)E + (\beta_1 S + \mu_1 P_2)I + \beta_2 ST \\ \alpha E - (\mu_2+\lambda)I \\ \lambda(1-p_1)I - (\mu_2+\sigma_4)T \end{bmatrix} \\ &+ \begin{bmatrix} -(\beta_1 I + \beta_2 T)S - \mu_1 P_2 I + (\alpha+\mu_2)E \\ (\lambda+\mu_2)I - \alpha E \\ (\sigma_4+\mu_2)I - \lambda(1-p_1)I \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \end{aligned}$$

Also, let $x' = (F-V)x - F(x,y)$ but $f(x,y) = 0 \therefore x' = (F-V)x$. Let $W^T \geq 0$ be defined as the left eigen vector of the nonnegative matrix $V^{-1}F$ corresponding to the eigen value $\rho(V^{-1}F) = \rho(FV^{-1}) = R_0$. We proceed to first find $V^{-1}F$ and $\rho(V^{-1}F)$

$$V^{-1} = \begin{bmatrix} \dfrac{\lambda+\mu_2}{[\alpha+\mu_2][\lambda+\mu_2] - \alpha\mu_1 P_2} & \dfrac{\mu_1 P_2}{[\alpha+\mu_2][\lambda+\mu_2] - \alpha\mu_1 P_2} & 0 \\ \dfrac{\alpha}{[\alpha+\mu_2][\lambda+\mu_2] - \alpha\mu_1 P_2} & \dfrac{\alpha+\mu_2}{[\alpha+\mu_2][\lambda+\mu_2] - \alpha\mu_1 P_2} & 0 \\ \dfrac{(1-P_1)\alpha\lambda}{D_0} & \dfrac{(1-P_1)\lambda(\alpha+\mu_2)}{D_0} & \dfrac{1}{\sigma_4+\mu_2} \end{bmatrix}$$

$$F = \begin{bmatrix} 0 & S^0\beta_1 & S^0\beta_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Now,

$$V^{-1}F = \begin{pmatrix} 0 & \dfrac{(\lambda+\mu_2)\beta_1 S}{[\alpha+\mu_2][\lambda+\mu_2]-\alpha\mu_1 P_2} & \dfrac{(\lambda+\mu_2)\beta_2 S}{[\alpha+\mu_2][\lambda+\mu_2]-\alpha\mu_1 P_2} \\ 0 & \dfrac{\beta_1 S\alpha}{[\alpha+\mu_2][\lambda+\mu_2]-\alpha\mu_1 P_2} & \dfrac{\beta_2 S\alpha}{[\alpha+\mu_2][\lambda+\mu_2]-\alpha\mu_1 P_2} \\ 0 & \dfrac{(1-P_1)\alpha\lambda\beta_1 S}{D_0} & \dfrac{(1-P_1)\alpha\lambda\beta_2 S}{D_0} \end{pmatrix}.$$

Define $g = \beta_1 S$, $h = \beta_2 S$, $j = \alpha+\mu_2$, $k = \lambda(1-p_1)$, $m = (\alpha+\mu_2)(\lambda+\mu_2)-\alpha\mu_1 P_2$, $n = \sigma_4+\mu_2$, $l = \lambda+\mu_2$, $S = S^0$. Hence

$$V^{-1}F = \begin{pmatrix} 0 & \dfrac{gl}{m} & \dfrac{hl}{m} \\ 0 & \dfrac{\alpha g}{m} & \dfrac{\alpha h}{m} \\ 0 & \dfrac{\alpha gk}{mn} & \dfrac{\alpha hk}{mn} \end{pmatrix}.$$

For $\rho(V^{-1}F)$, first determine eigenvalues, the characteristics equation $|(V^{-1}F)-\bar{\lambda}I| = 0$ is given by

$$\begin{aligned} &-\bar{\lambda}\left[(\frac{\alpha g}{m}-\bar{\lambda})(\frac{\alpha hk}{mn}-\bar{\lambda})-\frac{\alpha gk}{mn}\times\frac{\alpha h}{m}\right]=0, \\ &-\bar{\lambda}\left[\bar{\lambda}^2-\frac{\alpha g}{m}\bar{\lambda}-\frac{\alpha hk}{mn}\bar{\lambda}+\frac{\alpha^2 ghk}{m^2 n}-\frac{\alpha^2 ghk}{m^2 n}\right]=0, \\ &-\bar{\lambda}\left[\bar{\lambda}^2-(\frac{\alpha g}{m}+\frac{\alpha hk}{mn})\bar{\lambda}\right]=0, \end{aligned}$$

either

$$\begin{aligned} &-\bar{\lambda}=0, \quad or \quad \bar{\lambda}^2-\frac{\alpha gn+\alpha hk}{mn}\bar{\lambda}=0, \bar{\lambda}=\big(0,0,\frac{\alpha gn+\alpha hk}{mn}\big), \\ &\therefore V^{-1}F=\left(0,0,\frac{S^0\beta_1\alpha(\sigma_4+\mu_2)+S^0\beta_2\alpha(1-P_1)\lambda}{(\mu_2+\sigma_4)\big([\alpha+\mu_2][\lambda+\mu_2]-\alpha\mu_1 P_2\big)}\right), \\ &\therefore \rho\big(V^{-1}F\big)=\rho(FV^{-1})=R_0. \end{aligned}$$

The left eigen vector of the matrix $V^{-1}F$ corresponding to the eigen value R_0 can be obtained from

$$\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 0 & \frac{gl}{m} & \frac{hl}{m} \\ 0 & \frac{\alpha g}{m} & \frac{\alpha h}{m} \\ 0 & \frac{\alpha gk}{mn} & \frac{\alpha hk}{mn} \end{pmatrix} = R_0 \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix},$$

$$\left(0, \quad (\frac{gl}{m}x_1 + \frac{\alpha g}{m}x_2 + \frac{\alpha gk}{mn}x_3 \quad), (\frac{hl}{m}x_1 + \frac{\alpha h}{m}x_2 + \frac{\alpha hk}{mn}x_3)\right)$$
$$= R_0 \begin{pmatrix} x_1R_0 & x_2R_0 & x_3R_0 \end{pmatrix}.$$

Equating corresponding entries we have

$$\begin{aligned} R_0x_1 &= 0, \\ \frac{gl}{m}x_1 + \frac{\alpha g}{m}x_2 + \frac{\alpha gk}{mn}x_3 &= R_0x_2, \\ \frac{hl}{m}x_1 + \frac{\alpha h}{m}x_2 + \frac{\alpha hk}{mn}x_3 &= R_0x_3. \end{aligned}$$

Simplifying, we have

$$\begin{aligned} x_1 &= 0, \\ gnlx_1 + \alpha gnx_2 + \alpha gkx_3 &= mnR_0x_2, \\ hlnx_1 + \alpha hnx_2 + \alpha hkx_3 &= mnR_0x_3, \end{aligned}$$

but $x_1 = 0$, therefore

$$\begin{aligned} \alpha gnx_2 + \alpha gkx_3 = mnR_0x_2 &= (\alpha gn + \alpha hk)x_2, \\ \alpha hnx_2 + \alpha hkx_3 = mnR_0x_3 &= (\alpha gn + \alpha hk)x_3. \end{aligned}$$

Simplifying further and expressing x_3 in terms of x_2 yields

$$x_3 = \frac{h}{g}x_2 = \frac{\beta_2 S x_2}{\beta_1 S} = \frac{\beta_2 x_2}{\beta_1}.$$

Let $x_2 = t$, $(x_1, x_2, x_3) = (0, t, \frac{\beta_2}{\beta_1}t) = t\big(0, 1, \frac{\beta_2}{\beta_1}\big)$. Therefore, the left eigen vector of the matrix $V^{-1}F$ corresponding to the eigen value R_0 is $W^T = \big(0, 1, \frac{\beta_2}{\beta_1}\big)$ or any scalar multiple of $(0, 1, \frac{\beta_2}{\beta_1})$.

The Lyapunov function to prove the disease free equilibrium of the model is given by

$$A = W^T V^{-1} x = (0, 1, \frac{\beta_2}{\beta_1}) V^{-1} \begin{pmatrix} E \\ I \\ T \end{pmatrix} = (0, 1, \frac{\beta_2}{\beta_1}) \begin{pmatrix} \frac{l}{m} & \frac{\mu_1 P_2}{m} & 0 \\ \frac{\alpha}{m} & \frac{j}{m} & 0 \\ \frac{\alpha k}{mn} & \frac{\alpha j k}{mn} & \frac{1}{n} \end{pmatrix} \begin{pmatrix} E \\ I \\ T \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\alpha}{m} + \frac{\beta_2 \alpha k}{\beta_1 mn} & \frac{j}{m} + \frac{\beta_2 j k}{\beta_1 mn} & \frac{\beta_2}{\beta_1 n} \end{pmatrix} \begin{pmatrix} E \\ I \\ T \end{pmatrix}$$

$$= \left(\frac{\alpha}{m} + \frac{\beta_2 \alpha k}{\beta_1 mn} \right) E + \left(\frac{j}{m} + \frac{\beta_2 j k}{\beta_1 mn} \right) I + \frac{\beta_2}{\beta_1 n} T$$

$$= \frac{\alpha \beta_1 n + \beta_2 \alpha k}{\beta_1 mn} E + \frac{\beta_1 n j + \beta_2 j k}{\beta_1 mn} I + \frac{\beta_2}{\beta_1 n} T$$

$$\therefore A = \frac{\beta_1 \alpha(\sigma_4 + \mu_2) + \beta_2 \alpha \lambda (1 - p_1)}{\beta_1 (\sigma_4 + \mu_2) \big[(\alpha + \mu_2)(\lambda + \mu_2) - \alpha \mu_1 P_2 \big]} E \tag{26}$$

$$+ \frac{\beta_1 (\sigma_4 + \mu_2)(\alpha + \mu_2) + \beta_2 (\alpha + \mu_2)(1 - p_1) \lambda}{\beta_1 (\sigma_4 + \mu_2) \big[(\alpha + \mu_2)(\lambda + \mu_2) - \alpha \mu_1 P_2 \big]} I + \frac{\beta_2}{\beta_1 (\sigma_4 + \mu_2)} T. \tag{27}$$

Form this it is obvious that

(i) $A(x) > 0$ for all $x \in \upsilon$;

(ii) $||A(x)|| \to \infty \implies ||x|| \to \infty$ (that is radially unbounded).

We can directly find the derivative of the Lyapunov function from A. (Refer to the Appendix). However, we chose to find it from the equation as shown as

$$\begin{aligned} A' &= W^T V^{-1} x' \\ &= W^T V^{-1} \big[(F - V) x - f(x, y) \big] \\ &= W^T V^{-1} (F - V) x - W^T V^{-1} f(x, y) \\ &= W^T V^{-1} F x - W^T V^{-1} V x - 0 \\ &= W^T V^{-1} f x - W^T x \\ &= (0, 1, \frac{\beta_2}{\beta_1}) \begin{pmatrix} 0 & \frac{gl}{m} & \frac{hl}{m} \\ 0 & \frac{\alpha g}{m} & \frac{\alpha h}{m} \\ 0 & \frac{\alpha g k}{mn} & \frac{\alpha h k}{mn} \end{pmatrix} x - (0, 1, \frac{\beta_2}{\beta_1}) x \end{aligned}$$

$$= \begin{pmatrix} 0 & \frac{\alpha g}{m} + \frac{g\beta_2\alpha k}{\beta_1 mn} & \frac{\alpha h}{m} + \frac{\alpha\beta_2 hk}{\beta_1 mn} \end{pmatrix} \begin{pmatrix} E \\ I \\ T \end{pmatrix} - (0, 1, \frac{\beta_2}{\beta_1}) \begin{pmatrix} E \\ I \\ T \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \frac{\beta_1 n\alpha g + g\beta_2\alpha k}{\beta_1 mn} & \frac{\beta_1 n\alpha h + \alpha\beta_2 hk}{\beta_1 mn} \end{pmatrix} \begin{pmatrix} E \\ I \\ T \end{pmatrix} - (0, 1, \frac{\beta_2}{\beta_1}) \begin{pmatrix} E \\ I \\ T \end{pmatrix}.$$

Substituting model parameter for values of g and h

$$A' = \begin{pmatrix} 0 & \frac{\beta_1 n\alpha\beta_1 S + \beta_2\alpha k\beta_1 S}{\beta_1 mn} & \frac{\beta_1 n\alpha\beta_2 S + \alpha\beta_2\beta_2 Sk}{\beta_1 mn} \end{pmatrix} \begin{pmatrix} E \\ I \\ T \end{pmatrix} - (0, 1, \frac{\beta_2}{\beta_1}) \begin{pmatrix} E \\ I \\ T \end{pmatrix}$$

$$= \frac{n\alpha\beta_1 S + \alpha k\beta_2 S}{mn} (0, 1, \frac{\beta_2}{\beta_1}) \begin{pmatrix} E \\ I \\ T \end{pmatrix} - (0, 1, \frac{\beta_2}{\beta_1}) \begin{pmatrix} E \\ I \\ T \end{pmatrix}.$$

Substituting values of m, n and k in terms of parameters for the model

$$A' = \frac{\beta_1\alpha S^0(\sigma_4 + \mu_2) + \beta_2\alpha\lambda S^0(1 - p_1)}{(\sigma_4 + \mu_2)\big[(\alpha + \mu_2)(\lambda + \mu_2) - \alpha\mu_1 P_2\big]} (0, 1, \frac{\beta_2}{\beta_1}) \begin{pmatrix} E \\ I \\ T \end{pmatrix} - (0, 1, \frac{\beta_2}{\beta_1}) \begin{pmatrix} E \\ I \\ T \end{pmatrix}$$

$$= R_0 (0, 1, \frac{\beta_2}{\beta_1}) \begin{pmatrix} E \\ I \\ T \end{pmatrix} - (0, 1, \frac{\beta_2}{\beta_1}) \begin{pmatrix} E \\ I \\ T \end{pmatrix}$$

$$= R_0\big(I + \frac{\beta_2}{\beta_1} T\big) - \big(I + \frac{\beta_2}{\beta_1} T\big)$$

$$= \big(R_0 - 1\big)\big(I + \frac{\beta_2}{\beta_1} T\big).$$

It is easy to see that if $R_0 \leq 1$ then $A' \leq 0$, thus A is a Lyapunov function for the system. In addition, $A' = 0$ if $I = T = 0$ and $S = S^0$. Therefore the largest compact invariant set in $\{S(t), E(t), I(t), T(t), V(t), R(t) \in R^6_+ : A' = 0\}$ is the singleton Q^0, the disease free equilibrium of the system. Hence , by La Salle's invariance principle Q^0 is globally asymptotically stable in Υ if $R_0 \leq 1$.

4. Endemic equilibrium

It is a constant solution of the model where the disease per exist in the system. In our model the endemic equilibrium is denoted by Q^e. To obtained the endemic equilibrium we set the system in (2) to zero. Therefore, the endemic equilibrium

points are given by

$$\begin{cases} S^e = k_1, \\ E^e = \dfrac{(\lambda+\mu_2)(\sigma_4+\mu_2)(k_1k_2k_3 - k_1k_4 + k_5)}{\alpha\lambda(1-p_1)(k_6 + k_1k_7 - k_8)}, \\ I^e = \dfrac{(\sigma_4+\mu_2)(k_1k_2k_3 - k_1k_4 + k_5)}{\lambda(1-p_1)(k_6 + k_1k_7 - k_8)}, \\ T^e = \dfrac{k_1k_2k_3 - k_1k_4 + k_5}{k_6 + k_1k_7 - k_8}, \\ V^e = k_1k_2, \\ R^e = \dfrac{\sigma_2k_1k_2(1-p_1)(k_6 + k_1k_7 - k_8) + D_6(k_1k_2k_3 - k_1k_4 + k_5)}{(\sigma_3+\mu_2)(1-p_1)(k_6 + k_1k_7 - k_8)}, \end{cases}$$

where $D_6 = [p_1(\sigma_4+\mu_2) + \sigma_4(1-p_1)]$

$$k_1 = \frac{(\sigma_4+\mu_2)\big[(\mu_2+\alpha)(\lambda+\mu_2) - \alpha P_2\mu_1\big]}{\alpha\big[\beta_1(\sigma_4+\mu_2) + \beta_2\lambda(1-p_1)\big]}, \qquad k_2 = \frac{\sigma_1}{\eta+\mu_2},$$

$$k_3 = \lambda(1-p_1)(\sigma_3+\eta), \qquad k_4 = \lambda(1-p_1)(\sigma_3+\mu_2)(\sigma_1+\mu_2),$$

$$k_5 = \mu_1\lambda(1-p_1), \qquad k_7 = \beta_1(\sigma_3+\mu_2)(\sigma_4+\mu_2) + \lambda\beta_2(\sigma_3+\mu_2)(1-p_1),$$

$$k_6 = \mu_1P_2(\sigma_3+\mu_2)(\sigma_4+\mu_2), \qquad k_8 = \lambda\sigma_3\big[p_1(\sigma_4+\mu_2) + \sigma_4(1-p_1)\big].$$

Hence, $Q^e = \{S^e, E^e, I^e, T^e, V^e, R^e\}$.

4.1 Global stability of the endemic equilibrium point

Theorem 4.1. *If $R_0 > 1$, the endemic equilibrium Q^e of the model is globally asymptotically stable.*

Proof. The global asymptotic stability of the endemic equilibrium can be prooved using the Lyapunov function define as;

$$G(S^e, E^e, I^e, T^e, V^e, R^e) = G_1 + G_2 + G_3 + G_4 + G_5 + G_6 \tag{28}$$

$$\begin{cases} G_1 = S - S^e - S^e\ln\frac{S}{S^e}, & G_2 = E - E^e - E\ln\frac{E}{E^e} \\ G_3 = I - I^e - I^e\ln\frac{I}{I^e}, & G_4 = T - T^e - T^e\ln\frac{T}{T^e} \\ G_5 = V - V^e - V^e\ln\frac{V^e}{V}, & G_6 = R - R^e - R^e\ln\frac{R}{R^e} \end{cases}.$$

By setting the system of equation in (2) to zereo and taking the derivative of G along the solution of the equations, we obtain

$$\frac{dG}{dt} = \frac{dG_1}{dt} + \frac{dG_2}{dt} + \frac{dG_3}{dt} + \frac{dG_4}{dt} + \frac{dG_5}{dt} + \frac{dG_6}{dt}. \tag{29}$$

By substituting and simplifying all terms

$$
\begin{cases}
\frac{dG}{dt}=\mu_2 S^e\Big[-\mathcal{M}\Big(\frac{S^e}{S}\Big)-\mathcal{M}\Big(\frac{S}{S^e}\Big)\Big]+\mu_2 E^e\Big[-\mathcal{M}\Big(\frac{E^e}{E}\Big)-\mathcal{M}\Big(\frac{E}{E^e}\Big)\Big] \\
+\mu_2 I^e\Big[-\mathcal{M}\Big(\frac{I^e}{I}\Big)-\mathcal{M}\Big(\frac{I}{I^e}\Big)\Big]+\mu_2 T^e\Big[-\mathcal{M}\Big(\frac{T^e}{T}\Big)-\mathcal{M}\Big(\frac{T}{T^e}\Big)\Big] \\
+\mu_2 V^e\Big[-\mathcal{M}\Big(\frac{V^e}{V}\Big)-\mathcal{M}\Big(\frac{V}{V^e}\Big)\Big]+\mu_2 R^e\Big[-\mathcal{M}\Big(\frac{R^e}{R}\Big)-\mathcal{M}\Big(\frac{R}{R^e}\Big)\Big] \\
+\mu_1 P_2 I^e\Big[-\mathcal{M}\Big(\frac{S^e I}{SI^e}\Big)+\mathcal{M}\Big(\frac{E^e}{E}\Big)-\mathcal{M}\Big(\frac{E^e I}{EI^e}\Big)+\mathcal{M}\Big(\frac{S^e}{S}\Big)\Big] \\
+\sigma_1 S^e\Big[-\mathcal{M}\Big(\frac{S^e}{S}\Big)-\mathcal{M}\Big(\frac{V^e S}{VS^e}\Big)+\mathcal{M}\Big(\frac{V^e}{V}\Big)\Big]+\eta V^e\Big[-\mathcal{M}\Big(\frac{V^e}{V}\Big) \\
-\mathcal{M}\Big(\frac{S^e V}{SV^e}\Big)+\mathcal{M}\Big(\frac{S^e}{S}\Big)\Big]+\sigma_3 R^e\Big[-\mathcal{M}\Big(\frac{R^e}{R}\Big)-\mathcal{M}\Big(\frac{S^e R}{SR^e}\Big)+\mathcal{M}\Big(\frac{S^e}{S}\Big)\Big] \\
+\beta_1 I^e S^e\Big[-\mathcal{M}\Big(\frac{ISE^e}{I^e S^e E}\Big)-\mathcal{M}\Big(\frac{S^e}{S}\Big)+\mathcal{M}\Big(\frac{I}{I^e}\Big)+\mathcal{M}\Big(\frac{E^e}{E}\Big)\Big] \\
+\beta_2 T^e S^e\Big[-\mathcal{M}\Big(\frac{TSE^e}{T^e S^e E}\Big)+\mathcal{M}\Big(\frac{E^e}{E}\Big)-\mathcal{M}\Big(\frac{S^e}{S}\Big)+\mathcal{M}\Big(\frac{T}{T^e}\Big)\Big]
\end{cases} \tag{30}
$$

$$
\begin{cases}
+\ \alpha E^e\Big[-\mathcal{M}\Big(\frac{E^e}{E}\Big)-\mathcal{M}\Big(\frac{EI^e}{IE^e}\Big)+\mathcal{M}\Big(\frac{I^e}{I}\Big)\Big]+p_1\lambda I^e\Big[-\mathcal{M}\Big(\frac{I^e}{I}\Big) \\
-\mathcal{M}\Big(\frac{R^e I}{RI^e}\Big)+\mathcal{M}\Big(\frac{R^e}{R}\Big)\Big]+(1-p_1)\lambda I^e\Big[-\mathcal{M}\Big(\frac{I^e}{I}\Big)-\mathcal{M}\Big(\frac{T^e I}{TI^e}\Big) \\
+\mathcal{M}\Big(\frac{T^e}{T}\Big)\Big]+\sigma_4 T^e\Big[-\mathcal{M}\Big(\frac{T^e}{T}\Big)-\mathcal{M}\Big(\frac{R^e T}{RT^e}\Big)+\mathcal{M}\Big(\frac{R^e}{R}\Big)\Big] \\
+V^e\Big[-\mathcal{M}\Big(\frac{V^e}{V}\Big)-\mathcal{M}\Big(\frac{R^e V}{RV^e}\Big)+\mathcal{M}\Big(\frac{R^e}{R}\Big)\Big]
\end{cases} \tag{31}
$$

Let

$$
\frac{dG}{dt}=L_1+L_2, \tag{32}
$$

where

$$
\begin{cases}
L_1 &= -\mu_2 S^e\mathcal{M}\Big(\frac{S}{S^e}\Big)-\mu_2 E^e\mathcal{M}\Big(\frac{E}{E^e}\Big)-\mu_2 I^e\mathcal{M}\Big(\frac{I}{I^e}\Big)-\mu_2 T^e\mathcal{M}\Big(\frac{T}{T^e}\Big) \\
&-\mu_2 V^e\mathcal{M}\Big(\frac{V}{V^e}\Big)-\mu_2 R^e\mathcal{M}\Big(\frac{R}{R^e}\Big)-\mu_1 P_2 I^e\mathcal{M}\Big(\frac{E^e I}{EI^e}\Big)-\sigma_1 S^e\mathcal{M}\Big(\frac{V^e S}{VS^e}\Big) \\
&-\eta V^e\mathcal{M}\Big(\frac{S^e V}{SV^e}\Big)-\sigma_3 R^e\mathcal{M}\Big(\frac{S^e R}{SR^e}\Big)-\beta_1 I^e S^e\mathcal{M}\Big(\frac{ISE^e}{I^e S^e E}\Big)-\beta_2 T^e S^e\mathcal{M}\Big(\frac{TSE^e}{T^e S^e E}\Big) \\
&-\alpha E^e\mathcal{M}\Big(\frac{I^e E}{IE^e}\Big)-p_1\lambda I^e\mathcal{M}\Big(\frac{R^e I}{RI^e}\Big)-(1-p_1)\lambda I^e\mathcal{M}\Big(\frac{T^e I}{TI^e}\Big)-\sigma_4 T^e\mathcal{M}\Big(\frac{R^e T}{RT^e}\Big) \\
&-V^e\mathcal{M}\Big(\frac{R^e V}{RV^e}\Big)
\end{cases} \tag{33}
$$

and

$$
\begin{cases}
L_2 &= (\beta_1 I^e S^e+\beta_2 T^e S^e+\mu_1 P_2 I^e)\mathcal{M}\Big(\frac{E^e}{E}\Big)+(V^e+\sigma_4 T^e+p_1\lambda I^e)\mathcal{M}\Big(\frac{R^e}{R}\Big) \\
&+(1-p_1)\lambda I^e\mathcal{M}\Big(\frac{T^e}{T}\Big)+\sigma_1 S^e\mathcal{M}\Big(\frac{V^e}{V}\Big)+\alpha E^e\mathcal{M}\Big(\frac{I^e}{I}\Big)-(\mu_2 E^e+\alpha E^e) \\
&\mathcal{M}\Big(\frac{E^e}{E}\Big)-(\sigma_3 R^e+\mu_2 R^e)\mathcal{M}\Big(\frac{R^e}{R}\Big)-(\sigma_4 T^e+\mu_2 T^e)\mathcal{M}\Big(\frac{T^e}{T}\Big) \\
&-(\eta V^e+V^e+\mu_2 V^e)\mathcal{M}\Big(\frac{V^e}{V}\Big)-(p_1\lambda I^e+(1-p_1)\lambda I^e+\mu_2 I^e)\mathcal{M}\Big(\frac{I^e}{I}\Big)
\end{cases} \tag{34}
$$

$$
\begin{cases}
+\mu_1 P_2 I^e\mathcal{M}\Big(\frac{S^e I}{SI^e}\Big)+\beta_1 I^e S^e\mathcal{M}\Big(\frac{I}{I^e}\Big)+\beta_2 T^e S^e\mathcal{M}\Big(\frac{T}{T^e}\Big).
\end{cases} \tag{35}
$$

Using endemic equilibrium relations

$$L_2 = \mu_1 P_2 I^e \mathcal{M}\left(\frac{S^e I}{S I^e}\right) + \beta_1 I^e S^e \mathcal{M}\left(\frac{I}{I^e}\right) + \beta_2 T^e S^e \mathcal{M}\left(\frac{T}{T^e}\right) \tag{36}$$

Observation; $|L_1|$ is greater than $|L_2|$.

Hence, $\frac{dG}{dt} = L_1 + L_2 \leq 0$.

Noting that $L_1 + L_2 = 0$ if $(S, E, I, T, V, R) = (S^e, E^e, I^e, T^e, V^e.R^e)$. By La Salle's invariance principle the endemic equilibrium, Q^e is globally asymptotically stable [21]. □

5. Sensitivity analysis

In this section, we conducted an analysis of the contributions of each parameter to the basic reproductive number [22, 23]. Sensitivity analysis helps determine which parameter contributes positively or negatively to the basic reproduction number. Sensitivity index of $\mathcal{R}_0$ with respect to some parameter, say ρ is given by: $\Lambda_\rho^{\mathcal{R}_0} = \frac{\partial \mathcal{R}_0}{\partial \rho} \frac{\rho}{\mathcal{R}_0}$

Table 1: parameter and sensitivity indices.

Parameter	Sensitivity Index
β_1	0.6058
β_2	0.3942
μ_1	1.0001
μ_2	-3.7908
σ_1	-0.9599
σ_3	0.1511
σ_4	-0.0832
α	0.1415
η	0.0755
λ	-0.5931

From Table 1, the parameters β_1,β_2,σ_3, α and η have positive sensitivity indices. They have a high impact on the transmission dynamics and prevalence of Hepatitis B.

However,μ_2,σ_1,σ_4 and λ have negative sensitivity indices. Hence, they have high influence on controlling and combating Hepatitis B infections.

6. Bifurcation analysis

In this section, we carried out the phenomenon of backward bifurcation by aplying the centre manifold theory on the sysyem of equations in (2) as outlined in [24]. The phenomenon of backward bifurcation exists only in situations where

disease free equilibrium (DFE) and disease endemic equilibrium (EE) coexists [25, 26]. The epidemiological implications is that the idea that whenever $R_0 < 1$, the disease can be controled is no longer a sufficient condition. Our model exhibited the phenomenon of backward bifurcation and the bifurcation diagram in 2 supports our theoretical analysis.

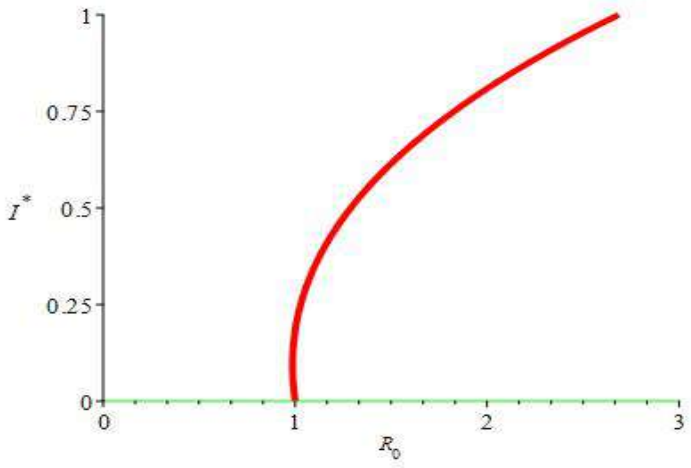


Figure 2: Backward Bifurcation

7. Numerical results

This section discusses the effect of limited medical resources on the spread of the disease. Controlling and preventing the spread of hepatitis B using treatment and vaccination require medical resources. The smaller the resources, the smaller the rates of vaccination and treatment. So in the simulation, we vary the rates of treatment and vaccination to determine the dynamics of the disease over time. The essence of controlling hepatitis B is to decrease the infectious population and increase recovery population. We examine the effects of different rates of vaccination and treatment on the performance criteria.

8. Population dynamics of susceptible, infectious and recovered

Figure 3 shows the population dynamics of susceptible, infectious and the recovered population with time. As the susceptible population decreases with, there is an increase in the number of people infected with hepatitis B. This implies that as the susceptible interact with the infected, the number of people getting infected with hepatitis B increases.

Moreover, as the number of infectious increases, there is a corresponding increase in the number of recovery. This shows that contact rate and recovery rate are directly proportional as shown in Figure 3.

Table 2: Numerical Values

Parameter	Value	Reference
β_1	0.0400	[27]
β_2	0.002	[28]
μ_1	0.0196	[28]
μ_2	0.0096	[28]
p_1	0.6500	Assumed
p_2	0.0025	Assumed
σ_1	0.2500	[27]
σ_3	0.0020	Assumed
σ_4	0.0025	[28]
α	0.0550	Assumed
η	0.0010	Assumed
λ	0.4500	Assumed

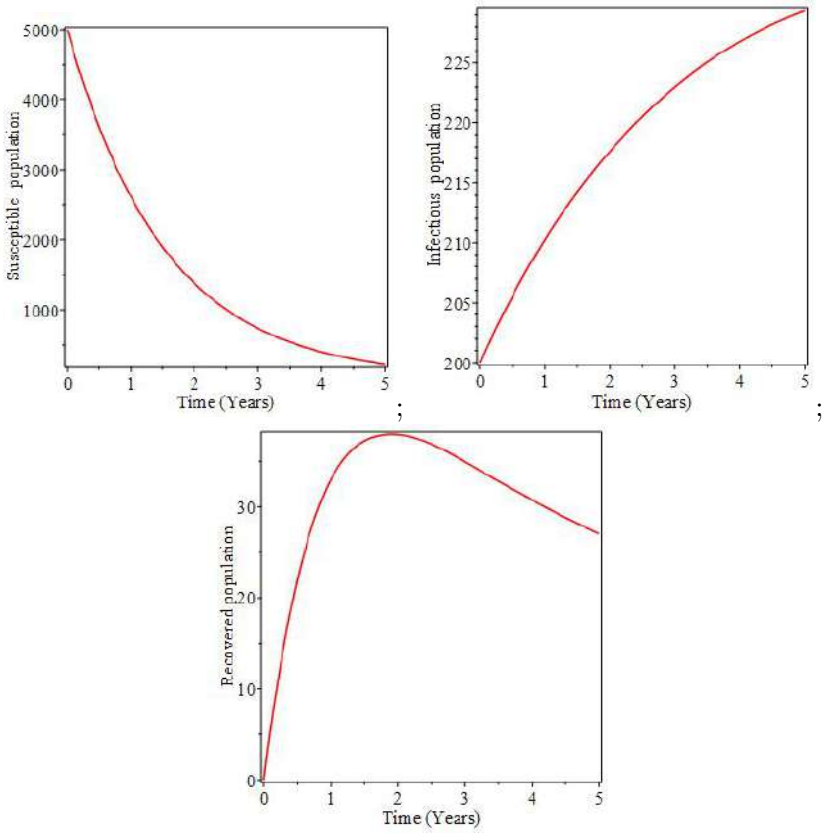


Figure 3: The population dynamics of susceptible, infectious and the recovered

8.1 Effects of contact rate on the infectious population

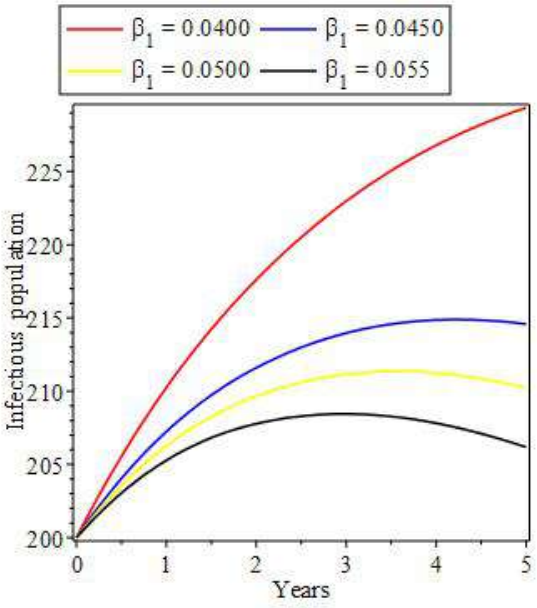


Figure 4: Model flow diagram

An analysis of the effects of interaction between the susceptible and infectious populations was investigated. This was done by varying the contact rate as shown in Figure 4.

As the value of the contact rate reduces, the population infected with hepatitis B decreases and vice versa. Hence, the infection can be controlled by ensuring that the contact rate reduces to the barest minimum as shown in Figure 4.

8.2 Effects of treatment rate on the recovered population

An analysis of the effects of treatment on the recovered populations was investigated. This was done by varying the contact rate as shown in Figure 5.

As the value of the treatment rate reduces, the population recovered from hepatitis B decreases and vice versa. Hence, the recovery can be achieved by ensuring that recovery rate increases to the barest minimum as shown in Figure 5.

9. Conclusion

A mathematical model of Hepatitis B with treatment and vaccination as compartments was developed and analysed as shown in Figure 1.

The local and global stability of the hepatitis B model was analysed using the Routh Hurwitz criteria. It was established that depending on the control

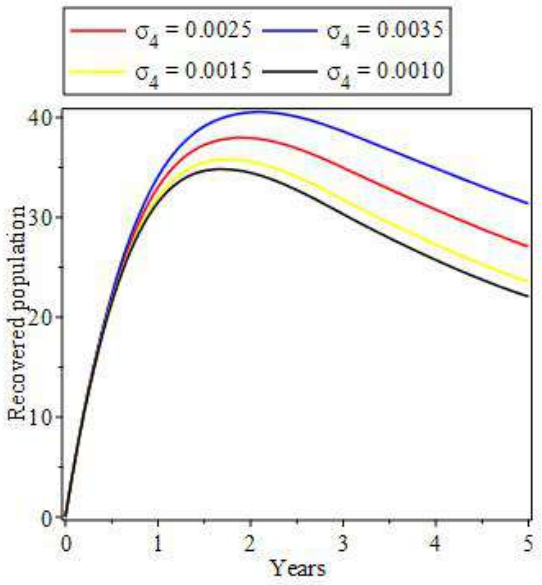


Figure 5: Model flow diagram

strategies authorities adopt, the disease can persist or get eliminated from the region.

The model was simulated to see the population dynamics of disease by plotting the susceptible, infectious and recovered populations inddividually. It was established the susceptible population decreases as the number of infected persons increases. Moreover, there is an increase in the number of persons recovered from hepatitis B as the number of infectious increases.

Additionally, the effects of treatment and contact rate was analysed numerically. As the value of the contact rate reduces, the population infected with hepatitis B decreases and vice versa. Hence, the infection can be controlled by ensuring that the contact rate reduces to the barest minimum as shown in Figure 4. As the value of the treatment rate reduces, the population recovered from hepatitis B decreases and vice versa. Hence, the recovery can be achieved by ensuring that recovery rate increases to the barest minimum as shown in Figure 5.

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sgp-resolvable and sgp-irresolvable spaces

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Abstract. The fundamental intent of this paper is to developing the idea of SSGP-continuous and SSGP-open functions using sgp-open sets. In addition sgp-resolvable and sgp-irresolvable spaces are introduced. Its various characterisations and properties are established.

Keywords: sgp-open set, SSGP-continuous, SSGP-irresolute , SSGP-open functoin, sgp-Resolvable space and sgp-Irresolvable space.

1. Introduction

The remarkable development in the field of general topology is the introduction of generalized closed sets by Levine [4] which plays a significant role. Many researchers studied extensively this notion. The investigation of generalized closed sets had paved the way for many interesting concepts. In [5] the idea of semigeneralized pre closed (briefly, sgp-closed) set was developed by G.B. Navalagi et al. In [6,7,8,9,10] authors continued the research using the set defined. Gentry and Hoyle [2] developed the concepts of "somewhat continuous and somewhat open functions".

In this paper, we will continue research related functions with sgp-closed and sgp-open sets. We introduce and characterize the concept of "somewhat

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sgp-continuous, Somewhat sgp-irresolute and Somewhat sgp-open functions". In addition to this sgp-resolvable and sgp-irresolvable spaces are introduced. Using these spaces characterization of defined open and continuous functions is done.

2. Preliminaries

In entire paper (X τ), (Y, σ) (or shortly X_τ, Y_σ) symbolize topological space(in short TS)s on which no separation axioms are assumed unless explicitly stated.

Definition 2.1. *A subset R of a TS X_τ is called:*

(1) *semi-open set [3] if $R \subset Cl(Int(R))$.*

(2) *semi-closed set [1] if $Int(Cl(Int(R))) \subset R$.*

Definition 2.2 ([5])**.** *A subset W of X_τ is semi generalized pre-closed (briefly, sgp-closed)set if $pCl(W) \subset \Delta$ whenever $W \subset \Delta$ and Δ is semi-open in X_τ. The complement of sgp-closed set(SGPCS in short) is semi generalized-pre open (briefly,sgp-open). The collection of all sgp-closed set (SGPCS in short) of X_τ is labelled as $SGPC(X_\tau)$ and sgp-open set(SGPOS in short)s by $SGPO(X_\tau)$.*

Definition 2.3. *A function $\mu\colon X_\tau \to Y_\sigma$ is named as:*

(i) semi generalized pre-irresolute (shortly, sgp-irresolute) [5] *if $\mu^{-1}(J)$ is SGPCS in X_τ for every SGPCS J of Y_σ;*

(ii) semi generalized pre-continuous (shortly, sgp-continuous) [5] *if $\mu^{-1}(C_1)$ is SGPCS in X_τ for each closed set C_1 of Y_σ;*

(iii) somewhat continuous [2] if for $S_1 \in \sigma$ along with $\mu^{-1}(S_1) \neq \phi$ there prevails an open set K in X_τ thereby $K \neq \phi$ together with $K \subseteq \mu^{-1}(S_1)$.

Definition 2.4 ([5])**.** *A function $\eta\colon X_\tau \to Y_\sigma$ is termed as sgp-open (resp., sgp-closed) if $\eta(N_0)$ is sgp-open (resp., sgp-closed) in Y_σ for each open set (resp., closed) N_0 in X_τ.*

3. Somewhat sgp-continuous functions

Definition 3.1. A function $\mu : X_\tau \to Y_\sigma$ is named as Somewhat sgp-continuous (in short, SSGP-continuous) if each $K \in Y_\sigma$ and $\mu^{-1}(K) \neq \phi$ there arises a SGPOS W in X_τ such that and $W \neq \phi$ and $V \subseteq \mu^{-1}(K)$.

Theorem 3.1. *Each somewhat continuous function is SSGP-continuous .*

Proof. Consider $\eta : X_\tau \to Y_\sigma$ is SSGP-continuous function. Take K as an open set in Y_σ thereby $\eta^{-1}(K) \neq \phi$. As η is somewhat continuous function, there exists an open set R in X_τ such that $R \neq \phi$ and $R \subseteq \eta^{-1}(K)$. As each open set is SGPOS, there prevails SGPOS, R so that $R \neq \phi$ and $R \subseteq \eta^{-1}(K)$, which gives η is SSGP-continuous function. □

Remark 3.2. The counter part of the above theorem is not possible.

Example 3.3. For $X_\tau = Y_\sigma = \{v_1, v_2, v_3\}$, $\tau = \{X, \phi, \{v_1\}\}$, $\sigma = \{Y, \phi, \{v_1\}, \{v_1, v_2\}\}$. We have SGPO(X) $= \{X, \phi, \{v_1\}, \{v_1, v_2\}, \{v_1, v_3\}\}$. Then, the identity map is SSGP-continuous yet it isn't somewhat continuous .

Theorem 3.4. *If $\lambda : X_\tau \to Y_\sigma$ and $\mu : Y_\sigma \to Z_\eta$ be any two functions. Whenever λ is SSGP-continuous and μ is continuous function, thereupon $\mu \circ \lambda$ is SSGP-continuous function.*

Proof. Take Δ be any open set in Z_η. Assume that $\mu^{-1}(\Delta) \neq \phi$. As $\Delta \in Z_\eta$ and μ is continuous function, $\mu^{-1}(\Delta) \in Z_\eta$. Guess that $\lambda^{-1}(\mu^{-1}(\Delta)) \neq \phi$. As par hypothesis λ is SSGP-continuous, there exists a SGPOS, K in X_τ such that $K \neq \phi$ and $K \subseteq \lambda^{-1}(\mu^{-1}(\Delta)) = (\mu \circ \lambda)^{-1}(\Delta)$, which implies that $K \subseteq (\mu \circ \lambda)^{-1}(\Delta)$. Therefore, $\mu \circ \lambda$ is SSGP-continuous function. □

Definition 3.2. *A subset W of a TS X is named as sgp-dense in X_τ whenever there is no proper SGPCS R in X_τ so as $W \subset R \subset X$.*

Theorem 3.5. *The consecutive are identical for $d : X_\tau \to Y_\sigma$:*

(*i*) *d is SSGP-continuous function;*

(*ii*) *Whenever $\mathcal{F}$ is a closed subset of Y_σ thereby $d^{-1}(\mathcal{F}) \neq X_\tau$, subsequently there is proper sgp-closed member D of X_τ so as $d^{-1}(\mathcal{F}) \subset D$;*

(*iii*) *Whenever $\mathcal{P}$ is a sgp-dense member of X_τ, then $d(\mathcal{P})$ is a dense subset of Y_σ.*

Proof. (i)⇒(ii): Consider $\mathcal{F}$ as a closed subset of Y_σ so, that $d^{-1}(\mathcal{F}) \neq X_\tau$. Thereupon $d^{-1}(Y_\sigma - \mathcal{F}) = X_\tau - d^{-1}(\mathcal{F}) \neq \phi$. From (i) there prevails a SGPOS V in X_τ thereby $V \neq \phi$ and $V \subset d^{-1}(Y - \mathcal{F}) = X_\tau - d^{-1}(\mathcal{F})$. This intend $d^{-1}(\mathcal{F}) \subset X_\tau - V$ and $X_\tau - V = D$ is a SGPCS in X_τ. Therefore (ii) holds.

(ii)⇒ (i): Take U as an open set in Y_σ and $d^{-1}(U) \neq \phi$. Accordingly, $d^{-1}(Y - U) = X - d^{-1}(U) = \phi$. By hypothesis, there arises a proper SGPCS, D such that $d^{-1}(Y - U) \subset D$. This implies that $X - D \subset d^{-1}(U)$ and $X_\tau - D$ is sgp-open and $X_\tau - D \neq \phi$.

(ii)⇒(iii): For a sgp-dense set $\mathcal{P}$ in X_τ. Suppose $d(\mathcal{P})$ is not dense subset of Y_σ, then there exists a proper SGPCS, D such that $\mathcal{P} \subset d^{-1}(\mathcal{F}) \subset D \subset X_\tau$. This disprove the fact that $\mathcal{P}$ is a sgp-dense set in X_τ. Therefore, (iii) holds.

(iii)⇒ (ii): Guess (iii) is doesn't holds. Then there arises a closed set $\mathcal{F}$ in Y_σ so long as $d^{-1}(\mathcal{F}) \neq X$. Though there is no proper SGPCS that $d^{-1}(\mathcal{F}) \subset D$. This means that $d^{-1}(\mathcal{F})$ is sgp-dense in X_τ. But from hypothesis $d(d^{-1}(\mathcal{F})) = \mathcal{F}$ must be dense in Y_σ, which negate the selection of $\mathcal{F}$. Hence (ii) hold. □

Theorem 3.6. *If $\mu : X_\tau \to Y_\sigma$ be a function and $X_\tau = N_1 \cup N_2$, N_1 and N_2 are open subsets of X_τ so as (μ/N_1) and (μ/N_2) are SSGP-continuous functions then μ is SSGP-continuous.*

Proof. Consider an open set $\mathcal{K}$ in Y_σ so as $\mu^{-1}(\mathcal{K}) \neq \phi$. Thereupon $(\mu/N_1)^{-1}(U) \neq \phi$ of $(\mu/N_2)^{-1}(\mathcal{K}) \neq \phi$ or both $(\mu/N_1)^{-1}(\mathcal{K}) \neq \phi$ and $(\mu/N_2)^{-1}(\mathcal{K}) \neq \phi$.
Case (i). Assume $(\mu/N_1)^{-1}(\mathcal{K}) \neq \phi$. Since (μ/N_1) is SSGP-continuous, then there exists a SGPOS C_1 in N_1 so that $C_1 \neq \phi$ and $C_1 \subset (\mu/N_1)^{-1}(\mathcal{K}) \subseteq \mu^{-1}(\mathcal{K})$. Since C_1 is SGPOS in $N_1$and N_1 is open in X_τ, C_1 is SGPOS in X_τ. Hence μ is SSGP-continuous function.
Case(ii). Presume that $(\mu/N_2)^{-1}(\mathcal{K}) \neq \phi$. As (μ/N_2) is SSGP-continuous function, then there exists a SGPOS C_1 in N_2 thereby $C_1 \neq \phi$ and $C_1 \subset (\mu/N_2)^{-1}(\mathcal{K}) \subset \mu^{-1}(\mathcal{K})$. As C_1 is SGPOS in N_2 as well as N_2 is open in X_τ, C_1 is SGPOS in X_τ. Hence μ is SSGP-continuous function.
Case(iii). Assume that $(\mu/N_1)^{-1}(\mathcal{K}) \neq \phi$ and $(\mu/N_2)^{-1}(\mathcal{K}) \neq \phi$. Follows from case(i) as well as case(ii). □

Definition 3.3. *A TS X_τ is termed as sgp-separable if there exists a countable subset R of X_τ whichever is sgp-dense in X_τ.*

Theorem 3.7. *Whenever η is SSGP-continuous from X_τ onto Y_σ along with X_τ as sgp-separable, thereupon Y_σ also separable.*

Proof. Consider η be SSGP-continuous function in such a way that X_τ is sgp-separable. Due to definition there prevails a countable member Δ of X_τ which is sgp-dense in X_τ. With reference to Theorem 3.5, $\eta(\Delta)$ is dense in Y_σ. As Δ is countable $\eta(\Delta)$ is also countable which is dense in Y_σ, so Y_σ is also separable. □

4. Somewhat sgp-irresolute function

Definition 4.1. A function $\mu : X_\tau \to Y_\sigma$ is termed as Somewhat sgp-irresolute (in short SSGP-irresolute) if for $J_0 \in SGPO(\sigma)$ and $\mu^{-1}(J_0) \neq \phi$, there exists a non-empty SGPOS set R in X_τ so that $R \subset \mu^{-1}(J_0)$.

Theorem 4.1. *If λ is SSGP-irresolute along with μ is sgp-irresolute function, then $(\mu \circ \lambda)$ is SSGP-irresolute.*

Proof. Consider $M \in SGPO(\sigma)$. Presume that $\mu^{-1}(M) \neq \phi$. As $M \in SGPO(\sigma)$ along with μ is SSGP-irresolute function, there exists a SGPOS W in X_τ so $W \neq \phi$ and $W \subseteq \lambda^{-1}(\mu^{-1}(M))$. However, $\lambda^{-1}(\mu^{-1}(M)) = (\mu \circ \lambda)^{-1}(M)$ $\Rightarrow W \subseteq (\mu \circ \lambda)^{-1}(M)$. Thus, $(\mu \circ \lambda)$ is SSGP-irresolute function. □

Theorem 4.2. *For $\alpha : X_\tau \to Y_\sigma$ subsequents are equal:*

(*i*) *α is SSGP-irresolute function.*

(*ii*) *Whenever K is a closed subset of Y_σ so as $\alpha^{-1}(K) \neq X$, then there is proper sgp-closed member D_1 of X thereby $\alpha^{-1}(K) \subset D_1$.*

(*iii*) *Whenever K is a sgp-dense member of X_τ, thereupon $\alpha(K)$ is a dense subset of Y_σ.*

Proof. Obvious. □

Theorem 4.3. *If $f : X_\tau \to Y_\sigma$ be a function and $X_\tau = K_1 \cup K_2$, K_1 and K_2 are open members of X_τ such that (f/K_1) and (f/K_2) are SSGP-irresolute function then f is also SSGP-irresolute .*

Proof. Obvious. □

Definition 4.2. *If X_τ is a set and τ as well as σ are topologies for X_τ, so τ is named as sgp-equivalent to σ provided whenever $J \in \tau$ along with $J \neq \phi$, then there is a SGPOS R in Y_σ thereby $R \neq \phi$ and $R \subset J$ and if $J \in \sigma$ and $J \neq \phi$ accordingly there is SGPOS R in X_τ so as $R \neq \phi$ and $R \subset J$.*

Theorem 4.4. *Let $\lambda : X_\tau \to Y_\sigma$ be a somewhat continuous function and τ^* be a topology for X_τ, which is sgp-equivalent to τ then $\lambda : X_\tau^* \to Y_\sigma$ is SSGP-continuous function.*

Proof. Consider an open set D_1 in Y_σ so as $\lambda^{-1}(D_1) \neq \phi$. In view of hypothesis λ is somewhat continuous function then there arises an open set in Δ in X_τ thereby $\Delta \neq \phi$ and $\Delta \subseteq \lambda^{-1}(D_1)$. As Δ is an open set in X_τ so that $\Delta \neq \phi$ and due to postulate τ is sgp-equivalent to τ^* so, there exists a SGPOS C_1 in X_τ^* so as $C_1 \neq \phi$ and $C_1 \subset \Delta \subset \lambda^{-1}(D_1)$. Henceforth $\Delta \subset \lambda^{-1}(D_1)$. Accordingly, for any open set D_1 in Y_σ so as $\lambda^{-1}(D_1) \neq \phi$ there prevails a SGPOS, C_1 in X_τ^* such that $C_1 \subset \lambda^{-1}(D_1)$. So $\lambda : X_\tau^* \to Y_\sigma$ is SSGP-continuous function. □

Theorem 4.5. *For SSGP-continuous function $\mu : X_\tau \to Y_\sigma$ along with σ^* be a topology for Y_σ, which is equivalent to σ. Then $\mu : X_\tau \to Y_\sigma^*$ is SSGP-continuous.*

Proof. Consider $\mathcal{G}$ be any open set in Y_σ^* thereby $\mu^{-1}(\mathcal{G}) \neq \phi \Rightarrow \mathcal{G} \neq \phi$. Since σ as well as σ^* are equivalent, then there arises an open set W in Y_σ thereby $W \neq \phi$ and $W \subset U$. Now, W is open set hereupon $W \neq \phi \Rightarrow \mu^{-1}(W) \neq \phi$. By the postulate $\mu : X_\tau \to Y_\sigma$ is SSGP-continuous. Therefore there prevails a SGPOS, V_1 in (X, τ) so that $V_1 \subseteq \mu^{-1}(W)$. Now, $W \subset \mathcal{G} \Rightarrow \mu^{-1}(W) \subset \mu^{-1}(\mathcal{G})$. This impart $V_1 \subset \mu^{-1}(W) \subset \mu^{-1}(\mathcal{G})$. So, we have $V_1 \subset \mu^{-1}(\mathcal{G}) \Rightarrow \mu : X_\tau \to Y_\sigma^*$ is SSGP-continuous. □

Theorem 4.6. *Let $\mu : X_\tau \to Y_\sigma$ be a SSGP-irresolute surjection function and let σ^* be a topology for Y_σ, which is equivalent to σ. Then, the function $\eta : X_\tau \to Y_\sigma^*$ is SSGP-irresolute function.*

Proof. Consider an open set $\mathcal{K}$ in Y_σ^* ther by $\eta^{-1}(\mathcal{K}) \neq \phi \Rightarrow \mathcal{K} \neq \phi$. As σ and σ^* are identical, thereupon there exists an open set W in Y_σ so as $W \neq \phi$ and $W \subset \mathcal{K}$. Now, W is open set such that $W \neq \phi$, which leads to $\mu^{-1}(W) \neq \phi$. Now, by postulation μ is SSGP-irresolute function. Therefore, there exists a SGPOS, S_1 in X_τ such that $S_1 \subseteq \mu^{-1}(W)$. Now, $W \subset \mathcal{K} \Rightarrow \mu^{-1}(W) \subset \eta^{-1}(\mathcal{K})$. This gives $S_1 \subset \mu^{-1}(W) \subset \eta^{-1}(\mathcal{K})$. So, we have $S_1 \subset \eta^{-1}(\mathcal{K})$, this leads to η is SSGP-irresolute function. □

5. Somewhat sgp-open functions

Definition 5.1. A function $\lambda : X_\tau \to Y_\sigma$ is termed as Somewhat sgp-open (SSGP-open, for short) provided that for each $K \in \tau$ with $K \neq \phi$, there exists a SGPOS, W in Y_σ such that and $W \neq \phi$ and $W \subseteq \lambda(Q)$.

Theorem 5.1. Each somewhat open function is SSGP-open.

Proof. Consider somewhat open function $\eta : X_\tau \to Y_\sigma$. Take $K \in \tau$ with $K \neq \phi$. As η is somewhat open function, there exists an open set J_1 in Y_σ such that $J_1 \neq \phi$ and $J_1 \subseteq \eta(K)$. However, each open is sgp-open. So, there exists a SGPOS, J_1 in Y_σ so that $J_1 \neq \phi$. Thus, η is SSGP-open function. □

Remark 5.2. The example below makes clear that reverse of the above theorem impossible.

Example 5.3. Let $X_\tau = Y_\sigma = \{k_1, k_2, k_3\}$, $\tau = \{X, \phi, \{k_1\}, \{k_2\}, \{k_1, k_2\}\}$, $\sigma = \{Y, \phi, \{k_1, k_2\}\}$. We have SGPO(Y) $= \{Y, \phi, \{k_1\}, \{k_2\}, \{k_1, k_2\}, \{k_1, k_3\}, \{k_2, k_3\}\}$. Then, the identity function is SSGP-open yet it isn't not somewhat open.

Theorem 5.4. *If $k : X_\tau \to Y_\sigma$ is an open function and $j : Y_\sigma \to Z_\eta$ is SSGP-open then $(j \circ k)$ is SSGP-open function.*

Proof. Consider $\mathcal{M} \in \tau$. Suppose $\mathcal{M} \neq \phi$. Since k is an open function, $k(\mathcal{M})$ is open and $k(\mathcal{M}) \neq \phi$. Consequently $k(\mathcal{M}) \in \sigma$ and $k(\mathcal{M}) \neq \phi$. As j is SSGP-open function and $k(\mathcal{M}) \in \sigma$ such that $k(\mathcal{M}) \neq \phi$, there exists a SGPOS $R_0 \in \eta$, $R_0 \subset j(k(\mathcal{M}))$, which indirect $(j \circ k)$ is SSGP-open function. □

Theorem 5.5. *For a bijective function $h : X_\tau \to Y_\sigma$ succeeding are equal;*
(i) h is SSGP-open function.
(ii) If $\mathcal{Q}$ is a closed subset of Y_σ such that $h(\mathcal{Q}) \neq Y$, then there exists a sgp-closed subset J of Y_σ such that $J \neq \phi$ and $h(\mathcal{Q}) \subset J$.

Proof. (i)⇒(ii): Consider closed subset $\mathcal{Q}$ of Y_σ such that $h(\mathcal{Q}) \neq Y_\sigma$. From (i), there exists a SGPOS W in X_τ such that $W \neq \phi$ such that $W \subset h(X_\tau - \mathcal{Q})$. Put $J = Y_\sigma - W$. Clearly J is a SGPCS in Y_σ and we claim that $J \neq \phi$. If $J = Y_\sigma$, then $W = \phi$ which is a negation. Since $W \subset h(X_\tau - \mathcal{Q})$, $J = Y_\sigma - W \subset Y_\sigma - [h(X_\tau - \mathcal{Q})] = h(\mathcal{Q})$.

(ii)⇒ (i): Take L_1 as a non empty open set in X_τ. Substitute $\mathcal{Q} = X_\tau - L_1$. Thereupon $\mathcal{Q}$ is a closed member of X_τ and $h(X - L_1) = h(\mathcal{Q}) = Y_\sigma - h(L_1)$ which imparts $h(\mathcal{Q}) \neq \phi$. With reference to (ii), there is a sgp-closed subset J of Y_σ so that $h(L_1) \subset J$. Put $V = X_\tau - J$, clearly V is SGPOS and $V \neq \phi$. Further $V = X_\tau - J \subset Y_\sigma - h(\mathcal{Q}) = Y_\sigma - [Y_\sigma - h(\mathcal{Q})] = h(L_1)$. □

Theorem 5.6. *If $k : X_\tau \to Y_\sigma$ be SSGP-open function and R be any open subset of X. Then $k/R : (R, \tau/R) \to (Y, \sigma)$ is also SSGP-open funtion.*

Proof. Consider $\mathcal{B} \in \tau/\mathcal{R}$ such that $\mathcal{B} \neq \phi$. Since $\mathcal{B}$ is open in $\mathcal{R}$ and $\mathcal{R}$ is open in X_τ, $\mathcal{B}$ is open in X_τ and due to the postulation k is SSGP-open function, then there exists a SGPOS, $\mathcal{L}$ in Y_σ, such that $\mathcal{L} \subset k(\mathcal{B})$. Thus, for any open set $\mathcal{B}$ in $(\mathcal{R}, \tau/\mathcal{R})$ with $\mathcal{B} \neq \phi$, there prevails a SGPOS, $\mathcal{L}$ in Y_σ thereby $\mathcal{L} \subset k(\mathcal{B})$ which imparts $k/\mathcal{R}$ is SSGP-open funtion. □

Theorem 5.7. *Let $\mu : X_\tau \to Y_\sigma$ be a function such that μ/K_1 and μ/K_2 are SSGP-open, then μ is SSGP-open function, where $X = K_1 \cup K_2$, K_1 and K_2 are open subsets of X.*

Proof. Obvious. □

Theorem 5.8. *The statements are equivalent:*

(1) *k be SSGP-open function.*

(2) *Whenever S_1 is a sgp-dense subset of Y_σ, then $k^{-1}(S_1)$ is a dense subset of X_τ.*

Proof. (i)⇒(ii): Presume S_1 is a sgp-dense set in Y_σ. If $k^{-1}(S_1)$ is not dense in X_τ, then there arises a closed set N_1 in X_τ so as $k^{-1} \subset N_1 \subset X_\tau$. As k is SSGP-open and $X_\tau - N_1$ is open, there exists a nonempty SGPOS H_1 in Y_σ so that $H_1 \subset k(X_\tau - N_1)$. Thereupon, $H_1 \subset k(k^{-1}(Y_\sigma - S_1)) \subset Y_\sigma - S_1$ implies $S_1 \subset Y_\sigma - H_1 \subset Y_\sigma$. Now, $Y_\sigma - H_1$ is a SGPCS along with $S_1 \subset Y_\sigma - H_1 \subset Y_\sigma$. This indicates that S_1 is not a sg-dense set in Y_σ, which is negation. So, $k^{-1}(S_1)$ is a dense in X_τ.

(ii)⇒(i): Suppose S_1 is a nonempty open subset of X_τ. We need to prove that $sgp(k(S_1))^0 \neq \phi$. Presume $sgp(k(S_1))^0 = \phi$. Thereupon, $sgp\overline{(k(S_1))} = Y_\sigma$. So by (ii), $k^{-1}Y_\sigma - k(S_1)$ is dense in X_τ. But $k^{-1}(Y_\sigma - k(S_1)) \subset X_\tau - S_1$. Now, $X_\tau - S_1$ is closed. Accordingly, $k^{-1}(Y_\sigma - k(S_1)) \subset X_\tau - S_1$ gives $X_\tau = \overline{k^{-1}(Y_\sigma - k(S_1))} \subset X_\tau - S_1$. This implies that $S_1 = \phi$ which contrary to $S_1 \neq \phi$. Therefore, $sgp(k(S_1))^0 \neq \phi$. Thereupon k is SSGP-open. □

6. sgp-resolvable spaces

Definition 6.1. *A TS X_τ is named as sgp-resolvable if there arises a sgp-dense set K in X_τ so that $X_\tau - K$ is also sgp-dense. Otherwise, X_τ is termed as sgp-irresolvable.*

Theorem 6.1. *The succeeding statements are similar*

(*i*) *X_τ is sgp-resolvable;*

(*ii*) *X_τ has a pair of sgp-dense sets Δ_1 as well as Δ_2 such that $\Delta_1 \subset \Delta_2$.*

Proof. (i)⇒(ii): Assume that X_τ is sgp-resolvable. There exists a sgp-dense set Δ_1 such that $X_\tau - \Delta_1$ is sgp-dense. Set $\Delta_1 = X_\tau - \Delta_2$.

(ii)⇒(i): Presume that (ii) is true. Take X_τ as sgp-irresolvable. A that time $X_\tau - \Delta_2$ is not sgp-dense and $sgp(\overline{\Delta_1}) \subset sgp(\overline{X_\tau - \Delta_2}) \neq X$. Hence, Δ_1 is not sgp-dense. This negates the presumption. □

Theorem 6.2. *For X_τ the succeeding statements are identical:*

(*i*) *X_τ is sgp-irresolvable;*

(*ii*) *For any sgp-dense set Δ in X_τ $sgp(\Delta)^o \neq \phi$.*

Proof. (i)⇒(ii): Consider Δ be any sgp-dense set of X_τ. Then, we have $sgp(\overline{X_\tau - \Delta}) \neq X_\tau$; then $sgp(\Delta)^o \neq \phi$.

(ii)⇒(i): Presume that X_τ is sgp-resolvable space. Then, there arises a sgp-dense set Δ in X_τ so that Δ^c is also sgp-dense in X_τ. It follows that $sgp(\Delta)^o = \phi$, which is a negation; henceforth X_τ is sgp-irresolvable. □

Definition 6.2. *A TS is termed as strongly sgp-irresolvable if for a nonempty set K in X_τ $sgp(K)^o = \phi$ implies $sgp(sgp\overline{K})^o = \phi$.*

Theorem 6.3. *Whenever X_τ is a strongly sgp-irresolvable space along with $sgp\overline{K} = X_\tau$ for a nonempty subset K of X, then $sgp(\overline{sgp(K)^o}) = X_\tau$.*

Theorem 6.4. *If X_τ is a strongly sgp-irresolvable space as well as $sgp\overline{K} = X_\tau$ along with $sgp(sgp\overline{K})^o \neq \phi$ for any nonempty subset K in X_τ, thence $sgp(K)^o \neq X_\tau$.*

Theorem 6.5. *Every strongly-irresolvable space is sgp-irresolvable. However converse is not true.*

Example 6.6. Consider $X_\tau = Y_\sigma = \{r_1, r_2, r_3\}$ with $\tau = \{X, \phi, \{r_1\}\}$. Then (X_τ, τ) is sgp-irresolvable space however it isn't strongly sgp-irresolvable.

Theorem 6.7. *If η is SSGP-open and $sgp(R)^o = \phi$ for a nonempty set R in Y_σ, then $(\eta^{-1}(R))^o = \phi$.*

Proof. Consider a nonempty set R in Y_σ so that $sgp(R)^o = \phi$. Thereupon $sgp(\overline{Y_\sigma - R}) = Y_\sigma$. As η is SSGP-open and $Y_\sigma - R$ is sgp-dense in Y_σ, by Theorem 5.9, $\eta^{-1}(Y_\sigma - R)$ is dense in X_τ. Accordingly $\overline{(X_\tau - \eta^{-1}(R))} = X_\tau$; so $(\eta^{-1}(R))^o = \phi$. □

Theorem 6.8. *For a function $\eta : X_\tau \to Y_\sigma$, whenever X is irresolvable, then Y_σ is sgp-irresolvable.*

Proof. Consider a nonempty set R in Y so that $sgp(\overline{R}) = Y$. We need to prove that $sgp(R)^o \neq \phi$. Suppose not, then $sgp(\overline{Y - R}) = Y$. As η is SSGP-open and $Y - R$ is sgp-dense in Y, with reference to Theorem 5.8, $\eta^{-1}(Y - R)$ is dense in X. Accordingly, $(\eta^{-1}(R))^o = \phi$. Now, since R is sgp-dense in Y, $\eta^{-1}(R)$ is dense in X. Therefore, for the dense set $\eta^{-1}(R)$, we have $(\eta^{-1}(R))^o = \phi$, which is a contradiction to theorem 6.3. So, we should have $sgp(R)^o \neq \phi$, for all sgp-dense sets R in Y. Hence, by Theorem 6.3, Y is sgp-irresolvable. □

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Continuous mappings in soft lattice topological spaces

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Abstract. The fuzzy set, soft set and their extensions have proven to be a fruitful bridge between precise classical mathematics and the imprecise real world. Soft lattices, which are a generalization of soft sets, are a novel mathematical approach to the study of uncertainty. Soft lattice topological spaces are introduced over an initial universe X with a fixed set of parameters P. In this paper, we introduce the concept of soft lattice continuous mappings(soft L-continuous mappings) in soft lattice topological spaces which are defined with a fixed set of parameters over an initial universe. Further we investigate some properties regarding the continuity of mappings in soft lattice topological spaces. Finally, open mappings, closed mappings and homeomorphism on soft L-topological spaces are defined and some interesting results are obtained.
Keywords: soft L-topology, soft L-continuity, soft L-homeomorphism.

1. Introduction

The concept of soft set theory begins with Molodtsov [17, 18] in the year 1999. It is completely new approach for modelling, vagueness and uncertainties. Few applications in many directions of soft set theory have been shown by Molodtsov in [17, 18]. Also Maji et.al [16, 20] studied soft sets introduced by Molodtsov [17, 18] and gave the definitions based on equality of two soft sets, subset and super set of a soft set, complement of a soft set, null soft set, and absolute soft set with examples and basic properties are also defined. The algebraic structure of set theory dealing with uncertainties has also been studied by some authors [1, 2, 3, 4, 5, 10, 11, 12, 13, 14, 23]. The concept of soft set has been extended to soft lattices and soft fuzzy sets by Li F [15] in the year 2010. Shabir and Naz [22]

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introduced the concept of soft topological spaces in the year 2011 and studied some basic properties. In our work, we use the notion of soft set initiated by Molodtsov [17, 18] and extend this idea to the field of soft lattices and obtain the topological properties of soft lattices. In 2016, Cigdem Gunduz Aras, Ayse Sonmez and Huseyin Cakalli [6] introduced soft continuous mappings. Some of its properties are studied. In 2012, Hazra, Majumdar and Samanta [8] gave the definition of continuity of soft mappings with their properties. Soft mappings have been studied by a variety of other authors [7, 9, 19].

Soft Lattice topological spaces (Soft L-topological spaces or Soft L-space) [21] are introduced with a fixed set of parameters P over an initial universe X. We have defined some basic properties of soft L-topological spaces and also gave the definition of soft L-open and soft L-closed sets. The soft L-closure of a soft lattice is also defined which is a generalization of closure of a set. The concept of parameters plays a major role with the set of parameterized topologies on the initial universe. We define a topological space corresponding to each parameter, and it is more essential. We show that a soft topological space gives a parameterized family of topologies on the initial universe whereas the converse need not be true. It means if we are given some topologies for each parameter, it is not possible to construct a soft topological space. Consequently, we tell that the soft topological spaces are more generalization of classical topological spaces. Similarly, soft lattice topological spaces are generalized than soft topological spaces.

In this paper, our purpose is to study soft L-continuity on soft L-topological spaces. Soft lattice continuous mappings(soft L-continuous mappings) are introduced with a fixed set of parameters P over an initial universe set X. We also discuss some algebraic properties of soft L-mappings and study their continuity properties under soft L-topology. Finally, we give the definitions of soft L-open mappings, soft L-closed mappings and soft L-homeomorphism and some interesting results are proved.

2. Preliminaries and basic definitions

Throughout this paper, we consider L as a complete lattice and we denote its universal bounds as $\perp$ and $\top$. Our assumption is L is consistent i.e. floor is different from top. Therefore, $\perp \leq \alpha \leq \top$ for every $\alpha \in L$. Also $\vee\phi = \perp$ and $\wedge\phi = \top$. The two point lattice $\{\perp, \top\}$ is denoted by 2. A unary operation $\prime\colon L \longrightarrow L$ is quasi complementation. It is an involution (i.e., $\alpha'' = \alpha$ for all $\alpha \in L$) that inverts the ordering. (i.e., $\alpha \leq \beta \Longrightarrow \beta' \leq \alpha'$). De Morgan's laws also hold in $(L, \prime)$. (i.e., $(\vee A)' = \wedge\{\alpha' : \alpha \in A\}$ and $(\wedge A)' = \vee\{\alpha' : \alpha \in A\}$ for every $A \subset L$). In addition, $\perp' = \top$ and $\top' = \perp$. Based on these concepts, we use a completely distributive lattice $(L, \prime)$ as a complete lattice equipped with an order reserving involution in this paper.

Definition 2.1 ([1])**.** Assume X as an initial universe set and P be a set of parameters. The power set of X is denoted as $\wp(X)$ and $B \subset P$. Then, a

pair (P, B) is said to be a *soft set* over X, where the mapping P is given by $P : B \to \wp(X)$ i.e., a soft set over X is regarded as a parametrized family of subsets of the universe X. For $b \in B$, the set of approximate elements of the soft set (P, B) denoted by $P(b)$.

Definition 2.2 ([1])**.** Let (P, A) and (Q, B) be soft sets over a common universe X, then (Q, B) is a soft subset of (P, A) if (i) $B \subset A$, and (ii) for all $b \in B$, $Q(b)$ and $P(b)$ are identical approximations.

We denote it by $(Q, B) \tilde{\subset} (P, A)$. If (P, A) is a soft subset of (Q, B) then (Q, B) is a soft super set of (P, A) and it is denoted by $(Q, B) \tilde{\supset} (P, A)$.

If (P, A) is a soft subset of (Q, B) and (Q, B) is a soft subset of (P, A), then these two soft sets (P, A) and (Q, B) over a common universe X are said to be soft equal.

Definition 2.3 ([1])**.** Let $P = \{g_1, g_2, g_3, ..., g_i\}$ be a set of parameters. The NOT set of P denoted by $\neg P$ is defined by $\neg P = \{\neg g_1, \neg g_2, \neg g_3, ..., \neg g_i\}$, where $\neg g_i = not g_n$ for all n.

Definition 2.4 ([1])**.** Let (Q, B) be a soft set, then $(Q, B)^c$ is complement or neg-complement of a soft set (Q, B) and is defined by $Q^c = (Q^c, \neg B)$, where $Q^c : \neg B \longrightarrow \wp(X)$ is a mapping given by $Q^c(\neg\beta) = X - Q(\beta)$, for every $\neg\beta \in \neg B$. Also $(Q^c)^c = Q$ and $((Q, B)^c)^c = (Q, B)$.

Definition 2.5 ([1])**.** Let (Q, B) be a soft set over X, then (Q, B) is said to be a NULL soft set if $\forall\ b \in B$, $Q(b) = \phi$,(null-set) and is denoted by Q_ϕ.

Definition 2.6 ([1])**.** A soft set (Q, B) over X is said to be absolute soft set denoted by $\overline{B}$, if $\forall\ b \in B$, $Q(b) = X$. Clearly $\overline{B}^c = Q_\phi$, $Q_\phi^c = \overline{B}$.

Definition 2.7 ([15])**.** Consider $M = (f, X, L)$, where L is a complete lattice, $f : X \longrightarrow \wp(L)$ is a mapping, X is a universe set, then M is called the soft lattice denoted by f_P^L, i.e., for every $x \in X$, f_P^L is a soft lattice over L, if $f(x)$ is a sub lattice of L.

Definition 2.8 ([21])**.** The relative complement of a soft lattice f_P^L is denoted by $(f_P^L)'$ and is defined as $(f_P^L)' = (f_P'^L)$ where $f' : P \longrightarrow \wp(L)$ is a mapping given by $f'(\alpha) = L - f(\alpha)$ for all $\alpha \in P$.

Definition 2.9 ([21])**.** Consider X as an initial universe set and P as the non-empty set of parameters.

Let τ be the set of complete, uniquely complemented soft lattices over L, then τ is said to be a soft lattice topology on L if:

(i) ϕ, L belongs to τ.

(ii) The arbitrary union of soft Lattices in τ belongs to τ.

(iii) The finite intersection of soft Lattices in τ belongs to τ.

Then, (L, τ, P) is called a soft lattice topological space (soft topological lattice space or soft L -space) over L.

Definition 2.10 ([21])**.** Consider (L, τ, P) as a soft lattice topological space over L, then the members of τ are called as soft L-open sets in L.

Definition 2.11 ([21])**.** Let (L, τ, P) be a soft lattice topological space over L. A soft lattice f_P^L over L is said to be a soft L-closed set in L, if its relative complement $(f_P^L)'$ belongs to τ.

Definition 2.12 ([21])**.** We consider L as a lattice, P be the set of parameters and $\tau = \{\phi, L\}$. Then τ is called the soft indiscrete lattice topology on L and (L, τ, P) is said to be a soft indiscrete lattice topological space over L.

Definition 2.13 ([21])**.** Consider L be a lattice, P be the set of parameters and let τ be the collection of all soft lattices which can be defined over L. Then τ is called the soft discrete lattice topology on L and (L, τ, P) is said to be a soft discrete lattice topological space over L.

Definition 2.14 ([21])**.** We consider (L, τ, P) as a soft lattice topological space over L and f_P^L be a soft lattice over L. Then the soft lattice closure of f_P^L, denoted by $\overline{f}_P^L$, is the intersection of all soft L-closed super sets of f_P^L .

Definition 2.15 ([21])**.** Let (L, τ, P) be a soft lattice topological space over L and f_P^L be a soft lattice over L. Then we associate with f_P^L, a soft lattice L, denoted by $\overline{f}_P^L$ and defined as $\overline{f}(\alpha) = \overline{f(\alpha)}$, where $\overline{f(\alpha)}$ is the soft L-closure of $f(\alpha)$ in τ_α for each $\alpha \in P$.

Definition 2.16 ([21])**.** Consider (L, τ, P) as a soft lattice topological space over L, g_P^L be a soft lattice over L and $x \in L$. Then x is said to be a soft L-interior point of g_P^L if there exists a soft L-open set f_P^L such that $x \in f_P^L \subset g_P^L$. It is denoted by $(f_P^L)^o$.

Definition 2.17 ([21])**.** Let (L, τ, P) be a soft lattice topological space over L, g_P^L be a soft lattice over L and $x \in L$. Then g_P^L is said to be a soft lattice neighbourhood of x if there exists a soft L-open set f_P^L such that $x \in f_P^L \subset g_P^L$.

2.1 Soft lattice mapping and soft lattice continuous mapping

Assume X as an initial universe set and P_1 and P_2 are two non-empty set of parameters.

Definition 2.18. Consider (L_1, τ_1, P_1) and (L_2, τ_2, P_2) as two soft lattice topological spaces over L_1 and L_2 respectively. The mapping f_g is called a soft L-mapping from L_1 to L_2 denoted by $f_g \colon (L_1, \tau_1, P_1) \longrightarrow (L_2, \tau_2, P_2)$, where $f \colon L_1 \longrightarrow L_2$ and $g \colon P_1 \longrightarrow P_2$ are two mappings.

Assume $F_{P_1}^L \in (L_1, \tau_1, P_1)$, then the image of $F_{P_1}^L$ under the soft L-mapping f_g is the soft lattice set over L_2 denoted by $f_g(F_{P_1}^L)$ and defined by

$$f_g(F_{P_1}^L)(p_2) = \begin{cases} \bigcup_{p_1 \in g^{-1}(p_2) \cap P_1} f(F_{P_1}^L(p_1)), & g^{-1}(p_2) \cap P_1 \neq \phi \\ \phi, & \text{otherwise} \end{cases}.$$

Let $G^L_{P_2} \in (L_2, \tau_2, P_2)$, then the pre-image of $G^L_{P_2}$ under the soft L-mapping f_g is the soft lattice set over L_1 denoted by $f_g(G^L_{P_2})$ and defined by

$$f_g^{-1}(G^L_{P_2})(p_1) = \begin{cases} f^{-1}(G^L_{P_2}(f(p_1))), & g(p_1) \in P_2 \\ \phi, & \text{otherwise} \end{cases}.$$

The soft L-mapping f_g is called injective if f and g are injective. Similarly, f_g is also surjective and bijective.

Example 2.1. Suppose $L_1 = \{l_1, l_2, l_3\}, L_2 = \{h_1, h_2, h_3\}, P_1 = \{p_1, p_2\}, P_2 = \{p_3, p_4\}$ and $\tau_1 = \{\phi, L_1, F^{L_1}_{1P_1}, F^{L_1}_{2P_1}, F^{L_1}_{3P_1}, F^{L_1}_{4P_1}\}$ is a soft L-topological space over L_1, $\tau_2 = \{\phi, L_2, G^{L_2}_{1P_2}, G^{L_2}_{2P_2}, G^{L_2}_{3P_2}, G^{L_2}_{4P_2}\}$ is a soft L-topological space over L_2, where $F^{L_1}_{1P_1}, F^{L_1}_{2P_1}, F^{L_1}_{3P_1}, F^{L_1}_{4P_1}$ are soft lattices over L_1 and $G^{L_2}_{1P_2}, G^{L_2}_{2P_2}, G^{L_2}_{3P_2}, G^{L_2}_{4P_2}$ are soft lattices over L_2, defined as follows,

$$\begin{aligned} &F_1(p_1) = \{l_2\}, F_1(p_2) = \{l_1\}, \\ &F_2(p_1) = \{l_2, l_3\}, F_2(p_2) = \{l_1, l_2\}, \\ &F_3(p_1) = \{l_1, l_2\}, F_3(p_2) = L_1, \\ &F_4(p_1) = \{l_1, l_2\}, F_4(p_2) = \{l_1, l_3\}, \end{aligned}$$

$$\begin{aligned} &G_1(p_3) = \{h_1\}, G_1(p_4) = \{h_2\}, \\ &G_2(p_3) = \{h_2, h_3\}, G_2(p_4) = \{h_1, h_2\}, \\ &G_3(p_3) = \{h_1, h_2\}, G_3(p_4) = \{h_1, h_2\}, \\ &G_4(p_3) = \{h_1\}, G_3(p_4) = \{h_1, h_3\}. \end{aligned}$$

Let f_g be a mapping, where $f\colon L_1 \longrightarrow L_2$ and $g\colon P_1 \longrightarrow P_2$.

Assume $f^L_{P_1} \in (L_1, \tau_1, P_1)$, then

$$f_g(F^L_{P_1})(p_2) = \begin{cases} \bigcup_{p_1 \in g^{-1}(p_2) \cap P_1} f(F^L_{P_1}(p_1)), & g^{-1}(p_2) \cap P_1 \neq \phi \\ \phi, & \text{otherwise} \end{cases}.$$

Then, f_g is a soft L-mapping over L_2.

But $f\colon L_1 \longrightarrow L_1$ is not soft L-mapping.

Let $G^L_{P_2} \in (L_2, \tau_2, P_2)$, then

$$f_g^{-1}(G^L_{P_2})(p_1) = \begin{cases} f^{-1}(G^L_{P_2}(f(p_1))), & g(p_1) \in P_2 \\ \phi, & \text{otherwise} \end{cases}.$$

Then, f_g is a soft L-mapping over L_1.

We consider that X as an initial universe set and P represents the non-empty set of parameters.

Definition 2.19. Consider f_P^L as a soft lattice over L. The soft lattice f_P^L is called a soft L-point, denoted by (l_p, P), for the element $p \in P$, $f(p) = \{l\}$ and $f(p') = \phi$ for all $p' \in P - \{l\}$.

Definition 2.20. Let (L_1, τ_1, P) and (L_2, τ_2, P) be two soft lattice topological spaces. The mapping f_g is called a soft L-mapping from L_1 to L_2 denoted by $f_g \colon (L_1, \tau_1, P) \longrightarrow (L_2, \tau_2, P)$, where $f \colon L_1 \longrightarrow L_2$ and $g \colon P \longrightarrow P$ are two mappings. For each soft L-neighbourhood g_P^L of $(f(l)_p, P)$, if there exist a soft L-neighbourhood f_P^L of (l_p, P) such that $f_g(f_P^L \subset g_P^L)$, then f_g is said to be soft L-continuous mapping at (l_p, P).

If f_g is soft L-continuous mapping for all (l_p, P), then f_g is called soft L-continuous mapping.

Definition 2.21. Let (L_1, τ_1, P) and (L_2, τ_2, P) be two soft lattice topological spaces, $f_g \colon (L_1, \tau_1, P) \longrightarrow (L_2, \tau_2, P)$ be a mapping. Then:

(a) If the image $f_g(F_P^L)$ of each soft L-open set F_P^L over L_1 is a soft L-open set in L_2, then f_g is said to be a soft L-open mapping.

(b) If the image $f_g(H_P^L)$ of each soft L-closed set H_P^L over L_1 is a soft L-closed set in L_2, then f_g is said to be a soft L-closed mapping.

Theorem 2.1. We know that (L_1, τ_1, P) and (L_2, τ_2, P) are two soft lattice topological spaces, $f_g \colon (L_1, \tau_1, P) \longrightarrow (L_2, \tau_2, P)$ be a mapping. Then the following conditions are equivalent:

(1) $f_g \colon (L_1, \tau_1, P) \longrightarrow (L_2, \tau_2, P)$ is a softL-continuous mapping.

(2) For each soft L-open set G_P^L over L_2, $f_g^{-1}(G_P^L)$ is a soft L-open set over L_1.

(3) For each soft L-closed set H_P^L over L_2, $f_g^{-1}(H_P^L)$ is a soft L-closed set over L_1.

(4) For each soft L-set F_P^L over L_1, $f_g(\overline{F_P^L}) \subset \overline{f_g(F_P^L)}$.

(5) For each soft L-set G_P^L over L_2, $\overline{f_g^{-1}(G_P^L)} \subset f_g(\overline{G_P^L})$.

(6) For each soft L-set G_P^L over L_2, $f_g^{-1}((G_P^L)^o) \subset (f_g^{-1}(G_P^L))^o$.

Proof. (1) $\Rightarrow$ (2):
Let G_P^L be a soft lattice open set over L_2 and $(l_p, P) \in f_g^{-1}(G_P^L)$ be an arbitrary soft L-point. Then $f_g(l_p, P) = (f_g(l)_p, P) \in F_P^L$.

Since f_g is soft L-continuous mapping, there exists $(l_p, P) \in F_P^L \in \tau$ such that $f_g(F_P^L) \subset G_P^L$. This implies $(l_p, P) \in F_P^L \subset f_g^{-1}(G_P^L)$, $f_g^{-1}(G_P^L)$ is a soft L-open set over L_1.

(2) $\Rightarrow$ (1):
Let (l_p, P) be a soft L-point and $(f(l)_p, P) \in G_P^L$ be an arbitrary soft L-neighbourhood. Then $(f(l)_p, P) \in f_g^{-1}(G_P^L)$ is a soft L-neighbourhood and $f_g(f_g^{-1}(G_P^L)) \subset G_P^L$.

(2) $\Rightarrow$ (3):
If for each soft L-open set G_P^L over L_2, $f_g^{-1}(G_P^L)$ is a soft L-open set L_1, then for each soft L-closed set H_P^L over L_2, $f_g^{-1}(H_P^L)$ is a soft L-closed set over L_1.

(3) $\Rightarrow$ (4):
Let F_P^L be a soft L-open set over L_1.

Since $F_P^L \subset f_g^{-1}(f_g(F_P^L))$ and $f_g(F_P^L) \subset \overline{(f_g(F_P^L))}$, we have $F_P^L \subset f_g^{-1}(f_g(F_P^L)) \subset f_g^{-1}\overline{(f_g(F_P^L))}$.

By (3), since $f_g^{-1}\overline{(f_g(F_P^L))}$ is a soft L-closed set over L_1, $\overline{(F_P^L)} \subset f_g^{-1}\overline{(f_g(F_P^L))}$. Thus, $f_g(\overline{(F_P^L)}) \subset f_g(\overline{f_g^{-1}(F_P^L)}) \subset \overline{f_g(F_P^L)}$ is obtained.

(4) $\Rightarrow$ (5):
Let G_P^L be a soft lattice open set over L_2 and $f_g^{-1}(G_P^L) = F_P^L$. By (4), $f_g(\overline{(F_P^L)}) = f_g(\overline{f_g^{-1}(G_P^L)}) \subset \overline{f_g(f_g^{-1}(G_P^L))} \subset \overline{(G_P^L)}$. Then, $\overline{f_g^{-1}(G_P^L)} = \overline{(F_P^L)} \subset f_g^{-1}\overline{(f_g(F_P^L))} \subset f_g^{-1}\overline{(G_P^L)}$.

(5) $\Rightarrow$ (6):
Let G_P^L be a soft lattice open set over L_2.

Substitute $(F_P^L)'$ in (5), then $\overline{f_g^{-1}((G_P^L)')} \subset f_g^{-1}\overline{((G_P^L)')}$.

Since $(G_P^L)^o = (\overline{(G_P^L)'})'$, then we have

$$\begin{aligned} f_g^{-1}((G_P^L)^o) &= f_g^{-1}((\overline{(\overline{(G_P^L)'})'}) \\ &= (f_g^{-1}((\overline{(\overline{(G_P^L)'})'}) \subset \overline{(f_g^{-1}(((G_P^L)')')'}) \\ &= (\overline{(f_g^{-1}(((G_P^L)'))}' = (f_g^{-1}(G_P^L))^o. \end{aligned}$$

(6) $\Rightarrow$ (2):
Let G_P^L be a soft lattice open set over L_2. Then, since $(f_g^{-1}((G_P^L))^o \subset f_g^{-1}(G_P^L) = f_g^{-1}(G_P^L)^o) \subset (f_g^{-1}((G_P^L))^o$, $(f_g^{-1}((G_P^L))^o = f_g^{-1}(G_P^L)$ is obtained. This implies $f_g^{-1}(G_P^L)$ is a soft lattice open set over L_1. □

Example 2.2. Suppose $L = \{l_1, l_2, l_3\}, P = \{p_1, p_2\}$ and $\tau_1 = \{\phi, L, F_{1P}^L, F_{2P}^L\}$, $\tau_2 = \{\phi, L, G_{1P}^L, G_{2P}^L\}$ be two soft L-topologies over L, where $F_{1P}^L, F_{2P}^L, G_{1P}^L, G_{2P}^L$ are soft lattices over L, defined as follows,

$$\begin{aligned} F_1(p_1) &= \{l_1, l_2\}, F_1(p_2) = L, \\ F_2(p_1) &= \{l_3\}, F_2(p_2) = \{l_3\}, \end{aligned}$$

and

$$\begin{aligned} G_1(p_1) &= \{l_1\}, G_1(p_2) = \{l_3\}, \\ G_2(p_1) &= \{l_1, l_3\}, G_2(p_2) = \{l_2, l_3\}. \end{aligned}$$

If $f_g : L \longrightarrow L$ defined as $f_g(h_1) = f_g(h_2) = h_1, f_g(h_3) = h_3$, then since $f_g^{-1}(G_{1P}^L) = F_{1P}^L$ and $f_g^{-1}(G_{2P}^L) = F_{2P}^L$, f_g is a soft L-continuous mapping.

Example 2.3. Suppose $L = \{l_1, l_2, l_3\}, P = \{p_1, p_2\}$ and $\tau_1 = \{\phi, L, F_{1P}^L, F_{2P}^L, F_{3P}^L, F_{4P}^L\}$, $\tau_2 = \{\phi, L, G_{1P}^L, G_{2P}^L, G_{3P}^L, G_{4P}^L\}$ be two soft L-topologies over

L, where $F^L_{1P}, F^L_{2P}, F^L_{3P}, F^L_{4P}, G^L_{1P}, G^L_{2P}, G^L_{3P}, G^L_{4P}$ are soft lattices over L, defined as follows,

$$F_1(p_1) = \{l_2\}, F_1(p_2) = \{l_1\},$$
$$F_2(p_1) = \{l_2, l_3\}, F_2(p_2) = \{l_1, l_2\},$$
$$F_3(p_1) = \{l_3\}, F_3(p_2) = \{l_1, l_2\},$$
$$F_4(p_1) = \{\phi\}, F_4(p_2) = L$$

and

$$G_1(p_1) = \{l_2\}, G_1(p_2) = \{l_1\},$$
$$G_2(p_1) = \{l_2, l_3\}, G_2(p_2) = \{l_1, l_2\},$$
$$G_3(p_1) = \{l_1, l_2\}, G_3(p_2) = L,$$
$$G_4(p_1) = \{l_2\}, G_4(p_2) = \{l_1, l_2\}.$$

Thus, (L, τ_1, P) and (L, τ_2, P) are two soft L-topological spaces and $f_g : 1_L : L \longrightarrow L$ is not soft L-continuous mapping.

Theorem 2.2. If $f_g \colon (L_1, \tau_1, P) \longrightarrow (L_2, \tau_2, P)$ is a soft L-continuous mapping, then for each $\alpha \in P$, $f_{g\alpha} \colon (L_1, \tau_{1\alpha}) \longrightarrow (L_2, \tau_{2\alpha})$ is a soft continuous mapping.

Proof. Let $V \in \tau_{2\alpha}$. Then there exist a soft L-open set G^L_P over L_2 such that $V = G(\alpha)$.

Since $f_g \colon (L_1, \tau_1, P) \longrightarrow (L_2, \tau_2, P)$ is a soft L-continuous mapping, $f_g^{-1}(G^L_P)$ is a soft L-open set over L_1 and $f_g^{-1}(G^L_P)(\alpha) = f_g^{-1}(G(\alpha)) = f^{-1}(V)$ is a soft open set.

This implies $f_{g\alpha}$ is a soft continuous mapping.

Now, this is an example given to show that the converse of above theorem does not hold. □

Example 2.4. Suppose $L_1 = \{l_1, l_2, l_3\}, L_2 = \{h_1, h_2, h_3\}, P = \{p_1, p_2\}$ and $\tau_1 = \{\phi, L_1, F^L_{1P}, F^L_{2P}, F^L_{3P}, F^L_{4P}, F^L_{5P}\}$ is a soft L-topological space over L_1, $\tau_2 = \{\phi, L_2, G^L_{1P}, G^L_{2P}, G^L_{3P}\}$ is a soft L-topological space over L_2, where $F^L_{1P}, F^L_{2P}, F^L_{3P}, F^L_{4P}, F^L_{5P}$ are soft lattices over $L_1, G^L_{1P}, G^L_{2P}, G^L_{3P}$ are soft lattices over L_2, defined as follows,

$$F_1(p_1) = \{l_1\}, F_1(p_2) = \{l_1, l_3\},$$
$$F_2(p_1) = \{l_2\}, F_2(p_2) = \{l_1\},$$
$$F_3(p_1) = \{l_1, l_2\}, F_3(p_2) = \{l_1, l_3\},$$
$$F_4(p_1) = \{\phi\}, F_4(p_2) = \{l_1\},$$
$$F_5(p_1) = \{l_1, l_2\}, F_5(p_2) = L_1$$

and

$$G_1(p_1) = \{L_2\}, G_1(p_2) = \{h_2\},$$
$$G_2(p_1) = \{h_1\}, G_2(p_2) = \{h_2\},$$
$$G_3(p_1) = \{h_1, h_2\}, G_3(p_2) = \{h_2\}.$$

If we define a mapping $f_g : L_1 \longrightarrow L_2$ as $f_g(l_1) = h_2, f_g(l_2) = h_1, f_g(l_3) = h_3$, then f_g is not a soft L-continuous mappings, since $f_g^{-1}(G_1) \in \tau_1$, where $f_g^{-1}(G_1)(p_1) = L_1$ and $f_g^{-1}(G_2)(p_2) = \{l_1\}$.

Also, $f_{gp_1} : (L_1, \tau_{1p_1}) \longrightarrow (L_2, \tau_{2p_1})$ and $f_{gp_2} : (L_1, \tau_{1p_2}) \longrightarrow (L_2, \tau_{2p_2})$ are soft L-continuous mappings.

Here,

$$\begin{aligned}
\tau_{1p_1} &= \{\phi, L_1, \{l_1\}, \{l_2\}, \{l_1, l_2\}\},\\
\tau_{1p_2} &= \{\phi, L_1, \{l_1\}, \{l_1, l_3\}\},\\
\tau_{2p_1} &= \{\phi, L_2, \{h_1\}, \{h_1, h_2\}\},\\
\tau_{2p_2} &= \{\phi, L_2, \{h_2\}\}.
\end{aligned}$$

Theorem 2.3. If $(\bar{F}_P^L)'$ is a soft L-open set over L_1, for each soft L-open set F_P^L, then $f_g : (L_1, \tau_1, P) \longrightarrow (L_2, \tau_2, P)$ is a soft L-continuous mapping if and only if $f_{g\alpha} : (L_1, \tau_{1\alpha}) \longrightarrow (L_2, \tau_{2\alpha})$ is a soft continuous mapping for each $\alpha \in P$.

Proof. Let $f_{g\alpha} : (L_1, \tau_{1\alpha}) \longrightarrow (L_2, \tau_{2\alpha})$ be a soft continuous mapping for each $\alpha \in P$ and let F_P^L be an arbitrary soft L-open set over L_1. Then $f_{g\alpha}(\bar{F}_P^L(\alpha)) \subset \overline{f_{g\alpha}(\overline{F_P}^L)}(\alpha)$ is satisfied for each $\alpha \in P$.

Since $(\bar{F}_P^L)' \in \tau_1$, $(\bar{F}_P^L)' = \overline{(F_P^{L'})}$. Thus, $f_g\overline{((F_P^L))} = \overline{f_g(F_P^L)}$ is obtained.

Let $f_g : (L_1, \tau_1, P) \longrightarrow (L_2, \tau_2, P)$ be a soft L-continuous mapping. Then for each $\alpha \in P$ and if F_P^L be an arbitrary soft L-open set over L_1. Then $f_g(\bar{F}_P^L(\alpha)) \subset \overline{f_g(\overline{F_P}^L)}(\alpha)$ is satisfied for each $\alpha \in P$. Thus, $f_{g\alpha} : (L_1, \tau_{1\alpha}) \longrightarrow (L_2, \tau_{2\alpha})$ is obtained.

This implies $f_{g\alpha}$ is a soft continuous mapping for each $\alpha \in P$. □

Proposition 2.1. If $f_{g\alpha} : (L_1, \tau_{1\alpha}) \longrightarrow (L_2, \tau_{2\alpha})$ is soft L-open(closed) mapping, then for each $\alpha \in P$, $f_{g\alpha} : (L_1, \tau_{1\alpha}) \longrightarrow (L_2, \tau_{2\alpha})$ is an soft open(closed) mapping.

Proof. If $f_{g\alpha} : (L_1, \tau_{1\alpha}) \longrightarrow (L_2, \tau_{2\alpha})$ is soft L-open(closed) mapping, then for each $\alpha \in P$, $f_{g\alpha}(F_P^L(\alpha)) \subset f_{g\alpha}(F_P{}^L)(\alpha)$.

This implies $f_{g\alpha} : (L_1, \tau_{1\alpha}) \longrightarrow (L_2, \tau_{2\alpha})$ is an soft open(closed) mapping. □

Note 2.1. The concepts of soft L-continuous, soft L-open and soft L-closed mappings are all independent of each other.

Example 2.5. Suppose (L, τ_1, P) be a soft discrete L-topological space and (L, τ_2, P) be soft indiscrete L-topological space. Then $f_g : 1_L : L \longrightarrow L$ is a soft L-open mapping and soft L-closed mapping. But it is not soft L-continuous mapping.

Example 2.6. Suppose $L = \{l_1, l_2, l_3\}, P = \{p_1, p_2\}$ and $\tau_1 = \{\phi, L, F_{1P}^L, F_{2P}^L, F_{3P}^L, F_{4P}^L, F_{5P}^L, F_{6P}^L, F_{7P}^L\}$, $\tau_2 = \{\phi, L, G_{1P}^L, G_{2P}^L, G_{3P}^L, G_{4P}^L\}$ be two soft L-topologies over L, where $F_{1P}^L, F_{2P}^L, F_{3P}^L, F_{4P}^L, F_{5P}^L, F_{6P}^L, F_{7P}^L, G_{1P}^L, G_{2P}^L, G_{3P}^L, G_{4P}^L$ are

soft lattices over L, defined as follows

$$F_1(p_1) = \{l_2\}, F_1(p_2) = \{l_1\},$$
$$F_2(p_1) = \{l_1, l_3\}, F_2(p_2) = \{l_2, l_3\},$$
$$F_3(p_1) = \{l_2\}, F_3(p_2) = L,$$
$$F_4(p_1) = \{\phi\}, F_4(p_2) = \{l_1\},$$
$$F_5(p_1) = \{l_1, l_3\}, F_5(p_2) = L,$$
$$F_6(p_1) = \{\phi\}, F_6(p_2) = \{l_2, l_3\},$$
$$F_7(p_1) = \{\phi\}, F_7(p_2) = L$$

and

$$G_1(p_1) = \{l_2\}, G_1(p_2) = \{l_1\},$$
$$G_2(p_1) = \{l_2, l_3\}, G_2(p_2) = \{l_1, l_2\},$$
$$G_3(p_1) = \{l_1, l_2\}, G_3(p_2) = L,$$
$$G_4(p_1) = \{l_2\}, G_4(p_2) = \{l_1, l_2\}.$$

If we define a mapping $f_g : 1_L : L \longrightarrow L$ as $f_g(l_i) = l_1$, for $1 \leq i \leq 3$, then $f_g^{-1}(G_1)(p_1) = f_g^{-1}(G_4)(p_1) = \phi$, $f_g^{-1}(G_1)(p_2) = f_g^{-1}(G_4)(p_2) = L$, $f_g^{-1}(G_3)(p_1) = f_g^{-1}(G_3)(p_2) = L$. Then, f_g is a soft L-continuous mapping, but $f_g(F_1)(p_1) = \{l_1\}$, $f_g(F_1)(p_2) = \{l_1\}$, $f_g(F_1')(p_1) = \{l_1\}$, $f_g(F_1')(p_2) = \{l_1\}$. Hence it is not both soft L-open and soft L-closed mapping.

Example 2.7. Suppose $L_1 = \{l_1, l_2, l_3\}, L_2 = \{h_1, h_2\}, P = \{p_1, p_2\}$ and $\tau_1 = \{\phi, L_1, F_{1P}^L, F_{2P}^L\}$ is a soft L-topological space over L_1, $\tau_2 = \{\phi, L_2, G_{1P}^L, G_{2P}^L\}$ is a soft L-topological space over L_2, where F_{1P}^L, F_{2P}^L are soft lattices over L_1, G_{1P}^L, G_{2P}^L are soft lattices over L_2, defined as follows $F_1(p_1) = \{l_1, l_2\}, F_1(p_2) = \{l_3\}$, $F_2(p_1) = L_1, F_2(p_2) = \{l_3\}$, $G_1(p_1) = \{L_2\}, G_1(p_2) = \{h_2\}$, $G_2(p_1) = \{h_1\}, G_2(p_2) = \{h_2\}$.

If we define a mapping $f_g : L_1 \longrightarrow L_2$ as $f_g(l_1) = h_1, f_g(l_2) = f_g(l_3) = h_2$, then it is clear that $f_g(F_1)(p_1) = L_2$, $f_g(F_1)(p_2) = \{h_2\}$, $f_g(F_2)(p_1) = L_2$, $f_g(F_2)(p_2) = \{h_2\}$. Then, the mapping $f_g : L_1 \longrightarrow L_2$ is a soft L-open mapping.

Also, since $f_g(F_1')(p_1) = \{h_1\}, f_g(F_1')(p_2) = L_2$, it is not soft L-closed mapping and $f_g(F_2')(p_1) = L_1, f_g(F_2')(p_2) = \{l_2, l_3\}$. Hence, it is not soft L-continuous mapping.

Example 2.8. Suppose $L_1 = \{l_1, l_2, l_3\}, L_2 = \{h_1, h_2\}, P = \{p_1, p_2\}$ and $\tau_1 = \{\phi, L_1, F_{1P}^L, F_{2P}^L, F_{3P}^L\}$ is a soft L-topological space over L_1, $\tau_2 = \{\phi, L_2, G_{1P}^L, G_{2P}^L\}$ is a soft L-topological space over L_2, where $F_{1P}^L, F_{2P}^L, F_{3P}^L$ are soft lattices over L_1, G_{1P}^L, G_{2P}^L are soft lattices over L_2, defined as follows, $F_1(p_1) = \{l_1, l_3\}, F_1(p_2) = \{l_2\}$, $F_2(p_1) = L_1, F_2(p_2) = \{l_2, l_3\}$, $F_3(p_1) = \{l_3\}, F_2(p_2) = \{l_2\}$, and $G_1(p_1) = \phi, G_1(p_2) = \{h_1\}$, $G_2(p_1) = \{h_1\}, G_2(p_2) = L_2$, Now, we define a mapping $f_g : L_1 \longrightarrow L_2$ as $f_g(l_1) = f_g(l_2) = h_1, f_g(l_3) = h_2$. Then, it is clear that $f_g(F_1')(p_1) = f_g(l_2) = \{h_1\}$, $f_g(F_1')(p_2) = f_g(\{l_1, l_3\}) = L_2$,

$f_g(F_2')(p_1) = \phi$, $f_g(F_2')(p_2) = f_g(\{l_1\}) = \{h_1\}$, $f_g(F_3')(p_1) = f_g(l_2) = \{h_1\}$, $f_g(F_3')(p_2) = f_g(\{l_1, l_3\}) = L_2$, This implies that f_g is not soft L-closed mapping. $f_g(F_2')(p_1) = L_1, f_g(F_2')(p_2) = \{l_2, l_3\}$,

Hence, it is not soft L-continuous mapping. Also, $f_g(F_1)(p_1) = L_2$, $f_g(F_1)(p_2) = \{h_1\}$, $f_g^{-1}(G_1)(p_1) = \phi$, $f_g^{-1}(G_1)(p_2) = \{l_1, l_2\}$. Hence it is not soft L-open and soft L-continuous mappings respectively.

Theorem 2.4. Let (L_1, τ_1, P) and (L_2, τ_2, P) be two soft lattice topological spaces, $f_g\colon (L_1, \tau_1, P) \longrightarrow (L_2, \tau_2, P)$ be a mapping. Then:

(a) f_g is a soft L-open mapping if for each soft L-set F_P^L over L_1, $f_g((F_P^L)^o) \subset (f_g(F_P^L))^o$ is satisfied.

(b) f_g is a soft L-closed mapping if for each soft L-set F_P^L over L_1, $\overline{f_g((F_P^L))} \subset f_g\overline{(F_P^L)}$ is satisfied.

Proof. (a) Let f_g be a soft L-open mapping and F_P^L be a soft L-set over L_1.

$(F_P^L)^o$ is a soft L-open set and $(F_P^L)^o \subset F_P^L$.

Since f_g is a soft L-open mapping, $f_g((F_P^L)^o)$ is a soft L-open set in L_2 and $f_g((F_P^L)^o) \subset (f_g(F_P^L))$.

Thus, $f_g((F_P^L)^o) \subset (f_g(F_P^L))^o$ is obtained. Conversely, let F_P^L be a soft L-open set over L_1. Then, $F_P^L = (F_P^L)^o$. From the given condition(theorem), $f_g((F_P^L)^o) \subset (f_g(F_P^L))^o$. Then, $f_g(F_P^L) = f_g((F_P^L)^o) \subset (f_g(F_P^L))^o \subset f_g(F_P^L)$. This implies $f_g(F_P^L) = (f_g(F_P^L))^o$.

(b) Let f_g be a soft L-closed mapping and F_P^L be a soft L-set over L_1. $\overline{(F_P^L)}$ is a soft L-closed set and $(F_P^L) \subset \overline{F_P^L}$.

Since f_g is a soft L-closed mapping, $f_g\overline{(F_P^L)}$ is a soft L-open set in L_2 and $f_g(F_P^L) \subset f_g\overline{(F_P^L)}$. Thus, $\overline{f_g((F_P^L))} \subset f_g\overline{(F_P^L)}$ is obtained.

Conversely, let F_P^L be a soft L-closed set over L_1. Then $F_P^L = \overline{(F_P^L)}$. From the given condition(theorem), $\overline{(f_g(F_P^L))} \subset f_g\overline{((F_P^L))} \subset f_g(F_P^L) = \overline{(f_g(F_P^L))}$. This implies $\overline{f_g((F_P^L))} = f_g(F_P^L)$. □

Definition 2.22. Consider (L_1, τ_1, P) and (L_2, τ_2, P) as two soft lattice topological spaces, $f_g\colon (L_1, \tau_1, P) \longrightarrow (L_2, \tau_2, P)$ be a mapping. If f_g is a bijection, soft L-continuous and f_g^{-1} is a soft L-continuous mapping, then f_g is said to be soft L-homeomorphism from L_1 to L_2.

When a soft homeomorphism f_g exists between L_1 and L_2, we say that L_1 is soft L-homeomorphic to L_2.

Theorem 2.5. Let (L_1, τ_1, P) and (L_2, τ_2, P) be two soft lattice topological spaces, $f_g\colon (L_1, \tau_1, P) \longrightarrow (L_2, \tau_2, P)$ be a bijection mapping. Then, the following conditions are equivalent:

(1) f_g is a homeomorphism on soft L-topological space;

(2) f_g is a continuous and closed mapping on soft L-topological space;

(3) f_g is a continuous and open mapping on soft L-topological space.

Proof. Proof of the statements are followed from Definition 2.22 and Theorem 2.4. □

3. Conclusion

Topological structures on soft sets are more generalized methods that can be used to assess the similarities and dissimilarities between the objects in a universe which are soft sets. The concept of soft L-topological spaces are defined over a soft lattice with a fixed set of parameter. This paper deals with the concept of soft L-continuous mappings in Soft Lattice topological spaces which are defined with a fixed set of parameters over an initial universe. Also we have given some properties regarding the continuity of mappings in Soft Lattice topological spaces. At the end, open mappings, closed mappings and homeomorphism on soft L-topological spaces are defined and some results are proved.

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Topologizing a hyper BCI-algebra using its hyper-order

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Abstract. In this paper, we topologize a given hyper BCI-algebra using the hyper-order associated with the hyper-structure. Specifically, we consider a family of subsets of the hyper BCI-algebra that serves as a basis for some topology on this structure. Properties of the topological spaces generated in this way are subsequently investigated.
Keywords: topology, basis, hyper BCI-algebra, hyper-order, hyperatom.

1. Introduction

In 1934, F. Marty [3] introduced the hyperstructure theory (also called multi-valued algebras) at the 8th congress of Scandinavian Mathematicians. Xin [9] applied the concept of hyperstructure theory to a BCI-algebra, a class of abstract algebra introduced by Iséki [1] in 1966, and introduced the notion of a hyper BCI-algebra.

In [2], Y. B. Jun et al. initiated the study of topological BCI-algebras (briefly, TBCI-algebras) and some properties of this structure, and gave a characterization of a TBCI-algebra in terms of neighborhoods. They also gave a filter base Ω generating a BCI-topology, and made a BCI-algebra X into a TBCI-algebra for which Ω is a fundamental system of neighborhoods of 0. Moreover, they introduced the notions of topological subalgebras, topological ideals and topo-

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logical homomorphisms in topological BCI-algebras and studied some related properties.

In this study, we topologize a given hyper BCI-algebra by right application of the hyper-order associated with it. In this manner, we obtain a family of subsets of the hyper BCI-algebra that serves as a basis for some topology on this structure. Properties of the induced topological space generated in this way are investigated. This method of topologizing a hyperstructure had been previously utilized in [6] and [7].

2. Preliminaries

A *hyperoperation* on a nonempty set H is a map from $H \times H$ into $P^*(H) = P(H) \setminus \{\emptyset\}$. Let $\circledast$ be a hyperoperation on H and $(x, y) \in H \times H$. Then its image under $\circledast$, denoted by $x \circledast y$, is called the *hyperproduct* of x and y. If A and B are nonempty subsets of H, then $A * B$ is given by $A \circledast B = \bigcup_{a \in A, b \in B} a \circledast b$. We shall use $x \circledast y$ instead of $x \circledast \{y\}$, $\{x\} \circledast y$, or $\{x\} \circledast \{y\}$. When $A \subseteq H$ and $x \in H$, we agree to write $A \circledast x$ instead of $A \circledast \{x\}$. Similarly, we write $x \circledast A$ for $\{x\} \circledast A$. In effect, $A \circledast x = \bigcup_{a \in A} a \circledast x$ and $x \circledast A = \bigcup_{a \in A} x \circledast a$.

A *hyper BCI-algebra* $(H, \circledast, 0)$ is a nonempty set H endowed with a hyperoperation "$\circledast$" and a constant 0 satisfying the following axioms: for all $x, y, z \in H$,

(B_1) $((x \circledast z) \circledast (y \circledast z)) \ll x \circledast y$,

(B_2) $(x \circledast y) \circledast z = (x \circledast z) \circledast y$,

(B_3) $x \ll x$,

(B_4) $x \ll y$ and $y \ll x$ imply $x = y$,

(B_5) $0 \circledast (0 \circledast x) \ll x$, $x \neq 0$,

where for every $A, B \subseteq H$, $A \ll B$ if and only if for each $a \in A$, there exists $b \in B$ such that $0 \in a \circledast b$. In particular, for every $x, y \in H$, $x \ll y$ if and only if $0 \in x \circledast y$. In such case, we call "$\ll$" the *hyper order* in H (see [9]).

A hyper BCI-algebra $(H, \circledast, 0)$ is said to be *ordered* if for each x, y, $z \in H$, $x \ll y$ and $y \ll z$ implies $x \ll z$.

Example 2.1. [9] Let $H = \{0, 1, 2\}$. Define the hyperoperation "$\circledast$" by the Cayley table shown below.

$\circledast$	0	1	2
0	$\{0, 1\}$	$\{0, 1\}$	$\{0, 1\}$
1	$\{1\}$	$\{0, 1\}$	$\{0, 1\}$
2	$\{2\}$	$\{1, 2\}$	$\{0, 1, 2\}$

Then by routine calculations, $(H, \circledast, 0)$ is an ordered hyper BCI-algebra.

All throughout, we denote a hyper BCI-algebra $(H, \circledast, 0)$ by H, unless otherwise specified.

Let H be a hyper BCI-algebra and $A \subseteq H$. The sets $L_H(A)$ and $R_H(A)$ are given by $L_H(A) = \{x \in H \mid x \ll a, \forall\, a \in A\} = \{x \in H \mid 0 \in x \circledast a, \forall\, a \in A\}$ and $R_H(A) = \{x \in H \mid a \ll x, \forall\, a \in A\} = \{x \in H \mid 0 \in a \circledast x, \forall\, a \in A\}$. If $A = \{a\}$, we write $L_H(\{a\}) = L_H(a)$, and $R_H(\{a\}) = R_H(a)$.

An element a of a hyper BCI-algebra H is called a *hyperatom* if for each $x \in H$, $x \ll a$ implies $x = 0$ or $x = a$. Denote by $A(H)$ the set of all hyperatoms of H, and by $A^*(H)$ the set of all nonzero hyperatoms of H; that is, $A^*(H) = A(H) \setminus \{0\}$. A hyper BCI-algebra H is said to be *hyperatomic* if each element of H is a hyperatom, that is, $A(H) = H$ (see [5]).

Let X be a nonempty set and $p \in X$. The topology τ_p given by $\tau_p = \{\emptyset\} \cup \{A \subseteq X : p \in A\}$ is called the *particular point p topology* on X (see [8]).

Example 2.2. Consider the hyper BCI-algebra defined in Example 2.1. Then $R_H(0) = \{0, 1, 2\} = H$, $R_H(1) = \{1, 2\}$, $R_H(2) = \{2\}$, $R_H(\{0, 1\}) = \{1, 2\}$, $R_H(\{0, 2\}) = \{2\}$, $R_H(\{1, 2\}) = \{2\}$, and $R_H(H) = \{2\}$.

Example 2.3. Consider $H = \{0, a, b\}$ with the hyperoperation "$\circledast$" defined as follows:

$\circledast$	0	a	b
0	$\{0, a\}$	$\{0, a\}$	$\{b\}$
a	$\{a\}$	$\{0, a\}$	$\{b\}$
b	$\{b\}$	$\{b\}$	$\{0, a\}$

H is a hyper BCI-algebra [4]. By the definition of R_H, $R_H(0) = \{0, a\}$, $R_H(a) = \{a\}$, $R_H(b) = \{b\}$, $R_H(\{0, a\}) = \{a\}$, $R_H(\{0, b\}) = \emptyset$, $R_H(\{a, b\}) = \emptyset$, and $R_H(H) = \emptyset$.

3. Results

The following properties of a hyper BCI-algebra H are taken from [9].

Proposition 3.1 ([9])**.** *In any hyper BCI-algebra H, the following hold:*

(*i*) $x \ll 0$ *implies* $x = 0$.

(*ii*) $0 \in x \circledast (x \circledast 0)$.

(*iii*) $x \ll x \circledast 0$.

(*iv*) $0 \circledast (x \circledast y) \ll y \circledast x$.

(*v*) $A \ll A$.

(*vi*) $A \subseteq B$ *implies* $A \ll B$.

(*vii*) $A \ll \{0\}$ *implies* $A = \{0\}$.

(*viii*) $x \circledast 0 \ll \{y\}$ *implies* $x \ll y$.

(*ix*) $y \ll z$ *implies* $x \circledast z \ll x \circledast y$.

(*x*) $x \circledast y = \{0\}$ *implies* $(x \circledast z) \circledast (y \circledast z) = \{0\}$ *and* $x \circledast z \ll y \circledast z$.

(*xi*) $A \circledast A = \{0\}$ *implies* A *is a singleton.*

(*xii*) $A \circledast \{0\} = \{0\}$ *implies* $A = \{0\}$,

for all $x, y, z \in H$ *and for all non-empty subsets* A *and* B *of* H.

The next result gives some properties of the operator L_H.

Proposition 3.2 ([5])**.** *Let* A *and* B *be subsets of* H. *Then the following hold:*

(*i*) $L_H(\emptyset) = H$.

(*ii*) $L_H(\{0\}) = \{0\}$.

(*iii*) *If* $A \subseteq B$, *then* $L_H(B) \subseteq L_H(A)$.

(*iv*) $L_H(A) = \bigcap_{a \in A} L_H(\{a\})$.

(*v*) *If* $x \in H$, *then* $x \in L_H(\{x\})$. *Furthermore,* $L_H(\{x\}) = \{0\}$ *if and only if* $x = 0$.

Theorem 3.1. *Let* A *and* B *be subsets of a hyper BCI-algebra* H. *Then the following hold:*

(*i*) $R_H(\emptyset) = H$.

(*ii*) *If* $A \subseteq B$, *then* $R_H(B) \subseteq R_H(A)$.

(*iii*) *If* H *is ordered and* $R_H(A) \neq \emptyset$, *then* $R_H(R_H(A)) \subseteq R_H(A)$.

Proof of Theorem 3.1. Suppose $R_H(\emptyset) \neq H$. Then there exists $h \in H$ such that $h \notin R_H(\emptyset)$. Thus, there exists $a \in \emptyset$ such that $a \not\ll h$, which is absurd. Therefore, $R_H(\emptyset) = H$. This shows (*i*).

For (*ii*), let $x \in R_H(B)$. Then $b \ll x$ for all $b \in B$. Since $A \subseteq B$, $a \ll x$ for all $a \in A$. Thus, $x \in R_H(A)$. Hence, $R_H(B) \subseteq R_H(A)$.

Finally, let $x \in R_H(R_H(A))$. Then $b \ll x$ for all $b \in R_H(A)$. Since $a \ll b$ for all $a \in A$ and H is ordered, it follows that $a \ll x$ for all $a \in A$. Thus, $x \in R_H(A)$. Hence, $R_H(R_H(A)) \subseteq R_H(A)$. This proves (*iii*). □

Theorem 3.2. *Let* H *be a hyper BCI-algebra and* $A \subseteq H$. *Then*

(*i*) $R_H(A) = \bigcap_{a \in A} R_H(a)$.

(*ii*) *For any* $x \in H$, $R_H(x) = \{z \in H : x \in L_H(z)\}$.

(*iii*) $R_H(x) \neq \emptyset$ *for all* $x \in H$. *In particular,* $x \in R_H(x)$.

Proof of Theorem 3.2. Note that $R_H = \{x \in H | a \ll x, \forall\, a \in A\} = \{x \in H \mid x \in R_H(a), \forall\, a \in A\} = \bigcap_{a\in A} R_H(a)$. Hence, (i) holds.

For (ii), let $x \in H$ and let $D = \{z \in H : x \in L_H(z)\}$. If $z \in R_H(x)$, then $x \ll z$, that is, $x \in L_H(z)$. Thus, $z \in D$. On the other hand, let $z \in D$. Then $x \in L_H(z)$. Hence, $x \ll z$. That is, $z \in R_H(x)$.

By (B_3), $x \ll x$ for all $x \in H$. It follows that $x \in R_H(x)$. That is, (iii) holds. □

Corollary 3.1. *Let H be a hyper BCI-algebra and $A \subseteq H$. Then*

(i) For any $x \in H \setminus \{0\}$, $0 \notin R_H(x)$. Consequently, for any $\emptyset \neq A \subseteq H$ such that $A \neq \{0\}$, $0 \notin R_H(A)$.

(ii) $R_H(0) = H$ if and only if $0 \in L_H(x)$ for all $x \in H$.

Proof of Corollary 3.1. For (i), let $x \in H \setminus \{0\}$. By Theorem 3.2(ii), $0 \in R_H(x) = \{z \in H : x \in L_H(z)\}$ means that $x \in L_H(0) = \{0\}$ by Proposition 3.2(ii), a contradiction. Consequently, by Theorem 3.2(i), for any $\emptyset \neq A \subseteq H$ such that $A \neq \{0\}$, $0 \notin R_H(A)$.

For (ii), by Theorem 3.2(ii), $R_H(0) = \{z \in H : 0 \in L_H(z)\}$. Hence, $R_H(0) = H$ if and only if $0 \in L_H(x)$ for all $x \in H$. □

Lemma 3.1. *Let H be a hyper BCI-algebra and A a subset of H. Then $A \subseteq L_H(R_H(A))$. In particular, $L_H(R_H(H)) = H$.*

Proof of Lemma 3.1. Let $A \subseteq H$. If $R_H(A) = \emptyset$, then $A \subseteq H = L_H(\emptyset) = L_H(R_H(A))$ by Proposition 3.2(i). Suppose $R_H(A) \neq \emptyset$ and let $x \in R_H(A)$. Then $a \ll x$ for all $a \in A$. Let $b \in A$. Then $b \ll x$ for every $x \in R_H(A)$. This means that $b \in L_H(x)$ for every $x \in R_H(A)$. Hence, by Proposition 3.2(iv), $b \in \bigcap_{x\in R_H(A)} L_H(x) = L_H(R_H(A))$. Therefore, $A \subseteq L_H(R_H(A))$. In particular, since $L_H(R_H(H)) \subseteq H$, it follows that $L_H(R_H(H)) = H$. □

Theorem 3.3. *Let H be a hyper BCI-algebra with $|H| \geq 2$. Then $R_H(H)$ is a singleton if and only if there exists $a \in H \setminus \{0\}$ such that $L_H(a) = H$. In particular, $R_H(H) = \{a\}$.*

Proof of Theorem 3.3. Suppose that $R_H(H)$ is a singleton. By Corollary 3.1(i), $R_H(H) = \{a\}$ for some $a \in H \setminus \{0\}$. Then by Lemma 3.1, $H = L_H(R_H(H)) = L_H(a)$.

For the converse, suppose that there exists $a \in H\setminus\{0\}$ such that $L_H(a) = H$. Then $x \ll a$ for all $x \in H$. Hence, $a \in R_H(x)$ for all $x \in H$. This implies that $a \in \bigcap_{x\in H} R_H(x) = R_H(H)$ by Theorem 3.2(i). Thus, $\{a\} \subseteq R_H(H)$. Suppose further that there exists $b \in R_H(H) \setminus \{a\}$. Then $x \ll b$ for all $x \in H$. Since a, $b \in H$, it follows that $a \ll b$ and $b \ll a$. By (B_4), $a = b$, contrary to the assumption that $a \neq b$. Thus, $R_H(H) = \{a\}$. □

Theorem 3.4. *Let H be a hyper BCI-algebra. Then either $|R_H(H)| = 0$ or $|R_H(H)| = 1$.*

Proof of Theorem 3.4. *If $R_H(H)$ is a singleton, then we are done. Suppose that $|R_H(H)| \neq 1$. Let $a \in H$. Then by assumption and Theorem 3.3, $L_H(a) \neq H$. Hence, there exists $y \in H$ such that $y \not\ll a$, that is, there exists $y \in H$ such that $a \notin R_H(y)$. Thus, $a \notin R_H(H) = \bigcap_{z\in H} R_H(z)$. Since a is arbitrary, $R_H(H) = \emptyset$.* □

Lemma 3.2. *Let $\{A_\alpha : \alpha \in I\}$ be a collection of subsets of a hyper BCI-algebra H. Then $\bigcap_{\alpha\in I} R_H(A_\alpha) = R_H\left(\bigcup_{\alpha\in I} A_\alpha\right)$.*

Proof of Lemma 3.2. Suppose $\bigcap_{\alpha\in I} R_H(A_\alpha) = \emptyset$. Since $A_\alpha \subseteq \bigcup_{\alpha\in I} A_\alpha$, by Theorem 3.1, $R_H\left(\bigcup_{\alpha\in I} A_\alpha\right) \subseteq \bigcap_{\alpha\in I} R_H(A_\alpha) = \emptyset$. Thus, $R_H\left(\bigcup_{\alpha\in I} A_\alpha\right) = \emptyset$. Suppose $\bigcap_{\alpha\in I} R_H(A_\alpha) \neq \emptyset$. Then

$$\begin{aligned} x \in \bigcap_{\alpha\in I} R_H(A_\alpha) &\Leftrightarrow x \in R_H(A_\alpha) \text{ for all } \alpha \in I \\ &\Leftrightarrow a \ll x \text{ for all } a \in A_\alpha \text{ and for all } \alpha \in I \\ &\Leftrightarrow a \ll x \text{ for all } a \in \bigcup_{\alpha\in I} A_\alpha \\ &\Leftrightarrow x \in R_H\left(\bigcup_{\alpha\in I} A_\alpha\right). \end{aligned}$$

This proves the assertion. □

Theorem 3.5. *Let H be a hyper BCI-algebra. Then the family $\mathcal{B}_R(H) = \{R_H(A) : \emptyset \neq A \subseteq H\}$ is a basis for some topology on H.*

Proof of Theorem 3.5. Clearly, $H = \bigcup_{a\in H} R_H(a)$. Let A and B be nonempty subsets of H. Then by Lemma 3.2, $R_H(A) \cap R_H(B) = R_H(A \cup B) \in \mathcal{B}_R(H)$. Therefore, $\mathcal{B}_R(H)$ is a basis for some topology on H. □

Denote by $\tau_R(H)$ the topology generated by $\mathcal{B}_R(H)$.

Example 3.1. Consider $H := [0, \infty)$ with the hyperoperation " $\circledast$ ", defined in [9]:

$$x \circledast y := \begin{cases} [0, x], & \text{if } x \leq y, \\ (0, y], & \text{if } x > y \neq 0, \\ \{x\}, & \text{if } y = 0 \end{cases}$$

for all $x, y \in H$. Then $(H, \circledast, 0)$ is a hyper BCI-algebra. Now, let $k \in H$. Then $R_H(k) = [k, \infty)$. Let $\emptyset \neq A \subseteq H$. First, suppose that A is bounded, and let $p = \sup A$. By Theorem 3.2(i), $R_H(A) = \bigcap_{a\in A} R_H(a) = \bigcap_{a\in A}[a, \infty)$. Since $a \leq p$ for all $a \in A$, $[p, \infty) \subseteq R_H(A)$. Now, let $r \in R_H(A)$. Then $a \leq r$ for all $a \in A$; hence, $p \leq r$. It follows that $r \in [p, \infty)$. Thus, $R_H(A) = [p, \infty)$. Suppose that A is not bounded. Let $m \in H$. Since A is not bounded, there exists $z \in A$ such that $m < z$. This implies that $m \notin R_H(z)$. Thus, $m \notin \bigcap_{w\in A} R_H(w) = R_H(A)$.

Hence, $R_H(A) = \emptyset$. Accordingly, $\mathcal{B}_R(H) = \{R_H(A) : A \subseteq H\} = \{R_H(p) : p \in H\} = \{[p, \infty) : p \in H\}$.

Next, let $G \in \tau_R(H) \setminus \{\emptyset\}$. Then $G = \bigcup_{a \in \mathcal{K}} R_H(a)$ for some $\mathcal{K} \subseteq H$. Since $a \in R_H(a) \subseteq G$ for all $a \in \mathcal{K}$, it follows that $\mathcal{K} \subseteq G$. Let $q = \inf G$ and consider the following cases:

Case 1. Suppose $q \in G$. Since $q \leq a$ for all $a \in \mathcal{K}$, it follows that $R_H(a) \subseteq R_H(q)$ for all $a \in \mathcal{K}$. This implies that $G \subseteq R_H(q)$. Further, since $q \in G$, there exists $a \in \mathcal{K}$ such that $q \in R_H(a)$. This implies that $a \leq q$. Since $q = \inf G$, $a = q$. Thus, $R_H(a) = R_H(q) \subseteq G$, showing that $G = R_H(q)$.

Case 2. Suppose $q \notin G$. Since $q \leq a$ for all $a \in \mathcal{K}$, $R_H(a) \subseteq (q, \infty)$ for all $a \in \mathcal{K}$. Hence, $G \subseteq (q, \infty)$. Now, let $p \in (q, \infty)$. Then $q < p$. This implies that there exists $z \in G$ such that $q < z < p$. Let $b \in \mathcal{K}$ such that $z \in R_H(b)$. Then $q < b \leq z < p$. Thus, $p \in R_H(b) \subseteq G$. Therefore, $G = (q, \infty)$.

Accordingly, $\tau_R(H) = \{\emptyset\} \cup \{R_H(x) : x \in H\} \cup \{(x, \infty) : x \in H\}$.

Example 3.2. Consider the hyper BCI-algebra defined in Example 2.3. Then by Theorem 3.5, $\mathcal{B}_R(H) = \{R_H(A) : A \subseteq H\} = \{\{0, a\}, \{a\}, \{b\}, \emptyset\}$. Thus, $\tau_R(H) = \{\emptyset, H, \{0, a\}, \{a\}, \{b\}, \{a, b\}\}$.

Observe that in Example 3.1, $(H, \tau_R(H))$ is connected, however, in Example 3.2, $H = \{0, a\} \cup \{b\}$. Hence, $(H, \tau_R(H))$ is disconnected.

Theorem 3.6. *Let H be a hyper BCI-algebra. If $R_H(0) = H$, then the topological space $(H, \tau_R(H))$ is connected.*

Proof of Theorem 3.6. Suppose that $R_H(0) = H$. It follows from Corollary 3.1(i) that there does not exist $G \in \tau_R(H) \setminus \{H\}$ such that $0 \in G$. Consequently, H cannot have a decomposition. Thus, $(H, \tau_R(H))$ is connected. □

Theorem 3.7. *If H is a finite hyper BCI-algebra, then the family $\mathcal{S}_R(H) = \{R_H(a) : a \in H\}$ is a subbase of $\tau_R(H)$.*

Proof of Theorem 3.7. Clearly, $\mathcal{S}_R(H) \subseteq \tau_R(H)$. By Theorem 3.2(i) and the assumption that H is finite, it follows that every element of $\mathcal{B}_R(H) \setminus \{\emptyset\}$ is a finite intersection of members of $\mathcal{S}_R(H)$. Therefore, $\mathcal{S}_R(H)$ is a subbase of $\tau_R(H)$. □

Remark 3.1. The converse of Theorem 3.7 is not always true. Consider Example 3.1. Notice that $\mathcal{B}_R(H) = \{R_H(A) : \emptyset \neq A \subseteq H\} = \{R_H(p) : p \in H\}$ is also a subbase of $\tau_R(H)$. However, H is an infinite hyper BCI-algebra.

Theorem 3.8. *Let H be a hyper BCI-algebra. The following statements are equivalent:*

(*i*) *H is hyperatomic.*

(*ii*) *$R_H(a) = \{a\}$ for all $a \in H \setminus \{0\}$.*

(iii) $a \notin R_H(A)$ *for all* $a \in H \setminus \{0\}$ *and* $A \notin \mathscr{P}(\{0, a\})$.

Proof of Theorem 3.8. *(i)* $\Rightarrow$ *(ii)* Suppose that H is hyperatomic. Let $a \in H \setminus \{0\} = A^*(H)$. Suppose further that $R_H(a) \neq \{a\}$ and let $x \in R_H(a) \setminus \{a\}$. Then $a \ll x$. Since a is a hyperatom, $a = 0$ or $a = x$, a contradiction. Hence, $R_H(a) = \{a\}$ for all $a \in H \setminus \{0\}$.

(ii) $\Rightarrow$ *(iii)* Suppose that $R_H(a) = \{a\}$ for all $a \in H \setminus \{0\}$. Let $b \in H \setminus \{0\}$ and let $A \in \mathscr{P}(H) \setminus \mathscr{P}(\{0, b\})$. Suppose that $b \in R_H(A)$. Then $x \ll b$ for all $x \in A$. Pick $x \in A \setminus \{b\}$. Then $b \in R_H(x)$, a contradiction. Hence, $b \notin R_H(A)$. This shows that *(iii)* holds.

(iii) $\Rightarrow$ *(i)* Let $b \in H \setminus \{0\}$. Then $b \notin R_H(A)$ for all $A \in \mathscr{P}(H) \setminus \mathscr{P}(\{0, b\})$ by assumption. Suppose that $x \ll b$. Then $b \in R_H(x)$. By assumption, $x \in \{0, b\}$. Hence, $x = 0$ or $x = b$. Therefore, H is hyperatomic. □

Corollary 3.2. *If* H *is a hyperatomic hyper BCI-algebra such that* $R_H(0) = H$, *then* $R_H(\{x, 0\}) = \{x\}$ *for all* $x \in H \setminus \{0\}$.

Proof of Corollary 3.2. Let $x \in H \setminus \{0\}$. By Theorem 3.2(i) and Theorem 3.8, $R_H(\{x, 0\}) = R_H(x) \cap R_H(0) = \{x\} \cap H = \{x\}$. □

Lemma 3.3. *Let* H *be a hyper BCI-algebra. If* H *is hyperatomic, then* $\mathcal{B}_R(H) = \{R_H(0)\} \cup \{\{a\} : a \in H \setminus \{0\}\}$.

Proof of Lemma 3.3. Suppose that H is hyperatomic and let $\emptyset \neq A \subseteq H$. Now, suppose that $|A \cap A^*(H)| \geq 2$. Let $a, b \in A \cap A^*(H)$ with $a \neq b$. Then by Theorem 3.1(ii) and Theorem 3.8, $R_H(A) = \emptyset$. Consequently,
$\mathcal{B}_R(H) = \{R_H(0)\} \cup \{R_H(a) : a \in H \setminus \{0\}\} = \{R_H(0)\} \cup \{\{a\} : a \in H \setminus \{0\}\}$.
This proves the assertion. □

Theorem 3.9. *Let* H *be a hyperatomic hyper BCI-algebra and* $A \subseteq H$. *Then* $A \in \tau_R(H)$ *if and only if* $0 \notin A$ *or* $R_H(0) \subseteq A$.

Proof of Theorem 3.9. Suppose $A \in \tau_R(H)$ and suppose that $0 \in A$. Then, by Lemma 3.3, $R_H(0) \subseteq A$.

For the converse, suppose first that $0 \notin A$. Then $\{a\} \in \mathcal{B}_R(H)$ for all $a \in A$ by Lemma 3.3. Hence, $A = \bigcup_{a \in A} \{a\} \in \tau_R(H)$. Next, suppose that $R_H(0) \subseteq A$. If $A = R_H(0)$, then $A \in \tau_R(H)$. Suppose that $A \neq R_H(0)$. Let $D = A \setminus R_H(0)$. Since $\{a\} \in \mathcal{B}_R(H)$ for all $a \in D$, it follows that $A = \left(\bigcup_{a \in D} \{a\}\right) \cup R_H(0) \in \tau_R(H)$. □

Corollary 3.3. *Let* H *be a hyperatomic hyper BCI-algebra and* $F \subseteq H$. *Then* F *is* $\tau_R(H)$*-closed if and only if* $0 \in F$ *or* $R_H(0) \cap F = \emptyset$.

Proof of Corollary 3.3. Suppose that F is $\tau_R(H)$-closed. Then $F^c \in \tau_R(H)$. Thus, $0 \notin F^c$ or $R_H(0) \subseteq F^c$ by Theorem 3.9. This implies that $0 \in F$ or $R_H(0) \cap F = \emptyset$.

Next, suppose that $0 \in F$. Then $0 \notin F^c$. If $R_H(0) \cap F = \emptyset$, then $R_H(0) \subseteq F^c$. In both cases, we find that $F^c \in \tau_R(H)$, that is, F is $\tau_R(H)$-closed. □

Theorem 3.10. *Let H be a hyperatomic hyper BCI-algebra with $|H| \geq 2$. Then $(H, \tau_R(H))$ is connected if and only if $R_H(0) = H$.*

Proof of Theorem 3.10. Suppose $(H, \tau_R(H))$ is connected. Suppose further that $R_H(0) \neq H$. Then $A = R_H(0)$ and $B = H \setminus R_H(0)$ are nonempty disjoint $\tau_R(H)$-open sets by Theorem 3.9. Further, since $A \cup B = H$, $A \cup B$ is a decomposition of H, contrary to our assumption that $(H, \tau_R(H))$ is connected. Therefore, $R_H(0) = H$.

Conversely, suppose that $R_H(0) = H$. Suppose further that $(H, \tau_R(H))$ is disconnected. Then there exist nonempty disjoint $\tau_R(H)$-open sets P and Q such that $H = P \cup Q$. Assume, without loss of generality, that $0 \in P$. Then, by Theorem 3.9, $R_H(0) \subseteq P$. Since $R_H(0) = H$, $P = H$ and $Q = \emptyset$, a contradiction. Therefore, $(H, \tau_R(H))$ is connected. □

Theorem 3.11. *Let H be hyperatomic hyper BCI-algebra such that $(H, \tau_R(H))$ is connected. Then $\tau_R(H) = \tau^0(H) = \{H\} \cup \{A \subseteq H : 0 \notin A\}$.*

Proof of Theorem 3.11. By Lemma 3.3 and Theorem 3.10, $\mathcal{B}_R(H) = \{H\} \cup \{\{a\} : a \in H \setminus \{0\}\}$. Let $G \in \tau_R(H)$. Then $G = H$ or $G = \emptyset$ or G is the union of singletons $\{a\}$ where $a \in H \setminus \{0\}$. Thus, $0 \notin G \neq H$. Therefore, $\tau_R(H) = \tau^0(H)$. □

Theorem 3.12. *Let H be an ordered hyper BCI-algebra and $D \subseteq H$. Then D is dense if and only if for each $x \in H \setminus D$, there exists $y \in D$ such that $x \ll y$.*

Proof of Theorem 3.12. Suppose that D is dense in H and let $x \in H \setminus D$. Then $x \in \overline{D} = H$. Since by Theorem 3.2(*iii*), $x \in R_H(x) \in \tau_R(H)$, we have $R_H(x) \cap D \neq \emptyset$. Let $y \in R_H(x) \cap D$. Then $y \in D$ and $x \ll y$.

Conversely, let $z \in H \setminus D$ and let $\emptyset \neq A \subseteq H$ such that $z \in R_H(A)$. Then $a \ll z$ for all $a \in A$. Since $z \in H \setminus D$, by assumption, there exists $y \in D$ such that $z \ll y$. Since H is ordered, it follows that $a \ll y$ for all $a \in A$. Consequently, $y \in R_H(A)$ implying that $R_H(A) \cap D \neq \emptyset$. Hence, $z \in \overline{D}$. Since z was arbitrarily chosen, it follows that D is dense in H. □

Theorem 3.13. *Let H be a hyperatomic hyper BCI-algebra and $A, F \subseteq H$. Then with respect to $\tau_R(H)$,*

(*i*)
$$int(A) = \begin{cases} A, & \text{if } 0 \notin A \text{ or } R_H(0) \subseteq A \\ A \setminus \{0\}, & \text{otherwise} \end{cases}$$

(*ii*)
$$\overline{F} = \begin{cases} F, & \text{if } 0 \in F \text{ or } R_H(0) \cap F = \emptyset \\ F \cup \{0\}, & \text{otherwise} \end{cases}$$

(*iii*) *F is dense if and only if $F = H$ or $F = H \setminus \{0\}$ and $R_H(0) \neq \{0\}$.*

Proof of Theorem 3.13. (*i*) If $0 \notin A$ or $R_H(0) \subseteq A$, then $A \in \tau_R(H)$ by Theorem 3.9. It follows that $intA = A$. Suppose that $A \notin \tau_R(H)$. Then $0 \in A$ and $A \setminus \{0\} \in \tau_R(H)$ by Theorem 3.9. Thus, $intA = A \setminus \{0\}$.

(*ii*) If $0 \in F$ or $R_H(0) \cap F = \emptyset$, then F is $\tau_R(H)$-closed by Corollary 3.3. Hence, $\overline{F} = F$. Suppose that F is not $\tau_R(H)$-closed. Then $0 \notin F$ and $F \cup \{0\}$ is $\tau_R(H)$-closed by Corollary 3.3. Therefore, $\overline{F} = F \cup \{0\}$.

(*iii*) Suppose that F is dense and $F \neq H$. Since F is not $\tau_R(H)$-closed, $0 \notin F$ and $R_H(0) \cap F \neq \emptyset$ by Corollary 3.3. Hence, $R_H(0) \neq \{0\}$ and $\overline{F} = H = F \cup \{0\}$ by (*ii*). Therefore, $F = H \setminus \{0\}$.

Conversely, suppose that $F = H \setminus \{0\}$ and $R_H(0) \neq \{0\}$. Since $R_H(0) \neq \{0\}$ and $0 \notin F$, F is not $\tau_R(H)$-closed. Hence, $\overline{F} = H$ by (*ii*). □

Lemma 3.4. *Let K be a hyper subalgebra of a hyper BCI-algebra H and let $D \subseteq H$. Then $R_H(D) \cap K \subseteq R_K(D \cap K)$. Moreover, if $D \subseteq K$, then $R_H(D) \cap K = R_K(D)$.*

Proof of Lemma 3.4. Let $D \subseteq H$ and let $z \in R_H(D) \cap K$. Then $z \in K$ and $d \ll z$ for all $d \in D$. In particular, $d \ll z$ for all $d \in D \cap K$. Thus, $z \in R_K(D \cap K)$. Hence, $R_H(D) \cap K \subseteq R_K(D \cap K)$.

If $D \subseteq K$, then $R_H(D) \cap K \subseteq R_K(D \cap K) = R_K(D)$. Now, let $z \in R_K(D)$. Then $z \in K$ and $d \ll z$ for all $d \in D \subseteq K \subseteq H$. Hence, $z \in R_H(D) \cap K$. Therefore, $R_H(D) \cap K = R_K(D)$. □

Lemma 3.5. *Let K be a hyper subalgebra of an ordered hyper BCI-algebra H. Then for any $\emptyset \neq A \subseteq H$, $R_H(A) \cap K = \bigcup_{x \in R_H(A) \cap K} R_K(x)$.*

Proof of Lemma 3.5. Let $x \in R_H(A) \cap K$ where $\emptyset \neq A \subseteq H$. Then $a \ll x$ for all $a \in A$ and $x \in K$. Let $y \in R_H(x) \cap K$. Then $x \ll y$ and $y \in K$. Since H is ordered, $a \ll y$ for all $a \in A$. Hence, $y \in R_H(A) \cap K$. That is, $R_H(x) \cap K \subseteq R_H(A) \cap K$. Consequently, $\bigcup_{x \in R_H(A) \cap K} (R_H(x) \cap K) \subseteq R_H(A) \cap K$.

Now, let $w \in R_H(A) \cap K$. By Theorem 3.2(iii), $w \in R_H(w)$. It follows that $w \in R_H(w) \cap K$. Hence, $R_H(A) \cap K \subseteq R_H(w) \cap K$. Thus, we have $R_H(A) \cap K \subseteq \bigcup_{x \in R_H(A) \cap K} (R_H(x) \cap K)$. Therefore, by Lemma 3.4,

$$R_H(A) \cap K = \bigcup_{x \in R_H(A) \cap K} (R_H(x) \cap K) = \bigcup_{x \in R_H(A) \cap K} R_K(x).$$

This proves the assertion. □

Theorem 3.14. *Let K be a hyper subalgebra of an ordered hyper BCI-algebra H. Then $\tau_R(K)$ coincides with the relative topology τ_K on K.*

Proof of Theorem 3.14. By Theorem 3.5 and Theorem 3.1, the families $\mathcal{B}_R(K) = \{R_K(A) : \emptyset \neq A \subseteq K\}$ and $\mathcal{B}_K = \{R_H(A) \cap K : \emptyset \neq A \subseteq H\}$ are bases for $\tau_R(K)$ and the relative topology τ_K, respectively.

Let $U \in \mathcal{B}_K$ and let $x \in U$. Then there exists $A \subseteq H$ such that $U = R_H(A) \cap K$. Since $x \in R_H(x)$ and $x \in K$, $x \in R_H(x) \cap K = R_K(x)$ by Lemma 3.4. Since H is ordered, $R_K(x) \subseteq R_H(A) \cap K$, by Lemma 3.5. Take $U' = R_K(x)$. Then for each $U \in \mathcal{B}_K$ and $x \in U$, there exists $U' \in \mathcal{B}_R(K)$ such that $x \in U' \subseteq U$. Thus, $\tau_K \subseteq \tau_R(K)$.

On the other hand, let $U \in \mathcal{B}_R(K)$. Then there exists $B \subseteq K$ such that $U = R_K(B)$. By Lemma 3.4, $U = R_K(B) = R_H(B) \cap K \in \mathcal{B}_K$. Hence, $\mathcal{B}_R(K) \subseteq \mathcal{B}_K$, that is, $\tau_L(K) \subseteq \tau_K$. Consequently, $\tau_R(K) = \tau_K$. □

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A penalty-free algorithm for solving nonlinear equality constrained optimization

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Abstract. In contrast to standard penalty type methods for nonlinear optimization, penalty-free type methods do not have to determine penalty parameters and have promising numerical results. So they have attracted the attention of many researchers. Filter method is a classical penalty-free method. In this paper, a different filter technique is introduced for solving nonlinear equality constrained optimization. The optimality condition of the nonlinear optimization problem is regarded as a new filter pair which is built in the backtracking line search framework. Then, the trial step size is accepted if one of the two measures in filter is improved after the search direction is determined. Under some reasonable assumptions, the global convergence of the algorithm is proved. Some preliminary numerical results are presented to show the usefulness of the proposed algorithm.

Keywords: nonlinear programming, equality constrained optimization, line search, filter method, global convergence.

1. Introduction

Nonlinear constrained optimization is one of the most fundamental models in scientific calculation because of the wide use of optimization in science, engineer-

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ing, economics, industry and so on [20, 17]. Furthermore, nonlinear optimization is a basic building block of more complex design and optimization paradigms. Many impressive methods have been established to tackle nonlinear constrained optimization [20, 5, 16].

Line search and trust region are two common methods for constrained programming. In these frameworks, it has to balance the twin goals of reducing the objective function and satisfying the constraints. Merit functions are typical tools for achieving this balance. However, the penalty parameters in merit functions are not easy to be determined for good performance. Penalty-free methods use different techniques to handle the balance problem. Penalty-free methods include filter methods and filter-free methods.

Fletcher and Leyffer [7] originally proposed filter methods which can be embedded in line search or trust region framework and replace the traditional merit function to ensure global convergence for nonlinear programming. Wächter and Biegler [30, 29] gave a filter line search method and proved its global convergence and local convergence. Leyffer and Vanaret [17] presented an augmented Lagrangian filter method. Nonmonotone filter SQP methods were proposed in [11,25]. As for primal-dual interior-point algorithms with a filter line search method, their effectiveness has been proved in [31, 15]. Chen and Sun [3] give a dwindling filter line search method for unconstrained optimization. Some dwindling filter line search methods were also introduced in [1, 13] for constrained optimization. Gu and Zhu [12] proposed a line search filter method with reduced Hessian for nonlinear programming. Zhu and Pu [35] presented a line search filter algorithm with inexact step computations for equality constrained optimization. A line search filter secant method was proposed in [33]. Based on trust-region framework, Fletcher et al. [6] and Wächter et al. [32] propose trust-region SQP-filter algorithms for nonlinear programming. Pei and Zhu [21] propose some algorithms combining line search filter technique. Li and Zhu [18] propose and analyzed two interior trust-region methods with line search filter technique. Recently, Sun et al. [26] introduced a trust region sequential quadratic programming approach for nonlinear systems based on nonlinear model predictive control.

There are also many researchers who proposed filter-free methods. For example, Ulbrich [27] proposed a non-monotone trust region methods for nonlinear equality constrained optimization without a penalty function. Bielschowsky [2] presented a type of trust-region algorithm for solving nonlinear programming problems with equality constraints. Xue, Shen and Pu [34] proposed a penalty-function-free line search SQP method for nonlinear programming. Liu and Yuan [19] presented a sequential quadratic programming method without a penalty function or a filter for nonlinear equality constrained optimization. Qiu and Chen [23] presented a class of trust region algorithms without using a penalty function or a filter for nonlinear inequality constrained optimization and analyze their global and local convergence. Chen and Dai constructed a new filter-free method in [4].

In this paper, we introduce a line search filter method which aims to obtain the stationary point of nonlinear equality constrained optimization. The search direction is computed based on the null space method in each iteration. Then, the step size is determined by the filter mechanism and the backtracking line search. The proof of global convergence is given. The numerical results show that our algorithm has achieved good results for some problems.

The most important difference from other filter methods is that the composition of the filter. Most of filter methods employ the objective function (or the Lagrange function) and the violation of the constraints as the filter pair, while the first-order optimality conditions of the nonlinear programming are used as the filter pair in this paper. So, it is not required to compute the value of objective function (or the Lagrange function) in every iteration to obtain a first order optimal point, while the objective function (or the Lagrange function) has to be evaluated in other filter methods. This is the main feature of the proposed filter method.

The outline of this paper is as follows. In section 2, we state the line search filter approach. The global convergence of the algorithm is proved in section 3. Finally, we report some numerical experiments in section 4.

The notation of this paper is as follows. Norms $\|\cdot\|$ denote a fixed vector norm and its compatible matrix norm unless otherwise noted. For brevity, we use the convention $(x, y) = (x^T, y^T)^T$ for vectors x, y. Given two vectors $v, w \in \mathbb{R}^n$, we define the convex segment $[v, w] := \{v + t(w - v) : t \in [0, 1]\}$. Finally, we denote by $O(t_k)$ a sequence $\{v_k\}$ satisfying $\|v_k\| \leq \beta t_k$ for some constant $\beta > 0$ independent of k, and by $o(t_k)$ a sequence $\{v_k\}$ satisfying $\|v_k\| \leq \beta_k t_k$ for some positive sequence $\{\beta_k\}$ with $\lim_k \beta_k = 0$.

2. A line search filter approach.

Consider the following nonlinear programming (NLP)

$$\text{(1a)} \qquad \underset{x}{\text{minimize}} \quad f(x)$$

$$\text{(1b)} \qquad \text{subject to} \quad c(x) = 0,$$

where the objective function $f(x) : \mathbb{R}^n \to \mathbb{R}$ and the equality constraints $c(x) : \mathbb{R}^n \to \mathbb{R}^m$ with $m < n$ are sufficiently smooth.

The Karush-Kuhn-Tucker (KKT) conditions for the NLP (1) are

$$\text{(2a)} \qquad g(x) - A(x)y = 0,$$

$$\text{(2b)} \qquad c(x) = 0,$$

where we denote with $A(x) := \nabla c(x)$ the transpose of the Jacobian of the constraints c, and with $g(x) := \nabla f(x)$ the gradient of the objective function. The vector y corresponds to the Lagrange multipliers for the equality constraints (1b). Under certain constraint qualifications, such as linear independence of the

constraint gradients, the KKT conditions are the first order optimality conditions for NLP (1).

Given an initial estimate x_0, the proposed algorithm of this paper generates a sequence of improved estimates x_k of the solution for the NLP(1). In every iteration k, the next iterate point can be determined if the search direction p_k and step size α_k have worked out, i.e, $x_{k+1} := x_k + \alpha_k p_k$, where $\alpha_k \in (0, 1]$.

As for the search direction, we have following system (see [20]):

$$\begin{bmatrix} H_k & -A_k \\ A_k^T & 0 \end{bmatrix} \begin{pmatrix} p_k \\ y_{k+1} \end{pmatrix} = - \begin{pmatrix} g_k \\ c_k \end{pmatrix}. \tag{3}$$

Here, $A_k := A(x_k)$, $g_k := g(x_k)$, and $c_k := c(x_k)$. The symmetric matrix H_k denotes the Hessian $\nabla^2_{xx} L(x_k, y_{k+1})$ of the Lagrangian

$$L(x, y) := f(x) - y^T c(x)$$

of the NLP (1), or an approximation to this Hessian. Here, we require that

$$\left(\nabla^2_{xx} L(x_k, y_{k+1}) - H_k\right) d = O(\|d\|^2). \tag{4}$$

The vector y_{k+1} is some estimate of the optimal multipliers corresponding to the equality constrains (1b), and its updating formula will be given later (see (6)).

Assume that the projection of the Hessian approximation H_k onto the null space of A_k^T is uniformly positive definite. In other words, there exists a constant λ_H so that

$$\lambda_{\min}(N_k^T H_k N_k) > \lambda_H,$$

where $\lambda_{\min}(N_k^T H_k N_k)$ is the smallest eigenvalue of matrix $N_k^T H_k N_k$ and the columns of matrices $N_k \in \mathbb{R}^{n\times(n-m)}$ form an orthonormal basis matrix of the null space of A_k^T.

Using the methods in [20], p_k can be decomposed into

$$p_k = Y_k p_Y + N_k p_N, \tag{5}$$

where columns of $Y_k \in \mathbb{R}^{n\times m}$ span the range space of A_k.

Substituting (5) into (3), the following system can be obtained to solve p_Y and p_N:

$$(A_k^T Y_k) p_Y = -c_k,$$

$$(N_k^T H_k N_k) p_N = -N_k^T H_k Y_k p_Y - N_k^T g_k.$$

We can choose $Y_k = A_k$, which is valid for Y_k when A_k^T has full row rank. Then a multiplier updating formula is obtained as follows:

$$y_{k+1} = [A_k^T A_k]^{-1} A_k^T (H_k p_k + g_k). \tag{6}$$

Along this direction p_k, the largest step size should be found to satisfy descent condition. Finally, the sequence $\{x_k\}$ of iterates converge to a stationary

point of the NLP(1). In this paper we consider a backtracking line search procedure to compute the largest step size. A decreasing sequence of step sizes $\alpha_{k,l} \in (0,1](l = 0,1,2,\ldots)$ is tried until some acceptance criterion is satisfied. In the remainder of this section we describe how the filter acceptance criterion is applied to the line search framework.

According to the first-order optimality condition, the filter pair is defined as $(\theta(x), \omega(x,y))$, where

$$\theta(x) = \|c(x)\|,$$

$$\omega(x,y) = \frac{1}{2}\|\nabla_x L(x,y)\|^2 = \frac{1}{2}\|g(x) - A(x)y\|^2.$$

The underlying idea is to interpret the NLP(1) as a biobjective optimization problem with two goals: minimizing the constraint violation $\theta(x)$ and minimizing the first order criticality measure $\omega(x,y)$ until they tend to zero. In the filter mechanism, the constraint violation measure is more important, because the optimal solution of NLP problem must be feasible. For both measurements, the trail point $x_k(\alpha_{k,l})$ can be accepted if it reduces either of the filter pair, i.e., if $\theta(x_k(\alpha_{k,l})) < \theta(x_k)$ or $\omega(x_k(\alpha_{k,l}), y_{k+1}) < \omega(x_k, y_{k+1})$. For convenience, we denote $\omega(x_k) = \omega(x_k, y_{k+1})$. Note that this criterion is not very demanding and therefore generally allows for a larger step size. However, this simple concept is not enough to ensure global convergence. Based on previous experience (see [6] and [8]), we have following descent criterion.

A trial step size $\alpha_{k,l}$ provides sufficient reduction with respect to the current iterate x_k, if

$$\theta(x_k(\alpha_{k,l})) \le (1 - \gamma_\theta)\theta(x_k) \tag{7a}$$

or

$$\omega(x_k(\alpha_{k,l})) \le \omega(x_k) - \gamma_w \theta(x_k) \tag{7b}$$

holds for fixed constants $\gamma_\theta, \gamma_\omega \in (0,1)$. In a practical implementation, the constants $\gamma_\theta, \gamma_\omega$ typically are chosen to be small. Sometimes, this descent criterion may cause the algorithm to converge to a feasible but not an optimal point. In order to prevent this, the following switching conditions should be given:

$$m_k(\alpha_{k,l}) < 0 \quad \text{and} \quad [-m_k(\alpha_{k,l})]^\tau [\alpha_{k,l}]^{1-\tau} > \delta[\theta(x_k)]^\phi \tag{8}$$

with fixed constants $\delta > 0, \phi > 1, \tau \ge 1$, where

$$m_k(\alpha) \overset{\text{def}}{=} \alpha(g_k - A_k y_{k+1})^T H_k p_k. \tag{9}$$

If the condition (8) holds, the step p_k is a descent direction. In order to make the current step size $\alpha_{k,l}$ acceptable, we require that $\alpha_{k,l}$ satisfies the Armijo-type condition

$$\omega(x_k(\alpha_{k,l})) \le \omega(x_k) + \eta_\omega m_k(\alpha_{k,l}). \tag{10}$$

Here, $\eta_\omega \in \left(0, \frac{1}{2}\right)$ is a fixed constant. If this Armijo-type condition (10) is not met, aiming to be accepted, we should decrease the step size and find the next trial point. But if the switching condition (8) does not hold, we can take the method revert to the acceptance criterion (7).

As we have known, the switching condition ensures that for the optimality enforced by the Armijo condition (10) is sufficiently large compared to the current constraint violation. Therefore, it prevents a trial point from making only a small progress when it is far away from the feasible region.

For two measures θ and ω, a trial point may improve one and worsen the other one. It happens between two such trial points. For preventing this cycle (like methods in [7, 30]), a filter is defined as a set $\mathcal{F}_k \subseteq [0,\infty) \times \mathbb{R}$ containing all $(\theta, \omega)-$ pairs that are prohibited in iteration k. We say that a trial point $x_k(\alpha_{k,l})$ is acceptable to the current filter $\mathcal{F}_k$ if

$$(\theta(x_k(\alpha_{k,l})), \omega(x_k(\alpha_{k,l}))) \notin \mathcal{F}_k. \tag{11}$$

During the algorithm we require that the current iterate x_k is always acceptable to the current filter $\mathcal{F}_k$. After some new iterates x_{k+1} have been accepted, the current filter can be augmented by

$$\mathcal{F}_{k+1} := \mathcal{F}_k \cup \left\{(\theta, \omega) \in \mathbb{R}^2 : \theta \geq (1-\gamma_\theta)\theta(x_k) \quad \text{and} \quad \omega \geq \omega(x_k) - \gamma_\omega \theta(x_k)\right\}. \tag{12}$$

If the filter is not augmented, it remains unchanged, i.e., $\mathcal{F}_{k+1} := \mathcal{F}_k$. Then $\mathcal{F}_k \subseteq \mathcal{F}_{k+1}$ holds for all k. In algorithm, if a trial point $x_k(\alpha_{k,l})$ does not satisfy the switching condition (8) but satisfies acceptance criterion (7), we need use (12) to augment filter. If a trial point makes the switching condition (8) hold and then makes the Armijo-type condition (10) hold, the filter remains unchanged. If the filter has been augmented in iteration k, x_k ensures to provide sufficient reduction for one of the measures.

The computed trial step $\alpha_{k,l}$ may be smaller than

$$\alpha_k^{\min} := \gamma_\alpha \begin{cases} \min\left\{\gamma_\theta, \frac{\gamma_\omega \theta(x_k)}{-(g_k - A_k y_{k+1})^T H_k p_k}, \frac{\delta[\theta(x_k)]^\phi}{[-(g_k - A_k y_{k+1})^T H_k p_k]^\tau}\right\}, \\ \gamma_\theta, \\ \text{if} \quad (g_k - A_k y_{k+1})^T H_k p_k < 0, \\ \text{otherwise.} \end{cases} \tag{13}$$

where $\gamma_\alpha \in (0, 1]$. Then there is no admissible step size can be found. So the algorithm switches to a feasibility restoration phase, whose purpose is to find a new iterate x_{k+1} that satisfies (7) and is also acceptable to the current filter by trying to decrease the constraint violation. Our approach is to compute a new iterate x_{k+1} by decreasing the infeasibility measure θ such that x_{k+1} satisfies the sufficient decrease conditions (7) and is acceptable to the filter, i.e., $(\theta(x_{k+1}), \omega(x_{k+1})) \notin \mathcal{F}_k$. This process can be called the feasibility restoration phase. In (13), γ_α is a safety factor. It is useful to avoid invoking the feasibility restoration phase unnecessarily in a practical implementation.

We are now ready to state the overall algorithm for solving the NLP (1) formally.

Algorithm I

Given

Set constants $\theta_{\max} \in (\theta(x_0), \infty]\,; \gamma_\theta, \gamma_\omega \in (0,1); \delta > 0; \gamma_\alpha \in (0,1]; \phi > 1; \tau \geq 1; \eta_\omega \in \left(0, \frac{1}{2}\right); 0 < \tau_1 \leq \tau_2 < 1$.

Initialize

Initialize the starting point x_0, the initial filter $\mathcal{F}_0{:=}\{(\theta,\omega){\in}\mathbb{R}^2 : \theta \geq \theta_{\max}\}$, the multiplier y_0 and the iteration counter $k = 0$.

Loop until x_k satisfies the KKT conditions (2) for some $y_{k+1} \in \mathbb{R}^m$

Compute the search direction p_k from (5);

Compute multiplier y_{k+1} with (6).

Set $\alpha_{k,0} = 1$ and $l = 0$.

Loop until find $x_k(\alpha_{k,l})$ is accepted by current filter $\mathcal{F}_k$

if $\alpha_{k,l} < \alpha_k^{\min}$ with $\alpha_k^{\min}$ defined by (13), go to the feasibility restoration phase;

else compute the new trial point $x_k(\alpha_{k,l}) = x_k + \alpha_{k,l} p_k$;

if $x_k(\alpha_{k,l})$ is acceptable to the current filter $\mathcal{F}_k$, **break**.

else choose $\alpha_{k,l+1} \in [\tau_1 \alpha_{k,l}, \tau_2 \alpha_{k,l}]$, set $l = l + 1$.

end

end

end

If (8) holds,

if (10) holds, set $\alpha_k := \alpha_{k,l}$ and $x_{k+1} := x_k(\alpha_k)$, $\mathcal{F}_{k+1} = \mathcal{F}_k$;

else choose $\alpha_{k,l+1} \in [\tau_1 \alpha_{k,l}, \tau_2 \alpha_{k,l}]$, set $l = l + 1$ and go to the inner "Loop";

end

else

if (7) holds, set $\alpha_k := \alpha_{k,l}$, $x_{k+1} := x_k(\alpha_k)$, and augment the filter using (12);

else choose $\alpha_{k,l+1} \in [\tau_1 \alpha_{k,l}, \tau_2 \alpha_{k,l}]$, set $l = l + 1$ and go to the inner "Loop".

end

end if
Set $k = k + 1$;

End

Feasibility restoration phase

Compute a new iterate x_{k+1}. Augment the filter using (12) (for x_k) and increase the iteration counter $k = k + 1$. Go to the outer "Loop".

Lemma 2.1. *Algorithm I is well defined.*

Proof. In the inner "Loop", it is clear that $\lim_l \alpha_{k,l} = 0$. If $\theta(x_k) > 0$, it can be seen from (13) that $\alpha_k^{\min} > 0$. So, the algorithm either accepts a new iterate or switches to the feasibility restoration phase. On the other hand, if $\theta(x_k) = 0$ and the algorithm does not stop at a KKT point, then $(g_k - A_k y_{k+1})^T H_k p_k < 0$ (see, Lemma 3.3 below). In that case, $\alpha_k^{\min} = 0$, and the Armijo condition (10) is satisfied, i.e., a new iterate is accepted. □

3. Global convergence

In the remainder of this paper we denote some index sets.

$\mathcal{Z} \subseteq \mathbb{N}$: the filter is augmented in the corresponding iterations, i.e.,

$$\mathcal{F}_k \subsetneqq \mathcal{F}_{k+1} \quad \Leftrightarrow \quad k \in \mathcal{Z}.$$

$\mathcal{R} \subseteq \mathcal{Z}$: the feasibility restoration phase is invoked in the corresponding iterations.

As is common for most line search methods, some reasonable assumptions are stated down below, which is necessary for the global convergence analysis of Algorithm I.

Assumptions G

Let $\{x_k\}$ be the sequence generated by Algorithm I. Assume that the feasibility restoration phase always terminates successfully.

(G1) There exists an open set $\mathcal{C} \subseteq \mathbb{R}^n$ with $[x_k, x_k + p_k] \subseteq \mathcal{C}$, for all k such that f and c are twice continuously differentiable over $\mathcal{C}$.

(G2) The matrices H_k approximating the Hessian of the Lagrangian in (3) are uniformly bounded for all k, that is, there exists a constant $M_H > 0$ so that

$$\|H_k\| \leq M_H$$

for all k.

(G3) There exists a constant $\lambda_H > 0$ such that

$$\lambda_{\min}(N_k^T H_k N_k) \geq \lambda_H, \tag{14}$$

where $\lambda_{\min}(N_k^T H_k N_k)$ is the smallest eigenvalue of matrix $N_k^T H_k N_k$.

(G4) There exists a constant $M_A > 0$ such that

$$\sigma_{\min}(A_k) \geq M_A, \tag{15}$$

where $\sigma_{\min}(A_k)$ is the smallest singular value of A_k.

In every iteration, it is possible to make sure that (14) is valid by monitoring and possibly modifying the eigenvalues of the reduced Hessian of (3) (see [28]). Similarly, we can guarantee that the entire sequence $\{H_k\}$ is uniformly bounded.

In the convergence analysis of the filter method, we employ

$$\chi(x_k) := \|g_k - A_k y_{k+1}\|_2 \tag{16}$$

as the criticality measure. If $\theta_k \to 0$ and $\chi(x_{k_i}) \to 0$, then there exists a y_* such that the KKT conditions (2) are satisfied for (x_*, y_*).

Let us prove the global convergence of Algorithm I in detail.

Firstly, we give some preliminary results.

Lemma 3.1. *Suppose Assumptions G hold. Then there exist constants M_p, M_y, $M_m > 0$, such that*

$$\|p_k\| \leq M_p, \quad \|y_{k+1}\| \leq M_y, \quad |m_k(\alpha)| \leq M_m \alpha,$$

for all k and $\alpha \in (0, 1]$.

Proof. From (G1) we have that the right-hand side of (3) is uniformly bounded. Additionally, Assumptions (G2), (G3), and (G4) guarantee that the inverse of the matrix in (3) exists and is uniformly bounded for all k. Consequently, the solution of (3) $(p_k, y_{k+1})^T$ is uniformly bounded, i.e., $\|p_k\| \leq M_p$ and $\|y_{k+1}\| \leq M_y$. From (9), we can get $m_k(\alpha)/\alpha = (g_k - A_k y_{k+1})^T H_k p_k$. So, $|m_k(\alpha)| \leq M_m \alpha$. □

Lemma 3.2. *Suppose Assumption (G1) holds. Then there exist constants $C_\theta, C_\omega > 0$ such that for all k and $\alpha \in (0, 1]$*

$$|\theta(x_k + \alpha p_k) - (1 - \alpha)\theta(x_k)| \leq C_\theta \alpha^2 \|p_k\|^2, \tag{17a}$$

$$|\omega(x_k + \alpha p_k) - \omega(x_k) - m_k(\alpha)| \leq C_\omega \alpha^2 \|p_k\|^2. \tag{17b}$$

Proof. The proof of (17a) can be found in Lemma 3.3 from [22].

Next, we give the proof of (17b). From second order Taylor expansions,

$$\begin{aligned}
& |\omega(x_k+\alpha p_k)-\omega(x_k)-m_k(\alpha)| \\
\overset{(9)}{=} & \left|\frac{1}{2}\|\nabla_x L(x_k+\alpha p_k, y_{k+1})\|^2-\frac{1}{2}\|\nabla_x L(x_k, y_{k+1})\|^2\right. \\
& \left.-\alpha(g_k-A_k y_{k+1})^T H_k p_k\right| \\
= & \left|\frac{1}{2}\nabla_x L(x_k+\alpha p_k, y_{k+1})^T\nabla_x L(x_k+\alpha p_k, y_{k+1})\right. \\
& \left.-\alpha(g_k-A_k y_{k+1})^T H_k p_k-\frac{1}{2}\nabla_x L(x_k, y_{k+1})^T\nabla_x L(x_k, y_{k+1})\right| \\
\leq & \left|\frac{1}{2}\nabla_x L(x_k, y_{k+1})^T\nabla_x L(x_k, y_{k+1})+\alpha\nabla_x L(x_k, y_{k+1})^T\nabla^2_{xx}L(x_k, y_{k+1})p_k\right. \\
& \left.+o(\alpha^2\|p_k\|^2)-\frac{1}{2}\nabla_x L(x_k, y_{k+1})^T\nabla_x L(x_k, y_{k+1})-\alpha(g_k-A_k y_{k+1})^T H_k p_k\right| \\
= & \left|\alpha(g_k-A_k y_{k+1})^T(\nabla^2_{xx}L(x_k, y_{k+1})-H_k)p_k+o(\alpha^2\|p_k\|^2)\right| \\
\overset{(4)}{=} & \left|O(\alpha^2\|p_k\|^2)+o(\alpha^2\|p_k\|^2)\right| \\
\leq & C_\omega\alpha^2\|p_k\|^2.
\end{aligned}$$

So, the conclusion is proved. □

Lemma 3.3. *Suppose Assumptions G hold. If $\{x_{k_i}\}$ is a subsequence of iterates for which $\chi(x_{k_i})\geq\epsilon_1$ with constants $\epsilon_1>0$ independent of i. Then there exists $\epsilon_2>0$ independent of i, we have*

$$\theta(x_{k_i})=0 \quad\Longrightarrow\quad m_{k_i}(\alpha)<-\alpha\epsilon_2<0, \tag{18}$$

for all i and $\alpha\in(0,1]$. Then,

$$\Theta_k\overset{\text{def}}{=}\min\{\theta:(\theta,\omega)\in\mathcal{F}_k\}>0, \tag{19}$$

for all k and $\alpha\in(0,1]$.

Proof. If $\theta(x_k)=0$, but $\chi(x_k)>0$, the Algorithm I would not terminate. From (3) and (16),

$$m_k(\alpha)/\alpha=(g_k-A_k y_{k+1})^T H_k p_k=-\|g_k-A_k y_{k+1}\|_2^2=-\chi_k^2<-\epsilon_1^2,$$

so

$$m_k(\alpha)<-\alpha\epsilon_1^2=-\alpha\epsilon_2<0,$$

where $\epsilon_2=\epsilon_1^2$, i.e., (18) holds.

The proof of (19) is by induction. From initialize of Algorithm I, it is clear that the statement is valid for $k = 0$ because $\theta_{\max} > 0$. Suppose the statement is true for k. If $\theta(x_k) > 0$ and the filter is augmented in iteration k, it is clear from the update rule (12) that $\Theta_{k+1} > 0$, since $\gamma_\theta \in (0, 1)$. On the other hand, if $\theta(x_k) = 0$, we have got $m_k(\alpha) < 0$, for all $\alpha \in (0, 1]$. So, the switching condition (8) is true for all trial step sizes. Therefore, algorithm always makes α_k have been accepted then it must have that α_k satisfies (10). Consequently, the filter is not augmented. Hence, $\Theta_{k+1} = \Theta_k > 0$. □

Under Assumptions G, the sequence $\theta(x_k)$ converges to zero can be proved. That is to say, all limit points of $\{x_k\}$ are feasible (see, Lemma 3.4, Lemma 3.5 and Theorem 3.1). Consider from whether the filter is augmented a finite number of times or not.

Lemma 3.4. *Suppose that Assumptions G hold. If the filter is augmented only a finite number of times, i.e., $|\mathcal{Z}| < \infty$. Then*

$$\lim_{k\to\infty} \theta(x_k) = 0. \tag{20}$$

Proof. Choose K such that for all iterations $k \geq K$ the filter is not augmented in iteration k. From the filter augmenting in Algorithm I we then have that for all $k \geq K$ both conditions (8) and (10) are satisfied for α_k. We now distinguish two cases, where $k \notin \mathcal{Z}$.

Case 1 ($\tau > 1$). From (8) it follows with M_m from Lemma 3.1 that

$$\delta[\theta(x_k)]^\phi < [-m_k(\alpha_k)]^\tau [\alpha_k]^{1-\tau} \leq M_m^\tau \alpha_k.$$

Hence, we can get

$$(\alpha_k)^{\tau-1} > \delta^{\tau-1}[\theta(x_k)]^{\phi(\tau-1)} M_m^{\tau(1-\tau)}$$

and

$$-m_k(\alpha_k) > \delta[\theta(x_k)]^\phi M_m^{1-\tau}.$$

This implies

$$\begin{aligned} \omega(x_k) - \omega(x_k(\alpha_k)) &\geq \eta_\omega(-m_k(\alpha_k)) \\ &> \eta_\omega \delta[\theta(x_k)]^\phi M_m^{1-\tau} \\ &= c_1[\theta(x_k)]^\phi \end{aligned}$$

with $c_1 = \eta_\omega \delta M_m^{1-\tau}$.

Case 2 ($\tau = 1$). From (8) we have $\delta[\theta(x_k)]^\phi < -m_k(\alpha_k)$ such that from (10) we immediately obtain

$$\begin{aligned} \omega(x_k) - \omega(x_k(\alpha_k)) &\geq \eta_\omega(-m_k(\alpha_k)) \\ &> \eta_\omega \delta[\theta(x_k)]^\phi. \end{aligned}$$

In either case, we have for all $k \notin \mathcal{Z}$ that

$$\omega(x_k(\alpha_k)) - \omega(x_k) < -\tilde{c}[\theta(x_k)]^{\phi} \tag{21}$$

for some $\tilde{c} > 0$. Hence, for all $i = 1, 2, ...,$

$$\begin{aligned}\omega(x_{K+i}) &= \omega(x_K) + \sum_{k=K}^{K+i-1} (\omega(x_{k+1}) - \omega(x_k)) \\ &< \omega(x_k) - \tilde{c} \sum_{k=K}^{K+i-1} [\theta(x_k)]^{\phi}.\end{aligned}$$

Since $\omega(x_{K+i})$ is bounded below as $i \to \infty$, the series on the right-hand side in the last line is bounded, which in turn implies (20). □

The following lemma considers a subsequence $\{x_{k_i}\}$ with $k_i \in \mathcal{Z}$, for all i, and its proof is borrowed from the method of [6].

Lemma 3.5. *Let $\{x_{k_i}\}$ be a subsequence of iterates generated by Algorithm I such that the filter is augmented in iteration k_i, i.e., $k_i \in \mathcal{Z}$, for all i. Furthermore, assume that there exist constants $c_\omega \in \mathbb{R}$ and $C_\theta > 0$ such that*

$$\omega(x_{k_i}) \geq c_\omega \quad \textit{and} \quad \theta(x_{k_i}) \leq C_\theta,$$

for all i (for example, if Assumption (G1) holds). It then follows that

$$\lim_{i \to \infty} \theta(x_{k_i}) = 0.$$

Proof. Suppose, for the purpose of obtaining a contradiction, that there exists an infinite subsequence $\{k_i\} \subseteq \mathcal{Z}$ such that

$$\theta_{k_i} \geq \epsilon, \tag{22}$$

for all i and for some $\epsilon > 0$. At each iteration k_i, the $(\theta, \omega) - pair$ associated with x_{k_i}, that is $(\theta_{k_i}, \omega_{k_i})$, is added to the filter. This means that no other (θ, ω)-pair can be added to the filter at a later stage within the square

$$[\theta_{k_i} - \gamma_\theta \epsilon, \theta_{k_i}] \times [\omega_{k_i} - \gamma_\omega \epsilon, \omega_{k_i}]$$

or with the intersection of this square with $\mathcal{F}_0$. Note that this holds, even if $(\theta_{k_i}, \omega_{k_i})$ is later removed from the filter, since the rule for removing entries, ensures that the envelope never shrinks. Now observe that the area of each of theses squares is $\gamma_\theta \gamma_\omega \epsilon^2$. As a consequence, the set $\mathcal{F}_0 \bigcap \{(\theta, \omega) | \omega \leq \kappa_\omega\}$ is completely covered by at most a finite number of such squares, for any choice of $\kappa_\omega \geq c_\omega$. Since the pairs $(\theta_{k_i}, \omega_{k_i})$ keep on being added to the filter, this implies that ω_{k_i} tends to infinity when i tends to infinity. Let us assume, without loss

of generality, that $\omega_{k_{i+1}} \geq \omega_{k_i}$, for all i sufficiently large. But (7) and (22) imply that

$$\theta_{k_{i+1}} \leq (1-\gamma_\theta)\theta_{k_i} \leq \theta_{k_i} - \gamma_\theta \epsilon,$$

and therefore that θ_{k_i} converges to zero, which contradicts (22). Hence this latter assumption is impossible and the conclusion follows. □

Based on the previous two lemmas, we can prove the following theorem similar to the proof in [21].

Theorem 3.1. *Suppose Assumptions G hold. Then*

$$\lim_{k\to\infty} \theta(x_k) = 0. \tag{23}$$

The next lemma shows that the Armijo condition (10) is satisfied under certain condition that there exists a step length bounded away from zero.

Lemma 3.6. *Suppose Assumptions G hold. Let $\{x_{k_i}\}$ be a subsequence. There exists certain constant $\bar{\alpha} > 0$ such that for all k_i and $\alpha \leq \bar{\alpha}$*

$$\omega(x_{k_i} + \alpha p_{k_i}) - \omega(x_{k_i}) \leq \eta_\omega m_{k_i}(\alpha). \tag{24}$$

Proof. Let M_p and C_ω be the constants from Lemmas 3.1 and 3.2. It then follows for all $\alpha \leq \bar{\alpha}$ with $\bar{\alpha} := \frac{(1-\eta_\omega)\epsilon_2}{C_\omega M_p^2}$ and (18) that

$$\begin{aligned} & \omega(x_{k_i} + \alpha p_{k_i}) - \omega(x_{k_i}) - m_{k_i}(\alpha) \\ \leq\ & C_\omega \alpha^2 \|p_{k_i}\|^2 \\ \leq\ & \alpha(1-\eta_\omega)\epsilon_2 \\ \leq\ & -(1-\eta_\omega) m_{k_i}(\alpha), \end{aligned}$$

which implies (24). □

The previous Theorem 3.1 has proved that a series of points generated by the Algorithm I make the constraint violation tend to zero. Next, we prove that Assumptions G guarantee that the optimality measure $\chi(x_k)$ is not bounded away from zero, i.e., if $\{x_k\}$ is bounded, at least one limit point is a first order optimal point for the NPL (1).

Lemma 3.7. *Suppose that Assumptions G hold and that the filter is augmented only a finite number of times, i.e., $|\mathcal{Z}| < \infty$. Then*

$$\lim_{k\to\infty} \chi(x_k) = 0.$$

Proof. Since $|\mathcal{Z}| < \infty$, there exists $K \in \mathbb{N}$ such that $k \notin \mathcal{Z}$, for all $k \geq K$. Suppose the claim is not true, i.e., there exist a subsequence $\{x_{k_i}\}$ and a constant

$\epsilon > 0$ such that $\chi(x_{k_i}) \geq \epsilon$, for all i. From (10), (18) and (23), there exist $\tilde{K} \geq K$, for $k_i \geq \tilde{K}$

$$\omega(x_{k_i+1}) - \omega(x_{k_i}) \leq \eta_\omega m_{k_i}(\alpha_{k_i}) \leq -\alpha_{k_i}\eta_\omega\epsilon_2.$$

Reasoning as in the proof of Lemma 3.4, one can conclude that $\lim_i \alpha_{k_i} = 0$, since $\omega(x_{k_i})$ is bounded below and $\omega(x_k)$ is monotonically decreasing (from (21)) for all $k \geq \tilde{K}$. We can now assume without loss of generality that $\tilde{K}$ is sufficiently large such that $\alpha_{k_i} < 1$. This means that for $k_i \geq \tilde{K}$ the first trial step $\alpha_{k,0} = 1$ has not been accepted. The last rejected trial step size

$$\alpha_{k_i,l_i} \in [\alpha_{k_i}/\tau_2, \alpha_{k_i}/\tau_1] \tag{25}$$

during the backtracking line search procedure then satisfies (8) since $k_i \notin \mathcal{Z}$ and $\alpha_{k_i,l_i} > \alpha_{k_i}$. Thus, it must have been rejected because it violates (10), i.e., it satisfies

$$\omega(x_{k_i} + \alpha_{k_i,l_i}p_{k_i}) - \omega(x_{k_i}) > \eta_\omega m_{k_i}(\alpha_{k_i,l_i}), \tag{26}$$

or it has been rejected because it is not acceptable to the current filter, i.e.,

$$(\theta(x_{k_i} + \alpha_{k_i,l_i}p_{k_i}), \omega(x_{k_i} + \alpha_{k_i,l_i}p_{k_i})) \in \mathcal{F}_{k_i} = \mathcal{F}_K. \tag{27}$$

We conclude the proof by showing that neither (26) nor (27) can be true for sufficiently large k_i.

As for (26), since $\lim_i \alpha_{k_i} = 0$, so $\lim_i \alpha_{k_i,l_i} = 0$ (see (25)). In particular, for sufficiently large k_i, $\alpha_{k_i,l_i} \leq \bar{\alpha}$ with $\bar{\alpha}$ from Lemma 3.6, i.e., (26) can not be satisfied for those k_i.

As for (27), from Lemma 3.3, $\Theta_K > 0$. Using Lemma 3.1 and Lemma 3.2, we see that

$$\theta(x_{k_i} + \alpha_{k_i,l_i}p_{k_i}) \leq (1 - \alpha_{k_i,l_i})\theta(x_{k,i}) + C_\theta M_p^2(\alpha_{k_i,l_i})^2.$$

Since $\lim_i \alpha_{k_i,l_i} = 0$ and from Theorem 3.1 also $\lim_i \theta(x_{k_i}) = 0$, it follows that $\theta(x_{k_i} + \alpha_{k_i,l_i}p_{k_i}) < \Theta_K$ for k_i sufficiently large, which contradicts (27). □

In order to find the acceptable step size of the current filter (see (11)), we establish a bound of step size as follows.

Lemma 3.8. *Suppose Assumptions G hold. Let $\{x_{k_i}\}$ be a subsequence and $m_{k_i}(\alpha) \leq -\alpha\epsilon_2$ for a constant $\epsilon_2 > 0$ independent of k_i and for all $\alpha \in (0, 1]$. Then there exist constants $c_2, c_3 > 0$ such that*

$$(\theta(x_{k_i} + \alpha p_{k_i}), \omega(x_{k_i} + \alpha p_{k_i})) \notin \mathcal{F}_{k_i},$$

for all k_i and $\alpha \leq \min\{c_2, c_3\theta(x_{k_i})\}$.

Proof. Let M_p, C_θ, and C_ω be the constants from Lemma 3.1 and Lemma 3.2. Define $c_2 := \min\{1, \epsilon_2/(M_p^2 C_\omega)\}$ and $c_3 := 1/(M_p^2 C_\theta)$.

Now choose an iterate x_{k_i}. The mechanisms of Algorithm I ensure that

$$(\theta(x_{k_i}), \omega(x_{k_i})) \notin \mathcal{F}_{k_i}.$$

For $\alpha \leq c_2$ we have $\alpha^2 \leq \frac{\alpha \epsilon_2}{M_p^2 C_\omega} \leq \frac{-m_{k_i}(\alpha)}{C_\omega \|p_{k_i}\|^2}$ or, equivalently,

$$m_{k_i}(\alpha) + C_\omega \alpha^2 \|p_{k_i}\|^2 \leq 0$$

and it follows with (17b) that

$$\omega(x_{k_i} + \alpha p_{k_i}) \leq \omega(x_{k_i}).$$

Similarly, for $\alpha \leq c_3 \theta(x_{k_i}) \leq \frac{\theta(x_{k_i})}{\|p_{k_i}\|^2 C_\theta}$, we have $-\alpha\theta(x_{k_i}) + C_\theta \alpha^2 \|p_{k_i}\|^2 \leq 0$ and thus from (17a)

$$\theta(x_{k_i} + \alpha p_{k_i}) \leq \theta(x_{k_i}).$$

The update rule (12) imply that for all k, the filter has the following property:

$$(\bar{\theta}, \bar{\omega}) \notin \mathcal{F}_k \;\Rightarrow\; (\theta, \omega) \notin \mathcal{F}_k \qquad \text{if} \qquad \theta \leq \bar{\theta} \;\text{ and }\; \omega \leq \bar{\omega}.$$

Then we obtain the conclusion. □

The last lemma explains that the filter is eventually not augmented when the iteration corresponds to a subsequence with only nonoptimal limit points. This result is used in the proof of the main global convergence theorem to yield a contradiction.

Lemma 3.9. *Suppose Assumption G hold. Let $\{x_{k_i}\}$ be a subsequence of $\{x_k\}$ with $\chi(x_{k_i}) \geq \epsilon$ for a constant $\epsilon > 0$ independent of k_i. The there exists $K \in \mathbb{N}$ such that for all $k_i \geq K$ the filter is not augmented in iteration k_i, i.e., $k_i \notin \mathcal{Z}$.*

Proof. Since by Theorem 3.1 we have $\lim_i \theta(x_{k_i}) = 0$, it follows from Lemma 3.3 that there exist constants $\epsilon_1, \epsilon_2 > 0$ such that

$$\theta(x_{k_i}) \leq \epsilon_1 \quad \textit{and} \quad m_{k_i}(\alpha) \leq -\epsilon_2 \alpha \tag{28}$$

for k_i sufficiently large and $\alpha \in (0, 1]$. Without loss of generality we can assume that (28) is valid for all k_i. We can now apply Lemma 3.6 and Lemma 3.8 to obtain the constants $\bar{\alpha}, c_2, c_3 > 0$. Choose $K > \mathbb{N}$ such that for all $k_i \geq K$

$$\theta(x_{k_i}) < \min\left\{\theta_{\text{inc}}, \frac{\bar{\alpha}}{c_3}, \frac{c_2}{c_3}, \left[\frac{\tau_1 c_3 \epsilon_2^\tau}{\delta}\right]^{\frac{1}{\phi - 1}}\right\} \tag{29}$$

with τ_1 from Algorithm I. For all $k_i \geq K$ with $\theta(x_{k_i}) = 0$ we can argue as in the proof of Lemma 3 that both (8) and (10) hold in iteration k_i such that $k_i \notin \mathcal{Z}$.

For the remaining iterations $k_i \geq K$ with $\theta(x_{k_i}) > 0$ we note that (29) implies that

$$\frac{\delta[\theta(x_{k_i})]^\phi}{\epsilon_2^\tau} < \tau_1 c_3 \theta(x_{k_i})$$

(since $\phi > 1$), as well as

$$c_3\theta(x_{k_i}) < \min\{\bar{\alpha}, c_2\}.$$

Now choose an arbitrary $k_i \geq K$ with $\theta(x_{k_i}) > 0$ and define

$$\beta_{k_i} = c_3\theta(x_{k_i}) = \min\{\bar{\alpha}, c_2, c_3\theta(x_{k_i})\}.$$

Lemma 3.6 and Lemma 3.8 then imply that a trial step size $\alpha_{k,l} \leq \beta_{k_i}$ satisfies both

$$\omega(x_{k_i}(\alpha_{k_i,l})) - \omega(x_{k_i}) \leq \eta_\omega m_k(\alpha_{k_i,l}) \tag{30}$$

and

$$(\theta(x_{k_i}(\alpha_{k_i,l}), \omega(x_{k_i}(\alpha_{k_i,l})))) \notin \mathcal{F}_{k_i}. \tag{31}$$

If we now denote with $\alpha_{k_i,L}$ the first trial step size satisfying both (30) and (31), then the backtracking line search procedure implies that for $\alpha \geq \alpha_{k_i,L}$

$$\alpha \geq \tau_1\beta_{k_i} = \tau_1 c_3\theta(x_{k_i}) > \frac{\delta[\theta(x_{k_i})]^\phi}{\epsilon_2^\tau}$$

and therefore for $\alpha \geq \alpha_{k_i,L}$

$$\delta[\theta(x_{k_i})]^\phi < \alpha\epsilon_2^\tau = \alpha^{1-\tau}(\alpha\epsilon_2)^\tau \leq \alpha^{1-\tau}[-m_{k_i}(\alpha)]^\tau.$$

This means that $\alpha_{k_i,L}$ and all previous trial step sizes satisfy (8) and (10). Consequently, for all trial step sizes $\alpha_{k_i,l} \geq \alpha_{k_i,L}$, we have $\alpha_{k_i,L} \geq \alpha_{k_i}^{\min}$. Hence, the method dose not switch to the feasibility phase for those trial step sizes. Therefore, $\alpha_{k_i,L}$ is indeed the accepted step size α_{k_i}. Since it satisfies both (8) and (30), the filter is not augmented in iteration k_i. □

Based on the lemmas and theorem above, we prove the main conclusion of this paper, that is, the global convergence result of the Algorithm I.

Theorem 3.2. *Suppose Assumptions G hold. Then*

$$\lim_{k\to\infty} \theta(x_k) = 0 \tag{32a}$$

and

$$\liminf_{k\to\infty} \chi(x_k) = 0. \tag{32b}$$

In other words, all limit points are feasible, and if $\{x_k\}$ is bounded, then there exists a limit point x_* of $\{x_k\}$ which is a first order optimal point for the equality constrained NLP(1).

Proof. (32a) follows from Theorem 3.1. In the case of the filter is augmented only a finite number of times, (32b) has been proved by Lemma 3.7. Otherwise, there exists a subsequence $\{x_{k_i}\}$ such that $k_i \in \mathcal{Z}$, for all i. Now, suppose that $\limsup_i \chi(x_{k_i}) > 0$. Then, there exist a subsequence $\{x_{k_{i_j}}\}$ of $\{x_{k_i}\}$ and a constant $\epsilon > 0$ such that $\lim_j \theta(x_{k_{i_j}}) = 0$ and $\chi(x_{k_{i_j}}) > \epsilon$, for all k_{i_j}. Applying Lemma 3.9 to $\{x_{k_{i_j}}\}$, we see that there is an iteration k_{i_j}, in which the filter is not augmented, i.e. $k_{i_j} \notin \mathcal{Z}$. This contradicts the choice of $\{x_{k_i}\}$ such that $\lim_i \chi(x_{k_i}) = 0$, which proves (32b). □

4. Numerical results

In this section, we present the numerical results of Algorithm I which have been performed on a desktop with Intel(R) Core(TM) i5-8250 CPU. Algorithm I was implemented as a MATLAB code and run under MATLAB version 9.2.0.518641(R2017a).

The selected parameter values are:

$$\begin{aligned}
&\epsilon = 10^{-6}, \gamma_\theta = 10^{-5},\\
&\gamma_\omega = 10^{-5}, \delta = 10^{-2},\\
&\gamma_\alpha = 10^{-4}, \phi = 2.01, \tau = 1.1,\\
&\eta_\omega = 0.25, \tau_1 = 0.25; \tau_2 = 0.75.
\end{aligned}$$

The computation terminates when

$$\max\{\|g(x_k) - A(x_k)y_{k+1}\|_2, \|c(x_k)\|\} \leq \epsilon$$

is satisfied. The update of B_k is implemented by using the methods recommended in Section 4.4 and Section 4.5 in [16].

The results are reported in Table 1 where the test problems are numbered in the same way as in [14] and in [24], respectively. For example, HS27 is the problem 27 in [14] and S252 is the problem 252 in [24]. For higher dimension problems, we compute a few versions of different dimensions from CUTEst collection [10] to see the efficiency of Algorithm I (see Table 2). The CPU times in Table 1 and Table 2 are counted in seconds. In these tables, NIT and NG are the numbers of iterations.

In addition, we compared Algorithm I in this paper with SNOPT [9] (Version 5.3). The results are listed in Table 3. We also compared Algorithm I with the algorithm in [13] and in [12]. Furthermore, we show their comparison figures (see, figure 1).

Table 1: Numerical results of Algorithm I

Problem	NIT	NG	CPU Time	$\|c(x_k)\|$	$\|g_k - A_k y_{k+1}\|_2$
byrdsphr	5	5	0.0469	1.7585e-13	1.9254e-09
himmelba	2	2	0.0781	0	7.8531e-08
himmelbc	2	2	0.0313	2.8424e-13	9.3043e-08
hs046	73	73	0.1094	1.3668e-11	6.1298e-07
hypcir	2	2	0.0156	1.1055e-13	6.4251e-08
maratos	6	6	0.0156	5.6520e-07	5.6520e-07
mwright	48	48	0.0469	9.4930e-07	2.3720e-07
powellbs	24	24	0.0313	2.1871e-08	1.5785e-12
supersim	2	2	0.0156	0	3.2034e-10
tame	2	2	0.0313	0	2.6859e-10
booth	2	2	0.0469	8.8818e-16	2.3054e-07
hong	7	7	0.0156	2.7704e-07	1.8421e-09
gottfr	5	5	0.0156	2.7446e-07	5.0986e-09
hatfldf	10	10	0.0313	5.3390e-07	6.9068e-11
hs051	6	6	0.0156	0	1.0400e-08
haifas	14	14	0.0625	7.0891e-07	1.7048e-08
recipe	2	2	0.0313	3.0307e-15	2.2964e-07
robot	4	4	0.0469	3.3815e-13	8.1110e-07
try-b	4	4	0.0156	9.2922e-08	4.8773e-07
cluster	9	9	0.0313	3.0766e-07	1.2776e-07
zangwil3	4	4	0.0625	7.9331e-09	4.2125e-10
bt2	20	20	0.0156	6.7502e-14	4.6037e-08
bt3	7	7	0.0156	1.5701e-16	9.4202e-08
bt5	6	6	0.0156	2.0367e-09	2.6323e-07
bt6	13	13	0.0313	8.8776e-07	4.3058e-07
bt7	12	12	0.0313	9.1551e-16	2.0754e-08
bt8	7	7	0.0156	5.1592e-10	3.7845e-07
bt9	13	13	0.0156	2.1202e-09	4.2764e-08
bt10	7	7	0.0156	2.5066e-10	7.4824e-09
bt11	16	16	0.0156	5.6726e-13	9.0841e-08
bt12	9	9	0.0156	8.2212e-10	6.7032e-10
HS07	10	10	0.0156	1.6101e-10	3.4446e-08
HS08	6	6	0.0313	4.6483e-07	2.1096e-10
HS09	6	6	0.0156	0	2.1181e-07
HS26	16	16	0.0156	8.7215e-07	3.4365e-08
HS27	30	30	1.2813	1.4955e-10	6.5668e-07
HS28	8	8	0.0156	2.2204e-16	1.2406e-07
HS39	13	13	0.0156	5.8040e-12	5.1408e-07
HS40	6	6	0.0156	8.5750e-10	2.2161e-08
HS42	8	8	0.0156	1.9229e-12	7.3685e-07
HS47	30	30	0.1406	7.0971e-09	4.7927e-07
HS49	29	29	0.0313	0	5.0716e-07
HS61	7	7	0.0156	3.9662e-13	7.3996e-11
HS77	12	12	0.0313	9.6157e-12	2.4020e-07
HS78	12	12	0.0469	8.4495e-07	3.7625e-08
HS79	15	15	0.0156	2.2486e-12	9.3840e-08
S216	8	8	0.1563	0	7.0458e-07
S219	18	18	0.1250	2.5963e-13	5.6842e-10
S252	43	43	0.0313	2.8200e-13	3.5788e-09
S254	13	13	0.0156	1.6703e-09	9.2910e-07
S269	8	8	0.0469	1.1102e-16	1.1357e-07
S316	3	3	0.0156	5.5120e-12	9.3910e-07
S317	4	4	0.0156	3.5741e-07	6.3540e-08
S335	24	24	0.0469	1.1196e-13	1.1547e-07
S336	7	7	0.0313	1.7902e-15	3.0976e-08
S338	11	11	0.0156	1.5776e-10	2.8493e-07
S344	11	11	0.0313	4.1567e-13	3.2734e-08
S345	18	18	0.0469	4.0980e-10	5.9741e-07
S378	61	61	0.1094	6.9241e-10	4.3425e-07

Table 2: Numerical results of Algorithm I for higher dimension problems

Problem	n	m	NIT	NG	CPU-time	$\|c(x_k)\|$	$\|g_k - A_k y_{k+1}\|_2$
bdvalue	50	50	3	3	0.0625	1.2862e-16	9.4625e-09
	100	100	3	3	0.1875	1.8105e-16	1.4843e-08
	200	200	3	3	0.9844	2.0619e-16	2.2062e-08
	300	300	4	4	2.8594	2.8180e-16	6.9444e-09
	500	500	4	4	10.3750	3.2170e-16	1.6849e-08
	600	600	4	4	9.0625	3.8500e-16	1.8518e-08
	700	700	4	4	13.0781	3.9056e-16	2.6413e-08
	800	800	4	4	27.7031	4.5237e-16	2.8287e-08
	900	900	4	4	29.7812	4.7032e-16	3.0043e-08
	1000	1000	4	4	44.3594	4.9907e-16	3.1703e-08
broydn3d	10	10	2	2	0.0156	3.9823e-14	3.0160e-08
	100	100	2	2	0.9375	2.4878e-12	4.2629e-07
	200	200	2	2	2.2969	1.5215e-12	3.5042e-07
	300	300	2	2	4.7500	2.0474e-12	5.1705e-07
	400	400	2	2	6.3750	7.2633e-12	9.6331e-07
	500	500	2	2	16.3594	1.4098e-12	4.5545e-07
	600	600	2	2	19.9688	3.6116e-12	8.1693e-07
	700	700	2	2	19.7813	2.2992e-12	6.2797e-07
	800	800	2	2	64.1250	3.0570e-12	8.2897e-07
	900	900	2	2	96.1875	1.3670e-12	5.0918e-07
	1000	1000	2	2	153.5156	1.8805e-12	4.9751e-07
gilbert	50	1	23	23	0.0625	6.7002e-12	9.5829e-07
	100	1	24	24	0.2344	3.6336e-12	7.5592e-07
	200	1	25	25	0.6719	2.1927e-12	5.7521e-07
	300	1	25	25	1.5938	3.3153e-12	7.0874e-07
	400	1	25	25	3.3438	4.4363e-12	8.2086e-07
	500	1	25	25	4.7500	5.5415e-12	9.1941e-07
	600	1	26	26	7.2188	2.1458e-12	5.3157e-07
	700	1	26	26	10.9688	2.4893e-12	5.7464e-07
	800	1	26	26	15.0000	2.9293e-12	6.1470e-07
	900	1	26	26	19.5781	3.2849e-12	6.5231e-07
	1000	1	26	26	27.4219	3.4908e-12	6.8786e-07
	1300	1	26	26	68.7344	4.5803e-12	7.8493e-07
	1500	1	26	26	121.8906	5.5477e-12	8.4345e-07
	1800	1	26	26	240.9063	6.2157e-12	9.2432e-07
	2000	1	26	26	355.7345	7.1335e-12	9.7451e-07

Table 3: Comparison of Algorithm I with SNOPT

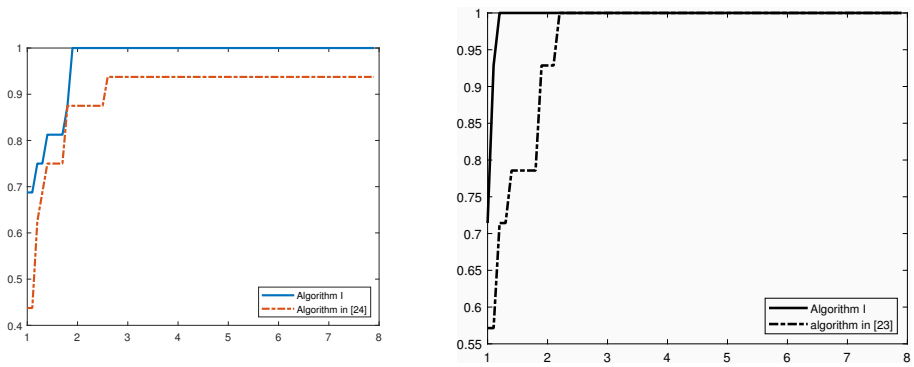
Problem	Algorithm I		SNOPT	
	NIT	NG	NIT	NG
bt2	20	20	16	20
bt3	7	7	5	5
bt5	8	8	10	10
bt6	15	15	17	16
bt7	12	12	21	36
bt8	7	7	13	13
bt9	18	18	20	30
bt10	7	7	23	23
bt11	19	19	15	14
bt12	9	9	9	9
hs07	10	10	18	30
hs08	5	5	2	2
hs09	7	7	9	8
hs26	16	16	25	24
hs27	30	30	22	23
hs28	8	8	4	4
hs39	13	13	20	30
hs40	6	6	7	7
hs42	8	8	8	8
hs47	29	29	24	31
hs49	29	29	37	32
hs61	7	7	69	168
hs77	12	12	15	14
hs78	12	12	9	7
hs79	15	15	14	14

Table 4: Comparison of Algorithm I with Algorithm in [13]

Problem	n	m	Algorithm I		Algorithm [13]	
			NIT	NG	NIT	NG
hs07	2	1	10	10	9	9
hs09	2	1	7	7	8	8
hs27	3	1	30	30	24	24
hs28	3	1	8	8	10	10
hs39	4	2	13	13	15	15
hs40	4	3	6	6	6	6
hs42	4	2	8	8	9	9
hs47	5	3	29	29	30	30
hs61	3	2	7	7	10	10
hs77	5	2	12	12	22	22
hs78	5	3	12	12	8	8
hs79	5	3	15	15	10	10
s219	4	2	18	18	26	26
s254	3	2	13	13	9	9
s269	5	3	8	8	9	9
s316	2	1	3	3	14	14

Table 5: Comparison of Algorithm I with Algorithm in [12]

Problem	n	m	Algorithm I		Algorithm in [12]	
			NIT	NG	NIT	NG
hs07	2	1	10	10	10	10
hs08	2	2	5	5	5	5
hs09	2	1	6	6	6	6
hs26	3	1	16	16	26	26
hs28	3	1	8	8	9	9
hs40	4	3	6	6	6	6
hs42	4	2	8	8	10	10
hs47	5	3	29	29	27	27
hs49	5	2	29	29	26	26
hs77	5	2	12	12	18	18
hs78	5	3	12	12	11	11
hs79	5	3	15	15	14	14
s216	2	1	8	8	12	12
s269	5	3	8	8	9	9



(a) Comparison of Algorithm I with [13]. (b) Comparison of Algorithm I with [12].

Figure 1: Comparison figure.

5. Conclusion

We have introduced a line search method combining filter technique for solving nonlinear equality constrained optimization. The global convergence analysis for this algorithm is presented under some reasonable assumptions. The preliminary numerical experiments show its practical performance. The comparison of numerical experiment data show that the algorithm in this paper has good effect. The main feature of the proposed filter method is that it is not required to compute the value of objective function (or the Lagrange function) in every iteration to obtain a first order optimal point. However, the analysis of local convergence rate is absent. That is what we are working on.

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Separation axioms in intuitionistic topological spaces

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Abstract. In this paper we study that every classical topological space is also an intuitionistic topological space but the converse is not true in general. This notion opens up a new conception of generalization of classical topological space. Besides this, by using the notions of separation axioms (T_0, T_1, T_2) under intuitionistic set we define the relations among them. The hereditary and topological properties of intuitionistic topological spaces have been also investigated. Finally, it is showed that under some conditions the images and homeomorphic images preserve in intuitionistic topological spaces.

Keywords: intuitionistic set, intuitionistic topological space, hereditary, separation axioms.

1. Introduction

Ever since the invention of topological spaces, many researchers have been paying remarkable contribution in this field. By investigating different properties on classical topological spaces, they also added new notions for its generalization. After the introduction of the fuzzy set by Zadeh [23] the classical topological space is directed to a new dimension namely "Fuzzy Topological Spaces" which is defined by Chang [8]. After that, Atanassov [5, 6] introduced the notion of intuitionistic fuzzy set which is a generalization of fuzzy set. Later, Coker et al. [9, 10, 11, 12] defined intuitionistic fuzzy topological spaces, intuitionistic sets and intuitionistic topological spaces. From the onwards, Coker and S. Bayhan [9, 10, 11, 12], Singh and Srivastava [22], S. J. Lee and E. P Lee [16], Saadati and Park [21], Estiaq Ahmed et al. [1, 2, 3] initiated different studies on intuitionistic fuzzy topological spaces by using intuitionistic fuzzy sets. In this paper, we

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investigate various properties of separation axioms in intuitionistic topological spaces.

2. Notation and preliminaries

In this paper X is noted as an non-empty set, $\mathcal{T}$ is topology, $(X, \mathcal{T})$ is topological space, $\check{\mathcal{T}}$ is an intuitionistic topology, $(X, \check{\mathcal{T}})$ is intuitionistic topological space, $G = (G_1, G_2)$ and $H = (H_1, H_2)$ are intuitionistic sets, f is a function, $\check{\mathcal{T}}_A$ is an intuitionistic relative topology on A where $A \subseteq X$, $\emptyset = (\emptyset, X)$ and $X = (X, \emptyset)$ are also noted as intuitionistic sets.

Definition 2.1 ([12]). *Suppose X is a non-empty set. An intuitionistic set A on X is an object having the form $A = (X, A_1, A_2)$ where A_1 and A_2 are subsets of X satisfying $A_1 \cap A_2 = \emptyset$. The set A_1 is called the set of member of A while A_2 is called the set of non-member of A. In this paper, we use the simpler notation $A = (A_1, A_2)$ instead of $A = (X, A_1, A_2)$ for an intuitionistic set.*

Remark 2.1 Every subset A of a nonempty set X may obviously be regarded as an intuitionistic set having the form $A = (A, A^c)$ where $A^c = X/A$

Definition 2.2 ([12]). *Let the intuitionistic sets A and B in X be of the forms $A = (A_1, A_2)$ and $B = (B_1, B_2)$ respectively. Furthermore, let $\{A_j, j \in J\}$ be an arbitrary family of intuitionistic sets in X, where $A_j = (A_j^{(1)}, A_j^{(2)})$. Then:*

a. $A \subseteq B$ if and only if $A_1 \subseteq B_1$ and $A_2 \supseteq B_2$;
b. $A = B$ if and only if $A \subseteq B$ and $B \supseteq A$;
c. $\bar{A} = (A_1, A_2)$ denotes the complement of A;
d. $\cap A_j = (\cap A_j^{(1)}, \cup A_j^{(2)})$;
e. $\cup A_j = (\cup A_j^{(1)}, \cap A_j^{(2)})$;
f. $\emptyset_\sim = (\emptyset, X)$ and $X_\sim = (X, \emptyset)$.

Definition 2.3 ([6]). *Let X be a non-empty set. A family $\check{\mathcal{T}}$ of intuitionistic sets in X is called an intuitionistic topology on X if the following conditions hold:*

a. $\emptyset_\sim, X_\sim \in \check{\mathcal{T}}$;
b. $A \cap B \in \check{\mathcal{T}}$ for all $A, B \in \check{\mathcal{T}}$;
c. $\cup A_j \in \check{\mathcal{T}}$ for any arbitrary family $\{A_j \in \check{\mathcal{T}}, j \in J\}$.

The pair $(X, \check{\mathcal{T}})$ is called an intuitionistic topological space (ITS, in short), member of $\check{\mathcal{T}}$ are called intuitionistic open sets (IOS, in short) in X and their complements are called intuitionistic closed sets (ICS, in short) in X.

Definition 2.4 ([3]). *let $(X, \check{\mathcal{T}})$ and (Y, δ) be IFTSs. A function $f : X \to Y$ is called continuous if $f^{-1}(B) \in \mathcal{T}$ for all $B \in \delta$ and f is called open if $f(A) \in \delta$ for all $A \in \check{\mathcal{T}}$.*

Definition 2.5. *Let $(X, \check{\mathcal{T}})$ be an intuitionistic topological space and $A \subseteq X$. If $\check{\mathcal{T}}_A$ is a topology generated by $\{G \cap (A, \emptyset); G \in \check{\mathcal{T}}\}$ then $\check{\mathcal{T}}_A$ is also an intuitionistic topology on A and the space $(A, \check{\mathcal{T}}_A)$ is a subspace of $(X, \check{\mathcal{T}})$.*

Definition 2.6 ([3]). *An intuitionistic topological space $(X, \check{\mathcal{T}})$ is called T_0 if for all $x, y \in X$ with $x \neq y$ there exists intuitionistic sets $A = (A_1, A_2) \in \check{\mathcal{T}}$ such that $x \in A_1, y \in A_2$ and $y \in A_1, x \in A_2$.*

Definition 2.7 ([6]). *An intuitioinistic topological space $(X, \check{\mathcal{T}})$ is called T_1 if for all $x, y \in X$ with $x \neq y$ there exists intuitionistic sets $A = (A_1, A_2), B = (B_1, B_2) \in \check{\mathcal{T}}$ such that $x \in A_1, y \notin A_1$ and $y \in B_1, x \notin B_2$.*

Definition 2.8 ([6]). *An intuitionistic topological space $(X, \check{\mathcal{T}})$ is called T_1 if for all $x, y \in X$ with $x \neq y$ there exists intuitionistic sets $A = (A_1, A_2), B = (B_1, B_2) \in \check{\mathcal{T}}$ such that $x \in A_1, y \notin A_1$ and $y \in B_1, x \notin B_1$ and $A \cap B = \emptyset$.*

Definition 2.9. *Let X and Y be two nonempty sets and $f : X \to Y$ be a function. If $A = (A_1, A_2) \subseteq X$ then $f(A)$ is defined as $f(A) = (f(A_1), f(A_2))$ and if $B = (B_1, B_2) \subseteq Y$ then $f^{-1}(B)$ is defined as $f^{-1}(B) = (f^{-1}(B_1), f^{-1}(B_2))$.*

3. Separation axioms in intuitionistic topological space

In this section, we will prove different properties of separation axioms in intuitionistic topological space. Beside this, by using some examples we will show that intuitionistic topological space is a generalization of classical topological space.

Theorem 3.1. *Every classical topological space can be represented as an intuitionistic topological space but the converse is not true.*

Proof of Theorem 3.1. We know that, if $(X, \mathcal{T})$ is a topological space then it follows the following axioms:

i. $X, \emptyset \in \mathcal{T}$;

ii. $A, B \in \mathcal{T} \Rightarrow A \cap B \in \mathcal{T}$;

iii. $A_j \in \mathcal{T} \Rightarrow \cup A_j \in \mathcal{T}$.

Now, we shall show that $(X, \mathcal{T})$ can be represented as an intuitionistic topological space $(X, \check{\mathcal{T}})$.

i. X can be represented as $(X, \emptyset) \in \check{\mathcal{T}}$, $\emptyset$ can be represented as $(\emptyset, X) \in \check{\mathcal{T}}$.

ii. $A, B \in \mathcal{T} \Rightarrow (A, A^c) \in \check{\mathcal{T}}$, $(B, B^c) \in \check{\mathcal{T}}$ and $A \cap B = (A \cap B, A^c \cup B^c) \in \check{\mathcal{T}}$.

iii. If $A_i \in \mathcal{T}$ then $(A_i, A_i^c) \in \check{\mathcal{T}}$ and $(\cup A_i, \cap A_i^c) \in \check{\mathcal{T}}$. Thus, $(X, \mathcal{T})$ is represented as an intuitionistic topological space $(X, \check{\mathcal{T}})$. But the converse is not true. For example, let $X = \{a, b, c\}$, $\check{\mathcal{T}} = \{A = (\{a\}, \{c\}), B = (\{b\}, \{c\}), C = (\{a, b\}, \{c\}), D = (\emptyset, \{c\}, \emptyset_\sim, X_\sim)\}$. Therefore, the intuitionistic topological space $(X, \check{\mathcal{T}})$ cannot be represented as a classical topological space.

Hence the proof.

Theorem 3.2. *Every classical T_0 topological space is an intuitionistic T_0 topological space but the converse is not true.*

Proof of Theorem 3.2. Let $(X, \mathcal{T})$ be a T_0 space. So according to the axioms of T_0 space we have $a, b \in X$ with $a \neq b$ $\exists$ a $G \in \mathcal{T}$ such that $a \in G, b \notin G$.

The above axioms imply that $\forall a, b \in X$ with $a \neq b \; \exists \; A = (G, G^c)$ and $A \in \check{\mathcal{T}}$ such that $a \in G, b \neq G^c$. Thus $(X, \check{\mathcal{T}})$ is an intuitionistic T_1 space.

But the converse is not true. For example, consider $X = \{a, b, c\}$ and $A = (\{a\}, \{b, c\})$, $B = (\{b\}, \{c\})$, $C = (\{c\}, \{a, b\})$. Therefore, $(X, \check{\mathcal{T}})$ is an intuitionistic T_0 space generated by $\{A, B, C\}$. But this topological space cannot be represented as a classical T_0 space. Because the element c in X bellongs to $B \in \check{\mathcal{T}}$ and $C \in \check{\mathcal{T}}$ as well.

Hence the proof.

Theorem 3.3. *Every classical T_1 space is an intuitionistic T_1 space but the converse is not true.*

Proof of Theorem 3.3. Let $(X, \mathcal{T})$ be a T_1 space. So according to the axioms of T_1 space we have $a, b \in X$ with $a \neq b \; \exists \; G, H \in T$ such that $a \in G, b \notin G$ and $b \in H, a \notin H$, G and H are not necessarily disjoint.

The above axioms imply that $\forall a, b \in X$ with $a \neq b \; \exists \; A = (G, G^c)$ and $B = (H, H^c) \in \check{\mathcal{T}}$ such that $a \in G, b \notin G$ and $b \in H, a \notin H$. Thus $(X, \check{\mathcal{T}})$ is an intuitionistic T_1 space.

But the converse is not true. For example, consider $X = \{a, b, c\}$ and $A = (\{a\}, \{b\})$, $B = (\{b\}, \{c\})$, $C = (\{c\}, \{a, b\})$. Therefore, $(X, \check{\mathcal{T}})$ is an intuitionistic T_1 space generated by $\{A, B, C\}$. But this cannot be represented as a classical T_1 space. Because the element b in X belongs to $A \in \check{\mathcal{T}}$ and $B \in \check{\mathcal{T}}$ as well.

Hence the proof.

Theorem 3.4. *Every classical T_2 space is an intuitionistic T_2 space but the converse is not true.*

Proof of Theorem 3.4. Let $(X, \mathcal{T})$ be a T_2 space. So according to the axioms of T_2 space we have $a, b \in X$ with $a \neq b \; \exists \; G, H \in \mathcal{T}$ such that $a \in G, b \notin G$ and $b \in H, a \notin H$, $G \cap H = \emptyset$.

The above axioms imply that $\forall a, b \in X$ with $a \neq b \; \exists \; A = (G, G^c)$ and $B = (H, H^c) \in \check{\mathcal{T}}$ such that $a \in G, b \notin G$ and $b \in H, a \notin H$ and $A \cap B = (G, G^c) \cap (H, H^c) = (G \cap H, G^c \cup H^c) = (\emptyset, D)$ where $G^c \cup H^c = D \subseteq X$. Thus, $(X, \check{\mathcal{T}})$ is an intuitionistic T_2 space. But, the converse is not true. For example, consider $X = \{p, q, r, s\}$ and $A = (\{p\}, \{q, s\})$, $B = (\{q\}, \{p, r, s\})$, $C = (\{s\}, \{p, r\})$, $D = (\{r\}, \{p, s\})$. Therefore, $(X, \check{\mathcal{T}})$ is an intuitionistic T_2 space generated by $\{A, B, C, D\}$. But this topological space cannot be represented as a classical T_2 space.

Hence the proof.

Theorem 3.5. *Let $(X, \check{\mathcal{T}})$ be an intuitionistic T_0 space then $(A, \check{\mathcal{T}}_A)$ is also T_0.*

Proof of Theorem 3.5. Let $a, b \in A$ with $a \neq b$. This implies $a, b \in X$ with $a \neq b$. Since $(X, \check{\mathcal{T}})$ is an intuitionistic T_0 space the $\exists \; G = (G_1, G_2) \in \check{\mathcal{T}}$ such that $a \in G_1, b \notin G_1$. Now we have $G \in \check{\mathcal{T}} \Rightarrow G \cap (A, \emptyset) \in \check{\mathcal{T}}_A$.

But $G \cap (A, \emptyset) = (G_1, G_2) \cap (A, \emptyset) = (G_1 \cap A, G_2 \cup \emptyset) = (G_1 \cap A, G_2)$. Furthermore, $a, b \in A$ and $a \in G_1, b \notin G_1 \Rightarrow a \in G_1 \cap A, b \notin G_1 \cap A$. Finally, we get $\forall a, b \in A$ with $a \neq b \; \exists$ a $(G_1 \cap A, G_2) \in \check{\mathcal{T}}_A$ such that $a \in G_1 \cap A, b \notin G_1 \cap A$. Therefore, $(A, \check{\mathcal{T}}_A)$ is an intuitionistic T_0 space.

Hence T_0 property is hereditary. (Proved)

Theorem 3.6. *Let $(X, \check{\mathcal{T}})$ be an intuitionistic T_1 space then $(A, \check{\mathcal{T}}_A)$ is also T_1.*

Proof of Theorem 3.6. Let $a, b \in A$ with $a \neq b$. This implies $a, b \in X$ with $a \neq b$. Since $(X, \check{\mathcal{T}})$ is intuitionistic T_1 space then $\exists \; G = (G_1, G_2) \in \check{\mathcal{T}}$. Therefore, $a \in G_1, b \notin G_1$ and $b \in H_1, a \notin H_1$. Now we have $G, H \in \check{\mathcal{T}} \Rightarrow G \cap (A, \emptyset), H \cap (A, \emptyset) \in \check{\mathcal{T}}_A$.

But $G \cap (A, \emptyset) = (G_1, G_1) \cap (A, \emptyset) = (G_1 \cap A, G_2 \cup \emptyset) = (G_1 \cap A, G_2)$ $H \cap (A, \emptyset) = (H_1, H_2) \cap (A, \emptyset) = (H_1 \cap A, H_2 \cup \emptyset) = (H_1 \cap A, H_2)$.

Further, we set $a, b \in A$ and $a \in G_1, b \notin G_1 \Rightarrow a \in G_1 \cap A, b \notin G_1 \cap A$. Again $b \in H_1, a \notin H_1 \Rightarrow b \in H_1 \cap A, a \notin H_1 \cap A$. Finally, we get $\forall a, b \in A$ with $a \neq b \; \exists \; (G_1 \cap A, G_2)$ and $(H_1 \cap A, H_2) \in \check{\mathcal{T}}_A$ such that $a \in G_1 \cap A, b \notin G_1 \cap A$ and $a \in H_1 \cap A, b \in H_1 \cap A$. Therefore, $(A, \check{\mathcal{T}}_A)$ is an intuitionistic T_1 space.

Hence, T_1 property is hereditary. (Showed)

Theorem 3.7. *Let $(X, \check{\mathcal{T}})$ be an intuitionistic T_2 space then $(A, \check{\mathcal{T}}_A)$ is also T_2.*

Proof of Theorem 3.7. Let $a, b \in A$ with $a \neq b$. This implies $a, b \in X$ with $a \neq b$. Since $(X, \check{\mathcal{T}})$ is intuitionistic T_1 space then $\exists \; G = (G_1, G_2) \in \check{\mathcal{T}}$ and $H = (H_1, H_2) \in \check{\mathcal{T}}$ such that $a \in G_1, b \notin G_1$ and $b \in H_1, a \notin H_1$ and $G \cap H = \emptyset$. Now we have $G, H \in \check{\mathcal{T}} \Rightarrow G \cap (A, \emptyset), H \cap (A, \emptyset) \in \check{\mathcal{T}}_A$.

But $G \cap (A, \emptyset) = (G_1, G_2) \cap (A, \emptyset) = (G_1 \cap A, G_2 \cup \emptyset) = (G_1 \cap A, G_2)$ $H \cap (A, \emptyset) = (H_1, H_2) \cap (A, \emptyset) = (H_1 \cap A, H_2 \cup \emptyset) = (H_1 \cap A, H_2)$.

Furthermore, $a, b \in A$ and $a \in G_1, b \notin G_1 \Rightarrow a \in G_1 \cap A, b \notin G_1 \cap A$. Again $b \in H_1, a \notin H_1 \Rightarrow b \in H_1 \cap A, b \notin H_1 \cap A$. Consider $(G \cap A) \cap (H \cap A) \neq (\emptyset \cap A)$, where $A \subseteq X \Rightarrow (G_1 \cap A, G_2) \cap (H_1 \cap A, H_2) \neq (\emptyset \cap A) \Rightarrow ((G_1 \cap A) \cap (H_1 \cap A), G_2 \cup H_2) \neq (\emptyset \cap A)$. Therefore, $(G_1 \cap A) \cap (H_1 \cap A) \neq \emptyset$ i.e. $\exists$ at least one $y \in A \Rightarrow y \in X$ such that $y \in G_1 \cap A$ and $y \in H_1 \cap A \Rightarrow y \in G_1$ and $y \in A$ and $y \in H_1, y \in A$. Since $(X, \check{\mathcal{T}})$ is intuitionistic T_1 space so $\nexists$ a y which belongs to G_1 and H_1 simultaneously. So, this is a contradiction to the fact that $(G_1 \cap A) \cap (H_1 \cap A) \neq \emptyset$. Therefore, $(G_1 \cap A) \cap (H_1 \cap A) = (\emptyset, A)$.

Finally, we get $\forall a, b \in A$ with $a \neq b \; \exists \; (G_1 \cap A, G_2)$ and $(H_1 \cap A, H_2) \in \check{\mathcal{T}}_A$ such that $a \in G_1 \cap A, b \notin G_1 \cap A$ and $a \notin H_1 \cap A, b \in H_1 \cap A$ and $(G_1 \cap A) \cap (H_1 \cap A) = (\emptyset, A)$. Therefore, $(A, \check{\mathcal{T}}_A)$ is an intuitionistic T_2 space. Hence, T_2 property is hereditary. (Proved)

Theorem 3.8. *Let $(X, \check{\mathcal{T}})$ and $(Y, \check{\delta})$ be two intuitionistic topological spaces and $f : X \to Y$ be a one-one, onto and open map. If $(X, \check{\mathcal{T}})$ is T_0 space then $(Y, \check{\delta})$ is also T_0 space.*

Proof of Theorem 3.8. Let $(X, \check{\mathcal{T}})$ and $(Y, \check{\delta})$ be two intuitionistic topological spaces and $f : X \to Y$ be one-one, onto and open map. Let $(X, \check{\mathcal{T}})$ be intuitioistic T_0 space, we shall show that $(Y, \check{\delta})$ is also intuitionistic T_0 space.

Suppose $a, b \in Y$ with $a \neq b$. Since f is onto then $\exists\ p, q \in X$ such that $f(p) = a$ and $f(q) = b$.

Again, since $a \neq b \Rightarrow f(p) \neq f(q) \Rightarrow p \neq q$ as f is one-one. Further since $p, q \in X$, $p \neq q$ and $(X, \check{\mathcal{T}})$ is T_0 space then $\exists\ G = (G_1, G_2) \in \check{\mathcal{T}}$ such that $p \in G_1, q \notin G_1$.

Since $G, H \in \check{\mathcal{T}} \Rightarrow f(G), f(H) \in \check{\delta}$ as f is open.

We know, $f(G) = (f(G_1), f(G_1))$. Furthermore $a = f(p) \in f(G_1)$ and $b = f(q) \notin f(G_1)$. Finally, we get $a, b \in Y$ with $a \neq b\ \exists\ f(G) \in \check{\delta}$ such that $a \in f(G_1), b \notin f(G_1)$. Therefore, $(Y, \check{\delta})$ is T_0 space. $\therefore$ Every homeomorphic image of intuitionistic T_0 space is also T_0 space.

Hence intuitionistic T_0 is a topological property. (Proved)

Theorem 3.9. *Let $(X, \check{\mathcal{T}})$ and $(Y, \check{\delta})$ be two intuitionistic topological spaces and $f : X \to Y$ be a one-one, onto and open map. If $(X, \check{\mathcal{T}})$ is T_1 space then $(Y, \check{\delta})$ is also T_1 space.*

Proof of Theorem 3.9. Let $(X, \check{\mathcal{T}})$ and $(Y, \check{\delta})$ be two intuitionistic topological spaces and $f : X \to Y$ be a one-one, onto and open map. Let $(X, \check{\mathcal{T}})$ be intuitionistic T_1 space, we shall show that $(Y, \check{\delta})$ is also T_1 space.

Suppose $a, b \in Y$ with $a \neq b$. Since f is onto then $\exists\ p, q \in X$ such that $f(p) = a$ and $f(q) = b$.

Again since $a \neq b \Rightarrow f(p) \neq f(q) \Rightarrow p \neq q$ as f is one-one. Further since $p, q \in X, p \neq q$ and $(X, \check{\mathcal{T}})$ is T_1 space then $\exists\ G = (G_1, G_2) \in \check{\mathcal{T}}$ and $H = (H_1, H_2) \in \check{\mathcal{T}}$ such that $p \in G_1, q \notin G_1$ and $q \in H_1, p \notin H_1$.

Since, $G, H \in \check{\mathcal{T}} \Rightarrow f(G), f(H) \in \check{\delta}$ as f is open.

Now, we have, $f(G) = (f(G_1), f(G_2))$ and $f(H) = (f(H_1), f(H_2))$. Furthermore, $a = f(p) \in f(G_1)$ and $b = f(q) \in f(H_1)$. Again, since $q \notin G_1 \Rightarrow b = f(q) \notin f(G_1)$. Finally, we get $a, b \in Y$ with $a \neq b\ \exists\ f(G)$ and $f(H) \in \check{\delta}$ such that $a \in f(G_1), b \notin f(G_1)$ and $b \in f(H_1), a \notin f(H_1)$.

$\therefore$ Every homeomorphic image of intuitionistic T_1 space is also T_1 space.

Hence, intuitionistic T_1 is a topological property. (Proved)

Theorem 3.10. *Let $(X, \check{\mathcal{T}})$ and $(Y, \check{\delta})$ be two intuitionistic topological spaces and $f : X \to Y$ be one-one, onto and open map. If $(X, \check{\mathcal{T}})$ is T_2 space then $(Y, \check{\delta})$ is also T_2 space.*

Proof of Theorem 3.10. Let $(X, \check{\mathcal{T}})$ and $(Y, \check{\delta})$ be two intuitionistic topological spaces and $f : X \to Y$ be a one-one, onto and open map. Let $(X, \check{\mathcal{T}})$ be intuitionistic T_2 space, we shall show that $(Y, \check{\delta})$ is also T_2 space.

Suppose $a, b \in Y$ with $a \neq b$. Since f is onto then $\exists\ p, q \in X$ such that $f(p) = a$ and $f(q) = b$.

Again since $a \neq b \Rightarrow f(p) \neq f(q) \Rightarrow p \neq q$ as f is one-one. Further since $p, q \in X, p \neq q$ and $(X, \check{\mathcal{T}})$ is T_2 space then $\exists\ G = (G_1, G_2) \in \check{\mathcal{T}}$ and $H = (H_1, H_2) \in \check{\mathcal{T}}$ such that $p \in G_1, q \notin G_1$ and $q \in H_1, p \notin H_1$ with $G \cap H = (\emptyset, A)$.

Since, $G, H \in \check{\mathcal{T}} \Rightarrow f(G), f(H) \in \check{\delta}$ as f is open.

Now, we have, $f(G) = (f(G_1), f(G_2))$ and $f(H) = (f(H_1), f(H_2))$. Furthermore $a = f(p) \in f(G_1)$ and $b = f(q) \in f(H_1)$. Consider $f(G) \cap f(H) \neq (\emptyset, A)$ where $A \subseteq X \Rightarrow (f(G_1) \cap f(H_1), f(G_2) \cup f(H_2) \neq (\emptyset, A) \Rightarrow f(G_1) \cap f(H_1) \neq \emptyset$ i.e. there exists at least one $y \in Y$ for which $y \in f(G_1) \cap f(H_1) \Rightarrow y \in f(G_1)$ and $y \in f(H_1)$. Then, there exists $m \in G_1$ and $n \in H_1$. Such that $f(m) = f(n) = y$ $\Rightarrow m = n$ as f is one-one $\Rightarrow m = n \in G_1 \cap H_1$ which is a contradiction to the fact that $G \cap H = \emptyset$. Therefore, we get $f(G) \cap f(H) = \emptyset$. Finally, we get $a, b \in Y$ with $a \neq b\ \exists\ f(G)$ and $f(H) \in \check{\delta}$ such that $a \in f(G_1), b \notin f(H_1)$ and $b \in f(H_1), a \notin f(H_1)$ and, $f(G) \cap f(H) = \emptyset$. Therefore, $(Y, \check{\delta})$ is T_2 space. $\therefore$ Every homeomorphic image of intuitionistic T_2 space is also T_2 space.

Hence intuitionistic T_2 is a topological property. (Proved)

Theorem 3.11. *Let $(X, \check{\mathcal{T}})$ and $(Y, \check{\delta})$ be two intuitionistic topological spaces and $f : X \to Y$ be one-one and continuous. If $(Y, \check{\delta})$ is T_2 space then $(X, \check{\mathcal{T}})$ is also T_2 space.*

Proof of Theorem 3.11. Let $a, b \in X$ with $a \neq b \Rightarrow f(a), f(b) \in Y$ with $f(a) \neq f(b)$ as f is one-one.

Since $f(a), f(b) \in Y$ and $(Y, \check{\delta})$ is T_2 space then $\exists\ G = (G_1, G_2),\ H = (H_1, H_2) \in \check{\delta}$ such that $f(a) \in G_1, f(b) \in H_1$ and $G \cap H = (\emptyset, A)$, where $A \subseteq X$. But we have $G \cap H = (G_1, G_2) \cap (H_1, H_2) = (G_1 \cap H_1, G_2 \cup H_2) = (\emptyset, A)$ i.e. $G_1 \cap H_1 = \emptyset, G_2 \cup H_2 = A$. We have, $f^{-1}(G) = (f^{-1}(G_1), f^{-1}(G_2))$ and $f^{-1}(H) = (f^{-1}(H_1), f^{-1}(H_2))$. Now, $f(a) \in G_1 \Rightarrow f^{-1}f(a) \in f^{-1}(G_1)$ $\Rightarrow a \in f^{-1}(G_1)$. And, $f(b) \in H_1 \Rightarrow f^{-1}f(b) \in f^{-1}(H_1) \Rightarrow b \in f^{-1}(H_1)$. Furthermore $f^{-1}(G) \cap f^{-1}(H) = (f^{-1}(G_1), f^{-1}(G_2)) \cap (f^{-1}(H_1), f^{-1}(H_2)) = (f^{-1}(G_1) \cap f^{-1}(H_1), f^{-1}(G_2) \cup f^{-1}(H_2))$. Suppose $f^{-1}(G_1) \cap f^{-1}(H_1) \neq \emptyset$ $\Rightarrow f(f^{-1}(G_1) \cap f^{-1}(H_1)) \neq f(\emptyset) \Rightarrow ff^{-1}(G_1) \cap ff^{-1}(H_1) \neq \emptyset \Rightarrow G_1 \cap H_1 \neq \emptyset$, which is a contradiction $\therefore f^{-1}(G_1) \cap f^{-1}(H_1) \neq \emptyset$. Again $f^{-1}(G_2) \subseteq X$, $f^{-1}(H_2) \subseteq X$. Thus $f^{-1}(G_2) \cup f^{-1}(H_2) \subseteq X$. Finally, $f^{-1}(G) \cap f^{-1}(H) \in \check{\mathcal{T}}$. Therefore, $f^{-1}(G) \cap f^{1}(H) = (\emptyset, A)$. Finally, we get $a, b \in X$ with $a \neq b\ \exists$ $f^{-1}(G)$ and $f^{-1}(H) \in \check{\mathcal{T}}$ such that $a \in f^{-1}(G), b \notin f^{-1}(G)$ and $b \in f^{-1}(H), a \notin f^{-1}(H)$ and $f^{-1}(G) \cap f^{-1}(H) = (\emptyset, A)$. Therefore, $(X, \check{\mathcal{T}})$ is T_2 space. (Proved)

Theorem 3.12. *Let $(X, \check{\mathcal{T}})$ and $(Y, \check{\delta})$ be two intuitionistic topological spaces and $f : X \to Y$ be one-one and continuous. If $(Y, \check{\delta})$ is T_1 space then $(X, \check{\mathcal{T}})$ is also T_1 space.*

Proof of Theorem 3.12. Let $a, b \in X$ with $a \neq b \Rightarrow f(a), f(b) \in Y$ with $f(a) \neq f(b)$ as f is one-one.

Since $f(a), f(b) \in Y$ and $(Y, \check{\delta})$ is T_1 space then $\exists\ G = (G_1, G_2),\ H = (H_1, H_2) \in \check{\delta}$ such that $f(a) \in G_1, f(b) \in H_1$.

Since f is continuous and $G, H \in \check{\delta}$ then $f^{-1}(G), f^{-1}(H) \in \check{\mathcal{T}}$.

We have, $f^{-1}(G) = (f^{-1}(G_1), f^{-1}(G_2))$ and $f^{-1}(H) = (f^{-1}(H_1), f^{-1}(H_2))$.

Now, $f(a) \in G_1 \Rightarrow f^{-1}f(a) \in f^{-1}(G_1) \Rightarrow a \in f^{-1}(G_1)$ and $f(b) \in H_1 \Rightarrow f^{-1}f(b) \in f^{-1}(H_1) \Rightarrow b \in f^{-1}(H_1)$. Finally, we have $a, b \in X$ with $a \neq b$ then there exists $f^{-1}(G_1), f^{-1}(G_1) \in \check{\mathcal{T}}$ such that $a \in f^{-1}(G), b \notin f^{-1}(G)$ and $a \notin f^{-1}(H), b \in f^{-1}(H)$.

Furthermore, $f(b) \notin G \Rightarrow f^{-1}f(b) \notin f^{-1}(G) \Rightarrow b \notin f^{-1}(G)$ and $f(a) \notin H \Rightarrow f^{-1}f(a) \notin f^{-1}(H) \Rightarrow a \notin f^{-1}(H)$. Thus, $(X, \check{\mathcal{T}})$ is a T_1 topological space. (Proved)

Theorem 3.13. *Let $(X, \check{\mathcal{T}})$ and $(Y, \check{\delta})$ be two intuitionistic topological spaces and $f : X \to Y$ be one-one and continuous. If $(Y, \check{\delta})$ is T_0 space then $(X, \check{\mathcal{T}})$ is also T_0 space.*

Proof of Theorem 3.13. This proof is obvious.

Theorem 3.14. *Show that every intuitionistic T_1 space is also intuitionistic T_0 space. But the converse is not true.*

Proof of Theorem 3.14. Let $(X, \check{\mathcal{T}})$ be an intuitionistic T_1 space. Then $\forall a, b \in X$ with $a \neq b \; \exists \; G = (G_1, G_2)$ and $H = (H_1, H_2) \in \check{\mathcal{T}}$ such that $a \in G_1, b \notin G_1$ and $b \in H_1, a \notin H_1$.

So, for $G = (G_1, G_2) \in \check{\mathcal{T}}$ we have $\forall a, b \in X$ with $a \neq b$ we get $a \in G_1, b \notin G_1$. Or, for $H = (H_1, H_2) \in \check{\mathcal{T}}$ we have $\forall a, b \in X$ with $a \neq b$ we get $b \in H_1, a \notin H_1$. Thus, this topological space follows the axioms of T_0 space. Hence, every intuitionistic T_1 space is also intuitionistic T_0 space.

To prove the converse, we have an exampple:

Consider $X = \{a, b, c\}$ and $A = (\{a\}, \{b, c\})$, $B = (\{a, b\}, \{c\})$, $C = (\{c\}, \{b\})$. Suppose $\check{\mathcal{T}}$ be an intuitionistic topology generated by $\{A, B, C\}$. Therefore, $(X, \check{\mathcal{T}})$ is an intuitionistic T_0 space. But this is not an intuitionistic T_1 space because $a, b \in X$ with $a \neq b$. We have an intuitionistic set A where $a \in \{a\}, b \notin \{a\}$, but we cannot have any intuitionistic set $D = (D_1, D_2)$ such that $a \notin D_1, b \in D_1$.

Theorem 3.15. *Show that every intuitionistic T_2 space is also intuitionistic T_1 space. But the converse is not true.*

Proof of Theorem 3.15. Let $(X, \check{\mathcal{T}})$ be an intuitionistic T_2 space. Then $\forall a, b \in X$ with $a \neq b \; \exists \; G = (G_1, G_2)$ and $H = (H_1, H_2) \in \check{\mathcal{T}}$ such that $a \in G_1, b \notin G_1$ and $b \in H_1, a \notin H_1$ and $G \cap H = (\emptyset, A)$ where $A \subseteq X$.

So, for $G \in \check{\mathcal{T}}$ we get $a \in G_1, b \notin G_1$ and for $H \in \check{\mathcal{T}}$ we have $b \in H_1, a \notin H_1$. Thus, this topological space is also intuitionistic T_1 space. Hence, every intuitionistic T_2 space is also T_1 space. But, the converse is not true.

For example, consider $X = \{a, b, c\}$ and $A = (\{a, c\}, \{b\})$, $B = (\{b, c\}, \{a\})$, $C = (\{a, b\}, \{c\})$. Suppose $\check{\mathcal{T}}$ be an intuitionistic topology generated by $\{A, B, C\}$. Therefore, $(X, \check{\mathcal{T}})$ is an intuitionistic T_1 space. But this is not an intuitionistic T_2 space. Because we cannot have two nonempty intuitionistic open set D, E such that $D \cap E = (\emptyset, M)$, where $M \subseteq X$.

4. Conclusion

In this paper, we observe that intuitionistic topological space is more general that the classical topological space. We also see that separation axioms defined under intuitionistic set satisfy hereditary and topological property. Moreover, they are preserve under one-one and open mapping. Thus, it is certain that this study will add a new dimension of further investigation in intuitionistic topological space.

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New remarks regarding positive implicative BCK-algebras

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Abstract. In this paper, we present a generalization of the Iseki Extension for a BCK-algebra. Furthermore, we prove that the Pseudo-Iseki extension is also a particular case of this generalization. Also, we introduce an algorithm for generating BCK-algebras using a computer and some new results regarding positive implicative BCK-algebras, results obtained with the help of this algorithm.

Keywords: BCK-algebras, positive-implicative BCK-algebras, BCK-trees, BCK-algebras extensions, generating algorithm.

1. Introduction

The concept of BCK-algebras appeared first in [6] as a generalization for the difference operation of two sets. Recently, in [7] the authors attached binary block codes to such an algebra. At the end of the article, they raised a question about the possibility to obtain a BCK-algebra starting from a binary block code. In [4], the authors have shown that in some special conditions, the answer to the question is yes, we can obtain BCK-algebras from binary block codes. Given these things, it has become interesting to study how BCK-algebras can be built and if there is a way to generate them using a computer. In [11], the author presents two new extensions for BCK-algebras and shows that the structure of the BCK-algebras class has the shape of a graph formed by three intersecting trees, named BCK-trees. The author also shows that BCK-algebras are divided in these trees by their properties. This means that we have the tree of commutative BCK-algebras, the tree of positive implicative BCK-algebras and that of implicative BCK-algebras. Moreover, the author shows that when two trees intersect each other, the BCK-algebra, which constitutes the intersection, loses properties, like in the Cayley Dickson derivation process for algebras over a field.

In ([8], pp. 249 - 256), the author constructs error-correcting codes attached to Hilbert algebras, based on the equivalence between positive implicative BCK-algebras and Hilbert algebras. In [5], starting from the equivalence between BCK bounded commutative algebras, Wajsberg and MV-algebras, the authors build binary block codes attached to Wajsberg and MV-algebras. In another paper, [1] authors give an algorithm for building Wajsberg Algebras, and, based on this,

in [2] the authors present an algorithm for building BCK bounded commutative algebras. In the first part of this paper, we present a new extension for BCK-algebras, a generalization for Iseki Extension and Pseudo-Iseki extension. In the second part, we show new identities that hold for positive implicative BCK-algebras. These identities were found using examples offered by a computer algorithm, which is presented in the last part of the paper.

2. Preliminaries

Definition 2.1 ([9]). *A set* $(X, *, 0)$ *that meets the following conditions:*

1. $((x * y) * (x * z)) * (z * y) = 0$, $\forall x, y, z \in X$;
2. $(x * (x * y)) * y = 0$, $x, y \in X$;
3. $x * x = 0$, $\forall x \in X$;
4. *If* $x * y = 0$ *and* $y * x = 0$, *then* $x = y$, $\forall x, y \in X$;

is a BCI-algebra

Definition 2.2 ([9]). *A BCI-algebra that meets the following relation is a BCK-algebra:*

5. $0 * x = 0$, $\forall x \in X$.

Proposition 2.1 ([10]). *Let* X *be a BCI-algebra. Then the following identities are true:*

6. $(x * y) * z = (x * z) * y$;
7. $(x * y) * x = 0$;
8. $((x * z) * (y * z)) * (x * y) = 0$;
9. $x * 0 = x$.

An alternative definition for the BCI-algebra and BCK-algebra is given in the following proposition:

Proposition 2.2 ([3], Proposition 3.1). *A BCI-algebra is defined also by the following axioms: 1, 4, 9,*
A BCK-algebra is defined by the following axioms: 1, 4, 9, 5.

Definition 2.3 ([9]). *A BCI-algebra* $(X, *, 0)$ *is commutative if the relation* $x * (x * y) = y * (y * x)$, *is true for all* $x, y \in X$.

Definition 2.4 ([10]). *A BCI-algebra* $(X, *, 0)$ *is positive-implicative, if the following relation is true:* $(x * z) * (y * z) = (x * y) * z$, *for all* $x, y, z \in X$.

Definition 2.5 ([9]). *A BCI-algebra $(X, *, 0)$ is implicative if the relation $x * (y * x) = x$ is true for all $x, y \in X$.*

Remark 2.1 ([9]). On a BCI-algebra, a partial order relation $<$ can be defined by $x < y$ if and only if $x * y = 0$.

Definition 2.6 ([9]). *Let X be a BCI / BCK-algebra. If there is an element 1 with $x < 1, \forall x \in X$, then X is called a bounded algebra.*

Definition 2.7 ([9]). *Let X be a BCK-algebra. It extends to a Bounded BCK-algebra if we add a 1 element that complies with the following rules:*

1. $1 * 1 = x * 1 = 0, \forall x \in X$;
2. $1 * x = 1, \forall x \in X$.

Remark 2.2 ([9]). If the BCK-algebra X is a positive implicative BCK-algebra, then the Iseki extension is a positive implicative BCK-algebra. If it is commutative, the Iseki extension is not commutative.

In [11], the author introduced a new Iseki like extension, named Pseudo-Iseki extension:

Definition 2.8 ([11]). *Let $(X, *, 0)$ be a BCK-algebra and I an element not in X. The $X[I]$ algebra, obtained using following rules is a BCK-algebra:*

1. $I * I = 0$;
2. $0 * I = 0$;
3. $x * I = x, \forall x \in X$;
4. $I * x = I, \forall x \in X$.

Remark 2.3 ([11]). If X is a commutative BCK-algebra, then the Pseudo-Iseki extension is commutative. If X is a positive implicative BCK-algebra, the Pseudo-Iseki extension is also positive implicative.

3. The generalized Iseki Construction of a BCK-algebra

Based on the new extension presented in [11], we will give in the following a generalized construction. We will show that the Iseki and Pseudo-Iseki extensions are particular cases of this generalized construction.

Proposition 3.1. *Let $(X, *, 0)$ be a BCK-algebra and x, y, z three elements in X with the following properties: $x * y = x$, $z * y = 0$. Then $x * z = x$.*

Proof. From axiom 1 of the BCK-algebra definition, we have: $((x * y) * (x * z)) * (z * y) = 0$. For the three elements mentioned, the expression is reduced to: $(x * (x * z)) = 0$, because $x * y = x$ and $z * y = 0$. In every BCK-algebra we have $(x * z) * x = 0$, whence $x = x * z$, according to the axiom 4 of BCK-algebras. □

Proposition 3.2. *Let X be a BCK-algebra and d an element not in X, with the following properties:*

1. $0 * d = 0$;
2. $d * d = 0$;
3. $d * x = d$, $\forall x \in X$;
4. $x * d = 0$ *or* $x * d = x$, $\forall x \in X$;
5. *If we have x in X with $x * d = 0$, then $(x * y) * d = 0$, $\forall y \in X$;*
6. *If there are two elements $x, z \in X$, with the following properties: $x * d = x$ and $z * d = 0$, then $x * z = x$.*

Algebra $X[D]$ is a BCK-algebra.

Proof. The axioms 3, 4, 5 are fulfilled from the construction process. It remains to prove the axioms 1 and 2. For the axiom 2, we have the following cases:

- $(x * (x * d)) * d$: $x * d$ can be x or 0. If $x * d = 0$, the entire expression becomes 0. If $x * d = x$, then the expression becomes $(x * x) * d = 0$;
- $(d * (d * x)) * x$: This is 0 because $d * x = d$ and $d * d = 0$.

For the axiom 1, we have the following expressions:

- $((d * y) * (d * z)) * (z * y)$: Is 0 because $d * y = d$, $d * z = d$ and $d * d = 0$;
- $((x * d) * (x * z)) * (z * d)$: We have multiple cases:
  If $x * d = 0$, the entire expression is 0. If $x * d = x$, the expression becomes: $(x * (x * z)) * (z * d)$. If $z * d = z$, then the expression is $(x * (x * z)) * z$, which is 0 from the axiom 2 of the algebra X. If $z * d = 0$ the expression is $x * (x * z)$. From the point 6 of the construction, if $x * d = x$ and $z * d = 0$, $x * z$ must be x, so the expression is $x * x = 0$;
- $((x * y) * (x * d)) * (d * y)$: $d * y = d$ and the expression is: $((x * y) * (x * d)) * d$. We have two cases: $x * d = x$ and $x * d = 0$. For $x * d = x$ the expression is: $((x * y) * x) * d$, which is 0, because in every BCK-algebra $(x * y) * x = 0$. If $x * d = 0$, the expression is $(x * y) * d$, which is 0 from the fifth point of the construction.
- The other cases are 0 because they contain identical elements on nearby positions. □

Definition 3.1. *The above construction will be named the generalized Iseki construction.*

Proposition 3.3. *If X is a positive implicative BCK-algebra, the BCK-algebra obtained through the generalized Iseki Construction, is a positive implicative BCK-algebra.*

Proof. We have the cases:

1. $(d*z)*(y*z) = (d*y)*z$: $d*y = d$, $d*z = d$ and the expression reduces to $d*(y*z) = d*z$. $d*z = d$, $d*(y*z) = d$, so the equality holds;

2. $(x*d)*(y*d) = (x*y)*d$: Here we have more cases: If $x*d = 0$, from the point 5 of the construction $(x*y)*d = 0$. If $x*d = x$ and $y*d = 0$, the expression reduces to $x = (x*y)*d$. But from point 6 of the construction, we have that: if $x*d = x$ and $y*d = 0$, then $x*y = x$. So the expression becomes $x = x*d$, having that $x*d = x$, the entire equality holds. If $x*d = x$ and $y*d = y$, the expression becomes $x*y = (x*y)*d$. But, we have already a BCK-algebra so the expression can be rewritten as $x*y = (x*d)*y$. But $x*d = x$, so the expression reduces to $x*y = x*y$;

3. $(x*z)*(d*z) = (x*d)*z$: $d*z = d$ and the expression becomes $(x*z)*d = (x*d)*z$. Which is equivalent with $(x*z)*d = (x*z)*d$;

4. $(d*d)*(y*d) = (d*y)*d$: $d*d = 0$, $d*y = d$, so the expression is 0 in both parts;

5. $(x*d)*(d*d) = (x*d)*d$: $d*d = 0$, and the expression becomes: $(x*d) = (x*d)*d$. We have two cases, $x*d = 0$, both parts becoming 0 and $x*d = x$, when the expression becomes $x = x*d$.

□

Proposition 3.4. *Let X be a commutative BCK-algebra. The BCK-algebra obtained through the generalized Iseki construction is not always commutative.*

Proof. We have the following expressions:

1. $x*(x*d) = d*(d*x)$: $d*x = d$, $d*d = 0$, so in the right hand we have 0. $x*d$ can be x, whence $x*x = 0$, and the expression holds. $x*d$ can be 0, but $x*0 = x$, and the expression does not hold anymore;

2. $d*(d*y) = y*(y*d)$: In this case $d*y = d$ and $d*d = 0$ and $y*d$ can be$y$, in which case $y*y = 0$ and the expression holds. But, if $y*d = 0$, $y*0 = y$ and the expression does not hold any more. □

Remark 3.1. Let X be a commutative BCK-algebra and Y the BCK-algebra obtained from X through the generalized Iseki Construction. Y is commutative if and only if for all $x \in X$, $x*d = x$. Reformulating, y is commutative if there is not any x for which $x*d = 0$.

Remark 3.2. Let X be a commutative BCK-algebra and Y the BCK-algebra obtained from X through the generalized Iseki Construction. If $x * d = 0$, for all $x \in X$, we have the Iseki Extension of X. If $x * d = x$ for all $x \in X$, we have the Pseudo-Iseki extension of X (see [11]).

Remark 3.3. From the above remarks, we observe that the Pseudo-Iseki extension is the only case that maintains commutativity. On the other hand, we have noticed that the generalized Iseki Constructions can not be applied to any BCK-algebra, in total contrast with its particular cases.

Indeed, to the following BCK-algebra, we can not apply one of the cases of the generalized Iseki construction:

Alg 1.

$*$	0	1	2
0	0	0	0
1	1	0	0
2	2	1	0

Alg 2.

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	1	0	2
3	3	3	3	0

, because this is not longer a BCK-algebra.

Remark 3.4. The generalized Iseki construction can be applied to all BCK-algebras from the 2nd and 3rd BCK-trees (see, [11]), the BCK-trees that contains positive implicative BCK-algebras while they do not intersect with the 1st BCK-tree. Thus, this construction can be applied while the source BCK-algebra does not contain a sub-algebra of order 3 isomorphic to the BCK-algebra from Alg 1.

4. New results about positive implicative BCK-algebras

In the following, we will present some new relations about elements in positive implicative BCK-algebras and an interesting way for building BCK-algebras. In [4], Proposition 2.1, a set with a binary relation is taken. There the author shows that, in some conditions, the set is a non-commutative, non-implicative BCK-algebra.

Proposition 4.1 ([4])**.** *Let A be a partial ordered set, and $*$ a binary relation on A with the following properties:*

1. $0 * x = 0$;
2. $x * x = 0$;
3. $x * y = 0$ *if and only if* $x \leq y$;

4. $x * y = x$ *otherwise;*

A together with $*$ *is a positive implicative BCK-algebra.*

Proof. First, we should prove that this set is a BCK-algebra, then we should prove that is a positive implicative one. From the conditions, we observe that the third, fourth and fifth axioms are fulfilled. The second axiom says that $(x * (x * y)) * y = 0$. Here we have two cases:

1. $x * y = 0$: $(x * (x * y)) * y = (x * 0) * y = x * y = 0$;
2. $x * y = x$: $(x * (x * y)) * y = (x * x) * y = 0 * y = 0$.

Now we arrive to the first axiom: $((x * y) * (x * z)) * (z * y) = 0$. There are two cases:

1. $x * y = 0$: $((x * y) * (x * z)) * (z * y) = (0 * (x * z)) * (z * y) = 0$;
2. $x * y = x$: $((x * y) * (x * z)) * (z * y) = (x * (x * z)) * (z * y)$. Here we also have two cases:
   (a) $x * z = 0$: $(x * (x * z)) * (z * y) = x * (z * y)$. In this moment we have that y is not greater than x, but z is greater than x. So y can not be greater than z, because then y would be greater than x. Whence $z * y = z$. $x * (z * y) = x * z = 0$;
   (b) $x * z = x$: $(x * (x * z)) * (z * y) = (x * x) * (z * y) = 0$.

Now we should prove that this is a positive implicative BCK-algebra. For this we should have $(x * z) * (y * z) = (x * y) * z$. We have two cases:

1. $x * z = 0$: $(x * z) * (y * z) = 0 * (y * z) = 0$. $(x * y) * z = (x * z) * y = 0 * y = 0$;
2. $x * z = x$: $(x * z) * (y * z) = x * (y * z)$. $(x * y) * z = (x * z) * y = x * y$. In this case, z is not greater than x. On the other hand, we can have $y * z = 0$ and $y * z = y$. In the last case, we obtain $x * y = x * y$. In the case when $y * z = 0$, we have $x = x * y$. But in this case $x * y$ can not be 0, because that means $x \leq y \leq z$ which contradicts our hypothesis $x * z = x$. So, $x * y = x$ in this case and the proof is complete. □

The interesting question is, when the converse is true, in what conditions a positive implicative BCK-algebra has this explicit general form. At a first sight, the answer is simple, using that in any BCK-algebra $x * y \leq x$, to obtain the desired form we should impose that $x * (x * y) = 0$ and $y * (y * x) = 0$, or $x * (x * y) = x$ and $y * (y * x) = 0$. For the last part the symmetric case is also possible: $x * (x * y) = 0$ and $y * (y * x) = y$. However, a deeper look shows us some interesting things about positive implicative BCK-algebras and the behaviour of their elements. In particular, these new things give us the possibility to build some higher-order positive implicative BCK-algebras. For this, we will give in the following a theorem with new relations regarding positive implicative BCK-algebras.

Theorem 4.1. *Let X be a positive implicative BCK-algebra. The following relations are true for any $x, y \in X$:*

1. $x * (x * y) \leq y * (x * y)$;
2. $x * (y * (x * y)) = x * y$;
3. $(y * (y * x)) * (y * (x * y)) = y * (y * (x * y))$;
4. $(x * (x * (y * x))) * (x * y) = (x * (x * (y * x)))$;
5. $(x * (x * y)) * (y * (y * x)) = x * (x * (y * x))$;
6. $x * (y * (y * (x * y))) = x$;
7. $X * (x * (y * x)) = x * (x * (y * (x * (x * y))))$;
8. $(x * (y * (x * (x * y)))) = x * (y * x)$.

Proof. 1. From the positive implicative BCK-algebra definition we have: $(x * (x * y)) * (y * (x * y)) = (x * y) * (x * y) = 0$. The symmetric relation is also true: $(y * (y * x)) \leq x * (y * x)$;

2. $y*(x*y) \leq y$. Hence $x*y \leq x*(y*(x*y))$. But $x*(y*(x*y))*(x*y) = (x*(x*y))*(y*(x*y))$. From the point 1, this is 0 and, by means of the fourth axiom, it results that $x * (y * (x * y)) = (x * y)$.

3. $(y*(y*(x*y)))*((y*(y*x))*(y*(x*y)))= (y*(y*(y*x)))*(y*(x*y))= (y * x) * (y * (x * y)) = 0$. The other inequality: $((y * (y * x)) * (y * (x * y))) * (y * (y * (x * y)))= (y * (y * x) * y) * (y * (x * y)) = 0$.

4. $(x * (x * y)) * (x * (y * x))= (x * (x * (y * x))) * (x * y)\leq x * (x * (y * x))$. Conversely:

   $(x*(x*(y*x)))*((x*(x*y))*(x*(y*x)))= (x*(x*(x*y)))*(x*(y*x))\leq (x * y) * (x * (y * x)) \leq 0$.

5. $(x * (x * y)) * (y * (y * x))$ can be written as: $(x * (y * (y * x))) * (x * y)$. $(x * (x * (y * x)))$ can be written as: $(x * (x * (y * x)) * (x * y)$. $(x * (y * (y * x)))*(x*y)*(x*(x*(y*x)))= ((x*(y*(y*x)))*(x*(x*(y*x))))*(x*y)= ((x*(x*(x*(y*x))))*(y*(y*x)))*(x*y)= ((x*(y*x))*(y*(y*x)))*(x*y)= ((x * y) * (y * x)) * (x * y)= ((x * y) * (x * y)) * (y * x) = 0$. Conversely: $((x*(x*(y*x)))*(x*y))*((x*(y*(y*x)))*(x*y)) = ((x*(x*(y*x)))*(x*(y*(y*x))))*(x*y) \leq ((y*(y*x))*(x*(y*x)))*(x*y) \leq 0*(x*y) = 0$.

6. According to the first BCK-algebra axiom, we have: $(y*(y*(x*y))) \leq y*(y*x)$. From [9] we have that: $x*(y*(y*x)) \leq x*(y*(y*(x*y))) \leq x$. From the same result we have that: $x*(x*(y*(y*(x*y)))) \leq x*(x*(y*(y*x)))$. On the other hand: $x*(x*(y*(y*x))) \leq x*(y*(y*(x*y)))$. So, we have

that: $(x*(x*(y*(y*(x*y)))))*(x*(x*(y*(y*x)))) = 0$. At the same time, $(x*(x*(y*(y*(x*y)))))*(x*(x*(y*(y*x)))) = (x*(x*(y*(y*(x*y))))))$. Therefore: $(x*(x*(y*(y*(x*y)))))) = 0$, whence $x = (x*(y*(y*(x*y))))$.

7. $(x*(x*(y*x)))*(x*(x*(y*(x*(x*y))))) \leq (x*(y*(x*(x*y))))*(x*(y*x)) \leq (y*x)*(y*(x*(x*y))) \leq (x*(x*y))*x = 0$. Conversely: $(x*(x*(y*(x*(x*y))))) = (x*(x*(y*(x*(x*y)))))*(x*y)$. $(x*(x*(y*x))) = (x*(x*(y*x)))*(x*y)$. $((x*(x*(y*(x*(x*y))))))*(x*y))*((x*(x*(y*x))))*(x*y) = ((x*(x*(y*(x*(x*y))))))*(x*(x*(y*x))))*(x*y)$. $((x*(x*(y*(x*(x*y))))))*(x*(x*(y*x))))*(x*y) \leq ((x*(y*x))*(x*(y*(x*(x*y)))))*(x*y) \leq ((y*(x*(x*y)))*(y*x))*(x*y) \leq (x*(x*(x*y)))*(x*y) \leq (x*y)*(x*y) = 0$.

8. It emerges from Point 7 if we multiply with x in both parts. □

Remark 4.1. As we have proven the above relations for any x, y, by changing x with y, we get the symmetric relations being true too.

Proposition 4.2. *Let X be a positive implicative BCK-algebra and $x, y \in X$ with the property that $x * y = 0$, then $y * (y * x) = x * (y * x)$.*

Proof. From Theorem 4.1 we have: $(y*(y*x)) \leq (x*(y*x))$. From the first axiom of BCK-algebras, we have already that: $((x*(y*x))*(y*(y*x))) \leq (x*y)$, thus $((x*(y*x))*(y*(y*x))) \leq 0$. From the fifth axiom, we have that the above expression is 0. From the fourth axiom, it results that the two expressions are equal. □

Remark 4.2. The equality can be obtained directly from the positive implicative BCK-algebra definition as follows: $((x*(y*x))*(y*(y*x))) = (x*y)*(y*x)$. Through the chosen proof, we have shown that the second inequality is true in any BCK-algebra, for any two elements x, y with the property that $x*y = 0$. this inequality becomes equality only when the BCK-algebra is a positive implicative one, for the elements x, y with the property that $x * y = 0$.

Remark 4.3. The result can also be obtained from Theorem 4.2, Point F from [9], where we have that in a positive implicative BCK-algebra $(y*(y*x))*(x*y) = (x*(x*y))*(y*x)$, by taking $(x*y) = 0$.

Now, We will see how the elements of a positive-implicative BCK-algebra behave in certain conditions.

Proposition 4.3. *Let X be a positive implicative BCK-algebra and $x, y \in X$ with $x*y = 0$, $y*x \neq 0$ and $x*(y*x) \neq 0$. The subalgebra generated by x, y is of order 6 and contains a subalgebra of order 4 which is bounded implicative.*

Proof. We have that $x*y = 0$, $y*x \neq 0$ and $x*(y*x) \neq 0$. We will use the following notations:

1. $1 = x * (y * x) = y * (y * x)$ from Proposition 4.2;
2. $2 = x * (x * (y * x))$;
3. $3 = x$;
4. $4 = y * x$;
5. $5 = y$;

Using these elements, together with 0 and the relations from theorem 4.1, we obtain a BCK-algebra of order 6 with the following multiplication table:

Alg 3.

$*$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	1	0	1	0	1	0
2	2	2	0	0	0	0
3	3	2	1	0	1	0
4	4	4	4	4	0	0
5	5	4	5	4	1	0

This algebra is a BCK-algebra of order 6. Moreover, we see that it contains a bounded implicative BCK-algebra of order 4. □

Remark 4.4. How we notice, the algebra from Alg 3, is not in the desired form, the one specified in 4.1. The condition used in the statement of the previous proposition made this happen. If we take $x, y \in X$, with $x * y = 0$ and $x * (y * x) = 0$, this implies that $y * (y * x) = 0$ and the multiplication table modifies as follows:

Alg 4.

$*$	0	3	5
0	0	0	0
3	3	0	0
5	5	5	0

This is a BCK-algebra of order 3, which has the desired form. Reformulating, when $x * y = 0$ $y * x$ must be y. Any other combination of elements does not offer us the desired form. For example if we take $x, y \in X$ with $x * y = 0$ and $x * (y * x) = x$ the table will look like:

Alg 5.

$*$	0	3	4	5
0	0	0	0	0
3	3	0	3	0
4	4	4	0	0
5	5	4	3	0

This is a bounded implicative BCK-algebra of order 4.

We have seen until now what happens in a positive implicative BCK-algebra when we have two elements x, y with the property that $x * y = 0$. Now, we should see which is the behaviour of two elements that have no relation between them.

Proposition 4.4. *Let X be a positive implicative BCK-algebra and $x, y \in X$ two elements with the property $x * y \neq 0$ and $y * x \neq 0$. If $x * (y * x) \neq (x * y)$ and $x * (y * x) \neq x$, then the subalgebra generated by x, y is a BCK-algebra of order 14 which contains a bounded implicative subalgebra of order 4.*

Proof. To prove this, we will use the relations from Theorem 4.1 and other known results about positive - implicative BCK-algebras, mentioned in [9]. We will use the following notations:

1. $1 = y * (y * (x * y))$;
2. $2 = (y * (y * x)) * (x * y)$;
3. $3 = y * (y * x)$;
4. $4 = (x * y)$;
5. $5 = x * (y * x)$;
6. $6 = x * (x * (y * x))$;
7. $7 = x * (x * y))$;
8. $8 = x * (y * (y * x))$;
9. $9 = x$;
10. $10 = y * x$;
11. $11 = y * (x * y)$;
12. $12 = y * (x * (x * y))$;
13. $13 = y$;

Together with the element 0, after some computations, we obtain the following multiplication table:

Alg 6.

$*$	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	0	1	0	0	0	1	1	0	0	1	1	0	0
2	2	2	0	0	2	0	2	0	2	0	2	0	2	0
3	3	2	1	0	2	0	3	1	2	0	3	1	2	0
4	4	4	4	4	0	0	4	4	0	0	4	4	4	4
5	5	5	4	4	2	0	5	4	2	0	5	4	5	4
6	6	6	6	6	6	6	0	0	0	0	0	0	0	0
7	7	7	6	6	7	6	2	0	2	0	2	0	2	0
8	8	8	88				4	4	0	0	4	4	4	4
9	9	9	8	8	7	6	5	4	2	0	5	4	5	4
10	10	10	10	10	10	10	10	10	10	10	0	0	0	0
11	11	11	10	10	11	10	11	10	11	10	2	0	2	0
12	12	10	12	10	10	10	12	12	10	10	1	1	0	0
13	13	11	12	10	11	10	13	12	11	10	3	1	2	0

This is a BCK-algebra of order 14, which contains a bounded implicative subalgebra of order 4. □

Remark 4.5. We notice that the algebra in Alg 6 is not in the desired form, the one specified in 4.1. The form specified there says that $x*y$ should be x if it is not 0. So, in the above table, we will take $x * y = x$ and see how the table modifies. If $x * y = x$ this means that $x * (x * y) = 0$ which implies that any element less than 7, is 0. This means that 2 and 6 are 0. Also, $x*(y*x) = x*(y*(y*x)) = x$ so $4 = 5 = 8 = 9$. The implied relation makes $1 = 3$, $11 = 10$ and $12 = 13$. Therefore $1, 2, 4, 5, 6, 7, 8, 11, 12$ will disappear from the table.

It will remain a BCK-algebra of order 5:

Alg 7.

$*$	0	3	9	10	13
0	0	0	0	0	0
3	3	0	0	3	0
9	9	9	0	9	9
10	10	10	10	0	0
13	13	10	10	3	0

How we see, the algebra is not in the desired form. For this, we should go further and impose that $y * (y * x) = 0$. This makes 3 and 10 disappear from the table.

Alg 8.

$*$	0	9	13
0	0	0	0
9	9	0	9
13	13	13	0

This way, we obtained an implicative BCK-algebra of order 3, which has the desired form.

5. An algorithm to generate BCK-algebras of order n with $n \leq 6$ using a computer

In building the results from the previous sections, we used a computer algorithm, which, up to our knowledge, is the first algorithm for generating BCK-algebras. We will present in this section the algorithm we used and some improvements for it. This algorithm is based on findings presented in [11], where the author shows that the higher-order BCK-algebras can be built through extensions from the lower-order ones. Also, the author shows that there are three BCK-algebras of order 3, so every higher-order BCK-algebra must contain one or more of these BCK-algebras of order 3. In conclusion, the author shows that the BCK-algebras are organized on three trees, named BCK-trees, each tree having as a root one of the tree BCK-algebras of order 3. An important thing about these trees is that every tree contains BCK-algebras with distinct properties. Thus, we have the tree of commutative BCK-algebras (tree NO. 1), the tree of positive implicative BCK-algebras (tree NO. 2) and the tree of implicative BCK-algebras (tree NO. 3). The author shows that these three trees intersect each other. The intersection points are represented by Isomorphic BCK-algebras. Moreover, from each point of intersection, the higher-order BCK-algebras lose one of their properties. Starting from this point, we concluded that we could write an algorithm that starts with one of the three algebras of order 3 and builds the BCK-algebras of order 4. Once it has found all BCK-algebras of order 4, it takes each BCK-algebra of order 4 and builds the BCK-algebras of order 5, which contains that BCK-algebra of order 4. The technique resembles a Deep-first search for graphs. In [11], the author discusses the complexity of such an algorithm, showing that it is $O((N^{n-2})^2)$ for each level of BCK-algebras, the total complexity being the product of complexities for each level.

5.1 Algorithm description

For finding the BCK-algebras of order N, starting from a BCK-algebra of order $N-1$, the algorithm takes the multiplication table of that BCK-algebra, which is a table with $N-1$ lines and $N-1$ columns, and borders it with one new line and one new column. On this new line and new column, the corner elements have fixed values, according to the BCK axioms: $[N-1,0] = N-1$, $[0,N-1] = 0$, $[N-1,N-1] = 0$. So, on the new line and new column there remain $(n-2)$ positions for each, positions that can vary between 0 and $N-1$ (see, [11]). Also, in establishing these corner values, we used the alternative definition for BCI and BCK-algebras, see 2.2. This way, we build a candidate table, with N lines and N columns, that satisfies the axioms $2, 3, 5$ and $x * 0 = x$ (see 9). In the next step, the algorithm iterates between 0 and $N-1$ for each of these $(N-2)*(N-2)$ positions, building this way all possible tables that can contain the multiplication table of the specified BCK-algebra. In this iteration process, the algorithm verifies that the built table satisfies the fourth axiom by checking if there are zeros on correspondent positions on the new line and column. This

way, even from the construction process of the table, the constructed tables satisfy the axioms 2, 3, 4, 5 of the BCK-algebra definition. The only axiom that must be checked is 1, but this axiom should be checked only for the newly introduced element in relation with the other elements from the BCK-algebra of order $N-1$. It is not necessary to check again that the $N-1$ elements satisfy the 1st axiom among them because we know that already from the previous steps.

The code is written in the Python language. It consists of two classes, one for representing logical algebras, one for representing BCK-trees and the main script. The class for logical algebras serves as a base class for all other logical algebras, useful if we will want to extend this software to work with other types of logical algebras too. At this moment, this class builds a logical algebra and can check if the algebra is a BCI or BCK-algebra. Also, the class can check if the BCI or BCK-algebra is commutative, positive implicative, or implicative.

The BCK tree class receives an algebra as a start point and builds all BCK-algebras of greater order starting from this algebra it received. It generates algebras up to an order received as a parameter for the generating function. For the generation process, two recursive functions are used, one which generates each order of BCK-algebras and one which generates all possible tables for a specific order.

The main script builds the 3 BCK-algebras of order 3, builds 3 instances of BCK Tree class, one for each 3 order algebra, and calls for each one the generating function with a parameter representing the desired order. The generating function calls itself recursively for each level up to specified order.

5.2 Algorithm performance

This is the first version of the algorithm. Various improvements are possible. The first improvement we made is that we used multiprocessing to parallelize the process of BCK-tree generation. These tree BCK-trees can be generated in parallel, independent one from another. The algorithm could be parallelized further at the table generation process level. Another improvement could be the introduction of a storage component to store all the generated BCK-algebras. This way, when someone asks for BCK-algebras of a specific order, the algorithm will check if it already generated them. If yes, it will serve them to the user. If not, it will check which is the last order it generated and will start to generate the BCK-algebras from the next order up to the order demanded by the user. For example, if the user asks for BCK-algebras of order 9 and the algorithm finds that it has generated all the BCK-algebras up to order 7, it will start to generate the BCK-algebras of order 8 and 9.

We have run the algorithm on four different types of machines and configurations to generate BCK-algebras of order 4, 5, 6, both in sequential and multiprocessing executions. There were two runs, one asking for BCK-algebras

up to order 5, another asking for BCK-algebras up to order 6. The machines used are:

1. Raspberry pi 4 - Arm 72 CPU, four cores at 1.5 GHz, 4 GB of RAM, with SD card Class 10 US3 Storage, having Raspbian 10 as OS;
2. Laptop - Intel I5 3210M CPU, two cores at 2.5 GHz, 12 GB of RAM, with SSD storage, having Windows 10 as OS;
3. PC - Intel I5 2400, four cores at 3.1 GHz, 8 GB of RAM, with SSD storage, having Windows 10 as OS;
4. VPS - Intel Xeon E5 2400 CPU, six cores at 2.2 GHz, 16 GB of ram, with SSD storage, having UBUNTU 20.04 as OS;

The runing times, in seconds, are presented in the following table, where (P) means parallel and (S) means sequential:

Machine	5 order (P)	5 order (S)	6 order (P)	6 order (S)
Raspberry Pi	2.71	6.90	5131.58	12442.49
Laptop	3.55	5.55	3884.66	7033.84
PC	2.49	4.42	3722.74	7407.54
VPS	1.16	2.94	2057.84	4676.01

5.3 Algorithm source code

In the following, we present the algorithm source code.

Listing 1: Logical Algebra class

```
class LAlgebra(object):
    '''
    classdocs
    '''

    def __init__(self,dim,den,alg=None):

#the dimension of the algebra
        self.n=dim
#the name of the algebra (id)
        self.den=den
        #initialises the table of the algebra with 0
        self.t= [x[:] for x in [[0]\ast dim]\ast dim]
#copy the source algebra if one is given
        if alg is not None:
            for i in range(0,alg.n):
                for j in range(0,alg.n):
                    self.t[i][j]=alg.t[i][j]
```

```
            for i in range(alg.n,dim):
                for j in range(0,dim):
                    self.t[i][j]=0;
                    self.t[j][i]=0;

#displays the algebra as simple text
    def __str__(self):
            return
            ('\n'+self.den+'\n')+('\ast|'+(''.join(
            ['{:4}'.format(i)\
            for i in range(0,self.n)])))
            +'\n'+('-|'+(''.join(
            ['----' for i in range(0,self.n)])))
            +'\n'+('\n'.join(
            [str(row[0])+'|'+(''.join(
            ['{:4}'.format(item) for item in row]))\
             for row in self.t]))

            #checks  if the logical algebra
#satisfies the 5th axiom
#and x\ast0=x in the same time
    def CheckAxiomBCI59(self):
        for i in range(0,self.n):
            if self.t[0][i]!=0 or  self.t[i][0]!=i:
                return False
        return True

            #checks if the logical algebra satisfies
           #x\ast0=x
    def CheckAxiomBCI9(self):
        for i in range(0,self.n):
            if self.t[i][0]!=i:
                return False
        return True

#checks if a logical algebra satisfies the 4th BCK axiom
    def CheckAxiomBCI4(self):
        for i in range(0,self.n):
            for j in range(0,self.n):
                if i!=j and
                self.t[i][j]==0 and self.t[j][i]==0:
                    return False
        return True
```

```
#checks if a logical algebra
#satisfies the first BCK axiom
    def CheckAxiomBCI1(self):
        for i in range(0,self.n):
            for j in range(0,self.n):
                for k in range(0,self.n):
                    if self.t[self.t[
                       self.t[i][j]][
                       self.t[i][k]]][self.t[k][j]]!=0:
                           return False
        return True

#checks if an extension table
#of a BCK-algebra satisfies the first BCK axiom
    def CheckAxiomBCI1ForExt(self):
        for i in range(0,self.n):
            for j in range(0,self.n):
                #if i!=j:
                if i!=j and (
                self.t[self.t[
                self.t[self.n-1][i]][
                self.t[self.n-1][j]]][
self.t[j][i]] or
                self.t[self.t[
                self.t[i][self.n-1]][
               self.t[i][j]]][self.t[j][self.n-1]] or
                self.t[self.t[
                self.t[i][j]][self.t[i][self.n-1]]][
                self.t[self.n-1][j]]):
                    return False
        return True

#checks if the logical algebra is a BCK-algebra
    def IsBCK(self):
        return self.CheckAxiomBCI59() and
        self.CheckAxiomBCI4() and self.CheckAxiomBCI1()

#check if the Logical Algebra
# is a BCI-algebra
    def IsBCI(self):
        return self.CheckAxiomBCI9() and
        self.CheckAxiomBCI4() and self.CheckAxiomBCI1()
```

```
#checks if the BCK-algebra is commutative
    def isBCKCom(self):
        for i in range(0, self.n):
            for j in range(0, self.n):
                if(self.t[i][self.t[i][j]]
               !=self.t[j][self.t[j][i]]):
                    return False
        return True

#checks if the BCK-algebra is implicative
    def IsBCKImp(self):
        for i in range(0, self.n):
            for j in range(0, self.n):
                if(self.t[i][self.t[j][i]]!=i):
                    return False
        return True

#checks if the BCK-algebra is positive implicative
    def IsBCKPoz(self):
        for i in range(0,self.n):
            for j in range(0,self.n):
                for k in range(0,self.n):
                    if self.t[self.t[i][k]][self.t[j][k]]
                    !=self.t[self.t[i][j]][k]:
                        return False
        return True

#Checks the properties of the BCK-algebra
# and set them accordingly
    def CheckBCKProperties(self):
        self._bckcom=False
        self._bckimp=False
        self._bckpoz=False
        if self.IsBCKImp():
            self._bckcom=True
            self._bckpoz=True
            self._bckimp=True
            self.den=self.den+"_[I]"
        elif self.isBCKCom():
            self._bckcom=True
            self.den=self.den+"_[C]"
        elif self.IsBCKPoz():
            self._bckpoz=True
            self.den=self.den+"_[P}"
```

Listing 2: BCK Tree class

```
class BCKTree(object):
    def __init__(self,alg):
#the starting algebra for which the BCK-tree is build
        self.alg=alg
#the list of next level (nl) BCK-algebras
        self.nl=[]
#recursive function used to
#generate a table with N lines and N columns
#starting from a BCK-algebra of order N-1
#The function iterates from 0 to N-1
#for each position from the last line and column,
#checking if it has found a BCK-algebra
#s is the current free position
    def GenerateTable(self,s,a):
#the dimension of the source algebra
        d=a.n
#iterates over each value from 0 to n-1
        for i in range(0,d):
#if the function is on the last column free positions
            if s>d-2:
#checks if the 4th axiom is satisfied, namely
#the corespondent positions
#on the last line and the last column
#should not be 0 in the same time
#if there is already 0 on the last line, skip
                if i==0 and a.t[d-1][s+2-d]==0:
                    continue;
                #set the position: s+2-d:d-1
#on the last column
                a.t[s+2-d][d-1]=i

            else:
#set the position: d-1:s
#on the last line
                a.t[d-1][s]=i
#check if all the free positions received a value
            if s==2\ast d-4:
#check if the axiom 1 is satisfied by the table build
                if a.CheckAxiomBCI1ForExt():
                    pos=len(self.nl)-1
                    e=len(self.nl[pos])+1
#put the new found BCK-algebra in the list of algebras
                    self.nl[pos].append(
```

```
                    LAlgebra(d,a.den+'-'+str(e),a))
            else:
#if not all the free positions received a value,
# the function call itself for the next free position
                self.GenerateTable(s+1,a)

        return

#a recursive function used to
#generate the list of BCK-algebras for the next level,
#up to the specified level
#parameters:
#alg - the source algebra,
#level the desired level
    def GenerateNextLevel(self,alg,level=4):
#the dimension of the new algebras
        d=alg.n+1
        #check if the specified level is processed
        if d>level: return
#add a empty list for the current level
        self.nl.append([])
        pos=len(self.nl)-1
#construct a new algebra of order n+1
        a=LAlgebra(d,alg.den,alg)
#set the fixed elements on the last line
# and on the last column
        a.t[d-1][0]=d-1

        a.t[d-1][d-1]=0
#call the generate table function
#to find all algebras of order N+1
        self.GenerateTable(1,a)
#print the header of the current level
        print('\n{:5} {:d}\n{:4}\n'.format(
        'Level',d,'Results'))
        e=len(self.nl[pos])
#print all found algebras on the current level
#together with the BCK-algebras of the next level
#that are build from them
        for i in range(0,e):
            self.nl[pos][i].CheckBCKProperties()
            print('\n{:6d}'.format(i+1))
            print(self.nl[pos][i])
#autocall to generate next level BCK-algebras
```

```
            self.GenerateNextLevel(self.nl[pos][i],level)
```

Listing 3: main script

```
if __name__ == '__main__':
#import lalgebra class
    from LAlgebra import \ast
#import the timer library
    from timeit import default_timer as timer
#import the multiprocessing library
    import multiprocessing
#import the defs module
#this module contains a definition for a function
# which starts the build process of a BCK tree
#this separation was needed
# to be able to run multiprocessing
    import defs as d
#start the timer
    start = timer()
#define a function
#to start the process of BCK tree generation
#using multiprocessing
#How python has a strange multithreading behaviour,
#we used multiprocessing to optain true parallelization
#parameters:
#algl - the start BCK-algebras list,
#level - the desired level
    def mgen(algl,level):
#print the third level header
        print('\n{:5} {:d}\n{:4}\n'.format(
        'Level',3,'Results'))
#define a list of processes,
#each process will generate a BCK tree
        ltree=[]
        for i in range(0,3):
#print the current start BCK-algebra
            print('\n{:6d}'.format(i+1))
            print(algl[i])
#create a new process for the current BCK-algebra
# to generate the desired BCK tree
            thread=multiprocessing.Process(
            target=d.talg,args=(algl[i],level, ))
#add the process in the process list
            ltree.append(thread)
```

```
#start the process
            thread.start()

        for i in range(0,3):
#make the main process wait the other processes to finish
            ltree[i].join()

    #define a sequential execution function
# to start the generate process for BCK-algebras
#parameters:
#algl - the list of start BCK-algebras,
#level - the desired level
    def gen(algl, level):
#define a list of BCK-trees
        ltree=[]
#print the level 3 header
        print('\n{:5} {:d}\n{:4}\n'.format(
        'Level',3,'Results'))
        for i in range(0,3):
#print the current start BCK-algebra
            print('\n{:6d}'.format(i+1))
            print(algl[i])
        #call the function
#which generates a BCK tree
#for current BCK-algebra
            ltree.append(d.talg(algl[i],level))

    #build the list of BCK-algebras of order 3
    algl=[
    LAlgebra(3,'alg-1'),
    LAlgebra(3,'alg-2'),
    LAlgebra(3,'alg-3')]
#set the values for the first BCK-algebra of order 3
    algl[0].t[1][0]=1
    algl[0].t[2][0]=2
    algl[0].t[2][1]=1
#set the values for the second BCK-algebra of order 3
    algl[1].t[1][0]=1
    algl[1].t[2][0]=2
    algl[1].t[2][1]=2
#set the values for the third BCK-algebra of order 3
    algl[2].t[1][0]=1
    algl[2].t[1][2]=1
    algl[2].t[2][0]=2
```

```
    algl[2].t[2][1]=2
#build a list with the two functions variants,
#one for sequential executing, other for multiprocessing
    f=[gen,mgen]
#execute the sequential one
#to generate the BCK-algebras
#of order 4 and 5
#to call the multiprocessing function, change 0 with 1
    f[0](algl,5)
    #stop the timer
    end = timer()
#print the execution time
    print(
'\n Execution time: {} seconds'.format(end - start))
```

Listing 4: Defs script

```
#import the logical algebra class
from LAlgebra import \ast
#define the function
#which will start the BCK-tree generation
#for a specified algebra and a desired order
def talg(alg,level):
#define a bck tree
    tr=BCKTree(alg)
#call the GenerateNextLevel function
    tr.GenerateNextLevel(alg,level)
#return the generated BCK tree
    return tr
```

6. Conclusions

We have introduced in this paper a new way of building higher-order BCK-algebras starting from the lower-order ones, namely the generalized Iseki construction. We have shown that both extensions, the Iseki extension and the new Pseudo-Iseki extension are particular cases of the generalized Iseki construction. We pointed that the generalized Iseki construction can not be applied entirely to all BCK-algebras. Also, we have shown that it preserves the property of positive implicative. Moreover, we have shown that the generalized Iseki construction preserves commutativity only in its particular case, namely the Pseudo-Iseki extension. In further research, it will be interesting to see if such extensions can be used to reduce the necessary time for generating all BCK-algebras of order $\leq N$, where n is a given natural number. Also, we introduced new identities for positive implicative BCK-algebras, and we have shown an algorithm for gener-

ating BCK-algebras. This algorithm was used in finding and proving some of the identities shown in this paper. With further research, the algorithm can be improved through various techniques.

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Geometry of warped product pseudo slant submanifolds in a nearly quasi-Sasakian manifold

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Abstract. In this paper we study distributions of pseudo slant submanifolds of a nearly quasi-Sasakian manifold and obtain some results related to integrability of the distributions and totally geodesic foliations determined by the distributions of pseudo-slant submanifold. Moreover, warped product pseudo slant submanifolds of a nearly quasi-Sasakian manifold are also studied and some characterization theorems for the existence of warped product isometrically immersed into nearly quasi-Sasakian manifolds have been proved.

Keywords: quasi-Sasakian manifold, pseudo slant submanifold, warped products.

1. Introduction

The geometry of warped product CR-submanifolds in a Kaehler manifold was studied by B. Y. Chen [8]. Further, within the different geometric aspects he tried to find the warping function in the sort of some partial differential equations. After that many research articles have been appeared on various kinds

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of warped product submanifolds of different class of structures [1-4, 9, 11-13, 15-17]. In [11], the author established general sharp inequality for the second fundamental form in terms of the warping function y in warped product submanifolds of a nearly Lorentzian para-Sasakian manifold. In 2012, S. Uddin et al. [14] obtained some characterization theorems on the existence or nonexistence of warped product pseudo-slant submanifolds of a nearly cosymplectic manifold in terms of endomorphisms. In the present work we investigate the properties of nontrivial warped product pseudo slant submanifold of the shape $M_\perp \times_y M_\theta$ which are the natural extension of CR-warped product submanifolds. It is clear that each CR-warped product submanifold is a non-trivial warped product pseudo slant submanifold of the shape $M_\perp \times_y M_\theta$ and $M_\theta \times_y M_\perp$ with slant angle $\theta = 0$. But the warped product pseudo slant submanifold never induces the CR-warped product submanifolds. Throughout the paper we consider M $=M_\perp \times_y M_\theta$, where M_θ and $M_\perp$ are proper slant and anti invariant submanifolds, respectively. Finally, we established some necessary and sufficient conditions involving some geometric conditions and properties of nearly quasi-Sasakian manifolds endowed with the warped product pseudo slant submanifolds.

2. Preliminaries

Let $\tilde{M}$ be an odd-dimensional almost contact manifold which carries a tensor field ϕ, a vector field ξ, called characteristic or Reeb vector field and a $1-$form υ satisfying

$$\phi^2 = -I + \upsilon \otimes \xi, \quad \upsilon(\xi) = 1, \tag{2.1}$$

where $I : T\tilde{M} \to T\tilde{M}$ is the identity map, $\phi\xi = 0$, $\upsilon o\phi = 0$ and the $(1,1)$-tensor field ϕ has constant rank $n-1$. An almost contact manifold $\tilde{M}(\phi, \xi, \upsilon)$ is called normal if $N_\phi(Y,Z) + 2d\upsilon(Y,Z)\xi = 0$ or $(\bar{\nabla}_{\phi Y}\phi)Z = \phi(\bar{\nabla}_Y\phi)Z - < \bar{\nabla}_Y\xi, Z > \xi$, where N is the Nijenhuis tensor of ϕ. An almost contact manifold $\tilde{M}(\phi, \xi, \upsilon)$ admits a Riemannian metric $<,>$ such that

$$< \phi Y, \phi Z > = < Y, Z > - \upsilon(Y)\upsilon(Z)\xi, \ \forall\ Y, Z \in \Gamma(T\tilde{M}). \tag{2.2}$$

The induced metric $<,>$ on $\tilde{M}$ with the structure $(\phi, \xi, \upsilon, <,>)$ is said to be an almost contact metric manifold. As an instant significance of (2.2), we have $\upsilon(X) = < X, \xi >$ and $< \phi X, Y > = - < X, \phi Y > = \psi(X,Y)$. Let ξ be a Killing vector field with respect to $<,>$, then the contact metric structure is called a K- contact structure. An almost contact metric manifold $(\tilde{M}, \phi, \xi, \upsilon, <,>)$ is called quasi-Sasakian manifold if

$$(\bar{\nabla}_Y\phi)Z = \upsilon(Z)AY - < AY, Z > \xi, \quad \phi AY = A\phi Y, \tag{2.3}$$

for all $Y, Z \in \Gamma(T\tilde{M})$, where A is the symmetric linear transformation field. From the formula (2.3), it follows that $\tilde{\nabla}_X\xi = \phi AX$. An almost contact metric structure $(\phi, \xi, \upsilon, <,>)$ on $\tilde{M}$ is called a nearly quasi-Sasakian manifold if

$$(\tilde{\nabla}_Y\phi)Z + (\tilde{\nabla}_Z\phi)Y = \upsilon(Z)AY + \upsilon(Y)AZ - 2 < AY, Z > \xi. \tag{2.4}$$

Let M be a Riemannian manifold isometrically immersed into an almost contact metric manifold $\tilde{M}$ and let $<,>$ denotes the Riemannian metric induced on M. Let $\Gamma(TM)$ $(resp., \Gamma(T^{\perp}M))$ be the Lie algebra of the vector fields tangent to M (resp., normal to M) and $\nabla^{\perp}$ be the induced connection on $(T^{\perp}M)$. If ∇ is the Levi-Civita connection of M, then the Gauss and Weingarten formulas are given respectively by

$$\tilde{\nabla}_Y Z = \nabla_Y Z + h(Y, Z), \tag{2.5}$$

$$\tilde{\nabla}_Y N = -A_N Y + \nabla^{\perp}_Y N, \tag{2.6}$$

for all $Y, Z \in \Gamma(TM)$ and $N \in \Gamma(T^{\perp}M)$, where h is the second fundamental form and A_N is the shape operator (corresponding to the normal vector field N), respectively and are related by

$$< h(Y, Z), N >=< A_N Y, Z > . \tag{2.7}$$

Now, for any $Y \in \Gamma(TM)$, we have

$$\phi Y = PY + FY, \tag{2.8}$$

where PY and FY are the tangential and normal components of ϕY, respectively. Similarly for any $N \in \Gamma(T^{\perp}M)$, we have

$$\phi N = tN + fN, \tag{2.9}$$

where tN and fN are tangential and normal components of ϕN.

$$< Y, PZ >= - < PY, Z > . \tag{2.10}$$

If we put $Q = P^2$, then we have

$$(\tilde{\nabla}_Y Q)Z = \nabla_Y QZ - Q\nabla_Y Z, \tag{2.11}$$

$$(\tilde{\nabla}_Y P)Z = \nabla_Y PZ - P\nabla_Y Z, \tag{2.12}$$

$$(\tilde{\nabla}_Y F)Z = \nabla^{\perp}_Y FZ - F\nabla_Y Z, \tag{2.13}$$

for any $Y, Z \in \Gamma(TM)$. From (2.5), (2.8) and $\tilde{\nabla}_X \xi = \phi AX$, it follows that

$$\tilde{\nabla}_Y \xi = PAY, \tag{2.14}$$

$$h(Y, \xi) = FAY. \tag{2.15}$$

A submanifold M is said to be totally geodesic if $h(Y, Z) = 0$ and it is totally umbilical if $h(Y, Z) =<, > H$, where H is the mean curvature vector defined by $H = \frac{1}{n} trace(h) = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i)$. Now we defined a class of submanifolds called the slant submanifold.

Definition 2.1. *For each non zero vector Y tangent to M at P, such that Y is not proportional to ξ_p, we denote by $0 \leq \theta(Y) \leq \frac{\pi}{2}$, the angle between ϕY and T_pM is called the Wirtinger angle. If the angle $\theta(Y)$ is constant for all $Y \in T_pM - \{\xi\}$ and $p \in M$, then M is said to be a slant submanifold* [10] *and the angle θ is called slant angle of M. Obviously if $\theta = 0$ $(resp., \theta = \frac{\pi}{2})$, then M is called invariant $(resp., anti - invariant)$ submanifold. A slant submanifold is said to be proper slant if it is neither invariant nor anti-invariant.*

Proposition 2.1. *Let M be a submanifold of an almost contact metric manifold $\tilde{M}$ such that $\xi \in TM$, then M is slant if and only if there exists a constant $\lambda \in [0, 1]$ such that* [7]

$$P^2 = \lambda(-I + \upsilon \otimes \xi). \tag{2.16}$$

Furthermore, in such a case, if θ is a slant angle, then it satisfies that λ=$\cos^2\theta$.

Thus, for a slant submanifold M of an almost contact metric manifold $\tilde{M}$, the following relations are the consequences of the Proposition 2.1.

$$< PY, PZ >= \cos^2\theta(< Y, Z > -\upsilon(Y)\upsilon(Z)), \tag{2.17}$$

$$< FY, FZ >= \sin^2\theta(< Y, Z > -\upsilon(Y)\upsilon(Z)), \tag{2.18}$$

for any $Y, Z \in \Gamma(TM)$. Also we proceed to give an another characterization which is directly related to the consequence of the Proposition 2.1.

Proposition 2.2. *Let M be a slant submanifold of an almost contact metric manifold $\tilde{M}$ such that $\xi \in TM$, then*

$$(i)\ tFY = \sin^2\theta(-Y + \upsilon(Y)\xi) \qquad and \qquad (ii)\ fFY = -FPY, \tag{2.19}$$

for any $Y \in \Gamma(TM)$.

3. Pseudo slant submanifolds of a nearly quasi-Sasakian manifold

In this section we define pseudo slant submanifolds of an almost contact metric manifold by using slant distribution given in [6]. We find the geometry of leaves of the distributions involved in the definition of pseudo slant submanifolds of a nearly quasi-Sasakian manifold. We also obtain some necessary and sufficient conditions for such sub-immersions to be totally geodesic foliations which are used later in characterization theorem.

Definition 3.1. *A submanifold M of an almost contact metric manifold $\tilde{M}$ is called pseudo slant submanifold, if there exist a pair of orthogonal distributions $D^{\perp}$ and D^{θ} such that*

(i) The tangent bundle TM admits the orthogonal direct decomposition $TM = D^{\perp} \oplus D^{\theta} \oplus \{\xi\}$, where $\{\xi\}$ is 1-dimensional distribution spanned by ξ,

(*ii*) *The distribution* $D^{\perp}$ *is anti-invariant distribution under* ϕ, *i.e.,* $\phi D^{\perp} \subseteq T^{\perp}M$,

(*iii*) *The distribution* D^{θ} *is slant distribution with slant angle* $\theta \neq 0, \frac{\pi}{2}$.

Let m_1 and m_2 be the dimensions of the distributions $D^{\perp}$ and D^{θ}, respectively. If $m_2 = 0$, then M is anti invariant submanifold. If $m_1 = 0$ and $\theta = 0$, then M is invariant submanifold. If $m_1 = 0$ and $\theta \neq 0, \frac{\pi}{2}$, then M is proper slant submanifold, or if $\theta = \frac{\pi}{2}$, then M is anti invariant submanifold and if $\theta = 0$, then M is semi-invariant submanifold. If μ is an invariant subspace of normal bundle $T^{\perp}M$, then $T^{\perp}M$ can be decomposed as follows:

$$T^{\perp}M = \phi D^{\perp} \oplus FD^{\theta} \oplus \mu, \tag{3.1}$$

where μ is the even dimensional invariant sub bundle of $T^{\perp}M$. Now we establish the following theorem:

Theorem 3.1. *Let* M *be a pseudo slant submanifold of a nearly quasi-Sasakian manifold* $\tilde{M}$, *then the distribution* $D^{\perp} \oplus \xi$ *defines as totally geodesic foliation of* M *if and only if*

$$\begin{aligned} &< A_{\phi W}Z, PX > + < A_{\phi Z}W, PX > \\ &= -\upsilon(Z) < AW, PX > -\upsilon(W) < AZ, PX > . \end{aligned} \tag{3.2}$$

Proof. By using (2.5) and (2.9) in the relation

$$< \phi\tilde{\nabla}_Z W, \phi X > = < \tilde{\nabla}_Z W, X > -\upsilon(\tilde{\nabla}_Z W)\upsilon(X) = < \tilde{\nabla}_Z W, X >,$$

we have

$$< \tilde{\nabla}_Z W, X > = < \tilde{\nabla}_Z \phi W, PX > - < (\tilde{\nabla}_Z \phi)W, PX > - < \tilde{\nabla}_Z W, \phi FX > .$$

Now, using (2.5) and (2.7) in the last equation, we get

$$\begin{aligned} &< \tilde{\nabla}_Z W, X > = - < A_{\phi W}Z, PX > + < (\bar{\nabla}_W \phi)Z, PX > -\upsilon(W) < AZ, PX > \\ &- \upsilon(Z) < AW, PX > +2 < AZ, W > < \xi, PX > - < \bar{\nabla}_Z W, \phi FX > \end{aligned}$$

in which applying covariant derivative of endomorphism ϕ and using proposition 2.1 we easily obtain

$$\begin{aligned} &< \tilde{\nabla}_Z W, X > = - < A_{\phi W}Z, PX > + < (\bar{\nabla}_W \phi)Z, PX > -\upsilon(W) < AZ, PX > \\ &- \upsilon(Z) < AW, PX > - < \bar{\nabla}_Z W, tFX > - < \bar{\nabla}_Z W, nFX > . \end{aligned}$$

This implies

$$\begin{aligned} < \tilde{\nabla}_Z W, X > = &- < A_{\phi W}Z, PX > + < \bar{\nabla}_W \phi Z, PX > - < \phi\bar{\nabla}_W Z, PX > \\ &- \upsilon(W) < AZ, PX > -\upsilon(Z) < AW, PX > + \sin^2\theta < \tilde{\nabla}_Z W, X > \\ &+ < \tilde{\nabla}_Z W, FPX > . \end{aligned}$$

Using (2.6), (2.8), (2.11) and (2.14) we finally arrive at

$$\cos^2\theta < \tilde{\nabla}_Z W, X >= - < A_{\phi W} Z, PX > - < A_{\phi Z} W, PX > - < \bar{\nabla}_W Z, \phi PX > - \upsilon(W) < AZ, PX > - \upsilon(Z) < AW, PX > + < \bar{\nabla}_Z W, FPX > .$$

Now, if this is totally geodesic foliation, then we have

$$\begin{aligned} 0 = & - < A_{\phi W} Z, PX > - < A_{\phi Z} W, PX > - \upsilon(W) < AZ, PX > \\ & - \upsilon(Z) < AW, PX >, \\ & < A_{\phi W} Z, PX > + < A_{\phi Z} W, PX >= - \upsilon(Z) < AW, PX > \\ & - \upsilon(W) < AZ, PX > \end{aligned}$$

the result follows immediately after applying it in the last expression. □

Theorem 3.2. *On a pseudo slant submanifold M of a nearly quasi-Sasakian manifold $\tilde{M}$, the distribution D^θ is integrable if and only if*

$$\begin{aligned} 2\cos^2\theta < \nabla_X Y, Z > = & < h(X, PY) + h(PX, Y), \phi Z > - < h(X, Z), FPY > \\ & - < FPX, h(Y, Z) > - \upsilon(Z) < \bar{\nabla}_X \xi, Y >, \end{aligned}$$

for every $Z \in \Gamma(D^\perp \otimes \xi)$ and $X, Y \in \Gamma(D^\theta)$.

Proof. By using (2.2), (2.5), (2.9) and the values $\upsilon(X) = 0, \upsilon(Y) = 0$, we have

$$\begin{aligned} < [X, Y], Z > = & < \bar{\nabla}_X PY, \phi Z > + < \bar{\nabla}_X FY, \phi Z > \\ & + < (\bar{\nabla}_Y \phi) X, \phi Z > - < \bar{\nabla}_Y X, Z > - \upsilon(Z) < \bar{\nabla}_X \xi, Y > . \end{aligned}$$

In view of (2.6), the last equation takes the form

$$\begin{aligned} < [X, Y], Z > = & < h(X, PY), \phi Z > - < FY, (\tilde{\nabla}_X \phi) Z > + < \phi FY, \tilde{\nabla}_X Z > \\ & + < \tilde{\nabla}_Y PX, \phi Z > + < \tilde{\nabla}_Y FX, \phi Z > - 2 < \tilde{\nabla}_Y X, Z > \\ & - \upsilon(Z) < \tilde{\nabla}_X \xi, Y > \end{aligned}$$

which immediately gives

$$\begin{aligned} < [X, Y], Z > = & < h(X, PY), \phi Z > + < h(PX, Y), \phi Z > \\ & - < FY, Q_X Z > + < tFY, \tilde{\nabla}_X Z > + < fFY, \bar{\nabla}_X Z > \\ & - < FX, \bar{\nabla}_Y \phi Z > - 2 < \bar{\nabla}_Y X, Z > - \upsilon(Z) < \tilde{\nabla}_X \xi, Y > . \end{aligned} \tag{3.3}$$

Now, since $< \phi Y, \mathcal{Q}_X Z >=< Y, \phi \mathcal{Q}_X Z >= 0$, which by virtue of (2.9) turns to $< FY, \mathcal{Q}_X Z >=< \phi Y, \mathcal{Q}_X Z >$.

Considering $(\tilde{\nabla}_X \phi) Y = \mathcal{P}_X Y + \mathcal{Q}_X Y$, where $\mathcal{P}_X Y$ and $\mathcal{Q}_X Y$ are the tangential and normal parts of $(\tilde{\nabla}_X \phi) Y$, also taking account of Proposition 2.2, (2.6), (2.10) and (2.14), (3.3) leads to

$$< [X, Y], Z >=< h(X, PY), \phi Z > + < h(PX, Y), \phi Z > - \sin^2\theta < Y, \tilde{\nabla}_X Z >$$

$$- < FPY, h(X,Z) > + < \phi FX, \tilde{\nabla}_Y Z > -2 < \tilde{\nabla}_Y X, Z > -\upsilon(Z) < \tilde{\nabla}_X \xi, Y > .$$

Also, taking an account that X, Y are orthogonal to Z, the above equation settle to

$$\begin{aligned} < [X,Y], Z > =& < h(X,PY) + h(PX,Y), \phi Z > - \cos^2\theta < [X,Y], Z > \\ & + \cos^2\theta < [X,Y], Z > - < h(X,Z), FPY > \\ & + < -\sin^2\theta (X - \upsilon(X)\xi), \tilde{\nabla}_Y Z > \\ & - \sin^2\theta < Y, \tilde{\nabla}_X Z > - < FPX, \tilde{\nabla}_Y Z > -2 < \tilde{\nabla}_Y X, Z > \\ & - \upsilon(Z) < \tilde{\nabla}_X \xi, Y > . \end{aligned}$$

Now, calculating for a while we arrive

$$\begin{aligned} \sin^2\theta < [X,Y], Z > =& < h(X,PY) + h(PX,Y), \phi Z > -2\cos^2\theta < Z, \tilde{\nabla}_X Y > \\ & - < h(X,Z), FPY > - < h(Y,Z), FPX > \\ & - \upsilon(Z) < \tilde{\nabla}_X \xi, Y > . \end{aligned}$$

If D^θ is integrable, then finally we find

$$\begin{aligned} 2\cos^2\theta < \nabla_X Y, Z > =& < h(X,PY) + h(PX,Y), \phi Z > - < h(X,Z), FPY > \\ & - < FPX, h(Y,Z) > - \upsilon(Z) < \bar{\nabla}_X \xi, Y > \end{aligned}$$

which proves our assumption. Converse part of the theorem can be proved easily. □

4. Warped product submanifold of the form $M_\perp \times_y M_\theta$

One of the most important part of Riemannian product is warped product with warping function y, and it was introduced by R.L. Bishop and B. O'Neill [5]. They described these manifolds as: let y be a positive differentiable function which always be defined on leaves and (M^*, g_*) and $(M', g_{\prime})$ are two Riemannian manifolds. Then the warped product of M^* and M' are the Riemannian manifolds $M^* \times_y M' = (M^* \times_y M', g)$, where $g = g_* + y^2 g_{\prime}$. For a warped product, we have

$$\nabla_X Z = \nabla_Z X = X \ln y Z \tag{4.1}$$

for any vector fields X, Z and are tangents to g_* and $g_{\prime}$, respectively, where ∇ denotes the Levi-Civita connection on M [5]. On the other hand, $\nabla \ln y$ is the gradient of $\ln y$ which is defined as $g(\nabla \ln y, X) = X \ln y$. If the warping function y is constant, then a warped product manifold $M^* \times_y M'$ is called simply Riemannian product or trivial warped product manifold. For a warped product manifold $M = M^* \times_y M'$, M^* is said to be totally geodesic and M' is totally umbilical submanifolds of M, respectively. Now, we obtain some basic useful propositions.

Proposition 4.1. *If $M = M_\perp \times_y M_\theta$ is a warped product pseudo slant submanifold of a nearly quasi-Sasakian manifold $\tilde{M}$ such that the structure vector field ξ is tangent to $M_\perp$, then for any $X \in \Gamma(TM_\theta)$ and $Z \in \Gamma(TM_\perp)$, we have*

$$\begin{aligned} 2 < h(X,Z), FPX > &= \cos^2\theta (Z \ln y) \parallel X \parallel^2 \\ &\quad - < h(X,PX), \phi Z > - < h(Z,PX), FX > . \end{aligned}$$

Proof. Let $M = M_\perp \times_y M_\theta$ be a warped product pseudo slant submanifold of a nearly quasi-Sasakian manifold $\tilde{M}$. Then from (2.6), we have

$$< \tilde{\nabla}_Z X, FPX > = < h(X,Z), FPX > .$$

Since ξ is tangent to $M_\perp$, then by using (2.9), (2.16) and (4.1), we have

$$\begin{aligned} < h(X,Z), FPX > = &< (\bar{\nabla}_Z \phi)X, PX > - < \bar{\nabla}_Z \phi X, PX > \\ &+ \cos^2\theta < \bar{\nabla}_Z X, X > \end{aligned}$$

which leads to

$$\begin{aligned} < h(X,Z), FPX > = &< (\bar{\nabla}_Z \phi)X, PX > - < \bar{\nabla}_Z PX, PX > \\ &- < \nabla_Z FX, PX > + \cos^2\theta (X \ln y) < Z, Z > . \end{aligned}$$

Now, using (2.1), (2.5)-(2.7), (2.9), (2.14) and (4.1), we arrive at

$$\begin{aligned} &< h(X,Z), FPX > = - < h(X,PX), \phi Z > + \cos^2\theta (Z \ln y) \parallel X \parallel^2 \\ &- < h(X,Z), FPX > + < h(Z,PX), FX >, \\ &2 < h(X,Z), FPX > = \cos^2\theta (Z \ln y) \parallel X \parallel^2 \\ &- < h(X,PX), \phi Z > - < h(Z,PX), FX >, \end{aligned}$$

which completes the proof. □

Proposition 4.2. *Assume that $M_\perp \times_y M_\theta$ is a warped product pseudo slant submanifold of a nearly quasi-Sasakian manifold $\tilde{M}$, then*

$$< h(X,Z), FPX > = < h(Z,PX), FX >,$$

for any $X \in \Gamma(TM_\theta)$ and $Z \in \Gamma(TM_\perp)$.

Proof. By using (2.9) and (2.12), we get

$$< h(Z,PX), FX > = < \tilde{\nabla}_Z PX, \phi X > - < \tilde{\nabla}_Z PX, PX > .$$

Now, using (2.11), (2.12) and (4.1), after performing few steps we have

$$\begin{aligned} < h(Z,PX), FX > = &< h(X,Z), FPX > - 2\cos^2\theta (Z \ln y) \parallel X \parallel^2 \\ &- < \tilde{\nabla}_{PX} Z, FX > - < \tilde{\nabla}_{PX} Z, PX > + < A_{\phi Z} PX, X > . \end{aligned}$$

Now, using (2.6)-(2.9) and (4.1), after some simple calculating steps, we get

$$\begin{aligned} 2 < h(Z, PX), FX > &= -3\cos^2\theta(Z\ln y) \parallel X \parallel^2 \\ &\quad + < h(X, Z), FPX > - < h(PX, X), \phi Z > . \end{aligned} \tag{4.2}$$

Now, replacing X by PX in (4.2) and taking an account of (2.14), we get

$$\begin{aligned} 2 < h(X, Z), FPX > &= -3\cos^2\theta(Z\ln y) \parallel X \parallel^2 \\ &\quad + < h(Z, PX), FX > - < h(PX, X), \phi Z > . \end{aligned} \tag{4.3}$$

Now, from (4.2) and (4.3), we get finally

$$< h(X, Z), FPX >=< h(Z, PX), FX >,$$

which completes the proof. □

Proposition 4.3. *In a warped product pseudo slant submanifold $M = M_\perp \times_y M_\theta$ of a nearly quasi-Sasakian manifold $\tilde{M}$, we have*

$$< h(X, PX), \phi Z >= \cos^2\theta(Z\ln y) \parallel U \parallel^2 -3 < h(Z, PX), FX >,$$

for any $X \in \Gamma(TM_\theta)$ and $Z \in \Gamma(TM_\perp)$.

Proof. From Propositions 4.1 and 4.2, Propositions 4.3 easily follows. □

Theorem 4.1. *Let $\tilde{M}$ be a nearly quasi-Sasakian manifold and M be a proper pseudo-slant submanifold of $\tilde{M}$ such that the slant distribution is integrable. Then $M = M_\perp \times_y M_\theta$ is a locally warped product of proper slant and anti-invariant submanifolds if and only if*

$$3A_{FPX}Z + A_{\phi Z}PX = \cos^2\theta(\lambda Z)X, \tag{4.4}$$

for any $Z \in \Gamma(D^\perp \oplus \xi)$ and $X \in \Gamma(D^\theta)$.

Proof. From Proposition 4.3, we have

$$< h(X, PX), \phi Z >= \cos^2\theta(Z\lambda) \parallel X \parallel^2 -3 < h(Z, X), FPX >$$

and hence we get

$$< A_{\phi Z}PX, X >= -3 < A_{FPX}Z, X > + \cos^2\theta(Z\lambda) < X, X > .$$

This immediately gives

$$A_{\phi Z}PX = -3A_{FPX}Z + \cos^2\theta(Z\lambda)X, \tag{4.5}$$

which proves the necessary part. Now conversely, if M is a proper pseudo slant submanifold of a nearly quasi-Sasakian manifold $\tilde{M}$ holding (4.5), then

taking the inner product in (4.5) with W and using the fact that X and W are orthogonal, we have

$$3 < h(Z,W), FPX >= - < h(PX,W), \phi Z > . \tag{4.6}$$

Interchanging Z and W, we have

$$3 < h(Z,W), FPX >= - < h(PX,Z), \phi W > . \tag{4.7}$$

Now, adding (4.6) and (4.7), we have

$$6 < h(Z,W), FPX >= - < h(PX,W), \phi Z > - < h(PX,Z), \phi W > . \tag{4.8}$$

Now, from

$$< h^{\theta}(X,Y), Z >=< \tilde{\nabla}_X Y, Z >=< \phi\tilde{\nabla}_X Y, \phi Z > + \upsilon(Z) < \xi, \tilde{\nabla}_X Y > .$$

Using (2.6), (2.8), (2.9), (2.11), (2.14) and supposing that X, Y are orthogonal to ξ, we arrive

$$\begin{aligned} < h^{\theta}(X,Y), Z > &= - < A_{\phi Y} PZ, X > + \sin^2\theta < \tilde{\nabla}_X Z, Y > \\ &\quad - \cos^2\theta \upsilon(Z) < \bar{\nabla}_X \xi, Y > + < A_{FPX} X, Y > \end{aligned} \tag{4.9}$$

which leads to

$$\begin{aligned} \cos^2\theta < h^{\theta}(X,Y), Z > &= - < A_{\phi Y} PZ, X > - \cos^2\theta \upsilon(Z) < \tilde{\nabla}_X \xi, Y > \\ &\quad + < A_{FPZ} X, Y > . \end{aligned} \tag{4.10}$$

Hence, we have

$$\begin{aligned} < h^{\theta}(X,Y), Z > &= (Z\lambda) < X, Y > - \upsilon(Z) < \tilde{\nabla}_X \xi, Y > \\ &=< X, Y >< \nabla\lambda, Z > - < \tilde{\nabla}_X \xi, Y >< \xi, Z > . \end{aligned} \tag{4.11}$$

This implies

$$h^{\theta}(X,Y) =< X, Y > \nabla\lambda - < \bar{\nabla}_X \xi, Y > \xi. \tag{4.12}$$

Interchanging X and Y in (4.11), we have

$$< h^{\theta}(X,Y), Z >=< X, Y >< \nabla\lambda, Z > - \upsilon(Z) < \bar{\nabla}_Y \xi, X > . \tag{4.13}$$

Now, by adding (4.11) and (4.13), we get

$$2 < h^{\theta}(X,Y), Z >= 2 < X, Y >< \nabla\lambda, Z > - \upsilon(Z)[< \bar{\nabla}_X \xi, Y > + < \bar{\nabla}_Y \xi, X >]$$

which gives

$$2 < h^{\theta}(X,Y), Z >= 2 < X, Y >< \nabla\lambda, Z > .$$

Thus, we get

$$< h^{\theta}(X,Y), Z >=< X, Y >< \nabla\lambda, Z > . \tag{4.14}$$

This gives

$$h^{\theta}(X,Y) =< X, Y > \nabla\lambda. \tag{4.15}$$

Thus, we have

$$H^{\theta} = \nabla\lambda \tag{4.16}$$

which is the mean curvature vector of M. □

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Decision making application based on parameterization of fuzzy hypersoft set with fuzzy setting

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Abstract. Fuzzy soft set is considered as an apt parameterization tool to deal with vagueness and uncertainties. There are many situations when attributes need to be further classified into attribute-valued disjoint sets. Such situations are not tackled by the existing fuzzy soft-like structures. Hypersoft set, an extension of soft set, employs multi-argument approximate function which addresses the inadequacy of existing structures for attribute-valued disjoint sets. In this study, theory of fuzzy parameterized fuzzy hypersoft set is characterized and some of its essential properties are discussed. In order to cope with the decision making problems, an algorithm is proposed which employs the fuzzy decision set of fuzzy parameterized fuzzy hypersoft set for dealing with decision making problems under uncertain scenarios. The proposed algorithm is validated with the help of a numerical example.

Keywords: fuzzy set, soft set, fuzzy parameterized soft set, hypersoft set, decision making.

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1. Introduction

In 1965, Zadeh [1] initiated the theory of fuzzy set (FST) as an emerging field of mathematics to cope with the uncertainty-based scenarios. There are many real life cases (i.e. decision making setting) where some sorts of parameterization tools are needed to handle problems having base on certain attributes. The existing theories including FST dealing uncertainties, are insufficient for such kind of setting. In 1999, Molodtsov [2] addressed this limitation with the introduction of new theory called soft set theory (SST) which is a parameterized family of subsets of universal set. Maji et al. [3] extended the concept and introduced some fundamental terminologies and operations of soft set. They also defined fuzzy soft set in [4] and successfully applied it in decision making. Some authors [5, 6, 7, 8, 9, 10, 11] discussed some more properties and relational features of soft set theory. Çağman et al. [12, 13] conceptualized fuzzy parameterized soft set which is the ordered pair of membership function of fuzzy set and approximate function of fuzzy soft. They discussed its properties and applied this theory to different fields. Kamacı [15] characterized the theory of interval-valued fuzzy parameterized intuitionistic fuzzy soft sets and discussed its applications.

In 2018, Smarandache [14] developed a new structure hypersoft set (HSS) to adequate the soft set for multi attribute-valued functions. In 2020, Saeed et al. [16, 17] characterized the necessary basic axioms, properties, laws and set theoretic operations of HSS with the help of appropriate examples. In 2020, Rahman et al. [18, 19] enhanced the study of HSS to develop fuzzy-like structures with complex sets and also studied certain properties of convexity under HSS-environment. In 2021, Rahman et al. [20, 21, 22] studied decision making applications based on neutrosophic parameterized hypersoft set, fuzzy parameterized hypersoft set and rough hypersoft set. Saeed et al. [23, 24, 25, 26, 27] discussed decision making techniques for mappings on hypersoft classes, neutrosophic hypersoft mapping and complex multi-fuzzy hypersoft set. They also developed hypersoft graphs with some properties. Ihsan et al. [28] investigated hypersoft expert set with application in decision making for the best selection of product. Yolcu et al. [29, 30] conceptualized the theories of fuzzy and intuitionistic fuzzy hypersoft sets with their employment in decision making. Ozturk et al. [31] introduced neutrosophic hypersoft topological spaces and discussed their important properties. Saqlain et al. [32, 33, 34] developed single and multi-valued neutrosophic hypersoft sets along with calculation of tangent similarity measure of single valued neutrosophic hypersoft sets, characterized aggregate operators of neutrosophic hypersoft set and employed TOPSIS method for neutrosophic hypersoft sets using accuracy function with application. Kamacı [35] made very valuable research on the hybrid structures of hypersoft sets and rough sets. Recently Kamacı et al. [36] investigated some new hybridized structures i.e. n-ary fuzzy hypersoft expert sets which are the extensions of existing structures (i.e. n-ary fuzzy soft expert sets).

1.1 Motivation

In many real life situations, distinct attributes are further partitioned into disjoint attribute-valued sets. Decision makers may suffer some kind of inclination and penchant while ignoring such partitioning of attributes during the judgment. The existing soft set theory is not projected for such sets. Therefore a new structure demands its place in literature for addressing such impediment, so hypersoft set theory is conceptualized to tackle such situations. This novel structure has increased the flexibility and reliability of decision making process. It not only addresses the inadequacy of existing soft-like structures for multi-argument approximate functions but also helps the decision makers to decide the matters with deep observation. Although this structure face some kind of challenge (limitation) regarding the consideration of overlapping attribute-valued sets. Motivating from the work discussed in [12, 14, 17], fuzzy parameterized fuzzy hypersoft set is characterized in order to adequate the literature regarding fuzzy parameterized soft set for multi attribute-valued approximate functions and its some essential elementary properties are discussed. A decision making based algorithm is proposed with successful application for the best choice of product.

1.2 Paper layout

The rest of the paper is organized as:

In section 2, some fundamental definitions and terms are recalled from already published relevant literature, section 3 formulates the theory of fuzzy parameterized fuzzy hypersoft set, section 4 proposes a decision-making algorithm based on decision system of fuzzy parameterized fuzzy hypersoft set with application for the best selection of an appropriate product, section 5 presents the validation for generalization and comparison analysis of proposed model and finally, section 6 concludes the paper with future directions.

2. Preliminary

In this section some basic terms are recalled from existing literature to support the proposed work. Throughout the paper, $\mathcal{U}$, $P(\mathcal{U})$ and $\mathbb{I}^{\bullet}$ will denote the universe of discourse, power set of $\mathcal{U}$ and closed unit interval respectively.

Definition 2.1 ([1]). *A* fuzzy set $\mathcal{X} \subseteq \mathcal{U}$ *is defined as* $\mathcal{X} = \{(\epsilon, \zeta_{\mathcal{X}}(\epsilon)) | \epsilon \in \mathcal{U}\}$ *such that* $\zeta_{\mathcal{X}} : \mathcal{U} \to \mathbb{I}^{\bullet}$ *where* $\zeta_{\mathcal{X}}(\epsilon)$ *denotes the belonging value of* $\epsilon \in \mathcal{X}$.

Definition 2.2 ([2]). *A pair* (ζ_S, Λ) *is called a* soft set *over* $\mathcal{U}$, *where* $\zeta_S : \Lambda \to P(\mathcal{U})$ *and* Λ *be a set of attributes.*

Definition 2.3 ([3]). *A soft set* (ζ_{S_1}, Λ_1) *is a* soft subset *of another soft set* (ζ_{S_2}, Λ_2) *if*

(*i*) $\Lambda_1 \subseteq \Lambda_2$;

(*ii*) $\forall \omega \in \Lambda_1, \zeta_{S_1}(\omega)$ *and* $\zeta_{S_2}(\omega)$ *are identical approximations.*

For more detail on soft set, see [3]-[11]

Definition 2.4 ([18])**.** *The pair* (Ψ, G) *is called a hypersoft set over* $\mathcal{U}$*, where* G *is the Cartesian product of* n *disjoint sets* $G_1, G_2, G_3, ..., G_n$ *having attribute values of* n *distinct attributes* $g_1, g_2, g_3, ..., g_n$ *respectively and* $\Psi : G \to P(\mathcal{U})$*.*

For more definitions and operations of hypersoft set (see, [16]-[18]).

3. Fuzzy parameterized fuzzy hypersoft set ($fpfhs$-set) theory

In this section, fuzzy parameterized fuzzy hypersoft set is conceptualized and some of its fundamentals are discussed.

Definition 3.1. *Let* $\mathcal{A} = \{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, ..., \mathcal{A}_n\}$ *be a collection of disjoint attribute-valued sets corresponding to* n *distinct attributes* $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n$ *respectively. An FPF-hypersoft set (*$fpfhs$*-set)* $\Psi_{\mathcal{W}}$ *over* $\mathcal{U}$ *is defined as*

$$\Psi_{\mathcal{W}} = \left\{ \begin{array}{l} (\zeta_{\mathcal{W}}(g)/g, \psi_{\mathcal{W}}(g)) : \\ g \in G,\ \psi_{\mathcal{W}}(g) \in F(\mathcal{U}),\ \zeta_{\mathcal{W}}(g) \in \mathbb{I}^{\bullet} \end{array} \right\}$$

where

(*i*) $F(\mathcal{U})$ *is a collection of all fuzzy sets over* $\mathcal{U}$

(*ii*) $G = \mathcal{A}_1 \times \mathcal{A}_2 \times \mathcal{A}_3 \times .... \times \mathcal{A}_n$

(*iii*) $\mathcal{W}$ *is a fuzzy set over* G *with* $\zeta_{\mathcal{W}} : G \to \mathbb{I}^{\bullet}$ *as membership function of* $fpfhs$*-set.*

(*iv*) $\psi_{\mathcal{W}}(g)$ *is a fuzzy set for all* $g \in G$ *with* $\psi_{\mathcal{W}} : G \to F(\mathcal{U})$ *and is called approximate function of* $fpfhs$*-set.*

Now, throughout the remaining part of the paper, $\Omega_{FPFHS}(\mathcal{U})$*,* $\zeta_{\mathcal{W}}$ *and* $\psi_{\mathcal{W}}$ *will represent the collection of all* $fpfhs$*-sets over* $\mathcal{U}$*, membership function and approximate function respectively.*

Example 3.1. Suppose that Mrs. Andrew wants to buy a washing machine from market. There are eight kinds of washing machines (options) which form the set of discourse $\mathbb{X} = \{\hat{W}_1, \hat{W}_2, \hat{W}_3, \hat{W}_4, \hat{W}_5, \hat{W}_6, \hat{W}_7, \hat{W}_8\}$. The best selection may be evaluated by observing the attributes i.e. $b_1 =$ Company, $b_2 =$ Power in Watts, $b_3 =$ Voltage, $b_4 =$ Capacity in kg, and $b_5 =$ Color. The attribute-valued sets corresponding to these attributes are:
$B_1 = \{b_{11} = National, b_{12} = Hier\}$,
$B_2 = \{b_{21} = 400, b_{22} = 500\}$,
$B_3 = \{b_{31} = 220, b_{32} = 240\}$,
$B_4 = \{b_{41} = 7, b_{42} = 10\}$,
$B_5 = \{b_{51} = White\}$,
then $\mathbb{Q} = B_1 \times B_2 \times B_3 \times B_4 \times B_5$,

$\mathbb{Q} = \{q_1, q_2, q_3, q_4, ..., q_{16}\}$ where each $q_i, i = 1, 2, ..., 16$, is a 5-tuples element. we can construct $fpfhs$-set $\Psi_{\mathcal{W}}$ as $\Psi_{\mathcal{W}} =$

$$\left\{ \begin{array}{l} (0.1/q_1, \{0.21/W_1, 0.51/W_2\}), (0.2/q_2, \{0.32/W_1, 0.16/W_2, 0.1/W_3\}), \\ (0.3/q_3, \{0.34/W_2, 0.46/W_6, 0.48/W_8\}), (0.4/q_4, \{0.63/W_4, 0.13/W_5, 0.91/W_6\}), \\ (0.5/q_5, \{0.42/W_4, 0.53/W_5, 0.84/W_8\}), (0.6/q_6, \{0.23/W_2, 0.54/W_4, 0.35/W_7\}), \\ (0.7/q_7, \{0.81/W_1, 0.72/W_3, 0.43/W_5\}), (0.8/q_8, \{0.14/W_2, 0.35/W_3, 0.36/W_7\}), \\ (0.9/q_9, \{0.34/W_2, 0.65/W_3, 0.86/W_6\}), (0.16/q_{10}, \{0.27/W_6, 0.68/W_7, 0.29/W_8\}), \\ (0.25/q_{11}, \{0.22/W_2, 0.63/W_4, 0.14/W_7\}), (0.45/q_{12}, \{0.15/W_1, 0.36/W_3, 0.77/W_7\}), \\ (0.35/q_{13}, \{0.11/W_2, 0.12/W_3, 0.93/W_5\}), (0.75/q_{14}, \{0.43/W_1, 0.75/W_5, 0.37/W_8\}), \\ (0.65/q_{15}, \{0.27/W_2, 0.15/W_3, 0.23/W_8\}), (0.85/q_{16}, \{0.46/W_4, 0.64/W_5, 0.27/W_6\}) \end{array} \right\}$$

Definition 3.2. *Let $\Psi_{\mathcal{W}} \in \Omega_{FPFHS}(\mathcal{U})$. If $\zeta_{\mathcal{W}}(g) = 0$ and $\psi_{\mathcal{W}}(g) = \phi$, for all $g \in G$, then $\Psi_{\mathcal{W}}$ is called an $\mathcal{W}$-empty $fpfhs$-set, denoted by $\Psi_{\Phi_{\mathcal{W}}}$. If $\mathcal{W} = \phi$, then $\Psi_{\mathcal{W}}$ is called an empty $fpfhs$-set, denoted by Ψ_{Φ}.*

Definition 3.3. *Let $\Psi_{\mathcal{W}} \in \Omega_{FPFHS}(\mathcal{U})$. If $\zeta_{\mathcal{W}}(g) = 1$ and $\psi_{\mathcal{W}}(g) = \mathcal{U}$ for all $g \in \mathcal{W}$, then $\Psi_{\mathcal{W}}$ is called $\mathcal{W}$-universal $fpfhs$-set, denoted by $\Psi_{\tilde{\mathcal{W}}}$. If $\mathcal{W} = G$, then the $\mathcal{W}$-universal $fpfhs$-set is called universal $fpfhs$-set, denoted by $\Psi_{\tilde{G}}$.*

Example 3.2. Consider $\mathcal{U} = \{u_1, u_2, u_3, u_4, u_5\}$ and $\mathcal{A} = \{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\}$ with $\mathcal{A}_1 = \{a_{11}, a_{12}\}$, $\mathcal{A}_2 = \{a_{21}, a_{22}\}, \mathcal{A}_3 = \{a_{31}\}$, then $G = \mathcal{A}_1 \times \mathcal{A}_2 \times \mathcal{A}_3$

$$G = \left\{ \begin{array}{l} (a_{11}, a_{21}, a_{31}), (a_{11}, a_{22}, a_{31}), \\ (a_{12}, a_{21}, a_{31}), (a_{12}, a_{22}, a_{31}) \end{array} \right\}$$
$$G = \{g_1, g_2, g_3, g_4\}.$$

Case 1. If $\mathcal{W}_1 = \{0.2/g_2, 0.5/g_3, 1.0/g_4\}$ and $\psi_{\mathcal{W}_1}(g_2) = \{0.4/u_2, 0.6/u_4\}$, $\psi_{\mathcal{W}_1}(g_3) = \phi$, and $\psi_{\mathcal{W}_1}(g_4) = \mathcal{U}$, then $\Psi_{\mathcal{W}_1} = \{(0.2/g_2, \{0.4/u_2, 0.6/u_4\}), (0.5/g_3, \phi), (1.0/g_4, U)\}$.
Case 2. If $\mathcal{W}_2 = \{0/g_2, 0/g_3\}, \psi_{\mathcal{W}_2}(g_2) = \phi$ and $\psi_{\mathcal{W}_2}(g_3) = \phi$, then $\Psi_{\mathcal{W}_2} = \Psi_{\Phi_{\mathcal{W}_2}}$.
Case 3. If $\mathcal{W}_3 = \phi$, then $\Psi_{\mathcal{W}_3} = \Psi_{\Phi}$.
Case 4. If $\mathcal{W}_4 = \{1.0/g_1, 1.0/g_2, 1.0/g_3, 1.0/g_4\}, \psi_{\mathcal{W}_4}(g_1) = \psi_{\mathcal{W}_4}(g_2) = \psi_{\mathcal{W}_4}(g_3) = \psi_{\mathcal{W}_4}(g_4) = \mathcal{U}$, then $\Psi_{\mathcal{W}_4} = \Psi_{\tilde{\mathcal{W}}_4}$.

Definition 3.4. *Let $\Psi_{\mathcal{W}_1}, \Psi_{\mathcal{W}_2} \in \Omega_{FPFHS}(\mathcal{U})$ then $\Psi_{\mathcal{W}_1}$ is an $fpfhs$-subset of $\Psi_{\mathcal{W}_2}$, denoted by $\Psi_{\mathcal{W}_1} \tilde{\subseteq} \Psi_{\mathcal{W}_2}$ if $\zeta_{\mathcal{W}_1}(g) \leq \zeta_{\mathcal{W}_2}(g)$ and $\psi_{\mathcal{W}_1}(g) \subseteq \psi_{\mathcal{W}_2}(g)$ for all $g \in G$.*

Proposition 3.1. *Let $\Psi_{\mathcal{W}_1}, \Psi_{\mathcal{W}_2}, \Psi_{\mathcal{W}_3} \in \Omega_{FPFHS}(\mathcal{U})$ then:*

(*i*) $\Psi_{\mathcal{W}_1} \tilde{\subseteq} \Psi_{\tilde{G}}$;

(*ii*) $\Psi_{\phi} \tilde{\subseteq} \Psi_{\mathcal{W}_1}$;

(*iii*) $\Psi_{\mathcal{W}_1} \tilde{\subseteq} \Psi_{\mathcal{W}_1}$;

(*iv*) $\Psi_{\mathcal{W}_1} \tilde{\subseteq} \Psi_{\mathcal{W}_2}$; $\Psi_{\mathcal{W}_2} \tilde{\subseteq} \Psi_{\mathcal{W}_3} \Rightarrow \Psi_{\mathcal{W}_1} \tilde{\subseteq} \Psi_{\mathcal{W}_3}$.

Proof. The proofs of (i)- (iv) can be obtained easily by using Definition 3.4. □

Definition 3.5. *Let $\Psi_{\mathcal{W}_1}, \Psi_{\mathcal{W}_2} \in \Omega_{FPFHS}(\mathcal{U})$ then, $\Psi_{\mathcal{W}_1}$ and $\Psi_{\mathcal{W}_2}$ are $fpfhs$-equal, represented as $\Psi_{\mathcal{W}_1} = \Psi_{\mathcal{W}_2}$, iff $\zeta_{\mathcal{W}_1}(x) = \zeta_{\mathcal{W}_2}(x)$ and $\psi_{\mathcal{W}_1}(g) = \psi_{\mathcal{W}_2}(g)$ for all $g \in G$.*

Proposition 3.2. *Let $\Psi_{\mathcal{W}_1}, \Psi_{\mathcal{W}_2}, \Psi_{\mathcal{W}_3} \in \Omega_{FPFHS}(\mathcal{U})$ then:*

(i) If $\Psi_{\mathcal{W}_1} = \Psi_{\mathcal{W}_2}$ and $\Psi_{\mathcal{W}_2} = \Psi_{\mathcal{W}_3} \Rightarrow \Psi_{\mathcal{W}_1} = \Psi_{\mathcal{W}_3}$.

(ii) $\Psi_{\mathcal{W}_1} \tilde{\subseteq} \Psi_{\mathcal{W}_2}$ and $\Psi_{\mathcal{W}_2} \tilde{\subseteq} \Psi_{\mathcal{W}_1} \Leftrightarrow \Psi_{\mathcal{W}_1} = \Psi_{\mathcal{W}_2}$.

Proof. The properties (i)- (ii) can be proved easily by using Definition 3.5. □

Definition 3.6. *Let $\Psi_{\mathcal{W}} \in \Omega_{FPFHS}(\mathcal{U})$ then, complement of $\Psi_{\mathcal{W}}$ (i.e. $\Psi^{\tilde{c}}_{\mathcal{W}}$) is an $fpfhs$-set given as $\zeta^{\tilde{c}}_{\mathcal{W}}(g) = 1 - \zeta_{\mathcal{W}}(g)$ and $\psi^{\tilde{c}}_{\mathcal{W}}(g) = \mathcal{U} \setminus \psi_{\mathcal{W}}(g)$*

Proposition 3.3. *Let $\Psi_{\mathcal{W}} \in \Omega_{FPFHS}(\mathcal{U})$ then,*

- $(\Psi^{\tilde{c}}_{\mathcal{W}})^{\tilde{c}} = \Psi_{\mathcal{W}}$.
- $\Psi^{\tilde{c}}_{\Phi} = \Psi_{\tilde{G}}$.

Definition 3.7. *Let $\Psi_{\mathcal{W}_1}, \Psi_{\mathcal{W}_2} \in \Omega_{FPFHS}(\mathcal{U})$ then, union of $\Psi_{\mathcal{W}_1}$ and $\Psi_{\mathcal{W}_2}$, denoted by $\Psi_{\mathcal{W}_1} \tilde{\cup} \Psi_{\mathcal{W}_2}$, is defined by*

- $\zeta_{\mathcal{W}_1 \tilde{\cup} \mathcal{W}_2}(g) = \max\{\zeta_{\mathcal{W}_1}(x), \zeta_{\mathcal{W}_2}(g)\}$ *and*
- $\psi_{\mathcal{W}_1 \tilde{\cup} \mathcal{W}_2}(g) = \psi_{\mathcal{W}_1}(g) \cup \psi_{\mathcal{W}_2}(g)$, *for all* $g \in G$.

Proposition 3.4. *Let $\Psi_{\mathcal{W}_1}, \Psi_{\mathcal{W}_2}, \Psi_{\mathcal{W}_3} \in \Omega_{FPFHS}(\mathcal{U})$ then,*

(i) $\Psi_{\mathcal{W}_1} \tilde{\cup} \Psi_{\mathcal{W}_1} = \Psi_{\mathcal{W}_1}$.

(ii) $\Psi_{\mathcal{W}_1} \tilde{\cup} \Psi_{\Phi} = \Psi_{\mathcal{W}_1}$.

(iii) $\Psi_{\mathcal{W}_1} \tilde{\cup} \Psi_{\tilde{G}} = \Psi_{\tilde{G}}$.

(iv) $\Psi_{\mathcal{W}_1} \tilde{\cup} \Psi_{\mathcal{W}_2} = \Psi_{\mathcal{W}_2} \tilde{\cup} \Psi_{\mathcal{W}_1}$.

(v) $(\Psi_{\mathcal{W}_1} \tilde{\cup} \Psi_{\mathcal{W}_2}) \tilde{\cup} \Psi_{\mathcal{W}_3} = \Psi_{\mathcal{W}_1} \tilde{\cup} (\Psi_{\mathcal{W}_2} \tilde{\cup} \Psi_{\mathcal{W}_3})$.

Proof. The proofs of (i)- (v) are obvious in accordance with Definition 3.7. □

Definition 3.8. *Let $\Psi_{\mathcal{W}_1}, \Psi_{\mathcal{W}_2} \in \Omega_{FPFHS}(\mathcal{U})$ then intersection of $\Psi_{\mathcal{W}_1}$ and $\Psi_{\mathcal{W}_2}$, denoted by $\Psi_{\mathcal{W}_1} \tilde{\cap} \Psi_{\mathcal{W}_2}$, is an $fpfhs$-set defined by $\zeta_{\mathcal{W}_1 \tilde{\cap} \mathcal{W}_2}(g) = \min\{\zeta_{\mathcal{W}_1}(g), \zeta_{\mathcal{W}_2}(g)\}$ and $\psi_{\mathcal{W}_1 \tilde{\cap} \mathcal{W}_2}(g) = \psi_{\mathcal{W}_1}(g) \cap \psi_{\mathcal{W}_2}(g)$*

Proposition 3.5. *Let $\Psi_{\mathcal{W}_1}, \Psi_{\mathcal{W}_2}, \Psi_{\mathcal{W}_3} \in \Omega_{FPFHS}(\mathcal{U})$ then*

(i) $\Psi_{\mathcal{W}_1} \tilde{\cap} \Psi_{\mathcal{W}_1} = \Psi_{\mathcal{W}_1}$;

(*ii*) $\Psi_{\mathcal{W}_1} \tilde{\cap} \Psi_{\Phi} = \Psi_{\Phi}$;

(*iii*) $\Psi_{\mathcal{W}_1} \tilde{\cap} \Psi_{\tilde{G}} = \Psi_{\tilde{\mathcal{W}}_1}$;

(*iv*) $\Psi_{\mathcal{W}_1} \tilde{\cap} \Psi_{\mathcal{W}_2} = \Psi_{\mathcal{W}_2} \tilde{\cap} \Psi_{\mathcal{W}_1}$;

(*v*) $(\Psi_{\mathcal{W}_1} \tilde{\cap} \Psi_{\mathcal{W}_2}) \tilde{\cap} \Psi_{\Psi_{\mathcal{W}_3}} = \Psi_X \tilde{\cap} (\Psi_{\mathcal{W}_2} \tilde{\cap} \Psi_{\Psi_{\mathcal{W}_3}})$.

Proof. The proofs of (i)- (v) are vivid by using Definition 3.8. □

Remark 3.3. Let $\Psi_{\mathcal{W}} \in \Omega_{FPFHS}(\mathcal{U})$. If $\Psi_{\mathcal{W}} \neq \Psi_{\tilde{G}}$, then $\Psi_{\mathcal{W}} \tilde{\cup} \Psi^{\tilde{c}}_{\mathcal{W}} \neq \Psi_{\tilde{G}}$ and $\Psi_{\mathcal{W}} \tilde{\cap} \Psi^{\tilde{c}}_{\mathcal{W}} \neq \Psi_{\Phi}$

Proposition 3.6. *Let* $\Psi_{\mathcal{W}_1}, \Psi_{\mathcal{W}_2} \in \Omega_{FPFHS}(\mathcal{U})$ *D Morgan's laws are valid*

(*i*) $(\Psi_{\mathcal{W}_1} \tilde{\cup} \Psi_{\mathcal{W}_2})^{\tilde{c}} = \Psi^{\tilde{c}}_{\mathcal{W}_1} \tilde{\cap} \Psi^{\tilde{c}}_{\mathcal{W}_2}$.

(*ii*) $(\Psi_{\mathcal{W}_1} \tilde{\cap} \Psi_{\mathcal{W}_2})^{\tilde{c}} = \Psi^{\tilde{c}}_{\mathcal{W}_1} \tilde{\cup} \Psi^{\tilde{c}}_{\mathcal{W}_2}$.

Proof. For all $g \in G$, (i) Since $(\zeta_{\mathcal{W}_1 \tilde{\cup} \mathcal{W}_2})^{\tilde{c}}(g) = 1 - \zeta_{\mathcal{W}_1 \tilde{\cup} \mathcal{W}_2}(g) = 1 - \max\{\zeta_{\mathcal{W}_1}(g), \zeta_{\mathcal{W}_2}(g)\} = \min\{1 - \zeta_{\mathcal{W}_1}(g), 1 - \zeta_{\mathcal{W}_2}(g)\} = \min\{\zeta_{\mathcal{W}_1}\tilde{c}(g), \zeta_{\mathcal{W}_2}\tilde{c}(g)\} = \zeta_{\mathcal{W}^{\tilde{c}}_1 \tilde{\cap} \mathcal{W}^{\tilde{c}}_2}(g)$ and $(\psi_{\mathcal{W}_1 \tilde{\cup} \mathcal{W}_2})^{\tilde{c}}(g) = \mathcal{U} \setminus \psi_{\mathcal{W}_1 \tilde{\cup} \mathcal{W}_2}(g) = \mathcal{U} \setminus (\psi_{\mathcal{W}_1}(g) \cup \psi_{\mathcal{W}_2}(g)) = (\mathcal{U} \setminus \psi_{\mathcal{W}_1}(g)) \cap (\mathcal{U} \setminus \psi_{\mathcal{W}_2}(g)) = \psi_{\mathcal{W}^{\tilde{c}}_1}(g) \tilde{\cap} \psi_{\mathcal{W}^{\tilde{c}}_2}(g) = \psi_{\mathcal{W}^{\tilde{c}}_1 \tilde{\cap} \mathcal{W}^{\tilde{c}}_2}(g)$. similarly (ii) can be proved easily. □

Proposition 3.7. *Let* $\Psi_{\mathcal{W}_1}, \Psi_{\mathcal{W}_2}, \Psi_{\mathcal{W}_3} \in \Omega_{FPFHS}(\mathcal{U})$ *then*

(*i*) $\Psi_{\mathcal{W}_1} \tilde{\cup} (\Psi_{\mathcal{W}_2} \tilde{\cap} \Psi_{\mathcal{W}_3}) = (\Psi_{\mathcal{W}_1} \tilde{\cup} \Psi_{\mathcal{W}_2}) \tilde{\cap} (\Psi_{\mathcal{W}_1} \tilde{\cup} \Psi_{\mathcal{W}_3})$.

(*ii*) $\Psi_{\mathcal{W}_1} \tilde{\cap} (\Psi_{\mathcal{W}_2} \tilde{\cup} \Psi_{\mathcal{W}_3}) = (\Psi_{\mathcal{W}_1} \tilde{\cap} \Psi_{\mathcal{W}_2}) \tilde{\cup} (\Psi_{\mathcal{W}_1} \tilde{\cap} \Psi_{\mathcal{W}_3})$.

Proof. For all $g \in G$, (i).

Since $\zeta_{\mathcal{W}_1 \tilde{\cup} (\mathcal{W}_2 \tilde{\cap} \mathcal{W}_3)}(g) = \max\{\zeta_{\mathcal{W}_1}(g), \zeta_{\mathcal{W}_2 \tilde{\cap} \mathcal{W}_3}(g)\} = \max\{\zeta_{\mathcal{W}_1}(g), \min\{\zeta_{\mathcal{W}_2}(g), \zeta_{\mathcal{W}_3}(g)\}\} = \min\{\max\{\zeta_{\mathcal{W}_1}(g), \zeta_{\mathcal{W}_2}(g)\}, \max\{\zeta_{\mathcal{W}_1}(g), \zeta_{\mathcal{W}_3}(g)\}\} = \min\{\zeta_{\mathcal{W}_1 \tilde{\cup} \mathcal{W}_2}(g), \zeta_{\mathcal{W}_1 \tilde{\cup} \mathcal{W}_3}(g)\} = \zeta_{(\mathcal{W}_1 \tilde{\cup} \mathcal{W}_2) \tilde{\cap} (\mathcal{W}_1 \tilde{\cup} \mathcal{W}_3)}(g)$ and $\psi_{\mathcal{W}_1 \tilde{\cup} (\mathcal{W}_2 \tilde{\cap} \mathcal{W}_3)}(g) = \psi_{\mathcal{W}_1}(g) \cup \psi_{\mathcal{W}_2 \tilde{\cap} \mathcal{W}_3}(g) = \psi_{\mathcal{W}_1}(g) \cup (\psi_{\mathcal{W}_2}(g) \cap \psi_{\mathcal{W}_3}(g)) = (\psi_{\mathcal{W}_1}(g) \cup \psi_{\mathcal{W}_2}(g)) \cap (\psi_{\mathcal{W}_1}(g) \cup \psi_{\mathcal{W}_3}(g)) = \psi_{\mathcal{W}_1 \tilde{\cup} \mathcal{W}_2}(g) \cap \psi_{\mathcal{W}_1 \tilde{\cup} \mathcal{W}_3}(g) = \psi_{(\mathcal{W}_1 \tilde{\cup} \mathcal{W}_2) \tilde{\cap} (\mathcal{W}_1 \tilde{\cup} \mathcal{W}_3)}(g)$. In the same way, (ii) can be proved. □

4. Fuzzy decision set of $fpfhs$-set

In this section novel algorithm is proposed with the help of characterization of fuzzy decision set on $fpfhs$-set which based on decision making technique and is explained with example.

Definition 4.1. *Let* $\Psi_{\mathcal{Z}} \in \Omega_{FPFHS}(\mathcal{U})$ *then a fuzzy decision set of* $\Psi_{\mathcal{Z}}$ (*i.e.* $\Psi^D_{\mathcal{Z}}$) *is represented as* $\Psi^D_{\mathcal{Z}} = \left\{\zeta^D_{\mathcal{Z}}(u)/u : u \in \mathcal{U}\right\}$, *where* $\zeta_{\Psi^D_{\mathcal{Z}}} : \mathcal{U} \to \mathbb{I}^{\bullet}$ *and*

$$\zeta_{\Psi^D_{\mathcal{Z}}}(u) = \frac{1}{|S(\mathcal{Z})|} \sum_{v \in S(\mathcal{Z})} \zeta_{\Psi}(v) \Gamma_{\psi_{\mathcal{Z}}(v)}(u),$$

where $S(\mathcal{Z})$ is the support set of $\mathcal{Z}$ with

$$\Gamma_{\psi_{\mathcal{Z}}(v)}(u) = \begin{cases} \psi_{\mathcal{Z}}(v), & u \in \Gamma_{\psi_{\mathcal{Z}}}(v), \\ 0, & u \notin \Gamma_{\psi_{\mathcal{Z}}}(v) \end{cases}.$$

4.1 Proposed algorithm

Once $\Psi_{\mathcal{Z}}^{D}$ has been established, it may be indispensable to select the best single substitute from the options. Therefore, decision can be set up with the help of following algorithm (see, Figure 1):

Step 1 Determine $\mathcal{Z} = \{\zeta_{\mathcal{Z}}(g)/g : \zeta_{\mathcal{Z}}(g) \in \mathbb{I}^{\bullet}, g \in G\}$,

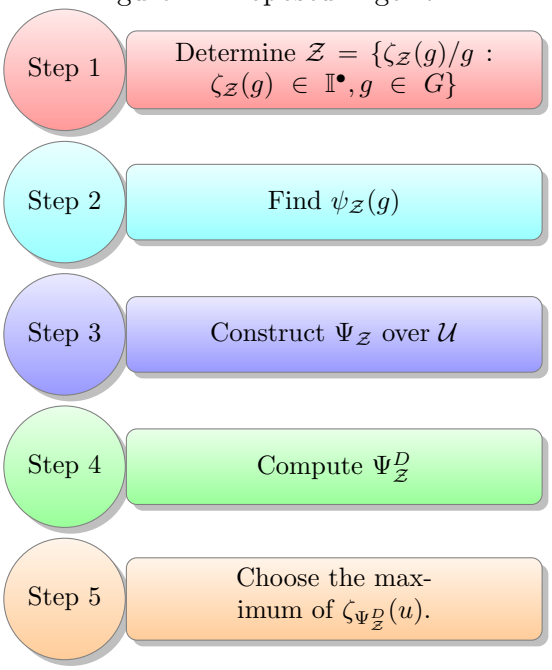
Step 2 Find $\psi_{\mathcal{Z}}(g)$,

Step 3 Construct $\Psi_{\mathcal{Z}}$ over $\mathcal{U}$,

Step 4 Compute $\Psi_{\mathcal{Z}}^{D}$,

Step 5 Choose the maximum of $\zeta_{\Psi_{\mathcal{Z}}^{D}}(u)$.

Figure 1: Proposed Algorithm



Example 4.1. Suppose that Mr. Smith wants to buy a mobile tablet from a mobile market. There are eight kinds of mobile tablets (options) which form the set of discourse $\mathcal{U} = \{T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8\}$. The best selection may be evaluated by observing the attributes i.e. a_1 = Company, a_2 = Camera Resolution (pixels), a_3 = Size (inches), a_4 = Storage (GB), and a_5 = Battery power (mAh). The attribute-valued sets corresponding to these attributes are: $B_1 = \{b_{11} = Samsung, b_{12} = Oppo\}$, $B_2 = \{b_{21} = 8, b_{22} = 16\}$, $B_3 = \{b_{31} = 9, b_{32} = 9.5\}$, $B_4 = \{b_{41} = 32, b_{42} = 64\}$, $B_5 = \{b_{51} = 4000\}$, then $G = B_1 \times B_2 \times B_3 \times B_4 \times B_5$, $G = \{g_1, g_2, g_3, g_4, ..., g_{16}\}$, where each $g_i, i = 1, 2, ..., 16$, is a 5-tuples element.
Step 1: From Table 1, we can construct $\mathcal{Z}$ as

Table 1: Degrees of Membership $\zeta_{\mathcal{Z}}(g_i)$

$\zeta_{\mathcal{Z}}(g_i)$	Degree	$\zeta_{\mathcal{Z}}(g_i)$	Degree
$\zeta_{\mathcal{Z}}(g_1)$	0.1	$\zeta_{\mathcal{Z}}(g_9)$	0.9
$\zeta_{\mathcal{Z}}(g_2)$	0.2	$\zeta_{\mathcal{Z}}(g_{10})$	0.16
$\zeta_{\mathcal{Z}}(g_3)$	0.3	$\zeta_{\mathcal{Z}}(g_{11})$	0.25
$\zeta_{\mathcal{Z}}(g_4)$	0.4	$\zeta_{\mathcal{Z}}(g_{12})$	0.45
$\zeta_{\mathcal{Z}}(g_5)$	0.5	$\zeta_{\mathcal{Z}}(g_{13})$	0.35
$\zeta_{\mathcal{Z}}(g_6)$	0.6	$\zeta_{\mathcal{Z}}(g_{14})$	0.75
$\zeta_{\mathcal{Z}}(g_7)$	0.7	$\zeta_{\mathcal{Z}}(g_{15})$	0.65
$\zeta_{\mathcal{Z}}(g_8)$	0.8	$\zeta_{\mathcal{Z}}(g_{16})$	0.85

$$\mathcal{Z} = \left\{ \begin{array}{l} 0.1/g_1, 0.2/g_2, 0.3/g_3, 0.4/g_4, \\ 0.5/g_5, 0.6/g_6, 0.7/g_7, 0.8/g_8, \\ 0.9/g_9, 0.16/g_{10}, 0.25/g_{11}, 0.45/g_{12}, \\ 0.35/g_{13}, 0.75/g_{14}, 0.65/g_{15}, 0.85/g_{16} \end{array} \right\}.$$

Step 2: Table 2 presents $\psi_{\mathcal{Z}}(g_i)$ corresponding to each element of G.

Table 2: Approximate functions $\psi_{\mathcal{Z}}(g_i)$

g_i	$\psi_{\mathcal{Z}}(g_i)$	g_i	$\psi_{\mathcal{Z}}(g_i)$
g_1	$\{0.2/T_1, 0.5/T_2\}$	g_9	$\{0.3/T_2, 0.6/T_3, 0.8/T_6\}$
g_2	$\{0.3/T_1, 0.1/T_2, 0.1/T_3\}$	g_{10}	$\{0.2/T_6, 0.6/T_7, 0.2/T_8\}$
g_3	$\{0.3/T_2, 0.4/T_6, 0.4/T_8\}$	g_{11}	$\{0.2/T_2, 0.6/T_4, 0.1/T_7\}$
g_4	$\{0.6/T_4, 0.1/T_5, 0.9/T_6\}$	g_{12}	$\{0.1/T_1, 0.3/T_3, 0.7/T_7\}$
g_5	$\{0.4/T_4, 0.5/T_5, 0.8/T_8\}$	g_{13}	$\{0.1/T_2, 0.1/T_3, 0.9/T_5\}$
g_6	$\{0.2/T_2, 0.5/T_4, 0.3/T_7\}$	g_{14}	$\{0.4/T_1, 0.7/T_5, 0.3/T_8\}$
g_7	$\{0.8/T_1, 0.7/T_3, 0.4/T_5\}$	g_{15}	$\{0.2/T_2, 0.1/T_3, 0.2/T_8\}$
g_8	$\{0.1/T_2, 0.3/T_3, 0.3/T_7\}$	g_{16}	$\{0.4/T_4, 0.6/T_5, 0.7/T_6\}$

Step 3: With the help of Tables 1 and 2, we can construct $\Psi_{\mathcal{Z}}$ as

$$\Psi_{\mathcal{Z}} = \left\{ \begin{array}{l} (0.1/g_1, \{0.2/T_1, 0.5/T_2\}), (0.2/g_2, \{0.3/T_1, 0.1/T_2, 0.1/T_3\}), \\ (0.3/g_3, \{0.3/T_2, 0.4/T_6, 0.4/T_8\}), (0.4/g_4, \{0.6/T_4, 0.1/T_5, 0.9/T_6\}), \\ (0.5/g_5, \{0.4/T_4, 0.5/T_5, 0.8/T_8\}), (0.6/g_6, \{0.2/T_2, 0.5/T_4, 0.3/T_7\}), \\ (0.7/g_7, \{0.8/T_1, 0.7/T_3, 0.4/T_5\}), (0.8/g_8, \{0.1/T_2, 0.3/T_3, 0.3/T_7\}), \\ (0.9/g_9, \{0.3/T_2, 0.6/T_3, 0.8/T_6\}), (0.16/g_{10}, \{0.2/T_6, 0.6/T_7, 0.2/T_8\}), \\ (0.25/g_{11}, \{0.2/T_2, 0.6/T_4, 0.1/T_7\}), (0.45/g_{12}, \{0.1/T_1, 0.3/T_3, 0.7/T_7\}), \\ (0.35/g_{13}, \{0.1/T_2, 0.1/T_3, 0.9/T_5\}), (0.75/g_{14}, \{0.4/T_1, 0.7/T_5, 0.3/T_8\}), \\ (0.65/g_{15}, \{0.2/T_2, 0.1/T_3, 0.2/T_8\}), (0.85/g_{16}, \{0.4/T_4, 0.6/T_5, 0.7/T_6\}) \end{array} \right\}.$$

Step 4:

Table 3: Membership values $\zeta_{\Psi_{\mathcal{Z}}^D}(T_i)$

T_i	$\zeta_{\Psi_{\mathcal{Z}}^D}(T_i)$	T_i	$\zeta_{\Psi_{\mathcal{Z}}^D}(T_i)$
T_1	0.0616	T_5	0.1200
T_2	0.0528	T_6	0.1142
T_3	0.2091	T_7	0.0535
T_4	0.0769	T_8	0.0567

From Table 3, we can construct $\Psi_{\mathcal{Z}}^D$ as

$$\Psi_{\mathcal{Z}}^D = \left\{ \begin{array}{l} 0.0616/T_1, 0.0528/T_2, 0.2091/T_3, 0.0769/T_4, \\ 0.1200/T_5, 0.1142/T_6, 0.0535/T_7, 0.0567/T_8 \end{array} \right\}.$$

Step 5: Since maximum of $\zeta_{\Psi_{\mathcal{Z}}^D}(T_i)$ is 0.2091 (see, Figure 2) so, the mobile tablet T_3 is selected.

Figure 2: Membership values of $\zeta_{\Psi_{\mathcal{Z}}^D}(T_i)$



5. Discussion

Our proposed structure $fpfhs$-set is very useful in dealing with many decisive systems and it is the generalization of:

(i) Fuzzy Parameterized Hypersoft Set (fphs-set) if the range $F(\mathcal{U})$ of approximate function $\psi_{\mathcal{W}}$ is replaced with $P(\mathcal{U})$.

(ii) Hypersoft Set (hs-set) if domain and range of approximate function $\psi_{\mathcal{W}}$ are replaced with G and $P(\mathcal{U})$ respectively.

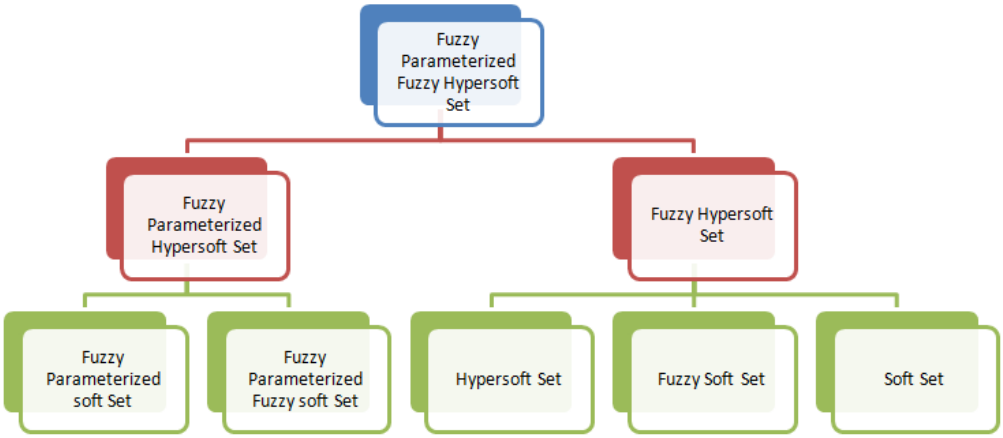
(iii) Fuzzy Parameterized Fuzzy Soft Set (fpfs-set) if disjoint attribute-valued sets are replaced with single set of distinct attributes.

(iv) Fuzzy Parameterized Soft Set (fpfs-set) if disjoint attribute-valued sets are replaced with single set of distinct attributes and the range $F(\mathcal{U})$ of approximate function $\psi_{\mathcal{W}}$ is replaced with $P(\mathcal{U})$.

(v) Fuzzy Soft Set (fs-set) if disjoint attribute-valued sets are replaced with single set E of distinct attributes and the domain G of approximate function $\psi_{\mathcal{W}}$ is replaced with E.

(vi) Soft Set (s-set) if disjoint attribute-valued sets are replaced with single set E of distinct attributes, the domain G and range $F(\mathcal{U})$ of approximate function $\psi_{\mathcal{W}}$ are replaced with E and $P(\mathcal{U})$ respectively.

Figure 3: Generalization of Proposed Model



5.1 Comparison

In this subsection, we compare our propose structure with the existing studies.

Authors	Structures	Remarks
Çağman et al. [13]	Fuzzy parameterized fuzzy soft set theory	Single set of attribute is employed to develop decision system via fuzzy parameterized fuzzy soft set theory,
Enginoğlu et al. [14]	Fuzzy parameterized fuzzy soft matrices	Single set of attribute is employed to develop decision system via fuzzy parameterized fuzzy soft matrix theory,
Kamacı et al. [15]	Interval-valued fuzzy parameterized intuitionistic fuzzy soft sets	Single set of attribute is employed to develop decision system via interval-valued fuzzy parameterized intuitionistic fuzzy soft sets theory,
Proposed structure	fuzzy parameterized fuzzy hypersoft sets theory	• Attributes are further classified into disjoint attribute-valued sets • Decision system is developed via employment of multi-argument approximate functions.

6. Conclusion

In this study, firstly. elementary notions of fuzzy parameterized fuzzy hypersoft set (i.e. an extension of fuzzy parameterized fuzzy soft set) are characterized with examples and generalized results. Secondly, theoretic operations i.e. union, intersection, of proposed structure are investigated with improved results and axiomatic properties. Thirdly, a novel algorithm is proposed for decision making and is validated with the help of an illustrative example for best purchasing of mobile tablet from mobile market. Lastly, comparison of proposed structure with the existing ones, is presented along with the generalization validation of proposed structure. Future work may include the extension of this work for other fuzzy-like environments such as

(i) interval valued fuzzy set;

(ii) intuitionistic fuzzy set;

(iii) pythagorean fuzzy set;

(iv) spherical fuzzy set;

(v) q-rung ortho pair fuzzy set;

(vi) neutrosophic set etc.

and for other soft set-like algebraic structures like

(i) soft topological space;

(ii) soft metric space;

(iii) soft group;

(iv) soft ring;

(v) soft vector space;

(vi) soft functional analysis etc.

with the implementation for solving more real life problems via decision making techniques i.e. TOPSIS etc.

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Applications of Fibonacci matrix to characterize some Banach spaces through weakly unconditional Cauchy series

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Abstract. In this article, we introduce new sequence spaces $U(\mathcal{T}(x), \mathcal{F}, p, u)$, $U(\mathcal{T}_w(x), \mathcal{F}, p, u)$ and $U(\mathcal{T}_{w^*}(h), \mathcal{F}, p, u)$ by using Fibonacci matrix and sequence of modulus functions. We make an effort to study some characterizations of weakly unconditionally Cauchy series associated to newly formed sequence spaces. Finally, the barreledness of a normed space Y is characterized by means of *weakly** unconditionally Cauchy series in Y^*.

Keywords: modulus function, weakly unconditionally Cauchy series, Fibonacci sequence spaces, barreledness.

1. Introduction

Fibonacci number satisfy the recurrence relation $t_k = t_{k-1} + t_{k-2};\ k \geq 2$ with initial condition $t_0 = 1$ and $t_1 = 1$, where t_k is the kth term of the sequence. As a result the sequence $1, 2, 3, 5, 8, 13, 21, 34, 55, \cdots$ is obtained which is called as Fibonacci sequence. Fibonacci numbers has many applications in arts, sciences

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and architecture. These numbers has many interesting properties like as division of two successive sequence of Fibonacci numbers converges to the golden ratio. In [8] Kara introduced the Fibonacci band matrix $\mathcal{T} = (t_{kn})$ as

$$t_{kn} = \begin{cases} -\frac{t_{k+1}}{t_k}, & (n = k-1), \\ \frac{t_k}{t_{k+1}}, & (n = k), \\ 0, & (0 \leq n < k-1 \text{ or } n > k), \end{cases}$$

where $k, n \in \mathbb{N}$. He studied some results related to Fibonacci sequence spaces $\ell_p(\hat{F})$ for $1 \leq p < \infty$ and $\ell_\infty(\hat{F})$. By ω, c, c_0 and ℓ_∞ we denote the space of all real valued sequences, convergent, null and bounded sequences respectively. In [2] the sequence spaces $c_0(\hat{F})$ and $c(\hat{F})$ were introduced and studied by Başarir et al.. Boussayoud et al. [4] have introduced new operator to derive new symmetric properties of Fibonacci numbers. Khan et al.[9] introduced intuitionistic fuzzy I-convergent Fibonacci difference sequence spaces. In [10] Kılınc and Candan generalized Fibonacci difference sequence spaces by using sequence of modulus functions. Recently, Raj et al. [20] introduced the following Fibonacci difference sequence spaces for modulus functions:

$$\ell_\infty(\mathcal{T}, \mathcal{F}, p, u) = \left\{ x = (x_k) \in \omega : \sup_{k \in \mathbb{N}} \left[u_k f_k \left(\left| \frac{t_k}{t_{k+1}} x_k - \frac{t_{k+1}}{t_k} x_{k-1} \right| \right) \right]^{p_k} < \infty \right\}$$

and

$$\ell(\mathcal{T}, \mathcal{F}, p, u) = \left\{ x = (x_k) \in \omega : \sum_k \left[u_k f_k \left(\left| \frac{t_k}{t_{k+1}} x_k - \frac{t_{k+1}}{t_k} x_{k-1} \right| \right) \right]^{p_k} < \infty \right\}.$$

Define a sequence space

$$c_0(\mathcal{T}, \mathcal{F}, p, u) = \left\{ x = (x_k) \in \omega : \lim_{k \to \infty} \left[u_k f_k \left(\left| \frac{t_k}{t_{k+1}} x_k - \frac{t_{k+1}}{t_k} x_{k-1} \right| \right) \right]^{p_k} = 0 \right\}.$$

Let $wuCs(Y)$, $wcs(Y)$, $ucs(Y)$, $\ell_1(Y)$ and $cs(Y)$ be the Y-valued sequence spaces of weakly unconditionally Cauchy series, weakly convergent series, unconditionally convergent series, absolutely convergent series and convergent series respectively.

Several authors have given different attribution of $wuCs$ ([1],[3],[6],[14]) as follows:
(i) A sequence $x = (x_k) \in wuCs(Y)$ iff $(s_k x_k) \in cs(Y)$ $\forall$ $s = (s_k) \in c_0$ and $x = (x_k) \in ucs(Y)$ iff $(s_k x_k) \in cs(Y)$ $\forall$ $s = (s_k) \in \ell_\infty$.
(ii) A sequence $x = (x_k) \in wuCs(Y)$ iff $B : c_0 \to Y$ is a bounded operator given by $B(s) = \sum_k s_k x_k$ with $Ze_m = x_m$, where e^m is the sequence with $e^m_m = 1$ and $e^m_k = 0$ for $k \neq m$ $(m \in \mathbb{N})$.
(iii) In a normed space Y, the series $\sum_k x_k$ is weakly unconditionally Cauchy series if and only if the set

$$G = \left\{ \sum_{k=1}^{m} s_k x_k : |s_k| \leq 1,\ k = 1, 2, \cdots, m; m \in \mathbb{N} \right\} \tag{1}$$

is bounded.
(iv) In a real Banach space Y a series $\sum_k x_k$ is called wuCs if $\sum_k |f(x_k)| < \infty$ for every $f \in Y^*$ and is called ucs if for every permutation ζ of $\mathbb{N}$ the series $\sum_k x_{\zeta(k)}$ is convergent.

A sequence space Y with a linear topology is called a K-space, provided each of the maps $q_n : Y \to \mathbb{R}$ defined by $q_n(x) = x_n$ is continuous for all $n \in \mathbb{N}$. A K-space Y is called an FK-space, if Y is a complete linear metric space. An FK-space Y has AD if the set c_{00} of all finitely non-zero sequences defined as $c_{00} = span\{e^m : m \in \mathbb{N}\}$ is dense in Y.

The idea of modulus was structured by Nakano [19]. A function $f : [0, \infty) \to [0, \infty)$ is said to be a modulus function if it satisfies the following conditions:-

1. $f(x) = 0$ if and only if $x = 0$,
2. $f(x + y) \leq f(x) + f(y)$, for all $x, y \geq 0$,
3. f is increasing,
4. f is continuous from the right at 0.

Ruckle [22] used the idea of modulus function to construct the sequence space

$$Y(f) = \{x = (x_k) : \sum_{k=1}^{\infty} f(|x_k|) < \infty\}.$$

Recently Mohiuddine et al.[16] introduced difference sequence spaces of Lucas band matrix and modulus function.

Let Y and Z be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real numbers a_{nk}, for $n, k \in \mathrm{N}$, where $\mathrm{N} = \mathbb{N} \cup \{0\}$. Then, we say that A defines a matrix transformation from Y into Z and we denote it by writing $A : Y \to Z$ if for every sequence $x = (x_k) \in Y$, the sequence $Ax = \{A_n(x)\}$ and the A-transform of x is in Z, where

$$A_n(x) = \sum_{k=0}^{\infty} a_{nk} x_k \quad (n \in \mathrm{N}).$$

The matrix domain Y_A of an infinite matrix A in a sequence space Y is defined by

$$Y_A = \{x = (x_k) \in \omega : Ax \in Y\}.$$

For a triangle infinite matrix many mathematicians have defined sequence spaces with the help of matrix domain see ([8], [11], [17], [18]). To know more about sequence spaces see ([5], [7], [12], [13], [15], [21], [23]) and references therein.
Let $\mathcal{T} = (t_{kn})$ denotes the Fibonacci band matrix, $\mathcal{T}(s_k)$ denotes a matrix transformation of a sequence (s_k), $\mathcal{F} = (f_k)$ be a sequence of modulus functions, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of positive real numbers. In normed spaces Y and Y^*(the dual spaces

of Y) consider two sequences $x = (x_k)$ and $h = (h_k)$ respectively. In this paper, we define the following sequence spaces:

$$U(\mathcal{T}(x), \mathcal{F}, p, u) = \left\{ s = (s_k) \in \ell_\infty(\mathcal{T}, \mathcal{F}, p, u) : \sum_k [u_k f_k\big(\big|\mathcal{T}(s_k)\big|\big)]^{p_k} x_k \text{ exists } \right\},$$

$$U(\mathcal{T}_w(x), \mathcal{F}, p, u) = \left\{ s = (s_k) \in \ell_\infty(\mathcal{T}, \mathcal{F}, p, u) : w - \sum_k [u_k f_k\big(\big|\mathcal{T}(s_k)\big|\big)]^{p_k} x_k \text{ exists } \right\}$$

and

$$U(\mathcal{T}_{w^*}(h), \mathcal{F}, p, u) = \left\{ s = (s_k) \in \ell_\infty(\mathcal{T}, \mathcal{F}, p, u) : w^* - \sum_k [u_k f_k\big(\big|\mathcal{T}(s_k)\big|\big)]^{p_k} h_k \text{ exists } \right\}.$$

These spaces are endowed with sup norm called the space of convergence, *weak* convergence and *weak** convergence of the series $\sum_k x_k$ respectively. Also, $w - \sum_k [u_k f_k\big(\big|\mathcal{T}(s_k)\big|\big)]^{p_k} x_k$ denotes the limit in weak topology and

$$w^* - \sum_k [u_k f_k\big(\big|\mathcal{T}(s_k)\big|\big)]^{p_k} h_k$$

denotes the limit in weak* topology.

2. Main results

Proposition 2.1. *If $U(\mathcal{T}(x), \mathcal{F}, p, u)$ is complete, then $c_0(\mathcal{T}, \mathcal{F}, p, u) \subseteq U(\mathcal{T}(x), \mathcal{F}, p, u)$.*

Proof. In order to show this let us consider $c_0(\mathcal{T}, \mathcal{F}, p, u) \nsubseteq U(\mathcal{T}(x), \mathcal{F}, p, u)$. Then, there exist a sequence $s^0 = (s_k^0) \in c_0(\mathcal{T}, \mathcal{F}, p, u)$ such that

$$\sum_k [u_k f_k\big(|\mathcal{T}(s_k^0)|\big)]^{p_k} x_k$$

is not convergent. Since $c_0(\mathcal{T}, \mathcal{F}, p, u)$ is a $AD-$space, there exist a Cauchy sequence $s = (s_k^n) \in c_{00}$ (also, in $U(\mathcal{T}(x), \mathcal{F}, p, u)$) such that

$$\lim_{n\to\infty} s_k^n = s_k^0.$$

Then, $U(\mathcal{T}(x), \mathcal{F}, p, u)$ is not complete. This implies that $c_0(\mathcal{T}, \mathcal{F}, p, u) \subseteq U(\mathcal{T}(x), \mathcal{F}, p, u)$ only if $U(\mathcal{T}(x), \mathcal{F}, p, u)$ is complete. □

Theorem 2.1. *Let $\mathcal{F} = (f_k)$ be a sequence of modulus functions, $u = (u_k)$ be a sequence of positive real numbers, $p = (p_k)$ be a bounded sequence of positive real numbers. Suppose Y is a Banach space with $x = (x_k) \in Y$. Then, $x \in wuCs(Y)$ iff $U(\mathcal{T}(x), \mathcal{F}, p, u)$ is a Banach space.*

Proof. The set G defined in (1) is bounded, as $x \in wuCs$. So, there exist $N > 0$ such that $\|G\| \leq N$. Let (s^n) be a Cauchy sequence in $U(\mathcal{T}(x), \mathcal{F}, p, u)$. Since $\ell_\infty(\mathcal{T}, \mathcal{F}, p, u)$ is a Banach space, there exist $s = (s_k^0) \in \ell_\infty(\mathcal{T}, \mathcal{F}, p, u)$ such that $\lim_n s^n = s^0 \in \ell_\infty(\mathcal{T}, \mathcal{F}, p, u)$. We need to show that $s^0 \in U(\mathcal{T}(x), \mathcal{F}, p, u)$. For $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ and $k \in \mathbb{N}$

$$\left|[u_k f_k\big(|\mathcal{T}(s_k^n)|\big)]^{p_k} - [u_k f_k\big(|\mathcal{T}(s_k^0)|\big)]^{p_k}\right| < \frac{\epsilon}{3N}.$$

This implies

$$\frac{3N}{\epsilon}\left|[u_k f_k\big(|\mathcal{T}(s^n)|\big)]^{p_k} - [u_k f_k\big(|\mathcal{T}(s^0)|\big)]^{p_k}\right| < 1.$$

Hence,

$$\frac{3N}{\epsilon}\sum_{k=1}^{m}\big([u_k f_k\big(|\mathcal{T}(s_k^n)|\big)]^{p_k} - [u_k f_k\big(|\mathcal{T}(s_k^0)|\big)]^{p_k}\big)x_j \in G.$$

Now, for $n > n_0$,

$$\left\|\sum_{k=1}^{m}\left([u_k f_k\big(|\mathcal{T}(s_k^n)|\big)]^{p_k} - [u_k f_k\big(|\mathcal{T}(s_k^0)|\big)]^{p_k}\right)x_k\right\| < \frac{\epsilon}{3}.$$

Since (s^n) is a Cauchy sequence in $U(\mathcal{T}(x), \mathcal{F}, p, u)$, there exists a sequence $(y_n) \subset Y$ such that for $m \geq m_0$

$$\left\|\sum_{k=1}^{m}[u_k f_k\big(|\mathcal{T}(s_k^n)|\big)]^{p_k}x_k - y_n\right\| < \frac{\epsilon}{3}.$$

Therefore, for $r > s > n_0$ and $m \in \mathbb{N}$, we have $\|y_r - y_s\| < \epsilon$. Hence (y_n) is a Cauchy sequence in Y. Then, for $\epsilon > 0$, there exists $y_0 \in Y$ such that for $n > n_1$

$$\|y_n - y_0\| < \frac{\epsilon}{3}.$$

Let us suppose $n_2 = \max\{n_0, n_1\}$. By using the above inequalities, we have

$$\begin{aligned}
&\left\|\sum_{k=1}^{m}[u_k f_k\big(|\mathcal{T}(s_k^0)|\big)]^{p_k}x_k - y_0\right\| \\
&\leq \left\|\sum_{k=1}^{m}\big([u_k f_k\big(|\mathcal{T}(s_k^0)|\big)]^{p_k} - [u_k f_k\big(|\mathcal{T}(s_k^n)|\big)]^{p_k}\big)x_k\right\| \\
&+ \left\|\sum_{k=1}^{m}[u_k f_k\big(|\mathcal{T}(s_k^n|\big)]^{p_k}x_k - y_n\right\| + \|y_n - y_0\| \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned}$$

This implies $s^0 \in U(\mathcal{T}(x), \mathcal{F}, p, u)$. Now, we show that if $U(\mathcal{T}(x), \mathcal{F}, p, u)$ is a Banach space, then x must belongs to weakly unconditionally Cauchy series. Consider a sequence x that does not belongs to wuCs(Y). Then $\exists$ a $h \in Y^*$ such that $\sum_k |h(x_k)| = \infty$. Now, let us construct a sequence $s = (s_k) \in c_0(\mathcal{T}, \mathcal{F}, p, u) \setminus U(\mathcal{T}(x), \mathcal{F}, p, u)$. Take $n_1 \in \mathbb{N}$ such that $\sum_{k=1}^{n_1} |h(x_k)| > 2.2$. Define

$$s_k = \begin{cases} \frac{1}{2} \sum_{j=0}^{k} \left[u_k f_k \left(\frac{t_{k+1}^2}{t_j t_{j+1}} \right) \right]^{p_k}, & \text{if } h(x_k) \geq 0, \\ -\frac{1}{2} \sum_{j=0}^{k} \left[u_k f_k \left(\frac{t_{k+1}^2}{t_j t_{j+1}} \right) \right]^{p_k}, & \text{if } h(x_k) < 0, \end{cases}$$

for $k \in \{1, 2, \cdots, n_1\}$. Similarly, we can choose $n_2 > n_1$ such that $\sum_{k=n_1+1}^{n_2} |h(x_k)| > 3.3$. Define

$$s_k = \begin{cases} \frac{1}{3} \sum_{j=0}^{k} \left[u_k f_k \left(\frac{t_{k+1}^2}{t_j t_{j+1}} \right) \right]^{p_k}, & \text{if } h(x_k) \geq 0, \\ -\frac{1}{3} \sum_{j=0}^{k} \left[u_k f_k \left(\frac{t_{k+1}^2}{t_j t_{j+1}} \right) \right]^{p_k}, & \text{if } h(x_k) < 0, \end{cases}$$

for $k \in \{n_1 + 1, 2, \cdots, n_2\}$. Proceeding in this way, we obtain an increasing sequence $(n_k) \in \mathbb{N}$ and the sequence $s = (s_k) \in c_0(\mathcal{T}, \mathcal{F}, p, u)$ such that

$$\sum_{k=1}^{\infty} [u_k f_k(|\mathcal{T}(s_k)|)]^{p_k} h(x_k) = \infty.$$

Then, $s \notin U(\mathcal{T}(x), \mathcal{F}, p, u)$ and hence $c_0(\mathcal{T}, \mathcal{F}, p, u) \nsubseteq U(\mathcal{T}(x), \mathcal{F}, p, u)$. From Proposition 2.1, we can conclude that $U(\mathcal{T}(x), \mathcal{F}, p, u)$ is not a Banach space. □

Remark 2.2. If Y is not a Banach space then $\exists$ a sequence $x = (x_k) \in \ell_1(Y) \setminus cs(Y)$ such that $\forall\, k \in \mathbb{N}$ and $x^{**} \in Y^{**} \setminus Y$. $\|x_k\| < \frac{1}{k2^k}$ and $\sum_k x_k = x^{**}$. Define the sequence $v = (v_k)$ by

$$v_k = \begin{cases} kx_k, & \text{if } k \text{ is odd}, \\ -kx_k, & \text{if } k \text{ is even}. \end{cases}$$

This implies $v = (v_k) \in wuCs(Y)$. Now, let us consider the sequence $s = (s_k) \in c_0(\mathcal{T}, \mathcal{F}, p, u)$ such that

$$s_k = \begin{cases} \frac{1}{2} \sum_{j=0}^{k} \frac{1}{k} \left[u_k f_k \left(\frac{t_{k+1}^2}{t_j t_{j+1}} \right) \right]^{p_k}, & \text{if } k \text{ is odd}, \\ -\frac{1}{2} \sum_{j=0}^{k} \frac{1}{k} \left[u_k f_k \left(\frac{t_{k+1}^2}{t_j t_{j+1}} \right) \right]^{p_k}, & \text{if } k \text{ is even}. \end{cases}$$

Then, $\sum_k [u_k f_k(|\mathcal{T}(s_k)|)]^{p_k} v_k = \frac{1}{2}x^{**} \in Y^{**} \setminus Y$. Thus, $s \notin U(\mathcal{T}(v), \mathcal{F}, p, u)$ and this implies $c_0(\mathcal{T}, \mathcal{F}, p, u) \nsubseteq U(\mathcal{T}(v), \mathcal{F}, p, u)$. Hence, $U(\mathcal{T}(v), \mathcal{F}, p, u)$ is not complete.

Theorem 2.3. *If Y be a normed space and $x = (x_k) \in Y$. Then, a linear operator $B : U(\mathcal{T}(x), \mathcal{F}, p, u) \to Y$, defined as $s \to B(s) = \sum_k [u_k f_k(|\mathcal{T}(s_k)|)]^{p_k} x_k$ is continuous iff $x = (x_k) \in wuCs(Y)$.*

Proof. Firstly, we show that if B is continuous, then $x = (x_k) \in wuCs$. Since B is continuous, there exists $L > 0$ such that $\|B(s_k)\| \leq L\|s_k\|$ for $s = (s_k) \in U(\mathcal{T}(x), \mathcal{F}, p, u)$. Let $d = (d_k) \in B_{c_{00}}$. Then, there exist a sequence $s = (s_k) \in c_{00}(\mathcal{T}, \mathcal{F}, p, u)$ such that $[u_k f_k(|\mathcal{T}(s_k)|)]^{p_k} = d_k$ for every $k \in \mathbb{N}$. Since $c_{00} \subseteq U(\mathcal{T}(x), \mathcal{F}, p, u)$, so we have

$$\begin{aligned} \left\| \sum_{k=1}^{m} d_k x_k \right\| &= \left\| \sum_{k=1}^{m} [u_k f_k(|\mathcal{T}(s_k)|)]^{p_k} x_k \right\| \\ &\leq L\|(s_k)\|. \end{aligned}$$

Hence, the set G defined in (1) is bounded and $x = (x_k) \in wuCs(Y)$.

Conversely, suppose $x \in wuCs(Y)$. As G is bounded, $\exists\ L > 0$ such that $\|g\| < L$ for every $g \in G$. Now, for $m \in \mathbb{N}$ and $(s_k) \in U(\mathcal{T}(x), \mathcal{F}, p, u)$, we have

$$\left\| \sum_{k=1}^{m} \frac{[u_k f_k(|\mathcal{T}(s_k)|)]^{p_k}}{\|[u_k f_k(|\mathcal{T}(s_k)|)]^{p_k}\|} x_k \right\| \leq L.$$

Hence, B is continuous. □

Theorem 2.4. *If Y is a Banach space, then $x \in wuCs(Y)$, iff $U(\mathcal{T}_w(x), \mathcal{F}, p, u)$ is a Banach space.*

Proof. As x is in weakly unconditionally Cauchy series. Suppose $\|g\| \leq N\ \forall$ $g \in G$ and (s^n) be a Cauchy sequence in $U(\mathcal{T}_w(x), \mathcal{F}, p, u)$ such that $(s^n) \to (s^0) \in \ell_\infty(\mathcal{T}, \mathcal{F}, p, u)$ as $n \to \infty$. Let $\epsilon > 0$ and $n_0 \in \mathbb{N}$ such that $\forall\ n \geq n_0$

$$\left|[u_k f_k(|\mathcal{T}(s_k^n)|)]^{p_k} - [u_k f_k(|\mathcal{T}(s_k^0)|)]^{p_k}\right| < \frac{\epsilon}{3N}.$$

Because $\frac{3N}{\epsilon}\left|[u_k f_k(|\mathcal{T}(s^n)|)]^{p_k} - [u_k f_k(|\mathcal{T}(s^0)|)]^{p_k}\right| < 1$ and

$$\frac{3N}{\epsilon} \sum_{k=1}^{m} \left([u_k f_k(|\mathcal{T}(s_k^n)|)]^{p_k} - [u_k f_k(|\mathcal{T}(s_k^0)|)]^{p_k}\right) x_j \in G.$$

So, for every $n > n_0$

$$\left\| \sum_{k=1}^{m} \left([u_k f_k(|\mathcal{T}(s_k^n)|)]^{p_k} - [u_k f_k(|\mathcal{T}(s_k^0)|)]^{p_k}\right) x_k \right\| < \frac{\epsilon}{3}.$$

Also (s^n) is a Cauchy sequence in $U(\mathcal{T}_w(x), \mathcal{F}, p, u)$, $\exists$ a sequence $(z_n) \subset Y$ such that for $m \geq m_0$ and $\forall\ f \in Y^*$

$$\left| \sum_{k=1}^{m} [u_k f_k\big(|\mathcal{T}(s_k^n)|\big)]^{p_k} f(x_k) - f(z_n) \right| < \frac{\epsilon}{3}.$$

By using Hahn- Banach theorem, we have $f \in Y^*$ such that

$$\|z_r - z_s\| = |f(z_r - z_s)|.$$

Hence, we have

$$\|z_r - z_s\| < \epsilon,$$

for $r > s > n_0$ and $m \in \mathbb{N}$. Therefore (z_n) is a Cauchy sequence in Y. Consider that $z_0 \in Y$, such that for $n > n_1$

$$\|z_n - z_0\| < \frac{\epsilon}{3}.$$

Take $n_2 = \max\{n_0, n_1\}$, then we have $|\sum_{k=1}^{m} [u_k f_k\big(|\mathcal{T}(s_k^0)|\big)]^{p_k} f(x_k) - f(z_0)|$

$$\begin{aligned}
&\leq \left| \sum_{k=1}^{m} \big([u_k f_k\big(|\mathcal{T}(s_k^0)|\big)]^{p_k} - [u_k f_k\big(|\mathcal{T}(s_k^n)|\big)]^{p_k}\big) f(x_k) \right| \\
&+ \left| \sum_{k=1}^{m} [u_k f_k\big(|\mathcal{T}(s_k^n)|\big)]^{p_k} f(x_k) - f(z_n) \right| \\
&+ |f(z_n) - f(z_0)| \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
&= \epsilon.
\end{aligned}$$

Thus, $(s^0) \in U(\mathcal{T}_w(x), \mathcal{F}, p, u)$. Now, we need to show that if $U(\mathcal{T}_w(x), \mathcal{F}, p, u)$ is a Banach space, then $x \in wuCs(Y)$. Consider that $h \in Y^*$ such that $\sum_k |h(x_k)| = \infty$.

We can establish $s = (s_k) \in c_0(\mathcal{T}, \mathcal{F}, p, u)$ such that

$$\sum_{k=1}^{\infty} [u_k f_k\big(|\mathcal{T}(s_k)|\big)]^{p_k} h(x_k) = \infty.$$

Hence, $s = (s_k) \notin U(\mathcal{T}_w(x), \mathcal{F}, p, u)$. Thus, $U(\mathcal{T}_w(x), \mathcal{F}, p, u)$ is not complete. □

Theorem 2.5. *Consider $\mathcal{F} = (f_k)$ be a sequence of modulus functions and Y be a normed space with $x{=}(x_k){\in}Y$. Then, a linear operator $B : U(\mathcal{T}_w(x), \mathcal{F}, p, u) \to Y$, defined as $s \to B(s) = w{-}\sum_k [u_k f_k\big(|\mathcal{T}(s_k)|\big)]^{p_k} x_k$ is continuous iff $x = (x_k) \in wuCs(Y)$.*

Proof. From Theorem 2.3, we get the result as desired. □

Theorem 2.6. *Consider a sequence $\mathcal{F} = (f_k)$ of modulus functions and a normed space Y with $h = (h_j) \in Y^*$. Then, we have:*

(i) $h \in wuCs(Y^*)$,

(ii) $U(\mathcal{T}_{w^*}(h), \mathcal{F}, p, u) = \ell_\infty(\mathcal{T}, \mathcal{F}, p, u)$ *and*

(iii) $h \in w^*ucs(Y^*)$

are equivalent iff Y is a barreled normed space.

Proof. Firstly, we prove $(i) \to (ii)$.

As we know $U(\mathcal{T}_{w^*}(h), \mathcal{F}, p, u) \subset \ell_\infty(\mathcal{T}, \mathcal{F}, p, u)$, now we only need to show that $\ell_\infty(\mathcal{T}, \mathcal{F}, p, u) \subset U(\mathcal{T}_{w^*}(h), \mathcal{F}, p, u)$.

Let $s = (s_k)$ be a sequence in $\ell_\infty(\mathcal{T}, \mathcal{F}, p, u)$, then $([u_k f_k\big(|\mathcal{T}(s_k)|\big)]^{p_k} h_k) \in wuCs(Y^*)$. Hence, $\sum_{k=1}^{\infty}[u_k f_k\big(|\mathcal{T}(s_k)|\ \big)]^{p_k} h_k$ is $weak^*$ convergent in Y^*. So, $s = (s_k) \in U(\mathcal{T}_{w^*}(h), \mathcal{F}, p, u)$.

Since $(ii) \to (iii)$ is clear. Now, to show $(iii) \to (i)$. For this define $G^{'}$ by

$$G^{'} = \left\{ \sum_{k=1}^{m} s_k h_k : |s_k| \leq 1,\ k = 1, 2, \cdots, m; m \in \mathbb{N} \right\}.$$

One can see that $G^{'}$ is pointwise bounded and we know that Y is barreled. Thus, $(h_k) \in wuCs(Y^*)$. Conversely, we need to show that Y is a barreled space if $(iii) \to (i)$. Consider that Y is not barreled space. Then $\exists$ a $weak^*$ bounded set $M \subseteq Y^*$ which is not bounded. Suppose $(h_k) \in M$ such that $\|h_k\| > 2^k.2^k$ for $k \in \mathbb{N}$. For $z_k = \frac{1}{2^k} h_k$ and $k \in \mathbb{N}, (z_k(x)) \in \ell_1\ \forall\ x \in Y$. The series $\sum_{k=1}^{\infty} \frac{1}{2^k} z_k$ does not converge because $\|z_k\| > 2^k$ for every $k \in \mathbb{N}$. Thus, we can conclude that $(z_k) \notin wuCs(Y^*)$. □

3. Conclusion

Not only in the field of Mathematics, but in other disciplines also the term "Modulus Function" attained many applications. In our day to day life, modulus function is used every now and then like if one travel anywhere the cost is positive. By geophysict to look at the total amount of the energy used, by scuba divers to discuss their location in regards to a sea level. Applied problems such as ranges of possible values can also be solved using the absolute value function. Similarly Fibonacci sequence serves many applications in real life, for example the number of petals in a flower consistently follows the Fibonacci sequences. Fibonacci numbers include computer algorithms such as the Fibonacci search technique and the Fibonacci heap data structure and graphs called Fibonacci cubes used for interconnecting parallel and distributed systems. In this paper we introduce some sequence spaces associated to sequence of modulus functions and Fibonacci sequence. We shall give certain characterizations of weakly unconditionally Cauchy series by means of completeness of these sequence spaces

and through continuity of operator $B : U(\mathcal{T}(x), \mathcal{F}, p, u)(U(\mathcal{T}_w(x), \mathcal{F}, p, u)) \to Y$. Also, we characterize the barreledness of a normed space Y through the behavior of its *weakly** unconditionally Cauchy series in Y^*.

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Fuzzy (left)topological β-algebras

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Abstract. In this paper we introduce the notions of fuzzy topological spaces on a β-algebra and (left) fuzzy topological β-algebras. We illustrate them with some examples and prove simple but elegant properties.

Keywords: fuzzy topological spaces, fuzzy continuity, β-algebras, (left)fuzzy topological β-algebras.

1. Introduction

In 1966, Imai and Isaki introduced two new classes of algebras namely, BCK algebras and BCI algebras [8]. As a generalization of these algebras sevaral other algebras were introduced by many authors. One such algebras are β algebras introduced by Neggeres and Kim [11]. In the year 1965, Zadeh [17] introduced the notion of fuzzy sets as a generalization of crisp sets, so as to include the uncertainities of the real world physical problems. This notion of fuzzy sets was studied by several authors in the areas of algebra, topology, etc.,. Three years after Zadeh's paper had appeared, Chang [5] made the first 'grafting' of the

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notion of a fuzzy set on general topology. He introduced the notion, what we call now a Chang fuzzy space and made an attempt to develop basic topological notions for such spaces. This paper was followed by others in which Chang fuzzy spaces and other topological type structures for fuzzy set systems were considered. It was further studied by Chang [5], Wong [14], and Lowen [10]. D.H.Foster [7] combined the structure of a fuzzy topological space with that of a fuzzy group, introduced by A.Rosenfeld [13] and defined the term fuzzy topological groups.

The notion of fuzzy topology on fuzzy sets was introduced by Chakrabarty and Ahsaunllah [3]. Chanduraty and Das [4] studied several fundamental properties of such fuzzy topologies. In 1993, Jun [9], combined the structure of a fuzzy topological spaces with that of fuzzy BCK-algebras to formulate the elements of the theory of fuzzy topological BCK-algebras. In [2] Akram and Dar introduced the notions of fuzzy topological subalgebras and ideals in K-algebras. In [1], the authors studied the notion of fuzzy subalgebras of a β-algebra. This paper deals with the notion of (left)fuzzy topological β algebras and investigate some simple properties.

2. Preliminaries

In this section we recall some basic definitions and results that are needed for our work.

Definition 2.1 ([8]). *A BCK-algebra $(X, *, e)$ is a non-empty set X with a constant e and a binary operation $'*'$ satisfying the following axioms: For all $x_1, x_2, x_3 \in X$,*

1. $((x_1 * x_2) * (x_1 * x_3)) * (x_3 * x_2) = e$.
2. $(x_1 * (x_1 * x_2)) * x_2 = e$.
3. $x_1 * x_1 = e$
4. $x_1 * x_2 = e$ *and* $x_2 * x_1 = e \Rightarrow x_1 = x_2$
5. $e * x_1 = e$

Definition 2.2 ([8]). *A BCI-algebra $(X, *, e)$ is a non-empty set X with a constant e and a binary operation $'*'$ satisfying the following axioms: For all $x_1, x_2, x_3 \in X$,*

1. $((x_1 * x_2) * (x_1 * x_3)) * (x_3 * x_2) = e$.
2. $(x_1 * (x_1 * x_2)) * x_2 = e$.
3. $x_1 * x_1 = e$
4. $x_1 * x_2 = e$ *and* $x_2 * x_1 = 0 \Rightarrow x_1 = x_2$

Definition 2.3 ([11]). *A β-algebra is a non-empty set X with a constant 0 and two binary operations $'+'$ and $'-'$ satisfying the following axioms: For all $x, y, z \in X$,*

1. $x - 0 = x$.
2. $(0 - x) + x = 0$.
3. $(x - y) - z = x - (z + y)$

Example 2.1. A β-algebra is the set $X = (\{0, 1, 2, 3\}, +, -, 0)$ with the following Cayley table

+	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

-	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Example 2.2. Consider the set of natural numbers $\mathbb{N}$. Define the binary operation $'*'$ as $a * b = \frac{a}{(a,b)}$ $\forall$ $a, b \in \mathbb{N}$ where (a, b) is the greatest common diviser of a and b. Then, $(\mathbb{N}, +, -, 1)$ where $x + y = x * y'$ with $y' = 0 * y$ and $x - y = x * y$ $\forall$ $x, y \in \mathbb{N}$, is an infinite β-algebra.

Definition 2.4 ([1]). *Let μ be a fuzzy set in a β-algebra X. Then μ is called a fuzzy β-subalgebra of X if foralll $x, y \in X$, we have*

1. $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$;
2. $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$.

Example 2.3. Consider the β-algebra $(X, +, -, 0)$ in Example 2.1.

Define $\mu : X \to [0, 1]$ such that

$$\mu(x) = \begin{cases} 0, & x = 0, \\ 0.5, & x = 1, \\ 1, & x = 2, 3 \end{cases}$$

then μ is a fuzzy β-subalgebra in X.

Definition 2.5 ([10]). *The constant fuzzy set, denoted by k_c is defined by the membership function, $\mu_{k_c}(x) = c$, for all $x \in X$ and $c \in [0, 1]$. The fuzzy set k_1 corresponds to the set X and the fuzzy set k_0 denotes the empty set Φ.*

Definition 2.6 ([10]). *A fuzzy topology is a family $\mathscr{T}$ of fuzzy sets in X which satisfies the following conditions*

1. *for all $c \in I$, $k_c \in \mathscr{T}$;*

2. $\bigcap_{i=1}^{n} A_i \in \mathscr{T}$;

3. $\bigcup_{i\in I} A_i \in \mathscr{T}$;

where $A_i \in \mathscr{T}$ for each $i \in I$, I an indexing set.

If X is a set with a fuzzy topology $\mathscr{T}$, then $(X, \mathscr{T})$ is called a fuzzy topological space and any element in $\mathscr{T}$ is called a $\mathscr{T}$-open fuzzy set in X.

Remark 2.1. Throughout this paper, by an open set in a fuzzy topological space $(X, \mathscr{T})$ we mean only a $\mathscr{T}$-open set.

3. Fuzzy topological spaces on β algebras

In this section, we introduce the notion of fuzzy topological spaces on a β algebra and fuzzy continuity on the fuzzy topological spaces on β algebras. Further, we shall prove some simple properties

Definition 3.1. *A fuzzy topology is a family $\mathscr{T}$ of fuzzy sets in the β-algebra $(X, +, -, 0)$ which satisfies the following conditions:*

1. *for all $c \in I, k_c \in \mathscr{T}$;*

2. $\bigcap_{i=1}^{n} A_i \in \mathscr{T}$;

3. $\bigcup_{i\in I} A_i \in \mathscr{T}$;

where $A_i \in \mathscr{T}$ for each $i \in I, I$ an indexing set.

If X is a β-algebra with a fuzzy topology $\mathscr{T}$, then $(X, \mathscr{T})$ is called a fuzzy topological space on β-algebra and any element in $\mathscr{T}$ is called a $\mathscr{T}$-open fuzzy set in X.

Example 3.1. 1. Let $X = \{0, x_1, x_2, x_3\}$ be a β-algebra with the following Cayley tables.

+	0	x_1	x_2	x_3
0	0	x_1	x_2	x_3
x_1	x_1	0	x_3	x_2
x_2	x_2	x_3	x_1	0
x_3	x_3	x_2	0	x_1

-	0	x_1	x_2	x_3
0	0	x_1	x_3	x_2
x_1	x_1	0	x_2	x_3
x_2	x_2	x_3	0	x_1
x_3	x_3	x_2	x_1	0

Let A be a fuzzy set on X defined by $A(0) = A(x_3) = 0$, $A(x_1) = 0.5$, $A(x_2) = 0.4$. Then $\mathscr{T} = \{0, A, 1\}$ is a fuzzy topology on X and $(X, \mathscr{T})$ is a fuzzy topological space on the β-algebra X.

2. For any β-algebra $X, \mathscr{T} = \{0, 1\}$ is called an indiscrete fuzzy topology on X.

3. For any β-algebra X the discrete fuzzy topology $\mathscr{T}$ on X contains all the fuzzy sets in X.

Definition 3.2. *Let $(X, \mathscr{T})$ be a fuzzy topological space on a β-algebra X. A fuzzy topology $\mathscr{U}$ on X is said to be coarser than $\mathscr{T}$ if and only if $\mathscr{U} \subseteq \mathscr{T}$.*

In ordinary topological space, we consider the neighbourhood of a point. However, in the fuzzy topological space $(X, \mathscr{T})$, on a β-algebra X, we consider the neighbourhood of a fuzzy set in X.

Definition 3.3. *Let $(X, \mathscr{T})$ be a fuzzy topological space on the β-algebra $(X, +, -, 0)$. Let A be a fuzzy set in X. A fuzzy set $U \in \mathscr{T}$ is said to a neighbourhood of A if there exists a $\mathscr{T}$-open fuzzy set O such that $A \subset O \subset U$. That is, $A(x) \leq O(x) \leq U(x)$, for all $x \in X$.*

From the definition the following result follows: Let $(X, \mathscr{T})$ be a fuzzy topological space on the β-algebra $(X, +, -, e)$. Let A be a fuzzy set in X. A is open if and only if for each fuzzy set B such that $B \subseteq A$, A is a neighbourhood of B.

Definition 3.4. *The neighbourhood system $\mathcal{N}$ of a fuzzy set A in a fuzzy topological space $(X, \mathscr{T})$ on the β-algebra $(X, +, -, e)$ is the family of all neighbourhoods of A.*

Theorem 3.1. *Let $(X, \mathscr{T})$ be a fuzzy topological space on the β-algebra and let A be a given fuzzy set in X. If $\mathcal{N}$ is the neighbourhood system of A, then finite intersections of members of $\mathcal{N}$ also belong to $\mathcal{N}$. Further, each fuzzy set of X that contains a member of $\mathcal{N}$ belongs to $\mathcal{N}$.*

Proof. Given that $(X, \mathscr{T})$ is a fuzzy topological space on the β-algebra X and A a fuzzy subset in X.

Since $\mathcal{N}$ is the neighbourhood system of A, it contains all the neighbourhoods of A.

Choose $R, S \in \mathcal{N}$. Then, both R and S are neighbourhoods of the fuzzy set A. Hence, there exists open sets $R_0 \subseteq R$ and $S_0 \subseteq S$ such that

$$A \subseteq R_0 \subseteq R \text{ and } A \subseteq S_0 \subseteq S$$

respectively, and hence $R_0 \cap S_0 \subseteq R \cap S$. That is, $R_0 \cap S_0$ is also a neighbourhood of A, thus proving our result. Further, if a fuzzy set R in X contains a neighbourhood of A, it contains an open neighbourhood of A. This proves that R itself is a neighbourhood of A. That is, $R \in \mathcal{N}$. □

Definition 3.5 ([5])**.** *Let A and B be fuzzy sets in a fuzzy topological space $(X, \mathscr{T})$ on the β-algebra$X$. Let $A \supset B$. Then B is called an interior of A if A is a neighbourhood of B. The union of all interior fuzzy sets of A is again an interior fuzzy set of A and is denoted by A^0*

Theorem 3.2. *Let $(X, \mathscr{T})$ be a fuzzy topological space on the β-algebra X and let A be a fuzzy set in X. Then, A^0 is open and is the largest open set contained in A. Moreover, the fuzzy set A is open if and only if $A^0 = A$.*

Proof. By Definition 3.5, A^0 itself is an interior fuzzy set of A. Therefore, there exists an open set O in X such that $A^0 \subseteq O \subseteq A$.

Since O is an interior fuzzy set of A, $O \subseteq A^0$, proving that $A^0 = O$. This proves that A^0 is an open set and is the largest open set contained in A. Further, if A is open then $A \subset A^0$ as A itself is an interior fuzzy set of A, forcing $A = A^0$.

On the other hand, if $A^0 = A$, then A^0 is the largest open set contained in A and hence A is open.

From the above theorem we define the $\mathscr{T}$-open fuzzy set and the $\mathscr{T}$-closed fuzzy set. □

Definition 3.6. *Let A be fuzzy a set in a fuzzy topological space $(X, \mathscr{T})$ on the β-algebraX. Any fuzzy subset $A \in X$ is called a $\mathscr{T}$-open fuzzy set, if $A^0 = A$. Any fuzzy subset $A \in X$ is called a $\mathscr{T}$-closed fuzzy set, if $A' \in \mathscr{T}$.*

Definition 3.7. *Let $(X, \mathscr{T})$ be a given fuzzy topological space on the β-algebra X. Let $\{A_i / i \in \mathbb{I}\}$ be a sequence of fuzzy sets in X where, $\mathbb{I}$ is an indexing set and let A be any fuzzy set in X. Then the sequence*

1. *$\{A_i\}$ is said to be eventually contained in A if and only if there exists an integer N such that $A_i \subset A$, $\forall\ i \geq N$.*

2. *$\{A_i\}$ is said to be frequently contained in A if and only if for each i there is an integer N such that $A_i \subset A$, $\forall\ i \geq N$.*

3. *$\{A_i\}$ is said to be fuzzy convergent to A if and only if $\{A_i\}$ is eventually contained in each neighbourhood of A.*

4. *A fuzzy set A in X is said to be a cluster fuzzy set of $\{A_i\}$ if and only if $\{A_i\}$ is frequently contained in every neighbourhood of A.*

From the definition one can prove the following.

Theorem 3.3. *Let $(X, \mathscr{T})$ be a fuzzy topological space on the β-algebra X. If the neighbourhood system of each fuzzy set A in X is countable, then*

1. *A is open if and only if each sequence of fuzzy sets $\{A_i\}$ converge to $B(\subset A)$ is eventually contained in A.*

2. *whenever A is a cluster fuzzy set of $\{A_i\}$ then there is a subsequence of $\{A_i\}$ converging to A.*

Definition 3.8. *Let $(X, \mathscr{T})$ and $(Y, \mathscr{U})$ be two fuzzy topological spaces on the β-algebras $(X, +, -, 0)$ and $(Y, +, -, 0)$ respectively. A function $f : X \to Y$ is called a fuzzy continuous (F-continuous) function if the inverse image of each $\mathscr{U}$- fuzzy open set is a $\mathscr{T}$- fuzzy open set.*

Example 3.2. Consider the β-algebra $(X, +, -, 0)$ where $X = \{0, 1, 2, 3, 4, 5\}$ and $+$ and-are defined by the following Cayley tables:

+	0	1	2	3	4	5
1	1	2	0	4	5	3
2	2	0	1	5	3	4
3	3	5	4	0	2	1
4	4	3	5	1	0	2
5	5	4	3	2	1	0

-	0	2	1	3	4	5
1	1	0	2	4	5	3
2	2	1	0	5	3	4
3	3	4	5	0	2	1
4	4	5	3	1	0	2
5	5	3	4	2	1	0

Let the fuzzy sets $\mu_i : X \to [0, 1], i = 1, 2, 3, 4, 5, 6, 7, 8, \ldots$ be given by

$$\mu_1(x) = \begin{cases} .7, & \text{if } x = 0 \\ .5, & \text{if } x = 1, 3 \\ .4, & \text{if } x = 2, 4, 5 \end{cases}, \quad \mu_2(x) = \begin{cases} .3, & \text{if } x = 0 \\ .2, & \text{if } x = 1, 3 \\ 0, & \text{if } x = 2, 4, 5 \end{cases}$$

$$\mu_3(x) = \begin{cases} .6, & \text{if } x = 0 \\ .4, & \text{if } x = 1, 3 \\ .2, & \text{if } x = 2, 4, 5 \end{cases}, \quad \mu_4(x) = \begin{cases} .5, & \text{if } x = 0 \\ .4, & \text{if } x = 1, 3 \\ .1, & \text{if } x = 2, 4, 5 \end{cases}$$

$$\mu_5(x) = \begin{cases} .8, & \text{if } x = 0 \\ .7, & \text{if } x = 1, 3 \\ .5, & \text{if } x = 2, 4, 5 \end{cases}, \quad \mu_6(x) = \begin{cases} .7, & \text{if } x = 0 \\ .4, & \text{if } x = 1, 3 \\ .3, & \text{if } x = 2, 4, 5 \end{cases}$$

$$\mu_7(x) = \begin{cases} .8, & \text{if } x = 0 \\ .6, & \text{if } x = 1, 3 \\ .5, & \text{if } x = 2, 4, 5 \end{cases}, \quad \mu_8(x) = \begin{cases} .8, & \text{if } x = 0 \\ .7, & \text{if } x = 1, 3 \\ .4, & \text{if } x = 2, 4, 5 \end{cases} \quad \ldots$$

Then, the collection $\mathscr{T} = \{0, 1, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6, \mu_7, \mu_8, \ldots\}$ is a fuzzy topology on X. Hence, $(X, \mathscr{T})$ is a fuzzy topological space on β-algebra, on X.

Consider the β-algebra $(Y, +, -, 0)$ where $Y = \{0, a, b, c\}$ and the binary operations $+$ and $-$ are difined by the following Cayley tables:

+	0	a	b	c
0	0	a	b	c
a	a	b	c	0
b	b	c	0	a
c	c	0	a	b

-	0	a	b	c
0	0	c	b	a
a	a	0	c	b
b	b	a	0	c
c	c	b	a	0

Let the fuzzy sets $\sigma_j : Y \to [0, 1], j = 1, 2, 3, 4, \ldots$ be given by

$$\sigma_1(y) = \begin{cases} .8, & \text{if } y = 0 \\ .5, & \text{if } y = a, b \\ .7, & \text{if } y = c \end{cases}, \quad \sigma_2(y) = \begin{cases} .5, & \text{if } y = 0 \\ .1, & \text{if } y = a, b \\ .4, & \text{if } y = c \end{cases}$$

$$\sigma_3(y) = \begin{cases} .6, & \text{if } y = 0 \\ .2, & \text{if } y = a, b \\ .4, & \text{if } y = c \end{cases}, \quad \sigma_4(y) = \begin{cases} .7, & \text{if } y = 0 \\ .3, & \text{if } y = a, b \\ .4, & \text{if } y = c \end{cases} \quad \ldots$$

Then, the collection $\mathscr{U} = \{0, 1, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \ldots\}$ is a fuzzy topology on Y, and hence $(Y, \mathscr{U})$ is a fuzzy topological space on β-algebra Y.

Let$f : X \to Y$ be the function given by, $f(0) = 0, f(1) = c, f(2) = a, f(3) = c, f(4) = a, f(5) = b$. Then, f is F-continuous.

Definition 3.9. *Let $(X, \mathscr{T})$, $(Y, \mathscr{U})$ be two fuzzy topological spaces on β-algebras X and Y respectively. A mapping $f : X \to Y$ is said to be fuzzy open if and only if the image of each $\mathscr{T}$- open fuzzy set is a $\mathscr{U}$- open fuzzy set.*

Theorem 3.4. *Let $(X, \mathscr{T})$ and $(Y, \mathscr{U})$ be the fuzzy topological spaces on β-algebras X and Y respectively. Let $f : X \to Y$ be a function from X to Y. Then the function f is F- continuous if and only if the inverse image of every fuzzy closed set in Y is a fuzzy closed set in X.*

Proof. Suppose that the function f is F- continuous. That is, the inverse image of each fuzzy $\mathscr{U}$- open set is $\mathscr{T}$- open.

Let U' be the set of fuzzy closed sets in Y. Then $\mu_{f^{-1}(U')}(x) = \mu'_U(f(x)) = \mu_{U'}(f(x)) = 1 - \mu_U(f(x)) = 1 - \mu_{f^{-1}(U)}(x) = \mu'_{f^{-1}(U)}(x) \Rightarrow f^{-1}(U') = \{f^{-1}(U)\}'$, for all x in X.

Since f is F- continuous, the inverse image of every fuzzy closed set in Y is a fuzzy closed set in X.

Conversely, assume that the inverse image of every fuzzy closed set of Y is fuzzy closed in X.

Let U be a fuzzy open set in Y. Then, U' is a fuzzy closed set in Y. By hypothesis, $f^{-1}(U')$ is a fuzzy closed set in X. That is, $(f^{-1}(U))'$ is fuzzy closed in X. Hence, $(f^{-1}(U))$ is a fuzzy open set in X. Since the inverse image of every fuzzy open set is open, f is F- continuous. □

We conclude this section with the definitions of restrictrd mapping and relatively fuzzy continuous mapping.

Definition 3.10. *Consider the fuzzy topological spaces $(X, \mathscr{T})$ and $(Y, \mathscr{U})$ on β-algebras X and Y respectively. Let $(A, \mathscr{T}_A)$ and $(B, \mathscr{U}_B)$ be fuzzy subspaces of $(X, \mathscr{T})$ and $(Y, \mathscr{U})$ respectively. The mapping $f : X \to Y$ is said to be a mapping of $(A, \mathscr{T}_A)$ into $(B, \mathscr{U}_B)$ if $f[A] \subset B$.*

Definition 3.11. *Let $(A, \mathscr{T}_A)$ and $(B, \mathscr{U}_B)$ be fuzzy subspaces of fuzzy topological spaces on β-algebras $(X, \mathscr{T})$ and $(Y, \mathscr{U})$ respectively. $f : (A, \mathscr{T}_A) \to (B, \mathscr{U})$ is said to be relatively F-continuous if and only if for each open fuzzy set V in $\mathscr{U}_B$ the intersection $f^{-1}[V] \cap A$ is in $\mathscr{T}_A$. f is said to be relatively fuzzy open if and only if for each fuzzy open set U in $\mathscr{T}_A$ the image $f[U]$ is in $\mathscr{U}_B$.*

4. Fuzzy (left) topological β-algebras

In this section, we introduce the notion of fuzzy left topological β-algebras and prove some simple properties. We recall that : A function $f : X \to Y$ of fuzzy topological spaces is called a fuzzy homeomorphism if and only if f is bijective, fuzzy continuous (F-continuous) and f^{-1} is fuzzy continuous (F-continuous).

Definition 4.1. *Let $(X, +, -, 0)$ and $(Y, +, -, 0)$ be two β- algebras. Let $(X, \mathscr{T})$, $(Y, \mathscr{U})$ be two fuzzy topological spaces on X and Y respectively. A function $f : X \times X \to Y$ is said to be fuzzy left continuous if f is fuzzy continuous with respect to the fuzzy topology on the product $X \times X$ generated by the collection $\{U \times V : U, V \in \mathscr{T}\}$ where*

$$(U \times V)(s,t) = \begin{cases} V(t), & \text{if } U(s) > 0 \\ 0, & \text{otherwise} \end{cases}.$$

Definition 4.2. *Let $(X, +, -, 0)$ be a β-algebra and $\mathscr{T}$ be a fuzzy topology on X such that, for all $x, y \in X$,*

1. *$(x, y) \mapsto x + y$ is fuzzy left continuous;*
2. *$(x, y) \mapsto x - y$ is fuzzy left continuous.*

The pair $(X, \mathscr{T})$ is called a fuzzy left topological β-algebra.

We define $U + V, U - V$ for non zero fuzzy sets U, V on X as follows:

$$(U+V)(x) = \sup\{V(x+s) : U(s) > 0\},$$

$$(U-V)(x) = \sup\{V(x-s) : U(s) > 0\}.$$

With this definition of $U + V$ and $U - V$ we modify the definition of fuzzy topological β-algebra as a fuzzy topological space $(X, \mathscr{T})$ on a β-algebra $(X, +, -, 0)$ that satisfies the following condition:

1. For each $x, y \in X$ and each fuzzy neighbourhood W of $x + y$ there exist fuzzy neighbourhoods U and V of x and y respectively such that $U + V \subseteq W$.
2. For each $x, y \in X$ and each fuzzy neighbourhood W of $x - y$ there exist fuzzy neighbourhoods U and V of x and y respectively such that $U - V \subseteq W$.

Example 4.1. Consider the β- algebra $(X, +, -, 0)$ where $X = \{0, 1, 2, 3\}$ + and $-$ are defined by the following Cayley Tables:

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

-	0	1	2	3
0	0	3	2	1
1	1	0	3	2
2	2	1	0	3
3	3	2	1	0

Define the fuzzy sets, $\mu_i : X \to [0,1]$ $i = 1, 2, \ldots$ by

$$\mu_1(x) = \begin{cases} 0.7, & \text{if } x = 0 \\ 0.4, & \text{if } x = 1, 2 \\ 0.5, & \text{if } x = 3 \end{cases}, \quad \mu_2(x) = \begin{cases} 0.3, & \text{if } x = 0 \\ 0, & \text{if } x = 1, 2 \\ 0.2, & \text{if } x = 3 \end{cases}$$

$$\mu_3(x) = \begin{cases} 0.6, & \text{if } x = 0 \\ 0.2, & \text{if } x = 1, 2 \\ 0.4, & \text{if } x = 3 \end{cases}, \quad \mu_4(x) = \begin{cases} 0.5, & \text{if } x = 0 \\ 0.1, & \text{if } x = 1, 2 \\ 0.4, & \text{if } x = 3 \end{cases},$$

$$\mu_5(x) = \begin{cases} 0.8, & \text{if } x = 0 \\ 0.5, & \text{if } x = 1 \\ 0.72, & \text{if } x = 3 \end{cases}, \quad \mu_6(x) = \begin{cases} 0.7, & \text{if } x = 0.3 \\ 0.4, & \text{if } x = 1 \\ 0.4, & \text{if } x = 2 \\ 0.6 \text{ if } x = 3 \end{cases} \quad \ldots$$

$0 : X \to [0,1]$ defined by $0(x) = 0$, for all $x \in X$. Similarly, $1 : X \to [0,1]$ defined by $1(x) = 1$, for all $x \in X$. $\Rightarrow 0 = \phi$ and $1 = X$. Then $(X, \mathscr{T})$ is a fuzzy topological system β-algebra, where $\mathscr{T}_X = \{0, 1, \mu_1, \mu_2, \mu_3, \ldots\}$.

Consider the set $Y = X \times X$, then Y is a β-algebra under componentwise operations. Y can be made into a topological space under the topology,

$$\mathscr{T}_Y = \{\mu_i \times \mu_j / \mu_i, \mu_j \in \mathscr{T}_X, i, j = 1, 2, 3, \ldots\}$$

by defining $(\mu_i, \mu_j)(x_1, x_2) = \begin{cases} \mu_j(x_2), & \mu_i(x_1) > 0 \\ 0, & \text{otherwise} \end{cases}$. Thus $(Y, \mathscr{T}_Y)$ becomes a topological space.

$+ : Y \to X$ is given by $+(x_1, x_2) = x_1 + x_2$ and $- : Y \to X$ is given by $-(x_1, x_2) = x_1 - x_2$. Now,

$$+^{-1}(\mu_1 \times \mu_2)(x_1, x_2) = (\mu_1 \times \mu_2)(+(x_1, x_2)) = (\mu_1 \times \mu_2)(x_1 + x_2) = \mu_2(x_1 + x_2).$$

This proves that $+$ defined on the β-algebra X is fuzzy left continuous on X.

Similarly, we can show that $-$ is fuzzy left continuous on X. Thus $(X, \mathscr{T})$ given above becomes a fuzzy left topological β-algebra.

Theorem 4.1. *Consider the fuzzy left topological β-algebra $(X, \mathscr{T})$. For any two fuzzy points x_α, y_α on X with $0 < \alpha < 1$, the following results hold:*

1. $x_\alpha + y_\alpha = (x + y)_\alpha$
2. $x_\alpha - y_\alpha = (x - y)_\alpha$

Proof. Consider the fuzzy points x_α and y_α respectively. Then, μ_{x_α} and μ_{y_α} are two fuzzy sets defined by

$$\mu_{x_\alpha}(z) = \begin{cases} \alpha, & \text{if } x = z \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \mu_{y_\alpha}(z) = \begin{cases} \alpha, & \text{if } y = z \\ 0, & \text{otherwise} \end{cases}$$

$(\mu_{x_\alpha} + \mu_{y_\alpha})(z) = \sup\{\mu_{y_\alpha}(y_1 + y_2)$, whenever $\mu_{x_\alpha}(z) > 0$ such that $z = y_1 + y_2, y_1, y_2 \in X\} = \sup\{\min\{\mu_{y_\alpha}(y_1), \mu_{y_\alpha}(y_2)\}\} = \alpha = \sup\{\mu_{y_\alpha}(y_2)\} = \sup\{(\mu_{x_\alpha} \times \mu_{y_\alpha})(y_1, y_2)/z = y_1 + y_2\} = \mu_{(x+y)_\alpha}(z)$. Hence, $(\mu_{x_\alpha} + \mu_{y_\alpha})(z) = \mu_{(x+y)_\alpha}(z)$. Since it is true, for all $z \in X$, $\mu_{x_\alpha} + \mu_{y_\alpha} = \mu_{(x+y)_\alpha}$.

In other words, $x_\alpha + y_\alpha = (x + y)_\alpha$. Analogously, we can prove, $x_\alpha - y_\alpha = (x - y)_\alpha$. □

Theorem 4.2. *In a fuzzy left topological β-algebra $(X, \mathscr{T})$, for any fuzzy sets, S_1, S_2, T_1, T_2 with $S_1 \leq S_2$ and $T_1 \leq T_2$, the following results hold.*

1. $S_1 + T_1 \leq S_2 + T_2$;
2. $S_1 - T_1 \leq S_2 - T_2$.

Proof. For all $(x, y) \in X \times X$, $(S_1 + T_1)(x, y) = \sup\{T_1(y)/S_1(x) > 0\} \leq \sup\{T_2(y)/0 < S_1(x) \leq S_2(x)\} = \sup\{T_2(y)/S_2(x) > 0\}$
$= (S_2 + T_2)(x, y)$. Hence, in general, $S_1 + T_1 \leq S_2 + T_2$. Analogously, we can prove $S_1 - T_1 \leq S_2 - T_2$. □

Corollary 4.1. *In a fuzzy left topological β-algebra $(X, \mathscr{T})$, for any fuzzy sets, S_1, S_2, T_1, T_2 with $S_1 \leq S_2$ and $T_1 \leq T_2$ and, for any $\alpha \in (0, 1]$,*

1. $x_\alpha + S_1 \leq x_\alpha + S_2$;
2. $S_1 + x_\alpha \leq S_2 + x_\alpha$;
3. $x_\alpha - S_1 \leq x_\alpha - S_2$;
4. $S_1 - x_\alpha \leq S_2 - x_\alpha$.

Proof. By taking $S_1 = S_2 = x_\alpha$ and $T_1 = S_1$, $T_2 = S_2$ in 4.2(1) and 4.2(2), respectively we obtain the results (1) and (3). Similarly, by taking, $T_1 = T_2 = x_\alpha$ in 4.2(1) and 4.2(2) respectively, we obtain the results (2) and (4). □

Theorem 4.3. *Let $(X, \mathscr{T})$ be a fuzzy left topological β-algebra then for each $a \in X$, $f_a : X \to X$, given by $f_a(x) = a + x$ is a fuzzy homeomorphism.*

Proof. Let $(X, \mathscr{T})$ be a fuzzy topological space on the β-algebra X and define a map $f : X \to X$ by $f_a(x) = a + x$ for every $a \in X$.

Let W be an open set such that, $(x+a)_\alpha \subseteq W$. Since, X is a fuzzy topological space, we have $(x + a)_\alpha = x_\alpha + a_\alpha$. Thus there exists open sets U and V of x_α and a_α respectively such that,

$$W(z) = (U + V)(z) = \sup_{z = x_\alpha + a_\alpha} \{\min\{U(x_\alpha), V(a_\alpha)\}\}.$$

By our choice $0 < \alpha \leq U(a)$, from we observe that $W(z) = V(y + a)$. Now,

$$f_a(V(z)) = \sup\{V(t)/f_a(t) = z\} = \sup\{V(t)/t + a = z\} = V(z).$$

Thus, $(U+V)(z) \geq f_a(V)(z)$, for all $z \in X$. Hence, $f_a(V) \subseteq U+V \subseteq W \Rightarrow V \subseteq f_a^{-1}(W)$. Hence, f_a is fuzzy continuous, since this is true for any element of X, we conclude that f_a is fuzzy continuous for all $a \in X$.

Now, define $g_a : X \to X$ by $g_a(x) = x - a$. Then, it easily follows that

$$f_a(g_a(x)) = x = 1(x) \text{ and } g_a(f_a(x)) = x = 1(x), \ \forall\ x \in X.$$

That is, $f_a^{-1} = g_a \Rightarrow g_a$ is bijective. Further $f_a^{-1} = f_{-a}$. Thus $g_a = f_{-a}$. Hence g_a is also fuzzy continuous, forcing us to conclude that f_a is a fuzzy homeomorphism. □

Since f_{-a} is a fuzzy homeomorphism, for each $a \in X$,

$$f_{-a}^{-1}(V(x)) = V(f_{-a}(x)) = V(-a+x) = a_\alpha + V(x),$$

for all $x \in X$. Hence V is fuzzy open if and only if $x_\alpha + V$ is fuzzy open for any fuzzy set V in a left fuzzy topological β-algebra $(X, \mathscr{T})$, for some α, $0 < \alpha \leq 1$, and $x \in X$. V is fuzzy closed if and only if V' is fuzzy open. Thus we obtain the following theorem.

Theorem 4.4. *Let $(X, \mathscr{T})$ be a left fuzzy topological β-algebra. Let V be a fuzzy subset of X for each α with $0 < \alpha \leq 1$ and $x \in X$, the following results hold.*

1. *V is fuzzy open if and only if $x_\alpha + V$ is fuzzy open.*
2. *V is fuzzy closed if and only if $x_\alpha - V$ is fuzzy closed.*

5. Conclusion

In this paper, we have introduced the notions of fuzzy topological spaces on a β-algebra and the fuzzy(left) topological β-algebras. One can extend this work to study different topologies on this structure, fuzzy compactness and fuzzy connectedness on fuzzy left topological β-algebra. Further one can identify some real life applications in signal processing, pattern classification and networks problem. In our future work we will introduce the concept of supra topology on the space of β-algebras that lead to the study of supra topological β-algebras, the concepts of compactness and separation axioms on supra topological spaces [6].

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Numerical approximation of vorticity transport equation using implicit-Euler finite difference scheme

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Abstract. This paper is devoted to study the numerical solutions of the two-dimensional vorticity transport equation (VTE) with homogenous Dirichlet boundary conditions. For this problem, we derive the implicit-Euler difference equation using Tayler expansion and with approximating the spatial partial derivatives by finite difference operators. In addition, the matrix formulation for the discrete problems are shown. The proof of consistency and the order of accuracy for the proposed scheme is provided by deriving the local truncation errors of the difference equations. Moreover, the necessary and sufficient condition for the stability of the proposed scheme is considered. Furthermore, the stability and convergence of the proposed method are tested by solving a numerical experiment for different space-steps. Additionally, the discrete graphs for the vorticity and stream functions are curried out to support the numerical results. The results show that the obtained numerical solutions are stable, convergent and decreasing over time.

Keywords: implicit-Euler scheme, vorticity transport equation, truncation-error, stream function, Reynolds number.

1. Introduction

The numerical solutions of time-dependent differential equations, used to model many phenomena in various scientific fields such as physics, engineering, population and fluid dynamics, have been studied by many authors, see for instance [1, 2, 3, 4, 5, 6, 7, 8]. Different scheme have been proposed to solve these problems, such as finite difference and finite element schemes. It known that the implicit finite difference methods are more recommended to solve many types of problems than explicit methods due to their strong stability properties and high convergence-order. However, the main drawback about using implicit schemes is that, at each advance time-level, we need to deal with complicated linear (non-

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linear) systems arising from the numerical approximations of the considered differential equations. One of the common-used implicit schemes for solving linear (semilinear) diffusion equations is Euler scheme [6, 9]. The reason behind that is the implicit Euler scheme is a one-step method and it is easier to use with less complicated computational-steps compared with other implicit methods, such as Crank-Nicolson and ADI methods.

In the present work, we consider the two-dimensional vorticity transport equation (VTE), which is also known as the streamfunction-vorticity formulation [10, 11]. It is well known that the two dimensional vorticity transport equation is used to represent the two-dimensional unsteady flow problem. Moreover, it has several applications in fluid dynamics such as analyzing of laminar to turbulent flow transition, solving the two-dimensional viscous incompressible flow, modeling of turbulent flows, studying on free and mixed convection. For further details regarding the importance, the derivation and the applications of this problem, see [12, 13].

Due to the complexity of nonlinear terms appearing in the two-dimensional vorticity transport equation, it cannot be solved analytically. Therefore, different schemes have been proposed to solve it numerically, subject to different boundary-conditions, such as upwind-Petrov-Galerkin method, boundary-domain integral method, ADI method and Crank-Nicolson scheme, see[10, 11, 14, 15, 16, 17, 18, 19, 20].

In this work, we propose the implicit-Euler finite difference scheme to solve the two-dimensional vorticity transport equation with homogenous Dirichlet boundary conditions. The aim of this work is to prove the consistency and to show the order of accuracy of the proposed scheme. In addition, to test the stability and convergence of the proposed scheme by solving a numerical experiment.

This paper contains eight sections, in section two the mathematical problem is presented, whereas its numerical approximation is presented in section three. Section four is devoted to the matrix formulation of the discrete problem, also the algorithm-steps are stated in this section. The theorems of consistency and stability are provided in section five and six, respectively. For a numerical experiment, the numerical results are presented in form of table and figures and discussed in section seven. The last section is devoted to point out the observed conclusions based on the numerical results.

2. The mathematical problem

The two-dimensional vorticity transport equation with homogenous Dirichlet boundary conditions is given as follows [19, 20]:

$$(1)\quad \frac{\partial \omega}{\partial t} = \frac{1}{R}\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right) - \left(\frac{\partial \psi}{\partial y}\right)\left(\frac{\partial \omega}{\partial x}\right) + \left(\frac{\partial \psi}{\partial x}\right)\left(\frac{\partial \omega}{\partial y}\right), (x, y) \in \Omega,\ t > 0,$$

$$\omega = -\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right), (x,y) \in \Omega, \tag{2}$$

$$\omega(x,y,0) = \omega_0(x,y), \quad \psi(x,y,0) = \psi_0(x,y), (x,y) \in \Omega, \tag{3}$$

$$\omega(x,y,t) = \psi(x,y,t) = 0, \qquad (x,y) \in \partial\Omega,\ t > 0, \tag{4}$$

where $\Omega = \{(x,y) : a < x < b; a < y < b\}$; $\partial\Omega = \{(a,y),(b,y),(x,a),(x,b)\}$; ω and ψ are the vorticity function and the stream function, respectively; ω_0, ψ_0 are smooth, nonnegative functions satisfying (2); and the positive value R is the Reynolds number.

3. Numerical discretization

Throughout this work, we choose equal space-steps (h) in x, y-directions and let k refers to the time-steps:

$$x_0 = a, \quad x_i = x_0 + ih, \quad x_m = b,$$
$$y_0 = a, \quad y_i = y_0 + jh, \quad y_m = b,$$

for $i, j = 1, 2, 3, \ldots m-1$, $h = (b-a)/m$

$$t_n = nk, \quad n = 0, 1, 2, \ldots, k > 0.$$

The approximate values to the $\omega(x_i, y_j, t_n)$ and $\psi(x_i, y_j, t_n)$, is denoted by $\omega^n_{i,j}$ and $\psi^n_{i,j}$, respectively.

In addition, the discrete space $\Omega^n_{i,j}$, takes the form:

$$\Omega^n_{i,j} = \{\ (x_i, y_j, t_n)\,;\ i,j = 0,1,2\ldots m\ ; n = 0,1,2,\ldots\ \}.$$

By using Taylor expansion for $\omega(x,y,(n+1)k)$ about $\omega(x,y,nk)$, it follows:

$$\omega(x,y,(n+1)k) = \left(1 + k\frac{\partial}{\partial t} + \frac{k^2}{2}\frac{\partial^2}{\partial t^2} + \ldots\right)\omega(x,y,nk).$$

Thus

$$\omega(x,y,(n+1)k) = \exp(k\frac{\partial}{\partial t})\omega(x,y,nk).$$

We can rewrite the last equation as follows:

$$\exp\left(-k\frac{\partial}{\partial t}\right)\ \omega(x,y,(n+1)k) = \omega(x,y,nk). \tag{5}$$

By equations (1), (5), we obtain:

$$\exp\left(-k\left[\frac{1}{R}\left(\frac{\partial^2}{x^2} + \frac{\partial^2}{y^2}\right) - \frac{\partial\psi}{\partial y}\frac{\partial}{\partial x} + \frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\right]\right)\omega(x,y,(n+1)k) = \omega(x,y,nk).$$

Now, we replace (ω, ψ), by $(\omega^n_{i,j}, \psi^n_{i,j})$, and approximate the partial derivatives ω_x , ψ_x ,ω_y, and ψ_y by using the central finite difference operators of first

order, δ_x , δ_y , while, the partial derivatives, ω_{xx} , ω_{yy} is approximated by the central finite difference operators of second order, ${\delta^2}_x$, ${\delta^2}_y$, then the last equation becomes:

$$\exp\left(-k\left[\frac{1}{R}\left(\frac{{\delta^2}_x}{h^2}+\frac{{\delta^2}_y}{h^2}\right)-\left(\frac{\delta_y\psi^n_{i,j}}{2h}\right)\left(\frac{\delta_x}{2h}\right)+\left(\frac{\delta_x\psi^n_{i,j}}{2h}\right)\left(\frac{\delta_y}{2h}\right)\right]\right)\omega^{n+1}_{i,j}=\omega^n_{i,j}.$$

For simplicity, we rewrite the last equation as follows:

$$\exp\left(-\frac{r}{R}\left({\delta^2}_x+{\delta^2}_y\right)+\frac{r}{4}\left(\delta_y\psi^n_{i,j}\right)\left(\delta_x\right)-\frac{r}{4}\left(\delta_x\psi^n_{i,j}\right)\left(\delta_y\right)\right)\omega^{n+1}_{i,j}=\omega^n_{i,j},$$

where $r=k/h^2$.

By taking the Taylor expansion for the left hand side of the above equation and truncating after the second term, it follows that

$$\left(1-\frac{r}{R}\left({\delta^2}_x+{\delta^2}_y\right)+\frac{r}{4}\left(\delta_y\psi^n_{i,j}\right)\left(\delta_x\right)-\frac{r}{4}\left(\delta_x\psi^n_{i,j}\right)\left(\delta_y\right)\right)\omega^{n+1}_{i,j}=\omega^n_{i,j},$$

$$i,j=1,2,3,\ldots..m-1;\quad n\geq 0.$$

Now, approximating the space-derivatives in equation (2) by the central finite difference operator of second order, we obtain

$$\omega^n_{i,j}=-\left(\frac{{\delta^2}_x\psi^n_{i,j}}{h^2}+\frac{{\delta^2}_y\psi^n_{i,j}}{h^2}\right).$$

For simplify, the last two equations can be rewritten as follows:

$$\begin{aligned}&\left(1+\frac{4r}{R}\right)\omega^{n+1}_{i,j}-\frac{r}{R}\left(\omega^{n+1}_{i+1,j}+\omega^{n+1}_{i-1,j}+\omega^{n+1}_{i,j+1}+\omega^{n+1}_{i,j-1}\right)\\&+\frac{r}{4}\left(\psi^n_{i,j+1}-\psi^n_{i,j-1}\right)\left(\omega^{n+1}_{i+1,j}-\omega^{n+1}_{i-1,j}\right)\\&-\frac{r}{4}\left(\psi^n_{i+1,j}-\psi^n_{i-1,j}\right)\left(\omega^{n+1}_{i,j+1}-\omega^{n+1}_{i,j-1}\right)=\omega^n_{i,j},\end{aligned}\tag{6}$$

$$-h^2\omega^n_{i,j}=\psi^n_{i+1,j}+\psi^n_{i-1,j}+\psi^n_{i,j+1}+\psi^n_{i,j-1}-4\psi^n_{i,j}.\tag{7}$$

For $i,j=1,2,3,\ldots..m-1, n=0,1,2,\ldots.$

The initial-boundary conditions (3) and (4), in the discrete space, $\Omega^n_{i,j}$, are written as follows:

$$\omega^0_{i,j}=\omega_0\left(x_i,y_i\right),\psi^0_{i,j}=\psi_0\left(x_i,y_i\right),\quad i,j=0,1,2,\ldots m,\tag{8}$$

$$\omega^n_{0,j}=\omega^n_{m,j}=\omega^n_{i,0}=\omega^n_{i,m}=0,\tag{9}$$

$$\psi^n_{0,j}=\psi^n_{m,j}=\psi^n_{i,0}=\psi^n_{i,m}=0,\quad i,j=1,2,3,\ldots m-1,n=1,2,3,\ldots\tag{10}$$

4. Matrix formulations and algorithm steps

4.1 Matrix formulations

We can write the implicit-Euler finite difference equation (6) and (7) in matrix forms as follows:

$$\left(I - \frac{r}{R}C + \frac{r}{4}\ V_2^n A - \frac{r}{4}\ V_1^n B\right)\omega^{n+1} = \omega^n, \tag{11}$$

$$-h^2\omega^n = C\psi^n, n = 0, 1, 2, \ldots \tag{12}$$

$$V_1^n = A\psi^n\ ,\ V_2^n = B\psi^n, \tag{13}$$

where

$$\omega^n = (\omega_{1,1}^n,\ \omega_{2,1}^n, \ldots, \omega_{m-1,1}^n;\ \omega_{1,2}^n,\ \omega_{2,2}^n, \ldots, \omega_{m-1,2}^n; \ldots;$$
$$\omega_{1,m-1}^n,\ \omega_{2,m-1}^n, \ldots, \omega_{m-1,m-1}^n),$$
$$\psi^n = (\psi_{1,1}^n,\ \psi_{2,1}^n, \ldots, \psi_{m-1,1}^n;\ \psi_{1,2}^n,\ w_{2,2}^n, \ldots, \psi_{m-1,2}^n; \ldots;$$
$$\psi_{1,m-1}^n,\ \psi_{2,m-1}^n, \ldots, \psi_{m-1,m-1}^n),$$

M_1, M_2, A and B take the following forms:

$$A = \begin{bmatrix} 0 & I_1 & & 0 \\ -I_1 & 0 & I_1 & \\ & & \ddots & \\ 0 & & -I_1 & 0 \end{bmatrix}_{(m-1)^2\times(m-1)^2}, \quad B = \begin{bmatrix} B_1 & & & 0 \\ & B_1 & & \\ & & \ddots & \\ 0 & & & B_1 \end{bmatrix}_{(m-1)^2\times(m-1)^2},$$

$$B_1 = \begin{bmatrix} 0 & 1 & & 0 \\ -1 & 0 & 1 & \\ & & \ddots & \\ 0 & & -1 & 0 \end{bmatrix}_{(m-1)\times(m-1)}, \ C = \begin{bmatrix} C_1 & I_1 & & 0 \\ I_1 & C_1 & I_1 & \\ & & \ddots & \\ 0 & & I_1 & C_1 \end{bmatrix}_{(m-1)^2\times(m-1)^2},$$

$$C_1 = \begin{bmatrix} -4 & 1 & & 0 \\ 1 & -4 & 1 & \\ & & \ddots & \\ 0 & & 1 & -4 \end{bmatrix}_{(m-1)\times(m-1)}$$

and $I,\ I_1$ are the identity matrices of orders $(m-1)^2$, $(m-1)$, respectively.

4.2 The proposed algorithm steps

The three required computational steps regarding using the implicit-Euler finite difference matrix forms (11)-(13), for solving problem (1)-(4) at each time-level $(n+1)$, are as follows:

First step: Find the vector ψ^n, by computing the solution of the linear system (12).

Second step: Compute V_1^n , V_2^n using (13) and then substitute them in system (11).

Third step: Solve the linear system (11), to get the solution-vector ω^{n+1} .

5. Consistency of the proposed method

The next theorem shows the order of accuracy of the implicit-Euler method by estimating the truncation-errors (consistency-errors) for both finite difference equations (6), (7).

Theorem 5.1. *Let $T^n_{i,j}$ and $\hat{T}^n_{i,j}$ be the local-truncation-errors at (x_i, y_j, t_n) for equations* (6) *and* (7)*, respectively. Then $|T^n_{i,j}| \leq C_1k^2 + C_2kh^2, |\widehat{T}^n_{i,j}| \leq Ch^2, C_1, C_2, C > 0$.*

Proof of Theorem 5.1. Set $\omega|^n_{i,j} = \omega(x_i, y_j, t_n)$, $\psi|^n_{i,j} = \psi(x_i, y_j, t_n)$. By the implicit-Euler discrete equation (6), we obtain

$$T^n_{i,j} = \left(1 - \frac{r}{R}\left(\delta^2{}_x + \delta^2{}_y\right) + \frac{r}{4}\left(\delta_y\psi|^n_{i,j}\right)(\delta_x) - \frac{r}{4}\left(\delta_x\psi|^n_{i,j}\right)(\delta_y)\right)\omega|^{n+1}_{i,j} - \omega|^n_{i,j}.$$

Thus

$$\begin{aligned} T^n_{i,j} &= \omega|^{n+1}_{i,j} - \omega|^n_{i,j} - \frac{r}{R}\left(\delta^2{}_x + \delta^2{}_y\right)\omega|^{n+1}_{i,j} \\ &\quad + \frac{r}{4}\left(\delta_y\psi|^n_{i,j}\right)\left(\delta_x\omega|^{n+1}_{i,j}\right) - \frac{r}{4}\left(\delta_x\psi|^n_{i,j}\right)\left(\delta_y\omega|^{n+1}_{i,j}\right). \end{aligned}$$

By truncating the Taylor expansion, the above equation of LTE becomes:

$$\begin{aligned} T^n_{i,j} &= k\left(\frac{\partial\omega}{\partial t}|^n_{i,j} + \frac{k}{2}\frac{\partial^2\omega}{\partial t^2} + O(k^2)\right) - \frac{k}{R}\left[\left(\frac{\partial^2\omega}{\partial x^2} + \frac{\partial^2\omega}{\partial y^2}\right)|^n_{i,j} + O\left(h^2\right)\right] \\ &\quad + k\left[\frac{\partial\psi}{\partial y}|^n_{i,j} + O\left(h^2\right)\right]\left[\frac{\partial\omega}{\partial x}|^n_{i,j} + O\left(h^2\right)\right] - k\left[\frac{\partial\psi}{\partial x}|^n_{i,j} + O\left(h^2\right)\right]\left[\frac{\partial\omega}{\partial y}|^n_{i,j} + O\left(h^2\right)\right]. \end{aligned}$$

By equation (1), we have

$$\left[\frac{\partial\omega}{\partial t} - \frac{1}{R}\left(\frac{\partial^2\omega}{\partial x^2} + \frac{\partial^2\omega}{\partial y^2}\right) + \left(\frac{\partial\psi}{\partial y}\right)\left(\frac{\partial\omega}{\partial x}\right) - \left(\frac{\partial\psi}{\partial x}\right)\left(\frac{\partial\omega}{\partial y}\right)\right]|^n_{i,j} = 0.$$

It follows that $T^n_{i,j} = O\left(k^2\right) + O\left(kh^2\right)$, or $T^n_{i,j} = O\left(k^2 + kh^2\right)$.

As the partial derivatives of ω, ψ are bounded in the domain, $D^n_{i,j}$, there are two positive constants, $C_1, C_2 \in R$, such that $|T^n_{i,j}| \leq C_1k^2 + C_2kh^2$.

Next, the local truncation error for the difference equation (7) at the mesh points (x_i, y_j, t_n), can be derived as follows:

$$\hat{T}^n_{i,j} = \omega|^n_{i,j} + \left(\frac{\delta^2{}_x\psi|^n_{i,j}}{h^2} + \frac{\delta^2{}_y\psi|^n_{i,j}}{h^2}\right).$$

By truncating the Taylor expansion, the above equation of LTE becomes $\hat{T}^n_{i,j} = \omega|^n_{i,j} + \left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right)|^n_{i,j} + O\left(h^2\right)$. By equation (2), we get $\omega|^n_{i,j} + (\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2})|^n_{i,j} = 0$. Thus $\hat{T}^n_{i,j} = O\left(h^2\right)$, and there exists $C > 0$ such that $|\widehat{T}^n_{i,j}| \leq Ch^2$.

Definition 5.1 ([9]). *The finite difference-equation $F(x_i, y_i, t_n)$ is said to be consistent if, $\frac{LTE}{k} \to 0$, as $h, k \to 0$.*

By Definition 1 and Theorem 1, we can prove the following theorem.

Theorem 5.2. *The difference equation of implicit-Euler scheme* (6) *and* (7) *is consistent.*

6. Stability of the proposed method

In this section, we discuss the stability for the matrix form (11) and (12).

The matrix form implicit-Euler scheme (11) and (12) can be rewritten as follows:

$$\omega^{n+1} = H_n\, \omega^n, \quad \forall\ n, \tag{14}$$

where $H_n = \left(I - \frac{r}{R}C + \frac{r}{4}\ V_2^n A - \frac{r}{4}\ V_1^n B\right)^{-1}$.

Theorem 6.1. *The necessary and sufficient condition, which guarantees that the implicit-Euler scheme* (11) *and* (12) *is stable, is given as follows:*

$$\|H_n\| \leq 1, \quad \textit{for all} \quad n, \tag{15}$$

where

$$\|H_n\|_2 = \max_s |\lambda_s|, \tag{16}$$

$\lambda_s (s = 1, 2, \ldots, (m-1)^2)$ *are the eigenvalues of* H_n .

The proof of this theorem is similar to that in [9], so it is omitted here.

7. Numerical example

Next, we apply the implicit-Euler finite difference algorithm, using (11)-(13), to find the numerical solution of problem (1)-(4), where $R = 50$, and the initial function for vorticity is given as follows:

$$\omega_0\,(x, y) = \left(1 - x^2\right)\left(1 - y^2\right), \qquad x, y \in [-1, 1]. \tag{17}$$

The numerical solutions are computed for $(h = 0.4, 0.2,\ 0.1)$, and $(k = 0.002)$.

Based on the type the chosen initial function (17), we know that the solution of problem (1)-(4) is symmetric and positive. Therefore, at any time-level, it is sufficient to find only the first β components of the approximate solution-vectors ω^n, ψ^n, where

$$\beta = \begin{cases} \frac{(m-1)^2}{2}, & \text{if } m \text{ is even,} \\ \frac{(m-1)^2+1}{2}, & \text{if } m \text{ is odd} \end{cases}.$$

Moreover, for each of $h = 0.4$, $h = 0.2$, and at the time-level n, we shall compute the errors -bounds that show the variations between the two numerical solutions (ω_h^n, ψ_h^n) and $(\omega_{h/2}^n,\ \psi_{h/2}^n)$ computed with h and $h/2$, respectively, at the points of the set:

$$\mathrm{H} = \{(x, y)\,,\ s.t.\ x = -1 + ih;\ y = -1 + jh;\ i = 1, 2, 3, 4, j = 1, 2, h = 0.4\}\,,$$

by using the errors-formulas:

$$E_h^n(\omega) = \frac{\sum_{(x,y)\in \mathrm{H}} \left|\omega_h^n(x, y) - \omega_{h/2}^n(x, y) \right|}{N = 8}, \tag{18}$$

$$E_h^n(\psi) = \frac{\sum_{(x,y)\in \mathrm{H}} \left|\psi_h^n(x, y) - \psi_{h/2}^n(x, y) \right|}{N = 8}. \tag{19}$$

7.1 Numerical results and discussion

The next tables show the numerical solutions of problem (1)-(4) with (17), for $(k = 0.002)$ and $(R = 50)$, obtained by using the implicit-Euler scheme, where Matlab software is used in the computational processes. In Tables 1, 2 and 3, the numerical results are shown, for $h = 0.4, 0.2, 0.1$, with respect to the time-levels 100, 200 and 400, respectively. In Table 4, the errors bounds of numerical solution are shown with $h = 0.4, 0.2$, at time levels: $n = 100, 200, 400$, using the formulas (18),(19). In Table 5, we show the values of $\|H_n\|_2$ defined in (16), for $h = 0.4, 0.2, 0.1$, at time levels: $n = 100, 200, 400$.

Table 1: Numerical solutions, where $n = 100$ $(t = 0.2)$

$\boldsymbol{h}$	**0.4**		**0.2**		**0.1**	
$\mathbf{(i, j)}$	$\boldsymbol{\psi}$	$\boldsymbol{\omega}$	$\boldsymbol{\psi}$	$\boldsymbol{\omega}$	$\boldsymbol{\psi}$	$\boldsymbol{\omega}$
(1,1)	0.0767	0.3989	0.0743	0.3974	0.0738	0.3975
(2,1)	0.1222	0.6068	0.1189	0.6067	0.1181	0.6069
(3,1)	0.1215	0.5937	0.1183	0.5936	0.1174	0.5935
(4,1)	0.0755	0.3773	0.0734	0.3765	0.0727	0.3763
(1,2)	0.1208	0.5795	0.1174	0.5769	0.1166	0.5763
(2,2)	0.1936	0.9057	0.1888	0.9055	0.1876	0.9055
(3,2)	0.1932	0.8990	0.1885	0.8990	0.1872	0.8987
(4,2)	0.1201	0.5689	0.1169	0.5671	0.1160	0.5662

From Tables 1-3, we see that the numerical results for the vorticity and stream are decreasing as time increases. Table 4 shows that at any fixed time-level, as we refine the space-grids, the corresponding errors-bounds are decreasing, which indicates that the resulting numerical solutions are convergent. On the other hand, at any fixed space-step, the corresponding errors are increasing as time increases. Table 5 shows that, at any time-level, the stability-condition (15) is held.

Table 2: Numerical solutions, where $n = 200$ $(t = 0.4)$

h	**0.4**		**0.2**		**0.1**	
(i, j)	ψ	ω	ψ	ω	ψ	ω
(1,1)	0.0742	0.3874	0.0715	0.3820	0.0709	0.3819
(2,1)	0.1188	0.5988	0.1151	0.5981	0.1143	0.5987
(3,1)	0.1174	0.5740	0.1141	0.5739	0.1131	0.5734
(4,1)	0.0720	0.3487	0.0699	0.3469	0.0691	0.3462
(1,2)	0.1161	0.5458	0.1123	0.5377	0.1114	0.5351
(2,2)	0.1878	0.8887	0.1825	0.8879	0.1813	0.8880
(3,2)	0.1870	0.8761	0.1820	0.8762	0.1806	0.8753
(4,2)	0.1149	0.5274	0.1116	0.5229	0.1104	0.5198

Table 3: Numerical solutions, where $n = 400$ $(t = 0.8)$

h	**0.4**		**0.2**		**0.1**	
(i, j)	ψ	ω	ψ	ω	ψ	ω
(1,1)	0.0693	0.3628	0.0659	0.3466	0.0652	0.3438
(2,1)	0.1122	0.5810	0.1077	0.5778	0.1067	0.5794
(3,1)	0.1098	0.5376	0.1061	0.5373	0.1049	0.5360
(4,1)	0.0658	0.3007	0.0636	0.2972	0.0627	0.2952
(1,2)	0.1071	0.4830	0.1026	0.4619	0.1015	0.4539
(2,2)	0.1766	0.8522	0.1703	0.8484	0.1687	0.8484
(3,2)	0.1753	0.8302	0.1695	0.8300	0.1677	0.8275
(4,2)	0.1053	0.4554	0.1016	0.4461	0.1003	0.4379

Table 4: Errors bounds

h	**0.4**		**0.2**	
n	$E_h^n(\psi)$	$E_h^n(\omega)$	$E_h^n(\psi)$	$E_h^n(\omega)$
100	0.0034	8.8750e-04	8.8750e-04	3.0000e-04
200	0.0037	0.0027	9.8750e-04	0.0011
400	0.0043	0.0072	0.0012	0.0033

Table 5: $\|H_n\|_2 = max_s|\lambda_s|$

n	100	200	400
h	$\|H_n\|_2$	$\|H_n\|_2$	$\|H_n\|_2$
0.4	0.999601	0.999606	0.999616
0.2	0.999570	0.999574	0.999584
0.1	0.999550	0.999555	0.999566

7.2 Numerical simulations

In Figures 1,2 and 3, we present the discrete graphs of vorticity and stream obtained by using the implicit-Euler scheme results with $R = 50$, $(k = 0.002)$ and $(h = 0.1)$, at time levels $n = 0,\ 200$ and 400 , respectively. It is observed that the shown discrete graphs of both vorticity and stream are decreasing as time increases.

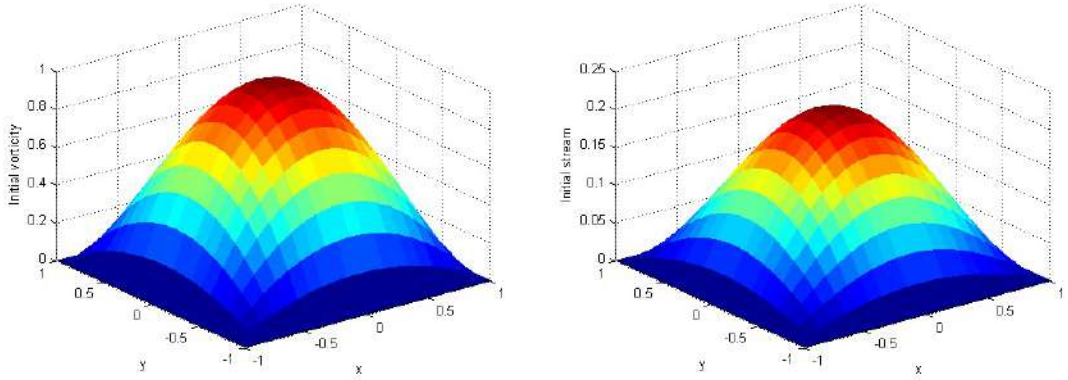


Figure 1: Numerical solutions at $t = 0$

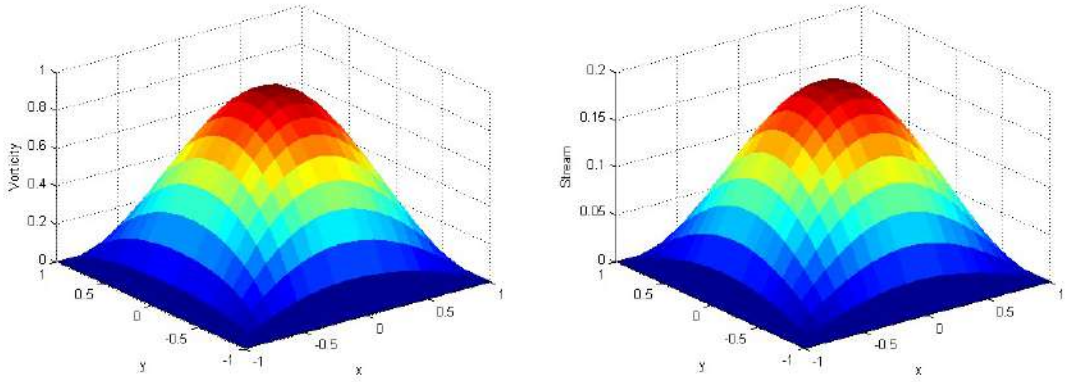


Figure 2: Numerical solutions at $t = 0.4$

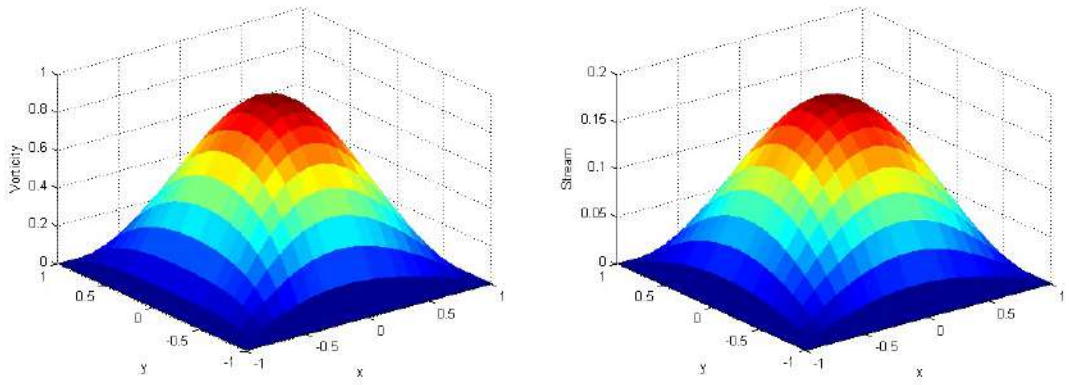


Figure 3: Numerical solutions at $\boldsymbol{t = 0.8}$

8. Conclusion

In this paper, the Implicit-Euler finite difference scheme is proposed to compute the numerical solution of the two-dimensional vorticity transport equation with homogenous Dirichlet boundary conditions. In addition, a numerical experiment is given, and the numerical results are presented and simulated in the form of tables and figures. From this work, we conclude that the Implicit-Euler finite difference scheme is consistent and stable, with any space-step and time level. Moreover, it is observed that the resulting numerical solutions are decreasing over time. Furthermore, at any fixed time-level, as we refine the space-grids, the corresponding errors-bounds are decreasing, which indicates that the resulting numerical solutions are convergent. On the other hand, at any fixed space-step, the corresponding errors are increasing as time increases. In order to overcome this problem, the space grids should be re-refined, as time goes farther.

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On some well covered graphs

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Abstract. If all maximal independent sets of a graph G have the same cardinality, then G is called a well covered graph. In this paper, we study the well covered property for several classes of graphs. We show that the line graph of a complete graphs $L(K_n)$ is a well covered graph. Furthermore, we show that (m, n)-lollipop graph $L_{m,n}$ is a well covered graph only if $n = 2$ and the line graph of (m, n)-lollipop graph $L(L_{m,n})$ is a well covered graph only if m is even and $n = 1$ or $n = 3$ or if m is odd and $n = 2$.

Keywords: maximal independent sets, well covered graphs, line graphs, complete graphs, lollipop graphs.

1. Introduction

All graphs in this paper are finite undirected simple graphs. For a graph G, we denote by $V(G)$ the vertex set of G, and $E(G)$ the edge set of G. The order of a graph G is equal to the cardinality of $V(G)$ and is denoted by $|G|$. The line graph $L(G)$ of a graph G, is a simple graph whose vertex set is the set of all edges of G, and two vertices of $L(G)$ are adjacent if the corresponding edges of G are adjacent. We follow Bondy and Murty [1] for any undefined terms.

For a graph G, a set $M \subseteq V(G)$ is said to be an independent set of G if any two vertices in M are non adjacent. An independent set M is said to be

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maximal if M cannot be included into another independent set. The set M is called a maximum independent set in G if it has the maximum cardinality among all maximal independent sets in G. It is not easy to determine the maximum and maximal independent sets of a graph G. This topic has been studied by different authors (see, [2] and [7]). A graph G is called well covered if all maximal independent sets in a graph G has the same cardinality. Equivalently, a graph G is well covered if every maximal independent set in G is a maximum independent set.

The study of the well covered property of a graph was introduced by Plummer [5]. Several studies on the subject have been published since then (see, [3] and [4]). Also, one of the most effective studies in this subject is Plummer and Michael (see, [6]).

However, there are certain families of well covered graphs can be discovered. Complete graphs K_n are well covered graphs, since each maximal independent set has cardinality one. Also, for a complete bipartite graphs $K_{m,n}$ with $m \neq n$, is not well covered because it has two maximal independent sets with different cardinality.

In this paper, we focus on the well covered property for several classes of graphs. This paper is structured as follows: In section 2, we show that the line graph of a complete graph $L(K_n)$ is well covered. In section 3, we discuss the well covered property of (m, n)-lollipop graph $L_{m,n}$, and its line graph $L(L_{m,n})$.

2. The graphs $L(K_n)$

In this section, we show that the line graph of a complete graph $L(K_n)$ is well covered.

Assume that the vertex set of K_n is $V(K_n) = \{u_1, u_2, ..., u_n\}$ and let the edge between u_i and u_j be denoted by $u_{_{i,j}}$. Thus, $V(L(K_n)) = \{u_{_{i,j}} : 1 \leq i < j \leq n\}$. Then $u_{_{i,j}}$ and $u_{k,l}$ are adjacent if $i = k$ or $j = l$.

Now, for $n \leq 3$, $L(K_n)$ is a well covered graph, since each maximal independent set has cardinality 1. In general, we show that the line graph of a complete graph $L(K_n)$ is well covered for each n.

Theorem 2.1. *For $n \geq 4$, the graph $L(K_n)$ is well covered.*

Proof. In $L(K_n)$ the vertices $u_{_{i,j}}$, $u_{k,l}$ are non adjacent iff $\{i, j\} \cap \{k, l\} = \phi$. So, all independent set of $L(K_n)$ have the form $A = \{u_{x,f(x)} : f : X \to Y\}$ where $f(x)$ is a one to one function, where $X = \{i_1, ..., i_k\}, Y = \{j_1, ..., j_h\}$ such that $X \cap Y = \phi$ and $X \cup Y \subseteq \{1, ..., n\}$.

Now, we will show that A is maximal if $|A| = \lfloor \frac{n}{2} \rfloor$. Indeed, we have two cases:

Case 1. n is even, we have $X = \{i_1, ..., i_{\frac{n}{2}}\}$ and $Y = \{j_1, ..., j_{\frac{n}{2}}\}$ such that $X \cap Y = \phi$ and $X \cup Y = \{1, ..., n\}$.

Let $u_{a,f(a)}, u_{b,f(b)} \in A$, with $a \neq b$. Since $f(x)$ is a one to one function, we have $f(a) \neq f(b)$. Therefore, $u_{a,f(a)}, u_{b,f(b)}$ are independent vertices. Now, let

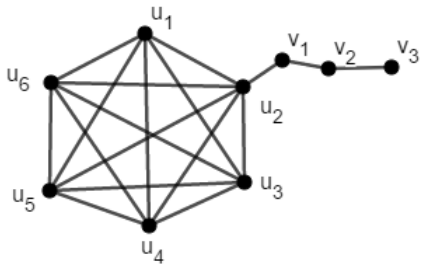
$u_{i,j} \notin A$, then $i, j \in X$ or $i, j \in Y$. Since $f(x)$ is a one to one and $|X| = |Y| = \frac{n}{2}$, we have $u_{i,j}$ is adjacent to one vertex in A which is $u_{i,f(i)}$ or $u_{f^{-1}(j),j}$ respectively. Hence A is maximal and $|A| = |X| = \frac{n}{2}$.
Case 2. n is odd, we have $X = \{i_1, ..., i_{\lfloor \frac{n}{2} \rfloor}\}, Y = \{j_1, ..., j_{\lceil \frac{n}{2} \rceil}\}$ such that $X \cap Y = \phi$ and $X \cup Y = \{1, ..., n\}$. Let $u_{a,f(a)}, u_{b,f(b)} \in A$, with $a \neq b$. Since $f(x)$ is a one to one function, we have $f(a) \neq f(b)$. Therefore, $u_{a,f(a)}, u_{b,f(b)}$ are independent vertices. Now, let $u_{i,j} \notin A$. Then $i, j \in X$ or $i \in X, j \in Y - f(X)$ or $i, j \in Y$. If i or j belongs to X, we have $u_{i,j}$ is adjacent to one vertex in A which is $u_{i,f(i)}$ or $u_{j,f(j)}$. Otherwise, if $i \in Y$ and $j \in Y$, we have $i \in f(X)$ or $j \in f(X)$ since $|Y| = |X| + 1$. Hence $u_{i,j}$ is adjacent to one vertex in A which is $u_{f^{-1}(i),i}$. Hence A is a maximal independent set in $L(K_n)$ and $|A| = |X| = \lfloor \frac{n}{2} \rfloor$. Since all maximal independent sets in $L(K_n)$ have the same cardinality; $|A| = \lfloor \frac{n}{2} \rfloor$, $L(K_n)$ is a well covered graph. □

3. The graph $L_{m,n}$ and its line graph $L(L_{m,n})$

In this section, we look at (m, n)-lollipop graph $L_{m,n}$ and their line graphs $L(L_{m,n})$ in terms of having the well covered property. We will show that the graph $L_{m,n}$ for all $n \neq 2$ are not well covered, while $L_{m,2}$ is a well covered graph for all m. Moreover, we will prove that the line graph of $L_{m,1}$; $L(L_{m,1})$ is well covered only if m is even, $L(L_{m,2})$ is well covered only if m is odd, and $L(L_{m,3})$ is well covered only if m is even. In the other cases the graph $L(L_{n,m})$ is not well covered.

An (m, n)-lollipop graph, denoted by $L_{m,n}$ is a graph on $m + n$ vertices obtained by connecting a complete graph K_m by a path graph P_n with an edge. Let $V(K_m) = \{u_1, u_2, ..., u_m\}$ be the set of vertex of K_m and $V(P_{n+1}) = \{u_i, v_1, ..., v_n\}$ be the set of vertex of P_{n+1} attached with a fixed vertex u_i for some $i : 1 \leq i \leq m$, of the complete graph K_m.

The following Figure shows the $(6, 3)$-lollipop graph where P_3 connected with the vertex u_2 of the complete graph K_6.



In the following theorem, we show that (m, n)-lollipop graph $L_{m,n}$ is well covered if $n = 2$.

Theorem 3.1. *The $(m, 2)$-lollipop graph $L_{m,2}$ is well covered.*

Proof. Let M be a maximal independent set of $(m,2)$-lollipop graph $L_{m,2}$. Let u_k represents the fixed vertex in $V(K_m)$ where P_2 has the vertex set $\{v_1, v_2\}$ and, P_2 is connected to a complete graph K_m. Then for any maximal independent set M, we have two cases.
Case 1. $u_k \in M$, then $v_1 \notin M$ since v_1 is adjacent to u_k and $u_i \notin M$, for all $i \neq k, 1 \leq i \leq m$. Therefore, $v_2 \in M$. Thus, $M = \{u_k, v_2\}$
Case 2. $u_k \notin M$, then $u_i \in M$ for some $i, 1 \leq i \leq m$ and $u_j \notin M$, for all $k \neq j \neq i, 1 \leq j \leq m$. Therefore, $v_1 \in M$ or $v_2 \in M$. That is $M = \{u_i, v_1\}$ or $M = \{u_i, v_2\}$. Thus, for any maximal independent set M of $L_{m,2}$, $|M| = 2$. This completes the proof. □

Now, for $n > 2$, the $(m,n)-$lollipop graph is not well covered. For, let u_k be the fixed vertex in $V(K_m)$ and $\{v_1, v_2, .., v_t : t > 2\}$, then $M_1 = \{u_k, v_i : i > 1, i$ is odd$\}$ and $M_2 = \{u_j, v_i : i$ is odd and $j \neq k\}$ are two maximal independent sets with cardinality $\lceil \frac{n}{2} \rceil$ and $\lceil \frac{n}{2} \rceil + 1$ respectively. Thus we have the following lemma.

Lemma 3.1. *For $m \geq 3$ and $n \neq 2$, the (m,n)-lollipop graph $L_{m,n}$ is not well covered.*

The line graph of a (m,n)-lollipop graph $L(L_{m,n})$ is a graph given by

$$V(L(L_{m,n})) = \{u_{i,j}, e_h : 1 \leq i < j \leq m, 1 \leq h \leq n\}.$$

Now, $L(L_{m,n})$ contains a line graph of complete graph $L(K_m)$ with $V(L(K_m)) = \{u_{i,j} : 1 \leq i < j \leq m\}$ and a path P_n with $V(P_n) = \{e_1, e_2, ..., e_n\}$. Suppose that in $L_{m,n}$, the vertex u_k is the fixed vertex in $V(K_m)$ where P_n is connected with the complete graph K_m. Then, e_1 is adjacent with $u_{1,k}, u_{2,k}, ..., u_{m,k}$ in $L(L_{m,n})$.

In the following theorem we show that if m is even then $L(L_{m,1})$ is a well covered graph.

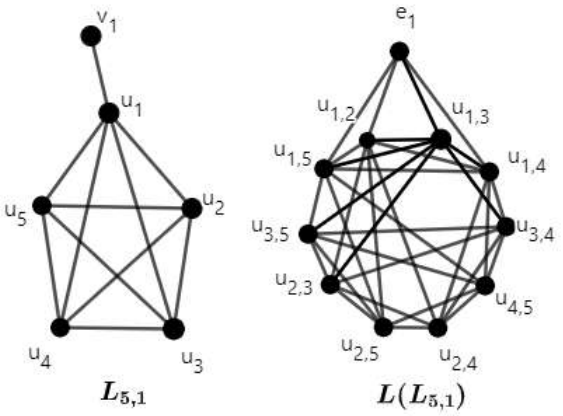
Theorem 3.2. *For any even number $m \geq 4$, the line graph of $(m,1)$-lollipop graph $L(L_{m,1})$ is well covered.*

Proof. Let u_k be the vertex of K_m that is adjacent with v_1 in P_1 by an edge e_1. Then, for any maximal independent set M in $L(L_{m,1})$, we have two cases; either $e_1 \in M$ or $e_1 \notin M$.

If $e_1 \notin M$, then M is a maximal independent set in $L(K_m)$, and hence by Theorem 2.1, $|M| = \frac{m}{2}$. If $e_1 \in M$, let $B = \{u_{k,j} : 1 \leq j \leq m, j \neq k\}$ be the set of all vertices with k is fixed. Since e_1 is incident with the vertices u_k and v_1, M and B have no common vertex. Thus, $M = \{e_1\} \cup M_k$, where $M_k = \{u_{r,s} : 1 \leq r < s \leq m, k \neq r, k \neq s\} \subset V(L(K_m))$. Now, let $D = \{r, s : u_{r,s} \in M_k\}$. Since m is even, k is excluded from D with exactly one j for some $j \neq k$ and $1 \leq j \leq m$. Thus $|D| = m - 2$, i.e. $|M_k| = \frac{m-2}{2}$ and $|M| = 1 + \frac{m-2}{2} = \frac{m}{2}$. Hence any maximal independent set of $L(L_{m,1})$ has a cardinality $\frac{m}{2}$. That is $L(L_{m,1})$ is a well covered graph. □

In general, if m is odd, the line graph of $(m,1)-$lollipop graph is not well covered graph.

For example, consider the line graph of the $(5,1)-$lollipop graph. Then $M_1 = \{e_1, u_{3,4}, u_{2,5}\}$ and $M_2 = \{u_{1,2}, u_{3,4}\}$ are two maximal independent sets with different cardinality (see, the figure below). Indeed, if m is odd, we can always able to find two maximal independent sets with different cardinality.



The following theorem shows that if m is odd, then $L(L_{m,2})$ is a well covered graph.

Theorem 3.3. *For any odd number $m \geq 3$, the line graph of $(m,2)$-lollipop graph $L(L_{m,2})$ is well covered.*

Proof. Let u_k be the vertex of K_m that adjacent with v_1 in P_2 by an edge e_1. Consider $L(L_{m,2})$ to be the line graph of $L_{m,2}$. Then, for any maximal independent set K in $L(L_{m,2})$, we have two cases;
Case 1. $e_1 \notin K$, then $K = \{e_2\} \cup M$, where M is a maximal independent set in $L(K_m)$ and hence by Theorem 2.1 $|M| = \lfloor \frac{m}{2} \rfloor + 1$.
Case 2. $e_1 \in K$, then $K = \{e_1\} \cup M_1$, where $M_1 = \{u_{r,s} : 1 \leq r < s \leq m\}$ and r, s are different from k. By Theorem 2.1 Since m is odd, M_1 is a maximal independent set in $L(K_m)$. So, we have $|M_1| = \lfloor \frac{m}{2} \rfloor$. Thus, for any maximal independent set K of $L(L_{m,2}), |K| = \lfloor \frac{m}{2} \rfloor + 1$. This completes the proof. □

In general, if m is even then $L(L_{m,2})$ is not a well covered graph. Let u_1 be the vertex of K_m that adjacent with v_1 in P_2 by an edge e_1. Then the sets $M_1 = \{e_1, u_{3,4}, u_{5,6}, ..., u_{m-1,m}\}$ and $M_2 = \{e_2, u_{1,2}, u_{3,4}, u_{5,6}, ..., u_{m-1,m}\}$ are two maximal independent sets with cardinality $\frac{m}{2}$ and $\frac{m}{2} + 1$ respectively.

The following theorem shows that if m is even, then $L(L_{m,3})$ is a well covered graph.

Theorem 3.4. *For any even number $m \geq 4$, the line graph of $(m,3)$-lollipop graph $L(L_{m,3})$ is well covered.*

Proof. Let u_k be the vertex of K_m that adjacent with v_1 in P_3 by an edge e_1. Consider $L(L_{m,3})$ to be the line graph of $L_{m,3}$. Then, for any maximal independent set H in $L(L_{m,3})$, we have two cases;
Case 1. $e_1 \notin H$, then $H = \{e_2\} \cup M$, or $H = \{e_3\} \cup M$ where M is a maximal independent set in $L(K_m)$ and hence by Theorem 2.1 $|M| = \frac{m}{2} + 1$.
Case 2. $e_1 \in H$, then $H = \{e_3, e_1\} \cup M_k$. Then we can complete the proof by the similar technique of Theorem 3.2. Hence, any maximal independent set of $L(L_{m,3})$ has a cardinality $\frac{m}{2} + 1$. Thus we have $L(L_{m,3})$ is a well covered graph. □

In general, if m is odd then $L(L_{m,3})$ is not well covered. Let u_1 be the vertex of K_m that adjacent with v_1 in P_3 by an edge e_1. Then the sets $M_1 = \{e_3, e_1, u_{3,4}, u_{5,6}, ..., u_{m-2,m-1}, u_{2,m}\}$ and $M_2 = \{e_3, u_{1,2}, u_{3,4}, u_{5,6}, ..., u_{m-2,m-1}\}$ are two maximal independent sets with different cardinality.

Now, if $n \geq 4$, then $L(L_{m,n})$ is not a well covered graph for all m.

Theorem 3.5. *For all $m \geq n \geq 4$, the line graph of (m, n)-lollipop graph $L(L_{m,n})$ is not well covered.*

Proof. Without loss of generality, suppose u_1 be the vertex of K_m such that P_n is connected to the complete graph K_m. We have two cases;
Case 1. m is even, then $L(L_{m,n})$ have two maximal independent sets with different cardinality $M_1 = \{u_{1,2}, u_{3,4}, u_{5,6}, ..., u_{m-1,m}\} \cup \{e_i : i \text{ is even } 2 \leq i \leq n-1\}$ and $M_2 = \{u_{3,4}, u_{5,6}, ..., u_{m-1,m}\} \cup \{e_1, e_i : i \text{ is even } 4 \leq i \leq n-1\}$.
Case 2. m is odd, then $L(L_{m,n})$ have two maximal independent sets with different cardinality $M_1 = \{u_{1,2}, u_{3,4}, u_{5,6}, ..., u_{m-2,m-1}\} \cup \{e_i : i \text{ is odd } 3 \leq i \leq n-1\}$ and $M_2 = \{u_{3,4}, u_{5,6}, ..., u_{m-2,m-1}, u_{2,m}\} \cup \{e_i : i \text{ is odd } 1 \leq i \leq n-1\}$.
Thus, $L(L_{m,n})$ is not well covered for all $m \geq n \geq 4$. □

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An application of improved method of fuzzy matrix composition in medical diagnosis

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Abstract. In this paper, the Thangaraj and Mallika's approach for the application of fuzzy matrix is studied. Also, we introduce an improved method of fuzzy matrix composition in medical diagnosis. The validity of the improved method is established in comparison to Thangaraj and Mallika's approach using operation of fuzzy matrices. Finally, an application of the improved method to medical diagnosis is carried out by the operations of fuzzy matrices and the improved method. This improved method could be used to solve other decision making problems.

Keywords: fuzzy set, fuzzy matrix, fuzzy matrix composition, relativity function, comparative matrix.

1. Introduction

Theory of fuzzy set initiated by Zadeh [20] as a generalization of theory of crisp sets plays a vital role in the field of decision making. In classical set, an element a of a subset A of set X has its membership degree 0 or 1, that is; either belongs or does not belong to A. Albeit, a fuzzy subset A of a fuzzy set X has its own membership degree $\mu_A(a)$ in the unit interval $\mathbb{I} = [0, 1]$. In order words, a fuzzy subset A of X is characterized by the membership function $\mu_A(a),\ a \in X$.

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The notion of fuzzy matrices (FMs) proposed by Thomoson [18] is interesting and resourceful. FM is the generalization of classical matrices. In fact, every fuzzy matrix is a classical matrix but classical matrix may not necessarily be a fuzzy matrix [19]. The notion of fuzzy matrix (FM) has been extensively studied (see [3, 5, 9, 10, 15, 16] for details).

Fuzzy matrices have been applied to many fields such as appointment procedure, learning techniques, medical diagnosis, (see [4, 7, 8, 13] for details). The results show that fuzzy matrix theory produces effective solutions to various decision-making problems.

Sanchez [11] developed a technique for medical diagnosis as a fuzzy relation between symptoms and diseases. Elizabeth and Sujath [1] developed the technique of Sanchez for medical diagnosis using the representation of triangular fuzzy membership matrix. Geetha and Usha [2] established the application of fuzzy matrix in yoga on obesity using max-average composition method of fuzzy matrices. Sapre et.al [12] applied fuzzy matrix to study the medical diagnosis of diabetes, anaemia and hypertension. Thangaraji and Mallika [17] established the application of fuzzy matrices in medical diagnosis using comparison matrix.

In this paper, we study Thangaraj and Mallika's approach for the application of fuzzy matrix, and also introduce an improved method of fuzzy matrix composition in medical diagnosis. The same hypothetical medical database studied by Thangaraj and Mallika [17] was considered and we arrived at the same conclusion using our proposed method as was established in [17]. The paper is organized as follows; section 2 presents some preliminaries, basic definitions, some algebraic operations and types of fuzzy matrices. In section 3, the improved method of fuzzy matrix composition in medical diagnosis is presented. Section 4, gives the application of the improved method using the fuzzy matrices in [17]. The paper is concluded in section 5.

2. Preliminaries

This section gives some preliminaries, basic definitions, some algebraic operations and types of FMs.

Definition 2.1 ([17]). *A matrix $A = (a_{ij})_{r\times s}$, $i \in [1, r]$ and $j \in [1, s]$ of size $(r \times s)$ whose entries are closed unit interval $\mathbb{I} = [0, 1]$ is said to be a fuzzy matrix.*

Example 2.1. Let

$$A = \begin{bmatrix} 0.1 & 1.0 \\ 0.5 & 0.4 \\ 1.0 & 0.3 \end{bmatrix}.$$

Then, A is a (3×2)-fuzzy matrix.

Let $\mathbb{M}at_{r,s}(\mathbb{F})$ be the set of all (r, s)-fuzzy matrices.

Definition 2.2 ([6]). *Let* $A = \begin{bmatrix} a_1 & a_2 & \cdots & a_s \end{bmatrix} \in \mathbb{M}at_{1,s}(\mathbb{F})$ *be a FM of size* $(1 \times s)$*, then* A *is said to be a fuzzy row matrix.*

Example 2.2. If $A = \begin{pmatrix} 0.1 & 1.0 & 0.6 & 0.2 \end{pmatrix}$, then A is a (1×4)-fuzzy row matrix.

Definition 2.3 ([6]). *Let*

$$A = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_r \end{bmatrix} \in \mathbb{M}_{r,1}(\mathbb{F})$$

be a FM of size $(r \times 1)$*. Then,* A *is called a fuzzy column matrix.*

Example 2.3. If

$$A = \begin{bmatrix} 0.3 \\ 0.1 \\ 0.2 \\ 1.0 \\ 0.0 \end{bmatrix}.$$

Then, A is a (5×1)-fuzzy column matrix.

Definition 2.4 ([17]). *Let* $A = (a_{ij}) \in \mathbb{M}at_{r,s}(\mathbb{F})$ *be a fuzzy matrix of order* $(r \times s)$*. Then, the fuzzy complement matrix of* A*, denoted by* A^c *is defined as* $A^c = (a_{ij})^c$*, where* $(a_{ij})^c = 1 - a_{ij}$*, for all* $i,\ j$*.*

Example 2.4. If

$$A = \begin{bmatrix} 0.1 & 1.0 \\ 0.5 & 0.4 \\ 1.0 & 0.3 \end{bmatrix}$$

then,

$$A^c = \begin{bmatrix} 0.9 & 0.0 \\ 0.5 & 0.6 \\ 0.0 & 0.7 \end{bmatrix}.$$

Definition 2.5 ([6]). *Let*

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1s} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2s} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{is} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{ss} \end{bmatrix} \in \mathbb{M}_{s,s}(\mathbb{F})$$

be a fuzzy matrix of order $(s \times s)$*. Then,* A *is called fuzzy square matrix.*

Example 2.5. Consider the fuzzy square matrix

$$A = \begin{bmatrix} 0.2 & 0.0 & 1.0 \\ 0.4 & 1.0 & 0.3 \\ 0.0 & 1.0 & 0.5 \end{bmatrix}.$$

Then, A is a fuzzy square matrix of order (3×3).

Definition 2.6 ([6])**.** *Let $A = (a_{ij}) \in \mathbb{M}at_{s,s}(\mathbb{F})$ be a fuzzy square matrix. Then, A is called a fuzzy diagonal matrix if $a_{ij} = 0$, where $i \neq j, a_{ij} \in [0, 1]$.*

Example 2.6. A fuzzy square matrix

$$A = \begin{bmatrix} 0.2 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.3 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.1 \end{bmatrix}$$

is a fuzzy diagonal matrix of order (4×4).

Definition 2.7. *Two fuzzy matrices $A = (a_{ij}) \in \mathbb{M}at_{r,s}(\mathbb{F})$ and $B = (b_{ij}) \in \mathbb{M}at_{r,s}(\mathbb{F})$, of the same order are said to be equal if their corresponding entries are equal.*

Example 2.7. If $A = \begin{bmatrix} 1.0 & 0.2 & 0.4 \\ 0.0 & 0.3 & 1.0 \end{bmatrix}$ and $B = \begin{bmatrix} 1.0 & 0.2 & 0.4 \\ 0.0 & 0.3 & 1.0 \end{bmatrix}$. Then, $A = B$.

Definition 2.8 ([6])**.** *Let $A = (a_{ij}) \in \mathbb{M}at_{r,s}(\mathbb{F})$ and $B = (b_{ij}) \in \mathbb{M}at_{r,s}(\mathbb{F})$ be fuzzy matrices of order $(r \times s)$. Then, the sum of A and B, denoted by $A + B$ is defined as $A + B = Max(a_{ij}, b_{ij})$, for all $i \in [1, r]$, $j \in [1, s]$.*

Example 2.8. Let $A = \begin{bmatrix} 0.0 & 0.9 & 0.2 & 1.0 \\ 0.4 & 0.2 & 0.3 & 0.0 \\ 1.0 & 0.8 & 0.5 & 0.3 \end{bmatrix}$ and $B = \begin{bmatrix} 0.1 & 0.6 & 0.0 & 1.0 \\ 0.0 & 0.2 & 0.5 & 0.1 \\ 1.0 & 0.7 & 0.4 & 0.2 \end{bmatrix}$.

Then

$$A + B = \begin{bmatrix} 0.1 & 0.9 & 0.2 & 1.0 \\ 0.4 & 0.2 & 0.5 & 0.1 \\ 1.0 & 0.8 & 0.5 & 0.3 \end{bmatrix}.$$

Definition 2.9. *Let $A = (a_{ij}) \in \mathbb{M}at_{r,s}(\mathbb{F})$ and $B = (b_{ij}) \in \mathbb{M}at_{r,s}(\mathbb{F})$ be fuzzy matrices of order $(r \times s)$. Then, the matrix difference of A and B, denoted by $A - B$ is defined as $A - B = Min(a_{ij}, b_{ij})$, for all $[1, r]$, $j \in [1, s]$.*

Example 2.9. If $A = \begin{bmatrix} 0.0 & 0.9 & 0.2 & 1.0 \\ 0.4 & 0.2 & 0.3 & 0.0 \\ 1.0 & 0.8 & 0.5 & 0.3 \end{bmatrix}$ and $B = \begin{bmatrix} 0.1 & 0.6 & 0.0 & 1.0 \\ 0.0 & 0.2 & 0.5 & 0.1 \\ 1.0 & 0.7 & 0.4 & 0.2 \end{bmatrix}$.

Then,

$$A - B = \begin{bmatrix} 0.0 & 0.6 & 0.0 & 1.0 \\ 0.0 & 0.2 & 0.3 & 0.0 \\ 1.0 & 0.7 & 0.4 & 0.2 \end{bmatrix}.$$

Definition 2.10. *Let $A = (a_{ij}) \in \mathbb{M}at_{r,s}(\mathbb{F})$ and $B = (b_{jk}) \in \mathbb{M}_{s,p}(\mathbb{F})$ be FMs of orders $(r \times s)$ and $(s \times p)$, respectively. Then, the product of A and B, denoted by $A * B$ is defined as*

$$A * B = (c_{ik}) \in \mathbb{M}_{r,p}(\mathbb{F}),$$

where $(c_{ik}) = Max(a_{ij}, b_{jk})$, for all $i \in [1, r]$, $j \in [1, p]$.

Example 2.10. If $A = \begin{bmatrix} 0.0 & 0.9 & 0.2 & 1.0 \\ 0.4 & 0.2 & 0.3 & 0.0 \\ 1.0 & 0.8 & 0.5 & 0.3 \end{bmatrix}$ and $B = \begin{bmatrix} 0.1 & 0.6 \\ 0.0 & 1.0 \\ 0.0 & 0.2 \\ 0.5 & 0.1 \end{bmatrix}$.

Then

$$A * B = \begin{bmatrix} 0.50 & 0.90 \\ 0.04 & 0.24 \\ 0.15 & 0.80 \end{bmatrix}.$$

Definition 2.11 ([17])**.** *Let $A = (a_{ij}) \in \mathbb{M}at_{r,s}(\mathbb{F})$ and $B = (b_{jk}) \in \mathbb{M}at_{s,p}(\mathbb{F})$ be FMs of orders $(r \times s)$ and $(s \times p)$, respectively. Then, the fuzzy Max-i composition of A and B, denoted by $A \circ^i B$ is defined as*

$$A \circ^i B = \max -i[A(x, y), B(y, z)],$$

where i is a t-norm.

Example 2.11. Using matrices A and B in example above

$$A \circ^i B = \begin{bmatrix} 0.50 & 0.90 \\ 0.04 & 0.24 \\ 0.15 & 0.80 \end{bmatrix}.$$

Definition 2.12. *Let $A = (a_{ij}) \in \mathbb{M}at_{r,s}(\mathbb{F})$ and $B = (b_{jk}) \in \mathbb{M}at_{s,p}(\mathbb{F})$ be FMs of orders $(r \times s)$ and $(s \times p)$, respectively. Then, the improved method for fuzzy composition of A and B, denoted by $A@B$ is defined as*

$$A@B = \left\{ Max \left(\frac{\mu_A(a_{ij}) + \mu_B(b_{jk})}{2} \right) \right\},$$

for all $1 \leq i \leq r$, $1 \leq k \leq p$.

Example 2.12. Using matrices A and B in the Example (2.10)

$$A@B = \begin{bmatrix} 0.75 & 0.95 \\ 0.25 & 0.60 \\ 0.55 & 0.90 \end{bmatrix}.$$

Definition 2.13 ([17])**.** *Given variables a and b defined on a universal set X. Then, the relativity function, denoted by $f\left(\frac{a}{b}\right)$ is defined as*

$$f\left(\frac{a}{b}\right) = \frac{\mu_b(a) - \mu_a(b)}{\max\{\mu_b(a), \mu_a(b)\}},$$

where $\mu_b(a)$ represents the membership function of a with respect to b.

Definition 2.14 ([17]). *Let $A = \{a_1, \cdots, a_{i-1}, a_i, a_{i+1}, \cdots, a_n\}$ be a set of n variables defined on a universal set X. Form a matrix of relativity values $f(\frac{a_i}{b_i})$ where $a_i's, i = 1, \cdots, n$ are n variables defined on a universal set X. Then, the $R = (r_{ij})$, a square matrix of order n is called the comparison matrix with $f\left(\frac{a_i}{b_i}\right) = \mu_{a_j}(a_i)$.*

3. Improved method for fuzzy matrix composition in medical diagnosis

The concept of max-i composition for FMs was applied in [17]. Notwithstanding, an improved method for fuzzy matrix composition is introduced in this section.

Let $P = \{P_1, \cdots, P_i\}$ be the set of i patients, $S = \{S_1, \cdots, S_j\}$ be the set of j symptoms and $D = \{D_1, \cdots, D_k\}$ be the set of k diseases.

Construct a fuzzy set (σ, D) of order $(m \times n)$ over S, where $\sigma : D \to \mathcal{S}, \mathcal{S}$ is the power set of S. This set represents a matrix A named as symptom-disease fuzzy matrix.

Construct another fuzzy set (τ, S) of order $(m \times n)$ over P, where $\tau : S \to \mathcal{P}, \mathcal{P}$ is the power set of P. This set represents a matrix B named as patient-symptom fuzzy matrix.

Using Definition 2.4, obtain the fuzzy complement matrices A^c and B^c.

Compute $A\Phi B$ and $A'\Phi B'$ using Definition (2.12).

Calculate the relativity value $f\left(\frac{p_i}{d_i}\right)$ and form the comparison matrix $R = (r_{ij})_{m\times n} = \left[f\left(\frac{x_i}{y_i}\right)\right]_{i=1,2,3}$ using Definitions (2.13) and (2.14), respectively.

Then, select the highest number for each patient P_i in R. Hence, the conclusion is that the patient P_i suffers from disease D_j.

Procedure

The procedure are presented as follow:

1. Construct the fuzzy matrices A and B of the same order $(m \times n)$ corresponding to (σ, D) and (τ, S), respectively.

2. Form the complement matrices A^c and B^c of the same order $(m \times n)$ by Definition (2.4).

3. Compute the composition matrices $D = A@B$ and $E = A^c@B^c$ by Definition (2.12).

4. Compute the difference fuzzy matrix $C = D - E$, using Definition (2.9).

5. Calculate the relativity values $f\left(\frac{p_i}{d_i}\right)$ and form the comparison matrix $R = (r_{ij})_{m\times n} = \left[f\left(\frac{p_i}{d_i}\right)\right]_{i=1,2,3}$ using Definitions (2.13) and (2.14), respectively.

6. Identify the highest number for each patient P_i in R. Hence, the patient P_i suffers from disease D_j.

4. Case study

Suppose there are three patients $P = \{P_1, P_2, P_3\}$ where P_1, P_2, P_3 represent Jesse, Faith and Paul in a hospital with symptoms $S = \{S_1, S_2, S_3\}$ where S_1, S_2, S_3 represent high temperature, headache and cough. Let the possible diseases relating to S be $D = \{D_1, D_2, D_3\}$ where D_1, D_2, D_3 represent Dengue, Viral fever and Malaria.

Suppose that fuzzy set (σ, D) over S gives an approximate description of fuzzy medical knowledge of the diseases and their symptoms and fuzzy set (τ, S) over P give a collection of an approximate description of patient-symptoms in the hospital.

Let

$$(\sigma, D) = \begin{cases} \sigma(D_1) = \{(S_1, 0.5), (S_2, 0.6), (S_3, 0.4)\}, \\ \sigma(D_2) = \{(S_1, 0.2), (S_2, 0.7), (S_3, 0.3)\}, \\ \sigma(D_3) = \{(S_1, 0.6), (S_2, 0.4), (S_3, 0.7)\} \end{cases}$$

and

$$(\tau, S) = \begin{cases} \tau(S_1) = \{(P_1, 0.4), (P_2, 0.2), (P_3, 0.7)\}, \\ \tau(S_2) = \{(P_1, 0.3), (P_2, 0.6), (P_3, 0.9)\}, \\ \tau(S_3) = \{(P_1, 0.7), (P_2, 0.5), (P_3, 0.8)\}. \end{cases}$$

The two fuzzy sets (σ, D) and (τ, S) are represented by the following fuzzy matrices A and B, respectively.

$$A = \begin{array}{c} \\ S_1 \\ S_2 \\ S_3 \end{array} \begin{array}{c} \begin{array}{ccc} D_1 & D_2 & D_3 \end{array} \\ \begin{bmatrix} 0.5 & 0.6 & 0.4 \\ 0.2 & 0.7 & 0.3 \\ 0.6 & 0.4 & 0.7 \end{bmatrix} \end{array}, \qquad B = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \end{array} \begin{array}{c} \begin{array}{ccc} S_1 & S_2 & S_3 \end{array} \\ \begin{bmatrix} 0.4 & 0.2 & 0.7 \\ 0.3 & 0.6 & 0.9 \\ 0.7 & 0.5 & 0.8 \end{bmatrix} \end{array}$$

Note that the matrices A and B are from [17].

Then, the fuzzy complement matrices A^c and B^c are

$$A^c = \begin{array}{c} \\ S_1 \\ S_2 \\ S_3 \end{array} \begin{array}{c} \begin{array}{ccc} D_1 & D_2 & D_3 \end{array} \\ \begin{bmatrix} 0.5 & 0.4 & 0.6 \\ 0.8 & 0.3 & 0.7 \\ 0.4 & 0.6 & 0.3 \end{bmatrix} \end{array}, \qquad B^c = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \end{array} \begin{array}{c} \begin{array}{ccc} S_1 & S_2 & S_3 \end{array} \\ \begin{bmatrix} 0.6 & 0.8 & 0.3 \\ 0.7 & 0.4 & 0.1 \\ 0.3 & 0.5 & 0.2 \end{bmatrix} \end{array}$$

Compute the fuzzy matrices D and E using Definition (2.12)

$$D = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \end{array} \begin{array}{c} \begin{array}{ccc} D_1 & D_2 & D_3 \end{array} \\ \begin{bmatrix} 0.55 & 0.60 & 0.75 \\ 0.50 & 0.65 & 0.80 \\ 0.70 & 0.60 & 0.75 \end{bmatrix} \end{array}$$

$$E = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \end{array} \begin{array}{c} \begin{array}{ccc} D_1 & D_2 & D_3 \end{array} \\ \begin{bmatrix} 0.55 & 0.65 & 0.40 \\ 0.70 & 0.80 & 0.55 \\ 0.65 & 0.60 & 0.35 \end{bmatrix} \end{array}$$

Compute the fuzzy matrix C using Definition (2.9).

$$C = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \end{array} \begin{array}{c} \begin{array}{ccc} D_1 & D_2 & D_3 \end{array} \\ \begin{bmatrix} 0.55 & \mathbf{0.60} & 0.40 \\ 0.50 & \mathbf{0.65} & 0.55 \\ \mathbf{0.65} & 0.60 & 0.35 \end{bmatrix} \end{array}$$

Calculate the relativity values $f\left(\frac{p_i}{d_i}\right)$ and form the comparison matrix $R = (r_{ij})_{m\times n} = \left[f\left(\frac{p_i}{d_i}\right)\right]_{i=1,2,3}$ using Definitions (2.13) and (2.14), respectively

$$f\left(\frac{p_i}{d_i}\right) = \frac{\mu_{d_i}(p_i) - \mu_{p_i}(d_i)}{\max\{\mu_{d_i}(p_i), \mu_{p_i}(d_i)\}},$$

$$f\left(\frac{p_1}{d_1}\right) = \frac{\mu_{d_1}(p_1) - \mu_{p_1}(d_1)}{\max\{\mu_{d_1}(p_1), \mu_{p_1}(d_1)\}} = \frac{0.55 - 0.55}{\max(0.55, 0.55)} = \frac{0.00}{0.55} = 0.00,$$

$$f\left(\frac{p_1}{d_2}\right) = \frac{\mu_{d_2}(p_1) - \mu_{p_1}(d_2)}{\max\{\mu_{d_2}(p_1), \mu_{p_1}(d_2)\}} = \frac{0.60 - 0.50}{\max(0.60, 0.50)} = \frac{0.10}{0.60} = 0.16,$$

$$f\left(\frac{p_1}{d_3}\right) = \frac{\mu_{d_3}(p_1) - \mu_{p_1}(d_3)}{\max\{\mu_{d_3}(p_1), \mu_{p_1}(d_3)\}} = \frac{0.40 - 0.65}{\max(0.40, 0.65)} = \frac{-0.25}{0.65} = -0.38,$$

$$f\left(\frac{p_2}{d_1}\right) = \frac{\mu_{d_1}(p_2) - \mu_{p_2}(d_1)}{\max\{\mu_{d_1}(p_2), \mu_{p_2}(d_1)\}} = \frac{0.50 - 0.60}{\max(0.50, 0.60)} = \frac{-0.10}{0.60} = -0.16,$$

$$f\left(\frac{p_2}{d_2}\right) = \frac{\mu_{d_2}(p_2) - \mu_{p_2}(d_2)}{\max\{\mu_{d_2}(p_2), \mu_{p_2}(d_2)\}} = \frac{0.65 - 0.65}{\max(0.65, 0.65)} = \frac{0.00}{0.65} = 0.00,$$

$$f\left(\frac{p_2}{d_3}\right) = \frac{\mu_{d_3}(p_2) - \mu_{p_2}(d_3)}{\max\{\mu_{d_3}(p_2), \mu_{p_2}(d_3)\}} = \frac{0.55 - 0.60}{\max(0.55, 0.60)} = \frac{-0.05}{0.60} = -0.08,$$

$$f\left(\frac{p_3}{d_1}\right) = \frac{\mu_{d_1}(p_3) - \mu_{p_3}(d_1)}{\max\{\mu_{d_1}(p_3), \mu_{p_3}(d_1)\}} = \frac{0.65 - 0.40}{\max(0.65, 0.40)} = \frac{0.25}{0.65} = 0.38,$$

$$f\left(\frac{p_3}{d_2}\right) = \frac{\mu_{d_2}(p_3) - \mu_{p_3}(d_2)}{\{\max\{\mu_{d_2}(p_3), \mu_{p_3}(d_2)\}} = \frac{0.60 - 0.55}{\max(0.60, 0.55)} = \frac{0.05}{0.60} = 0.08,$$

$$f\left(\frac{p_3}{d_3}\right) = \frac{\mu_{d_3}(p_3) - \mu_{p_3}(d_3)}{\max\{\mu_{d_3}(p_3), \mu_{p_3}(d_3)\}} = \frac{0.35 - 0.35}{\max(0.35, 0.35)} = \frac{0.00}{0.35} = 0.00.$$

Hence, the comparison matrix R is

$$R = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \end{array} \begin{array}{c} \begin{array}{ccc} D_1 & D_2 & D_3 \end{array} \\ \begin{bmatrix} 0.00 & \mathbf{0.16} & -0.38 \\ -0.16 & \mathbf{0.00} & -0.08 \\ \mathbf{0.38} & 0.08 & 0.00 \end{bmatrix} \end{array}$$

We observe from matrices C and R that, patients Jesse and Faith (P_1 and P_2) are suffering from Viral fever (D_2) and patient Paul (P_3) is suffering from Dengue (D_1).

5. Conclusion

The theory of fuzzy matrices are interesting and useful. The studies have shown that, fuzzy matrix yields efficient solutions to various decision-making problems. In this paper, the Thangaraj and Mallika's [17] approach for medical diagnosis was studied and contribution was made to extend the approach. The same hypothetical medical database studied by Thangaraj and Mallika [17] was considered and we arrived at the same conclusion using our proposed method as was established in [17]. Albeit, our improved method of fuzzy matrix composition is more reliable in the sense that matrix C shows a greater relational value when compared to matrix M in [17]. Also, we can deduce from the comparison matrix R that, the potency of the immune systems of the patients were stronger in fighting against the potency of the diseases compare to comparison matrix R in [17]. Hence, the need for its application to other related problems can not be overemphasized since it is an effective tool for decision-making problems. Furthermore, the improved method put forward in this paper together with relativity function and comparison matrix could be adopted in solving other multi-criteria decision-making problems in medicine and other related discipline to get quick output in future work.

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Certain fixed point theorems on partial metric spaces

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Abstract. The existence and uniqueness of common fixed point theorems for two pairs of weakly compatible self mappings satisfying some contractive condition on partial metric space is given. Further proved a theorem for a pair of self mappings satisfying rational type contraction condition without using weakly compatibility. Some examples are also given to support our results.
Keywords: common fixed point, weakly compatible mappings, partial metric space.

1. Introduction

Contraction is one of the main tool, to prove the existence and uniqueness of fixed point. Banach [2] contraction principle gives an answer on is the existence and uniqueness of the solution of an equation $Tx = x$. This principle is one of the most useful tool in the study of nonlinear equations. There are many

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generalizations of Banach contraction mappings in the literature (see [5]-[9],[13]-[14] etc.).

Matthews [10] generalize the Banach contraction principle and introduced the concept of partial metric space. The fixed point result given by matthews [10] is generalized by several authors (see, [11], [3]-[4], [8] and others). It is widely recognized that the partial metric spaces play an important role in constructing models in the theory of computation. Partial metric has applications in the branch of science, where the size of the data point is represented by its self distance.

It is known that "Every metric space is a partial metric space with zero self distance that is partial metric spaces are the generalization of metric spaces." In this direction recently, using rational contraction, Arshed et. al [1] proved some fixed point theorems in the setting of partially ordered metric space. Motivated by [1] in this paper, some coincidence and common fixed point results in partial metric space are given.

The definition of partial metric space is given by Matthews [10] as follows:

Definition 1.1. *Let X be a nonempty set and let $p : X \times X \to \mathbb{R}_0^+$ satisfy:*

1. $x = y \iff p(x,x) = p(y,y) = p(x,y)$;
2. $p(x,x) \leq p(x,y)$;
3. $p(x,y) = p(y,x)$;
4. $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$,

for all $x, y \in X$, where $\mathbb{R}_0^+ = [0,\infty)$. Then, the pair (X,p) is called a partial metric space and p is called a partial metric on X. It is clear that if $p(x,y) = 0$, then from (1) and (2) $x = y$. But if $x = y$, $p(x,y)$ may not be zero.

Example 1.1. Let $X = \mathbb{R}_0^+$ and $p : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ defined by $p(x,y) = \max\{x,y\}$ for all $x, y \in \mathbb{R}_0^+$, then $(\mathbb{R}_0^+, p)$ is a partial metric space.

Example 1.2. Let $X = \{[a,b] : a, b \in \mathbb{R}, a \leq b\}$ and define $p([a,b],[c,d]) = \max\{b,d\} - \min\{a,c\}$. Then (X,p) is a partial metric space.

Let (X,p) be a partial metric space. Then, the function $d_p : X \times X \to \mathbb{R}_0^+$ given by $d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y)$ is a (usual) metric on X.Each partial metric p on X generates a T_0 topology τ_p on X with a base of the family of open p- balls $\{B_p(x,\epsilon) : x \in X, \epsilon > 0\}$, where $B_p(x,\epsilon) = \{y \in X : p(x,y) < p(x,x) + \epsilon\}$ for all $x \in X$ and $\epsilon > 0$.

Definition 1.2. *Let (X,p) be a partial metric space.*

(1) *A sequence $\{x_n\}$ in a partial metric space (X,p) converges to a point $x \in X$ if $\lim_{n\to\infty} p(x,x_n) = p(x,x)$.*

(2) *A sequence* $\{x_n\}$ *in a partial metric space* (X, p) *is called a Cauchy sequence if and only if* $\lim_{n,m\to\infty} p(x_n, x_m)$ *exists and finite.*

(3) *A partial metric space* (X, p) *is said to be complete if every Cauchy sequence* $\{x_n\}$ *in* X *converges with respect to* τ_p*, to a point* $x \in X$ *such that* $p(x, x) = lim_{n,m\to\infty} p(x_n, x_m)$.

Lemma 1.1. *Let* (X, p) *be a partial metric space. Then:*

(1) *A sequence* $\{x_n\}$ *in a partial metric space* (X, p) *is a Cauchy sequence if and only if it is a Cauchy sequence in the metric space* (X, d_p).

(2) *A partial metric space* (X, p) *is complete if and only if the metric space* (X, d_p) *is complete. Moreover,* $\lim_{n\to\infty} d_p(x_n, x) = 0$ *iff* $\lim_{n\to\infty} p(x, x_n) = \lim_{n,m\to\infty} p(x_n, x_m) = p(x, x)$.

(3) *A subset* E *of a partial metric space* (X, p) *is closed if whenever* $\{x_n\}$ *is a sequence in* E *such that* $\{x_n\}$ *converges to some* $x \in X$*, then* $x \in E$.

Definition 1.3. *Let* f *and* g *are self mappings on a set* X*. A point* $x \in X$ *is called a coincidence point of* f *and* g *if* $fx = gx = w$*, where* w *is called a point of coincidence of* f *and* g.

Definition 1.4. *Two self mappings* f *and* g *on a set* X *are said to be weakly compatible if* f *and* g *commute at their coincidence points, that is, if* $fx = gx$ *for some* $x \in X$*, then* $fgx = gfx$.

Now, we present our main results.

2. Main results

Theorem 2.1. *Let* (X, p) *be a complete partial metric space. Suppose that* f, g, F *and* G *are self mappings satisfying the following conditions:*

(*i*) $f(X) \subseteq g(X)$ *and* $F(X) \subseteq G(X)$*;*

(*ii*) *there exist* α, β*, and* L *in* $(0, 1)$ *with* $2\alpha + \beta + 2L < 1$ *such that*

$$p(Fx, fy) \leq \alpha \frac{\max\{p(gx, Fx).p(gy, Fy), p(gx, Gy).p(fy, Fx), p(gx, fy).p(Gy, Fx)\}}{p(gx, Gy)} + \beta p(gx, Gy) + L \min\{p(gx, Fx), p(fy, Gy), p(gx, fy), p(Fx, Gy)\},$$

for all $x, y \in X$;

(*iii*) $f(X)$ *or* $g(X)$ *is closed.*

If the pairs (f, G) *and* (g, F) *are weakly compatible, then* f, g, F *and* G *have unique common fixed point in* X.

Proof. Suppose $x_0 \in X$ be an arbitrary element of X. Since $f(X) \subseteq g(X)$ and $F(X) \subseteq G(X)$. Construct a sequence $\{y_n\}$ in X satisfying $y_n = Fx_n = Gx_{n+1}$ and $y_{n+1} = fx_{n+1} = gx_{n+2}$, for all $n \in \mathbb{N} \cup \{0\}$. Taking $x = x_n$ and $y = x_{n+1}$ and Using condition (ii), we have

$$\begin{aligned}
p(Fx_n, fx_{n+1}) &\leq \alpha \frac{1}{p(gx_n, Gx_{n+1})} \max\{p(gx_n, Fx_n).p(fx_{n+1}, Gx_{n+1}),\\
&\quad p(gx_n, Gx_{n+1}).p(fx_{n+1}, Fx_n), p(gx_n, fx_{n+1}).p(Gx_{n+1}, Fx_n)\}\\
&+ \beta p(gx_n, Gx_{n+1}) + L \min\{p(gx_n, Fx_n), p(Gx_{n+1}, fx_{n+1}),\\
&\quad p(gx_n, fx_{n+1}), p(Fx_n, Gx_{n+1})\}\\
&= \alpha \frac{\max\{p(y_{n-1}, y_n).p(y_n, y_{n+1}), p(y_{n-1}, y_n).p(y_n, y_{n+1}), p(y_{n-1}, y_{n+1}).p(y_n, y_n)\}}{p(y_{n-1}, y_n)}\\
&+ \beta p(y_{n-1}, y_n)\\
&+ L \min\{p(y_{n-1}, y_n), p(y_n, y_{n+1}), p(y_{n-1}, y_{n+1}), p(y_n, y_n)\}\\
&= \alpha \frac{\max\{p(y_{n-1}, y_n).p(y_n, y_{n+1}), p(y_{n-1}, y_{n+1}).p(y_n, y_n)\}}{p(y_{n-1}, y_n)}\\
&+ \beta p(y_{n-1}, y_n) + Lp(y_n, y_n).
\end{aligned}$$

Case I. If

$$\max\{p(y_{n-1}, y_n).p(y_n, y_{n+1}), p(y_{n-1}, y_{n+1}).p(y_n, y_n)\} = p(y_{n-1}, y_n).p(y_n, y_{n+1})$$

and

$$\min\{p(y_{n-1}, y_{n+1}), p(y_n, y_n)\} = p(y_n, y_n),$$

then we have

$$p(y_n, y_{n+1}) \leq \alpha \frac{p(y_{n-1}, y_n).p(y_n, y_{n+1})}{p(y_{n-1}, y_n)} + \beta p(y_{n-1}, y_n) + Lp(y_n, y_n).$$

Since $p(y_n, y_n) \leq p(y_n, y_{n+1})$, we find that

$$p(y_n, y_{n+1}) \leq \alpha p(y_n, y_{n+1}) + \beta p(y_{n-1}, y_n) + Lp(y_n, y_{n+1}).$$

which implies

$$p(y_n, y_{n+1}) \leq \frac{\beta}{(1 - \alpha - L)} p(y_{n-1}, y_n).$$

Therefore,

$$p(y_n, y_{n+1}) \leq K_1 p(y_{n-1}, y_n).$$

where $K_1 = \frac{\beta}{(1-\alpha-L)}$.

Case II. If

$$\max\{p(y_{n-1}, y_n).p(y_n, y_{n+1}), p(y_{n-1}, y_{n+1}).p(y_n, y_n)\} = p(y_{n-1}, y_{n+1}).p(y_n, y_n)$$

and $\min\{p(y_{n-1}, y_{n+1}), p(y_n, y_n)\} = p(y_n, y_n)$ then, we have

$$p(y_n, y_{n+1}) \leq \alpha \frac{p(y_{n-1}, y_{n+1}).p(y_n, y_n)}{p(y_{n-1}, y_n)} + \beta p(y_{n-1}, y_n) + Lp(y_n, y_n).$$

Since $p(y_n, y_n) \leq p(y_n, y_{n-1})$ and $p(y_n, y_n) \leq p(y_n, y_{n+1})$ we obtain

$$\begin{aligned} p(y_n, y_{n+1}) &\leq \alpha p(y_{n-1}, y_{n+1}) + \beta p(y_{n-1}, y_n) + Lp(y_n, y_{n+1}) \\ &\leq \alpha p(y_{n-1}, y_n) + \alpha p(y_n, y_{n+1}) - \alpha p(y_n, y_n) + \beta p(y_{n-1}, y_n) + Lp(y_n, y_{n+1}) \\ &\leq \alpha p(y_{n-1}, y_n) + \alpha p(y_n, y_{n+1}) + \beta p(y_{n-1}, y_n) + Lp(y_n, y_{n+1}). \end{aligned}$$

Consequently, we get

$$p(y_n, y_{n+1}) \leq \frac{\alpha + \beta}{(1 - \alpha - L)} p(y_{n-1}, y_n)$$

which in turn implies that $p(y_n, y_{n+1}) \leq K_2 p(y_{n-1}, y_n)$, where $K_2 = \frac{\alpha+\beta}{(1-\alpha-L)}$.
Case III. If

$$\max\{p(y_{n-1}, y_n).p(y_n, y_{n+1}), p(y_{n-1}, y_{n+1}).p(y_n, y_n)\} = p(y_{n-1}, y_n).p(y_n, y_{n+1})$$

and

$$\min\{p(y_{n-1}, y_{n+1}), p(y_n, y_n)\} = p(y_{n-1}, y_{n+1}),$$

then, we have

$$\begin{aligned} p(y_n, y_{n+1}) &\leq \alpha \frac{p(y_{n-1}, y_n).p(y_n, y_{n+1})}{p(y_{n-1}, y_n)} + \beta p(y_{n-1}, y_n) + Lp(y_{n-1}, y_{n+1}) \\ &\leq \alpha p(y_n, y_{n+1}) + \beta p(y_{n-1}, y_n) + Lp(y_{n-1}, y_n) + Lp(y_n, y_{n+1}) - Lp(y_n, y_n) \\ &\leq \alpha p(y_n, y_{n+1}) + \beta p(y_{n-1}, y_n) + Lp(y_{n-1}, y_n) + Lp(y_n, y_{n+1}). \end{aligned}$$

We obtain

$$p(y_n, y_{n+1}) \leq \frac{\beta + L}{1 - \alpha - L} p(y_{n-1}, y_n) = K_3 p(y_{n-1}, y_n),$$

where $K_3 = \frac{\beta+L}{1-\alpha-L}$.
Case IV. If

$$\max\{p(y_{n-1}, y_n).p(y_n, y_{n+1}), p(y_{n-1}, y_{n+1}).p(y_n, y_n)\} = p(y_{n-1}, y_{n+1}).p(y_n, y_n)$$

and $\min\{p(y_{n-1}, y_{n+1}), p(y_n, y_n)\} = p(y_{n-1}, y_{n+1})$, then, we have

$$p(y_n, y_{n+1}) \leq \alpha \frac{p(y_{n-1}, y_{n+1}).p(y_n, y_n)}{p(y_{n-1}, y_n)} + \beta p(y_{n-1}, y_n) + Lp(y_{n-1}, y_{n+1}).$$

Since $p(y_n, y_n) \leq p(y_{n-1}, y_n)$, we have

$$\begin{aligned} p(y_n, y_{n+1}) &\leq \alpha p(y_{n-1}, y_n) + \alpha p(y_n, y_{n+1}) - \alpha p(y_n, y_n) + \beta p(y_{n-1}, y_n) \\ &\quad + Lp(y_{n-1}, y_n) + Lp(y_n, y_{n+1}) - Lp(y_n, y_n) \\ &\leq (\alpha + L) p(y_n, y_{n+1}) + (\alpha + \beta) p(y_{n-1}, y_n). \end{aligned}$$

From this inequality we obtain

$$p(y_n, y_{n+1}) \leq \frac{\beta + L}{1 - \alpha - L} p(y_{n-1}, y_n) = K_4 p(y_{n-1}, y_n),$$

where $K_4 = \frac{\beta+L}{1-\alpha-L}$.

Let $K = \max\{K_1, K_2, K_3, K_4\}$, so, $0 < K < 1$. Therefore, for each $n \in \mathbb{N}$, we obtain that $p(y_n, y_{n+1}) \leq K^n p(y_0, y_1)$. Now, we shall show that $\{y_n\}$ is a Cauchy sequence in (X, d_p). Let $m, n \in \mathbb{N}$ with $m > n$, we have $p(y_n, y_m) \leq p(y_n, y_{n+1}) + p(y_{n+1}, y_m) - p(y_{n+1}, y_{n+1}) \leq p(y_n, y_{n+1}) + p(y_{n+1}, y_m)$. Also, $p(y_{n+1}, y_m) \leq p(y_{n+1}, y_{n+2}) + p(y_{n+2}, y_m) - p(y_{n+2}, y_{n+2}) \leq p(y_{n+1}, y_{n+2}) + p(y_{n+2}, y_m)$. Therefore, we have $p(y_n, y_m) \leq p(y_n, y_{n+1}) + p(y_{n+1}, y_{n+2}) + ... + p(y)m - 1, y_m) \leq (K^n + K^{n+1} + ... + K^{m-1})p(y_0, y_1)$. By straight forward calculations, we have

$$\lim_{n,m\to\infty} p(y_n, y_m) = 0. \tag{1}$$

We observe that

$$d_p(y_n, y_m) = 2p(y_n, y_m) - p(y_m, y_m) - p(y_n, y_n) \leq 2p(y_n, y_m).$$

By (1), we conclude that $d_p(y_n, y_m) = 0$. Hence $\{y_n\}$ is a Cauchy sequence in (X, d_p). Since X is complete, we have $\lim_{n\to\infty} y_n = z$ for some $z \in X$. By Lemma 1.1, we obtain

$$p(z, z) = \lim_{n\to\infty} p(y_n, z) = \lim_{n,m\to\infty} p(y_n, y_m).$$

we conclude that

$$p(z, z) = 0. \tag{2}$$

Assume that $g(X)$ is closed, therefore there exists a point $u \in X$ such that $z = gu$. Now,

$$\begin{aligned}
&p(z, Fu) \leq p(z, y_{n+1}) + p(y_{n+1}, Fu) - p(y_{n+1}, y_{n+1}) \\
&\leq p(z, y_{n+1}) + p(y_{n+1}, Fu) = p(z, y_{n+1}) + p(Fu, fx_{n+1}) \leq p(z, y_{n+1}) \\
&\quad + \alpha \frac{\max\{p(gu, Fu).p(Gx_{n+1}, fx_{n+1}), p(gu, Gx_{n+1}).p(fx_{n+1}, Fu), p(gu, fx_{n+1}).p(Fu, Gx_{n+1})\}}{p(gu, Gx_{n+1})} \\
&\quad + \beta p(gu, Gx_{n+1}) + L \min\{p(gu, Fu), p(Gx_{n+1}, fx_{n+1}), p(gu, fx_{n+1}), p(Fu, Gx_{n+1})\} \\
&= p(z, y_{n+1}) + \alpha \frac{\max\{p(gu, Fu).p(y_n, y_{n+1}), p(gu, gu).p(Fu, y_{n+1}), p(gu, y_{n+1}).p(Fu, y_n)\}}{p(gu, y_n)} \\
&\quad + \beta p(gu, y_n) + L \min\{p(gu, Fu), p(y_n, y_{n+1}), p(gu, y_{n+1}), p(Fu, y_n)\}.
\end{aligned}$$

Taking $n \to \infty$ and using (2), we have $p(z, Fu) = 0$. So, $Z = Fu$, and hence $z = Fu = gu$. Since F and g are weakly compatible, we obtain $gFu = Fgu$. Hence $gz = Fz$. Therefore, z is a coincidence point of F and g. Again $F(X) \subseteq G(X)$, there exists a point $v \in X$ such that $z = Gv$.

$$\begin{aligned}
p(z, fv) &= p(Fu, fv) \\
&\leq \alpha \frac{\max\{p(gu, Fu).p(Gv, fv), p(gu, Gu).p(fv, Fu), p(gu, fv).p(Fu, Gv)\}}{p(gu, Gv)} \\
&\quad + \beta p(gu, Gv) + L \min\{p(gu, Fu), p(Gv, fv), p(gu, fv), p(Fu, Gv)\}.
\end{aligned}$$

Taking $n \to \infty$, we have $p(z, fv) \leq Lp(z, fv)$ implies $z = fv$. Hence, $fv = z = Gv$. Since G and f are weakly compatible, we obtain $fGv = Gfv$ implies $fz = Gz$. Therefore, z is a coincidence point of f, g, F and G.

Now, we show that z is a fixed point of F and g.

$$\begin{aligned}
p(Fz, z) &= p(Fz, fv) \\
&\leq \alpha \frac{\max\{p(gz, Fz).p(Gv, fv), p(gz, Gv).p(fv, Fz), p(gz, fv).p(Fz, Gv)\}}{p(gu, Gv)} \\
&\quad + \beta p(gz, Gv) + L \min\{p(gz, Fz), p(Gv, fv), p(gz, fv), p(Fz, Gv)\} \\
&= \alpha \frac{\max\{p(Fz, Fz).p(z, z), p(z, Fz).p(z, Fz), p(Fz, z).p(Fz, z)\}}{p(Fz, z)} \\
&\quad + \beta p(Fz, z) + L \min\{p(Fz, Fz), p(z, z), p(Fz, z), p(Fz, z)\} \\
&= (\alpha + \beta) p(Fz, z),
\end{aligned}$$

which gives $p(Fz, z) = 0$ implies $Fz = z$, and so $gz = Fz = z$. Thus, z is a common fixed point of F and g. Again

$$\begin{aligned}
p(z, fz) &= p(Fz, fz) \\
&\leq \alpha \frac{\max\{p(gz, Fz).p(Gz, fz), p(gz, Gz).p(fz, Fz), p(gz, fz).p(Fz, Gz)\}}{p(gz, Gz)} \\
&\quad + \beta p(gz, Gz) + L \min\{p(gz, Fz), p(Gz, fz), p(gz, fz), p(Fz, Gz)\}. \\
&= \alpha \frac{\max\{p(z, Fz).p(fz, fz), p(z, fz).p(z, fz), p(z, fz).p(z, fz)\}}{p(z, fz)} \\
&\quad + \beta p(z, fz) + L \min\{p(z, z), p(fz, z), p(z, fz), p(z, fz)\} \\
&= (\alpha + \beta) p(z, fz),
\end{aligned}$$

implies that $p(z, fz) = 0$, so $fz = z = Gz$.

Next, we shall show that z is unique common fixed point of $f, g, F$and G. On contrary, assume that w is another common fixed point in X.

$$\begin{aligned}
p(z, w) &= p(Fz, fw) \\
&\leq \alpha \frac{\max\{p(gz, Fz).p(Gw, fw), p(gz, Gw).p(fw, Fz), p(gz, fw).p(Fz, Gw)\}}{p(gz, Gw)}
\end{aligned}$$

$$
\begin{aligned}
&+ \beta p(gz, Gw) + L \min\{p(gz, Fz), p(Gw, fw), p(gz, fw), p(Fz, Gw)\} \\
&= \alpha \frac{\max\{p(z,z).p(w,w), p(z,w).p(z,w), p(z,w).p(z,w)\}}{p(z,w)} \\
&+ \beta p(z,w) + L \min\{p(z,z), p(w,w), p(z,w), p(z,w)\} \\
&= (\alpha + \beta) p(z,w),
\end{aligned}
$$

implies that $p(z,w) = 0$ implies $z = w$. Hence z is a unique common fixed point of f, g, F and G. □

Corollary 2.1. *Let (X,p) be a complete partial metric space. Suppose f and g be two self mappings on X, satisfying the following conditions:*

(i) $f(X) \subseteq g(X)$;

(ii) there exist α, β and L in $(0,1)$ with $2\alpha + \beta + 2L < 1$ such that

$$
\begin{aligned}
p(fx, fy) \leq \alpha \frac{\max\{p(gx, fx).p(gy, fy), p(gx, gy).p(fx, fy), p(gx, fy).p(gy, fx)\}}{p(gx, gy)} \\
+ \beta p(gx, gy) + L \min\{p(fx, gx), p(fy, gy), p(gx, fy), p(fx, gy)\},
\end{aligned}
$$

for all $x, y \in X$;

(iii) $f(X)$ or $g(X)$ is complete.

If f and g are weakly compatible, then f and g have a unique common fixed point in X.

Corollary 2.2. *Let (X,p) be a complete partial metric space. Suppose f be a self mapping on X. Suppose that there exist α, β and L in $(0,1)$ with $2\alpha + \beta + 2L < 1$ such that*

$$
\begin{aligned}
p(fx, fy) \leq \alpha \frac{\max\{p(x, fx).p(y, fy), p(x, y).p(fx, fy), p(x, fy).p(y, fx)\}}{p(x, y)} \\
+ \beta p(x, y) + L \min\{p(x, fx), p(y, fy), p(x, fy), p(y, fx)\},
\end{aligned}
$$

for all $x, y \in X$. If $f(X)$ is complete, then f has a unique fixed point in X.

Example 2.1. Let $X = [0,6]$ and $p : X \times X \to \mathbb{R}$ be defined by $p(x,y) = \max\{x, y\}$. Then (X,p) is a complete partial metric space. Let $f, g : X \to X$ be defined by

$$
fx = \begin{cases} 2x - 1, & x \in [1,3], \\ 3, & x \in]3,6]. \end{cases}
$$

and

$$
gx = \begin{cases} 1, & x = 1, \\ 2, & x \in]1, \frac{4}{3}[\\ 2x, & x \in [\frac{4}{3}, 3] \\ 6, & x \in]3,6]. \end{cases}
$$

Now, consider the following cases:

Case I. When $x = y = 1$, we have

$$\begin{aligned} p(fx, fy) &= 1 = p(gx, gy) \\ &\leq \alpha \frac{\max\{p(gx, fx).p(gy, fy), p(gx, gy).p(fx, fy), p(gx, fy).p(gy, fx)\}}{p(gx, gy)} \\ &+ \beta p(gx, gy) + L \min\{p(fx, gx), p(fy, gy), p(gx, fy), p(fx, gy)\}. \end{aligned}$$

Case II. For $x, y \in [1, \frac{4}{3}[$ and $x < y$, we have

$$\begin{aligned} p(fx, fy) &= 2y - 1 \leq \frac{5}{6} 2 = \frac{5}{6} p(gx, gy) \\ &\leq \alpha \frac{\max\{p(gx, fx).p(gy, fy), p(gx, gy).p(fx, fy), p(gx, fy).p(gy, fx)\}}{p(gx, gy)} \\ &+ \beta p(gx, gy) + L \min\{p(fx, gx), p(fy, gy), p(gx, fy), p(fx, gy)\}. \end{aligned}$$

Case III. If $x \in [1, 3]$,$y \in [\frac{4}{3}, 3]$ and $x < y$, then we have

$$\begin{aligned} p(fx, fy) &= 2y - 1 \leq \frac{5}{6} 2y = \frac{5}{6} p(gx, gy) \\ &\leq \alpha \frac{\max\{p(gx, fx).p(gy, fy), p(gx, gy).p(fx, fy), p(gx, fy).p(gy, fx)\}}{p(gx, gy)} \\ &+ \beta p(gx, gy) + L \min\{p(fx, gx), p(fy, gy), p(gx, fy), p(fx, gy)\}. \end{aligned}$$

Case IV. For $x \in [1, 3]$,$y \in [3, 6]$ and $x < y$, we have

$$\begin{aligned} p(fx, fy) &= \max\{2x - 1, 3\} \leq \frac{5}{6} 6 = \frac{5}{6} p(gx, gy) \\ &\leq \alpha \frac{\max\{p(gx, fx).p(gy, fy), p(gx, gy).p(fx, fy), p(gx, fy).p(gy, fx)\}}{p(gx, gy)} \\ &+ \beta p(gx, gy) + L \min\{p(fx, gx), p(fy, gy), p(gx, fy), p(fx, gy)\}. \end{aligned}$$

Case V. $x, y \in [3, 6]$ and $x < y$, we have

$$\begin{aligned} p(fx, fy) &= 3 \leq \frac{5}{6} 6 = \frac{5}{6} p(gx, gy) \\ &\leq \alpha \frac{\max\{p(gx, fx).p(gy, fy), p(gx, gy).p(fx, fy), p(gx, fy).p(gy, fx)\}}{p(gx, gy)} \\ &+ \beta p(gx, gy) + L \min\{p(fx, gx), p(fy, gy), p(gx, fy), p(fx, gy)\}. \end{aligned}$$

Thus, in all cases the contractive conditions of Corollary 2.1 are satisfied and (f, g) weakly compatible, by Corollary 2.1 $f\&g$ have a unique common fixed point $x = 1$ in X.

Theorem 2.2. *Let (X,p) be a complete partial metric space and let f and g be self mappings on X. If there exist α,β,γ and δ in $(0,1)$ with $\alpha+\beta+2\gamma+3\delta<1$ such that*

$$p(fx,gy)\le \alpha p(x,y)+\beta\frac{p(x,fx).p(y,gy)}{1+p(x,y)}+\gamma\frac{p(x,gy).p(y,fx)}{1+p(1+p(x,y))}+\delta\frac{p(x,fx).p(x,gy)+p(y,fx).p(y,gy)}{1+p(x,y)}, \tag{3}$$

for all $x,y\in X$. Then, there exist $z\in X$ such that $fz=gz=z$.

Proof. Let $x_0\in X$ be arbitrary element of X. Define a sequence in X such that $x_{2k+2}=fx_{2k+1}$ and $x_{2k+1}=gx_{2k}$ for $k=0,1,2,3,...$ If there exist a positive integer N such that $x_{2N}=x_{2N+1}$, then x_{2n} is a fixed point of g and hence fixed point of f. Since $x_{2N}=x_{2N+1}=Sx_{2N}$, using equation (3), we have

$$\begin{aligned}
&p(x_{2N+2},x_{2N+1})\\
&=p(fx_{2N+1},gx_{2N})\\
&\le \alpha p(x_{2N+1},x_{2N})+\beta\frac{p(x_{2N+1},fx_{2N+1}).p(x_{2N},gx_{2N})}{1+p(x_{2N+1},x_{2N})}\\
&+\gamma\frac{p(x_{2N+1},gx_{2N}).p(x_{2N},fx_{2N+1})}{1+p(x_{2N+1},x_{2N})}\\
&+\delta\frac{p(x_{2N+1},fx_{2N+1}).p(x_{2N+1},gx_{2N})+p(x_{2N},fx_{2N+1}).p(x_{2N},gx_{2N})}{1+p(x_{2N+1},x_{2N})}\\
&\le \alpha p(x_{2N+1},x_{2N})+\beta\frac{p(x_{2N+1},x_{2N+2}).p(x_{2N},x_{2N+1})}{1+p(x_{2N+1},x_{2N})}\\
&+\gamma\frac{p(x_{2N+1},x_{2N+1}).p(x_{2N},x_{2N+2})}{1+p(x_{2N+1},x_{2N})}\\
&+\delta\frac{p(x_{2N+1},x_{2N+2}).p(x_{2N+1},x_{2N+1})+p(x_{2N},x_{2N+2}).p(x_{2N},x_{2N+1})}{1+p(x_{2N+1},x_{2N})}.
\end{aligned}$$

Using the fact $p(x_{2N+1},x_{2N+1})\le p(x_{2N},x_{2N+1})$ and $\frac{p(x_{2N},x_{2N+1})}{1+p(x_{2N},x_{2N+1})}<1$, we obtain

$$\begin{aligned}
p(x_{2N+2},x_{2N+1})&\le \alpha p(x_{2N+1},x_{2N})+\beta p(x_{2N+1},x_{2N+2})+\gamma p(x_{2N},x_{2N+2})\\
&+\delta(p(x_{2N+1},x_{2N+2})+p(x_{2N},x_{2N+2}))\\
&\le(\alpha+\gamma+\delta)p(x_{2N},x_{2N+1})+(\beta+\gamma+2\delta)p(x_{2N+1},x_{2N+2})\\
&-(\gamma+\delta)p(x_{2N+1},x_{2N+1})\\
&\le(\alpha+\gamma+\delta)p(x_{2N},x_{2N+1})+(\beta+\gamma+2\delta)p(x_{2N+1},x_{2N+2})
\end{aligned}$$

implies that

$$p(x_{2N+2},x_{2N+1})\le\Big(\frac{\alpha+\gamma+\delta}{1-\beta-\gamma-2\delta}\Big)p(x_{2N+1},x_{2N})=Kp(x_{2N+1},x_{2N}),$$

where $K = (\frac{\alpha+\gamma+\delta}{1-\beta-\gamma-2\delta}) < 1$. Since $x_{2N} = x_{2N+1}$, we have $p(x_{2N+2}, x_{2N+1}) \leq Kp(x_{2N+1}, x_{2N+1}) \leq Kp(x_{2N+1}, x_{2N+2})$, which given $p(x_{2N+1}, x_{2N+1}) = 0$, implies $x_{2N+1} = x_{2N+2} = Tx_{2N+1}$.

Note that, $Sx_{2N} = x_{2N} = x_{2N+1} = Tx_{2N+1}$. Thus, in this case $x_{2N} = x_{2N+1}$ is a fixed point of S and T. A similar conclusion holds if $x_{2N+1} = x_{2N+2}$ for some $N \in \mathbb{Z}^+$. Therefore, we may assume that $x_k \neq x_{k+1}$ for all k.

Case I. If k is odd, using (3), we have

$$\begin{aligned}
p(x_{k+1}, x_{k+2}) &= p(fx_K, gx_{k+1}) \\
&\leq \alpha p(x_k, x_{k+1}) + \beta \frac{p(x_k, fx_k).p(x_{k+1}), gx_{k+1}}{1 + p(x_k, x_{k+1})} \\
&+ \gamma \frac{p(x_k, gx_{k+1}).p(x_{k+1}), fx_k}{1 + p(x_k, x_{k+1})} \\
&+ \delta \frac{p(x_k, fx_k).p(x_k, gx_{k+1}) + p(x_{k+1}, fx_k).p(x_{k+1}, gx_{k+1})}{1 + p(x_k, x_{k+1}} \\
&= \alpha p(x_k, x_{k+1}) + \beta \frac{p(x_k, x_{k+1}).p(x_{k+1}, x_{k+2})}{1 + p(x_k, x_{k+1})} \\
&+ \gamma \frac{p(x_k, x_{k+2}).p(x_{k+1}, x_{k+1})}{1 + p(x_k, x_{k+1})} \\
&+ \delta \frac{p(x_k, x_{k+1}).p(x_k, x_{k+2}) + p(x_{k+1}, x_{k+1}).p(x_{k+1}, x_{k+2})}{1 + p(x_k, x_{k+1})}.
\end{aligned}$$

Using the fact $p(x_{k+1}, x_{k+1}) \leq p(x_k, x_{k+1})$ and $\frac{p(x_k, x_{k+1})}{1+p(x_k, x_{k+1})} < 1$, we have

$$\begin{aligned}
p(x_{k+1}, x_{k+2}) &\leq \alpha p(x_k, x_{k+1}) + \beta p(x_{k+1}, x_{k+2}) + \gamma p(x_k, x_{k+2}) \\
&+ \delta p(x_k, x_{k+2}) + \delta p(x_{k+1}, x_{k+2}) \\
&= \alpha p(x_k, x_{k+1}) + (\beta + \delta) p(x_{k+1}, x_{k+2}) + (\gamma + \delta) p(x_k, x_{k+2}) \\
&\leq \alpha p(x_k, x_{k+1}) + (\beta + \delta) p(x_{k+1}, x_{k+2}) + (\gamma + \delta)(p(x_k, x_{k+1}) \\
&+ p(x_k, x_{k+1}) - p(x_{k+1}, x_{k+1}))
\end{aligned}$$

implies

$$p(x_{k+1}, x_{k+2}) \leq \Big(\frac{\alpha + \gamma + \delta}{1 - \beta - \gamma - 2\delta}\Big) p(x_k, x_{k+1}) = rp(x_k, x_{k+1}), \tag{4}$$

where $r = \frac{\alpha+\gamma+\delta}{1-\beta-\gamma-2\delta}$.

Case II. If k is even, using (2.3), we have

$$\begin{aligned}
p(x_{k+1}, x_{k+2}) &= p(gx_k, fx_{k+1}) \\
&\leq \alpha p(x_k, x_{k+1}) + \beta \frac{p(x_{k+1}, fx_{k+1}).p(x_k), gx_k}{1 + p(x_k, x_{k+1})} \\
&+ \gamma \frac{p(x_{k+1}, gx_k).p(x_k), fx_{k+1}}{1 + p(x_k, x_{k+1})}
\end{aligned}$$

$$
\begin{aligned}
&+ \delta\frac{p(x_{k+1}, fx_{k+1}).p(x_{k+1}, gx_k) + p(x_k, fx_{k+1}).p(x_k, gx_k)}{1 + p(x_k, x_{k+1}} \\
&= \alpha p(x_k, x_{k+1}) + \beta\frac{p(x_{k+1}, x_{k+2}).p(x_k, x_{k+1})}{1 + p(x_k, x_{k+1})} \\
&+ \gamma\frac{p(x_{k+1}, x_{k+1}).p(x_k, x_{k+2})}{1 + p(x_k, x_{k+1})} \\
&+ \delta\frac{p(x_{k+1}, x_{k+2}).p(x_{k+1}, x_{k+1}) + p(x_k, x_{k+2}).p(x_k, x_{k+1})}{1 + p(x_k, x_{k+1})}.
\end{aligned}
$$

Using the fact $p(x_{k+1}, x_{k+1}) \leq p(x_k, x_{k+1})$, $p(x_{k+1}, x_{k+1}) \leq p(x_{k+1}, x_{k+2})$ and $\frac{p(x_k, x_{k+1})}{1+p(x_k, x_{k+1})} < 1$, we have

$$
\begin{aligned}
p(x_{k+1}, x_{k+2}) &\leq \alpha p(x_k, x_{k+1}) + \beta p(x_{k+1}, x_{k+2}) + \gamma p(x_k, x_{k+2}) \\
&\quad + \delta p(x_k, x_{k+2}) + \delta p(x_{k+1}, x_{k+2}) \\
&= \alpha p(x_k, x_{k+1}) + (\beta + \delta)p(x_{k+1}, x_{k+2}) + (\gamma + \delta)p(x_k, x_{k+2}) \\
&\leq \alpha p(x_k, x_{k+1}) + (\beta + \delta)p(x_{k+1}, x_{k+2}) \\
&\quad + (\gamma + \delta)(p(x_k, x_{k+1}) + p(x_k, x_{k+1}) - p(x_{k+1}, x_{k+1}))
\end{aligned}
$$

implies

$$
p(x_{k+1}, x_{k+2}) \leq \Big(\frac{\alpha + \gamma + \delta}{1 - \beta - \gamma - 2\delta}\Big)p(x_k, x_{k+1}) = rp(x_k, x_{k+1}), \tag{5}
$$

where $r = \frac{\alpha+\gamma+\delta}{1-\beta-\gamma-2\delta} < 1$.

Regarding (4) and (5), we conclude that $\{p(x_k, x_{k+1})\}$ is a nonnegative nonincreasing sequence of real numbers and we obtain that

$$
p(x_k, x_{k+1}) \leq r^n p(x_0, x_1), \tag{6}
$$

for all $r = 0, 1, 2, ...$

Consider

$$
\begin{aligned}
d_p(x_{k+1}, x_{k+2}) &= 2p(x_{k+1}, x_{k+2}) - p(x_{k+1}, x_{k+1}) - p(x_{k+2}, x_{k+2}) \\
&\leq 2p(x_{k+1}, x_{k+2}) \leq 2r^{k+1}p(x_0, x_1). \qquad (7)
\end{aligned}
$$

We conclude that

$$
\lim_{k\to\infty} d_p(x_{k+1}, x_{k+2}) = 0. \tag{8}
$$

Moreover,

$$
\begin{aligned}
d_p(x_{k+1}, x_{k+s}) &\leq d_p(x_{k+s}, x_{k+s-1}) + ... + d_p(x_{k+2}, x_{k+1}) \\
&\leq 2(r^{k+s-1} + ... + r^{k+1})p(x_0, x_1). \qquad (9)
\end{aligned}
$$

After simple calculation we obtain $\{x_k\}$ a Cauchy sequence in (X, d_p). Since (X, p) is complete, by Lemma 1.1 (X, d_p) is also complete,hence the sequence $\{x_k\}$ converges to a point say $z \in X$. Again, by Lemma 1.1, we have

$$p(z,z) = \lim_{k\to\infty} p(x_k, z) = \lim_{k,m\to\infty} p(x_k, x_m). \tag{10}$$

Since $\{x_k\}$ a Cauchy sequence in (X, d_p), we have

$$\lim_{k,m\to\infty} d_p(x_k, x_m) = 0$$

implies that

$$\lim_{k,m\to\infty} p(x_k, x_m) = 0. \tag{11}$$

Without loss of generality, we may assume that $n > m$, we have

$$\begin{aligned} p(x_m, x_{m+2}) &\leq p(x_m, x_{m+1}) + p(x_{m+1}, x_{m+2}) - p(x_{m+1}, x_{m+1}) \\ &\leq p(x_n, x_{m+1}) + p(x_{m+1}, x_{m+2}) \end{aligned}$$

Similarly,

$$\begin{aligned} p(x_m, x_{m+3}) &\leq p(x_m, x_{m+1}) + p(x_{m+1}, x_{m+2}) + p(x_{m+2}, x_{m+3}) \\ &\quad - p(x_{m+1}, x_{m+1}) - p(x_{m+2}, x_{m+2}) \\ &\leq p(x_m, x_{m+1}) + p(x_{m+1}, x_{m+2}) + p(x_{m+2}, x_{m+3}). \end{aligned}$$

Inductively, we obtain

$$p(x_n, x_m) \leq p(x_m, x_{m+1}) + ... + p(x_{n-1}, x_n) \leq (r^m + ... + r^{n-1}) p(x_0, x_1).$$

As $r < 1$, we can observe that $\lim_{n,m\to\infty} p(x_n, x_m) = 0$. Regarding (10), we have

$$p(z,z) = \lim_{k\to\infty} p(x_k, z) = \lim_{k,m\to\infty} p(x_k, x_m) = 0. \tag{12}$$

Now, we shall show that z is a fixed point of f. On contrary, assume that $fz \neq z$, then $p(z, fz) > 0$. Let $\{x_{2k_i}\}$ be a subsequence of $\{x_{2k}\}$, so of $\{x_k\}$, we have

$$\begin{aligned} p(Sx_{2k_i}, fz) \leq \alpha p(x_{2k_i}, z) + \beta \frac{p(x_{2k_i}, gx_{2k_i}).p(z, fz)}{1 + p(x_{2k_i}, z)} + \gamma \frac{p(x_{2k_i}, fz).p(z, gx_2k_i)}{1 + p(x_{2k_i}, z)} \\ + \delta \frac{p(x_{2k_i}, gx_2k_i).p(x_{2k_i}, fz) + p(z, gx_2k_i).p(z, fz)}{1 + p(x_{2k_i}, z)}. \end{aligned}$$

Taking $i \to \infty$, we have $p(z, fz) = 0$ implies that $fz = z$. Similarly, we can obtain $gz = z$. Hence, $fz = gz = z$. Now, we can easily show that z is a unique fixed point of $f\&g$. □

Example 2.2. Let $X = [0, 1]$ and $p(x, y) = \max\{x, y\}$.Then (X, p) is a complete partial metric space. Clearly p is not metric on X. Let $f, g : X \to \infty$ defined by $fx = gx = \frac{x}{6}$, for all $x \in X$. Without loss of generality, we may assume $x > y$, then

$$\begin{aligned} p(fx, gy) &= \max\{\frac{x}{6}, \frac{y}{6}\} = \frac{x}{6} \leq \frac{1}{3}\max\{x, y\} \\ &\leq \alpha p(x, y) + \beta\frac{p(x, fx).p(y, gy)}{1 + p(x, y)} + \gamma\frac{p(x, gy).p(y, fx)}{1 + p(1 + p(x, y))} \\ &+ \delta\frac{p(x, fx).p(x, gy) + p(y, fx).p(y, gy)}{1 + p(x, y)}, \end{aligned}$$

with $\alpha = \frac{1}{3}$. Thus, all the conditions of the Theorem 2.2 are satisfied and 0 is a unique common fixed point of f and g.

Corollary 2.3. *Let (X, p) be a complete partial metric space. Let f be a self map on X. If there exists α, β, γ and δ in $(0, 1)$ with $\alpha + \beta + 2\gamma + 3\delta < 1$ such that*

$$\begin{aligned} p(fx, fy) &\leq \alpha p(x, y) + \beta\frac{p(x, fx).p(y, fy)}{1 + p(x, y)} + \gamma\frac{p(x, fy).p(y, fx)}{1 + p(x, y)} \\ &+ \delta\frac{p(x, fx).p(x, fy) + p(y, fy).p(y, fx)}{1 + p(x, y)}, \end{aligned}$$

for all $x, y \in X$. Then, there exists $z \in X$ such that $fz = z$.

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Frames and topological spaces in MSet

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Abstract. In this paper we give a definition for frames in the category **MSet** of actions of a monoid M on sets. We then show that, like in the classical case, frames and (internally) complete Heyting algebras are the same. Further, we give a definition of a topological space in **MSet** and study the relation between frames and topological spaces in this category. We show that the well known adjunction between the two categories still exists in **MSet**.

Keywords: frame, topological space, topos, M-set.

1. Introduction

Algebraic study of topological spaces through their lattice of open sets also known as *pointless* topology or theory of *frames* was initiated by M. Stone's representation theorem for Boolean algebras where a distinguished class of topological spaces known as *Stone* spaces arise as the representation of Boolean algebras. Further there was established a duality between frames and topological spaces in general and a class of spaces namely *sober* spaces were classified which could be studied algebraically through the lattice of their open sets and regardless of their geometric entity. In this paper frames and topological objects are defined in the context of the topos of actions of a monoid on sets and it is shown that such an adjunction exists in this context.

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2. Preliminaries

In this section we recall some basic facts and definitions needed in the sequel from [2], [3], and [5].

Definition 2.1. *A topos is a cartesian closed category (a category with finite limits and exponentiation) $\mathcal{E}$ together with a subobject classifier which is an object $\Omega \in \mathcal{E}$ together with a morphism $true : \mathbf{1} \to \Omega$ such that for every monomorphism $f : A \to B$, there exists a unique morphism $\chi_f : B \to \Omega$ making the diagram below a pullback square:*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle !} & & \downarrow{\scriptstyle \chi_f} \\ \mathbf{1} & \xrightarrow{true} & \Omega \end{array}$$

Definition 2.2. *Let M be a monoid with the identity element e. A (left) M-Set X is a set equipped with a map $\lambda : M \times X \to X$, called the (left) action of M on X, such that, denoting $\lambda(s, x) = sx$,*

- $ex = x \quad (\forall x \in X)$,
- $(st)x = s(tx) \quad (\forall s, t \in M, \forall x \in X)$.

A map $f : X \to Y$ between M-sets X and Y is said to be action-preserving or an equivariant map if $f(sx) = sf(x)$ for all $s \in M$ and $x \in X$. The category so obtained is denoted by **MSet**.

Recall (from, for example, [3, p.100] or [2]) that, **MSet** is a topos. Limits in **MSet** are computed just as they are in **Set**. For $X, Y \in \mathbf{MSet}$, $X^Y = \{f : M \times Y \to X \mid f \text{ is equivariant}\}$ is the exponential object with the evaluation arrow $ev : X^Y \times Y \to X$ defined by $ev(f, y) = f(e, y)$ and with the action of M on X^Y defined as $(sf)(m, y) = f(ms, y)$ for all $s, m \in M$ and $y \in Y$. Also, the subobject classifier Ω is the set L_M of all left ideals of M ($I \subseteq M$ is a left ideal if for all $s \in M$ and $x \in I$ we have $sx \in I$) with the action $sI = \{m \in M : ms \in I\}$, for every $s \in M$ and $I \in \Omega$. Further, notice that monomorphisms and epimorphisms in **MSet** are, respectively, one-one and onto equivariant maps.

Remark 2.1. Every $f \in X^Y$ can be equivalently seen as $f = (f_s)_{s \in M}$ where $f_s : Y \to X$ is a map, f being equivariant imposes a compatibility rule on the set $\{f_s : s \in M\}$ such that for all $t \in M$ and for all $y \in Y$, $tf_s(y) = f_{ts}(ty)$ or for short, $tf_s = f_{ts}t$. We will use this interpretation of elements of X^Y throughout this paper. Also, as a matter of convenience and in some definitions and proofs, for every $X \in \mathbf{MSet}$ and every $Y \in \Omega^X$, regarding the fact that in **MSet**, $Sub_{\mathbf{MSet}}(M \times X) \simeq hom_{\mathbf{MSet}}(M \times X \to \Omega) \simeq \Omega^X$, Y will be equivalently regarded as $Y = (Y_s)_{s \in M}$ where $Y_s \subseteq X$ and for every $t \in M$, $tY_s = \{ty : y \in Y_s\} \subseteq Y_{ts}$ (see, for example, [1]).

Definition 2.3. *A frame is a complete lattice A satisfying the infinite distributive law. That is, for every $X \subseteq A$ and every $a \in A$ we have that*

$$a \wedge \bigvee X = \bigvee \{a \wedge x : x \in X\}.$$

We denote the category of frames and maps preserving infinite joins and finite meets between them by **Frm**.

Remark 2.2. We recall that the category **Frm** is related to the category **Top** of topological spaces and continuous maps in the following way (for more information see, for example, [4] or [7]).

There exists an adjunction

$$pt : \mathbf{Frm} \leftrightarrows \mathbf{Top} : O$$

in which O takes every topological space to its frame of open subsets and every continuous map $f : X \to Y$ to the frame homomorphism $f^{-1} : O(Y) \to O(X)$. The functor pt takes every frame A to the set of so called "points" of A which are frame homomorphisms from A to the two-element frame **2**. The set of points of A comes with a natural topology $\{pt(a) : a \in A\}$ where $pt(a) = \{p \in pt(A) : p^{-1}(a) = 1\}$. The functor pt takes every frame homomorphism $\gamma : A \to B$ to the continuous map $pt(\gamma) : pt(B) \to pt(A)$ by composition.

3. Frames in MSet

In this section we introduce the notion of frames and Heyting algebras in the category **MSet** and show that frames and complete Heyting algebras coincide in this category, just as in the classical context.

Definition 3.1. *A lattice in the category* **MSet** *is a lattice with action preserving maps $\vee, \wedge : A \times A \to A$ satisfying the equalities of a lattice. A homomorphism of lattices in* **MSet** *is that of a lattice which is also action preserving. We denote the category of lattices and lattice homomorphisms in* **Set** *by* **Latt** *and the category of lattices in* **MSet** *and action-preserving lattice homomorphisms (or M-lattice homomorphisms) by* **MLatt**. *We also denote the (full) subcategory of* **MLatt** *of bounded lattices in* **MSet** *by* $\mathbf{MLatt_{01}}$.

Definition 3.2 ([1])**.** *A lattice A in* **MSet** *is said to be internally complete if there exists an equivariant, order-preserving map $\bigvee : \Omega^A \to A$ such that for every $X = (X_s)_{s \in M} \in \Omega^A$ and every $a \in A$ we have that*

$$\bigvee X \leq a \Leftrightarrow X \leq [a],$$

where $[a]_s = \{b \in A : b \leq sa\} = {\downarrow} sa$.

Definition 3.3. *A frame in* **MSet** *is an internally complete lattice A in* **MSet** *such that*

$$a \wedge \bigvee X = \bigvee (\{sa \wedge x : x \in X_s\})_{s \in M},$$

for every $a \in A$ and $X \in \Omega^A$.

Remark 3.1 ([6])**.** Let A be an internally complete lattice in **MSet**. Then, for each $X \in \Omega^A$, $\bigvee X = \bigvee \langle X \rangle$, where $\langle X \rangle \in \Omega^A$ is such that for every $s \in M$, $\langle X \rangle_s$ is the ideal of A generated by X_s.

Lemma 3.1. *Let $A \in$ **MFrm**. Then, for $a \in A$ we have*

$$\bigvee [a] = \bigvee (sa)_{s \in M} = a.$$

Proof. Since $[a] \leq [a]$, by definition of internal supremum, we have that $\bigvee [a] \leq a$. On the other side we have that $\bigvee [a] \leq \bigvee [a]$ so again by the definition of the internal supremum we have that $[a] \leq [\bigvee [a]]$. So, for every $s \in M$ we have that $\downarrow sa \subseteq \downarrow s \bigvee [a]$ and therefore by substituting s with e, we have $a \leq \bigvee [a]$. So, $\bigvee [a] = a$. Also, $\bigvee (sa)_{s \in M} = \bigvee [a]$ follows from the previous remark. □

Definition 3.4. *An M-frame homomorphism between two M-frames A and B is an internally complete M-lattice homomorphism. That is $h : A \to B$ is said to be internally complete if it commutes with the internal join as in the diagram*

$$\begin{array}{ccc} \Omega^A & \xrightarrow{\exists h} & \Omega^B \\ \big\downarrow \bigvee & & \big\downarrow \bigvee \\ A & \xrightarrow{h} & B \end{array}$$

where $(\exists h)(X) = (h(X_s))_{s \in M}$ for every $X \in \Omega^A$.

The category **MFrm** is consisted of M-frames and M-frame homomorphisms between them.

Definition 3.5. *Let $A \in$ **MLatt$_{01}$**. Then, A is called an M-Heyting algebra if there exists an equivariant map $\to$: $A \times A \to A$ such that*

$$x \wedge a \leq b \quad \Leftrightarrow \quad x \leq a \to b \quad \forall x, a, b \in A.$$

We denote the category of Heyting algebras in **MSet** *and $\to$-preserving equivariant lattice homomorphisms by* **MHeyt**.

The following is a straightforward consequence of the definition of Heyting algebras in **MSet** just as in the classical case. See [5, p.200].

Lemma 3.2. **MHeyt** *is the subcategory of* **MLatt$_{01}$** *the objects of which satisfy the equations*

- $a \to a = 1$,
- $a \wedge (a \to b) = a \wedge b$,
- $b \wedge (a \to b) = b$,
- $a \to (b \wedge c) = (a \to b) \wedge (a \to c)$.

Just as the case for frames in **Set**, (internal) completeness in **MFrm** makes defining pseudocomplements and implication operation possible. Hence making a connection with Heyting algebras.

Lemma 3.3. *Let $A \in \mathbf{MFrm}$ be given with the internal join $\bigvee : \Omega^A \to A$. Define $\to\colon A \times A \to A$ as*

$$a \to b = \bigvee(\{x : x \wedge sa \leq sb\})_{s\in M}.$$

Then, $\to$ is equivariant and it satisfies the condition stated in 3.5.

Proof. First observe that if $x \wedge sa \leq sb$ for $s \in M$ and $a, b \in A$, then for every $t \in M$ we have that $tx \wedge tsa \leq tsb$ and therefore $(\{x : x \wedge sa \leq sb\})_{s\in M} \in \Omega^A$. Now, let $m \in M$. We have that

$$\begin{aligned} m(a \to b) &= m \bigvee(\{x : x \wedge sa \leq sb\})_{s\in M} \\ &= \bigvee m(\{x : x \wedge sa \leq sb\})_{s\in M} \\ &= \bigvee(\{x : x \wedge sma \leq smb\})_{s\in M} \\ &= ma \to mb. \end{aligned}$$

So, $\to$ is equivariant. Now, suppose that $x, a, b \in A$ and $x \wedge a \leq b$. Then, we have that

$$\begin{aligned} &(\forall s \in M)(sx \wedge sa \leq sb) \\ &\Rightarrow (\forall s \in M)(\downarrow sx \subseteq \{x \in A : x \wedge sa \leq sb\}) \\ &\Rightarrow \bigvee(\downarrow sx)_{s\in M} \leq \bigvee(\{x :\in A : x \wedge sa \leq sb\})_{s\in M} \\ &\Rightarrow x \leq (a \to b). \end{aligned}$$

Now, conversely suppose that $x \leq a \to b$. Then

$$\begin{aligned} &a \wedge x \leq a \wedge (a \to b) \\ &\Rightarrow a \wedge x \leq a \wedge \bigvee(\{x : x \wedge sa \leq sb\})_{s\in M} \\ &\Rightarrow a \wedge x \leq \bigvee(\{sa \wedge x : x \wedge sa \leq sb\})_{s\in M} \\ &\Rightarrow a \wedge x \leq \bigvee(\downarrow sb)_{s\in M} \\ &\Rightarrow a \wedge x \leq b. \end{aligned}$$

□

Remark 3.2. Let $A \in \mathbf{MFrm}$ and $x \in A$. Then

$$x^* = \bigvee(\{y \in A : y \wedge sx = 0\})_{s\in M}$$

is the largest element of A such that $x^* \wedge x = 0$.

Proof. We have that $x \wedge x^* = 0$. For

$$\begin{aligned} x \wedge x^* &= x \wedge \bigvee(\{y \in A : y \wedge sx = 0\})_{s\in M} \\ &= \bigvee(\{sx \wedge y : y \wedge sx = 0\})_{s\in M} \\ &= \bigvee \mathbf{0} \\ &= 0, \end{aligned}$$

where by $\mathbf{0}$ we imply $(0_s)_{s\in M} \in \Omega^A$ with $0_s = \{0_A\}$ for every $s \in M$. The last line being due to the fact that $0_s = s\{0_A\}$ for every $s \in M$. Now, let $a \in A$ and $a \wedge x = 0$. Then, for every $s \in M$, we have $sa \wedge sx = 0$, and so $\downarrow sa \subseteq \{y \in A : y \wedge sx = 0\}$ for every $s \in M$. Thus $\bigvee[a] = \bigvee(\downarrow sa)_{s\in M} \leq \bigvee\{y \in A : y \wedge sx = 0\} = x^*$. But by Lemma 3.1, $a = \bigvee[a]$, and therefore $a \leq x^*$. □

Lemma 3.4. *An object $A \in$ **MSet** is an M-frame if and only if it is an internally complete M-Heyting algebra.*

Proof. Necessity. By 3.3, it is evident.

Sufficiency. Suppose $A \in$ **MHeyt** and is internally complete. We need to show that for all $a \in A$ and $X \in \Omega^A$,

$$a \wedge \bigvee X = \bigvee(\{ma \wedge x : x \in X_m\})_{m\in M}.$$

We have that

$$\begin{aligned} &a \wedge \bigvee X \leq \bigvee(\{ma \wedge x : x \in X_m\})_{m\in M} \\ &\Leftrightarrow \bigvee X \leq (a \to \bigvee(\{ma \wedge x : x \in X_m\})_{m\in M}) \\ &\Leftrightarrow (\forall s \in M)(X_s \subseteq\downarrow s(a \to \bigvee(\{ma \wedge x : x \in X_m\})_{m\in M})) \\ &\Leftrightarrow (\forall s \in M)(X_s \subseteq\downarrow (sa \to \bigvee(\{msa \wedge x : x \in X_{ms}\})_{m\in M})) \\ &\Leftrightarrow (\forall s \in M)(\forall x \in X_s)(x \leq sa \to (\bigvee(\{msa \wedge x : x \in X_{ms}\})_{m\in M})) \\ &\Leftrightarrow (\forall s \in M)(\forall x \in X_s)(x \wedge sa \leq \bigvee(\{msa \wedge x : x \in X_{ms}\})_{m\in M}) \\ &\Leftrightarrow (\forall s \in M)(\forall x \in X_s)(\bigvee(\downarrow (mx \wedge msa))_{m\in M} \leq \bigvee(\{msa \wedge x : x \in X_{ms}\})_{m\in M}). \end{aligned}$$

the last line being due to lemma 3.1 and the fact that

$$(\forall s, m \in M)(mX_s \subseteq X_{ms}).$$

For the other side we have that $\{ma \wedge x : x \in X_m\} \subseteq\downarrow ma$ for every $m \in M$. So, $\bigvee(\{ma \wedge x : x \in X_m\})_{m\in M} \leq \bigvee[a] = a$. Since $\{ma \wedge x : x \in X_m\} \subseteq \langle X_m\rangle$ for every $m \in M$, we conclude that

$$\bigvee(\{ma \wedge x : x \in X_m\})_{m\in M} \leq \bigvee\langle X\rangle = \bigvee X.$$

So, $\bigvee(\{ma \wedge x : x \in X_m\})_{m\in M} \leq a \wedge \bigvee X$. □

Lemma 3.5. *For every $X \in \mathbf{MSet}$, Ω^X is a frame with the internal join*

$$\bigvee : \Omega^{\Omega^X} \to \Omega^X,$$
$$\bigvee \Theta = (\bigcup\{X_e : X = (X_s)_{s\in M} \in \Theta_m\})_{m\in M}.$$

for every $\Theta = (\Theta_s)_{s\in M} \in \Omega^{\Omega^X}$.

Proof. It is proven in [6] that Ω^X is an internally complete lattice with the stated internal join. So, we just need to show that for every $\Theta \in \Omega^{\Omega^X}$ and every $U \in \Omega^X$,

$$U \wedge \bigvee \Theta = \bigvee(\{sU \wedge \Theta_s\})_{s\in M}.$$

We have

$$\begin{aligned} U \wedge \bigvee \Theta &= U \wedge (\bigcup\{X_e : X = (X_s)_{s\in M} \in \Theta_m\})_{m\in M} \\ &= (U_m \cap \bigcup\{X_e : X = (X_s)_{s\in M} \in \Theta_m\})_{m\in M}. \end{aligned}$$

On the other hand

$$\begin{aligned} \bigvee(\{sU \wedge \Theta_s\})_{s\in M} &= (\bigcup\{X_e : X = (X_s)_{s\in M} \in mU \cap \Theta_m\})_{m\in M} \\ &= (\bigcup\{X_e : X = (X_s)_{s\in M} \in U_m \cap \Theta_m\})_{m\in M} \\ &= (U_m \cap \bigcup\{X_e : X = (X_s)_{s\in M} \in \Theta_m\})_{m\in M}. \end{aligned}$$

So, both sides are equal and Ω^{Ω^X} is an M-frame as required. □

4. Topological object in the category MSet

In this section we introduce the notion of a *topological object* in the category **MSet** using the concept of internal completeness. For a general notion of a topological space in a topos one can refer to [8].

Remark 4.1. For every $X \in \mathbf{MSet}$, Ω^X is an externally complete distributive lattice with arbitrary join defined as

$$(\bigvee_{i\in I}(\{f_i : i \in I)\})_m(x) = \bigcup_{i\in I}(f_i)_m(x),$$

for every family $\{f_i : i \in I\} \subseteq \Omega^X$. Meet is also defined as

$$(\bigwedge_{i\in I}(\{f_i : i \in I\}))_m(x) = \bigcap_{i\in I}(f_i)_m(x),$$

or, by the alternative interpretation of Ω^X,

$$\bigvee_{i\in I} X^i = (\bigcup_{i\in I} X^i_s)_{s\in M},$$

for every family $\{X^i : i \in I\}$ in Ω^X and

$$\bigwedge_{i\in I} X^i = (\bigcap_{i\in I} X^i_s)_{s\in M}.$$

Ω^X is also bounded with the least member being $f^{\varnothing} = (f^{\varnothing}_m)_{m\in M}$, where for every $m \in M$ and every $x \in X$, $f_m(x) = \varnothing$, and with the greatest element $f^M = (f^M_m)_{m\in M}$, where for every $m \in M$ and for every $x \in X$, $f^M_m(x) = M$.

Definition 4.1. *A topological object in the category* **MSet** *is a pair* $(X, O(X))$ *such that* $X \in$ **MSet** *and* $O(X) \subseteq \Omega^X$ *is a sub* M*-frame of* Ω^X *or, equivalently, is a sub* M*-set of* Ω^X *containing the least and the greatest elements of* Ω^X *and is closed under finite meets (intersections) and internal joins in* Ω^X*. This is summerized as:*

- $f^{\varnothing}$ *and* f^M *belong to* $O(X)$*.*
- *If* $\Theta = (\Theta_s)_{s\in M} \in \Omega^{O(X)}$ *then* $\bigvee \Theta$ *as defined in lemma 3.5 belongs to* $O(X)$*.*
- *Any binary (external) meet of elements of* $O(X)$ *belongs to* $O(X)$*.*

We call every element of $O(X)$ *an open* M*-subset of the topology which can be regarded both as some* $U = (U_s)_{s\in M}, U_s \subseteq X$ *or, as used in this paper, as some* $f^U = (f^U_s)_{s\in M}, f^U_s : X \to \Omega$*.*

Definition and Lemma 4.1. *Let* $(X, O(X))$ *and* $(Y, O(Y))$ *be two topological spaces in* **MSet***. Every equivariant map* $f : X \to Y$ *induces an* M*-frame homomorphism* $f^{-1} : \Omega^Y \to \Omega^X$ *such that for every* $(f^U) \in \Omega^Y$*,* $f^{-1}((f^U)) = (f^U_s \circ f)_{s\in M}$*. Also* f *is said to be* M*-continuous if* $f^{-1}((f^U_s)_{s\in M}) \in O(X)$ *for every* $(f^U_s)_{s\in M} \in O(Y)$*.*

Note that considering the isomorphism $\Omega^X \simeq Sub_{\mathbf{MSet}}(M \times X)$*,* f^{-1} *can be alternatively viewed as* $f^{-1}((Y_s)_{s\in M}) = (f^{-1}(Y_s))_{s\in M}$ *for* $(Y_s)_{s\in M} \in \Omega^Y$ *where the right-hand side* f^{-1} *is the set-theoretical inverse image.*

Proof. It is well-known that f^{-1} is a morphism of Heyting algebras in any topos (see, [5, p. 201]). We just need to show that it is internally complete. Let $\Theta \in \Omega^{\Omega^Y}$. We need to show that $\mathbf{f}^{-1}(\bigvee \Theta) = \bigvee(f^{-1}(\Theta_s))_{s\in M}$. We have

$$\begin{aligned}
\mathbf{f}^{-1}(\bigvee \Theta) &= (f^{-1}(\bigvee \Theta)_s)_{s\in M} \\
&= (f^{-1}(\cup\{Y_e : Y = (Y_m)_{m\in M} \in \Theta_s\}))_{s\in M} \\
&= (\cup\{X_e : (X_m)_{m\in M} \in f^{-1}(\Theta_s)\})_{s\in M} \\
&= \bigvee(f^{-1}(\Theta_s))_{s\in M}.
\end{aligned}$$

□

We denote the category of topological M-sets and M-continuous maps as **MTop**. In the classical setting, all maps from the one-point topological space (with the obvious topology) to a topological space $(X, O(X))$ are continuous. These maps are in one-one correspondence with actual points of the set X. We show that in **MSet**, the topological object (M, Ω^M) has such property.

Lemma 4.1. *For every monoid M, (M, Ω^M) is a topology in* **MSet**. *Moreover, for every topology $(X, O(X))$ in* **MSet**, *every equivariant map $f : M \to X$ is continuous. In fact, there exists a one-one correspondence between actual points of X and continuous maps from (M, Ω^M) to $(X, O(X))$.*

Proof. Let $f : M \to X$ be equivariant. In fact every equivariant map from M to X is identified with some $x \in X$ where $x = f(e)$. Then, by Lemma 4.1, f induces the M-frame homomorphism $f^{-1} : \Omega^X \to \Omega^M$, taking $(f_s^U)_{s\in M} \in \Omega^X$ to $(f_s^U \circ f)_{s\in M}$. It is trivial that f is continuous, since M is equipped with the largest topology. □

5. Adjunction between MTop and MFrm

As we know for topological spaces and frames in the classical setting, there exists an adjunction between the two categories. In this section we show the existence of such an adjunction between the two categories in **MSet**.

Lemma 5.1. *Let $A \in$* **MFrm** *and put $pt(A) = \{\gamma : A \to \Omega^M : \gamma$ is equivariant$\}$. Then $pt(A)$ is an M-set with the action of M on it defined as*

$$((t\gamma)(a))_m(n) = (\gamma(a))_m(nt), \forall a \in A, \forall t, m, n \in M.$$

Proof. First we need to show that $(t\gamma)(a) \in \Omega^M$ for every $t \in M$ and every $a \in A$ or, equivalently, for every $s, m, n \in M$,

$$s((t\gamma)(a))_m(n) = ((t\gamma)(a))_{sm}(sn).$$

But, we have

$$\begin{aligned} s((t\gamma)(a))_m(n) &= s(\gamma(a))_m(nt) \\ &= (\gamma(a))_{sm}(snt) \\ &= ((t\gamma)(a))_{sm}(sn). \end{aligned}$$

Now, to show that $t\gamma$ is equivariant, for every $s, t, m, n \in M$ we have

$$\begin{aligned} ((t\gamma)(sa))_m(n) &= (\gamma(sa))_m(nt) \\ &= (s\gamma(a))_m(nt) \\ &= (\gamma(a))_{ms}(nt) \\ &= (t\gamma(a))_{ms}(n) \\ &= (s(t\gamma(a)))_m(n). \end{aligned}$$

So, $t\gamma \in pt(A)$. Also,

$$\begin{aligned}(((st)\gamma)(a))_m(n) &= (\gamma(a))_m(n(st))\\ &= (\gamma(a))_m((ns)t)\\ &= (t\gamma(a))_m(ns)\\ &= (s(t\gamma(a)))_m(n).\end{aligned}$$ □

Lemma 5.2. *For every $A \in \mathbf{MFrm}$ and $a \in A$, put $pt(a) = (\{\gamma \in pt(A) : (\gamma(a))_m(e) = M\})_{m\in M}$. Then $pt(a) \in \Omega^{pt(A)}$ and $\{pt(a) : a \in A\}$ is a sub M-set of $\Omega^{pt(A)}$ and is an M-frame.*

Proof. Let $a \in A$ and $\gamma \in (pt(a))_s$. So, we have $(\gamma(a))_s(e) = M$. Then, for every $t \in M$

$$\begin{aligned}((t\gamma)(a))_{ts}(e) &= (\gamma(a))_{ts}(t)\\ &= t(\gamma(a))_s(e),\end{aligned}$$

and therefore $t\gamma \in (pt(a))_{ts}$ and $pt(a) \in \Omega^{pt(A)}$. Now, we need to show that $\{pt(a) : a \in A\}$ is an M-set. Let $t \in M$. We have

$$\begin{aligned}tpt(a) &= (\{\gamma \in pt(A) : (\gamma(a))_{mt}(e) = M\})_{m\in M}\\ &= (\{\gamma \in pt(A) : (t\gamma(a))_m(e) = M\})_{m\in M}\\ &= (\{\gamma \in pt(A) : (\gamma(ta))_m(e) = M\})_{m\in M}\\ &= pt(ta).\end{aligned}$$

Now, we want to show that $\{pt(a) : a \in A\}$ is an M-frame. Suppose $\Theta \in \Omega^{\{pt(a):a\in A\}}$. Then, by Lemma 3.5,

$$\begin{aligned}(\bigvee \Theta)_m &= \bigcup\{(pt(a))_e : pt(a) \in \Theta_m\}\\ &= \{\gamma \in pt(A) : \exists a \in A : pt(a) \in \Theta_m, (\gamma(a))_e(e) = M\}.\end{aligned}$$

Now, put $Y = (\{a \in A : pt(a) \in \Theta_s\})_{s\in M}$. It is easy to verify that $Y \in \Omega^A$. Now, since γ is an internally complete lattice homomorphism, we have that

$$\gamma(\bigvee Y) = \bigvee(\gamma(Y)),$$

where $\gamma(Y) = (\gamma(Y_s))_{s\in M}$ and $\gamma(Y) \in \Omega^{\Omega^M}$. So, again, by Lemma 3.5, for every $m \in M$ we have

$$\begin{aligned}(\bigvee \gamma(Y))_m &= \bigcup\{X_e : X \in (\gamma(Y))_m\}\\ &= \bigcup\{X_e : X \in \{\gamma(a) : pt(a) \in \Theta_m\}\}\\ &= \bigcup\{(\gamma(a))_e : pt(a) \in \Theta_m\}.\end{aligned}$$

And

$$\begin{aligned}(pt(\gamma(\bigvee Y)))_m &= \{\gamma \in pt(A) : (\bigvee \gamma(Y))_m(e) = M\}\\ &= \{\gamma \in pt(A) : \cup\{(\gamma(a))_e : pt(a) \in \Theta_m\} = M\}\\ &= \{\gamma \in pt(A) : \exists a \in A : pt(a) \in \Theta_m, (\gamma(a))_e(e) = M\}\\ &= (\bigvee \Theta)_m.\end{aligned}$$

So, $\{pt(a) : a \in A\}$ is closed under internal join and therefore is a frame. □

Remark 5.1. Considering that for a monoid M and an M-set X, $Sub_{\mathbf{MSet}}(M \times X) \simeq \Omega^X$, $pt(a)$ can be equivalently regarded as

$$(pt(a))_s(\gamma) = \{x \in M : x \in (\gamma(a))_s(e)\}.$$

Lemma 5.3. *Let $A \in \mathbf{MFrm}$. Then $(pt(A), \{pt(a) : a \in A\})$ is a topology in* **MSet**.

Proof. By Lemma 5.1, $pt(A)$ is an M-set and, by Lemma 5.2, $\{pt(a) : a \in A\}$ is a frame in **MSet** and a subobject of $\Omega^{pt(A)}$. □

Lemma 5.4. *The mapping $A \mapsto pt(A)$, is functorial from $\mathbf{MFrm}^{op}$ to* **MTop**. *In fact, it takes an M-frame A to the M-topological space $(pt(A), \{pt(a) : a \in A\})$, and an M-frame homomorphism $\gamma : A \to B$ to M-continuous map $pt(\gamma) : pt(B) \to pt(A)$ taking $f : B \to \Omega^M$ to $f \circ \gamma$.*

Proof. For every M-frame homomorphism $\gamma : A \to B$, $pt(\gamma)$ is equivariant. Let $m \in M$. Then

$$\begin{aligned}((pt\gamma(mf))(a))_t(n) &= ((mf)(\gamma(a)))_t(n)\\ &= (f(\gamma(a)))_t(nm)\\ &= ((m(f \circ \gamma))(a))_t(m)\\ &= (m(pt\gamma f))_t(n).\end{aligned}$$

We also have that $pt\gamma$ is continuous. Let $\Theta \in \Omega^{\{pt(a):a\in A\}}$ and $f \in pt(B)$. Then

$$\begin{aligned}((pt\gamma)^{-1}(\bigvee \Theta))_s(f) &= (\bigvee \Theta)_s(pt\gamma(f))\\ &= (\bigvee \Theta)_s(f \circ \gamma)\\ &= \{x \in M : \exists a \in A : pt(a) \in \Theta_{xs}, ((f \circ \gamma)(a))_e(x) = M\}.\end{aligned}$$

Now, let $\Gamma = (\{pt(\gamma(a)) : pt(a) \in \Theta_m\})_{m\in M}$. It is straightforward to see that $\Gamma \in O(pt(B))$. Then

$$\begin{aligned}(\bigvee \Gamma)_s(f) &= \{x \in M : \exists b \in B : pt(b) \in \Gamma\}\\ &= \{x \in M : \exists b \in B : pt(b) \in \Gamma_{xs}, (f(b))_e(x) = M\}\\ &= \{x \in M : \exists a \in A : pt(a) \in \Theta_{xs}, ((f \circ \gamma)(a))_e(x) = M\}.\end{aligned}$$

So, $pt\gamma(\bigvee \Theta) = \bigvee \Gamma$ and $pt\gamma$ is continuous. □

Proposition 5.1. *There exists an adjunction between the categories* $\mathbf{MFrm}^{\mathbf{op}}$ *and* **MTop**.

Proof. Let $X \in \mathbf{MTop}$. First note that, by Lemma 4.1, for every nonempty X, $pt(O(X))$ is nonempty. Define $\alpha : X \to pt(O(X))$ as $((\alpha(x))(f^U)))_m(n) = f^U_m(nx)$.

- For every $x \in X$ and every $f^U \in O(X)$, $(\alpha(x))(f^U) \in \Omega^M$. For, let $t, m, n \in M$. Then

$$\begin{aligned} t((\alpha(x))(f^U))_m(n) &= t(f^U)_m(nx) \\ &= (f^U)_{tm}(tnx) \\ &= ((\alpha(x))(f^U))_{tm}(tn). \end{aligned}$$

- For every $x \in X$, $\alpha(x) \in pt(O(X))$ or $\alpha(x) : O(X) \to \Omega^M$ is equivariant. For let $t, m, n \in M$ and $f^U \in O(X)$. Then

$$\begin{aligned} ((\alpha(x))(tf^U))_m(n) &= (tf^U)_m(nx) \\ &= (f^U)_{mt}(nx) \\ &= (\alpha(x)(f^U))_{mt}(n) \\ &= (t\alpha(x)(f_U))_m(n). \end{aligned}$$

- α is equivariant. Let $t, m, n \in M$ and $f^U \in O(X)$. Then

$$\begin{aligned} ((\alpha(tx))(f^U))_m(n) &= f^U_m(ntx) \\ &= ((\alpha(x))(f^U))_m(nt) \\ &= ((t\alpha(x))(f^U))_m(n). \end{aligned}$$

- α is continuous. Let $f^U = (f^U_s)_{s\in M} \in O(X)$. Then $pt(f^U) \in O(pt(O(X)))$. For every $x \in X$ we have

$$\begin{aligned} (\alpha^{-1}(pt(f^U)))_s(x) &= ((pt(f^U))_s(\alpha(x))_s \\ &= \{y \in M : y \in (\alpha(x)(f^U))_s(e)\} \\ &= \{y \in M : y \in f^U_s(x)\} \\ &= f^U_s(x). \end{aligned}$$

Now, we show that for every $A \in \mathbf{MFrm}$, and every continuous $\beta : X \to pt(A)$, there exists a unique M-frame homomorphism $\gamma : A \to O(X)$ such that $pt(\gamma) \circ \alpha = \beta$. For every $m \in M, x \in X$, $a \in A$ put

$$(\gamma(a))_m(x) = ((\beta(x))(a))_m(e).$$

- $\gamma(a) \in O(X)$. We have that

$$\begin{aligned}(\beta^{-1}(pt(a)))_m(x) &= (pt(a))_m(\beta(x)) \\ &= \{y \in M : y \in ((\beta(x))(a))_m(e)\} \\ &= (\gamma(a))_m(x).\end{aligned}$$

  So, we have that $\gamma(a) = \beta^{-1}(pt(a))$ and by continuity of β, $\gamma(a) \in O(X)$.

- γ is equivariant. Let $m, n \in M, a \in A$ and $x \in X$. Then

$$\begin{aligned}(\gamma(na))_m(x) &= ((\beta(x))(na))_m(e) \\ &= (n(\beta(x))(a))_m(e) \\ &= ((\beta(x))(a))_{mn}(e) \\ &= (n\gamma(a))_m(x).\end{aligned}$$

- $pt\gamma \circ \alpha = \beta$. Let $x \in X$ and $a \in A$. Then

$$\begin{aligned}(pt\gamma(\alpha(x))(a))_m(n) &= ((\alpha(x))(\gamma(a)))_n(m) \\ &= (\gamma(a))_m(nx) \\ &= ((\beta(nx))(a))_m(e) \\ &= ((n\beta(x))(a))_m(e) \\ &= ((\beta(x))(a))_m(n).\end{aligned}$$

- γ, as defined above, is the unique M-frame homorphism from A to $O(X)$ such that $pt\gamma \circ \alpha = \beta$. Suppose $\lambda : A \to O(X)$ is an M-frame homomorphism such that $pt\lambda(\alpha(x)) = \beta(x)$. So, we have that $(\alpha(x))(\lambda) = \beta(x)$, which yields $(\lambda(a))_m(nx) = (\beta(x)(a))_m(n)$ and therefore $(\lambda(a))_m(x) = (\beta(x)(a))_m(e)$. □

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Some results on the total graph of a commutative ring

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Abstract. In this paper, we study the total graph of a ring and determine its matching, factors and arboricity. We find the independence number of the line graph of the total graph of a ring and give complete characterization of rings for which diam $L(T(\Gamma(R))) =, <$ or $>$ diam $T(\Gamma(R))$. We find the crossing number of all local rings with some bounds in the cardinality of zero divisors and also determine the crossing number of all finite rings with the genus at most two.

Keywords: crossing number, zero divisors, matching, factors, arboricity.

1. Introduction

The idea of associating a graph to an algebraic object initiated by I. Beck [1] lead to many interesting characterization of algebraic objects. Following the same path many mathematicians have derived a number of interesting results by assigning graph to an algebraic object. All rings we consider in this paper are commutative rings with identity. In this paper, we try to obtain some characterization of a commutative ring by considering the total graph of the ring. The total graph of a ring was introduced by Anderson and Badawi in [2], defined as "an undirected graph with all elements of R as vertices and for distinct $x, y \in R$, x is adjacent to y whenever $x + y$ belongs to the set of $Z(R)$ and this graph is denoted by $T(\Gamma(R))$". In 1944, Pal. Turán [3] posed the problem of finding crossing number of a graph and still after 75 years, very little is known about this simple yet intricate nonplanarity measure. The crossing number of a complete graph is conjectured by Hill's [4] as

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$cr(K_n) =: \frac{1}{4}\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{n-2}{2}\rfloor\lfloor\frac{n-3}{2}\rfloor$ and crossing number of a complete bipartite graph by Zarankiewicz [5] as $cr(K_{m,n}) =: \lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor$. However, these conjectures remain open and some partial results are known. The former have been verified for $n \leq 10$, while the later holds for all n with $m \leq 6$ and for $m = 7$ with $n \leq 10$. $Reg(\Gamma(R))$ is the (induced) subgraph of $T(\Gamma(R))$ with vertex set Reg(R) and $Z(\Gamma(R))$ is the (induced) subgraph of $T(\Gamma(R))$ with vertex set Z(R). The study of a total graph breaks into two parts depending on whether R is a finite local ring or not.

In Section 2 of this paper we find the maximum matching, the perfect matching and the number of perfect matching when R is a finite ring of even order. In Section 3, we decompose the total graph into line-disjoint spanning subgraphs to find its factorisation. In Section 4, we find arboricity of the total graph of a ring. In Section 5, we determine some properties of the line graph associated with the total graph of a ring. The study of the line graph is interesting as it enables us to phrase questions on the edges of the graph which are viewed as the vertices in its corresponding line graph. In Section 6, we find all local rings up to isomorphism whose crossing number of the total graph can be computed from Hill's conjecture and Zarankiewicz conjecture with some bound on the cardinality of the set of zero divisors. At last in Section 7, we draw the total graph of all rings with the genus at most 2 to find the best possible drawing with the least crossing of edges. For basic graph-theoretic terminologies and results we refer the reader to [6] and [7].

2. Matching of total graph of ring

A matching in a graph G is a set of non-loop edges with no shared endpoints. The vertices incident to the edges of a matching M are saturated by M; the other vertices are unsaturated. A perfect matching in a graph is a matching that saturates every vertex. In a graph a maximal matching is a matching which cannot be increased by adding an extra edge. A matching with maximum cardinality is called maximum matching. A vertex cover of a graph G is a set $Q \subseteq V(Q)$ that contains at least one endpoint of every edge. The vertices in Q cover $E(Q)$. An edge cover of G is a set L of edges such that every vertex of G is incident to some edge of L. We denote "the maximum size of the independent set, the maximum size of matching, the minimum size of vertex cover and the minimum size of edge cover by $\alpha(G)$, $\alpha'(G)$, $\beta(G)$, $\beta'(G)$ respectively ".

In proposition 2.1 we find the existance of a perfect matching in the total graph of a finite commutative ring of even order.

Proposition 2.1. *If R is a finite ring of even order, then $T(\Gamma(R))$ has a perfect matching.*

Proof. First, we consider the case when $Z(R)$ is an ideal and $2 \in Z(R)$. Let $|Z(R)| = \lambda$ and $\frac{|R|}{|Z(R)|} = \beta$. Then, the total graph of the finite local ring with $2 \in Z(R)$ is $T(\Gamma(R)) = \sqcup_\beta K_\lambda$. So $|R| = 2^k$ for some k $\in \mathbb{N}$ since $2 \in Z(R)$ and

λ is a power of 2. As K_{2n} has a perfect matching and the number of perfect matching is $\frac{(2n)!}{(2^n n!)}$ by example 3.1.3 [6], $T(\Gamma(R))$ has a perfect matching. Now, we consider when $Z(R)$ is an ideal and $2 \notin Z(R)$. In this case $|R|$ is even and $\exists$ an element of order 2 in R. This implies that $\exists\ x \neq 0 \in R$ such that $2x \in Z(R)$ which is a contradiction. Next, we consider the case when $Z(R)$ is not an ideal. Let I be a maximal ideal contained in $Z(R)$. If $2 \in I$, then we construct a spanning subgraph H of the ring $\frac{R}{I}$ where the vertex set is $\frac{R}{I}$ and two vertices x and y are adjacent whenever $x + y$ belongs to an ideal I. The subgraph H is the disjoint union of complete graphs as in the proof of theorem 3.2 [8]. If the cardinality of I is even, then as the complete graph has a perfect matching, H has a perfect matching. If $|I|$ is odd, then each component of H is $|I| - 1$ regular and $|I| < |Z(R)|$, so $T(\Gamma(R))$ is obtained by adding at least 1 edge to every vertex. Now, these components are complete graphs and the addition of an extra edge makes the resultant graph connected with a perfect matching. At last, we consider I a maximal ideal with $2 \notin I$. Then, the subgraph H of the ring $\frac{R}{I}$ where the vertex set is $\frac{R}{I}$ is a disjoint union of $K_{|I|}$ and these $\frac{(\frac{|R|}{|I|}-1)}{2}$ number of disjoint $K_{|I|,|I|}$'s. Then, $\frac{|R|}{|I|}$ is odd and $|I|$ is even and hence, there is a perfect matching. □

In the following two propositions we find the maximum size of a perfect matching in the total graph of a ring.

Proposition 2.2. *If R is a finite ring of even order, then $\alpha'(T(\Gamma(R)))= \frac{|R|}{2}$. In particular, if $Z(R)$ is an ideal of R and $2 \in Z(R)$, then the number of perfect matching is equal to $((((n!)!)!)!...)!$, factorial upto l times, where l is given by $\frac{|R|}{|Z(R)|} = 2^l$ and n is the number of perfect matching in $K_{|Z(R)|}$.*

Proof. Let $|Z(R)| = \lambda$, $\frac{|R|}{|Z(R)|} = \beta$. As $2 \in Z(R)$, we have $T(\Gamma(R)) = \sqcup_\beta K_\lambda$. Clearly by proposition 2.1, $T(\Gamma(R))$ has a perfect matching and $\alpha'(T(\Gamma(R)))= \frac{|R|}{2}$. Let n be the number of perfect matching in each K_λ. Now we consider a pair of two distinct sets of perfect matchings of any K_λ say V_i, V_j having cardinality n each, where $1 \leq i, j \leq \beta$. Then, the total numbers of one-one map from V_i to V_j is $n!$. So, $K_\lambda \sqcup K_\lambda$ has $n!$ number of perfect matchings. Similarly, taking two more $K'_\lambda s$ and repeating the process we see that there are $(n!)!$ number of perfect matching of $K_\lambda \sqcup K_\lambda \sqcup K_\lambda \sqcup K_\lambda$. Continuing this process we find that the total number of perfect matchings is $((((n!)!)!)!...)!$, factorial taken l times, where l is given by $\frac{|R|}{|Z(R)|} = 2^l$. □

Proposition 2.3. *If R is a finite ring of odd order, then $\alpha'(T(\Gamma(R)))= \frac{|R|-1}{2}$.*

Proof. First, we consider the case when $Z(R)$ is an ideal and $|R|$ is odd. The total graph of the finite local ring with $2 \notin Z(R)$ is K_λ union $\frac{(\beta-1)}{2}$ number of disjoint $K_{\lambda,\lambda}$'s where $|Z(R)| = \lambda$ and $\frac{|R|}{|Z(R)|} = \beta$. Hence, we get $\alpha'(T(\Gamma(R)))=$

$(\frac{\lambda-1}{2} + (\frac{\beta-1}{2})\lambda) = \frac{\lambda\beta-1}{2} = \frac{|R|-1}{2}$. Next, we consider the case when $Z(R)$ is not an ideal and $|R|$ is odd. Let I be a maximal ideal contained in $Z(R)$. We construct a spanning subgraph H of the ring $\frac{R}{I}$ with the vertex set $\frac{R}{I}$ and the vertices x and y are adjacent whenever $x+y$ belongs to an ideal I. The subgraph H of the ring $\frac{R}{I}$ is a disjoint union of $K_{|I|}$ and $\frac{(\frac{|R|}{|I|}-1)}{2}$ number of disjoint $K_{|I|,|I|}$'s. Thus, $\alpha'(T(\Gamma(R)))=(\frac{|I|-1}{2} + (\frac{(\frac{|R|}{|I|}-1)}{2})|I|) = \frac{|R|-1}{2}$. □

Proposition 2.4. *If R is an integral domain, then $\alpha'(T(\Gamma(R))) = \beta(T(\Gamma(R)))$.*

Proof. Let $V(T(\Gamma(R))) = V_1 \sqcup V_2$ such that whenever $x \in V_1$, $-x \in V_2$ and also the zero element, self inverse elements we include in any of the set V_1 or V_2. Then, $T(\Gamma(R))$ is a bipartite graph with partite sets V_1 and V_2. Now, by Köning - Egerváry theorem [9, 10] $\alpha'(T(\Gamma(R))) = \beta(T(\Gamma(R)))$. □

Remark 2.1. Converse of the above result is false. For example $\mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2$ are not integral domains but their total graphs are bipartite.

Proposition 2.5. $\alpha(T(\Gamma(R))) = \alpha'(T(\Gamma(R))) = \beta(T(\Gamma(R))) = \beta'(T(\Gamma(R))) = 2$ *if and only if* $R \cong \mathbb{Z}_4,\ \mathbb{Z}_2 \times \mathbb{Z}_2,\ \frac{\mathbb{Z}_2[X]}{(X^2)}$.

Proof. We assume, $\alpha(T(\Gamma(R))) = \alpha'(T(\Gamma(R))) = \beta(T(\Gamma(R))) = \beta'(T(\Gamma(R))) = 2$. Then, $|R|= 4$ and the only possible ring of order four having $Z(R) \neq \{0\}$ are $\mathbb{Z}_4,\ \mathbb{Z}_2 \times \mathbb{Z}_2,\ \frac{\mathbb{Z}_2[X]}{(X^2)}$.

Conversely, suppose $R \cong \mathbb{Z}_4,\ \mathbb{Z}_2 \times \mathbb{Z}_2,\ \frac{\mathbb{Z}_2[X]}{(X^2)}$. Then, $\mathbb{Z}_4,\ \frac{\mathbb{Z}_2[X]}{(X^2)}$ are local rings of order 4 whose total graph is $2K_2$. Also $\mathbb{Z}_2 \times \mathbb{Z}_2$ is a non local ring whose total graph is C_4. Hence, $\alpha(T(\Gamma(R))) = \alpha'(T(\Gamma(R))) = \beta(T(\Gamma(R))) = \beta'(T(\Gamma(R))) = 2$. □

3. Factorisation of total graph of ring

A factor of a graph G is a spanning subgraph of G. A k-factor is a spanning k-regular subgraph. We say that G is the sum of factors G_i if it is line disjoint union, and such a union is called a factorization of G. If G is sum of $n-$ factors, their union is called n-factorization and G itself $n-$factorable. In this section we find when $T(\Gamma(R))$ has a factorisation. For our further computation we shall refer to [7].

Remark 3.1. $K_{2k+1,2k+1}$ is a spanning regular subgraph of the complete graph $K_{2(2k+1)}$ and we know that the complete graph K_{2n} is a sum of $1-$factor and $n-1$ spanning cycles by Theorem 9.7 [7]. As degree of a vertex in K_{2n} is $2n-1$ it is a sum of $1-$factor and $n-1$ spanning cycles. Deleting $1-$spanning cycle from K_{2n} we get a subgraph of degree $2n-3$ which is a sum of $1-$factor and $n-2$ spanning cycles. So, as we continue reducing degree by 2, number of spanning cycles is reduced by 1.

If $Z(R)$ is an ideal of R and $2 \in Z(R)$, then by Theorem 9.1 [7], we have the following:

Proposition 3.1. *Let R be a finite ring such that $Z(R)$ is an ideal of R and $2 \in Z(R)$. Then, $T(\Gamma(R))$ has a $1-$factorisation.*

Proposition 3.2. *Let R be a finite ring such that $Z(R)$ is an ideal of R and $2 \notin Z(R)$. Then, $T(\Gamma(R))$ has a factorisation.*

Proof. Let $Z(R)$ be an ideal of R. If $2 \notin Z(R)$, then the total graph of R is disjoint union of K_λ and $\frac{(\beta-1)}{2}$ number of disjoint $K_{\lambda,\lambda}$'s, where $|Z(R)| = \lambda$ and $\frac{|R|}{|Z(R)|} = \beta$. As λ is odd by Theorem 9.6 [7] K_{2k+1} is a sum of k spanning cycles. Each component $K_{2k+1,2k+1}$ is a spanning regular subgraph of the complete graph $K_{2(2k+1)}$. Therefore by remark 3.1, $K_{2k+1,2k+1}$ is a sum of $1-$factor and k spanning cycles. So, $T(\Gamma(R)) = F_1 \cup F_2 \cup F_3 ... \cup F_{k+1}$, where $F_i's$ are the set of $2-$factor in 2k+1 vertices from $Z(R)$ union $\frac{(\beta-1)}{2}$ number of 2-factor in 2(2k+1) vertices from $Reg(R)$, for all $1 \leq i \leq k$ and F_{k+1} is the graph of spanning 2k+1 vertices from $Z(R)$ union $\frac{(\beta-1)}{2}$ number of $1-$factor in 2(2k+1) vertices from $Reg(R)$. Hence, $T(\Gamma(R))$ has a factorisation. □

Proposition 3.3. *Let R be a finite ring such that $Z(R)$ is not an ideal of R and $2 \in Z(R)$. Then, $T(\Gamma(R))$ is $2-$factorable if $|Z(R)|$ is odd otherwise it is sum of a $1-$factor and spanning cycles.*

Proof. Let $2 \in Z(R)$. If $|Z(R)|$ is odd, then $T(\Gamma(R))$ is a regular graph. By Theorem 9.9 [7], it is $2-$factorable. If $|Z(R)|$ is even, then $T(\Gamma(R))$ is a regular graph of odd degree and it is the spanning subgraph of the complete graph K_{2n}. By Theorem 9.7 [7] and remark 3.1, $T(\Gamma(R))$ is a sum of $1-$factor and spanning cycles. □

Proposition 3.4. *Let R be a finite ring such that $Z(R)$ is not an ideal of R and $2 \notin Z(R)$. Then, $T(\Gamma(R)) = F_1 \cup F_2 \cup F_3 ... \cup F_{k+1}$, where $F_i's$ are factors and $k = \frac{(|Z(R)|-1)}{2}$.*

Proof. Let $2 \notin Z(R)$. Then, each element will have distinct additive inverse. For each $u \in Reg(R)$, $-u \in Reg(R)$. Therefore, $|Reg(R)|$ is even and $|Z(R)|$ is odd. Let H be a $|Z(R)|$–1 regular subgraph of $T(\Gamma(R))$ excluding the edges between regular elements with their respective additive inverse. Then, H is a connected regular graph of even degree and so it is $2-$factorable. Therefore, $H = F_1 \cup F_2 \cup F_3 ... \cup F_k$, where $k = \frac{(|Z(R)|-1)}{2}$, for all $1 \leq i \leq k$ and each $F_i's$ are spanning cycles. Next, the factor F_{k+1} is the graph with the set of spanning vertices from $Z(R)$ union $1-$factor in $Reg(R)$. Thus, $T(\Gamma(R)) = H \cup F_{k+1}$. □

4. Arboricity

Any graph G can be expressed as a sum of spanning forests. A natural problem is to determine minimum number of line-disjoint spanning forest into which G can be decomposed. This number is called arboricity of G and is denoted by $\gamma(G)$. For example, $\gamma(K_4) = 2$ and $\gamma(K_5) = 3$. Nash-Williams [11] has determined a precise formula for arboricity of graph; namely, "$\gamma(G) = \lceil \frac{e_H}{n_H - 1} \rceil$", where maximum is taken over all nontrivial subgraphs H and e_H, n_H denote the number of edges and vertices of H respectively. For complete graph $\gamma(K_n) = \lceil \frac{n}{2} \rceil$ and for complete bipartite $\gamma(K_{m,n}) = \lceil \frac{mn}{m+n-1} \rceil$.

Proposition 4.1. *Let R be a finite ring such that $Z(R)$ is an ideal of R. Then, $\gamma(T(\Gamma(R))) = \lceil \frac{\lambda}{2} \rceil$, where $|Z(R)| = \lambda$.*

Proof. If $Z(R)$ is an ideal of R and $2 \in Z(R)$, then the total graph of $T(\Gamma(R))$ is a disjoint union of the complete graphs K_λ. Thus by Nash-Williams formula [11], $\gamma(T(\Gamma(R))) = \lceil \frac{\lambda}{2} \rceil$. If $Z(R)$ is an ideal of R and $2 \notin Z(R)$, then the total graph of $T(\Gamma(R))$ is K_λ union $\frac{(\beta-1)}{2}$ number of disjoint $K_{\lambda,\lambda}$'s, where $|Z(R)| = \lambda$ and $\frac{|R|}{|Z(R)|} = \beta$. As λ is odd, say $\lambda = 2k + 1$, for complete bipartite graph $\gamma(K_{\lambda,\lambda}) = \lceil \frac{\lambda\lambda}{2\lambda-1} \rceil = \lceil \frac{4k^2+4k+1}{4k+1} \rceil$ = k+$\lceil \frac{3k+1}{4k+1} \rceil$ =$k + 1$ =$\lceil \frac{2k+1}{2} \rceil$ =$\lceil \frac{\lambda}{2} \rceil$. Since, arboricity of all the components are equal, $\gamma(T(\Gamma(R))) = \lceil \frac{\lambda}{2} \rceil$. □

By Nash-Williams formula we have the following:

Proposition 4.2. *Let R be a finite ring such that $Z(R)$ is not an ideal of R. Then, $\gamma(T(\Gamma(R))) = Max\{\lceil \frac{e_H}{n_H - 1} \rceil\}$, where e_H, n_H denote the number of edges and vertices of subgraph H respectively.*

5. Line Graph of the total graph of ring

The line graph of a graph G is a graph whose vertices are the set of edges of G and two vertices are adjacent if and only if the corresponding edges are adjacent in G. If G is connected, then its line graph is also connected. A connected graph is isomorphic to its line graph if and only if it is a cycle. We state here a theorem due to Ramane et.al [12] which we shall use to prove our result.

Theorem 5.1 ([12])**.** *For a connected graph G diam $L((G)) \leq 2$ if and only if none of the three graphs of figure* (1) *is an induced subgraph of G.*

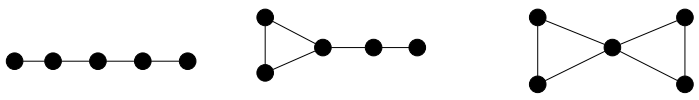


Figure 1:

The maximum matching of a graph is equal to the maximum independent set of its corresponding line graph. So from propositions 2.1 and 2.2, the following holds:

Proposition 5.1. *If R is a finite commutative ring, then*

$$\alpha(L(T(\Gamma(R))) = \begin{cases} \frac{|R|}{2}, & \text{if } |R| \text{ is even} \\ \frac{|R|-1}{2}, & \text{if } |R| \text{ is odd.} \end{cases}$$

Proposition 5.2. *$L(T(\Gamma(R))) \cong T(\Gamma(R))$ if and only if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.*

Proof. We assume $L(T(\Gamma(R))) \cong T(\Gamma(R))$. Then, $T(\Gamma(R))$ is a cycle with $|Z(R)| = 3$ and $|R| \leq 9$. Thus if $|R| = 9$, then $T(\Gamma(R))$ is not regular. Next, if $|R| = 8$, then from the list of Section 5 [13] except $\mathbb{F}_8$ there exist 5 local rings with $|Z(R)| = 4$ and others are non-local rings with $|Z(R)| \geq 4$. When $|R| = 6$ we know $R \cong \mathbb{Z}_6$ with $|Z(R)| = 4$. If $|R| = 4$, then there are two local rings $\mathbb{Z}_4$, $\frac{\mathbb{Z}_2[X]}{(X^2)}$ whose $T(\Gamma(R)) \cong 2K_2$. If R is non local, then $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $T(\Gamma(R)) \cong C_4$. Conversely if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, then $L(T(\Gamma(R))) \cong T(\Gamma(R))$. Hence, the proposition hold. □

Remark 5.1. A line graph is complete iff $G \cong K_3$ *or* $K_{1,n-1}$. So it follows that $L(T(\Gamma(R)))$ is complete iff $R \cong \mathbb{Z}_3$.

Proposition 5.3. *Let R be a finite ring. Then, diam $L(T(\Gamma(R))) =$ diam $T(\Gamma(R)) = 2$ if and only if $R \cong \mathbb{Z}_2^{n+1}$ or $\mathbb{Z}_2^n \times \mathbb{Z}_3$, where n is natural number.*

Proof. Let $R \cong R_1 \times R_2 ... \times R_k$, where $R_i's$ are finite local rings such that diam $L(T(\Gamma(R))) =$ diam $T(\Gamma(R)) = 2$. Then, $T(\Gamma(R))$ is connected and $Z(R)$ is not an ideal.

Case 1. Let $|Reg(R)| \geq 3$ and $u, v \in Reg(R)$ such that $u + v \in Z(R)$. Let $u+v = z$. As diam $T(\Gamma(R)) = 2$, $\exists\, c \in R$ and a path on five vertices $(0)-(-z)-(c)-(u)-(v)$, where 0 and $-z$ are not adjacent to u and v. If $u + v \in Reg(R)$, then $\exists\ z_1,\ z_2 \in Z(R)$ such that $z_1 + z_2 = 1$. Let $u = (u_1, u_2, ..., u_k)$, $v = (v_1, v_2, ..., v_k)$, $z_1 = (1, 0, ..., 0)$ and $z_2 = (0, 1, ..., 1)$. Then, the graph obtained by identifying a vertex of the triangle having vertices $uz_1,\ vz_1,\ 0$ with a vertex of another triangle having vertices $uz_2,\ vz_2,\ 0$ is an induced subgraph with 5 vertices. Hence by theorem 5.1, diam $L(T(\Gamma(R))) \neq 2$.

Case 2. Let $|Reg(R)| \leq 2$. Then, by Lemma 1 [14], $R \cong \mathbb{Z}_3,\ \mathbb{Z}_4, \frac{\mathbb{Z}_2[X]}{(X^2)}, \mathbb{Z}_2^{n+1}$, $\mathbb{Z}_2^n \times \mathbb{Z}_3$, $\mathbb{Z}_2^n \times \mathbb{Z}_4$, $\mathbb{Z}_2^n \times \frac{\mathbb{Z}_2[X]}{(X^2)}$. If $R \cong \mathbb{Z}_2^n \times \mathbb{Z}_4$, then the total graph of $\mathbb{Z}_2 \times \mathbb{Z}_4$ has an induced subgraph obtained by identifying a vertex of the triangle formed by the vertices $(0, 2),\ (0, 0),\ (1, 0)$ with a vertex of another triangle formed by the vertices $(1, 0),\ (1, 1),\ (1, 3)$. Hence, the total graph of $\mathbb{Z}_2^n \times \mathbb{Z}_4$ contains a graph of figure 1. as an induced subgraph. Again if $R \cong \mathbb{Z}_2 \times \frac{\mathbb{Z}_2[X]}{(X^2)}$, then the total graph of R has an induced subgraph obtained by identifying a vertex of the triangle formed by the vertices $(0, x),\ (0, 0),\ (1, 0)$ with a vertex of another

triangle formed by the vertices $(1,0),\ (1,1),\ (1,1+x)$. Therefore, the total graph of $\mathbb{Z}_2^n \times \frac{\mathbb{Z}_2[X]}{(X^2)}$ contains a graph of figure 1. as an induced subgraph. Thus, diam $L(T(\Gamma(R))) \neq 2$. If $R \cong \mathbb{Z}_2^{n+1}$, then $|Reg(R)| = 1$. Suppose $\exists$ a path on 5 vertices $a,\ b,\ c,\ d,\ e$ such that a is not adjacent to d and e, then we get a contradiction as degree of each vertex is $|R| - 2$. So $\nexists$ any induced subgraph of figure 1. Hence, diam $L(T(\Gamma(R))) = 2$. At last, if $R \cong \mathbb{Z}_2^n \times \mathbb{Z}_3$ and if $\exists$ an induced subgraph on 5 vertices $a,\ b,\ c,\ d,\ e$ such that $a,\ b$ are not adjacent to d and e, without loss of generality we can assume $a,\ b \in \{0\} \times \{0\} \times ... \times \mathbb{Z}_6$ and $d,\ e \in \{1\} \times \{1\} \times ... \times \mathbb{Z}_6$. Let $a = (0,0,...,n),\ b = (0,0,...,n'),\ d = (1,1,...,m),\ e = (1,1,...,m')$, where $n,\ n',\ m,\ m' \in \mathbb{Z}_6$. Then, $a+d,\ b+e,\ a+e,\ b+d$=$(1,1,...,5)$ or $(1,1,...,1)$. This leads us to $0 = 2$, which is a contradiction. Thus by theorem 5.1, diam $L(T(\Gamma(R))) = 2$. □

Remark 5.2. From remark 5.1 and proposition 5.3 it follows that there does not exist any ring with diam $L(T(\Gamma(R))) = 1$. If $T(\Gamma(R))$ is connected and $R \ncong \mathbb{Z}_2^{n+1}$ or $\mathbb{Z}_2^n \times \mathbb{Z}_3$, then diam $L(T(\Gamma(R))) = 3$.

6. Crossing number of total graph of some rings

In the following two propositions 6.1 and 6.2 we find crossing number of the total graph of rings with $2 \in Z(R)$ and $2 \notin Z(R)$ with restriction on $|Z(R)|$.

Proposition 6.1. *Let R be a finite local ring with $2 \in Z(R)$ and $|Z(R)| \leq 15$. Then, the total graph of R with crossing number 36 or 144 is isomorphic to one of the following rings*

$$\mathbb{Z}_{16},\ \frac{\mathbb{Z}_2[X]}{(X^4)},\ \frac{\mathbb{Z}_4[X]}{(X^2+2)},$$
$$\frac{\mathbb{Z}_4[X]}{(X^2+3X)},\ \frac{\mathbb{Z}_4[x]}{(X^3-2,2X^2,2X)},\ \frac{\mathbb{Z}_2[X,Y]}{(X^3,XY,Y^2)},\ \frac{\mathbb{Z}_8[x]}{(2X,X^2)},\ \frac{\mathbb{Z}_4[x]}{(X^3,2X^2,2X)},$$
$$\frac{\mathbb{Z}_4[X]}{(X^2+2X)},\ \frac{\mathbb{Z}_8[x]}{(2X,X^2+4)},\ \frac{\mathbb{Z}_2[X,Y]}{(X^2,Y^2-XY)},\ \frac{\mathbb{Z}_4[X,Y]}{(X^2,Y^2-XY,XY-2,2X,2Y)},$$
$$\frac{\mathbb{Z}_4[X,Y]}{(X^2,Y^2,XY-2,2X,2Y)},$$
$$\frac{\mathbb{Z}_2[X,Y]}{(X^2,Y^2)},\ \frac{\mathbb{Z}_4[X]}{(X^2)},\ \frac{\mathbb{Z}_4[X]}{(X^3-X^2-2,2X^2,2X)},$$
$$\frac{\mathbb{Z}_2[X,Y,Z]}{((X,Y,Z)^2)},\ \frac{\mathbb{Z}_4[X,Y]}{(X^2,Y^2,XY,2X,2Y)},\ \frac{\mathbb{F}_8[X]}{(X^2)},\ \frac{\mathbb{Z}_4[X]}{(X^3+x+1)}.$$

Proof. Let $|Z(R)| = \lambda$ and $\frac{|R|}{|Z(R)|} = \beta$. Then, the total graph of the finite local ring R with $2 \in Z(R)$ is $T(\Gamma(R)) = \sqcup_\beta K_\lambda$ by Theorem 2.2 [2]. So, $|R| = 2^k$ since $2 \in Z(R)$ and λ is power of 2. Let $5 \leq \lambda \leq 15$. Then, we must have λ= 2^3 and by Theorem 1 [15] we get $|R| \leq 64$. If $|R| = 64$, then $R \cong \frac{\mathbb{F}_8[X]}{(X^2)}$ or

$\frac{\mathbb{Z}_4[X]}{(X^3+x+1)}$, by discussion in introductory section of [16]. As $T(\Gamma(R)) = \cup\beta K_8$, $cr(T(\Gamma(R))) = \beta cr(K_8) = 8 \times 18 = 144$. Next, if $|R| = 32 = 2^5$ by Theorem 2 [17] $|Z(R)| = 2^{(5-1)1} = 16$ or 1. So, $\nexists$ any local ring of order 32 with $|Z(R)| = 8$. If $|R| = 16$, by the list in Section 5 [13] we have 18 non isomorphic rings which are as follows:

$$\mathbb{Z}_{16},\ \frac{\mathbb{Z}_2[X]}{(X^4)},\ \frac{\mathbb{Z}_4[X]}{(X^2+2)},\ \frac{\mathbb{Z}_4[X]}{(X^2+3X)},$$
$$\frac{\mathbb{Z}_4[x]}{(X^3-2, 2X^2, 2X)},\ \frac{\mathbb{Z}_2[X,Y]}{(X^3, XY, Y^2)},\ \frac{\mathbb{Z}_8[x]}{(2X, X^2)},\ \frac{\mathbb{Z}_4[x]}{(X^3, 2X^2, 2X)},$$
$$\frac{\mathbb{Z}_4[X]}{(X^2+2X)},\ \frac{\mathbb{Z}_8[x]}{(2X, X^2+4)},\ \frac{\mathbb{Z}_2[X,Y]}{(X^2, Y^2-XY)},$$
$$\frac{\mathbb{Z}_4[X,Y]}{(X^2, Y^2-XY, XY-2, 2X, 2Y)},\ \frac{\mathbb{Z}_4[X,Y]}{(X^2, Y^2, XY-2, 2X, 2Y)},$$
$$\frac{\mathbb{Z}_2[X,Y]}{(X^2, Y^2)},\ \frac{\mathbb{Z}_4[X]}{(X^2)},\ \frac{\mathbb{Z}_4[X]}{(X^3-X^2-2, 2X^2, 2X)},$$
$$\frac{\mathbb{Z}_2[X,Y,Z]}{((X,Y,Z)^2)},\ \frac{\mathbb{Z}_4[X,Y]}{(X^2, Y^2, XY, 2X, 2Y)}$$

So, $T(\Gamma(R)) = \cup\beta K_8 \Rightarrow cr(T(\Gamma(R))) = \beta cr(K_8) = 2 \times 18 = 36$. □

Proposition 6.2. *Let R be a finite local ring with $2 \notin Z(R)$ and $|Z(R)| \leq 7$. Then, the total graph of R with crossing number 1 or 33 or 171 is isomorphic to one of the following rings $\mathbb{Z}_9$, $\frac{\mathbb{Z}_3[X]}{(X^2)}$, $\mathbb{Z}_{25}$, $\frac{\mathbb{Z}_5[X]}{(X^2)}$, $\mathbb{Z}_{49}$, $\frac{\mathbb{Z}_7[X]}{(X^2)}$.*

Proof. Let $|Z(R)| = \lambda$ and $\frac{|R|}{|Z(R)|} = \beta$. Then, by Theorem 2.2 [2] the total graph of the finite local ring R with $2 \notin Z(R)$ is $T(\Gamma(R)) = K_\lambda \cup \frac{(\beta-1)}{2} K_{\lambda,\lambda}$. So, if $\exists\ x(\neq 0) \in \frac{R}{Z(R)}$ such that $2x = 0$, then $2x \in Z(R)$ which is a contradiction as $Z(R)$ is a prime ideal. Thus, every non zero element has distinct inverse such that $\frac{|R|}{|Z(R)|}$ is odd and by Theorem 2 [17], $|R|$ is power of odd prime. Let $3 \leq \lambda \leq 7$. Then, we must have λ= 3 or 5 or 7. By the discussion, in Section 3 [13] for $p = 3$ or 5 or 7, R is isomorphic to $\mathbb{Z}_{p^2}$ and $\frac{\mathbb{Z}_p[x]}{(X^2)}$.

Now, $cr(T(\Gamma(R))) = cr(K_\lambda) + \frac{(\beta-1)}{2}$ cr($K_{\lambda,\lambda}$). Therefore, for λ= 3 we find that $cr(T(\Gamma(R))) = 1$, for $\lambda = 5$, $cr(T(\Gamma(R))) = 33$ and for $\lambda = 7$, $cr(T(\Gamma(R))) = 171$. □

Remark 6.1. Further, if the conjecture of the crossing number for the complete and complete bipartite graph is solved, then a generalization to the above results are as follows:

- If R is a finite local ring with $2 \in Z(R)$, then the crossing number of $T(\Gamma(R)) = \beta\ cr(K_\lambda)$ and also if R is a finite local ring with $2 \notin Z(R)$, then the crossing number of $T(\Gamma(R)) = cr(K_\lambda) + \frac{(\beta-1)}{2}\ cr(K_{\lambda,\lambda})$.

7. $T(\Gamma(R))$ with genus atmost 2

The total graph of rings upto isomorphism with genus atmost 2 are
$\mathbb{Z}_2 \times \mathbb{F}_4$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \frac{\mathbb{Z}_2[X]}{(X^2)}$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_{10}$, $\mathbb{Z}_3 \times \mathbb{F}_4$.
In this section we find the crossing number of $T(\Gamma(R))$ for these rings. We have $T(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_4)) \cong T(\Gamma(\mathbb{Z}_2 \times \frac{\mathbb{Z}_2[X]}{(X^2)}))$. The graph in Figure 2. is drawn by plotting $2K_4$ and adding other edges. Let $I = \{(0,0),(0,1),(0,x)(0,x+1)\}$ be the maximal ideal in $Z(R)$ with $2 \in I$. We first construct the graph by taking $x, y \in R$ as vertices and x and y are adjacent whenever $x + y$ belongs to I. Therefore, $T(\Gamma(I \cup \frac{R}{I})) = K_4 \cup K_4$ which is the spanning subgraph of $T(\Gamma(R))$. Next, we add the remaining edges to obtain the total graph. This shows that $cr(\mathbb{Z}_2 \times \mathbb{F}_4) = 2$.

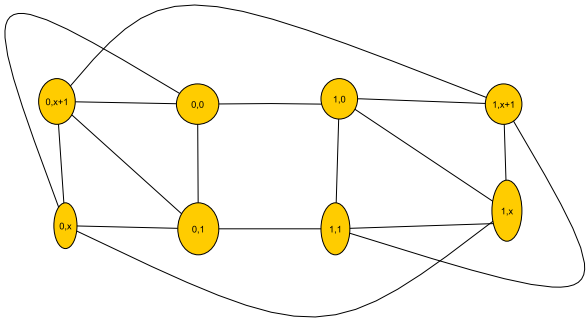


Figure 2: $cr(\mathbb{Z}_2 \times \mathbb{F}_4) = 2$

Similarly, in Figure 3. we draw $2K_4$ taking $I = \{(0,0),(0,1),(0,2),(0,3)\}$ to get the spanning subgraph of $T(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_4))$ and add the remaining edges to obtain the total graph. This shows that $cr(\mathbb{Z}_2 \times \mathbb{Z}_4) = 4$.

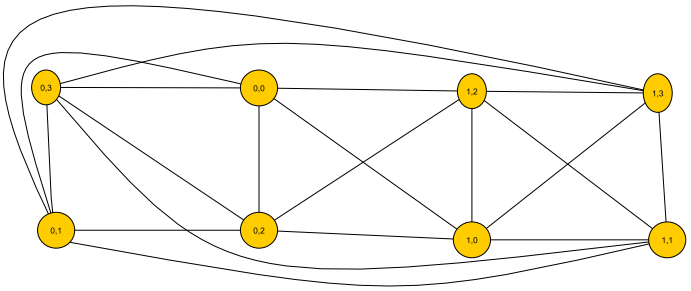


Figure 3: $cr(\mathbb{Z}_2 \times \mathbb{Z}_4) = 4$

In Figure 4, first, we draw $K_3 \cup K_{3,3}$ taking $I = \{(0,0),(0,1),(0,2)\}$ to get the spanning subgraph of $T(\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3))$ and add the remaining edges to obtain the total graph. This shows that $cr(\mathbb{Z}_3 \times \mathbb{Z}_3) = 6$.

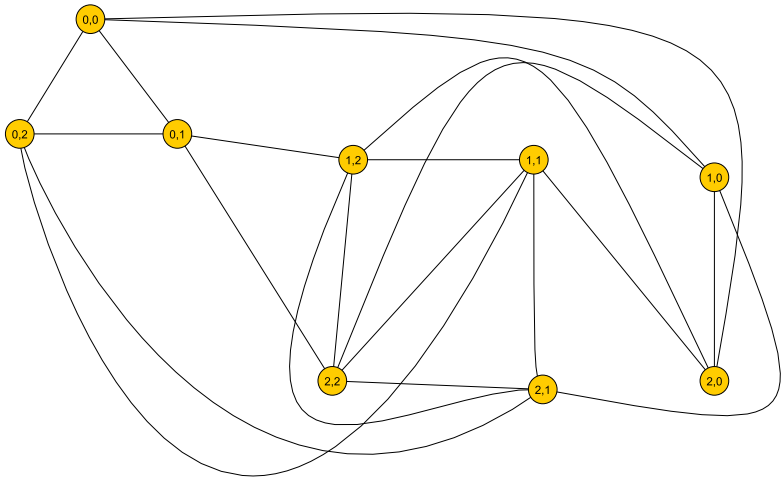


Figure 4: $cr(\mathbb{Z}_3 \times \mathbb{Z}_3) = 6$

Next, Figure 5 is a complete 4 partite graph with each partite set containing 2 elements and the resultant $cr(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) = 10$.

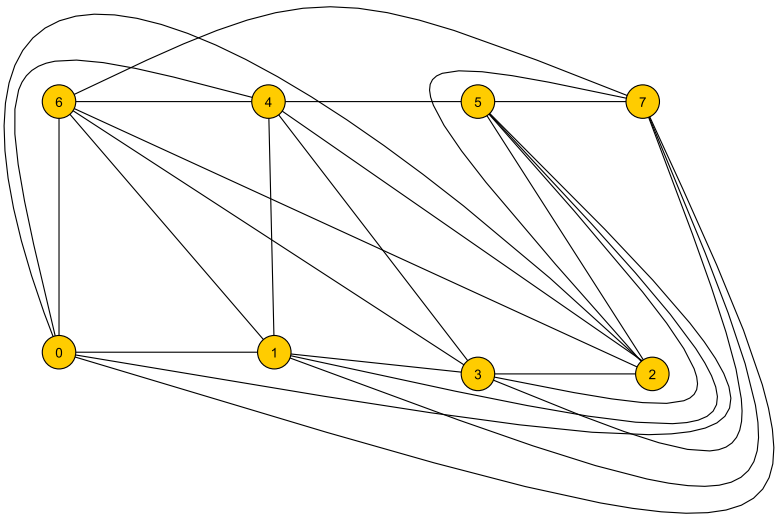


Figure 5: $cr(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) = 10$

In Figure 6, we take $I = \{0,2,4,6,8\}$ and draw $2K_5$ to get the spanning subgraph of $T(\Gamma(\mathbb{Z}_{10}))$ and add the remaining edges to obtain the total graph. This shows that $cr(\mathbb{Z}_{10}) = 8$.

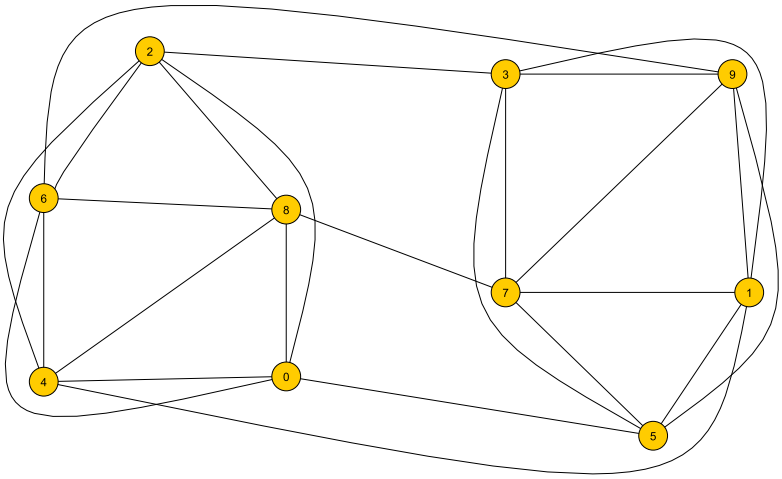


Figure 6: $cr(\mathbb{Z}_{10}) = 8$

At last, in Figure 7. we take $I = \{(0,0),(0,1),(0,x),(0,x+1)\}$ and draw $K_4 \cup K_{4,4}$ which is spanning subgraph of $T(\Gamma(\mathbb{Z}_3 \times \mathbb{F}_4))$ and then add the remaining edges to form the total graph. This shows that $cr(\mathbb{Z}_3 \times \mathbb{F}_4) = 13$.

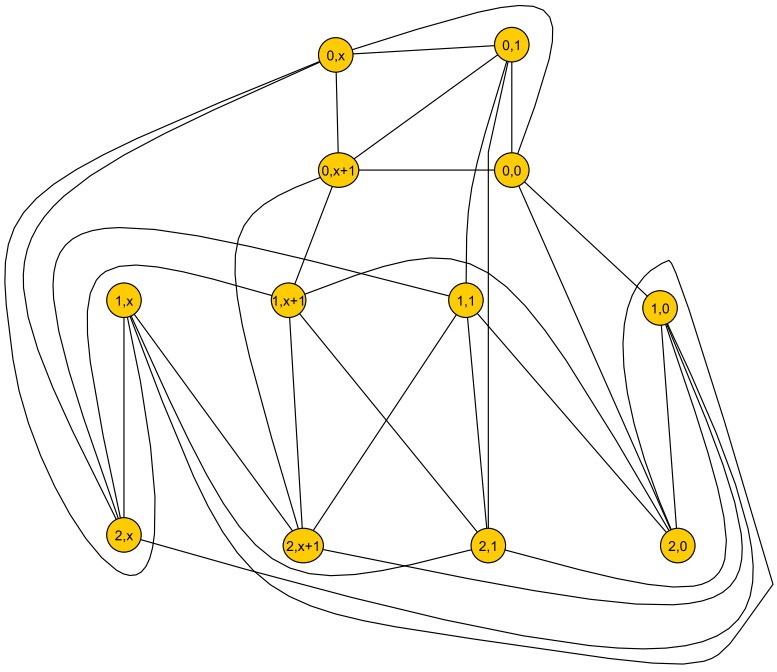


Figure 7: $cr(\mathbb{Z}_3 \times \mathbb{F}_4) = 13$

We conclude that the results may be useful for characterization of other algebraic objects such as modules over a ring, vector spaces over a field.

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Photothermoelastic deformation in dual phase lag model due to concentrated inclined load

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Abstract. Existing problem is regarding the deformation of photothermoelastic solid with dual phase lag (DPL) by applying inclined load.Here inclined is considered as the consolidation of horizontal and vertical strengths. The displacement, stress, temperature distribution and carrier density are obtained by using Laplace and Fourier transform. These are also computed numerically for silicon material and presented graphically to analyze the photothermal effect on considered physical variables. Notable cases are drawn out from this problem.

Keywords: photo-thermal, semiconductor, inclined load, Laplace and Fourier transforms.

1. Introduction

Generalized thermoelasticity have many applications in various fields of engineering including soil dynamics, oil extraction, mineral exploration, earthquake prediction and many more. With the advancement in the technologies, semiconducting materials (as silicon (Si)) were used in various branches of engineering and material science.

Over the past few years, the study of semiconductor materials through both photoacoustic and photothermal technologies [1,2] are regarded as accepted approach. Mandelis and Hess [3] explored the effective personification in the analysis of photoacoustic and photothermal technologies. The new law of heat conduction take the place of the classical Fourier law and was established as the generalized thermoelastic theory with one relaxation time suggested by Lord and Shulman [4]. Green and Lindsay [5] popularized another generalized thermoelasticity theory with two relaxation times. For an anisotropic body the theories

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of generalized thermoelasticity were continued by Sherief and Dhaliwal [6]. Tzou [7] examined the heat conduction from imperceptible to perceptible extend. The analytical and empirical sense of the Dual phase lag model have been supported in Tzou [8]. Abbas and Zenkor [9] explored the deformation problem with special function applied on temperature in dual phase lag model. Numerical study of heat transfer and Hall current impact on peristaltic propulsion of particle-fluid suspension with compliant wall properties have been examined by Bhatti et al. [10]. Many authors [11-13] studied the effects of electronic deformation in photothermoelastic materials.

Zenkor, Abouelregal and Aifantis [14] discussed the deformation problem in two-temperature dual-phase-lags theory due to inclined load. Marin, Agarwal and Codarcea [15] studied a mathematical model for three-phase-lag dipolar thermoelastic bodies. By applying fractional order photo-thermoelasticity, Hobiny and Abbas [16] studied problem of wave propagation in photothermoelastic medium. Photothermal interactions with dual phase lag in a semiconductor medium containing a sphere with a hole using spherical polar coordinates was discussed by Hobiny and Abbas [17]. Lotfy [18] studied photo-thermal elastic waves in semiconductor medium with pluse heat flux. The study of Photo-thermal-elastic waves in a functionally graded material was investigated by Lotfy and Tantawi [19].

Different authors [20-24] discussed different problems in various mediums such as elastic, thermoelastic and micropolar elastic by applying inclined load. Regardless of this, the problem of photothermoelastic medium due to inclined load was not done before.

2. Basic equations

Following [4, 7, 25, 26] as

$$K\left(1+t_T\frac{\partial}{\partial t}\right)\theta_{,jj}$$

$$(1)\qquad =-\frac{E_g}{\tau}N+\left(1+t_q\frac{\partial}{\partial t}+m\frac{t_q^2}{2}\frac{\partial^2}{\partial t^2}\right)\left(\rho C_e\frac{\partial\theta}{\partial t}+\gamma_t T_0\frac{\partial u_{j,j}}{\partial t}\right),$$

$$(2)\qquad D_eN_{,jj}=\frac{\partial N}{\partial t}+\frac{N}{\tau}-k\frac{\theta}{\tau},$$

$$(\lambda+\mu)u_{j,ij}+\mu u_{i,jj}-\gamma_n N_{,i}-\gamma_t\theta_{,i}=\rho\ddot{u}_i,$$

$$(3)\qquad \sigma_{ij}=\left(\lambda u_{j,j}-\gamma_n N-\gamma_t\theta\right)\delta_{ij}+\mu\left(u_{i,j}+u_{j,i}\right),$$

$0\prec t_T\prec t_q, m=1$ (DPL model), $t_q=t_0\succ 0, t_T=m=0$ (LS theory), $t_q=t_T=m=0$ (CT theory), where $N=n-n_0$, n_0-initial carrier concentration, $\theta=T-T_0$, T_0-initial temperature, u_i-displacement components,$t_q$and t_T-times, t_0-thermal relaxation time, C_e- specific heat, K-thermal conductivity, ρ-density, σ_{ij}-stress tensors, $\gamma_t=(3\lambda+2\mu)\,\alpha_t$, α_t is the linear thermal expansion coefficient, D_e-carrier diffusion coefficient, λ and μ-Lame's constants,

τ-photo-generated carrier lifetime, E_g-semiconductor energy gap, E-excitation energy, $\gamma_n = (3\lambda + 2\mu)\, d_n$, d_n-electronic deformation coefficient and $k = \frac{\partial n_0}{\partial \theta}$, t-time. A superposed dot represents differentiation with respect to time variable t. δ_{ij}-kronecker delta.

3. Construction of the problem and solution methodology

A semiconductor medium with reference temperature T_0 permeated by an inclined line load F_0 acting on the y-axis is considered Figure (a). For two dimensional problem.

The components of displacement vector $\vec{u}$as $u_1 = u_1(x, z, t)$ and $u_3 = u_3(x, z, t)$.

N as $N = N(x, z, t)$, θ as $\theta = \theta(x, z, t)$.

In two dimensional $x - z$ plane (1)-(3) can be written as

$$K\left(1 + t_T \frac{\partial}{\partial t}\right)\left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial z^2}\right)$$

$$= -\frac{E_g}{\tau} N + \left(1 + t_q \frac{\partial}{\partial t} + m \frac{t_q^2}{2} \frac{\partial^2}{\partial t^2}\right)\left(\rho C_e \frac{\partial \theta}{\partial t} + \gamma_t T_0 \left(\frac{\partial^2 u_1}{\partial t \partial x} + \frac{\partial^2 u_3}{\partial t \partial z}\right)\right), \tag{4}$$

$$D_e\left(\frac{\partial^2 N}{\partial x^2} + \frac{\partial^2 N}{\partial z^2}\right) = \frac{\partial N}{\partial t} + \frac{N}{\tau} - k\frac{\theta}{\tau}, \tag{5}$$

$$(\lambda + 2\mu)\frac{\partial^2 u_1}{\partial x^2} + \mu \frac{\partial^2 u_1}{\partial z^2} + (\lambda + \mu)\frac{\partial^2 u_3}{\partial x \partial z} - \gamma_n \frac{\partial N}{\partial x} - \gamma_t \frac{\partial \theta}{\partial x} = \rho \frac{\partial^2 u_1}{\partial t^2}, \tag{6}$$

$$(\lambda + 2\mu)\frac{\partial^2 u_3}{\partial z^2} + \mu \frac{\partial^2 u_3}{\partial x^2} + (\lambda + \mu)\frac{\partial^2 u_1}{\partial x \partial z} - \gamma_n \frac{\partial N}{\partial z} - \gamma_t \frac{\partial \theta}{\partial z} = \rho \frac{\partial^2 u_3}{\partial t^2}. \tag{7}$$

To smoothen the solution, the non-dimensional variables are proposed

$$\left(x', z', u_1', u_3'\right) = \eta C\left(x, z, u_1, u_3\right), \left(t_{11}', t_{31}', t_{33}'\right) = \frac{1}{\mu}\left(t_{11}, t_{31}, t_{33}\right),$$

$$\theta' = \frac{\theta}{T_0}, \left(t', \tau', \tau_0'\right) = \eta c^2 \left(t, \tau, \tau_0\right), N' = \frac{N}{n_0}, F_n' = \frac{F_n}{\mu}, F_t' = \frac{F_t}{\mu}, \tag{8}$$

where $c^2 = \frac{\lambda + 2\mu}{\rho}, \eta = \frac{\rho C_E}{K}$.

In non-dimensional form, (4)-(7) takes the form

$$\left(1 + t_T \frac{\partial}{\partial t}\right)\left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial z^2}\right)$$

$$= -\in_2 \frac{N}{\tau} + \left(1 + t_q \frac{\partial}{\partial t} + m \frac{t_q^2}{2} \frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial \theta}{\partial t} + \in_1 \left(\frac{\partial^2 u_1}{\partial t \partial x} + \frac{\partial^2 u_3}{\partial t \partial z}\right)\right), \tag{9}$$

$$\left(\frac{\partial^2 N}{\partial x^2} + \frac{\partial^2 N}{\partial z^2}\right) = \alpha \frac{\partial N}{\partial t} + \alpha \frac{N}{\tau} - \beta \frac{\theta}{\tau}, \tag{10}$$

$$\frac{\partial^2 u_1}{\partial x^2} + g_2 \frac{\partial^2 u_1}{\partial z^2} + g_1 \frac{\partial^2 u_3}{\partial x \partial z} - \beta_n \frac{\partial N}{\partial x} - \beta_t \frac{\partial \theta}{\partial x} = \frac{\partial^2 u_1}{\partial t^2}, \tag{11}$$

$$\frac{\partial^2 u_3}{\partial z^2} + g_2 \frac{\partial^2 u_3}{\partial x^2} + g_1 \frac{\partial^2 u_1}{\partial x \partial z} - \beta_n \frac{\partial N}{\partial z} - \beta_t \frac{\partial \theta}{\partial z} = \frac{\partial^2 u_3}{\partial t^2} \tag{12}$$

where $g_1 = \frac{\lambda+\mu}{\lambda+2\mu},\ g_1 = \frac{\mu}{\lambda+2\mu},\ \ \beta_n = \frac{\gamma_n n_0}{\lambda+2\mu},\ \beta_n = \frac{\gamma_t T_0}{\lambda+2\mu},\ \alpha = \frac{1}{D_e \eta},\ \beta = \frac{kT_0}{D_e \eta n_0},$

$$\varepsilon_1 = \frac{\gamma_t}{K\eta}, \quad \varepsilon_2 = \frac{E_g n_0}{T_0 \eta K}.$$

The expressions relating displacement components $u_1(x,z,t)$and$u_3(x,z,t)$to the scalar potential functions$\phi(x,z,t)$and $\psi(x,z,t)$ in a dimensionless form are given by

$$u_1 = \frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial z}, u_3 = \frac{\partial \phi}{\partial z} + \frac{\partial \psi}{\partial x}. \tag{13}$$

Laplace and Fourier transforms determined as

$$\bar{f}(x,z,s) = \int_0^\infty f(x,z,t) e^{-st} dt \tag{14}$$

$$\hat{f}(\xi,z,s) = \int_{-\infty}^\infty \bar{f}(x,z,s)\ e^{i\xi x}, \tag{15}$$

applied on (9)-(12) along with (13) yields

$$\left\{ \left(\frac{d^2}{\partial z^2} - \xi^2 \right) (1 + t_T s) - \left(1 + t_q s + m \frac{t_q^2}{2} s^2 \right) s \right\} \hat{\theta}$$

$$+ \in_2 \frac{\hat{N}}{\tau} - \left\{ \in_1 s \left(\frac{d^2}{dz^2} - \xi^2 \right) \left(1 + t_q s + m \frac{t_q^2}{2} s^2 \right) \right\} \hat{\phi} = 0, \tag{16}$$

$$\left(\frac{\partial^2}{\partial z^2} - \xi^2 - \alpha s - \frac{\alpha}{\tau} \right) \hat{N} + \frac{\beta}{\tau} \hat{\theta} = 0, \tag{17}$$

$$\left\{ \frac{d^2}{\partial z^2} - \left(\xi^2 + s^2 \right) \right\} \hat{\phi} - \beta_n \hat{N} - \beta_t \hat{\theta} = 0, \tag{18}$$

$$\left\{ \frac{d^2 \hat{\psi}}{\partial z^2} - \left(\xi^2 + \frac{s^2}{g_2} \right) \right\} \hat{\psi} = 0. \tag{19}$$

After simplification

$$\left(\frac{d^6}{dz^6} + L \frac{d^4}{dz^4} + M \frac{d^2}{dz^2} + N \right) (\hat{\phi}, \hat{\theta}, \hat{N}) = 0, \tag{20}$$

$$\left(\frac{d^2}{dz^2} - \lambda_4^2 \right) \hat{\psi} = 0, \tag{21}$$

where L, M and N are listed (Appendix A).

$\pm\lambda_j$ $(j = 1, 2, 3)$ specifies roots of (20) whereas$\pm\lambda_4$ give roots of (21).
The initial and regularity conditions are given by

$$\begin{aligned}
&u_1(x, z, 0) = 0 = \dot{u}_1(x, z, 0),\\
&u_3(x, z, 0) = 0 = \dot{u}_3(x, z, 0),\\
&\theta(x, z, 0) = 0 = \dot{\theta}(x, z, 0),\\
&N(x, z, 0) = 0 = \dot{N}(x, z, 0) \text{ for } z \geq 0, -\infty < x < \infty,\\
&u_1(x, z, t) = u_3(x, z, t) = \theta(x, z, t) = N(x, z, t) = 0,
\end{aligned}$$

for $t > 0$ when $z \to \infty$.

Using the radiation condition $\hat{\phi}, \hat{\theta}, \hat{N}$ and $\hat{\psi} \to 0$ as $z \to \infty$, the solution of (20) and (21) takes the form

$$\hat{\phi} = A_1 e^{-\lambda_1 z} + A_2 e^{-\lambda_2 z} + A_3 e^{-\lambda_3 z}, \tag{22}$$

$$\hat{\theta} = s_1 A_1 e^{-\lambda_1 z} + s_2 A_2 e^{-\lambda_2 z} + s_3 A_3 e^{-\lambda_3 z}, \tag{23}$$

$$\hat{N} = t_1 A_1 e^{-\lambda_1 z} + t_2 A_2 e^{-\lambda_2 z} + t_3 A_3 e^{-\lambda_3 z}, \tag{24}$$

$$\hat{\psi} = A_4 e^{-\lambda_4 z}. \tag{25}$$

$A_i (i = 1, 2, 3, 4)$ arbitrary constants. The coupling constants $s_l$and t_l are listed (Appendix B) .

4. Boundary conditions

$$\begin{aligned}
&(i)\quad t_{33} = -F_n \phi_1(x)\delta(t), \quad (ii)\quad t_{31} = -F_t \phi_2(x)\delta(t),\\
&(iii)\quad \theta = 0\,, \qquad\qquad\qquad (iv)\quad N = 0,
\end{aligned} \tag{26}$$

where F_n and F_t-magnitude of forces, $\delta(t)$-Kronecker delta function, $\phi_1(x)$-vertical load distribution $\phi_2(x)$- horizontal load distribution.

Making use of (3), (8) and (13) in (26), applying (14)-(15) then substituting the values of $\hat{\phi}, \hat{\theta}, \hat{N}$and $\hat{\psi}$ from (22)-(25) in emerging mathematical statements, we acquire displacement, stress, temperature distribution and carrier density in Appendix C (C.1)-(C.6).

4.1 Concentrated force

In this case

(*) $\phi_1 = \delta(x), \phi_2(x) = \delta(x)$, with $\hat{\phi}_1(\xi) = 1, \hat{\phi}_2(\xi) = 1$.

Using (*) in (C.1)-(C.6).the required quantities are established for this source.

5. Utilization

F_0 Inclined load with angle of inclination δ is applied on y-axis

(**) $F_n = F_0 \cos\delta, F_t = F_0 \sin\delta.$

Using equation (**) in equations (C.1)-(C.6) and along with (*), we find the components for concentrated force applied on the surface of photothermoelastic medium (semiconductor medium).

6. Particular case

Leaving the photothermal reaction (i.e., $E_g = 0$, D_e=0, τ=0), in (C.1)-(C.6), with (*) and (**) we find the components in generalized thermoelastic half-space.

7. Inversion of the transformation

To acquire the results in the physical domain, invert the transforms in (C.1)-(C.6) by

$$(27)\quad \bar{f}(x,z,s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi x}\hat{f}(\xi,z,s)d\xi = \frac{1}{\pi}\int_{0}^{\infty}(\cos(\xi x)f_e - i\sin(\xi x)f_0)d\xi,$$

where f_e and f_0 are, respectively, even and odd parts of the function $\hat{f}(\xi,z,s)$.

8. Numerical results and discussion

We choose silicon material for numerical purpose to show the theoretical results as by [27] $C_e = 695Jkg^{-1}k^{-1}$, $\mu = 5.46\times10^{10}Nm^{-2}$, $\lambda = 3.64\times10^{10}Nm^{-2}$, $E_g = 1.11eV$, $s_0 = 2ms^{-1}$, $E = 2.33eV$, $D_e = 2.5\times10^{-3}m^2s^{-1}$, $\alpha_t = 3\times10^{-6}k^{-1}$, $K = 150Wm^{-1}k^{-1}$, $\rho = 2330kgm^{-3}$, $\tau = 5\times10^{-5}s$, $n_0 = 10^{20}m^{-3}$, $d_n = -9\times10^{-31}m^3$, $T_0 = 300k$, u_3, t_{33}, θ and N for Dual phase phtothermoelasticity (DP) and without photothermelasticity with dual phase(WPD) (for $\delta = 0^0$ and $\delta = 90^0$ and variations of u_3, t_{33}, θ and N with x. Taking $t = 0.5$ (dimensionless). The solid line and the small dashed without center symbol predict the variations DP for different values of δ whereas the solid and the small dashed and with center symbol predict the variations of WDP for different $\delta's$.

Figure 1 it is noted that for WPD, u_3 starts with its maximum values at $\delta = 0^0$ and shows small variation at $\delta = 90^0$ whereas for DP it follows oscillatory pattern at both angles. Gradually u_3 decrease to lowest values for all the cases. The values of u_3 for WPD are greater DP in the entire range at both angles.

Figure 2 t_{33} follows an oscillatory pattern for both DP and WPD at both initial and extreme angles of inclination. Further the values of t_{33} terminates to zero obeying the boundary conditions. Near the loading surface, t_{33} for WPD at extreme angle is greater than all other cases but with advancement in$x$, attains minimum values with oscillatory pattern.

It is evident from Figure 3 that the trend of variation θ is similar for both DP and WPD at both $\delta's$. Near the loading surface at $\delta = 0^0$the values of $\theta$for

DP is more than for WPD whereas the behavior is reverse at $\delta = 90^0$. Further with increase in x the values start oscillating but magnitude of oscillation is small for DP, showing the photothermal effect. Also, with advancement in x the curves converges to boundary.

The variation of carrier density N as in fig.4.At the point where source is applied, for both initial and extreme angel there is a sharp decrease in N. With increase in x, at $\delta = 0^0$its start increasing in $2 \leq x \leq 5$then follow oscillatory pattern whereas at $\delta = 90^0$ shows small variations near zero.

9. Conclusion

A two-dimensional problem of phothothermoelastic model with dual phase lag has been examined by employing dimensionless quantities. The problem is simplified by using potential functions and then by applying Laplace and Fourier transform to examine the problem. The deformation due to inclined load has been considered to display the application of the problem. From the numerical result obtained the following conclusions are drawn:

1. As viewed figures that photothermo have oscillatory result on the physical quantities obtained after computational process.

2. It is noticed that the field quantities are affected by different angle of inclination with and without dual phage lag.

3. The results obtained in this study are advantageous to analyst involved in material science, engineering, physicists in addition to people active in the area of continuum mechanics.

Appendix A

L, M and N as:

$$L = \frac{1}{E}[F - 3\xi^2 E] \ , \ \ M = \frac{1}{E}[G - 2F\xi^2 + 3\xi^4 E],$$
$$N = \frac{1}{E}[F\xi^4 - G\xi^2 + H - E\xi^6], \lambda_4^2 = \xi^2 + \frac{s^2}{g_2},$$
$$E = n_1, F = n_2 s\left(-1 + \varepsilon_1 \beta_t\right) + n_1\left(\frac{\alpha}{\tau} - \alpha s + s^2\right),$$
$$H = \frac{\beta \varepsilon_2 s^2}{\tau^2} + \frac{\alpha n_2 s^2}{\tau} + \alpha s^3 n_2,$$
$$G = \frac{\beta}{\tau}\left(-\varepsilon_2 + \varepsilon_1 s n_2 \beta_n\right) - \alpha n_2\left(s + \frac{1}{\tau}\right) + n_2 s^3 + n_1 \alpha s^2\left(s - \frac{1}{\tau}\right)$$
$$+ \varepsilon_1 s n_2 \beta_t\left(1 - \alpha s + \frac{1}{\tau}\right),$$

$$n_1 = (1 + t_T s)\,, n_2 = \left(1 + t_q s + m s^2 \frac{t_q^2}{2}\right).$$

Appendix B

s_l and t_l are $s_l = \frac{P^*\lambda_l^2 + Q^*}{R^*\lambda_l^2 + S^*}, t_l = \frac{U^*\lambda_l^2 + V^*}{X^*\lambda_l^2 + T^*}, l = 1, 2, 3,$

$$\begin{aligned}
P^* &= \frac{a_1 + 1}{a_1}, U^* = n_1, Q^* = -\left[(\xi^2 + s^2) + \frac{\xi^2}{a_1}\right], \\
V^* &= \xi^2\,(-n_1 + n_2) - \frac{\varepsilon_2 \beta_t}{\tau \beta_n}, \\
T^* &= -\frac{\varepsilon_2 \xi^2}{\beta_n \tau}\,(1 + \varepsilon_1 s n_2)\,, \\
R^* &= \frac{1}{\varepsilon_1 a_1 s(1 + \tau_0 s)}, X^* = \frac{\varepsilon_2}{\beta_n \tau}\left(s^2 + \varepsilon_1 s n_2\right), \\
S^* &= 1 - \frac{1}{\varepsilon_1 a_1 s(1 + \tau_0 s)}(\xi^2 + (1 + \tau_0 s)s).
\end{aligned}$$

Appendix C

Displacement, stress, and temperature distribution and carrier density are:

(C.1) $\hat{u}_1 = \frac{1}{\Delta}\{F_n\hat{\phi}_1(\xi)[(-i\xi)(\Delta_1 e^{-\lambda_1 z} - \Delta_3 e^{-\lambda_2 z} + \Delta_5 e^{-\lambda_3 z}) + \lambda_4\Delta_7 e^{-\lambda_4 z}] + F_2\hat{\phi}_2(\xi)[(-i\xi)(\Delta_2 e^{-\lambda_1 z} - \Delta_4 e^{-\lambda_2 z} + \Delta_6 e^{-\lambda_3 z}) - \lambda_4\Delta_8 e^{-\lambda_4 z}]\};$

(C.2) $\hat{u}_3 = \frac{-1}{\Delta}\{F_n\hat{\phi}_1(\xi)(\xi)[\lambda_1\Delta_1 e^{-\lambda_1 z} - \lambda_2\Delta_3 e^{-\lambda_2 z} + \lambda_3\Delta_5 e^{-\lambda_3 z}) + i\xi\Delta_7 e^{-\lambda_4 z}] + F_t\hat{\phi}_2(\xi)[\lambda_1\Delta_2 e^{-\lambda_1 z} - \lambda_2\Delta_4 e^{-\lambda_2 z} + \lambda_3\Delta_6 e^{-\lambda_3 z}) - i\xi\Delta_8 e^{-\lambda_4 z}]\};$

(C.3) $\hat{t}_{33} = \frac{1}{\Delta}\{F_n\hat{\phi}_1(\xi)(\xi)[s_1\Delta_1 e^{-\lambda_1 z} - s_2\Delta_3 e^{-\lambda_2 z} + s_3\Delta_5 e^{-\lambda_3 z} + s_4\Delta_7 e^{-\lambda_4 z}] + F_t\hat{\phi}_2(\xi)\xi)[s_1\Delta_2 e^{-\lambda_1 z} - s_2\Delta_4 e^{-\lambda_2 z} + s_3\Delta_6 e^{-\lambda_3 z} - s_4\Delta_8 e^{-\lambda_4 z}]\};$

(C.4) $\hat{t}_{31} = \frac{1}{\Delta}\{F_n\hat{\phi}_1(\xi)[\lambda_1\Delta_1 e^{-\lambda_1 z} - \lambda_2\Delta_3 e^{-\lambda_2 z} + \lambda_3\Delta_5 e^{-\lambda_3 z} - m_1\Delta_7 e^{-\lambda_4 z}] + F_t\hat{\phi}_1(\xi)[\lambda_1\Delta_2 e^{-\lambda_1 z} - \lambda_2\Delta_4 e^{-\lambda_2 z} + \lambda_3\Delta_6 e^{-\lambda_3 z} + m_1\Delta_8 e^{-\lambda_4 z}]\};$

(C.5) $\hat{\theta} = \frac{1}{\Delta}\{F_n\hat{\phi}_1(\xi)[d_1\Delta_1 e^{-\lambda_1 z} - d_2\Delta_3 e^{-\lambda_2 z} + d_3\Delta_5 e^{-\lambda_3 z}] + F_t\hat{\phi}_2(\xi)[d_1\Delta_2 e^{-\lambda_1 z} - d_2\Delta_4 e^{-\lambda_2 z} + d_3\Delta_6 e^{-\lambda_3 z}]\};$

(C.6) $\hat{N} = \frac{1}{\Delta}\{F_n\hat{\phi}_1(\xi)[t_1\Delta_1 e^{-\lambda_1 z} - t_2\Delta_3 e^{-\lambda_2 z} + t_3\Delta_5 e^{-\lambda_3 z}] + F_t\hat{\phi}_2(\xi)(\xi)[t_1\Delta_2 e^{-\lambda_1 z} - t_2\Delta_4 e^{-\lambda_2 z} + t_3\Delta_6 e^{-\lambda_3 z}]\}.$

where

$$\begin{aligned}
\Delta &= \mu\{[-s_4\lambda_1 - m_1 s_1]\Delta_{10} + [s_4\lambda_2 + m_1 s_2]\Delta_{20} \\
&\quad + [-s_4\lambda_3 - m_1 s_3]\Delta_{30}\}, \\
\Delta_{1,2} &= (\mu\lambda_3, s_4)\Delta_{10}, \Delta_{3,4} = (\mu\lambda_4, s_4)\Delta_{20}, \\
\Delta_{5,6} &= (\mu\lambda_4, s_4)\Delta_{30}, \Delta_7 = \mu[\lambda_1\Delta_{10} - \lambda_2\Delta_{20} + \lambda_3\Delta_{30}], \\
s_l &= b_1\lambda_l^2 - b_2 i\xi - b_3 e_l - b_4 d_l, \quad (l = 1, 2, 3), s_4 = (i\xi b_1 + b_2)\lambda_4, \\
b_1 &= \frac{(\lambda + 2\mu)}{\mu}, b_2 = \frac{\lambda}{\mu}, b_3 = \frac{\gamma_n n_0}{\mu}, b_4 = \frac{\gamma_t T_0}{\mu}, m_1 = \frac{\lambda_4^2 + \xi^2}{2i\xi}.
\end{aligned}$$

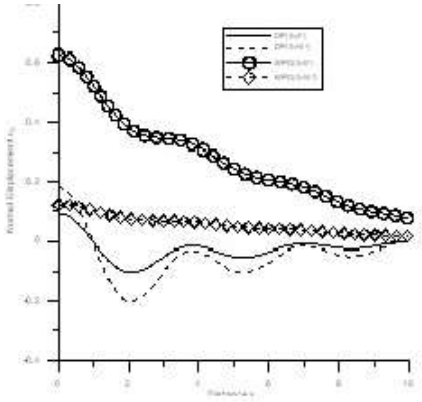


Figure 1: Variation of Normal Displacement with x

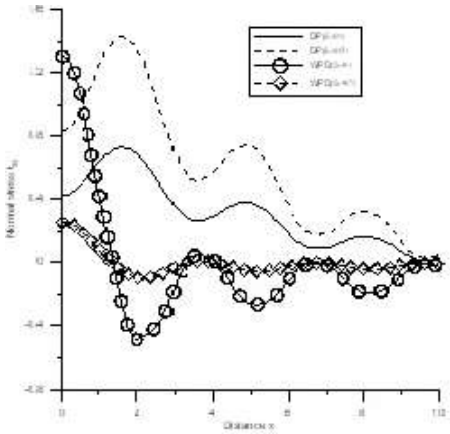


Figure 2: Variation of Normal Stress with x

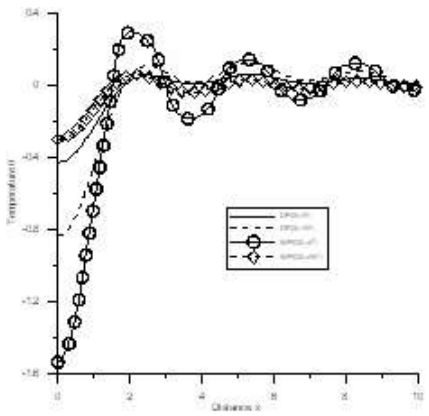


Figure 3: Variation of Temperature with x

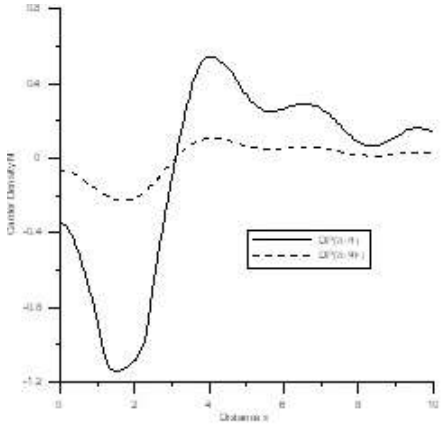


Figure 4: Variation of Carrier Density with x

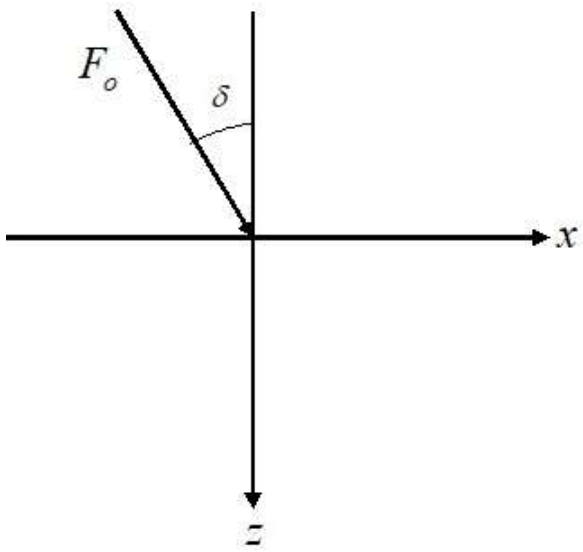


Figure 5: Inclined load over a semiconductor medium

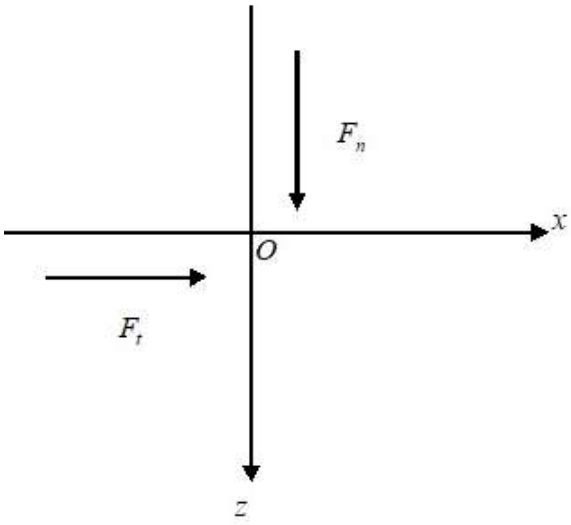


Figure 6: Normal and tangential loadings

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Schur power convex functions with applications to a class of conditional inequalities

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Abstract. In this paper, by the use of Schur power convexity, we prove some inequalities involving the condition that the sum of powers of variables is a fixed value. It is shown that the approach proposed is useful for establishing certain conditional inequalities.

Keywords: Schur power convex function, Schur convexity, majorization, power mean, conditional inequality.

1. Introduction

Schur convexity was introduced by Schur in 1923, which brought out new perspectives for researchers to study the convexity of functions. In particular, the Schur-Ostrowski theorem provides a distinct characterization of Schur convex functions through local two-variables conditions (see [1]). As is known to us, Schur convexity has many important applications in linear regression, graphs,

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matrices, combinatorial optimization, special functions, stochastic orderings, reliability, and other related fields (see [2, 3]). In recent years, the applications of Schur convexity and majorization to discover and prove various kinds of analytic inequalities has been a hot research topic. Also, the Schur convexity was generalized and extended to a variety of types by adding several parameters. Among these generalization results, Schur power convexity is a class of useful convexity and has attracted great attention and interest from researchers (see [4–14]).

It is our usual practice that the conditional inequalities are processed by the Lagrangian multiplier method, but it often results in complicated computations (see [15–19]). In this paper, we provide a novel and concise method to deal with conditional inequalities, more specifically, we will use Schur power convexity to prove some inequalities involving the condition that the sum of powers of variables is a fixed value.

We begin with introducing some essential definitions and lemmas. Throughout the paper, we assume that the set of n-dimensional row vector by $\mathbb{R}^n$.

$$\mathbb{R}^n_+ = \{\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_i \geq 0,\ i = 1, 2, \ldots, n\},$$

$$\mathbb{R}^n_{++} = \{\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_i > 0,\ i = 1, 2, \ldots, n\}.$$

In particular, $\mathbb{R}^1$, $\mathbb{R}^1_+$ and $\mathbb{R}^1_{++}$ denoted by $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{R}_{++}$ respectively.

Definition 1.1 ([1, 2]). *Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$.*

(*i*) *The vector $\boldsymbol{x}$ is said to be majorized by $\boldsymbol{y}$ (in symbols $\boldsymbol{x} \prec \boldsymbol{y}$) if $\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}$ for $k = 1, 2, \ldots, n-1$ and $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$, where $x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[n]}$ and $y_{[1]} \geq y_{[2]} \geq \cdots \geq y_{[n]}$ are rearrangements of $\boldsymbol{x}$ and $\boldsymbol{y}$ in a descending order.*

(*ii*) *A set $\Omega \subset \mathbb{R}^n$ is called a convex set if $(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \ldots, \alpha x_n + \beta y_n) \in \Omega$ for any $\boldsymbol{x}$ and $\boldsymbol{y} \in \Omega$, where α and $\beta \in [0, 1]$ with $\alpha + \beta = 1$.*

(*iii*) *Let $\Omega \subset \mathbb{R}^n$, φ: $\Omega \to \mathbb{R}$ is said to be a Schur-convex function on Ω if $\boldsymbol{x} \prec \boldsymbol{y}$ on Ω implies $\varphi(\boldsymbol{x}) \leq \varphi(\boldsymbol{y})$. A function φ is said to be a Schur-concave function on Ω if and only if $-\varphi$ is a Schur-convex function.*

Definition 1.2 ([3]). *Let $\Omega \subset \mathbb{R}^n_+$.*

(*i*) *A set Ω is said to be a harmonically convex set if $\frac{\boldsymbol{xy}}{\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y}} \in \Omega$ for every $\boldsymbol{x}, \boldsymbol{y} \in \Omega$ and $\lambda \in [0, 1]$, where $\frac{1}{\boldsymbol{x}} = \left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_n}\right)$ and*

$$\frac{\boldsymbol{xy}}{\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y}} = \left(\frac{x_1 y_1}{\lambda x_1 + (1-\lambda) y_1}, \frac{x_2 y_2}{\lambda x_2 + (1-\lambda) y_2}, \cdots, \frac{x_n y_n}{\lambda x_n + (1-\lambda) y_n}\right).$$

(*ii*) *A function $\varphi : \Omega \to \mathbb{R}_+$ is said to be a Schur harmonically convex function on Ω if $\frac{1}{\boldsymbol{x}} \prec \frac{1}{\boldsymbol{y}}$ implies $\varphi(\boldsymbol{x}) \leq \varphi(\boldsymbol{y})$. A function φ is said to be a Schur harmonically concave function on Ω if and only if $-\varphi$ is a Schur harmonically convex function.*

Definition 1.3 ([4]). *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a function defined by*

$$f(x) = \begin{cases} \dfrac{x^m - 1}{m}, & m \neq 0; \\ \ln x, & m = 0. \end{cases} \tag{1}$$

Then a function $\phi : \Omega \subset \mathbb{R}_+^n \to \mathbb{R}$ is said to be Schur m-power convex on Ω if

$$(f(x_1), f(x_2), \ldots, f(x_n)) \prec (f(y_1), f(y_2), \ldots, f(y_n)),$$

for all $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \Omega$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_n) \in \Omega$ implies $\phi(\boldsymbol{x}) \leq \phi(\boldsymbol{y})$.

If $-\phi$ is Schur m-power convex, then we say that ϕ is Schur m-power concave.

If we put $f(x) = x$ and $f(x) = \frac{1}{x}$ in Definition 1.3, respectively, then the concepts of Schur-convex function and Schur-harmonically convex function can be derived.

Lemma 1.1 ([1, 2]). *Suppose that $\Omega \subset \mathbb{R}^n$ is convex set, and has a nonempty interior set Ω°. Let $\varphi : \Omega \to \mathbb{R}$ be continuous on Ω and differentiable in Ω°. Then φ is the Schur-convex (or Schur-concave; respectively) function, if and only if it is symmetric on Ω and*

$$(x_1 - x_2)\left(\frac{\partial \varphi}{\partial x_1} - \frac{\partial \varphi}{\partial x_2}\right) \geq 0 (\text{ or } \leq 0; \text{ respectively})$$

holds for any $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \Omega^\circ$.

Lemma 1.2 ([3]). *Let $\Omega \subset \mathbb{R}_+^n$ be a symmetric harmonically convex set with a nonempty interior Ω°. Let $\varphi : \Omega \to \mathbb{R}_+$ be continuous on Ω and differentiable on Ω°. Then φ is a Schur harmonically convex (or Schur harmonically concave; respectively) function if and only if φ is symmetric on Ω and*

$$(x_1 - x_2)\left(x_1^2 \frac{\partial \varphi}{\partial x_1} - x_2^2 \frac{\partial \varphi}{\partial x_2}\right) \geq 0 \quad (or \ \leq 0; \ respectively)$$

holds for any $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \Omega^\circ$.

Lemma 1.3 ([4]). *Let $\Omega \subset \mathbb{R}_+^n$ be a symmetric set with nonempty interior Ω°, and $\varphi : \Omega \to \mathbb{R}_+$ be continuous on Ω and differentiable in Ω°. Then φ is Schur m-power convex on Ω if and only if φ is symmetric on Ω and*

$$\frac{x_1^m - x_2^m}{m}\left[x_1^{1-m}\frac{\partial \varphi(\boldsymbol{x})}{\partial x_1} - x_2^{1-m}\frac{\partial \varphi(\boldsymbol{x})}{\partial x_2}\right] \geq 0, \quad \text{if } m \neq 0 \tag{2}$$

and

$$(\ln x_1 - \ln x_2)\left[x_1\frac{\partial \varphi(\boldsymbol{x})}{\partial x_1} - x_2\frac{\partial \varphi(\boldsymbol{x})}{\partial x_2}\right] \geq 0, \quad \text{if } m = 0 \tag{3}$$

for all $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \Omega^\circ$.

Lemma 1.4 ([1, 2])**.** *Let* $\boldsymbol{x} = (x_1, x_2, \cdots, x_n) \in \mathbb{R}_+^n$ *and* $A_n(\boldsymbol{x}) = \frac{1}{n}\sum_{i=1}^n x_i$. *Then*

$$\mathbf{u} = \left(\underbrace{A_n(\boldsymbol{x}), A_n(\boldsymbol{x}), \cdots, A_n(\boldsymbol{x})}_{n}\right) \prec (x_1, x_2, \cdots, x_n) = \boldsymbol{x}. \tag{4}$$

For $m \neq 0$, we have

$$\left(\frac{1}{n}\sum_{i=1}^n \frac{x_i^m - 1}{m}, \ldots, \frac{1}{n}\sum_{i=1}^n \frac{x_i^m - 1}{m}\right) \prec \left(\frac{x_1^m - 1}{m}, \ldots, \frac{x_n^m - 1}{m}\right)$$

$\Longleftrightarrow$

$$\left(\frac{\left(\left(\frac{1}{n}\sum_{i=1}^n x_i^m\right)^{\frac{1}{m}}\right)^m - 1}{m}, \ldots, \frac{\left(\left(\frac{1}{n}\sum_{i=1}^n x_i^m\right)^{\frac{1}{m}}\right)^m - 1}{m}\right) \prec \left(\frac{x_1^m - 1}{m}, \ldots, \frac{x_n^m - 1}{m}\right),$$

As a consequence of Lemma 1.4, we have the following result:

Lemma 1.5. *If* φ *is Schur* m*-power convex function (or Schur* m*-power concave function; respectively) on* Ω*, then*

$$\varphi\left(M_m(\boldsymbol{x}), M_m(\boldsymbol{x}), \ldots, M_m(\boldsymbol{x})\right) \leq (\text{or} \geq; \text{respectively})\ \varphi\left(x_1, x_2, \ldots, x_n\right), \tag{5}$$

where $M_m(\boldsymbol{x}) = \left(\frac{1}{n}\sum_{i=1}^n x_i^m\right)^{\frac{1}{m}}$ *is the power mean of* $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$.

2. Main results

Theorem 2.1. *Let* n *be a natural number, and let* a, b *be positive real numbers with* $a^{-1} + b^{-1} = 1$. *Then*

$$(a+b)^n - a^n - b^n \geq 2^{2n} - 2^{n+1}. \tag{6}$$

Proof. The assumption condition, $a^{-1} + b^{-1} = 1$, implies that $a + b = ab$.

Let

$$\varphi(a, b) = (a+b)^n - a^n - b^n.$$

Then

$$\frac{\partial \varphi}{\partial a} = n(a+b)^{n-1} - na^{n-1}, \quad \frac{\partial \varphi}{\partial b} = n(a+b)^{n-1} - nb^{n-1}.$$

We have

$$\begin{aligned}
&(a-b)\left(a^2\frac{\partial \varphi}{\partial a} - b^2\frac{\partial \varphi}{\partial b}\right)\\
=&n(a-b)\left[(a^2-b^2)(a+b)^{n-1} - (a^{n+1} - b^{n+1})\right]\\
=&n(a-b)^2\left[(a+b)^n - (a^n + a^{n-1}b + \cdots + ab^{n-1} + b^n)\right]\\
\geq&0.
\end{aligned}$$

Thereby $\varphi(a,b)$ is a Schur-harmonically convex function on $\mathbb{R}^2_{++}$. In the light of Lemma (1.4), one has

$$\Big(\frac{a^{-1}+b^{-1}}{2}, \frac{a^{-1}+b^{-1}}{2}\Big) \prec \left(a^{-1}, b^{-1}\right),$$

then by $a^{-1}+b^{-1}=1$, we get $\left(\frac{1}{2},\frac{1}{2}\right) \prec \left(\frac{1}{a},\frac{1}{b}\right)$. From the Schur-harmonically convexity, it follows that $\varphi(2,2) \leq \varphi(a,b)$, which implies the desired inequality (6) in Theorem 2.1. □

Similarly, we can prove the following generalization of Theorem 2.1 in the case of m-dimension.

Theorem 2.2. *Let $x_i > 0$ $(i = 1, 2, \ldots, m)$ and $\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_m} = 1$. Then*

$$(x_1 + x_2 + \cdots + x_m)^n - x_1^n - x_2^n - \cdots - x_m^n \geq m^{2n} - m^{n+1}. \tag{7}$$

Theorem 2.3. *Let a, b, c be positive real numbers with $a^2+b^2+c^2 = 1$. Then*

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geq \frac{2(a^3+b^3+c^3)}{abc} + 3. \tag{8}$$

Proof. Let $\varphi(a,b,c) = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} - \frac{2(a^3+b^3+c^3)}{abc} - 3$. Then

$$\frac{\partial\varphi}{\partial a} = \frac{-2}{a^3} - \frac{6a^3bc - 2(a^3+b^3+c^3)bc}{a^2b^2c^2}, \quad \frac{\partial\varphi}{\partial b} = \frac{-2}{b^3} - \frac{6ab^3c - 2(a^3+b^3+c^3)ac}{a^2b^2c^2},$$

$$\begin{aligned}
\Delta :&= \frac{a^2-b^2}{2}\left(\frac{1}{a}\cdot\frac{\partial\varphi}{\partial a} - \frac{1}{b}\cdot\frac{\partial\varphi}{\partial b}\right) \\
&= (a^2-b^2)\left[\frac{a^4-b^4}{a^4b^4} + \frac{1}{a^2b^2c^2}\Big(\frac{c(a^2+b^2+c^2)(a^2-b^2) - 3a^2b^2c(a-b)}{ab}\Big)\right] \\
&= (a+b)(a-b)^2\left[\frac{a^3+a^2b+ab^2+b^3}{a^4b^4} + \frac{1}{a^2b^2c}\Big(\frac{a+b-3a^2b^2}{ab}\Big)\right].
\end{aligned}$$

From $a^2+b^2+c^2 = 1$, we get $2ab \leq a^2+b^2 < 1$, thus $a+b-3a^2b^2 \geq 2\sqrt{ab} - \frac{3}{2}ab = \sqrt{ab}(2 - \frac{3}{2}\sqrt{ab}) > 0$. We conclude that $\Delta > 0$. Therefore, φ is the Schur 2-power convex function on $\mathbb{R}^3_{++}$. Now, using inequality (5), it follows that

$$\begin{aligned}
\varphi(a,b,c) \geq& \varphi\Big(\Big(\frac{a^2+b^2+c^2}{3}\Big)^{\frac{1}{2}}, \Big(\frac{a^2+b^2+c^2}{3}\Big)^{\frac{1}{2}}, \Big(\frac{a^2+b^2+c^2}{3}\Big)^{\frac{1}{2}}\Big) \\
=& \varphi(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}) = 0.
\end{aligned}$$

This proves the required inequality in Theorem 2.3. □

Theorem 2.4. *Let x, y, z be positive real numbers with $x^2+y^2+z^2=1$. Then*

$$x^2yz + xy^2z + xyz^2 \le \frac{1}{3}. \tag{9}$$

Proof. Let $\varphi(x,y,z) = x^2yz + xy^2z + xyz^2$. Then

$$\begin{aligned}\Delta :=& \frac{x^2-y^2}{2}\left(\frac{1}{x}\cdot\frac{\partial\varphi}{\partial x} - \frac{1}{y}\cdot\frac{\partial\varphi}{\partial y}\right)\\ =&\frac{x^2-y^2}{2}\left[\frac{1}{x}(2xyz+y^2z+yz^2) - \frac{1}{y}(x^2z+2xyz+xz^2)\right]\\ =&\frac{x^2-y^2}{2xy}\left[-2xyz(x-y) - z(x^3-y^3) - z^2(x^2-y^2)\right].\end{aligned}$$

Note that $(x^2-y^2)(x^k-y^k)\ge 0$ for $k=1,2,3$, we obtain $\Delta\le 0$. Thus φ is the Schur 2-power concave function on $\mathbb{R}^3_{++}$. Now, utilizing inequality (5), we have

$$\varphi(x,y,z) \le \varphi\left(M_2(x,y,z), M_2(x,y,z), M_2(x,y,z)\right) = \varphi\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \frac{1}{3}.$$

This completes the proof of Theorem 2.4. □

Theorem 2.5. *Let x, y be nonnegative real numbers with $x^2+y^2=1$. Then*

$$\frac{x}{1+y} + \frac{y}{1+x} \ge 2(\sqrt{2}-1). \tag{10}$$

Proof. Let $\varphi(x,y) = \frac{x}{1+y} + \frac{y}{1+x}$. Then

$$\begin{aligned}&\frac{x^2-y^2}{2}\left(\frac{1}{x}\cdot\frac{\partial\varphi}{\partial x} - \frac{1}{y}\cdot\frac{\partial\varphi}{\partial y}\right)\\ =&\frac{x^2-y^2}{2}\left[\frac{1}{x}\left(\frac{1}{1+y} - \frac{y}{(1+x)^2}\right) - \frac{1}{y}\left(\frac{1}{1+x} - \frac{x}{(1+y)^2}\right)\right]\\ =&\frac{(x-y)^2(x+y)}{2}\left[\frac{x^2+xy+y^2+x+y}{xy(1+y)^2(1+x)^2}\right]\\ \ge&0.\end{aligned}$$

Hence φ is the Schur 2-power convex function on $\mathbb{R}^2_+$. By means of inequality (5), we derive that

$$\varphi(x,y) \ge \varphi\left(M_2(x,y), M_2(x,y)\right) = \varphi\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = 2(\sqrt{2}-1),$$

which implies the inequality asserted by Theorem 2.5. □

Theorem 2.6. *Let x, y be positive real numbers, and n be positive integer. If $x^{2n+1} + y^{2n+1} \geq 2$, then*

$$x^{n+1} + y^{n+1} \geq x^n + y^n. \tag{11}$$

Proof. Let $\varphi(x, y) = x^{n+1} + y^{n+1} - x^n - y^n$.

Case (I). If $x^{2n+1} + y^{2n+1} = 2$, then

$$(xy)^{n+\frac{1}{2}} \leq \frac{x^{2n+1} + y^{2n+1}}{2} = 1 \iff xy \leq 1.$$

Note that

$$\frac{\partial \varphi}{\partial x} = (n+1)x^n - nx^{n-1}, \quad \frac{\partial \varphi}{\partial y} = (n+1)y^n - ny^{n-1}.$$

$$\begin{aligned}
\Delta :&= \frac{x^{2n+1} - y^{2n+1}}{2n+1} \left(\frac{1}{x^{2n}} \frac{\partial \varphi}{\partial x} - \frac{1}{y^{2n}} \frac{\partial \varphi}{\partial y} \right) \\
&= \frac{x^{2n+1} - y^{2n+1}}{2n+1} \left[(n+1) \left(\frac{1}{x^n} - \frac{1}{y^n} \right) - n \left(\frac{1}{x^{n+1}} - \frac{1}{y^{n+1}} \right) \right] \\
&= \frac{x^{2n+1} - y^{2n+1}}{2n+1} \left[\frac{n(x^{n+1} - y^{n+1}) - (n+1)xy(x^n - y^n)}{x^{n+1}y^{n+1}} \right].
\end{aligned}$$

To prove $\Delta \geq 0$, it is enough to show that $n(x^{n+1} - y^{n+1}) \geq (n+1)xy(x^n - y^n)$ for $x > y$, namely,

$$\frac{1}{xy} E(n, n+1; x, y) = \frac{1}{xy} \left(\left(\frac{n}{n+1} \right) \frac{x^{n+1} - y^{n+1}}{x^n - y^n} \right)^{\frac{1}{(n+1)-n}} \geq 1.$$

On the other hand, it was proved in ([7], Theorem 1.1) that $E(n, n+1; x, y)$ is Schur convex on $\mathbb{R}^2_{++}$, therefore

$$E(n, n+1; x, y) \geq E\left(n, n+1; \frac{x+y}{2}, \frac{x+y}{2} \right) = \frac{x+y}{2},$$

this yields

$$\frac{1}{xy} E(n, n+1; x, y) \geq \frac{\frac{x+y}{2}}{xy} \geq \frac{\sqrt{xy}}{xy} = \frac{1}{\sqrt{xy}} \geq 1.$$

Thus φ is the Schur $(2n+1)$-power convex function on $\mathbb{R}^2_{++}$. By employing inequality (5), we have

$$\varphi(x, y) \geq \varphi\left(\left(\frac{x^{2n+1} + y^{2n+1}}{2} \right)^{\frac{1}{2n+1}}, \left(\frac{x^{2n+1} + y^{2n+1}}{2} \right)^{\frac{1}{2n+1}} \right) = \varphi(1, 1) = 0.$$

Case (II). If $x^{2n+1}+y^{2n+1}>2$, setting $x^{2n+1}+y^{2n+1}=a$ $(a>2)$, then

$$\left[\left(\frac{2}{a}\right)^{\frac{1}{2n+1}}x\right]^{2n+1}+\left[\left(\frac{2}{a}\right)^{\frac{1}{2n+1}}y\right]^{2n+1}=2.$$

By virtue of (I), we get

$$\left[\left(\frac{2}{a}\right)^{\frac{1}{2n+1}}x\right]^{n+1}+\left[\left(\frac{2}{a}\right)^{\frac{1}{2n+1}}y\right]^{n+1}\geq\left[\left(\frac{2}{a}\right)^{\frac{1}{2n+1}}x\right]^{n}+\left[\left(\frac{2}{a}\right)^{\frac{1}{2n+1}}y\right]^{n},$$

that is

$$x^{n+1}\left(\frac{2}{a}\right)^{\frac{n+1}{2n+1}}+y^{n+1}\left(\frac{2}{a}\right)^{\frac{n+1}{2n+1}}\geq x^{n}\left(\frac{2}{a}\right)^{\frac{n}{2n+1}}+y^{n}\left(\frac{2}{a}\right)^{\frac{n}{2n+1}},$$

this yields

$$x^{n+1}\left(\frac{2}{a}\right)^{\frac{1}{2n+1}}+y^{n+1}\left(\frac{2}{a}\right)^{\frac{1}{2n+1}}\geq x^{n}+y^{n},$$

further, we obtain $x^{n+1}+y^{n+1}>x^{n}+y^{n}$ with the help of $\frac{2}{a}<1$. The proof of Theorem 2.6 is complete. □

Theorem 2.7. *Let a,b,c be positive real numbers with $a^3+b^3+c^3=3$. Then*

$$\frac{1}{a^2+a+1}+\frac{1}{b^2+b+1}+\frac{1}{c^2+c+1}\geq 1. \tag{12}$$

Proof. Let $\varphi(a,b,c)=\frac{1}{a^2+a+1}+\frac{1}{b^2+b+1}+\frac{1}{c^2+c+1}$. Then

$$\begin{aligned}\Lambda :=&\frac{a^3-b^3}{3}\left(\frac{1}{a^2}\cdot\frac{\partial\varphi}{\partial a}-\frac{1}{b^2}\cdot\frac{\partial\varphi}{\partial b}\right)\\ =&-\frac{a^3-b^3}{3}\left[\frac{2a+1}{a^2(a^2+a+1)^2}-\frac{2b+1}{b^2(b^2+b+1)^2}\right]\\ =&\frac{(a-b)^2(a^2+ab+b^2)}{3}\Big[\frac{Q(a,b)}{a^2b^2(a^2+a+1)^2(b^2+b+1)^2}\Big],\end{aligned}$$

where

$$\begin{aligned}Q(a,b)=&\,a+b+7ab^2+7a^2b+8ab^3+8a^3b+5ab^4+5a^4b+2ab^5+2a^5b+8a^2b^2+5a^2b^3\\ &+5a^3b^2+2a^2b^4+2a^3b^3+2a^4b^2+4ab+2a^2+3a^3+2a^4+2b^2+a^5+3b^3+2b^4+b^5.\end{aligned}$$

Hence, $\Lambda\geq 0$, we deduce that $\varphi(a,b,c)$ is Schur 3-power convex on $\mathbb{R}^3_{++}$. Using inequality (5) gives

$$\varphi(a,b,c)\geq\varphi\Big(M_3(a,b,c),M_3(a,b,c),M_3(a,b,c)\Big)=\varphi(1,1,1)=1.$$

This completes the proof of Theorem 2.7. □

Theorem 2.8. *If $a, b, c > 0$ and $a^2 + b^2 + c^2 = 1$, then*

$$a + b + c + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 4\sqrt{3}. \tag{13}$$

Proof. Let $\varphi(a, b, c) = a + b + c + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$. Then

$$\begin{aligned}\Lambda :&= \frac{a^2 - b^2}{2}\left(\frac{1}{a}\cdot\frac{\partial\varphi}{\partial a} - \frac{1}{b}\cdot\frac{\partial\varphi}{\partial b}\right)\\ &= \frac{a^2 - b^2}{2}\left[\left(\frac{1}{a} - \frac{1}{a^3}\right) - \left(\frac{1}{b} - \frac{1}{b^3}\right)\right]\\ &= \frac{(a-b)^2(a+b)}{2a^3b^3}\left(a^2 + ab + b^2 - a^2b^2\right).\end{aligned}$$

From the condition $a^2 + b^2 + c^2 = 1$, it follows that $2ab \leq a^2 + b^2 < 1$. Hence $a^2 + ab + b^2 - a^2b^2 \geq 3ab - a^2b^2 = ab(3 - ab) > 0$. Then we get $\Lambda \geq 0$. Therefore, φ is the Schur 2-power convex function on $\mathbb{R}^3_{++}$. Applying inequality (5), we obtain

$$\varphi(a, b, c) \geq \varphi\left(M_2(a, b, c), M_2(a, b, c), M_2(a, b, c)\right) = \varphi\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = 4\sqrt{3},$$

which implies the required inequality in Theorem 2.8. □

Theorem 2.9. *Let $x_i > 0$ $(i = 1, 2, \ldots, n)$ with $x_1^2 + x_2^2 + \cdots + x_n^2 = n$. Then*

$$\frac{x_1}{x_1 + n} + \frac{x_2}{x_2 + n} + \cdots + \frac{x_n}{x_n + n} \leq \frac{n}{n + 1}. \tag{14}$$

Proof. Let $\varphi(x_1, x_2, \ldots, x_n) = \frac{x_1}{x_1+n} + \frac{x_2}{x_2+n} + \cdots + \frac{x_n}{x_n+n}$. Then

$$\begin{aligned}\Lambda :&= \frac{x_1^2 - x_2^2}{2}\left(\frac{1}{x_1}\cdot\frac{\partial\varphi}{\partial x_1} - \frac{1}{x_2}\cdot\frac{\partial\varphi}{\partial x_2}\right)\\ &= \frac{x_1^2 - x_2^2}{2}\left[\frac{n}{x_1(x_1 + n)^2} - \frac{n}{x_2(x_2 + n)^2}\right]\\ &= \frac{-n(x_1 - x_2)^2(x_1 + x_2)(2nx_1 + 2nx_2 + x_1^2 + x_2^2 + x_1x_2 + n^2)}{2x_1x_2(x_1 + n)^2(x_2 + n)^2}\\ &\leq 0.\end{aligned}$$

Thus φ is the Schur 2-power concave function on $\mathbb{R}^n_{++}$. By virtue of inequality (5), it follows that

$$\begin{aligned}\varphi(x_1, x_2, \ldots, x_n) &\leq \varphi\left(M_2(x_1, x_2, \ldots, x_n), \ldots, M_2(x_1, x_2, \ldots, x_n)\right)\\ &= \varphi(1, 1, \ldots, 1) = \frac{n}{n + 1}.\end{aligned}$$

The Theorem 2.9 is proved. □

Theorem 2.10. *Let a, b, c, m, n be positive real numbers with $\sqrt{a}+\sqrt{b}+\sqrt{c}=3$ and $m \geq 2n$. Then*

$$(ab)^{\frac{n}{m}} + (bc)^{\frac{n}{m}} + (ca)^{\frac{n}{m}} \leq 3. \tag{15}$$

Proof. Setting $a = x^{2m}, b = y^{2m}, c = z^{2m}$, then $x^m + y^m + z^m = 3$. We just need to prove that

$$x^{2n}y^{2n} + y^{2n}z^{2n} + z^{2n}x^{2n} \leq 3. \tag{16}$$

Let $\varphi(x, y, z) = x^{2n}y^{2n} + y^{2n}z^{2n} + z^{2n}x^{2n}$. Since

$$\begin{aligned}\Lambda :&= \frac{x^m - y^m}{m}\left(\frac{1}{x^{m-1}} \cdot \frac{\partial \varphi}{\partial x} - \frac{1}{y^{m-1}} \cdot \frac{\partial \varphi}{\partial y}\right) \\ &= \frac{-2n(x^m - y^m)}{m}\left[\frac{(x^m - y^m) + z^{2n}(x^{m-2n} - y^{m-2n})}{x^{m-2n}y^{m-2n}}\right] \leq 0.\end{aligned}$$

We conclude that φ is the Schur m-power concave function on $\mathbb{R}^3_{++}$. Applying inequality (5), it follows that

$$\varphi(x, y, z) \leq \varphi\left(M_m(x, y, z), M_m(x, y, z), M_m(x, y, z)\right) = \varphi(1, 1, 1) = 3.$$

The proof of Theorem 2.10 is complete. □

Theorem 2.11. *If x, y, z are all positive numbers, and $x^2 + y^2 + z^2 = 8$, then*

$$x^3 + y^3 + z^3 \geq \frac{16}{3}\sqrt{6}. \tag{17}$$

Proof. Let $\varphi(x, y, z) = x^3 + y^3 + z^3$. Note that

$$\Lambda := \frac{x^2 - y^2}{2}\left(\frac{1}{x} \cdot \frac{\partial \varphi}{\partial x} - \frac{1}{y} \cdot \frac{\partial \varphi}{\partial y}\right) = \frac{3(x^2 - y^2)(x - y)}{2} \geq 0.$$

So φ is the Schur 2-power convex function on $\mathbb{R}^3_{++}$, by using inequality (5), we obtain

$$\varphi(x, y, z) \geq \varphi(M_2(x, y, z), M_2(x, y, z), M_2(x, y, z)) = \varphi\Big(\sqrt{\frac{8}{3}}, \sqrt{\frac{8}{3}}, \sqrt{\frac{8}{3}}\Big) = \frac{16}{3}\sqrt{6}.$$

This completes the proof of Theorem 2.11. □

Theorem 2.12. *Let x, y be positive real numbers, $x^2 + y^2 = 1$ and $p > 2$. Then*

$$x^{p-1}y^{p-1}(x^{-p} + y^{-p}) \geq 2^{2-\frac{p}{2}}. \tag{18}$$

Proof. Let $\varphi(x,y) = x^{p-1}y^{p-1}(x^{-p}+y^{-p})$. Since

$$\begin{aligned}\Lambda :&= \frac{x^2-y^2}{2}\left(\frac{1}{x}\cdot\frac{\partial\varphi}{\partial x}-\frac{1}{y}\cdot\frac{\partial\varphi}{\partial y}\right)\\ &= \frac{x^2-y^2}{2}\left[(p-1)x^{-1}y^{-1}(x^{p-2}-y^{p-2})+x^{-3}y^{-3}(x^{2+p}-y^{2+p})\right]\geq 0,\end{aligned}$$

we conclude that φ is the Schur 2-power convex function on $\mathbb{R}^2_{++}$. From the inequality (5), we deduce that

$$\varphi(x,y)\geq\varphi\left(M_2(x,y),M_2(x,y)\right)=\varphi(2^{-\frac{1}{2}},2^{-\frac{1}{2}})=2^{2-\frac{p}{2}}.$$

The Theorem 2.12 is proved. □

Theorem 2.13. *Let $a_i>0$ $(i=1,2,\ldots,n)$ with $\sum_{i=1}^n a_i^2=S$. Then*

$$\begin{aligned}&\frac{a_1^3}{a_2+a_3+\cdots+a_n}+\frac{a_2^3}{a_1+a_3+\cdots+a_n}\\ &+\cdots+\frac{a_n^3}{a_1+a_2+\cdots+a_{n-1}}\geq\frac{S}{n-1}.\end{aligned}\tag{19}$$

Proof. Let

$$p=\sum_{i=1}^n a_i,\quad q=\sum_{i=1}^n\frac{a_i^3}{(p-a_i)^2},\quad \varphi(a_1,a_2,\ldots,a_n)=\sum_{i=1}^n\frac{a_i^3}{p-a_i}.$$

Then

$$\frac{\partial\varphi}{\partial a_1}=\frac{a_1^2(3p-2a_1)}{(p-a_1)^2}-q,\quad \frac{\partial\varphi}{\partial a_2}=\frac{a_2^2(3p-2a_2)}{(p-a_2)^2}-q.$$

We have

$$\begin{aligned}\Lambda :&= \frac{a_1^2-a_2^2}{2}\left(\frac{1}{a_1}\cdot\frac{\partial\varphi}{\partial a_1}-\frac{1}{a_2}\cdot\frac{\partial\varphi}{\partial a_2}\right)\\ &= \frac{a_1^2-a_2^2}{2}\left(\frac{a_1(3p-2a_1)}{(p-a_1)^2}-\frac{a_2(3p-2a_2)}{(p-a_2)^2}-q\Big(\frac{1}{a_1}-\frac{1}{a_2}\Big)\right)\\ &= \frac{a_1^2-a_2^2}{2}\left[\frac{p(a_1-a_2)(a_1a_2+p(a_1+a_2+3a_3))}{(p-a_1)^2(p-a_2)^2}+\frac{q(a_1-a_2)}{a_1a_2}\right]\\ &\geq 0.\end{aligned}$$

Thus φ is the Schur 2-power convex function on $\mathbb{R}^n_{++}$. By utilizing inequality (5), we get

$$\begin{aligned}&\varphi(a_1,a_2,\ldots,a_n)\geq\varphi\left(M_2(a_1,a_2,\ldots,a_n),\ldots,M_2(a_1,a_2,\ldots,a_n)\right)\\ &\geq\varphi\Big(\sqrt{\frac{S}{n}},\sqrt{\frac{S}{n}},\ldots,\sqrt{\frac{S}{n}}\Big)=\frac{S}{n-1}.\end{aligned}$$

This completes the proof of Theorem 2.13. □

Theorem 2.14. *If $x_i > 0$ $(i = 1, 2, \ldots, n)$ and $k > m > 0$, then*

$$\left(\frac{\sum_{i=1}^{n} x_i^k}{n}\right)^{\frac{1}{k}} \geq \left(\frac{\sum_{i=1}^{n} x_i^m}{n}\right)^{\frac{1}{m}}. \tag{20}$$

Proof. The inequality (20) can be rewritten as a conditional inequalities:

$$\sum_{i=1}^{n} x_i^k \geq n\Big(\frac{\ell}{n}\Big)^{\frac{k}{m}} \quad \text{where} \quad \sum_{i=1}^{n} x_i^m = \ell.$$

Let $\varphi(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} x_i^k$. Note that

$$\frac{x_1^m - x_2^m}{m}\left(\frac{1}{x_1^{m-1}} \cdot \frac{\partial \varphi}{\partial x_1} - \frac{1}{x_2^{m-1}} \cdot \frac{\partial \varphi}{\partial x_2}\right) = \frac{k}{m}(x_1^m - x_2^m)(x_1^{k-m} - x_2^{k-m}) \geq 0.$$

Hence φ is the Schur m-power convex function on $\mathbb{R}_{++}^n$. By making use of the inequality (5), we obtain

$$\varphi(x_1, x_2, \ldots, x_n) \geq \varphi\left(M_m(x_1, x_2, \ldots, x_n), \ldots, M_m(x_1, x_2, \ldots, x_n)\right) = n\Big(\frac{\ell}{n}\Big)^{\frac{k}{m}},$$

which is equivalent to the inequality (20). The proof of Theorem 2.14 is complete. □

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Multi-monad graph

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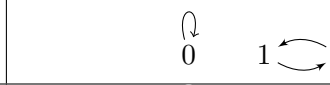
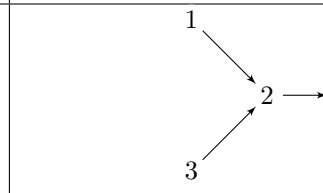
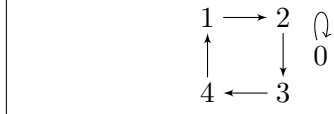
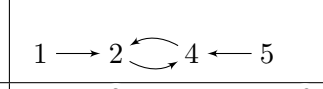
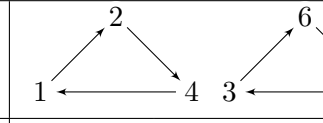
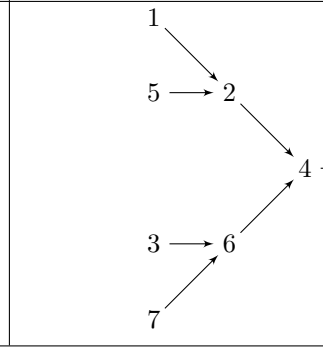
Abstract. Monad graph is a directed graph related to finite group G where every vertex of the elements of the correspond group G is adjacent with it's image by directed connected edge under the action of given map. In this paper, we introduce multi-monad graph, which is a joint union of finite number of monad graphs.

Keywords: discrete dynamical systems, Euler function, quadratic residues.

1. Introduction

Dynamics group theory is one of the most active area in group theory, which allowing understand the behavior of each elements of the given group. In [1], 2003 was introduced by V.I. Arnold a very interested phenomena termed by **monad.** Let G be finite group. **A monad** is a mapping of each element from G into itself, i.e. $f : G \to G$ for all $g \in G$. **The monad graph** $\Gamma(G)$ is adjacent every vertex of elements of the correspond G with it's image by directed connected edge under the action of f. For more details, we refer to [2], [4], [5], [6], [7]. Below, by using additive notation for the group operation, we show some monad graphs with "squaring" doubles the residue of simplest cyclic groups $n \leq 8$:

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$\Gamma(\mathbb{Z}_2)$		A_1
$\Gamma(\mathbb{Z}_3)$		$O_1 + O_2$
$\Gamma(\mathbb{Z}_4)$		T_4
$\Gamma(\mathbb{Z}_5)$		$O_1 + O_4$
$\Gamma(\mathbb{Z}_6)$		$A_1 + A_2$
$\Gamma(\mathbb{Z}_7)$		$O_1 + 2O_3$
$\Gamma(\mathbb{Z}_8)$		T_8

Let us remind about the meaning of the following symbols:

O_n : a directed cycle on n vertices;

D_n : the $4n$-vertex graph consisting of the cycle O_n of length n equipped, at each vertex, with three edges entering the vertex;

T_{2^n}, : the binary rooted tree with $2n$ vertices and n levels double branching at the levels $1, \ldots, n-1$ (exactly two edges enter each vertex at such a level); the root forms the zeroth level of the tree, and two edges enter the root as well: one from itself, and the other from the unique vertex of the first level.

In short, monad is dynamical system contains the following elements: finite action group G, monad function f and monad graph $\Gamma(G)$. In [1] was considered a special case where G is a finite group and f is squaring map. Result of this kind is the following:

Theorem 1.1 ([1])**.** *Each connected component of a monad graph is a forest of rooted trees directed towards their roots, which lie on a directed cycle formed by the edges connecting the roots.*

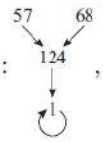
Moreover, Arnold showed that if $\gamma > 0$ and η is odd number, then the monad graphs defined by a finite commutative groups G_n of order $n = 2^\gamma \times \eta$, are sums of products of monads of the form T_{2^γ} and O_η. Notice that the number of cycles does not exceed n.

To show particular details, let's consider $\mathbb{Z}_{100} = \mathbb{Z}_4 \times \mathbb{Z}_{25}$ (see, [1]). The monad graph of $\Gamma(\mathbb{Z}_4) = T_{2^2}$. For $\mathbb{Z}_{25}$, we begin from calculating the graphs of the factors. The monad graph of $\mathbb{Z}_{25}$ of odd order splits into cycles of orders 1, 4 and 20. Hence the monad of $\mathbb{Z}_{25}$ is $O_1 + O_4 + O_{20}$. Therefore, the product has the form

$$\Gamma(\mathbb{Z}_{100}) = T_4 + (T_4 * O_4) + (T_4 * O_{20}) \tag{1}$$

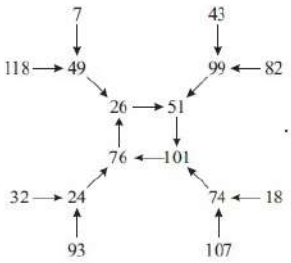
and

$|T_4| = 4$:



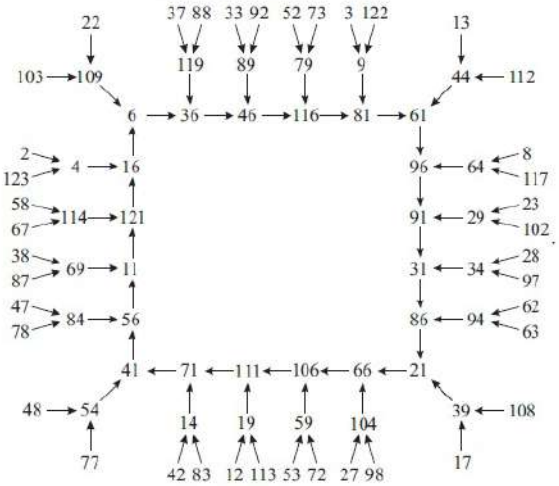
,

$|T_4 * O_4| = 16$:



.

$|T_4 * O_{20}| = 80$:



.

V.I. Arnold also proved that in the case of non commutative finite, the trees framing each attracting cycle are homogeneous, but it is unknown if the framing trees have some special structure:

Homogeneity theorem. Given a component of the squaring monad $g \to g^2$ of any finite group, the framing trees of the attracting cycle are homogeneous (the rooted trees attracted by the vertices of the cycle are all isomorphic – as directed graphs – along that component).

Now, let n be a positive integer, the Euler group $E(n)$ is the multiplicative group of coprime residue classes modulo n. Nontrivial Euler groups begins from

$$E(2) = \{1\},\ E(3) \approx \mathbb{Z}_2,\ E(5) \approx \mathbb{Z}_4,\ E(6) \approx \mathbb{Z}_2,$$
$$E(7) \approx \mathbb{Z}_6,\ E(8) \approx \mathbb{Z}_2 \times \mathbb{Z}_2\ E(9) \approx \mathbb{Z}_6, E(13) \approx \mathbb{Z}_{12}.$$

In [3], was obtained that he Euler group $E(n)$ is cyclic whenever n is a prime or a degree of q prime such as

$$E(p^q) \approx \mathbb{Z}_{\alpha(p^q)},$$

where $\alpha(p^q) = (p-1)p^{q-1}$.

In the following table, we show the monad graphs defined by the Euler groups $E(n)$, $n \leq 22$ with multiplication operation:

n	$\Gamma(E(n))$	n	$\Gamma(E(n))$
2	O_1	12	D_1
3	A_1	13	$T_4 + (T_4 * O_2)$
4	A_1	14	$A_1 + A_2$
5	T_4	15	$T_4 * A_1$
6	A_1	16	$T_4 * A_1$
7	$A_1 + A_2$	17	T_{16}
8	D_1	18	$A_1 + A_2$
9	$A_1 + A_2$	19	$A_1 + A_2 + A_6$
10	T_4	20	$T_4 * A_1$
11	$A_1 + A_4$	21	$A_1 + A_2 + D_1 + D_2$

Table 2.2. The monad graph defined by Euler group

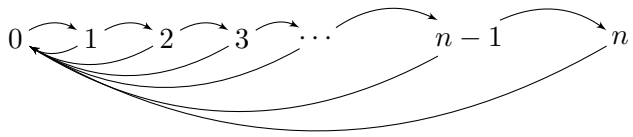
After brief introduction to monad graphs, let us introduce the following system:

Definition 1.1. *Let G be finite group and $f : G \to G$ for $g \in G$. The graph generated by the following system:*

$$\Gamma(G_1, f) \cup \Gamma(G_2, f) \cup \cdots \cup \Gamma(G_n, f) = \bigcup_{i=1}^{n} \Gamma(G_i, f) := \Psi_{n,f}$$

we called **multi-monad graph** *and denoted by $\Gamma(\Psi_{n,f})$.*

Obviously, if f is a shift function, i.e. $f_+(g) = g + 1$ for all $g \in G$, then the multi-monad $\Gamma(\Psi_{n,f_+})$ can be represented by $\bigcup_{i=2}^{n} O_i$ such as:



.

In this paper, we are interested in studying the properties of multi-monad graph with respect to a squaring map.

2. Main result

To obtain our result we will review some supportive material:

Definition 2.1. *The graph of the product monad will be called the product of the graphs of the factors; the multiplication of graphs will be denoted by the same symbol* $*$*:*

$$[graph(A * B)] = [graph(A)] * [graph(B)].$$

Lemma 2.1 ([1])**.** *The graph of multiplication in an additive cyclic group of odd order is a disjoint union of cycles* O_n*.*

Lemma 2.2 ([1])**.** *The graph of multiplication in the additive cyclic group of residues modulo* 2^n *is the binary rooted tree* T_{2^n}*.*

The combinatorics of squaring monad graph showed that even the simplest algebraic theories has complicated topological structure while the combinatorics of squaring multi-monad graph shows more complicated structure and its dynamical behavior is unpredictable (chaotic behavior), as we show in next example:

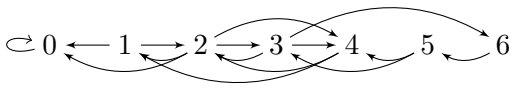
Example 2.1. Let f be a squaring map and

$$\Psi_{7,f} = \Gamma(Z_2, f) \cup (Z_3, f) \cup \cdots \cup \Gamma(Z_7, f).$$

Then, we have

$$\Gamma(\Psi_{7,f}) = A_1 \cup (O_1 + O_2) \cup T_4 \cup (O_1 + O_4) \cup (A_1 + A_2) \cup (O_1 + 2O_3)$$

and $\Gamma(\Psi_{7,f})$ is pertaining as:



.

By $T_{2^r,\tau,\zeta}$ we denote the binary rooted trees T_{2^r}, which contain a troubled branches, i.e. if there are some edges connecting the branches at levels $1, \ldots, n-1$ and there are some edges from levels $n > 1$ entering the root except one from itself, and the other from the unique vertex of the first level. In other

words, troubled binary rooted trees $T_{2^r,\tau,\zeta}$ contain τ, which is the number of edges connecting between the tree branches and ζ which is the number of edges entering the root (except one from itself, and the other from the unique vertex of the first level).

By T_{2^r,τ,ζ,O_i} we denote troubled binary rooted tree which included a cycle O_i of finite order.

2.1 Algorithm of multi-monad graphs

In particular, we will compute the algorithm of multi-monad graphs generated by the following system

$$\Psi_{2^r,f} = \bigcup_{m=1}^{r} \Gamma(\mathbb{Z}_{2^m}), \quad r > 0. \tag{2}$$

Theorem 2.1. *Let $\Psi_{n,f}$ is given by* (2)*. Then, the multi-monad graph pertaining to system $\Psi_n(\mathbb{Z}_n)$ is $T_{2^r,\tau,\zeta}$.*

Proof. From Lemma 2, we have that each group of $\Psi_n(\mathbb{Z}_n) = \bigcup_{m=1}^{r} \mathbb{Z}_{2^m}$ pertaining a binary tree T_{2^n}. According to joint union of them, one can see that each element belongs to $\mathbb{Z}_{2^{r-1}} \bigcap \mathbb{Z}_{2^r}$ has two action under the considered map with one image obtained by the residue of 2^{r-1} and another image, which belongs to the group of 2^r. This implies the following algorithm:

- each vertex of form 2^k for $k = 1, 2, \ldots, r$ have a set of entering edges defined as:
$$E(\Gamma|_{2^k}) = \bigcup_{k=1}^{r-1}\{ \bigcup_{j=k-1}^{r-2} \{2^k, 2^k + \sum_{d=k-1}^{j} 2^d\}\}.$$
- each vertex of form $2^k m$ for $k = 1, 2, \ldots, r$ and m is odd, have two sets of outgoing edges defined as:
$$E(\Gamma|_{2^k m}) = \{\{2^k m, 2^k m + 2^{k-1}\}, \{2^k m, 2^{k-1} * m\}\},\ k, m > 1.$$

Its easy to check that some vertices has two outgoing edges belong to $E(\Gamma|_{2^k m})$, which connect to vertices in groups of 2^k and 2^k+1 for $k = 1, 2, \ldots, r$, i.e. connecting the branches of T_{2^k} and $T_{2^{k+1}}$. Moreover, each vertex of form 2^k have zero residue in each group of (2), so they has outgoing edge to zero, which proves the theorem. □

Theorem 2.2. *Let $\Psi_{n,f} = \left[\bigcup_{m=1}^{r} \Gamma(\mathbb{Z}_{2^m})\right] \cup \Gamma(\mathbb{Z}_p)$ where p is odd number. Then the multi-monad graph pertaining to system $\Psi_n(\mathbb{Z}_n)$ is T_{2^r,τ,ζ,O_i}, where O_i be cyclic of order $i \leq p$.*

Directly from Lemma 2.1 and Theorem 2.1

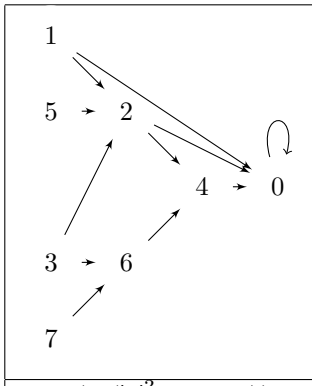
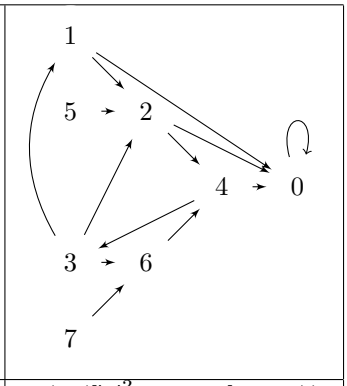
	
$\Gamma(\Psi(\bigcup_{m=1}^{3}\mathbb{Z}_{2^m}))$	$\Gamma(\Psi([\bigcup_{m=1}^{3}\mathbb{Z}_{2^m}]\cup\mathbb{Z}_5))$
$T_{2^3,1,2}$	$T_{2^3,1,2,O_4}$

Table 2. The multi-monad graphs of Theorems 2.1. and 2.2

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Fixed point results in a complex-valued generalized G_b-metric space

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Abstract. In this paper, we introduce the concept of a complex $G_b^{\alpha\beta}$-metric space and prove the Banach fixed point principle and Kannan's fixed point theorem. Our results generalize the idea of a complex valued G_b-metric space.
Keywords: G_b- metric, partial order, Kannan's fixed point theorem.

1. Introduction

Bakhtin [3] presented b-metric spaces as a generalization of metric spaces. In the frame work of b-metric spaces in [7, 6], authors Mebawondu et al., introduced some classes of mappings called TAC-Suzuki Berinde Type F-contractions and TAC-Suzuki type rational F-contractions. Azam et al. [2] defined a complex valued metric space. Rao et al. [10] then introduced the concept of a complex valued b-metric space. Similarly, G-metric spaces were defined by Mustafa and Sims [8]. Thereafter followed the concept of G_b-metric spaces by Aghajani et al. [1]. Recently as an application Mebawondu et al. [5] discussed the existence of solutions for integral equations for Z-contraction mappings in the frame work of a complete G_b metric spaces. Ozgur [9] extended this idea to define a complex valued G_b-metric space. Likewise, Singh [11] introduced the idea of a α, β b-metric space and thereafter a $G_b^{\alpha\beta}$-metric space. Here we extend this concept to a complex valued $G_b^{\alpha\beta}$-metric space and prove the Banach contraction principle and Kannan's [4] fixed point theorem in this space.

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2. Preliminaries

Here we state some properties of $G_b^{\alpha\beta}$-metric spaces.

Definition 2.1 ([11]). *Let $X \neq \emptyset$ be a set and $\alpha, \beta \geq 1$ be real numbers. A function $G_b^{\alpha\beta} : X \times X \times X \to \mathbb{R}^+$ is called a $G_b^{\alpha\beta}$-metric if the following hold, for every $x, y, z, a \in X$:*

(i) $G_b^{\alpha\beta}(x, y, z) = 0$ *if* $x = y = z$;

(ii) $G_b^{\alpha\beta}(x, x, y) > 0$ *with* $x \neq y$;

(iii) $G_b^{\alpha\beta}(x, x, y) \leq G_b^{\alpha\beta}(x, y, z)$ *with* $y \neq z$;

(iv) $G_b^{\alpha\beta}(x, y, z) = G_b^{\alpha\beta}(\rho\{x, y, z\})$, *where ρ is any permutation of $\{x, y, z\}$;*

(v) $G_b^{\alpha\beta}(x, y, z) \leq \alpha G_b^{\alpha\beta}(x, a, a) + \beta G_b^{\alpha\beta}(a, y, z)$.

Then, $(X, G_b^{\alpha\beta})$ is a $G_b^{\alpha\beta}$-metric space, when $\alpha = \beta$, $G_b^{\alpha\beta}$ is a G_b metric.

Proposition 2.1. *Let X be a $G_b^{\alpha\beta}$-metric space, then for $x, y, z \in X$ it follows that:*

(i) if $G_b^{\alpha\beta}(x, y, z) = 0$ *then* $x = y = z$;

(ii) $G_b^{\alpha\beta}(x, y, z) \leq \alpha G_b^{\alpha\beta}(x, x, y) + \beta G_b^{\alpha\beta}(x, x, z)$;

(iii) $G_b^{\alpha\beta}(x, y, y) \leq (\alpha + \beta) G_b^{\alpha\beta}(y, x, x)$;

(iv) $G_b^{\alpha\beta}(x, y, z) \leq \alpha G_b^{\alpha\beta}(x, a, z) + \beta G_b^{\alpha\beta}(a, y, z)$.

Definition 2.2. *Let X be a $G_b^{\alpha\beta}$-metric space. A sequence $\{x_n\}$ is said to be:*

(i) $G_b^{\alpha\beta}$-Cauchy if for each $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $m, n, l \geq N$, $G_b^{\alpha\beta}(x_n, x_m, x_l) < \epsilon$;

(ii) $G_b^{\alpha\beta}$-convergent to $x^\star \in X$ if for each $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$, $G_b^{\alpha,\beta}(x_m, x_n, x^\star) < \epsilon$.

Proposition 2.2. *Let X be a $G_b^{\alpha\beta}$-metric space and $\{x_n\}$ a sequence in X. Then, the following are equivalent:*

(i) The sequence $\{x_n\}$ is $G_b^{\alpha\beta}$-Cauchy;

(ii) For any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $G_b^{\alpha\beta}(x_n, x_m, x_m) < \epsilon$, for all $m, n \geq N$.

Proposition 2.3. *Let X be a $G_b^{\alpha\beta}$-metric space. The the following are equivalent:*

(i) $\{x_n\}$ *is* $G_b^{\alpha\beta}$*-convergent to* $x^\star$*;*

(ii) $G_b^{\alpha\beta}(x_n, x_n, x^\star) \to 0$ *as* $n \to \infty$*;*

(iii) $G_b^{\alpha\beta}(x_n, x^\star, x^\star) \to 0$ *as* $n \to \infty$*.*

Definition 2.3. *A* $G_b^{\alpha\beta}$*-metric space* X *is complete if every* $G_b^{\alpha\beta}$*-Cauchy sequence is* $G_b^{\alpha\beta}$*-convergent in* X*.*

Definition 2.4. *Let* $\mathbb{C}$ *be the set of complex numbers and for* $z_1, z_2 \in \mathbb{C}$ *define a partial order* $\precsim$ *on* $\mathbb{C}$ *by*

$$z_1 \precsim z_2 \text{ if and only if } \Re(z_1) \leq \Re(z_2) \text{ and } \Im(z1) \leq \Im(z_2).$$

Proposition 2.4. *If* $z_1 \precsim z_2$ *then one of the following are true:*

(i) $\Re(z_1) = \Re(z_2)$ *and* $\Im(z_1) = \Im(z_2)$*;*

(ii) $\Re(z_1) < \Re(z_2)$ *and* $\Im(z_1) = \Im(z_2)$*;*

(iii) $\Re(z_1) = \Re(z_2)$ *and* $\Im(z_1) < \Im(z_2)$*;*

(iv) $\Re(z_1) < \Re(z_2)$ *and* $\Im(z_1) < \Im(z_2)$*.*

We write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (ii),(iii) or (iv) holds. If only (iv) is satisfied then we write $z_1 \prec z_2$.

Proposition 2.5. *The following statements are true:*

(i) if $a, b \in \mathbb{R}$ *with* $a \leq b$ *then* $az \precsim bz$*;*

(ii) if $0 \precsim z_1 \precnsim z_2$ *then* $|z_1| < |z_2|$*;*

(iii) if $z_1 \precsim z_2$ *and* $z_2 \prec z_3$*, then* $z_1 \prec z_3$*.*

3. Complex-valued $G_b^{\alpha\beta}$-metric spaces

Definition 3.1. *Let* $X \neq \emptyset$ *be a set and* $\alpha, \beta \geq 1$ *be real numbers. A function* $G_b^{\alpha\beta} : X \times X \times X \to \mathbb{C}$ *is called a* $G_b^{\alpha\beta}$*-complex metric if the following hold, for every* $x, y, z, a \in X$

(i) $G_b^{\alpha\beta}(x, y, z) = 0$ *if* $x = y = z$*;*

(ii) $0 \prec G_b^{\alpha\beta}(x, x, y)$ *with* $x \neq y$*;*

(iii) $G_b^{\alpha\beta}(x, x, y) \precsim G_b^{\alpha\beta}(x, y, z)$ *with* $y \neq z$*;*

(iv) $G_b^{\alpha\beta}(x, y, z) = G_b^{\alpha\beta}(\rho\{x, y, z\})$ *where* ρ *is any permutation of* $\{x, y, z\}$*;*

(v) $G_b^{\alpha\beta}(x, y, z) \precsim \alpha G_b^{\alpha\beta}(x, a, a) + \beta G_b^{\alpha\beta}(a, y, z)$*.*

Then $(X, G_b^{\alpha\beta})$ is a $G_b^{\alpha\beta}$-complex metric space. When $\alpha = \beta$, $G_b^{\alpha\beta}$ is a G_b-complex metric.

Example 3.1. Let $X = (1, 3)$ then define $G_b^{\alpha\beta} : X \times X \times X \to \mathbb{C}$ by

$$G_b^{\alpha\beta}(x, y, z) = \begin{cases} (1+i)e^{|x-y|+|y-z|+|z-x|}, & x \neq y \neq z \\ 0, & x = y = z. \end{cases}$$

To show that $G_b^{\alpha\beta}(x, y, z)$ is a $G_b^{\alpha\beta}$-metric we verify properties (i)-(v) of definition 3.1. Properties (i)-(iv) are easily verified. We only verify property (v) of definition 3.1. Let $x, y, z \in X$ such that $x \neq y \neq z$ then from proposition 2.5 it follows that

$$\begin{aligned} &G_b^{\alpha\beta}(x, y, z) \\ &\precsim (1+i)e^{|x-a|+|y-a|+|y-z|+|z-a|+|x-a|} \\ &= (1+i)e^{2|x-a|+|y-a|+|y-z|+|z-a|} \\ &\precsim \sup_{x,y,z\in X} e^{\frac{2}{3}|2(x-a)|+\frac{1}{3}[|y-a|+|y-z|+|z-a|]}(1+i)e^{\frac{1}{3}|2(x-a)|+\frac{2}{3}[|y-a|+|y-z|+|z-a|]} \\ &\precsim \frac{e^{\frac{14}{3}}}{3}(1+i)e^{|2(x-a)|} + \frac{2e^{\frac{14}{3}}}{3}(1+i)e^{[|y-a|+|y-z|+|z-a|]} \\ &= \frac{e^{\frac{14}{3}}}{3}G_b^{\alpha\beta}(x, a, a) + \frac{2e^{\frac{14}{3}}}{3}G_b^{\alpha\beta}(a, y, z) \end{aligned}$$

with $\alpha = \frac{e^{\frac{14}{3}}}{3} \geq 1$ and $\beta = \frac{2e^{\frac{14}{3}}}{3} \geq 1$, $\alpha \neq \beta$ and $\alpha < \beta$.

Proposition 3.1. *Let $(X, G_b^{\alpha\beta})$ be a $G_b^{\alpha\beta}$-complex metric space, then:*

1. $G_b^{\alpha\beta}(x, y, z) \precsim \alpha G_b^{\alpha\beta}(x, x, y) + \beta G_b^{\alpha\beta}(x, x, z)$;
2. $G_b^{\alpha\beta}(x, y, y) \precsim (\alpha + \beta) G_b^{\alpha\beta}(y, x, x)$.

Proof. 1. Interchanging variables x and y in property (v) of definition 3.1 we get $G_b^{\alpha\beta}(y, x, z) \precsim \alpha G_b^{\alpha\beta}(y, a, a) + \beta G_b^{\alpha\beta}(a, x, z)$. Setting $a = x$, we get $G_b^{\alpha\beta}(y, x, z) \precsim \alpha G_b^{\alpha\beta}(y, x, x) + \beta G_b^{\alpha\beta}(x, x, z)$. Finally, using property (iv) of definition 3.1 we get $G_b^{\alpha\beta}(x, y, z) \precsim \alpha G_b^{\alpha\beta}(x, x, y) + \beta G_b^{\alpha\beta}(x, x, z)$.

2. Interchanging variables x and y in property (v) of definition 3.1, we get $G_b^{\alpha\beta}(y, x, z) \precsim \alpha G_b^{\alpha\beta}(y, a, a) + \beta G_b^{\alpha\beta}(a, x, z)$ for all $x, y, z \in X$. Taking $z = y$, we get $G_b^{\alpha\beta}(y, x, y) \precsim \alpha G_b^{\alpha\beta}(y, a, a) + \beta G_b^{\alpha\beta}(a, x, y)$. Finally, setting $a = x$ we get $G_b^{\alpha\beta}(y, x, y) \precsim \alpha G_b^{\alpha\beta}(y, x, x) + \beta G_b^{\alpha\beta}(x, x, y)$. Using property (iv) of definition 3.1 we obtain $G_b^{\alpha\beta}(x, y, y) \precsim (\alpha + \beta) G_b^{\alpha\beta}(y, x, x)$. □

Definition 3.2. *Let X be a $G_b^{\alpha\beta}$-complex metric space. A sequence $\{x_n\}$ is said to be:*

1. *$G_b^{\alpha\beta}$-Cauchy if for each $0 \prec \epsilon$ there exists $N \in \mathbb{N}$ such that for all $m, n, l \geq N$, $G_b^{\alpha\beta}(x_n, x_m, x_l) \prec \epsilon$;*

2. *$G_b^{\alpha\beta}$-convergent to $x^\star \in X$ if for each $0 \prec \epsilon$ there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$, $G_b^{\alpha\beta}(x_m, x_n, x^\star) \prec \epsilon$.*

Proposition 3.2. *Let $(X, G_b^{\alpha\beta})$ be a $G_b^{\alpha\beta}$-complex metric space and $\{x_n\}$ a sequence in X. Then $\{x_n\}$ is complex valued $G_b^{\alpha\beta}$-convergent to $x^\star$ if and only if $|G_b^{\alpha\beta}(x_m, x_n, x^\star)| \to 0$ as $n, m \to \infty$.*

Proof. ($\Rightarrow$) Let $\epsilon' > 0$ be given and consider $\epsilon = \frac{\epsilon'}{\sqrt{2}}(1+i)$, then $0 \prec \epsilon \in \mathbb{C}$ and there exists a $N \in \mathbb{N}$ such that for all $n, m > N$,

$$0 \prec G_b^{\alpha\beta}(x_n, x_n, x^\star) \preceq G_b^{\alpha\beta}(x_m, x_n, x^\star) \prec \epsilon.$$

Hence, $|G_b^{\alpha\beta}(x_m, x_n, x^\star)| < |\epsilon| = \epsilon'$, which implies that $|G_b^{\alpha\beta}(x_m, x_n, x^\star)| \to 0$ as $n, m \to \infty$.

($\Leftarrow$) Suppose $|G_b^{\alpha\beta}(x_m, x_n, x^\star)| \to 0$ as $n, m \to \infty$. Given $\epsilon \in \mathbb{C}$ with $0 \prec \epsilon$, it follows that $0 < |\epsilon|$. By the convergence of $|G_b^{\alpha\beta}(x_m, x_n, x^\star)|$ to 0 there exists $N \in \mathbb{N}$ such that for all $n, m > N$, $|G_b^{\alpha\beta}(x_m, x_n, x^\star)| < |\epsilon|$, which implies that $G_b^{\alpha\beta}(x_m, x_n, x^\star) \prec \epsilon$ for all $n, m > N$. □

Theorem 3.1. *Let $(X, G_b^{\alpha,\beta})$ be a $G_b^{\alpha\beta}$-complex metric space and $\{x_n\}$ a sequence in X. Then the following are equivalent:*

(i) $\{x_n\}$ is complex valued $G_b^{\alpha\beta}$-convergent to $x^\star$;

(ii) $|G_b^{\alpha\beta}(x_n, x_n, x^\star)| \to 0$ as $n \to \infty$;

(iii) $|G_b^{\alpha\beta}(x_n, x^\star, x^\star)| \to 0$ as $n \to \infty$;

(iv) $|G_b^{\alpha\beta}(x_m, x_n, x^\star)| \to 0$ as $n, m \to \infty$.

Proof. (i) $\Rightarrow$ (ii) Replace m by n in Proposition 3.2.

(ii) $\Rightarrow$ (iii) By Proposition 3.1 (ii) it follows that

$$G_b^{\alpha\beta}(x_n, x^\star, x^\star) \precsim (\alpha + \beta) G_b^{\alpha\beta}(x_n, x_n, x^\star).$$

Hence, $|G_b^{\alpha\beta}(x_n, x^\star, x^\star)| \leq (\alpha+\beta)|G_b^{\alpha\beta}(x_n, x_n, x^\star)|$ and taking the limit as $n \to \infty$ the result is proved.

(iii) $\Rightarrow$ (iv) By Proposition 3.1 (i) it follows that

$$G_b^{\alpha,\beta}(x_m, x_n, x^\star) = G_b^{\alpha\beta}(x^\star, x_m, x_n) \precsim \alpha G_b^{\alpha\beta}(x^\star, x^\star, x_m) + \beta G_b^{\alpha\beta}(x^\star, x^\star, x_n),$$

which implies that

$$|G_b^{\alpha\beta}(x_m, x_n, x^\star)| \leq |G_b^{\alpha\beta}(x^\star, x_m, x_n)| \leq \alpha |G_b^{\alpha\beta}(x^\star, x^\star, x_m)| + \beta |G_b^{\alpha\beta}(x^\star, x^\star, x_n)|.$$

Taking the limit as $m, n \to \infty$ the result is obtained.

(iv)⇒(i) This is by Proposition 3.2 (ii). □

Theorem 3.2. *Let $(X, G_b^{\alpha\beta})$ be a complete complex valued $G_b^{\alpha\beta}$ metric space and $T : X \to X$ be a mapping that satisfies*

$$G_b^{\alpha\beta}(Tx, Ty, Tz) \precsim \lambda G_b^{\alpha\beta}(x, y, z), \tag{1}$$

for all $x, y, z \in X$, where $\lambda \in [0, \frac{1}{\beta})$. Then T has a unique fixed point.

Proof. Consider the sequence $\{x_n\}$ generated by $x_n = Tx_{n-1}$, where $x_0 \in X$ is arbitrary. Using (1) recursively it may be shown that

$$\begin{aligned} G_b^{\alpha\beta}(x_n, x_{n+1}, x_{n+1}) &= G_b^{\alpha,\beta}(Tx_{n-1}, Tx_n, Tx_n) \\ &\precsim \lambda G_b^{\alpha\beta}(x_{n-1}, x_n, x_n) \\ &\precsim \lambda G_b^{\alpha\beta}(Tx_{n-2}, Tx_{n-1}, Tx_{n-1}) \\ &= \lambda^2 G_b^{\alpha\beta}(x_{n-2}, x_{n-1}, x_{n-1}) \\ &\precsim \lambda^n G_b^{\alpha\beta}(x_0, x_1, x_1). \end{aligned} \tag{2}$$

For $m, n \in \mathbb{N}$ with $m > n$ we have

$$\begin{aligned} G_b^{\alpha\beta}(x_n, x_m, x_m) &\precsim \alpha G_b^{\alpha\beta}(x_n, x_{n+1}, x_{n+1}) + \beta G_b^{\alpha\beta}(x_{n+1}, x_m, x_m) \\ &\precsim \alpha G_b^{\alpha\beta}(x_n, x_{n+1}, x_{n+1}) \\ &+ \beta \left[\alpha G_b^{\alpha\beta}(x_{n+1}, x_{n+2}, x_{n+2}) + \beta G_b^{\alpha\beta}(x_{n+2}, x_m, x_m)\right] \\ &= \alpha G_b^{\alpha\beta}(x_n, x_{n+1}, x_{n+1}) + \alpha\beta G_b^{\alpha\beta}(x_{n+1}, x_{n+2}, x_{n+2}) \\ &+ \beta^2 G_b^{\alpha\beta}(x_{n+2}, x_m, x_m) \\ &\precsim \alpha G_b^{\alpha\beta}(x_n, x_{n+1}, x_{n+1}) + \alpha\beta G_b^{\alpha\beta}(x_{n+1}, x_{n+2}, x_{n+2}) \\ &+ \alpha\beta^2 G_b^{\alpha\beta}(x_{n+2}, x_{n+3}, x_{n+3}) \\ &+ \cdots + \alpha\beta^{m-n-2} G_b^{\alpha\beta}(x_{m-2}, x_{m-2}, x_{m-1}) \\ &+ \beta^{m-n-1} G_b^{\alpha\beta}(x_{m-1}, x_m, x_m). \end{aligned} \tag{3}$$

Using (2) in (3) we obtain

$$\begin{aligned} G_b^{\alpha\beta}(x_n, x_m, x_m) &\precsim \left(\alpha\lambda^n + \alpha\beta\lambda^{n+1} + \cdots + \alpha\beta^{m-n-2}\lambda^{m-2} + \beta^{m-n-1}\lambda^{m-1}\right) \\ &\times G_b^{\alpha\beta}(x_0, x_1, x_1) \\ &\precsim \alpha\lambda^n \left(1 + \beta\lambda + \cdots + (\beta\lambda)^{m-n-2} + (\beta\lambda)^{m-n-1}\right) \\ &\times G_b^{\alpha\beta}(x_0, x_1, x_1) \\ &\precsim \frac{\alpha\lambda^n}{1 - \beta\lambda} G_b^{\alpha\beta}(x_0, x_1, x_1). \end{aligned} \tag{4}$$

Thus, we have

$$|G_b^{\alpha\beta}(x_n, x_m, x_m)| \leq \frac{\alpha\lambda^n}{1-\beta\lambda}|G_b^{\alpha\beta}(x_0, x_1, x_1)|. \tag{5}$$

Taking the limit as $n \to \infty$ we obtain $G_b^{\alpha\beta}(x_n, x_m, x_m) \to \infty$. From Proposition 3.1 we have

$$G_b^{\alpha\beta}(x_n, x_m, x_l) \precsim \alpha G_b^{\alpha\beta}(x_l, x_m, x_m) + \beta G_b^{\alpha\beta}(x_n, x_m, x_m), \tag{6}$$

for $n, m, l \in \mathbb{N}$, which yields

$$|G_b^{\alpha\beta}(x_n, x_m, x_l)| \leq \alpha|G_b^{\alpha\beta}(x_n, x_m, x_m)| + \beta|G_b^{\alpha\beta}(x_l, x_m, x_m)|. \tag{7}$$

Taking the limit as $n, m, l \to \infty$ we obtain $G_b^{\alpha\beta}(x_n, x_m, x_l) = 0$, so $\{x_n\}$ is $G_b^{\alpha\beta}$-complex Cauchy. By the completion of $(X, G_b^{\alpha\beta})$ there is $x^\star \in X$ such that $\{x_n\}$ is $G_b^{\alpha\beta}$-convergent to $x^\star$. To show that $x^\star$ is a fixed point of T consider

$$G_b^{\alpha\beta}(x_{n+1}, Tx^\star, Tx^\star) \precsim \lambda G_b^{\alpha\beta}(x_n, x^\star, x^\star),$$

which implies that

$$|G_b^{\alpha\beta}(x_{n+1}, Tx^\star, Tx^\star)| \leq \lambda|G_b^{\alpha\beta}(x_n, x^\star, x^\star)|.$$

Taking the limit as $n \to \infty$, we have

$$|G_b^{\alpha\beta}(x^\star, Tx^\star, Tx^\star)| \leq \lambda G_b^{\alpha\beta}(x^\star, x^\star, x^\star) = 0,$$

hence $G_b^{\alpha\beta}(x^\star, Tx^\star, Tx^\star) = 0$ and $Tx^\star = x^\star$. To prove uniqueness of the fixed point assume that $x^{\star\star}$ is another fixed point of T then

$$\begin{aligned} G_b^{\alpha\beta}(x^\star, x^{\star\star}, x^{\star\star}) &= G_b^{\alpha\beta}(Tx^\star, Tx^{\star\star}, Tx^{\star\star}) \\ &\precsim \lambda G_b^{\alpha\beta}(x^\star, x^{\star\star}, x^{\star\star}). \end{aligned} \tag{8}$$

Hence,

$$|G_b^{\alpha\beta}(x^\star, x^{\star\star}, x^{\star\star})| \leq \lambda|G_b^{\alpha\beta}(x^\star, x^{\star\star}, x^{\star\star})|,$$

and as $\lambda < 1$ this implies that $G_b^{\alpha,\beta}(x^\star, x^{\star\star}, x^{\star\star)} = 0$ or equivalently $x^\star = x^{\star\star}$. □

We prove Kannan's fixed point theorem for $G_b^{\alpha\beta}$-complex metric spaces.

Theorem 3.3. *Let $(X, G_b^{\alpha\beta})$ be a complete $G_b^{\alpha\beta}$-complex metric space and $T : X \to X$ satisfies*

$$G_b^{\alpha\beta}(Tx, Ty, Ty) \precsim \lambda[G_b^{\alpha\beta}(x, Tx, Tx) + G_b^{\alpha\beta}(y, Ty, Ty)], \tag{9}$$

for all $x, y \in X$, where $0 < \lambda < \frac{1}{\alpha+\beta}$. Then T has a unique fixed point.

Proof. Define the sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \geq 0$. We shall show that $\{x_n\}$ is $G_b^{\alpha\beta}$-Cauchy. We also assume that $x_{n+1} \neq x_n$ for $n \geq 0$, otherwise x_n is a fixed point of T. It follows from (9) that

$$\begin{aligned} G_b^{\alpha\beta}(x_n, x_{n+1}, x_{n+1}) &= G_b^{\alpha\beta}(Tx_{n-1}, Tx_n, Tx_n) \\ &\precsim \lambda[G_b^{\alpha,\beta}(x_{n-1}, Tx_{n-1}, Tx_{n-1}) + G_b^{\alpha\beta}(x_n, Tx_n, Tx_n)] \\ &= \lambda[G_b^{\alpha\beta}(x_{n-1}, x_n, x_n) + G_b^{\alpha\beta}(x_n, x_{n+1}, x_{n+1}]. \end{aligned} \tag{10}$$

Then

$$G_b^{\alpha\beta}(x_n, x_{n+1}, x_{n+1}) \precsim \frac{\lambda}{1-\lambda} G_b^{\alpha\beta}(x_{n-1}, x_n, x_n). \tag{11}$$

Repeating the process we have

$$G_b^{\alpha\beta}(x_n, x_{n+1}, x_{n+1}) \precsim \left(\frac{\lambda}{1-\lambda}\right)^n G_b^{\alpha\beta}(x_0, x_1, x_1). \tag{12}$$

Now,

$$\begin{aligned} G_b^{\alpha\beta}(x_n, x_m, x_m) &= G_b^{\alpha\beta}(T^n x_0, T^m x_0, T^m x_0) \\ &\precsim G_b^{\alpha\beta}(T^{n-1} x_0, T^n x_0, T^n x_0) + G_b^{\alpha\beta}(T^{m-1} x_0, T^m x_0, T^m x_0) \\ &= G_b^{\alpha\beta}(x_{n-1}, x_n, x_n) + G_b^{\alpha\beta}(x_{m-1}, x_m, x_m) \\ &\precsim \left[\left(\frac{\lambda}{1-\lambda}\right)^{n-1} + \left(\frac{\lambda}{1-\lambda}\right)^{m-1}\right] G_b^{\alpha\beta}(x_0, x_1, x_1). \end{aligned} \tag{13}$$

Hence, $|G_b^{\alpha\beta}(x_n, x_m, x_m)| \to 0$ as $n, m \to \infty$, which implies that $\{x_n\}$ is $G_b^{\alpha\beta}$-Cauchy in X. Since $(X, G_b^{\alpha\beta})$ is complete there exists $x^\star \in X$ such that $x_n \to x$. Consider

$$\begin{aligned} G_b^{\alpha\beta}(Tx^\star, x^\star, x^\star) &\precsim \alpha G_b^{\alpha\beta}(Tx^\star, Tx_n, Tx_n) + \beta G_b^{\alpha\beta}(Tx_n, x^\star, x^\star) \\ &\precsim \alpha\lambda[G_b^{\alpha\beta}(x^\star, Tx^\star, Tx^\star) + G_b^{\alpha\beta}(x_n, Tx_n, Tx_n)] \\ &\quad + \beta G_b^{\alpha\beta}(Tx_n, x^\star, x^\star) \\ &= \alpha\lambda[G_b^{\alpha\beta}(x^\star, Tx^\star, Tx^\star) + G_b^{\alpha\beta}(x_n, x_{n+1}, x_{n+1})] \\ &\quad + \beta G_b^{\alpha\beta}(x_{n+1}, x^\star, x^\star). \end{aligned} \tag{14}$$

Taking the limit as $n \to \infty$ we have $|G_b^{\alpha\beta}(x^\star, Tx^\star, Tx^\star)| = 0$ and hence $G_b^{\alpha\beta}(x^\star, Tx^\star, Tx^\star) = 0$ as $\alpha\lambda < 1$, proving that $x^\star$ is a fixed point of T. To prove uniqueness assume that $x^{\star\star}$ is another fixed point of T, then

$$\begin{aligned} G_b^{\alpha\beta}(x^\star, x^{\star\star}, x^{\star\star}) &= G_b^{\alpha\beta}(Tx^\star, Tx^{\star\star}, Tx^{\star\star}) \\ &\precsim \lambda[G_b^{\alpha\beta}(x^\star, Tx^\star, Tx^\star) + G_b^{\alpha\beta}(x^{\star\star}, Tx^{\star\star}, Tx^{\star\star})] \\ &\precsim \lambda[G_b^{\alpha,\beta}(x^\star, x^\star, x^\star) + G_b^{\alpha\beta}(x^{\star\star}, x^{\star\star}, x^{\star\star})] \\ &\precsim 0. \end{aligned} \tag{15}$$

Hence, we conclude that $x^\star = x^{\star\star}$. □

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Inverse int-fuzzy soft (left, right) ideals over semigroups

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Abstract. In this paper, we define inverse int-soft sets, inverse int-fuzzy soft sets, inverse int-soft subsemigroups, inverse int-fuzzy soft subsemigroups and inverse int-soft (left, right) ideals, inverse int-fuzzy soft (left, right) ideals over a semigroup. We also give some of their properties such as AND operation, OR operation, union and intersection of them over semigroups with supported examples. Moreover, we show that the images of inverse int-fuzzy soft (left, right) ideals over semigroups are the inverse int-fuzzy soft (left, right) ideals over semigroups under some conditions.

Keywords: inverse int-fuzzy soft set, inverse int-fuzzy soft subsemigroups, inverse int-fuzzy soft (left, right) ideals.

1. Introduction

The solution to real-world problems in many fields such as engineering, economics, computer science, environment, medical science involve data that contain uncertainties. There is a wide range of theories that can be used when dealing with uncertainties in data, such as the theory of fuzzy sets [22], rough sets [20], intuitionistic fuzzy sets [1], interval mathematics [4], as well as other mathematic tools. In 1999, Molodtsov [14] defined soft set theory as a new mathematical tool for dealing with uncertainties that are are free from the difficulties. He pointed out directions of soft sets for the applications like soft analysis, operation research, probability and game theory. Later, Maji et al. [13] defined soft subset and soft super set, equality of two soft sets. They presented soft binary operation, such as AND, OR, intersection, union and studied De morgan's Laws of soft set.

Jun [6] applied the notion of soft sets to BCK/BCI-algebras. Next, Muhiuddin and Al-Roqi [16] introduced the notions of (internal, external) cubic soft sets, P-cubic (resp. R-cubic) soft subsets, R-union (resp. R-intersection, P-union, P-intersection) of cubic soft sets, and the complement of a cubic soft set and investigated some of their properties. They applied the notion of cubic soft sets to BCK/BCI-algebras. Later, Muhiuddin et al.[17] considered some operations of cubic soft sets, such as AND operation and OR operation based on

the P-order and the R-order. They discussed conditions for the R-union of two internal cubic soft sets to be an internal cubic soft set and investigated some properties of cubic soft subalgebras in BCK/BCI-algebras. Jun et al. [7] introduced the notions of cubic soft $\circ$-subalgebras and (closed) cubic soft ideals in BCK/BCI-algebras and investigated some of their properties. They considered relations between cubic soft subalgebras, cubic soft $\circ$-subalgebras and (closed) cubic soft ideals and discussed characterizations of cubic soft ideals.

In 2001, Maji et al. [12] extended the soft sets to fuzzy soft sets. They introduced the concept of fuzzy soft sets and defined fuzzy soft subsets, the intersection and union of fuzzy soft sets over common universe. The fuzzy soft sets are developed to fuzzy soft semigroups by Yang [21] (2011). He defined fuzzy soft [left, right] ideals over semigroups and fuzzy soft semigroups, and studied sufficient and necessary conditions for α-level set, intersection and union of fuzzy soft [left, right] ideals. In 2014, Song et al. [19] introduced int-soft semigroups and int-soft left (right) ideals of semigroups. They proved that the soft intersection of int-soft left (right) ideals (int-soft semigroups) is also int-soft left (right) ideals (int-soft semigroups).

The soft sets and fuzzy soft sets are developed to inverse soft sets and inverse fuzzy soft sets over parameters by Çetkin et al. [3] (2016). They defined inverse soft sets and inverse fuzzy soft sets over parameters, and discussed the application in decision making problems of them. In 2019, Khalil and Hassan [8] studied inverse soft sets and inverse fuzzy soft sets and defined operations, such as AND operation, OR operation and investigated some of their properties. They constructed an algorithm using max-min and min-max decision of inverse fuzzy soft set for a fuzzy decision making problem. Motivated and inspired by the works above, we applied the notion of inverse soft sets and inverse fuzzy soft sets to int-soft sets and int-fuzzy soft sets over semigroups.

In this paper, we define inverse int-soft sets, inverse int-fuzzy soft sets, inverse int-soft subsemigroups, inverse int-fuzzy soft subsemigroups and inverse int-soft (left, right) ideals, inverse int-fuzzy soft (left, right) over a semigroup. We investigate some of their properties such as AND operation, OR operation, union and intersection of them over semigroups with supported examples. Finally, we show that the images of inverse int-fuzzy soft (left, right) ideals over semigroups are the inverse int-fuzzy soft (left, right) ideals over semigroups under some conditions.

2. Preliminaries

In this section, we shall give some of basic definitions and results that will be used.

Let S be a semigroup. A non-empty subset T of a semigroup S is called a subsemigroup if $T^2 \subseteq T$. A subsemigroup T of a semigroup S is called a left (right) ideal of S if $ST \subseteq T (TS \subseteq T)$. A subsemigroup T on a semigroup S is called an ideal on S if it is both a left and a right ideal on S [5].

The concept of fuzzy semigroups are introduced by Kuroki [10, 11].

For any $a, b \in [0,1]$, define $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$. Then $a \wedge b$ and $a \vee b$ are element in $[0,1]$. A function f from a semigroup S to the unit interval $[0,1]$ is called a fuzzy set on S. A fuzzy set f on S is called a fuzzy subsemigroup on S if $f(xy) \geq f(x) \wedge f(y)$ for all $x, y \in S$.

A fuzzy set f on a semigroup S is called a fuzzy left [right] ideal on S if $f(xy) \geq f(y)[f(xy) \geq f(x)]$ for all $x, y \in S$. A fuzzy set f on a semigroup S is called a fuzzy ideal on S if it is both a fuzzy left and a fuzzy right ideal on S. We note that any fuzzy left [right] ideal on S is a fuzzy subsemigroup on S. Let T be a nonempty subset of a semigroup S, the characteristic function on T is the function $\chi_T : S \to [0,1]$ defined by $\chi_T(x) = 1$ if $x \in T$ and $\chi_T(x) = 0$ if $x \notin T$.

If f and g are fuzzy sets on a semigroup S then $f \leq g$, $f \vee g$ and $f \wedge g$ (some authors [15, 11] use notations $f \subseteq g$, $f \cup g$ and $f \cap g$, respectively), respectively, are defined as follows:

$$\begin{aligned} f \leq g \text{ if } f(x) \leq g(x), \quad \text{for all} \quad x \in S, \\ (f \vee g)(x) = f(x) \vee g(x), \text{ for all } x \in S, \\ (f \wedge g)(x) = f(x) \wedge g(x), \text{ for all } x \in S. \end{aligned}$$

Lemma 2.1. [15] *Let T be a nonempty subset of a semigroup S. Then, the following properties hold.:*

(*i*) *T is a subsemigroup of S if and only if χ_T is a fuzzy subsemigroup on S.*

(*ii*) *T is a (left, right) ideal of S if and only if χ_T is a fuzzy (left, right) ideal on S.*

Lemma 2.2. [15] *Let S be a semigroup. Then, the following properties hold:*

(*i*) *Let f and g be two fuzzy subsemigroups on S. Then, $f \cap g$ is also a fuzzy subsemigroup on S.*

(*ii*) *Let f and g be two fuzzy (left, right) ideals of S. Then, $f \cap g$ is also a fuzzy (left, right) ideal on S.*

Now, we give some definition of soft sets and fuzzy soft sets over an initial universe set.

Definition 2.1 ([14])**.** *Let U be an initial universe set and let E be a set of parameters. Let $P(U)$ denotes the power set of U and let $\emptyset \neq A \subseteq E$. A pair $(\widehat{F}, A)$ is called a soft set over U, where $\widehat{F}$ is a mapping given by $\widehat{F} : A \to P(U)$.*

Definition 2.2 ([12])**.** *Let E be a set of parameters and let $\emptyset \neq A \subseteq E$. A pair (F, A) is called a fuzzy soft set over U, where F is a mapping given by $F : A \to Fuz(U)$ and $Fuz(U)$ is the set of all fuzzy sets on U.*

The concept of the inverse fuzzy soft set over an initial universe set are introduced by Çetkin et al. [3].

Definition 2.3 ([3]). *Let U be an initial universe set and let E be a set of parameters. A pair $(\widehat{F}, U)$ is called an inverse soft set over E, where $\widehat{F}$ is a mapping given by $\widehat{F} : U \to P(E)$ and $P(E)$ is the power set of E.*

Definition 2.4 ([3]). *Let E be a set of parameters. A pair $(\mathcal{F}, U)$ is called an inverse fuzzy soft set over E, where $\mathcal{F}$ is a mapping given by $\mathcal{F} : U \to Fuz(E)$ is a mapping and $Fuz(E)$ is the set of all fuzzy parameter set on E.*

Song et al. [19] provided new definitions and various results on int-soft set theory. In what follows, we take $E = S$, as a set of parameters, which is a semigroup unless otherwise specified.

Definition 2.5 ([19]). *A soft set $(\widehat{f}, S)$ over U is called an int-soft semigroup over U if it satisfies:*

$$\widehat{f}(xy) \supseteq \widehat{f}(x) \cap \widehat{f}(y), \text{ for all } x, y \in S.$$

Definition 2.6 ([19]). *A soft set $(\widehat{f}, S)$ over U is called an int-soft left [right] ideal over U if it satisfies:*

$$\widehat{f}(xy) \supseteq \widehat{f}(y)[\widehat{f}(xy) \supseteq \widehat{f}(x)], \text{ for all } x, y \in S.$$

If a soft set $(\widehat{f}, S)$ over U is both an int-soft left ideal and an int-soft right ideal over U, we say that $(\widehat{f}, S)$ is an int-soft [two-sided] ideal over U.

Next, we give some definition of fuzzy soft sets over semigroups.

Definition 2.7 ([21]). *Let E be a set of parameters and let $\emptyset \neq A \subseteq E$. A pair (F, A) is called a fuzzy soft set over a semigroup S, where $F : A \to Fuz(S)$ is a mapping and $Fuz(S)$ is the set of all fuzzy sets on S.*

Let (F, A) be a fuzzy soft set over a semigroup S. For $p \in A, F(p) \in Fuz(S)$. Set $F_p := F(p)$. Then $F_p \in Fuz(S)$.

Definition 2.8 ([18]). *Let T be a subset of a semigroup S and A be a set of parameter. A fuzzy soft characteristic function $(\mathcal{C}^T, A)$ over S is defined by $(\mathcal{C}^T)_p = \chi_T$ for all $p \in A$.*

Definition 2.9 ([2]). *Let (F, A) be a fuzzy soft set over a semigroup S. For each $\alpha \in [0, 1]$, the set $(F, A)^\alpha$ is called an α-level set of (F, A), where*

$$(F_p)^\alpha = \{x \in S \mid F_p(x) \geq \alpha\}, \text{ for all } p \in A.$$

Definition 2.10 ([12]). *Let (F, A) and (G, B) be two fuzzy soft sets over a semigroup S. Define $(F, A) \leq (G, B)$ if*

(*i*) *$A \subseteq B$ and*

(ii) $F_p \leq G_p$ *for all* $p \in A$.

Definition 2.11 ([12])**.** *Let* (F, A) *and* (G, B) *be two fuzzy soft sets over a semigroup* S *with* $A \cap B \neq \emptyset$. *Define* $(F, A)\tilde{\wedge}(G, B) := (F \wedge G, A \cap B)$. *For each* $p \in A \cap B$, $(F \wedge G)_p = F_p \wedge G_p$. *Then,* $(F, A)\tilde{\wedge}(G, B)$ *is a fuzzy soft set over* S.

Some authors call $(F, A)\tilde{\wedge}(G, B)$, *the intersection of* (F, A) *and* (G, B).

Definition 2.12 ([12])**.** *Let* (F, A) *and* (G, B) *be two fuzzy soft sets over a semigroup* S. *Define* $(F, A)\tilde{\vee}(G, B) := (F \vee G, A \cup B)$, *where for all* $p \in A \cup B$,

$$(F \vee G)_p = \begin{cases} F_p, & \text{if } p \in A - B, \\ G_p, & \text{if } p \in B - A, \\ F_p \vee G_p, & \text{if } p \in A \cap B. \end{cases}$$

Then $(F, A)\tilde{\vee}(G, B)$ *is a fuzzy soft set over* S.

Some authors call $(F, A)\tilde{\vee}(G, B)$, *the union of* (F, A) *and* (G, B).

Definition 2.13 ([9])**.** *Let* U *be an initial universe set and let* E *be a set of parameters. A collection of all fuzzy soft sets over* U *with parameters from* E *is called a fuzzy soft class and is denoted by* $FS(U, E)$.

That is, $FS(U, E) = \{(F, A) | A \subseteq E \text{ and } F : A \to Fuz(U)\}$.

3. Inverse int-fuzzy soft (left, right) ideals over semigroups

In this section, we define inverse int-soft subsemigroup, inverse int-soft (left, right) ideals, inverse int-fuzzy soft subsemigroup and inverse int-fuzzy soft (left, right) ideals over semigroups and investigate some of their properties and support examples.

Definition 3.1. *Let* U *be an initial universe and let* S *be a semigroup. A pair* $(\widehat{F}, U)$ *is called an inverse int-soft set over* S, *where* $\widehat{\mathcal{F}}$ *is a mapping given by* $\widehat{\mathcal{F}} : U \to P(S)$ *and* $P(S)$ *is the power set of* S.

Definition 3.2. *Let* $(\widehat{\mathcal{F}}, U)$ *be an inverse int-soft set over a semigroup* S. $(\widehat{\mathcal{F}}, U)$ *is called an inverse int-soft subsemigroup if and only if* $\widehat{\mathcal{F}}(u)$ *is a subsemigroup of* S *for all* $u \in U$.

Definition 3.3. *Let* $(\widehat{\mathcal{F}}, U)$ *be an inverse int-soft set over a semigroup* S. $(\widehat{\mathcal{F}}, U)$ *is called an inverse int-soft left (right) ideal if and only if* $\widehat{\mathcal{F}}(u)$ *is a left (right) ideal of* S *for all* $u \in U$.

Definition 3.4. *Let* $(\widehat{\mathcal{F}}, U)$ *be an inverse int-soft set over* S. $(\widehat{\mathcal{F}}, U)$ *is called an inverse int-soft ideal if and only if* $(\widehat{\mathcal{F}}, U)$ *is both an inverse int-soft left and an inverse int-soft right ideal over* S.

Definition 3.5. *Let* U *be an initial universe and let* S *be a semigroup. A pair* $(\widehat{\mathbf{F}}, S)$ *is called an int-fuzzy soft set over* S, *where* $\widehat{\mathbf{F}}$ *is a mapping given by* $\widehat{\mathbf{F}} : S \to Fuz(U)$ *is a mapping and* $Fuz(U)$ *is the set of all fuzzy sets on* U.

Table 1: Multiplication table of a semigroup S

$\cdot$	x	y	z
x	x	z	z
y	z	y	z
z	z	z	z

Example 3.1. Let $U = \{u_1, u_2, u_3\}$ and $S = \{x, y, z\}$ be a semigroup having the following multiplication table (Table 1):

We choose the int-fuzzy soft set over U by Definition 3.5 as follows:

$$\widetilde{\mathbf{F}}_x = \{u_1/0.1, u_2/0.4, u_3/0.8\},$$
$$\widetilde{\mathbf{F}}_y = \{u_1/0.2, u_2/0.5, u_3/0.9\},$$
$$\widetilde{\mathbf{F}}_z = \{u_1/0.3, u_2/0.6, u_3/0.9\}.$$

Thus, we can view the int-fuzzy soft set $(\widetilde{\mathbf{F}}, S)$ as a collection of approximations as follow:

$$(\widetilde{\mathbf{F}}, S) = \{\widetilde{\mathbf{F}}_x = \{u_1/0.1, u_2/0.4, u_3/0.8\},$$
$$\widetilde{\mathbf{F}}_y = \{u_1/0.2, u_2/0.5, u_3/0.9\},$$
$$\widetilde{\mathbf{F}}_z = \{u_1/0.3, u_2/0.6, u_3/0.9\}\}.$$

Now, we define an inverse int-fuzzy soft set over a semigroup and give some their operations.

Definition 3.6. *Let U be an initial universe and let S be a semigroup. A pair $(\widetilde{\mathcal{F}}, U)$ is called an inverse int-fuzzy soft set over S, where $\widetilde{\mathcal{F}}$ is a mapping given by $\widetilde{\mathcal{F}} : U \to Fuz(S)$ is a mapping and $Fuz(S)$ is the set of all fuzzy set on S.*

Example 3.2. Let $U = \{u_1, u_2, u_3\}$ and $S = \{x, y, z\}$ be a semigroup having the same multiplication table as Example 3 (Table 1): We choose the inverse int-fuzzy soft set over S by Definition 3.6 as follows:

$$\widetilde{\mathcal{F}}_{u_1} = \{x/0.1, y/0.4, z/0.8\},$$
$$\widetilde{\mathcal{F}}_{u_2} = \{x/0.2, y/0.5, z/0.9\},$$
$$\widetilde{\mathcal{F}}_{u_3} = \{x/0.3, y/0.6, z/0.9\}.$$

Thus, we can view the inverse int-fuzzy soft set $(\widetilde{\mathcal{F}}, U)$ as a collection of approximations as follow:

$$(\widetilde{\mathcal{F}}, U) = \{\widetilde{\mathcal{F}}_{u_1} = \{x/0.1, y/0.4, z/0.8\},$$
$$\widetilde{\mathcal{F}}_{u_2} = \{x/0.2, y/0.5, z/0.9\},$$
$$\widetilde{\mathcal{F}}_{u_3} = \{x/0.3, y/0.6, z/0.9\}\}.$$

Definition 3.7. *Let $(\widetilde{\mathcal{F}}, U)$ and $(\widetilde{\mathcal{G}}, V)$ be two inverse int-fuzzy soft sets over S. The "$(\widetilde{\mathcal{F}}, U)$ and $(\widetilde{\mathcal{G}}, V)$", denoted by $(\widetilde{\mathcal{F}}, U)\widetilde{\sqcap}(\widetilde{\mathcal{G}}, V)$ is defined as*

$$(\widetilde{\mathcal{F}}, U)\widetilde{\sqcap}(\widetilde{\mathcal{G}}, V) = (\widetilde{\mathcal{H}}, U \times V),$$

where $\widetilde{\mathcal{H}}(u, v) = \widetilde{\mathcal{F}}(u) \wedge \widetilde{\mathcal{G}}(v)$ for all $(u, v) \in U \times V$ and $\wedge$ is the operation "fuzzy intersection" of two of fuzzy parameter sets.

Definition 3.8. *Let $(\widetilde{\mathcal{F}}, U)$ and $(\widetilde{\mathcal{G}}, V)$ be two inverse int-fuzzy soft sets over S. The "$(\widetilde{\mathcal{F}}, U)$ OR $(\widetilde{\mathcal{G}}, V)$", denoted by $(\widetilde{\mathcal{F}}, U)\widetilde{\sqcup}(\widetilde{\mathcal{G}}, V)$ is defined as*

$$(\widetilde{\mathcal{F}}, U)\widetilde{\sqcup}(\widetilde{\mathcal{G}}, V) = (\widetilde{\mathcal{K}}, U \times V),$$

where $\widetilde{\mathcal{K}}(u, v) = \widetilde{\mathcal{F}}(u) \vee \widetilde{\mathcal{G}}(v)$ for all $(u, v) \in U \times V$ and $\vee$ is the operation "fuzzy union" of two of fuzzy parameter sets.

Theorem 3.1. *Let $(\widetilde{\mathcal{F}}, U), (\widetilde{\mathcal{G}}, V)$ and $(\widetilde{\mathcal{H}}, W)$ be three inverse int-fuzzy soft sets over a semigroup S. Then:*

(i) $(\widetilde{\mathcal{F}}, U)\widetilde{\sqcap}((\widetilde{\mathcal{G}}, V)\widetilde{\sqcap}(\widetilde{\mathcal{H}}, W)) = ((\widetilde{\mathcal{F}}, U)\widetilde{\sqcap}(\widetilde{\mathcal{G}}, V))\widetilde{\sqcap}(\widetilde{\mathcal{H}}, W)$.
(ii) $(\widetilde{\mathcal{F}}, U)\widetilde{\sqcup}((\widetilde{\mathcal{G}}, V)\widetilde{\sqcup}(\widetilde{\mathcal{H}}, W)) = ((\widetilde{\mathcal{F}}, U)\widetilde{\sqcup}(\widetilde{\mathcal{G}}, V))\widetilde{\sqcup}(\widetilde{\mathcal{H}}, W)$.
(iii) $(\widetilde{\mathcal{F}}, U)\widetilde{\sqcap}((\widetilde{\mathcal{G}}, V)\widetilde{\sqcup}(\widetilde{\mathcal{H}}, W)) = ((\widetilde{\mathcal{F}}, U)\widetilde{\sqcap}(\widetilde{\mathcal{G}}, V))\widetilde{\sqcup}((\widetilde{\mathcal{F}}, U)\widetilde{\sqcap}(\widetilde{\mathcal{H}}, W))$.
(iv) $(\widetilde{\mathcal{F}}, U)\widetilde{\sqcup}((\widetilde{\mathcal{G}}, V)\widetilde{\sqcap}(\widetilde{\mathcal{H}}, W)) = ((\widetilde{\mathcal{F}}, U)\widetilde{\sqcup}(\widetilde{\mathcal{G}}, V))\widetilde{\sqcap}((\widetilde{\mathcal{F}}, U)\widetilde{\sqcup}(\widetilde{\mathcal{H}}, W))$.

Proof. For any $u \in U, v \in V$ and $w \in W$, we have

$$\widetilde{\mathcal{F}}_u \wedge (\widetilde{\mathcal{G}}_v \wedge \widetilde{\mathcal{H}}_w) = (\widetilde{\mathcal{F}}_u \wedge \widetilde{\mathcal{G}}_v) \wedge \widetilde{\mathcal{H}}_w.$$

Thus, $(\widetilde{\mathcal{F}}, U)\widetilde{\sqcap}((\widetilde{\mathcal{G}}, V)\widetilde{\sqcap}(\widetilde{\mathcal{H}}, W)) = ((\widetilde{\mathcal{F}}, U)\widetilde{\sqcap}(\widetilde{\mathcal{G}}, V))\widetilde{\sqcap}(\widetilde{\mathcal{H}}, W)$.

Similarly, we have $(\widetilde{\mathcal{F}}, U)\widetilde{\sqcup}((\widetilde{\mathcal{G}}, V)\widetilde{\sqcup}(\widetilde{\mathcal{H}}, W)) = ((\widetilde{\mathcal{F}}, U)\widetilde{\sqcup}(\widetilde{\mathcal{G}}, V))\widetilde{\sqcup}(\widetilde{\mathcal{H}}, W)$. For any $u \in U, v \in V$ and $w \in W$, we have

$$\widetilde{\mathcal{F}}_u \wedge (\widetilde{\mathcal{G}}_v \vee \widetilde{\mathcal{H}}_w) = (\widetilde{\mathcal{F}}_u \wedge \widetilde{\mathcal{G}}_v) \vee (\widetilde{\mathcal{F}}_u \wedge \widetilde{\mathcal{H}}_w).$$

Thus, $(\widetilde{\mathcal{F}}, U)\widetilde{\sqcap}((\widetilde{\mathcal{G}}, V)\widetilde{\sqcup}(\widetilde{\mathcal{H}}, W)) = ((\widetilde{\mathcal{F}}, U)\widetilde{\sqcap}(\widetilde{\mathcal{G}}, V))\widetilde{\sqcup}((\widetilde{\mathcal{F}}, U)\widetilde{\sqcap}(\widetilde{\mathcal{H}}, W))$.

Similarly, we have

$$(\widetilde{\mathcal{F}}, U)\widetilde{\sqcup}((\widetilde{\mathcal{G}}, V)\widetilde{\sqcap}(\widetilde{\mathcal{H}}, W)) = ((\widetilde{\mathcal{F}}, U)\widetilde{\sqcup}(\widetilde{\mathcal{G}}, V))\widetilde{\sqcap}((\widetilde{\mathcal{F}}, U)\widetilde{\sqcup}(\widetilde{\mathcal{H}}, W)). \qquad \square$$

Now, we define an inverse int-fuzzy soft subsemigroup and an inverse int-fuzzy soft (left, right) ideal over a semigroup and give some their properties with supported examples.

Definition 3.9. *Let $(\widetilde{\mathcal{F}}, U)$ be an inverse int-fuzzy soft set over a semigroup S. $(\widetilde{\mathcal{F}}, U)$ is called an inverse int-fuzzy soft subsemigroup if and only if $\widetilde{\mathcal{F}}_u$ is a fuzzy subsemigroup on S for all $u \in U$.*

Definition 3.10. *Let $(\widetilde{\mathcal{F}}, U)$ be an inverse int-fuzzy soft set over a semigroup S. $(\widetilde{\mathcal{F}}, U)$ is called an inverse int-fuzzy soft left (right) ideal if and only if $\widetilde{\mathcal{F}}_u$ is a fuzzy left (right) on S for all $u \in U$.*

Example 3.3. Let $U = \{u_1, u_2, u_3\}$ and $S = \{x, y, z\}$ be a semigroup having the same multiplication table as Example 3 (Table 1):

Choose $(\widetilde{\mathcal{F}}, U)$ is an inverse int-fuzzy soft set over S as:

$$\widetilde{\mathcal{F}}_{u_1} = \{x/0.1, y/0.4, z/0.8\},$$
$$\widetilde{\mathcal{F}}_{u_2} = \{x/0.2, y/0.5, z/0.9\},$$
$$\widetilde{\mathcal{F}}_{u_3} = \{x/0.3, y/0.6, z/0.9\}.$$

It is easy to verify that $\widetilde{\mathcal{F}}_{u_1}, \widetilde{\mathcal{F}}_{u_2}, \widetilde{\mathcal{F}}_{u_3}$ are both fuzzy left ideals on S.

Hence, $(\widetilde{\mathcal{F}}, U)$ is an inverse int-fuzzy soft left ideal over S.

Definition 3.11. *Let $(\widetilde{\mathcal{F}}, U)$ be an inverse int-fuzzy soft set over S. $(\widetilde{\mathcal{F}}, U)$ is called an inverse int-fuzzy soft ideal if and only if $(\widetilde{\mathcal{F}}, U)$ is both an inverse int-fuzzy soft left and an inverse int-fuzzy soft right ideal over S.*

Example 3.4. Let $U = \{u_1, u_2, u_3\}$ and $S = \{x, y, z\}$ be a semigroup having the same multiplication table as Example 3 (Table 1):

Choose $(\widetilde{\mathcal{F}}, U)$ is an inverse int-fuzzy soft set over S as:

$$\widetilde{\mathcal{F}}(u_1) = \{x/0.1, y/0.3, z/0.5\},$$
$$\widetilde{\mathcal{F}}(u_2) = \{x/0.2, y/0.5, z/0.9\},$$
$$\widetilde{\mathcal{F}}(u_3) = \{x/0.2, y/0.4, z/0.8\}.$$

It is easy to verify that $\widetilde{\mathcal{F}}_{u_1}, \widetilde{\mathcal{F}}_{u_2}, \widetilde{\mathcal{F}}_{u_3}$ are both fuzzy left ideals on S, at the same time, $\widetilde{\mathcal{F}}_{u_1}, \widetilde{\mathcal{F}}_{u_2}, \widetilde{\mathcal{F}}_{u_3}$ are both fuzzy right ideals on S. Hence $(\widetilde{\mathcal{F}}, U)$ is an inverse int-fuzzy soft ideal over S.

Definition 3.12. *Let T be a nonempty subset of a semigroup S and let U be an initial universe. An inverse int-fuzzy soft characteristic function $(\widetilde{\mathcal{C}^T}, U)$ over S if $(\widetilde{\mathcal{C}^T})_u = \chi_T$ for all $u \in U$.*

Definition 3.13. *Let $(\widetilde{\mathcal{F}}, U)$ and $(\widetilde{\mathcal{G}}, V)$ be two inverse int-fuzzy soft sets over a semigroup S. $(\widetilde{\mathcal{F}}, U)$ is called an inverse int-fuzzy soft subset of $(\widetilde{\mathcal{G}}, V)$, denoted by $(\widetilde{\mathcal{F}}, U)\widetilde{\leq}(\widetilde{\mathcal{G}}, V)$, if*

(i) $U \subseteq V$;

(ii) for all $u \in U$ there exists $v \in V$ such that $\widetilde{\mathcal{F}}_u \leq \widetilde{\mathcal{G}}_v$.

Definition 3.14. *Two inverse int-fuzzy soft sets $(\widetilde{\mathcal{F}}, U)$ and $(\widetilde{\mathcal{G}}, V)$ over a semigroup S are called an inverse int-fuzzy soft equal if $(\widetilde{\mathcal{F}}, U)\widetilde{\leq}(\widetilde{\mathcal{G}}, V)$ and $(\widetilde{\mathcal{G}}, V)\widetilde{\leq}(\widetilde{\mathcal{F}}, U)$. We denote by $(\widetilde{\mathcal{F}}, U) = (\widetilde{\mathcal{G}}, V)$.*

Definition 3.15. *Let $(\widetilde{\mathcal{F}}, U)$ and $(\widetilde{\mathcal{G}}, V)$ be two inverse int-fuzzy soft sets over a semigroup S with $U \cap V \neq \emptyset$. The intersection of them denoted by*

$$(\widetilde{\mathcal{F}}, U)\tilde{\wedge}(\widetilde{\mathcal{G}}, V) := (\widetilde{\mathcal{F}} \wedge \widetilde{\mathcal{G}}, U \cap V).$$

Define for each $u \in U \cap V$, $(\widetilde{\mathcal{F}} \wedge \widetilde{\mathcal{G}})_u = \widetilde{\mathcal{F}}_u \wedge \widetilde{\mathcal{G}}_u$. Then, $(\widetilde{\mathcal{F}}, U)\tilde{\wedge}(\widetilde{\mathcal{G}}, V)$ is an inverse int-fuzzy soft set over S.

Definition 3.16. *Let $(\widetilde{\mathcal{F}}, U)$ and $(\widetilde{\mathcal{G}}, V)$ be two inverse int-fuzzy soft sets over a semigroup S. The union of them denoted by $(\widetilde{\mathcal{F}}, U)\tilde{\vee}(\widetilde{\mathcal{G}}, V) := (\widetilde{\mathcal{F}} \vee \widetilde{\mathcal{G}}, U \cup V)$, where for all $u \in U \cup V$,*

$$(\widetilde{\mathcal{F}} \vee \widetilde{\mathcal{G}})_u = \begin{cases} \widetilde{\mathcal{F}}_u, & \text{if } u \in U - V, \\ \widetilde{\mathcal{G}}_u, & \text{if } u \in V - U, \\ \widetilde{\mathcal{F}}_u \vee \widetilde{\mathcal{G}}_u, & \text{if } u \in U \cap V. \end{cases}$$

Then, $(\widetilde{\mathcal{F}}, U)\tilde{\vee}(\widetilde{\mathcal{G}}, V)$ is an inverse int-fuzzy soft set over S.

Lemma 3.1. *Let T be a nonempty subset of a semigroup S. Then, the following properties hold:*

(*i*) *T is a subsemigroup of S if and only if $(\widetilde{\mathcal{C}^T}, U)$ is an inverse int-fuzzy soft subsemigroup over S.*

(*ii*) *T is a (left, right) ideal of S if and only if $(\widetilde{\mathcal{C}^T}, U)$ is an inverse int-fuzzy soft (left, right) ideal on S.*

Proof. It follows from Lemma 2.1. □

In the following theorem, we show that an inverse int-fuzzy soft set over a semigroup is an inverse int-fuzzy soft subsemigroup over a semigroup if and only if an α-level set of inverse int-fuzzy soft set over the semigroup is an inverse int-soft subsemigroup over the semigroup for all $\alpha \in [0, 1]$.

Theorem 3.2. *Let $(\widetilde{\mathcal{F}}, U)$ is an inverse int-fuzzy soft set over a semigroup S. Then $(\widetilde{\mathcal{F}}, U)$ is an inverse int-fuzzy soft subsemigroup over S if and only if $(\widetilde{\mathcal{F}}, U)^\alpha$ is an inverse int-soft subsemigroup over S for all $\alpha \in [0, 1]$.*

Proof. Assume that $(\widetilde{\mathcal{F}}, U)$ is an inverse int-fuzzy soft subsemigroup over S. Let $\alpha \in [0, 1]$, $u \in U, y, z \in (\widetilde{\mathcal{F}}_u)^\alpha$. Then $\widetilde{\mathcal{F}}_u(y) \geq \alpha$ and $\widetilde{\mathcal{F}}_u(z) \geq \alpha$. By the assumption, $\widetilde{\mathcal{F}}_u$ is a fuzzy subsemigroup on S. We then have

$$\widetilde{\mathcal{F}}_u(y \cdot z) \geq \widetilde{\mathcal{F}}_u(y) \wedge \widetilde{\mathcal{F}}_u(z) \geq \alpha.$$

It follows that $y \cdot z \in (\widetilde{\mathcal{F}}_u)^\alpha$. Thus $(\widetilde{\mathcal{F}}_u)^\alpha$ is a subsemigroup of S. Therefore, $(\widetilde{\mathcal{F}}, U)^\alpha$ is an inverse int-soft subsemigroup over S.

Conversely, suppose that $(\widetilde{\mathcal{F}}, U)^\alpha$ is an inverse int-soft subsemigroup over S for all $\alpha \in [0, 1]$. Let $u \in U$ and $y, z \in S$. Choose $\alpha := \widetilde{\mathcal{F}}_u(y) \wedge \widetilde{\mathcal{F}}_u(z)$. Then $y, z \in (\widetilde{\mathcal{F}}_u)^\alpha$. Since $(\widetilde{\mathcal{F}}_u)^\alpha$ is a subsemigroup of S, we then have $y \cdot z \in (\widetilde{\mathcal{F}}_u)^\alpha$. This implies that $\widetilde{\mathcal{F}}_u(y \cdot z) \geq \alpha = \widetilde{\mathcal{F}}_u(y) \wedge \widetilde{\mathcal{F}}_u(z)$. Thus $\widetilde{\mathcal{F}}_u$ is a fuzzy subsemigroup on S. By the assumption, $y \cdot z \in (\widetilde{\mathcal{F}})^\alpha$. Thus

$$\widetilde{\mathcal{F}}_u(y \cdot z) \geq \alpha = \widetilde{\mathcal{F}}_u(y) \wedge \widetilde{\mathcal{F}}_u(z).$$

Hence $\widetilde{\mathcal{F}}_u$ is a fuzzy subsemigroup on S. Therefore $(\widetilde{\mathcal{F}}, U)$ is an inverse int-fuzzy soft subsemigroup over S. □

In the following theorem, we show that an inverse int-fuzzy soft set over a semigroup is an inverse int-fuzzy soft left (right) ideal over a semigroup if and only if an α-level set of inverse int-fuzzy soft set over the semigroup is an inverse int-soft left (right) ideal over the semigroup for all $\alpha \in [0,1]$.

Theorem 3.3. *Let $(\widetilde{\mathcal{F}}, U)$ be an inverse int-fuzzy soft set over a semigroup S. Then $(\widetilde{\mathcal{F}}, U)$ is an inverse int-fuzzy soft left (right) ideal over S if and only if $(\widetilde{\mathcal{F}}, U)^\alpha$ is an inverse int-soft left (right) ideal over S for all $\alpha \in [0,1]$.*

Proof. We prove only an inverse int-fuzzy soft left ideal, an inverse int-soft right ideal is similar. Assume that $(\widetilde{\mathcal{F}}, U)$ is an inverse int-fuzzy soft left ideal over S. Let $\alpha \in [0,1]$, $u \in U, y \in (\widetilde{\mathcal{F}}_u)^\alpha$ and $x \in S$. Then $\widetilde{\mathcal{F}}_u(y) \geq \alpha$. By the assumption, $\widetilde{\mathcal{F}}_u$ is a fuzzy left ideal on S. We then have

$$\widetilde{\mathcal{F}}_u(x \cdot y) \geq \widetilde{\mathcal{F}}_u(y) \geq \alpha.$$

It follows that $x \cdot y \in (\widetilde{\mathcal{F}}_u)^\alpha$. Thus $(\widetilde{\mathcal{F}}_u)^\alpha$ is a left ideal of S. Therefore $(\widetilde{\mathcal{F}}, U)^\alpha$ is an inverse int-soft left ideal over S.

Conversely, suppose that $(\widetilde{\mathcal{F}}, U)^\alpha$ is an inverse int-soft left ideal over S for all $\alpha \in [0,1]$. Let $u \in U$ and $y \in S$. Choose $\alpha := \widetilde{\mathcal{F}}_u(y)$. Then $y \in (\widetilde{\mathcal{F}}_u)^\alpha$. Since $(\widetilde{\mathcal{F}}_u)^\alpha$ is a left ideal of S, we then have $x \cdot y \in (\widetilde{\mathcal{F}}_u)^\alpha$ for all $x \in S$. This implies that

$$\widetilde{\mathcal{F}}_u(x \cdot y) \geq \alpha = \widetilde{\mathcal{F}}_u(y).$$

Thus $\widetilde{\mathcal{F}}_u$ is a fuzzy left ideal on S. Therefore $(\widetilde{\mathcal{F}}, U)$ is an inverse int-fuzzy soft left ideal over S. □

Theorem 3.4. *Let $(\widetilde{\mathcal{F}}, U)$ be an inverse int-fuzzy soft set over a semigroup S. Then $(\widetilde{\mathcal{F}}, U)$ is an inverse int-fuzzy soft ideal over S if and only if $(\widetilde{\mathcal{F}}, U)^\alpha$ is an inverse int-soft ideal over S for all $\alpha \in [0,1]$.*

Proof. It follows from Theorem 3.3. □

The following theorem shows that $(\widetilde{\mathcal{F}}, U)\tilde{\wedge}(\widetilde{\mathcal{G}}, V)$ with $U \cap V \neq \emptyset$ is an inverse int-fuzzy soft subsemigroup over a semigroup.

Theorem 3.5. *Let $(\widetilde{\mathcal{F}}, U)$ and $(\widetilde{\mathcal{G}}, V)$ be two inverse int-fuzzy soft subsemigroups over a semigroup S. If $U \cap V \neq \emptyset$ then $(\widetilde{\mathcal{F}}, U)\tilde{\wedge}(\widetilde{\mathcal{G}}, V)$ is an inverse int-fuzzy soft subsemigroup over S.*

Proof. Let $u \in U \cap V \neq \emptyset$. Thus $(\widetilde{\mathcal{F}} \wedge \widetilde{\mathcal{G}})_u = \widetilde{\mathcal{F}}_u \wedge \widetilde{\mathcal{G}}_u$. Since $\widetilde{\mathcal{F}}_u$ and $\widetilde{\mathcal{G}}_u$ are both fuzzy subsemigroup on S and the intersection of two fuzzy subsemigroups is a fuzzy subsemigroup (Lemma 2.2), we have $(\widetilde{\mathcal{F}} \wedge \widetilde{\mathcal{G}})_u$ is a fuzzy subsemigroup on S. Therefore $(\widetilde{\mathcal{F}}, U)\tilde{\wedge}(\widetilde{\mathcal{G}}, V)$ is an inverse int-fuzzy soft subsemigroups over S. □

The following theorem shows that $(\widetilde{\mathcal{F}}, U)\tilde{\wedge}(\widetilde{\mathcal{G}}, V)$ with $U \cap V \neq \emptyset$ is an inverse int-fuzzy soft (left, right) ideal over a semigroup.

Theorem 3.6. *Let $(\widetilde{\mathcal{F}}, U)$ and $(\widetilde{\mathcal{G}}, V)$ be two inverse int-fuzzy (left, right) ideals over a semigroup S. If $U \cap V \neq \emptyset$ then $(\widetilde{\mathcal{F}}, U)\tilde{\wedge}(\widetilde{\mathcal{G}}, V)$ is an inverse int-fuzzy soft (left, right) ideal over S.*

Proof. Let $u \in U \cap V \neq \emptyset$. Thus $(\widetilde{\mathcal{F}} \wedge \widetilde{\mathcal{G}})_u = \widetilde{\mathcal{F}}_u \wedge \widetilde{\mathcal{G}}_u$. Since $\widetilde{\mathcal{F}}_u$ and $\widetilde{\mathcal{G}}_u$ are both fuzzy (left, right) ideals on S and the intersection of two fuzzy (left, right) ideal is a fuzzy (left, right) ideal (Lemma 2.2), we have $(\widetilde{\mathcal{F}} \wedge \widetilde{\mathcal{G}})_u$ is a fuzzy (left, right) ideal on S. Therefore $(\widetilde{\mathcal{F}}, U)\tilde{\wedge}(\widetilde{\mathcal{G}}, V)$ is an inverse int-fuzzy soft (left, right) ideal over S. □

In general, the union of two inverse int-fuzzy soft semigroups is not an inverse int-fuzzy soft semigroup, as is shown in the following:

Example 3.5. Let $U = \{u_1\}, V = \{u_1\}$ and $S = \{x, y, z\}$ be a semigroup having the same multiplication table as Example 3.1 (Table 1): Choose $(\widetilde{\mathcal{F}}, U)$ and $(\widetilde{\mathcal{G}}, V)$ are inverse int-fuzzy soft sets over S as:

$$\widetilde{\mathcal{F}}(u_1) = \{x/0.2, y/0.7, z/0.3\}$$

and

$$\widetilde{\mathcal{G}}(u_1) = \{x/0.8, y/0.3, z/0.4\}.$$

It is easy to verify that $(\widetilde{\mathcal{F}}, U)$ and $(\widetilde{\mathcal{G}}, V)$ are both inverse int-fuzzy soft subsemigroups over S. Then, $(\widetilde{\mathcal{F}}, U)\tilde{\vee}(\widetilde{\mathcal{G}}, V)$ as $\widetilde{\mathcal{F}}_u \vee \widetilde{\mathcal{G}}_{u_1} = \{x/0.8, y/0.7, z/0.4\}$. We see that

$$(\widetilde{\mathcal{F}} \vee \widetilde{\mathcal{G}})_{u_1}(x \cdot y) = (\widetilde{\mathcal{F}} \vee \widetilde{\mathcal{G}})_{u_1}(z) = 0.4 < 0.7 = (\widetilde{\mathcal{F}} \vee \widetilde{\mathcal{G}})_{u_1}(x) \cap (\widetilde{\mathcal{F}} \vee \widetilde{\mathcal{G}})_{u_1}(y).$$

Thus, $(\widetilde{\mathcal{F}} \vee \widetilde{\mathcal{G}})_{u_1}$ is not a fuzzy subsemigroup on S. Hence, the union of two inverse int-fuzzy soft semigroups is not an inverse int-fuzzy soft semigroup.

The next theorem shows that $(\widetilde{\mathcal{F}}, U)\tilde{\vee}(\widetilde{\mathcal{G}}, V)$ is an inverse int-fuzzy soft subsemigroup over a semigroup under some conditions.

Theorem 3.7. *Let $(\widetilde{\mathcal{F}}, U)$ and $(\widetilde{\mathcal{G}}, V)$ be two inverse int-fuzzy soft subsemigroups over a semigroup S. If $U \cap V = \emptyset$ then $(\widetilde{\mathcal{F}}, U)\tilde{\vee}(\widetilde{\mathcal{G}}, V)$ is an inverse int-fuzzy soft subsemigroup over S.*

Proof. Suppose that $U \cap V = \emptyset$. Thus $(\widetilde{\mathcal{F}} \vee \widetilde{\mathcal{G}})_u = \widetilde{\mathcal{F}}_u$ or $\widetilde{\mathcal{G}}_u$.

Since $\widetilde{\mathcal{F}}_u$ and $\widetilde{\mathcal{G}}_u$ are both fuzzy subsemigroup on S, we have $(\widetilde{\mathcal{F}} \vee \widetilde{\mathcal{G}})_u$ is a fuzzy subsemigroup on S. Therefore, $(\widetilde{\mathcal{F}}, U)\tilde{\vee}(\widetilde{\mathcal{G}}, V)$ is an inverse int-fuzzy soft subsemigroups over S. □

Theorem 3.8. *Let $(\widetilde{\mathcal{F}}, U)$ and $(\widetilde{\mathcal{G}}, V)$ be two inverse int-fuzzy soft subsemigroups over a semigroup S. If $U \subseteq V$ or $V \subseteq U$ then $(\widetilde{\mathcal{F}}, U)\tilde{\vee}(\widetilde{\mathcal{G}}, V)$ is an inverse int-fuzzy soft subsemigroup over S.*

Proof. It suffices to show the theorem for the case $U \subseteq V$.

Let $u \in U \cup V$. Since $U \subseteq V$, we have $U \cup V = V$.

If $u \in V - U$ then $(\widetilde{\mathcal{F}} \vee \widetilde{\mathcal{G}})_u = \widetilde{\mathcal{G}}_u$.

If $u \in U \cap V = U$ then $(\widetilde{\mathcal{F}} \vee \widetilde{\mathcal{G}})_u = \widetilde{\mathcal{F}}_u$. Thus, $(\widetilde{\mathcal{F}} \vee \widetilde{\mathcal{G}})_u$ is a fuzzy subsemigroups on S. Hence $(\widetilde{\mathcal{F}}, U)\tilde{\vee}(\widetilde{\mathcal{G}}, V)$ is an inverse int-fuzzy soft subsemigroup over S. □

The next theorem shows that $(\widetilde{\mathcal{F}}, U)\tilde{\vee}(\widetilde{\mathcal{G}}, V)$ is an inverse int-fuzzy soft (left, right) ideals over a semigroup.

Theorem 3.9. *Let $(\widetilde{\mathcal{F}}, U)$ and $(\widetilde{\mathcal{G}}, V)$ be two inverse int-fuzzy (left, right) ideals over a semigroup S. Then $(\widetilde{\mathcal{F}}, U)\tilde{\vee}(\widetilde{\mathcal{G}}, V)$ is an inverse int-fuzzy soft (left, right) ideal over S.*

Proof. Let $u \in U \cup V$.

If $u \in U - V$ or $u \in V - U$ then $(\widetilde{\mathcal{F}} \vee \widetilde{\mathcal{G}})_u = \widetilde{\mathcal{F}}_u$ or $\widetilde{\mathcal{G}}_u$.

Since $\widetilde{\mathcal{F}}_u$ and $\widetilde{\mathcal{G}}_u$ are both fuzzy (left, right) ideals on S, we have $(\widetilde{\mathcal{F}} \vee \widetilde{\mathcal{G}})_u$ is a fuzzy (left, right) ideal on S.

If $u \in U \cap V$ then $(\widetilde{\mathcal{F}} \vee \widetilde{\mathcal{G}})_u = \widetilde{\mathcal{F}}_u \vee \widetilde{\mathcal{G}}_u$. Since $\widetilde{\mathcal{F}}_u$ and $\widetilde{\mathcal{G}}_u$ are both fuzzy (left, right) ideals on S for all $x, y \in S$, we have

$$(\widetilde{\mathcal{F}} \vee \widetilde{\mathcal{G}})_u(x \cdot y) = (\widetilde{\mathcal{F}})_u(x \cdot y) \vee (\widetilde{\mathcal{G}})_u(x \cdot y) \geq (\widetilde{\mathcal{F}})_u(y) \vee (\widetilde{\mathcal{G}})_u(y) = (\widetilde{\mathcal{F}} \vee \widetilde{\mathcal{G}})_u(y).$$

Hence $(\widetilde{\mathcal{F}} \vee \widetilde{\mathcal{G}})_u$ is a fuzzy (left, right) ideal on S. Therefore, $(\widetilde{\mathcal{F}}, U)\tilde{\vee}(\widetilde{\mathcal{G}}, V)$ is an inverse int-fuzzy soft (left, right) ideal over S. □

4. Mapping on classes of inverse int-fuzzy soft (left, right) ideals over semigroups

In this section, we construct the image of inverse int-fuzzy soft subsemigroups and int-fuzzy soft (left, right) ideals over semigroups.

Now, we define notations and definitions of the images of functions as follows.

Definition 4.1. *Let U be an initial universe set and let S be a semigroup. A collection of all inverse int-fuzzy soft sets over S is called an inverse int-fuzzy soft class and is denoted by $\widetilde{FS}(S, U)$. That is, $\widetilde{FS}(S, U) = \{(\widetilde{\mathcal{F}}, U) | \widetilde{\mathcal{F}} : U \to Fuz(S)\}$.*

For any sets A and B, define $M(U, V) := \{f | f : U \to V\}$.

Let $\widetilde{FS}(S, U)$ and $\widetilde{FS}(T, V)$ be inverse int-fuzzy soft classes over a semigroup S and a semigroup T with the initial universes from U and V, respectively. That is,

$\widetilde{FS}(S, U) = \{(\widetilde{\mathcal{F}}, U) | \widetilde{\mathcal{F}} : U \to \text{Fuz}(S)\}$,

$\widetilde{FS}(T, V) = \{(\widetilde{\mathcal{G}}, V) | \widetilde{\mathcal{G}} : V \to \text{Fuz}(T)\}$.

Let $u : S \to T$ and $v : U \to V$ be two mappings.

We want to construct two functions:

(i) $\Psi : \widetilde{FS}(S, U) \to \widetilde{FS}(T, V)$;

(ii) $\Sigma : \widetilde{FS}(T,V) \to \widetilde{FS}(S,U)$.

(i) Let $(\widetilde{\mathcal{F}},U) \in \widetilde{FS}(S,U)$.

Let $\Omega : M(U,\mathrm{Fuz}(S)) \to M(v(U),\mathrm{Fuz}(T))$ be defined as follows:

Given $y \in T$ and $\beta \in v(U)$, define

$$[\Omega(\widetilde{\mathcal{F}})]_\beta(y) = \begin{cases} \bigvee\limits_{x\in u^{-1}(y)} [\bigvee\limits_{\delta\in v^{-1}(\beta)\cap U} \widetilde{\mathcal{F}}_\delta(x)], & \text{if } u^{-1}(y) \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\Psi(\widetilde{\mathcal{F}},U) := (\Omega(\widetilde{\mathcal{F}}), v(U))$. Then, $(\Omega(\widetilde{\mathcal{F}}), v(U))$ is an image of $(\widetilde{\mathcal{F}},U)$ under the function Ψ.

(ii) Let $(\widetilde{\mathcal{G}},V) \in \widetilde{FS}(V,T)$.

Let $\Gamma : M(V,\mathrm{Fuz}(T)) \to M(v^{-1}(V),\mathrm{Fuz}(S))$ be defined as follows:

Given $x \in S$ and $\delta \in v^{-1}(V)$, define

$$[\Gamma(\widetilde{\mathcal{G}})]_\delta(x) = \widetilde{\mathcal{G}}_{v(\delta)}(u(x)).$$

Let $\Sigma(\widetilde{\mathcal{G}},V) := (\Gamma(\widetilde{\mathcal{G}}), v^{-1}(S))$.

Remark 4.1. Let $\widetilde{\mathcal{FS}}(S,U)$ and $\widetilde{FS}(T,V)$ be two inverse int-fuzzy soft classes over semigroups S and T, respectively. Suppose that $u : S \to T$ and $v : U \to V$ are two mappings and $\Psi : \widetilde{FS}(S,U) \to \widetilde{FS}(T,V)$ and $\Sigma : \widetilde{FS}(T,V) \to \widetilde{FS}(S,U)$ are also two mappings. If u and v are bijections, then $\Sigma \circ \Psi = I$ and $\Psi \circ \Sigma = I$, where $\circ$ is a composite mapping and I is an identity mapping.(Hence Ψ and Σ are bijections.)

The following theorem shows that the image of an inverse int-fuzzy soft subsemigroup over a semigroup is an inverse int-fuzzy soft subsemigroup over the semigroup under some conditions.

Theorem 4.1. *Let $\widetilde{FS}(S,U)$ and $\widetilde{FS}(T,V)$ be two inverse int-fuzzy soft classes over semigroups S and T, respectively. Suppose that $\Psi : \widetilde{FS}(S,U) \to \widetilde{FS}(T,V)$ is a mapping, $v : U \to V$ is an injection and u is an isomorphism mapping from S to T. If $(\widetilde{\mathcal{F}},U)$ is an inverse int-fuzzy soft subsemigroup over S then $\Psi(\widetilde{\mathcal{F}},U)$ is an inverse int-fuzzy soft subsemigroup over T.*

Proof. Suppose that $(\widetilde{\mathcal{F}},U)$ is an inverse int-fuzzy soft subsemigroup over S. By Theorem 3, $(\widetilde{\mathcal{F}},U)^\alpha$ is an inverse int-soft subsemigroup over S for all $\alpha \in [0,1]$. We shall show that $\Psi(\widetilde{\mathcal{F}},U)^\alpha = (\Omega(\widetilde{\mathcal{F}}), v(U))^\alpha$ is an inverse int-soft subsemigroup over T for all $\alpha \in [0,1]$. Let $p \in v(U)$. There exists $q \in U$ such that $p = v(q)$. Let $z_1, z_2 \in ([\Omega(\widetilde{\mathcal{F}})]_p)^\alpha$. Thus

$$\alpha \leq [\Omega(\widetilde{\mathcal{F}})]_p(z_1) = \bigvee_{s\in u^{-1}(z_1)} [\bigvee_{\beta\in v^{-1}(p)\cap U} \widetilde{\mathcal{F}}_\beta(s)] = \bigvee_{s\in u^{-1}(z_1)} \widetilde{\mathcal{F}}_q(s),$$

and

$$\alpha \leq [\Omega(\widetilde{\mathcal{F}})]_p(z_2) = \bigvee_{s\in u^{-1}(z_2)} [\bigvee_{\beta\in v^{-1}(p)\cap U} \widetilde{\mathcal{F}}_\beta(s)] = \bigvee_{s\in u^{-1}(z_2)} \widetilde{\mathcal{F}}_q(s).$$

This implies that there exist $s_1, s_2 \in S$ such that $u(s_1) = z_1, u(s_2) = z_2$ and $\widetilde{\mathcal{F}}_q(s_1) \geq \alpha, \widetilde{\mathcal{F}}_q(s_2) \geq \alpha$. Then,

$$\begin{aligned}
[\Omega(\widetilde{\mathcal{F}})]_p(z_1 \cdot z_2) &= \bigvee_{s\in u^{-1}(z_1\cdot z_2)} [\bigvee_{\beta\in v^{-1}(p)\cap U} \widetilde{\mathcal{F}}_\beta(s)] \\
&= \bigvee_{s\in u^{-1}(z_1\cdot z_2)} \widetilde{\mathcal{F}}_q(s) \\
&\geq \widetilde{\mathcal{F}}_q(s_1 \cdot s_2) \\
&\geq \widetilde{\mathcal{F}}_q(s_1) \wedge \widetilde{\mathcal{F}}_q(s_2) \geq \alpha.
\end{aligned}$$

Thus $z_1 \cdot z_2 \in ([\Omega(\widetilde{\mathcal{F}})]_p)^\alpha$. Hence $([\Omega(F)]_p)^\alpha$ is an inverse int-soft subsemigroup of T. Therefore $\Psi(\widetilde{\mathcal{F}}, U)^\alpha = (\Omega(\widetilde{\mathcal{F}}), v(U))^\alpha$ is an inverse int-fuzzy soft subsemigroup over T for all $\alpha \in [0, 1]$. By Theorem 3.2, $\Psi(\widetilde{\mathcal{F}}, U)$ is an inverse int-fuzzy soft subsemigroup over T. □

The following theorem shows that $\Sigma(\widetilde{\mathcal{G}}, V)$ of an inverse int-fuzzy soft subsemigroup over a semigroup is an inverse int-fuzzy soft subsemigroup over the semigroup under some conditions.

Theorem 4.2. *Let $\widetilde{FS}(S, U)$ and $\widetilde{FS}(T, V)$ be two inverse int-fuzzy soft classes over semigroups S and T, respectively. Suppose that $\Sigma : \widetilde{FS}(T, V) \to \widetilde{FS}(S, V)$ is a mapping, $v : U \to V$ is a mapping and u is a homomorphism from S to T. If $(\widetilde{\mathcal{G}}, V)$ is an inverse int-fuzzy soft subsemigroup over T then $\Sigma(\widetilde{\mathcal{G}}, V)$ is an inverse int-fuzzy soft subsemigroup over S.*

Proof. Assume that $(\widetilde{\mathcal{G}}, V)$ is an inverse int-fuzzy soft subsemigroup over T. Let $\delta \in v^{-1}(V)$. We then have $v(\delta) \in V$. This implies that $\widetilde{\mathcal{G}}_{v(\delta)}$ is an inverse int-fuzzy subsemigroup on T. Let $x, y \in S$. Then

$$\begin{aligned}
[\Gamma(\widetilde{\mathcal{G}})]_\delta(x \cdot y) &= \widetilde{\mathcal{G}}_{v(\delta)}(u(x \cdot y)) \\
&= \widetilde{\mathcal{G}}_{v(\delta)}(u(x) \cdot u(y)) \\
&\geq \widetilde{\mathcal{G}}_{v(\delta)}(u(x)) \wedge \widetilde{\mathcal{G}}_{v(\delta)}(u(y)) \\
&= [\Gamma(\widetilde{\mathcal{G}})]_\delta(x) \wedge [\Gamma(\widetilde{\mathcal{G}})]_\delta(y).
\end{aligned}$$

This implies that $[\Gamma(\widetilde{\mathcal{G}})]_\delta$ is an inverse int-fuzzy subsemigroup on S. Hence $\Sigma(\widetilde{\mathcal{G}}, V)$ is an inverse int-fuzzy soft subsemigroup over S. □

The following corollary shows that $(\Sigma\circ\Psi)(\widetilde{\mathcal{F}}, U) = (\widetilde{\mathcal{F}}, U)$ and $(\Psi\circ\Sigma)(\widetilde{\mathcal{G}}, V) = (\widetilde{\mathcal{G}}, V)$ for every inverse int-fuzzy soft subsemigroups $(\widetilde{\mathcal{F}}, U)$ and $(\widetilde{\mathcal{G}}, V)$, where $\circ$ is a composite mapping, under certain conditions.

Corollary 4.1. *Let $\widetilde{FS}(S,U)$ and $\widetilde{FS}(T,V)$ be two inverse int-fuzzy soft classes over semigroups S and T, respectively. Suppose that $u : S \to T$ and $v : U \to V$ are two mappings and $\Psi : \widetilde{FS}(S,U) \to \widetilde{FS}(T,V)$ and $\Sigma : \widetilde{FS}(T,V) \to \widetilde{FS}(S,U)$ are also two mappings. If u and v are bijections, then $(\Sigma \circ \Psi)(\widetilde{\mathcal{F}},U) = (\widetilde{\mathcal{F}},U)$ and $(\Psi \circ \Sigma)(\widetilde{\mathcal{G}},V) = (\widetilde{\mathcal{G}},V)$ for every inverse int-fuzzy soft subsemigroups $(\widetilde{\mathcal{F}},U)$ and $(\widetilde{\mathcal{G}},V)$, where $\circ$ is a composite mapping.*

Proof. It follows from Remark 4.1. □

5. Conclusions

In this paper, we defined inverse int-soft sets, inverse int-fuzzy soft sets, inverse int-soft subsemigroups, inverse int-fuzzy soft subsemigroups and inverse int-soft (left, right) ideals, inverse int-fuzzy soft (left, right) ideal over a semigroup. We obtained some of their properties such as AND operation, OR operation, union and intersection of them over semigroups with supported examples. We also proved that the images of inverse int-fuzzy soft (left, right) ideals over semigroups are the inverse int-fuzzy soft (left, right) ideals over semigroups under some conditions.

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Ordered almost ideals and fuzzy ordered almost ideals in ordered semigroups

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Abstract. In this paper, we define ordered almost ideals and fuzzy ordered almost ideals in ordered semigroups and we study some of their properties. We give relationship between fuzzy ordered almost ideals and ordered almost ideals of ordered semigroups.
Keywords: ordered semigroups, ordered almost ideals, fuzzy ordered almost ideals.

1. Introduction

The notion of almost ideals (or A-ideals) of semigroups was introduced by Grosek and Satko [1] in 1980. Afterwards, they studied minimal almost ideals and smallest almost ideals of semigroups in [6] and [2], respectively.

In 1965, Zadeh [10] introduced the concept of fuzzy subsets. Rosenfeld [4] applied the concept of Zadeh to define fuzzy subgroups and fuzzy ideals in groups. In [3], Kuroki studied various kinds of fuzzy ideals in semigroups and characterized them. In 2018, Wattanatripop, Chinram and Changphas [9] introduced the notion of fuzzy almost ideals in semigroups by using the concept of almost ideals and fuzzy ideals of semigroups. In [7], Suebsung, Wattanatripop and Chinram defined and studied some properties of almost ideals and fuzzy almost ideals of ternary semigroups. Moreover, they introduced the notion of minimal fuzzy almost ideals of ternary semigroups and studied properties of them. Recently, Solano, Suebsung and Chinram [8] extended the ideal of almost ideals to n-ary semigroups.

In this paper, we define ordered almost ideals and fuzzy ordered almost ideals in ordered semigroups and we study some of their properties. We give

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relationship between fuzzy ordered almost ideals and ordered almost ideals of ordered semigroups.

2. Preliminaries

In this section, we give some basic definitions and results, which are necessary for the subsequent sections.

Definition 2.1. Let S be a set with a binary operation $\cdot$ and a binary relation $\leq$. Then $(S, \cdot, \leq)$ is called an *ordered semigroup* if:

1. $(S, \cdot)$ is a semigroup;
2. $(S, \leq)$ is a partially ordered set;
3. for all $x, y, z \in S$, if $x \leq y$, then $xz \leq yz$ and $zx \leq zy$.

Let S be an ordered semigroup. For a nonempty subset A of S, we denote

$$(A] := \{x \in S \mid x \leq a \text{ for some a } \in A\}.$$

Proposition 2.2. *Let A and B be any two nonempty subsets of an ordered semigroup S.*

(1) $A \subseteq (A]$.

(2) *If $A \subseteq B$, then $(A] \subseteq (B]$.*

For nonempty subsets A and B of S, denote

$$AB := \{ab \mid a \in A \text{ and } b \in B\}.$$

If $s \in S$, we let $As := A\{s\}$, $sA := \{s\}A$.

Definition 2.3. ([1]) Let S be a semigroup.

1. A nonempty subset L of S is called a *left almost ideal* of S if $sL \cap L \neq \emptyset$ for any $s \in S$.
2. A nonempty subset R of S is called a *right almost ideal* of S if $Rs \cap R \neq \emptyset$ for any $s \in S$.
3. A nonempty subset I of S is called an *almost ideal* of S if I is both a left almost ideal and a right almost ideal of S.

Definition 2.4. Let S be an ordered semigroup. A nonempty subset A is called a *left (resp. right) ideal* of S if:

1. $SA \subseteq A$ (resp. $AS \subseteq A$);
2. if $a \in A$ and $s \in S$ such that $s \leq a$, then $s \in A$, that is $(A] \subseteq A$.

If A is both a left ideal and a right ideal of S, then it is called an *ideal* of S.

Definition 2.5. Let S be an ordered semigroup. The element $a \in S$ is called an *idempotent* if $a \leq a^2$.

A function f from a set S to the unit interval $[0,1]$ is called a *fuzzy subset* of S. For any two fuzzy subsets f and g of a nonempty set S, define the *union* and *intersection* of f and g, denoted by $f \cup g$ and $f \cap g$, by for all $x \in S$,

$$(f \cup g)(x) = \max\{f(x), g(x)\},$$
$$(f \cap g)(x) = \min\{f(x), g(x)\}.$$

Let f and g be two fuzzy subsets of an ordered semigroup S. Define $f \subseteq g$ by

$$f \subseteq g \Longleftrightarrow f(x) \leq g(x), \quad \text{for all} \quad x \in S.$$

For a fuzzy subset f of S, the *support* of f is defined by

$$supp(f) = \{x \in S \mid f(x) \neq 0\}.$$

The *characteristic mapping* of a subset A of S is a fuzzy subset of S defined by

$$C_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases}$$

The definition of fuzzy points was given by Pu and Liu [5]. Let $s \in S$ and $\alpha \in (0,1]$. A *fuzzy point* s_α of a set S is a fuzzy subset of S defined by

$$s_\alpha(x) = \begin{cases} \alpha, & x = s, \\ 0, & x \neq s. \end{cases}$$

Let $F(S)$ be the set of all fuzzy subsets of an ordered semigroup S. For any $f, g \in F(S)$, we define the *product* of f and g by

$$(f \circ g)(x) := \begin{cases} \sup_{x \leq uv} \min\{f(u), g(v)\}, & \text{if } x \leq uv \text{ where } u, v \in S, \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 2.6. *Let f, g and h be fuzzy subsets of an ordered semigroup S.*

(1) *If $f \subseteq g$, then $f \circ h \subseteq g \circ h$ and $h \circ f \subseteq h \circ g$.*

(2) *If $f \subseteq g$, then $f \cap h \subseteq g \cap h$.*

(3) *If $f \subseteq g$, then $f \cup h \subseteq g \cup h$.*

(4) *If $f \subseteq g$, then $supp(f) \subseteq supp(g)$.*

Definition 2.7. Let S be an ordered semigroup. A fuzzy subset f of S is called a *fuzzy left (resp. right) ideal* of S if for all $x, y \in S$,

1. if $x \leq y$, then $f(x) \geq f(y)$;
2. $f(xy) \geq f(y)$ (resp. $f(xy) \geq f(x)$).

A fuzzy subset f of S is called a *fuzzy ideal* of S if it is both a fuzzy left ideal and a fuzzy right ideal of S, that is, for all $x, y \in S$,

1. if $x \leq y$, then $f(x) \geq f(y)$;
2. $f(xy) \geq \max\{f(x), f(y)\}$.

3. Ordered almost ideals

In this section, we introduce the definition of ordered almost ideals in ordered semigroups and we study some of their properties.

Definition 3.1. Let S be an ordered semigroup.

1. A nonempty subset L of S is called a *left ordered almost ideal* of S if $(sL] \cap L \neq \emptyset$, for all $s \in S$.
2. A nonempty subset R of S is called a *right ordered almost ideal* of S if $(Rs] \cap R \neq \emptyset$, for all $s \in S$.
3. A nonempty subset I of S is called an *ordered almost ideal* of S if it is both a left ordered almost ideal and a right ordered almost ideal of S.

Remark 3.2. Let S be an ordered semigroup.

(i) Every left ideal of S is a left ordered almost ideal of S.

(ii) Every right ideal of S is a right ordered almost ideal of S.

(iii) Every ideal of S is an ordered almost ideal of S.

Proof. We prove that (i) holds. Let L be a left ideal of S and $s \in S$. Then, $sL \neq \emptyset$ and $sL \subseteq L$. So, $\emptyset \neq (sL] \subseteq (L] \subseteq L$. Thus $(sL] \cap L \neq \emptyset$. Hence, L is a left ordered almost ideal of S. (ii) and (iii) can be seen in the same manner. □

The converse of Remark 3.2(i), does not hold true in general, as Example 3.3 shows.

Example 3.3. Consider the ordered semigroup $\mathbb{Z}_6 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$ under the addition on $\mathbb{Z}_6$ and the order $\leq := \{(\bar{0}, \bar{0}), (\bar{1}, \bar{1}), (\bar{2}, \bar{2}), (\bar{3}, \bar{3}), (\bar{4}, \bar{4}), (\bar{5}, \bar{5})\}$. Let $L = \{\bar{1}, \bar{3}, \bar{4}\}$. We have
$(\bar{0} + L] \cap L = L \cap L = L$,
$(\bar{1} + L] \cap L = \{\bar{2}, \bar{4}, \bar{5}\} \cap L = \{\bar{4}\}$,
$(\bar{2} + L] \cap L = \{\bar{0}, \bar{3}, \bar{5}\} \cap L = \{\bar{3}\}$,
$(\bar{3} + L] \cap L = \{\bar{0}, \bar{1}, \bar{4}\} \cap L = \{\bar{1}, \bar{4}\}$,
$(\bar{4} + L] \cap L = \{\bar{1}, \bar{2}, \bar{5}\} \cap L = \{\bar{1}\}$,

$(\bar{5}+L]\cap L=\{\bar{0},\bar{2},\bar{3}\}\cap L=\{\bar{3}\}$.
Then $(s+L]\cap L\neq\emptyset$ for all $s\in\mathbb{Z}_6$. Thus L is a left ordered almost ideal of $\mathbb{Z}_6$. However, L is not a left ideal of $\mathbb{Z}_6$ because $\bar{0}=\bar{6}=\bar{5}+\bar{1}\in\bar{5}+L$ but $\bar{0}\notin L$.

Theorem 3.4. *Let S be an ordered semigroup. Then the following conditions hold.*

(1) *If L is a left ordered almost ideal of S, then every subset L' of S such that $L\subseteq L'$ is a left ordered almost ideal of S.*

(2) *If R is a right ordered almost ideal of S, then every subset R' of S such that $R\subseteq R'$ is a right ordered almost ideal of S.*

(3) *If I is an ordered almost ideal of S, then every subset I' of S such that $I\subseteq I'$ is an ordered almost ideal of S.*

Proof. Assume that L is a left ordered almost ideal of S. Let L' be a subset of S such that $L\subseteq L'$ and $s\in S$. Then $sL\subseteq sL'$, so $(sL]\subseteq(sL']$. This implies that $\emptyset\neq(sL]\cap L\subseteq(sL']\cap L'$. Thus L' is a left ordered almost ideal of S. This proves (1). In the case R is a right ordered almost ideal of S, the proof follows similarly. By conditions (1) and (2), it follows that the condition (3) holds. □

Corollary 3.5. *Let S be an ordered semigroup.*

(1) *If L_1 and L_2 are left ordered almost ideals of S, then $L_1\cup L_2$ is a left ordered almost ideal of S.*

(2) *If R_1 and R_2 are right ordered almost ideals of S, then $R_1\cup R_2$ is a right ordered almost ideal of S.*

(3) *If I_1 and I_2 are ordered almost ideals of S, then $I_1\cup I_2$ is an ordered almost ideal of S.*

Proof. Each of (1), (2), and (3) follows from the corresponding item in Theorem 3.4. □

For in case the intersection of two left ordered almost ideals is not true as in the following example:

Example 3.6. Consider the ordered semigroup $\mathbb{Z}_6=\{\bar{0},\bar{1},\bar{2},\bar{3},\bar{4},\bar{5}\}$ under the addition on $\mathbb{Z}_6$ and the order $\leq:=\{(\bar{0},\bar{0}),(\bar{1},\bar{1}),(\bar{2},\bar{2}),(\bar{3},\bar{3}),(\bar{4},\bar{4}),(\bar{5},\bar{5})\}$. Let $L_1=\{\bar{1},\bar{3},\bar{4}\}$ and $L_2=\{\bar{2},\bar{3},\bar{5}\}$. We have

$(\bar{0}+L_1]\cap L_1=L_1$ and $(\bar{0}+L_2]\cap L_2=L_2$,
$(\bar{1}+L_1]\cap L_1=\{\bar{4}\}$ and $(\bar{1}+L_2]\cap L_2=\{\bar{3}\}$,
$(\bar{2}+L_1]\cap L_1=\{\bar{3}\}$ and $(\bar{2}+L_2]\cap L_2=\{\bar{5}\}$,
$(\bar{3}+L_1]\cap L_1=\{\bar{1},\bar{4}\}$ and $(\bar{3}+L_2]\cap L_2=\{\bar{2},\bar{5}\}$,
$(\bar{4}+L_1]\cap L_1=\{\bar{1}\}$ and $(\bar{4}+L_2]\cap L_2=\{\bar{3}\}$,

$$(\bar{5} + L_1] \cap L_1 = \{\bar{3}\} \quad \text{and} \quad (\bar{5} + L_2] \cap L_2 = \{\bar{2}\}.$$

Thus $(s + L_1] \cap L_1 \neq \emptyset$ and $(s + L_2] \cap L_2 \neq \emptyset$ for all $s \in \mathbb{Z}_6$. Hence L_1 and L_2 are left ordered almost ideals of $\mathbb{Z}_6$. But $L_1 \cap L_2 = \{\bar{3}\}$ is not a left ordered almost ideal of $\mathbb{Z}_6$ because $(\bar{3} + \{\bar{3}\}] \cap \{\bar{3}\} = \{\bar{0}\} \cap \{\bar{3}\} = \emptyset$.

Lemma 3.7. *Let S be an ordered semigroup and $|S| > 1$. Then the following properties hold.*

(1) *S has no proper left ordered almost ideals if and only if for all $a \in S$, there exists an element $s_a \in S$ such that $\big(s_a(S - \{a\})\big] = \{a\}$.*

(2) *S has no proper right ordered almost ideals if and only if for all $a \in S$, there exists an element $k_a \in S$ such that $\big((S - \{a\})k_a\big] = \{a\}$.*

(3) *S has no proper ordered almost ideals if and only if for all $a \in S$, there exist elements $s_a, k_a \in S$ such that $(s_a(S - \{a\})] = \{a\}$ and $((S - \{a\})k_a] = \{a\}$.*

Proof. We prove that property (1) holds. Assume S has no proper left ordered almost ideals and let $a \in S$. Then $S - \{a\}$ is not a left ordered almost ideal of S. That is there exists $s_a \in S$ such that $\big(s_a(S - \{a\})\big] \cap (S - \{a\}) = \emptyset$. Thus $\big(s_a(S - \{a\})\big] = \{a\}$.

Conversely, let L be a proper nonempty subset of S. Then $L \subseteq S - \{a\}$ for some $a \in S$. By assumption, there exists $s_a \in S$ such that $\big(s_a(S - \{a\})\big] = \{a\}$. Since $s_aL \subseteq s_a(S - \{a\})$, it follows that $(s_aL] \subseteq \big(s_a(S - \{a\})\big]$. We have

$$(s_aL] \cap L \subseteq \big(s_a(S - \{a\})\big] \cap (S - \{a\}) = \{a\} \cap (S - \{a\}) = \emptyset.$$

Thus $(s_aL] \cap L = \emptyset$ which implies that L is not a left ordered almost ideal of S. Therefore S has no proper left ordered almost ideals. Property (2) and (3) can be proved in a similar manner. □

Theorem 3.8. *Let S be an ordered semigroup such that $|S| > 1$ and $a \in S$. Then the following statements hold.*

(1) *If S has no proper left ordered almost ideals, then either a or a^2 is an idempotent.*

(2) *If S has no proper right ordered almost ideals, then either a or a^2 is an idempotent.*

(3) *If S has no proper ordered almost ideals, then either a or a^2 is an idempotent.*

Proof. (1) Assume S has no proper left ordered almost ideals. By Lemma 3.7(1), there exists an element $s_a \in S$ such that $\big(s_a(S - \{a\})\big] = \{a\}$.
Case 1. $a = a^2$. Then, $a = a^2 \leq a^2$, so a is an idempotent.
Case 2. $a \neq a^2$. That is $a^2 \in S - \{a\}$. So, $s_aa^2 = a$.
Case 2.1. $s_a \leq a$. Then, $a = s_aa^2 \leq a^3$, so $a^2 \leq a^4$. Thus, a^2 is an idempotent.

Case 2.2. $s_a \nleq a$. That is $s_a \in S - \{a\}$. Thus, $s_a s_a = a$. If $s_a a \nleq a$, then $s_a a \in S - \{a\}$, so $s_a s_a a = a$. Then, we have $a^2 = aa = (s_a s_a)a = a$, this is a contradiction. Thus, $s_a a \leq a$. Hence, $a = s_a a^2 \leq a^2$. Therefore, a is an idempotent. For statements (2) and (3), we can prove similarly. □

Example 3.9. Consider the ordered semigroup $S = \{-1, 1\}$ under the usual multiplication on $\mathbb{Z}$ and the order $\leq := \{(-1,-1),(1,1)\}$. We have $\{-1\}, \{1\}$ and S are nonempty subsets of S. We see that $(-1 \cdot \{-1\}] \cap \{-1\} = \{1\} \cap \{-1\} = \emptyset$, $(-1 \cdot \{1\}] \cap \{1\} = \{-1\} \cap \{1\} = \emptyset$. Thus, $\{-1\}$ and $\{1\}$ are not left ordered almost ideals of S. Hence S has no proper left ordered almost ideals. We can see that $\big(-1 \cdot (S - \{-1\})\big] = \big(-1 \cdot \{1\}\big] = \{-1\}$, $\big(-1 \cdot (S - \{1\})\big] = \big(-1 \cdot \{-1\}\big] = \{1\}$. Thus, for any $a \in S$, there exists $s_a \in S$ such that $\big(s_a(S - \{a\})\big] = \{a\}$.

Example 3.9 shows that there is an ordered semigroup which correspond to Theorem 3.8

4. Fuzzy ordered almost ideals

In this section, we define fuzzy ordered almost ideals of ordered semigroups and we study some of their properties. Moreover, we study the relationship between fuzzy ordered almost ideals and ordered almost ideals of ordered semigroups.

Let S be an ordered semigroup. For a fuzzy subset f of S, we defined $(f] : S \longrightarrow [0,1]$ by

$$(f](x) = \sup_{x \leq y} f(y), \text{ for all } x \in S.$$

Proposition 4.1. *Let f, g and h be fuzzy subsets of an ordered semigroup S.*

(1) $f \subseteq (f]$.

(2) *If $f \subseteq g$, then $(f] \subseteq (g]$.*

(3) *If $f \subseteq g$, then $(f \circ h] \subseteq (g \circ h]$ and $(h \circ f] \subseteq (h \circ g]$.*

Proof. (1) Let $x \in S$. We have $x \leq x$. Then, $(f](x) = \sup_{x \leq y} f(y) \geq f(x)$. Thus $f \subseteq (f]$.

(2) Assume that $f \subseteq g$. Then $f(x) \leq g(x)$ for all $x \in S$. Let $x \in S$. Thus $(f](x) = \sup_{x \leq y} f(y) \leq \sup_{x \leq y} g(y) = (g](x)$. Hence $(f] \subseteq (g]$.

(3) Assume that $f \subseteq g$. By Proposition 2.6(1), $f \circ h \subseteq g \circ h$ and $h \circ f \subseteq h \circ g$. It follows from (1) that, $(f \circ h] \subseteq (g \circ h]$ and $(h \circ f] \subseteq (h \circ g]$. □

Proposition 4.2. *Let f be a fuzzy subset of an ordered semigroup S. The following statements are equivalent.*

(1) *If $x \leq y$, then $f(x) \geq f(y)$.*

(2) $(f] = f$.

Proof. Let $x \in S$. By assumption, $f(x) \geq f(y)$ for all $y \in S$ where $x \leq y$. Then $(f](x) = \sup_{x \leq y} f(y) = f(x)$. Hence, $(f] = f$. Conversely, assume that $x, y \in S$ and $x \leq y$. Then, we have $f(x) = (f](x) = \sup_{x \leq y} f(y) \geq f(y)$. Thus $f(x) \geq f(y)$. □

Definition 4.3. Let S be an ordered semigroup.

1. A fuzzy subset f of S is called a *fuzzy left ordered almost ideal* of S if $(s_\alpha \circ f] \cap f \neq 0$ for all $s \in S$ and $\alpha \in (0, 1]$.
2. A fuzzy subset f of S is called a *fuzzy right ordered almost ideal* of S if $(f \circ s_\alpha] \cap f \neq 0$ for all $s \in S$ and $\alpha \in (0, 1]$.
3. A fuzzy subset f of S is called a *fuzzy ordered almost ideal* of S if f is both a fuzzy left ordered almost ideal and a fuzzy right ordered almost ideal of S.

Remark 4.4. Let f be a nonzero fuzzy subset of an ordered semigroup S.

(i) Every fuzzy left ideal of S is a fuzzy left ordered almost ideal of S.

(ii) Every fuzzy right ideal of S is a fuzzy right ordered almost ideal of S.

(iii) Every fuzzy ideal of S is a fuzzy ordered almost ideal of S.

Proof. Assume that f is a fuzzy left ideal of S. Let $s \in S$ and $\alpha \in (0, 1]$. Since f is a nonzero function, there exists an element $a \in S$ such that $f(a) \neq 0$. Let $x = sa$. Since f is a fuzzy left ideal, it follows that $f(x) = f(sa) \geq f(a) \neq 0$, so $f(x) \neq 0$. Then we see that

$$\begin{aligned}(s_\alpha \circ f](x) &\geq \sup_{x \leq u}(s_\alpha \circ f)(u) \\ &\geq (s_\alpha \circ f)(x) \\ &= \sup_{x \leq uv} \min\{s_\alpha(u), f(v)\} \\ &\geq \min\{s_\alpha(s), f(a)\} \\ &= \min\{\alpha, f(a)\} \neq 0.\end{aligned}$$

Thus, $\big((s_\alpha \circ f] \cap f\big)(x) = \min\{(s_\alpha \circ f](x), f(x)\} \neq 0$. Hence, f is a fuzzy left ordered almost ideal of S. This prove (i). (ii) and (iii) can be seen in the same manner. □

Example 4.5. Consider the ordered semigroup $\mathbb{Z}_6 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$ under the addition on $\mathbb{Z}_6$ and the order $\leq := \{(a, a) \mid a \in \mathbb{Z}_6\}$. Define a function $f : \mathbb{Z}_6 \longrightarrow [0, 1]$ by

$$f(\bar{0}) = 0, f(\bar{1}) = 0.3, f(\bar{2}) = 0, f(\bar{3}) = 0, f(\bar{4}) = 0.2, f(\bar{5}) = 0.1.$$

Then, for each $\alpha \in (0,1]$, we have

$$\begin{aligned}
\big((\bar{0}_\alpha \circ f] \cap f\big)(\bar{1}) &\neq 0,\\
\big((\bar{1}_\alpha \circ f] \cap f\big)(\bar{5}) &\neq 0,\\
\big((\bar{2}_\alpha \circ f] \cap f\big)(\bar{1}) &\neq 0,\\
\big((\bar{3}_\alpha \circ f] \cap f\big)(\bar{4}) &\neq 0,\\
\big((\bar{4}_\alpha \circ f] \cap f\big)(\bar{5}) &\neq 0,\\
\big((\bar{5}_\alpha \circ f] \cap f\big)(\bar{4}) &\neq 0.
\end{aligned}$$

Thus, for each $s \in \mathbb{Z}_6$ and $\alpha \in (0,1]$, there exists an element $x \in \mathbb{Z}_6$ such that $\big((s_\alpha \circ f] \cap f\big)(x) \neq 0$. Hence f is a fuzzy left ordered almost ideal of $\mathbb{Z}_6$ but not a fuzzy left ideal of $\mathbb{Z}_6$ because $f(\bar{2}+\bar{1}) = f(\bar{3}) = 0 \ngeq 0.3 = f(\bar{1})$.

From the example above, shows that the converse of the Remark 4.4(i) does not holds true.

Theorem 4.6. *Let S be an ordered semigroup.*

(1) *If f is a fuzzy left ordered almost ideal of S, then every fuzzy subset g of S such that $f \subseteq g$ is a fuzzy left ordered almost ideal of S.*

(2) *If f is a fuzzy right ordered almost ideal of S, then every fuzzy subset g of S such that $f \subseteq g$ is a fuzzy right ordered almost ideal of S.*

(3) *If f is a fuzzy ordered almost ideal of S, then every fuzzy subset g of S such that $f \subseteq g$ is a fuzzy ordered almost ideal of S.*

Proof. Assume f is a fuzzy left ordered almost ideal of S. Let g be a fuzzy subset of S such that $f \subseteq g$ and let $s \in S$ and $\alpha \in (0,1]$. Then we have $s_\alpha \circ f \subseteq s_\alpha \circ g$ and so $(s_\alpha \circ f] \subseteq (s_\alpha \circ g]$. Thus $(s_\alpha \circ f] \cap f \subseteq (s_\alpha \circ g] \cap g$. Since f is a fuzzy left ordered almost ideal, $(s_\alpha \circ f] \cap f \neq \emptyset$. Hence $(s_\alpha \circ g] \cap g \neq \emptyset$. Therefore g is a fuzzy left ordered almost ideal of S. This proves (1), and (2) and (3) follows in the same way. □

Corollary 4.7. *Let S be an ordered semigroup.*

(1) *If f and g are fuzzy left ordered almost ideals of S, then $f \cup g$ is a fuzzy left ordered almost ideal of S.*

(2) *If f and g are fuzzy right ordered almost ideals of S, then $f \cup g$ is a fuzzy right ordered almost ideal of S.*

(3) *If f and g are fuzzy ordered almost ideals of S, then $f \cup g$ is a fuzzy ordered almost ideal of S.*

Proof. The proof follows from Theorem 4.6. □

Example 4.8. Consider the ordered semigroup $\mathbb{Z}_6 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$ under the addition on $\mathbb{Z}_6$ and the order $\leq := \{(a, a) \mid a \in \mathbb{Z}_6\}$. Define functions $f : \mathbb{Z}_6 \longrightarrow [0, 1]$ by

$$f(\bar{0}) = 0, f(\bar{1}) = 0.3, f(\bar{2}) = 0, f(\bar{3}) = 0, f(\bar{4}) = 0.2, f(\bar{5}) = 0.1,$$

and $g : \mathbb{Z}_6 \longrightarrow [0, 1]$ by

$$g(\bar{0}) = 0, g(\bar{1}) = 0.3, g(\bar{2}) = 0.1, g(\bar{3}) = 0, g(\bar{4}) = 0, g(\bar{5}) = 0.3.$$

Then, for each $\alpha \in (0, 1]$, we have

$$\begin{array}{llll}
\big((\bar{0}_\alpha \circ f] \cap f\big)(\bar{1}) \neq 0 & \text{and} & \big((\bar{0}_\alpha \circ g] \cap g\big)(\bar{1}) \neq 0, \\
\big((\bar{1}_\alpha \circ f] \cap f\big)(\bar{5}) \neq 0 & \text{and} & \big((\bar{1}_\alpha \circ g] \cap g\big)(\bar{2}) \neq 0, \\
\big((\bar{2}_\alpha \circ f] \cap f\big)(\bar{1}) \neq 0 & \text{and} & \big((\bar{2}_\alpha \circ g] \cap g\big)(\bar{1}) \neq 0, \\
\big((\bar{3}_\alpha \circ f] \cap f\big)(\bar{4}) \neq 0 & \text{and} & \big((\bar{3}_\alpha \circ g] \cap g\big)(\bar{5}) \neq 0, \\
\big((\bar{4}_\alpha \circ f] \cap f\big)(\bar{5}) \neq 0 & \text{and} & \big((\bar{4}_\alpha \circ g] \cap g\big)(\bar{5}) \neq 0, \\
\big((\bar{5}_\alpha \circ f] \cap f\big)(\bar{4}) \neq 0 & \text{and} & \big((\bar{5}_\alpha \circ g] \cap g\big)(\bar{1}) \neq 0.
\end{array}$$

Thus, for each $s \in \mathbb{Z}_6$ and $\alpha \in (0, 1]$, there are elements $x, y \in \mathbb{Z}_6$ such that $\big((s_\alpha \circ f] \cap f\big)(x) \neq 0$ and $\big((s_\alpha \circ g] \cap g\big)(y) \neq 0$. Hence f and g are fuzzy left ordered almost ideals of $\mathbb{Z}_6$.

Next, consider $f \cap g : \mathbb{Z}_6 \longrightarrow [0, 1]$ where

$$\begin{array}{lll}
(f \cap g)(\bar{0}) = 0, & (f \cap g)(\bar{1}) = 0.3, & (f \cap g)(\bar{2}) = 0, \\
(f \cap g)(\bar{3}) = 0, & (f \cap g)(\bar{4}) = 0, & (f \cap g)(\bar{5}) = 0.1.
\end{array}$$

Then, $f \cap g$ is not a fuzzy left ordered almost ideal of $\mathbb{Z}_6$ because $(\bar{1}_\alpha \circ (f \cap g)] \cap (f \cap g) = 0$ for all $\alpha \in (0, 1]$.

Example 4.8 implies that the intersection of two fuzzy left ordered almost ideals need not be a fuzzy left ordered almost ideal.

Theorem 4.9. *Let A be a nonempty subset of an ordered semigroup S. Then, the following properties hold:*

(1) *A is a left ordered almost ideal of S if and only if C_A is a fuzzy left ordered almost ideal of S;*

(2) *A is a right ordered almost ideal of S if and only if C_A is a fuzzy right ordered almost ideal of S;*

(3) *A is an ordered almost ideal of S if and only if C_A is a fuzzy ordered almost ideal of S.*

Proof. Assume A is a left ordered almost ideal of S. Let $s \in S$ and $\alpha \in (0,1]$. Then, $(sA] \cap A \neq \emptyset$. That is there exists $x \in A$ and $x \in (sA]$. So, $C_A(x) = 1 \neq 0$ and $x \leq sa$, for some $a \in A$. We have

$$\begin{aligned}(s_\alpha \circ C_A](x) &= \sup_{x \leq y}(s_\alpha \circ C_A)(y) \geq (s_\alpha \circ C_A)(x) \\ &= \sup_{x \leq uv} \min\{s_\alpha(u), C_A(v)\} \\ &\geq \min\{s_\alpha(s), C_A(a)\} = \min\{\alpha, 1\} \neq 0.\end{aligned}$$

Hence $\big((s_\alpha \circ C_A] \cap C_A\big)(x) \neq 0$. Therefore, C_A is a fuzzy left ordered almost ideal of S.

Conversely, assume that C_A is a fuzzy left ordered almost ideal of S. Let $s \in S$. There exists $x \in S$ such that $\big((s_\alpha \circ C_A] \cap C_A\big)(x) \neq 0$ for all $\alpha \in (0,1]$. That is $C_A(x) \neq 0$ and $(s_\alpha \circ C_A](x) \neq 0$ for all $\alpha \in (0,1]$. So, $x \in A$. Since $(s_\alpha \circ C_A](x) = \sup_{x \leq y}(s_\alpha \circ C_A)(y) \neq 0$, there is $y \in S$ such that $x \leq y$ and

$$(s_\alpha \circ C_A)(y) = \sup_{y \leq uv} \min\{(s_\alpha(u), C_A(v)\} \neq 0.$$

Thus, $y \leq sa$ for some $a \in A$. Since $x \leq y$, we get $x \leq sa$, so $x \in (sA]$. Hence $x \in (sA] \cap A$. Therefore, A is a left ordered almost ideal of S. In the case property (2), the proof follows similarly. By property (1) and (2), it can see that property (3) holds. □

Theorem 4.10. *Let S be an ordered semigroup. Then the following properties hold.*

(1) *f is a fuzzy left ordered almost ideal of S if and only if $supp(f)$ is a left ordered almost ideal of S.*

(2) *f is a fuzzy right ordered almost ideal of S if and only if $supp(f)$ is a right ordered almost ideal of S.*

(3) *f is a fuzzy ordered almost ideal of S if and only if $supp(f)$ is an ordered almost ideal of S.*

Proof. Assume that f is a fuzzy left ordered almost ideal of S. Let $s \in S$. There exists $x \in S$ such that $\big((s_\alpha \circ f] \cap f\big)(x) \neq 0$ for all $\alpha \in (0,1]$. That is $f(x) \neq 0$ and $(s_\alpha \circ f](x) \neq 0$ for all $\alpha \in (0,1]$. So $x \in supp(f)$. For all $\alpha \in (0,1]$, there is an element $y \in S$ such that $x \leq y$ and $(s_\alpha \circ f)(y) \neq 0$. Since $(s_\alpha \circ f)(y) \neq 0$, we obtain $y \leq sa$ for some $a \in S$ such that $f(a) \neq 0$. Thus $x \leq sa$ for some $a \in supp(f)$. This implies that $x \in \big(s(supp(f))\big]$. Hence $x \in \big(s(supp(f))\big] \cap supp(f)$. Therefore, $supp(f)$ is a left ordered almost ideal of S.

Conversely, assume that $supp(f)$ is a left ordered almost ideal of S. By Theorem 4.9 (1), $C_{supp(f)}$ is a fuzzy left ordered almost ideal of S. Let $s \in S$ and

$\alpha \in (0,1]$. Then, there exists $x \in S$ such that $\big((s_\alpha \circ C_{supp(f)}] \cap C_{supp(f)}\big)(x) \neq 0$ which implies that $C_{supp(f)}(x) \neq 0$ and $(s_\alpha \circ C_{supp(f)}](x) \neq 0$. So, $x \in supp(f)$ and there exists $y \in S$ such that $x \leq y$ and $(s_\alpha \circ C_{supp(f)})(y) \neq 0$. Thus, $y \leq sa$ for some $a \in supp(f)$. This implies that $x \leq sa$ where $f(x) \neq 0$ and $f(a) \neq 0$. Hence $\big((s_\alpha \circ f] \cap f\big)(x) \neq 0$. Therefore, f is a fuzzy left ordered almost ideal of S. Property (2), we can prove similarly. The proof of property (3) follows from properties (1) and (2). □

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Synchronization of chaotic systems via periodically piecewise feedback control

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Abstract. In this paper, the synchronization of chaotic systems via periodically piecewise feedback control is investigated. An exponential synchronization criterion is given by means of Lyapunov function and inequalities. The effectiveness of theoretical result is confirmed via an example based on Chua's circuit.

Keywords: chaos, synchronization, periodically piecewise feedback control.

1. Introduction

The synchronization of chaotic systems was presented by Fujisaka and Yamada [1]. Since then, chaos synchronization has many applications in biological chemical reactions, secure communications, artificial neural networks and so on. Unfortunately, chaotic systems are difficult to synchronize due to sensitive dependence on initial conditions. Nevertheless, many fundamental and meaningful results have been obtained by scholars from mathematics, physics, engineering, etc. There are many methods to synchronize of chaotic systems. For example, fuzzy control [2], switch control [3], impulsive control [4-6], sliding mode control [7], adaptive control [8], intermittent control [9], feedback control [10, 11]. For more information on stabilization and synchronization of chaotic systems have been investigated in the literature (see, [12-19]).

This paper is organized as follows. In 2, some preliminaries are introduced. The problem synchronization of chaotic systems via periodically piecewise feedback control is described. In Section 3, the criterion of synchronization is strictly derived. In Section 4, an example is given to illustrate the effectiveness of the theoretical result.

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2. Problem formulation and preliminaries

Notations. Let R be the set of real numbers, R^n the $n-$ dimensional real Euclidean space with norm $\|\cdot\|$, $R^{m\times n}$ the set of all $m\times n-$ dimensional real matrices. The notation $\lambda_{\max}(A)$, $\lambda_{\min}(A)$ and A^T denote the maximum, the minimum eigenvalue and the transpose of real symmetric matrix A, respectively. $A>0$ means that A is a positive definite matrix. I stands for the identity matrix with appropriate dimensions. Set $f\left(x\left(t_0^-\right)\right)=\lim\limits_{t\to t_0^-} f(x(t))$.

The chaotic master (driver) system considered in [20] can be shown by:

$$\begin{cases}\dot{x}(t)=Ax(t)+Qf(x(t)), & t>0,\\ x(t_0)=x_0.\end{cases} \tag{2.1}$$

In order to synchronize system (2.1) via periodically piecewise feedback control, the slave (response) system is described as

$$\begin{cases}\dot{y}(t)=Ay(t)+Qf(y(t))+g(t)(x(t)-y(t)), & t>0,\\ y(t_0)=y_0,\end{cases} \tag{2.2}$$

where $x(t),y(t)\in R^n$ denote the state vectors of systems (2.1) and (2.2), respectively, $A,Q\in R^{n\times n}$ are known matrices, $f:R^n\to R^n$ is said to be a continuous nonlinear function satisfying $f(0)=0$ and $||f(x)-f(y)||\le l||x-y||$, $l\ge 0$ is a constant, $g(t)$ is the periodically piecewise feedback gain defined as follows:

$$g(t)=\begin{cases}G_1, & kT+t_0\le t<kT+\tau+t_0,\\ G_2, & kT+\tau+t_0\le t<(k+1)T+t_0,\end{cases}$$

where $G_1,G_2\in R^{n\times n}$ are two constant control gains, $T>0$ represents control period, $\tau>0$ is the control width. Our target is to design suitable T,τ, G_1 and G_2 such that system (2.2) synchronizes system (2.1).

Define $e(t)=y(t)-x(t)$. $e(t)$ is the periodically synchronization error between the states of drive system (2.1) and response system (2.2). Then, the error system can be written as follows:

$$\dot{e}(t)=\begin{cases}Ae(t)+Q(f(y(t))-f(x(t)))-G_1e(t),\\ \qquad kT+t_0\le t<kT+\tau+t_0,\\ Ae(t)+Q(f(y(t))-f(x(t)))-G_2e(t),\\ \qquad kT+\tau+t_0\le t<(k+1)T+t_0.\end{cases} \tag{2.3}$$

Obviously (2.3) is a classical switched system where the switching rule depends on time.

Remark 2.1. When $G_2e(t)=0$, the error system (2.3) becomes the classical intermittent error system [20].

In this paper, we show a new sufficient exponential synchronization of system (2.1) and system (2.2) criterion condition. The computational cost of solving conditions of our result is lower than ones shown in [20].

We need a lemma which plays a major role in the proof of theorem.

Lemma 2.1 ([21]). *Suppose that $A \in R^{n\times n}$ is a symmetric matrix. Then, for all $x \in R^n$, $\lambda_{\min}(A)x^Tx \le x^TAx \le \lambda_{\max}(A)x^Tx$.*

3. Main result

In what follows, we will obtain the main result.

Theorem 3.1. *Let $0 < P \in R^{n\times n}$ such that the following two conditions are satisfied*:

(1) $r_1 < 0$,

(2) $r_1\tau + r_2(T-\tau) < 0$,

where $\beta_1 = \lambda_{\max}\left(P^{-1}\left(PA + A^TP - PG_1 - G_1^TP\right)\right)$, $\beta_2 = \lambda_{\max}\left(Q^TPQ\right)$, $\beta_3 = \lambda_{\min}(P)$, $\beta_4 = \lambda_{\max}\left(P^{-1}\left(PA + A^TP - PG_2 - G_2^TP\right)\right)$, $r_1 = \beta_1 + 2l\sqrt{\frac{\beta_2}{\beta_3}}$, $r_2 = \beta_4 + 2l\sqrt{\frac{\beta_2}{\beta_3}}$. Then, master system (2.1) and slave system (2.2) are exponential synchronization via periodically piecewise feedback control.

Proof. Define $V(e(t)) = e^T(t)Pe(t)$. For $t \in [kT + t_0, kT + \tau + t_0)$, by Cauchy-Schwarz inequality and Lemma 2.1, we obtain

$$\begin{aligned}
D^+\left(V\left(e\left(t\right)\right)\right) &= 2e^T(t)P[Ae(t) + Q(f(y(t)) - f(x(t))) - G_1e(t)] \\
&= 2e^T(t)PQ(f(y(t)) - f(x(t))) \\
&\quad + e^T\left(t\right)\left(PA + A^TP - PG_1 - G_1^TP\right)e\left(t\right) \\
&\le 2\sqrt{e^T(t)Pe(t)(f(y(t)) - f(x(t)))^TQ^TPQ(f(y(t)) - f(x(t)))} \\
&\quad + \beta_1e^T\left(t\right)Pe\left(t\right) \\
&\le 2\sqrt{e^T(t)Pe(t)\beta_2(f(y(t)) - f(x(t)))^T(f(y(t)) - f(x(t)))} \\
&\quad + \beta_1e^T\left(t\right)Pe\left(t\right) \\
&\le \beta_1e^T\left(t\right)Pe\left(t\right) + 2l\sqrt{e^T(t)Pe(t)\beta_2e^T(t)e(t)} \\
&\le \beta_1e^T\left(t\right)Pe\left(t\right) + 2l\sqrt{e^T\left(t\right)Pe\left(t\right)\frac{\beta_2}{\beta_3}e^T\left(t\right)Pe\left(t\right)} \\
&= r_1V\left(e\left(t\right)\right),
\end{aligned}$$

which deduces

$$V\left(e\left(t\right)\right) \le V\left(e\left(\left(kT + t_0\right)^-\right)\right)e^{r_1(t-kT-t_0)}. \tag{3.1}$$

In the same way, for $t \in [kT+\tau+t_0, (k+1)T+t_0)$, we also obtain

$$\begin{aligned} D^+\left(V\left(e\left(t\right)\right)\right) &= 2e^T(t)P[Ae(t)+Q(f(y(t))-f(x(t)))-G_2e(t)] \\ &= e^T\left(t\right)\left(PA+A^TP-PG_2-G_2^TP\right)e\left(t\right) \\ &+ 2e^T(t)PQ(f(y(t))-f(x(t))) \\ &\leq \beta_4 e^T\left(t\right)Pe\left(t\right)+2l\sqrt{e^T\left(t\right)Pe\left(t\right)\frac{\beta_2}{\beta_3}e^T\left(t\right)Pe\left(t\right)} \\ &= r_2V\left(e\left(t\right)\right), \end{aligned}$$

which means

$$V\left(e\left(t\right)\right) \leq V\left(e\left(\left(kT+\tau+t_0\right)^-\right)\right)e^{r_2(t-kT-\tau-t_0)}. \tag{3.2}$$

When $k=0$, for $t \in [t_0, \tau+t_0)$, from (3.1), we have

$$V\left(e\left(t\right)\right) \leq V\left(e\left(t_0\right)\right)e^{r_1(t-t_0)},$$

hence

$$V\left(e\left(\left(\tau+t_0\right)^-\right)\right) \leq V\left(e\left(t_0\right)\right)e^{r_1\tau}. \tag{3.3}$$

For $t \in [\tau+t_0, T+t_0)$, by (3.2) and (3.3), we have

$$V\left(e\left(t\right)\right) \leq V\left(e\left(\left(\tau+t_0\right)^-\right)\right)e^{r_2(t-\tau-t_0)} \leq V\left(e\left(t_0\right)\right)e^{r_1\tau+r_2(t-\tau-t_0)},$$

so

$$V\left(e\left(\left(T+t_0\right)^-\right)\right) \leq V\left(e\left(t_0\right)\right)e^{r_1\tau+r_2(T-\tau)}. \tag{3.4}$$

When $k=1$, for $t \in [T+t_0, T+\tau+t_0)$, by (3.1) and (3.4), we have

$$\begin{aligned} V\left(e\left(t\right)\right) &\leq V\left(e\left(\left(T+t_0\right)^-\right)\right)e^{r_1(t-T-t_0)} \\ &\leq V\left(e\left(t_0\right)\right)e^{r_1\tau+r_2(T-\tau)+r_1(t-T-t_0)}, \end{aligned}$$

so

$$V\left(e\left(\left(T+\tau+t_0\right)^-\right)\right) \leq V\left(e\left(t_0\right)\right)e^{2r_1\tau+r_2(T-\tau)}. \tag{3.5}$$

For $t \in [T+\tau+t_0, 2T+t_0)$, by (3.2) and (3.5), we have

$$\begin{aligned} V\left(e\left(t\right)\right) &\leq V\left(e\left(\left(T+\tau+t_0\right)^-\right)\right)e^{r_2(t-T-\tau-t_0)} \\ &\leq V\left(e\left(t_0\right)\right)e^{2r_1\tau+r_2(T-\tau)+r_2(t-T-\tau-t_0)}. \end{aligned}$$

By induction, when $k{=}m, m{=}0,1,\ldots$, for $t \in [mT+t_0, mT+\tau+t_0)$, we have

$$V\left(e\left(t\right)\right) \leq V\left(e\left(t_0\right)\right)e^{mr_1\tau+mr_2(T-\tau)+r_1(t-mT-t_0)}, \tag{3.6}$$

so

$$V\left(e\left(\left(mT+\tau+t_0\right)^-\right)\right) \leq V\left(e\left(t_0\right)\right)e^{(m+1)r_1\tau+mr_2(T-\tau)}. \tag{3.7}$$

For $t \in [mT+\tau+t_0, (m+1)T+t_0)$, by (3.2) and (3.7), we have

$$\begin{aligned} V(e(t)) &\leq V\left(e\left((mT+\tau+t_0)^-\right)\right) e^{r_2(t-mT-\tau-t_0)} \\ &\leq V(e(t_0))\, e^{(m+1)r_1\tau+mr_2(T-\tau)+r_2(t-mT-\tau-t_0)}. \end{aligned} \tag{3.8}$$

By (3.6), we have

$$\begin{aligned} V(e(t)) &\leq V(e(t_0))\, e^{mr_1\tau+mr_2(T-\tau)} \\ &< V(e(t_0))\, e^{\frac{t-\tau-t_0}{T}(r_1\tau+r_2(T-\tau))} \\ &< V(e(t_0))\, e^{\frac{t-T-t_0}{T}(r_1\tau+r_2(T-\tau))}, \end{aligned} \tag{3.9}$$

where $t \in [mT+t_0, mT+\tau+t_0)$. By (3.8), we know
Case 1. When $r_2 > 0$, we have

$$\begin{aligned} V(e(t)) &< V(e(t_0))\, e^{(m+1)r_1\tau+(m+1)r_2(T-\tau)} \\ &< V(e(t_0))\, e^{\frac{t-t_0}{T}(r_1\tau+r_2(T-\tau))} \\ &< V(e(t_0))\, e^{\frac{t-T-t_0}{T}(r_1\tau+r_2(T-\tau))}. \end{aligned} \tag{3.10}$$

Case 2. When $r_2 \leq 0$, we have

$$\begin{aligned} V(e(t)) &\leq V(e(t_0))\, e^{(m+1)r_1\tau+mr_2(T-\tau)} \\ &< V(e(t_0))\, e^{mr_1\tau+mr_2(T-\tau)} \\ &< V(e(t_0))\, e^{\frac{t-T-t_0}{T}(r_1\tau+r_2(T-\tau))}. \end{aligned} \tag{3.11}$$

So, for any $r_2 \in R$, by (3.10) and (3.11), we have

$$V(e(t)) < V(e(t_0))\, e^{\frac{t-T-t_0}{T}(r_1\tau+r_2(T-\tau))}, \tag{3.12}$$

where $t \in [mT+\tau+t_0, (m+1)T+t_0)$. For all $t > 0$, we can conclude from (3.9) and (3.12) that

$$V(e(t)) < V(e(t_0))\, e^{\frac{t-T-t_0}{T}(r_1\tau+r_2(T-\tau))}. \tag{3.13}$$

Let $\eta = V(e(t_0))e^{-[r_1\tau+r_2(T-\tau)]}e^{\frac{-[r_1\tau+r_2(T-\tau)]t_0}{T}}$. By Lemma 2.1 and (3.13), we get

$$\lambda_{\min}(P)\,\|e(t)\|^2 < \eta e^{\frac{[r_1\tau+r_2(T-\tau)]t}{T}}.$$

Namely,

$$\|e(t)\| < \sqrt{\frac{\eta}{\lambda_{\min}(P)}}\, e^{\frac{[r_1\tau+r_2(T-\tau)]t}{2T}},$$

which concludes that system (2.1) and system (2.2) synchronize exponentially. This completes the proof. □

Remark 3.1. Theorem 1 of [20] needs to solve the linear matrix inequalities, however, Theorem 3.1 does not need to solve them.

4. A numerical example

In this section, we study the synchronization of Chua's oscillator via employing above theoretical result with the initial condition $x(0) = (0.2, -0.1, 0.2)^T$ for the master system and $y(0) = (-0.2, 0.1, -0.2)^T$ for the slave system.

Example 4.1. The Chua's system [22] is given as follows:

$$\begin{cases} \dot{x}_1 = \alpha(x_2 - x_1 - g(x_1)), \\ \dot{x}_2 = x_1 - x_2 + x_3, \\ \dot{x}_3 = -\eta x_2, \end{cases} \tag{4.1}$$

where α and η are two parameters $g(x_1) = bx_1 + 0.5(a-b)(|x_1+1| - |x_1-1|)$, where a and b are two given constants satisfying $a < b < 0$.

In order to use the above result, we rewrite system (4.1) as follows $\dot{x}(t) = Ax + f(x)$, where

$$A = \begin{bmatrix} -\alpha - \alpha b & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\eta & 0 \end{bmatrix},$$

$$f(x) = \begin{bmatrix} -0.5\alpha(a-b)(|x_1+1| - |x_1-1|) \\ 0 \\ 0 \end{bmatrix}.$$

By simple calculation, we can choose $l^2 = \alpha^2(a-b)^2$.

The Chua's system exhibits chaotic phenomenon when $\alpha = 9.2156$, $\eta = 15.9946$, $a = -1.24905$, $b = -0.75735$ (see, [20]).

The two Chua systems are not synchronization without periodically piecewise feedback control as shown in Figure 1.

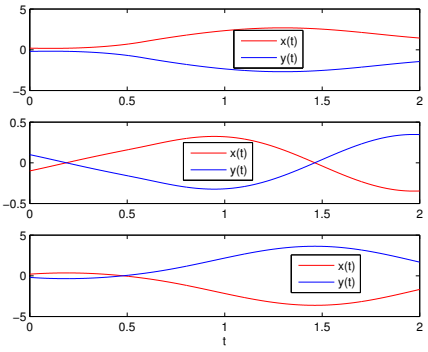


Figure 1: The two Chua systems are not synchronization without periodically piecewise feedback control .

At the same time, for convenience of calculations, we choose $Q = P = I$, $G_1 = diag(18, 18, 18)$, $G_2 = diag(8, 8, 8)$, $T = 1$ and $\tau = 0.5$. A small calculations show that $\beta_1 = -19.4502$, $\beta_2 = \beta_3 = 1$, $\beta_4 = 0.5498$, $l = 4.5313$, $r_1 = -10.3876$, $r_2 = 9.6124$ and $r_1\tau + r_2(T - \tau) = -0.3876 < 0$. Thus, system (2.1) and system (2.2) are exponential synchronization by Theorem 3.1, as shown in Figure 2.

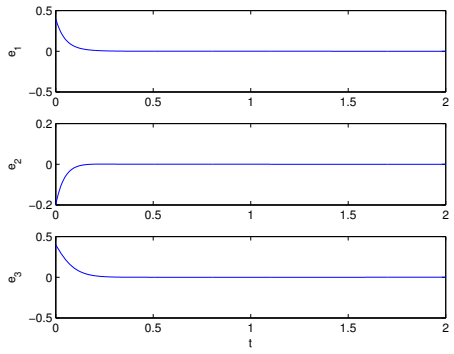


Figure 2: The two Chua systems synchronize via periodically piecewise feedback control.

5. Conclusions

This paper first introduces a generalized model of chaotic systems. We obtain a new exponential synchronization criterion of chaotic systems via periodically piecewise feedback control. Finally, an example based on Chua's circuit is provided to show the validity and superiority of the result.

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Wrapped Aradhana distribution

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Abstract. In this paper, we introduce a new circular distribution generated from Aradhana distribution. We obtained the probability densities and characteristic functions of wrapped Aradhana distribution. The maximum likelihood estimator of the new distribution parameter is also obtained. Then, some properties of the distribution are studied such as means, skewness, and kurtosis.

Keywords: Aradhana distribution, wrapped distribution, maximum likelihood estimation, circular variance, circular standard deviation, trigonometric moments, mean direction, mean resultant length, skewness, kurtosis.

1. Introduction

R. Shanker in [7] proposed a new lifetime distribution which is a combination of three component mixture of exponential distributions, an exponential λ distribution with mixing proportion $\lambda^2/(\lambda^2+2\lambda+2)$, a gamma $(2,\lambda)$ distribution with mixing proportion $2\lambda/(\lambda^2+2\lambda+2)$ and a gamma $(3,\lambda)$ distribution with mixing proportion $2/(\lambda^2+2\lambda+2)$. The new distribution entitled Aradhana distribution (AD) is more flexible for modeling lifetime data than other well known distributions. Also, R. Shanker in [9] obtained the generalized distribution of AD of which AD is a special case. The two parameter quasi AD was proposed by R. Shanker in [12]. A two parameter power AD was studied by R. Shanker in [10]. R. Ganaie, V. Rajagopalan, A. Rather in [1] proposed a generalization of AD called weighted AD, and M. Gharaibieh in [2] used the quadratic rank transmutation map to introduce the transmuted AD as a new generalization of AD. Moreover, R. Shanker in [8] proposed the discrete Poisson-AD by compounding the Poisson distribution with AD and studied its properties, and the size-biased Poisson-AD was proposed by R. Shanker in [13]. Recently, R. Shanker in [11] proposed a quasi Poisson-AD by compounding a Poisson distribution with a quasi AD.

The probability density function (pdf) of AD is given by

$$f(x;\lambda)=\frac{\lambda^3}{\lambda^2+2\lambda+2}(1+x)^2e^{-x\lambda};\quad x>0,\quad \lambda>0, \tag{1}$$

with cumulative distribution function (cdf)

$$F(x;\lambda) = 1 - \left[1 + \frac{x\lambda(x\lambda + 2\lambda + 2)}{\lambda^2 + 2\lambda + 2}\right] e^{-x\lambda}; \;\; x > 0, \quad \lambda > 0. \tag{2}$$

The characteristic function AD is defined as

$$\varphi_X(t) = \frac{\lambda^3}{\lambda^2 + 2\lambda + 2}\left[\frac{1}{\lambda - it} + \frac{2}{(\lambda - it)^2} + \frac{2}{(\lambda - it)^3}\right], ; \;\; i = \sqrt{-1}. \tag{3}$$

Direction data are data measured by two or three dimensions and they are used in the natural and physical sciences. Researchers use circular distributions to analyze and model directional data, and many of these circular distributions are generated from known probability distribution using different methods, such as, wrapping the distribution around the unite circle (see, for example, [6, 3]).

In this paper, wrapped Aradhana distribution (WAD) has been proposed and its statistical properties including probability densities functions for varying values of the parameters, characteristic function, moments trigonometric moments, means, skewness and kurtosis has been studied. The maximum likelihood estimation has been discussed for estimating its parameters.

2. Wrapped Aradhana Distribution (WAD)

Let X be an Aradhana random variable and θ be the circular random variable defined by $\theta = X(mod\ 2\pi)$. The probability density function for θ is given by

$$g(\theta) = \sum_{m=-\infty}^{\infty} f(\theta + 2m\pi), \qquad 0 \le \theta < 2\pi.$$

The probability density function $g(\theta)$ satisfies the following conditions:

(1) $g(\theta) \ge 0$, for all θ,

(2) $\int_0^{2\pi} g(\theta)d\theta = 1$,

(3) $g(\theta)$ is periodic function, such that, $g(\theta) = g(\theta + 2\pi m)$ for $m \in \mathbb{Z}$.

Hence, we can say that θ is a wrapped Aradhana random variable with probability density function (pdf) $g(\theta;\lambda)$ given by

$$\begin{aligned} & g(\theta;\lambda) \\ & = \sum_{m=0}^{\infty} f(\theta + 2m\pi) \\ & = \sum_{m=0}^{\infty} \frac{\lambda^3}{\lambda^2 + 2\lambda + 2}(1 + \theta + 2m\pi)^2 e^{-(\theta + 2m\pi)\lambda} \\ & = \frac{\lambda^3\ e^{-\theta\lambda}}{\lambda^2 + 2\lambda + 2}\left[\frac{(1+\theta)^2}{1 - e^{-2\pi\lambda}} + \frac{4(\theta+1)\pi e^{-2\pi\lambda}}{(1 - e^{-2\pi\lambda})^2} + \frac{4\pi^2 e^{-2\pi\lambda}(1 + e^{-2\pi\lambda})}{(1 - e^{-2\pi\lambda})^3}\right]. \end{aligned} \tag{4}$$

Moreover, the cumulative distribution function $G(\theta;\lambda)$, for $\theta \in [0, 2\pi)$ and $\lambda > 0$ is given by

$$
\begin{aligned}
G(\theta;\lambda) &= \sum_{m=0}^{\infty} [F(\theta + 2m\pi) - F(2m\pi)] \\
&= \sum_{m=0}^{\infty} \Bigg[1 - \left(1 + \frac{(\theta + 2m\pi)\lambda[(\theta + 2m\pi)\lambda + 2\lambda + 2]}{\lambda^2 + 2\lambda + 2} \right) e^{-(\theta+2m\pi)\lambda} \\
&\qquad - 1 + \left(1 + \frac{2m\pi\lambda[2m\pi\lambda + 2\lambda + 2]}{\lambda^2 + 2\lambda + 2} \right) e^{-(2m\pi)\lambda} \Bigg] \\
&= \frac{1 - e^{-\theta\lambda}}{1 - e^{-2\pi\lambda}} - \frac{\theta\lambda(\theta\lambda + 2\lambda + 2)e^{-\theta\lambda}}{(\lambda^2 + 2\lambda + 2)(1 - e^{-2\pi\lambda})} \\
&\qquad + \frac{4\pi\lambda \left[(1 + \lambda)(1 - e^{-\theta\lambda}) - \theta\lambda e^{-\theta\lambda} \right] e^{-2\pi\lambda}}{(\lambda^2 + 2\lambda + 2)(1 - e^{-2\pi\lambda})^2} \\
&\qquad + \frac{4\pi^2\lambda^2(1 - e^{-\theta\lambda})(1 + e^{-2\pi\lambda})e^{-2\pi\lambda}}{(\lambda^2 + 2\lambda + 2)(1 - e^{-2\pi\lambda})^3}.
\end{aligned} \tag{5}
$$

Figure 1 shows the pdf and cdf of WAD for different values of λ.

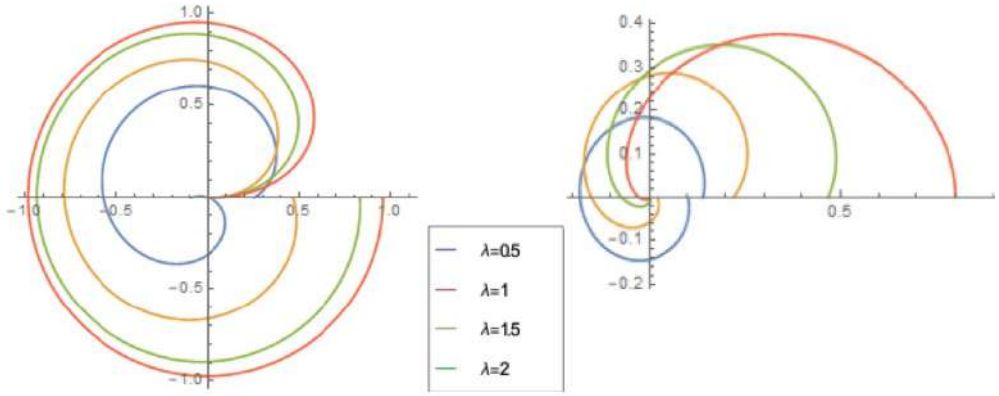


Figure 1: The Circular representation for WAD for $\lambda = 0.5, 1, 1.5, 2$.

3. Characteristic function and trigonometric moments

The pth trigonometric moment of WAD is given by

$$
\begin{aligned}
\varphi_\theta(p) &= \varphi_X(p) \\
&= \frac{\lambda^3}{\lambda^2 + 2\lambda + 2} \left[\frac{1}{\lambda - ip} + \frac{2}{(\lambda - ip)^2} + \frac{2}{(\lambda - ip)^3} \right]; \\
& i = \sqrt{-1}, p = \mp 1, \mp 2, \ldots
\end{aligned} \tag{6}
$$

$$= \frac{\lambda^3}{\lambda^2+2}\left[\frac{(\lambda-ip)^2+2}{(\lambda-ip)^3}\right]$$
$$= \frac{\lambda^3}{\lambda^2+2}\left[(\lambda^2+2\lambda-p^2+2)-2\ ip(1+\lambda)\right](\lambda-ip)^{-3},$$

using the fact that $(a-ib)^{-r} = (a^2+b^2)^{\frac{-r}{2}} e^{ir\arctan(b/a)}$, for any $a, b, r \in \Re$, we can write the pth trigonometric moment for WAD $\varphi_\theta(p)$ as

$$\varphi_\theta(p) = \rho_p e^{i\mu_p}; \quad i=\sqrt{-1},\ p=\mp 1, \mp 2, \ldots, \tag{7}$$

where

$$\rho_p = \frac{\lambda^3}{\lambda^2+2\lambda+2}\sqrt{\frac{(\lambda^2+2\lambda-p^2+2)^2+4p^2(1+\lambda^2)}{(\lambda^2+p^2)^3}},$$

and

$$\mu_p = 3\arctan\left(\frac{p}{\lambda}\right) - \arctan\left(\frac{2p(1+\lambda)}{\lambda^2+2\lambda-p^2+2}\right).$$

Since $\varphi_\theta(p) = \alpha_p + i\beta_p$ with $\alpha_p = E(\cos\ p\theta)$ and $\beta_p = E(\sin\ p\theta)$ the non-central trigonometric moments for WAD are given by

$$\begin{aligned}\alpha_p &= \rho_p \cos\ \mu_p \\ &= \frac{\lambda^3}{\lambda^2+2\lambda+2}\sqrt{\frac{(\lambda^2+2\lambda-p^2+2)^2+4p^2(1+\lambda^2)}{(\lambda^2+p^2)^3}}\cos\left[3\arctan\left(\frac{p}{\lambda}\right)\right. \\ &\qquad \left. - \arctan\left(\frac{2p(1+\lambda)}{\lambda^2+2\lambda-p^2+2}\right)\right],\end{aligned} \tag{8}$$

and

$$\begin{aligned}\beta_p &= \rho_p \sin\ \mu_p \\ &= \frac{\lambda^3}{\lambda^2+2\lambda+2}\sqrt{\frac{(\lambda^2+2\lambda-p^2+2)^2+4p^2(1+\lambda^2)}{(\lambda^2+p^2)^3}}\sin\left[3\arctan\left(\frac{p}{\lambda}\right)\right. \\ &\qquad \left. - \arctan\left(\frac{2p(1+\lambda)}{\lambda^2+2\lambda-p^2+2}\right)\right].\end{aligned} \tag{9}$$

The central trigonometric moments for WAD are given by

$$\begin{aligned}\bar{\alpha}_p &= \rho_p \cos(\mu_p - p\mu_1) \\ &= \frac{\lambda^3}{\lambda^2+2\lambda+2}\sqrt{\frac{(\lambda^2+2\lambda-p^2+2)^2+4p^2(1+\lambda)^2}{(\lambda^2+p^2)^3}}\cos\left[3\arctan\left(\frac{p}{\lambda}\right)\right. \\ &\quad \left. - \arctan\left(\frac{2p(1+\lambda)}{\lambda^2+2\lambda-p^2+2}\right) - 3p\arctan\left(\frac{1}{\lambda}\right) + p\arctan\left(\frac{2(1+\lambda)}{\lambda^2+2\lambda+1}\right)\right],\end{aligned} \tag{10}$$

and

$$\begin{aligned}\bar{\beta}_p &= \rho_p \sin(\mu_p - p\mu_1) \\ &= \frac{\lambda^3}{\lambda^2+2\lambda+2}\sqrt{\frac{(\lambda^2+2\lambda-p^2+2)^2+4p^2(1+\lambda)^2}{(\lambda^2+p^2)^3}}\sin\left[3\arctan\left(\frac{p}{\lambda}\right)\right. \\ &\left. -\arctan\left(\frac{2p(1+\lambda)}{\lambda^2+2\lambda-p^2+2}\right) -3p\arctan\left(\frac{1}{\lambda}\right) +p\arctan\left(\frac{2(1+\lambda)}{\lambda^2+2\lambda+1}\right)\right].\end{aligned} \tag{11}$$

Table 1 shows the values of non-central and central trigonometric moments for $p = 1, 2$ and $\lambda = 0.5, 1, 1.5, 2, 4$.

Parameter	**Non-central Trigonometric Moments**				**Central Trigonometric Moments**			
λ	α_1	α_2	β_1	β_2	$\bar{\alpha}_1$	$\bar{\alpha}_2$	$\bar{\beta}_1$	$\bar{\beta}_2$
0.5	-0.064	0.013	0.059	-0.015	0.087	0.016	0	0.012
1	0	-0.031	0.346	0.098	0.346	0.031	0	-0.098
1.5	0.266	0.011	0.508	0.235	0.573	0.187	0	-0.143
2	0.499	0.120	0.517	0.361	0.719	0.357	0	-0.133
4	0.867	0.601	0.321	0.460	0.924	0.756	0	-0.042

Table 1: Trigonometric Moments for WAD for $p = 1, 2$ and $\lambda = 0.5, 1, 1.5, 2, 4$.

4. Means and related measures

The mean direction and resultant length for WAD are given respectively by

$$\mu = \mu_1 = 3\arctan\left(\frac{1}{\lambda}\right) - \arctan\left(\frac{2(1+\lambda)}{\lambda^2+2\lambda+1}\right), \tag{12}$$

and

$$\rho = \rho_1 = \frac{\lambda^3}{\lambda^2+2\lambda+2}\sqrt{\frac{(\lambda^2+2\lambda+1)^2+4(1+\lambda^2)}{(\lambda^2+1)^3}}. \tag{13}$$

Table 2 shows the values of mean direction and mean resultant for $\lambda = 0.5, 1, 1.5, 2, 4$.

Means	**Parameter λ**				
	0.5	1	1.5	2	4
Mean Direction μ	2.394	1.571	1.089	0.803	0.354
Mean Resultant Length ρ	0.087	0.346	0.573	0.719	0.924

Table 2: Mean Direction and Mean Resultant Length of WAD for $\lambda = 0.5, 1, 1.5, 2, 4$.

While, the circular variance and standard deviation for WAD are given respectively by

$$\begin{aligned} V &= 1-\rho \\ &= 1-\frac{\lambda^3}{\lambda^2+2\lambda+2}\sqrt{\frac{(\lambda^2+2\lambda+1)^2+4(1+\lambda^2)}{(\lambda^2+1)^3}}, \end{aligned} \tag{14}$$

and

$$\begin{aligned} \sigma &= \sqrt{-2\ln\rho} \\ &= \sqrt{-2\ln\left(\frac{\lambda^3}{\lambda^2+2\lambda+2}\sqrt{\frac{(\lambda^2+2\lambda+1)^2+4(1+\lambda^2)}{(\lambda^2+1)^3}}\right)}. \end{aligned} \tag{15}$$

Table 3 shows the values of circular variance and circular standard deviation for $\lambda = 0.5, 1, 1.5, 2, 4$.

Measures of Variation	**Parameter λ**				
	0.5	1	1.5	2	4
Circular Variance V	0.913	0.654	0.427	0.281	0.076
Circular Standard Deviation σ	2.208	1.456	1.055	0.812	0.396

Table 3: Circular Variance and Standard Deviation for WAD for $\lambda = 0.5, 1, 1.5, 2, 4$.

5. Skewness and kurtosis

The skewness coefficient ζ_1 and the kurtosis coefficient ζ_2 for WAD are given respectively by

$$\begin{aligned} \zeta_1 &= \bar{\beta}_2 V^{-3/2} \\ &= \frac{\lambda^3}{\lambda^2+2\lambda+2}\sqrt{\frac{(\lambda^2+2\lambda-2)^2+16(1+\lambda)^2}{(\lambda^2+4)^3}}\sin\left[3\arctan\left(\frac{2}{\lambda}\right)\right. \\ &\left.-\arctan\left(\frac{4(1+\lambda)}{\lambda^2+2\lambda-2}\right)-6\arctan\left(\frac{1}{\lambda}\right)+2\arctan\left(\frac{4\lambda}{\lambda^2+2\lambda+1}\right)\right] \\ &\left(1-\frac{\lambda^3}{\lambda^2+2\lambda+2}\sqrt{\frac{(\lambda^2+2\lambda+1)^2+4(1+\lambda^2)}{(\lambda^2+1)^3}}\right)^{-3/2}, \end{aligned} \tag{16}$$

and

$$\begin{aligned} \zeta_2 &= \left(\bar{\alpha_2}-(1-V)^4\right)V^{-2} \\ &= \left(1-\frac{\lambda^3}{\lambda^2+2\lambda+2}\sqrt{\frac{(\lambda^2+2\lambda+1)^2+4(1+\lambda^2)}{(\lambda^2+1)^3}}\right)^{-2} \end{aligned} \tag{17}$$

$$\left[\frac{\lambda^3}{\lambda^2+2\lambda+2}\sqrt{\frac{(\lambda^2+2\lambda-2)^2+16(1+\lambda)^2}{(\lambda^2+4)^3}}\cos\left[3\arctan\left(\frac{2}{\lambda}\right)\right.\right.$$
$$\left.-\arctan\left(\frac{4(1+\lambda)}{\lambda^2+2\lambda-2}\right)-6\arctan\left(\frac{1}{\lambda}\right)+2\arctan\left(\frac{4\lambda}{\lambda^2+2\lambda+1}\right)\right]$$
$$\left.\left(\frac{\lambda^3}{\lambda^2+2\lambda+2}\sqrt{\frac{(\lambda^2+2\lambda+1)^2+4(1+\lambda^2)}{(\lambda^2+1)^3}}\right)^4\right].$$

Table 4 shows the values of skewness and kurtosis coefficients for $\lambda = 0.5, 1, 1.5, 2, 4$.

Coefficients	**Parameter** λ				
	0.5	1	1.5	2	4
Skewness ζ_1	0.014	-0.186	-0.514	-0.893	-2.031
Kurtosis ζ_2	0.019	0.038	0.435	1.133	4.394

Table 4: Skewness and Kurtosis Coefficients for WAD for $\lambda = 0.5, 1, 1.5, 2, 4$.

Conclusions

We introduced a new circular distribution generated from wrapping Aradhana distribution around the unit circle. The explicit expressions for the probability density function and the cumulative density function for WAD were obtained. Moreover, the trigonometric moments, measures of variation, and some special characteristics for the new distribution were studied.

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Matrices of infinite dimensions and their applications

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Abstract. Matrices are very popular and widely used in mathematics and other fields of science. Every mathematician has known the properties of finite-sized matrices since the time of study. In this paper, we consider the basic theory of infinite matrices. So far, there have been references and few results in certain scientific fields, but they have not been thoroughly researched.

Keywords: matrix, determinant, inverse matrix, rank.

1. Introduction

Infinite matrices, the forerunner and a main constituent of many branches of classical mathematics (infinite quadratic forms, integral equations, differential equations, etc.) and of the modern operator theory, is revisited to demonstrate its deep influence on thedevelopment of many branches of mathematics, classical and modern, replete with applications.

It is known that we can add matrices, multiply by scalar, multiply matrices, calculate determinant, calculate inverse matrix, determine rank matrix. We can find all these properties of matrices in many basic academic books, for example in [1], [2], [3].

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Matrices with an infinite number of rows and/or columns are also considered - formally, it is sufficient that for any elements indexing rows and columns there is a well-defined matrix element (index sets do not even have to be subsets of natural numbers). Similarly to the finite case, we can define addition, subtraction, multiplication by scalar or matrix shifting, although matrix multiplication requires some assumptions.

Applications of matrices are found in most scientific fields ([5]). In every branch of physics, including classical mechanics, optics, electromagnetism, quantum mechanics, and quantum electrodynamics, they are used to study physical phenomena, such as the motion of rigid bodies. In computer graphics, they are used to manipulate 3D models and project them onto a 2-dimensional screen. In probability theory and statistics, stochastic matrices are used to describe sets of probabilities. For example, they are used within the PageRank algorithm that ranks the pages in a Google search. ([4]) Matrix calculus generalizes classical analytical notions such as derivatives and exponentials to higher dimensions. Matrices are used in economics to describe systems of economic relationships.

A major branch of numerical analysis is devoted to the development of efficient algorithms for matrix computations, a subject that is centuries old and is today an expanding area of research. Matrix decomposition methods simplify computations, both theoretically and practically. Algorithms that are tailored to particular matrix structures, such as sparse matrices and near-diagonal matrices, expedite computations in finite element method and other computations. Infinite matrices occur in planetary theory and in atomic theory. A simple example of an infinite matrix is the matrix representing the derivative operator, which acts on the Taylor series of a function.

In this paper, we formalize and develop the basic theory of infinite matrices. So far, they have not been thoroughly researched, despite their significant use in some fields of science.

2. Results

By an infinite dimension matrix we call a matrix for which the number of rows is infinite or the number of columns is infinite.

We define zero, triangular, diagonal, unitary and transposed matrices of an infinite dimension very analogously.

A square matrix of an infinite dimension is a matrix in which the number of rows is equinumerous to the number of columns.

Matrix sum and by scalar multiplication are also analogous.

Corollary 2.1. *If we try to multiply matrix $A_{m\times n}$ with matrix $B_{n\times k}$, we get the following conclusions:*

(*a*) *If $m=\infty$, $k=\infty$, then $AB=C_{\infty\times\infty}$.*

(*b*) *If $n=\infty$, then $AB=C_{m\times k}=[c_{ij}]$, where $c_{ij}=\sum_{l=1}^{\infty}a_{il}b_{lj}$ $(1\leqslant i\leqslant m,$ $1\leqslant j\leqslant k)$ be a convergent series.*

(c) *If A, B be square matrices of an infinite dimension, then $AB = C$ holds.*

(d) *If A, B be matrices of an infinite dimension and the number of rows in A is equinumerous to the number of columns in B, then we can multiply matrices A and B only if rows of A and columns of B be a convergent series.*

Let $M_1(\infty, R) = M_1(R)$ be denote the set of all square matrices of an infinite dimension with coefficients from any integral domain R, where all rows and columns are convergent series. Then $M_1(\infty, R)$ be a ring. Easy to check that $\{A \in M_1(\infty, \mathbb{Z})\colon \det A \in \{-1, 1\}\}$ and $\{A \in M_1(\infty, \mathbb{Z})\colon \det A = 1\}$ are multiplicative groups.

The determinant of a square matrix A of finite dimension can be easily determined by the formula:

$$\det A = \det(\exp(\log A)) = \exp(\operatorname{tr}(\log A)),$$

where $\log A = \sum_{k=1}^{\infty}(-1)^{k+1}\dfrac{A^k}{k}$. For an infinite dimension we must add the assumption that $\operatorname{tr}(\log A)$ be a convergent series.

Proposition 2.1. *Let A be an $m \times n$ matrix, and let B be an matrix $n \times m$, where $m, n \in \mathbb{N} \cup \{\infty\}$. Let $1 \leqslant j_1, j_2, \ldots, j_m \leqslant n$. Let $A_{j_1 j_2 \ldots j_m}$ denote the $m \times m$ matrix consisting of columns $j_1, j_2, \ldots, j_m$ of A. Let $B_{j_1 j_2 \ldots j_m}$ denote the $m \times m$ matrix consisting of rows $j_1, j_2, \ldots, j_m$ of B. Then*

$$\det(AB) = \sum_{1 \leqslant j_1 < j_2 < \cdots < j_m \leqslant n} \det(A_{j_1 j_2 \ldots j_m}) \det(B_{j_1 j_2 \ldots j_m}).$$

Proof. First we will show the proof in the finite version.

Let $(k_1, k_2, \ldots, k_m)$ be an ordered m-tuple of integers. Let $\eta(k_1, k_2, \ldots, k_m)$ denote the sign of $(k_1, k_2, \ldots, k_m)$. Let $(l_1, l_2, \ldots, l_m)$ be the same as $(k_1, k_2, \ldots, k_m)$ except for k_i and k_j having been transposed. Then, from Transposistion is of Odd Parity:

$$\eta(l_1, l_2, \ldots, l_m) = -\eta(k_1, k_2, \ldots, k_m).$$

Let $(j_1, j_2, \ldots, j_m)$ be the same as $(k_1, k_2, \ldots, k_m)$ by arranged into non-decreasing order. That is $j_1 \leqslant j_2 \leqslant \cdots \leqslant j_m$. Then, it follows that:

$$\det(B_{k_1 \ldots k_m}) = \eta(k_1, k_2, \ldots, k_m) \det(B_{j_1 \ldots j_m}).$$

Hence:

$$\begin{aligned}
\det(AB) &= \sum_{1\leqslant l_1,\ldots,l_m\leqslant m} \eta(l_1,\ldots,l_m)(\sum_{k=1}^{n} a_{1k}b_{kl_1})\ldots(\sum_{k=1}^{n} a_{mk}b_{kl_m}) = \\
&= \sum_{1\leqslant k_1,\ldots,k_m\leqslant n} a_{1k_1}\ldots a_{mk_m} \sum_{1\leqslant l_1,\ldots,l_m\leqslant m} \eta(l_1,\ldots,l_m)b_{k_1l_1}\ldots b_{k_ml_m} = \\
&= \sum_{1\leqslant k_1,\ldots,k_m\leqslant n} a_{1k_1}\ldots a_{mk_m}\det(B_{k_1\ldots k_m}) = \\
&= \sum_{1\leqslant k_1,\ldots,k_m\leqslant n} a_{1k_1}\eta(k_1,\ldots,k_m)\ldots a_{mk_m}\det(B_{j_1\ldots j_m}) = \\
&= \sum_{1\leqslant j_1\leqslant j_2\leqslant\cdots\leqslant j_m\leqslant n} \det(A_{j_1\ldots j_m})\det(B_{j_1\ldots j_m}).
\end{aligned}$$

If two js are equal:

$$\det(A_{j_1\ldots j_m}) = 0.$$

For an infinite matrices we put ∞-tuple in proof in the form $(k_1, k_2, k_3, \ldots)$. And put $1 \leqslant j_1, j_2, j_3, \cdots < n = \infty$. □

Corollary 2.2. *If $m = n$ ($m, n \in \mathbb{N}\cup\{\infty\}$), then*

$$\det(AB) = \det(A)\det(B).$$

The following two Propositions give us a way to compute the inverse matrix.

Proposition 2.2. *Let A be a matrix in which every rows and colums form convergent series such that $||I - A|| < 1$, where $||\cdot||$ is a submultiplicative norm. Then*

$$A^{-1} = I + (I - B) + (I - B)^2 + \ldots$$

Proof. A matrix $A \in M_1(K)$ (where $n \in \mathbb{N}\cup\{\infty\}$, K be a field) is invertible if and only if the map $f\colon K^n \to K^n$ defined by $f(x) = Ax$ is invertible, where elements of K^n are considered as column vectors. □

3. Applications

Let A be an $m \times n$ matrix over an arbitrary field F ($m, n \in \mathbb{N}\cup\{\infty\}$). There is an associated linear mapping $f\colon F^n \to F^m$ defined by $f(x) = Ax$. The rank of A is the dimension of the image f. This definition has the advantage that it can be applied to any linear map without need for a specific matrix.

Let

$$\begin{cases}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots &= b_3 \\
\ldots
\end{cases}$$

$(AX = B)$ be a system of equations. By Cramer's system we mean a system in which the number of equations is equinumerous to the number of unknowns. Then, Cramer's theorem states that in finite case the system has a unique solution provided we have n equations. Hence individual values for the unknowns are given by:

$$x_i = \frac{\det A_i}{\det A},$$

for $i = 1, 2, \dots$, where A_i is the matrix formed by replacing the i-th column of A by the column vector B. If $\operatorname{tr} A_i$, $\operatorname{tr} A$ are convergent series, then Cramer's formula holds for infinity case.

In other hand, system of equations $AX = B$ (A, X, B can have an infinite dimension) implies $X = A^{-1}B$.

Theorem 3.1 (Rouché-Capelli Theorem (Kronecker-Capelli Theorem))**.** *Let $m, n \in \mathbb{N} \cup \{\infty\}$. A system of m linear equations in n variables $Ax = b$ is compatible if and only if both the incomplete and complete matrices (A and $[A|b]$ respectively) are characterised by the same rank $A = rank\,[A|b]$.*

Proof. Let m, $n \in \mathbb{N} \cup \{\infty\}$. The system of linear equations $Ax = b$ can be interpreted as a linear mapping $f : F^n \to F^m$, by $f(x) = Ax$, such that $A \in M(n, R)$, R be an integral domain.

This system is determined if one solution exists, t.e. if there exists x_0 such that $f(x_0) = b$. This means that the system is determined if $b \in Im(f)$.

The basis spanning the image vector space $(Im(f), +, \cdot)$ is composed of the column vectors of the matrix A:

$$B_{Im(f)} = \{I_1, I_2, \dots, I_n\}, A = (I_1 I_2 \dots I_n).$$

Thus, the fact that $b \in Im(f)$ is equivalent to the fact that b belongs to the span of the column vectors of the matrix A:

$$b = (I_1, I_2, I_3, \dots).$$

This is equivalent to say that the rank of

$$A = (I_1 I_2 I_3 \dots)$$

and

$$[A|b] = (I_1 I_2 I_3 \dots b)$$

have the same rank. Thus, the system is compatible if $\operatorname{rank} A = \operatorname{rank} [A|B]$. □

Let $B = \{v_1, v_2, v_3, \dots\}$, $B' = \{u_1, u_2, u_3, \dots\}$. Then, for $i = 1, 2, 3, \dots$ we compute coordinates $\alpha_1^{(i)}$, $\alpha_2^{(i)}$, $\alpha_3^{(i)}$, $\dots$ of the basis vector B' in basis B:

$$u_i = \sum_{j=1}^{\infty} \alpha_j^{(i)} v_j.$$

Hence a transition matrix is of the form:

$$\begin{pmatrix} \alpha_1^{(1)} & \alpha_1^{(2)} & \alpha_1^{(3)} & \dots \\ \alpha_2^{(1)} & \alpha_2^{(2)} & \alpha_2^{(3)} & \dots \\ \alpha_3^{(1)} & \alpha_3^{(2)} & \alpha_3^{(3)} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Let $L\colon U \to V$ be a linear transformation, where U, V be a linear spaces such that $\dim U = m$, $\dim V = n$ (m, n can be ∞) and a basis of U be $\{u_1, u_2, \dots, u_m\}$, a basis of V be $\{v_1, v_2, \dots, v_n\}$. For $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$ compute

$$L(u_i) = \sum_{j=1}^{n} \alpha_j^{(i)} v_j.$$

Then a transformation matrix of transformation L is of the form:

$$\begin{pmatrix} \alpha_1^{(1)} & \alpha_1^{(2)} & \dots & \alpha_1^{(n)} \\ \alpha_2^{(1)} & \alpha_2^{(2)} & \dots & \alpha_2^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_m^{(1)} & \alpha_m^{(2)} & \dots & \alpha_m^{(n)} \end{pmatrix}.$$

We will try to find the eigenvalues and eigenvectors of the infinity matrix.

Solve a characteristic equation:

$$\det(A - \lambda I) = 0.$$

So, we have to calculate

$$\det(A - \lambda I) = \exp(\operatorname{tr}(\log(A - \lambda I))),$$

where $\operatorname{tr}\log(A - \lambda I)$ be a convergent series.

For an appropriate eigenvalue λ, we find the corresponding eigenvector $v = (x_1, x_2, x_3, \dots)$ from the system of equations:

$$(A - \lambda I) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}.$$

Let A be a matrix of any dimension, whose rows are given linearly independent vectors. We are building a block matrix $[AA^T \mid A]$. Applying elementary row operations we bring it to the block matrix of the form $[G \mid A']$, where G be the upper triangular matrix. The rows of A' form orthogonal vectors.

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New conformable fractional HPT for solving systems of linear and nonlinear conformable fractional PDEs

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Abstract. In this work, an analytical solution for a linear and nonlinear system of conformable fractional PDEs is obtained by applying a New conformable fractional homotopy perturbation technique (NCFHPT) with the help of fractional series expansion. In fact, the current method is a natural extension of NHPM for partial differential equation of real order in conformable sense. By constructing the homotopy transformation for a given system and gathering the coefficients with the identical power of p, a system of recursive equations is established. In addition, by a convenient assumption on the initial approximate solution. We can consecutively starting with this solution and working upward until getting general term of the intended coefficients to deduce a closed form series solution. The NCFHPT gives an analytical solution without making any linearization, rough conditions or discretization, especially for non-linear problems. This technique shows a powerful and a promise tool to solve linear and nonlinear systems of CFPDEs. The practicable validation and effectiveness of this method are demonstrated by three typical examples.

Keywords: system of conformable fractional PDEs, new conformable fractional HPT, conformable fractional derivative, fractional power series.

1. Introduction

In the two decades before, the researchers have paid close attention to the analytical and numerical solutions of the nonlinear problems which have been arising in the present scientific and technological fields, such as physics, mathematics, biology, engineering, fluid mechanics, fluid dynamic, and viscoelasticity etc.. Such problems are not easy to be solved directly and obtained an exact solution so that suitable methods for finding exact or numerical solutions are required. Thereafter, analytical and numerical methods have been proposed by mathematical, physical and engineering experts, for instance, homotopy technique and its various modifications [1, 2, 3, 4], He variational iteration method [5], He-Laplace Method [6], parameter-expansion method [7], Wavelet-Galerkin method [8], Adomian decomposition method[9], Differential transform method [10] and the cited references.

Many problems that may happen in the real life world can be modelled only by differential equations of fractional order, we refer the readers to [11, 12] and the references therein. Actually, there is no physical meaning to the fractional derivative in order to unify its definition as ordinary derivatives [13]. Therefore, in this study we use the definition of Khalil et al. [14] which looks very similar to the definition of ordinary derivative and it is named a conformable fractional derivative. This novel definition has attracted the awareness of numerous scholars of mathematics and engineering to extend different solution techniques from ordinary differential equations to conformable fractional differential equations [15, 16, 17, 18, 19, 20, 21, 22, 23, 24].

More recently, the HPM in fractional version has showed a vigorous capability in solving conformable fractional PDEs see [25, 26] and references therein. We have motivated by these beneficial results and the conducive outcomes in the works of Biazar and Eslami [27] and Mirzazadeh and Ayati [28], to apply NCFHPT on several systems of CFPDEs with given initial and boundary conditions. In this work, we shall address a system of CFPDEs in the standard form as:

$$\begin{cases} \partial_t^q u_1 = N_1^q(t, X, U, \partial_{x_i} U, \partial_t^q U) + f_1^q(t, X), \\ \partial_t^q u_2 = N_2^q(t, X, U, \partial_{x_i} U, \partial_t^q U) + f_2^q(t, X), \\ \vdots \\ \partial_t^q u_n = N_n^q(t, X, U, \partial_{x_i} U, \partial_t^q U) + f_n^q(t, X), \end{cases} \tag{1}$$

subject to I.C.

$$\Big(u_1(t_0, X), u_2(t_0, X), \ldots, u_n(t_0, X)\Big) = \Big(\phi_1(X,), \phi_2(X), \ldots, \phi_n(X)\Big), \tag{2}$$

where $i = 1, 2 \ldots, n$, ∂_t^q stands for the conformable partial fractional derivative of order $0 < q \leq 1$ with respect to t , $\phi_1(X), \phi_2(X), \ldots, \phi_n(X)$ are real-valued functions depend only on the vector $X = (x_1, x_2, \ldots x_{n-1})$, $f_1^q, f_2^q, \ldots f_n^q$ are unknown functions depending on the parameter t and $x_1, x_2, \ldots, x_{n-1}$, and $N_1^q, N_2^q \ldots, N_n^q$ are the nonlinear parts of the equations of the system (1).

The rest of this article is arranged as the following: the appropriate adaptation of NHPM, which is called NCFHPT, for obtaining solutions of a system (1) is presented in the second section. NCFHPT solutions for three special systems of non-linear CFPDEs with given initial and boundary conditions are given in the third section. Finally, In Section 5, we briefly summarize the results in a short conclusion.

2. New conformable fractional homotopy technique

In this section, we present the appropriate adaptation of NHPM, which is called NCFHPT, for obtaining solutions of a system (1).

By the homotopy method [1, 2, 3, 4], we can build a system of homotopies as follows:

$$(3)\quad \begin{cases} (1-p)\big(\partial_t^q v_1 - u_{10}(t,X)\big)+p\Big(-N_1^q(t,X,V,\partial_{x_i}V,\partial_t^q V)-f_1^q(t,X)\Big)=0,\\ (1-p)\big(\partial_t^q v_2 - u_{20}(t,X)\big)+p\Big(-N_2^q(t,X,V,\partial_{x_i}V,\partial_t^q V)-f_2^q(t,X)\Big)=0,\\ \vdots\\ (1-p)\big(\partial_t^q v_n - u_{n0}(t,X)\big)+p\Big(-N_n^q(t,X,V,\partial_{x_i}V,\partial_t^q V)-f_n^q(t,X)\Big)=0 \end{cases}$$

or, equivalently

$$(4)\quad \begin{cases} \partial_t^q v_1 = u_{10}(t,X) - p\Big(u_{10}(t,X) - N_1^q(t,X,V,\partial_{x_i}V,\partial_t^q V) - f_1^q(t,X)\Big),\\ \partial_t^q v_2 = u_{20}(t,X) - p\Big(u_{20}(t,X) - N_2^q(t,X,V,\partial_{x_i}V,\partial_t^q V) - f_2^q(t,X)\Big),\\ \vdots\\ \partial_t^q v_n = u_{n0}(t,X) - p\Big(u_{n0}(t,X) - N_n^q(t,X,V,\partial_{x_i}V,\partial_t^q V) - f_n^q(t,X)\Big), \end{cases}$$

where $V=(v_1,v_2,\dots,v_n)$, X, $f_i^q, i=1\dots n$, are as defined above, $0\le p\le 1$ is an embedding parameter and $u_0=(u_{10}(t,X),u_{20}(t,X),\dots,u_{n0}(t,X))$ is a vector initial approximation of solution for the system (1).

Applying the fractional integral $\int_{t_0}^t (t-t_0)^{q-1}dt$ at all equations of the system (4), one can obtain

$$(5)\quad \begin{cases} v_1 = v_1(t_0,X) + \int_{t_0}^t t^{q-1}u_{10}(t,X)dt \\ \qquad\qquad -p\int_{t_0}^t t^{q-1}\big(u_{10}(t,X) - N_1^q - f_1^q(t,X)\big)dt,\\ v_2 = v_2(t_0,X) + \int_{t_0}^t t^{q-1}u_{20}(t,X)dt \\ \qquad\qquad -p\int_{t_0}^t t^{q-1}\big(u_{20}(t,X) - N_2^q - f_2^q(t,X)\big)dt,\\ \vdots\\ v_n = v_n(t_0,X) + \int_{t_0}^t t^{q-1}u_{n0}(t,X)dt \\ \qquad\qquad -p\int_{t_0}^t t^{q-1}\big(u_{n0}(t,X) - N_n^q - f_n^q(t,X)\big)dt, \end{cases}$$

with $V(t_0,X)=U(t_0,X)$.

Suppose that the willful solution of the given system (1) is presented in a power series of p, i.e.

$$(6)\quad \begin{cases} v_1 = v_{10} + pv_{11} + p^2v_{12} + \dots\\ v_2 = v_{20} + pv_{21} + p^2v_{22} + \dots\\ \vdots\\ v_n = v_{n0} + pv_{n1} + p^2v_{n2} + \dots, \end{cases}$$

where, the $v_{ij}, i=1,\dots n, j=1,\dots n$, are the seeking functions which are needed to be calculated. Assume that for $i=1,\dots n$, f_i^q and the initial approximate

solution u_{i0} are infinitely q partial differentiable functions with respect to t, around a point t_0, then their fractional power series can be given as:

$$\begin{cases} u_{10} = \sum_{k=0}^{\infty} \frac{a_{1k}(X)(t-t_0)^{qk}}{q^k k!}, \\ u_{20} = \sum_{k=0}^{\infty} \frac{a_{2k}(X)(t-t_0)^{qk}}{q^k k!}, \\ \vdots \\ u_{n0} = \sum_{k=0}^{\infty} \frac{a_{nk}(X)(t-t_0)^{qk}}{q^k k!} \end{cases} \tag{7}$$

and

$$\begin{cases} f_1^q = \sum_{k=0}^{\infty} \frac{b_{1k}(X)(t-t_0)^{qk}}{q^k k!}, \\ f_2^q = \sum_{k=0}^{\infty} \frac{b_{2k}(X)(t-t_0)^{qk}}{q^k k!}, \\ \vdots \\ f_n^q = \sum_{k=0}^{\infty} \frac{b_{nk}(X)(t-t_0)^{qk}}{q^k k!}, \end{cases} \tag{8}$$

where $a_{ik'}s$ are unknown functions which must be computed and $b_{ik'}s$ are given ones.

Now, substituting (6) and (7) into (5) and equating the coefficients of terms with identical powers of p, we get

$$p^0 : \begin{cases} v_{10} = \phi_1(X) + \sum_{k=0}^{\infty} \frac{a_{1k}(X)(t-t_0)^{q(k+1)}}{q^{k+1}(k+1)!} \\ v_{20} = \phi_2(X) + \sum_{k=0}^{\infty} \frac{a_{2k}(X)(t-t_0)^{q(k+1)}}{q^{k+1}(k+1)!} \\ \vdots \\ v_{n0} = \phi_n(X) + \sum_{k=0}^{\infty} \frac{a_{nk}(X)(t-t_0)^{q(k+1)}}{q^{k+1}(k+1)!} \end{cases}$$

$$p^1 : \begin{cases} v_{11} = -\sum_{k=0}^{\infty} \frac{a_{1k}(X)(t-t_0)^{q(k+1)}}{q^{k+1}(k+1)!} \\ \qquad + \int_{t_0}^{t} t^{q-1}\big(N_1^q(t, X, v_{.0}, \partial_{x_i} v_{.0}, \partial_t^q v_{.0}) - f_1^q\big)dt \\ v_{21} = -\sum_{k=0}^{\infty} \frac{a_{2k}(X)(t-t_0)^{q(k+1)}}{q^{k+1}(k+1)!} \\ \qquad + \int_{t_0}^{t} t^{q-1}\big(N_2^q(t, X, v_{.0}, \partial_{x_i} v_{.0}, \partial_t^q v_{.0}) - f_2^q\big)dt \\ \vdots \\ v_{n1} = -\sum_{k=0}^{\infty} \frac{a_{nk}(X)(t-t_0)^{q(k+1)}}{q^{k+1}(k+1)!} \\ \qquad + \int_{t_0}^{t} t^{q-1}\big(N_n^q(t, X, v_{.0}, \partial_{x_i} v_{.0}, \partial_t^q v_{.0}) - f_n^q\big)dt \end{cases}$$

$$\vdots$$

$$
(9)\qquad p^m:\begin{cases} v_{1m}=\int_{t_0}^{t} t^{q-1}\big(N_1^q(t,X,v_{.0},\dots v_{.m-1},\partial_{x_i}v_{.0},\dots,\partial_{x_i}v_{.m-1},\\ \qquad \partial_t^q v_{.0},\dots,\partial_t^q v_{.m-1}\big)dt,\\ v_{2m}=\int_{t_0}^{t} t^{q-1}\big(N_2^q(t,X,v_{.0},\dots v_{.m-1},\partial_{x_i}v_{.0},\dots,\partial_{x_i}v_{.m-1},\\ \qquad \partial_t^q v_{.0},\dots,\partial_t^q v_{.m-1}\big)dt,\\ \vdots\\ v_{nm}=\int_{t_0}^{t} t^{q-1}\big(N_n^q(t,X,v_{.0},\dots v_{.m-1},\partial_{x_i}v_{.0},\dots,\partial_{x_i}v_{.m-1},\\ \qquad \partial_t^q v_{.0},\dots,\partial_t^q v_{.m-1}\big)dt. \end{cases}
$$

$$\vdots$$

Solving these equations under the condition that $v_{i1}=0, i=1,\dots n$, then (9) gives $v_{i2}=v_{i3}=\ldots=0, i=1,\dots n$. Thus the approximate solution which constructed from the presented method can be obtained as

$$
(10)\qquad \begin{cases} u_1=v_{10}=\phi_1(X)+\sum_{k=0}^{\infty}\frac{a_{1k}(X)(t-t_0)^{q(k+1)}}{q^{k+1}(k+1)!}\\ u_2=v_{20}=\phi_2(X)+\sum_{k=0}^{\infty}\frac{a_{2k}(X)(t-t_0)^{q(k+1)}}{q^{k+1}(k+1)!}\\ \vdots\\ u_n=v_{n0}=\phi_n(X)+\sum_{k=0}^{\infty}\frac{a_{nk}(X)(t-t_0)^{q(k+1)}}{q^{k+1}(k+1)!}. \end{cases}
$$

This method can be easily adapted to the system with given boundary conditions. In the following section, one of the examples will be given with boundary conditions.

3. Numerical results

To illustrate the efficiency of NCFHPT, we applied it to three special systems of non-linear CFPDEs with given initial conditions and boundary conditions as well. This method shows high capabilities for obtaining closed form series solutions. These solutions coincide with the exact solutions in the given systems here.

Example 1. Given a system of coupled time conformable fractional Burgers equations

$$
(11)\qquad \begin{cases} \partial_t^q u_1=\partial_x^2 u_1-u_1\partial_x u_1-\partial_x(u_1u_2)+x^2-2\frac{t^q}{q}+2x^3\frac{t^{2q}}{q^2}+\frac{t^{2q}}{q^2},\\ \partial_t^q u_2=\partial_x^2 u_2-u_2\partial_x u_2+\partial_x(u_1u_2)+x^{-1}-2x^{-3}\frac{t^q}{q}-2x^{-3}\frac{t^{2q}}{q^2}-\frac{t^{2q}}{q^2} \end{cases}
$$

subject to I.C.

$$
(12)\qquad \begin{cases} u_1(0,x)=0,\\ u_2(0,x)=0. \end{cases}
$$

Thanks to NCFHPT, we have

$$
(13)\qquad \begin{cases} v_1(t,x) = v_1(0,x) + \int_0^t t^{q-1}u_{10}(t,x)dt - p\int_0^t t^{q-1}u_{10}(t,x)dt \\ \quad -p\int_0^t t^{q-1}\big(-\partial_x^2 v_1 + v_1\partial_x v_1 + \partial_x(v_1v_2) \\ \quad -x^2 + 2\frac{t^q}{q} - 2x^3\frac{t^{2q}}{q^2} - \frac{t^{2q}}{q^2}\big)dt, \\ v_2(t,x) = v_2(0,x) + \int_0^t t^{q-1}u_{20}(t,x)dt - p\int_0^t t^{q-1}u_{20}(t,x)dt \\ \quad -p\int_0^t t^{q-1}\big(-\partial_x^2 v_2 + v_2\partial_x v_2 - \partial_x(v_1v_2) - x^{-1} + 2x^{-3}\frac{t^q}{q} \\ \quad +2x^{-3}\frac{t^{2q}}{q^2} + \frac{t^{2q}}{q^2}\big)dt. \end{cases}
$$

Substituting (6) and (7) into (13) and equating the coefficients with the same power of p we obtain

$$
p^0 : \begin{cases} v_{10} = v_1(0,x) + \sum_{k=0}^{\infty}\frac{a_{1k}(x)t^{q(k+1)}}{q^{k+1}(k+1)!}, \\ v_{20} = v_2(0,x) + \sum_{k=0}^{\infty}\frac{a_{2k}(x)t^{q(k+1)}}{q^{k+1}(k+1)!}, \end{cases}
$$

$$
p^1 : \begin{cases} v_{11} = -\sum_{k=0}^{\infty}\frac{a_{1k}(x)t^{q(k+1)}}{q^{k+1}(k+1)!} \\ -\int_0^t t^{q-1}\big(-\partial_x^2 v_{10} + v_{10}\partial_x v_{10} + \partial_x(v_{10}v_{20}) \\ \quad -x^2 + 2\frac{t^q}{q} - 2x^3\frac{t^{2q}}{q^2} - \frac{t^{2q}}{q^2}\big)dt \\ v_{21} = -\sum_{k=0}^{\infty}\frac{a_{2k}(x)t^{q(k+1)}}{q^{k+1}(k+1)!} \\ -\int_0^t t^{q-1}\big(-\partial_x^2 v_{20} + v_{20}\partial_x v_{20} - \partial_x(v_{10}v_{20}) \\ \quad -x^{-1} + 2x^{-3}\frac{t^q}{q} + 2x^{-3}\frac{t^{2q}}{q^2} + \frac{t^{2q}}{q^2}\big)dt \end{cases},
$$

$$
\vdots
$$

Assume that $v_1(0,x) = u_1(0,x), v_2(0,x) = u_2(0,x)$, and we obtain

$$
\begin{aligned} v_{11} &= \big(-a_{10} + x^2\big)\frac{t^q}{q} + \big(-a_{11} + a_{10}'' - 2\big)\frac{t^{2q}}{2q^2} \\ &\quad + \Big(-2a_{12} + a_{11}'' - 2a_{10}a_{10}' - 2a_{20}a_{10}' - 2a_{10}a_{20}' + 4x^3 + 2\Big)\frac{t^{3q}}{3!q^3} + \dots \end{aligned}
$$

and

$$
\begin{aligned} v_{21} &= \big(-a_{20} + x^{-1}\big)\frac{t^q}{q} + \big(-a_{21} + a_{20}'' - 2x^{-3}\big)\frac{t^{2q}}{2q^2} \\ &\quad + \Big(-2a_{22} + a_{11}'' - 2a_{20}a_{20}' + 2a_{20}a_{10}' + 2a_{10}a_{20}' - 2x^{-3} - 2\Big)\frac{t^{3q}}{3!q^3} + \dots \end{aligned}
$$

Now, setting $v_{11} = v_{21} = 0$ we get:

$$
\begin{aligned} &a_{10}(x) = x^2,\ a_{11}(x) = a_{12}(x) =, \dots, = 0, \\ &a_{20}(x) = x^{-1},\ a_{21}(x) = a_{22}(x) =, \dots, = 0. \end{aligned}
$$

Therefore, we obtain the new homotopy solution of the given system as

$$\begin{cases} u_1 = v_{10} = \phi_1(x) + \sum_{k=0}^{\infty} \frac{a_{1k}(x) t^{q(k+1)}}{q^{k+1}(k+1)!} = x^2 \frac{t^q}{q}, \\ u_2 = v_{20} = \phi_2(x) + \sum_{k=0}^{\infty} \frac{a_{2k}(x) t^{q(k+1)}}{q^{k+1}(k+1)!} = x^{-1} \frac{t^q}{q}, \end{cases} \tag{14}$$

which gives the exact solution of the system (11).

Example 2. Given a system of two space-conformable fractional partial differential equations

$$\begin{cases} \partial_x^q u_1 = \partial_t(u_1) u_2 - u_1 \partial_t u_2 + e^{\frac{x^q}{q}} \sin t - 1, \\ \partial_x^q u_2 = -\partial_t u_1 \partial_x^q u_2 - \partial_x^q u_1 \partial_t u_2 - e^{-\frac{x^q}{q}} \cos t - 1, \end{cases} \tag{15}$$

subject to

$$\begin{cases} u_1(t,0) = \sin t, \\ u_2(t,0) = \cos t. \end{cases} \tag{16}$$

Thanks to NCFHPT, we have

$$\begin{cases} v_1(t,x) = v_1(t,0) + \int_0^x x^{q-1} u_{10}(t,x) dx \\ \qquad -p \int_0^x x^{q-1} \big(u_{10}(t,x) - \partial_t(v_1) v_2 + v_1 \partial_t v_2 - e^{\frac{x^q}{q}} \sin t + 1 \big) dx, \\ v_2(t,x) = v_2(t,0) + \int_0^x x^{q-1} u_{20}(t,x) dt \\ \qquad -p \int_0^x x^{q-1} \big(u_{20}(t,x) + \partial_t v_{10} \partial_x^q v_{20} + \partial_x^q v_{10} \partial_t v_{20} + e^{-\frac{x^q}{q}} \cos t + 1 \big) dx. \end{cases} \tag{17}$$

Substituting (7) and (6) into (17) and equating the terms with the same power of p we obtain

$$p^0 : \begin{cases} v_{10} = v_1(t,0) + \sum_{k=0}^{\infty} \frac{a_{1k}(t) x^{q(k+1)}}{q^{k+1}(k+1)!}, \\ v_{20} = v_2(t,0) + \sum_{k=0}^{\infty} \frac{a_{2k}(t) x^{q(k+1)}}{q^{k+1}(k+1)!}, \end{cases}$$

$$p^1 : \begin{cases} v_{11} = -\sum_{k=0}^{\infty} \frac{a_{1k}(t) x^{q(k+1)}}{q^{k+1}(k+1)!} \\ \qquad - \int_0^x x^{q-1} \big(- \partial_t(v_{10}) v_{20} + v_{10} \partial_t v_{20} - e^{\frac{x^q}{q}} \sin t + 1 \big) dx, \\ v_{21} = -\sum_{k=0}^{\infty} \frac{a_{2k}(t) x^{q(k+1)}}{q^{k+1}(k+1)!} \\ \qquad - \int_0^x x^{q-1} \big(\partial_t v_1 \partial_x^q v_2 + \partial_x^q v_1 \partial_t v_2 + e^{-\frac{x^q}{q}} \cos t + 1 \big) dx, \end{cases}$$

$$\vdots$$

Assume that $v_1(t,0) = u_1(t,0), v_2(t,0) = u_2(t,0)$, and evaluate v_{11} and v_{21} we obtain

$$\begin{aligned} v_{11} &= (\sin t - a_{10}(t))\frac{x^q}{q} + \big(- a_{11}(t) + (a'_{10}(t) \\ &\quad + a_{20}(t))\cos t + (a_{10}(t) - a'_{20}(t))\sin t + \sin t\big)\frac{x^{2q}}{2q^2}\Big(- 2a_{12}(t) \\ &\quad + (a'_{11}(t) + a_{21}(t))\cos t + (a_{11}(t) \\ &\quad - a'_{21}(t))\sin t + 2(a_{20}(t)a'_{10}(t) - a'_{20}(t)a_{10}(t)) + \sin t\Big)\frac{x^{3q}}{3!q^3} + \dots \end{aligned}$$

and

$$\begin{aligned} v_{21} &= \Big(a_{20}(t)(-1-\cos t) + a_{10}(t)\sin t - \cos t - 1\Big)\frac{x^q}{q} \\ &\quad + \Big(a_{21}(t)(-1-\cos t) - a_{20}(t)a'_{10}(t) \\ &\quad + a_{11}(t)\sin t - a_{10}(t)a'_{20}(t) + \cos t\Big)\frac{x^{2q}}{2q^2} \\ &\quad + \Big(- 2a_{22}(t)(1+\cos t) - a'_{11}(t)a_{20}(t) - 2a_{21}(t)a'_{10}(t) \\ &\quad + 2a_{12}(t)\sin t - a'_{21}(t)a_{10}(t) - 2a'_{20}(t)a_{11}(t) - \cos t\Big)\frac{x^{3q}}{3!q^3} + \dots \end{aligned}$$

Now, setting $v_{11} = v_{12} = 0$ we get:

$$\begin{aligned} &a_{10}(t) = \sin t,\ a_{11}(t) = \sin t,\ a_{12}(t) = \sin t,\ a_{13}(t) = \sin t, \dots, a_{1n}(t) = \sin t, \dots, \\ &a_{20}(t) = -\cos t,\ a_{21}(t) = \cos t,\ a_{22}(t) = -\cos t, \\ &a_{23}(t) = \cos t, \dots, a_{2n}(t) = (-1)^{n+1}\cos t, \dots . \end{aligned}$$

Therefore, we obtain the new homotopy solution of the given system as

$$\begin{cases} u_1 = v_{10} = \phi_1(x) + \sum_{k=0}^{\infty} \frac{a_{1k}(t)x^{q(k+1)}}{q^{k+1}(k+1)!} = e^{\frac{x^q}{q}}\sin t, \\ u_2 = v_{20} = \phi_2(x) + \sum_{k=0}^{\infty} \frac{a_{2k}(t)x^{q(k+1)}}{q^{k+1}(k+1)!} = e^{-\frac{x^q}{q}}\cos t, \end{cases} \tag{18}$$

which is the exact solution of the system (15).

Example 3. Given a system of two dimensional time-conformable fractional partial differential equations

$$\begin{cases} \partial_t^q u_1 = \partial_x(u_1)u_2 + \partial_t u_2 \partial_y u_1 + 1 - x + y + \frac{t^q}{q}, \\ \partial_t^q u_2 = \partial_x(u_2)u_1 + \partial_t u_1 \partial_y u_2 + 1 - x - y - \frac{t^q}{q}, \end{cases} \tag{19}$$

subject to I.C.

$$\begin{cases} u_1(0,x,y) = x + y - 1, \\ u_2(0,x,y) = x - y + 1. \end{cases} \tag{20}$$

According to NCFHPT, we have

$$
(21)\qquad \begin{cases} v_1(x,y,t) = v_1(0,x,y) + \int_0^t t^{q-1}u_{10}(x,y,t)dt \\ \qquad -p\int_0^t t^{q-1}\big(u_{10}(x,y,t) - \partial_x(v_1)v_2 \\ \qquad -\partial_t v_2\partial_y v_1 - 1 + x - y - \frac{t^q}{q}\big)dt, \\ v_2(x,y,t) = v_2(0,x,y) + \int_0^t t^{q-1}u_{20}(x,y,t)dt \\ \qquad -p\int_0^t t^{q-1}\big(u_{20}(x,y,t) - \partial_x(v_2)v_1 \\ \qquad -\partial_t v_1\partial_y v_2 - 1 + x + y + \frac{t^q}{q}\big)dt. \end{cases}
$$

Substituting (7) and (6) into (21) and comparing the equating the terms with the same power of p we have

$$
p^0 : \begin{cases} v_{10} = v_1(0,x,y) + \sum_{k=0}^{\infty}\frac{a_{1k}(x,y)t^{q(k+1)}}{q^{k+1}(k+1)!}, \\ v_{20} = v_2(0,x,y) + \sum_{k=0}^{\infty}\frac{a_{2k}(x,y)t^{q(k+1)}}{q^{k+1}(k+1)!}, \end{cases}
$$

$$
p^1 : \begin{cases} v_{11} = -\sum_{k=0}^{\infty}\frac{a_{1k}((x,y))t^{q(k+1)}}{q^{k+1}(k+1)!} \\ -\int_0^t t^{q-1}\big(-\partial_x(v_{10})v_{20} - \partial_t v_{20}\partial_y v_{10} - 1 + x - y - \frac{t^q}{q}\big)dt, \\ v_{21} = -\sum_{k=0}^{\infty}\frac{a_{2k}(x,y)t^{q(k+1)}}{q^{k+1}(k+1)!} \\ -\int_0^t t^{q-1}\big(-\partial_x(v_{20})v_{10} - \partial_t v_{10}\partial_y v_{20} - 1 + x + y + \frac{t^q}{q}\big)dt, \end{cases}
$$

$$
p^2 : \begin{cases} v_{12} = \int_0^t t^{q-1}\big(\partial_x(v_{11})v_{20} + v_{21}\partial_x v_{10} + \partial_t v_{20}\partial_y v_{11} + \partial_t v_{21}\partial_y v_{10}\big)dt, \\ v_{22} = \int_0^t t^{q-1}\big(\partial_x(v_{21})v_{10} + v_{11}\partial_x v_{20} + \partial_t v_{10}\partial_y v_{21} + \partial_t v_{11}\partial_y v_{20}\big)dt, \end{cases}
$$

$$
\vdots
$$

Assume that $v_1(0,x,y) = u_1(0,x,y), v_2(0,x,y) = u_2(0,x,y)$, and evaluate v_{11} and v_{21} we obtain

$$
\begin{aligned} v_{11} = {} & \big(-a_{10} + a_{20} + 2\big)\frac{t^q}{q} + \big(-a_{11} \\ & + a_{20} + (x-y+1)\partial_x a_{10} + a_{21} + a_{20}\partial_y a_{10} + 1\big)\frac{t^{2q}}{2q^2} \\ & + \Big(-2a_{12} + a_{21} + (x-y+1)\partial_x a_{11} \\ & + 2a_{20}\partial_x a_{10} + 2a_{22} + 2a_{21}\partial_y a_{10} + a_{20}\partial_y a_{11}\Big)\frac{t^{3q}}{3!q^3} + \dots \end{aligned}
$$

and

$$
\begin{aligned} v_{21} = {} & \big(-a_{10} - a_{20}\big)\frac{t^q}{q} + \big(-a_{21} + a_{10} + (x+y-1)\partial_x a_{20} \\ & - a_{11} + a_{10}\partial_y a_{20} - 1\big)\frac{t^{2q}}{2q^2} \end{aligned}
$$

$$+\Big(a_{11}-2a_{21}+(x+y-1)\partial_x a_{21}+2a_{10}\partial_x a_{20}$$
$$-2a_{12}+2a_{11}\partial_y a_{20}+a_{10}\partial_y a_{21}\Big)\frac{t^{3q}}{3!q^3}+\dots$$

Now, setting $v_{11}=v_{21}=0$ we get:

$$a_{10}(x,y)=1,\ a_{11}(x,y)=a_{12}(x,y)=a_{13}(x,y)=\dots=0,$$
$$a_{20}(x,y)=-1,\ a_{21}(x,y)=a_{22}(x,y)=a_{23}(x,y)=\dots=0.$$

Therefore, we obtain the new homotopy solution of the given system as

$$\begin{cases} u_1=v_{10}=\phi_1(x,y)+\sum_{k=0}^{\infty}\frac{a_{1k}(x,y)t^{q(k+1)}}{q^{k+1}(k+1)!}=x+y-1+\frac{t^q}{q}, \\ u_2=v_{20}=\phi_2(x,y)+\sum_{k=0}^{\infty}\frac{a_{2k}(x,y)t^{q(k+1)}}{q^{k+1}(k+1)!}=x-y+1-\frac{t^q}{q}, \end{cases} \tag{22}$$

which is the exact solution of the system (19).

4. Conclusion

In this paper, we carried out a convenient adaptation of NHPM to obtain a solution for linear and nonlinear systems of conformable fractional PDEs subject to initial and boundary conditions. The proposed method seems to be designed for working without discretization, linearization or any computational complexity. At the end, we successfully apply this new method to three nonlinear systems and it provides exact solutions in direct way and simple computations. By applying NCFHPT on such systems, we obtained a new system of recurrence relations which can be easily solved starting with the given initial condition and then iteratively getting the general form of each required coefficient. As we have seen in the forgoing examples, the NCFHPT solutions of the given three systems were coincide with the exact ones. It is observed that NCFHPT is an effective and promise tool for solving nonlinear systems of CFPDEs, especially when exact solutions of such systems can be expressed as a closed form of a summation.

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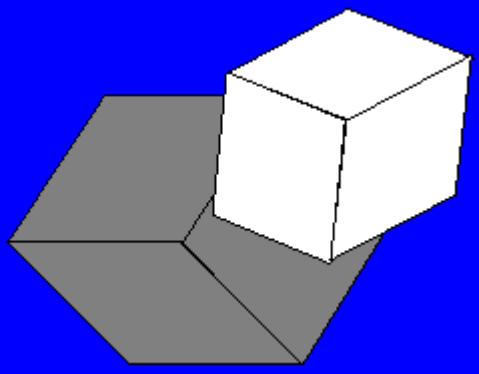
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