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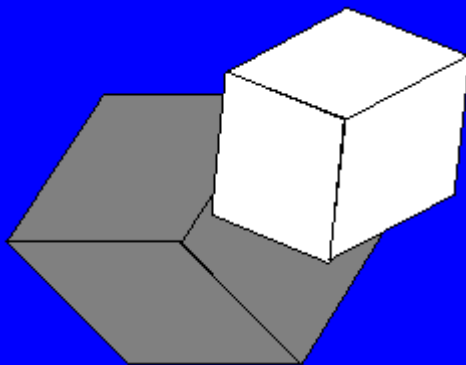
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Quadratic residues and squares of the ring of dual numbers modulo n

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Abstract. For an integer $n \geq 1$, we denote the ring of dual numbers over the integers modulo n by $\mathbb{Z}_n[\alpha]$. The ring $\mathbb{Z}_n[\alpha]$ is a commutative extension for the ring $\mathbb{Z}_n$. We introduce and study square and quadratic residue elements in $\mathbb{Z}_n[\alpha]$. We also find multiplicative functions that count squares and quadratic residues for the ring $\mathbb{Z}_n[\alpha]$.

Keywords: squares, quadratic residues, ring of dual numbers.

1. Introduction

Let $\mathbb{Z}_n$ be the ring of integers modulo n , where $n \geq 1$. An element $a \in \mathbb{Z}_n$ is called a square element if and only if there exists $c \in \mathbb{Z}_n$ such that $a \equiv c^2 \pmod{n}$. Square elements of $\mathbb{Z}_n$ that are units are called quadratic residues. Quadratic residues and squares have many interested applications in cryptography, factoring of large numbers and in acoustical engineering, see for example [6, 5]. W.D. Stangl in [1] obtained multiplicative functions, $s(n)$ and $q(n)$, that count number of squares and number of quadratic residues in $\mathbb{Z}_n$, respectively. In [2], a full characterization for the quadratic residues in $\mathbb{Z}_n$ has been given.

In this paper, our work is based on the well-studied ring $\mathbb{Z}_n[\alpha]$, the ring of dual numbers of $\mathbb{Z}_n$, see [7, 8]. Every element in $\mathbb{Z}_n[\alpha]$ is of the form $a + b\alpha$ where $a, b \in \mathbb{Z}_n$, and $\alpha^2 = 0$, with addition and multiplication defined as

$$\begin{aligned}(a + b\alpha) + (c + d\alpha) &= (a + c) + (b + d)\alpha, \\ (a + b\alpha)(c + d\alpha) &= (ac) + (ad + bc)\alpha\end{aligned}$$

$\mathbb{Z}_n[\alpha]$ is a commutative ring extension of $\mathbb{Z}_n$ with unity 1. Motivated by W.D. Stangl work in [1], we want to make an analogous attempt to study and count squares and quadratic residues for the ring $\mathbb{Z}_n[\alpha]$. Thus, we will see the strong relationship between squares and quadratic residues in $\mathbb{Z}_n$ and those over the ring of dual numbers $\mathbb{Z}_n[\alpha]$. For definitions and terms of general number theory consult [2],[4] and [3].

2. The ring $\mathbb{Z}_n[\alpha]$

Our purpose in this section is only to collect results in one place concerning the ring $\mathbb{Z}_n[\alpha]$.

The ring of dual numbers modulo n , $\mathbb{Z}_n[\alpha] = \{a + b\alpha : a, b \in \mathbb{Z}_n, \text{ with } \alpha^2 = 0\}$, is a commutative ring under addition and multiplication modulo n . This ring is a natural commutative extension of the ring $\mathbb{Z}_n$. Hence we will generalize some concepts in a natural way concerning the ring $\mathbb{Z}_n$ to be suitable for the ring $\mathbb{Z}_n[\alpha]$.

The following definition is a natural generalization of the congruence concept over the ring $\mathbb{Z}_n[\alpha]$, you can see [9].

Definition 2.1. For $n > 1$, let $a + b\alpha$ and $c + d\alpha$ be two elements of $\mathbb{Z}_n[\alpha]$. Then $a + b\alpha$ is congruent to $c + d\alpha$ modulo n if and only if $a \equiv c \pmod{n}$ and $b \equiv d \pmod{n}$. We will use the notation $a + b\alpha \stackrel{\alpha}{\equiv} c + d\alpha \pmod{n}$ to say that $a + b\alpha$ is congruent to $c + d\alpha$ modulo n .

The following proposition talks about the units of $\mathbb{Z}_n[\alpha]$, and it is probably well known.

Proposition 2.1. Let $n > 1$. Then $a + b\alpha$ is a unit of $\mathbb{Z}_n[\alpha]$ if and only if a is a unit of $\mathbb{Z}_n$, that is $a + b\alpha$ is a unit of $\mathbb{Z}_n[\alpha]$ if and only if a is relatively prime to n .

Now, we can generalize the squares and quadratic residues concepts for the ring $\mathbb{Z}_n[\alpha]$

Definition 2.2. Let $n > 1$. Then

1. An element $a + b\alpha$ is a square in $\mathbb{Z}_n[\alpha]$ if and only if

$$a + b\alpha \stackrel{\alpha}{\equiv} (c + d\alpha)^2 \pmod{n}$$

for some $c + d\alpha \in \mathbb{Z}_n[\alpha]$.

2. An element $a + b\alpha$ is a quadratic residue in $\mathbb{Z}_n[\alpha]$ if and only if $a + b\alpha$ is a square in $\mathbb{Z}_n[\alpha]$ and a is relatively prime to n .

We will use the notations: $s_\alpha(n)$ to indicate number of squares in $\mathbb{Z}_n[\alpha]$, and $q_\alpha(n)$ to indicate number of quadratic residues in $\mathbb{Z}_n[\alpha]$.

Recall that the number-theoretic function f is multiplicative if $f(mn) = f(m)f(n)$, for any two relatively prime integers m and n .

Theorem 2.1. Let $n \in \mathbb{Z}$. Then $s_\alpha(n)$ and $q_\alpha(n)$ are multiplicative functions.

Proof. Suppose that $\gcd(m, n) = 1$. Then one can check that $\mathbb{Z}_{mn}[\alpha]$ is isomorphic to $\mathbb{Z}_m[\alpha] \times \mathbb{Z}_n[\alpha]$ under the ring isomorphism $f : \mathbb{Z}_{mn}[\alpha] \rightarrow \mathbb{Z}_m[\alpha] \times \mathbb{Z}_n[\alpha]$ which is defined by $f(a + b\alpha \pmod{mn}) = (a + b\alpha \pmod{m}, a + b\alpha \pmod{n})$.

Now, suppose $a+b\alpha \in \mathbb{Z}_m[\alpha]$ and $c+d\alpha \in \mathbb{Z}_n[\alpha]$ are squares. Then there exist $x_0+x_1\alpha \in \mathbb{Z}_m[\alpha]$ and $y_0+y_1\alpha \in \mathbb{Z}_n[\alpha]$ such that $a+b\alpha \equiv (x_0+x_1\alpha)^2 \pmod{m}$ and $c+d\alpha \equiv (y_0+y_1\alpha)^2 \pmod{n}$. So $((x_0+x_1\alpha)^2 \pmod{m}, (y_0+y_1\alpha)^2 \pmod{n}) = (a+b\alpha \pmod{m}, c+d\alpha \pmod{n}) \in \mathbb{Z}_m[\alpha] \times \mathbb{Z}_n[\alpha]$. Hence

$$\begin{aligned} & f^{-1}(a+b\alpha \pmod{m}, c+d\alpha \pmod{n}) \\ &= f^{-1}((x_0+x_1\alpha)^2 \pmod{m}, (y_0+y_1\alpha)^2 \pmod{n}) \\ &= [f^{-1}((x_0+x_1\alpha) \pmod{m}, (y_0+y_1\alpha) \pmod{n})]^2 \end{aligned}$$

Thus, $f^{-1}(a+b\alpha \pmod{m}, c+d\alpha \pmod{n})$ is square in $\mathbb{Z}_{mn}[\alpha]$. Therefore, $s_\alpha(n) \cdot s_\alpha(m) \leq s_\alpha(nm)$.

To prove the other inclusion, let $a+b\alpha$ be square in $\mathbb{Z}_{mn}[\alpha]$. Then, there exists $x+y\alpha \in \mathbb{Z}_{mn}[\alpha]$, such that $a+b\alpha \equiv (x+y\alpha)^2 \pmod{mn}$

Now, since f is a function from $\mathbb{Z}_{mn}[\alpha]$ onto $\mathbb{Z}_m[\alpha] \times \mathbb{Z}_n[\alpha]$, we have

$$(w_0+w_1\alpha \pmod{m}, z_0+z_1\alpha \pmod{n}) \in \mathbb{Z}_m[\alpha] \times \mathbb{Z}_n[\alpha]$$

such that

$$\begin{aligned} (w_0+w_1\alpha \pmod{m}, z_0+z_1\alpha \pmod{n}) &= f(a+b\alpha \pmod{mn}) \\ &= f((x+y\alpha)^2 \pmod{mn}) \\ &= [f((x+y\alpha) \pmod{mn})]^2, \end{aligned}$$

so, $(w_0+w_1\alpha \pmod{m}, z_0+z_1\alpha \pmod{n})$ is square in $\mathbb{Z}_m[\alpha] \times \mathbb{Z}_n[\alpha]$. Thus, $s_\alpha(nm) \leq s_\alpha(n) \cdot s_\alpha(m)$. The previous two inclusions lead us to the result $s_\alpha(nm) = s_\alpha(n) \cdot s_\alpha(m)$.

It is clear that any quadratic residue is a square modulo n . Hence to prove that $q_\alpha(n)$ is a multiplicative function, we used Proposition [2.1] and the fact that $\gcd(a, nm) = 1$ if and only if $\gcd(a, n) = 1$ and $\gcd(a, m) = 1$. $\square$

Since both $s_\alpha(n)$ and $q_\alpha(n)$ are multiplicative functions, we can compute $s_\alpha(n)$ and $q_\alpha(n)$ for any n , based on the prime factorization of n . So if we can find a closed-form formulas of these functions on the powers of primes, then our job will be hold.

3. Quadratic residues in $\mathbb{Z}_n[\alpha]$

In this section, we will study and count the quadratic residues in the ring $\mathbb{Z}_n[\alpha]$.

The quadratic residues in the ring $\mathbb{Z}_n$ have been completely characterized in [2]. W.D. Stangl in [1] derived a closed-form formula for the function $q(n)$, for any n , based on the prime factorization of n .

Proposition 3.1. *Let p be a prime number. Then:*

$$q(p^k) = \begin{cases} 1, & \text{if } p = 2 \text{ and } k = 1 \text{ or } 2, \\ 2^{k-3}, & \text{if } p = 2 \text{ and } k \geq 3 \\ \frac{(p^k - p^{k-1})}{2}, & \text{if } p \neq 2 \text{ and } k \geq 1. \end{cases}$$

The following theorem gives us the strong relationship between the quadratic residues in $\mathbb{Z}_{p^k}$ and $\mathbb{Z}_{p^k}[\alpha]$, where p is an odd prime.

Theorem 3.1. *Let p be an odd prime and $k \geq 1$, then $a + b\alpha$ is a quadratic residue in $\mathbb{Z}_{p^k}[\alpha]$ if and only if a is a quadratic residue in $\mathbb{Z}_{p^k}$.*

Proof. Suppose a is a quadratic residue in $\mathbb{Z}_{p^k}$, then $a \equiv c^2 \pmod{p^k}$ for some $c \in \mathbb{Z}_{p^k}$. Since $\gcd(a, p^k) = 1$, one can easily prove that $\gcd(c, p^k) = 1$, so c is a unit in $\mathbb{Z}_{p^k}$. Thus, $a + b\alpha \equiv (c^2 + b\alpha) \pmod{p^k} \equiv (c + (2c)^{-1}b\alpha)^2 \pmod{p^k}$.

It is clear that $a + b\alpha$ is a quadratic residue in $\mathbb{Z}_n[\alpha]$. For the other direction, we leave it as an easy exercise to the reader. $\square$

The following corollary is a natural consequence of Theorem [3.1].

Corollary 3.1. *Let p be an odd prime. Then $q_\alpha(p^k) = p^k q(p^k)$ for all $k \geq 1$.*

Theorem 3.2. *For all $k \geq 1$, $a + b\alpha$ is a quadratic residue in $\mathbb{Z}_{2^k}[\alpha]$ if and only if a is a quadratic residue in $\mathbb{Z}_{2^k}$ and b is even.*

Proof. Suppose that $a + b\alpha$ is a quadratic residue in $\mathbb{Z}_{2^k}[\alpha]$. Then, there exists a unit $c + d\alpha$ in $\mathbb{Z}_{2^k}[\alpha]$ such that

$$a + b\alpha \equiv [c + d\alpha]^2 \pmod{2^k}$$

so,

$$\begin{aligned} a &\equiv c^2 \pmod{2^k} \quad \text{and} \\ b &\equiv 2cd \pmod{2^k}. \end{aligned}$$

It is obvious that a is a quadratic residue in $\mathbb{Z}_{2^k}$ and $2|b$, so b is even. Conversely, suppose that a is a quadratic residue in $\mathbb{Z}_{2^k}$ and b is even. Then there exists a unit c in $\mathbb{Z}_{2^k}$ such that $a \equiv c^2 \pmod{2^k}$. Also, since b is even, $b = 2t$ for some $t \in \mathbb{Z}_{2^k}$. Hence

$$\begin{aligned} a + b\alpha &\equiv c^2 + b\alpha \pmod{2^k} \\ &\equiv c^2 + 2t\alpha \pmod{2^k} \\ &\equiv [c + c^{-1}t\alpha]^2 \pmod{2^k}. \end{aligned}$$

So, $a + b\alpha$ is a quadratic residue in $\mathbb{Z}_{2^k}[\alpha]$. $\square$

Corollary 3.2. $q_\alpha(2^k) = 2^{k-1}q(2^k)$ for all $k \geq 1$.

Proof. By Theorem [3.2], one can deduce that

$$\begin{aligned} q_\alpha(2^k) &= [\text{number of even elements in } \mathbb{Z}_{2^k}[\alpha]].q(2^k) \\ &= [2^k - \phi(2^k)].q(2^k) \\ &= 2^{k-1}q(2^k). \quad \square \end{aligned}$$

Using Proposition [3.1], we can summarize the results in Corollary [3.1] and Corollary [3.2] in the following theorem

Theorem 3.3. *Let p be a prime number. Then:*

$$q_\alpha(p^k) = \begin{cases} 1, & \text{if } p = 2 \text{ and } k = 1, \\ 2, & \text{if } p = 2 \text{ and } k = 2, \\ 2^{2k-4}, & \text{if } p = 2 \text{ and } k \geq 3, \\ \frac{p^k(p^k - p^{k-1})}{2}, & \text{if } p \neq 2 \text{ and } k \geq 1. \end{cases}$$

Hence, if we write n as product of its prime factors, then it is easy to count all quadratic residues in $\mathbb{Z}_n[\alpha]$.

4. Squares in $\mathbb{Z}_n[\alpha]$

In this section, we will study and count the squares in the ring $\mathbb{Z}_n[\alpha]$.

Firstly, we introduce a proposition, see [1], concerning the number of squares $s(p^k)$ in the ring $\mathbb{Z}_{p^k}$ for any prime p .

Proposition 4.1. 1. *If p is an odd prime, then*

$$(a) \ s(p) = \frac{p+1}{2};$$

$$(b) \ s(p^2) = \frac{p^2 - p + 2}{2};$$

$$(c) \ \text{For } k \geq 3, \ s(p^k) = \begin{cases} \frac{p^{k+1} + p + 2}{2(p+1)}, & \text{if } k \text{ is even,} \\ \frac{p^{k+1} + 2p + 1}{2(p+1)}, & \text{if } k \text{ is odd.} \end{cases};$$

$$2. \ s(2^k) = \begin{cases} \frac{2^{k-1} + 4}{3}, & \text{if } k \text{ is even,} \\ \frac{2^{k-1} + 5}{3}, & \text{if } k \text{ is odd and } k \geq 3. \end{cases}$$

The following lemma is obtained directly and it will help us to investigate the squares in the ring $\mathbb{Z}_n[\alpha]$.

Lemma 4.1. *Let p be any prime integer. Then the sets $A_i = \{x \in \mathbb{Z}_{p^k} : \gcd(x, p^k) = p^i\}$ where $0 \leq i \leq k-1$, and $A_k = \{0\}$ form a partition for the ring $\mathbb{Z}_{p^k}$.*

Recall that the set A_0 is called the set of units, and its group under multiplication modulo p^k .

Now, we will use the notation: $s(A_i)$ to indicate number of squares of $\mathbb{Z}_{p^k}$ in the set A_i .

Lemma 4.2. *Let p be any prime integer and let k be an even integer. Then for*

$$0 \leq i \leq k-1, \ \text{we have } s(A_i) = \begin{cases} q(p^{k-i}), & \text{if } i \text{ is even,} \\ 0, & \text{if } i \text{ is odd.} \end{cases}$$

Proof. By Lemma [4.1], $A_i = \{x \in \mathbb{Z}_{p^k} : \gcd(x, p^k) = p^i\}$. Hence

$$\begin{aligned} A_i &= \{x \in \mathbb{Z}_{p^k} : \gcd\left(\frac{x}{p^i}, p^{k-i}\right) = 1\} \\ &= \{p^i y \in \mathbb{Z}_{p^k} : \gcd(y, p^{k-i}) = 1\} \\ &= p^i \{y \in \mathbb{Z}_{p^k} : \gcd(y, p^{k-i}) = 1\} \\ &= p^i U(\mathbb{Z}_{p^{k-i}}). \end{aligned}$$

Thus, it is obvious that if i is even, then the squares in A_i are identical with the squares in $p^i U(\mathbb{Z}_{p^{k-i}})$. that is because p^i is square in $\mathbb{Z}_{p^k}$. So, number of squares in A_i is equivalent to the number of quadratic residues in $\mathbb{Z}_{p^{k-i}}$, i.e. $s(A_i) = q(p^{k-i})$. On the other hand, if i is odd, then A_i will never have squares. Since p^i is not square in $\mathbb{Z}_{p^k}$. $\square$

Lemma 4.3. *Let p be an odd prime, k be an even integer and let $a \in A_{2j}$ (as defined in Lemma [4.1]) for $1 \leq j \leq \frac{k}{2}$. Then $a + b\alpha$ is a square in $\mathbb{Z}_{p^k}[\alpha]$ if and only if a is a square in $\mathbb{Z}_{p^k}$ and $p^j | b$.*

Proof. Suppose that $a + b\alpha$ is a square in $\mathbb{Z}_{p^k}[\alpha]$. Then, there exists $c + d\alpha \in \mathbb{Z}_{p^k}[\alpha]$ such that $a + b\alpha \equiv [c + d\alpha]^2 \pmod{p^k}$. Hence

$$\begin{aligned} a &\equiv c^2 \pmod{p^k} \quad \text{and} \\ b &\equiv 2cd \pmod{p^k}. \end{aligned}$$

Obviously, a is a square in $\mathbb{Z}_{p^k}$. Now, because $a \in A_{2j}$, we have $p^{2j} | a$. So $p^j | c$. Thus, by the second congruence above, $p^j | b$.

Conversely, suppose a is a square in $\mathbb{Z}_{p^k}$ and $p^j | b$. Then $a \equiv c^2 \pmod{p^k}$ for some $c \in \mathbb{Z}_{p^k}$ and $b = p^j t$ for some $t \in \mathbb{Z}$. Since $a \in A_{2j}$, one can deduce that $c = p^j r$ with $\gcd(r, p) = 1$. Thus, we get

$$\begin{aligned} a + b\alpha &\equiv c^2 + b\alpha \pmod{p^k} \\ &\equiv (p^j r)^2 + p^j t\alpha \pmod{p^k} \\ &\equiv [p^j r + 2^{-1} r^{-1} t\alpha]^2 \pmod{p^k}. \end{aligned}$$

Hence, $a + b\alpha$ is a square in $\mathbb{Z}_{p^k}[\alpha]$. $\square$

The previous lemmas (Lemma [4.1], Lemma [4.2] and Lemma [4.3]) will play a key role in counting the number of squares in $\mathbb{Z}_{p^k}[\alpha]$. We will use the notation: $s_\alpha(A_i)$ to denote number of the squares in $\mathbb{Z}_{p^k}[\alpha]$ that has the form $a + b\alpha$, where $a \in A_i$, (Note that by Lemma [4.2] and Lemma [4.3]: If i is odd, then $s_\alpha(A_i) = 0$). Now, let us define the set $D_i = \{b \in \mathbb{Z}_{p^k} : p^i | b\}$. It is clear that $D_i = \bigcup_{r=i}^k A_r$. Note that $D_0 = \mathbb{Z}_{p^k}$. Also, since the sets $A_0, A_1, \dots, A_k$ are disjoint, then $D_i = D_{i-1} - A_{i-1}$, for $i \geq 1$. Hence, we have the following lemma.

Lemma 4.4. *For any positive integer k , $|D_i| = p^{k-i}$, for $i \geq 1$.*

Proof. By the definition of A_i , one can prove $|A_i| = \phi(p^{k-i})$. Thus, the result can be obtained easily by induction. $\square$

Theorem 4.1. *Let p be an odd prime. If k is even, then*

$$s_\alpha(p^k) = \frac{p^{2(k+1)} + p^{\frac{k}{2}}(p^2 + 2p + 2)}{2(p^2 + p + 1)}.$$

Proof.

$$\begin{aligned} s_\alpha(p^k) &= \sum_{i=0}^k s_\alpha(A_i), \\ &= s_\alpha(A_0) + s_\alpha(A_2) + s_\alpha(A_4) + \cdots + s_\alpha(A_{k-2}) + s_\alpha(A_k) \\ &= s_\alpha(A_0) + |D_1| s(A_2) + |D_2| s(A_4) + \cdots + \left| D_{\frac{k-2}{2}} \right| s(A_{k-2}) + \left| D_{\frac{k}{2}} \right| s(A_k) \\ &= q_\alpha(p^k) + p^{k-1} s(A_2) + p^{k-2} s(A_4) + \cdots + p^{k-(\frac{k-2}{2})} s(A_{k-2}) + p^{\frac{k}{2}} \\ &= p^k q(p^k) + p^{k-1} q(p^{k-2}) + p^{k-2} q(p^{k-4}) + \cdots + p^{\frac{k+2}{2}} q(p^2) + p^{\frac{k}{2}} \\ &= \frac{(p^{2k} - p^{2k-1}) + (p^{2k-3} - p^{2k-4}) + \cdots + \left(p^{\frac{k+6}{2}} - p^{\frac{k+4}{2}} \right) + 2p^{\frac{k}{2}}}{2} \\ &= \frac{p^{2k+2} + p^{\frac{k+4}{2}} + 2p^{\frac{k+2}{2}} + 2p^{\frac{k}{2}}}{2(p^2 + p + 1)} \\ &= \frac{p^{2(k+1)} + p^{\frac{k}{2}}(p^2 + 2p + 2)}{2(p^2 + p + 1)}. \end{aligned} \quad \square$$

To count number of squares in $\mathbb{Z}_{p^k}[\alpha]$ where p is an odd prime and k is odd, we need the following lemmas.

The following lemma is probably well known but does not seem to appear in the literature.

Lemma 4.5. *Let p be any prime. If $p^{2t+1} | c^2$, then $p^{t+1} | c$.*

Lemma 4.6. *Let p be an odd prime, $k = 2t+1$ be an odd integer and let $a \in A_{2j}$ (as defined in Lemma [4.1]). Then, for $1 \leq j \leq t$, we have*

1. $a + b\alpha$ is a square in $\mathbb{Z}_{p^k}[\alpha]$ if and only if a is a square in $\mathbb{Z}_{p^k}$ and $p^j | b$.
2. $b\alpha$ is a square in $\mathbb{Z}_{p^k}[\alpha]$ if and only if $p^{\frac{k+1}{2}} | b$.

Proof. The proof is similar to the proof of Lemma [4.3]. Hence it is omitted. $\square$

Lemma 4.7. *Let p be any prime integer and let k be an odd integer. Then, for $1 \leq j \leq \frac{k-1}{2}$, we have $s(A_i) = \begin{cases} q(p^{k-i}), & \text{if } i \text{ is even,} \\ 0, & \text{if } i \text{ is odd.} \end{cases}$*

Proof. Similar to the proof of Lemma [4.2]. $\square$

Theorem 4.2. *Let p be an odd prime. If k is odd, then*

$$s_\alpha(p^k) = \frac{p^{2(k+1)} + p^{\frac{k-1}{2}}(2p^2 + p + 2)}{2(p^2 + p + 1)}.$$

Proof.

$$\begin{aligned} s_\alpha(p^k) &= \sum_{i=0}^k s_\alpha(A_i), \\ &= s_\alpha(A_0) + s_\alpha(A_2) + s_\alpha(A_4) + \cdots + s_\alpha(A_{k-1}) + s_\alpha(A_k) \\ &= s_\alpha(A_0) + |D_1| s(A_2) + |D_2| s(A_4) + \cdots + \left| D_{\frac{k-1}{2}} \right| s(A_{k-1}) + \left| D_{\frac{k+1}{2}} \right| s(A_k) \\ &= q_\alpha(p^k) + p^{k-1} s(A_2) + p^{k-2} s(A_4) + \cdots + p^{k-(\frac{k-1}{2})} s(A_{k-1}) + p^{k-(\frac{k+1}{2})} s(A_k) \\ &= p^k q(p^k) + p^{k-1} q(p^{k-2}) + p^{k-2} q(p^{k-4}) + \cdots + p^{\frac{k+1}{2}} q(p) + p^{\frac{k-1}{2}} \\ &\quad \frac{(p^{2k} - p^{2k-1}) + (p^{2k-3} - p^{2k-4}) + \cdots + \left(p^{\frac{k+3}{2}} - p^{\frac{k+1}{2}} \right) + 2p^{\frac{k-1}{2}}}{2} \\ &= \frac{p^{2k+2} + 2p^{\frac{k+3}{2}} + p^{\frac{k+1}{2}} + 2p^{\frac{k-1}{2}}}{2(p^2 + p + 1)} \\ &= \frac{p^{2(k+1)} + p^{\frac{k-1}{2}}(2p^2 + p + 2)}{2(p^2 + p + 1)}. \end{aligned} \quad \square$$

For completeness, let us study the case when $p = 2$. To do this, we present the following lemmas.

Lemma 4.8. *Let k be an even integer and let $a \in A_{2j}$ (as defined in Lemma [4.1]), for $1 \leq j \leq \frac{k}{2}$. Then, $a + b\alpha$ is a square in $\mathbb{Z}_{2^k}[\alpha]$ if and only if a is a square in $\mathbb{Z}_{2^k}$ and $2^{j+1}|b$.*

Proof. Suppose that $a + b\alpha$ is a square in $\mathbb{Z}_{2^k}[\alpha]$. Then, there exists $c + d\alpha \in \mathbb{Z}_{2^k}[\alpha]$ such that $a + b\alpha \equiv [c + d\alpha]^2 \pmod{2^k}$. Hence

$$\begin{aligned} a &\equiv c^2 \pmod{2^k} \quad \text{and} \\ b &\equiv 2cd \pmod{2^k} \end{aligned}$$

Obviously, a is a square in $\mathbb{Z}_{2^k}$. Now, because $a \in A_{2j}$, we have $2^{2j}|a$. So, $2^j|c$. Thus, by the second congruence above, $2^{j+1}|b$.

Conversely, suppose a is a square in $\mathbb{Z}_{2^k}$ and $2^{j+1}|b$. Then, $a \equiv c^2 \pmod{2^k}$ for some $c \in \mathbb{Z}_{2^k}$ and $b = 2^{j+1}t$ for some $t \in \mathbb{Z}$. Since $a \in A_{2j}$, one can deduce that $c = 2^j r$ with $\gcd(r, 2) = 1$. Thus, we get

$$\begin{aligned} a + b\alpha &\equiv c^2 + b\alpha \pmod{2^k} \\ &\equiv (2^j r)^2 + 2^{j+1}t\alpha \pmod{2^k} \\ &\equiv [2^j r + r^{-1}t\alpha]^2 \pmod{2^k}. \end{aligned}$$

Hence, $a + b\alpha$ is a square in $\mathbb{Z}_{2^k}[\alpha]$. $\square$

Lemma 4.9. *Let $k = 2t + 1$ be an odd integer and let $a \in A_{2j}$ (as defined in Lemma [4.1]) for $1 \leq j \leq t$ ($= \frac{k-1}{2}$). Then*

1. $a + b\alpha$ is a square in $\mathbb{Z}_{2^k}[\alpha]$ if and only if a is a square in $\mathbb{Z}_{2^k}$ and $2^{j+1} | b$.
2. $b\alpha$ is a square in $\mathbb{Z}_{2^k}[\alpha]$ if and only if $2^{\frac{k+3}{2}} | b$.

Proof. Using Lemma [4.5], then the idea of the proof of (1) is similar to the proof of Lemma [4.6] above. So, we skip it. Thus, we will only prove (2). Suppose that $b\alpha$ is a square in $\mathbb{Z}_{2^k}[\alpha]$. So, there exists $c + d\alpha \in \mathbb{Z}_{2^k}[\alpha]$ such that $b\alpha \stackrel{\alpha}{\equiv} [c + d\alpha]^2 \pmod{2^k}$. Hence $b\alpha \stackrel{\alpha}{\equiv} c^2 + 2cd\alpha \pmod{2^k}$. It is clear that $2^k | c^2$, and $b \equiv 2cd \pmod{2^k}$. Since k is odd and by Lemma [4.5], we get $2^{\frac{k+1}{2}} | c$. Now because of $b \equiv 2cd \pmod{2^k}$, we have $2^{\frac{k+1}{2}+1} | b$. Thus $2^{\frac{k+3}{2}} | b$. To prove the other implication, suppose that $2^{\frac{k+3}{2}} | b$. So $b = 2^{\frac{k+1}{2}+1}r$ for some $r \in \mathbb{Z}$. Hence

$$\begin{aligned} b\alpha &\stackrel{\alpha}{\equiv} 2^{\frac{k+1}{2}+1}r\alpha \pmod{2^k} \\ &\stackrel{\alpha}{\equiv} [2^{\frac{k+1}{2}} + r\alpha]^2 \pmod{2^k}. \end{aligned}$$

Thus, $b\alpha$ is a square in $\mathbb{Z}_{2^k}[\alpha]$. $\square$

Now, we are ready to count the number of squares in the ring $\mathbb{Z}_{2^k}[\alpha]$.

Theorem 4.3. *Let k be a positive integer. Then*

1. $s_\alpha(2) = 2$, and $s_\alpha(2^2) = 3$.
2. If k is even and $k \geq 3$, then $s_\alpha(2^k) = \frac{2^{2k-1} - 2^{2k - (\frac{3k}{2} - 2)}}{7} + 3(2^{\frac{k}{2}-1})$.
3. If k is odd and $k \geq 3$, then $s_\alpha(2^k) = \frac{2^{2k-1} - 2^{2k - (\frac{3k-1}{2})}}{7} + 3(2^{\frac{k-3}{2}})$.

Proof. (1) Obvious

(2) Suppose that, k is even. Then:

$$\begin{aligned} s_\alpha(2^k) &= \sum_{i=0}^k s_\alpha(A_i), \text{ by Lemma [4.2], we have} \\ &= \sum_{j=0}^{\frac{k-2}{2}} s_\alpha(A_{2j}) + s_\alpha(A_k) \\ &= s_\alpha(A_0) + s_\alpha(A_2) + s_\alpha(A_4) + \cdots s_\alpha(A_{k-4}) + s_\alpha(A_{k-2}) + s_\alpha(A_k) \\ &= q_\alpha(2^k) + |D_2| s(A_2) + |D_3| s(A_4) + \cdots \left| D_{\frac{k-4}{2}+1} \right| s(A_{k-4}) + \\ &\quad \left| D_{\frac{k-2}{2}+1} \right| s(A_{k-2}) + \left| D_{\frac{k}{2}+1} \right| s(A_k) \\ &= q_\alpha(2^k) + 2^{k-2} s(A_2) + 2^{k-3} s(A_4) + \cdots 2^{k - (\frac{k}{2}-1)} s(A_{k-4}) + \\ &\quad 2^{k - (\frac{k}{2})} s(A_{k-2}) + 2^{k - (\frac{k+2}{2})} \end{aligned}$$

$$\begin{aligned}
&= 2^{2k-4} + 2^{k-2}q(2^{k-2}) + 2^{k-3}q(2^{k-4}) + \dots + 2^{k-(\frac{k}{2}-1)}q(2^{k-(k-4)}) + \\
&2^{\frac{k}{2}}q(2^2) + 2^{k-(\frac{k+2}{2})} \\
&= 2^{2k-4} + 2^{k-2}2^{(k-2)-3} + 2^{k-3}2^{(k-4)-3} + \dots + 2^{k-(\frac{k}{2}-1)}2 + \\
&2^{\frac{k}{2}} + 2^{\frac{k}{2}-1} \\
&= 2^{2k-4} + 2^{2k-7} + 2^{2k-10} + \dots + 2^{2k-(\frac{3k}{2}-2)} + 2^{\frac{k}{2}} + 2^{\frac{k}{2}-1} \\
&= \frac{2^{2k-2} + 2^{2k-3} + 2^{2k-4} + 2^{2k-5} + 2^{2k-6} + 2^{2k-7}}{7} + \dots + \\
&\frac{2^{2k-(\frac{3k}{2}-4)} + 2^{2k-(\frac{3k}{2}-3)} + 2^{2k-(\frac{3k}{2}-2)}}{7} + 2^{\frac{k}{2}} + 2^{\frac{k}{2}-1} \\
&= \frac{2^{2k-(\frac{3k}{2}-2)}[2^{(\frac{3k}{2}-4)} + 2^{(\frac{3k}{2}-5)} + 2^{(\frac{3k}{2}-6)} + \dots + 2 + 1]}{7} + 2^{\frac{k}{2}} + 2^{\frac{k}{2}-1} \\
&= \frac{2^{2k-(\frac{3k}{2}-2)}[2^{(\frac{3k}{2}-4)+1} - 1]}{7} + 2^{\frac{k}{2}} + 2^{\frac{k}{2}-1} \\
&= \frac{2^{2k-(\frac{3k}{2}-2)}[2^{(\frac{3k}{2}-3)} - 1]}{7} + 2^{\frac{k}{2}} + 2^{\frac{k}{2}-1} \\
&= \frac{2^{2k-1} - 2^{2k-(\frac{3k}{2}-2)}}{7} + 2^{\frac{k}{2}} + 2^{\frac{k}{2}-1} \\
&= \frac{2^{2k-1} - 2^{2k-(\frac{3k}{2}-2)}}{7} + 3(2^{\frac{k}{2}-1}).
\end{aligned}$$

(3) Suppose that k is odd. Then

$$\begin{aligned}
s_\alpha(2^k) &= \sum_{i=0}^k s_\alpha(A_i), \quad \text{by Lemma [4.2], we have} \\
&= \sum_{j=0}^{\frac{k-1}{2}} s_\alpha(A_{2j}) + s_\alpha(A_k) \\
&= s_\alpha(A_0) + s_\alpha(A_2) + s_\alpha(A_4) + \dots + s_\alpha(A_{k-3}) + s_\alpha(A_{k-1}) + s_\alpha(A_k) \\
&= q_\alpha(2^k) + |D_2|s(A_2) + |D_3|s(A_4) + \dots + \left|D_{\frac{k-3}{2}+1}\right|s(A_{k-3}) \\
&+ \left|D_{\frac{k-1}{2}+1}\right|s(A_{k-1}) + \left|D_{\frac{k+3}{2}}\right|s(A_k) \\
&= q_\alpha(2^k) + 2^{k-2}s(A_2) + 2^{k-3}s(A_4) + \dots + 2^{k-(\frac{k-1}{2})}s(A_{k-3}) \\
&+ 2^{k-(\frac{k+1}{2})}s(A_{k-1}) + 2^{k-(\frac{k+3}{2})} \\
&= 2^{2k-4} + 2^{k-2}q(2^{k-2}) + 2^{k-3}q(2^{k-4}) + \dots + 2^{k-(\frac{k-1}{2})}q(2^{k-(k-3)}) \\
&+ 2^{\frac{k-1}{2}}q(2) + 2^{k-(\frac{k+3}{2})} \\
&= 2^{2k-4} + 2^{k-2}2^{(k-2)-3} + 2^{k-3}2^{(k-4)-3} + \dots + 2^{k-(\frac{k-1}{2})}.
\end{aligned}$$

$$\begin{aligned}
 & + 2^{\frac{k-1}{2}} + 2^{\frac{k-3}{2}} \\
 & = 2^{2k-4} + 2^{2k-7} + 2^{2k-10} + \dots + 2^{2k-(\frac{3k-1}{2})} + 2^{\frac{k-1}{2}} + 2^{\frac{k-3}{2}} \\
 & = \frac{2^{2k-2} + 2^{2k-3} + 2^{2k-4} + 2^{2k-5} + 2^{2k-6} + 2^{2k-7}}{7} \\
 & + \dots + \frac{2^{2k-(\frac{3k-5}{2})} + 2^{2k-(\frac{3k-3}{2})} + 2^{2k-(\frac{3k-1}{2})}}{7} + 2^{\frac{k-1}{2}} + 2^{\frac{k-3}{2}} \\
 & = \frac{2^{2k-(\frac{3k-1}{2})} [2^{(\frac{3k-1}{2}-2)} + 2^{(\frac{3k-1}{2}-3)} + 2^{(\frac{3k-1}{2}-5)} + \dots + 2+1]}{7} + 2^{\frac{k-1}{2}} + 2^{\frac{k-3}{2}} \\
 & = \frac{2^{2k-(\frac{3k-1}{2})} [2^{(\frac{3k-1}{2}-2)+1} - 1]}{7} + 2^{\frac{k-1}{2}} + 2^{\frac{k-3}{2}} \\
 & = \frac{2^{2k-(\frac{3k-1}{2})} [2^{(\frac{3k-1}{2}-1)} - 1]}{7} + 2^{\frac{k-1}{2}} + 2^{\frac{k-3}{2}} \\
 & = \frac{2^{2k-1} - 2^{2k-(\frac{3k-1}{2})}}{7} + 2^{\frac{k-1}{2}} + 2^{\frac{k-3}{2}} \\
 & = \frac{2^{2k-1} - 2^{2k-(\frac{3k-1}{2})}}{7} + 3(2^{\frac{k-3}{2}}). \quad \square
 \end{aligned}$$

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Predicting Jordanian's GDP based on ARIMA modeling

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Abstract. Gross Domestic Product (GDP) is the market value of all goods and services that are produced within the country's national borders in a year. Our study aims to estimate and predict Jordanian's GDP using a time series data for the period from 1978 till 2017, the data has been taken from Jordanian's department of statistics, Minitab and Matlab statistical software's are used, we deploy a wavelet transform (WT) model to decomposes the time series data then detecting the fluctuations and outliers values, also an ARIMA (autoregressive integrated moving average) is established, the fitted ARIMA (2, 2, 1) time series model is the best for modeling the Jordanian's GDP according to the recognition rules and stationary test of time series. The results show that the predicted values are within the range of 5%, and the prediction capability of this model is relatively adequate and efficient in modeling the annual GDP for the next 20 years, thus, the prediction accuracy is considered high. It is concluded that Jordanian's GDP is an upward trend for upcoming 20 fiscal years, Furthermore, Jordanian's government has to follow more comprehensive economic policies and should implement key growth-enhancing reforms to strengthen its economy, also it has to stimulate job-creating growth and creates conditions to increase private investment and improve country's competitiveness.

Keywords: prediction, WT, ARIMA, GDP.

1. Introduction

Gross Domestic Product (GDP) is the market value of all goods and services that are produced within the country's national borders in a year. GDP is a measure that is used to evaluate the overall economic performance of a country; it includes all goods and services produced by the economy including personal consumption, government purchases, private inventories, paid in construction

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costs and the foreign trade balance (net exports). The subject of GDP became of high importance among macro economic variables. Data on GDP is regarded as an important indicator for evaluating the national economic development and growth of the entire economy (Ahmad and Harnhirun, 1996).

Most popular definitions of GDP conceptually identical and can be categorized in three ways: First, it is equal to the total expenditures for all final goods and services produced within the country in a specified period of time (a fiscal year). Second, it is equal to the sum of the value added at every phase of production by all the industries in a country, plus taxes minus grants on products in a fiscal year. Third, it is equal to the sum of the income generated by production in the country in the fiscal year, which is compensation of employees, taxes on production and imports less grants, and profits (Ahmad and Harnhirun, 1996). GDP aggregates the entire economic motion. It is frequently used as the finest measure of how is the performance of the economy. GDP is generally measured in one of three approaches. First, the Expenditure approach, it involves the market value of all domestic expenditures made on final goods and services of the year, including consumption expenditures, investment expenditures, government expenditures, and net exports the domestic purchases of goods and services and net exports. Second, the Production approach, it is involving the summation of all value-added activities at every phase of production by all industries inside the country, plus taxes and product's subsidies of the period. A third is the Income approach, it is the summation of all aspect of the income made by production within the economy as the remuneration of employees, capital income, and gross operating surplus of enterprises i.e. profit, taxes on production and imports less grants of the period (Reijer, 2010)

The aim of this research is to estimate and predict future Jordanian's GDP, using a time series data for the period from 1978 till 2017, this study is supposed to offer a valuable understanding for the Jordanian's expected GDP. Prediction of GDP involves the application of statistical and mathematical models to predict upcoming developments in the economy. It allows reviewing previous economic movements and predicting how current economic changes will amend the patterns of the previous trend, therefore, a more accurate prediction would provide significant help to the government in setting up economic development goals, strategies and policies. Consequently, an accurate GDP prediction presents a leading insight and an understanding of future economics' trend.

The research question addressed in this study: "Is the ARIMA model a significant model for predicting the future GDP values? ", consequently, we need to attain the subsequent objectives:

- To know the Jordanian's GDP trend.
- To select the best ARIMA model that fits the Jordanian's GDP.
- To predict the Jordanian's GDP.
- To develop ARIMA model for predicting future GDP of the Jordanian's economy.

This paper is organized as follows. Section 2 describes the literature review. Section 3 introduces the methodology and framework. Sections 4 describe the data descriptions and analytical results. Finally, Section 5 presents the conclusions.

2. Literature review

2.1 Gross Domestic Product (GDP)

Economic size is measured by its output, the most widely-used measure of economic output is the GDP, it is generally measured using one of three approaches, and these approaches of measuring GDP should result in the same number, with some possible discrepancies caused by usual mathematical and statistical figures rounding. These approaches as follow: First, the Expenditure approach, it involves the domestic purchases of goods and services plus net exports. Second, the Production approach, sometimes called Net Product or Value-added method, it involves the summation of all value-added activities at every phase of production by all industries inside the country, plus taxes and product's subsidies of the period. A third is the Income approach, it is the summation of all aspect of the income made by production within the economy as the remuneration of employees, capital income, and gross operating surplus of enterprises i.e. profit, taxes on production and imports less grants of the period (Ahmad and Harnhirun, 1996).

Predicting GDP is a vital issue if it is able to understand and capture the future developments of the economy, so it is meaningful to review the economic trends and predict the effect of current economic circumstances on the future GDP trend. This can be done through using time series data of GDP, which consists of observations consecutively generated over time. Such data are ordered with respect to the time, which shows the trend related to the time period observed. The trend may be increasing, decreasing, constant or having a cyclical fluctuation an ups and downs pattern the over time. Also, the data may show that the underlying process has periodic fluctuations of constant length, which is seasonal behavior. Therefore, Modeling would capture this underlying process using the observed time series data so that it could be possible to forecast what would likely be realized at a specific point in time in the future. In predicting macroeconomic time series variables like GDP, there are many possible types of models that are used in literature, in our study we will use ARIMA (Litterman, 1986; Stockton and Glassman, 1987).

In predicting a time series data of GDP as a macroeconomic variable, ARIMA model has been proven to be reliable and an accurate model. (Tsay and Tiao, 1985) used ARIMA modeling, they fitted a non-seasonal data by identifying autoregressive and moving average terms with the help of partial autocorrelation and autocorrelation functions (Box and Jenkins, 1970). Also (Topolewski, et al., 1995) deployed automatic methods were developed to identify as well as es-

timate the parameters of ARIMA model by utilizing time-series data for a single variable. Furthermore (Mait and Chatterjee, 2012) used a similar methodology to model macroeconomic variable like GDP. However, both the studies were limited to only non-seasonal time series and such modeling needs a long time-series data on the macroeconomic variable in question. (Mei, et. al, 2011) constructed a multi-factor dynamic system VAR prediction model of GDP by selecting six main economic indicators, using time series data extracted from the Shanghai region in China, also (Wang and Wang, 2011) deployed ARIMA for predicting the GDP of China based on time series data. They set up an ARIMA model of the GDP of China from 1978 to 2006. They then choose the best ARIMA model based on statistical tests and predict the GDP from 2007 to 2011. The result shows that the error between the actual value and the predicted value is insignificant which shows that the ARIMA model is highly accurate, Additionally (Wei and Yuan, 2010) used ARIMA model for GDP data series from 1952 to 2007 to predict the GDP of the Shanxi province in China, they found that error between the real GDP value and the predicted value is within 5% range.

2.2 Jordanian's economy

Jordan exists in the most volatile regions in the world. Given the country's high exposure to exogenous shocks, it has been affected by the regional conflicts, the fluctuations of commodity prices, and shifts in geopolitical relations which all compound the country's existing vulnerabilities. A combination of rising external and internal pressures challenges Jordan's balance, like a tightrope walker subject to gusts of winds. Over the past decade, Jordan has pursued structural economic reforms in many sectors like education, health, as well as privatization and liberalization.

The Government of Jordan has introduced social protection systems and reformed subsidies, creating the circumstances for public-private partnerships in infrastructure and making tax reforms. However, further progress is needed so that reforms aimed at enhancing the investment climate and ease of doing business can lead to enhance economic growth and increase its GDP. A major challenge facing Jordan remains to strengthen the economy in the context of a challenging external environment. Adverse regional developments, in particular, the Syria and Iraq crises, remain the largest recent shock affecting Jordan. Continued regional uncertainty and reduced external assistance will continue to put pressure on Jordan. Additionally, a High unemployment rate, high dependency on grants and declining remittances from Gulf economies pose a serious challenge. More comprehensive economic policies and the quick implementation of key growth-enhancing reforms are necessary to reduce the country's sensitivity to external shocks and help to strengthen the economy, also creating conditions for increased private investment and improve country's competitiveness will remain crucial for Jordan to stimulate job-creating growth (World Bank, 2018).

As seen in Figure 1 the annual growth of GDP during the period between 1978 -2017, the average growth rate was 7.6%, and it seems that in the last 10 years the growth is declining due to political instability in the surrounding countries, the 2017 growth rate declined to 3.6%.

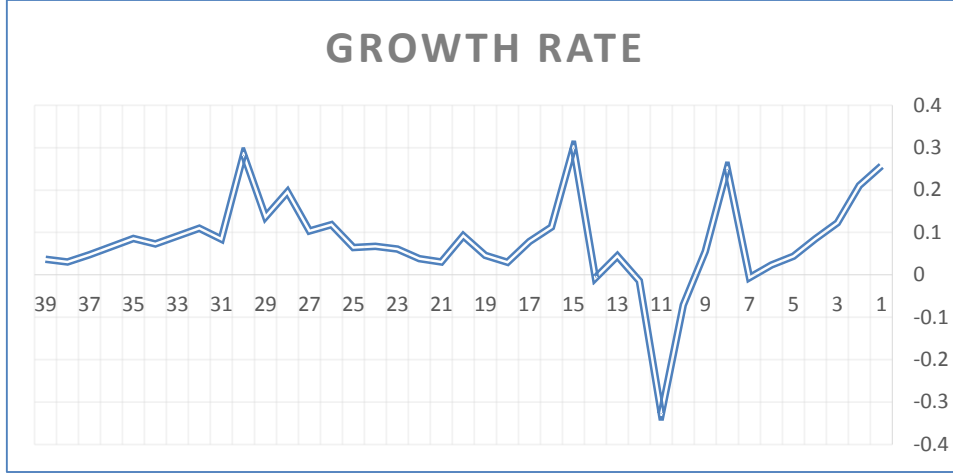


Figure 1: GDP Annual Growth Rate

3. Methodology

This section consists of the research framework, the mathematical model that is used to achieve the purpose of this research and mathematical criteria used is shown in Figure 2.

3.1 Arima model

An Auto regressive (AR) process is a series depends on its lagged values. The AR (p) model is a regression model which defined as appear in equation 1:

$$(1) \quad Y_t = \alpha_0 + \alpha_1 Y_{t-1} + \alpha_2 Y_{t-2} + \dots + \alpha_p Y_{t-p}.$$

Moving average (MA) model is related if the AR process is not the only mechanism that generates Y, but it contains past values with its error terms. MA (q) process is defined in equation 2:

$$(2) \quad \epsilon_t = \beta_1 \epsilon_t + \beta_2 \epsilon_{t-2} + \beta_3 \epsilon_{t-3} + \dots + \beta_q \epsilon_{t-q}.$$

Which contains the white noise errors; When Y has both the features of AR and MA, it is called as ARMA (p,q) process. (Gujarathi, et al., 2012) ARIMA

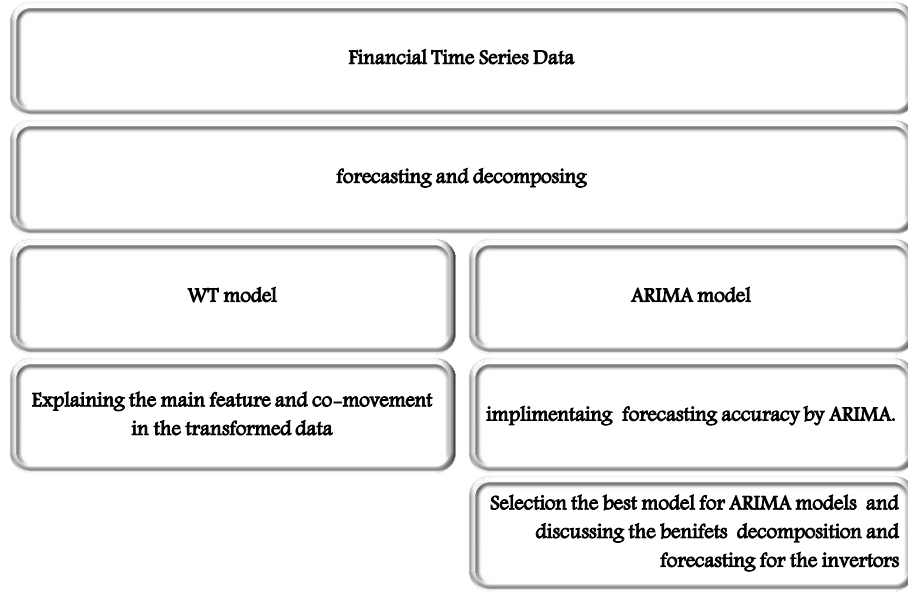


Figure 2: the flowchart of the paper

(Box-Jenkins model) is to classify and estimate a statistical model which can be explained as having generated the sample data. Since the financial time series data are type of non-stationary, therefore differencing the series will yield a stationary time series.

If the financial data becomes stationary when differenced d times, we name the series as $I(d)$. Consequently, if $ARMA(p, q)$ is applied to a series financial data which is $I(d)$, then the original time series is $ARIMA(p, d, q)$. The ARIMA methodology proposed that finding the values of p and q for AR and MA respectively by referring to the correlogram. In MA (q) model, moving average of order q , ACF Dies Down or Cuts off after lag q while for AR (p), autoregressive of order p PACF Dies Down or Cuts off after lag p . (Princeton, 2008). Model diagnosis can be applied based on the values of Root mean Square Error (RMSE) and Mean Absolute percent Error (MAPE).

3.2 Wavelet transform formula

WT is a mathematical model employed to convert the original observations into a time-scale domain. The model is very appropriate with the non-stationary data since most of the financial data are non-stationary. WT can be divided into Discrete Wavelet transform (DWT) and continuous wavelet transforms (CWT). DWT consists of many functions such as Haar, Daubechies, and Maximum overlapping Wavelet transform (MODWT) and others. All of these functions have the same properties with different applications. In this article, the WT will

be presented with its equation for all functions. For more details please refer to (Daubechies, 1992; Chiann and Morettin, 1998; Gençay et al., 2002; Al Wadi, 2010)

Wavelets theory is based on Fourier analysis, which represents any function as the sum of the sine and cosine functions. A wavelet is simply a function of time t that obeys a basic rule, known as the wavelet admissibility condition (Gençay et al., 2002):

WT is based on Fourier analysis (FT), which represents any function as the sum of the sine and cosine functions. WT is simply a function of time t that obeys a basic rule, known as the wavelet admissibility condition (Gençay et al., 2002):

$$(3) \quad C_\varphi = \int_0^\infty \left(\frac{|\varphi f|}{f} \right) df < \infty,$$

where QF is the Fourier transform and a function of frequency f , of . The WT is a mathematical tool that can be applied to numerous applications, such as image analysis and signal processing. It was introduced to solve problems associated with the Fourier transform, when dealing with non-stationary signals, or signals that are localized in time, space, or frequency.

There are two types of wavelets within a given function/family. Father wavelets describe the smooth and low-frequency parts of a signal, and mother wavelets describe the detailed and high-frequency components. Equation (4) represents the father wavelet and mother wavelet respectively, with $j=1,2,3,\dots, J$ in the J -level wavelet decomposition (Gençay et al., 2002):

$$(4) \quad \Phi_j, k = 2^{-\frac{j}{2}} \phi\left(t - \frac{2^j k}{2^j}\right),$$

where J denotes the maximum scale sustainable by the number of data points and the two types of wavelets stated above, namely father wavelets and mother wavelets and satisfies:

$$(5) \quad \int \phi(t) dt = 1, \quad \int \varphi(t) t dt = 0$$

time series data, i.e., function $f(t)$, is an input represented by wavelet analysis, and can be built up as a sequence of projections onto father and mother wavelets indexed by both $k, k = 0, 1, 2, \dots$ and by $S = 2^j, j = 1, 2, 3, \dots, J$.

Analyzing real discretely sampled data requires creating a lattice for making calculations. Mathematically, it is convenient to use a dyadic expansion, as shown in equation (6). The expansion coefficients are given by the projections:

$$(6) \quad S_{J,K} = \int \phi(t) dt = 1, \quad d_{J,K} = \int \varphi(t) t dt = 0.$$

The orthogonal wavelet series approximation to $f(t)$ is defined by:

$$(7) \quad F(t) = \sum S_{J,K} \Phi_{J,K}(t) + \sum d_{J,K} \varphi_{J,K}(t) + \sum d_{J-1,K} \varphi_{J-1,K}(t) + \dots + \sum d_{1,K} \varphi_{1,K}(t),$$

$$(8) \quad S_J(t) = \sum S_{J,K} \Phi_{J,K}(t),$$

$$(9) \quad D_J(t) = \sum d_{J,K} \Phi_{J,K}(t).$$

The WT is used to calculate the coefficient of the wavelet series approximation in Eq. (6) for a discrete signal, Where $S_J(t)$, and $D_J(t)$ are introducing the smooth and details coefficients respectively. The smooth coefficients dives the most important features of the data set and the details coefficients are used to detect the main features in the dataset. For more details about the WT and its functions please refer to (Al-Khazaleh et al., 2015). When the data pattern is very rough, the wavelet process is repeatedly applied. The aim of preprocessing is to minimize the Root Mean Squared Error (RMSE) between the signal before and after transformation. The noise in the original data can thus be removed. Importantly, the adaptive noise in the training pattern may reduce the risk of over fitting in training phase. Thus, we adopt WT twice for the preprocessing of training data in this study.

3.2.1 Accuracy Criteria

This section consists of two subsections. Firstly, we will present the criteria which have been used to make a fair comparison, and then the framework comparison will be presented with more details. The researchers have been adopted to compare the performance of the models within three types of accuracy criteria which are Mean square error (MSE), Root mean squared error (RMSE) and Mean absolute error (MAE). For more details about the mathematical model refer to (Aggarwal et al., 2008; Wadi et al., 2011).

4. Results

4.1 Data description

In this research, we use a time series data that represent the Jordanian's GDP for the period from 1978 till 2017, the data has been taken from Jordanian's department of statistics. The Matlab and Minitab statistical software were used to analyze the data.

As shown in the Figure 4 a histogram, accumulated histogram and descriptive statistics of the time series show a non-linear path; therefore, it is none stationary homogeneous type, characterized by random changes from one period to another.

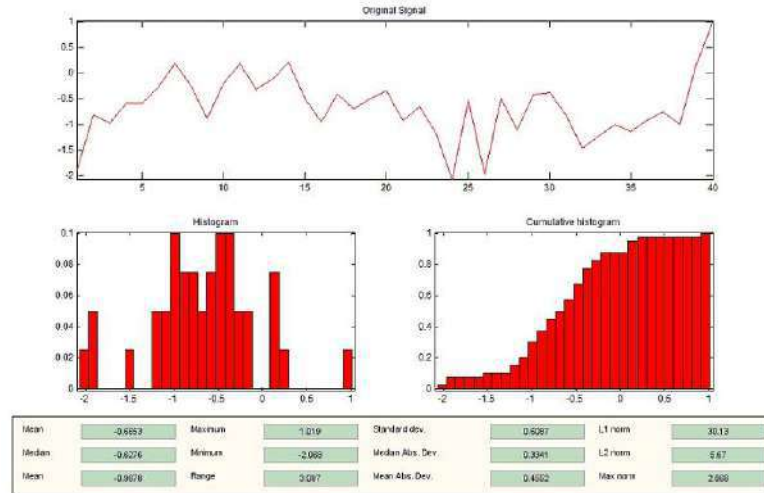


Figure 3: Data analysis matrix

4.1.1 Decomposing time series

A time series generally has three components that are a trend, noise and seasonal components. Decomposition of the time series means separating original time series into these components: First-Trend: The increasing or decreasing values in any time series. Second-Seasonal: The repeating cycle over a specific period in any time series. Third-Noise: The random of values in any time series. Figure 1 shows the decomposition based on WT. Before proceed to make data decomposition the data has been transformed using the log function in second order since the data is highly fluctuated, the decomposition procedure consists of a_1 , which is the approximated coefficients used for the proper forecasting and d_1 which shows the fluctuations of data. Mathematically, the equation can be represented as $S = a_1 + d_1$ where S is the original data.

Refer to the Figure 5 a visual inspection of the time plots shows that GDP data time series has a trend of random fluctuations, which means that the data are non-stationary and it is not constant around mean and variance. This type of non-stationary time series data contains a seasonal trend can be carried out by spectral analysis function which is WT. which lead to transform a random trend into a linear trend. Since d_1 explains the main features and fluctuations of the time series data, it has been clear that there were many fluctuations that the Jordanian's economy faced during the study period.

After we have done the decomposing process of time series data, we applied the ARIMA forecasting process.

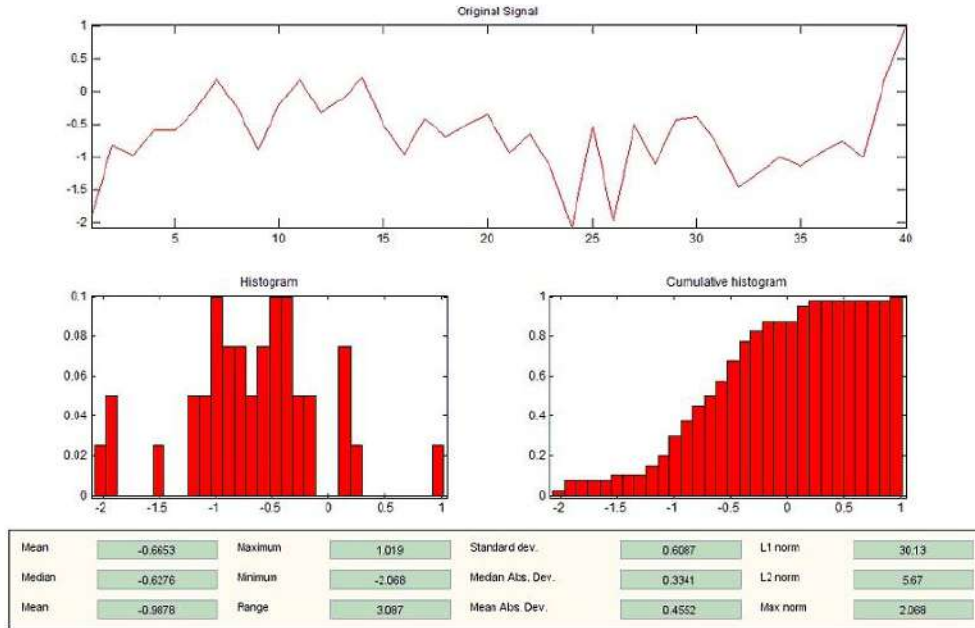


Figure 4: Data Description of GDP

After we have done the decomposing process of time series data, we applied the ARIMA forecasting process.

Table 1: ARIMA Model for Jordanian's GDP

	Fitted ARIMA Model
Model	(2,2,1)
MASE	0.3708
RMSE	0.0380

The fitted ARIMA models were diagnosed using MASE and RMSE. Parameter estimation for the ARIMA models was done using the Gaussian MLE criterion. The ARIMA models fitted based on the lowest value of MASE (0.3708) and RMSE (0.0380) as appear in Table 2, with a fit ARIMA is ARIMA (2,2,1). Where y_t represents the value of GDP.

Figures 6 and 7 show the GDP original and predicted values, the data series-stationary from (1978 till 2017) and the forecasted values of GDP for the coming twenty years (years 2017 till 2036), this suggests that the GDP long term trend is

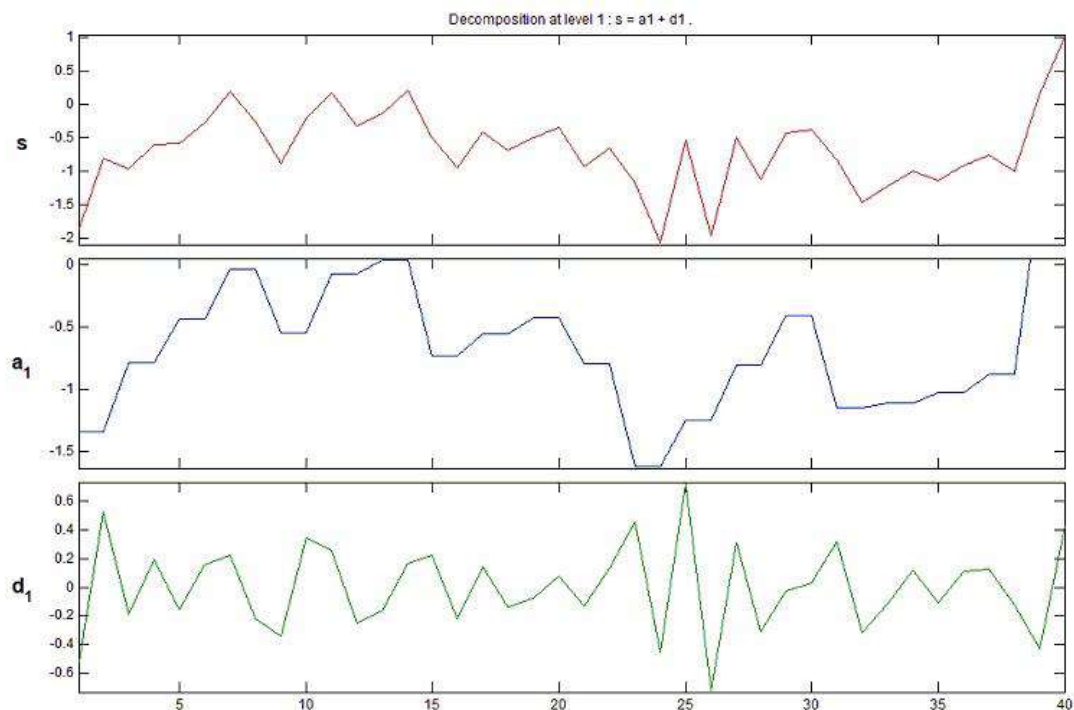


Figure 5: WT decomposition

up ward slopping, which gives an indication that Jordanian's economy long-term view considered positive and will continue its growing.

5. Conclusion

In this paper we deployed a WT model to decompose the data to detect the fluctuation and outlier values, then we utilized ARIMA model in predicting Jordanian's GDP, using GDP time series of the years 1978 to 2017. It is clear that the ARIMA model offers an excellent technique to predict any single data variable like GDP as an important macroeconomic indicator. Its strength lies in its fitting varieties of different types of time series with any pattern. In the process of model building, the original data is found a non-stationary then transformed to be stationary. An ARIMA (2,2,1) model is developed for analyzing and predict GDP among all of various tentative ARIMA models as it has the lowest BIC values. From the results, it can be observed that influence R Square value is (95%) high and Mean Absolute Percentage Error is very small for the fitted model. Therefore, the prediction accuracy is high. It is concluded that Jordanian's GDP will continue growing in the same growth pattern in

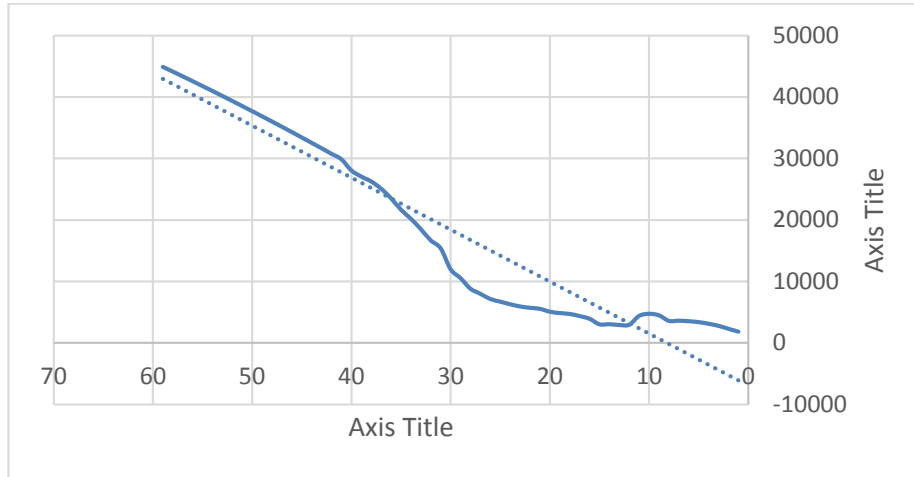


Figure 6: GDP, Billions of JD

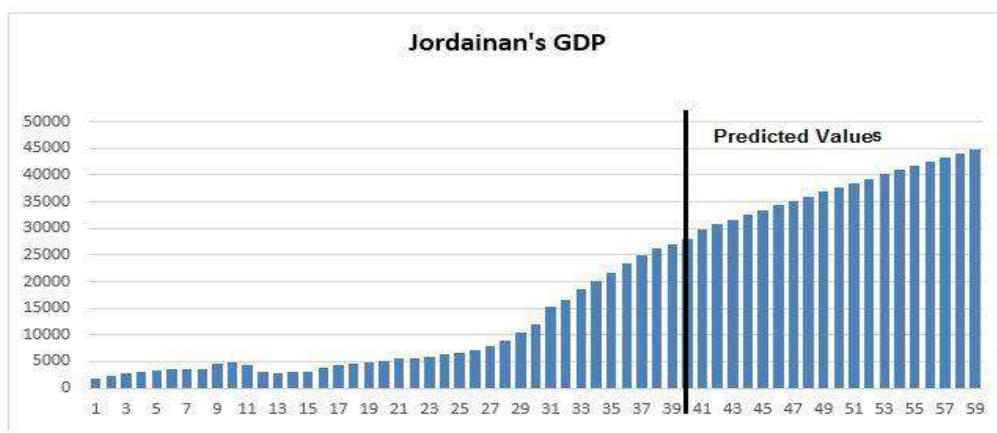


Figure 7: jordainians GDP prediction

the coming 20 years, Jordanian's government has to follow more comprehensive economic policies and implement key growth-enhancing reforms to strengthen the economy, also it has to stimulate job-creating growth and creates conditions to increase private investment and improve country's competitiveness.

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Oscillation of fractional Emden-Fowler type neutral vector partial differential equations with mixed nonlinearities and deviating arguments

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Abstract. The main purpose of this paper is to extend and improve existing oscillation criteria of mixed fractional order Emden-Fowler type neutral vector partial differential equations with mixed nonlinearities subject to Robin and Dirichlet boundary conditions. Several sufficient conditions are obtained for oscillation of solutions of such class of equations by using the generalized Riccati substitution and integral average method. We support our results by an illustrative example.

Keywords: fractional differential equations, Riccati technique, oscillation, Emden-Fowler equation, neutral.

1. Introduction

The problem of oscillation and nonoscillation of third order originated by Hanan [19] in his monumental paper published in 1961. Since then a number of researches contributed to the subject investigating various classes of differential equations and applying variety of techniques[15-18,27,42-44]. Third order differential equations arise in modelling an entry flow phenomenon in a problem of hydro dynamics or of the propagation of electrical pulses in the nerve of a squid approximated by the famous Nagumous equation; see the papers [6,25] for more details regarding the applications of third order (neutral) differential equations. A systematic survey of the most significant efforts in this theory can be found

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in the excellent monographs of Swanson [45], Greguš and the very recent-one of Padhi and Pati [38].

Fractional differential equations are now recognized as an excellent source of knowledge in modelling dynamical processes involving self similar and porous structures, probability and statistics, electrical networks, viscoelasticity, electro dynamics of complex medium, electro chemistry of corrosion, polymer rheology, industrial robotics, biotechnology, economics etc. For the theory and applications of fractional differential equations are refer the monographs and journals in the literature [1,4,20,21,33,42] and the reference cited there in.

In the middle of the nineteenth century, the Emden-Fowler equations emerged from theories deals with gaseous dynamics in astrophysics. The Emden-Fowler equations are considered to be one of the most important classical objects in the theory of differential equations. This type of equations has variety of interesting physical applications occuring in astrophysics in the form of Emden equation and in atomic physics in the form of Thomas-Fermi equation.

The Emden-Fowler type of equation has significant applications in many fields of scientific and technical world and this equation has been investigated by many researchers [2,3,5,9-12,24,28-31,40,48,50,52,54] and the reference cited there in.

The Emden-Fowler equations were first considered only for second order equations of the form $(p(t)u')' + q(t)u^\gamma = 0, t \geq 0$. By a mixed type Emden-Fowler equation we mean the equation contains a finite sum of powers of x and if there exists in sum exponents of which are both greater than and less than 1. These type of equations arises for instant in the growth of bacteria population with competitive species. As we have known, almost all existing oscillation criteria in the literature, see for the example [32,46,51] are established for Emden-Fowler type equations with mixed nonlinearities of second order.

Partial differential equations are used to model a number of real world problems arising in various branches of science and engineering [13,22,23,47,49,53]. In 1970, Domšlak introduced the concept of H -oscillation to study the oscillation of the solution of vector differential equations, where H is a unit vector in $\mathbb{R}^M$. We refer the reader to [7,8] for vector ordinary differential equations, [34-37] for vector partial differential equations, [26,14,39] for impulsive vector partial differential equations, [41] for fractional vector partial differential equations. There are essentially less results on oscillation of fractional order vector partial differential equations. Motivated by this, we initiate the oscillation criteria for fractional order Emden-Fowler type neutral vector partial differential equations with mixed nonlinearities and deviating arguments of the form

$$\begin{aligned} & \frac{\partial^{\alpha_1}}{\partial t^{\alpha_1}} \left(r_2(t) \frac{\partial^{\alpha_2}}{\partial t^{\alpha_2}} \left(r_1(t) |D_{+,t}^{\alpha_3} Z(x,t)|^{\gamma-1} D_{+,t}^{\alpha_3} Z(x,t) \right) \right) \\ & + p_1(x,t) |U(x,t-\delta_1)|^{\gamma_1-1} U(x,t-\delta_1) + p_2(x,t) |U(x,t-\delta_2)|^{\gamma_2-1} U(x,t-\delta_2) \end{aligned}$$

$$\begin{aligned}
&= a(t)\Delta U(x, t) + \sum_{i=1}^m a_i(t)\Delta U(x, \rho_i(t)) \\
(1) \quad &- \sum_{j=1}^k b_j(x, t)f_j \left(\int_0^t (t-s)^{-\alpha_3} \|U(x, \sigma_j(s))\| ds \right) U(x, \sigma_j(t)) + F(x, t),
\end{aligned}$$

$(x, t) \in \Omega \times \mathbb{R}_+ = G$ where $Z(x, t) = U(x, t) + q(t)U(x, t-\tau)$. $\mathbb{R}_+ = (0, \infty)$, where Ω is bounded domain in $\mathbb{R}^N$ with a piecewise smooth boundary $\partial\Omega$, $0 < \alpha_i \leq 1$ ($i = 1, 2$) is a constant. $\frac{\partial^{\alpha_i}}{\partial t^{\alpha_i}}$ is the conformable fractional partial derivative of order α_i with respect to t and $D_{+,t}^{\alpha_3}$ is the Riemann-Liouville fractional partial derivative of order α_3 with respect to t . $\alpha_3 \in (0, 1)$. Δ is the Laplacian operator in the Euclidean N space $\mathbb{R}^N$, i.e., $\Delta u(x, t) = \sum_{r=1}^N \frac{\partial^2 u(x, t)}{\partial x_r^2}$. $\|U(x, \sigma_j(t))\|$ is the usual Euclidean norm in $\mathbb{R}^N$.

Equation (1) is supplemented with the Robin boundary condition with

$$(2) \quad \frac{\partial U(x, t)}{\partial \mu} + g(x, t)U(x, t) = 0, \quad (x, t) \in \partial\Omega \times \mathbb{R}_+,$$

where μ is the unit exterior normal vector to $\partial\Omega$ and $g(x, t)$ is a non-negative continuous function on $\partial\Omega \times \mathbb{R}_+$. And the Dirichlet boundary condition

$$(3) \quad U(x, t) = 0, \quad (x, t) \in \partial\Omega \times \mathbb{R}_+.$$

Throughout this paper, we will suppose that the following conditions hold:

- (A₁) τ, δ_1 and δ_2 are positive constants, γ_1, γ_2 and γ are nonnegative constants with $0 < \gamma_1 < \gamma < \gamma_2$;
- (A₂) $r_1(t) \in C^{\alpha_1+\alpha_2}(\mathbb{R}_+, \mathbb{R}_+), r_2(t) \in C^{\alpha_1}(\mathbb{R}_+, \mathbb{R}_+)$;
- (A₃) $q(t) \in C^{\alpha_1+\alpha_2+\alpha_3}(\mathbb{R}_+, \mathbb{R})$; $0 \leq q(t) \leq 1$;
- (A₄) $p_1(x, t), p_2(x, t) \in C(\bar{G}, \mathbb{R}_+), p_i(t) = \min_{x \in \bar{\Omega}} p_i(x, t)$, where $i=1, 2$, $p_i(t)$ is not identically zero on any ray from $[t_*, \infty)$ for any $t_* \geq 0$;
- (A₅) $a(t), a_i(t) \in C(\mathbb{R}_+, \mathbb{R}_+), i = 1, 2, \dots, m$;
- (A₆) $\sigma_j(t), \rho_j(t) \in C(\mathbb{R}_+, \mathbb{R}_+), \lim_{t \rightarrow \infty} \sigma_j(t) = \lim_{t \rightarrow \infty} \rho_i(t) = \infty, j = 1, 2, \dots, k, i = 1, 2, \dots, m$;
- (A₇) $b_j(x, t) \in C(\bar{G}, \mathbb{R})$ and $\min_{x \in \bar{\Omega}} b_j(x, t), j = 1, 2, \dots, k$;
- (A₈) $F(x, t) \in C(\bar{G}, \mathbb{R}^n), f_H(x, t) \in C(\bar{G}, \mathbb{R})$ such that $\int_{\Omega} f_H(x, t) dx \leq 0$;
- (A₉) $f_j \in C(\mathbb{R}_+, \mathbb{R})$ are convex and nondecreasing in $\mathbb{R}$ with $uf_j(u)$ for $u \neq 0$ and there exist positive constants ν_j such that $\frac{f_j(u)}{u} \geq \nu_j$ for all $u \neq 0, j = 1, 2, \dots, k$.

The following notations will be used for our convenience.

$$(4) \quad u_H(x, t) = \langle U(x, t), H \rangle,$$

$$(5) \quad f_H(x, t) = \langle F(x, t), H \rangle,$$

$$(6) \quad V_H(t) = \frac{1}{|\Omega|} \int_{\Omega} u_H(x, t) dx, \quad \text{where } |\Omega| = \int_{\Omega} dx,$$

$$(7) \quad z_H(x, t) = u_H(x, t) + q(t)u_H(x, t - \tau),$$

$$(8) \quad Z_H(t) = V_H(t) + q(t)V_H(t - \tau).$$

This paper is organized as follows: In Section 2, we recall the basic definitions of the Riemann-Liouville fractional derivative and conformable fractional derivatives together with basic lemmas concerning the above set of operators. In Section 3, we present some new sufficient conditions for the H -oscillation of the solutions of (1),(2) and (1), (3). In Section 4, an example is provided to illustrate the main results.

2. Preliminaries

In this section, we present some definitions and review some note worthy results from the literature, which we will use throughout the paper.

Definition 2.1. *By a solution of (1), (2) or (1), (3) we mean a nontrivial function $U(x, t) \in C^{\alpha_1+\alpha_2+\alpha_3}(\Omega \times [G, \mathbb{R}^M]) \cap C^2(\Omega \times [t_1, \infty), \mathbb{R}^M) \cap C(\Omega \times [\hat{t}_1, \infty), \mathbb{R}^M)$ satisfying (1) on the domain G and the boundary condition (2), (3), where*

$$t_1 := \min \left\{ 0, \min_{1 \leq i \leq m} \left\{ \inf_{t \geq 0} \rho_i(t) \right\} \right\},$$

$$\hat{t}_1 := \min \left\{ 0, \min_{1 \leq j \leq m} \left\{ \inf_{t \geq 0} \sigma_j(t) \right\} \right\}.$$

Denoted by $\mathbb{P}$ the set of all solutions $U(x, t)$ of (1), (2) or (1), (3). The set $\mathbb{P}$ is said to be proper if there is a $T_1 > t_0$ such that

$$\sup \left\{ \sum_{k=1}^n |U_k(x, s)|, t \leq s < \infty \right\} > 0,$$

for any $t \geq T$, we make a standing hypothesis is that of (1) has such a solution. A proper solution $U(x, t) \in \mathbb{P}$ of (1), (2) or (1), (3) is said to be oscillatory in G if it is neither eventually positive nor eventually negative. Otherwise, it is nonoscillatory. Equation (1) is said to be oscillatory if all its proper solutions are oscillatory.

Definition 2.2 ([53]). *Let H be a fixed unit vector in $\mathbb{R}^n$. A solution $U(x, t)$ of (1) is said to be H -oscillatory in G if the inner product $\langle U(x, t), H \rangle$ has a zero in $\Omega \times (t_0, \infty)$ for $t_0 > 0$. Otherwise, it is H -nonoscillatory.*

Definition 2.3 ([33]). *The Riemann-Liouville fractional partial derivative of order $0 < \alpha < 1$ with respect to t of a function $u(x, t)$ is given by*

$$(9) \quad (D_{+,t}^\alpha u)(x, t) := \frac{\partial}{\partial t} \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\nu)^{-\alpha} u(x, \nu) d\nu$$

provided the right hand side is pointwise defined on $\mathbb{R}_+$, where Γ is the gamma function.

Definition 2.4 ([33]). *The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $y : \mathbb{R}_+ \rightarrow \mathbb{R}$ on the half-axis $\mathbb{R}_+$ is given by*

$$(10) \quad (I_+^\alpha y)(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t - \nu)^{\alpha-1} y(\nu) d\nu, \quad \text{for } t > 0$$

provided the right hand side is pointwise defined on $\mathbb{R}_+$.

Definition 2.5 ([33]). *The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $y : \mathbb{R}_+ \rightarrow \mathbb{R}$ on the half-axis $\mathbb{R}_+$ is given by*

$$(11) \quad (D_+^\alpha y)(t) := \frac{d^{[\alpha]}}{dt^{[\alpha]}} \left(I_+^{[\alpha]-\alpha} y \right) (t), \quad \text{for } t > 0$$

provided the right hand side is pointwise defined on $\mathbb{R}_+$, where $[\alpha]$ is the ceiling function of α .

Lemma 2.1. *Let*

$$(12) \quad K(t) := \int_0^t (t - \nu)^{-\alpha} y(\nu) d\nu \quad \text{for } \alpha \in (0, 1) \quad \text{and } t > 0.$$

Then, $K'(t) = \Gamma(1 - \alpha)(D_+^\alpha y)(t)$.

Definition 2.6 ([4]). *Let f be a function of m variables $x_1, x_2, \dots, x_m$. Then the conformable partial derivative of f of order α , $0 < \alpha \leq 1$ in x_j is defined as follows*

$$\begin{aligned} & \frac{\partial^\alpha}{\partial x_j^\alpha} f(x_1, x_2, \dots, x_m) \\ &= \lim_{\epsilon \rightarrow 0} \frac{f(x_1, x_2, \dots, x_{j-1}, x_j + \epsilon x_j^{1-\alpha}, \dots, x_m) - f(x_1, x_2, \dots, x_m)}{\epsilon}. \end{aligned}$$

Definition 2.7 ([20]). *Given a function $f : [0, \infty) \rightarrow \mathbb{R}$. Then the conformable fractional derivative of f of order α , is defined by*

$$T_\alpha(f)(t) = \lim_{\epsilon \rightarrow 0} \frac{f(t + \epsilon t^{1-\alpha}) - f(t)}{\epsilon},$$

for all $t > 0, \alpha \in (0, 1]$. If f is α -differentiable in some $(0, a), a > 0$, and $\lim_{t \rightarrow 0^+} f^{(\alpha)}(t)$ exists, then define

$$f^{(\alpha)}(0) = \lim_{t \rightarrow 0^+} f^{(\alpha)}(t).$$

We will sometimes write $f^{(\alpha)}(t)$ for $T_\alpha(f)(t)$, to denote the conformable fractional derivatives of f of order α .

Some properties of conformable fractional derivative:

Let $\alpha \in (0, 1]$ and f and g be α - differentiable at a point $t > 0$. Then:

$$(P_1) \quad T_\alpha(t^p) = pt^{p-\alpha} \text{ for all } p \in \mathbb{R}.$$

$$(P_2) \quad T_\alpha(\lambda) = 0, \text{ for all constant functions } f(t) = \lambda.$$

$$(P_3) \quad T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f).$$

$$(P_4) \quad T_\alpha\left(\frac{f}{g}\right) = \frac{gT_\alpha(f) - fT_\alpha(g)}{g^2}.$$

$$(P_5) \quad \text{If, in addition, } f \text{ is differentiable, then } T_\alpha(f)(t) = t^{1-\alpha} \frac{df}{dt}(t).$$

Definition 2.8 ([44]). Let $\alpha \in (0, 1]$ and $0 \leq a < b$. A function $f : [a, b] \rightarrow \mathbb{R}$ is α -fractional integrable on $[a, b]$ if the integral

$$\int_a^b f(x) d_\alpha x := \int_a^b f(x) x^{\alpha-1} dx$$

exists and is finite.

Lemma 2.2 ([1]). Let $f : (a, b) \rightarrow \mathbb{R}$ be differentiable and $0 < \alpha \leq 1$. Then, for all $t > a$ we have

$$(13) \quad I_\alpha^\alpha T_\alpha(f)(t) = f(t) - f(a).$$

3. Main results

In this section, we will give the sufficient conditions for the H -oscillation of all solutions of problem (1), (2). In the sequel, we need the following lemmas which will be used to prove our main theorems.

Lemma 3.1. Assume that $(A_1) - (A_9)$ hold. Let H be a fixed unit vector in $\mathbb{R}^n$ and $U(x, t)$ be a solution of (1), (2).

(i) If $u_H(x, t)$ is eventually positive, then $u_H(x, t)$ satisfies the scalar mixed fractional partial inequality

$$(14) \quad \begin{aligned} & \frac{\partial^{\alpha_1}}{\partial t^{\alpha_1}} \left(r_2(t) \frac{\partial^{\alpha_2}}{\partial t^{\alpha_2}} \left(r_1(t) |D_{+,t}^{\alpha_3} z_H(x, t)|^{\gamma-1} D_{+,t}^{\alpha_3} z_H(x, t) \right) \right) \\ & + p_1(t) |u_H(x, t - \delta_1)|^{\gamma_1-1} u_H(x, t - \delta_1) + p_2(t) |u_H(x, t - \delta_2)|^{\gamma_2-1} u_H(x, t - \delta_2) \\ & - a(t) \Delta u_H(x, t) - \sum_{i=1}^m a_i(t) \Delta u_H(x, \rho_i(t)) \\ & + \sum_{j=1}^k b_j(t) f_j \left(\int_0^t (t-s)^{-\alpha_3} u_H(x, \sigma_j(s)) ds \right) u_H(x, \sigma_j(t)) \leq f_H(x, t). \end{aligned}$$

(ii) If $u_H(x, t)$ is eventually negative, then $u_H(x, t)$ satisfies the scalar mixed fractional partial inequality

$$\begin{aligned}
 & \frac{\partial^{\alpha_1}}{\partial t^{\alpha_1}} \left(r_2(t) \frac{\partial^{\alpha_2}}{\partial t^{\alpha_2}} \left(r_1(t) |D_{+,t}^{\alpha_3} z_H(x, t)|^{\gamma-1} D_{+,t}^{\alpha_3} z_H(x, t) \right) \right) \\
 & + p_1(t) |u_H(x, t - \delta_1)|^{\gamma_1-1} u_H(x, t - \delta_1) + p_2(t) |u_H(x, t - \delta_2)|^{\gamma_2-1} u_H(x, t - \delta_2) \\
 & - a(t) \Delta u_H(x, t) - \sum_{i=1}^m a_i(t) \Delta u_H(x, \rho_i(t)) \\
 (15) \quad & + \sum_{j=1}^k b_j(t) f_j \left(\int_0^t (t-s)^{-\alpha_3} u_H(x, \sigma_j(s)) ds \right) u_H(x, \sigma_j(t)) \geq f_H(x, t).
 \end{aligned}$$

Proof. Assume that $u_H(x, t)$ is an eventually positive solution. Taking the inner product (1) and H , we get

$$\begin{aligned}
 & \frac{\partial^{\alpha_1}}{\partial t^{\alpha_1}} \left(r_2(t) \frac{\partial^{\alpha_2}}{\partial t^{\alpha_2}} \left(r_1(t) |D_{+,t}^{\alpha_3} \langle U(x, t), H \rangle|^{\gamma-1} D_{+,t}^{\alpha_3} \langle U(x, t), H \rangle \right) \right) \\
 & + p_1(x, t) | \langle U(x, t - \delta_1), H \rangle |^{\gamma_1-1} \langle U(x, t - \delta_1), H \rangle \\
 & + p_2(x, t) | \langle U(x, t - \delta_2), H \rangle |^{\gamma_2-1} \langle U(x, t - \delta_2), H \rangle \\
 & = a(t) \Delta \langle U(x, t), H \rangle + \sum_{i=1}^m a_i(t) \Delta \langle U(x, \rho_i(t)), H \rangle \\
 & - \sum_{j=1}^k b_j(x, t) f_j \left(\int_0^t (t-s)^{-\alpha_3} \|U(x, \sigma_j(s))\| ds \right) \langle U(x, \sigma_j(t)), H \rangle \\
 (16) \quad & + \langle F(x, t), H \rangle.
 \end{aligned}$$

Using (4), (5) and (7), we have

$$\begin{aligned}
 & \frac{\partial^{\alpha_1}}{\partial t^{\alpha_1}} \left(r_2(t) \frac{\partial^{\alpha_2}}{\partial t^{\alpha_2}} \left(r_1(t) |D_{+,t}^{\alpha_3} z_H(x, t)|^{\gamma-1} D_{+,t}^{\alpha_3} z_H(x, t) \right) \right) \\
 & + p_1(x, t) |u_H(x, t - \delta_1)|^{\gamma_1-1} u_H(x, t - \delta_1) \\
 & + p_2(x, t) |u_H(x, t - \delta_2)|^{\gamma_2-1} u_H(x, t - \delta_2) \\
 & = a(t) \Delta u_H(x, t) + \sum_{i=1}^m a_i(t) \Delta u_H(x, \rho_i(t)) \\
 (17) \quad & - \sum_{j=1}^k b_j(x, t) f_j \left(\int_0^t (t-s)^{-\alpha_3} \|U(x, \sigma_j(s))\| ds \right) u_H(x, \sigma_j(t)) + f_H(x, t).
 \end{aligned}$$

Since $f_j \in C(\mathbb{R}_+, \mathbb{R})$, $j = 1, 2, \dots, k$. We have $u_H(x, \sigma_j(t)) \leq \|U(x, \sigma_j(t))\|$ and using (A7), we get

$$(18) \quad \begin{aligned} & b_j(x, t) f_j \left(\int_0^t (t-s)^{-\alpha_3} \|U(x, \sigma_j(s))\| ds \right) u_H(x, \sigma_j(t)) \\ & \geq b_j(t) f_j \left(\int_0^t (t-s)^{-\alpha_3} u_H(x, \sigma_j(s)) ds \right) u_H(x, \sigma_j(t)). \end{aligned}$$

Applying (A4),

$$(19) \quad \begin{aligned} & p_1(x, t) |u_H(x, t - \delta_1)|^{\gamma_1 - 1} u_H(x, t - \delta_1) \\ & \geq p_1(t) |u_H(x, t - \delta_1)|^{\gamma_1 - 1} u_H(x, t - \delta_1) \\ & p_2(x, t) |u_H(x, t - \delta_2)|^{\gamma_2 - 1} u_H(x, t - \delta_2) \\ (20) \quad & \geq p_2(t) |u_H(x, t - \delta_2)|^{\gamma_2 - 1} u_H(x, t - \delta_2). \end{aligned}$$

Applying the above results (19), (20) and (21) in (18), we get the inequality

$$\begin{aligned} & \frac{\partial^{\alpha_1}}{\partial t^{\alpha_1}} \left(r_2(t) \frac{\partial^{\alpha_2}}{\partial t^{\alpha_2}} \left(r_1(t) |D_{+,t}^{\alpha_3} z_H(x, t)|^{\gamma-1} D_{+,t}^{\alpha_3} z_H(x, t) \right) \right) \\ & + p_1(t) |u_H(x, t - \delta_1)|^{\gamma_1 - 1} u_H(x, t - \delta_1) + p_2(t) |u_H(x, t - \delta_2)|^{\gamma_2 - 1} u_H(x, t - \delta_2) \\ & - a(t) \Delta u_H(x, t) - \sum_{i=1}^m a_i(t) \Delta u_H(x, \rho_i(t)) \\ & + \sum_{j=1}^k b_j(t) f_j \left(\int_0^t (t-s)^{-\alpha_3} u_H(x, \sigma_j(s)) ds \right) u_H(x, \sigma_j(t)) \leq f_H(x, t). \end{aligned}$$

Similarly, letting $u_H(x, t)$ be eventually negative, we easily obtain (16). The proof is complete. $\square$

The inner product of (2), (3) with H yield the following boundary conditions

$$(21) \quad \frac{\partial u_H(x, t)}{\partial \mu} + g(x, t) u_H(x, t) = 0, \quad (x, t) \in \partial\Omega \times \mathbb{R}_+,$$

$$(22) \quad u_H(x, t) = 0, \quad (x, t) \in \partial\Omega \times \mathbb{R}_+.$$

Lemma 3.2. *Assume that (A1)-(A9) hold. Let H be a fixed unit vector in $\mathbb{R}^n$. If the scalar mixed fractional partial inequality (15) has no eventually positive solutions and the scalar mixed fractional partial inequality (16) has no eventually negative solutions satisfying the boundary conditions (22) or (23), then every solution $U(x, t)$ of the problem (1), (2) or (1), (3) is H -oscillatory in G .*

Theorem 3.1. *Assume that (A1)-(A9) and*

(A10) $\min_{j \in 1, 2, \dots, k} \sigma_j(t) = \sigma(t) \geq t$;

(A11) $u_H(x, t) \geq L$ hold.

If the mixed fractional differential inequality

$$(23) \quad \begin{aligned} & T_{\alpha_1} \left(r_2(t) T_{\alpha_2} \left(r_1(t) (D_+^{\alpha_3} Z_H(t))^{\gamma} \right) \right) \\ & + p_1(t) (V_H(t - \delta_1))^{\gamma_1} + p_2(t) (V_H(t - \delta_2))^{\gamma_2} \\ & + L \sum_{j=1}^k b_j(t) f_j(K_H(t)) \leq 0 \end{aligned}$$

has no eventually positive solutions and the mixed fractional inequality

$$(24) \quad \begin{aligned} & T_{\alpha_1} \left(r_2(t) T_{\alpha_2} \left(r_1(t) (D_+^{\alpha_3} Z_H(t))^{\gamma} \right) \right) \\ & + p_1(t) (V_H(t - \delta_1))^{\gamma_1} + p_2(t) (V_H(t - \delta_2))^{\gamma_2} \\ & + L \sum_{j=1}^k b_j(t) f_j(K_H(t)) \geq 0 \end{aligned}$$

has no eventually negative solutions, then the solution $U(x, t)$ of equations (1), (2) is H -oscillatory in G .

Proof. Let us suppose to the contrary that there exists a solution $U(x, t)$ of Equations (1), (2) which is not H -oscillatory. Integrating (1) with respect to x over Ω , we have

$$(25) \quad \begin{aligned} & \int_{\Omega} \frac{\partial^{\alpha_1}}{\partial t^{\alpha_1}} \left(r_2(t) \frac{\partial^{\alpha_2}}{\partial t^{\alpha_2}} \left(r_1(t) |D_{+,t}^{\alpha_3} z_H(x, t)|^{\gamma-1} D_{+,t}^{\alpha_3} z_H(x, t) \right) \right) dx \\ & + p_1(t) \int_{\Omega} |u_H(x, t - \delta_1)|^{\gamma_1-1} u_H(x, t - \delta_1) dx \\ & + p_2(t) \int_{\Omega} |u_H(x, t - \delta_2)|^{\gamma_2-1} u_H(x, t - \delta_2) dx - a(t) \int_{\Omega} \Delta u_H(x, t) dx \\ & - \sum_{i=1}^m a_i(t) \int_{\Omega} \Delta u_H(x, \rho_i(t)) dx \\ & + \sum_{j=1}^k b_j(t) \int_{\Omega} f_j \left(\int_0^t (t-s)^{-\alpha_3} u_H(x, \sigma_j(s)) ds \right) u_H(x, \sigma_j(t)) dx \\ & \leq \int_{\Omega} f_H(x, t) dx, \quad t \geq t_0. \end{aligned}$$

Using Green's formula and boundary condition (22) yield that

$$(26) \quad \int_{\Omega} \Delta u_H(x, t) dx = \int_{\partial\Omega} \frac{\partial u_H(x, t)}{\partial \mu} dS = - \int_{\partial\Omega} g(x, t) u_H(x, t) dS \leq 0, \quad t \geq t_0$$

and

$$(27) \quad \begin{aligned} & \int_{\Omega} \Delta u_H(x, \rho_i(t)) dx = \int_{\partial\Omega} \frac{\partial u_H(x, \rho_i(t))}{\partial \mu} dS \\ & = - \int_{\partial\Omega} g(x, t) u_H(x, \rho_i(t)) dS \leq 0, \quad t \geq t_0, \quad i = 1, 2, \dots, m. \end{aligned}$$

Applying Jensen's inequality, it follows that

$$\begin{aligned}
 p_1(t) \int_{\Omega} |u_H(x, t - \delta_1)|^{\gamma_1 - 1} u_H(x, t - \delta_1) dx &\geq p_1(t) \int_{\Omega} (u_H(x, t - \delta_1))^{\gamma_1} dx \\
 &\geq p_1(t) \left(\int_{\Omega} u_H(x, t - \delta_1) dx \right)^{\gamma_1} \\
 (28) \quad &\geq p_1(t) (V_H(t - \delta_1))^{\gamma_1}, \quad t \geq t_0,
 \end{aligned}$$

$$\begin{aligned}
 p_2(t) \int_{\Omega} |u_H(x, t - \delta_2)|^{\gamma_2 - 1} u_H(x, t - \delta_2) dx &\geq p_2(t) \int_{\Omega} (u_H(x, t - \delta_2))^{\gamma_2} dx \\
 &\geq p_2(t) \left(\int_{\Omega} u_H(x, t - \delta_2) dx \right)^{\gamma_2} \\
 (29) \quad &\geq p_2(t) (V_H(t - \delta_2))^{\gamma_2}, \quad t \geq t_0.
 \end{aligned}$$

Using (A_{10}) , (A_{11}) and Jensen's inequality, we get

$$\begin{aligned}
 &\int_{\Omega} f_j \left(\int_0^t (t-s)^{-\alpha_3} u_H(x, \sigma_j(s)) ds \right) u_H(x, \sigma_j(t)) dx \\
 &\geq L f_j \left(\int_{\Omega} \left(\int_0^t (t-s)^{-\alpha_3} u_H(x, \sigma_j(s)) ds \right) dx \right) \\
 &\geq L f_j \left(\int_0^t (t-s)^{-\alpha_3} \left(\int_{\Omega} u_H(x, \sigma_j(s)) dx \right) ds \right) \\
 &\geq L \int_{\Omega} dx f_j \left(\int_0^t (t-s)^{-\alpha_3} \left(\int_{\Omega} u_H(x, \sigma_j(s)) dx \right) \left(\int_{\Omega} dx \right)^{-1} ds \right) \\
 &\geq L \int_{\Omega} dx f_j \left(\int_0^t (t-s)^{-\alpha_3} (V_H(\sigma_j(s))) ds \right) \\
 (30) \quad &\geq L \int_{\Omega} dx f_j(K_H(t)), \quad t \geq t_0.
 \end{aligned}$$

Also, by (A_8) ,

$$(31) \quad \int_{\Omega} f_H(x, t) dx \leq 0.$$

In view of (6), (8) and (26)-(32) yield

$$\begin{aligned}
 &T_{\alpha_1} \left(r_2(t) T_{\alpha_2} \left(r_1(t) (D_+^{\alpha_3} Z_H(t))^{\gamma} \right) \right) + p_1(t) (V_H(t - \delta_1))^{\gamma_1} \\
 (32) \quad &+ p_2(t) (V_H(t - \delta_2))^{\gamma_2} + L \sum_{j=1}^k b_j(t) f_j(K_H(t)) \leq 0.
 \end{aligned}$$

Therefore, $V_H(t)$ is an eventually positive solution of (24). This contradicts the hypothesis. The case where $u_H(x, t) < 0$ in $\Omega \times [t_0, \infty)$ can be treated similarly and we are also getting a contradiction. This completes the proof. $\square$

Lemma 3.3. *Assume that (A_1) -(A_{11}) hold and $u_H(x, t) > 0$. If*

$$(33) \quad \int_0^\infty \frac{1}{r_2(s)} d_{\alpha_2} s = \infty,$$

$$(34) \quad \int_0^\infty \frac{1}{r_1(s)} ds = \infty,$$

$$(35) \quad \int_0^\infty \left(\frac{1}{r_1(\theta)} \int_\theta^\infty \frac{1}{r_2(\xi)} \left(\int_\xi^\infty P(s) d_{\alpha_1} s \right) d_{\alpha_2} \xi \right)^{\frac{1}{\gamma}} d\theta = \infty,$$

where

$$(36) \quad \begin{aligned} P(t) &= p_1(t)(1 - q(t - \delta_1))^{\gamma_1} M^{\gamma_1} + p_2(t)(1 - q(t - \delta_2))^{\gamma_2} M^{\gamma_2} \\ &+ L \sum_{j=1}^k b_j(t) \nu_j \beta, \end{aligned}$$

then there exists a sufficiently large T such that $T_{\alpha_2}(r_1(t)(D_+^{\alpha_3} Z_H(t))^\gamma) > 0$ on $[T, \infty)$ and either $D_+^{\alpha_3} Z_H(t) > 0$ on $[T, \infty)$ or $\lim_{t \rightarrow \infty} Z_H(t) = 0$.

Proof. Suppose to the contrary that there exists a solution $U(x, t)$ of (1),(2) which is not H -oscillatory in G . Without loss of generality we may assume that $u_H(x, t)$ in $\Omega \times [t_0, \infty)$ for some $t_0 > 0$. That is, $V_H(t)$ is an eventually positive solution of (24). There exists a $t_1 \geq t_0$ such that $V_H(t) > 0, V_H(t - \tau) > 0, V_H(t - \delta_1) > 0, V_H(t - \delta_2) > 0$ for $t \geq t_1$ and from (24) we have,

$$(37) \quad \begin{aligned} T_{\alpha_1}(r_2(t)T_{\alpha_2}(r_1(t)(D_+^{\alpha_3} Z_H(t))^\gamma)) &\leq -p_1(t)(V_H(t - \delta_1))^{\gamma_1} - p_2(t)(V_H(t - \delta_2))^{\gamma_2} \\ &- L \sum_{j=1}^k b_j(t) f_j(K_H(t)) < 0, \quad t \geq t_1. \end{aligned}$$

Then, $T_{\alpha_1}(r_2(t)T_{\alpha_2}(r_1(t)(D_+^{\alpha_3} Z_H(t))^\gamma))$ is strictly decreasing on $[t_1, \infty)$, and thus $T_{\alpha_2}(r_1(t)(D_+^{\alpha_3} Z_H(t))^\gamma)$ is eventually of one sign. For $t_2 > t_1$ is sufficiently large on $[t_1, \infty)$, we claim $T_{\alpha_2}(r_1(t)(D_+^{\alpha_3} Z_H(t))^\gamma) > 0$ on $[t_2, \infty)$. Otherwise, assume that there exists a sufficiently large $t_3 > t_2$ such that $T_{\alpha_2}(r_1(t)(D_+^{\alpha_3} Z_H(t))^\gamma) < 0$ on $[t_3, \infty)$. Then, $r_1(t)(D_+^{\alpha_3} Z_H(t))^\gamma$ is decreasing on $[t_3, \infty)$, and we have

$$\begin{aligned} r_1(t)(D_+^{\alpha_3} Z_H(t))^\gamma - r_1(t_3)(D_+^{\alpha_3} Z_H(t_3))^\gamma &= \int_{t_3}^t \frac{r_2(s)}{r_2(s)} T_{\alpha_2}(r_1(s)(D_+^{\alpha_3} Z_H(s))^\gamma) d_{\alpha_2} s \\ &\leq r_2(t_3) T_{\alpha_2}(r_1(t_3)(D_+^{\alpha_3} Z_H(t_3))^\gamma) \int_{t_3}^t \frac{1}{r_2(s)} d_{\alpha_2} s. \end{aligned}$$

By (34), we have $\lim_{t \rightarrow \infty} r_1(t)(D_+^{\alpha_3} Z_H(t))^\gamma = -\infty$. So, there exists a sufficiently large t_4 with $t_4 > t_3$ such that $D_+^{\alpha_3} Z_H(t) < 0, t \in [t_4, \infty)$. Further more

$$D_+^{\alpha_3} Z_H(t) = \frac{K'_H(t)}{\Gamma(1 - \alpha_3)} < 0,$$

$$\begin{aligned}
(38) \quad \frac{1}{\Gamma(1-\alpha_3)}[K_H(t) - K_H(t_4)] &= \frac{1}{\Gamma(1-\alpha_3)} \int_{t_4}^t K'_H(s) ds \\
&\leq r_1(t_4) K'_H(t_4) \int_{t_4}^t \frac{1}{r_1(s)} ds.
\end{aligned}$$

By (35), we deduce that $\lim_{t \rightarrow \infty} K_H(t) = -\infty$, which contradicts the fact that $K_H(t)$ is an eventually positive solution of (35). So $T_{\alpha_2}(r_1(t)(D_+^{\alpha_3} Z_H(t))^\gamma) > 0$ on $[t_2, \infty)$. Thus $D_+^{\alpha_3} Z_H(t)$ is eventually of one sign. Now, we assume that $D_+^{\alpha_3} Z_H(t) < 0, t \in [t_5, \infty)$, for sufficiently large $t_5 > t_4$. Since $K'_H(t) > 0$, further more, we have $\lim_{t \rightarrow \infty} K_H(t) = \beta \geq 0$. We claim that $\beta = 0$. Otherwise assume that $\beta > 0$. Then $K_H(t) \geq \beta$ on $[t_5, \infty)$ and for $t \in [t_5, \infty)$ by (24) and (A₉), we have

$$\begin{aligned}
(39) \quad T_{\alpha_1}(r_2(t)T_{\alpha_2}(r_1(t)(D_+^{\alpha_3} Z_H(t))^\gamma)) &+ p_1(t)(V_H(t - \delta_1))^{\gamma_1} + p_2(t)(V_H(t - \delta_2))^{\gamma_2} \\
&+ L \sum_{j=1}^k b_j(t)\nu_j\beta \leq 0.
\end{aligned}$$

By (8), we get

$$(40) \quad V_H(t) \geq Z_H(t) - q(t)V_H(t - \tau) \geq Z_H(t)(1 - q(t)).$$

Then, for all $t \geq t_5$,

$$(41) \quad V_H(t - \delta_1) \geq (1 - q(t - \delta_1))Z_H(t - \delta_1),$$

$$(42) \quad V_H(t - \delta_2) \geq (1 - q(t - \delta_2))Z_H(t - \delta_2).$$

Using (41)-(43) in (40), implies that,

$$\begin{aligned}
(43) \quad T_{\alpha_1}(r_2(t)T_{\alpha_2}(r_1(t)(D_+^{\alpha_3} Z_H(t))^\gamma)) &+ p_1(t)(1 - q(t - \delta_1))^{\gamma_1} Z_H^{\gamma_1}(t - \delta_1) \\
&+ p_2(t)(1 - q(t - \delta_2))^{\gamma_2} Z_H^{\gamma_2}(t - \delta_2) + L \sum_{j=1}^k b_j(t)\nu_j\beta \leq 0.
\end{aligned}$$

Since $z(t) \geq M$ on $[t_5, \infty)$, we get

$$\begin{aligned}
(44) \quad T_{\alpha_1}(r_2(t)T_{\alpha_2}(r_1(t)(D_+^{\alpha_3} Z_H(t))^\gamma)) &+ p_1(t)(1 - q(t - \delta_1))^{\gamma_1} M^{\gamma_1} \\
&+ p_2(t)(1 - q(t - \delta_2))^{\gamma_2} M^{\gamma_2} + L \sum_{j=1}^k b_j(t)\nu_j\beta \leq 0, \quad t \geq t_5.
\end{aligned}$$

Using (37), we have

$$(45) \quad T_{\alpha_1}(r_2(t)T_{\alpha_2}(r_1(t)(D_+^{\alpha_3} Z_H(t))^\gamma)) + P(t) \leq 0, \quad t \geq t_5.$$

α_1 - integrating from t to ∞ yields,

$$\int_t^\infty T_{\alpha_1} (r_2(s) T_{\alpha_2} (r_1(s) (D_+^{\alpha_3} Z_H(s))^\gamma)) d_{\alpha_1} s \leq - \int_t^\infty P(s) d_{\alpha_1} s,$$

$$T_{\alpha_2} (r_1(t) (D_+^{\alpha_3} Z_H(t))^\gamma) \geq \frac{1}{r_2(t)} \int_t^\infty P(s) d_{\alpha_1} s$$

α_2 - integrating from t to ∞ yields,

$$\int_t^\infty T_{\alpha_2} (r_1(s) (D_+^{\alpha_3} Z_H(s))^\gamma) d_{\alpha_2} s \geq \int_t^\infty \frac{1}{r_2(\xi)} \int_\xi^\infty P(s) d_{\alpha_1} s d_{\alpha_2} \xi,$$

$$r_1(t) (D_+^{\alpha_3} Z_H(t))^\gamma \leq - \int_t^\infty \frac{1}{r_2(\xi)} \int_\xi^\infty P(s) d_{\alpha_1} s d_{\alpha_2} \xi,$$

$$D_+^{\alpha_3} Z_H(t) \leq \left(-\frac{1}{r_1(t)} \int_t^\infty \frac{1}{r_2(\xi)} \int_\xi^\infty P(s) d_{\alpha_1} s d_{\alpha_2} \xi \right)^{\frac{1}{\gamma}}.$$

Using Lemma 2.1, we have

$$K'_H(t) \leq \Gamma(1 - \alpha_3) \left(-\frac{1}{r_1(t)} \int_t^\infty \frac{1}{r_2(\xi)} \int_\xi^\infty P(s) d_{\alpha_1} s d_{\alpha_2} \xi \right)^{\frac{1}{\gamma}}.$$

Once again, integrating with respect to s from t_5 to ∞ yields,

$$K_H(t) \leq K_H(t_5) - \int_{t_5}^t \left[\frac{1}{r_1(\theta)} \int_\theta^\infty \frac{1}{r_2(\xi)} \int_\xi^\infty P(s) d_{\alpha_1} s d_{\alpha_2} \xi \right]^{\frac{1}{\gamma}} d\theta.$$

Letting $t \rightarrow \infty$, from (36) we get $\lim_{t \rightarrow \infty} K_H(t) = -\infty$, which causes a contradiction. So the proof is complete. $\square$

Lemma 3.4. Assume that (A_1) - (A_{11}) hold and $V(t)$ is an eventually positive solution of Eq.(1) such that $T_{\alpha_2} (r_1(t) (D_+^{\alpha_3} Z_H(t))^\gamma) > 0, D_+^{\alpha_3} Z_H(t) > 0$ on $[t_1, \infty)$, where $t_1 \geq t_0$ is sufficiently large. Then one has

$$(46) \quad K'_H(t) \geq \left\{ \frac{1}{r_1(t)r_2(t)} T_{\alpha_2} (r_1(t) (D_+^{\alpha_3} Z_H(t))^\gamma) R_1(t_1, t) \right\}^{\frac{1}{\gamma}}$$

or

$$(47) \quad K_H(t) \geq [T_{\alpha_2} (r_1(t) (D_+^{\alpha_3} Z_H(t))^\gamma)]^{\frac{1}{\gamma}} R_2(t_1, t),$$

where $R_1(t_1, t) = \int_{t_1}^t \frac{1}{r_2(s)} d_{\alpha_2} s, R_2(t_1, t) = \int_{t_1}^t \left(\frac{R_1(t_1, s)}{r_1(s)r_2(s)} \right)^{\frac{1}{\gamma}} ds.$

Proof. By (45), we obtain that $r_2(t) T_{\alpha_2} (r_1(t) (D_+^{\alpha_3} Z_H(t))^\gamma)$ is strictly decreasing on $[t_1, \infty)$. So,

$$\int_{t_1}^t T_{\alpha_2} (r_1(s) (D_+^{\alpha_3} Z_H(s))^\gamma) d_{\alpha_2} s = r_1(t) (D_+^{\alpha_3} Z_H(t))^\gamma - r_1(t_1) (D_+^{\alpha_3} Z_H(t_1))^\gamma.$$

Multiplying and divided by $r_2(t)$,

$$\begin{aligned}
r_1(t)(D_+^{\alpha_3} Z_H(t))^\gamma &\geq r_1(t)(D_+^{\alpha_3} Z_H(t))^\gamma - r_1(t_1)(D_+^{\alpha_3} Z_H(t_1))^\gamma \\
&= \int_{t_1}^t \frac{r_2(s)}{r_2(s)} T_{\alpha_2} (r_1(s)(D_+^{\alpha_3} Z_H(s))^\gamma) d_{\alpha_2} s, \\
r_1(t)(D_+^{\alpha_3} Z_H(t))^\gamma &\geq \frac{1}{r_2(t)} T_{\alpha_2} (r_1(t)(D_+^{\alpha_3} Z_H(t))^\gamma) \int_{t_1}^t \frac{1}{r_2(s)} d_{\alpha_2} s, \\
(D_+^{\alpha_3} Z_H(t))^\gamma &\geq \frac{1}{r_1(t)r_2(t)} T_{\alpha_2} (r_1(t)(D_+^{\alpha_3} Z_H(t))^\gamma) R_1(t_1, t), \\
D_+^{\alpha_3} Z_H(t) &\geq \left\{ \frac{1}{r_1(t)r_2(t)} T_{\alpha_2} (r_1(t)(D_+^{\alpha_3} Z_H(t))^\gamma) R_1(t_1, t) \right\}^{\frac{1}{\gamma}}.
\end{aligned}$$

Integrating with respect to s from t_1 to t we obtain,

$$\begin{aligned}
K'_H(t) &\geq \Gamma(1 - \alpha_3) \left\{ \frac{1}{r_1(t)r_2(t)} T_{\alpha_2} (r_1(t)(D_+^{\alpha_3} Z_H(t))^\gamma) R_1(t_1, t) \right\}^{\frac{1}{\gamma}}, \\
\int_{t_1}^t K'_H(s) ds &\geq \Gamma(1 - \alpha_3) \int_{t_1}^t \left\{ \frac{1}{r_1(s)r_2(s)} T_{\alpha_2} (r_1(s)(D_+^{\alpha_3} Z_H(s))^\gamma) R_1(t_1, s) \right\}^{\frac{1}{\gamma}} ds, \\
K_H(t) &\geq \Gamma(1 - \alpha_3) [T_{\alpha_2} (r_1(t)(D_+^{\alpha_3} Z_H(t))^\gamma)]^{\frac{1}{\gamma}} \int_{t_1}^t \left(\frac{R_1(t_1, s)}{r_1(s)r_2(s)} \right)^{\frac{1}{\gamma}} ds, \\
K_H(t) &\geq \Gamma(1 - \alpha_3) [T_{\alpha_2} (r_1(t)(D_+^{\alpha_3} Z_H(t))^\gamma)]^{\frac{1}{\gamma}} R_2(t_1, t).
\end{aligned}$$

This completes the proof. $\square$

In this section, we will obtain Philos - type oscillation criteria for (1),(2) under the case when $0 \leq q(t) \leq 1$. The following notations are used in the sequel.

Denote

$$\lambda = \min \left\{ \frac{\gamma_2 - \gamma_1}{\gamma_2 - \gamma}, \frac{\gamma_2 - \gamma_1}{\gamma - \gamma_1} \right\},$$

$$P_1(t)$$

$$= \lambda \left[\left(\frac{p_1(t)}{r_1(t)} \right)^{\gamma_2 - \gamma} \left(\frac{p_2(t)}{r_1(t)} \right)^{\gamma - \gamma_1} (1 - q(t - \delta_1))^{\gamma_1(\gamma_2 - \gamma)} (1 - q(t - \delta_2))^{\gamma_2(\gamma - \gamma_1)} \right]^{\frac{1}{\gamma_2 - \gamma_1}}.$$

Let us define the following Philos functions $\mathbb{J}$.

Let $\mathbb{D}_0 = \{(t, s) : t > s \geq t_0\}$ and $\mathbb{D} = \{(t, s) : t \geq s \geq t_0\}$. We say that the functions $B \in C(\mathbb{D}, \mathbb{R})$ belongs to the class $\mathbb{J}$, denoted by $B \in \mathbb{J}$, if

(H₁) $B(t, t) = 0$ for $t \geq t_0$. $B(t, s) > 0$ on $(t, s) \in \mathbb{D}_0$;

(H₂) B has a continuous and nonpositive partial derivative on $\mathbb{D}_0$ with respect to the second variable, such that

$$\frac{\partial}{\partial s} B(t, s) = -h(t, s)B(t, s) \quad \text{for } (t, s) \in \mathbb{D}_0,$$

where $h \in C(\mathbb{D}, \mathbb{R})$.

For given functions $h \in C(\mathbb{D}, \mathbb{R})$, $\phi \in C'([\mathbb{R}_+, \infty), \mathbb{R}_+)$ and $\eta \in C'([\mathbb{R}_+, \infty), \mathbb{R})$ we set

$$\begin{aligned}\psi(t, s) &= h(t, s) - \frac{\phi'(s)}{\phi(s)}, \\ Q_1(t, s) &= P_1(s) - \eta'(s) + \psi(t, s)\eta(s).\end{aligned}$$

Theorem 3.2. *Let $(A_1) - (A_{13})$ hold. Then (1), (2) is H -oscillatory provided that the following condition holds*

$$(48) \quad \limsup_{t \rightarrow \infty} \frac{1}{B(t, t_0)} \times \int_{t_0}^t B(t, s) \phi(s) \left\{ Q_1(t, s) - |\Lambda(s)| |\psi(t, s)|^{\gamma+1} - \left| \frac{s^{\alpha_1-1} L \sum_{j=1}^k b_j(s) \nu_j}{r_1(s) Z_H^\gamma(t - \delta_1)} \right| \right\} ds = \infty,$$

where

$$(49) \quad \Lambda(t) = \frac{\gamma^{\gamma+1}}{(\gamma+1)^{\gamma+1}} \frac{(r_2(t))^2}{\gamma R_1(t_1, t) Z_H(t - \delta_1) K_H(t)}.$$

Proof. Suppose to the contrary that there exists a solution $U(x, t)$ of (1), (2) which is not H -oscillatory in G . Without loss of generality we may assume that $u_H(x, t)$ in $\Omega \times [t_0, \infty)$ for some $t_0 > 0$. That is, $V_H(t)$ is an eventually positive solution of (24). There exists a $t_1 \geq t_0$ such that $V_H(t) > 0$, $V_H(t - \tau) > 0$, $V_H(t - \delta_1) > 0$, $V_H(t - \delta_2) > 0$ for $t \geq t_1$ (The Case $U(x, t) < 0$ can be considered by the same method). Therefore, we get (24). Now, define

$$(50) \quad W(t) = \phi(t) \left(\frac{r_2(t) T_{\alpha_2} (r_1(t) (D_+^{\alpha_3} Z_H(t))^\gamma)}{r_1(t) Z_H^\gamma(t - \delta_1) K_H(t)} + \eta(t) \right) \quad \text{for } t \geq t_1.$$

α_1 -differentiating (51) and (38), we have

$$\begin{aligned}T_{\alpha_1} W(t) &\leq \frac{T_{\alpha_1} \phi(t)}{\phi(t)} W(t) - \phi(t) \left[\frac{p_1(t) (1 - q(t - \delta_1))^{\gamma_1} Z_H^{\gamma_1}(t - \delta_1)}{r_1(t) Z_H^\gamma(t - \delta_1) K_H(t)} \right] \\ &- \phi(t) \left[\frac{p_2(t) (1 - q(t - \delta_2))^{\gamma_2} Z_H^{\gamma_2}(t - \delta_2)}{r_1(t) Z_H^\gamma(t - \delta_1) K_H(t)} \right] - \phi(t) \left[\frac{L \sum_{j=1}^k b_j(t) \nu_j}{r_1(t) Z_H^\gamma(t - \delta_1)} \right] \\ &- \frac{\phi(t) r_2(t) T_{\alpha_2} (r_1(t) (D_+^{\alpha_3} Z_H(t))^\gamma) T_{\alpha_1} (r_1(t) Z_H^\gamma(t - \delta_1) K_H(t))}{(r_1(t) Z_H^\gamma(t - \delta_1) K_H(t))^2} + \phi(t) T_{\alpha_1} \eta(t).\end{aligned}$$

Applying (p5), we get

$$W'(t) \leq \frac{\phi'(t)}{\phi(t)} W(t) - \phi(t) t^{\alpha_1-1} \left[\frac{p_1(t) (1 - q(t - \delta_1))^{\gamma_1} Z_H^{\gamma_1}(t - \delta_1)}{r_1(t) Z_H^\gamma(t - \delta_1) K_H(t)} \right]$$

$$\begin{aligned}
& - \phi(t)t^{\alpha_1-1} \left[\frac{p_2(t)(1-q(t-\delta_2))^{\gamma_2} Z_H^{\gamma_2}(t-\delta_2)}{r_1(t)Z_H^\gamma(t-\delta_1)K_H(t)} \right] - \phi(t)t^{\alpha_1-1} \left[\frac{L \sum_{j=1}^k b_j(t)\nu_j}{r_1(t)Z_H^\gamma(t-\delta_1)} \right] \\
& - \frac{\phi(t)r_2(t)T_{\alpha_2} \left(r_1(t) \left(D_+^{\alpha_3} Z_H(t) \right)^\gamma \right) \left(r_1(t)(Z_H^\gamma(t-\delta_1))K_H'(t) \right)}{\left(r_1(t)Z_H^\gamma(t-\delta_1)K_H(t) \right)^2} \\
& + \phi(t)\eta'(t).
\end{aligned}$$

Applying Lemma 3.4, we have

$$\begin{aligned}
W'(t) & \leq \frac{\phi'(t)}{\phi(t)} W(t) - \phi(t)t^{\alpha_1-1} \left[\frac{p_1(t)(1-q(t-\delta_1))^{\gamma_1} Z_H^{\gamma_1}(t-\delta_1)}{r_1(t)Z_H^\gamma(t-\delta_1)K_H(t)} \right] \\
& - \phi(t)t^{\alpha_1-1} \left[\frac{p_2(t)(1-q(t-\delta_2))^{\gamma_2} Z_H^{\gamma_2}(t-\delta_2)}{r_1(t)Z_H^\gamma(t-\delta_1)K_H(t)} \right] - \phi(t)t^{\alpha_1-1} \left[\frac{L \sum_{j=1}^k b_j(t)\nu_j}{r_1(t)Z_H^\gamma(t-\delta_1)} \right] \\
& \quad \phi(t)r_2(t)T_{\alpha_2} \left(r_1(t) \left(D_+^{\alpha_3} Z_H(t) \right)^\gamma \right) \left(r_1(t)(Z_H^\gamma(t-\delta_1)) \right) \\
& - \frac{\left\{ \frac{1}{r_1(t)r_2(t)} T_{\alpha_2} \left(r_1(t) \left(D_+^{\alpha_3} Z_H(t) \right)^\gamma \right) R_1(t_1, t) \right\}^{\frac{1}{\gamma}}}{\left(r_1(t)Z_H^\gamma(t-\delta_1)K_H(t) \right)^2} \\
& + \phi(t)\eta'(t).
\end{aligned}$$

Therefore,

$$\begin{aligned}
W'(t) & \leq \frac{\phi'(t)}{\phi(t)} W(t) - \phi(t)t^{\alpha_1-1} \left[\frac{p_1(t)(1-q(t-\delta_1))^{\gamma_1} Z_H^{\gamma_1}(t-\delta_1)}{r_1(t)Z_H^\gamma(t-\delta_1)K_H(t)} \right] \\
& - \phi(t)t^{\alpha_1-1} \left[\frac{p_2(t)(1-q(t-\delta_2))^{\gamma_2} Z_H^{\gamma_2}(t-\delta_2)}{r_1(t)Z_H^\gamma(t-\delta_1)K_H(t)} \right] - \phi(t)t^{\alpha_1-1} \left[\frac{L \sum_{j=1}^k b_j(t)\nu_j}{r_1(t)Z_H^\gamma(t-\delta_1)} \right] \\
(51) \quad & - \frac{\phi(t)R_1^{\frac{1}{\gamma}}(t_1, t)Z_H^{\frac{1}{\gamma}}(t-\delta_1)K_H^{\frac{1}{\gamma}}(t)}{r_2^{\frac{2}{\gamma}}(t)} \left(\frac{W(t)}{\phi(t)} - \eta(t) \right)^{1+\frac{1}{\gamma}} + \phi(t)\eta'(t).
\end{aligned}$$

For simplicity $\delta_1 \geq \delta_1$ (a similar argument holds for $\delta_1 < \delta_1$) then $V_H(t-\delta_1) \leq V_H(t-\delta_1)$ and using Young's inequality ($\frac{l}{p} + \frac{m}{q} \geq l^{\frac{1}{p}} m^{\frac{1}{q}}$), we obtain that

$$\begin{aligned}
& \frac{\gamma_2 - \gamma}{\gamma_2 - \gamma_1} \left[\frac{p_1(t)}{r_1(t)} (1-q(t-\delta_1))^{\gamma_1} z^{\gamma_1-\gamma}(t-\delta_1) \right] \\
& + \frac{\gamma - \gamma_1}{\gamma_2 - \gamma_1} \left[\frac{p_2(t)}{r_1(t)} (1-q(t-\delta_2))^{\gamma_2} z^{\gamma_2-\gamma}(t-\delta_1) \right] \\
& \geq \left\{ \left(\frac{p_1(t)}{r_1(t)} \right)^{\gamma_2-\gamma} \left(\frac{p_2(t)}{r_1(t)} \right)^{\gamma-\gamma_1} (1-q(t-\delta_1))^{\gamma_1(\gamma_2-\gamma)} (1 \right. \\
(52) \quad & \left. - q(t-\delta_2) \right)^{\gamma_2(\gamma-\gamma_1)} \right\}^{\frac{1}{\gamma_2-\gamma_1}} = P_1(t).
\end{aligned}$$

Combining (52) and (53) for $t \geq T_0$, we have

$$(53) \quad \begin{aligned} W'(t) &\leq \frac{\phi'(t)}{\phi(t)} W(t) - \phi(t) t^{\alpha_1-1} \left[\frac{L \sum_{j=1}^k b_j(t) \nu_j}{r_1(t) Z_H^\gamma(t - \delta_1)} \right] \\ &- \phi(t) [P_1(t) - \eta'(t)] - \frac{\phi(t) R_1^{\frac{1}{\gamma}}(t_1, t) Z_H^{\frac{1}{\gamma}}(t - \delta_1) K_H^{\frac{1}{\gamma}}(t)}{r_2^{\frac{2}{\gamma}}(t)} \left| \frac{W(t)}{\phi(t)} - \eta(t) \right|^{1+\frac{1}{\gamma}}. \end{aligned}$$

Replacing t in (54) by s , then multiplying (54) by $B(t, s)$ and integrating on $[T, t]$, it follows from (H_2) that for all $t \geq T \geq T_0$,

$$(54) \quad \begin{aligned} &\int_T^t B(t, s) \phi(s) [P_1(s) - \eta'(s)] ds \leq - \int_T^t B(t, s) W'(s) ds \\ &+ \int_T^t B(t, s) \frac{\phi'(s)}{\phi(s)} W(s) ds \\ &- \int_T^t \frac{\phi(s) R_1^{\frac{1}{\gamma}}(t_1, s) Z_H^{\frac{1}{\gamma}}(s - \delta_1) K_H^{\frac{1}{\gamma}}(s)}{r_2^{\frac{2}{\gamma}}(s)} B(t, s) \left| \frac{W(s)}{\phi(s)} - \eta(s) \right|^{\frac{\gamma+1}{\gamma}} ds \\ &- \int_T^t \phi(s) s^{\alpha_1-1} B(t, s) \left[\frac{L \sum_{j=1}^k b_j(s) \nu_j}{r_1(s) Z_H^\gamma(s - \delta_1)} \right] ds \\ &= B(t, T) W(T) - \int_T^t B(t, s) \psi(s) W(s) ds \\ &- \int_T^t \frac{\phi(s) R_1^{\frac{1}{\gamma}}(t_1, s) Z_H^{\frac{1}{\gamma}}(s - \delta_1) K_H^{\frac{1}{\gamma}}(s)}{r_2^{\frac{2}{\gamma}}(s)} B(t, s) \left| \frac{W(s)}{\phi(s)} - \eta(s) \right|^{\frac{\gamma+1}{\gamma}} ds \\ &- \int_T^t \phi(s) s^{\alpha_1-1} B(t, s) \left[\frac{L \sum_{j=1}^k b_j(s) \nu_j}{r_1(s) Z_H^\gamma(s - \delta_1)} \right] ds \\ &\int_T^t B(t, s) \phi(s) Q_1(t, s) ds \leq B(t, T) W(T) \\ &+ \int_T^t B(t, s) \phi(s) |\psi(t, s)| \left| \frac{W(s)}{\phi(s)} - \eta(s) \right| ds \\ &- \int_T^t B(t, s) \phi(s) \left| \frac{R_1^{\frac{1}{\gamma}}(t_1, s) Z_H^{\frac{1}{\gamma}}(s - \delta_1) K_H^{\frac{1}{\gamma}}(s)}{r_2^{\frac{2}{\gamma}}(s)} \right| \left| \frac{W(s)}{\phi(s)} - \eta(s) \right|^{\frac{\gamma+1}{\gamma}} ds \\ &- \int_T^t \phi(s) s^{\alpha_1-1} B(t, s) \left[\frac{L \sum_{j=1}^k b_j(s) \nu_j}{r_1(s) Z_H^\gamma(s - \delta_1)} \right] ds. \end{aligned}$$

For given t and s , $t \neq s$, set

$$G(\omega) := |\psi| |\omega| - \left| \frac{R_1^{\frac{1}{\gamma}}(t_1, t) Z_H^{\frac{1}{\gamma}}(t - \delta_1) K_H^{\frac{1}{\gamma}}(t)}{r_2^{\frac{2}{\gamma}}(t)} \right| |\omega|^{\frac{\gamma+1}{\gamma}}, \quad \omega > 0.$$

$G(\omega)$ attains its maximum at $|\psi|^\gamma \frac{\gamma^\gamma}{(\gamma+1)^\gamma} \left| \frac{r_2^2(t)}{R_1(t_1, t) Z_H^\gamma(t-\delta_1) K_H(t)} \right|$ and

$$(55) \quad G(\omega) \leq G_{\max} \leq |\Lambda(t)| |\psi|^{\gamma+1}.$$

Substituting (56) into (55), we have

$$(56) \quad \begin{aligned} & \int_T^t B(t, s) \phi(s) Q_1(t, s) ds \leq B(t, T) W(T) \\ & + \int_T^t B(t, s) \phi(s) |\Lambda(s)| |\psi(t, s)|^{\gamma+1} ds \\ & - \int_T^t \phi(s) s^{\alpha_1-1} B(t, s) \left[\frac{L \sum_{j=1}^k b_j(s) \nu_j}{r_1(s) Z_H^\gamma(t-\delta_1)} \right] ds \end{aligned}$$

Set $T = T_0$, so

$$\begin{aligned} & \int_{T_0}^t B(t, s) \phi(s) \left[Q_1(t, s) - |\Lambda(s)| |\psi(t, s)|^{\gamma+1} - s^{\alpha_1-1} \left[\frac{L \sum_{j=1}^k b_j(s) \nu_j}{r_1(s) Z_H^\gamma(t-\delta_1)} \right] \right] ds \\ & \leq H(t, T_0) W(T_0). \end{aligned}$$

Thus, by (A_{13}) , we obtain

$$(57) \quad \begin{aligned} & \int_{T_0}^t B(t, s) \phi(s) \left[Q_1(t, s) - |\Lambda(s)| |\psi(t, s)|^{\gamma+1} - s^{\alpha_1-1} \left[\frac{L \sum_{j=1}^k b_j(s) \nu_j}{r_1(s) Z_H^\gamma(t-\delta_1)} \right] \right] ds \\ & = \left(\int_{t_0}^{T_0} + \int_{T_0}^t \right) B(t, s) \phi(s) \left[Q_1(t, s) - |\Lambda(s)| |\psi(t, s)|^{\gamma+1} \right. \\ & \quad \left. - s^{\alpha_1-1} \left[\frac{L \sum_{j=1}^k b_j(s) \nu_j}{r_1(s) Z_H^\gamma(t-\delta_1)} \right] \right] ds \\ & \leq B(t, t_0) \left(\int_{t_0}^{T_0} \phi(s) \left[Q_1(t, s) - |\Lambda(s)| |\rho(t, s)|^{\gamma+1} \right. \right. \\ & \quad \left. \left. - s^{\alpha_1-1} \left[\frac{L \sum_{j=1}^k b_j(s) \nu_j}{r_1(s) Z_H^\gamma(t-\delta_1)} \right] \right] ds + |W(T_0)| \right), \end{aligned}$$

we divide (58) through by $B(t, t_0)$ and take $\limsup$ in it as $t \rightarrow \infty$. Eq. (49) gives a desired contradiction in (58). This proves the theorem. $\square$

Theorem 3.3. *Let H, ϕ, η be as in Theorem 3.1. Suppose that*

$$(58) \quad 0 < \inf_{s \geq t_0} \left\{ \liminf_{t \rightarrow \infty} \frac{B(t, s)}{B(t, t_0)} \right\} \leq \infty$$

and

$$(59) \quad \limsup_{t \rightarrow \infty} \frac{1}{B(t, t_0)} \int_{t_0}^t B(t, s) \phi(s) |\psi(t, s)|^{\gamma+1} ds < \infty.$$

Then (1), (2) is H -oscillatory provided the following condition holds. There exists $\Theta \in C([t_0, \infty), \mathbb{R})$

$$(60) \quad \int^{\infty} \phi(s) \left(\frac{\Theta(s)}{\phi(s)} - \eta(s) \right)_+^{\frac{\gamma+1}{\gamma}} ds = \infty,$$

and for any $T \geq t_0$,

$$(61) \quad \limsup_{t \rightarrow \infty} \frac{1}{B(t, T)} \int_T^t B(t, s) \phi(s) [Q_1(t, s) - |\Lambda(s)| |\psi(t, s)|^{\gamma+1} - \left| s^{\alpha_1-1} \frac{L \sum_{j=1}^k b_j(s) \nu_j}{r_1(s) Z_H^\gamma(t - \delta_1)} \right|] ds \geq \Theta(t),$$

where $\Theta_+(s) = \max \{\Theta(s), 0\}$.

Proof. Proceeding as in the proof of Theorem 3.2, we have that (55) and (57) hold. Therefore, from (57), for all $t > T \geq T_0$,

$$\limsup_{t \rightarrow \infty} \frac{1}{B(t, T)} \int_T^t B(t, s) \phi(s) [Q_1(t, s) - |\Lambda(s)| |\psi(t, s)|^{\gamma+1} - \left| s^{\alpha_1-1} \frac{L \sum_{j=1}^k b_j(s) \nu_j}{r_1(s) Z_H^\gamma(t - \delta_1)} \right|] ds \leq W(T)$$

Also, by (62), we have

$$(62) \quad \Theta(t) \leq W(T), \quad T \geq T_0.$$

Define

$$E_1(t) = \frac{1}{B(t, T_0)} \int_{T_0}^t B(t, s) \phi(s) |\rho(s)| \left| \frac{W(s)}{\phi(s)} - \eta(s) \right| ds$$

and

$$E_2(t) = \frac{1}{B(t, T_0)} \int_{T_0}^t B(t, s) \phi(s) \left| \frac{R_1^{\frac{1}{\gamma}}(t_1, s) Z_H^{\frac{1}{\gamma}}(t - \delta_1) K_H^{\frac{1}{\gamma}}(s)}{r_2^{\frac{2}{\gamma}}(s)} \right|^{\frac{\gamma+1}{\gamma}} \left| \frac{W(s)}{\phi(s)} - \eta(s) \right|^{\frac{\gamma+1}{\gamma}} ds.$$

Then, by (55) and (62), we see that

$$(63) \quad \begin{aligned} & \liminf_{t \rightarrow \infty} [E_2(t) - E_1(t)] \\ & \leq W(T_0) - \limsup_{t \rightarrow \infty} \frac{1}{B(t, t_0)} \int_{T_0}^t B(t, s) \phi(s) Q_1(t, s) ds \\ & \leq W(T_0) - \Theta(T_0) < \infty. \end{aligned}$$

Now, we claim that

$$(64) \quad \int_{T_0}^{\infty} \phi(s) \left| \frac{R_1^{\frac{1}{\gamma}}(t_1, s) Z_H^{\frac{1}{\gamma}}(t - \delta_1) K_H^{\frac{1}{\gamma}}(s)}{r_2^{\frac{2}{\gamma}}(s)} \right| \left| \frac{W(s)}{\phi(s)} - \eta(s) \right|^{\frac{\gamma+1}{\gamma}} ds < \infty.$$

Suppose to the contrary that

$$(65) \quad \int_{T_0}^{\infty} \phi(s) \left| \frac{R_1^{\frac{1}{\gamma}}(t_1, s) Z_H^{\frac{1}{\gamma}}(t - \delta_1) K_H^{\frac{1}{\gamma}}(s)}{r_2^{\frac{2}{\gamma}}(s)} \right| \left| \frac{W(s)}{\phi(s)} - \eta(s) \right|^{\frac{\gamma+1}{\gamma}} ds = \infty.$$

By (59), there exists a positive constant l_1 such that

$$(66) \quad \inf_{s \geq t_0} \left\{ \liminf_{t \rightarrow \infty} \frac{B(t, s)}{B(t, t_0)} \right\} \geq l_1.$$

Letting l_2 be an arbitrary positive number, then it follows from (66) that there exists a $T_1 \geq T_0$ such that

$$(67) \quad \int_{T_0}^t \phi(s) \left| \frac{R_1^{\frac{1}{\gamma}}(t_1, s) Z_H^{\frac{1}{\gamma}}(t - \delta_1) K_H^{\frac{1}{\gamma}}(s)}{r_2^{\frac{2}{\gamma}}(s)} \right| \left| \frac{W(s)}{\phi(s)} - \eta(s) \right|^{\frac{\gamma+1}{\gamma}} ds \geq \frac{l_2}{l_1}, \quad t \geq T_1.$$

Therefore,

$$\begin{aligned} E_2(t) &= \frac{1}{B(t, T_0)} \int_{T_0}^t B(t, s) ds \\ &\quad \cdot \left(\int_{T_0}^s \phi(\xi) \left| \frac{R_1^{\frac{1}{\gamma}}(t_1, \xi) Z_H^{\frac{1}{\gamma}}(t - \delta_1) K_H^{\frac{1}{\gamma}}(\xi)}{r_2^{\frac{2}{\gamma}}(\xi)} \right| \left| \frac{W(\xi)}{\phi(\xi)} - \eta(\xi) \right|^{\frac{\gamma+1}{\gamma}} d\xi \right) \\ &\geq \frac{1}{B(t, T_0)} \int_{T_1}^t \left(\frac{-\partial B(t, s)}{\partial s} \right) ds \\ &\quad \cdot \int_{T_0}^s \phi(\xi) \left| \frac{R_1^{\frac{1}{\gamma}}(t_1, \xi) Z_H^{\frac{1}{\gamma}}(t - \delta_1) K_H^{\frac{1}{\gamma}}(\xi)}{r_2^{\frac{2}{\gamma}}(\xi)} \right| \left| \frac{W(\xi)}{\phi(\xi)} - \eta(\xi) \right|^{\frac{\gamma+1}{\gamma}} d\xi ds \\ &\geq \frac{l_2}{l_1} \frac{1}{B(t, T_0)} \int_{T_1}^t \left(\frac{-\partial B(t, s)}{\partial s} \right) ds \\ &\geq \frac{l_2}{l_1} \frac{1}{B(t, T_0)} B(t, T_1). \end{aligned}$$

By (67), there exists a $T_2 \geq T_1$ such that $\frac{B(t, T_1)}{B(t, T_0)} \geq l_1$ for all $t \geq T_2$, which implies that $E_2(t) \geq l_2$. Since l_2 is arbitrary, then

$$(68) \quad \lim_{t \rightarrow \infty} E_2(t) = \infty.$$

Next, in view of (64), we may consider a sequence $\{T_n\}_{n=1}^\infty$ in $[t_0, \infty)$ satisfying

$$\lim_{n \rightarrow \infty} [E_2(T_n) - E_1(T_n)] = \liminf_{t \rightarrow \infty} [E_2(t) - E_1(t)] < \infty.$$

Then, there exists a constants M such that

$$(69) \quad E_2(T_n) - E_1(T_n) \leq M$$

for all sufficiently large $n \in \mathbb{N}$. Since (69) ensure that

$$(70) \quad \lim_{n \rightarrow \infty} E_2(T_n) = \infty$$

and we have (70) implies that

$$(71) \quad \lim_{n \rightarrow \infty} E_1(T_n) = \infty.$$

Further, (70) and (72) yield the inequalities

$$\frac{E_1(T_n)}{E_2(T_n)} - 1 \geq -\frac{M}{E_2(T_n)} > -\frac{1}{2} \quad \text{or} \quad \frac{E_1(T_n)}{E_2(T_n)} \geq \frac{1}{2}$$

hold for all sufficiently large $n \in \mathbb{N}$. In view of this and (72) we have

$$(72) \quad \lim_{n \rightarrow \infty} \frac{E_1^{\gamma+1}(T_n)}{E_2^\gamma(T_n)} = \infty$$

On the other hand, from the definition of E_1 we can obtain by Hölder's inequality

$$\begin{aligned} E_1(T_n) &\leq \left(\frac{1}{B(T_n, T_0)} \int_{T_0}^{T_n} B(T_n, s) \phi(s) \right. \\ &\quad \left. \left| \frac{R_1^\gamma(t_1, s) Z_H^{\frac{1}{\gamma}}(t - \delta_1) K_H^{\frac{1}{\gamma}}(s)}{r_2^{\frac{2}{\gamma}}(s)} \right| \left| \frac{W(s)}{\phi(s)} - \eta(s) \right|^{\frac{\gamma+1}{\gamma}} ds \right)^{\frac{\gamma}{\gamma+1}} \\ &\quad \times \left(\frac{1}{B(T_n, T_0)} \int_{T_0}^{T_n} B(T_n, s) \phi(s) |\psi(t, s)|^{\gamma+1} ds \right)^{\frac{1}{\gamma+1}} \end{aligned}$$

and accordingly

$$\frac{E_1^{\gamma+1}(T_n)}{E_2^\gamma(T_n)} \leq \frac{1}{B(T_n, T_0)} \int_{T_0}^{T_n} B(T_n, s) \phi(s) |\psi(t, s)|^{\gamma+1} ds.$$

So, because of (73), we have

$$\lim_{n \rightarrow \infty} \frac{1}{B(T_n, T_0)} \int_{T_0}^{T_n} B(T_n, s) \phi(s) |\psi(t, s)|^{\gamma+1} ds = \infty$$

which gives that

$$\limsup_{n \rightarrow \infty} \frac{1}{B(t, T_0)} \int_{T_0}^t B(t, s) \phi(s) |\psi(t, s)|^{\gamma+1} ds = \infty.$$

Contradicting (60). Therefore (65) holds. Now, in view of (63) and (65) we obtain

$$\int_{T_0}^{\infty} \phi(s) \left(\frac{\Theta(s)}{\phi(s)} - \eta(s) \right)_+^{\frac{\gamma+1}{\gamma}} ds \leq \int_{T_0}^{\infty} \phi(s) \left| \frac{W(s)}{\phi(s)} - \eta(s) \right|^{\frac{\gamma+1}{\gamma}} ds < \infty,$$

which contradicts (61). This completes the proof. $\square$

Theorem 3.4. *Let H, ϕ and η be as in Theorem 3.2, suppose that (59) holds and*

$$(73) \quad \liminf_{t \rightarrow \infty} \frac{1}{B(t, t_0)} \int_{t_0}^t B(t, s) \phi(s) |\psi(t, s)|^{\gamma+1} ds < \infty.$$

then (1) is H -oscillatory provided the following condition holds.

There exists $\Theta \in C([t_0, \infty), \mathbb{R})$ such that (61) holds, and for any $T \geq t_0$,

$$(74) \quad \liminf_{t \rightarrow \infty} \frac{1}{B(t, T)} \int_T^t B(t, s) \phi(s) [Q_1(t, s) - |\Lambda(s)| |\psi(t, s)|^{\gamma+1}] ds \geq \Theta(t).$$

Theorem 3.5. *Let H, ϕ and η be as in Theorem 3.1, suppose that (59) holds. Then (1) is H -oscillatory provided the following condition holds*

$$(75) \quad \liminf_{t \rightarrow \infty} \frac{1}{B(t, T)} \int_T^t B(t, s) \phi(s) Q_1(t, s) ds < \infty.$$

Further, suppose that there exists $\phi \in C([t_0, \infty), \mathbb{R})$ such that (61), (75) hold.

Corollary 3.1. *Let H, ϕ and η be as in Theorem 3.2. Then Eq. (1), (2) is H -oscillatory provided that the following condition holds:*

$$\limsup_{t \rightarrow \infty} \frac{1}{B(t, t_0)} \int_{t_0}^t B(t, s) \phi(s) Q_1(t, s) ds = \infty$$

and

$$\limsup_{t \rightarrow \infty} \frac{1}{B(t, t_0)} \int_T^t B(t, s) \left[|\Lambda(s)| |\psi(t, s)| - |s^{\alpha_1-1} \frac{L \sum_{j=1}^k b_j(s) \nu_j}{r_1(s) Z_H^\gamma(t - \delta_1)}| \right] ds \leq \infty.$$

Corollary 3.2. *Eq. (1), (2) is H -oscillatory provided that the following condition holds:*

$$\limsup_{t \rightarrow \infty} \frac{1}{t^n} \int_{t_0}^t (t-s)^n Q_1(t, s) ds = \infty,$$

for some $n > \gamma$.

In this section we establish sufficient condition for the oscillation of all solutions of equations (1), (3). For this we need the following.

The smallest eigen value β_0 of the Dirichlet problem

$$\begin{aligned}\Delta\omega(x) + \beta\omega(x) &= 0 \quad \text{in } \Omega \\ \omega(x) &= 0 \quad \text{on } \partial\Omega\end{aligned}$$

is positive and the corresponding eigen function $\varphi(x)$ is positive in Ω .

Theorem 3.6. *Let all conditions of Theorem 3.2 hold. Assume that $|\varphi(x)| \leq M$ for $x \in \bar{\Omega}$. Then every solution of equations (1), (3) H -oscillatory in G .*

Proof. Suppose to the contrary that there exists a solution $U(x, t)$ of (1), (3) which is not H -oscillatory in G . Without loss of generality we may assume that $u_H(x, t)$ in $\Omega \times [t_0, \infty)$ for some $t_0 > 0$. That is, $V_H(t)$ is an eventually positive solution of (24). There exists a $t_1 \geq t_0$ such that $V_H(t) > 0, V_H(t - \tau) > 0, V_H(t - \delta_1) > 0, V_H(t - \delta_2) > 0$ for $t \geq t_1$. Multiplying both sides of (15) by $\varphi > 0$ and integrating with respect to x over Ω , we obtain for $t \geq t_1$,

$$\begin{aligned}& \int_{\Omega} \frac{\partial^{\alpha_1}}{\partial t^{\alpha_1}} \left(r_2(t) \frac{\partial^{\alpha_2}}{\partial t^{\alpha_2}} \left(r_1(t) |D_{+,t}^{\alpha_3} z_H(x, t)|^{\gamma-1} D_{+,t}^{\alpha_3} z_H(x, t) \right) \right) \varphi(x) dx \\& + \int_{\Omega} p_1(t) |u_H(x, t - \delta_1)|^{\gamma_1-1} u_H(x, t - \delta_1) \varphi(x) dx \\& + \int_{\Omega} p_2(t) |u_H(x, t - \delta_2)|^{\gamma_2-1} u_H(x, t - \delta_2) \varphi(x) dx = \int_{\Omega} a(t) \Delta u_H(x, t) \varphi(x) dx \\& - \int_{\Omega} \sum_{i=1}^m a_i(t) \Delta u_H(x, \rho_i(t)) \varphi(x) dx \\& + \int_{\Omega} \sum_{j=1}^k b_j(t) f_j \left(\int_0^t (t-s)^{-\alpha_3} u_H(x, \sigma_j(s)) \varphi(x) ds \right) u_H(x, \sigma_j(t)) \varphi(x) dx \\(76) \quad & \leq \int_{\Omega} f_H(x, t) \varphi(x) dx.\end{aligned}$$

Using Green's formula and boundary condition (23), it follows that

$$\begin{aligned}& \int_{\Omega} \Delta u_H(x, t) \varphi(x) dx = \int_{\Omega} u_H(x, t) \Delta \varphi(x) dx \\(77) \quad & = -\beta_0 \int_{\Omega} u_H(x, t) \varphi(x) dx \leq 0, \quad t \geq t_1.\end{aligned}$$

Also, from Jensen's inequality, it follows that

$$\begin{aligned}& \int_{\Omega} p_1(t) |u_H(x, t - \delta_1)|^{\gamma_1-1} u_H(x, t - \delta_1) \varphi(x) dx \\& \geq p_1(t) \int_{\Omega} (u_H(x, t - \delta_1))^{\gamma_1} \varphi(x) dx \\& \geq p_1(t) \left(\int_{\Omega} u_H(x, t - \delta_1)^{\gamma_1} (\varphi(x))^{1-\gamma_1+\gamma_1} dx \right)\end{aligned}$$

$$\begin{aligned}
(78) \quad & \geq M^{1-\gamma_1} p_1(t) (V_H(t - \delta_1))^{\gamma_1}, \quad t \geq T, \\
& \int_{\Omega} p_2(t) |u_H(x, t - \delta_2)|^{\gamma_2-1} u_H(x, t - \delta_2) \varphi(x) dx \\
(79) \quad & \geq M^{1-\gamma_2} p_2(t) (V_H(t - \delta_2))^{\gamma_2}, \quad t \geq T.
\end{aligned}$$

Set

$$(80) \quad V_H(t) = \int_{\Omega} u_H(x, t) \varphi(x) dx, \quad t \geq t_1.$$

In view of (78)-(81) and (A_8) , (77) yield

$$\begin{aligned}
& \left(r_2(t) (r_1(t) (z'(t))^{\gamma})' \right)' \\
(81) \quad & + M^{1-\gamma_1} p_1(t) (V(t - \delta_1))^{\gamma_1} + M^{1-\gamma_2} p_2(t) (V(t - \delta_2))^{\gamma_2} \leq 0, \quad t \geq T.
\end{aligned}$$

Rest of the proof is similar to that of Theorem 2.2 and hence the details are omitted. $\square$

Theorem 3.7. *Let the conditions of Theorem 3.2 hold, then every solution $U(x, t)$ of (1), (3) is H -oscillatory in G .*

Theorem 3.8. *Let the conditions of Theorem 3.3 hold, then every solution $U(x, t)$ of (1), (3) is H -oscillatory in G .*

Theorem 3.9. *Let the conditions of Theorem 3.4 hold, then every solution $U(x, t)$ of (1), (3) is H -oscillatory in G .*

Theorem 3.10. *Let the conditions of Theorem 3.5 hold, then every solution $U(x, t)$ of (1), (3) is H -oscillatory in G .*

4. Example

In this section, we give an example to illustrate our main results.

Example 4.1.

$$\begin{aligned}
& \frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}} \left(\frac{\partial^{\frac{1}{4}}}{\partial t^{\frac{1}{4}}} \left(t^{\frac{2}{3}} D_{+,t}^{\frac{1}{3}} Z(x, t) \right) \right) + \frac{\sqrt{3}+1}{2} \frac{17}{12} t^{\frac{11}{12}} |U(x, t - \pi)|^{\frac{2}{3}-1} U(x, t - \pi) \\
& + \frac{\sqrt{3}+1}{2} t^{\frac{23}{12}} |U(x, t - \frac{\pi}{2})|^{\frac{5}{3}-1} U(x, t - \frac{\pi}{2}) + \frac{\sqrt{3}-1}{2} t^{\frac{23}{12}} \Delta U(x, t) \\
& + \frac{\sqrt{3}-1}{3} \frac{5}{12} t^{\frac{-1}{12}} \Delta U(x, t - \pi) \\
& - \frac{2}{9} t^{\frac{1}{4}} \left(\int_0^t (t-s)^{-\frac{1}{3}} ||U(x, t - \frac{\pi}{2})|| ds \right) U(x, t - \frac{\pi}{2}) \\
(82) \quad & + F(x, t), \quad (x, t) \in \Omega \times \mathbb{R}_+,
\end{aligned}$$

$(x, t) \in G$ where $G = (0, \pi) \times (0, \pi) \times (0, \infty)$, with the boundary condition

$$U(0, t) = \begin{pmatrix} u_1(0, t) \\ u_2(0, t) \end{pmatrix} = U(\pi, t) = \begin{pmatrix} u_1(\pi, t) \\ u_2(\pi, t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$t \geq 0$. Here $\alpha_1 = \frac{1}{2}, \alpha_2 = \frac{1}{4}, \alpha_3 = \frac{1}{3}, r_1(t) = t^{\frac{2}{3}}, r_2(t) = 1, \gamma_1 = \frac{2}{3}, \gamma_2 = 1, q(t) = 1, \tau = \frac{3\pi}{2}, f_1(u) = u, a(t) = \frac{\sqrt{3}-1}{2}t^{\frac{23}{12}}, a_1(t) = \frac{\sqrt{3}-1}{3}\frac{5}{12}t^{\frac{-1}{12}}, b(t) = \frac{2}{9}t^{\frac{1}{4}}, \delta_1, \rho_1 = \pi, \delta_2, \sigma_1 = t - \frac{\pi}{2}, p_1(t) = \min_{x \in \bar{\Omega}} p_1(x, t) = \frac{\sqrt{3}+1}{2}\frac{17}{12}t^{\frac{11}{12}}, p_2(t) = \min_{x \in \bar{\Omega}} p_2(x, t) = \frac{\sqrt{3}+1}{2}t^{\frac{23}{12}},$

$$H = e_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

we observe that

$$\begin{aligned} f_{e_1}(x, t) &= \left[\frac{\sqrt{3}}{3}t^{\frac{11}{12}} + \frac{\sqrt{3}-1}{2}\frac{17}{12}t^{\frac{11}{12}} + \frac{\sqrt{3}+1}{3}\frac{5}{12}t^{\frac{-1}{12}} \right] \sin x \cos t - \frac{\sqrt{3}+1}{3}t^{\frac{11}{12}} \sin x \sin t, \\ \int_{\Omega} f_{e_1}(x, t) dx &\leq \left[\frac{2\sqrt{3}}{3}t^{\frac{11}{12}} + \sqrt{3} - 1\frac{17}{12}t^{\frac{11}{12}} + \frac{\sqrt{3}+1}{3}\frac{5}{6}t^{\frac{-1}{12}} \right] \cos t \leq 0, \frac{\pi}{2} \leq t \leq \frac{3\pi}{2}, \\ F(x, t) &= \begin{pmatrix} \left[\frac{\sqrt{3}}{3}t^{\frac{11}{12}} + \frac{\sqrt{3}-1}{2}\frac{17}{12}t^{\frac{11}{12}} + \frac{\sqrt{3}+1}{3}\frac{5}{12}t^{\frac{-1}{12}} \right] \sin x \cos t - \frac{\sqrt{3}+1}{3}t^{\frac{11}{12}} \sin x \sin t \\ \frac{\Gamma_{\frac{1}{3}}}{12\pi\sqrt{3}}t^{\frac{-5}{12}} + 3t^{\frac{2}{3}} \end{pmatrix}. \end{aligned}$$

Take $n = 2, h(t, s) = (t - s), \phi(t) = 1, \phi'(t) = 0, \eta(t) = 1$. It is clear that $(A_1) - (A_{11})$ hold. Thus all the conditions of Corollary 3.2 are satisfied. Therefore, every solution $U(x, t)$ of (83) with the boundary condition is e_1 -oscillatory in G . Infact,

$$U(x, t) = \begin{pmatrix} \sin x \sin t \\ 2 \end{pmatrix}$$

is one such solution of the problem (82) with the boundary condition. We note that the above solution $U(x, t)$ is not e_2 -oscillatory in G , where

$$e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

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Numerical solution of nonlinear time-fractional Cable equation by finite volume element method

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Abstract. In this paper, we study the use of finite volume element method and see how we can employ finite element bases to numerical solution of nonlinear time fractional Cable equation in one dimension. Finite element bases are as linear polynomials which are compactly supported. Caputo in the proposed scheme constitutes the fractional derivative. This scheme will be analyzed and the stability of the method will be discussed. Furthermore, the numerical results are compared with analytical solution to ascertain how accurate the presented method is.

Keywords: finite volume element method, nonlinear systems, Cable equation, Caputo derivative, Von-Neumann stability analysis.

1. Introduction

In this paper, the nonlinear fractional Cable equation is denoted:

$$(1) \quad \begin{cases} u_t + {}_0^C D_t^\vartheta u(x, t) - {}_0^C D_t^\vartheta u_{xx}(x, t) + f(u) = s(x, t), & \text{in } I \times (0, T], \\ u(0, t) = g_1(t), & t \in (0, T], \\ u(1, t) = g_2(t), & t \in (0, T], \\ u(x, 0) = u_0(x), & \text{in } I. \end{cases}$$

In $0 < \vartheta < 1$ and $I = (0, 1)$ be the computational domain $s(x, t)$, where $s(x, t)$ is a continuous function, that we define as source term. Let subscripts x and t are the space and time differentiations, respectively, and superscript ϑ is the order of fractional derivative. Here the fractional derivatives is in the Caputo

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sense [6], i.e.

$$(2) \quad \frac{\partial^\vartheta u(x, t)}{\partial t^\vartheta} = \frac{1}{\Gamma(1 - \vartheta)} \int_0^t \frac{\partial u(x, \xi)}{\partial \xi} \frac{d\xi}{(t - \xi)^\vartheta}, \quad 0 < \vartheta < 1$$

in which $\Gamma(\vartheta)$ is the Gamma function.

Different initial-boundary value problems including Cable equation, have been solved analytically by many authors [9, 8, 7, 2, 3].

In this paper, the time fractional Cable equation is considered as an initial-boundary value problem to solve it numerically. The scheme illustrated here is finite difference method to discretize the time fractional derivative, along with finite volume element method with linear B-spline basis function to discretize in space scale. The time fractional derivative is in the Caputo sense, and will discretize by a backward finite difference formula. The resulting proposed scheme will be shown to be unconditionally stable.

This paper is outlined as follows. In Section 2, the numerical method is described. In Section 3, the stability of the scheme is investigated. In Section 4, the recommended scheme is demonstrated with the help of some numerical test examples, and in the end, Section 5 provides us with conclusions.

2. Finite volume element method

The finite volume element method (FVEM), which in fact is a variation of the finite volume method (FVM), is also regarded as a Petrov-Galerkin finite element method (FEM).

Here, we can denote some preliminaries around FVEM and Sobolev spaces.

Let $I = (a, b) \subset \mathbf{R}$ be a domain, we will employ the standard notation for Sobolev spaces $H^s(I)$, $s \geq 0$, making up functions v which include generalized derivatives of order s in the spaces $L_2(I)$ for all $|\alpha| \leq s$. The norm and seminorm of $H^s(I)$ are defined by

$$\|v\|_s = \left(\sum_{|\alpha| \leq s} |D^\alpha v|^2 \right)^{\frac{1}{2}}, \quad |v|_s = \left(\sum_{|\alpha|=s} |D^\alpha v|^2 \right)^{\frac{1}{2}}.$$

Parallel with the Sobolev space $H^s(I)$, we have the space $H_0^1(I)$ the subspace of $H^1(I)$ which include functions that disappear on the boundary of I , i.e.,

$$H_0^1(I) = \{v \in H^1(I), I = [a, b], v|_{\partial I} = 0\}.$$

To clarify the proposed numerical method [1], let's focus on the equation (1). We divide the domain I into finite elements $I_i = [x_{i-1}, x_i]$ with the nodes $x_i, i = 0, 1, 2, \dots, M$ and set $h_i = x_i - x_{i-1}$. Let a primal partition of I with elements I_i denotes by $\zeta_h : a = x_0 < x_1 < x_2 < \dots < x_M = b$. Set element $I_i = [x_{i-1}, x_i]$, midpoint $x_{i-\frac{1}{2}} = (x_i + x_{i-1})/2$. Let $I_0^* = [0, x_{\frac{1}{2}}]$, $I_i^* = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$ ($i = 1, \dots, M-1$) and $I_M^* = [x_{M-\frac{1}{2}}, 1]$ compose the dual partition ζ_i^* of ζ_i . All I_i^* ($i = 0, 1, 2, \dots, M$) called control volumes.

Consider the trial and test spaces S_h as the linear element space with basis functions at the point x_i

$$(3) \quad \phi_i(x) = \begin{cases} \frac{x - x_{i-1}}{h_i}, & x_{i-1} \leq x \leq x_i, \\ \frac{x_{i+1} - x}{h_{i+1}}, & x_i \leq x \leq x_{i+1}, \\ 0, & \text{otherwise,} \end{cases} \quad 1 \leq i \leq M-1$$

and

$$(4) \quad \begin{aligned} \phi_0(x) &= \begin{cases} \frac{x - x_1}{x_0 - x_1}, & x_0 \leq x \leq x_1, \\ 0, & \text{otherwise,} \end{cases} \\ \phi_M(x) &= \begin{cases} \frac{x - x_{M-1}}{x_M - x_{M-1}}, & x_{M-1} \leq x \leq x_M, \\ 0, & \text{otherwise,} \end{cases} \end{aligned}$$

and S_h^* as piecewise constant function space with basis function of the point x_i

$$(5) \quad \chi_i(x_j) = \begin{cases} 1, & x_j \in I_i^*, \\ 0, & \text{otherwise.} \end{cases}$$

The choice of different solution spaces, test spaces, and dual partitions leads to different FVE methods [4].

The linear basis functions $\phi_i(x)$ and their first derivatives vanish outside the interval $[x_{i-1}, x_{i+1}]$. We can observe the values of $\phi_i(x)$ and its first derivative $\phi'_i(x)$ at the knots in Table 1.

Table 1: Quantities of basis functions and their first derivatives in points $x_{i-\frac{1}{2}}$ and $x_{i+\frac{1}{2}}$.

	$x_{i-\frac{1}{2}}$	$x_{i+\frac{1}{2}}$		$x_{i-\frac{1}{2}}$	$x_{i+\frac{1}{2}}$
ϕ_{i-1}	$\frac{1}{2}$	0	ϕ'_{i-1}	$\frac{1}{h}$	0
ϕ_i	$\frac{1}{2}$	$\frac{1}{2}$	ϕ'_i	$\frac{1}{h}$	$-\frac{1}{h}$
ϕ_{i+1}	0	$\frac{1}{2}$	ϕ'_{i+1}	0	$-\frac{1}{h}$

The standard Lagrangian interpolation operator is also defined

$$(6) \quad I_h : C(I) \rightarrow S_h, \quad s.t. \quad I_h u = \sum_{i=0}^M u_i \phi_i(x), \quad \forall u \in C(I),$$

$$(7) \quad I_h^* : C(I) \rightarrow S_h^*, \quad s.t. \quad I_h^* u = \sum_{i=0}^M u_i \chi_i(x), \quad \forall u \in C(I),$$

where $u_i = u(x_i)$.

Regarding basis functions ϕ_i , each $u_h(x, t) \in S_h$ may be written as

$$(8) \quad u_h(x, t) = \sum_{i=0}^M \delta_i(t) \phi_i(x), \quad t > 0,$$

where the time-dependent coefficients $\delta_i(t)$ are unknown and, with the help of the finite volume element method, we can compute them from the initial and boundary conditions.

Afterwards, we can define the FVE approximation u_h of (1) as a solution to the following problem: Find $u_h \in S_h$, for each $t > 0$, such that for all control volume I_i^* , $i = 0, 1, 2, \dots, M$

$$(9) \quad \left(\frac{\partial u_h}{\partial t}, I_h^* v_h \right) + \left(\frac{\partial^\vartheta u_h}{\partial t^\vartheta}, I_h^* v_h \right) + a(u_h, I_h^* v_h) = (s, I_h^* v_h), \quad I_h^* \in S_h,$$

We here define the bilinear form $a(u_h, I_h^* v_h)$ as follows:

$$a(u_h, I_h^* v_h) = (I_h f(u_h), I_h^* v_h) - \left(\frac{\partial^\vartheta (u_h)_{xx}}{\partial t^\vartheta}, I_h^* v_h \right),$$

where $(\cdot, \cdot)$ is inner product in $L_2(I)$ and $f(u) = u^2$.

Explaining the FVE method for problem (1), we can verify that

$$\begin{aligned} \int_I \frac{\partial u_h}{\partial t} \chi_i dx + \int_I \frac{\partial^\vartheta u_h}{\partial t^\vartheta} \chi_i dx + \int_I I_h(u_h)^2 \chi_i dx - \int_I \frac{\partial^\vartheta (u_h)_{xx}}{\partial t^\vartheta} \chi_i dx \\ = \int_I s \chi_i dx, \quad i = 0, 1, 2, \dots, M. \end{aligned}$$

According to definition of χ_i , $i = 0, 1, 2, \dots, M$, we have

$$(10) \quad \begin{aligned} \int_{I_i^*} \frac{\partial u_h}{\partial t} dx + \int_{I_i^*} \frac{\partial^\vartheta u_h}{\partial t^\vartheta} dx + \int_{I_i^*} I_h(u_h)^2 dx - \int_{I_i^*} \frac{\partial^\vartheta (u_h)_{xx}}{\partial t^\vartheta} dx \\ = \int_{I_i^*} s dx, \quad i = 0, 1, 2, \dots, M. \end{aligned}$$

Let $\delta_i = \delta_i(t)$, $\delta_i^{(\vartheta)} = \frac{\partial^\vartheta \delta_i(t)}{\partial t^\vartheta}$ and $f_i(t) = f(u_h(x_i, t))$. If we apply (8) to every control volume I_i^* and use (2) and Table 1, we have

$$(11) \quad \int_{I_i^*} \frac{\partial u_h}{\partial t} dx = \frac{h}{8} \{ \delta'_{i-1} + 6\delta'_i + \delta'_{i+1} \},$$

and

$$(12) \quad \int_{I_i^*} \frac{\partial^\vartheta u_h}{\partial t^\vartheta} dx = \frac{h}{8} \{ \delta_i^\vartheta_{i-1} + 6\delta_i^\vartheta + \delta_i^\vartheta_{i+1} \},$$

and

$$(13) \quad \int_{I_i^*} \frac{\partial^\vartheta (u_h)_{xx}}{\partial t^\vartheta} dx = \frac{1}{h} \{\delta_{i-1}^\vartheta - 2\delta_i^\vartheta + \delta_{i+1}^\vartheta\},$$

and

$$(14) \quad \int_{I_i^*} I_h(u_h)^2 dx = \frac{h}{24} \{\delta_{i-1}^2 + 14\delta_i^2 + \delta_{i+1}^2\}.$$

If we substitute (11), (12), (13) and (14) in (10), we can reach the semi-discrete form of (10) as a formula we see below

$$(15) \quad \begin{aligned} & \frac{h}{8} \{(\delta'_{i-1} + \delta_{i-1}^\vartheta) + 6(\delta'_i + \delta_i^\vartheta) + (\delta'_{i+1} + \delta_{i+1}^\vartheta)\} \\ & + \frac{h}{24} \{\delta_{i-1}^2 + 14\delta_i^2 + \delta_{i+1}^2\} - \frac{1}{h} \{\delta_{i-1}^\vartheta - 2\delta_i^\vartheta + \delta_{i+1}^\vartheta\} \\ & = \int_{I_i^*} s dx, \quad i = 1, 2, \dots, M-1. \end{aligned}$$

We have also for I_0^* and I_M^* , respectively,

$$(16) \quad \begin{aligned} & \frac{3h}{8} (\delta'_0 + \delta_0^{(\vartheta)}) + \frac{h}{8} (\delta'_1 + \delta_1^{(\vartheta)}) + \frac{h}{24} \{7\delta_0^2 + \delta_1^2\} - \frac{1}{h} \{\delta_1^\vartheta - \delta_0^\vartheta\} = \int_{I_0^*} s dx, \\ & \frac{h}{8} (\delta'_{M-1} + \delta_{M-1}^{(\vartheta)}) + \frac{3h}{8} (\delta'_M + \delta_M^{(\vartheta)}) + \frac{h}{24} \{7\delta_M^2 + \delta_{M-1}^2\} - \frac{1}{h} \{\delta_{M-1}^\vartheta - \delta_M^\vartheta\} \\ & = \int_{I_M^*} s dx. \end{aligned}$$

We define $\Delta t = \tau = t_k - t_{k-1}, k = 1, 2, \dots, n, t_k = k\tau$ and $x_i = ih, i = 0, 1, \dots, M$ where τ and h are time and space steps, respectively. We can consider $\delta_i^k := u_h(x_i, t_k)$ as the approximation value of $u(x_i, t_k)$ by FVEM.

The time fractional derivative is discretized with the help of the Caputo derivative formulation for $\delta_i^{(\vartheta)}(t)$, as seen below

$$(17) \quad \begin{aligned} & \frac{\partial^\vartheta \delta_i(t_{k+1})}{\partial t^\vartheta} = \frac{1}{\Gamma(1-\vartheta)} \sum_{j=0}^k \int_{t_j}^{t_{j+1}} (t_{k+1} - \xi)^{-\vartheta} \frac{\partial \delta(x_i, \xi)}{\partial \xi} d\xi \\ & \simeq \frac{1}{\Gamma(1-\vartheta)} \sum_{j=0}^k \frac{1}{\tau} [\delta(x_i, t_{j+1}) - \delta(x_i, t_j)] \int_{t_j}^{t_{j+1}} (t_{k+1} - \xi)^{-\vartheta} d\xi \\ & = \frac{1}{(1-\vartheta)\Gamma(1-\vartheta)\tau^\vartheta} \sum_{j=0}^k \Theta_{k-j} [\delta(x_i, t_{j+1}) - \delta(x_i, t_j)] \\ & = \psi_{\vartheta, \tau} \sum_{s=0}^k \Theta_s [\delta_i^{k-s+1} - \delta_i^{k-s}], \end{aligned}$$

where $\psi_{\vartheta,\tau} = \frac{\tau^{-\vartheta}}{\Gamma(2-\vartheta)}$ and $\Theta_s = (s+1)^{1-\vartheta} - s^{1-\vartheta}$, $s = 0, 1, \dots, n$.

Truncation error of the approximation (17) is denoted by

$$(18) \quad R_{\tau}^{k+1} := \frac{\partial^{\vartheta} \delta_i(t_{k+1})}{\partial t^{\vartheta}} - \psi_{\vartheta,\tau} \sum_{s=0}^k \Theta_s [\delta_i^{k-s+1} - \delta_i^{k-s}],$$

and we have [5]

$$(19) \quad R_{\tau}^{k+1} \leq C_u \tau^{2-\vartheta},$$

in which C_u is a constant only related to u .

Lemma 1 ([7]). For coefficients Θ_s , we have

1. $\Theta_0 = 1$,
2. $\Theta_s > 0$, $s = 0, 1, \dots, n$,
3. $\Theta_s > \Theta_{s+1}$, $s = 0, 1, \dots, n-1$.

If we substitute (17) in (15), for a uniform partition $h_i = h$, the fully-discrete formula of the equation (1) can be obtained as the form below

$$(20) \quad \begin{aligned} & \frac{h}{8\tau} \{(\delta_{i-1}^{k+1} - \delta_{i-1}^k) + 6(\delta_i^{k+1} - \delta_i^k) + (\delta_{i+1}^{k+1} - \delta_{i+1}^k)\} \\ & + r \sum_{s=0}^k \Theta_s \{[\delta_{i-1}^{k-s+1} - \delta_{i-1}^{k-s}] - 2[\delta_i^{k-s+1} - \delta_i^{k-s}] + [\delta_{i+1}^{k-s+1} - \delta_{i+1}^{k-s}]\} \\ & - \iota \sum_{s=0}^k \Theta_s \{[\delta_{i-1}^{k-s+1} - \delta_{i-1}^{k-s}] + 6[\delta_i^{k-s+1} - \delta_i^{k-s}] + [\delta_{i+1}^{k-s+1} - \delta_{i+1}^{k-s}]\} \\ & + \frac{h}{24} \{(\delta^2)_{i-1}^{k+1} + 14(\delta^2)_i^{k+1} + (\delta^2)_{i+1}^{k+1}\} = \int_{I_i^*} s(x, t_{k+1}), i=1, 2, \dots, M-1, \end{aligned}$$

where $r = \frac{h\tau^{-\vartheta}}{8\Gamma(2-\vartheta)}$ and $\iota = \frac{\tau^{-\vartheta}}{h\Gamma(2-\vartheta)}$.

Then, (20) can result in the following recurrence difference formula based on the time-dependent parameters δ_i^k

$$(21) \quad \begin{aligned} & \left(\frac{h}{8\tau} + r - \iota\right) \delta_{i-1}^{k+1} + \left(\frac{6h}{8\tau} + 6r + 2\iota\right) \delta_i^{k+1} + \left(\frac{h}{8\tau} + r - \iota\right) \delta_{i+1}^{k+1} \\ & + \frac{h}{24} \{(\delta^2)_{i-1}^{k+1} + 14(\delta^2)_i^{k+1} + (\delta^2)_{i+1}^{k+1}\} \\ & = \left(\frac{h}{8\tau} + r - \iota\right) \delta_{i-1}^k + \left(\frac{6h}{8\tau} + 6r + 2\iota\right) \delta_i^k + \left(\frac{h}{8\tau} + r - \iota\right) \delta_{i+1}^k \\ & + r \sum_{s=1}^k \Theta_s \{[\delta_{i-1}^{k-s+1} - \delta_{i-1}^{k-s}] + 6[\delta_i^{k-s+1} - \delta_i^{k-s}] + [\delta_{i+1}^{k-s+1} - \delta_{i+1}^{k-s}]\} \\ & - \iota \sum_{s=1}^k \Theta_s \{[\delta_{i-1}^{k-s+1} - \delta_{i-1}^{k-s}] - 2[\delta_i^{k-s+1} - \delta_i^{k-s}] \\ & + [\delta_{i+1}^{k-s+1} - \delta_{i+1}^{k-s}]\} + \int_{I_i^*} s(x, t_{k+1}). \end{aligned}$$

When $k = 0$, i.e. in the first temporal step, the scheme simply reads

$$\begin{aligned}
 & \left(\frac{h}{8\tau} + r - \iota \right) \delta_{i-1}^1 + \left(\frac{6h}{8\tau} + 6r + 2\iota \right) \delta_i^1 + \left(\frac{h}{8\tau} + r - \iota \right) \delta_{i+1}^1 \\
 (22) \quad & + \frac{h}{24} \{ (\delta^2)_{i-1}^1 + 14(\delta^2)_i^1 + (\delta^2)_{i+1}^1 \} = \left(\frac{h}{8\tau} + r - \iota \right) \delta_{i-1}^0 \\
 & + \left(\frac{6h}{8\tau} + 6r + 2\iota \right) \delta_i^0 + \left(\frac{h}{8\tau} + r - \iota \right) \delta_{i+1}^0 + \int_{I_i^*} s(x, t_1), \quad i = 1, 2, \dots, M-1.
 \end{aligned}$$

Similarly, for $i = 0$ and $i = M$ we have respectively

$$\begin{aligned}
 & \left(\frac{3h}{8\tau} + 3r + \iota \right) \delta_0^1 + \left(\frac{h}{8\tau} + r - \iota \right) \delta_1^1 + \frac{h}{24} \{ (7(\delta^2)_0^1 + (\delta^2)_1^1) \} \\
 (23) \quad & = \left(\frac{3h}{8\tau} + 3r + \iota \right) \delta_0^0 + \left(\frac{h}{8\tau} + r - \iota \right) \delta_1^0 + \int_{I_0^*} s(x, t_1),
 \end{aligned}$$

$$\begin{aligned}
 & \left(\frac{h}{8\tau} + r - \iota \right) \delta_{M-1}^1 + \left(\frac{3h}{8\tau} + 3r + \iota \right) \delta_M^1 + \frac{h}{24} \{ (\delta^2)_{M-1}^1 + 7(\delta^2)_M^1 \} \\
 (24) \quad & = \left(\frac{h}{8\tau} + r - \iota \right) \delta_{M-1}^0 + \left(\frac{3h}{8\tau} + 3r + \iota \right) \delta_M^0 + \int_{I_M^*} s(x, t_1).
 \end{aligned}$$

Afterwards, the nonlinear system of equations is obtained as a complete discrete formula, which we can see below

$$(25) \quad \left\{ \begin{aligned}
 & \left(\frac{3h}{8\tau} + 3r + \iota \right) \delta_0^1 + \left(\frac{h}{8\tau} + r - \iota \right) \delta_1^1 + \frac{h}{24} \{ (7(\delta^2)_0^1 + (\delta^2)_1^1) \} \\
 & = \left(\frac{3h}{8\tau} + 3r + \iota \right) \delta_0^0 + \left(\frac{h}{8\tau} + r - \iota \right) \delta_1^0 + \int_{I_0^*} s(x, t_1), \quad i = 0, \\
 & \left(\frac{h}{8\tau} + r - \iota \right) \delta_{i-1}^1 + \left(\frac{6h}{8\tau} + 6r + 2\iota \right) \delta_i^1 + \left(\frac{h}{8\tau} + r - \iota \right) \delta_{i+1}^1 \\
 & + \frac{h}{24} \{ (\delta^2)_{i-1}^1 + 14(\delta^2)_i^1 + (\delta^2)_{i+1}^1 \} \\
 & = \left(\frac{h}{8\tau} + r - \iota \right) \delta_{i-1}^0 + \left(\frac{6h}{8\tau} + 6r + 2\iota \right) \delta_i^0 + \left(\frac{h}{8\tau} + r - \iota \right) \delta_{i+1}^0 \\
 & + \int_{I_i^*} s(x, t_1), \quad i = 1, 2, \dots, M-1, \\
 & \left(\frac{h}{8\tau} + r - \iota \right) \delta_{M-1}^1 + \left(\frac{3h}{8\tau} + 3r + \iota \right) \delta_M^1 + \frac{h}{24} \{ (\delta^2)_{M-1}^1 + 7(\delta^2)_M^1 \} \\
 & = \left(\frac{h}{8\tau} + r - \iota \right) \delta_{M-1}^0 + \left(\frac{3h}{8\tau} + 3r + \iota \right) \delta_M^0 + \int_{I_M^*} s(x, t_1), \quad i = M.
 \end{aligned} \right.$$

We can iteratively solve the system (25) by using Newton iteration method for the nonlinear systems. To begin, we must calculate the initial value δ^0 with

the help of the initial condition of the equation (1). Afterwards, we will be capable of determining the approximate solution at new time levels by using the time-dependent parameters δ^1 which are computed from (25).

By rearranging the equation (21) and $\lambda = \frac{\tau^{-\vartheta}}{\Gamma(2-\vartheta)}$ we have

$$\begin{aligned}
 & \left(\frac{h}{8\tau} + \left(\beta + \frac{2}{h}\right)\lambda\right)\delta_{i-1}^{k+1} + \left(\frac{6h}{8\tau} + \left(6\beta + \frac{8}{h}\right)\lambda\right)\delta_i^{k+1} + \left(\frac{h}{8\tau} + \beta\lambda\right)\delta_{i+1}^{k+1} \\
 & + \frac{h}{24}\{(\delta^2)_{i-1}^{k+1} + 14(\delta^2)_i^{k+1} + (\delta^2)_{i+1}^{k+1}\} \\
 (26) \quad & = \left(\frac{h}{8\tau} + \left(\beta + \frac{2}{h}\right)\lambda\right)\delta_{i-1}^k + \left(\frac{6h}{8\tau} + \left(6\beta + \frac{8}{h}\right)\lambda\right)\delta_i^k + \left(\frac{h}{8\tau} + \beta\lambda\right)\delta_{i+1}^k \\
 & + \lambda \sum_{s=1}^k \Theta_s \{ \beta[\delta_{i-1}^{k-s+1} - \delta_{i-1}^{k-s}] + (6\beta + \frac{8}{h})[\delta_i^{k-s+1} - \delta_i^{k-s}] + \beta[\delta_{i+1}^{k-s+1} - \delta_{i+1}^{k-s}] \} \\
 & + \int_{I_i^*} s(x, t_{k+1}),
 \end{aligned}$$

where $\beta = \frac{h^2-8}{8h}$.

For $k > 0$ we have

$$\begin{aligned}
 & \left(\frac{h}{8\tau} + \left(\beta + \frac{2}{h}\right)\lambda\right)\delta_{i-1}^{k+1} + \left(\frac{6h}{8\tau} + \left(6\beta + \frac{8}{h}\right)\lambda\right)\delta_i^{k+1} + \left(\frac{h}{8\tau} + \beta\lambda\right)\delta_{i+1}^{k+1} \\
 & + \frac{h}{24}\{(\delta^2)_{i-1}^{k+1} + 14(\delta^2)_i^{k+1} + (\delta^2)_{i+1}^{k+1}\} \\
 (27) \quad & = \left(\frac{h}{8\tau} + \left(\frac{2}{h} + \beta(1 + \Theta_1)\right)\lambda\right)\delta_{i-1}^k + \left(\frac{6h}{8\tau} + \left(6\beta + \frac{8}{h} + \Theta_1\beta\right)\lambda\right)\delta_i^k \\
 & + \left(\frac{h}{8\tau} + \lambda\beta(1 + \Theta_1)\right)\delta_{i+1}^k \\
 & - \lambda \sum_{s=2}^k (\Theta_{s-1} - \Theta_s) \left(\beta\delta_{i-1}^{k-s+1} + \left(6\beta + \frac{8}{h}\right)\delta_i^{k-s+1} + \beta\delta_{i+1}^{k-s+1} \right) \\
 & - \lambda\Theta_k \left(\left(\beta\delta_{i-1}^0 + \left(6\beta + \frac{8}{h}\right)\delta_i^0 + \beta\delta_{i+1}^0\right) + \int_{I_i^*} s(x, t_{k+1}) \right).
 \end{aligned}$$

3. Stability analysis

Now the Von-Neumann stability analysis [10] and mathematical induction can be used to investigate the stability conditions of the numerical method at hand.

Theorem 3.1. *The numerical scheme (21) to solve the initial and boundary value problem (1) is unconditionally stable.*

Proof. To investigate the stability conditions of the present numerical method, we use the Von-Neumann stability analysis and mathematical induction. First of all, the nonlinear term u^2 is linearized by taking the solution σ as a constant and, after that, we let $s(x, t) = 0$. As the error of the method has only to do

with the time parameters δ_m^k , denoting $e^k := \delta^{k+1} - \delta^k$ as the error of scheme at time level k , we can rewrite the numerical scheme (21) as

$$\begin{aligned}
 & \left(\frac{h}{8\tau} + r - \iota + \frac{h\sigma}{24}\right)\delta_{m-1}^{k+1} + \left(\frac{6h}{8\tau} + 6r + 2\iota + \frac{14h\sigma}{24}\right)\delta_m^{k+1} + \left(\frac{h}{8\tau} + r - \iota + \frac{h\sigma}{24}\right)\delta_{m+1}^{k+1} \\
 &= \left(\frac{h}{8\tau} + r - \iota\right)\delta_{m-1}^k + \left(\frac{6h}{8\tau} + 6r + 2\iota\right)\delta_m^k + \left(\frac{h}{8\tau} + r - \iota\right)\delta_{m+1}^k \\
 (28) \quad &+ r \sum_{s=1}^k \Theta_s \{[\delta_{m-1}^{k-s+1} - \delta_{m-1}^{k-s}] + 6[\delta_m^{k-s+1} - \delta_m^{k-s}] + [\delta_{m+1}^{k-s+1} - \delta_{m+1}^{k-s}]\} \\
 &- \iota \sum_{s=1}^k \Theta_s \{[\delta_{m-1}^{k-s+1} - \delta_{m-1}^{k-s}] - 2[\delta_m^{k-s+1} - \delta_m^{k-s}] + [\delta_{m+1}^{k-s+1} - \delta_{m+1}^{k-s}]\}.
 \end{aligned}$$

By definition $a = (\frac{6h}{8\tau} + 6r + 2\iota + \frac{14h\sigma}{24})$, $b = (\frac{h}{8\tau} + r - \iota + \frac{h\sigma}{24})$ and $c = (\frac{h}{8\tau} + r - \iota + \frac{h\sigma}{24})$, $d = (6r - 2\iota)$, $e = (r - \iota)$ and $f = (r - \iota)$ the system of (28) can be converted into the matrix form as follows

$$(29) \quad \mathbf{T}\delta^{k+1} = \mathbf{U}[\Theta_k\delta^0 + \sum_{s=0}^{k-1}(\Theta_s - \Theta_{s+1})\delta^{k-s}],$$

where

$$\mathbf{T} = \begin{pmatrix} a & c & & & \\ b & a & c & & \\ & b & a & c & \\ & & \ddots & \ddots & \ddots \\ & & & b & a & c \\ & & & & b & a & c \\ & & & & & b & a \end{pmatrix}, \mathbf{U} = \begin{pmatrix} d & f & & & \\ e & d & f & & \\ & e & d & f & \\ & & \ddots & \ddots & \ddots \\ & & & e & d & f \\ & & & & e & d & f \\ & & & & & e & d \end{pmatrix}.$$

Then substituting the Fourier mode $e_m^k = v^k e^{im\rho}$ ($i := \sqrt{-1}$) into (28). We have

$$\begin{aligned}
 & v^{k+1} \left\{ \left(\frac{h}{8\tau} + r - \iota + \frac{h\sigma}{24}\right)e^{-i\rho} + \left(\frac{6h}{8\tau} + 6r + 2\iota + \frac{14h\sigma}{24}\right) + \left(\frac{h}{8\tau} + r - \iota + \frac{h\sigma}{24}\right)e^{i\rho} \right\} \\
 &= v^k \{ (r - \iota)e^{-i\rho} + (6r - 2\iota) + (r - \iota)e^{i\rho} \} \\
 &+ \sum_{s=1}^{k-1} \Theta_s \{ (v^{k-j+1} - v^{k-j}) \times ((r - \iota)e^{-i\rho} + (6r - 2\iota) + (r - \iota)e^{i\rho}), \}
 \end{aligned}$$

or

$$(30) \quad v^{k+1} = Q \{ \Theta_k v^0 + \sum_{s=0}^{k-1} (\Theta_s - \Theta_{s+1}) v^{k-s} \},$$

where

$$Q = \frac{2\alpha \cos \rho + \beta}{(2\alpha + \frac{h\sigma}{24}) \cos \rho + (\beta + \frac{14h\sigma}{24})},$$

where $\alpha = \frac{h}{8\tau} + r - \iota$ and $\beta = \frac{6h}{8\tau} + 6r + 2\iota$.

It can easily be verified that

$$|Q|^2 \leq 1.$$

As a result, for $k = 0$, (21) gives the following form:

$$v^1 = Q\Theta_0 v^0,$$

and then

$$|v^1| = |Q||\Theta_0||v^0| \leq |v^0|.$$

Let $|v^j| \leq |v^0|$, $j = 1, 2, \dots, k$. Hence, by equation (30) we have,

$$\begin{aligned} |v^{k+1}| &\leq \Theta_k |v^0| + \sum_{s=0}^{k-1} (\Theta_s - \Theta_{s+1}) |v^{k-s}| \\ &\leq \Theta_k |v^0| + \sum_{s=0}^{k-1} (\Theta_s - \Theta_{s+1}) |v^0| \\ &= |v^0| (\Theta_k + \sum_{s=0}^{k-1} (\Theta_s - \Theta_{s+1})) \\ &= |v^0|. \end{aligned}$$

Thus, for every k , we have $|v^{k+1}| \leq |v^0|$. This relation demonstrates that $|e^{k+1}| \leq |e^0|$, i.e., the error of this method in time level k , for every k , does not grow and is either less than or equal to its initial error. So, the unconditional stability of the method is verified. $\square$

4. Numerical examples

In this section, we try to demonstrate, with some examples, that the numerical solution is the agreement to the theoretical analysis. We can measure the precision of the numerical method by both the L_2 error norm and the L_∞ error norm, where

$$\|u^{exact} - u_h\|_2^2 \simeq h \sum_{j=0}^N |(u^{exact})_j - (u_h)_j|^2,$$

and

$$\|u^{exact} - u_h\|_\infty \simeq \max_j |(u^{exact})_j - (u_h)_j|.$$

All computations were conducted with the help of MATLAB software.

Example 1. Considering the following the fractional initial and boundary value problem below

$$(31) \quad \begin{cases} u_t + {}_0^C D_t^\vartheta u(x, t) - {}_0^C D_t^\vartheta u_{xx}(x, t) + f(u) = s(x, t), & \text{in } I \times (0, T], \\ u(0, t) = u(1, t) = 0, & t \in (0, T], \\ u(x, 0) = 0, & \text{in } I, \end{cases}$$

with $I = (0, 1)$ and $0 < \vartheta < 1$, where $s(x, t) = x(1 - x) + \frac{t^{1-\vartheta}}{\Gamma(2-\vartheta)}(x(1 - x) + 2) + t^2 x^2(1 - x)^2$ and the exact solution is $u(x, t) = tx(1 - x)$.

For the time and space stepsize sufficiently small, Table 2 and Table 3 show the numerical errors at $\vartheta = 0.05$ and $\vartheta = 0.95$ between the numerical solution and the exact solutions in time t_1 . We denote that in Table 2, $\tau = 0.001$ and in Table 3, $h = 0.001$.

Table 2: Error of approximate solutions in $\tau = 0.001$, $h_i = \frac{1}{2^{i+1}}$ and $\vartheta = 0.05$ in t_1 .

i	h	L_2 -norm	L_∞ -norm
1	0.25	0.00494547	0.00344572
2	0.125	0.00081220	0.00057444
3	0.0625	0.000124167	0.00008648
4	0.03125	0.00002070	0.00001312
5	0.015625	0.00000374	0.00000195

Table 3: Error of approximate solutions in $h = 0.001$, $\tau_i = \frac{1}{2^i}$ and $\vartheta = 0.95$ in t_1 .

i	τ	L_2 -norm	L_∞ -norm
1	0.5	0.00437715	0.00019922
2	0.25	0.00056749	0.00002583
3	0.125	0.00022010	0.00010014
4	0.0625	0.00002842	0.00000129
5	0.03125	0.00000368	0.00000016

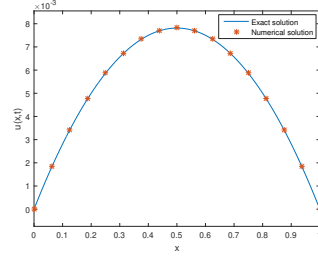
In Table 2, errors of approximate solutions in L_2 -norm and L_∞ -norm at $\tau = 0.001$, $h_i = \frac{1}{2^{i+1}}$ and $\vartheta = 0.05$ are presented.

Also, in Table 3, errors of approximate solutions in L_2 -norm and L_∞ -norm at $h = 0.001$, $\tau_i = \frac{1}{2^i}$ and $\vartheta = 0.95$ are presented.

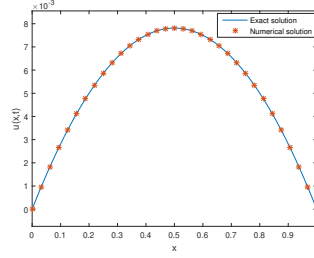
Table 2 compares the analytical solution and numerical solutions obtained for $\vartheta = 0.05$ different number of divisions of the region. Obviously, there is harmony between the analytical and numerical solutions obtained by the present scheme. As the value of M increases, the values of error norms L_2 and L_∞ decrease. Table

3 illustrates the error norms L_2 and L_∞ for $\vartheta = 0.95$ at different time steps. Table 3 demonstrates that there is a correspondence between the decrease in the time steps, on the one hand, and the higher accuracy of the obtained numerical results, on the other. This can be seen from the decreasing values of the error norms L_2 and L_∞ .

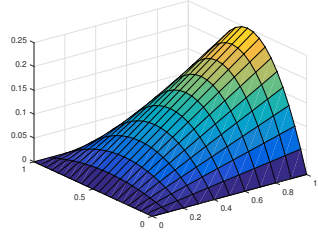
Figure 1 depicts the numerical solutions of Eq. (1) at the interval (0,1) for $\vartheta = 0.95$ and various values of M, n . We can see striking similarities in the graphs of the numerical and analytical solutions.



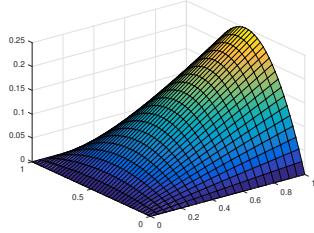
(a) $n = 32$ and $M = 16$ for t_1



(b) $n = 32$ and $M = 32$ for t_1



(c) $n = 10$ and $M = 20$ for all t



(d) $n = 20$ and $M = 40$ for all t

Figure 1: Numerical solutions of (31) in Example 1 for $\vartheta = 0.95$.

Example 2. As the second example, consider the nonhomogeneous fractional Cable equation (31) with

$$s(x, t) = [(1 + (1 + 4\pi^2) \frac{t^{1-\vartheta}}{\Gamma(2-\vartheta)} + \sin(2\pi x)] \sin(2\pi x)$$

and the exact solution is $u(x, t) = t \sin(2\pi x)$.

We can observe, in Table 4, the numerical errors at $\vartheta = 0.5$ for several h and τ .

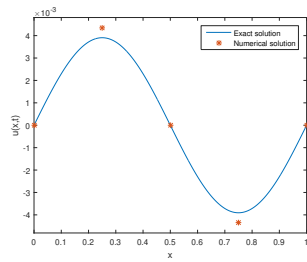
Table 4 represents errors of approximate solutions in L_2 -norm and L_∞ -norm at $h = \tau_i = \frac{1}{2^{i+1}}$ and $\vartheta = 0.5$.

At this stage, we need to compare the results. Table 4 shows the comparison made for $\vartheta = 0.5$ and various mesh sizes and time steps. Obviously, there exists a good agreement between the solutions. An increase in M and n results in a

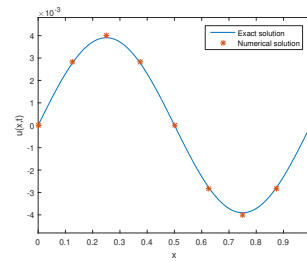
Table 4: Comparison of errors of approximate solution at $\tau_i = h_i = \frac{1}{2^{i+1}}$ and $\vartheta = 0.5$ in t_1 .

i	$\tau = h$	L_2 -norm	L_∞ -norm
1	0.25	0.0563156	0.0435957
2	0.125	0.0113709	0.0069766
3	0.0625	0.0033776	0.0017897
4	0.03125	0.0026571	0.0008616
5	0.015625	0.0024747	0.0005352

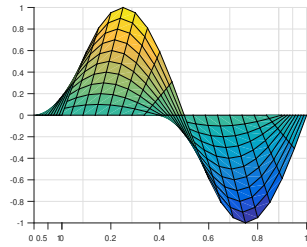
decrease in L_2 and L_∞ . The decrease in the error norms shows the accuracy of the newly obtained results. The accuracy of the newly obtained results can be shown by the decrease in the error norms. Finally, in Figure 2 we manifest the numerical solutions of Example 1 at the interval $(0,1)$ for $\vartheta = 0.5$. Clearly, their graphs show a lot of similarities to each other.



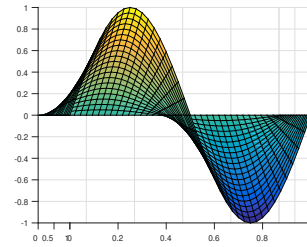
(a) $n = 16$ and $M = 4$ for t_1



(b) $n = 16$ and $M = 8$ for t_1



(c) $n = 4$ and $M = 16$ for all t



(d) $n = 8$ and $M = 16$ for all t

Figure 2: Numerical solutions of (31) in Example 2 for $\vartheta = 0.05$.

5. Conclusions

In the paper under study, we see how we can apply a numerical scheme based on finite volume element method to solve a class of nonlinear time-fractional Cable equation with initial and boundary condition. At the beginning, we utilize

the finite volume element method and Newton's iterative algorithm in order to approximate the numerical solutions of the initial and boundary value problem. While doing this, the backward Euler method is employed to obtain the fully discrete form of the problem. Then we present the stability of the scheme at hand and, in the end, include some numerical results in order to demonstrate the scheme's applicability.

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Coefficient estimates for subclasses $B_{\Sigma}^m(\alpha, \lambda)$ and $B_{\Sigma}^m(\beta, \lambda)$ of analytic and bi-univalent functions defined by a differential operator

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Abstract. In this paper two subclasses $B_{\Sigma}(\alpha, \lambda)$ and $B_{\Sigma}(\beta, \lambda)$ are studied and a new estimation for the initial coefficients $|a_2|$ and $|a_3|$ in these subclasses are found.

Keywords: analytic functions, close-to-convex functions, differential operator, integral operator.

1. Introduction

Let $\mathbb{R} = (-\infty, \infty)$ be the set of real numbers, $\mathbb{C}$ the set of complex numbers, and $\mathbb{N} = \{1, 2, 3, \dots\} = \mathbb{N}_0 \setminus \{0\}$ the set of positive integers.

Let $\mathcal{A}$ denote the class of all functions of the form

$$(1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.

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We also denote by $\mathcal{S}$ the class of all functions in the normalized analytic function class $\mathcal{A}$ which are univalent in $\mathbb{U}$ (for details, see [8]; see also some of the recent investigations [2, 3, 4, 5, 11, 15]).

For $f \in \mathcal{A}$, we utilize the following differential operator studied by Darus and Ibrahim [9].

$$(2) \quad T^0 f(z) = f(z),$$

$$(3) \quad T_{\sigma, \delta, \gamma}^1 f(z) = [1 - \delta(\gamma - \sigma)]f(z) + \delta(\gamma - \sigma)zf'(z),$$

$$(4) \quad T_{\sigma, \delta, \gamma}^m f(z) = z + \sum_{n=2}^{\infty} [1 + (n-1)\delta(\gamma - \sigma)]^m a_n z^n,$$

for $\sigma \geq 0, \delta \geq 0, \gamma \geq 0$ and $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ with $T_{\sigma, \delta, \gamma}^m f(0) = 0$.

When $\delta = \sigma = 1$ and $\gamma = 2$, we get Sălăgean's differential operator [13].

Univalent functions are injective they do not require to be defined on the entire unit disk $\mathbb{U}$ since they are reversible, indeed, the image of $\mathbb{U}$ under every univalent function $f \in \mathcal{S}$ contains a disk of radius $1/4$, which verified by Koebe one-quarter theorem [8]. According to this, every function $f \in \mathcal{S}$ has an inverse map f^{-1} that satisfies the following conditions:

$$f^{-1}(f(z)) = z, \quad (z \in \mathbb{U}),$$

and

$$f(f^{-1}(w)) = w, \quad \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4}\right).$$

In fact, the inverse function f^{-1} is given by

$$(5) \quad f^{-1}(w) = w - a_2 w^2 + (2a_2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \dots$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1). The class Σ has been investigated and the Taylor-Maclaurin coefficient inequalities for various classes of bi-univalent functions have been studied, see Srivastava et al. [14], see also [1, 6, 7, 10, 16, 17, 18, 19].

In this paper two new subclasses of the class Σ are introduced and the estimation for the coefficients $|a_2|$ and $|a_3|$ for the functions in the above new subclasses of the class Σ are found.

Firstly, to achieve our main objectives, we need the following lemma.

Lemma 1.1 ([12]). *If $p \in \mathcal{P}$, then $|c_k| \leq 2$ for each k , where $\mathcal{P}$ is the family of all functions p analytic in $\mathbb{U}$ for which $\Re(p(z)) > 0$, $p(z) = 1 + c_1 z + c_2 z^2 + \dots$ for $z \in \mathbb{U}$.*

2. Coefficient bounds for the class $B_{\Sigma}^m(\alpha, \lambda)$

Definition 2.1 ([20]). A function f given by (1) is said to be in the class $B_{\Sigma}^m(\alpha, \lambda)$ if the following conditions are satisfied:

$$(6) \quad \left| \arg \left((1 - \lambda) \frac{T_{\sigma, \delta, \gamma}^m f(z)}{z} + \lambda (T_{\sigma, \delta, \gamma}^m f(z))' \right) \right| < \frac{\alpha\pi}{2},$$

($0 < \alpha \leq 1, \lambda \geq 1$ and $\sigma, \delta, \gamma \geq 0, z \in \mathbb{U}$)

and

$$(7) \quad \left| \arg \left((1 - \lambda) \frac{T_{\sigma, \delta, \gamma}^m g(w)}{w} + \lambda (T_{\sigma, \delta, \gamma}^m g(w))' \right) \right| < \frac{\alpha\pi}{2},$$

($0 < \alpha \leq 1, \lambda \geq 1$ and $\sigma, \delta, \gamma \geq 0, w \in \mathbb{U}$),

where the function g is given by (5).

Theorem 2.1. Let f given by (1) be in the class $B_{\Sigma}^m(\alpha, \lambda)$, $0 < \alpha \leq 1$ and $\lambda \geq 1$. Then

$$(8) \quad |a_2| \leq \frac{2\alpha}{\sqrt{2\alpha(2\lambda + 1)[1 + 2\delta(\gamma - \sigma)]^m - (\alpha - 1)(\lambda + 1)^2[1 + \delta(\gamma - \sigma)]^{2m}}},$$

and

$$(9) \quad |a_3| \leq \frac{4\alpha^2}{(\lambda + 1)^2[1 + 2\delta(\gamma - \sigma)]^{2m}} + \frac{2\alpha}{(2\lambda + 1)[1 + 2\delta(\gamma - \sigma)]^m}.$$

Proof. It follows from (6) and (7) that

$$(10) \quad (1 - \lambda) \frac{T_{\sigma, \delta, \gamma}^m f(z)}{z} + \lambda T_{\sigma, \delta, \gamma}^m f'(z) = [p(z)]^{\alpha},$$

$$(11) \quad (1 - \lambda) \frac{T_{\sigma, \delta, \gamma}^m g(w)}{w} + \lambda T_{\sigma, \delta, \gamma}^m g'(w) = [q(w)]^{\alpha},$$

where $p(z)$ and $q(w)$ are in $\mathcal{P}$ have the forms

$$(12) \quad p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \dots,$$

$$(13) \quad q(w) = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \dots$$

Now, equating the coefficients in (10) and (11), we get

$$(14) \quad (\lambda + 1)[1 + \delta(\gamma - \sigma)]^m a_2 = \alpha p_1,$$

$$(15) \quad (2\lambda + 1)[1 + 2\delta(\gamma - \sigma)]^m a_3 = \alpha p_2 + \frac{\alpha(\alpha - 1)}{2} p_1^2,$$

$$(16) \quad -(\lambda + 1)[1 + \delta(\gamma - \sigma)]^m a_2 = \alpha q_1$$

and

$$(17) \quad (2\lambda + 1)(2a_2^2 - a_3)[1 + 2\delta(\gamma - \sigma)]^m = \alpha q_2 + \frac{\alpha(\alpha - 1)}{2} q_1^2.$$

From (14) and (16), we get

$$(18) \quad p_1 = -q_1$$

and

$$(19) \quad 2(\lambda + 1)^2[1 + \delta(\gamma - \sigma)]^{2m} a_2^2 = \alpha^2(p_1^2 + q_1^2).$$

Now, from (15), (17) and (19), we obtain

$$(20) \quad \begin{aligned} 2(\lambda + 1)[1 + \delta(\gamma - \sigma)]^m a_2^2 &= \alpha(p_2 + q_2) + \frac{\alpha(\alpha - 1)}{2}(p_1^2 + q_1^2) \\ &= \alpha(p_2 + q_2) + \frac{\alpha(\alpha - 1)}{2} \frac{2(\lambda + 1)^2[1 + \delta(\gamma - \sigma)]^{2m}}{\alpha^2} a_2^2. \end{aligned}$$

Therefore, we have

$$(21) \quad a_2^2 = \frac{\alpha^2(p_2 + q_2)}{2\alpha(2\lambda + 1)[1 + 2\delta(\gamma - \sigma)]^m - (\alpha - 1)(\lambda + 1)^2[1 + \delta(\gamma - \sigma)]^{2m}}.$$

Applying Lemma 1.1 for the coefficients p_2 and q_2 , we immediately have

$$(22) \quad |a_2| \leq \frac{2\alpha}{\sqrt{2\alpha(2\lambda + 1)[1 + 2\delta(\gamma - \sigma)]^m - (\alpha - 1)(\lambda + 1)^2[1 + \delta(\gamma - \sigma)]^{2m}}}.$$

This gives the bound of $|a_2|$ as asserted in (8).

Next, in order to find the bound of $|a_3|$, by subtracting (17) from (15), we get

$$(23) \quad \begin{aligned} &2(2\lambda + 1)[1 + 2\delta(\gamma - \sigma)]^m a_3 - 2(2\lambda + 1)[1 + 2\delta(\gamma - \sigma)]^m a_2^2 \\ &= \alpha(p_2 - q_2) + \frac{\alpha(\alpha - 1)}{2}(p_1^2 - q_1^2). \end{aligned}$$

It follows from (17)-(23) that

$$(24) \quad |a_3| = \frac{4\alpha^2(p_1^2 + q_1^2)}{(\lambda + 1)^2[1 + 2\delta(\gamma - \sigma)]^{2m}} + \frac{2\alpha(p_2 + q_2)}{(2\lambda + 1)[1 + 2\delta(\gamma - \sigma)]^m}.$$

Applying Lemma 1.1 for the coefficients p_1, p_2, q_1 and q_2 , we get

$$(25) \quad |a_3| \leq \frac{4\alpha^2}{(\lambda + 1)^2[1 + 2\delta(\gamma - \sigma)]^{2m}} + \frac{2\alpha}{(2\lambda + 1)[1 + 2\delta(\gamma - \sigma)]^m}.$$

This gives the required condition. Hence, the theorem follows. $\square$

If we take $\lambda = 1$ in Theorem 2.1, then we have the following corollary.

Corollary 2.1. *Let the function f given by (1) be in the class $B_\Sigma(\alpha, \lambda)$, ($0 < \alpha \leq 1$), then*

$$(26) \quad |a_2| \leq \frac{2\alpha}{\sqrt{6\alpha[1 + 2\delta(\gamma - \sigma)]^m - 4(\alpha - 1)[1 + \delta(\gamma - \sigma)]^{2m}}}$$

and

$$(27) \quad |a_3| \leq \frac{4\alpha^2}{4[1 + 2\delta(\gamma - \sigma)]^{2m}} + \frac{2\alpha}{3[1 + 2\delta(\gamma - \sigma)]^m}.$$

If we take $\alpha = 1$ in Theorem 2.1, then we have the following corollary.

Corollary 2.2. *Let the function f given by (1) be in the class $B_\Sigma(\alpha, \lambda)$, ($\lambda \geq 1$), then*

$$(28) \quad |a_2| \leq \frac{2}{\sqrt{2(2\lambda + 1)[1 + 2\delta(\gamma - \sigma)]^m}}$$

and

$$(29) \quad |a_3| \leq \frac{4}{(\lambda + 1)^2[1 + 2\delta(\gamma - \sigma)]^{2m}} + \frac{2}{(2\lambda + 1)[1 + 2\delta(\gamma - \sigma)]^m}.$$

3. Coefficient bounds for the class $B_\Sigma^m(\beta, \lambda)$

Definition 3.1. *A function f given by (1) is said to be in the class $B_\Sigma(\beta, \lambda)$ if the following conditions are satisfied:*

$$(30) \quad \Re \left((1 - \lambda) \frac{T_{\sigma, \delta, \gamma}^m f(z)}{z} + \lambda T_{\sigma, \delta, \gamma}^m f'(z) \right) < \beta, \quad (0 \leq \beta < 1, \\ \lambda \geq 1 \text{ and } \sigma, \delta, \gamma \geq 0, z \in \mathbb{U})$$

and

$$(31) \quad \Re \left((1 - \lambda) \frac{T_{\sigma, \delta, \gamma}^m g(w)}{w} + \lambda T_{\sigma, \delta, \gamma}^m g'(w) \right) < \beta, \quad (0 \leq \beta < 1, \\ \lambda \geq 1 \text{ and } \sigma, \delta, \gamma \geq 0, w \in \mathbb{U}),$$

where the function g is defined by (5).

Theorem 3.1. *Let $f(z)$ given by (1) be in the class $B_\Sigma^m(\beta, \lambda)$, $0 \leq \beta < 1$ and $\lambda \geq 1$. Then*

$$(32) \quad |a_2| \leq \sqrt{\frac{2(1 - \beta)}{(2\lambda + 1)[1 + 2\delta(\gamma - \sigma)]^m}}$$

and

$$(33) \quad |a_3| \leq \frac{4(1 - \beta)^2}{(\lambda + 1)^2[1 + \delta(\gamma - \sigma)]^{2m}} + \frac{2(1 - \beta)}{(2\lambda + 1)[1 + 2\delta(\gamma - \sigma)]^m}.$$

Proof. It follows from (30) and (31) that there exist p and $q \in \mathcal{P}$ such that

$$(34) \quad (1 - \lambda) \frac{T_{\sigma, \delta, \gamma}^m f(z)}{z} + \lambda T_{\sigma, \delta, \gamma}^m f'(z) = \beta + (1 - \beta)p(z)$$

and

$$(35) \quad (1 - \lambda) \frac{T_{\sigma, \delta, \gamma}^m g(w)}{w} + \lambda T_{\sigma, \delta, \gamma}^m g'(w) = \beta + (1 - \beta)q(w),$$

where $p(z)$ and $q(w)$ have the forms (12) and (13), respectively. Equating coefficients in (34) and (35) yields

$$(36) \quad (\lambda + 1)[1 + \delta(\gamma - \sigma)]^m a_2 = (1 - \beta)p_1,$$

$$(37) \quad (2\lambda + 1)[1 + 2\delta(\gamma - \sigma)]^m a_3 = (1 - \beta)p_2,$$

$$(38) \quad -(\lambda + 1)[1 + \delta(\gamma - \sigma)]^m a_2 = (1 - \beta)q_1$$

and

$$(39) \quad (2\lambda + 1)(2a_2^2 - a_3)[1 + 2\delta(\gamma - \sigma)]^m = (1 - \beta)q_2.$$

From (36) and (38), we get

$$(40) \quad p_1 = -q_1$$

and

$$(41) \quad 2(\lambda + 1)^2[1 + \delta(\gamma - \sigma)]^{2m} a_2^2 = (1 - \beta)^2(p_1^2 + q_1^2).$$

Also, from (37) and (39), we find that

$$(42) \quad 2(2\lambda + 1)[1 + 2\delta(\gamma - \sigma)]^m a_2^2 = (1 - \beta)(p_2 + q_2).$$

Thus, we have

$$(43) \quad |a_2^2| \leq \frac{(1 - \beta)}{2(2\lambda + 1)[1 + 2\delta(\gamma - \sigma)]^m} (|p_1| + |q_2|) \\ = \frac{2(1 - \beta)}{(2\lambda + 1)[1 + 2\delta(\gamma - \sigma)]^m},$$

which is the bound of $|a_2|$ as given in (32).

Next, in order to find the bound of $|a_3|$, by subtracting (39) from (37), we get

$$(44) \quad (2\lambda + 1)[1 + 2\delta(\gamma - \sigma)]^m a_3 - (2\lambda + 1)(2a_2^2 - a_3)[1 + 2\delta(\gamma - \sigma)]^m \\ = (1 - \beta)(p_2 - q_2)$$

or, equivalently,

$$(45) \quad a_3 = a_2^2 + \frac{(1-\beta)(p_2 - q_2)}{2(2\lambda + 1)[1 + 2\delta(\gamma - \sigma)]^m}.$$

Upon substituting the value of a_2^2 from (41), we obtain

$$(46) \quad a_3 = \frac{(1-\beta)^2(p_1^2 + q_1^2)}{2(\lambda + 1)^2[1 + \delta(\gamma - \sigma)]^{2m}} + \frac{(1-\beta)(p_2 - q_2)}{2(2\lambda + 1)[1 + 2\delta(\gamma - \sigma)]^m}.$$

Applying Lemma 1.1 for the coefficients p_1, p_2, q_1 and q_2 , we readily get

$$(47) \quad |a_3| \leq \frac{4(1-\beta)^2}{(\lambda + 1)^2[1 + \delta(\gamma - \sigma)]^{2m}} + \frac{2(1-\beta)}{(2\lambda + 1)[1 + 2\delta(\gamma - \sigma)]^m},$$

which is the bound on $|a_3|$ as asserted in (33). $\square$

If we take $\lambda = 1$ in Theorem 3.1, then we get the following corollary.

Corollary 3.1. *Let the function $f(z)$ given by (1) be in the class $B_\Sigma(\beta, \lambda)$ ($0 \leq \beta < 1$). Then*

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{3[1 + 2\delta(\gamma - \sigma)]^m}}$$

and

$$|a_3| \leq \frac{(1-\beta)^2}{[1 + \delta(\gamma - \sigma)]^{2m}} + \frac{2(1-\beta)}{3[1 + 2\delta(\gamma - \sigma)]^m}.$$

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SDA-COVID-19: Social Distancing App for COVID-19 track and control

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Abstract. COVID-19 is a highly contagious and lethal virus-based disease. Social distancing is the only way to stop the virus from spreading. In this context, the re-

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searchers suggest a social distancing application named SDA-COVID-19. The suggested App (SDA-COVID-19) will help individuals maintain social distancing by the exchange of data among phones about potentially infected and/or contaminated people with COVID-19, with whom an individual socialized or came in contact, whereby, an individual will be alerted if a COVID-19 infected person is in close proximity. Two versions are suggested for SDA-COVID-19 one is Service-Oriented and the other is Bluetooth oriented.

Keywords: COVID-19, SDA-COVID-19, tracking, warning, disease control, social distancing.

1. Introduction

COVID-19 is a virus that brought the world to a dead halt. COVID-19 is highly contagious and lethal. Social distancing is the only way to stop the virus from spreading. Although, governments across the globe asked the people to practice social distancing still people responded slowly and with disregard to government advisement. Accordingly, some government escalated the advisement to lock down in some countries. However, long-term lock down is not a viable solution, due to the anticipated economic harmful repercussions, as well as plummeting productivity of people and organizations, particularly in view that reports stipulate that COVID-19 predicament will not end soon.

Evidently, not only would COVID-19 pandemic freeze the economic cycle, but the accompanying lock-down would also restrict people's mobility regardless of being sick or healthy. Thereby, allowing COVID-19 to control all aspects of life. As such, Governments have two options either enforcing lock down and save people or sacrifice people for economic reasons. Nevertheless, the researchers suggest another alternative, produce a technological solution that will allow healthy people to continue their lives by working while sick or risky people can rest until they feel better. The suggested solution will be used as an application developed to be used on smart phones. The application will be distributed in nature and has no intervention with the privacy issue as will be explained later. Since the current situation with COVID-19 will be long term such a solution will be a fair choice to bring the economy back to life and allow countries and nations to go back to normal life as much as possible.

The goal of the SDA-COVID-19 is to collect data from people that may be within 1.8M perimeter of a person. SDA-COVID-19 is installed on any smart phone, then it will collect the phone numbers of people that are within 1.8M proximity of other people. SDA-COVID-19 will accumulate the numbers in a storage on the phone. If a person become infected of COVID-19 then either personnel from hospital or the infected person can broadcast such information to the collected list. Hence, alerting all the list of phones stored on her/his list of the contamination. The alerted persons can then take the right measure of self-quarantine and receiving the right treatment. Furthermore, rather than locking down all people, only people who carry the virus or are sick of the virus can be quarantined.

This paper is composed of literature review that will consider the applications similar to the suggested SDA-COVID-19. The next section will shed light on the issues and challenges facing the implementation of the SDA-COVID-19 such as privacy, security, smart phone access, undocumented immigrant's data availability, and laws and regulations. Then, the paper will present the first and second versions of SDA-COVID-19. Further, the paper will produce the data structure of the SDA-COVID-19, as well as the Input and output Screen. Consequently, the paper will present the implementation policy of SDA-COVID-19, as well as the Future work for the suggested Framework.

2. Literature review

This section first presents two Apps resembling the suggested SDA-COVID-19. Next present the practice of some countries like Israel, China, Taiwan, South Korea, and EU.

The researchers found two Apps resembling the suggested App namely: Private Kit: Safe Paths and FluPhone. According to [1] there is an App named Private Kit: Safe Paths that tracks COVID-19 positively tested patients and alarms anyone who crosses paths with the patient. The same source discuss the software FluPhone developed by Cambridge scientist in 2011 but failed since only 1% of the Cambridge community used it.

Another App TraceTogether app Developed by Singapore Government Agency Website [2]. The App trades contacts of previously met people in case a person gets sick. The App works on Bluetooth, when a person gets infected with COVID-19, the App will send a warning message to others. The App hinge on heavily on the cooperation of the people and works as a waning system.

According to [3] countries like Israel invoked an emergency in March-16-2020 [4] and [5] act to track COVID-19 infected people by tracking their smart phones. The same source mentioned that Iran asked people to download an App to locate their citizens, and China used smart phones and face recognition to locate infected people. South Korea sent message about the movement of people without using their names. According to [1] Taiwan used smart phone technology to build an electronic fence around quarantined patients. According to [5] the EU has General Data Protection Regulation (GDPR) which limits Mobile carriers use of customers personal data whether sharing or managing. Although in Italy's Lombardy region such data was used "helping authorities get a clearer picture of how well people are observing the rules" [5]. Previous work of the researchers were conducted in [6] [7] [8] [9] [10] [11] [12] [13] [14].

3. Issues and challenges facing SDA-COVID-19

There were several issues and challenges raised in the previously mentioned articles [8] [9] [10] [11] [12] [13] [14]:

1. Privacy

2. Security
3. Smart phone access.
4. Undocumented immigrant's data availability.
5. Laws and regulations.

3.1 Privacy

Although many are concerned of disclosing their data privacy, still SDA-COVID-19 can be considered as a validation tool for people applying for work, and current staff to maintain a healthy workspace environment, synonym to a driver's license or electronic Identification Card. In addition, all data is stored on the person's own phone hence no one will be able to access such data. Furthermore, data required from Phone-Owner is not more than what a person would usually provide to his/her doctor when visiting for the first time. In this context, the researchers suggested two versions of SDA-COVID-19.

3.2 Security

There are many aspects to security issue in SDA-COVID-19. First, data theft, the data of each phone owner resides on their own phone and is encrypted and distributed. Thief cannot follow people around to steal their data. And cannot decrypt each data from each phone. Hence security in numbers is the name of the game in this case.

3.3 Smart phone access

Albeit, many do not own smart phones which is a personal choice to people. However, according to [15] the number of smart phone owners is 3.5 billion people, which "means 45.12% of the world's population owns a smartphone" [16]. Additionally, SDA-COVID-19 could entice people to purchase smart phones since the App will be their health detector and an early warning system of potential risk to be exposed to such a pandemic that is readily available.

3.4 Undocumented immigrant's data availability

Many countries still suffer from illegal/ Undocumented immigrants. As such, Undocumented immigrants will not volunteer their data, still no one is willing to die from COVID-19. Taking into consideration that undocumented immigrants still need to work and practice their lives as normal as possible, as such, employers of illegal immigrants may turn a blind eye regarding immigration status but will not expose themselves and others to a deadly virus. Furthermore, SDA-COVID-19 does not require immigration status, nor cares to. SDA-COVID-19 will work between two smart phones as a pass of healthiness, seen in the second version of SDA-COVID-19.

3.5 Laws and regulations

Regarding the fifth issue, Laws and regulations, SDA-COVID-19 will benefit all stakeholders whether organization, government, or individual. An individual (aka smart phone owner) is the one to be at risk of getting the disease. The use of SDA-COVID-19 is most beneficial for a person since it will keep the individual informed about his/her health status. Furthermore, the issue of regulations and laws will not apply when a person is trying to detect a safe environment and to be informed about his/her life and health. In the same token, this research provides two version of the App.

4. Working steps of SDA-COVID-19 service oriented version

The First version of SDA-COVID-10 is described in this section. This version of SDA-COVID-19 is service oriented and needs to be in contact with communication service provider through internet cloud in order to exchange data. In this version of SDA-COVID-19 the data is transmitted to COVID-19 database and the communication service provider can manipulate and use the data to track COVID-19 infected people, see Figure 1. Such framework is duly connected to medical center. Medical centers can be any governmental health organizations like ministry of health. In the following the paper presents the essential 9 steps of SDA-COVID-19 as follows:

- 1- Global COVID-19 database accessed via the Internet will be updated by the COVID-19 health care agents responsible to record all the cases in each country around the world.

- 2- The SDA-COVID-19 will be downloaded and installed on smart phone from iPhone APP store, or Google play or Samsung Play Store. Each downloaded SDA-COVID-19 will give the smart phone a number (primary number) based on the smart phone serial number and phone number.

- 3- A person must register with SDA-COVID-19 using name, address, national ID, Date of birth, Health condition, as shown in the data structure section. Data given will be encrypted using phone owner password.

- 4- Using the Internet and the GPS, SDA-COVID-19 will track all the users' movements and locations and record all the places and the people who have within 1.8M proximity of the user all the time.

- 5- SDA-COVID-19 will not only notify the user in case there is an infected or potentially infected person by scanning the people around the user, but will also alert health care agents in case the user were close enough (less than 1.8M), and so health agents would follow up her/his case as s/he will be recorded as a suspected case in the COVID-19 database, and will be notified to stay home for 14 days.

- 6- When the health care agent discover that someone is infected with COVID-19, all the people who were in close proximity of the infected case during the last 3-4 weeks must be advised to go into self-quarantine and to be tested to make

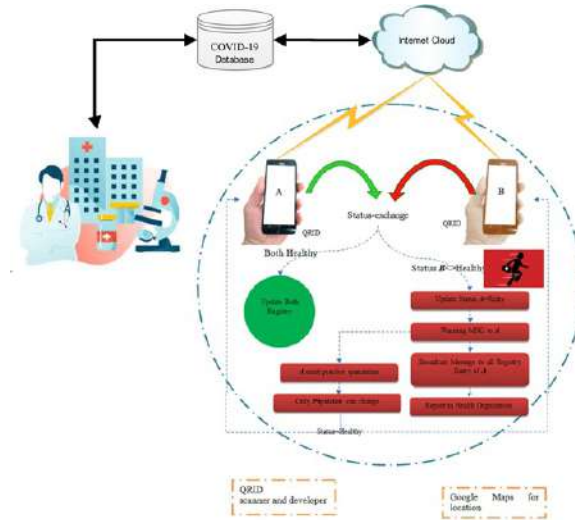


Figure 1: SDA-COVID-19 Service-Oriented Version.

sure that they are not infected with COVID-19 virus. 7- In case of a person hits a Contaminate Button on smart phone that he might be infected with the virus, SDA-COVID-19 will send warning message to all the people who were around during the last 3-4 weeks to notify them to take the necessary action.



Figure 2: Simplified view of SDA-COVID-19 interaction.

8- When a warning message is received the status of the phone will change to status “Risk”, hence the phone owner must be quarantined for 14 days and contact the nearest health facility. Phone owner must practice precautionary measures so as not to infect others. The phone will also send contamination alert message to others on its registry in the form of a chain message.

9- Once the phone owner is cured and is no longer a threat, his/her status will be changed by a physician from an accredited health facility. The physician

will sign off the phone owner with his/her credentials using two-way factor authentication method.

5. Working steps of SDA-COVID-19 bluetooth version

The second version of SDA-COVID-19 is Bluetooth based. SDA-COVID-19 in this case is a stand-alone application with the option that is not connected to COVID-9 database, hence the communication service provider has no access to the data, see Figure 3 . The data in SDA-COVID-19 is distributed and each owner of smart-phone can voluntarily provide the her/his data to health care facilities. Such a version of SDA-COVID-19 will be more acceptable to public and people will not have a privacy issue or “Big Brother Watching” problem. In the following the paper presents the essential 9 steps of SDA-COVID-19 as follows:

1- The SDA-COVID-19 will be downloaded and installed on smart phone from iPhone APP store, or Google play or Samsung Play Store.

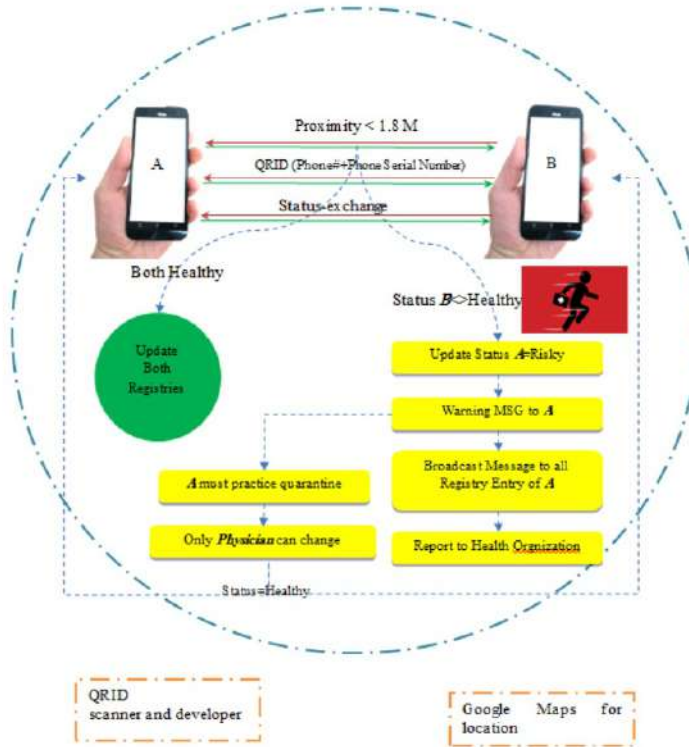


Figure 3: SDA-COVID-19 Bluetooth Oriented Version.

2- Each downloaded SDA-COVID-19 will give the smart phone a number (primary number) based on the smart phone serial number and phone number.

3- A person must register with SDA-COVID-19 using name, address, national ID, Date of birth, Health condition, as shown in the data structure section. Data given will be encrypted using phone owner password.

4- SDA-COVID-19 through phone will radiate its number to other phones withing 1.8M perimeter, see Figure 2.

5- SDA-COVID-19 also receives the radiated numbers and store it in a registry. Registry is composed of Phone Number and phone serial number, and proximity (distance) as shown in the data structure section.

6- In case of a person hits a Contaminate Button on smart phone. Smart phone will send warning message to all the registry entries.

6. Data structure

The underlying of the suggested SDA-COVID-19 is composed of two basic database table: Phone-owner database table which resides on the smart phone of the owner. The second database table is called Registry which also resides on the smart phone of the owner. The database table phone-owner is made of 12 attributes and is reflected in Figure 2. The phone-owner table will be filled by the phone owner using credentials as downloaded. The user may access two application QRID for building a QRID for identification purposes and Google maps for location. The data pertaining to database table will be filled only once. The same table is used when the role is physician where data is filled by medical institution that has credentials and uses two-way factor authentication scheme. Physician role has more privileges than user role since the person with physician role has the privilege to change the status attribute in the Registry database table from Risky/Sick to Healthy.

Phone-Owner	
Role	User – Physician (reserved to be set by institution)
QRID	Phone Serial Number+ Phone number (13 digits)
Name	Varchar 30
National ID Number	Varchar (24) may be validated
Sex	Male/Female
Date of birth	Date
Address- building, street, area, city, country	or location google map
Pres existing - Diabetes	Yes/No
Pres existing – Cardiovascular disease	Yes/No
Pres existing – Chronic respiratory disease	Yes/No
Pres existing – Hypertension	Yes/No
Pres existing – Cancer	Yes/No

Figure 4: Database table named phone-owner.

The second database table is Registry which is made of 6 attributes, see Figure 5. The attributes are filled automatically by SDA-COVID-19. The registry is updated automatically when two smart phones are in the proximity of 1.8m. The attribute Status reflects initially is Figure 5: The Registry database table and attributes.

Registry	
Sequence Number	Integer number
QRID	From the others <i>Phone-Owner</i>
Proximity	0-1.8 meter
Timestamp	Time + Date
Status	Healthy/Risky/Sick (only physician role can change status)
Location	Based on google maps

Figure 5: The Registry database table and attributes

Healthy, once the person is within proximity of 1.8m of a person with Risky or Sick smart phone the status will change to risky. Only a person with role of Physician can change the attribute Status back to Healthy. The change of attribute Status is protected by two-way factor security. Further, the database table Registry will be automatically updated daily by erasing all entries that are 24 days old, hence the Timestamp attribute.

7. Input and output screens of SDA-COVID-19

There are two types of screens in SDA-COVID-19: first the initial registration screen. This input screen, see Figure 6, allow the registration in SDA-COVID-19 and is based on the phone-owner database table previously explained. The second interface is an output screen shown in Figure 7 that reflects two things: QRID and the Status of the phone-owner. The Status of the phone-owner is three types: Healthy, Risky, Sick. The Healthy status is colored green, the Risky status is color blue, and the Sick status is colored red as shown in Figure 7.

8. Implementation policy

To succeed, this App must adhere to the following suggested implementation policy:

- 1- All organizations, companies, corporations, institutions will enforce the use of this App to be used by employees.
- 2- A person may not do the following if STATUS not equal HEALTHY
 - a. Enter working place
 - b. Use public transport
 - c. Enter a mall/ supermarket/shops/banks/ schools/universities/etc.
- 3- The registry will be updated by deleting all entries of 24 days old.
- 4- If a Sick/Risky person comes too close to Healthy person his/her phone will beep to warn others and their phones will beep also to warn healthy people from risky/sick person.
- 5- Only an accredited physician can change the status of a sick person to healthy, using two-way factor authentication.

Role: User ☐ Physician ☐
QRID:
Name:
National ID Number:
Sex: Male ☐ Female ☐
Date of birth:
Address:
Building:
Street:
Area:
City:
Country:
Do you have:
Diabetes: Yes ☐ No ☐
Cardiovascular disease: Yes ☐ No ☐
Chronic respiratory disease: Yes ☐ No ☐
Hypertension: Yes ☐ No ☐
Cancer: Yes ☐ No ☐

Figure 6: Screen of Initial Registration



Figure 7: Status of Phone Owner Sick, Risky, Healthy

6- The password for the App is either face recognition or fingerprint so as not allow phone trading among users.

9. Future work

This suggested SDA-COVID-19 need to be implemented by programming the SDA-COVID-19 and everyone must cooperate for their safety and well-being. Such use can be encouraged by government adoption of SDA-COVID-19.

10. Conclusion

In conclusion this paper presented SDA-COVID-19 mobile application. The SDA-COVID-19 will allow the user to be notified and warned about COVID-19

infected people that may cross her/his path. The SDA-COVID-19 will keep track of all the people that may come in proximity of the phone owner for the past period. Once an infected person is discovered all people that met her/him will be warned. Hence, precautionary measures to be taken.

SDA-COVID-19 is suggested as two versions: Service based, and Bluetooth based. The service-based version of SDA-COVID-19 is to be used with data exchanged through communication service provider. The Bluetooth SDA-COVID-19 version is totally distributed system with no need for intermediate for communication other than Bluetooth. Although this version is more likely the popular option since privacy issue is resolved, still, battery consumption and storage are to be raised as problem.

In this paper, the researchers, provided literature review of other application similar to SDA-COVID-19, then discussed all issue and challenges that may face SDA-COVID-19, then presented the design of SDA-COVID-19 input, output, data structure, framework of both Bluetooth and service versions, and implementation policy.

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On quaternion algebras that split over specific quadratic number fields

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Abstract. Let d and m be two distinct squarefree integers and $\mathcal{O}_K$ the ring of integers of the quadratic field $K = \mathbb{Q}(\sqrt{d})$. Denote by $H_K(\alpha, m)$ a quaternion algebra over K , where $\alpha \in \mathcal{O}_K$. In this paper we give necessary and sufficient conditions for $H_K(\alpha, m)$ to split over K for some values of α , and we obtain a complete characterization of division quaternion algebras $H_K(\alpha, m)$ over K whenever α and m are two distinct positive prime integers. Examples are given involving prime Fibonacci numbers.

Keywords: quaternion algebras, Hilbert symbol, quadratic fields, Fibonacci numbers

1. Introduction

Let K be a field of characteristic $\neq 2$. If $\alpha, \beta \in K^* = K \setminus \{0\}$, then there exists a unique unital associative K -algebra of dimension 4 with K -basis $\{1, i, j, k\}$

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such that $i^2 = \alpha$, $j^2 = \beta$ and $ij = -ji = k$. This K -algebra will be denoted by $H_K(\alpha, \beta)$. Its presentation, as a K -algebra, is given by $K\{i, j\}/(i^2 - \alpha, j^2 - \beta, ij = -ji)$. A quaternion algebra over K is a K -algebra isomorphic to such an algebra for some $\alpha, \beta \in K^*$. The classical instance, where $K = \mathbb{R}$, is $H_K(-1, -1) = \mathbb{H}$, the Hamilton's quaternions ($\alpha = \beta = -1$).

The classification of quaternion algebras over K can be rephrased in terms of quadratic forms, and a more detailed description depends on the field K . In this vein, the most important question one may ask about a quaternion algebra $H_K(\alpha, \beta)$ is whether it is isomorphic to the matrix ring $M_2(K)$; if so, we say that $H_K(\alpha, \beta)$ splits over K . For example, every quaternion algebra over $\mathbb{C}$ (or an algebraically closed field) splits, and a quaternion algebra $H_{\mathbb{R}}(\alpha, \beta)$ over $\mathbb{R}$ splits if and only if $\alpha > 0$ or $\beta > 0$ (see [14]).

Quaternion algebras are central simple algebras over K (i.e. associative and non-commutative algebras without two sided ideals whose center is precisely K , see [14]) of dimension 4 over K . Recall that the dimension d of a central simple algebra A over a field K is always a perfect square, and its square root n is defined to be the degree of A .

The theory of central simple algebras (and thus in particular quaternion algebras and cyclic algebras) has strong connections with algebraic number theory, combinatorics, algebraic geometry, coding theory, computer science and signal theory (see [6, 9, 14]). Quaternion algebras have been studied in many papers that deal with conditions for an algebra to be split or a division algebra (e.g. [15, 16, 17]).

To decide whether a quaternion algebra is a division algebra or it splits, different approaches are used: quadratic forms, the associated conics (which are projective plane curves defined by the homogeneous equation $\alpha x^2 + \beta y^2 = z^2$), cyclotomic fields, p -adic fields and some other properties of associative algebras (e.g. [15, 16, 17]).

In this paper we adopt two distinct approaches. The first consists of studying the ramification of certain integral primes, and we obtain a nice characterization of quaternion division algebras $H_K(p, q)$ solely in terms of quadratic residues, assuming that p and q are positive primes and K is quadratic number field. The second one, makes use of the Hilbert symbol in order to obtain necessary and sufficient conditions for a quaternion algebra $H_K(\alpha, m)$ to split over a quadratic field $K = \mathbb{Q}(\sqrt{d})$, for some integers α of $\mathcal{O}_K$, the ring of integers of the quadratic field K , where m and d are squarefree integers.

2. Preliminaries

Let us collect some results that we will use in the sequel. We begin by recalling the definition of the ring of integers of a number field.

Definition 2.1. *Let K be a number field.*

1. The ring of integers $\mathcal{O}_K$ of K is defined as follows

$$\mathcal{O}_K = \{\alpha \in K : P(\alpha) = 0 \text{ for some monic polynomial } P(X) \in \mathbb{Z}[X]\}.$$

2. $\mathcal{O}_K^\times$, the unit group of K , is the set of invertible elements of $\mathcal{O}_K$.
3. If $K = \mathbb{Q}(\sqrt{d})$ is a quadratic field, where $d > 0$ is a squarefree integer, then there exists a unit ε_d , called the fundamental unit of K , and it is a generator (modulo the roots of unity) of the unit group $\mathcal{O}_K^\times$.

Let K be a number field and q an element of the quaternion algebra $H_K(\alpha, \beta)$ over K ; then we can write $q = a + bi + cj + ek$, where a, b, c and e are in K . The conjugate $\bar{q}$ of q is defined as $\bar{q} = a - bi - cj - ek$. The norm map is defined by $N: q \mapsto N(q) = q\bar{q} = a^2 - \alpha b^2 - \beta c^2 + \alpha\beta e^2 \in K$. So N may be regarded as a quadratic form in the four variables a, b, c and e .

Definition 2.2. A non-zero vector $q \in H_K(\alpha, \beta)$ is said to be isotropic if $N(q) = 0$.

Definition 2.3. An associative algebra A over a field is called a division algebra if it has a multiplicative identity element $1 \neq 0$ and every non-zero element in A has a left and a right multiplicative inverses. If A is a finite-dimensional algebra, then A is a division algebra if and only if A has no nontrivial zero divisors.

Definition 2.4. A central simple algebra A over a field K is called split by K if A is isomorphic to the full matrix algebra $M_n(K)$ over K .

Theorem 2.1 ([6, 20, 21]). Let K be a field with $\text{char } K \neq 2$ and let $\alpha, \beta \in K \setminus \{0\}$. Then the quaternion algebra $H = H_K(\alpha, \beta)$ is either split or a division algebra. Furthermore, the following statements are equivalent:

1. $H \cong H_K(1, 1) \cong M_2(K)$;
2. H is not a division algebra;
3. H has an isotropic element;
4. the equation $\alpha x^2 + \beta y^2 = 1$ has a solution $(x, y) \in K \times K$;
5. α (resp. β) is a norm from $K(\sqrt{\beta})$ (resp. from $K(\sqrt{\alpha})$).

The above statements can be checked by using local techniques. For a number field K , it is well known that its ring of integers $\mathcal{O}_K$ is a Dedekind ring. If $\alpha \in K$ and $\mathcal{P}$ is a prime ideal of $\mathcal{O}_K$, we have $\alpha\mathcal{O}_K = \mathcal{P}^{v_{\mathcal{P}}(\alpha)}I$, where $v_{\mathcal{P}}(\alpha)$ is the highest power of $\mathcal{P}$ dividing $\alpha\mathcal{O}_K$ and I is an ideal of $\mathcal{O}_K$. The map $\alpha \mapsto N(\mathcal{P})^{-v_{\mathcal{P}}(\alpha)}$ is the non-Archimedean absolute value, where N denotes the absolute norm map. We denote by $K_{\mathcal{P}}$, the completion of K with respect to this absolute value. We conclude this section by defining the concept of ramification of a quaternion algebra at a prime ideal.

Definition 2.5. Let $\mathcal{P}$ be a prime ideal of $\mathcal{O}_K$. We say that $H_K(\alpha, \beta)$ is ramified at $\mathcal{P}$ if $H_{K_{\mathcal{P}}}(\alpha, \beta)$ is a division ring. The set of ramified primes of $H_K(\alpha, \beta)$ will be denoted by $\text{Ram}(H_K(\alpha, \beta))$. The reduced discriminant $D_{H_K(\alpha, \beta)}$ of the quaternion algebra $H_K(\alpha, \beta)$ is defined as the product of those prime ideals of $\mathcal{O}_K$ that ramify in $H_K(\alpha, \beta)$.

3. Symbols over number fields

If K is a number field of degree n , with signature (r, s) and L is a quadratic extension of K , then $L = K(\sqrt{\alpha})$ for some $\alpha \in \mathcal{O}_K$. If $\mathcal{P}$ is a prime ideal of $\mathcal{O}_K$, then the behavior of its extension $\mathcal{P}\mathcal{O}_L$ to L is as follows:

$$\mathcal{P}\mathcal{O}_L = \begin{cases} \mathcal{B}_1\mathcal{B}_2, & \mathcal{P} \text{ splits into 2 different prime ideals of } \mathcal{O}_L; \\ \mathcal{P}, & \mathcal{P} \text{ remains prime } (\mathcal{P} \text{ is called inert in } L); \\ \mathcal{B}^2, & \mathcal{P} \text{ ramifies in } L. \end{cases}$$

Note that $\mathcal{B}_i \cap K = \mathcal{B} \cap K = \mathcal{P}$ and $\mathcal{P} \cap \mathbb{Q} = p\mathbb{Z}$, where p is a prime number. We say that $\mathcal{B}$ (resp. $\mathcal{B}_i$) lies above $\mathcal{P}$. The following properties hold:

- $\mathcal{O}_K/\mathcal{P}$ is a finite field and $N(\mathcal{P}) := |\mathcal{O}_K/\mathcal{P}| = p^{f_{\mathcal{P}}}$ with $f_{\mathcal{P}} \in \mathbb{N}$. $N(\mathcal{P})$ is called the absolute norm of $\mathcal{P}$ and $f_{\mathcal{P}}$ is its inertia degree;
- if $v_{\mathcal{P}}(\alpha)$ is odd, then $\mathcal{P}$ is ramified in $L = K(\sqrt{\alpha})$;
- if $v_{\mathcal{P}}(\alpha)$ is even and $\mathcal{P}$ does not divide 2, then $\mathcal{P}$ is unramified in $L = K(\sqrt{\alpha})$;
- the infinite primes are just the $r + s$ non equivalent archimedean absolute values coming from the r real and s pairs of complex embeddings.

I. The quadratic residue symbol. The quadratic residue symbol $(\frac{\alpha}{\mathcal{P}})$ is classically defined as follows. Let $\mathcal{P}$ be a prime ideal of $\mathcal{O}_K$. If α is a square in K , we let $(\frac{\alpha}{\mathcal{P}}) = 1$. Otherwise, we let

$$\left(\frac{\alpha}{\mathcal{P}}\right) = \begin{cases} 1, & \text{if } \mathcal{P} \text{ splits in } K(\sqrt{\alpha}); \\ -1, & \text{if } \mathcal{P} \text{ is inert in } K(\sqrt{\alpha}); \\ 0, & \text{if } \mathcal{P} \text{ ramifies in } K(\sqrt{\alpha}). \end{cases}$$

Theorem 3.1 ([7, 12]). *If $\mathcal{P}$ is prime to the ideal generated by α and to the ideal generated by 2, then*

$$\left(\frac{\alpha}{\mathcal{P}}\right) \equiv \alpha^{\frac{N(\mathcal{P})-1}{2}} \pmod{\mathcal{P}}.$$

Remark 3.1. 1. If $K = \mathbb{Q}$, then $\mathcal{P} = p\mathbb{Z}$ where p is a prime integer. In this case the symbol $(\frac{\alpha}{\mathcal{P}})$ is denoted by $(\frac{\alpha}{p})$ and called the Legendre symbol. If $(\frac{\alpha}{p}) = 1$ (in this case, we say that α is a quadratic residue modulo p) and $p \equiv 1 \pmod{4}$, then $(\frac{\alpha}{p})_4$ will denote the rational biquadratic symbol which is equal to 1 or -1 , according as $(\alpha)^{\frac{p-1}{4}} \equiv 1$ or $-1 \pmod{p}$.

2. If $K = \mathbb{Q}(\sqrt{d})$ is a quadratic field, $\mathcal{P}$ is a prime ideal of $\mathcal{O}_K$ above a prime number p and $\alpha \in \mathcal{O}_K$, then according to [12], we have :

$$\left(\frac{\alpha}{\mathcal{P}}\right) = \begin{cases} \left(\frac{N(\alpha)}{p}\right), & \text{if } p \text{ is inert in } K; \\ \left(\frac{\alpha}{p}\right)^{f_{\mathcal{P}}}, & \text{if } \alpha \in \mathbb{Q}; \\ \left(\frac{\alpha}{\mathcal{P}'}\right) \left(\frac{N(\alpha)}{p}\right), & \text{if } p\mathcal{O}_K = \mathcal{P}\mathcal{P}', \mathcal{P} \neq \mathcal{P}'. \end{cases}$$

II. The Hilbert symbol. Let $\mathcal{P}$ be a prime of K (finite or infinite) and let $K_{\mathcal{P}}$ be the completion of K at $\mathcal{P}$. For $\alpha, \beta \in K_{\mathcal{P}}$, we define the local Hilbert Symbol as

$$(\alpha, \beta)_{\mathcal{P}} = \begin{cases} 1, & \text{if } H_{K_{\mathcal{P}}}(\alpha, \beta) \text{ splits;} \\ -1, & \text{if not.} \end{cases}$$

Recall that the Hilbert symbol $(a, b)_{\mathcal{P}}$ is equal to -1 if and only if the equation $ax^2 + by^2 = 1$ has no solutions in $K_{\mathcal{P}}$.

If $i_{\mathcal{P}}$ is the natural injection from K to $K_{\mathcal{P}}$, then for $\alpha, \beta \in K$ we define the global Hilbert Symbol as

$$\left(\frac{\alpha, \beta}{\mathcal{P}}\right) = i_{\mathcal{P}}^{-1}((i_{\mathcal{P}}(\alpha), i_{\mathcal{P}}(\beta))_{\mathcal{P}}).$$

Theorem 3.2 ([7]). *The Hilbert symbol satisfies the following conditions:*

- i. $(\frac{\alpha, \beta}{\mathcal{P}}) = (\frac{\beta, \alpha}{\mathcal{P}})$;
- ii. If $\mathcal{P}$ is a prime ideal of K unramified in $K(\sqrt{\alpha})$, then $(\frac{\alpha, \beta}{\mathcal{P}}) = (\frac{\alpha}{\mathcal{P}})^{v_{\mathcal{P}}(\beta)}$;
- iii. if $\mathcal{P}_{\infty}$ denotes an infinite prime, then $(\frac{\alpha, \beta}{\mathcal{P}_{\infty}}) = -1$ if and only if $i_{\mathcal{P}}(\alpha) < 0$ and $i_{\mathcal{P}}(\beta) < 0$;
- iv. β is a norm in $K(\sqrt{\alpha})$ if and only if $(\frac{\alpha, \beta}{\mathcal{P}}) = 1$ for all prime $\mathcal{P}$;
- v. $\prod_{\mathcal{P}} (\frac{\alpha, \beta}{\mathcal{P}}) = 1$ (the product formula).

Remark 3.2. Let K be a number field and $\alpha, \beta \in K^* = K \setminus \{0\}$. Then $H_K(\alpha, \beta)$ splits if and only if $\text{Ram}(H_K(\alpha, \beta)) = \emptyset$. More generally, we have

$$\text{Ram}(H_K(\alpha, \beta)) = \left\{ \mathcal{P} \text{ the prime ideal of } K : \left(\frac{\alpha, \beta}{\mathcal{P}}\right) = -1 \right\}.$$

Note also that the following splitting criterion for a quaternion algebras is well known [1, Corollary 1.10]: the quaternion algebra $H_K(\alpha, \beta)$ is split if and only if its discriminant $D_{H_K(\alpha, \beta)}$ is equal to the ring of integers $\mathcal{O}_K$ of K .

4. Main results

Let $K = \mathbb{Q}(\sqrt{d})$ be a quadratic number field and $\mathcal{O}_K$ its ring of integers, where $d \neq 1$ is a squarefree integer.

4.1 First case: $\alpha \in \mathcal{O}_K$ and $m \equiv 1 \pmod{4}$ a squarefree integer

Let us begin by defining an algebraic integer $\alpha \in \mathcal{O}_K$ to be odd if any prime ideal appearing in its factorization into prime ideals in $\mathcal{O}_K$ does not lie above 2.

The numbers m and α are said to satisfy hypotheses (H) if they satisfy the following two conditions:

1. α is an odd integer of $\mathcal{O}_K$, $m \equiv 1 \pmod{4}$ and d are two distinct square-free integers with $m > 0$;
2. α and m are relatively prime.

Moreover, we adopt, in the sequel, the following notation:

1. $\frac{m}{\gcd(m, d)} = (q_1 q_2 \cdots q_t)^\lambda (r_1 r_2 \cdots r_s)^\gamma$, where $\lambda, \gamma \in \{0, 1\}$ and q_i, r_i are prime numbers satisfying $(\frac{d}{q_i}) = -1$ if $\lambda = 1$, and $(\frac{d}{r_i}) = 1$ if $\gamma = 1$ for all i ;
2. $\mathcal{Q}_i$ (resp. $\mathcal{R}_i, \mathcal{R}'_i$) denotes the prime ideal (resp. ideals) of $\mathcal{O}_K$ above q_i (resp. r_i);
3. $\mathcal{I}$ denotes the set of all prime ideals $\mathcal{P}$ of $\mathcal{O}_K$ such that $v_{\mathcal{P}}(\alpha)$ is odd and whose absolute norm is a rational prime.

We can now establish our first main result.

Theorem 4.1. *If m and α satisfy the hypotheses (H), then $H_K(\alpha, m)$ splits if and only if the following conditions are satisfied:*

1. $(\frac{N(\alpha)}{q_i}) = (\frac{\alpha}{\mathcal{Q}_i}) = 1$, if $\lambda = 1$;
2. $(\frac{N(\alpha)}{r_j}) = (\frac{\alpha}{\mathcal{R}_j}) = 1$, if $\gamma = 1$;
3. for any prime $\mathcal{P} \in \mathcal{I}$ we have $(\frac{m}{N(\mathcal{P})}) = 1$.

Proof. Let $\mathcal{P}$ be an odd prime ideal of K . Lemma 1 of [3] implies that the discriminant $D(L/K) = (\mathcal{Q}_1 \mathcal{Q}_2 \cdots \mathcal{Q}_t)^\lambda (\mathcal{R}_1 \mathcal{R}'_1 \mathcal{R}_2 \mathcal{R}'_2 \cdots \mathcal{R}_s \mathcal{R}'_s)^\gamma$, where $L = K(\sqrt{m})$.

If $\mathcal{P} \neq \mathcal{Q}_i$, $\mathcal{P} \neq \mathcal{R}_j$ and $\mathcal{P} \neq \mathcal{R}'_j$, then $\mathcal{P}$ is unramified in $K(\sqrt{m})$. Hence by putting $N(\mathcal{P}) = p^{f_{\mathcal{P}}}$, Remark 3.1 implies that

$$\left(\frac{\alpha, m}{\mathcal{P}}\right) = \left(\frac{m}{\mathcal{P}}\right)^{v_{\mathcal{P}}(\alpha)} = \left(\frac{m}{p}\right)^{f_{\mathcal{P}} v_{\mathcal{P}}(\alpha)} = \begin{cases} \left(\frac{m}{p}\right), & \text{if } \mathcal{P} \in \mathcal{I}; \\ 1, & \text{if not.} \end{cases}$$

If $\mathcal{P} = \mathcal{Q}_i$, $\mathcal{R}_j$ or $\mathcal{R}'_j$, then, since α and m are relatively prime, $v_{\mathcal{P}}(\alpha) = 0$ and $v_{\mathcal{P}}(m) = 1$. Hence $\mathcal{P}$ is unramified in $K(\sqrt{\alpha})$ and thus

$$\left(\frac{\alpha, m}{\mathcal{P}}\right) = \left(\frac{\alpha}{\mathcal{P}}\right)^{v_{\mathcal{P}}(m)} = \begin{cases} \left(\frac{N(\alpha)}{q_i}\right), & \text{if } \mathcal{P} = \mathcal{Q}_i; \\ \left(\frac{\alpha}{\mathcal{P}}\right), & \text{if not.} \end{cases}$$

• As $m \equiv 1 \pmod{4}$, then the prime ideal $\mathcal{P}_2$ of $\mathcal{O}_K$ above 2 is unramified in $K(\sqrt{m})$, and since α is an odd integer (hence $v_{\mathcal{P}_2}(\alpha) = 0$), we have:

$$\left(\frac{\alpha, m}{\mathcal{P}_2}\right) = \left(\frac{m}{\mathcal{P}_2}\right)^0 = 1.$$

• Now, let $\mathcal{P}_{\infty}$ be an infinite prime ideal, then $\left(\frac{\alpha, m}{\mathcal{P}_{\infty}}\right) = 1$, because $i_{\mathcal{P}_{\infty}}(m) = m > 0$. Finally, Remarks 3.1 and 3.2 imply the assertions. $\square$

Remark 4.1. Thanks to Hilbert's symbol product formula, we get the same results if we consider the conditions $m \equiv 1 \pmod{4}$ and $\alpha \in \mathcal{O}_K$ with $d \not\equiv 1 \pmod{8}$ instead of the conditions $m \equiv 1 \pmod{4}$ and α is an odd integer of $\mathcal{O}_K$. In this case we will say that m and α satisfy the hypothesis $\hat{H}$.

In [15, Theorem 3.6 and Proposition 3.7] the second author gave conditions for a quaternion algebra $H_K(\alpha, p)$ (where p is a prime positive integer) to split over the quadratic field $K = \mathbb{Q}(i)$. The following corollary generalizes Theorem 3.6 and Proposition 3.7 from [15]. With similar proof ideas as in Theorem 3.6 and Proposition 3.7 from [15], we obtain in the following corollary necessary and sufficient conditions for a quaternion algebra $H_K(\alpha, m)$ to split over any quadratic number field $K = \mathbb{Q}(\sqrt{d})$, where $d \neq m$ is a squarefree integer.

Corollary 4.1. *If m and α satisfy hypotheses $\hat{H}$, $\alpha \in \mathbb{Z}$ and m is prime, then $H_K(\alpha, m)$ splits if and only if one of the following conditions holds:*

1. α is a square in $\mathbb{N}$;
2. $-\alpha$ is a square in $\mathbb{N}$ and m divides d or d is a quadratic residue modulo m ;
3. $\alpha = \pm \delta t^2$ with $\delta > 1$ and each prime divisor of δ , which splits in K , is a quadratic residue modulo m and
 - either m divides d or d is not a quadratic residue modulo m ;

- or d and α are quadratic residue modulo m .

Corollary 4.2. *If α is a unit of K and m divides d , then $H_K(\alpha, m)$ splits.*

Proof. It is enough to note that if α is a unit of K , then $\mathcal{I} = \emptyset$; and since m divides d we have $\lambda = \gamma = 0$. $\square$

4.2 Applications of Theorem 4.1

In this subsection, we state some applications of the first main theorem. We start with a classical lemma [10]:

Lemma 4.1. *Let K be a quadratic field.*

1. $K(\sqrt{\alpha})$ is a biquadratic number field if and only if $N(\alpha)$ is a square in $\mathbb{N}$.
2. $K(\sqrt{\alpha})$ is a quartic cyclic number field if and only if $dN(\alpha)$ is a square in $\mathbb{N}$.

The first application of the main result is the following theorem.

Theorem 4.2. *If m and α satisfy the hypotheses (H) and $K(\sqrt{\alpha}) = \mathbb{Q}(\sqrt{d}, \sqrt{d_1})$ is a biquadratic number field, then $H_K(\alpha, m)$ splits if and only if the following conditions are satisfied:*

1. $(\frac{d_1}{r_j}) = 1$, if $\gamma = 1$;
2. For any prime $p|d_1$ such that $(p, d) = 1$ and $(\frac{d}{p}) = 1$, we have $(\frac{m}{p}) = 1$.

Proof. As $K(\sqrt{\alpha}) = \mathbb{Q}(\sqrt{d}, \sqrt{d_1})$ is a biquadratic number field, then the previous Lemma imply the $N(\alpha)$ is a square; moreover $\alpha = t^2 d_1$ with $t \in \mathbb{Q}(\sqrt{d})$. So $H_K(\alpha, m)$ splits if and only if the following conditions are satisfied:

1. $(\frac{\alpha}{\mathcal{R}_j}) = (\frac{t^2 d_1}{\mathcal{R}_j}) = (\frac{d_1}{r_j}) = 1$, if $\gamma = 1$;
2. For any prime $\mathcal{P} \in \mathcal{I}$, we have $(\frac{m}{N(\mathcal{P})}) = 1$.

If a prime $\mathcal{P} \in \mathcal{I}$, then $v_{\mathcal{P}}(\alpha) = 2v_{\mathcal{P}}(t) + v_{\mathcal{P}}(d_1) = 2v_{\mathcal{P}}(t) + v_p(d_1)v_{\mathcal{P}}(p)$ is odd and $N(\mathcal{P}) = p$ is a prime integer, which is equivalent to $p|d_1$, $(p, d) = 1$, $(\frac{d}{p}) = 1$ and $(\frac{m}{p}) = 1$. $\square$

As a second application of the main Theorem, we assume that $K(\sqrt{\alpha})$ is a quartic cyclic number field.

Theorem 4.3. *If m and α satisfy hypotheses (H), $d \equiv 1 \pmod{4}$ and $K(\sqrt{\alpha}) = \mathbb{Q}(\sqrt{a + b\sqrt{d}})$ is quartic cyclic number field ($a^2 = d(b^2 + c^2)$), then $H_K(\alpha, m)$ splits if and only if the following conditions are satisfied:*

1. $\lambda = 0$ and $(-1)^{(r_j-1)(d-1)/8} (\frac{2}{r_j})^b (\frac{r_j}{d})_4 = 1$, if $\gamma = 1$;
2. For any prime $\mathcal{P} \in \mathcal{I}$, we have $(\frac{m}{N(\mathcal{P})}) = 1$.

Proof. It is an immediate consequence of the main Theorem (i.e. Theorem 4.1) and of the main theorem of [22]. $\square$

As a third application of the main Theorem, we give necessary and sufficient conditions for $H_K(\alpha, m)$ to split over K whenever $\alpha = \varepsilon_d$ is a fundamental unit of $K = \mathbb{Q}(\sqrt{d})$.

Theorem 4.4. *Let us assume that the norm of ε_d is -1 , and let $d = p_1 \cdots p_n$. Suppose that p_j and m are primes $\equiv 1 \pmod{4}$ such that $\gamma = 1$ and $(\frac{p_i}{r_i}) = 1$ for all $1 \leq j \leq n$ and $1 \leq i \leq s$. Then $H_K(\varepsilon_d, m)$ splits if and only if*

$$r_i \equiv 1 \pmod{4} \text{ and } \left(\frac{r_i}{d}\right)_4 \left(\frac{d}{r_i}\right)_4 = 1,$$

for all $1 \leq i \leq s$.

Proof. Note that m and $\alpha = \varepsilon_d$ satisfy the hypotheses (H), $\mathcal{I} = \emptyset$. Then [5, Corollary at page 143] gives the claimed result. $\square$

We now suppose that the norm of ε_d is 1. It is known that ε_d does not always belong to $\mathbb{Z}[\sqrt{d}]$. Actually, $\varepsilon_d \in \mathbb{Z}[\sqrt{d}]$ if $d \equiv 3, 2 \pmod{4}$ or $d \equiv 1 \pmod{8}$, but if $d \equiv 5 \pmod{8}$, ε_d may not belong to $\mathbb{Z}[\sqrt{d}]$. To simplify, we set $\varepsilon = \varepsilon_d^3$ if $d \equiv 5 \pmod{8}$ and $\varepsilon_d \notin \mathbb{Z}[\sqrt{d}]$, and $\varepsilon = \varepsilon_d$ otherwise.

Theorem 4.5. *Let m and d be squarefree integers such that $\gamma = 1$, let the norm of $\varepsilon = x + y\sqrt{d}$ be 1. Then $H_K(\varepsilon_d, m)$ splits if and only if one of the following conditions holds:*

1. $(\frac{d_1}{r_j}) = 1$ where d_1 divides d and $2d_1(x+1)$ is a square in $\mathbb{N}$ for all $1 \leq i \leq s$;
2. $(\frac{2}{r_j}) = (\frac{d_1}{r_j})$ where d_1 divides d and $d_1(x+1)$ is a square in $\mathbb{N}$ for all $1 \leq i \leq s$.

Proof. As the norm of ε_d is 1, then ε is too, i.e., $(x+1)(x-1) = dy^2$. Furthermore $K(\sqrt{\varepsilon})$ is a biquadratic field.

- If x is odd, then according to the unique prime factorization in $\mathbb{Z}$, there exist two squarefree integers d_1 and d_2 such that $d = d_1 d_2$, $(x+1) = 2d_1 y_1^2$ and $(x-1) = 2d_2 y_2^2$ with $y_1, y_2 \in \mathbb{N}$ and $2y_1 y_2 = y$. In this case, one can easily check that $2d_1(x+1)$ is a square in $\mathbb{N}$ and that $d_1 \varepsilon = (y_1 d_1 + y_2 \sqrt{d})^2$. With Theorem 4.2, we have $H_K(\varepsilon_d, m)$ splits if and only if $(\frac{d_1}{r_j}) = 1$.

- If x is even, then proceeding similarly, we get that $d_1(x+1)$ is a square in $\mathbb{N}$ and $2d_1 \varepsilon = (y_1 d_1 + y_2 \sqrt{d})^2$. As in the first case $H_K(\varepsilon_d, m)$ splits if and only if $(\frac{2}{r_j}) = (\frac{d_1}{r_j})$. $\square$

Remark 4.2. Note that if the norm of ε_d is 1, then d_1 exists and it is unique.

4.3 Second case: m and α are prime integers

In this subsection, we replace $\{\alpha, m\}$ by $\{p, q\}$, where p and q are rational primes. The purpose is then to establish the following theorem that classify division quaternion algebras $H_{\mathbb{Q}(\sqrt{d})}(p, q)$ over quadratic fields where $d \neq 1$ is a squarefree integer:

Theorem 4.6. *Let $d \neq 1$ be a squarefree integer and let $K = \mathbb{Q}(\sqrt{d})$, with discriminant Δ_K . Let p and q be two positive primes. Then the quaternion algebra $H_K(p, q)$ is a division algebra if and only if one of the following conditions holds:*

1. p and q are odd and distinct, p or $q \equiv 1 \pmod{4}$, $(\frac{p}{q}) = -1$ and $(\frac{\Delta_K}{p}) = 1$ or $(\frac{\Delta_K}{q}) = 1$;
2. $q = 2$, $p \equiv 3 \pmod{8}$ or $p \equiv 5 \pmod{8}$ and either $(\frac{\Delta_K}{p}) = 1$ or $d \equiv 1 \pmod{8}$;
3. p and q are odd, with $p \equiv q \equiv 3 \pmod{4}$, and
 - $(\frac{q}{p}) \neq 1$ and either $(\frac{\Delta_K}{p}) = 1$ or $d \equiv 1 \pmod{8}$;
 - or
 - $(\frac{p}{q}) \neq 1$ and either $(\frac{\Delta_K}{q}) = 1$ or $d \equiv 1 \pmod{8}$.

Proof. The proof of the first assertion is a simple deduction from Theorem 4.2. It suffices to take $\alpha, m \in \{p, q\}$ such that $\alpha \neq m$ and $m \equiv 1 \pmod{4}$. Assertions 2 and 3 follow easily applying other results which we obtained in this article, namely Propositions 4.5, 4.6 and 4.7 below, after taking into account Lemma 4.2 and Lemma 4.4 below. $\square$

To complete the proof of this second main result, we need some preliminary results. Let K be a number field and $\mathcal{O}_K$ its ring of integers. If $\mathcal{O}_K$ is a principal ideal domain, then we may identify the ideals of $\mathcal{O}_K$ with their generators, up to units. Thus, in a quaternion algebra H over $\mathbb{Q}$, the element D_H (the discriminant of H) turns out to be an integer, and H is split if and only if $D_H = 1$. On the other hand, a quaternion algebra $H_{\mathbb{Q}}(\alpha, \beta)$ is a division algebra if and only if there is a prime p such that $p | D_{H_{\mathbb{Q}}(\alpha, \beta)}$. We continue this note with two statements following from the classical Albert-Brauer-Hasse-Noether theorem. Proofs of specific formulations of this theorem can be found in [13, 4].

Theorem 4.7. *Let H_F be a quaternion algebra over a number field F and let K be a quadratic extension of F . Then there is an embedding of K into H_F if and only if no prime of F which ramifies in H_F splits in K .*

Proposition 4.1. *Let F be a number field and let K be a quadratic extension of F . Let H_F be a quaternion algebra over F . Then K splits H_F if and only if there exists an F -embedding $K \hookrightarrow H_F$.*

In [15] the second author obtained the following result about quaternion algebras over the field $\mathbb{Q}(i)$ which is also a simple deduction from Corollary 4.1.

Proposition 4.2. *Let $p \equiv 1 \pmod{4}$ be a prime integer and let m be an integer which is not a quadratic residue modulo p . Then the quaternion algebra $H_{\mathbb{Q}(i)}(m, p)$ is a division algebra.*

In [16] the second author obtained some sufficient conditions for a quaternion algebra $H_{\mathbb{Q}(i)}(p, q)$ to split, where p and q are two distinct primes:

Proposition 4.3. *Let $d \neq 0, 1$ be a squarefree integer such that $d \not\equiv 1 \pmod{8}$, and let p and q be two primes, with $q \geq 3$ and $p \neq q$. Let $\mathcal{O}_K$ be the ring of integers of the quadratic field $K = \mathbb{Q}(\sqrt{d})$ and let Δ_K be its discriminant.*

1. *If $p \geq 3$ and both $(\frac{\Delta_K}{p})$ and $(\frac{\Delta_K}{q})$ are not equal to 1, then the quaternion algebra $H_{\mathbb{Q}(\sqrt{d})}(p, q)$ splits;*
2. *If $p = 2$ and $(\frac{\Delta_K}{q}) \neq 1$, then the quaternion algebra $H_{\mathbb{Q}(\sqrt{d})}(2, q)$ splits.*

From the aforementioned results we deduce easily a necessary and sufficient condition for a quaternion algebra $H_{\mathbb{Q}(i)}(p, q)$ to be a division algebra:

Proposition 4.4. *Let p and q be two distinct odd primes, such that $(\frac{q}{p}) \neq 1$. Then the quaternion algebra $H_{\mathbb{Q}(i)}(p, q)$ is a division algebra if and only if $p \equiv 1 \pmod{4}$ or $q \equiv 1 \pmod{4}$.*

Proof. To prove the necessity, note that if $H_{\mathbb{Q}(i)}(p, q)$ is a division algebra, then Proposition 4.3 and Theorem 2.1 tell us that $(\frac{\Delta_K}{p}) = 1$ or $(\frac{\Delta_K}{q}) = 1$. This is equivalent to $p \equiv 1 \pmod{4}$ or $q \equiv 1 \pmod{4}$.

To prove the sufficiency, we must distinguish amongst two cases:

- $p \equiv 1 \pmod{4}$:
Since $(\frac{q}{p}) \neq 1$, Proposition 4.2 tells us that $H_{\mathbb{Q}(i)}(p, q)$ is a division algebra.
- $q \equiv 1 \pmod{4}$:
Since $(\frac{q}{p}) = -1$, the quadratic reciprocity law implies $(\frac{p}{q}) = -1$. Proposition 4.2 then tells us that $H_{\mathbb{Q}(i)}(p, q)$ is a division algebra. $\square$

We ask ourselves whether we can obtain a necessary and sufficient explicit condition for $H_{\mathbb{Q}(\sqrt{d})}(p, q)$ to be a division algebra when d is arbitrary. From Proposition 4.3 we obtain a necessary explicit condition for $H_{\mathbb{Q}(\sqrt{d})}(p, q)$ to be a division algebra, namely: if $H_{\mathbb{Q}(\sqrt{d})}(p, q)$ is a division algebra, then $(\frac{\Delta_K}{p}) = 1$ or $(\frac{\Delta_K}{q}) = 1$. However this condition is not sufficient: for example, if we let $K = \mathbb{Q}(\sqrt{3})$, $p = 7$, $q = 47$, then $(\frac{\Delta_K}{p}) \neq 1$ and $(\frac{\Delta_K}{q}) = 1$, the quaternion algebra $H_{\mathbb{Q}}(7, 47)$ is a division algebra, but the quaternion algebra $H_{\mathbb{Q}(\sqrt{3})}(7, 47)$ is not a division algebra.

It is known [11] that if a prime integer p divides $D_{H(\alpha,\beta)}$, then it must divide $2\alpha\beta$, hence we may restrict our attention to these primes. In other words, in order to obtain a sufficient condition for a quaternion algebra $H_{\mathbb{Q}(\sqrt{d})}(p, q)$ to be a division algebra, it is important to study the ramification of the primes $2, p, q$ in the algebra $H_{\mathbb{Q}}(p, q)$. The following lemma [1, Lemma 1.21] gives us a hint:

Lemma 4.2. *Let p and q be two primes, and let $H_{\mathbb{Q}}(p, q)$ be a quaternion algebra of discriminant D_H .*

1. *If $p \equiv q \equiv 3 \pmod{4}$ and $(\frac{q}{p}) \neq 1$, then $D_H = 2p$;*
2. *If $q = 2$ and $p \equiv 3 \pmod{8}$, then $D_H = pq = 2p$;*
3. *If p or $q \equiv 1 \pmod{4}$, with $p \neq q$ and $(\frac{p}{q}) = -1$, then $D_H = pq$.*

In addition, the following lemma [1, Lemma 1.20] tells us precisely when a quaternion algebra $H_{\mathbb{Q}}(p, q)$ splits.

Lemma 4.3. *Let p and q be two prime integers. Then, $H_{\mathbb{Q}}(p, q)$ is a matrix algebra if and only if one of the following conditions is satisfied:*

1. $p = q = 2$;
2. $p = q \equiv 1 \pmod{4}$;
3. $q = 2$ and $p \equiv \pm 1 \pmod{8}$;
4. $p \neq q$, $p \neq 2$, $q \neq 2$, $(\frac{q}{p}) = 1$ and either p or q is congruent to 1 $\pmod{4}$.

The next theorem [1, Theorem 1.22] describes the discriminant of $H_{\mathbb{Q}}(p, q)$, where p and q are primes:

Theorem 4.8. *Let $H = (\frac{a,b}{\mathbb{Q}})$ be a quaternion algebra.*

1. *If $D_H = 1$, then H splits.*
2. *If $D_H = 2p$, p prime and $p \equiv 3 \pmod{4}$, then $H \cong (\frac{p,-1}{\mathbb{Q}})$.*
3. *If $D_H = pq$, p, q primes, $q \equiv 1 \pmod{4}$ and $(\frac{p}{q}) = -1$, then $H \cong (\frac{p,q}{\mathbb{Q}})$.*

If a and b are prime numbers, then the algebra H satisfies one and only one of the above statements.

Note that when $q = 2$ and p is a prime such that $p \equiv 3 \pmod{8}$, then, according to Lemma 4.2 the discriminant $D_{H_{\mathbb{Q}}(p,q)}$ is equal to $2p$, so $H_{\mathbb{Q}}(p, q)$ is a division algebra. The next proposition shows what happens when we extend the field of scalars from $\mathbb{Q}$ to $\mathbb{Q}(\sqrt{d})$.

Proposition 4.5. *Let p be an odd prime, with $p \equiv 3 \pmod{8}$. Let $K = \mathbb{Q}(\sqrt{d})$ and let Δ_K be the discriminant of K . Then $H_{\mathbb{Q}(\sqrt{d})}(p, 2)$ is a division algebra if and only if $(\frac{\Delta_K}{p}) = 1$ or $d \equiv 1 \pmod{8}$.*

Proof. If $H_{\mathbb{Q}(\sqrt{d})}(p, 2)$ is a division algebra then, from Proposition 4.3, Theorem 2.1, Theorem 4.7 and Proposition 4.1, we conclude that $(\frac{\Delta_K}{p}) = 1$ or $d \equiv 1 \pmod{8}$.

Conversely, since $p \equiv 3 \pmod{8}$ then, according to Lemma 4.2(ii) we must have $D_H = 2p$. It follows that the primes which ramify in $H_{\mathbb{Q}}(p, 2)$ are precisely p and 2. Since either $(\frac{\Delta_K}{p}) = 1$ or $d \equiv 1 \pmod{8}$ then, by the decomposition of primes in quadratic fields, we obtain that either p or 2 splits in the ring or integers of K . From Theorem 4.7, Proposition 4.1 and Theorem 2.1, we conclude that $H_{\mathbb{Q}(\sqrt{d})}(p, 2)$ is a division algebra. $\square$

We next study the case when p and q are primes, both congruent to 3 modulo 4. If $(\frac{q}{p}) \neq 1$, then, according to Lemma 4.2(i), the discriminant $D_{H_{\mathbb{Q}}(p, q)}$ is equal to $2p$, so $H_{\mathbb{Q}}(p, q)$ is a division algebra. The next proposition tells us when the quaternion algebra $H_{\mathbb{Q}(\sqrt{d})}(p, q)$ is still a division algebra.

Proposition 4.6. *Let p and q be two odd prime integers, with $p \equiv q \equiv 3 \pmod{4}$ and $(\frac{q}{p}) \neq 1$. Let $K = \mathbb{Q}(\sqrt{d})$ and let Δ_K be the discriminant of K . Then the quaternion algebra $H_{\mathbb{Q}(\sqrt{d})}(p, q)$ is a division algebra if and only if $(\frac{\Delta_K}{p}) = 1$ or $d \equiv 1 \pmod{8}$.*

Proof. If $H_{\mathbb{Q}(\sqrt{d})}(p, q)$ is a division algebra then from Proposition 4.3(i) it follows that either $(\frac{\Delta_K}{p}) = 1$ or $(\frac{\Delta_K}{q}) = 1$. But, according to Lemma 4.2(i) we must have $D_{H_{\mathbb{Q}}(p, q)} = 2p$. So the integral primes which ramify in $H_{\mathbb{Q}}(p, q)$ and could split in K are precisely p and 2. Finally, after applying Theorem 2.1, Theorem 4.7, Proposition 4.1 and the decomposition of primes in quadratic fields, we obtain that either $(\frac{\Delta_K}{p}) = 1$ or $d \equiv 1 \pmod{8}$. The proof of the converse is similar to the proof of sufficiency of Proposition 4.5. $\square$

The only case left out is $q = 2$, $p \equiv 5 \pmod{8}$. We consider first the quaternion algebra $H_{\mathbb{Q}}(p, 2)$, and we get the following result:

Lemma 4.4. *Let $p \equiv 5 \pmod{8}$ be a prime integer. Then the discriminant of the quaternion algebra $H_{\mathbb{Q}}(p, 2)$ is equal to $2p$, and hence $H_{\mathbb{Q}}(p, 2)$ is a division algebra.*

Proof. We give here a simple proof which is independent of the theorems stated above. We know that if a prime divides the discriminant of $H_{\mathbb{Q}}(a, b)$ then it must divide $2ab$. Since $p \equiv 5 \pmod{8}$, from the properties of the Hilbert symbol and of the Legendre symbol we obtain: $(2, p)_p = (\frac{2}{p}) = (-1)^{\frac{p^2-1}{8}} = -1$ and $(2, p)_2 = (-1)^{\frac{p-1}{2} \cdot \frac{1-1}{2} + \frac{p^2-1}{8}} = -1$. Hence the primes which ramify in $H_{\mathbb{Q}}(p, 2)$ are exactly 2 and p . Therefore, the reduced discriminant of $H_{\mathbb{Q}}(p, 2)$ must be equal to $2p$. $\square$

We turn now our attention to the quaternion algebra $H_{\mathbb{Q}(\sqrt{d})}(p, q)$, where $q = 2$ and $p \equiv 5 \pmod{8}$.

Proposition 4.7. *Let p be an odd prime, with $p \equiv 5 \pmod{8}$. Let $K = \mathbb{Q}(\sqrt{d})$ and let Δ_K be the discriminant of K . Then $H_{\mathbb{Q}(\sqrt{d})}(p, 2)$ is a division algebra if and only if $(\frac{\Delta_K}{p}) = 1$ or $d \equiv 1 \pmod{8}$.*

Proof. The proof is similar to the proof of Proposition 4.5, after replacing Lemma 4.2 with Lemma 4.4. $\square$

Taking into account these results and Theorems 2.1 and 4.6, we are able to understand when a quaternion algebra $H_{\mathbb{Q}(\sqrt{d})}(p, q)$ splits. It is clear also that in each one of the cases covered by Lemma 4.3, a quaternion algebra $H_{\mathbb{Q}(\sqrt{d})}(p, q)$ splits.

5. Examples involving prime Fibonacci numbers

Let $(F_n)_{n \geq 0}$ be the Fibonacci sequence which is defined by the following recurrence relation:

$$F_0 = 0, F_1 = 1 \text{ and } F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 2.$$

Binet's formula, which has been discovered by the mathematician J. P. Marie Binet (1786- 1856), states that:

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n \right].$$

By means of this formula, one can show that (see [8]):

$$F_{2n+1} = F_{n+1}^2 + F_n^2 \text{ and } 4^n F_{2n+1} = \sum_{k=0}^n 5^k C_{2n+1}^{2k+1}.$$

where C_m denotes the m -th Catalan number. From this formula we deduce that if p is an odd prime number > 5 , then $F_p \equiv 1 \pmod{4}$ and $F_p \equiv \left(\frac{p}{5}\right) \pmod{p}$. This implies the following lemma.

Lemma 5.1. *Let p be a prime > 5 . Then the Fibonacci number F_p has the following properties:*

1. *if $p \equiv 1 \pmod{4}$, then $(\frac{F_p}{p}) = 1$;*
2. *if $p \equiv 3 \pmod{4}$, then $(\frac{F_p}{p}) = (\frac{p}{5})$.*

To prove the last theorem of this paper, we also need the following lemma.

Lemma 5.2 ([19]). *Let F_p be a Fibonacci number with prime index $p \equiv 1 \pmod{4}$. Then*

$$\left(\frac{F_p}{p}\right)_4 \left(\frac{p}{F_p}\right)_4 = \begin{cases} 1, & \text{if } p \equiv 1 \pmod{3}, \\ \left(\frac{2}{p}\right), & \text{if not.} \end{cases}$$

Theorem 5.1. *Let F_p be a Fibonacci prime number with $p > 5$ and let $K = \mathbb{Q}(\sqrt{p})$ be a quadratic number field. Then $H_K(\varepsilon_p, F_p)$ splits if and only if one of the following conditions holds:*

1. $p \equiv 1 \pmod{12}$;
2. $p \equiv 17 \pmod{24}$;
3. $p \equiv 3 \pmod{4}$, $p^2 \equiv 1 \pmod{5}$ and $(\frac{2}{F_p}) = 1$;
4. $p \equiv 3 \pmod{4}$ and $p^2 \equiv 4 \pmod{5}$.

Proof. First of all, recall that if F_n is prime, then n is prime; and n divides m if and only if F_n divides F_m , hence since F_p is a prime Fibonacci number such that $p > 5$, it follows that p is an odd prime.

• If $p \equiv 1 \pmod{4}$, then $N(\varepsilon_p) = -1$ and $(\frac{p}{F_p}) = (\frac{F_p}{p}) = 1$ ((1) of Lemma 5.1). We can then apply Corollary 4.4 and find that $H_K(\varepsilon_p, F_p)$ splits if and only if $(\frac{F_p}{p})_4(\frac{p}{F_p})_4 = 1$. According to Lemma 5.2, this is equivalent to $p \equiv 1 \pmod{3}$ or $p \equiv 1 \pmod{8}$. By the Chinese remainder theorem, we deduce that $p \equiv 1 \pmod{12}$ or $p \equiv 17 \pmod{24}$.

• If $p \equiv 3 \pmod{4}$, then $N(\varepsilon_p) = 1$. We must now study the value of the symbol $(\frac{F_p}{p})$. If $(\frac{F_p}{p}) = 1$, then the second assertion of Lemma 5.1 implies that $(\frac{5}{p}) = 1$, i.e., $p^2 \equiv 1 \pmod{5}$. In this case $H_K(\varepsilon_p, F_p)$ splits if and only if $(\frac{2}{F_p}) = 1$ ([2, Lemma 6] and 2. of Theorem 4.5). Finally, if $p^2 \equiv 4 \pmod{5}$, then Theorem 4.2 shows that $H_K(\varepsilon_p, F_p)$ splits. $\square$

As an application of this theorem, we used the SageMath software [18] to find all prime or probably prime Fibonacci numbers F_p ($p > 5$) known until today (see sequence A001605 in the OEIS or see Wikipedia) such that $H_K(\varepsilon_p, F_p)$ and p satisfy conditions 1, 2, 3 and 4 above:

1. $p \in \{13, 433, 25561, 30757, 50833, 130021, 590041, 593689, 1968721\}$;
2. $p \in \{17, 137, 449, 569, 1049897, 2904353\}$;
3. $p \in \{11, 131, 359, 431, 9311, 35999, 37511, 81839, 148091, 397379, 1803059\}$;
4. $p \in \{7, 23, 43, 47, 83, 4723, 5387, 201107, 1285607, 1636007\}$.

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An hybrid block method for direct integration of first, second and third order IVPs

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Abstract. Development of numerical methods for the solution of initial value problems (IVPs) in ordinary differential equations (ODEs) has been considered overwhelmingly in literature. However, the use of a single numerical method for the integration of ODEs of different order has not been commonly reported.

In this paper, we focus on development of a numerical method capable of obtaining the numerical solution of first, second and third order IVPs. The method is formulated from continuous schemes obtained via collocation and interpolation techniques and applied in a block-by-block manner as numerical integrator for first, second and third order ODEs. The convergence properties of the method are discussed via zero-stability and consistency. Numerical examples are included and comparisons are made with existing methods in the literature.

Keywords: consistency, convergence, hybrid block method, initial value problems, zero-stability.

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1. Introduction

In science and engineering, many physical problems are modelled by ordinary differential equations of the form

$$(1) \quad y^m(x) = f(x, y(x), y'(x), \dots, y^{m-1}(x)), \quad y^{m-1}(x_0) = \gamma, \quad a \leq x \leq b.$$

Many of the equations resulting from these problems have no analytical solutions hence, the need for efficient numerical methods to address this difficulty as the quest for the solution is on the increase.

Several numerical methods have been proposed for the solution of (1) for $m = 1, 2, 3, 4$ with the ultimate goal of improving on the efficiency and convergence of the existing methods (see Butcher (2003), Yahaya *et al.* (2016), Adeniyi and Alabi (2011), Jator (2007, 2010a, 2010b), Kuboye (2015), Kuboye *et al.* (2018), Lambert (1973, 1991), Ismail (2009), Ramos *et al.* (2016), Simos (2002)). The advent of the self-starting method popularly called block method has been adopted greatly by researchers and discussed extensively in literature (see Sagir (2014), Vigo-Aguiar and Ramos (2006), Olabode (2009, 2013), Ramos *et al.* (2016), Adeyefa (2017)) among others. The formulation of block method to integrate (1) has been widely reported in the literature to be self starting and efficient. However, to use a formulated block method for integration of several order IVPs, say first, second and third order ODEs has not been commonly reported. Thus, the focus of this paper is to formulate a self-starting method for the numerical solution of the ODE (1) for $m = 1, 2$ and 3.

In what immediately follows in Section two, we derive the proposed method and discuss the basic properties of the method in Section three. Numerical examples are given to show the efficiency of the method in Section four while the discussion of results is given in Section five. Finally, the conclusion of the paper is discussed in Section six.

2. Materials and methods

In this section, we formulate new one-step hybrid method capable of solving first, second and third order ODEs.

Our approximate solution is formulated using constructed orthogonal polynomials $q_0(x) = 1$, $q_1(x) = \frac{1}{2}(3x - 1)$, $q_2(x) = \frac{1}{2}(5x^2 - 2x - 1)$, $q_3(x) = \frac{1}{8}(35x^3 - 15x^2 - 15x + 3)$, $q_4(x) = \frac{1}{8}(63x^4 - 28x^3 - 42x^2 + 12x + 3)$, $q_5(x) = \frac{1}{16}(231x^5 - 105x^4 - 210x^3 + 70x^2 + 35x - 5)$, $q_6(x) = \frac{1}{16}(429x^6 - 198x^5 - 495x^4 + 180x^3 + 135x^2 - 30x - 5)$ constructed in Adeyefa and Adeniyi (2015) as our basis function. Thus, we introduce

$$(2) \quad y(x) = \sum_{j=0}^{k+8} a_j q^j$$

as an approximate solution to first, second and third order ODEs of the form

$$(3) \quad y'(x) = f(x, y(x)), \quad y(x_0) = y_0,$$

$$(4) \quad y''(x) = f(x, y(x), y'(x)), \quad y(x_0) = y_0, \quad y'(x_0) = y'_0,$$

$$(5) \quad y'''(x) = f(x, y(x), y'(x), y''(x)), \quad y(x_0) = y_0, \quad y'(x_0) = y'_0, \quad y''(x_0) = y''_0,$$

where a and q in (2) are constants to be determined and the constructed orthogonal polynomials respectively.

Equation (2) is interpolated at $x = x_n$, its first and second derivatives are collocated at $x = x_{n+v}$, $v = 0, \frac{1}{3}, 1$ while its third derivative is collocated at $x = x_{n+r}$, $r = 0, \frac{1}{3}$.

As a result, we have

$$(6) \quad \begin{aligned} \sum_{j=0}^{k+8} \alpha_j q_n^j &= y_n, \\ \sum_{j=1}^{k+8} j \alpha_j q_{n+r}^{j-1} &= f_{n+r}, \\ \sum_{j=2}^{k+8} j(j-1) \alpha_j q_{n+r}^{j-2} &= g_{n+r}, \\ \sum_{j=3}^{k+8} j(j-1)(j-2) \alpha_j q_{n+r}^{j-3} &= k_{n+r}. \end{aligned}$$

Solving equation (6) using Gaussian elimination approach in order to get the unknown variables a 's which are substituted into equation (2). This yields a continuous implicit scheme of the form:

$$(7) \quad \begin{aligned} \alpha_0(t) y_n &= h \left(\sum_{j=0}^k \beta_j(t) f_{n+j} + \beta_{\frac{1}{3}}(t) f_{n+\frac{1}{3}} \right) \\ &+ h^2 \left(\sum_{j=0}^k \lambda_j(t) g_{n+j} + \lambda_{\frac{1}{3}}(t) g_{n+\frac{1}{3}} \right) + h^3 \left(\delta_0(t) k_n + \delta_{\frac{1}{3}}(t) k_{n+\frac{1}{3}} \right) \end{aligned}$$

where

$$\begin{aligned} t &= \frac{2x - 2x_n - h}{h}, \quad (\alpha_0(t) = 1, \\ \beta_0 &= -\frac{1}{71680} h(t+1)(70875t^7 + 33885t^6 - 186345t^5 - 97743t^4 + 127633t^3 \\ &\quad + 86007t^2 + 37053t - 8773), \\ \beta_{\frac{1}{3}} &= \frac{243}{1146880} h(4725t^4 - 11820t^3 + 8990t^2 - 2924t + 1781)(t+1)^4, \\ \beta_1 &= -\frac{1}{1146880} h(14175t^4 - 12420t^3 - 11430t^2 - 4164t - 289)(t+1)^4, \\ \lambda_0 &= -\frac{1}{71680} h^2(10395t^6 - 6750t^5 - 19995t^4 + 9052t^3 + 7701t^2 \\ &\quad + 4386t - 373)(t+1)^2, \end{aligned}$$

$$\begin{aligned}
\lambda_{\frac{1}{3}} &= -\frac{81}{143360}h^2(315t^4 - 780t^3 + 530t^2 - 44t + 11)(t+1)^4, \\
\lambda_1 &= \frac{1}{114688}h^1(189t^4 - 108t^3 - 114t^2 - 44t - 3)(t+1)^4, \\
\delta_0 &= -\frac{1}{430080}h^3(2835t^5 - 5265t^4 + 990t^3 + 1086t^2 + 927t - 29)(t+1)^3, \\
\delta_{\frac{1}{3}} &= \frac{9}{573440}h^3(945t^4 - 1980t^3 + 710t^2 + 388t + 133)(t+1)^4.
\end{aligned}$$

Equation (7) is evaluated at $x = x_{n+1}$ (i.e. $t = 1$) and $x = x_{n+\frac{1}{3}}$ (i.e. $t = -\frac{1}{3}$) to obtain the following schemes

$$(8) \quad \begin{pmatrix} y \\ n+\frac{1}{3} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} y_n + hD \begin{pmatrix} f_n \\ f_{n+\frac{1}{3}} \\ f_{n+1} \end{pmatrix} + h^2E \begin{pmatrix} g_n \\ g_{n+\frac{1}{3}} \\ g_{n+1} \end{pmatrix} + h^3F \begin{pmatrix} k_n \\ k_{n+\frac{1}{3}} \\ k_{n+1} \end{pmatrix}$$

where the values of D, E and F are

$$\begin{aligned}
D &= \begin{pmatrix} \frac{3527}{22680} & \frac{797}{4480} & -\frac{29}{362880} \\ \frac{489}{280} & \frac{4480}{4480} & \frac{883}{4480} \\ -\frac{1}{280} & -\frac{1}{280} & -\frac{1}{448} \end{pmatrix}, E = \begin{pmatrix} \frac{641}{68040} & -\frac{11}{840} & \frac{1}{108864} \\ \frac{69}{280} & -\frac{81}{280} & -\frac{5}{448} \\ -\frac{1}{280} & -\frac{1}{280} & -\frac{1}{448} \end{pmatrix}, \\
F &= \begin{pmatrix} \frac{31}{136080} & \frac{83}{181440} \\ \frac{17}{1680} & \frac{99}{2240} \\ -\frac{1}{1680} & -\frac{1}{2240} \end{pmatrix}.
\end{aligned}$$

Equation (8) is our proposed first, second and third order IVPs solver.

3. Basic properties of the method

We shall consider in this section, the analysis of basic properties of this method such as order, error constant, zero stability and consistency is investigated.

3.1 Order and error constant

Equation (8) derived is a discrete scheme belonging to the class of LMMs of the form

$$(9) \quad \sum_{j=0}^k \alpha_j y_{n+j} = h \sum_{j=0}^k \omega_j f_{n+j} + h^2 \sum_{j=0}^k \beta_j g_{n+j} + h^3 \sum_{j=0}^k \gamma_j G_{n+j}.$$

Following, Fatunla (1988) and Lambert (1991), we define the local truncation error associated with equation (9) by the difference operator

$$\begin{aligned}
L[y(x) : h] &= \sum_{j=0}^k [\alpha_j y(x_n + jh) - h\omega_j f(x_n + jh) - h^2\beta_j g(x_n + jh) \\
(10) \quad &\quad - h^3\gamma_j G(x_n + jh)],
\end{aligned}$$

where $y(x)$ is an arbitrary function, continuously differentiable on $[a, b]$.

Expanding (10) in Taylor series about the point x , we obtain the expression

$$L[y(x); h] = C_0 y(x) + C_1 h y'(x) + C_2 h^2 y''(x) + \cdots + C_{p+2} h^{p+2} y^{(p+2)}(x),$$

where the $C_0, C_1, C_2, \dots, C_p, \dots, C_{p+2}$ are obtained as

$$\begin{aligned} C_0 &= \sum_{j=0}^k \alpha_j, C_1 = \sum_{j=1}^k j \alpha_j, \\ C_2 &= \frac{1}{2} \sum_{j=0}^k j^2 \alpha_j, \\ C_q &= \frac{1}{q!} \left[\sum_{j=1}^k j^q \alpha_j - q(q-1) \sum_{j=1}^k \beta_j j^{q-2} - q(q-1)(q-2) \sum_{j=1}^k \gamma_j j^{q-3} \right]. \end{aligned}$$

In the spirit of Lambert (1991), equation (10) is of order p if $C_0 = C_1 = C_2 = \cdots = C_p = C_{p+1}$ and $C_{p+2} \neq 0$. The $C_{p+2} \neq 0$ is called the error constant and $C_{p+2} h^{p+2} y^{(p+2)}(x_n)$ is the principal local truncation error at the point x_n .

Thus, the block (8) is of order $p = 6$ and error constant

$$C_{p+3} = \left[\frac{113}{1999918771200}, \frac{13}{914457600} \right]^T.$$

3.2 Zero stability of the method

The linear multistep method (9) is said to be zero-stable if no root of the first characteristic polynomial $\rho(R)$ has modulus greater than one and if every root of modulus one has multiplicity not greater than the order of the differential equation.

To analyze the zero-stability of the method, we present (9) in vector notation form of column vectors $e = (e_1, \dots, e_\gamma)^T$, $d = (d_1, \dots, d_\gamma)^T$, $y_m = (y_{n+1}, \dots, y_{n+\gamma})^T$, $F(y_m) = (f_{n+1}, \dots, f_{n+\gamma})^T$, $G(y_m) = (g_{n+1}, \dots, g_{n+\gamma})^T$, $W(y_m) = (k_{n+1}, \dots, k_{n+\gamma})^T$ and matrices $A = (a_{ij})$, $B = (b_{ij})$. Thus, equation (8) forms the block formula

$$(11) \quad A_0 y_m = h B F(y_m) + A^1 y_n + h b f_n + h^2 D G(y_m) + h^2 d g_n + h^3 V W(y_m) + h^3 u T_n,$$

where h is a fixed mesh size within a block.

In line with equation (11), $A^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $A^1 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$. The first characteristic polynomial of the block hybrid method is given by

$$(12) \quad \rho(R) = \det(RA^0 - A^1).$$

Substituting $A^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $A^1 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ in equation (12) and solving for R , the values of R are obtained as 0 and 1.

According to Fatunla in (1988, 1991), the block method represented by equation (8) are zero-stable, since from equation (12), $\rho(R) = 0$, satisfy $|R_j| \leq 1$, $j = 1$ and for those roots with $|R_j| = 1$, the multiplicity does not exceed two.

3.3 Consistency and convergence of the method

The linear multistep method (9) is said to be consistent if it has order $p \geq 1$. The method is consistent being of order 6.

According to the theorem of Dahlquist in Dahlquist (1979), the necessary and sufficient condition for a LMM to be convergent is to be consistent and zero stable. Since the method satisfies the two conditions hence it is convergent.

4. Numerical experiment

We consider in this section, four test problems which includes first, second and third order ordinary differential equations to test the effectiveness of this new scheme.

Problem 1. We consider the IVP $y'' = y'$, $y(0) = 0$, $y'(0) = -1$, $h = 0.1$ with exact solution, $y(x) = 1 - e^x$ which has been solved in Mohammed and Adeniyi (2014) with step number $k = 5$ and in Kuboye (2015) with step number $k = 5$.

Problem 2. We consider the IVP $y''' = e^x$, $y(0) = 3$, $y'(0) = 1$, $y''(0) = 5$, $h = 0.1$ for which the exact solution is given by $y(x) = 2 + 2x^2 + e^x$.

This example was solved in Abdullahi *et al.* (2016) and in Olabode (2009) using step number $k = 5$ methods.

Problem 3. Here we consider another IVP $y''' = 3 \sin x$, $y(0) = 1$, $y'(0) = 0$, $y''(0) = -2$, $h = 0.1$ solved by Olabode (2013) and Kuboye (2015) with step number $k = 6$ and for which the exact solution is given by $y(x) = 3 \cos x + \frac{x^2}{2} - 2$.

Problem 4. We consider first order IVP $y' = 0.5(1 - y)$, $y(0) = 0.5$, $h = 0.1$ with the exact solution $y(x) = 1 - 0.5e^{-0.5x}$. This IVP was solved by Ajileye *et al.* (2018) and Sunday *et al.* (2013).

Problem 5. We consider non-linear IVP $y'' - x(y')^2 = 0$, $y(0) = 1$, $y'(0) = 0.5$, $h = 0.003125$ whose exact solution is $1 + \frac{1}{2} \ln(\frac{2+x}{2-x})$.

Details of the numerical results for Problems 1, 2, 3, 4 and 5 are shown in Figures 1, 2, 3, 4 and 5 respectively and discussed in Section five.

5. Discussion of results

Figures 1, 2, 3 and 4 summarize the results obtained for the four test problems considered. In Figures 1a, 2a, 3a and 4a, comparisons of the exact and approximate solutions are displayed while the Errors are compared in Figures 1b, 2b,

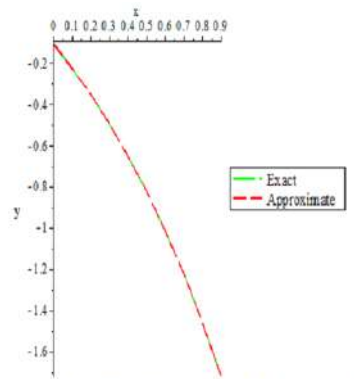


Fig 1a: Comparing exact and approximate solutions of Problem 1

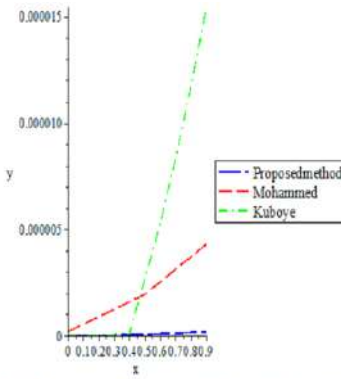


Fig 1b: Error comparison of existing methods and proposed method for Problem 1

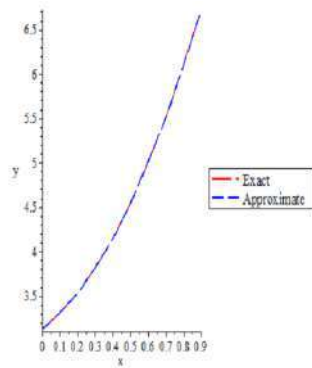


Fig 2a: Comparing exact and approximate solutions of Problem 2

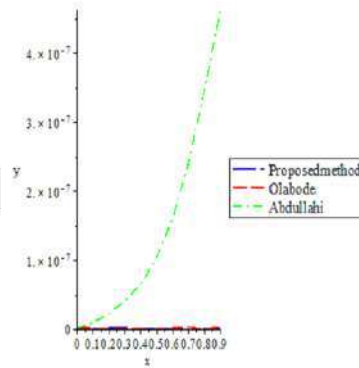


Fig 2b: Error comparison of existing methods and proposed method for Problem 2

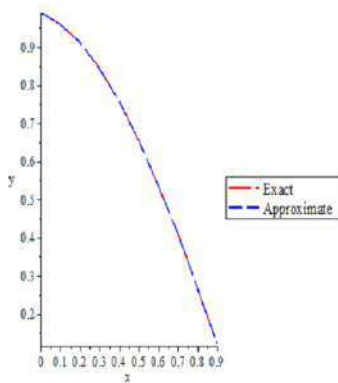


Fig 3a: Comparing exact and approximate solutions of Problem 3

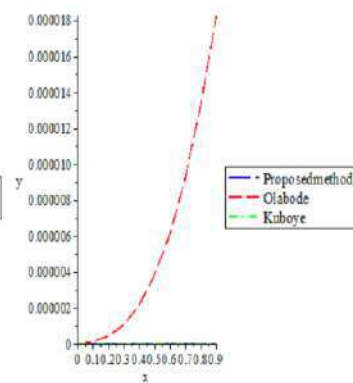


Fig 3b: Error comparison of existing methods and proposed method for Problem 3

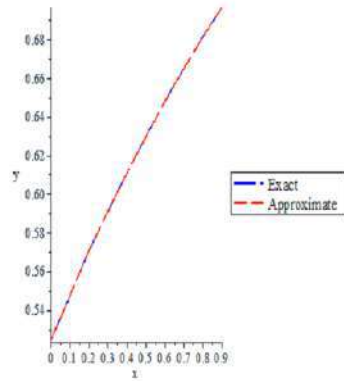


Fig 4a: Comparing exact and approximate solutions of Problem 4

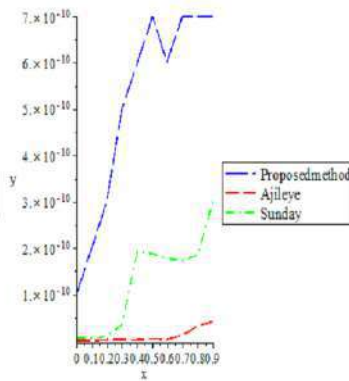
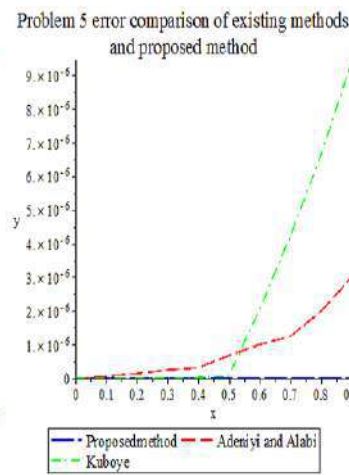
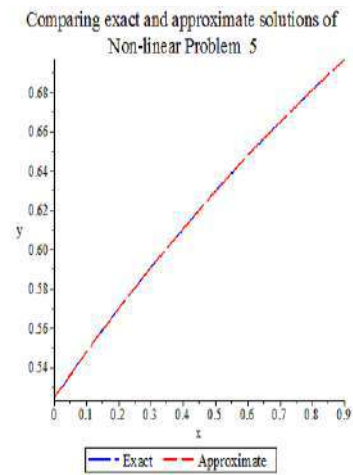


Fig 4b: Error comparison of existing methods and proposed method for Problem 4



3b and 4b. The proposed method is of step number $k = 1$ and it compared favourably with existing methods despite their $k > 1$ methods. In Problem 1, our step length is $h = 0.1$ against $h = 0.01$ used in Mohammed and Adeniyi (2014) and Kuboye (2015). The proposed method still gives better accuracy even with larger h . In Figure 4, the methods developed by Ajileye *et al.* (2018) and Sunday *et al.* (2013) performed better than the new method in terms of accuracy but their methods do not have the ability to solve higher order ordinary differential equations.

Figure 5 displays the comparison of the exact and the approximate solutions and the error between the existing methods and the proposed method. It is obvious that the new method compares favourably with the existing methods.

6. Conclusion

A block method has been applied to solve first, second and third order ordinary differential equations directly without construction of additional schemes or employing existing predictors for implementation. Numerical experiments performed using this method show that the method is consistent, efficient and accurate. We therefore recommend the method for direct integration of first, second and third order ordinary differential equations.

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Ordinary differential equations of the probability functions of the Benktander kind II distribution with $b = 1$ and its properties

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Abstract. The idea of convolution is the sum of independent and identically distributed (iid) random variables and the structure of linear combination of random variables. The cases of ordinary differential equation (ODE) of the convolution of probability distributions of mixture of Benktander distribution of the second kind have been studied. Moreover, the ODE of quantile function (QF), survival function (SF), hazard function (HF) and reversed hazard function (RHF) of convoluted probability distributions has been considered. We obtain explicit forms for the densities and distribution functions for Benktander distribution of the second kind, as well as their moments and related parameters. We derive basic properties of these laws and illustrate their modeling possible using a simulated data.

Keywords: convolution, Benktander distribution, ordinary differential equations.

1. Introduction

A differential equation is an equation with unknown function, the function and its derivatives may possibly appear in the equation. Differential equations are crucial for a mathematical explanation of nature. They are positioned at the center of many physical theories. A classic application of differential equations proceeds along the solution of mathematical model and interpreting the model. The ordinary differential equations (ODE) according to whether or not they have partial derivatives. The order of a differential equation is the highest order derivative happening. A solution (or particular solution) of a differential equation of order n consists of a function defined and n times differentiable on a domain D having the property that the functional equation achieved by substi-

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tuting the function and its n derivatives into the differential equation holds for all points in D [1]. The idea of convolution is the sum of iid random variables and the structure of linear combination of random variables. The primary motivation of convolution is to derive probability density function (PDF) of a sum of random variables. Moreover, the motivation is to conclude if the convolution random variables are more flexible than the base random variables. The general formula for the sum $Z = X + Y$ of continuous probability distribution is defined as

$$(1) \quad f(Z) = \int_{-\infty}^z f(z-x)f(x)dx = \int_{-\infty}^z f(z-y)f(y)dy.$$

In statistical distributions theory, numerous convolutions are studied for both discrete and continuous random variables. For examples, the sum of exponential random variables is gamma random variables; the sum of Bernoulli random variables is Binomial random variable, the sum of geometric random variables is Negative Binomial random variable and many others. In mathematical modeling uncertainties is very useful in many areas such as; ecology, engineering, meteorology, astronomy, medicine, geology, finance, ecology, economics and so on. There are many researchers introduced many convoluted distributions for examples; [2] introduced a convoluted Poisson Distributions with applications to Estimation, F distribution and Beta Prime distribution or Inverted Beta distribution pioneered in the text books by [3] and [4] respectively. [5] studied the products and convolutions of Gaussian probability density functions. The convoluted Beta-Exponential Distribution proposed by [6]. Moreover, [7] introduced Quantile approximation of the Chi-Square Distribution using the Quantile Mechanics. Again they are found the solutions of Chis-Quare quantile differential equation [8]. [9] studied the ODEs of probability functions of convoluted distributions. The ODE of Lomax distribution introduced by [10]. The beta distribution initiated by [11]. Raised Cosine distribution recovered by [12] and [13] performed the convoluted exponential distribution. ODE has supportive in statistical and probabilistic representation. It can also be seen as a tool for measuring uncertainties and forecasting. The cases of ODE of the convolution of probability distributions of mixture of Benktander distribution of the second kind have not been studied in literature. Moreover, the ODE of quantile function (QF), survival function (SF), hazard function (HF) and reversed hazard function (RHF) of convoluted probability distributions has not been considered.

2. The convolution Benktander distribution of the second kind

The Benktander distribution of the second kind is one of two distributions introduced by [14] to model heavy-tailed losses commonly found in non-life Actuarial sciences, using various forms of mean excess functions. The Benktander distribution of the first kind is "close" to the Weibull distribution [15].

The pdf and cdf of this distribution defined as follows respectively

$$(2) \quad f(x) = e^{\frac{a}{b}(1-x)} x^{b-2} (ax^b - b + 1); \quad a > 0, 0 < b < 1 \text{ and } x > 0,$$

$$(3) \quad F(x) = 1 - x^{b-1} e^{\frac{a}{b}(1-x^b)}; \quad a > 0, 0 < b < 1 \text{ and } x > 0.$$

By assuming $b = 1$ the benktander distribution becomes

$$(4) \quad f(x) = ae^{a(1-x)}; \quad a > 0 \text{ and } x > 0,$$

$$(5) \quad F(x) = 1 - e^{a(1-x)}; \quad a > 0 \text{ and } x > 0.$$

Definition 2.1. Let X and Y be independent random variable belong to benktander distribution with constant parameter, for $x > 1$, $b = 1$ and $a > 0$. Then the PDF of the convolution of benktander distribution (CBD) is given by:

$$(6) \quad f(z) = \int_1^{z-1} f(z-x)f(x)dx,$$

$$(7) \quad f(z) = a^2 \int_1^{z-1} e^{a(1-z+x)} e^{a(1-x)} dx,$$

$$(8) \quad f(z) = a^2 e^{a(2-z)}(z-2); \quad a > 0 \text{ and } z \geq 2.$$

Definition 2.2. Let X and Y be independent distributed random variables belong to benktander distribution with constant parameter, for $x \geq 1$, $y \geq 1$, $b = 1$ and $a > 0$. Then the CDF of the convolution of benktander distribution (CBD) is given by:

$$(9) \quad F(z) = \int_2^z a^2 e^{a(2-t)}(t-2)dt = 1 + (2a - az - 1)e^{2a-az}; \quad a > 0 \text{ and } z \geq 2.$$

The graphs of the pdf and the cdf of convoluted benktander distribution with $b=1$ as defined in (9) and (10) for different values of a are shown in Figures 1 and 2 respectively.

3. Constant parameter convoluted benktander distribution (CBD)

The differential calculus used to find the various ODEs. The quantile mechanics approach used to find the nonlinear second order ODEs [16], [17] and [18]. It is used to find the quantile functions of the constant parameter convoluted benktander distribution. In this section, we derived various ODEs for the CBD as in the following propositions:

Proposition 1. The first ordinary differential equation whose solution is the pdf of CBD is defined as

$$(10) \quad f'(x)(z-2) - f(z)[1 - a(z-2)] = 0.$$

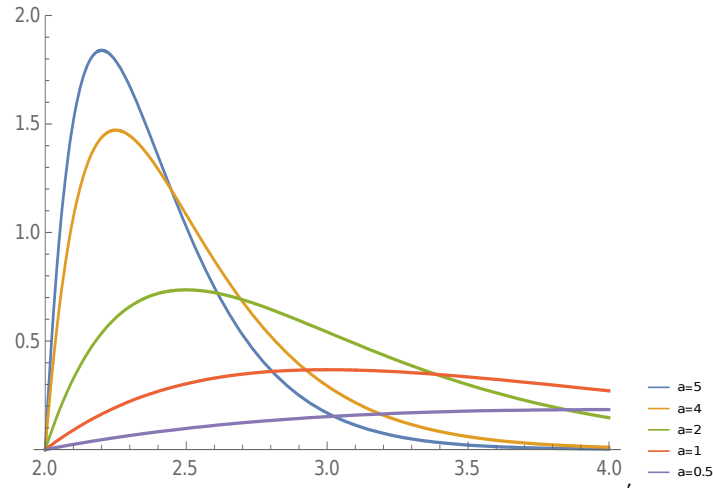


Figure 1: Graphs of pdf of the convoluted benktander distribution with $b=1$ for selected values of parameter

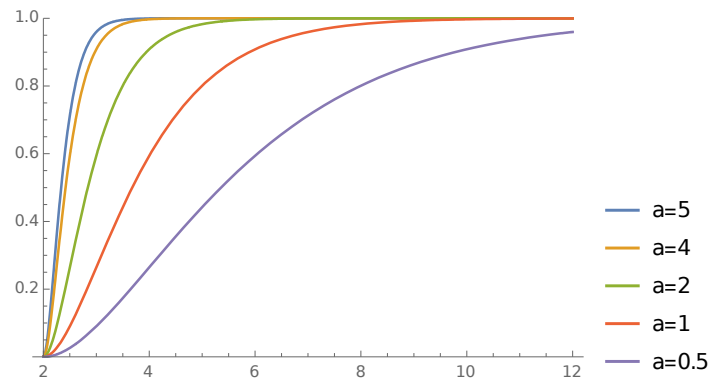


Figure 2: Graphs of cdf. of the convoluted benktander distribution with $b=1$ for selected values of parameter

Proof. The first derivative of CBD is given as

$$(11) \quad f'(x) = a^2 e^{a(2-z)} [1 - a(z-2)]$$

by using equation (8) the term $e^{a(2-z)}$ can be written as

$$(12) \quad e^{a(2-z)} = \frac{f(z)}{a^2(z-2)}$$

substitute equation (12) in equation (11) and simplify we have the first ODE whose solution is the PDF of CBD. $\square$

Proposition 2. The second ordinary differential equation whose solution is the pdf of CBD is defined as

$$(13) \quad (z-2)f''(z) - (a[a(z-2)-2])f(z) = 0.$$

Proof. The second derivative of CBD is given as

$$(14) \quad f''(z) = a^3[a(z-2)-2]e^{a(2-z)}.$$

Simplify and substitute Eq (14) in Eq (7), we have the second ODE whose solution is the PDF of CBD. $\square$

Proposition 3. The mixture of the first and second ordinary differential equation whose solution is the pdf of CBD is defined as

$$(15) \quad (z-2)f''(z) + a(z-2)f'(z) + af(z) = 0.$$

Proof. The first and second derivatives of CBD are given by equations (11) and (14) respectively. Simplify and substitute equation (11) in Eq. (14), we have the mixture first and second ODE whose solution is the PDF of CBD. $\square$

Proposition 4. The ODE whose solution is the QF $w(u)$ of CBD is given by:

$$(16) \quad (w-2)\frac{d^2w}{du^2} - \left(\frac{dw}{du}\right)^2 [a(w-2)-1] = 0.$$

Proof. Regarding to the definition of quantile functions (for more details see [16], [17] and [18]) the quantile function defined as $\frac{dw}{du} = \frac{1}{f(w)}$. Moreover, the first derivative of $w(u)$ can be evaluated as follows

$$(17) \quad \frac{dw}{du} = \frac{1}{a^2e^{a(2-w)}(w-2)} = \frac{e^{aw}}{a^2e^{2a}(w-2)}.$$

Now, we differentiate to obtain a second order of the nonlinear ODE:

$$(18) \quad \begin{aligned} \frac{d^2w}{du^2} &= \frac{1}{a^2e^{2a}} \left[\frac{(w-2)ae^{aw} - e^{aw}}{(w-2)^2} \right] \frac{dw}{du} \\ &= \left[\frac{(w-2)ae^{aw}}{(a^2e^{2a})(w-2)^2} - \frac{e^{aw}}{(a^2e^{2a})(w-2)^2} \right] \frac{dw}{du}. \end{aligned}$$

Simplify Eq (18) as follows

$$(19) \quad \frac{d^2w}{du^2} = \left(\frac{dw}{du}\right)^2 \left[\frac{a(w-2)-1}{w-2} \right].$$

and rearrange Eq (19), we have the ODE whose solution is the QF of CBD. $\square$

Proposition 5. The ODE whose solution is the survivor function (SF) of CBD is given by:

$$(20) \quad (1 + a(z - 2))S'(z) + a^2(z - 2)S(z) = 0.$$

Proof. The survival function of CBD is:

$$(21) \quad S(z) = 1 - F(z) = (1 + a(z - 2))e^{a(2-a)}.$$

From Eq. (21) we have

$$(22) \quad e^{a(2-a)} = \frac{S(z)}{1 + a(z - 2)}.$$

Differentiate Eq. (21), we have

$$(23) \quad S'(z) = (-a^2(z - 2))e^{a(2-a)}$$

and substitute Eq. (22) in Eq (23), we have

$$(24) \quad S'(z) = \frac{-a^2(z - 2)S(z)}{1 + a(z - 2)}.$$

Rearrange Eq. (24) we get the ODE whose solution is the (SF) of CBD. $\square$

Proposition 6. The ODE whose solution is the hazard function (HF) of CBD is given by

$$(25) \quad a^2(z - 2)^2h'(z) - (h(z))^2 = 0.$$

Proof. The hazard function of CBD is:

$$(26) \quad h(z) = \frac{f(z)}{S(z)} = \frac{a^2(z - 2)}{1 + a(z - 2)}.$$

Differentiate Eq. (26) we get

$$(27) \quad h'(z) = \frac{a^2}{(az - 2a + 1)^2}.$$

Substitute Eq. (26) in Eq. (27), we have $a^2(z - 2)^2h'(z) - (h(z))^2 = 0$ with the condition $h(2) = 0$. $\square$

4. Order statistics of CBD

In this section, we drive the single order statistics for CBD. The probability density function of $f_{(j)}(z)$ is given as:

$$(28) \quad f_{(j)}(z) = \frac{n!}{(j-1)!(n-j)!} g(z) [G(z)]^{j-1} [1 - G(z)]^{n-j}; i = 1, 2, \dots$$

$$(29) \quad \begin{aligned} f_{(j)}(z) &= \frac{n!}{(j-1)!(n-j)!} [a^2 e^{a(2-z)} (z-2)] \\ &\cdot [1 + (2a - az - 1)e^{a(2-z)}]^{j-1} [1 - (1 - (2a - az - 1)e^{a(2-z)})]^{n-j} \\ &= \frac{n!}{(j-1)!(n-j)!} [a^2 e^{a(2-z)} [1 + (n-j)(z-2)]] \\ &\cdot [1 + (2a - az - 1)e^{a(2-z)}]^{j-1} [az - 2a + 1]^{n-j}. \end{aligned}$$

Let $Z_{[1]} = \min(Z_1, Z_2, \dots, Z_n)$ and $Z_{[n]} = \max(Z_1, Z_2, \dots, Z_n)$ with pdfs $f_{(1)}(z)$ and $f_{(n)}(z)$ defined as follows $f_{(1)}(z) = nf(z)[1 - F(z)]^{n-1}$ and $f_{(n)}(z) = nf(z)[1 - F(z)]^{n-1}$. As a special case of Eq. (28), the pdf of the minimum and maximum order statistics, respectively, are

$$(30) \quad f_{(1)}(z) = na^2 e^{an(2-z)} (az - 2a + 1)^{n-1},$$

$$(31) \quad f_{(n)}(z) = na^2 e^{a(2-z)} (z-2) [1 + (2a - az - 1)e^{a(2-z)}]^{n-1}$$

5. Some asymptotic properties of the CBD

Here, we show some asymptotic properties of the random variable z , where $z \sim CBD(a)$. We derive the asymptotic r^{th} moment, asymptotic expected value and asymptotic standard deviation of z as special cases. The r^{th} moment of the CBD is given by the following theorem.

Lemma 5.1. *If $z \sim CBD(a)$, then the r th moment is given by*

$$(32) \quad E(Z^r) = \sum_{k=0}^r r C k 2^{r-k} \left(\frac{1}{a}\right)^k \gamma(k+2).$$

Proof. By using the definition of the r th moment as follows:

$$(33) \quad E(Z^r) = \lim_{c \rightarrow \infty} \int_2^c z^r f(z) dz.$$

By substitute $f(z)$ in Eq. 32, we get

$$(34) \quad E(Z^r) = \lim_{c \rightarrow \infty} \int_2^c z^r (a^2 e^{a(2-z)} (z-2)) dz.$$

Assume that $x = z - 2$ and we have $dx = dz$, at $z = 2$ then $x = 0$ and as z goes to ∞ then x goes to ∞ . Now Eq. (34) becomes as

$$(35) \quad E(Z^r) = a^2 \lim_{c \rightarrow \infty} \int_2^c x(x+2)^r e^{-ax} dx.$$

By using binomial theorem we have

$$(36) \quad (x+2)^r = \sum_{k=0}^r rCk x^k 2^{r-k}.$$

Substitute Eq. (36) in Eq. (35) and some algebraic manipulation, we have

$$(37) \quad \begin{aligned} E(Z^r) &= a^2 \lim_{c \rightarrow \infty} \int_2^c x \sum_{k=0}^r rCk x^k 2^{r-k} e^{-ax} dx \\ &= a^2 \sum_{k=0}^r 2^{r-k} rCk \lim_{c \rightarrow \infty} \int_2^c x^{k+1} e^{-ax} dx. \end{aligned}$$

Let $ax = y$ and $x = \frac{y}{a}$, $dx = \frac{dy}{a}$ then we get

$$(38) \quad \begin{aligned} E(Z^r) &= a^2 \sum_{k=0}^r 2^{r-k} rCk \lim_{c \rightarrow \infty} \int_0^c \left(\frac{y}{a}\right)^{k+1} e^{-y} dy \\ &= a^2 \sum_{k=0}^r 2^{r-k} rCk \left(\frac{1}{a}\right)^{k+2} \lim_{c \rightarrow \infty} \int_0^c y^{k+1} e^{-y} dy \end{aligned}$$

since $\lim_{c \rightarrow \infty} \int_0^c y^{\alpha-1} e^{-y} dy = \Gamma(\alpha)$, so $\alpha - 1 = k + 1$, $\alpha = k + 2$ and substitute in Eq. (37) then the proof is complete. $\square$

As a special case of the r^{th} moments, we have the following remarks:

Remark 1. When $r = 1$, we found the asymptotic expected value as follows:

$$(39) \quad \mu = E(Z) = \left(2 + \frac{2}{a}\right).$$

Remark 2. When $r = 2$, we found the asymptotic expected value of z^2 as follows:

$$(40) \quad E(Z^2) = 4 + 8\frac{1}{a} + 6\left(\frac{1}{a}\right)^2.$$

Remark 3. The asymptotic variance of Z can be derived by using the formula $var(Z) = \sigma^2 = E(Z^2) - [E(Z)]^2$ and substitute Eqs. (38) and (39) in it, we have the asymptotic variance of z as follows

$$(41) \quad var(z) = \sigma^2 = \frac{2}{a^2}.$$

Remark 4. The asymptotic skewness of z may be computed by

$$(42) \quad \text{Skewness} = E\left(\frac{Z - \mu}{\sigma}\right)^3 = \sqrt{2},$$

where $E\left(\frac{z - \mu}{\sigma}\right)^3 = \frac{E[z^3] - 3\mu\sigma^2 - \mu^3}{\sigma^3}$ and $E[z^3] = 8 + \frac{24}{a} + 36\left(\frac{1}{a}\right)^2 + 24\left(\frac{1}{a}\right)^3$.

Remark 5. The asymptotic kurtosis of z may be computed by

$$(43) \quad \text{Kurtosis} = E\left(\frac{Z - \mu}{\sigma}\right)^4,$$

since $E\left(\frac{Z - \mu}{\sigma}\right)^4 = \frac{E[Z^4] - 4\mu E[Z^3] + 6(\mu)^2 E[Z^2] - 4\mu^3 E[Z] + \mu^4}{\sigma^4}$ just we need to find the $E[Z^4]$ and it can be computed as $E[Z^4] = 16 + 32\frac{1}{a} + 24\left(\frac{1}{a}\right)^2 + 8\left(\frac{1}{a}\right)^3 + \left(\frac{1}{a}\right)^4$ and we get the asymptotic kurtosis of z .

Lemma 5.2. If $Z \sim CBD(a)$, then the moment generating function is given as

$$(44) \quad E[e^{tz}] = \frac{a^2 e^{2t}}{(a - t)^2}; t < a.$$

Proof. The moment generating function can be defined as follows

$$(45) \quad E[e^{tz}] = \int_2^\infty e^{tz} f(z) dz = \int_2^\infty e^{tz} (a^2 e^{a(2-z)} (z - 2)) dz.$$

Let $x = z - 2$ then at $z = 2$, $x = 0$ and at x goes to ∞ then z goes to ∞ . We have

$$(46) \quad E[e^{tz}] = a^2 e^{a(2-z)} \int_0^\infty x e^{-x(a-t)} dx.$$

Let $y = x(a - t)$, $dy = (a - t)dx$ and $x = \frac{y}{a-t}$. We computed the integral and simplified as follows

$$(47) \quad E[e^{tz}] = a^2 e^{2t} \frac{1}{(a - t)^2} \int_0^\infty y e^{-y} dy = a^2 e^{2t} \left(\frac{1}{(a - t)^2} \Gamma(2) \right)$$

and the proof is completed. $\square$

6. Maximum likelihood estimation of the parameter

The maximum likelihood estimator (MLE) is a well famous estimator. It is defined by take care of our parameters as unknown values and computing the joint density of all members of a data set, which are suggested to be independent and identically distributed (iid). Let $Z_1, Z_2, \dots, Z_N$ be random sample of n (iid) random variables each from CBD. The likelihood function is given by

$$(48) \quad L(a, Z_1, Z_2, \dots, Z_N) = \prod_{i=1}^N f(z; a) = \prod_{i=1}^N a^2 e^{a(2-z_i)} (z_i - 2).$$

By taking the natural logarithm of both sides, the log-likelihood function is defined to be:

$$(49) \quad l(a, Z_1, Z_2, \dots, Z_N) = \ln[a^{2N} (e^{a \sum_{i=1}^N (2-z_i)})^N \prod_{i=1}^N (z_i - 2)].$$

After simplify we get

$$(50) \quad l(a, Z_1, Z_2, \dots, Z_N) = 2N \ln a + a \sum_{i=1}^N (2 - z_i) + \sum_{i=1}^N (z_i - 2).$$

Taking the partial derivative with respect to a we have

$$(51) \quad \frac{\partial l(a, Z_1, Z_2, \dots, Z_N)}{\partial \hat{a}} = \frac{2N}{a} + \sum_{i=1}^N (2 - z_i).$$

Set the derivative function to zero and then rearrange the equation to make the parameter of interest the subject of the equation as follows:

$$(52) \quad \frac{2N}{a} + \sum_{i=1}^N (2 - z_i) = 0.$$

Solve it for $\hat{a}$ we get

$$(53) \quad \hat{a} = \frac{2N}{\sum_{i=1}^N (z_i - 2)} = \frac{2}{\bar{Z} - 2},$$

which is the exact solution for the estimated parameter $\hat{a}$.

7. Asymptotic behaviors of CBD

The asymptotic properties of the CBD when $Z \rightarrow 2$ and when $Z \rightarrow \infty$, the behavior as follows $\lim_{z \rightarrow 2} f(z) = \lim_{z \rightarrow 2} a^2 e^{a(2-z)} (z - 2) = 0$ and $\lim_{z \rightarrow \infty} f(z) = \lim_{z \rightarrow \infty} a^2 e^{a(2-z)} (z - 2) = 0$. This confirms the unimodality of the distribution; this means there is only a single highest value.

8. Renyi entropy

There is wide literature on the applications of the Renyi entropy in many fields such as biology, medicine, and economics, computer science, chemistry and physics. The Rényi entropy has been widely used in the study of quantum systems. Especially, it used in the analysis of quantum correlations and quantum measurement and quantum statistical mechanics [19]. In the paper, in view of these numerous and successful applications, it seems valuable to formulate the Rényi entropies in the following theorem. Because the entropic measure

of uncertainty and it is adequate in quantum mechanical measurements. The Rényi entropy is defined as

$$(54) \quad E_R = \frac{1}{1-\rho} \int_2^\infty [g(z)]^\rho dz.$$

Lemma 8.1. *If $Z \sim CBD(a)$, then the Rényi entropy function of CBD is given by*

$$(55) \quad E_R = \frac{1}{1-\rho} (a^2)^\rho \left(\frac{1}{a\rho}\right)^{\rho+1} \Gamma(\rho+1).$$

Proof. Since

$$(56) \quad E_R = \frac{1}{1-\rho} \int_2^\infty (a^2 e^{a(2-z)} (z-2))^\rho dz$$

let $x = z - 2$, $dx = dz$ and as $z = 2 \rightarrow x = 0$, $z \rightarrow \infty$, $x \rightarrow \infty$ then the integral becomes

$$(57) \quad E_R = \frac{1}{1-\rho} (a^2)^\rho \int_0^\infty e^{-a\rho x} x^\rho dx.$$

Now, assume that $y = a\rho x$, $x = \frac{y}{a\rho}$ and $dx = \frac{dy}{a\rho}$ then the integral becomes

$$(58) \quad E_R = \frac{1}{1-\rho} (a^2)^\rho \frac{1}{a\rho} \int_0^\infty e^{-y} y^\rho dy.$$

Simplify, rearrange and the integral $\int_0^\infty e^{-y} y^\rho dy = \Gamma(\rho+1)$ and the proof is completed. $\square$

9. Simulation study

Simulation is a way to model random events, such that simulated outcomes closely match real-world outcomes. By observing simulated outcomes, researchers gain insight on the real world. An estimator is a numerical function of the data. There are many ways to specify the form of an estimator for a particular parameter of a given distribution, and many ways to evaluate the quality of an estimator. The following algorithm is used to generate data from the CBD:

- Generate U random numbers from $U(0, 1)$ distribution.
- Since Z has a continuous distribution function $F(z)$ belongs to CBD (i.e. $Pr(Z \leq z)$) then $F(Z) \sim Uniform(0, 1)$.
- Set $F(z) = u$ provided the inverse as follows $z = F^{-1}(u)$ and

$$(59) \quad 1 + (2a - az - 1)e^{2a-az} = u.$$

Since no exact solution for Eq. (59), so we used Newton Raphson method. beginning with some arbitrary z_0 , calculate the equation of the line tangent to

Table 1: Simulated results of average, variance, skewness, kurtosis and bias of the estimator $a = 0.5$.

$a = 0.5$						
n	average	variance	skewness	kurtosis	a	bias
5	6.328125	6.095517	-0.61743	0.087814	0.462094	0.037906
10	5.94627	10.45756	0.997421	-0.11516	0.506808	-0.00681
20	6.398513	7.867614	0.789465	-0.10424	0.454699	0.045301
30	5.600391	5.925499	1.077468	1.203318	0.555495	-0.0555
50	6.628457	8.511992	1.311712	2.292154	0.432109	0.067891
100	5.493748	5.268311	0.659865	-0.64601	0.572451	-0.07245
200	6.070159	9.222475	1.554787	4.649339	0.491381	0.008619
2000	5.999317	8.363794	1.502168	3.485188	0.500085	-8.5E-05
5000	6.024985	7.929047	1.391688	3.020997	0.496896	0.003104

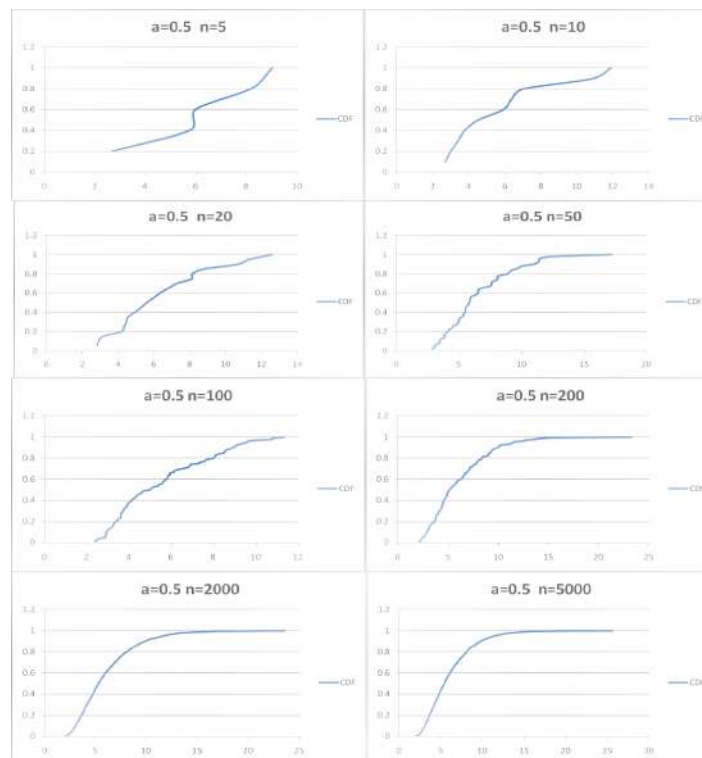
Table 2: Simulated results of average, variance, skewness, kurtosis and bias of the estimator $a = 1$.

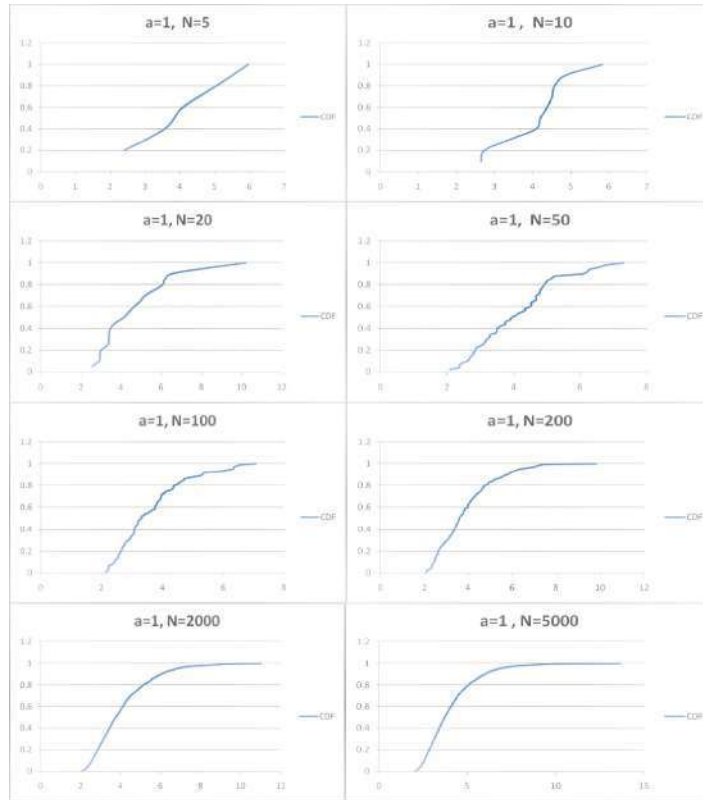
$a = 1$						
n	average	variance	skewness	kurtosis	a	bias
5	4.212014	1.858279	-0.02157	-0.55717	0.904154	0.095846
10	4.114667	0.956655	-0.09912	-0.05631	0.945776	0.054224
20	4.730873	3.809207	1.383953	1.99232	0.732367	0.267633
25	3.840373	1.290374	0.843757	-0.32851	1.086736	-0.08674
50	4.106315	1.567502	0.582919	-0.15735	0.949525	0.050475
100	3.624332	1.322742	1.053599	0.607814	1.231276	-0.23128
200	3.857998	1.707839	1.155441	1.745109	1.076427	-0.07643
1000	4.071128	2.128726	1.581991	4.794034	0.965657	0.034343
2000	4.063872	2.022965	1.227163	1.771559	0.969052	0.030948
5000	4.032718	2.050063	1.367185	2.555873	0.983904	0.016096

f at z_0 , and see where that tangent line crosses 0. The equation of the line tangent is $g(z) = f(z_0) + f'(z_0)(z - z_0)$ also it is the first two terms Taylor approximation at z_0 . Rearranging terms as $z = z_0 - \frac{f(z_0)}{f'(z_0)}$, then Now setting $z_0 = z$ and repeating until convergence is essentially Newton's method. The simulation study is based on generating $N = 10,000$ samples of size $n = 5, 10, 20, 50, 100, 200, 2000$ and 5000 for $a = 0.5, 1, 2$ using above algorithm. Then we calculate the following measures; average, variance, skewness, kurtosis and bias of the estimator. All the results for different values of $a = 0.5, 1, 2$ are illustrated in Tables 1,2 and 3. As well as the result of the simulated data that derived from the new distribution CBD for the CDF are shown in Figures 3, 4 and 5 for $a = 0.5, 1, 2$.

Table 3: Simulated results of average, variance, skewness, kurtosis and bias of the estimator $a = 2$.

$a = 2$						
n	average	variance	skewness	kurtosis	a	bias
5	3.183778	0.835779	1.409327	2.15818	1.689505	0.310495
10	3.33264	0.535555	0.105309	-1.30659	1.500781	0.499219
20	3.398707	0.90538	0.790657	-0.49655	1.429892	0.570108
30	2.991535	0.36166	1.016148	0.361959	2.017075	-0.01708
50	2.994086	0.515195	1.020777	1.482307	2.011898	-0.0119
100	3.126318	0.65937	1.294906	2.017583	1.775698	0.224302
200	3.09809	0.57787	1.141619	1.174657	1.821344	0.178656
1000	2.993328	0.449259	1.184761	1.712884	2.013434	-0.01343
2000	3.025623	0.525183	1.432676	2.906886	1.950035	0.049965
5000	2.974395	0.467879	1.368338	2.589046	2.052555	-0.05255

Figure 3: The CDF of CBD when $a = 0.5$

Figure 4: The CDF of CBD when $a = 1$

10. Conclusions

In this paper, we proposed the convolution $Z = X + Y$ of the benktander distribution, of which the parameter $b = 1$. Several properties of the CBD such as moments, skewness and kurtosis have been discussed. Moreover, the ODE of (QF), (SF), (HF) and (RHF) of convoluted probability distributions has been illustrated. The Graphs of CDF of the simulated data derived from CBD for selected values of parameter are shown. These graphs show the behavior of the distribution function more fit when the size of the sample increased. In addition, the biases are decreased when the size of the sample increased.

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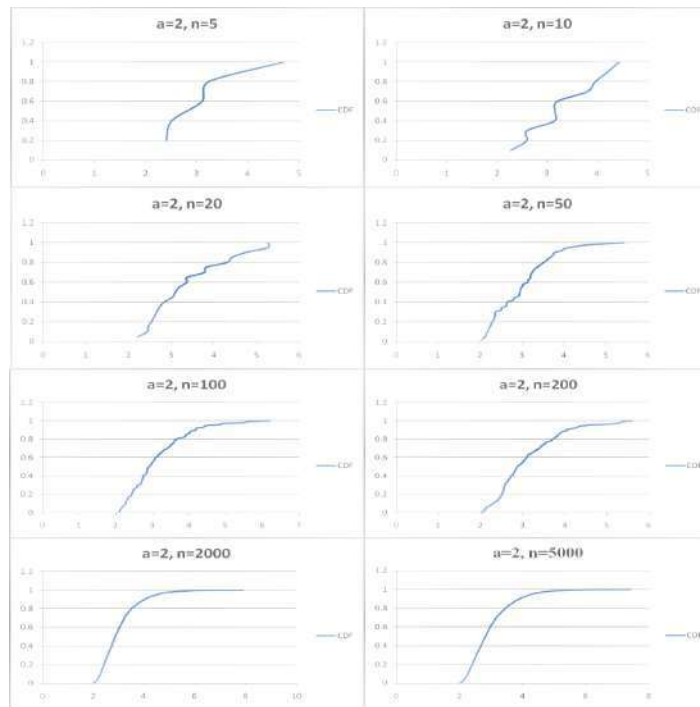


Figure 5: The CDF of CBD when $a = 2$

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Hadamard product of meromorphic multivalent functions with positive coefficient

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Abstract. In the present paper, we introduced a class $S(\gamma, \tau, \delta, p)$ of a meromorphic multivalent function and used this class of function Hadamard product (or convolution) to prove some nice and attractive properties such that the coefficient inequalities, distortion theorem, radii of starlikeness and convexity. Moreover, we verified that the class $S(\gamma, \tau, \delta, p)$ is closed under convex linear combination.

Keywords: multivalent function, analytic functions, negative coefficients, convexity, distortion theorem.

1. Introduction

Let L_p denoted the class of functions given by the formula

$$(1) \quad f_1(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} a_{p+k-1} z^{p+k-1}, p \in N$$

and

$$(2) \quad f_2(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} b_{p+k-1} z^{p+k-1},$$

*. Corresponding author

which are normalized meromorphic p -valent in the unit disk $U = \{z : 0 < |z| < 1\}$ ([9],[20]). Let $f_1, f_2 \in L_p$, then the Hadamard product of $f_1 * f_2$ is defined by

$$(3) \quad W(z) = (f_1 * f_2)(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} a_{p+k-1} b_{p+k-1} z^{p+k-1}.$$

The class L_p of analytic p -valent functions, have been studied by many authors, for Significant properties and characteristics of various subclasses of the class L_p , such as: Owa [22], Saibah [21], Aouf ([12],[13]), Uralgeddi [3], Moa'ath ([10], [16]), Srivastava ([7], [8]), Morga ([14], [15]), Kulkarni [23], Osama ([17], [18], [19]), Aabed [1], Gahmin [5], Kamali [11] and Markinde [4]. The class of functions $S(\gamma, \tau, \delta, p)$ define as follows:

A function $W(z)$ given by (3) is said to be a member of the class $S(\gamma, \tau, \delta, p)$ if it satisfies

$$(4) \quad \left| \frac{(1-2\delta)zW'(z) - \delta z^2 W''(z)}{(\tau-1)W(z) + \tau z W'(z)} \right| \leq \gamma,$$

where $\gamma > 0$, $0 \leq \tau < 1$, $\delta \geq 0$, $\forall z \in U$. This class of function is motivated from the work done by Alghalary and Joshi [2].

2. Coefficient inequalities

Alghalary and Joshi [2] presented a basic properties such as coefficient inequality, growth and distortion and several others. In this Section, we provide a sufficient condition for a function W , analytic in a punctured disk U in $S(\gamma, \tau, \delta, p)$.

Theorem 2.1. *If $W \in L_p$ satisfies*

$$(5) \quad \sum_{k=1}^{\infty} \{(p+k-1)[1-\tau\gamma-\delta(p+k)] + \gamma(1-\tau)\} a_{p+k-1} b_{p+k-1} \leq \gamma(\tau+1-\tau p) - p(1-\delta) - \delta p^2,$$

then $W \in S(\gamma, \tau, \delta, P)$, where $\gamma > 0, 0 \leq \tau < 1, \delta \geq 0$ for all $z \in U$.

Proof. Let us suppose that

$$(6) \quad \sum_{k=1}^{\infty} \{(p+k-1)[1-\tau\gamma-\delta(p+k)] + \gamma(1-\tau)\} a_{p+k-1} b_{p+k-1} \leq \gamma(\tau+1-\tau p) - p(1-\delta) - \delta p^2,$$

for $W \in S(\gamma, \tau, \delta, p)$. It suffices to show that

$$(7) \quad \left| \frac{(1-2\delta)zW'(z) - \delta z^2 W''(z)}{(\tau-1)W(z) + \tau z W'(z)} \right| \leq \gamma,$$

where $\gamma > 0, 0 \leq \tau < 1, \delta \geq 0$ for all $z \in U$. We know

$$\begin{aligned}
 & (1 - 2\delta)zW'(z) - \delta z^2W''(z) \\
 &= (1 - 2\delta)[-pz^{-p-1} + \sum_{k=1}^{\infty} (p+k-1)a_{p+k-1}b_{p+k-1}z^{p+k-2}] \\
 & \quad - \delta z^2[-p(-p-1)z^{-p-2}] + \sum_{k=1}^{\infty} (p+k-1)(p+k-2)a_{p+k-1}b_{p+k-1}z^{p+k-3}) \\
 &= -pz^{-p}(1 - \delta) - \delta p^2 + \sum_{k=1}^{\infty} [(p+k-1)(1 - \delta(p+k))]a_{p+k-1}b_{p+k-1},
 \end{aligned}$$

and

$$\begin{aligned}
 & (\tau - 1)[z^{-p} + \sum_{k=1}^{\infty} a_{p+k-1}b_{p+k-1}z^{p+k-1}] + \tau z - pz^{-p} + \sum_{k=1}^{\infty} a_{p+k-1}b_{p+k-1}z^{p+k-1} \\
 &= \tau z^{-p} - z^{-p} - \tau pz^{-p+1} + \sum_{k=1}^{\infty} [(\tau - 1) + \tau(p+k-1)]a_{p+k-1}b_{p+k-1}.
 \end{aligned}$$

Then

$$\begin{aligned}
 & \left| \frac{-pz^{-p}(1 - \delta) - \delta p^2 + \sum_{k=1}^{\infty} [(p+k-1)(1 - \delta(p+k))]a_{p+k-1}b_{p+k-1}z^{p+k-1}}{\tau z^{-p} - z^{-p} - \tau pz^{-p+1} + \sum_{k=1}^{\infty} [(\tau - 1) + \tau(p+k-1)]a_{p+k-1}b_{p+k-1}z^{p+k-1}} \right| \\
 & \leq \frac{p(1 - \delta) + \sum_{k=1}^{\infty} [(p+k-1)(1 - \delta)(p+k)]a_{p+k-1}b_{p+k-1}}{(\tau + 1 - \tau p) - \sum_{k=1}^{\infty} [(1 - \tau) - \tau(p+k-1)]a_{p+k-1}b_{p+k-1}},
 \end{aligned}$$

that is

$$\begin{aligned}
 & \sum_{k=1}^{\infty} [(p+k-1)(1 - \tau\gamma - \delta(p+k)) + \gamma(1 - \tau)]a_{p+k-1}b_{p+k-1} \\
 (8) \quad & \leq \gamma(\tau + 1 - \tau p) - p(1 - \delta) - \delta p^2.
 \end{aligned}$$

When $k = 1$, we have

$$(9) \quad a_p b_p \leq \frac{\gamma(\tau + 1 - \tau p) - p(1 - \delta) - \delta p^2}{p[(1 - \tau\gamma - \delta(p+1))] + \gamma(1 - \tau)},$$

which is equivalent to our condition of the theorem, so that $W \in S(\gamma, \tau, \delta, p)$.

Theorem 2.2. *Let the function W defined by 3, then $W \in S(\gamma, \tau, \delta, p)$ if and only if*

$$(10) \quad \left| \frac{(1 - 2\delta)zW'(z) - \delta z^2W''(z)}{(\tau - 1)W(z) + \tau zW'(z)} \right| \leq \gamma$$

is satisfied.

Proof. With the aid of Theorem 2.1 it suffices to show the 'only if' $W \in S(\gamma, \tau, \delta, p)$ then

$$\begin{aligned} & \left| \frac{(1-2\delta)zW'(z) - \delta z^2W''(z)}{(\tau-1)W(z) + \tau zW'(z)} \right| \\ &= \left| \frac{-pz^{-p}(1-\delta) - \delta p^2 + \sum_{k=1}^{\infty} [(p+k-1)(1-\delta(p+k))]a_{p+k-1}b_{p+k-1}z^{p+k-2}}{\tau z^{-p} - z^{-p} + \sum_{k=1}^{\infty} [(\tau-1) + \tau(p+k-1)]a_{p+k-1}b_{p+k-1}z^{p+k-1}} \right| \\ &\leq \frac{p(1-\delta) + \delta p^2 + \sum_{k=1}^{\infty} (p+k-1)a_{p+k-1}b_{p+k-1}}{\tau + 1 - \tau p - \sum_{k=1}^{\infty} \gamma(p+k-1)a_{p+k-1}b_{p+k-1}} \leq \gamma, \end{aligned}$$

this implies that

$$(11) \quad \sum_{k=1}^{\infty} (p+k-1)[1 - \tau\gamma - \delta(p+k)] + \gamma(1-\tau)a_{p+k-1}b_{p+k-1} \leq \gamma(\tau+1-\tau p) - p(1-\delta) - \delta p^2.$$

Similarly, the method of proving in Theorem 2.1 applied and obtained the required result. The result is sharp for function W of the form

$$(12) \quad W_{p+k-1}(z) = z^{-p} + \frac{\gamma(\tau+1-\tau p) - p(1-\delta) - \delta p^2}{(p+k-1)[1 - \tau\gamma - \delta(p+k)] + \gamma(1-\tau)} z^{p+k-1}, k \geq 1.$$

Corollary 2.1. Let the function W be defined by (12). If $W \in S(\gamma, \tau, \delta, p)$, then

$$(13) \quad a_{p+k-1}b_{p+k-1} \leq \frac{\gamma(\tau+1-\tau p) - p(1-\delta) - \delta p^2}{(p+k-1)[1 - \tau\gamma - \delta(p+k)] + \gamma(1-\tau)}.$$

The result is sharp for functions W_{p+k-1} given by (12).

3. Distortion theorem

In this Section, a Hadamard product applied to the distortion property, in the class $W \in S(\gamma, \tau, \delta, p)$ by the following Theorem:

Theorem 3.1. If the function W defined by (5) is in the class $W \in S(\gamma, \tau, \delta, p)$, then for $0 < |z| = r < 1$, we have

$$\begin{aligned} r^{-p} &= \frac{\gamma(\tau+1-\tau p) - p(1-\delta) - \delta p^2}{p[1 - \tau\gamma - \delta(p+1)] + \gamma(1-\tau)} r^p \\ &\leq |(f_1 * f_2)(z)| \leq r^{-p} + \frac{\gamma(\tau+1-\tau p) - p(1-\delta) - \delta p^2}{p[1 - \tau\gamma - \delta(p+1)] + \gamma(1-\tau)} r^p, \end{aligned}$$

with equality holds for

$$W_1 = z^{-p} + \frac{\gamma(\tau + 1 - \tau p) - p(1 - \delta) - \delta p^2}{p[1 - \tau\gamma - \delta(p + 1)] + \gamma(1 - \tau)} z^p (z = \pm ir, \pm r),$$

and

$$pr^{-p-1} - \frac{p[\gamma(\tau + 1 - \tau p) - p(1 - \delta) - \delta p^2]}{p[1 - \tau\gamma - \delta(p + 1)] + \gamma(1 - \tau)} r^{p-1},$$

with equality holds for

$$W_1 = z^{-p} + \frac{\gamma(\tau + 1 - \tau p) - p(1 - \delta) - \delta p^2}{p[1 - \tau\gamma - \delta(p + 1)] + \gamma(1 - \tau)} z^p (z = \pm ir, \pm r).$$

Proof. Since $(f_1 * f_2)(z) \in S(\gamma, \tau, \delta, p)$, Theorem 5.2 yields the inequality

$$(14) \quad \sum a_{p+k-1} b_{p+k-1} \leq \frac{\gamma(\tau + 1 - \tau p) - p(1 - \delta) - \delta p^2}{(p + k - 1)[1 - \tau\gamma - \delta(p + k)] + \gamma(1 - \tau)},$$

thus, for $0 < |z| = r < 1$, and making use of (14) we have

$$\begin{aligned} |(f_1 * f_2)(z)| &= \left| z^{-p} + \sum_{k=1}^{\infty} a_{p+k-1} b_{p+k-1} z^{p+k-1} \right| \\ &\leq |z|^{-p} + \frac{\gamma(\tau + 1 - \tau p) - p(1 - \delta) - \delta p^2}{(p + k - 1)[1 - \tau\gamma - \delta(p + k)] + \gamma(1 - \tau)} |z|^{p+k-1}. \end{aligned}$$

(Note: we substitute in (14) when $k=1$)

$$\begin{aligned} &\leq r^{-p} + \frac{\gamma(\tau + 1 - \tau p) - p(1 - \delta) - \delta p^2}{(p + k - 1)[1 - \tau\gamma - \delta(p + k)] + \gamma(1 - \tau)} \\ &= r^{-p} + \frac{\gamma(\tau + 1 - \tau p) - p(1 - \delta) - \delta p^2}{(p + k - 1)[1 - \tau\gamma - \delta(p + k)] + \gamma(1 - \tau)} \end{aligned}$$

and

$$\begin{aligned} |(f_1 * f_2)(z)| &= \left| z^{-p} + \sum_{k=1}^{\infty} a_{p+k-1} b_{p+k-1} z^{p+k-1} \right| \\ &\geq |z|^{-p} - \sum_{k=1}^{\infty} a_{p+k-1} b_{p+k-1} |z|^{p+k-1} \\ &\geq r^{-p} - \frac{\gamma(\tau + 1 - \tau p) - p(1 - \delta) - \delta p^2}{(p + k - 1)[1 - \tau\gamma - \delta(p + k)] + \gamma(1 - \tau)} \end{aligned}$$

(Note: we substitute in (14) when $k=1$)

$$r^{-p} - \frac{\gamma(\tau + 1 - \tau p) - p(1 - \delta) - \delta p^2}{p[1 - \tau\gamma - \delta(p + 1)] + \gamma(1 - \tau)}.$$

Also from Theorem 2.1, it follows that

$$(15) \quad \sum_{k=1}^{\infty} (p+k-1)a_{p+k-1}b_{p+k-1} \leq \frac{\gamma(\tau+1-\tau p) - p(1-\delta) - \delta p^2}{[1-\tau\gamma - \delta(p+k)] + \gamma(1-\tau)},$$

thus

$$\begin{aligned} |(f_1 * f_2)'|(z) &= \left| -pz^{-p-1} + \sum_{k=1}^{\infty} (p+k-1)a_{p+k-1}b_{p+k-1}z^{p+k-2} \right| \\ &\leq \left| -pz^{-p-1} \right| + \left| \sum_{k=1}^{\infty} (p+k-1)a_{p+k-1}b_{p+k-1}z^{p+k-2} \right| \\ &\leq pr^{-p-1} + \frac{\gamma(\tau+1-\tau p) - p(1-\delta) - \delta p^2}{1-\tau\gamma - \delta(p+k) + \gamma(1-\tau)}. \end{aligned}$$

(Note: we substitute in 15 when $k=1$)

$$\leq pr^{-p-1} + \frac{\gamma(\tau+1-\tau p) - p(1-\delta) - \delta p^2}{1-\tau\gamma - \delta(p+k) + \gamma(1-\tau)}$$

and

$$\begin{aligned} |(f_1 * f_2)'|(z) &= \left| -pz^{-p-1} + \sum_{k=1}^{\infty} (p+k-1)a_{p+k-1}b_{p+k-1}z^{p+k-2} \right| \\ &\geq \left| -pz^{-p-1} \right| + \left| -\sum_{k=1}^{\infty} (p+k-1)a_{p+k-1}b_{p+k-1}z^{p+k-2} \right| \\ &\geq pr^{-p-1} - \frac{\gamma(\tau+1-\tau p) - p(1-\delta) - \delta p^2}{1-\tau\gamma - \delta(p+k) + \gamma(1-\tau)}. \end{aligned}$$

(Note: we substitute in 15 when $k=1$)

$$\geq pr^{-p-1} + \frac{\gamma(\tau+1-\tau p) - p(1-\delta) - \delta p^2}{1-\tau\gamma - \delta(p+1) + \gamma(1-\mu)}.$$

Hence, completes the proof of theorem.

4. Radii of starlikeness and convexity

In this Section, we show that the radii of starlikeness and convexity for the class $S(\gamma, \tau, \delta, p)$ is given by the following theorem:

Theorem 4.1. *If the function $(f_1 * f_2)(z)$ defined by (3) is in the class $S(\gamma, \tau, \delta, p)$, then $(f_1 * f_2)(z)$ is starlike of order ρ ($0 \leq \rho < p$) in the disk $|z| \leq r_1(\gamma, \tau, \delta, p, \rho)$ where $r_1(\gamma, \tau, \delta, p, \rho)$ is the largest value for which*

$$\begin{aligned} r_1 &= r_1(\gamma, \tau, \delta, p, \rho) \\ &= \inf_{k \geq 1} \left(\frac{(p-\rho)(p+k-1)[1-\tau\gamma - \delta(p+k)] + \gamma(1-\tau)}{(p+k-1-\rho)[\gamma(\tau+1-\tau p) - p(1-\delta) - \delta p^2]} \right)^{\frac{1}{2p+k-1}}. \end{aligned}$$

The result is sharp for functions W_{p+k-1} given by (12).

Proof. It suffices to show that

$$\left| \frac{zW'(z)}{W(z)} + 1 \right| \leq (1 - \rho) \text{ for } |z| \leq r_1.$$

We have

$$\begin{aligned} (4.1) \quad & \left| \frac{zW'(z)}{W(z)} + 1 \right| \\ &= \frac{(-p-1) + \sum_{k=1}^{\infty} (p+k)a_{p+k-1}b_{p+k-1}z^{2p+k-1}}{1 + \sum_{k=1}^{\infty} a_{p+k-1}b_{p+k-1}z^{2p+k-1}} \\ &\leq \frac{(-p+1) + \sum_{k=1}^{\infty} \frac{(p+k)[\gamma(\tau+1-\tau p) - p(1-\delta) - \delta p^2]}{(p+k-1)[1-\tau\gamma - \delta(p+k)] + \gamma(1-\tau)} |z|^{2p+k-1}}{1 - \sum_{k=1}^{\infty} \frac{[\gamma(\tau+1-\tau p) - p(1-\delta) - \delta p^2]}{(p+k-1)[1-\tau\gamma - \delta(p+k)] + \gamma(1-\tau)} |z|^{2p+k-1}} \leq 1 - \rho, \end{aligned}$$

hence (4.1) holds true if

$$\sum_{k=1}^{\infty} \left(\frac{(p+k-1-\rho)[\gamma(\tau+1-\tau p) - p(1-\delta) - \delta p^2]}{(p+k-1)[1-\tau\gamma - \delta(p+k)] + \gamma(1-\tau)} \right) |z|^{2p+k-1} \leq p - \rho,$$

and it follows that

$$|z| \leq \left(\frac{(p-\rho)(p+k-1)[1-\tau\gamma - \delta(p+k)] + \gamma(1-\tau)}{(p+k-1-\rho)[\gamma(\tau+1-\tau p) - p(1-\delta) - \delta p^2]} \right)^{\frac{1}{2p+k-1}}, \quad k \geq 1$$

then

$$r_1 = \inf_{k \geq 1} \left(\frac{(p-\rho)(p+k-1)[1-\tau\gamma - \delta(p+k)] + \gamma(1-\tau)}{(p+k-1-\rho)[\gamma(\tau+1-\tau p) - p(1-\delta) - \delta p^2]} \right)^{\frac{1}{2p+k-1}}$$

as required.

Theorem 4.2. If the function $(f_1 * f_2)(z)$ be defined by (3) is in the class $S(\gamma, \tau, \delta, p)$, then $(f_1 * f_2)(z)$ is convex of order ρ ($0 \leq \rho < p$) in the disk $|z| \leq r_2(\gamma, \tau, \delta, p, \rho)$ where $r_2(\gamma, \tau, \delta, p, \rho)$ is the largest value for which r_2 ,

$$\begin{aligned} (16) \quad & r_2(\gamma, \tau, \delta, p, \rho) \\ &= \inf_{k \geq 1} \left(\frac{p(p-\rho)(p+k-1)[1-\tau\gamma - \delta(p+k)] + \gamma(1-\tau)}{(p+k-1)(p+k-1-\rho)[\gamma(\tau+1-\tau p) - p(1-\delta) - \delta p^2]} \right)^{\frac{1}{2p+k-1}}. \end{aligned}$$

The result is sharp for functions W_{p+k-1} given by (12).

Proof. It suffices to show that $\left| \frac{z(f_1 * f_2)''(z)}{(f_1 * f_2)'(z)} + 2 \right| \leq 1 - \rho$, for $|z| \leq r_2$. We have

$$\begin{aligned}
 (4.2) \quad & \left| \frac{z(f_1 * f_2)''(z)}{(f_1 * f_2)'(z)} + 2 \right| \\
 &= \left| \frac{(p^2 - p) + \sum_{k=1}^{\infty} (p+k-1)(p+k)a_{p+k-1}b_{p+k-1}z^{2p+k-1}}{-p + \sum_{k=1}^{\infty} (p+k-1)a_{p+k-1}b_{p+k-1}z^{2p+k-1}} \right|, \\
 &\leq \frac{(p-p^2) + \sum_{k=1}^{\infty} (p+k-1)(p+k)a_{p+k-1}b_{p+k-1}|z|^{2p+k-1}}{p - \sum_{k=1}^{\infty} (p+k-1)a_{p+k-1}b_{p+k-1}|z|^{2p+k-1}},
 \end{aligned}$$

hence (4.2) holds true if

$$\begin{aligned}
 & \sum_{k=1}^{\infty} \frac{(p+k-1)(p+k+1-\rho)[\gamma(\tau+1-\tau p) - p(1-\delta) - \delta p^2]}{(p+k-1)[1-\tau\gamma - \delta(p+k)] + \gamma(1-\tau)} \leq p(p-\rho), k \geq 1, \\
 & |z| \leq \left(\frac{p(p-\rho)(p+k-1)[1-\tau\gamma - \delta(p+k)] + \gamma(1-\tau)}{(p+k-1-\rho)[\gamma(\tau+1-\tau p) - p(1-\delta) - \delta p^2]} \right)^{\frac{1}{2p+k-1}}, k \geq 1.
 \end{aligned}$$

Then,

$$r_2 = \inf_{k \geq 1} \left(\frac{p(p-\rho)(p+k-1)[1-\tau\gamma - \delta(p+k)] + \gamma(1-\tau)}{(p+k-1)(p+k+1-\rho)[\gamma(\tau+1-\tau p) - p(1-\delta) - \delta p^2]} \right)^{\frac{1}{2p+k-1}}$$

as required.

5. Convex linear combination

Our next result involves a linear combination of function W of the type (12).

Theorem 5.1. *Let*

$$(17) \quad W_p = z^{-p} + \frac{\gamma(\tau+1-\tau p) - p(1-\delta) - \delta p^2}{p[(1-\tau\gamma - \delta(p+1))] + \gamma(1-\tau)} z^p,$$

and

$$(18) \quad W_{p+k-1} = z^{-p} + \frac{\gamma(\tau+1-\tau p) - p(1-\delta) - \delta p^2}{(p+k-1)[(1-\tau\gamma - \delta(p+1))] + \gamma(1-\tau)} z^{p+k-1}, k \geq 1$$

then $(f_1 * f_2)(z) \in S(\gamma, \tau, \delta, P)$, if and only if, it can be expressed in the form

$$(f_1 * f_2)(z) = \sum_{k=1}^{\infty} \vartheta_{p+k-1} W_{p+k-1}(z), \text{ where } \vartheta_{p+k-1} \geq 0 \text{ and } \sum_{k=1}^{\infty} \vartheta_{p+k-1} \leq 1.$$

Proof. From (17) and (18), it is easily seen that

$$\begin{aligned}
 (f_1 * f_2)(z) &= \sum_{k=1}^{\infty} (\vartheta_{p+k-1} W_{p+k-1}(z)) \\
 (19) \quad &= z^{-p} + \sum_{k=1}^{\infty} \frac{\gamma(\tau+1-\tau p) - p(1-\delta) - \delta p^2 \vartheta_{p+k-1}}{(p+k-1)[(1-\tau\gamma-\delta(p+k))] + \gamma(1-\tau)} z^{p+k-1}.
 \end{aligned}$$

Since

$$\begin{aligned}
 &\Rightarrow \sum_{k=1}^{\infty} \frac{(p+k-1)[(1-\tau\gamma-\delta(p+k))] + \gamma(1-\tau)}{\gamma(\tau+1-\tau p) - p(1-\delta) - \delta p^2} \\
 &\times \frac{\gamma(\tau+1-\tau p) - p(1-\delta) - \delta p^2 \vartheta_{p+k-1}}{(p+k-1)[(1-\tau\gamma-\delta(p+k))] + \gamma(1-\tau)} \\
 &= \sum_{k=1}^{\infty} \vartheta_{p+k-1} \leq 1,
 \end{aligned}$$

It follows from Theorem 2.1 that the function $f_1 \in S(\gamma, \tau, \delta, p)$.

Conversely, suppose that $(f_1 * f_2)(z) \in S(\gamma, \tau, \delta, p)$, Since

$$a_{p+k-1} b_{p+k-1} \leq \frac{\gamma(\tau+1-\tau p) - p(1-\delta) - \delta p^2}{(p+k-1)[1-\tau\gamma-\delta(p+1)] + \gamma(1-\tau)}, \quad k \geq 1$$

we can write

$$\begin{aligned}
 (f_1 * f_2)(z) &= \sum_{k=1}^{\infty} (\vartheta_{p+k-1} W_{p+k-1}(z)) \\
 &= z^{-p} + \sum_{k=1}^{\infty} \frac{\gamma(\tau+1-\tau p) - p(1-\delta) - \delta p^2 \vartheta_{p+k-1}}{(p+k-1)[(1-\tau\gamma-\delta(p+1))] + \gamma(1-\tau)} z^{p+k-1},
 \end{aligned}$$

then

$$a_{p+k-1} b_{p+k-1} = \frac{\gamma(\tau+1-\tau p) - p(1-\delta) - \delta p^2}{(p+k-1)[1-\tau\gamma-\delta(p+1)] + \gamma(1-\tau)} \vartheta_{p+k-1}, \quad k \geq 1.$$

Setting

$$\vartheta_{p+k-1} = \frac{(p+k-1)[1-\tau\gamma-\delta(p+1)] + \gamma(1-\tau)}{\gamma(\tau+1-\tau p) - p(1-\delta) - \delta p^2}, \quad k \geq 1$$

it follows that $(f * f_1)(z) = \sum_{k=1}^{\infty} \vartheta_{p+k-1} W_{p+k-1}(z)$, thus the theorem is complete.

Finally we prove the following theorem:

Theorem 5.2. *The class is closed under convex linear combinations.*

Proof. Suppose that the functions W_1 and W_2 defined by

$$(20) \quad W_i = z^{-p} + \sum_{k=1}^{\infty} a_{p+k-1,i} b_{p+k-1,i} z^{p+k-1}, i = 1, 2; z \in U$$

are in the class $S(\gamma, \tau, \delta, P)$. Setting

$$(21) \quad W(z) = \sigma W_1(z) + (1 - \sigma) W_2(z), 0 \leq \sigma \leq 1$$

we want to show that $W \in S(\gamma, \tau, \delta, P)$, we find that from (20)

$$\begin{aligned} W(z) &= \sigma W_1(z) + (1 - \sigma) W_2(z), \\ &= z^{-p} + \sum_{k=1}^{\infty} \sigma a_{p+k-1,1} b_{p+k-1,1} + (1 - \sigma) a_{p+k-1,2} b_{p+k-1,2} z^{p+k-1}, (0 \leq \sigma \leq 1), \end{aligned}$$

where $z \in U$. In view of Theorem 2.1, we have

$$\begin{aligned} &\sum_{k=1}^{\infty} [(p+k-1)(1 - \tau\gamma - \delta(p+k) + \gamma(1 - \tau))] \sigma a_{p+k-1,1} b_{p+k-1,1} \\ &+ (1 - \sigma) a_{p+k-1,2} b_{p+k-1,2} \\ &\leq \sigma [\gamma(\tau + 1 - \tau p) - p(1 - \delta) - \delta p^2] + (1 - \sigma) [\gamma(\tau + 1 - \tau p) - p(1 - \delta) - \delta p^2] \\ &= \gamma(\tau + 1 - \tau p) - p(1 - \delta) - \delta p^2, \end{aligned}$$

which shows that $W \in S(\gamma, \tau, \delta, P)$.

6. Conclusion

In this paper, we proved properties of the coefficient inequalities, distortion theorem and radii of starlikeness and convexity for the Hadamard product of the class $S(\gamma, \tau, \delta, p)$ of multivalent function. Also, it is shown that the class $S(\gamma, \tau, \delta, p)$ is closed under convex linear combination.

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A mathematical approach for linking/integrating management plans: Marine Spatial Planning (MSP) and Integrated Coastal Zone Management (ICZM) as a case study

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Abstract. This research has proposed the Management Plan's Linking/Integrating Methodology (MPLIM), that is articulated around the fundamental differences between the management plans to be linked. MPLIM consists of five main steps: identification of similar sectors or actions between the two management plans; evaluating management plans; identifying and classifying the differences between the management plans into themes; analysing the differences and making recommendations; and lastly, linking all desirable sectors or actions. MPLIM approach had been justified with the help of geometrical shapes and mathematical formulae, based on the concept of union and intersection sets. However, it is the preparation for linkage that is the challenge and not the link itself, as the link is taking decisions that are in favour of all parties, supported by the harmony generated from the recommendations step. When implementing MPLIM approach on two-management plans concepts (i.e., Integrated Coastal Zone Management, ICZM and Marine Spatial Planning, MSP), it was noted that the proposed methodology could be used as a promising tool for linking/integrating various management plans. In addition, it turned out that it facilitated understanding of stubborn conceptual issues related to ICZM and MSP and their relations.

Keywords: maritime spatial planning, integrated coastal zone management, linking, integrating.

1. Introduction

Some of the industries regarded the linkage of management plans very important as it helped in conserving efforts and costs. The “merging plans” concept is usually used for the mobile service robots, where it refers to the merger of all software algorithm-based activities, which had to be simultaneously performed

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(Foulser et al., 1992; Alami et al., 1997). From another perspective, in the case of management plans, different terms had been used for indicating a similar or same meaning, such as merging, integrating or linking which are generally used interchangeably.

For instance, a linkage between departments and their management plans ensures that every entity maintains its identity, however, the linkage, means that every management plan or entity can coordinate its activities with other plans, systems or departments. During integration, one of the systems integrates or becomes a part of a different system, however, its identity is maintained. Nevertheless, during a merging, every component loses its identity when it forms a different system, agency/department or plan. In this chapter, the main focus is on linking/integrating; since both are generally used in the context of management plans (Figure 1). These terms have been properly defined below (Dictionaries, 2013).

Link is defined as “making, forming, or suggesting a connection between two entities”. *Integrate* indicates “combining (one entity) with another entity for forming a whole new component”. Here, though one system became a part of another system, every system was able to maintain its original identity. *Merge* indicates “combining or presenting causes for combining and forming one new entity”.

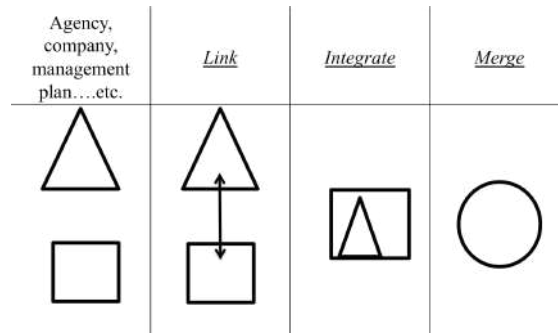


Figure 1: Description of how the linkage, integration, and merger functions operate; it has been developed based on the definitions mentioned above. It can be seen that in *link* each entity is staying as is yet linked with the other entity, whereas in *integrate* both entities are combined together, nevertheless, each entity still enjoy its own shape and characteristics and maintain its original identity; and finally, in *merge* both entities lost their shape and uniqueness and they formed one new entity with new shape and characteristics.

However, speaking of two plans or authorities that are managing two domains Marine Spatial Planning (MSP) and Integrated Coastal Zone Management (ICZM) is a vivid example, as both those concepts manage sometimes similar areas owing to the overlaps noted between the Territorial Waters (TWs) (under the ICZM focus) and Exclusive Economic Zones (EEZ) (under the MSP focus). Hence, the overlapped regions must be addressed appropriately for avoiding any arising conflicts. Add to that, the fuzziness of borders due to the dynamic nature of the coast makes it difficult to be clearly defined (Douvere and Maes, 2010). Moreover, the boundaries for management and planning efforts are often established based on political considerations and not necessarily meaningful from an ecological perspective (Douvere and Maes, 2010), as effects may originate outside the boundary of the management areas (Alvarez-Romero, 2011), such as the negative consequences that arise from the trans-boundary nature of ecosystems (Beger et al., 2010), not to forget the dynamicity of marine waters and some of its habitats. Thus calls were made to link or integrate ICZM and MSP plans (see... Stoms et al., 2005; Cao and Wong 2007; Ehler 2008; Alvarez-Romero et al., 2011; Flannery and Cinnéide 2012; Kidd and Ellis 2012; Queffelec and Maes 2013; Ferreira et al., 2014; Guneroglu et al., 2014; Kerr et al., 2014; Becker-Weinberg 2015; Saunders et al., 2016), thus, this research aimed to propose a methodology that can be used for linking or integrating management plans. And the Hypothesis is as follows: in case of two management plans ICZM and MSP that include similar elements, those elements and activities can be linked while, each management plan still works independently.

This study has several limitations; where it only focused on a theoretical review of the ICZM and MSP on a regional scale of EU and the differences between them.

2. Literature review

Many researchers tried different approaches while investigating the linkage or integration related to the management plans (Stoms et al., 2005; Cao and Wong 2007; Ehler 2008; Alvarez-Romero et al., 2011; Flannery and Cinnéide 2012; Kidd and Ellis 2012; Queffelec and Maes 2013; Ferreira et al., 2014; Guneroglu et al., 2014; Kerr et al., 2014; Becker-Weinberg 2015; Saunders et al., 2016). A few of these papers focused on the integration between the land and sea regions, while some others investigated the management of marine regions. However, it was not clear what the researchers meant exactly by (linkage or integration) as those words are used interchangeably. As there was no specific explanation to any of those terms when presented, or in other words, what should be done in details to accomplish the linkage or integration process.

In more details, Alvarez-Romero et al. (2011) highlighted the need to study the land-sea linkages for conservation planning. They recommended a new operational framework that included three stages regional scoping (like planning domain, evaluation of the conservation context, etc.), analysis and inte-

gration (cross-system warnings, ecosystem-based services, biodiversity in the areas, etc.), and synthesis (schedule regions and activities, general technical results, etc.). In addition, Smith et al. (2011) argued that the integration of the land-sea planning was handled using two contexts – one was related to the planning framework; while the second was related to the environmental management context, like conservation planning. Moreover, Queffelec and Maes (2013), used a framework for coastal and marine strategies at the national or sub-national levels. In this framework, they developed the MSP into the operational or strategic stages. MSP was seen to be an operational device and was an important component of the ICZM plan, as noted in Belgium. Here the MSP zoning was included in the ICZM. Add to that, Ferreira et al. (2014) presented questions that have to be addressed before initiating the integration process, like “Are there (effective) national policies or plans for oceans and coasts? Are the ICM strategies and MSP policies aligned properly? Are all agencies that are responsible for the MSP and ICM properly coordinated? Do the different policies/plans share common indicators, objectives and integrative measures?”. And finally, Saunders et al. (2016) highlighted the significance of using a horizontal integration approach between the sectors, as it was vital for the explicit management along with vertical integration. For example, the German authorities divided the different administrative responsibilities for marine spaces, which improved the border planning between regions with different planning systems and jurisdictions that existed simultaneously.

In a summary, the (linkage and/or integration) approaches were addressed based on four perspectives. The first perspective considered the points that need to be taken into consideration before or during the linking process, though this approach of vital importance, yet, it only focused at what to consider before initiating the linkage/integration process, and it lack details. The second approach addressed the context of the linking procedure (which lay in the planning/conservation planning framework), here, its clear that the focus is on conservation only, and this may ignore economical domains within the plans. The third approach highlighted the importance of including the MSP and ICZM strategies within each other, here; the concentration is on the strategies, which are related to the big picture, not the details of the MSP and ICZM plans or concepts. And finally the fourth approach presented the significance of including the horizontal and vertical integration strategies before linking the plans, though considering vertical (hierarchy) and horizontal (between entities at the same level) is of vital importance, but again it can be seen as an organizational approach rather than being micro (considering all details of the plans). Although these approaches are of vital importance in addressing different issues, however, when implemented, they resulted certain difficulties (Queffelec and Maes, 2013) such as duplication of efforts and lack of implementation (Ferreira et al., 2014), overlap of jurisdictions (Lau, 2005; Cao and Wong, 2007; Saunders et al., 2016) and a fragmented approach. In addition, none of the studies has proposed a

linkage or integration methodology that addresses the management plans based on their outputs (the contents of the plans themselves).

In other fields, the linkage of an archaeological site's artefacts is being considered vital in identifying possible linkage through time and space (Cardinal, 2019). Though the linking of the physical artefacts differs from the linking of management plans, the approach used is interesting as it considers the spatial and temporal components of these pieces, based on the periods and location's sites.

3. Management Plan'S Linking/Integrating Methodology (MPLIM)

In case of two different management plans manage different domains and environments, and they consist of some similar and differing components (physical components, physical factors, human applications, constructed structures, services, etc.), linking them could be an option to overcome many challenges related to their implementation, particularly when the management actions of both of them are considered to be boundary-less.

The procedure to link the management plans can be initiated by establishing a connection node between all similar elements of the plans. Thus, one needs to understand which factors related to the elements and data of the plans display an empirical association. Thereafter the initial structural linkage needs to be established based on the observed results of the plans, which are going to be linked. However, because the plans have different components/elements and structures; the conceptual differences between these management plans need to be understood before any further progress. Furthermore, since all the plans have to undergo a preparation cycle before their actual implementation and their assessment, it is recommended to evaluate the plans before the initiation of any linkage. This helps in identifying the gaps and the missing features within the plans, which could hinder the linking process in its advanced stage. This evaluation would also help in determining the missing features in the theoretical components of the plan.

3.1 Analytical approach to the methodology

Assume two different management plans (i.e., plan1 and plan2) as shown in (Figure 2).

Plan1 and Plan2 manage different geographical regions/domains; wherein every plan consists of several components (physical components, physical factors, human applications, constructed structures, services, etc.).

Assume each of this plan is represented by geometrical shape. Plan1 was represented by a square shape; whereas Plan2 was represented by a triangle (*the usage of different geometrical shapes is to indicate that the management plans were not identical, as they consisted of different geographical domains, different areas of interest, differing legislations, etc. However, these differences were not*

definitive and the plans could share some of the features of the same geographical domains or sectors).

It is assumed that every plan consisted of different elements and components that are represented by different letters (a, b, c, d) for plan1; while plan2 was represented by (c, d, e, f) as shown in (Table 1).

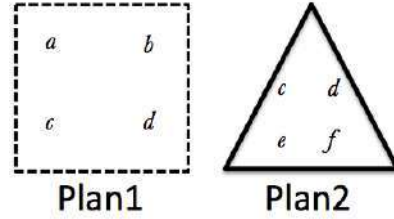


Figure 2: The different hypothetical shapes represented different management plans (i.e., 1 and 2). The different geometrical shapes indicated that both the management plans were not identical. The letters (c, d) represented the similar elements that exist in both plans, however, (d) represented the elements that appeared after evaluation. The letters (a, b, e, f) represented the differing elements/components in the plans, however, The elements (b, f) represented the missing elements/components in the management that appeared after evaluation.

Table 1: Elements of both plans and what do they represent. The elements that appeared after evaluation (b, f) meant to be the elements that are present on papers, yet, missing on ground, due to unknown reasons

Plan1 elements	Plan2 elements	Status
a	e	Exist (different and unique elements/components existed in one plan and not the other)
b	f	Appeared after evaluation (those elements were missing) (different and unique elements/components existed in one plan and not the other)
c	c	Exist (the similar elements that exist in both plans)
d	d	Appeared after evaluation (this element was missing) (the similar elements that exist in both plans)

Mathematically, the proposed methodology MPLIM could be described and proven as follows:

Assume $Plan1 = A = a, b, c, d$; $Plan2 = B = c, d, e, f$. As assumed earlier that the evaluation of the plans showed that b, d were missing from A and d, f from B .

The union of two sets A and B is the set of elements which are in A , in B , or in both A and B .

In symbols, $A \sqcup B \{x : x \in A \text{ or } x \in B\}$ then $A \sqcup B = \{a, b, c, d, e, f\}$. (It could be noted that no repetition of element appeared, as well as, the source is neglected). However, a novel concept known as multiset allows the repetition of the elements. It is defined as the “*collection of objects (or elements), where the elements can occur multiple times*” (Blizard, 1988). Thus, this approach will be called (union+) ($\sqcup+$). It can be applied to overcome the limitations of the set union concept, which does not allow any repetition of the elements

$$\therefore A (\sqcup+) B = \{a, b, c, d, c, d, e, f\};$$

The intersection of two sets A and B , denoted by $A \cap B$, is the set of all objects that are members of both the sets A and B .

In symbols, $A \cap B = \{x : x \in A \text{ or } x \in B\}$ then $A \cap B = \{c, d\}$. It was seen that only two elements (c, d) appeared in the intersection set, owing to the limitations of the classical set functions. However, based on (Blizard, 1988) multiset concept

$$\therefore \text{The (intersection+) } (\cap+) \text{ of } A \text{ and } B = \{c, d, c, d\}$$

The symmetric difference, also known as the disjunctive union, of two sets is the set of elements that are in either of the sets and not in their intersection. The symmetric difference of the sets A and B is $A \ominus B = \{x (x \in A) \ominus (x \in B)\}$ or the symmetric difference can also be expressed as the union of the two sets, minus their intersection

$$\therefore A \ominus B = \{(A \cup B) \setminus (A \cap B)\}, \text{ Then } A \ominus B = \{a, b, e, f\}$$

Therefore, based on the above-mentioned information, and Figure 2, the MPLIM could be described as illustrated in (Figure 3).

3.2 Steps of MPLIM

Based on the above-mentioned mathematical approach the steps of MPLIM are as follows

1. **Step 1: Identifying the potential sectors of linkage.** This step must be carried out first. If the plans show potential sectors to be linked, the process can be continued further. Or else, the process needs to be stopped. *However, there is a wide list of sectors to choose from*

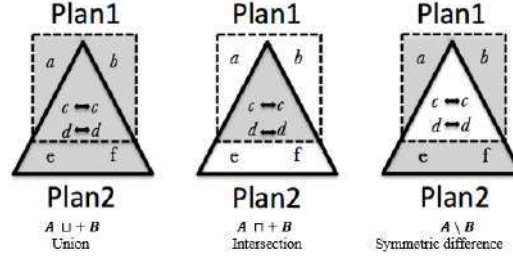


Figure 3: The Union, Intersection and Symmetric difference of two management plans are shown in grey color (shaded). These plans share the c, d, c, d elements and could be effectively linked. The elements a, b, e, f represent the different elements which have to be handled independently. Figure 2 presents additional details regarding this Figure.

and could include hard or soft elements/components. The hard elements include those which are based on their physical presence of any element; whereas the soft elements are those which are associated with the performance, knowledge or actions of associative agencies.

- The intersection + set represent all possible linkage nodes between both management plans or in other words the potential sectors or elements that could be linked.
 - $A(\cup+)B = \{c, c\}$; **before the evaluation of management plans**, thus there are possible sectors that could be linked and we may proceed to the next step.
2. **Step 2: Evaluating the management plans** which to be linked, it is recommended to fulfill and address those gaps before initiating the linkage.
- Appeared after the evaluation of management plans $\{d, d, b, f\}$.
3. **Step 3: Identifying the differences** seen between the management plans and then categorising them into various themes; as this will facilitate addressing it.
- The symmetric difference set, which is represented as
- $$A\Theta B = \{(A \cup B) \setminus (A \cap B)\}, \text{ Then } A\Theta B = \{a, b, e, f\}$$
- The differing elements to be classified into themes according to their shape or the topic they address.
4. **Step 4: Recommendations.**

- At this stage recommendation regarding linkage should be addressed, as for instance in case of two similar elements to be linked, the differences between management plans should be respected and considered. In our case, the elements $\{a, b, e, f\}$ could be soft (actions or performances) or hard elements (sectors).
 - Thus, while linking, for instance, (c element) from *Plan1* and (c element) from *Plan2* or d and d , we should keep in mind the differences in a, b, e, f . The reason to say that is because differences are generally affecting the similarities, as (elements c or d) are not independent rather they are an elements that generally cannot be found solely without being part of all (affecting and being affected). For instance, in case of the fishing sector its affecting other sectors such as shipping lanes and being affected by the seasonal and climate changes in addition to the humans actions and behaviours.
5. **Step 5: Link the desired sectors.** This step addresses the link between the two sectors from a management plan perspective and not from outside these plans. Hence, the results of the linkage between the plans would be compatible with the plans. In this context, the representatives of the other sectors can enrich the database during the linkage process. It was stated that by addressing this linkage, the researchers could save a lot of time and energy (Figure 4).

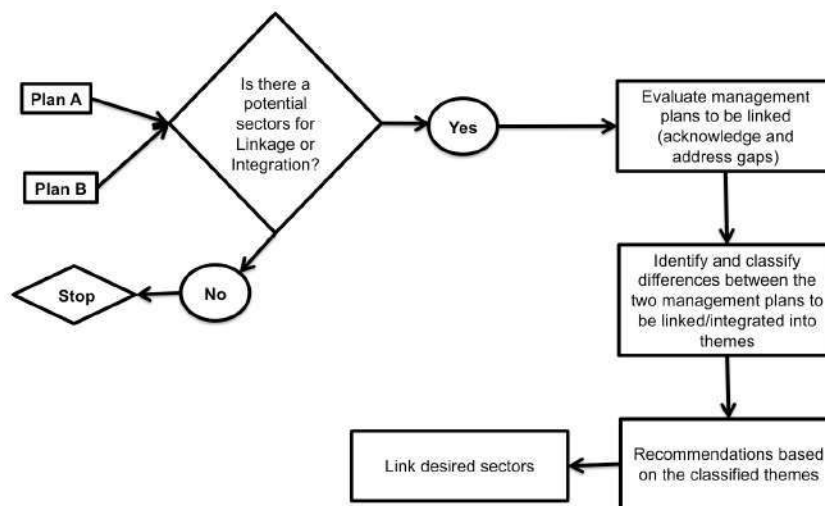


Figure 4: Summary of the applied MPLIM Steps

The following **MPLIM principles** should be observed during the process:

- **Sequentially**; this technique included the determination and definition of empirical interrelationships. Hence, this step must be carried out before

evaluating the plans that to be linked. For example, if no connection nodes are seen between the plans, the evaluation plans step needs to be stopped. Thus in determining the potential linkage nodes the following questions should be answered

- How many potential connection nodes can be established?
- Which of the spatial, temporal and cognitive elements in *Plan1* correspond to similar elements in *Plan2*?
- **Participatory**; this considers the interests of all stakeholders since many activities are conducted in the management plan sector and they could negatively affect other domains. This will ensure higher chances of linkage's success.
- **Local specificity**; this addresses the plan and their differences since every region uses a unique management plan which differs from plans in the other regions. For example, management plans may have different priorities in different regions.
- Significance of **addressing the gaps** Determining the gaps in the management plans, will ensure that they are included in the following steps (i.e., identifying the differences), thus, decreases chances of failure of the linkage process.
- **Emphasis on the differences instead of challenges**; the challenges are embedded-within and being caused-by the differences; and not vice versa.
- **Respect the differences** between the management plans, along with the limitations related to the different activities. As much as energy and time is spent to ensure this, will safeguard that all the parties are equally benefited.
- **Thematic**; the classification of the differences into themes helps in addressing all differences in a predefined path, comprehensive, harmonized, and levelled.
- The linkage step must be carried out within the **context** of all management plans, and not between the sectors directly.

The following section presents a case study that explains the MPLIM in more detail, by attempting to link ICZM and MSP. Currently, main focus is on Steps 1, 3 and 4. However, a detailed explanation of Step 2 (evaluation of management plans) by discussing the various evaluations tools is presented. Nevertheless, step 5 was not addressed, as there is no management plans (within the scope of this study) to be linked.

4. Implementing MPLIM on ICZM and MSP

4.1 Calls and reasons for attempting to link MSP and ICZM

The following are the calls for and reasons behind attempting to link between ICZM and MSP as described in (Table 2). The supporting opinions show that many authors highlighted the importance of the linkage/integration between ICZM and MSP with a variety of opinions and reasons.

4.2 Linking the ICZM and MSP plans using MPLIM

4.2.1 Identifying potential sectors for linkage

MSP resembles ICZM in several ways (Douvere and Maes 2010; Queffelec and Maes 2013). For example, both the plans use strategic, adaptive and participatory approaches, which have been integrated into the government agencies and economic sectors (Douvere and Maes 2010). These concepts help in acquiring an ecosystem-based management plan (Queffelec and Maes 2013), which can maximise the compatibilities noted between the human activities and decrease the spatial conflicts noted between nature and human users (Kidd and Ellis 2012).

The similarities noted between the MSP and the ICZM principles were described by the 2002 European Commission (EC) and the Baltic Marine Environment Protection Commission (Helsinki Commission - HELCOM) as long-term perspective and objectives; participation of all stakeholders and relevant administration; adaptive management (learning by doing) and a broad overall perspective; high quality data and information basis such as GIS (use of a combination of instruments); and coherent terrestrial and maritime spatial planning.

Both these concepts also manage similar areas owing to the overlaps noted between the Territorial Waters (TWs) (under the ICZM focus) and Exclusive Economic Zones (EEZ) (under the MSP focus). Hence, the overlapped regions must be addressed appropriately for avoiding any arising conflicts.

With regards to the hard elements (that are physically present such as the different sectors) (Joliffe and Patman (1985) as reported by Clark (1992)) to be linked, the two concepts included similar sectors. In addition to, various soft elements such as (i.e., performance, participation) as will be discussed later within the following chapter (Table 3).

To summaries, there are many potential sectors which can be linked, and this was seen to be a good reason for initiating the linkage between ICZM and MSP concepts. Additionally, an overlap of management areas was noted between these concepts. Thus, there is a need to ensure that the management should be conducted in a harmonised manner.

Table 2: Calls and supporting opinions for linking MSP and ICZM plans

Reason	Supporting opinions
(1) The two concepts manage interconnected domain generally (no ecological boundary could be determined).	• Historically, the ICZM and MSP concepts have been proposed to reduce coastal (land and sea) conflicts (Douvere and Maes, 2010). However, the fuzziness of borders due to the dynamic nature of the coast makes it difficult to be clearly defined. • “Not only are coastal and ocean ecosystems connected, the boundaries of these systems often transition slowly, and the boundaries are hard to detect even from the surface of the water. This makes it difficult to delineate definitive boundaries between ecosystems; therefore, the boundaries that are drawn often seem like indiscriminate lines drawn on a map that mean nothing in the context of the ecosystems themselves” (Bareford, 2016). • Boundaries for management and planning efforts are often established based on political considerations and not necessarily meaningful from an ecological perspective (Douvere and Maes, 2010), as effects may originate outside the boundary of the management areas (Alvarez-Romero, 2011), such as the negative consequences that arise from the trans-boundary nature of ecosystems (Beger et al., 2010). • Dynamicity of marine waters and some of its habitats.
Continued on next page	

Table 2 – continued from previous page

Reason	Supporting opinions
(2) Calls to link the two concepts.	• “Ultimately, marine spatial plans should be consistent with plans developed in the coastal zone. It is therefore important to stop discussing MSP in isolation, and instead, explore the linkages among relevant plans” (Douvere and Maes, 2010). • Past research has revealed that one such linkage might be the potential of MSP in making ICZM more operational, which advances its implementation accordingly by utilising MSP’s focus on spatial planning; the ICZM would thus produce more meaningful results (Douvere and Maes, 2010). • The first European High-Level Forum on Community Strategies for ICZM highlighted a similar viewpoint, where the spatial planning had been recognised as one of the priority themes for the further implementation of ICZM in European coastal zones (Douvere and Maes, 2010). • The PLANCOAST handbook has even illustrated how ICZM can support the implementation of MSP (Smith et al., 2011). • Kidd and Ellis (2012) and Kerr et al. (2014) had emphasised on the importance of the close relationship between marine and coastal planning. For this reason, the European Economic and Social Committee of the Regions had recommended a one-stop shop concept to overcome the challenges in managing marine areas that are up to 200 Nautical Miles (NM) (Becker-Weinberg, 2015).
Continued on next page	

Table 2 – continued from previous page

Reason	Supporting opinions
(3) Calls to integrate the two concepts or inclusion of one of them in the other.	• Guneroglu et al. (2014) argued that MSP should be an integral part of ICZM, wherein MSP can be regarded as an instrument to address integration issues of all scales/levels as a means of supporting the implementation of policy objectives (Queffelec and Maes, 2013). • Since MSP is focused on maritime and ICZM is land oriented, the former could fill the gaps in ICZM concerning the maritime domain (Queffelec and Maes, 2013), while ICZM could do the same for MSP. • Ehler (2008) argued that MSP has become one of the most powerful tools for the integrated management of coastal and marine environment concerns. • Agardy (2010) stated that there should be harmony between land and sea planning due to the continuity of activities, and it is believed that a regional planning would help in overcoming the fragmentation issues faced by both the marine environment and coastal zone. • Although the EU had not referred MSP in its recommendations on ICZM that was adopted in 2002, it does, however, provide a basis as part of a requirement for all EU countries to develop national ICZM strategies (European Commission, 2002; Douvere and Ehler, 2009), which could be utilised by MSP and for its integration with ICZM. • Directive 2014/89/EU of the European Parliament and the Council of 23 July 2014, establishing a framework for MSP, emphasised “member states shall aim to promote coherence between MSP and the resulting plan or plans and other processes, such as integrated coastal management or equivalent formal or informal practices”. • Directive 2013/0074 (COD) established a framework on how to implement ICZM and MSP and what are the legal bases and principles. Although it is mentioned that coherence should be established between ICZM and MSP of a state, it did not mention how to execute it. Thus, the following section will attempt to link both concepts using MPLM.

Potential (sectors) elements to be linked that generally appear in ICZM and MSP plans			
Human usages			Ecosystem/Habitats
Multiple water space use	International sea trade	Coastal settlement and hotels	Oceanic Hydrothermal Vents
Power line approach	Renewable energy	Shipwreck	Sea floor
Dredged channel	Port Terminals	Buoys, e.g., water skiing	Coral reefs
Estuarine reclamation	Dumping of toxic wastes	Coastal trade	Ocean's surface
Marine park	Artificial reef, e.g., fishing	Drainage/irrigation	Intertidal systems
Effluent discharge	Scientific interest	Offshore navigation	Wetland conservation/nature reserve
Flood area	Groin field	Lighthouses	Dune conservation area
Coastal industry/power stations	Artificial beach	Fishing sector	Salt marshes
Estuarine urbanization	Beach mining	Floating or submerged storage tanks	Mudflats
Buoys	Cross-channel ferries	Artificial island	Seagrass
Marina	Military activities, e.g., air-to-sea firing range	Coastal airport	Meadows
Mariculture	Offshore oil/gas rig and undersea pipelines		Mangroves

4.2.2 Evaluation of ICZM and MSP plans which are to be linked or integrated

Evaluation help in learning from successes, failures and in determining the gaps which affect the implementation of a plan. It was presumed that all adjustments made in the planning and decision-making stages could help in managing the coastal sector (Oslen 2003; Jacobson et al., 2014). However, since there are no specific management plans (ICZM and MSP) that have to be linked within the scope of this thesis, the approach had focused only on introducing the evaluation tools that can be used. This step aims at determining the various missing elements along with the non-performing activities of these management plans. However, if the linkage was carried out before determining the missing or immature components, the whole linkage process could be severely affected and the whole process will revert to the first step. Additionally, by evaluating the plans the whole linkage procedure could be improved and accelerated. In addition, there is a need to consider the vulnerable situation of the coastal and marine ecosystems, as these ecosystems are constantly changing. Many approaches have been described for evaluating the ICZM efforts such as generic framework for determining the success (Burbridge 1997), understanding the process indicators (Henocque 2003), progress framework (Olsen 2003), progress indicators (Pickaver et al., 2004), factors affecting the success (Stojanovic et al., 2004), dual-level framework (Billé 2007), coastal sustainability standards (Gallagher 2010) and assessing the ICZM components (Pérez-Cayeiro and Chica-Ruiz 2015). Most of the above research studies classified the assessment into two different categories, i.e., *results-based or progress-based*. The result-based assessment compares the results and the desired objectives; evaluates the three pillars of sustainability; and, determines the different types of outcomes, (which were presented by Olsen (2003) as 1. community activities which improve the coastal approach; 2. different indicators which indicate the successful application of the plans; 3. environmental or economic indicators which highlight the environment or economic which was benefited due to the implementation of a certain plans; and 4. creation of a balance between the social and ecological systems). The progress-based assessments are carried out by determining the actual progress of the plans based on the ICZM cycle and can be measured based on the level of accomplishment for each step (the progress-based evaluations include several outputs like guidelines, management plans and the direction in the criteria).

One of the major issues that should be noted while choosing an evaluation tool is the local specificity; as irrespective of the selected methodology, it must be designed and developed for suiting the coastal initiatives and efforts selected for evaluations. For example, the differences noted between the coastal regions at the national or regional levels with regards to the governance hierarchy, society, nature and interests should be considered before selecting the evaluation technique. In addition, a demonstration of the historical legislations and boundaries during the evaluation process will help identify gaps in the legislations and

governance (if any). These gaps should be addressed separately and resolved before any initiation of a linkage process. With regards to the MSP evaluation, Ehler (2014) proposed a guide for evaluating marine spatial plans. He endorsed it by saying,. The guide designed the evaluation methodology in 8 different steps. However, the fundamental steps include evaluate the effectiveness of the management action, develop the baseline information for the marine spatial plan, choose suitable indicators to be monitored on a regular and continuing basis, and communicate the results to the decision-makers. It should be noted that this methodology employs a results-based approach.

Bareford (2016) in her Ph.D. dissertation entitled *An Evaluation of Coastal and Marine Spatial Plans*. It is clear from the title that the proposed evaluation methodology can be used to address any plan within this domain, and that it appeared from the case studies discussed in the dissertation, such as those that comprised marine bioregional plans from the year 1999. However, the evaluation only assesses what is contained within the written document, not the process that created the document or the outcomes that result from the document. It is important to critically assess the content and quality of plans because they are the major elements of the process which lives into the future (Bareford, 2016).

Hence, the selection of an appropriate methodology for evaluating is based on when and what needs to be further evaluated. This could be carried out after the preparation of the plan, during the implementation of the plans or the comparison of the results with the desirable objectives. Along with the different evaluation methodologies, the selection of an appropriate methodology is also the responsibility of the planners and governments. Furthermore, it must be noted that during the evaluation of the MSP, the planners need to use the ICZM evaluation methodologies (as they may be more advanced) if considered appropriate.

4.2.3 Identification of the differences between the MSP and ICZM plans

Speaking of the differences between the two concepts (ICZM and MSP), many studies have addressed their main characteristics; however, by using the *Cmap-Tools software toolkit* (Cañas et al., 2004), the differences have been identified and divided into themes.

The first theme is related to knowledge, as there are differences between ICZM and MSP. For instance the ICZM knowledge is is tacit in locals minds and it encompass different expertise of the administrative agencies because they are generally realm-specific (Beger et al., 2010). Speaking of MSP knowledge, it is expert knowledge (Saunders et al., 2016). And there is incomplete knowledge of the consequences of human activities on the marine environment (Kerr et al., 2014), even as it can be studied and understood with regards to the coastal land.

The second theme is related to stakeholder. Speaking of ICZM the nature of property rights differences between marine and terrestrial (coastal land) (Kidd and Ellis, 2012) and the sense of belonging to the land for the locals (Kidd and Ellis, 2012; Kerr et al., 2014) are featured aspects. In addition to the private ownership and the sense of belonging to the land by the locals. However, People believe sea is a public property (Bareford, 2016). Add to that, land planners see the sea from the context of visual scenery only, and the sea planners believe that land is their invader (Barale et al., 2009). As a matter of fact, the inhabitants from the sea expect nothing from the land, while the land inhabitants are dependent on the sea and its resources. Nevertheless, There is limited scope for the public to influence decisions within the marine and seabed areas, which are managed by the state on behalf of the public (Kerr et al., 2014). In other words, Flannery et al. (2018) believed that MSP is not facilitating public engagement, except for a few groups, to legitimise the process, and it keeps the power in the hands of the elite or the government.

The third theme is related to the used approach. ICZM approaches are more of landward-oriented (Queffelec and Maes, 2013). Add to that, ICZM principals focus on local specificity. And ICZM is a process-oriented (sectorial) approach that emphasises integration across agencies and sectors, and it rarely addresses the allocation of coastal space for efficient economic development and effective protection of valuable ecological and biological areas (Douvere and Maes, 2010; Smith et al., 2011; Ferreira et al., 2014). Thus, it is driven by the significance of competing for environmental protection and economic development (Kerr et al., 2014). However, MSP is Seaward-oriented. And its principles are based on an ecosystem approach and try not to limit themselves within the political boundary as this approach is trans-boundary (European Commission, 2007; Douvere and Maes, 2010). Moreover, MSP approach is systemic (Ferreira et al., 2014). While acknowledging the need for interagency and cross-sectoral integration, MSP has focused on determining a basis for the efficient and effective allocation of ocean space to economic activities and the designation of areas for conservation and protection (Ehler and Douvere, 2009; Smith et al., 2011; Ferreira et al., 2014). In brief, management priorities are different (Ferreira et al., 2014).

The fourth theme is related to jurisdictional. As the coastal area is generally under the jurisdiction of the local government. However, EEZ water is national jurisdiction (Agardy, 2010) and, in some cases, it is international jurisdiction according to article 86 of UNCLOS (UNCLOS, 1982). Thus, there is a possibility of jurisdictional conflict between authorities (Saunders et al., 2016). Within the same context, ICZM is typically applied to marine zones less than five kilometers from the coastline, while MSP can be applied to much wider areas up to EEZ. However, it is worth noting that the geographic scope of the ICZM Protocol in the Mediterranean extends seaward to the external limit of the TWs that overlaps with the MSP area of interest (Agardy, 2010).

The fifth theme is related to legislation. For instance, the earth Summit UN Conference on Environment and Development (UNCED) that was held in Rio de

Janeiro, Brazil, it had introduced ICZM as an integrated approach for managing coasts in a sustainable manner, and yet it is a non-binding instrument (Cicin-Sain et al., 1998; Meltzer and Canada Dept. of Fisheries and Oceans, 1998; Wheeler et al., 2011; Tiller et al., 2012). Moreover, the EC, in its Recommendation of the European Parliament and of the Council of 30 May 2002 concerning the implementation of ICZM in Europe, reemphasised the importance of ICZM principles and entailed all member states to report to the Commission about the experience with implementation of the Recommendation 45 months after its adoption. However, one of the challenges faced by MSP is the weak international and national legislation. For instance, it is not explicitly specified in UNCLOS or the CBD (Ardron et al., 2008; Saunders et al., 2016). Yet, at the EU regional level, the Directive 2014/89/EU of the European Parliament and of the Council of 23 July 2014, while establishing a framework for MSP, played a significant role in recognising MSP at the regional level.

The sixth theme is related to the outputs. Since the expected outputs of both plans are different. While the ICZM strategy, plan or programme is simply an integrated set of desired and integrated outcomes along with an action plan to realise them. However, MSP uses zoning maps and/or a permit system to plan for a marine area or ecosystem that can influence the spatial and temporal distribution of human activities.

4.2.4 Recommendations and link

This research has highlighted the major differences between the two concepts (ICZM and MSP within the European experience), which could act as a guide for linking the plans.

Now at this stage, linkage between any two similar sectors could be initiated as shown in (Figure 5). The attempt here is to link fishing sector from MSP's plan/concept and ICZM's plan/concept; (the word concept used to indicate that there are no particular plans to be linked). However, other sectors such as (navy, ships traffic lanes, renewable energy... etc), needs and concerns also need to be respected and addressed if needed. This would ensure that the decisions made, will be in favour of the two management plans/concepts, and not only one, and that was the sole purpose of linking the plans.

As shown earlier, the differences between these concepts were classified into various themes, i.e., **knowledge-related, perspective-based, approach-based, stakeholders-related, jurisdictional-related, legislation-related, and output-related**. And it's believed this classification will help planners to focus on a particular theme, and address it solely without the interference of other themes.

The approach here could be an elucidation approach and capacity building of planners and to try to harmonise and explain the differences to stakeholders and the possible consequences to their areas/sectors of interest. In addition, it would help them present a recommended technique based on these differences.

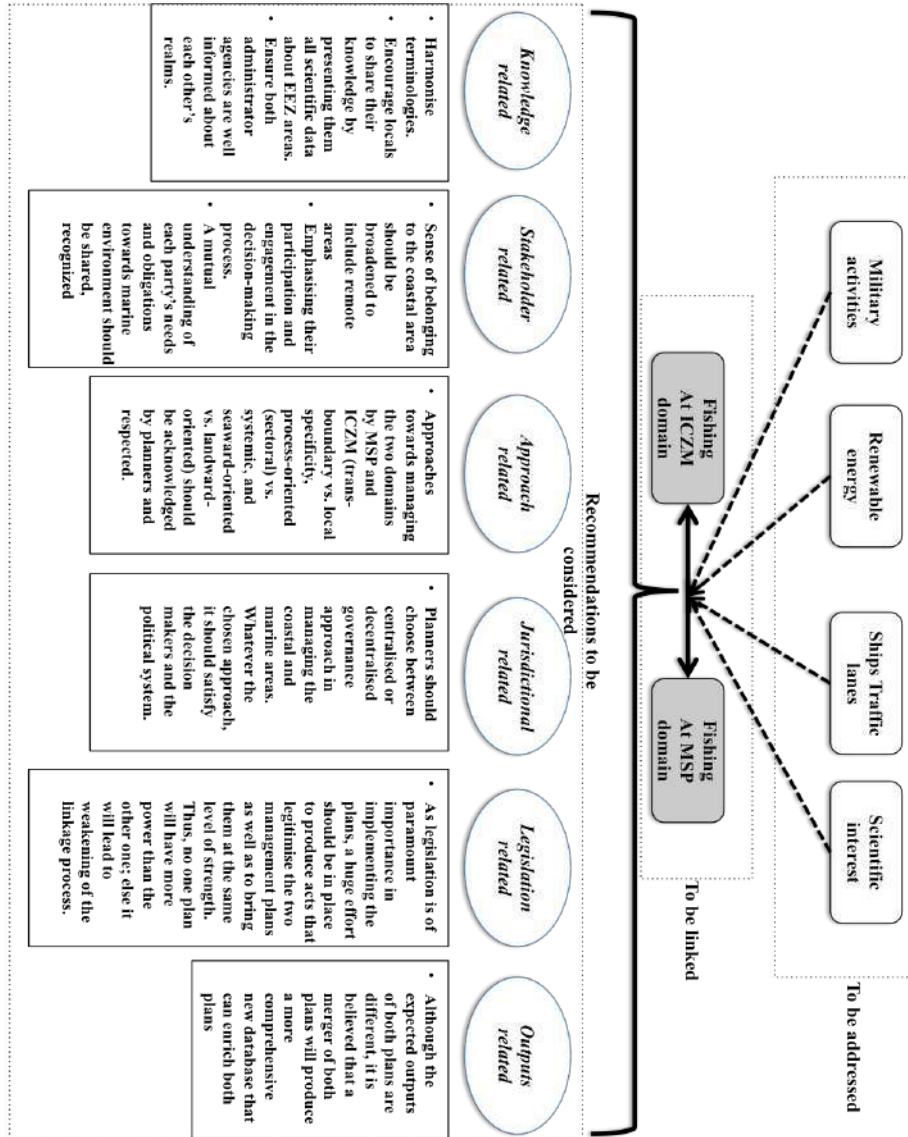


Figure 5: The linkage between the fishing sectors is seen to respect and also address the various sectors. It also considers all recommendations related to the differences.

5. Discussion

The absence of a practical and concise technique for linking management plans encouraged the approach to develop a novel methodology for linking the management plans. The proposed methodology focused on the technique for linking different management plans and was known as the Management Plan Link-

ing/Integrating Methodology (MPLIM). Though there were attempts to link or integrate MSP and ICZM or land and sea plans as argued and discussed by (Stoms et al., 2005; Cao and Wong 2007; Ehler 2008; Alvarez-Romero et al., 2011; Flannery and Cinnéide 2012; Kidd and Ellis 2012; Queffelec and Maes 2013; Ferreira et al., 2014; Guneroglu et al., 2014; Kerr et al., 2014; Becker-Weinberg 2015; Saunders et al., 2016), yet, none of those researches introduced a methodology to be followed. Thus, it's believed that the methodology introduced is a novel technique.

The MPLIM included five steps; these steps were articulated around the fundamental differences noted between the management plans. Though, the importance of identifying differences between the various management plans were also described earlier by Stanwick (2000), yet, it was not specified exactly, how to address those differences.

The justification of the MPLIM was based on the geometric shapes that presented an analytical proof as mentioned earlier. It must be highlighted here that the reason behind using different geometric shapes to represent the two management plans, is to give indication that the plans were not identical, though they may share same elements. This approach is different from previous approaches in mathematical calculation with sets; as they used the circles only to represent sets. Maybe this approach is acceptable with math, but not with management plans. It is believed that this may be misleading as it indicates that the sets are identical. All in all the geometric shapes assisted in visualisation and helped in determining the different steps of MPLIM, which eventually led to the acceptance of the methodology by author.

When the MPLIM had been implemented on management plans (ICZM and MSP), this technique can help in linking these plans. This technique was further characterised by various features that addressed the linking based on the original plans. This revealed the potential sectors to be linked and the differences between management plans concepts and approaches. However, it was realised that the significance of classifying these differences in various themes as these themes further helped in grouping and addressing the differences more objectively. These differences also highlighted the vital components of both concepts that could not be easily identified before the implementation of the MPLIM. In addition, this study indicated that the differences between both concepts (ICZM and MSP) were more of shape-based and far from being core-based. The reason behind saying that can be justified by the appearance of the many similarities between the two concepts, add to that, the differences between them are not huge, rather, they are basically on the way things were addressed or appeared.

With regards to the recommendation section, it focused on the remediation of all differences seen in the predefined themes. For example, in the case study, the focus was on *knowledge, approaches, stakeholders, jurisdictions, legislation, and outputs themes*; however, the study yielded many notable issues. If these

issues were properly considered, an effective and proper linkage process could be determined.

The approach described in this study differs from other techniques that attempted to link ICZM and MSP plans. As previously researchers developed a new framework and recommended some questions that have to be addressed (Ferreira et al., 2014), while, some used a vertical or horizontal integration process for linking the plans (Saunders et al., 2016). However, these techniques were not benefited from the presence of the original plans, while this study considered the original plans as the start point for linking. It is believed that this will help in saving a lot of time and efforts; as starting from the available plans is of paramount importance in finding details in a fast paced Furthermore, a higher reliance on the differences between the management plans presented a systematic and distinctive approach as it helped in developing a strategy which highlighted what needs to be addressed before linking the plans. However, if no difference is noted between the plans, then the linkage will take place between identical plans, which should not cause any issues. Furthermore, by addressing the differences all challenges could be overridden, since it is believed that *the challenges are embedded-within and being caused-by the differences; and not vice versa*.

This study has several limitations; where it only focused on a theoretical review of the ICZM and MSP on a regional scale (EU) and the differences between them. Furthermore, it presented only the evaluation methodologies that could be used without conducting any evaluation, as no specific management plans were selected for linking within the scope of this study.

6. Conclusions

It's believed that the MPLIM introduced in this thesis could help the planners and governments to link various management plans in various similar sectors. Furthermore, this technique is important as it presented simple steps for initiating the linkage without requiring any new laws or entities. It is perceived that this will help in saving a lot of time and efforts, in addition to preventing the development of any issues during the linkage process.

This study focused on introducing and testing the MPLIM by implementing it on ICZM and MSP, however, it was able to identify potential linkage sectors and analyse the differences between these management plans. The results of this study could help in understanding all issues related to management plans under investigation and linkage.

It is believed that understanding of the cumulative impact on the coastal environment will require the development of a novel and innovative methodological approach that can melts all components and impacts in coastal and marine areas in one crucible. This could help the planners and decision-makers to make a more scientifically informed and better decisions, as it is believed that ICZM or MSP cannot be completed individually unless the other is completed.

Though this methodology was designed to suit MSP and ICZM management plans, yet, it is anticipated that it could be used for linking management plans in other fields.

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Share price forecast: using ARIMA model special study on Amman Stock Exchange (ASE)

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Abstract. The goal of this research is to explore the events and Stock price prediction of Amman Stock Exchange (ASE) so for their scientific importance in economics for predict financial time series, using the ARIMA model for the active Participation's of stock market as well as macro decision makers to be able to predict trend of the market. The data examined in the paper had been collected from Amman Stock Exchange (ASE) from 2012 to 2018. 15 well-known banks on the ASE have thus been selected for stock price forecasting. The data set examined information about: opening price, closing price Concerning the results of previous researches, which confirm the relative superiority of linear models in price index prediction, an appropriate model has been offered in this research ARIMA model, in order to predict price index of Amman Stock Exchange. The results show ability of a to predict stock price time series data using ARIMA model, but more variables should be studied to support results and non-impartiality.

Keywords: price, ARIMA model, time series, forecasting, Amman stock exchange (ASE).

1. Introduction

All investors, investment banks, hedge funds, etc., they financial decision makers. They will need to predict prices of financial assets such as stocks, bonds, etc. in order to make accurate financial decisions. It is the main reason that efforts for improving the efficiency for forecasting models has never been stopped in the finance. Banking time series data and financial time series in general is very hard in decomposition and forecasting because the data are non-stationary and non-linear with high heteroscedasticity (Pai and Lin, 2005; Wang et al, 2012; Wei, 2013). The next plan of the investors is to enhance their profits less risk. This idea force the researchers to develop many predicting models such as hybrid models and ARIMA models. The ARIMA procedure analyzes and forecasts equally spaced univariate time series data, transfer function data, and intervention data by using the autoregressive integrated moving average (ARIMA) or autoregressive moving average (ARMA) model.

An ARIMA model predict a value in a response time series as a linear combination of its own past values, past errors and current and past values of other time series. Pankratz (1991) refers to the ARIMA model as dynamic regression

Whereas, rarely to find any article that utilize Amman Stock Exchange using ARIMA model in forecasting such as (Alwadi, 2015).

This paper is organized as follows: The following section describes the literature review of the method used. Section 2 provides a description of the data set. Section 3 provides the methodology and research framework. Section 4 describes the ARIMA Box-Jenkins. In Section 5, we mention the conclusion and recommendations.

2. Literature review

Stock price prediction is regarded as an important topic in finance and economics which has spurred the interest of researchers over the years to develop better predictive models Ayo, Adewumi, Adebisi [17]. And it's been autoregressive integrated moving average (ARIMA) models have been explored in literature for time series prediction. This research came to progress extensive process of building stock price predictive model using the ARIMA model. the Data were collected stock data obtained from New York Stock Exchange (NYSE) and Nigeria Stock Exchange (NSE) are used with stock price predictive model developed. The results showed obtained revealed in this research that the ARIMA model has a strong potential for short-term prediction and can compete favourably with existing techniques for stock price prediction.

The idea of research came to clarify demonstrate the usefulness for Stock price prediction over time is a problem of practical concern in economics and of scientific interest in financial time series forecasting. The matter in this research also expands toward detecting the variables that play an important role in its behaviour Stoean, Catalin Stoean, Sandita [13]. the Researchers' studied appoints an ARIMA model with regressors to predict the daily return of ten companies enlisted in the Romanian stock market on the base of nine exogenous predictors. The experimental results showed justify the benefits of the model with the evolutionary selector. The focus of research is to clarify the usefulness Banking time series forecasting gains a main rule in finance and economics which has encouraged the researchers to introduce a fit models in forecasting (Almasarweh and AL Wadi, [12]). The purpose of this study was the researchers present the advantages of the autoregressive integrated moving average (ARIMA) model forecasting accuracy. Data were collected from Banking data from Amman stock market (ASE) in Jordan was selected as a tool to show the ability of ARIMA in forecasting banking data. Therefore, Daily data from 1993 until 2017. the results shows that the ARIMA model has significant results for short-term prediction.

Stock price prediction has always attracted interest because of the direct financial benefit and the associated complexity (Mondal, et al [19]). Was check the effectiveness of Autoregressive Integrated Moving Average (ARIMA) model, on fifty six Indian stocks from different sectors. Was selected chosen ARIMA model, because of its simplicity and wide acceptability of the model. We studied

in research the effect on prediction accuracy based on various possible previous period data taken. The comparison and parameterization of the ARIMA model have been done using Akaike information criterion(AIC) (Mondal, Shit, Goswami, 2014). The result Accuracy of ARIMA model in predicting stock prices is above 85%, which indicates that ARIMA gives good accuracy of prediction.

In 2017 the autoregressive integrated moving average (ARIMA) models have been especially popular in time series prediction. The presents extensive process of building stock price predictive model using the ARIMA method, was selected from Stock data of Apple Inc obtained from Yahoo Finance are used (Dong, Li et al, 2017). Results revealed that the ARIMA model has a strong potential for short run forecast and is competitive with other prediction methods in guiding investment decision.

Stock price prediction is an important topic in finance and economics which has spurred the interest of researchers over the years to develop better predictive models (Adebiyi et al, [21]). The published stock data obtained from New York Stock Exchange (NYSE) and Nigeria Stock Exchange (NSE) are used with stock price predictive model developed. the Results show revealed that the ARIMA model has a strong potential for short-term prediction and can compete favourably with existing for stock price prediction (Adebiyi, Adewumi, K. Ayo, 2014).

3. Methodology

Previous studies have used statistics, technical analysis, fundamental analysis, and linear regression to forecast share price. Models in Time Series Analysis enable the user to generate: forecasts of a (dependent) time series that is based upon the information of its own past, explain events that occurred in the past, and provide insight into the dynamical interrelationships between variables. ARIMA model is a procedure of forecasting future values of time series that same as statistical models by using historical data generate forecasting value of variables. ARIMA is an Abbreviation that stands for Autoregressive Integrated Moving Average. This Abbreviation is described are:

AR: Auto regression. That uses the dependent relationship between an observation and some number from lagged observations. MA: Moving Average That uses relationship between an observation and a residual error from a moving average model applied to lagged observations. The data examined in the paper had been collected from the of Amman Stock Exchange (ASE) from 2012 to 2018. 15 well-known banks on the ASE have thus been selected for stock price forecasting. Two exogenous predictors are also taken into account in the process: opening price, closing price. Their values were recorded yearly (obviously excluding the weekends) for 7 years, about 105 observations were accumulated for each variable from related in period.

3.1 Variables

Table 1: list definition variables

variables	Define
x_1 x_2 Opening price	price of the first trade of any listed stock
x_1x_2 Closing price	final price a trades during regular market hours

3.2 ARIMA model

Application of nonlinear regression to price forecasting has not been reported so far. ARIMA is a suitable model for the stationary and non-stationary time series data, although most of the software uses least square estimation which requires stationary. The modeling procedure of ARIMA models based on the Box–Jenkins (Zhang, Patuwo, and Hu, 1998) methodology always contain three iterative steps, including model identification, parameter estimation and diagnostic checking. These three steps are described as follows:

1) Identification: In this step we look for the actual values of p (the number of auto regressive), d (the number of referencing) and q (the number of moving average). For this purpose Box and Jenkins (Zhang, Patuwo, and Hu 1998) proposed the autocorrelation function (ACF) and the partial autocorrelation function (PACF) of the sample data as the basic tools to identify the order of the autoregressive integrated moving average models.

2) Estimation: After choosing a specify ARIMA (p, d, q), the parameters which were identified in the previous phase, should be estimated by the ordinary least squared (OLS) method.

3) Diagnosis checking: The last step in model building is the diagnostic checking of model adequacy. If the model result is not adequate, a new structure of ARIMA model will be identified, and the three previous steps should be repeated until the best structure is found.

The formula of ARIMA model for time series forecasting is shown in equation (1).

The forecast (with forecast lead l) of a time series Y_t at a forecast origin t is denoted by F_{t+l} or $F_t(l)$ which is, in our case, always a combination of previous observations. The forecast can obviously also be written in terms of previous error-shocks.

$$(1) \quad F_t(1) = \sum_{i=0}^{\varphi} \psi^* 1 + 1^e t - 1.$$

The general ARMA(p,q) can be defined as in equation 2:

$$(1 - \phi_1 B - \phi_2 B^2 - \dots - \phi_p B^p) W_t = e_t W_t - \phi_2 W_{t-1} - \phi_2 W_{t-2} - \dots - \phi_p W_{t-p} = e_t \quad (2)$$

$$W_t - e_t = \phi_1 W_{t-1} + \phi_2 W_{t-2} + \dots + \phi_p W_{t-p} - F_t = \sum_{t=1}^p \phi_1 W_{t-1},$$

where W_t is a stationary time series, e_t is a white noise error element, and F_t is the forecasting. the AR(p) process can be written as in equations 3 and 4.

$$(3) \quad \phi(B) W_t = \alpha, \quad W_t = \psi(B) e_t (\phi(B)^{-1} = \psi(B))$$

the MA(q) process is defined by

$$(4) \quad \begin{aligned} W_t &= (1 - \theta_1 B - \theta_2 B^2 - \theta_2 B^3 - \dots - \theta_q B^q) \alpha \\ W_t - e_t &= -\theta_1 e_t - \theta_2 e_t - \theta_1 e_t - \dots - \theta_q e_t \\ F_t &= -\sum_{t=1}^q \theta_1 e_t, \end{aligned}$$

where W_t is a stationary time series, e_t is a white noise error element, and F_t is the forecasting. Remark that this description of the $MA(q)$ process is not simple to use for forecasting purposes due to its recursive character. The theoretical ACF and PACF are equations in 5 and 66 on replacing $\psi_1 = -\theta_1, \psi_2 = -\theta_2, \psi_3 = -\theta_3, \dots, \psi_q = -\theta_q$ and $\psi_j = 0$, for all

$$(5) \quad j > q,$$

$$(6) \quad \rho_k - \sum_{j=1}^p \phi_j \rho_{k-j} = 0 \text{ for } k > q.$$

4. Results

The statistical analysis of ARIMA variables is presented in table 2.

Table 2: The sample distribution for field of specialization

	N	MEAN	MEDIAN	STD.DEV	STD.ERR
$X_1 \ X_1$	105	2.611	1.52	0.654	0.063
$X_2 \ X_2$	102	2.577	1.52	0.669	0.065

ARIMA Parameter Estimation and Backward Selection are presented in Figure 1.

The AR(1) coefficient is fairly significant and the “Mean” (which is the estimated trend per period when a single order of differences has been used).

	AR1	MA1	SAR1	SMA1
Estimates (1)	0.4567	-0.93	0.4567	-0.93
(p-val.)	(0.0507)	(0)	(0.0507)	(0)
Estimates (2)	0	-1.064	0.7686	-1.064
(p-val.)	(NA)	(0)	(0)	(0)
Estimates (3)	0.4667	-0.9257	0.4667	-0.9257
(p-val.)	(0.1298)	(0)	(0.1298)	(0)
Estimates (4)	0	-1.0001	0.8804	-1.0001
(p-val.)	(NA)	(0)	(0)	(0)

Figure 1: ARIMA Parameter Estimation and Backward Selection

The AR1 for $x_1 = 0.45$ and the AR1 for $x_2 = 0.46$ is a tiny numbers reflecting the smoothness of the series. the MA(1) there is a negatives for x_1 and x_2 in the ACF plot and a decay pattern (from below) in the PACF plot.

The residual and partial autocorrelation function is presented in Figure 2.

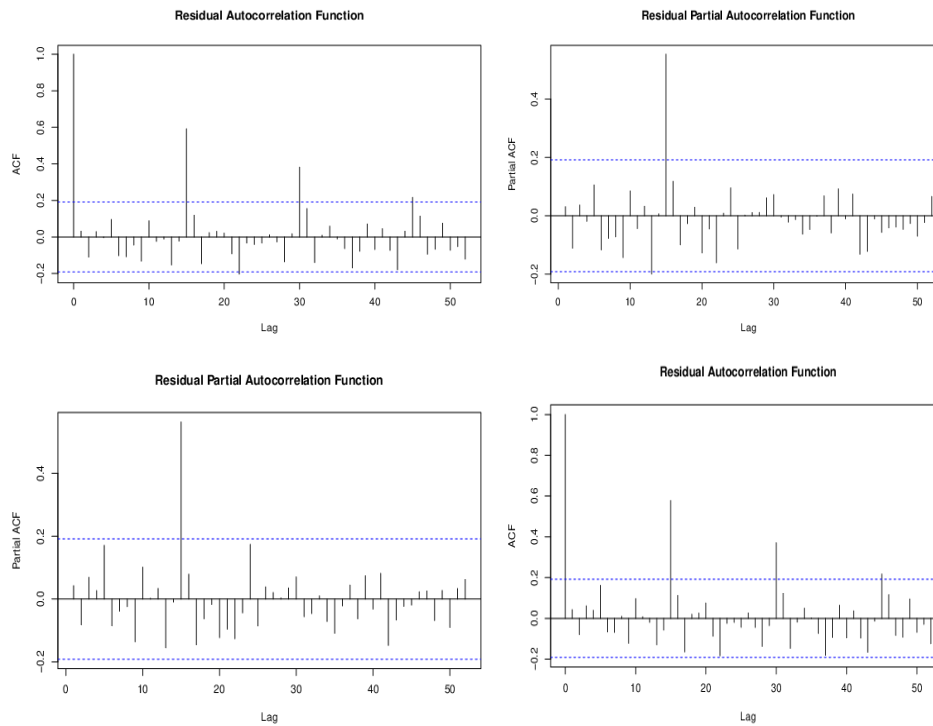


Figure 2: residual and partial autocorrelation function

Results refer to the ACF and PACF plots of the different series, we see our first significant value in lag 16 for ACF and in the lag 16 for PACF which suggest to use $p = 1$ and $q = 1$. We also have a big value at lag 16 in the ACF and PACF plot for all variables which suggests our season is $S = 6$ and since this lag is a positive it suggests $P = 1$ and $Q = 0$. Since this is a differentiated series for ARIMA, we have identified $d = 1$, and since the seasonal pattern is unstable over time we set $D = 0$. All this gives us together the ARIMA model $(1, 1, 0)$, with the autoregressive coefficient $\alpha = 0.5$.

4.1 Conclusion and recommendation

All investors, investment banks, hedge funds, etc., they financial decision makers. They will need to predict prices of financial assets such as stocks, bonds, etc. in order to make accurate financial decisions. It is the main reason that efforts for improving the efficiency for forecasting models has never been stopped in the finance. This study came to was conducted to obtain a suitable forecasting model for stock prices of ASE banks sector. The data examined in the paper had been collected from the of Amman Stock Exchange (ASE) from 2012 to 2018. 15 well-known banks on the ASE have thus been selected for stock price forecasting. The data set examined information about: opening price, closing price. Concerning the results of previous researches, which confirm the relative superiority of linear models in price index prediction, an appropriate model has been offered in this research ARIMA model, in order to predict price index of Amman Stock Exchange.

ARIMA model has been selected for forecasting stock prices. From the Structural Time Series, it could be seen that the highest value of stock price occurred in 2017 and the lowest total value of stock price occurred in the year 2018. It is suggested that in future for forecasting the time series a Hybrid method and study more variables such as: average closing price, exchange rate, highest price, lowest price. However, for the scope of the present study ARIMA $(1, 1, 0)$ is the most appropriate model for forecasting purposes.

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Internal audit practices consistency with international internal auditing standards for entities with government contributions

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Abstract. The study aims to investigate the consistency of internal auditing practices with international internal auditing standards for Jordanian companies with government contributions. The study employs both qualitative and quantitative research methods in addressing the problem of the study. To achieve the study objective, the researchers distributed a questionnaire to (460) employees of (115) Jordanian companies with government contribution, (434), (90) percent of the questionnaires were returned. On order to analyze the data, the researchers used the Mean, Standard Deviation, Percentages, and T-test. The general findings of the study revealed that Jordanian companies with government contributions apply internal auditing practices that are enacted by government agencies but unfortunately these practices are not fully consistent with international internal auditing standards. the results showed weak commitment from sampled companies to such necessary standards. In the light of the finding, the researchers gave a number of recommendations that are necessary to achieve sufficient and effective internal auditing standards. the originality of this study is steamed from addressing the importance of adopting internal auditing standards for performance and their role in increasing the control over organizations activities.

Keywords: internal audit, internal audit standards, owned companies and government contributions.

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1. Introduction

Internal auditing is an important activity for any organization; It represents one of the important functions that provide management with permanent, objective, neutral evaluation and reports for various activities in organization. Internal auditing departments usually control the conduct of various operations and activities within the organization. As well as mitigate the high risk associated with the activities of business organizations, in addition to their advisory role that helps organizations to improve their operations. Many argued that internal audit has become a crucial function within organizations; however, insufficient attention has been given to study the internal audit compared to external audit. This view is supported by Gendron and Bedard (2006) who noted that academic researchers have tended to focus on external audit as a main control function mandatory while ignoring internal audit. Internal audit effectiveness has not been extensively studied yet, and several researchers have recommended the need for more research on the internal audit effectiveness especially in developing countries (Al-Twaijry et al., 2003; Mihret and Yismaw, 2007; Arena and Azzone, 2007; Yee et al., 2008; Ahmad et al., 2009), where the internal auditing might play an important role against both the fraud and corruption.

Companies are continuously demanded to protect and guarantee the rights of shareholders and all interested parties associated with the company's business through control the performance of management. Therefore, this requires the availability of internal auditing elements that support and promote the implementation of organization strategy according to predetermined plans. Unfortunately, when organization conduct their operations they lack of the commitment culture to comply with auditing standards; this in-compliance can be attributed to the weak interest in internal regulations, instructions and controls, even though they claim the compliance with such standards.

Internal auditing is applied in different environments with different objectives, sizes and regulations; this mixed environment follows a variety of local laws and regulations that differ in practice from original internal auditing standards. This contradictory environment affects the productivity and nature of the internal auditor's work. Therefore, in order to achieve effective internal auditing activities of organizations must be harmonized with the applied practices of internal auditing and with the internal auditing standards, Thus if organizations adopted real auditing standards internal auditor's reports appears to be more reliable and effective.

The growing attention for the adoption of internal auditing standards and their role in accounting have always attracted and motivated researchers into investigate the degree of company's commitment to such standards from different aspects such as legislation and internal auditing applications; hence, due to this increased attention for internal auditing issues and their role in organizations success raised the need to investigate whether such organizations are complying with international internal auditing standards and whether the local enacted

laws and regulations concerning internal auditing are consistent with right performance auditing standards or not. Therefore, the originality of this study is steamed from addressing the importance of adopting internal auditing standards for performance and their role in increasing the control over organizations activities.

However, the rest of our study is constructed as follows: the literature review and hypotheses development (Section 2); methodology of the study (Section 3). In section 4 the data and analysis of the, results and finally the conclusion and recommendations of the study are addressed in Section 5.

2. Literature review and hypotheses development

Internal auditing in any economic unit gained a great importance for effective procedures and controls that provides the road map to carry out activity within the unit. There is no doubt that the general objective of the internal control system is to protect and assets the performance of institutions in addition to ensuring the accuracy of the accounting data, achieve maximum productivity, and moreover to ensure adherence of staff policies and plans of management. Thus, it is clear that the system of internal control is not limited to monitoring the financial and accounting aspects only, but also deals with the institution as a big unit. Internal auditing is also regarded as one of the essential components for internal control system and works as an independent internal function of individuals and systems within the organization that serve the administration by ensuring efficient and effective operations to achieve the objectives of the regulatory system as defined by the administration.

2.1 The concept of internal audit

Historically the audit concept has been limited to a narrow scope in detection of errors, fraud and manipulation in financial operations; but recently this concept become broader with a wider mean in delivering the results of organization and staff commitment to procedures and standards of effective work. Furthermore, internal audit is regarded as effective continuous financial control system aims to detect errors and fraud, and broader as a comprehensive qualitative and consultative system that controls and improve performance.

The Institute of Internal Auditors (IIA) (2004) has stated that “Internal auditing is an independent, objective assurance and consulting activity designed to add value and improve an organization’s operations, Thus it helps organization to accomplish its objectives by bringing a systematic, disciplined approach to evaluate and improve the effectiveness of risk management, control and governance processes”. Sawyer (2003) stated that internal auditing is “a systematic, objective appraisal by internal auditors of the diverse operations and controls within an organization to determine whether (1) financial and operating information is accurate and reliable, (2) risks to the enterprise are identified and

minimized, (3) external regulations and acceptable internal policies and procedures are followed, (4) satisfactory operating criteria are met, (5) resources are used efficiently and economically and (6) the organization's objectives are effectively achieved for the purpose of consulting with management and for assisting members of the organization in the effective discharge of their governance responsibilities".

2.2 The need for internal auditing by government agencies

Government agencies need accurate data on the various operating units within the country, such data is used in economic planning, government control, pricing, scientific competencies, for this reasons, government agencies seek accurate, periodic data to follow up the activity of these units and their effect on the national economy. Government owned organizations highly contribute in executing governmental development plans and helps in carrying a fair share of society's burdens in form of taxes, absorbing a measure of employment; these contributions is something that cannot be achieved unless there is a strong and effective internal control system.

A company's accounting control practices (such as internal auditing) is widely believed to be crucial to the success of an enterprise as it acts as a powerful brake on the possible deviations from the predetermined objectives and policies. This means that an organization that put in place an appropriate and adequate system of accounting controls is likely to perform better (in financial terms) than those that do not. As Okezie (2004) puts it, "an enterprise's internal audit function can significantly affect the operations of the enterprise and may have an impact on the ability of the entity to remain a going-concern. Conrad (2003) had portrayed Enron's demise as the consequence of a "few unethical "rogues" or "bad eggs" acting in the absence of any control". Thus inadequate control systems may negatively affect an organization's success. According to Hermanson and Rittenberg (2003) the existence of an effective internal audit function is associated with superior organizational performance.

2.3 International standards for the professional practice of internal auditing and their objectives

The Internal audit function is conducted in diverse legal and cultural environments; within organizations that vary in purpose, size, complexity, and structure; and by persons within or outside the organization. While differences may affect the practice of internal auditing in each environment, conformance with The IIA's International Standards for the Professional Practice of Internal Auditing (Standards) is essential in meeting the responsibilities of internal auditors and the internal audit activity. If internal auditors or the internal audit activity is prohibited by law or regulation from conformance with certain parts of the Standards, conformance with all other parts of the Standards and appropriate disclosures are needed (IIA, 2012).

It had been agreed upon that Internal Audit department must apply a complete set of internal audit standards issued by the Institute of Internal Auditors. The (IIA) defines the Standard as “an official declaration of internal audit standards that defines the requirements for performing a wide range of internal audit activities and evaluating Internal Audit Performance” (IIA, 1957). So in order for the internal audit to fully perform its functions, all employees of the establishment must assist them and cooperate fully with them, and give priority to answering their requests without restriction, in order to benefit from the internal audit of the establishment in achievement of internal audit standards is aimed to achieve the following objectives: 1. Direct compliance with the mandatory elements of the international professional framework for internal audit practice 2. Provide a reference frame to perform and promote a wide range of value-added internal auditing services. 3. Lay the basis for evaluating internal audit performance. 4. Enhance the improvement of the operations and work of the institution.

The Institute of Internal Auditors (IIA) issues the International Professional Practices Framework (IPPF). The professional practice framework contains the basic principles of conducting profession. The IPPF consists of three sets of standards, all of which are mandatory: A. Attribute standards, B. Performance Standards (Kontogeorgis and Filos, 2012).

The structure of the Standards is divided between Attribute and Performance Standards. Attribute Standards address the attributes of organizations and individuals performing internal auditing. The Performance Standards describe the nature of internal auditing and provide quality criteria against which the performance of these services can be measured. The Attribute and Performance Standards are also provided to apply to all internal audit services. Table (1) represents the Institute of Internal Auditors the criteria for internal auditing standards and the Jordanian internal auditing articles that enacted as a substitute of performance standards.

Table 1: Internal Auditing Standards

Attr. St		Performance. St		Jor. artic
label	N	label	N	
Purpose, power, resp	1000	Manage internal audit	2000	Art 3, 4, 5
Indep,objectivity	1100	work nature	2100	Art 8, 10
Skill, care of crisis	1200	Planning	2200	Art 6, 9
Programming,quality	2300	Perform functions	2300	Art 8
		Report results	2400	Art 9
		Review the results	2500	Art 8, 11
		Resolve and accept risk	2600	Art 8

According to The Institute of Internal Auditors (IIA) entities are required to comply with these two sets of criteria; but if compliance for some conditions is restricted to certain parts of the standards due to enforced local laws or regulations, the internal auditors or the activity of internal audit must comply with other parts of the standards with appropriate disclosures. Moreover, if the criteria are used in conjunction with requirements issued by other official bodies, the internal audit may indicate the use of these requirements in their communications. Also if internal audit activity refers to compliance with "standards" and there is a discrepancy between "criteria" and other requirements, internal auditors and the activity of internal audit must adhere to "standards" and can comply with other requirements if they are more restrictive.

2.4 Theoretical background

Until recently many nations did not have national requirements for internal audit function in public sector organizations. This is not just the case of developing nations but some developed nations did not have internal audit requirements, for example as of 2004, New Zealand had no requirement for internal audit for public sector organizations and Australia had less clear cut requirements (Goodwin, 2004). Though the audit function has always been seen as playing a key role in government financial management, the emphasis has mostly been placed on external audit and not internal audit. However, the interest for internal audit in public sector organizations has been on the ascendancy in responds to calls from users of public goods and donor agencies for improved accountability, transparency and increased consumer choice (Goodwin, 2004; Diamond, 2002).

In accordance with the needs of the IIA as the professional body that takes care of internal auditing, several studies have conducted to investigate the benefits of internal auditing, effect on performance, efficiency of organizations, factors effect internal audit. Selim et al. (2009) conducted a comparative study of factors affected internal audit consulting activities carried out by a sample of The Institute of Internal Auditors members in the UK/Ireland and Italy. The results of study show that a noteworthy growth in consulting activities carried out in Italian companies, and internal auditors are experiencing both benefits and drawbacks in both countries. Furthermore, he stated that the consulting part of internal auditors is voluntary rather than mandatory for them to create value added for the benefits of the company to which they are committed. In attempt to study and investigate the organizational and drivers of internal audit effectiveness, Arena and Azzon (2009) surveyed 153 Italian companies. Their study revealed that there are many issues and factors influenced internal audit effectiveness such as the characteristics of the internal audit team, the Audit process and activities and the organizational link.

In the same vine, Anderson et al. (2012) investigated the internal audit function. They stated that the internal audit function is influenced by many factors audit committee governance, experience of the chief audit executive, the

use of audit technologies, organization size. Moreover, the percentage of internal audit staff that are certificated internal auditor and the extent of assurance and compliance activities outsourced to an outsider play a vital role in the process of decreases the amount of work performed by the internal auditors and is an indicator of performance. Okezie (2004) assured that the main objective of internal auditing is “to assist management in the effective discharge of their responsibilities by furnishing them with analysis, appraisal, recommendations and pertinent comments concerning the activities reviewed”.

Internal auditing which is often seen as constituting a large and significant aspect of an organization’s financial control system is a vehicle to success and survival. According to Rittenberg and Schwieger (1997). “Internal auditing is taking on increased importance in many of today’s global organizations by assisting management in evaluating controls and operations and thereby providing an Important element of global control”. Venables and Impey (1991) also recognized the control role of internal auditing when they stated: It is generally recognized that the proper organization, staffing and methodology of internal audit presents the board with the best means of focusing on its obligation to ensure proper controls in the business. However, the need for an internal audit function will vary depending on company specific factors including the scale, diversity and complexity of the company’s activities and the number of employees as well as cost-benefit considerations (ICAEW, 1999). Moreover, Venables and Impey (1991) had argued that for an internal audit function to be effective; organizations must realize its full benefits, clearly define objectives, authority, independence and appropriate resources.

At the empirical level, a survey conducted by KPMG (1999) found that the internal audit function in organizations where it exists, contributes substantially to performance improvement and assist in identifying profit evidence in corporate disasters, particularly financial fraud consistently documents an association between weak governance (e.g. less independent boards or the absence of an internal audit function) and the incidence of problems (e.g Dechow, et al 1996; Beasley , 1996, Beasley et al 2000; Abott et al. 2000). Thus, internal audit by acting as a watchdog could save the organization from malpractices and irregularities thus enabling the organization to achieve its objectives of ensuring high level of productivity and profit.

Organizations with government contributions are regarded as entities created by government and government play the role of shareholders; thus holding these shares in trust for the general taxpaying public. Although, it is true that these organizations may be evaluated in the same way like their private counterparts (Mazzolini, 1979); it is equally important to remember that these organizations were established also to promote government’s socioeconomic policies. Viewing it from this dimension, some scholars (for example, Lal, 1980) have argued that the evaluation of government investment should employ social and cost benefit analysis.

In a global study about the role of internal auditing, the report of IIA's Supplemental Guidance in 2012 for the role of auditing in public sector governance, this document defined the following nine elements as essential for an effective public sector audit activity: 1) Organizational independence; 2) a formal mandate; 3) unrestricted access; 4) sufficient funding; 5) competent leadership; 6) objective staff ;7) competent staff; 8) stakeholder support and 9) professional audit standards. These nine elements were compared to findings from The IIA's 2010 Global Internal Audit Survey, which included responses from 2,824 public sector practitioners from 107 countries (out of a total of 13,500 individual respondents). The IIA Research Foundation (IIARF) conducts this global survey every five years as one of the projects for The IIA's Global Internal Audit Common Body of Knowledge (CBOK), Highlights from the findings include: 1- More than two-thirds of public sector respondents indicated that internal auditing was required by law or regulation in the jurisdictions in which their organizations were based. Internal auditing was required by law or regulation the most in the Europe-Central Asia region, and the least in the United States and Canada, followed closely by the Middle East and the Asia Pacific regions. 2- About a fifth of public sector respondents indicated that they had been subject to coercion to change a rating or assessment or withdraw a finding in an internal audit report. Internal auditor respondents from Africa indicated that they had experienced coercion the most, while respondents from the United States and Canada indicated that they had experienced it the least. 3- About half the public sector chief audit executive (CAE) respondents indicated that an audit committee or equivalent was established in their organizations, but only a quarter of them indicated that the audit committee was involved in their appointments. 4- About 40 percent of the CAE respondents indicated that they had more than six years' senior experience as a CAE or equivalent. CAE respondents from Latin America and the Caribbean had the most senior experience (60) percent. By contrast, CAE respondents from Europe-Central Asia had the least senior experience (25) percent. Public sector internal auditors and their stakeholders can use the results of this study to compare the conditions in their organizations to other public sector organizations in their region and around the world. This information can then be used to target the most beneficial areas for improvement.

Similar to this study Kasim and Hanafi in (2012) investigated the availability of construct a valid and reliable instrument to quantitatively measure the level of conformance by internal auditors towards the International Professional Practice Framework (IPPF). Their study involved the distribution of the new instrument to 400 internal auditors, auditors, accountants and account executives in Kuala Lumpur, Malaysia. The study concludes that most of the existing instruments measuring internal audit quality suffer major weaknesses that limit its value. The development process of the new instrument was clearly highlighted in the methodology section. A new approach of measurement was used as a solution to existing instruments. The results indicated that the new instrument satisfies the

criteria for a valid and reliable research instrument and conform to the existing framework suggested by the Institute of Internal Auditors (IIA). The instrument could serve as additional assessment tools for audit committee in assessing the quality of internal audit functions in line with the new Bursa Malaysia Listing Requirements.

In Jordan Alhawtmeh Study (2018) clarified the possibility of applying the basic principles of financial control in accordance with the requirements of ISSAI 200 of the international standards of the Supreme Audit Institutions (INTOSAI) and its impact on the internal control in the public sector in addition to identifying the obstacles that limit the application of the basic principles of financial control according to the requirements of the standard ISSAI 200 of the International Standards of the Supreme Audit Institutions (INTOSAI). The study results showed that the application of the basic principles of financial control in accordance with the requirements of ISSAI 200 of the international standards of the Supreme Audit Institutions (INTOSAI) and its impact on internal control in the public sector in general was high; these results were attributed to the commitment of such organizations to laws, internal control regulations, financial and accounting regulations prescribed for the financial control operations. The results also showed that the commitment of internal control units in the government sector regulatory standards, field standards, reporting standards of the international organization of Supreme Audit controls and accountability (Alhawtmeh, 2016).

In Jordan internal audit standards of (IIA) are not applied, instead Jordanian organizations apply local enacted articles concerning internal auditing. According to these articles the Jordanian government represented by the ministry of finance obligates all government units (owned or independent units) to establish an internal audit unit its main function is to provide government with annual reports in accordance with specific models or specific internal audit instructions, and often these reports are not fully consistent with the requirements of international internal audit standards. The establishment of internal audit unit is clearly stated in the articles as appear in the statement of Article (7) that The internal control criteria and the conditions to be met for those holding the position of head of the internal control unit shall be determined by instructions issued by the Council of Ministers upon the recommendation of the Minister. based on the recommendation of the Central Committee for Internal Control Standards. Also this appears in Article (12) that internal audit units perform their work based on the internal regulations of the institutions.

The Jordanian economy composed of two primary sectors the first one is completely owned and managed by the government the second one is private owned and managed by public. In the second sector the Jordan Government has also considered in many publically traded companies as main shareholders regardless of the percentage of contribution Many of these companies' actions are governed by government laws and regulations such as the internal audit regulations. These companies are obligated to comply with internal audit work

according to its internal audit articles issued by Ministry of Industry and Trade, and approved by the Ministry of Labor. These articles provided guidelines how to manage internal audit activity concerning performance similar to international internal auditing standards, but the question here is whether these articles of internal audit practices has succeeded in action as a substitute of international standards or not. Based on previous discussion our main hypothesis is:

H1: There is no consistency between the practice of internal auditing in Jordan and international internal auditing standards (performance criteria).

Our main hypothesis will be subdivided into seven sub-hypotheses to examine the consistency between the seven international internal auditing standards for performance with their opposite practiced articles issued in Jordan, hence our first sub-hypothesis is:

H1-1: There is no consistency between internal auditing practices in Jordan and International Internal Auditing Standard 2000 (Manage internal audit activity).

Some researcher in their investigation such as Wood and Wilson (1989) has tested the impact of stress on performance of audit work. The researchers concluded those internal auditors who perform as well as other not experiencing such stress. The researcher noticed that measuring the performance of internal audit function is a quite complex and difficult task for practical implementation. Planning for audit work is also regarded as essential part for success of internal auditing and in order to do it successfully, Wynne (1999) suggests that organizations must consider following three objectives into account when doing the audit planning process assurance, risk management, and external audit reliability. The IIAs 2100 of internal audit activity assured the evaluation, improve of governance, risk management, and control processes using a systematic and disciplined approach (IIA, 2012). Similarly, Agencies in Jordan carrying out internal control system and internal audit of financial, administrative and technical transactions assured the same purpose with their regulations and the instructions. The Internal audit systems for these Jordanian companies are built according to the nature of the company's work and usually according to the financial system of companies. The internal audit department/ section in such companies are found either through the company's bylaws or through the financial system or the administrative system. For example, Article (2) of internal control system 2011 defined Internal Audit as: independent objective activity that achieves the quality of the department or the government unit in its operations and enables it to achieve its objectives by adopting a systematic approach to evaluate effectiveness, risk management processes and their development in accordance with corporate governance (JICS, 2011). Thus our second sub-hypothesis is:

H1-2: There is no consistency between internal auditing practices in Jordan and International Internal Auditing Standard 2100 (work nature).

According to the statement of IIAs 2200 internal auditors must develop and document a plan for each engagement, including the engagement's objectives,

scope, timing, and resource allocations. Likewise, The Jordanian internal auditing system regulations that enacted by the government in 2011; articles 6 and 9 assured for the internal control and auditing unit responsibilities such as; drawing up and updating the general policy related to internal control, preparation of the internal control procedures manual, Prepare an annual plan for financial audit and submit it to the competent minister or the Council for approval before the end of November of each year prepare the financial audit procedures manual and submit them to the Minister for approval, integrity of financial and administrative transactions, budgets and financial statements, verifies compliance with the applicable laws and regulations, ensures the expenditure on the various items of the fund's budget and the use of resources has been in accordance with the objectives and policies. In general, the overall internal audit unit purposes are monitoring, evaluation and accountability (JICS, 2011). The third hypothesis related to planning internal audit functions is that:

H1-3: There is no consistency between internal auditing practices in Jordan and International Internal Auditing Standard 2200 (Planning internal audit functions).

The work of internal auditors requires sufficient, reliable, relevant, and useful information to achieve the engagement's objectives, therefore, IIAs 2300 assured that internal auditors must identify, analyze, evaluate, and document sufficient information to achieve the engagement's purposes. Similarly, in Jordan article 8 and 9 of JICS assured also that a monthly report, whenever necessary, on internal control, achievements and observations, including observations not addressed to the competent minister or the council; and submit a similar annual report to the Minister during the first four months of the following year; prepare the financial audit procedures manual and submit them to the minister for approval. Furthermore, articles assure the exercise of administrative control and technical control and ensure that the department and the government unit achieve the objectives and general policies of the department and the government unit through reviewing the decisions administrative issues. In article 12 the internal audit unit work plan should consist of the strategic objective of the company, the strategic objective of the directorate, the annual objective (SMART Objective), performance index, measurement equation, program / project, sample worksheet checker, working papers worksheet / task execution model, warehouse inventory model, and the record of the advances examination through a set of other models, which represent the working mechanism of the internal audit unit. Based on prior comparison our fourth sub-hypothesis is:

H1-4: There is no consistency between internal auditing practices in Jordan and international internal auditing standard 2300 (Perform internal audit functions). As internal auditors must communicate the results of engagements to proper users, the communication must include the engagement's objectives and scope as well as applicable conclusions, recommendations, and action plans (IIAs, 2012). The final communication of engagement results must, where appropriate, contain the internal auditors' opinion and/or conclusions. On the

other side; internal auditing practices in Jordan covered this standard requirement through article 9 for internal auditing; the article assured that organization must prepare an annual plan for financial audit and submit it to the competent minister or the Council for approval, before the end of November of each year, further a monthly report must be submitted, whenever necessary, on internal control, achievements and observations, including observations not addressed to the competent minister or the Council; and submit a similar annual report to the minister during the first four months of the following year. Consistency with Standard 2400 on internal auditing will be examined by the following sub-hypothesis:

H1-5: There is no consistency between internal auditing practices in Jordan and international internal auditing standard 2400 (Report the results).

Monitoring the progress of internal audit procedures is a vital step in completing the efficient, complete internal control. International auditing standard 2500 obligated the chief audit executive to establish and maintain a system to monitor the disposition of results communicated to management. That means the chief audit executive must establish a follow-up process to monitor and ensure that management actions have been effectively implemented or that senior management has accepted the risk of not taking action (IIAS, 2012). Concerning this, the Jordanian regulation through article 8 similarly obligated departments to exercise administrative control and technical control and ensure that the department and the government unit achieve the objectives and general policies of the department; so our sixth sub-hypothesis to ensure the consistency of Jordanian practices with standard 2500 is:

H1-6: There is no consistency between internal auditing practices in Jordan and international internal auditing standard 2500 (Review the results).

As for the final international internal auditing standard 2600 the ability of management to accept the risk; the content of this standard requested that chief audit executive must concludes that management has accepted a level of risk that may be unacceptable to the organization, and he must discuss the matter with senior management. If the chief audit executive determines that the matter has not been resolved, the chief audit executive must communicate the matter to the board (IIAS, 2012). With comparison of this content to our local practices we can observe that article 8 Jordanian internal control system obligated management to reduce the risk through ensuring compliance with the legislation in force to reduce errors and discovery in the event of occurrence. Thus, our final sub-hypothesis is:

H1-7: There is no consistency between internal auditing practices in Jordan and international internal auditing standard 2600 (management's acceptance of risk).

3. Methodology of the study

3.1 The study method

This study follows both descriptive and field research methodologies. The descriptive method review and discuss the different literature about consistency of internal auditing practices in companies with government contributions with reference to journals, reports, textbooks, thesis and any related sources. On the other hand, the field investigation was based on a structured questionnaire prepared and distributed to a sample of Jordanian companies' staff to obtain the necessary information that serve the aim of the study.

3.2 Data and sample characteristics

Since we were interested in organizations with government contributions, the sample of the study consisted of 115 Jordanian Companies with Government Contributions. The reason for such sample selection is that; these types of companies are affected by government regulations and committed to internal control standards. A structured 460 questionnaires are distributed to staff in these companies, these individuals comprising accountants, Chief Accountants, Chief internal auditors, internal auditors and finance managers constituted the respondents of this study; four questionnaires send for each company. 434 questionnaires were returned and 416 (90%) of them where valid for analysis purposes.

Responses received from respondents were based on a five-point scale, ranging from strongly disagrees to strongly agree, where 1 represented strongly disagree (SD) and 5 represented strongly agree (SA). Statistical Package for Social Sciences (SPSS) was used to analyze data and presented in the form of means and standard deviations for each question and each category or section of the questionnaire. The study acknowledges that Likert scale suffer from the danger of extremes, however Likert scale is used most often in social sciences to measure attitudes and perception of respondents.

Table 2 shows the level of education for the sample. The results show that holders of bachelor's degree are the highest with 40% percentage; the diploma was in the second order by 28% while holders of master degree were 14%. Over viewing education results reveal that about 82% of the sample is well educated and have the knowledge to response for the required information for the study.

Table 3 results show the respondent's age; the results show that 40% of respondent's age was between 31 to 40 and 19% of respondent's age was below 30 while 21% of the respondent's age was between 41 and 50. These results indicate that majority of respondents are in the middle age (80%) and they cohabit the new concepts and knowledge about Internal Auditing practices.

Table 4 shows the results of respondent's specialization field. The highest specialization was for internal auditing with 33% then financial managers by 21%. Chief internal auditors were 18%. It's has been argued that the most fa-

Table 2: The level of education for the sample

Qualification	Frequency	Percentage
Less than diploma	44	10%
Diploma	119	28%
Bachelor	165	40%
Master	57	14%
PHD	18	5%
Others	13	3%
total	416	100%

Table 3: The sample distribution according to age

Age(years)	Frequency	Percentage
30 and below	78	19%
31-40	193	46%
41-50	85	21%
51-60	41	10%
More than 60	19	4%
total	416	100%

miliar specializations with Internal Auditing practices are those who are directly to internal control field such as financial managers, accountants and auditing or internal control; their percentage together compromise 72% of the sample.

Table 4: The sample distribution for field of specialization

Specialization	Frequency	Percentage
Financial managers	88	21%
Internal auditors	137	33%
Chief accountant	51	12%
Chief internal auditors	76	18%
Accountant	64	16%
total	416	100%

Table 5 shows the years of experience for respondents. The experience results showed that the majority of respondents have more than 6 years of experience; 80% of respondents have an experience between 6 and 15 years which is reasonable expertise about Internal auditing practices and its implications.

Table 5: The sample distribution for years of experience

Years of Experience	Frequency	Percentage
1-5	68	16%
6-10	137	33%
11-15	112	27%
16-20	82	20%
More than 20	17	4%
total	416	100%

3.3 Validity and reliability of the study instrument

The internal consistency for the study investigating instrument (questionnaire) was tested using Cronbach's Alpha test to examine the reliability of the measurement instrument according to the answers of respondents. Table 6 shows the results for this test; the value of Cronbach's alpha value for all paragraphs was (85.7). this result is acceptable to analyse the collected data.

Table 6: Cronbach Alpha test

Sample number	Cronbach's Alpha
416	(85.7)%

4. Results of the study

The examination of the study hypotheses requires an initial examination of frequencies and standard deviation for each paragraph and then one sample t- Test will be employed to test each hypothesis independently. The following are the results of frequencies and standard deviation for the sub-hypotheses paragraphs that were addressed by the questionnaire questions.

4.1 Hypothesis Testing

4.1.1 Hypothesis 1-1: Manage internal audit activity

Table 7 shows the mean and the standard deviations for the first sub-hypothesis paragraphs that related to managing internal audit activity standard. As appears from the results; the arithmetic mean for all paragraphs is less than 3 except for paragraphs 3 are more than 3. These results assure that the requirements for managing internal audit activity are unavailable in the sampled companies; this conclusion is evidenced through the arithmetic mean for all paragraphs of (2.483); Paragraphs mean is less than 3, this result shows a weakness of companies to commitment for internal auditing standard 2000.

Table 7: Arithmetic mean and standard deviation for Standard 2000: Manage internal audit activity

No	Paragraph	mean	std.dev
1	Internal auditing adds value to the institution	2.310	1.150
2	plan developed to establish priorities for objec.	2.833	0.718
3	plan sent to manag and Dir. Board for approval	3.021	1.047
4	Allocate Resources to implement plan effectively	2.333	1.148
5	Policies, are developed to implement the audit plan	2.259	0.874
6	plan information shared internally and externally	2.144	1.358
	Average arithmetic mean for all paragraphs	2.483	

4.1.2 Hypothesis 1-2: The Work Nature

In Table 8 the mean and standard deviations for the second sub-hypothesis paragraphs concerned with requirements for work nature; it is evidenced from all paragraphs results that the mean is below 3. This result also confirms our expectations for the failure in commitment to internal auditing standards. The average mean for all paragraphs was below 3 (2.593); this result also in line with the previous hypothesis results.

Table 8: mean and standard deviation for Standard 2100: Work Nature

No	Paragraph	mean	std.dev
1	Internal audit activity improves governance	2.165	1.025
2	Internal audit activity improves risk management	1.941	0.732
3	Internal audit activity raise the level of control	2.985	1.147
4	Audit leads to commitment to values and ethics	2.214	1.222
5	Internal audit contributes to performance	2.663	1.415
	Average arithmetic mean for all paragraphs	2.483	

4.1.3 Hypothesis1- 3: Internal Auditor

Table 9 shows the mean and the standard deviations for the third sub-hypothesis paragraphs that related to Standard 2200 (Internal Auditor). As appears from the paragraphs results the arithmetic mean for almost all paragraphs is less than 3. These results assures that internal auditing departments do not comply with standard 2200 requirements; this conclusion is evidenced through the arithmetic mean for all paragraphs of (2.875). Such result also is likewise previous results that showed weakness in commitment to internal auditing standards.

Table 9: mean and standard deviation for Standard 2200: Internal Auditor

No	Paragraph	mean	std.dev
1	Internal audit implement, strategies ,objectives	2.856	1.015
2	internal auditor performs risk assessment	2.742	0.957
3	internal auditor considers the possible errors, fraud	3.153	1.108
4	Auditor provide metrics to assess govern,control	2.691	0.837
5	int. auditor ensure existence of records, documents	3.133	1.658
6	auditor determines resources he needs to do his work	2.481	1.235
7	auditor documents work procedures files	3.075	0.658
	Average arithmetic mean for all paragraphs	2.875	

4.1.4 Hypothesis1- 4: Internal Auditor director

Similar to prior results, the results in Table 10 also revealed weakness and failure in commitment to Standard 2300 (Internal Auditor Director); the mean and the standard deviations for all paragraphs is less than 3 with average arithmetic mean for all paragraphs of (2.755).

Table 10: mean and standard deviation for Standard 2300: Internal Auditor Director

No	Paragraph	mean	std.dev
1	auditor identify, analyze information of objectives	2.268	0.753
2	auditor verifies information reliably, complete	2.992	1.584
3	int. auditor documents information support results	3.153	0.708
4	int auditor performs supervision and follow-up	2.240	0.966
5	auditor accounts the quality of the results	3.122	1.016
	Average arithmetic mean for all paragraphs	2.755	

4.1.5 Hypothesis1-5: Report the results (table 11)

Tables 11, 12 and 13 concerned with internal auditing standards 2400, 2500 and 2600 also shows failure in commitment to internal auditing standards requirements. The arithmetic mean for all of these standards was below 3. These results also appear similar to the results reached in previous tables.

Table 11: mean and standard deviation for Standard 2400: Report the results

No	Paragraph	mean	std.dev
1	internal auditor shall report all his results	2.116	1.003
2	auditor provide information on work plans	1.992	0.741
3	Auditor recommends the results of his work	2.235	1.658
4	presents his opinion to Directors on results	2.402	1.133
5	auditor's reports are objective, correct and clear	2.870	1.555
6	auditor determines outcomes of the activities only	2.590	0.159
	Average arithmetic mean for all paragraphs	2.367	

Table 12: mean and standard deviation for Standard 2500: Review the results

No	Paragraph	mean	std.dev
1	auditor express his opinion about objectives	2.412	1.333
2	auditor support final results with documentation	3.047	0.541
3	auditor adhere by rules for giving his opinion	2.865	1.278
4	auditor send findings to administration	2.135	1.166
5	auditor verifies that management review risks	2.149	0.936
	Average arithmetic mean for all paragraphs	2.521	

4.1.6 Hypothesis1-6: Review the results (table 12)

4.1.7 Hypothesis1- 7: Resolve management's acceptance of risk (table 13)

Table 13: mean and standard deviation for Standard 2600: Resolve management's acceptance of risk

No	Paragraph	mean	std.dev
1	audit department discuss risks with management	2.713	0.628
2	Audit Dep. inform BOD for dispute with Manag.	1.932	0.746
3	audit department provide solutions for risks	2.153	0.951
4	audit dep. ensures measures fit risk direction	2.359	0.755
	Average arithmetic mean for all paragraphs	2.289	

Based on previous results of all sub-hypothesis we conclude that, there is no commitment from companies with government contributions to all internal auditing standard of performance. All of the results for the seven dimensions of performance showed weaknesses and failure to apply the standards requirements.

4.2 One sample t- Test

As mentioned earlier the methodology of the study includes conducting one sample t-test to examine the main and the sub-hypotheses of the study which are:

Table 13 shows the results of one sample t- test; the results show that the two hypotheses are significantly valid. As appears in the table all of calculated-t results are higher than that of statistical-t at a degree of freedom of 415 with significance below 5%.

Table 14: results of one sample t- test

Hypothesis	T-calculated	T-stat	Sig.	Df	decision
H1	4.61	1.96	0.000	415	accepted
H1-1	5.25	1.96	0.000	415	accepted
H1-2	4.79	1.96	0.000	415	accepted
H1-3	4.56	1.96	0.000	415	accepted
H1-4	3.92	1.96	0.000	415	accepted
H1-5	4.18	1.96	0.000	415	accepted
H1-6	4.33	1.96	0.000	415	accepted
H1-7	4.27	1.96	0.000	415	accepted
			Sig.below 0.05		

According to the rule of one sample t-test; if calculated-t is higher than statistical-t the null hypothesis is rejected. Therefore, and based on these results we accept the seven null sub-hypotheses and the main hypothesis of the study; because all of the results previously discussed proved that Jordanian companies with government contributions do not comply with internal auditing standards of performance and they do not commit to all standards requirements and to the proper methods and concepts concerned with internal auditing.

5. Conclusion and recommendations

5.1 Conclusion

In the last 20 years the concept of internal auditing has a great importance in policy-making process and considered a major turning point for our societies. Although of complexity of this concept it's vital for both present and future actions; the importance of this concept raised the need for adoption of internal auditing standards.

This study was constructed to address the issue of consistency between local practice of internal auditing and the international internal auditing standards. The study was aimed to investigate whether Jordanian companies with government contributions comply and commit to such standards or not. Recent economic development regarded internal auditing as one of the main func-

tions of companies to achieve its strategies and desired objectives; therefore, it's necessary to investigate our commitment to such standards similar to other economies. The methodology of the study followed two scientific methods to investigate this problem; both descriptive and field investigation was applied. The study reviewed the previous literature to address the importance for implementing internal auditing concept; and a questionnaire was also used to examine and capture the companies' commitment to internal auditing standards. Commitment and consistency to standards was tested from seven dimensions; the results of the arithmetic mean for almost all of paragraphs concerning this consistency proved that Jordanian companies are not committed to the requirements of internal auditing. When testing the seven predetermined sub-hypotheses of the study using one sample t-test the results also confirmed this inconsistency of local practice of internal auditing to international internal auditing standards. Consequently, the final conclusion of the study is consistent with many results of similar conclusions reached by local and international studies.

5.2 Recommendations of the study

1. Providing more awareness for companies' staff about the concept of internal auditing and their general framework in order to achieve the desired benefits of internal auditing standards. 2. Adopting proper legislation and adopting the necessary internal auditing standards that guide companies' activities concerning monitoring company's activities. 3. Obligating the internal audit departments to collect measure and disclose accurate, relevant information for interested parties. 4. More education and training to enhance the qualification, skills and expertise of accountants, auditors, and administrators to meet the requirement of internal auditing activities.

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On binary block codes associated to UP-algebras

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Abstract. In this article, we define the notion of a UP-algebra valued function on a set and investigate related properties. We establish the binary block codes generated by UP-algebras valued function. We have also shown that for every binary block-code C , there exists a UP-algebra and UP-valued function which determines C whereas UP-algebras associated to a binary block code are not unique up to isomorphism.

Keywords: UP-algebras, binary block codes, partially ordered set.

1. Introduction

Logical algebras like BCI/BCK, BE and KU-algebras with their fuzzy and Intuitionistic concepts have become the keen interest for researchers in recent years and widely considered as a strong tool for information system and many other branches of Computer Sciences including fuzzy informatics with rough and soft concepts. Imai and Iseki [12] introduced BCK/BCI algebras as a generalization of the concept of set-theoretic difference and proportional calculi. In computer science, a block code is a type of channel coding. It adds redundancy to a message so that, at the receiver, one can decode the message with a minimum number of errors, where it is already provided that the information rate would

not exceed the channel capacity. Block code simply encodes strings formed by an alphabet set $\mathcal{B}$ into code words by encoding each letter of $\mathcal{B}$ separately. Block codes can be source codes used in data compression, or channel codes used for detection and correction of channel errors [14]. Codes based on family of algorithms was constructed by Lempel and Ziv [20], which is applicable for real-world problems and sequences. A detailed terminology based on codes and decoding through graphs are discussed in [19].

Block codes in different logical algebras are being studied by numerous researchers in past few years. Jun and Song [13] introduced codes based on BCK-algebras whereas Surdive et al. studied coding theory in hyper BCK-algebras [18]. Further Fu and Xin [6] defined the concept of block codes in lattices and gave some properties based on them. Next Mostafa et al. [16] applied coding theory and gave some relation and connection between coding theory and KU-algebras.

UP-algebras was introduced by Iampan [7]. Further Iampan et. al contributed on different aspects related to UP-algebras in ([10], [8], [9], [11]). Recently Mosrijai and Iampan have introduced a new branch of bialgebraic structures: UP-bialgebras [15]. Senapati et. al [17] represented UP-algebras in intervalued intuitionistic fuzzy environment. Moin et al. [5] introduced roughness in UP-algebras. Graphs of UP-algebras are studied by Moin et al. [4]. Further Moin et al. defined and studied concepts based on KU-algebras containing (α, β) -US soft sets in [1] whereas Koam et al. introduced the concept of n -ary block codes related to KU-algebras in [2]. Furthermore JU-algebras and p -closure ideals is introduced and studied by Moin et al. in [3].

In this paper, we introduce the UP-valued functions and investigate its several properties. Also, we establish block-codes by using the notion of UP-valued functions. We show that every finite UP-algebra determines a block-code.

Section 1, we have given introduction. Section 2 contains preliminaries and related definitions with some examples. Section 3 is based on main results. Section 4 is the conclusion of our work.

2. Preliminaries

In this section, we shall consider concepts based on UP-algebras, UP-subalgebras, UP-ideals, UP-valued function (cut function) and other important terminologies with examples and some related results. An UP-algebra is a type of logical algebras which is under investigation by several researchers.

Definition 2.1 ([7]). *By a UP-algebra we mean an algebra $(U, *, \emptyset)$ of type $(2, 0)$ with a single binary operation $*$ that satisfies the following identities: for any $x, y, z \in U$,*

$$(UP-1): (y * z) * [(x * y) * (x * z)] = \emptyset,$$

$$(UP-2): \emptyset * x = x,$$

$$(UP-3): x * \emptyset = \emptyset,$$

$$(UP-4): x * y = y * x = \emptyset \text{ implies } x = y.$$

An UP-algebras U is said to be commutative if $x * (x * y) = y * (y * x)$.

We define partial order relation in a UP-algebra U as $y \leq x$ if and only if $x * y = \emptyset$. If $(U, *, \emptyset)$ and $(V, \circ, \emptyset)$ are two UP-algebras then a map $f : U \rightarrow V$ with the property $f(x * y) = f(x) \circ f(y)$, for all $x, y \in U$, is called a UP-algebra morphism. If f is bijective map, then f is an isomorphism of UP-algebras.

Example 2.1. Let $U = \{\emptyset, a, b, c\}$ be a set in which $*$ is defined by the following Cayley table

$*$	$\emptyset$	a	b	c
$\emptyset$	$\emptyset$	a	b	c
a	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
b	$\emptyset$	a	$\emptyset$	c
c	$\emptyset$	a	b	$\emptyset$

It is easy to see that $U = \{\emptyset, a, b, c\}$ is a UP-algebra.

Example 2.2. Let $U = \{\emptyset, a, b, c, d\}$ be a set in which $*$ is defined by the following Cayley table

$*$	$\emptyset$	a	b	c	d
$\emptyset$	$\emptyset$	a	b	c	d
a	$\emptyset$	$\emptyset$	b	c	d
b	$\emptyset$	$\emptyset$	$\emptyset$	c	d
c	$\emptyset$	$\emptyset$	b	$\emptyset$	d
d	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

Here $U = \{\emptyset, a, b, c, d\}$ is UP-algebra.

Example 2.3. Let $U = \{\emptyset, a, b, c, d\}$ be a set in which $*$ is defined by the following Cayley table

$*$	$\emptyset$	a	b	c	d
$\emptyset$	$\emptyset$	a	b	c	d
a	$\emptyset$	$\emptyset$	b	a	d
b	$\emptyset$	a	$\emptyset$	c	d
c	$\emptyset$	$\emptyset$	b	$\emptyset$	d
d	$\emptyset$	a	b	c	$\emptyset$

Here $U = \{\emptyset, a, b, c, d\}$ is UP-algebra.

Example 2.4. Let $X = \{t_n | n = 1, 2, 3, \dots, 9\}$ and define a binary operation $*$ on X as $t_i * t_j = t_k \forall t_i, t_j \in X$ where $k = \frac{lcm(i,j)}{j}$. Then $(X, *, t_1)$ is a UP-algebra. The following table represents this operation:

*	t_1	t_2	t_3	t_4	t_5	t_6
t_1	t_1	t_2	t_3	t_4	t_5	t_6
t_2	t_1	t_1	t_3	t_2	t_5	t_3
t_3	t_1	t_2	t_1	t_4	t_5	t_2
t_4	t_1	t_1	t_3	t_1	t_5	t_3
t_5	t_1	t_2	t_3	t_4	t_1	t_6
t_6	t_1	t_1	t_1	t_2	t_5	t_1

Lemma 2.1 ([8]). *In a UP-algebra U the following properties hold for any $x, y, z \in U$:*

- (UP-5) $x * x = \emptyset$,
- (UP-6) $x * y = \emptyset$ and $y * z = \emptyset \Rightarrow x * z = \emptyset$,
- (UP-7) $x * y = \emptyset \Rightarrow (z * x) * (z * y) = \emptyset$,
- (UP-8) $x * y = \emptyset \Rightarrow (y * z) * (x * z) = \emptyset$,
- (UP-9) $x * (y * x) = \emptyset$,
- (UP-10) $(y * x) * x = \emptyset \iff x = y * x$, and
- (UP-11) $x * (y * y) = \emptyset$

Lemma 2.2. *Let $U = (A, *, \emptyset)$ be UP-algebras, then define a binary relation $\leq$ on U as follows: for all $x, y, z \in A$*

- (UP-12) $x \leq x$,
- (UP-13) $\emptyset \leq x$.
- (UP-14) $y * x \leq x$.
- (UP-15) $x \leq y$ and $y \leq x \Rightarrow x = y$,
- (UP-16) $y \leq x$ and $z \leq y \Rightarrow z \leq x$,
- (UP-17) $y \leq x \Rightarrow z * y \leq z * x$,
- (UP-18) $y \leq x \Rightarrow x * z \leq y * z$,
- (UP-19) $(x * y) * (x * z) \leq y * z$.

3. UP-valued functions

Throughout we consider U to be a non-empty set and X to be a UP-algebra, respectively.

Definition 3.1. *A mapping $\tilde{U} : U \rightarrow X$ is called a UP-valued function or simply UP-function on U .*

Definition 3.2. *A cut function of $\tilde{U}$, for $t \in X$, is defined to be a mapping $\tilde{U}_t : U \rightarrow \{0, 1\}$, such that $(\tilde{U}_t(x) = 1 \iff \tilde{U}(x) * t = \emptyset) \forall x \in U$. It is obvious here that $\tilde{U}_t$ is the characteristics function of the following subset of U , called a cut subset or a t -cut of $\tilde{U}$: Here $U_t := \{x \in U \mid \tilde{U}(x) * t = \emptyset\}$. It is clear that $U_\emptyset = U$.*

Example 3.1. Let $U = \{l, m, n\}$ and $X = \{\emptyset, a, b, c, d\}$ is the UP-algebras given in example 2.

The function $\tilde{U} : U \rightarrow X$ given by $\tilde{U}(l) = a$, $\tilde{U}(m) = b$, $\tilde{U}(n) = c$, is a UP-function on U , and its cut subsets are $U_\emptyset = U$, $U_a = U$, $U_b = \{m\}$, $U_c = \{n\}$ and $U_d = \{\}$,

Proposition 3.1. *Every UP-function $\tilde{U} : U \rightarrow X$ on U is represented by the supremum of the set $\{t \in X | \tilde{U}_t(x) = 1\}$, that is $(\tilde{U}(x) = \sup\{t \in X | \tilde{U}_t(x) = 1\})$ for all $x \in U$.*

Proof. For any $x \in U$, let $\tilde{U}(x) = r \in X$. Then $\tilde{U}(x) * r = \emptyset$, and so $\tilde{U}_r(x) = 1$. Assume that $\tilde{U}_t(x) = 1$ for $t \in X$. Then $\emptyset = \tilde{U}(x) * t = r * t$, i.e., $t \leq r$. Since $r \in \{t \in X | \tilde{U}_t(x) = 1\}$, it follows that $\tilde{U}(x) = r = \sup\{t \in X | \tilde{U}_t(x) = 1\}$. $\square$

Proposition 3.2. *If $\tilde{U} : U \rightarrow X$ is a UP-function on U , then $(\tilde{U}(x) = \sup\{\tilde{U}_t(x) \diamond t | t \in X\})$ where*

$$\tilde{U}_t(x) \diamond t = \begin{cases} t, & \text{if } \tilde{U}_t(x) = 1, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Proof. Follows from Proposition 3.1. $\square$

Proposition 3.3. *Let $\tilde{U} : U \rightarrow X$ be a UP-function on U . Then $t * s = \emptyset \Rightarrow U_t \subseteq U_s$ for all $s, t \in X$.*

Proof. Let $s, t \in X$ be such that $t * s = \emptyset$. For any $x \in U_t$, we have $\tilde{U}(x) * t = \emptyset$. Now by Lemma 1(4) we have that $(t * s) * (\tilde{U}(x) * s) = \emptyset \Rightarrow \emptyset * (\tilde{U}(x) * s) = \emptyset \Rightarrow \tilde{U}(x) * s = \emptyset$ (by UP-2) $\Rightarrow x \in U_s$. Hence $U_t \subseteq U_s$. $\square$

Proposition 3.4. *Let $\tilde{U} : U \rightarrow X$ be a UP-algebra on U . Then:*

1. $\tilde{U}(x) \neq \tilde{U}(y) \iff U_{\tilde{U}(x)} \neq U_{\tilde{U}(y)}$ for all $x, y \in U$.
2. $\tilde{U}(x) * t = \emptyset \iff U_{\tilde{U}(x)} \subseteq U_t$ for all $t \in X, x \in U$.

Proof. 1. If $U_{\tilde{U}(x)} \neq U_{\tilde{U}(y)}$, then it is obvious that $\tilde{U}(x) \neq \tilde{U}(y)$ for all $x, y \in U$. Then $\tilde{U}(x) * \tilde{U}(y) \neq \emptyset$ or $\tilde{U}(y) * \tilde{U}(x) \neq \emptyset$. Thus $U_{\tilde{U}(x)} = \{z \in U | \tilde{U}(z) * \tilde{U}(x) = \emptyset\} \neq \{z \in U | \tilde{U}(y) * \tilde{U}(z) = \emptyset\} = U_{\tilde{U}(y)}$.

2. If $\tilde{U}(x) * t = \emptyset$, then by Proposition 3.3 $U_{\tilde{U}(x)} \subseteq U_t$. Let $t \in X$ and $x \in U$ be such that $U_{\tilde{U}(x)} \subseteq U_t$. If $\tilde{U}(x) * t \neq \emptyset$, then $x \notin U_t$. Since $\tilde{U}(x) * \tilde{U}(x) = \emptyset$, it follows that $x \in U_{\tilde{U}(x)}$ so that $U_{\tilde{U}(x)} \not\subseteq U_t$ which is a contradiction. $\square$

Corollary 3.1. *Let $\tilde{U} : U \rightarrow X$ be a UP-function on U . Then $\tilde{U}(x) * \tilde{U}(y) = \emptyset \iff U_{\tilde{U}(x)} \subseteq U_{\tilde{U}(y)}$ for all $x, y \in U$.*

Proof. It follows directly from Proposition 3.3 and Proposition 3.4. $\square$

For a UP-function $\tilde{U} : U \rightarrow X$ on U , consider the following sets:
 $U_X := \{U_t | t \in X\}$, $\tilde{U}_X := \{\tilde{U}_t | t \in X\}$.

Proposition 3.5. *Let $\tilde{U} : U \rightarrow X$ be a UP-function on U then there exists sup Y in X such that $U_{\sup\{t|t \in Y\}} = \bigcap \{U_t | t \in Y\}$, for all $Y \subseteq X$.*

Proof. We suppose that for every subset Y of X , there exists a sup Y in X . We have $x \in U_{\sup\{t|t \in Y\}} \iff \tilde{U}(x) * \sup\{t|t \in Y\} = \emptyset \iff \tilde{U}(x) * r = \emptyset \forall r \in Y \iff x \in U_r \forall r \in Y \iff x \in \bigcap \{U_t | t \in Y\}$. $\square$

Corollary 3.2. *Let $\tilde{U} : U \rightarrow X$ be a UP-function on U , where X is a bounded UP-algebra. Then, $U_{\sup\{t|t \in S\}} = \bigcap \{U_t | t \in S\} \forall S \subseteq X$.*

Corollary 3.3. *Let $\tilde{U} : U \rightarrow X$ be a UP-function on U . Suppose that for any subset Y of X , there exists a sup of Y . Then, $U_s \cap U_t \in U_X \forall s, t \in Y$.*

Converse of the above corollary may not be true as shown in the following example:

Example 3.2. Let $U = \{x, y\}$ be a set and $X = \{\emptyset, a, b, c, d\}$ be a UP-algebra given in Example 2.3. Then, for a function $\tilde{U} : U \rightarrow X$ such that $\tilde{U}(x) = a$ and $\tilde{U}(y) = b$. Here, we obtained cut sets of $\tilde{U}$ are as follows: $U_\emptyset = U, U_a = \{x\}, U_b = \{y\}, U_c = U_d = \emptyset$.

It is clear here that $U_a \cap U_b \in U_X$ but $\sup\{a, b\}$ does not exist in X .

Proposition 3.6. *Let $\tilde{U} : U \rightarrow X$ be a UP-function on U . Then, $\bigcup_{t \in X} U_t = U$.*

Proof. For every $x \in U$, let $\tilde{U}(x) = t \in X$. Then $x \in U_t$, hence $x \in \bigcup_{t \in X} U_t$. Thus $U \subseteq \bigcup_{t \in X} U_t$. $\square$

Proposition 3.7. *Let $\tilde{U} : U \rightarrow X$ be a UP-function on U . Then $\bigcap_{x \in U_t} U_t \in U_x \forall x \in U$.*

Proof. For any $x \in U$, $x \in U_t \iff \tilde{U}_t(x) = 1$. It follows from Proposition 5 that $\bigcap \{U_t | x \in U_t\} = \bigcap \{U_t | \tilde{U}_t(x) = 1\} = U_{\sup\{t | \tilde{U}_t(x) = 1\}} \in U_X$. $\square$

Next we define equivalence relation $\sim$ on X . Let $\tilde{U} : U \rightarrow X$ be a UP-function on U and let $\sim$ be a binary relation on X defined by $s \sim t \iff U_s = U_t \forall s, t \in X$. This relation is an equivalence relation on X . The corresponding equivalence class of an element $t \in X$ is denoted by $[t]^\sim$.

Let $\tilde{U}(U) := \{t \in X | \tilde{U}(x) = t \text{ for some } x \in U\}$ and for $t \in X$, let $[t] := \{x \in X | x * t = \emptyset\}$ and $(t) := \{x \in X | t * x = \emptyset\}$.

Proposition 3.8. *For a UP-function $\tilde{U} : U \rightarrow X$ on U , we have $s \sim t \iff [s] \cap \tilde{U}(U) = [t] \cap \tilde{U}(U) \forall s, t \in X$.*

Proof. If $s \sim t \iff U_s = U_t \iff \tilde{U}(x) * s = \tilde{U}(x) * t = \emptyset \iff \tilde{U}(x) \in \tilde{U}(U) \iff s < \tilde{U}(x) \iff t < \tilde{U}(x) \iff \tilde{U}(x) \in \tilde{U}(U) \iff [s] \cap \tilde{U}(U) = [t] \cap \tilde{U}(U)$

Given $s \sim s \iff U_s = U_t \iff s * \tilde{U}(x) = \emptyset \forall x \in U \iff \tilde{U}(x) \in [s] \forall x \in U = \tilde{U}(U) \iff [s] \cap \tilde{U}(U) = [t] \cap \tilde{U}(U)$. $\square$

Example 3.3. Consider $U = \{a, b, c, d\}$ and X be a UP-algebra given in Example 2.4. Define a UP-function $\tilde{U} : U \rightarrow X$ by $\tilde{U}(a) = t_2, \tilde{U}(b) = t_1, \tilde{U}(c) = t_5, \tilde{U}(d) = t_3$. Then corresponding cut sets of $\tilde{U}$ are given as:

$$U_{t_1} = U, U_{t_2} = \{a\}, U_{t_3} = \{d\}, U_{t_5} = \{c\}, U_{t_4} = U_{t_6} = \{\}.$$

Lemma 3.1. Let $\tilde{U} : U \rightarrow X$ be a UP-function on U . For every $x \in U$, we have $\tilde{U}(x) = \sup([\tilde{U}(x)]^\sim)$, that is, $\tilde{U}(x)$ is the maximal element of the $[[\tilde{U}(x)]^\sim]$ where ever it belongs.

4. Codes based on UP-algebras

Consider $U = \{1, 2, 3, \dots, n\}$ and let X be a UP-algebra. Every UP-function $\tilde{U} : U \rightarrow X$ on U determines a binary block-code C of length n in such a way that to every $[x]^\sim$, where $x \in X$, there would be codewords $c_x = x_1 x_2 x_3 \dots x_n$ with $x_i = j \iff \tilde{U}_x(i) = j$, for $i \in U$ and $j \in \{0, 1\}$. Let $c_x = x_1 x_2 x_3 \dots x_n$ and $c_y = y_1 y_2 y_3 \dots y_n$ be two codewords belonging to a binary block-code C . We define an order relation $\preceq$ on the set of codewords belonging to a binary block-code C as $c_x \preceq c_y \iff y_i \leq x_i$ for $i = 1, 2, 3, \dots, n$.

Example 4.1. Let $U = X = \{\emptyset, a, b, c\}$ be a UP-algebra given in Example 2.1. Let $\tilde{U} : U \rightarrow X$ be a UP-function on X given by $\tilde{U}(\emptyset) = \emptyset, \tilde{U}(a) = a, \tilde{U}(b) = b, \tilde{U}(c) = c$ with its cut sets as: $U_\emptyset = U, U_a = \{a\}, U_b = \{a, b\}$ and $U_c = \{a, c\}$. We have the following table

$\tilde{U}_x$	$\emptyset$	a	b	c
$\tilde{U}_\emptyset$	1	1	1	1
$\tilde{U}_a$	0	1	0	0
$\tilde{U}_b$	0	1	1	0
$\tilde{U}_c$	0	1	0	1

Therefore, $C = \{1111, 0100, 0110, 0101\}$ and graphically it is represented as:

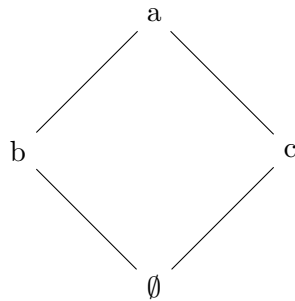
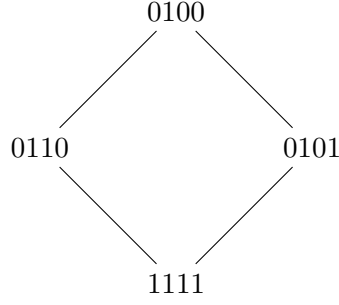


Fig. (i), $(X, \leq)$

Fig. (ii), $(C, \preceq)$.

Now, we have the following theorem:

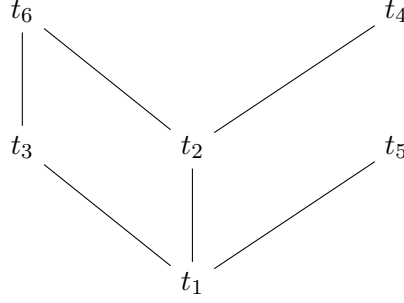
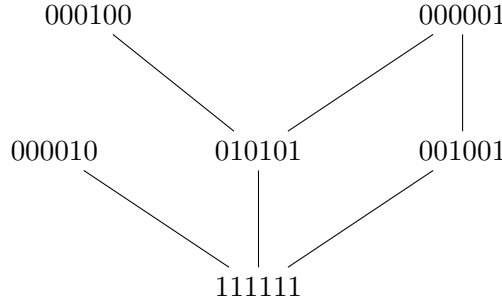
Theorem 4.1. *Every finite UP-algebra X determines a block-code C such that $(X, \leq) \simeq (C, \preceq)$.*

Proof. Let $X = \{t_1, t_2, \dots, t_n\}$ be a finite UP-algebra in which t_1 is the smallest element and $\tilde{U} : U \rightarrow X$ be the identity UP-function on X . The decomposition of $\tilde{U}$ gives a family $\{\tilde{U}_t | t \in X\}$ which is the required code under the order defined by the relation $c_x \preceq c_y \iff y_i \leq x_i$ for $i = 1, 2, 3, \dots, n$. Next suppose that $f : X \rightarrow \{\tilde{U}_t | t \in X\}$ be a function defined by $f(t) = \tilde{U}_t$ for all $t \in X$. Hence by Lemma 3.1, every equivalence class contains exactly one element. Thus f is one-to-one. Let $x, y \in X$ be such that $y * x = t_1$. Then $U_y \subseteq U_x$ by using Proposition 3.3. Therefore $\tilde{U}_x \leq \tilde{U}_y$. Hence f is an isomorphism. $\square$

Example 4.2. Consider $U = \{a_1, a_2, a_3, a_4, a_5, a_6\}$ and X be a UP-algebra given in Example 2.4. Define a UP-function $\tilde{U} : U \rightarrow X$ by $\tilde{U}(a_i) = t_i$, $i = 1, 2, \dots, 6$. Then we have the table below:

	a_1	a_2	a_3	a_4	a_5	a_6
$\tilde{U}_{t_1}$	1	1	1	1	1	1
$\tilde{U}_{t_2}$	0	1	0	1	0	1
$\tilde{U}_{t_3}$	0	0	1	0	0	1
$\tilde{U}_{t_4}$	0	0	0	1	0	0
$\tilde{U}_{t_5}$	0	0	0	0	1	0
$\tilde{U}_{t_6}$	0	0	0	0	0	1

Cut sets of $\tilde{U}$ are: $U_{t_1} = U$, $U_{t_2} = \{a_2, a_4, a_6\}$, $U_{t_3} = \{a_3, a_6\}$, $U_{t_4} = \{a_4\}$, $U_{t_5} = \{a_5\}$, $U_{t_6} = \{a_6\}$. Therefore, $C = \{111111, 010101, 001001, 000100, 000010, 000001\}$ and its graph representation is as below

Fig. (iii), $(X, \leq)$ Fig. (iv), $(C, \leq)$

Proposition 4.1. Suppose that $(X, \leq)$ be a finite partial order set with the minimum element $\emptyset$. We define a binary relation $*$ on X as follows:

1. $\emptyset * x = x, x * x = \emptyset, x \in X,$
2. $x * y = \emptyset$ if $y \leq x, x, y \in X,$
3. $x * y = y$, otherwise

Then the structure $(X, *, \emptyset)$ is a UP-algebra.

Proof. We just need to show that $(y * z) * ((x * y) * (x * z)) = \emptyset$, for all $z, y, x \in X$. We have three cases here:

Case i At least one element is $\emptyset$.

1. $x = \emptyset; (y * z) * ((\emptyset * y) * (\emptyset * z)) = (y * z) * (y * z) = \emptyset.$
2. $y = \emptyset; (\emptyset * z) * ((x * \emptyset) * (x * z)) = z * (x * z) = z * z = \emptyset.$
3. $z = \emptyset; (y * \emptyset) * ((x * y) * (x * \emptyset)) = \emptyset * ((x * y) * \emptyset) = \emptyset * \emptyset = \emptyset.$

Case ii One element is comparable with another.

1. $x \leq y; (y * z) * ((x * y) * (x * z)) = (y * z) * (y * z) = \emptyset,$
2. $x \leq z; (y * z) * ((x * y) * (x * z)) = (y * z) * (y * z) = \emptyset,$
3. $y \leq z; (y * z) * ((x * y) * (x * z)) = (y * z) * (y * z) = \emptyset,$
4. $y \leq x; (y * z) * ((x * y) * (x * z)) = (y * z) * (\emptyset * (x * z)) = (y * z) * (x * z) = z * z = \emptyset,$
5. $z \leq x; (y * z) * ((x * y) * (x * z)) = (y * z) * ((x * y) * \emptyset) = (y * z) * \emptyset = \emptyset,$
6. $z \leq y; (y * z) * ((x * y) * (x * z)) = \emptyset * (y * z) = y * z = \emptyset.$

Case iii Two elements are comparable with the third.

1. $x \leq y$ and $z \leq y$; $(y * z) * ((x * y) * (x * z)) = \emptyset * (y * z) = y * z = \emptyset$,
2. $x \leq z$ and $y \leq z$; $(y * z) * ((x * y) * (x * z)) = (y * z) * (y * z) = \emptyset$,
3. $y \leq x$ and $z \leq x$; $(y * z) * ((x * y) * (x * z)) = (y * z) * (\emptyset * \emptyset) = (y * z) * \emptyset = \emptyset$.

Rest cases will follow from transitive property of partial order set $(X, \leq)$. $\square$

Let C be a binary block code with n codewords of length n . Consider the matrix $M_C = (m_{i,j}) \in \mathbb{M}_C(\{0,1\})$ with the rows consisting of the codewords of C . We say $\mathbb{M}_C(\{0,1\})$ associated matrix to code C .

A question was asked by Jun and Song in [13] for BCK-algebras that for every block code C do we have a BCK-algebra for which we can determine C . Here we have answered that question in positive sense but partially for a UP-algebra in the following results.

Theorem 4.2. *Let $\mathbb{M}_C(\{0,1\})$ be associated matrix to code C , if the codeword $111\dots 1$ is in C and the matrix M_C is upper triangular with $m_{ii} = 1$, for all $i \in \{1, 2, 3, \dots, n\}$, then there exists a set U with n elements, a UP-algebra X and a UP-function $f : U \rightarrow X$ such that f determines C .*

Proof. Considering the lexicographic order $\leq_{lex}$ in C we find that $(C, \leq_{lex})$ is

a totally ordered set. Let $C = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ with the lexi order $c_n \leq_{lex} c_{n-1} \leq_{lex}$

$\dots \leq_{lex} c_2 \leq_{lex} c_1$. Hence $c_1 = 111\dots 1$ and $c_n = 00\dots 01$. Then by partial order in C , $(C, \preceq)$ is a POS with $c_1 \preceq c_i, i \in \{1, 2, 3, \dots, n\}$. Here zero element and maximal element in $(C, \preceq)$ is $\emptyset = c_1$ and c_n respectively. We define a binary operation $\circ$ on $X = C$ as defined in Proposition 4.1. Then we get that $X = (C, \circ, c_1)$ is a UP-algebra and C is isomorphic to $\mathcal{C}_n$ as a UP-algebra, where $\mathcal{C}_n$ is a UP-algebra with n elements defined in Proposition 4.1. We suppose that $U = C$ and the identity map $f : U \rightarrow C, f(t) = t$ as a UP-function. The decomposition of f provides a family of maps $C_{c_n} = \{f_r : A \rightarrow \{0,1\} | f_r(a) = 1, \text{ if and only if } f(a) \circ r = \emptyset, \forall a \in A, r \in X\}$. This set forms a binary block code C relative to the partial order relation $\preceq$. Further let $c_k \in C, 1 < k < n, c_k = 00\dots 0a_{i_k}\dots a_{i_n}, a_{i_j} \in \{0,1\}$. If $a_{i_j} = 0$, then $c_k \preceq c_{i_j}$. Also $c_{i_j} \circ c_k = \emptyset$. Again $a_{i_j} = 1$, we get that $c_{i_j} \preceq c_k$ or a_{i_j} and c_k can't be compared, hence $c_{i_j} \circ c_k = c_k$. $\square$

Two different UP-algebras having same binary block code are said to be code similar algebras and denoted by $U_1 \sim U_2$. Let $\mathfrak{U}_n$ be set of all finite UP-algebra with n elements then the relation code similar is an equivalence relation on $\mathfrak{U}_n$ and that $\mathfrak{Q}_n$ denotes the quotient set then for $C \in \mathfrak{C}_n$, where $\mathfrak{C}_n$ is the set of block codes having similar properties given in Theorem 4.2. An equivalence class in $\mathfrak{Q}_n$ is denoted by $\check{C} = \{U \in \mathfrak{U}_n | U \text{ determines the binary block code } C\}$.

Proposition 4.2. *The quotient set $\mathfrak{Q}_n$ has $2^{\frac{(n-1)(n-2)}{2}}$ elements, which is the same cardinal as in the case of $\mathfrak{C}_n$.*

Proof. Let M_C be the associated matrix of $C \in \mathfrak{C}_n$ then we shall compute the cardinal of the set M_C which is an upper triangular matrix with $m_{ii} = 1$ for all $i \in \{1, 2, 3, \dots, n\}$. Second row of the matrix M_C has the form $(0, 1, a_3, \dots, a_n)$, where $a_3, \dots, a_n \in \{0, 1\}$. Hence number of different rows of this type of matrix is 2^{n-2} . The third row of the matrix M_C has the form $(0, 0, 1, a_4, a_5, \dots, a_n)$, where $a_4 \dots a_n \in \{0, 1\}$. Hence number of different rows of this type is 2^{n-3} . Therefore we get that the cardinal of the set $\mathfrak{C}_n$, is $2^{n-2} 2^{n-3} \dots 2 = 2^{\frac{(n-1)(n-2)}{2}}$. $\square$

For finite non-isomorphic UP-algebras with n elements cardinality is $\geq 2^{\frac{(n-1)(n-2)}{2}}$. We take the following definitions:

Definition 4.1. *A totally ordered relation $\succeq_{lex}$ on $\mathfrak{C}_n$ for the matrix $M_{C_j}, j \in \{1, 2, \dots, n\}$ is defined as $C_1 \succeq_{lex} C_2$ if there is $i \in \{2, 3, \dots, n\}$ such that $r_1^{C_1} = r_2^{C_2} \dots r_{i-1}^{C_1} = r_{i-1}^{C_2}$ and $r_i^{C_1} \succeq_{lex} r_i^{C_2}$ where $\succeq_{lex}$ represents lexicographic order.*

Definition 4.2. *For two associated matrix M_{C_1} and M_{C_2} , the partial order on $\mathfrak{C}_n$ for $C_1, C_2 \in \mathfrak{C}_n$ is defined as $C_2 \gg C_1$ such that $r_1^{C_1} = r_1^{C_2} \dots r_{i-1}^{C_1} = r_{i-1}^{C_2}$ and $r_i^{C_1} \preceq r_i^{C_2}$.*

Let $\Phi = (\phi_{ij}) \in \mathbb{M}\{0, 1\}$ be a matrix such that $\phi_{ij} = 1, i \leq j$, for all $i, j \in \{1, 2, 3, \dots, n\}$ and $\phi_{ij} = 0$ in the rest part. We get a minimum element $M_\omega = \Phi$ for the code ω in the partial order set $(\mathfrak{C}_n, \gg)$ where elements in $\mathfrak{C}_n$ are ascending order relative to $\succeq_{lex}$. Then we see that $(\mathfrak{C}_n, \circ, \omega)$ is a non-commutative and non-implicative UP-algebra. This UP-algebra determines a binary block code $C_{\mathfrak{C}_n}$ of length $2^{\frac{(n-1)(n-2)}{2}}$. It is clear that $\check{C}_{\mathfrak{C}_n} \in \mathfrak{C}_{2^{\frac{(n-1)(n-2)}{2}}}$.

Proposition 4.3. *Let $A = (a_{i,j}) \in \mathbb{M}_{n \times m}(\{0, 1\})$ be a matrix with rows lexicographic in ascending order. Consider a matrix $B = (b_{i,j}) \in \mathbb{M}_q(\{0, 1\})$, $q = n + m$, such that B is an upper triangular matrix with $b_{ii} = 1$, $i \in \{1, 2, \dots, q\}$. In this way we see that A becomes a submatrix of B .*

Proof. We introduce new columns in the matrix A on the left side (from right

to left) $\overbrace{00 \dots 01}^n, \overbrace{00 \dots 10}^n, \dots, \overbrace{10 \dots 00}^n$ and obtained a new matrix E with n rows and $n + m$ columns. Now introducing the following m rows in the bottom of the matrix E :

$\overbrace{00 \dots 01}^n \overbrace{10 \dots 00}^m, \overbrace{00 \dots 00}^{n+1} \overbrace{01 \dots 00}^{m-1}, \dots, \overbrace{00 \dots 01}^{n+m-1}$ to get matrix B . $\square$

Theorem 4.3. *Assuming the above notations, let C be a binary block code with n codewords of length $m, n \neq m$, or a block code with n codewords of length n such that the codeword $\overbrace{11 \dots 1}^{n \text{ times}}$ is not in C , or a block code with n codewords of*

length n such that the matrix M_C is not upper triangular. There are a natural number $q \geq \max\{m, n\}$, a set A with m elements and a UP-function $f : A \rightarrow \mathcal{C}_q$ such that we obtain block code C_{C_n} contains the block code C as a subset.

Proof. Assume C to be a binary block code, $C = \{c_1, c_2, \dots, c_n\}$, with codeword of length m and with its lexicographic order $c_1 \geq_{lex} c_2 \geq_{lex} \dots \geq_{lex} c_n$. Let $M \in \mathbb{M}_{n,m}(\{0,1\})$ be the associated matrix with rows $c_1 \dots c_n$ in this order. By using Theorem 4.2 we can extend the matrix M to a square matrix $B \in \mathbb{M}_q(\{0,1\})$, $q = m + n$, such that $B = (m'_{i,j})_{i,j \in \{1,2,3,\dots,q\}}$ is an upper triangular

matrix with $m_{ii} = 1$, for all $i \in \{1, 2, \dots, q\}$. If the first row of B is not $\overbrace{11 \dots 1}^q$

then we insert the first row $\overbrace{11 \dots 1}^{q+1}$ and first column $\overbrace{10 \dots 0}^q$ to obtain a new matrix M' . Now for the matrix M' , we obtain a UP-algebra $\mathcal{C}_q = \{c_1, c_2, \dots, c_q\}$, with $c_1 = \phi$ the zero of $\mathcal{C}_q$ and a binary code C_{C_q} . Assuming that initial columns of M have in the new matrix M' positions $i_{j_1}, i_{j_2}, \dots, i_{j_m} \in \{1, 2, 3, \dots, q\}$, let $A = \{x_{j_1}, x_{j_2}, \dots, x_{j_m}\} \subseteq \mathcal{C}_q$. The UP-function $f : A \rightarrow \mathcal{C}_q$, $f(x_{j_i}) = x_{j_i}$, $i \in \{1, 2, 3, \dots, m\}$, determines the binary block code C_{C_q} such that $C \subseteq C_{C_q}$. $\square$

Example 4.3. Let $C = \{0110, 0010, 1111, 0001\}$ be a binary block code. Code C can be written as $C = \{1111, 0110, 0010, 0001\} = \{c_1, c_2, c_3, c_4\}$ by lexicographic order. From Theorem 4.2, $c_1 \preceq c_i$, $i \in \{2, 3, 4\}$, $c_2 \preceq c_3$ where as c_2 can't be compared with c_4 and c_3 can't be compared with c_4 . The $\circ$ operation on C is given in the table below

$\circ$	c_1	c_2	c_3	c_4
c_1	c_1	c_2	c_3	c_4
c_2	c_1	c_1	c_3	c_4
c_3	c_1	c_1	c_1	c_4
c_4	c_1	c_2	c_3	c_1

It is clear that C with $\circ$ is a UP-algebra.

For $A = \{\emptyset, 1, 2, 3\}$ the same binary block code C can be obtained from UP-algebra $(A, *, \emptyset)$ as given in the table below with UP-function $f : A \rightarrow A$, $f(x) = x$.

$*$	$\emptyset$	1	2	3
$\emptyset$	$\emptyset$	1	2	3
1	$\emptyset$	$\emptyset$	1	3
2	$\emptyset$	$\emptyset$	$\emptyset$	3
3	$\emptyset$	1	2	$\emptyset$

Since $(A, *, \emptyset)$ is a commutative UP-algebra and $(C, \circ, c_1)$ is a non-commutative UP-algebra. Therefore, $(A, *, \emptyset)$ is not isomorphic to $(C, \circ, c_1)$. Hence we can say that a UP-algebra associated to a binary block code as in Theorem 4.2 is

not unique up to an isomorphism. If we start from the commutative UP-algebra $(A, *, c_1)$ to obtain the code C , and construct UP-algebra $(C, \circ, c_1)$ as in the Theorem 4.2, then we get that the algebra $(C, \circ, c_1)$ lost the commutative property even they are code similar.

Example 4.4. Let $C = \{11110, 10010, 10011, 00001\}$ be a binary block code. Using lexicographic order, we can write $C = \{11110, 10011, 10010, 00000\} = \{c_1, c_2, c_3, c_4\}$. Let $M_c \in \mathcal{M}_{4 \times 5}(\{0, 1\})$ be the associated matrix,

$$\mathcal{M}_c = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Using Proposition 4.3 we have the following upper triangular matrix which starts from the matrix M_C as given below

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Since the first 1st row is not $\overbrace{11 \dots 1}^9$. Using Proposition 4.3 we insert a new row $\overbrace{11 \dots 1}^{10}$ as a first row and a new column $\overbrace{10 \dots 0}^{10}$ as a fixed column, then we

obtain the following matrix:

$$M' = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The binary block code $C = \{c_1, c_2, c_3, c_4\}$ whose codewords are the row of the matrix M' , determines a UP-algebra $(C, \circ, c_1)$. Let $V = \{c_6, c_7, c_8, c_9, c_{10}\}$ and $f : V \rightarrow X, f(c_i) = c_i, i \in \{6, 7, 8, 9, 10\}$ be a UP-function which determines the binary block code $U = \{11111, 11110, 10011, 10010, 00000, 10000, 01000, 00100, 00010, 00001\}$ the code C is a subset of the code U .

5. Conclusion

In this work we have studied a branch of coding theory namely, *Binary Block Codes* associated to UP-algebras. At first we have defined and investigated UP-valued functions and their properties. By using these UP-functions we have associated every UP-algebra X to a block code C through cut-function. We have also partially answered the question raised by Jun and Song [13] in case of a UP-algebra that for each binary block code C we determined a UP-algebra X such that the binary block code generated by X , C_X contains code C . We have also shown that UP-algebras associated to a binary block code are not unique up to isomorphism.

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Behaviour of motion of infinitesimal variable mass oblate body in the generalized perturbed circular restricted three-body problem

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Abstract. The main goal of the present study is to reveal the behaviour of motion of the infinitesimal body in case of circular restricted three-body problem where all the participating bodies have oblate shapes and both primaries have the effect of solar radiation pressure. The third infinitesimal body is varying its mass according to Jeans law. We also consider that the system is affected by the small perturbations in Coriolis and centrifugal forces. We evaluate the equations of motion of the infinitesimal oblate body under the generalized sense in the perturbed circular restricted three-body problem by using the Meshcherskii-space time transformations to preserve the dimensions of the position as well as time, and then determine the Jacobi-integral. Further we numerically illustrated the equilibrium points, Poincaré surfaces of section, regions of possible and forbidden motion and then basins of the attracting domain by supposing the effects of all the parameters used. Further more, we examine the stability of these

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equilibrium points with the help of Meshcherskii space-time inverse transformations and found them unstable.

Keywords: infinitesimal oblate body, variable mass, regions of forbidden motion, stability of equilibrium points.

1. Introduction

Plagiarism is a serious issue at the time of writing a research paper. We should directly focus towards our problem. Our research field is a part of Mechanics i.e. Celestial mechanics which also lies in the Applied Mathematics and theoretical Physics. Here, we have focused on the motion of the small particle (especially Satellite in comparison to the Earth) under the gravitational attraction of the other two celestial bodies i.e restricted three-body problem. Restricted means the two bigger primary bodies are influencing the third infinitesimal body with their forces but the third body will not affect the motion of the primaries. We also consider that both the primaries have solar radiation pressure as well as oblateness effect while the third infinitesimal body is taken as oblate in shape and varies its mass according to Jeans law. The system is perturbed by the Coriolis and centrifugal forces. The two primaries are moving in circular orbits around their common center of mass which is taken as origin. All these effects and perturbations are studied by many researchers separately as follows:

[25] numerically investigated the effect of oblateness of the more massive primary on the location of five equilibrium points in the circular restricted 3-body problem. They noticed that the patterns of angular frequencies have interesting differences. [17] investigated the effects of perturbations in the frame of restricted three-body problem and compared there work with the previous works. They found that these perturbations have significant impact on the stability of equilibrium points. [26, 27] investigated the effects of radiation pressure, oblateness, variable mass and asteroid belts in the framework of the circular restricted three-body problem by finding Lagrangian function with the help of kinetic energy and potential energy. [1] studied the effects of radiation pressure, oblateness as well as Coriolis and centrifugal forces on the stability of equilibrium points in the frame of circular restricted three-body problem. They observed that the Coriolis force has stabilizing tendency while other forces have destabilizing tendency. They also found that collinear equilibrium points are always unstable while non-collinear points are week in stability due to these external factors.

[21] investigated the effect of variable mass in the frame of circular restricted three-body problem by considering that sum of the masses of all bodies are always constant. They obtained the zero-velocity surfaces using Jacobi-quasi-integral. They also presented an example for their model as binary stars system with conservation of mass transfer. [18] revealed the existence, position and stability of the collinear equilibrium points in the frame of generalized Hill's problem by hypothesizing primary as radiating and secondary as oblate. They

also plotted the basins of attraction by using fast and simple Newton-Raphson iterative method for several cases of parameters. [23] investigated the effect of oblateness on the periodic orbits as well as on the regions of quasi-periodic motion around both the primaries in the circular restricted three-body problem by using the numerical technique of Poincaré surfaces of section. They found some variations in the stability of orbits due to oblateness. [29] studied the motion of infinitesimal variable mass in the circular restricted three-body problem by taking on account to one of the primaries having photo-gravitational effect using Jeans law [19] and Meshcharskii space-time transformations [22].

[20] analysed the motion of infinitesimal body in the frame of restricted four-body problem with solar wind drag and radiation pressure. They numerically studied the equilibrium points, zero-velocity curves, Poincaré surfaces of sections and Lyapunov Characteristic exponents (LCEs). [4, 2, 5, 6] studied the effect of oblateness, radiation pressure of the primaries as well as the effect of variable mass of the infinitesimal body in the frame of restricted problem. [24] explored the motion of the infinitesimal body around one of the triangular equilibrium points (L_4) in the circular restricted three-body problem by choosing one of the primary as Heterogeneous triaxial body. [8, 7, 9, 10, 11, 12, 13, 14, 16] investigated the effects of the perturbations and variable mass in the frame of restricted three-body problem. They also studied the effects of these perturbations on the locations of equilibrium points, Poincaré surfaces of section, regions of possible and forbidden motion, basins of attraction and examined the stability of these equilibrium points where most of the cases they found that these points are unstable. [30] investigated the basins of convergence of attracting points by considering the modified gravitational potential using well-known Newton-Raphson iterative method in the frame of restricted three-body problem. They also computed the degree of fractality using the basin entropy.

This work is organised in various sections and subsections which are as follows: The overview of literature is given in section 1. The determination of equations of motion and Jacobi-integral are given in section 2 while section 3 represent the numerical works with subsections 3.1, 3.2, 3.3 and 3.4. The stability of equilibrium points are numerically studied in section 4. Finally the conclusion of the work is presented in section 5.

2. Determination of equations of motion

Let there be three masses m_i and m where two masses (m_i) are primaries of oblate shapes with oblateness factors σ_i and radiation pressures with radiation factors q_i , ($i = 1, 2$) respectively while the third body has also oblate shape with oblateness factor σ and varies its mass according to Jeans law. We also considered that the motion is perturbed by the coriolis and centrifugal forces with factors ϕ and ψ respectively. In the synodic coordinate system xyz , the line joining both the primaries, is taken as x-axis while the line perpendicular to this line is considered as y-axis. The two primary bodies are moving in circular orbits

around their common center of mass which is taken as origin. The mean motion n of the system is taken about z - axis which is perpendicular to the plane of motion of the primaries. The gravitational forces of primaries are influencing to the third infinitesimal variable mass oblate body while these two bodies are not influenced by the third body. To fix the variables as non-dimensional, assume that the total mass of the primaries is unity, the separation distance between the primaries (R) is unity and the time of unit is so chosen as the gravitational constant is also unity. i.e. $m_1 + m_2 = 1$, $R = 1$ and $G = 1$. Let $\mu = \frac{m_2}{m_1 + m_2}$ and $m_1 = 1 - \mu$. The coordinates of the primaries(m_i) and the infinitesimal body (m) are given by $(-\mu, 0, 0)$, $(1 - \mu, 0, 0)$ and (x, y, z) respectively. Following the procedure given by [3] and [15], we can illustrate the equations of motion of the infinitesimal variable mass oblate body with non-dimensional variables where the variation of mass of this body originates from one point and have zero momenta as:

$$(1) \quad \begin{aligned} V_x &= \frac{\dot{m}}{m}(\dot{x} - n\phi y) + (\ddot{x} - 2n\phi\dot{y}), \\ V_y &= \frac{\dot{m}}{m}(\dot{y} + n\phi x) + (\ddot{y} + 2n\phi\dot{x}), \\ V_z &= \frac{\dot{m}}{m}\dot{z} + \ddot{z}, \end{aligned}$$

where

$$(2) \quad V = \frac{n^2\psi}{2}(x^2 + y^2) + (1 - \mu) \left(\frac{q_1}{r_1} + \frac{(q_1\sigma_1 + \sigma)}{2r_1^3} \right) + \mu \left(\frac{q_2}{r_2} + \frac{(q_2\sigma_2 + \sigma)}{2r_2^3} \right),$$

with

$$\begin{aligned} r_1^2 &= (x + \mu)^2 + y^2 + z^2, \\ r_2^2 &= (x - 1 + \mu)^2 + y^2 + z^2, \\ n^2 &= 1 + \frac{3}{2}(\sigma_1 + \sigma_2), \\ \phi &= 1 + \epsilon_1, \\ \psi &= 1 + \epsilon_2, \\ \epsilon_1, \epsilon_2 &\ll 1. \end{aligned}$$

Due to variation of mass, we should preserve the dimension of the space and time therefore we will use Jean's law [19] and Meshcherskii space time transformations [22] as,

$$(3) \quad \begin{aligned} m &= m_0 e^{-\alpha_1 t}, dt = d\tau \\ x &= \alpha_2^{-1/2} \xi, \dot{x} = \alpha_2^{-1/2} \left(\dot{\xi} + \frac{\alpha_1}{2} \xi \right), \ddot{x} = \alpha_2^{-1/2} \left(\ddot{\xi} + \alpha_1 \dot{\xi} + \frac{\alpha_1^2}{4} \xi \right) \\ y &= \alpha_2^{-1/2} \eta, \dot{y} = \alpha_2^{-1/2} \left(\dot{\eta} + \frac{\alpha_1}{2} \eta \right), \ddot{y} = \alpha_2^{-1/2} \left(\ddot{\eta} + \alpha_1 \dot{\eta} + \frac{\alpha_1^2}{4} \eta \right) \\ z &= \alpha_2^{-1/2} \zeta, \dot{z} = \alpha_2^{-1/2} \left(\dot{\zeta} + \frac{\alpha_1}{2} \zeta \right), \ddot{z} = \alpha_2^{-1/2} \left(\ddot{\zeta} + \alpha_1 \dot{\zeta} + \frac{\alpha_1^2}{4} \zeta \right), \end{aligned}$$

where α_1 is constant coefficient, $\alpha_2 = \frac{m}{m_0}$ and m_0 is the mass of infinitesimal oblate body at time $t = 0$.

After using Eq.(3) in Eq. (1), we get

$$(4) \quad \begin{aligned} \frac{\partial \Omega}{\partial \xi} &= \ddot{\xi} - 2n\phi\dot{\eta}, \\ \frac{\partial \Omega}{\partial \eta} &= \ddot{\eta} + 2n\phi\dot{\xi}, \\ \frac{\partial \Omega}{\partial \zeta} &= \ddot{\zeta}, \end{aligned}$$

where

$$(5) \quad \begin{aligned} \Omega &= \frac{\alpha_1^2}{8}(\xi^2 + \eta^2 + \zeta^2) + \frac{n^2\psi}{2}(\xi^2 + \eta^2) \\ &+ \alpha_2^{3/2} \left[(1-\mu) \left\{ \frac{q_1}{\rho_1} + \frac{(q_1\sigma_1 + \sigma)\alpha_2}{2\rho_3^3} \right\} + \mu \left\{ \frac{q_2}{\rho_2} + \frac{(q_2\sigma_2 + \sigma)\alpha_2}{2\rho_2^3} \right\} \right], \end{aligned}$$

with

$$(6) \quad \begin{aligned} \rho_1^2 &= (\xi + \mu\sqrt{\alpha_2})^2 + \eta^2 + \zeta^2, \\ \rho_2^2 &= (\xi - \sqrt{\alpha_2}(1-\mu))^2 + \eta^2 + \zeta^2. \end{aligned}$$

The equations of motion (4) admits as

$$(7) \quad \dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2 = 2\Omega - C - 2 \int_{t_0}^t \frac{\partial \Omega}{\partial t} dt,$$

where left hand side of Eq. (7) represents the velocity of the infinitesimal oblate body and C is the Jacobian constant for the system.

3. Numerical performances

3.1 Location of equilibrium points

We can find the locations of equilibrium points after putting all the derivatives to zero and solving the obtained equations of the system (4), i.e. $\Omega_\xi = 0$, $\Omega_\eta = 0$ and $\Omega_\zeta = 0$, and hence

$$(8) \quad \begin{aligned} \xi(n^2\psi + \frac{\alpha_1^2}{4}) - \alpha_2^{3/2} \left(\frac{q_1(1-\mu)(\xi + \sqrt{\alpha_2}\mu)}{\rho_1^3} + \frac{3(\sigma + \sigma_1 q_1)\alpha_2(1-\mu)(\xi + \sqrt{\alpha_2}\mu)}{2\rho_1^5} \right) \\ - \alpha_2^{3/2} \left(\frac{q_2\mu(\xi + \sqrt{\alpha_2}(-1+\mu))}{\rho_2^3} + \frac{3(\sigma + \sigma_2 q_2)\alpha_2\mu(\xi + \sqrt{\alpha_2}(-1+\mu))}{2\rho_2^5} \right) = 0, \end{aligned}$$

$$(9) \quad \eta(n^2\psi + \frac{\alpha_1^2}{4}) - \eta\alpha_2^{3/2} \left(\frac{q_1(1-\mu)}{\rho_1^3} + \frac{3(\sigma + \sigma_1 q_1)\alpha_2(1-\mu)}{2\rho_1^5} \right) - \eta\alpha_2^{3/2} \left(\frac{q_2\mu}{\rho_2^3} + \frac{3(\sigma + \sigma_2 q_2)\alpha_2\mu}{2\rho_2^5} \right) = 0,$$

$$(10) \quad \zeta \frac{\alpha_1^2}{4} - \zeta\alpha_2^{3/2} \left(\frac{q_1(1-\mu)}{\rho_1^3} + \frac{3(\sigma + \sigma_1 q_1)\alpha_2(1-\mu)}{2\rho_1^5} \right) - \zeta\alpha_2^{3/2} \left(\frac{q_2\mu}{\rho_2^3} + \frac{3(\sigma + \sigma_2 q_2)\alpha_2\mu}{2\rho_2^5} \right) = 0.$$

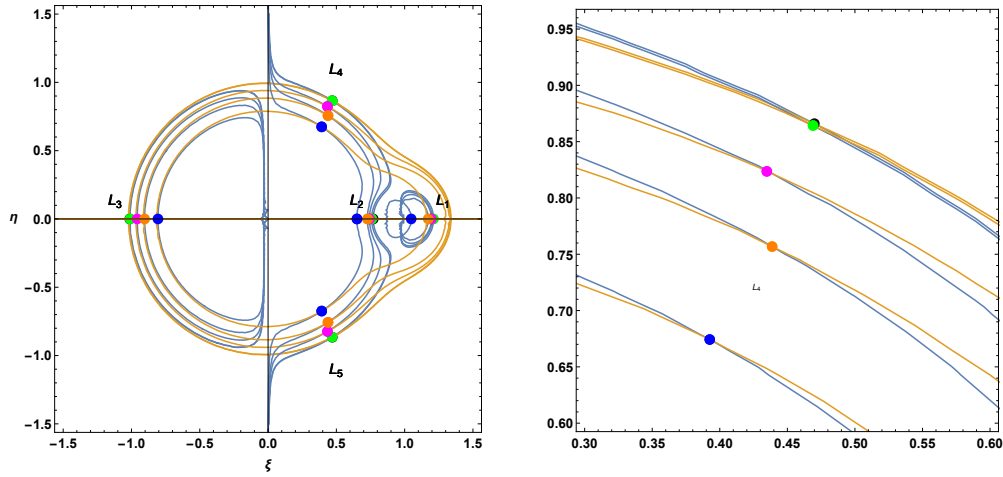
After numerically solving Eqs. (8), (9) and (10) for the variations of the parameters used with the help of well known software, Mathematica, we perform these graphs in Figures 1,2,3 and 4. We have illustrated these points in five cases as:

1. The Classical case:
($\sigma = 0, \sigma_1 = 0, \sigma_2 = 0, q_1 = 1, q_2 = 1, \epsilon_1 = 0, \epsilon_2 = 0, \alpha_1 = 0, \alpha_2 = 1$)
2. Taken the effect of oblateness of all three participating bodies:
($\sigma = 0.001, \sigma_1 = 0.003, \sigma_2 = 0.005, q_1 = 1, q_2 = 1, \epsilon_1 = 0, \epsilon_2 = 0, \alpha_1 = 0, \alpha_2 = 1$)
3. Taken the effect of oblateness and radiation pressure:
($\sigma = 0.001, \sigma_1 = 0.003, \sigma_2 = 0.005, q_1 = 0.85, q_2 = 0.95, \epsilon_1 = 0, \epsilon_2 = 0, \alpha_1 = 0, \alpha_2 = 1$)
4. Taken the effect of oblateness, radiation pressure, Coriolis and centrifugal forces:
($\sigma = 0.001, \sigma_1 = 0.003, \sigma_2 = 0.005, q_1 = 0.85, q_2 = 0.95, \epsilon_1 = 0.2, \epsilon_2 = 0.2, \alpha_1 = 0, \alpha_2 = 1$)
5. Taken the effect of all perturbations given in case 4 including the variation of mass parameters:
($\sigma = 0.001, \sigma_1 = 0.003, \sigma_2 = 0.005, q_1 = 0.85, q_2 = 0.95, \epsilon_1 = 0.2, \epsilon_2 = 0.2, \alpha_1 = 0.2, \alpha_2 \neq 1$.)

Figure (1) represents the variation of the cases from case 1 to case 5 in $\xi - \eta$ -plane. Here the points due to the cases 1, 2, 3, 4 and 5 are denoted by black, green, magenta, orange and blue colors respectively. We introduced the cases one by one, further we observed from Figure (1(a)) that the system has five equilibrium points ($L_{1,2,3,4,5}$) out of which three ($L_{1,2,3}$) are collinear and two ($L_{4,5}$) are non-collinear (also known as triangular points), all these equilibrium points are moving towards origin. This effect can be easily seen in Figure 1(b) which is representing the zoomed part of the Figure 1(a) near L_4 . Figure 2 presents the equilibrium points due to these five cases in $\xi - \zeta$ -plane. From this

Figure we observed that due to first four cases there are only three collinear equilibrium points given in Figure 2(a) and found that due to these cases three equilibrium points ($L_{1,2,3}$) are moving towards origin while due to case 5 we receive two more equilibrium points ($L_{4,5}$) including the collinear equilibrium points ($L_{1,2,3}$) (Figure 2(b)).

In Figure 3, we perform the case 5 where we have seen the effect of variation of mass parameter α_1 (0.2 (Black), 0.4 (Green), 0.6 (Magenta)) in $\xi - \eta$ -plane (Figure 3(a)) and in $\xi - \zeta$ -plane (Figure 3(b)). The Figure 3(a) shows only slight variation in equilibrium points $L_{3,4,5}$ while there are no effect on the equilibrium points $L_{1,2}$ due to increase in the value of α_1 . The Figure 3(b) shows as the value of α_1 increases, the three equilibrium points ($L_{1,2,3}$) remain unchanged while two equilibrium points $L_{4,5}$ are moving towards the origin. Figure 4 reveals the case 5 where we have seen the effect of variation of mass parameter α_2 (0.4 (black), 0.8 (green), 1.4 (magenta)) in $\xi - \eta$ -plane (Figure 4(a)) and in $\xi - \zeta$ -plane (Figure 4(b)). These two Figures 4(a) and 4(b) shows that as the value of α_2 increases, all five equilibrium points ($L_{1,2,3,4,5}$) move away from the origin. We also noticed that there is no effect of Coriolis force on the location of equilibrium points.



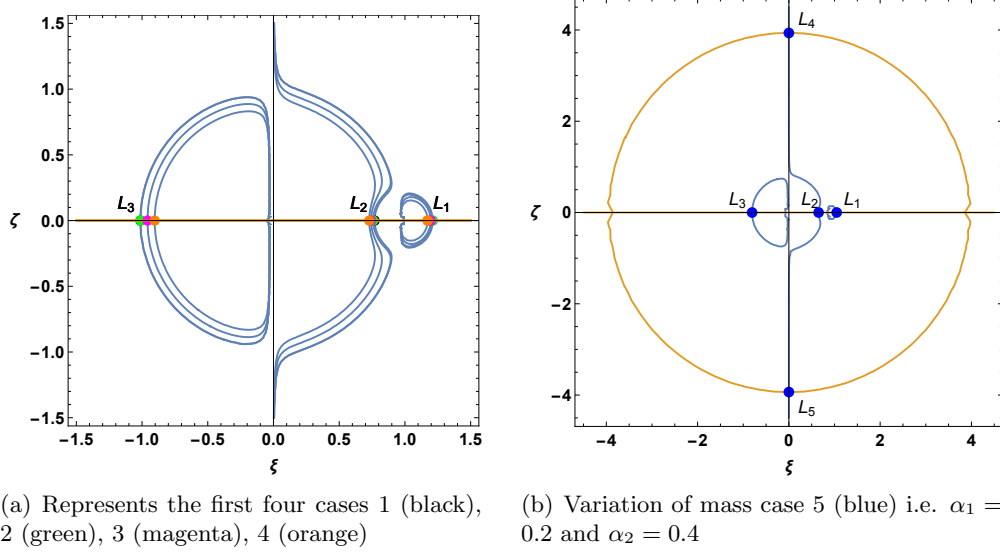
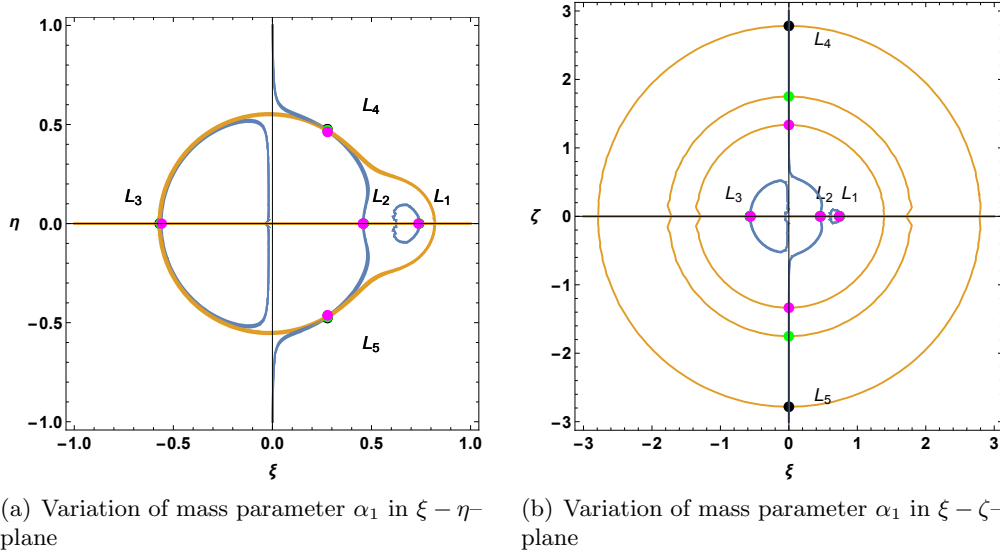
(a) Different color points show the variation of cases 1 (black), 2 (green), 3 (magenta), 4 (orange) and 5 (blue)

(b) Zoomed part of figure (1(a)) near L_4

Figure 1: Locations of equilibrium points in $\xi - \eta$ -plane at $\mu = 0.03$

3.2 Poincaré surfaces of section

The aim of plotting the Poincaré surfaces of section is to examine the chaotic or regular behaviour of the path of the infinitesimal oblate body. The graph between ξ and $\dot{\xi}$ must be drawn when $\eta = 0$, whenever the orbit intersects the plane at $\dot{\eta} \geq 0$. Here we have plotted the Poincaré surfaces of section by creating a phase space in case 5 and given in Figures 5(a) and 5(b) in $\xi - \dot{\xi}$ and

Figure 2: Locations of equilibrium points in $\xi - \zeta$ -plane at $\mu = 0.03$ Figure 3: Locations of equilibrium points at $\alpha_2 = 0.4, \alpha_1 = 0.2$ (black), 0.4 (green), 0.6 (magenta)

$\eta - \dot{\eta}$ -planes respectively. We observed from these two figures that as the value of α_2 increases, the surfaces of section expanded and these appear as butterfly on the phase plane. In this way there are no chaos found in this study.

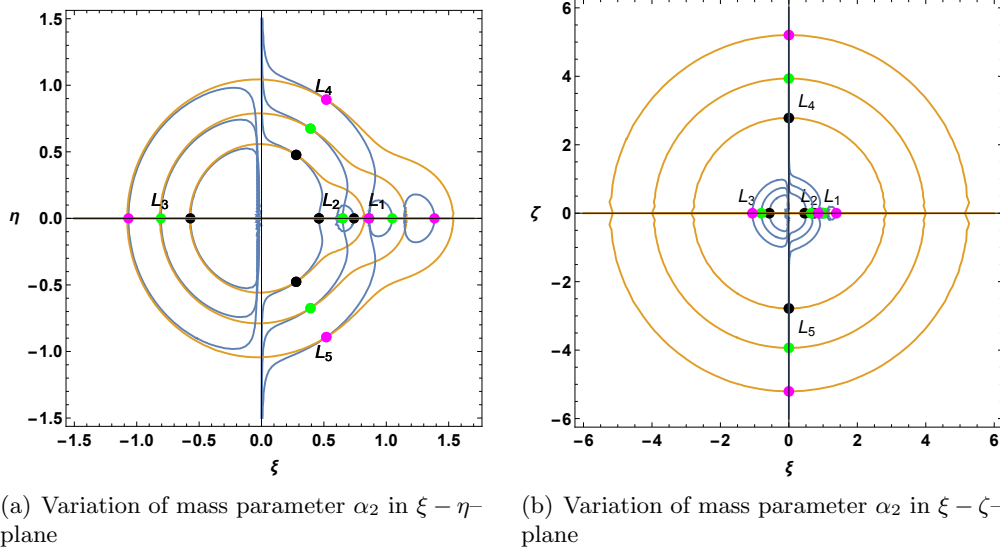


Figure 4: Locations of equilibrium points at $\alpha_1 = 0.2, \alpha_2 = 0.4$ (black), 0.8 (green), 1.4 (magenta)

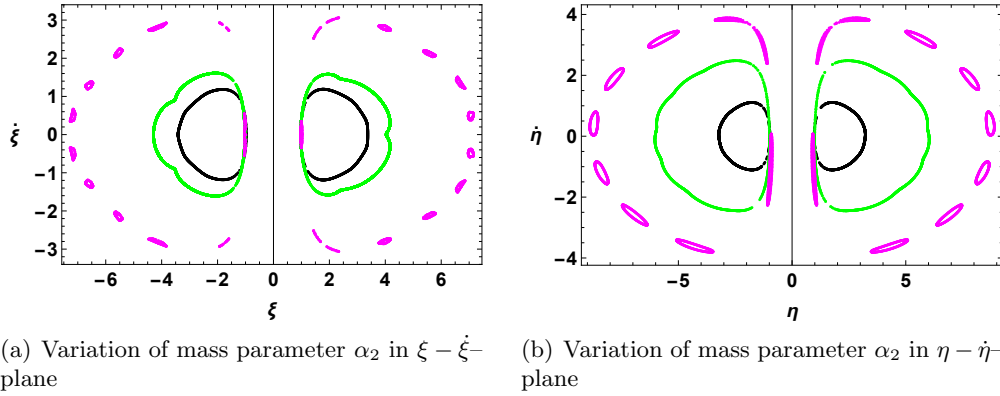


Figure 5: Poincaré surfaces of section at $\alpha_1 = 0.2, \alpha_2 = 0.4$ (black), 0.8 (green), 1.4 (magenta)

3.3 Regions of possible motion

To illustrate the regions of possible or forbidden motion, we follow the procedure and terminology used by [15]. To determine the each value of Jacobian constant corresponding to the each equilibrium point we use Eq. (7) and plotted gray shaded regions as the forbidden regions which are given in Figure (6). These figures reveal very important dynamical properties of the motion of the infinitesimal variable mass oblate body. Figure (6(a)) represents the region cor-

responding to the equilibrium point L_1 and shows that infinitesimal body can move near primaries except the gray color shaded region i. e. it can not move near $L_{2,3,4,5}$. It can move only near L_1 which is a gateway for the circular region. Figure (6(b)) represents the region corresponding to the equilibrium point L_2 and shows that the small particle can move freely except near the equilibrium points $L_{3,4,5}$. Figure (6(c)) represents the region corresponding to the equilibrium point L_3 and shows that the small particle can move near the three equilibrium points L_1 , L_2 and L_3 where as it can not move in gray shaded regions i.e. near the equilibrium points $L_{4,5}$. Figure (6(d)) represents the region corresponding to the equilibrium points $L_{4,5}$ and shows that the small particle can move freely and there is no forbidden region. Blue points are indicating the locations of the equilibrium points.

3.4 Basins of attraction

One of the most important qualitative behaviour of the dynamical systems is the basins of attracting domain. To study this we use very simple and well known Newton-Raphson iterative method. Using this iterative method, we have illustrated the attracting domain in $\xi - \eta$ -plane for different values of variation parameter (α_2). The algorithm of this problem is given as:

$$(11) \quad \xi_{n+1} = \xi_n - \left(\frac{\Omega_\xi \Omega_{\eta\eta} - \Omega_\eta \Omega_{\xi\eta}}{\Omega_{\xi\xi} \Omega_{\eta\eta} - \Omega_{\xi\eta} \Omega_{\eta\xi}} \right)_{(\xi_n, \eta_n)},$$

$$(12) \quad \eta_{n+1} = \eta_n - \left(\frac{\Omega_\eta \Omega_{\xi\xi} - \Omega_\xi \Omega_{\eta\xi}}{\Omega_{\xi\xi} \Omega_{\eta\eta} - \Omega_{\xi\eta} \Omega_{\eta\xi}} \right)_{(\xi_n, \eta_n)},$$

where ξ_n, η_n are the values of ξ and η coordinates of the n^{th} step of iterative process. The point (ξ, η) will be a member of the attracting domain, if the initial point converges rapidly to one of the equilibrium points. This process stops when the successive approximation converges to an equilibrium point. We also declare that the basins of attracting domain is not related with the classical basins of attracting domain in dissipative system. We used a color code for the classification of different equilibrium points on the $\xi - \eta$ -plane.

We have performed the basins of attracting domain for three different values of variation parameter α_2 (0.4, 0.8 and 1.4 corresponds to the Figures 7, 8 and 9 respectively) in $\xi - \eta$ -plane for the variable mass case 5. Figures 7(b), 8(b) and 9(b) are the zoomed parts of the Figures 7(a), 8(a) and 9(a) respectively. In all three figures, we observed that there are five attracting points $L_{1,2,3,4,5}$. From the Figure (7(b)), we found that $L_{1,2,3}$ corresponds to the light blue color region, L_4 corresponds to the blue color region and L_5 corresponds to the green color region, all these regions extended to infinity. From the Figure (8(b)), we found that $L_{1,2,3}$ corresponds to the green color region, L_4 corresponds to the cyan color region while L_5 corresponds to the red color region, all these regions extended to infinity. Further from the Figure (9(b)), we found that

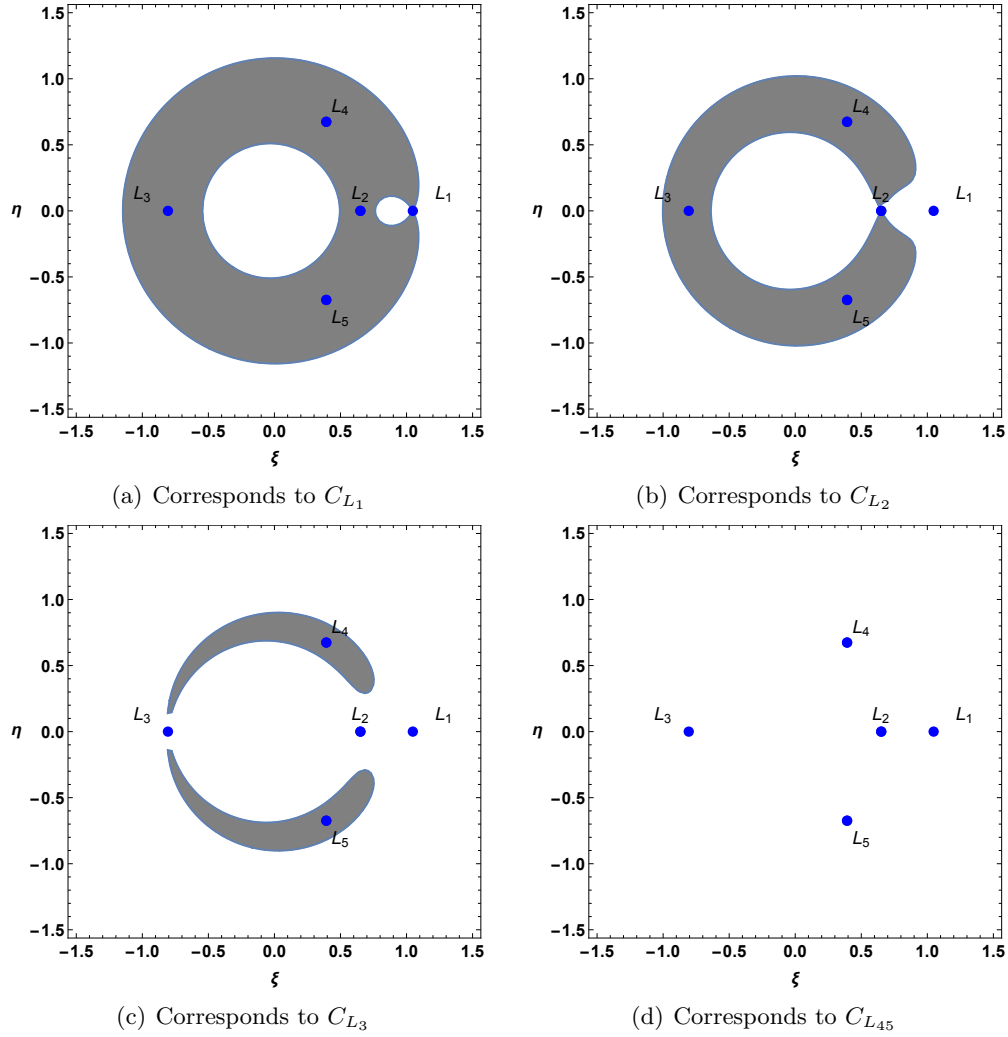


Figure 6: Regions of motion in $\xi - \eta$ -plane corresponding to the variable mass case 5

$L_{1,2,3}$ corresponds to the green color region, L_4 corresponds to the purple color region and L_5 corresponds to the red color region, all these regions extended to infinity. In all the figures black dots are denoting the locations of the attracting points.

4. Stability of equilibrium points

In this section we examine the stability of equilibrium points of the infinitesimal variable mass oblate body in the neighbourhood of $(\xi_0 + \xi_1, \eta_0 + \eta_1, \zeta_0 + \zeta_1)$ under the effect of the oblate radiating primaries and the perturbing factors of

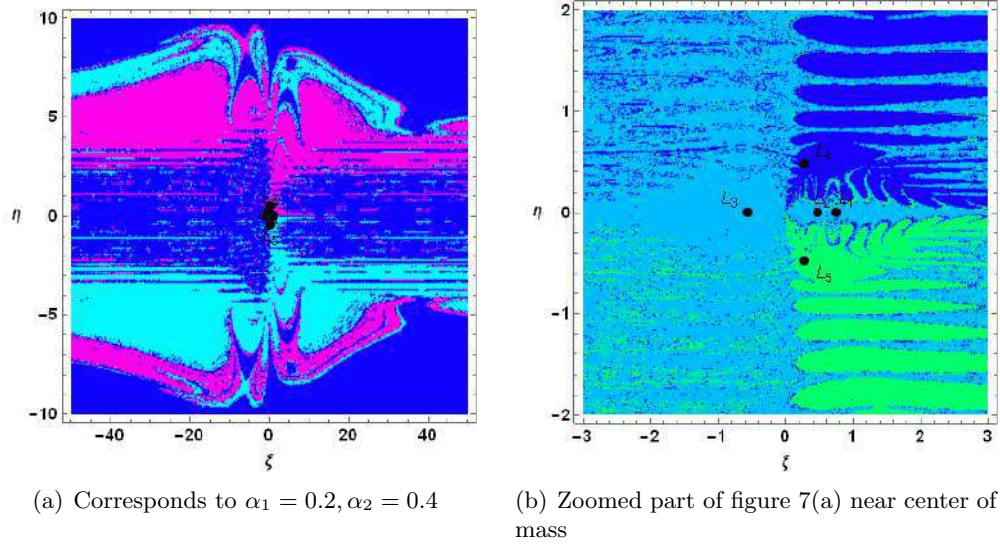


Figure 7: Basins of Attracting domain corresponding to variable mass case 5 in $\xi - \eta$ -plane

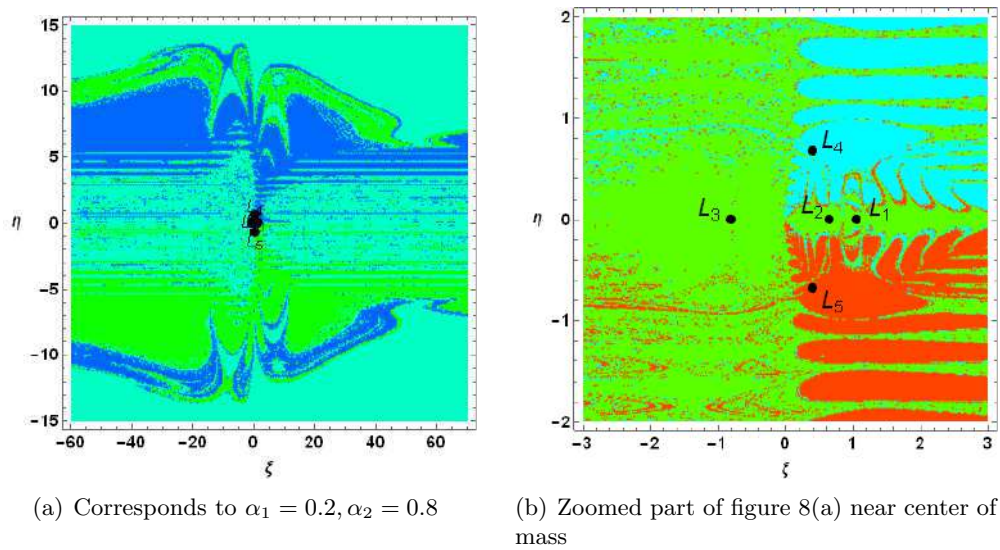


Figure 8: Basins of Attracting domain corresponding to variable mass case 5 in $\xi - \eta$ -plane

Coriolis and centrifugal forces. Where (ξ_1, η_1, ζ_1) are small displacements from the equilibrium points (ξ_0, η_0, ζ_0) .

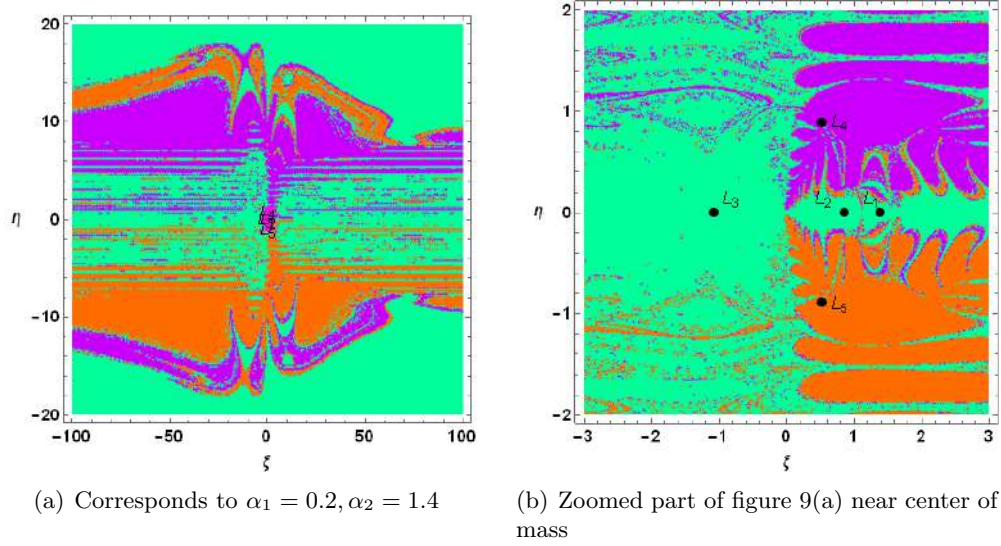


Figure 9: Basins of Attracting domain corresponding to variable mass case 5 in $\xi - \eta$ -plane

The system (4) can be rewritten in the phase space as:

$$\begin{aligned}
 \dot{\xi}_1 &= \xi_2, \\
 \dot{\eta}_1 &= \eta_2, \\
 \dot{\zeta}_1 &= \zeta_2, \\
 \dot{\xi}_2 &= 2n\phi\eta_2 + (\Omega_{\xi\xi})^0\xi_1 + (\Omega_{\xi\eta})^0\eta_1 + (\Omega_{\xi\zeta})^0\zeta_1, \\
 \dot{\eta}_2 &= -2n\phi\xi_2 + (\Omega_{\eta\xi})^0\xi_1 + (\Omega_{\eta\eta})^0\eta_1 + (\Omega_{\eta\zeta})^0\zeta_1, \\
 \dot{\zeta}_2 &= (\Omega_{\zeta\xi})^0\xi_1 + (\Omega_{\zeta\eta})^0\eta_1 + (\Omega_{\zeta\zeta})^0\zeta_1,
 \end{aligned}
 \tag{13}$$

where the superscript 0 indicates the value of the second derivative of potential function at the corresponding equilibrium point (ξ_0, η_0, ζ_0) .

When $\alpha_1 = 0$, then the above system (13) will reduce to the constant mass, while when $\alpha_1 \neq 0$, then we can not examine the stability by ordinary method because the distances of the equilibrium point to the primaries vary with time. Therefore we will use Meshcherskii-space-time inverse transformations which are as follows:

$$\begin{aligned}
 \xi_3 &= \alpha_2^{-1/2}\xi_1, \eta_3 = \alpha_2^{-1/2}\eta_1, \zeta_3 = \alpha_2^{-1/2}\zeta_1, \\
 \xi_4 &= \alpha_2^{-1/2}\xi_2, \eta_4 = \alpha_2^{-1/2}\eta_2, \zeta_4 = \alpha_2^{-1/2}\zeta_2.
 \end{aligned}
 \tag{14}$$

Using the transformations given in Eq. (14), the system (13) can be written as follows:

$$\dot{\chi} = A \chi,
 \tag{15}$$

where

$$(16) \quad \dot{\chi} = \begin{pmatrix} \dot{\xi}_3 \\ \dot{\eta}_3 \\ \dot{\zeta}_3 \\ \dot{\xi}_4 \\ \dot{\eta}_4 \\ \dot{\zeta}_4 \end{pmatrix}, \chi = \begin{pmatrix} \xi_3 \\ \eta_3 \\ \zeta_3 \\ \xi_4 \\ \eta_4 \\ \zeta_4 \end{pmatrix}$$

and

$$(17) \quad A = \begin{pmatrix} \frac{1}{2}\alpha_1 & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{2}\alpha_1 & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2}\alpha_1 & 0 & 0 & 1 \\ (\Omega_{\xi\xi})^0 & (\Omega_{\xi\eta})^0 & (\Omega_{\xi\zeta})^0 & \frac{1}{2}\alpha_1 & 2n\phi & 0 \\ (\Omega_{\eta\xi})^0 & (\Omega_{\eta\eta})^0 & (\Omega_{\eta\zeta})^0 & -2n\phi & \frac{1}{2}\alpha_1 & 0 \\ (\Omega_{\zeta\xi})^0 & (\Omega_{\zeta\eta})^0 & (\Omega_{\zeta\zeta})^0 & 0 & 0 & \frac{1}{2}\alpha_1 \end{pmatrix}$$

The characteristic equation for the matrix A is

$$(18) \quad \lambda^6 + \beta_5 \lambda^5 + \beta_4 \lambda^4 + \beta_3 \lambda^3 + \beta_2 \lambda^2 + \beta_1 \lambda + \beta_0 = 0,$$

where

$$(19) \quad \begin{aligned} \beta_5 &= -3\alpha_1, \\ \beta_4 &= 4n^2\phi^2 - (\Omega_{\xi\xi})^0 - (\Omega_{\eta\eta})^0 - (\Omega_{\zeta\zeta})^0 + \frac{15}{4}\alpha_1^2, \\ \beta_3 &= 2\alpha_1((\Omega_{\xi\xi})^0 + (\Omega_{\eta\eta})^0 + (\Omega_{\zeta\zeta})^0 - 8n^2\phi^2 - \frac{5}{4}\alpha_1^2), \\ \beta_2 &= \frac{1}{64}\alpha_1^2(384n^2\phi^2 + 60\alpha_1^2 - 96((\Omega_{\xi\xi})^0 + (\Omega_{\eta\eta})^0 + (\Omega_{\zeta\zeta})^0)) \\ &\quad - (4n^2\phi^2(\Omega_{\zeta\zeta})^0 + ((\Omega_{\xi\eta})^0)^2 + ((\Omega_{\xi\zeta})^0)^2 - (\Omega_{\xi\xi})^0(\Omega_{\eta\eta})^0 \\ &\quad + ((\Omega_{\eta\zeta})^0)^2 - (\Omega_{\xi\xi})^0(\Omega_{\zeta\zeta})^0 - (\Omega_{\eta\eta})^0(\Omega_{\zeta\zeta})^0), \\ \beta_1 &= \frac{1}{64}\alpha_1^3(32((\Omega_{\xi\xi})^0 + (\Omega_{\eta\eta})^0 + (\Omega_{\zeta\zeta})^0 - 4n^2\phi^2) - 12\alpha_1^2) \\ &\quad + \alpha_1(((\Omega_{\xi\eta})^0)^2 + ((\Omega_{\xi\zeta})^0)^2 - (\Omega_{\xi\xi})^0(\Omega_{\eta\eta})^0 \end{aligned}$$

$$\begin{aligned}
& + ((\Omega_{\eta\zeta})^0)^2 - (\Omega_{\xi\xi})^0(\Omega_{\zeta\zeta})^0 - (\Omega_{\eta\eta})^0(\Omega_{\zeta\zeta})^0 + 4n^2\phi^2(\Omega_{\zeta\zeta})^0, \\
\beta_0 = & \frac{\alpha_1^4}{64}(-4((\Omega_{\xi\xi})^0 + (\Omega_{\eta\eta})^0 + (\Omega_{\zeta\zeta})^0 - 4n^2\phi^2) + \alpha^2) \\
& + \frac{\alpha_1^2}{4}(((\Omega_{\xi\zeta})^0)^2 - ((\Omega_{\xi\eta})^0)^2 + (\Omega_{\xi\xi})^0(\Omega_{\eta\eta})^0 \\
& - ((\Omega_{\eta\zeta})^0)^2 + (\Omega_{\xi\xi})^0(\Omega_{\zeta\zeta})^0 + (\Omega_{\eta\eta})^0(\Omega_{\zeta\zeta})^0 - n^2\phi^2(\Omega_{\zeta\zeta})^0) \\
& + ((\Omega_{\xi\zeta})^0)^2(\Omega_{\eta\eta})^0 - 2(\Omega_{\xi\eta})^0(\Omega_{\xi\zeta})^0(\Omega_{\eta\zeta})^0 + (\Omega_{\xi\xi})^0((\Omega_{\eta\zeta})^0)^2 \\
& + ((\Omega_{\xi\eta})^0)^2(\Omega_{\zeta\zeta})^0 - (\Omega_{\xi\xi})^0(\Omega_{\eta\eta})^0(\Omega_{\zeta\zeta})^0.
\end{aligned}$$

We solved equation (18) numerically for different values of parameters used and evaluated the characteristic roots for each equilibrium points which are given in table 1. From this table we observed that all the equilibrium points are unstable because at-least one characteristic root is either positive real number or positive real part of the complex characteristic root. While in the classical case collinear equilibrium points are always unstable and triangular equilibrium points are stable [28]. Therefore, the variation parameters changes the stability of equilibrium points to instability.

Table 1: The nature of equilibrium points in $\xi - \eta$ -plane at $\alpha_1 = 0.2$, $\alpha_2 = 0.4$, $\mu = 0.03$, $q_1 = 0.85$, $q_2 = 0.95$, $\sigma = 0.001$, $\sigma_1 = 0.003$, $\sigma_2 = 0.005$, $\epsilon_1 = 0.2$, $\epsilon_2 = 0.2$.

<i>EquilibriumPoint</i>		<i>Roots</i>	<i>Nature</i>
$\xi - Co.$	$\eta - Co.$		
0.7408249862	0.0000000000	$0.0999999999 \pm 2.2972215331i$ $0.0999999999 \pm 2.1773781596i$ $- 2.7663246283$ 2.9663246283	<i>Unstable</i>
0.4602562445	0.0000000000	$0.1000000002 \pm 2.0542780618i$ $0.1000000009 \pm 2.2072142439i$ $- 2.4264415713$ 2.6264415713	<i>Unstable</i>
-0.5699707318	0.0000000000	$0.0999999992 \pm 1.1187298304i$ $0.1000000007 \pm 1.4706767196i$ $- 0.1544263628$ 0.3544263628	<i>Unstable</i>
0.2775860568	± 0.4767832700	$0.0999999995 \pm 0.4141521317i$ $0.1000000006 \pm 1.1019981851i$ $0.1000000009 \pm 1.4017456238i$	<i>Unstable</i>

5. Conclusion

The effects of parameters used are studied in the perturbed circular restricted three-body problem where infinitesimal oblate body varies its mass according to Jean's law and both the primaries are having radiation as well as oblateness effects. And the system is also affected by the perturbing factor in Coriolis and centrifugal forces. The equations of motion are evaluated by using Jeans law and Meshcherskii-space time transformations, our determined system differs by the variation parameters (α_1 and α_2) from the classical case 1. We have numerically and graphically shown the equilibrium points, Poincaré surfaces of section, regions of possible and forbidden motion and also basins of attracting domain with the different values of parameters used. Here at most five equilibrium points are obtained, out of which three are collinear (L_1 , L_2 and L_3) and other two are non-collinear (L_4 and L_5)(Figures 1, 2, 3 and 4). As we introduce one by one cases (from case 1 to case 5) in the system (4) for equilibrium points, Figures 1 and 2 show that all the equilibrium points are moving towards origin. Figure 3 represents the locations of equilibrium points with the change in variation parameter α_1 , from here we found that all the equilibrium points are moving toward origin in $\xi - \zeta$ -plane while in $\xi - \eta$ -plane, it has little effect. Figure 4 performs the change in the variation parameter α_2 , and we observed that equilibrium points are moving away from the origin in both $\xi - \eta$ and $\xi - \zeta$ -planes. To study the dynamical behaviour of the infinitesimal oblate body, we have plotted the Poincaré surfaces of section and found regular curves with many islands which appear like a butterfly (Figure 5). Next we have determined the value of Jacobi-constant corresponding to each equilibrium points by using the method given by [15] and illustrated the prohibited and allowed regions for the infinitesimal oblate body. Here gray shaded regions are the forbidden regions (Figure 6) and in the rest part infinitesimal body can move freely. Further more we have drawn the basins of attracting domain for three different values of α_2 and given in Figures (7, 8 and 9). From these figures we obtained different color regions which are showing the different attracting domain. Finally we performed the stability of equilibrium points numerically. This numerical study is given in the table 1. From this table we observed that at least one of the roots have either positive real part of the complex roots or only positive real root. Which confirm that all the equilibrium points are unstable. This result is different from the result obtained in classical case [28], where we found that three equilibrium points are unstable while two are stable. Therefore these variation parameters have great impact on the dynamical behaviour of the motion of the infinitesimal variable mass oblate body.

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Soft ii -mappings in soft topological spaces

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Abstract. In this paper, we have presented new ideas of soft mappings from a soft topological space into another are called soft i -open, soft inter-open and soft ii -open mappings, (soft i -continuous, soft inter-continuous and soft ii -continuous mappings), (soft topological i -homeomorphisms, soft topological inter-homeomorphisms and soft topological ii -homeomorphisms. The relations among these concepts and some different concepts of soft mappings as soft open, soft semi-open and soft α -open mappings (separately, soft continuous, soft semi-continuous and soft α -continuous mappings), (separately, soft topological homeomorphisms, soft topological semi-homeomorphisms and soft topological α -homeomorphisms) are examined by utilizing evidences and guides to clarify and explain it.

Keywords: soft ii -open mappings, soft ii -continuity, soft open sets in soft topological spaces.

1. Introduction

The concepts of semi-open sets, α -open sets were presented in 1963, 1965 (see [17], [22]). Askandar [4] had presented the concept of i -open sets in standard topological spaces. In 2019 (see [19]) the ideas of inter-open, ii -open sets have been presented by Mohammed A.A. and Abdullah B.S. the idea of soft sets and its properties has been presented by Molodtsov and numerous different specialists in 1999, 2003, 2009, 2011, 2014 and 2015 (see [20], [18], [3], [26], [24], [10]). Chen [9] and Kannan [15] introduced the concept of soft semi-open sets and soft α -open sets individually in soft topological spaces. In 2008, Fing [11] defined soft semi-rings to establish the connection between soft sets and semi-rings.

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In 2009, Shabir and Ali [25] studied soft ideals over semi-groups. In 2011, Shabir and Naz [26] initiated the study of soft topological spaces. Many other studies of soft topological spaces have also been studied by many researchers (see [27], [6], [28], [21], [8] and [14]). In 2013, (see [7]) different soft point concepts from the studies in [27], [6], [28], [21], [8] and [14] were introduced. In 2014, Ozturk and Bayramov [23] have used the concepts of soft point in the study [7].

In this work, soft point concepts in the study [28] have been used.

All through this paper (X, τ, E) and (Y, ρ, H) , and continuously indicate soft topological spaces ${}_sT_S$ and $f_{pu} : SS(X_E) \rightarrow SS(Y_H)$ is a soft mapping from (X, τ, E) into (Y, ρ, H) and $u : X \rightarrow Y$, $P : E \rightarrow H$, are mappings. We indicates by $({}_sS)$ to the soft set, $int(K, E)$ and $Cl(K, E)$ denotes soft interior and soft closure of ${}_sS(K, E)$ individually. The individuals from τ are called soft open sets (SOS) of X_S , what's more, its supplements are called soft closed sets (SCS) of X_E . ϕ_E and X_E denote soft null and soft absolute sets individually.

In the fragment 1, we give known fundamental thoughts and aftereffects of the hypothesis of soft sets and soft topological spaces. Additionally, we give basic meanings of soft inter-open; soft i -open and soft ii -open sets (see [5]). In the segment 2, we characterize new ideas of soft open mappings as soft i -open, soft inter-open and soft ii -open mappings and explore its properties. In the third and fourth segment, soft i -continuous, soft inter-continuous and soft ii -continuous mappings, soft topological i -homeomorphisms, soft topological inter-homeomorphisms and soft topological ii -homeomorphisms are characterized and numerous significant outcomes are determined.

Definition 1.1. A ${}_sS$ is named a soft point (sP) in X_E designated by e_K if $\exists x \in X$ and $\exists e \in E$, $K(e) \neq \phi$, $K(e^c = \phi \forall e^c \in E - e$. E_k is said to have a place the soft set (G, E) , $e_K \tilde{\in} (G, E)$, if $\forall e \in E$, $e_K \subseteq G(e)$, the arrangement of each single soft point of X meant by $SP(X)$ ([28]).

Proposition 1.1 ([26]). *Let $(K, A), (L, A) \tilde{\in} SS(X_A)$, then:*

- i) $((K, A) \tilde{\cup} (L, A))^c = (K, A)^c \tilde{\cup} (L, A)^c$
- ii) $((K, A) \tilde{\cap} (L, A))^c = (K, A)^c \tilde{\cap} (L, A)^c$

Theorem 1.1 ([14]). *(K, E) be a soft set in (X, τ, E) , then:*

- i) $Int(K, E)^c = (Cl(K, E))^c$
- ii) $Cl(K, E)^c = (Int(K, E))^c$
- iii) $Int(K, E) = (Cl(K, E))^c$

Proposition 1.2 ([2],[12]). *Let $(F, A), (F_1, A) \tilde{\in} SS(X_E)$ and $(G, B), (G_1, B) \tilde{\in} SS(Y_B)$ at that point the following proclamations are valid:*

- i) *If $(F, A) \tilde{\subseteq} (F_1, A)$, then $f_{pu}(F, A) \tilde{\subseteq} f_{pu}(F_1, A)$*

- ii) $(G, B) \widetilde{\subseteq} (G_1, B)$, then $f_{pu}^{-1}(G, B) \widetilde{\subseteq} f_{pu}^{-1}(G_1, B)$
- iii) $(F, A) \widetilde{\subseteq} f_{pu}^{-1}(f_{pu}(F, A))$
- iv) $f_{pu}(f_{pu}^{-1}(G, B)) \widetilde{\subseteq} (G, B)$
- v) $f_{pu}^{-1}((G, B)^c) \widetilde{\subseteq} (f_{pu}^{-1}(G, B))^c$
- vi) $f_{pu}((F, A) \widetilde{\cap} (F_1, A)) = f_{pu}(F, A) \widetilde{\cap} f_{pu}(F_1, A)$
- vii) $f_{pu}((F, A) \widetilde{\cap} (F_1, A)) \widetilde{\subseteq} f_{pu}(F, A) \widetilde{\cap} f_{pu}(F_1, A)$
- viii) $f_{pu}^{-1}((G, B) \widetilde{\cap} (G_1, B)) = f_{pu}^{-1}(G, B) \widetilde{\cap} f_{pu}^{-1}(G_1, B)$
- ix) $f_{pu}^{-1}((G, B) \widetilde{\cap} (G_1, B)) = f_{pu}^{-1}(G, B) \widetilde{\cap} f_{pu}^{-1}(G_1, B)$

Definition 1.2. Let (F, E) be a soft set $({}_S S)$ in (X, τ, E) , then (F, E) considers:

- i) Soft Inter-open set (${}_S IntO_S$) if there exists a ${}_S O_S(O, E) \neq \phi, X$ wherein $Int(F, E) = (O, E)$ ([5]).
- ii) Soft i-open set (${}_S IO_S$) if there exists a ${}_S O_S(O, E) \neq \phi, X$ wherein $(F, E) \widetilde{\subseteq} Cl((F, E) \widetilde{\cap} (O, E))$ (see [5]).
- iii) Soft ii-open set (${}_S IIO_S$) if there exists a ${}_S O_S(O, E) \neq \phi, X$ wherein:
 - a) $(F, E) \widetilde{\subseteq} Cl((F, E) \widetilde{\cap} (O, E))$
 - b) $Int(F, E) = (O, E)$. As it were an ${}_S S(F, E)$ is named ${}_S IIO_S$ if it is ${}_S IntO_S$ and ${}_S IO_S$ together (see [5]).
- iv) Soft semi-open set ${}_S SO_S$ if:
 - a) $(F, E) \widetilde{\subseteq} Cl(Int(F, E))$.
 - b) If there exists a ${}_S O_S(O, E) \neq \phi, X$ wherein $(O, E) \widetilde{\subseteq} (F, E) \widetilde{\subseteq} Cl(O, E)$ (see [9]).
- v) Soft α -open set (${}_S \alpha O_S$, if $(F, E) \widetilde{\subseteq} Int(Cl(Int(F, E)))$ (see [15]).

The complement of ${}_S IIO_S$ (resp., ${}_S IntO_S, {}_S IO_S, {}_S SO_S$ and ${}_S \alpha O_S$) is called soft ii-closed (${}_S IIC_S$) (resp., soft int-closed (${}_S IntC_S$), soft i-closed (${}_S IC_S$), soft semi-closed (${}_S SC_S$) and soft α -closed (${}_S \alpha C_S$)). The intersection of all ${}_S IIC_S$ (resp., ${}_S IntC_S, {}_S IC_S, {}_S SC_S$ and ${}_S \alpha C_S$) over X containing (F, E) is called the soft ii-closure (resp., soft int-closure, soft i-closure, soft semi-closure and soft α -closure) of (F, E) and designated by $IICl(F, E)$ (resp., $INTCl(F, E), IC_l(F, E), SC_l(F, E)$ and $\alpha Cl(F, E)$). The union of all ${}_S IIO_S$ (resp., ${}_S IntO_S, {}_S IO_S, {}_S SO_S$ and ${}_S \alpha O_S$) over X contained in (F, E) is named a soft II-interior (resp., soft INT-interior, soft I-interior, soft Semi-interior and soft α -interior) of a soft set (F, E) and denoted by $IIInt(F, E)$ (resp., $INTInt(F, E), IInt(F, E), SInt(F, E)$ and

$\alpha Int(F, E)$. The collection of all SO_S (resp., $sIIO_S$, $sIntO_S$, sIO_S , sSO_S and $s\alpha O_S$, sC_S , $sIIC_S$, $sIntC_S$, sIC_S , sSC_S and $s\alpha C_S$) in (X, τ, E) are denoted by $(SO_S(X_E))$ (resp., $sIntO_S(X_E)$, $sIntO_S(X_E)$, $sIO_S(X_E)$, $sSO_S(X_E)$, $s\alpha O_S(X_E)$, $sC_S(X_E)$, $sIIC_S(X_E)$, $sIntC_S(X_E)$, $sIC_S(X_E)$, $sSC_S(X_E)$ and $s\alpha C_S(X_E)$).

Definition 1.3. An $sS(G, E)$ in $sT_S(X, \tau, E)$ is named a soft ii-neighborhood of an $sP, F(E)$ if there exists a $sIIO_S(G_1, E)$ wherein $F(e) \widetilde{\in} (G_1, E) \widetilde{\subseteq} (G_1, E)$. An $sS(G, E)$ in a $sT_S(X, \tau, E)$, is named a soft ii-neighborhood of an $sS(F, E)$ if there exists a $sIIO_S(G_1, E)$ wherein $(F, E) \widetilde{\subseteq} (G_1, E) \widetilde{\subseteq} (G_1, E)$. The ii-neighborhood system of a $sPF(e)$ designated by $IIN_\tau(F(e))$ is the group of all its ii-neighborhoods.

Definition 1.4. Consider $f_{pu} : SS(X_E) \rightarrow SS(Y_H), u : X \rightarrow Y, p : E \rightarrow H$, be mappings. Let $e_F \in SP(X)$. Then f_{pu} is a soft ii-pu-continuous at e_F if for each $(G, H) \widetilde{\in} N_\rho(f_{pu}(e_F))$, there exists $(G_1, E) \widetilde{\in} IIN_\tau(e_F)$ wherein $f_{pu}(G_1, E) \widetilde{\subseteq} (G, H)$, f_{pu} is a soft ii-pu-continuous on X if it is a soft ii-pu-continuous at each soft point in X .

2. Soft ii-open mappings

Definition 2.1. Consider a mappings $f_{pu} : SS(X_E) \rightarrow SS(Y_H), u : X \rightarrow Y, p : E \rightarrow H$. Then a mapping f_{pu} is named:

- i) Soft open (sOm) if $f_{pu}(F, E) \widetilde{\in} SO_S(Y_H)$, for each $(F, E) \widetilde{\in} SO_S(X_E)$ ([28]).
- ii) Soft closed (sCm) if $f_{pu}(F, E) \widetilde{\in} SC_S(Y_H)$, for each $(F, E) \widetilde{\in} SC_S(X_E)$ ([28]).
- iii) Soft semi-open ($sSOM$) if $f_{pu}(F, E) \widetilde{\in} SO_S(Y_H)$, for each $(F, E) \widetilde{\in} SO_S(X_E)$ ([16]).
- iv) Soft semi-closed ($sSCm$) if $f_{pu}(F, E) \widetilde{\in} SC_S(Y_H)$, for each $(F, E) \widetilde{\in} SC_S(X_E)$ ([16]).
- v) Soft α -open ($s\alpha Om$) if $f_{pu}(F, E) \widetilde{\in} s\alpha O_S(Y_H)$, for each $(F, E) \widetilde{\in} SO_S(X_E)$ ([16]).
- vi) Soft α -closed ($s\alpha Cm$) if $f_{pu}(F, E) \widetilde{\in} s\alpha C_S(Y_H)$, for each $(F, E) \widetilde{\in} SC_S(X_E)$ ([16]).
- vii) Soft i-open ($sIOM$) if $f_{pu}(F, E) \widetilde{\in} IO_S(Y_H)$, for each $(F, E) \widetilde{\in} SO_S(X_E)$.
- viii) Soft i-closed ($sICm$) if $f_{pu}(F, E) \widetilde{\in} IC_S(Y_H)$, for each $(F, E) \widetilde{\in} SC_S(X_E)$.
- ix) Soft inter-open ($sIntOm$) if $f_{pu}(F, E) \widetilde{\in} sIntO_S(Y_H)$, for each $(F, E) \widetilde{\in} SO_S(X_E)$.
- x) Soft inter-closed ($sIntCm$) if $f_{pu}(F, E) \widetilde{\in} sIntC_S(Y_H)$, for each $(F, E) \widetilde{\in} SC_S(X_E)$.

- xi) Soft ii-open ($_{S}IIOm$) if $f_{pu}(F, E) \widetilde{\in}_{S} IIOS(Y_H)$, for each $(F, E) \widetilde{\in}_{S} OS(X_E)$.
- xii) Soft ii-closed ($_{S}IICm$) if $f_{pu}(F, E) \widetilde{\in}_{S} IICS(Y_H)$, for each $(F, E) \widetilde{\in}_{S} CS(X_E)$.

Theorem 2.1. *Each $_{S}Om$ is a $_{S}IOm$.*

Proof. Assume that a function $f_{pu} : SS(X_E) \rightarrow SS(Y_H)$ be a $_{S}Om$. Let (G, E) be an $_{S}OS$ in (X, τ, E) , we have $f_{pu}(G, E)$ is a $_{S}OS$ in (Y, ρ, H) (by assume). Since each $_{S}OS$ is a $_{S}IOS$ ([5], Theorem 3.2) we have $f_{pu}(G, E)$ is a $_{S}IOS$ in (Y, ρ, H) . Hence f_{pu} is a $_{S}IOm$. $\square$

Example 2.1. Let $X = \{1, 3, 5\}, Y = \{2, 4, 6\}, E = \{l, w\}$ and $H = \{l', w'\}$. Obviously $\tau = \{\phi_E, X_E, (F_1, E)\}$ and $\rho = \{\phi_H, Y_H, (G_1, H), (G_2, H)\}$ are $_{S}TS$ over X and Y respectively, where $(F_1, E) = \{(l, \{1, 3\}), (w, \{1, 3\})\}$, $(G_1, H) = \{(l', \{4\}), (w', \{4\})\}$, $(G_2, H) = \{(l', \{2, 4\}), (w', \{2, 4\})\}$. Presently characterize the mapping $f_{pu} : SS(X_E) \rightarrow SS(Y_H)$ where $u : X \rightarrow Y, p : E \rightarrow H$ are characterized by $p(l) = l', p(w) = w', u(1) = 4, u(3) = 6, u(5) = 2$. $_{S}IO - OS(Y_H) = \{\phi_H, Y_H, (G_1, H), (G_2, H), \{(l', 2), (w', 2)\}, \{(l', \{4, 6\}), (w', \{4, 6\})\}\}$. Apparently, f_{pu} isn't a $_{S}Om$ since (F_1, E) is $_{S}OS$ in (X, τ, E) but $f_{pu}(F_1, E) = \{(l', \{4, 6\}), (w', \{4, 6\})\}$ isn't $_{S}OS$ in (Y, ρ, H) . Hence f_{pu} is a $_{S}IOm$.

Theorem 2.2. *Each $_{S}SOM$ is $_{S}IOm$*

Proof. Assume that a function $f_{pu} : SS(X_E) \rightarrow SS(Y_H)$ be a $_{S}SOM$. Let (G, E) be an $_{S}OS$ in (X, τ, E) , we have $f_{pu}(G, E)$ is a $_{S}SOS$ in (Y, ρ, H) (by assume). Since each $_{S}SOS$ is a $_{S}IOS$ ([5], Theorem 3.5), we have $f_{pu}(G, E)$ is a $_{S}IOS$ in (Y, ρ, H) . Hence f_{pu} is a $_{S}IOm$. $\square$

Example 2.2. Let $X = \{1, 3, 5\}, Y = \{2, 4, 6\}, E = \{l, w\}$ and $H = \{l', w'\}$. Obviously $\tau = \{\phi_E, X_E, (F_1, E)\}$ and $\rho = \{\phi_H, Y_H, (G_1, H)\}$, where $(F_1, E) = \{(l, \{1, 3\}), (w, \{1, 3\})\}$, $(G_1, H) = \{(l', \{2, 4\}), (w', \{2, 4\})\}$. Presently characterize the mapping $f_{pu} : SS(X_E) \rightarrow SS(Y_H)$ where $u : X \rightarrow Y, p : E \rightarrow H$ are characterized by $p(l) = l', p(w) = w', u(1) = 2, u(3) = 6, u(5) = 4$. Essentially as in the example referenced previously, f_{pu} is not a $_{S}SOM$ since (F_1, E) is $_{S}OS$ in (X, τ, E) but $f_{pu}(F_1, E) = \{(l', \{2, 6\}), (w', \{2, 6\})\}$ isn't $_{S}SOS$ in (Y, ρ, H) . Hence, f_{pu} is a $_{S}IOm$.

Theorem 2.3. *Any $_{S}\alpha Om$ is $_{S}SOM$*

Proof. Assume that a function $f_{pu} : SS(X_E) \rightarrow SS(Y_H)$ be a $_{S}\alpha Om$. Let (G, E) be an $_{S}OS$ in (X, τ, E) , we have $f_{pu}(G, E)$ is a $_{S}\alpha OS$ in (Y, ρ, H) (by assume). Since each $_{S}\alpha OS$ is a $_{S}SOS$ ([5], Theorem 3.7), we have $f_{pu}(G, E)$ is a $_{S}SOS$ in (Y, ρ, H) . Hence f_{pu} is a $_{S}SOM$ (see [16]). $\square$

Example 2.3. Let $X = \{1, 3, 5\}, Y = \{2, 4, 6\}, E = \{l, w\}$ and $H = \{l', w'\}$. Obviously $\tau = \{\phi_E, X_E, (F_1, E)\}$ and $\rho = \{\phi_H, Y_H, (G_1, H), (G_2, H), (G_3, H)\}$, where $(F_1, E) = \{(l, \{1, 3\}), (w, \{1, 3\})\}$, $(G_1, H) = \{(l', \{2\}), (w', \{2\})\}$, $(G_2, H) = \{(l', \{4\}), (w', \{4\})\}$, $(G_3, H) = \{(l', \{2, 4\}), (w', \{2, 4\})\}$. Presently

characterize the mapping $f_{pu} : SS(X_E) \rightarrow SS(Y_H)$ where $u : X \rightarrow Y$, $p : E \rightarrow H$ are characterized by $p(l) = l', p(w) = w', u(1) = 2, u(3) = 6, u(5) = 4$. Obviously, f_{pu} is not a $s\alpha Om$ since (F_1, E) is sOs in (X, τ, E) but $f_{pu}(F_1, E) = \{(l', \{2, 6\}), (w', \{2, 6\})\}$ isn't $s\alpha Os$ in (Y, ρ, H) . Hence f_{pu} is a $sSOM$.

Corollary 2.1. *Each $s\alpha Om$ is $sIOM$*

Proof. Assume that a function $f_{pu} : SS(X_E) \rightarrow SS(Y_H)$ be a $s\alpha Om$. We have f_{pu} is an $sSOM$. Hence f_{pu} is a $sIOM$. $\square$

Example 2.4. Let $X = \{1, 3, 5\}, Y = \{2, 4, 6\}, E = \{l, w\}$ and $H = \{l', w'\}$. Obviously $\tau = \{\phi_E, X_E, (F_1, E)\}$ and $\rho = \{\phi_H, Y_H, (G_1, H)\}$, where $(F_1, E) = \{(l, \{1, 3\}), (w, \{1, 3\})\}$, $(G_1, H) = \{(l', \{2, 4\}), (w', \{2, 4\})\}$. Presently characterize the mapping $f_{pu} : SS(X_E) \rightarrow SS(Y_H)$ where $u : X \rightarrow Y$, $p : E \rightarrow H$ are characterized by $p(l) = l', p(w) = w', u(1) = 2, u(3) = 6, u(5) = 4$. Obviously, f_{pu} is not a $s\alpha Om$ since (F_1, E) is sOs in (X, τ, E) but $f_{pu}(F_1, E) = \{(l', \{2, 6\}), (w', \{2, 6\})\}$ isn't $s\alpha Os$ in (Y, ρ, H) . Hence f_{pu} is a $sIOM$.

Theorem 2.4. *Any $sIIOM$ is $sIOM$, also it is a $sIntOm$*

Proof. Assume that a function $f_{pu} : SS(X_E) \rightarrow SS(Y_H)$ be a $sIIOM$. Let (G, E) be an sOs in (X, τ, E) , we have $f_{pu}(G, E)$ is a $sIIOS$ in (Y, ρ, H) (by assume). Since each $sIIOS$ is a $sIOS$ and $sIntOS$ ([5], Definition 3.1(3) and Remark 3.19), we have $f_{pu}(G, E)$ is a $sIOS$ and $sIntOS$ in (Y, ρ, H) . Hence f_{pu} is a $sIOM$, also it is a $sIntOm$. $\square$

Example 2.5. Let $X = \{1, 3, 5\}, Y = \{2, 4, 6\}, E = \{l, w\}$ and $H = \{l', w'\}$. Obviously $\tau = \{\phi_E, X_E, (F_1, E)\}$ and $\rho = \{\phi_H, Y_H, (G_1, H), (G_2, H)\}$, where $(F_1, E) = \{(l, \{1, 5\}), (w, \{1, 5\})\}$, $(G_1, H) = \{(l', \{2\}), (w', \{2\})\}$ and $(G_2, H) = \{(l', \{2, 6\}), (w', \{2, 6\})\}$. Presently characterize the mapping $f_{pu} : SS(X_E) \rightarrow SS(Y_H)$ where $u : X \rightarrow Y$, $p : E \rightarrow H$ are characterized by $p(l) = l', p(w) = w', u(1) = 4, u(3) = 2, u(5) = 6$. Obviously, f_{pu} is not a $sIIOM$ since (F_1, E) is sOs in (X, τ, E) but $f_{pu}(F_1, E) = \{(l', \{4, 6\}), (w', \{4, 6\})\}$ isn't $sIIOS$ in (Y, ρ, H) . Hence f_{pu} is a $sIOM$.

Example 2.6. Let $X = \{1, 3, 5\}, Y = \{2, 4, 6\}, E = \{l, w\}$ and $H = \{l', w'\}$. Obviously $\tau = \{\phi_E, X_E, (F_1, E)\}$ and $\rho = \{\phi_H, Y_H, (G_1, H), (G_2, H)\}$, where $(F_1, E) = \{(l, \{3, 5\}), (w, \{3, 5\})\}$, $(G_1, H) = \{(l', \{2\}), (w', \{2\})\}$ and $(G_2, H) = \{(l', \{4, 6\}), (w', \{4, 6\})\}$. Presently characterize the mapping $f_{pu} : SS(X_E) \rightarrow SS(Y_H)$ where $u : X \rightarrow Y$, $p : E \rightarrow H$ are characterized by $p(l) = l', p(w) = w', u(1) = 4, u(3) = 2, u(5) = 6$. Obviously, f_{pu} is not a $sIIOM$ since (F_1, E) is sOs in (X, τ, E) , but $f_{pu}(F_1, E) = \{(l', \{2, 6\}), (w', \{2, 6\})\}$ isn't $sIIOS$ in (Y, ρ, H) . Hence f_{pu} is a $sIntOm$.

Theorem 2.5. *Each $sSOM$ is $sIIOM$.*

Proof. Assume that a function $f_{pu} : SS(X_E) \rightarrow SS(Y_H)$ be a $sSom$. Let (G, E) be an sOS in (X, τ, E) , we have $f_{pu}(G, E)$ is a $sSOS$ in (Y, ρ, H) (by assume). Since each $sSOS$ is a $sIIOS$ ([5], Theorem 3.11), we have $f_{pu}(G, E)$ is a $sIIOS$ in (Y, ρ, H) . Hence f_{pu} is a $sIIOm$. $\square$

Theorem 2.6. Any sOm is $sIIOm$.

Proof. Assume that a function $f_{pu} : SS(X_E) \rightarrow SS(Y_H)$ be a sOm . Let (G, E) be an sOS in (X, τ, E) , we have $f_{pu}(G, E)$ is a sOS in (Y, ρ, H) (by assume). Since each sOS is a $sIIOS$ ([5], Theorem 3.12), we have $f_{pu}(G, E)$ is a $sIIOS$ in (Y, ρ, H) . Hence f_{pu} is a $sIIOm$. $\square$

Example 2.7. Let $X = \{1, 3, 5\}, Y = \{2, 4, 6\}, E = \{l, w\}$ and $H = \{l', w'\}$. Obviously $\tau = \{\phi_E, X_E, (F_1, E)\}$ and $\rho = \{\phi_H, Y_H, (G_1, H), (G_2, H)\}$, where $(F_1, E) = \{(l, \{1, 5\}), (w, \{1, 5\})\}$, $(G_1, H) = \{(l', \{4\}), (w', \{4\})\}$ and $(G_2, H) = \{(l', \{4, 6\}), (w', \{4, 6\})\}$. Presently characterize the mapping $f_{pu} : SS(X_E) \rightarrow SS(Y_H)$ where $u : X \rightarrow Y, p : E \rightarrow H$ are characterized by $p(l) = l', p(w) = w', u(1) = 2, u(3) = 6, u(5) = 4$. Obviously, f_{pu} is not a sOm since (F_1, E) is sOS in (X, τ, E) , but $f_{pu}(F_1, E) = \{(l', \{2, 4\}), (w', \{2, 4\})\}$ isn't sOS in (Y, ρ, H) . Hence f_{pu} is a $sIIOm$.

Theorem 2.7. Each $s\alpha Om$ is $sIIOm$.

Proof. Assume that a function $f_{pu} : SS(X_E) \rightarrow SS(Y_H)$ be a $s\alpha Om$. Let (G, E) be an sOS in (X, τ, E) , we have $f_{pu}(G, E)$ is a $s\alpha OS$ in (Y, ρ, H) (by assume). Since each $s\alpha OS$ is a $sIIOS$ ([5], Theorem 3.15), we have $f_{pu}(G, E)$ is a $sIIOS$ in (Y, ρ, H) . Hence f_{pu} is a $sIIOm$. $\square$

Example 2.8. Let $X = \{1, 3, 5, 7\}, Y = \{2, 4, 6, 8\}, E = \{l, w\}$ and $H = \{l', w'\}$.

Obviously $\tau = \{\phi_E, X_E, (F_1, E)\}$ and $\rho = \{\phi_H, Y_H, (G_1, H), (G_2, H), (G_3, H)\}$, where $(F_1, E) = \{(l, \{3, 5, 7\}), (w, \{3, 5, 7\})\}$, $(G_1, H) = \{(l', \{2\}), (w', \{2\})\}$ and $(G_2, H) = \{(l', \{6, 8\}), (w', \{6, 8\})\}$, $(G_3, H) = \{(l', \{2, 6, 8\}), (w', \{2, 6, 8\})\}$.

Presently characterize the mapping $f_{pu} : SS(X_E) \rightarrow SS(Y_H)$ where $u : X \rightarrow Y, p : E \rightarrow H$ are characterized by $p(l) = l', p(w) = w', u(1) = 4, u(3) = 2, u(5) = 6$ and $u(7) = 8$. Obviously, f_{pu} is not a $s\alpha Om$ since (F_1, E) is sOS in (X, τ, E) , but $f_{pu}(F_1, E) = \{(l', \{4, 6, 8\}), (w', \{4, 6, 8\})\}$ isn't $s\alpha OS$ in (Y, ρ, H) . Hence f_{pu} is a $sIIOm$.

Corollary 2.2. Any $s\alpha Om$ is a $sIntOm$.

Proof. Clear. $\square$

Example 2.9. Let $X = \{1, 3, 5\}, Y = \{2, 4, 6\}, E = \{l, w\}$ and $H = \{l', w'\}$. Obviously $\tau = \{\phi_E, X_E, (F_1, E)\}$ and $\rho = \{\phi_H, Y_H, (G_1, H), (G_2, H)\}$, where $(F_1, E) = \{(l, \{1, 5\}), (w, \{1, 5\})\}$, $(G_1, H) = \{(l', \{2\}), (w', \{2\})\}$ and $(G_2, H) = \{(l', \{4, 6\}), (w', \{4, 6\})\}$. Presently characterize the mapping $f_{pu} : SS(X_E) \rightarrow$

$SS(Y_H)$ where $u : X \rightarrow Y$, $p : E \rightarrow H$ are characterized by $p(l) = l'$, $p(w) = w'$, $u(1) = 2$, $u(3) = 6$, $u(5) = 4$. Obviously, f_{pu} is not a $s\alpha Om$ since (F_1, E) is sOs in (X, τ, E) , but $f_{pu}(F_1, E) = \{(l', \{2, 4\}), (w', \{2, 4\})\}$ isn't $s\alpha Os$ in (Y, ρ, H) . Hence f_{pu} is a $sIntOm$.

3. Soft ii-continuous mappings

Definition 3.1. Consider a mappings $f_{pu} : SS(X_E) \rightarrow SS(Y_H)$, $u : X \rightarrow Y$, $p : E \rightarrow H$. Then a mapping f_{pu} is called:

- i) Soft continuous ($sCONm$) if $f_{pu}^{-1}(G, H) \tilde{\in} sOs(X_E)$, $\forall (G, H) \tilde{\in} sOs(Y_H)$ (see [28]).
- ii) Soft α -continuous ($s\alpha CONm$) if $f_{pu}^{-1}(G, H) \tilde{\in} s\alpha Os(X_E)$, $\forall (G, H) \tilde{\in} sOs(Y_H)$ (see [16]).
- iii) Soft semi-continuous ($sSCONm$) if $f_{pu}^{-1}(G, H) \tilde{\in} sSOs(X_E)$, $\forall (G, H) \tilde{\in} sOs(Y_H)$ (see [16]).
- iv) Soft i-continuous ($sICONm$) if $f_{pu}^{-1}(G, H) \tilde{\in} sIOs(X_E)$, $\forall (G, H) \tilde{\in} sOs(Y_H)$.
- v) Soft inter-continuous ($sINTCONm$) if $f_{pu}^{-1}(G, H) \tilde{\in} sIntOs(X_E)$, $\forall (G, H) \tilde{\in} sOs(Y_H)$.
- vi) Soft ii-continuous ($sIICONm$) if $f_{pu}^{-1}(G, H) \tilde{\in} sIIOs(X_E)$, $\forall (G, H) \tilde{\in} sOs(Y_H)$.

Theorem 3.1. Each $sCONm$ is $sICONm$

Proof. Assume that a function $f_{pu} : SS(X_E) \rightarrow SS(Y_H)$ be a $sCONm$. Let (G, H) be an sOs in (Y, ρ, H) , we have $f_{pu}^{-1}(G, H)$ is a sOs in (X, τ, E) (by assume). Since each sOs is a $sIOs$ ([5], Theorem 3.2), we have $f_{pu}^{-1}(G, H)$ is a $sIOs$ in (X, τ, E) . Hence f_{pu} is a $sICONm$. $\square$

Example 3.1. Let $X = \{1, 3, 5, 7\}$, $Y = \{2, 4\}$, $E = \{l, w\}$ and $H = \{l', w'\}$. Obviously $\tau = \{\phi_E, X_E, (F_1, E), (F_2, E), (F_3, E)\}$ and $\rho = \{\phi_H, Y_H, (G_1, H), (G_2, H)\}$, are sTs over X and Y individually where $(F_1, E) = \{(l, \{3\}), (w, \{3\})\}$, $(F_2, E) = \{(l, \{5, 7\}), (w, \{5, 7\})\}$, $(F_3, E) = \{(l, \{3, 5, 7\}), (w, \{3, 5, 7\})\}$, $(G_1, H) = \{(l', \{2\}), (w', \{2\})\}$ and $(G_2, H) = \{(l', \{4\}), (w', \{4\})\}$. Presently characterize the mapping $f_{pu} : SS(X_E) \rightarrow SS(Y_H)$ where $u : X \rightarrow Y$, $p : E \rightarrow H$ are characterized by $p(l) = l'$, $p(w) = w'$, $u(1) = 2$, $u(5) = 2$, $u(7) = 2$, $u(3) = 4$. $sIOs(X_E) = \{\phi_E, X_E, (F_1, E), (F_2, E), (F_3, E), \{(l, \{1, 5, 7\}), (w, \{1, 5, 7\})\}\}$.

Plainly, f_{pu} is not a $sCONm$ since (G_1, H) is sOs in (Y, ρ, H) , but $f_{pu}^{-1}(G_1, H) = \{(l, u^{-1}(G_1(p(l))))\} = \{(l, u^{-1}(G_1(p(w))))\} = \{(l, u^{-1}(G_1(p(l'))))\}$, $\{(w, u^{-1}(G_1(p(w'))))\} = \{(l, u^{-1}(\{2\}))\}, \{(w, u^{-1}(\{2\}))\} = \{(l, \{1, 5, 7\})\}, \{(w, \{1, 5, 7\})\}$, which isn't a sOs in (X, τ, E) . Be that as it may f_{pu} is a $sIICONm$.

Theorem 3.2. *Each $sSCONm$ is $sICONm$*

Proof. Consider $f_{pu} : SS(X_E) \rightarrow SS(Y_H)$ be a $sSCONm$. Let (G, H) be an sOS in (Y, ρ, H) , we have $f_{pu}^{-1}(G, H)$ is a sSO_S in (X, τ, E) (by consider). Since each sSO_S is a $sIOS$ ([5], Theorem 3.5), we have $f_{pu}^{-1}(G, H)$ is a $sIOS$ in (X, τ, E) . Hence f_{pu} is a $sICONm$. $\square$

Example 3.2. Let $X = \{1, 3, 5, 7\}, Y = \{2, 4\}, E = \{l, w\}$ and $H = \{l', w'\}$. Obviously $\tau = \{\phi_E, X_E, (F_1, E)\}$ and $\rho = \{\phi_H, Y_H, (G_1, H)\}$, where $(F_1, E) = \{(l, \{5, 7\}), (w, \{5, 7\})\}$, and $(G_1, H) = \{(l', \{4\}), (w', \{4\})\}$.

Presently characterize the mapping $f_{pu} : SS(X_E) \rightarrow SS(Y_H)$ where $u : X \rightarrow Y, p : E \rightarrow H$ are characterized by $p(l) = l', p(w) = w', u(1) = 2, u(3) = 2, u(7) = 2, u(5) = 4$. Plainly, f_{pu} is not a $sSCONm$ since (G_1, H) is sOS in (Y, ρ, H) , but $f_{pu}^{-1}(G_1, H) = \{(l, u^{-1}(G_1(p(l))))\}, (w, u^{-1}(G_1(p(w))))\} = \{(l, u^{-1}(G_1(p(l'))))\}, (w, u^{-1}(G_1(p(w'))))\} = \{(l, u^{-1}(\{4\}))\}, (w, u^{-1}(\{4\}))\} = \{(l, \{5\}), (w, \{5\})\}$, which isn't a sSO_S in (X, τ, E) . Hence f_{pu} is a $sICONm$.

Theorem 3.3. *Any $s\alpha CONm$ is $sICONm$*

Proof. Assume that a function $f_{pu} : SS(X_E) \rightarrow SS(Y_H)$ be a $s\alpha CONm$. Let (G, H) be an sOS in (Y, ρ, H) , we have $f_{pu}^{-1}(G, H)$ is a $s\alpha OS$ in (X, τ, E) (by assume). Since each $s\alpha OS$ is a $sIOS$ ([5], Corollary 3.9), we have $f_{pu}^{-1}(G, H)$ is a $sIOS$ in (X, τ, E) . Hence f_{pu} is a $sICONm$. $\square$

In Example 3.2, we have f_{pu} isn't a $s\alpha CONm$ since (G_1, H) is a sOS in (Y, ρ, H) , but $f_{pu}^{-1}(G_1, H) = \{(l, u^{-1}(G_1(p(l))))\}, (w, u^{-1}(G_1(p(w))))\} = \{(l, u^{-1}(G_1(p(l'))))\}, (w, u^{-1}(G_1(p(w'))))\} = \{(l, u^{-1}(\{4\}))\}, (w, u^{-1}(\{4\}))\} = \{(l, \{5\}), (w, \{5\})\}$, which isn't a $s\alpha OS$ in (X, τ, E) . But f_{pu} is a $sICONm$.

Theorem 3.4. *Each $s\alpha CONm$ is $sSCONm$.*

Proof. Consider $f_{pu} : SS(X_E) \rightarrow SS(Y_H)$ be a $s\alpha CONm$. Let (G, H) be an sOS in (Y, ρ, H) , we have $f_{pu}^{-1}(G, H)$ is a $s\alpha OS$ in (X, τ, E) (by consider). Since each $s\alpha OS$ is a sSO_S ([5], Theorem 3.7), we have $f_{pu}^{-1}(G, H)$ is a sSO_S in (X, τ, E) . Hence f_{pu} is a $sSCONm$ (see [16]). $\square$

Example 3.3. Let $X = \{1, 3, 5\}, Y = \{2, 4, 6\}, E = \{l, w\}$ and $H = \{l', w'\}$. Obviously $\tau = \{\phi_E, X_E, (F_1, E), (F_2, E), (F_3, E)\}$ and $\rho = \{\phi_H, Y_H, (G_1, H)\}$, where $(F_1, E) = \{(l, \{1\}), (w, \{1\})\}, (F_2, E) = \{(l, \{3\}), (w, \{3\})\}, (F_3, E) = \{(l, \{1, 3\}), (w, \{1, 3\})\}$ and $(G_1, H) = \{(l', \{2, 4\}), (w', \{2, 4\})\}$. Presently characterize the mapping $f_{pu} : SS(X_E) \rightarrow SS(Y_H)$ where $u : X \rightarrow Y, p : E \rightarrow H$ are characterized by $p(l) = l', p(w) = w', u(1) = 2, u(3) = 6, u(5) = 4$.

Obviously, f_{pu} is not a $s\alpha CONm$ since (G_1, H) is sOS in (Y, ρ, H) , but $f_{pu}^{-1}(G_1, H) = \{(l, u^{-1}(G_1(p(l))))\}, (w, u^{-1}(G_1(p(w))))\} = \{(l, u^{-1}(G_1(p(l'))))\}, (w, u^{-1}(G_1(p(w'))))\} = \{(l, u^{-1}(\{2, 4\}))\}, (w, u^{-1}(\{2, 4\}))\} = \{(l, \{1, 5\}), (w, \{1, 5\})\}$, which isn't a $s\alpha OS$ in (X, τ, E) . Hence f_{pu} is a $sSCONm$.

Theorem 3.5. *Each $sIICONm$ is $sICONm$ and a $sINTCONm$*

Proof. Assume that a function $f_{pu} : SS(X_E) \rightarrow SS(Y_H)$ be a $sIICONm$. Let (G, H) be an sOS in (Y, ρ, H) , we have $f_{pu}^{-1}(G, H)$ is a $sIIOS$ in (X, τ, E) (by assume). Since each $sIIOS$ is a $sIOS$ and a $sIntOS$ ([5], Definition 3.1(1, 2 and 3)) and "Remark 3.19), we have $f_{pu}^{-1}(G, H)$ is a $sIOS$ and a $sIntOS$ in (X, τ, E) . Hence f_{pu} is a $sICONm$ and a $sINTCONm$. $\square$

Example 3.4. Let $X = \{1, 3, 5\}, Y = \{2, 4, 6\}, E = \{l, w\}$ and $H = \{l', w'\}$. Obviously $\tau = \{\phi_E, X_E, (F_1, E), (F_2, E)\}$ and $\rho = \{\phi_H, Y_H, (G_1, H)\}$, where $(F_1, E) = \{(l, \{1\}), (w, \{1\})\}, (F_2, E) = \{(l, \{1, 5\}), (w, \{1, 5\})\}$ and $(G_1, H) = \{(l', \{2, 6\}), (w', \{2, 6\})\}$. Presently characterize the mapping $f_{pu} : SS(X_E) \rightarrow SS(Y_H)$, where $u : X \rightarrow Y, p : E \rightarrow H$ are characterized by $p(l) = l', p(w) = w', u(1) = 4, u(3) = 2, u(5) = 6$.

Obviously, f_{pu} is not a $sIICONm$ since (G_1, H) is sOS in (Y, ρ, H) , but $f_{pu}^{-1}(G_1, H) = \{(l, u^{-1}(G_1(p(l)))), (w, u^{-1}(G_1(p(w))))\} = \{(l, u^{-1}(G_1(p(l')))), (w, u^{-1}(G_1(p(w'))))\} = \{(l, u^{-1}(\{2, 6\})), (w, u^{-1}(\{2, 6\}))\} = \{(l, \{3, 5\}), (w, \{3, 5\})\}$, which isn't a $sIIOS$ in (X, τ, E) . Hence f_{pu} is a $sICONm$.

Example 3.5. Let $X = \{1, 3, 5\}, Y = \{2, 4, 6\}, E = \{l, w\}$ and $H = \{l', w'\}$. Obviously $\tau = \{\phi_E, X_E, (F_1, E), (F_2, E)\}$ and $\rho = \{\phi_H, Y_H, (G_1, H)\}$, where $(F_1, E) = \{(l, \{1\}), (w, \{1\})\}, (F_2, E) = \{(l, \{3, 5\}), (w, \{3, 5\})\}$ and $(G_1, H) = \{(l', \{4, 6\}), (w', \{4, 6\})\}$. Presently characterize the mapping $f_{pu} : SS(X_E) \rightarrow SS(Y_H)$ where $u : X \rightarrow Y, p : E \rightarrow H$ are characterized by $p(l) = l', p(w) = w', u(1) = 4, u(3) = 2, u(5) = 6$.

Obviously, f_{pu} is not a $sIICONm$ since (G_1, H) is sOS in (Y, ρ, H) , but $f_{pu}^{-1}(G_1, H) = \{(l, u^{-1}(G_1(p(l))))\} = \{(l, u^{-1}(G_1(p(l'))))\}, (w, u^{-1}(G_1(p(w'))))\} = \{(l, u^{-1}(\{4, 6\})), (w, u^{-1}(\{4, 6\}))\} = \{(l, \{1, 5\}), (w, \{1, 5\})\}$, which isn't a $sIIOS$ in (X, τ, E) . Hence f_{pu} is a $sINTCONm$.

Theorem 3.6. *Any $sSCONm$ is $sIICONm$.*

Proof. Assume that a function $f_{pu} : SS(X_E) \rightarrow SS(Y_H)$ be a $sSCONm$. Let (G, H) be an sOS in (Y, ρ, H) , we have $f_{pu}^{-1}(G, H)$ is a sSO_S in (X, τ, E) (by assume). Since each sSO_S is a $sIIOS$ ([5], Theorem 3.11), we have $f_{pu}^{-1}(G, H)$ is a $sIIOS$ in (X, τ, E) . Hence f_{pu} is a $sIICONm$. $\square$

Theorem 3.7. *Any $sCONm$ is $sIICONm$.*

Proof. Assume that a function $f_{pu} : SS(X_E) \rightarrow SS(Y_H)$ be a $sCONm$. Let (G, H) be an sOS in (Y, ρ, H) , we have $f_{pu}^{-1}(G, H)$ is a sOS in (X, τ, E) (by assume). Since each sOS is a $sIIOS$ ([5], Theorem 3.12), we have $f_{pu}^{-1}(G, H)$ is a $sIIOS$ in (X, τ, E) . Hence f_{pu} is a $sIICONm$. $\square$

Example 3.6. Let $X = \{1, 3, 5\}, Y = \{2, 4, 6\}, E = \{l, w\}$ and $H = \{l', w'\}$. Obviously $\tau = \{\phi_E, X_E, (F_1, E), (F_2, E)\}$ and $\rho = \{\phi_H, Y_H, (G_1, H)\}$, where $(F_1, E) = \{(l, \{3\}), (w, \{3\})\}, (F_2, E) = \{(l, \{3, 5\}), (w, \{3, 5\})\}$ and $(G_1, H) =$

$\{(l', \{2, 6\}), (w', \{2, 6\})\}$. Presently characterize the mapping $f_{pu} : SS(X_E) \rightarrow SS(Y_H)$ where $u : X \rightarrow Y$, $p : E \rightarrow H$ are characterized by $p(l) = l'$, $p(w) = w'$, $u(1) = 2$, $u(3) = 6$, $u(5) = 4$. Obviously, f_{pu} is not a $s\alpha CONm$ since (G_1, H) is sOS in (Y, ρ, H) , but $f_{pu}^{-1}(G_1, H) = \{(l, u^{-1}(G_1(p(l))))\} = \{(l, u^{-1}(G_1(p(w))))\} = \{(l, u^{-1}(G_1(p(l'))))\} = \{(l, u^{-1}(G_1(p(w'))))\} = \{(l, u^{-1}(\{2, 6\}))\} = \{(l, \{1, 3\}), (w, \{1, 3\})\}$, which isn't a sOS in (X, τ, E) . Hence f_{pu} is a $sIICONm$.

Theorem 3.8. *Any $s\alpha CONm$ is $sIICONm$.*

Proof. Assume that a function $f_{pu} : SS(X_E) \rightarrow SS(Y_H)$ be a $s\alpha CONm$. Let (G, H) be an sOS in (Y, ρ, H) , we have $f_{pu}^{-1}(G, H)$ is a $s\alpha OS$ in (X, τ, E) (by assume). Since each $s\alpha OS$ is a $sIIOs$ ([5], Theorem 3.15), we have $f_{pu}^{-1}(G, H)$ is a $sIIOs$ in (X, τ, E) . Hence f_{pu} is a $sIICONm$. $\square$

Example 3.7. Let $X = \{1, 3, 5, 7\}$, $Y = \{2, 4, 6, 8\}$, $E = \{l, w\}$ and $H = \{l', w'\}$.

Obviously $\tau = \{\phi_E, X_E, (F_1, E), (F_2, E), (F_3, E)\}$ and $\rho = \{\phi_H, Y_H, (G_1, H)\}$, where $(F_1, E) = \{(l, \{1\}), (w, \{1\})\}$, $(F_2, E) = \{(l, \{5, 7\}), (w, \{5, 7\})\}$, $(F_3, E) = \{(l, \{1, 5, 7\}), (w, \{1, 5, 7\})\}$ and $(G_1, H) = \{(l', \{2, 6, 8\}), (w', \{2, 6, 8\})\}$. Presently characterize the mapping $f_{pu} : SS(X_E) \rightarrow SS(Y_H)$ where $u : X \rightarrow Y$, $p : E \rightarrow H$ are characterized by $p(l) = l'$, $p(w) = w'$, $u(1) = 4$, $u(3) = 2$, $u(5) = 6$, $u(7) = 8$. Obviously, f_{pu} is not a $s\alpha CONm$ since (G_1, H) is sOS in (Y, ρ, H) , but $f_{pu}^{-1}(G_1, H) = \{(l, u^{-1}(G_1(p(l))))\} = \{(l, u^{-1}(G_1(p(w))))\} = \{(l, u^{-1}(G_1(p(l'))))\} = \{(l, u^{-1}(G_1(p(w'))))\} = \{(l, u^{-1}(\{2, 6, 8\}))\} = \{(l, \{3, 5, 7\}), (w, \{3, 5, 7\})\}$, which isn't a $s\alpha OS$ in (X, τ, E) . Hence f_{pu} is a $sIICONm$.

Theorem 3.9. *Any $s\alpha CONm$ is $sINTCONm$.*

Proof. Assume that a function $f_{pu} : SS(X_E) \rightarrow SS(Y_H)$ be a $s\alpha CONm$. Let (G, H) be an sOS in (Y, ρ, H) , we have $f_{pu}^{-1}(G, H)$ is a $s\alpha OS$ in (X, τ, E) (by assume). Since each $s\alpha OS$ is a $sIntOS$ ([5], Corollary 3.17), we have $f_{pu}^{-1}(G, H)$ is a $sIntOS$ in (X, τ, E) . Hence f_{pu} is a $sINTCONm$. $\square$

Example 3.8. Let $X = \{1, 3, 5\}$, $Y = \{2, 4, 6\}$, $E = \{l, w\}$ and $H = \{l', w'\}$. Obviously $\tau = \{\phi_E, X_E, (F_1, E), (F_2, E)\}$ and $\rho = \{\phi_H, Y_H, (G_1, H)\}$, where $(F_1, E) = \{(l, \{1\}), (w, \{1\})\}$, $(F_2, E) = \{(l, \{3, 5\}), (w, \{3, 5\})\}$ and $(G_1, H) = \{(l', \{2, 6\}), (w', \{2, 6\})\}$. Presently characterize the mapping $f_{pu} : SS(X_E) \rightarrow SS(Y_H)$ where $u : X \rightarrow Y$, $p : E \rightarrow H$ are characterized by $p(l) = l'$, $p(w) = w'$, $u(1) = 2$, $u(3) = 6$, $u(5) = 4$. Obviously, f_{pu} is not a $s\alpha CONm$ since (G_1, H) is sOS in (Y, ρ, H) , but $f_{pu}^{-1}(G_1, H) = \{(l, u^{-1}(G_1(p(l))))\} = \{(l, u^{-1}(G_1(p(w))))\} = \{(l, u^{-1}(G_1(p(l'))))\} = \{(l, u^{-1}(G_1(p(w'))))\} = \{(l, u^{-1}(\{2, 6\}))\} = \{(l, \{1, 3\}), (w, \{1, 3\})\}$, which isn't a $s\alpha OS$ in (X, τ, E) .

Hence f_{pu} is a $sINTCONm$.

4. Soft topological ii-homeomorphisms

Definition 4.1. Consider $f_{pu} : SS(X_E) \rightarrow SS(Y_H)$ be mapping from (X, τ, E) into (Y, ρ, H) where $u : X \rightarrow Y$, $p : E \rightarrow H$ are two mappings, then f_{pu} is called bijection mapping (BM) if it is:

1. Onto wherein $f_{pu}(X_E) = Y_H$.
2. One-to-one if $(F_1, E) \neq (F_2, E)$, then $f_{pu}(F_1, E) \neq f_{pu}(F_2, E)$, $\forall (F_1, E), (F_2, E) \in SS(X_E)$ (see [13]).

Definition 4.2. Consider $f_{pu} : SS(X_E) \rightarrow SS(Y_H)$ be a BM from (X, τ, E) into (Y, ρ, H) where $u : X \rightarrow Y$, $p : E \rightarrow H$ are two mappings, then f_{pu} is called:

- i) Soft topological homeomorphism (sTh) if it is a $sCONm$ and sOm (see [13]).
- ii) Soft topological semi-homeomorphism ($sTSh$) if it is a $sSCONm$ and $sSOM$ (see [13]).
- iii) Soft topological α -homeomorphism ($sT\alpha h$) if it is a $s\alpha CONm$ and $s\alpha Om$ (see [16]).
- iv) Soft topological i-homeomorphism ($sTIh$) if it is a $sICONm$ and $sIOM$.
- v) Soft topological inter-homeomorphism ($sTINT h$) if it is a $sINTCONm$ and $sIntOm$.
- vi) Soft topological ii-homeomorphism ($sTIIh$) if it is a $sIICONm$ and $sIIOM$.

Theorem 4.1. Each sTh is $sTIh$.

Proof. Assume that a function $f_{pu} : SS(X_E) \rightarrow SS(Y_H)$ be a sTh . We have f_{pu} is a $sCONm, sOm$ and a BM (by assume). By Theorem 3.1, we have that f_{pu} is a $sICONm$ and Theorem 2.1 f_{pu} is a $sIOM$. Hence, f_{pu} is a $sTIh$. $\square$

Example 4.1. Let $X = \{1, 3, 5\}, Y = \{2, 4, 6\}, E = \{l, w\}$ and $H = \{l', w'\}$. Obviously $\tau = \{\phi_E, X_E, (F_1, E)\}$ and $\rho = \{\phi_H, Y_H, (G_1, H)\}$, are sTs over X and Y in dividually where $(F_1, E) = \{(l, \{1, 3\}), (w, \{1, 3\})\}$ and $(G_1, H) = \{(l', \{2\}), (w', \{2\})\}$. Presently characterize the mapping $f_{pu} : SS(X_E) \rightarrow SS(Y_H)$ where $u : X \rightarrow Y$, $p : E \rightarrow H$ are characterized by $p(l) = l', p(w) = w', u(1) = 2, u(3) = 6, u(5) = 4$.
 $sIOS(X_E) = \{\phi_E, X_E, (F_1, E), \{(l, \{1\}), (w, \{1\})\}, \{(l, \{3\}), (w, \{3\})\}, \{(l, \{1, 5\}), (w, \{1, 5\})\}, \{(l, \{3, 5\}), (w, \{3, 5\})\}\}$. $sIOS(Y_H) = \{\phi_H, Y_H, (G_1, H), \{(l', \{2, 4\}), (w', \{2, 4\})\}, \{(l', \{2, 6\}), (w', \{2, 6\})\}\}$.

Apparently, f_{pu} is not a $sCONm$ since (G_1, H) is sOs in (Y, ρ, H) , but $f_{pu}^{-1}(G_1, H) = \{(l, u^{-1}(G_1(p(l)))), (w, u^{-1}(G_1(p(w))))\} = \{(l, u^{-1}(G_1(p(l')))),$

$(w, u^{-1}(G_1(p(w'))))\} = \{(l, u^{-1}(\{2\})), (w, u^{-1}(\{2\}))\} = \{(l, \{1\}), (w, \{1\})\}$, which isn't a sOS in (X, τ, E) . Hence f_{pu} is a $sICONm$. f_{pu} Isn't a sOm since (F_1, E) is an sOS in (X, τ, E) but $f_{pu}(F_1, E) = \{(l', \{2, 6\}), (w', \{2, 6\})\}$ isn't a sOS in (Y, ρ, H) . f_{pu} is a $sIOM$. Since f_{pu} is BM . Hence, f_{pu} is $sTIh$.

Theorem 4.2. *Any $sTSh$ is $sTIh$.*

Proof. Assume that a function $f_{pu} : SS(X_E) \rightarrow SS(Y_H)$ be a $sTSh$. We have f_{pu} is a $sSCONm, sSOM$ and a BM (by assume). By Theorem 3.2, we have that f_{pu} is a $sICONm$ and by Theorem 2.2 f_{pu} is a $sIOM$. Hence, f_{pu} is a $sTIh$. $\square$

Example 4.2. Let $X = \{1, 3, 5\}, Y = \{2, 4, 6\}, E = \{l, w\}$ and $H = \{l', w'\}$. Obviously $\tau = \{\phi_E, X_E, (F_1, E)\}$ and $\rho = \{\phi_H, Y_H, (G_1, H)\}$, where $(F_1, E) = \{(l, \{1\}), (w, \{1\})\}$ and $(G_1, H) = \{(l', \{2, 4\}), (w', \{2, 4\})\}$. Presently characterize the mapping $f_{pu} : SS(X_E) \rightarrow SS(Y_H)$ where $u : X \rightarrow Y, p : E \rightarrow H$ are characterized by $p(l) = l', p(w) = w', u(1) = 2, u(3) = 6, u(5) = 4$. Essentially as in above example f_{pu} is a $sSCONm$ Definition 3.1 (iii). f_{pu} is a $sICONm$ Definition 3.1 (iv), f_{pu} isn't a $sSOM$ since (F_1, E) is an sOS in (X, τ, E) but $f_{pu}(F_1, E) = \{(l', \{2\}), (w', \{2\})\}$ isn't a sOS in (Y, ρ, H) . f_{pu} is a $sIOM$ Definition 2.1 (vii) and since f_{pu} is BM . Hence, f_{pu} is $sTIh$ but it isn't a $sTSh$.

Theorem 4.3. *Each $sT\alpha h$ is $sTIh$.*

Proof. Assume that a function $f_{pu} : SS(X_E) \rightarrow SS(Y_H)$ be a $sT\alpha h$. We have f_{pu} is a $s\alpha CONm, s\alpha OM$ and a BM (by assume). By Theorem 3.3, we have that f_{pu} is a $sICONm$ and by Corollary 2.1 f_{pu} is a $sIOM$. Hence, f_{pu} is a $sTIh$. $\square$

In Example 4.2, we have f_{pu} is a $s\alpha CONm$ Definition 3.1 (ii), f_{pu} is $sICONm$ Definition 3.1(iv), but f_{pu} isn't a $s\alpha OM$ since (F_1, E) is an sOS in (X, τ, E) but $f_{pu}(F_1, E) = \{(l', \{2\}), (w', \{2\})\}$ isn't a $s\alpha OS$ in (Y, ρ, H) . f_{pu} is a $sIOM$ Definition 2.1 (vii) and since f_{pu} is BM . Hence, f_{pu} is $sTIh$ but it isn't a $sT\alpha h$.

Theorem 4.4. *Each $sT\alpha h$ is $sTSh$.*

Proof. Assume that a function $f_{pu} : SS(X_E) \rightarrow SS(Y_H)$ be a $sT\alpha h$. We have f_{pu} is a $s\alpha CONm, s\alpha OM$ and a BM (by assume). By Theorem 3.4, we have that f_{pu} is a $sSCONm$ and by Theorem 2.3 f_{pu} is a $sSOM$. Hence, f_{pu} is a $sTSh$ (see [16]). $\square$

Example 4.3. Let $X = \{1, 3, 5\}, Y = \{2, 4, 6\}, E = \{l, w\}$ and $H = \{l', w'\}$. Obviously $\tau = \{\phi_E, X_E, (F_1, E), (F_2, E), (F_3, E)\}$ and $\rho = \{\phi_H, Y_H, (G_1, H), (G_2, H), (G_3, H), (G_4, H)\}$, where $(F_1, E) = \{(l, \{1\}), (w, \{1\})\}$, $(F_2, E) = \{(l, \{3\}), (w, \{3\})\}$, $(F_3, E) = \{(l, \{1, 3\}), (w, \{1, 3\})\}$, $(G_1, H) = \{(l', \{2\}), (w', \{2\})\}$, $(G_2, H) = \{(l', \{6\}), (w', \{6\})\}$, $(G_3, H) = \{(l', \{2, 6\}), (w', \{2, 6\})\}$.

and $(G_4, H) = \{(l', \{4, 6\}), (w', \{4, 6\})\}$. Presently characterize the mapping $f_{pu} : SS(X_E) \rightarrow SS(Y_H)$ where $u : X \rightarrow Y$, $p : E \rightarrow H$ are characterized by $p(l) = l', p(w) = w', u(1) = 2, u(3) = 6, u(5) = 4$. Obviously, f_{pu} is a $sSCONm$ Definition 3.1(iii). f_{pu} isn't a $s\alpha CONm$ since (G_4, H) is an sOS in (Y, ρ, H) but $f_{pu}^{-1}(G_4, H) = \{(l, u^{-1}(G_4(p(l))))\} = \{(l, u^{-1}(G_4(p(w))))\} = \{(l, u^{-1}(G_4(p(l'))))\} = \{(l, u^{-1}(G_4(p(w'))))\} = \{(l, u^{-1}(\{4, 6\}))\} = \{(l, \{3, 5\}), (w, \{3, 5\})\}$ isn't a $s\alpha OS$ in (X, τ, E) . f_{pu} is a $sSOM$ Definition 2.1 (iii) and f_{pu} is a $s\alpha Om$ Definition 2.1 (v), since f_{pu} is BM . Hence, f_{pu} is $sTSh$ but it isn't a $sT\alpha h$

Theorem 4.5. Any $sTIIh$ is $sTIh$ and it is a $sTINTh$

Proof. Assume that a function $f_{pu} : SS(X_E) \rightarrow SS(Y_H)$ be a $sTIIh$. We have f_{pu} is a $sIICONm, sIIOM$ and a BM (by assume). By Theorem 3.5, we have that f_{pu} is a $sICONm$ and $sINTCONm$ by Theorem 2.4 f_{pu} is a $sIOM$ and $sIntOm$. Hence, f_{pu} is a $sTIh$ and it is a $sTINTh$. $\square$

Example 4.4. Let $X = \{1, 3, 5\}, Y = \{2, 4, 6\}, E = \{l, w\}$ and $H = \{l', w'\}$. Obviously $\tau = \{\phi_E, X_E, (F_1, E)\}$ and $\rho = \{\phi_H, Y_H, (G_1, H)\}$, where $(F_1, E) = \{(l, \{1, 3\}), (w, \{1, 3\})\}$, and $(G_1, H) = \{(l', \{2, 6\}), (w', \{2, 6\})\}$. Presently characterize the mapping $f_{pu} : SS(X_E) \rightarrow SS(Y_H)$ where $u : X \rightarrow Y$, $p : E \rightarrow H$ are characterized by $p(l) = l', p(w) = w', u(1) = 4, u(3) = 2, u(5) = 6$.

Obviously, f_{pu} is a $sICONm$ Definition 3.1(iv). f_{pu} isn't a $sIICONm$ since (G_1, H) is an sOS in (Y, ρ, H) but $f_{pu}^{-1}(G_1, H) = \{(l, u^{-1}(G_1(p(l))))\} = \{(l, u^{-1}(G_1(p(w))))\} = \{(l, u^{-1}(G_1(p(l'))))\} = \{(l, u^{-1}(G_1(p(w'))))\} = \{(l, u^{-1}(\{2, 6\}))\} = \{(l, \{3, 5\}), (w, \{3, 5\})\}$ isn't a $sIIOs$ in (X, τ, E) . f_{pu} is a $sIOM$ Definition 2.1 (vii) and f_{pu} is not a $sIIOM$, since (F_1, E) is an sOS in (X, τ, E) , also $f_{pu}(F_1, E) = \{(l', \{2, 4\}), (w', \{2, 4\})\}$ isn't a $sIIOs$ in (Y, ρ, H) . f_{pu} is BM . Hence, f_{pu} is $sTIh$ but it isn't a $sTIIh$

Example 4.5. Let $X = \{1, 3, 5\}, Y = \{2, 4, 6\}, E = \{l, w\}$ and $H = \{l', w'\}$. Obviously $\tau = \{\phi_E, X_E, (F_1, E), (F_2, E), (F_3, E)\}$ and $\rho = \{\phi_H, Y_H, (G_1, H), (G_2, H)\}$, where $(F_1, E) = \{(l, \{3\}), (w, \{3\})\}, (F_2, E) = \{(l, \{1, 5\}), (w, \{1, 5\})\}, (F_3, E) = \{(l, \{1, 3\}), (w, \{1, 3\})\}, (G_1, H) = \{(l', \{4\}), (w', \{4\})\}$ and $(G_2, H) = \{(l', \{2, 6\}), (w', \{2, 6\})\}$. Presently characterize the mapping $f_{pu} : SS(X_E) \rightarrow SS(Y_H)$ where $u : X \rightarrow Y$, $p : E \rightarrow H$ are characterized by $p(l) = l', p(w) = w', u(1) = 6, u(3) = 4, u(5) = 2$. Obviously, f_{pu} is a $sINTCONm$ Definition 3.1 (v). f_{pu} is a $sIICONm$ Definition 3.1 (vi). f_{pu} is a $sIntOm$ Definition 2.1 (ix) f_{pu} isn't a $sIIOM$ since (F_3, E) is an sOS in (X, τ, E) $f_{pu}(F_3, E) = \{(l', \{4, 6\}), (w', \{4, 6\})\}$ isn't a $sIIOs$ in (Y, ρ, H) . f_{pu} is BM . Hence, f_{pu} is $sTINTh$ but it isn't a $sTIIh$

Theorem 4.6. Any $sTSh$ is $sTIIh$.

Proof. Assume that a function $f_{pu} : SS(X_E) \rightarrow SS(Y_H)$ be a $sTSh$. We have f_{pu} is a $sSCONm, sSOM$ and a BM (by assume). By Theorem 3.5, we have

that f_{pu} is a $sICONm$ and $sINTCONm$ by Theorem 3.6 f_{pu} is a $sIICONm$ and by Theorem 2.5 f_{pu} is a $sIIOM$. Hence, f_{pu} is a $sTIIh$. $\square$

Theorem 4.7. Any sTh is $sTIIh$.

Proof. Assume that a function $f_{pu} : SS(X_E) \rightarrow SS(Y_H)$ be a sTh . We have f_{pu} is a $sCONm, sOm$ and a BM (by assume). By Theorem 3.5, we have that f_{pu} is a $sICONm$ and $sINTCONm$ by Theorem 3.7 f_{pu} is a $sIICONm$ and by Theorem 2.6 f_{pu} is a $sIIOM$. Hence, f_{pu} is a $sTIIh$. $\square$

Example 4.6. Let $X = \{1, 3, 5\}, Y = \{2, 4, 6\}, E = \{l, w\}$ and $H = \{l', w'\}$. Obviously $\tau = \{\phi_E, X_E, (F_1, E), (F_2, E)\}$ and $\rho = \{\phi_H, Y_H, (G_1, H), (G_2, H)\}$, where $(F_1, E) = \{(l, \{3\}), (w, \{3\})\}, (F_2, E) = \{(l, \{3, 5\}), (w, \{3, 5\})\}, (G_1, H) = \{(l', \{2\}), (w', \{2\})\}$ and $(G_2, H) = \{(l', \{2, 6\}), (w', \{2, 6\})\}$. Presently characterize the mapping $f_{pu} : SS(X_E) \rightarrow SS(Y_H)$ where $u : X \rightarrow Y, p : E \rightarrow H$ are characterized by $p(l) = l', p(w) = w', u(1) = 6, u(3) = 2, u(5) = 4$. $sOS(X_E) = \{\phi_E, X_E, (F_1, E), (F_2, E)\}, sOS(Y_H) = \{\phi_H, Y_H, (G_1, H), (G_2, H)\}$. Apparently, f_{pu} is a $sIICONm$ Definition 3.1 (vi). f_{pu} is a $sIIOM$ Definition 2.1 (xi). f_{pu} isn't a sOm since (F_2, E) is an sOS in $(X, \tau, E), f_{pu}(F_2, E) = \{(l', \{2, 4\}), (w', \{2, 4\})\}$ isn't a sOS in (Y, ρ, H) . f_{pu} is BM . Hence, f_{pu} is $sTIIh$ but it isn't a sTh .

Theorem 4.8. Any $sTah$ is $sTIIh$.

Proof. Assume that a function $f_{pu} : SS(X_E) \rightarrow SS(Y_H)$ be a $sTah$. We have f_{pu} is a $s\alpha CONm, s\alpha Om$ and a BM (by assume). By Theorem 3.8, we have that f_{pu} is a $sIICONm$ and by Theorem 2.7 f_{pu} is a $sIIOM$. Hence, f_{pu} is a $sTIIh$. $\square$

Example 4.7. Let $X = \{1, 3, 5, 7\}, Y = \{2, 4, 6, 8\}, E = \{l, w\}$ and $H = \{l', w'\}$. Obviously $\tau = \{\phi_E, X_E, (F_1, E), (F_2, E), (F_3, E), (F_4, E)\}$ and $\rho = \{\phi_H, Y_H, (G_1, H), (G_2, H), (G_3, H)\}$, where $(F_1, E) = \{(l, \{5\}), (w, \{5\})\}, (F_2, E) = \{(l, \{3, 7\}), (w, \{3, 7\})\}, (F_3, E) = \{(l, \{3, 5, 7\}), (w, \{3, 5, 7\})\}, (F_4, E) = \{(l, \{1, 3, 7\}), (w, \{1, 3, 7\})\}, (G_1, H) = \{(l', \{2\}), (w', \{2\})\}, (G_2, H) = \{(l', \{6, 8\}), (w', \{6, 8\})\}$ and $(G_3, H) = \{(l', \{2, 6, 8\}), (w', \{2, 6, 8\})\}$. Presently characterize the mapping $f_{pu} : SS(X_E) \rightarrow SS(Y_H)$ where $u : X \rightarrow Y, p : E \rightarrow H$ are characterized by $p(l) = l', p(w) = w', u(1) = 4, u(3) = 6, u(5) = 2$. Obviously, f_{pu} is a $sIICONm$ Definition 3.1 (vi). f_{pu} is a $sIIOM$ Definition 2.1 (xi). f_{pu} isn't a $s\alpha Om$ since (F_4, E) is an sOS in $(X, \tau, E), f_{pu}(F_4, E) = \{(l', \{4, 6, 8\}), (w', \{4, 6, 8\})\}$ isn't a $s\alpha OS$ in (Y, ρ, H) . f_{pu} is BM . Hence, f_{pu} is $sTIIh$ but it isn't a $sTah$.

Theorem 4.9. Any $sTah$ is $sTINTh$.

Proof. Assume that a function $f_{pu} : SS(X_E) \rightarrow SS(Y_H)$ be a $sTah$. We have f_{pu} is a $s\alpha CONm, s\alpha Om$ and a BM (by assume). By Theorem 3.9, we have that f_{pu} is a $sIICONm$ and by Corollary 2.2 f_{pu} is a $sIIOM$. Hence, f_{pu} is a $sTIIh$. $\square$

In Example 4.7, we have f_{pu} is a $sINTCONm$ Definition 3.1 (v). f_{pu} is a $sIntOm$ Definition 2.1 (ix). f_{pu} isn't a $s\alpha Om$ since (F_4, E) is an sOS in (X, τ, E) , $f_{pu}(F_4, E) = \{(l', \{4, 6, 8\}), (w', \{4, 6, 8\})\}$ isn't a $s\alpha OS$ in (Y, ρ, H) . f_{pu} is BM . Hence, f_{pu} is $sTINTTh$ but it isn't a $sT\alpha h$

5. Conclusions

1. The relations among $(sOm, sIOm, sINTOm, sIIOm, s\alpha Om$ and $sSOM)$ depends on the relations among $(sOS, sIOS, sINTOS, sIIOS, s\alpha OS$ and $sSOs)$.
2. The relations among $(sCONm, sICONm, sINTCONm, sIICONm, s\alpha CONm$ and $sSCONm)$ depends on the relations among $(sOS, sIOS, sINTOS, sIIOS, s\alpha OS$ and $sSOs)$.
3. The relations among $(sTh, sTIh, sTINTTh, sTIIh, sT\alpha h$ and $sTSh)$ depends on the relations among $(sOm, sIOm, sINTOm, sIIOm, s\alpha Om$ and $sSOM)$ together with $(sCONm, sICONm, sINTCONm, sIICONm, s\alpha CONm$ and $sSCONm$ and $sSCONm)$.

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Comparison of two types of rough approximation via grill

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Abstract. Rough set theory and topology are now branching far into applied areas, such as economics, data processing, imaging and chemistry. As a consequence of this importance gained from topology after the advent of a rough set theory which helps to quantify things that were previously difficult to measure. It was important to work on the extension of the topological space with new concepts such as grill and ideal. In this paper, we present new approximations of rough sets via a grill concept which has helped to extend the topological spaces. In addition, the topology created by the present method is finer than other methods. Finally, grill topological spaces will be obtained in terms of relations and grills aimed at minimizing the boundary regions.

Keywords: rough sets, grill topological space, lower approximation, upper approximation.

1. Introduction

Since the advent of the rough sets theory, which aims at reduce the boundary region of a universe set, it has attracted the attention of many mathematicians, computer specialists, and others, especially those interested in topology. As a consequence of this interest in the theory of rough sets. This theory has many applications in many fields that are used to process control [17], computer [10], data analysis and classification [38], economics [6], such as medical diagnosis, chemistry, psychology, finance, marketing, biochemistry, environmental science, intelligent agents, image analysis, biology, telecommunication and other fields (see [4], [7], [18], [19], ([20]-[24]), ([28]-[33]), [34], [41], [42], [43], ([44]-[48]) and the bibliography throughout these article). The emergence of the value of topological space and its applications besides the theory of rough sets, it had to be expanded by identifying terms such as ideals and grills. Many approaches of

rough sets depended on the relation on a universe set. Although the topological grill structures will be used in this article. So, the purpose of this paper is to add new approximations to the rough set theory using the grill definition, the eliminates the difference between the upper and lower approximations compared to previous approximations. We also contrasted this new definition in terms of grills with the upper and lower approximations of [25], [40] and [5], as shown in Theorem 3.1 below. Finally, we strengthened the relation with a variety of different instances.

2. Preliminaries

Two Φ and Ψ mappings that are the basis of the grill concept were first introduced in [9]. There's a resemblance expansion of the topological space between the concept of grills, ideals, filters and nets. There have been some hypotheses and features run by [30, 6, 38, 36]. It was helping to general the topological structure used to understand description rather than the quantity, e.g. diet, intelligence, appearance, standard of education and so on. It also extended topological layout by using the idea of grill adjustments approaches of rough sets and the boundary regions. Throughout this article, if (Y, ζ) is a topological space with a grill g on Y . Then we must call it a grill topological space, denote it by (Y, ζ, G) and abbreviate it by GTS .

Definition 2.1 ([9]). *A nonempty collection G of subsets of a set Y is said to be a grill on space Y , if satisfying the following conditions:*

- (1) $\Phi \notin G$,
- (2) $B \in G$ and $B \subseteq C \subseteq Y \Rightarrow C \in G$,
- (3) if $B \cup C \in G$ for $B, C \subseteq Y$, then $B \in G$ or $C \in G$.

Remark 2.1. (1) The minimal grill is $G = \{Y\}$ in any space Y which carries the topology ζ .

- (2) The maximal grill is $G = P(Y) \setminus \{\Phi\}$ in any topological space (Y, ζ) .

Definition 2.2 ([30]). *Let (Y, ζ, G) be a GTS . Considering the operator $\Phi : P(Y) \rightarrow P(Y)$ which is denoted by $\Phi_G(B)$ for $B \subseteq Y$, which defined as $\Phi_G(B) = \Phi_G(B, \zeta) = \{y \in Y : B \cap U \in G \text{ for every open set } U \text{ containing } y\}$ for each $B \in P(Y)$.*

Definition 2.3 ([30]). *Let (Y, ζ, G) be a GTS . Considering the operator $\Psi : P(Y) \rightarrow P(Y)$ by $\Psi(B) = B \cup \Phi(B)$ for all $B \in P(Y)$. The operator Φ is a Kuratowski closure axioms. Based on a grill G on a topological space (Y, ζ) , there exists a unique topology ζ_G on Y given by $\zeta_G = \{U \subseteq Y : \Psi(Y \setminus U) = Y \setminus U\}$, where for any $B \subseteq Y$, $\Psi(B) = B \cup \Phi(B) = \zeta_G - Cl(B)$. For any grill G on a topological space (Y, ζ) , we have $\zeta \subseteq \zeta_G$.*

Definition 2.4 ([25]). *If R is an equivalence relation on a miss-null set Y , $[y]_R$ is an equivalence classes of y , then for $B \subseteq Y$, a pair of upper and lower*

approximations, $\overline{R}(B)$ and $\underline{R}(B)$, are defined respectively as:

$$\overline{R}(B) = \{y \in Y : [y]_R \cap B \neq \Phi\};$$

$$\underline{R}(A) = \{y \in Y : [y]_R \subseteq B\}.$$

Theorem 2.2 ([25]). *If R is an equivalence relation on a miss-null set Y , and $B, C \subseteq Y$, then the following properties are satisfied concerning for the upper approximation:*

- (1) $\overline{R}(\Phi) = \Phi$;
- (2) $B \subseteq C \Rightarrow \overline{R}(B) \subseteq \overline{R}(C)$;
- (3) $B \subseteq \overline{R}(B)$;
- (4) $\overline{R}(B \cup C) = \overline{R}(B) \cup \overline{R}(C)$;
- (5) $\overline{R}(B \cap C) \subseteq \overline{R}(B) \cup \overline{R}(C)$;
- (6) $\overline{R}(\overline{R}(B)) = \overline{R}(B)$;
- (7) $\overline{R}(B) = [\overline{R}(B^c)]^c$, where B^c denote the complement of B .

Corollary 2.1 ([25]). *If R is an equivalence relation on a miss-null set Y . Then, the operator $\overline{R}$ on $P(Y)$ satisfies the Kuratowski's axioms and deduces the topology on Y called τ_R is given by $\zeta_R = \{B \subseteq Y : \overline{R}(B^c) = B^c\}$.*

Definition 2.5 ([39]). *Let R be a binary relation on a nonempty set Y . For $B \subseteq Y$, we notice that, the rough approximations of the lower and upper are defined as, $\underline{R}(B) = \{y \in Y : yR \cap B \subseteq B\}$, $\overline{R}(B) = \{y \in Y : yR \cap B \neq \Phi\}$. Where $yR = \{c \in Y : yRc\}$ is called after set.*

Theorem 2.3 ([25]). *If R is a pre-order relation on Y , then the upper approximation satisfies the properties of Theorem 2.6.*

Definition 2.6 ([2]). *If R is a reflexive relation on a nonempty set Y . Then, $B \subseteq Y$. $\underline{R}(B)$ and $\overline{R}(B)$ are defined by $\underline{R}(B) = \{y \in Y : \langle y \rangle_R \subseteq B\}$, $\overline{R}(B) = \{y \in Y : \langle y \rangle_R \cap B \neq \Phi\}$. Where, $\langle y \rangle_R = \cap \{pR : y \in pR\}$.*

Proposition 2.1 ([2]). *If R is a binary relation on a nonempty set Y and $z \in \langle y \rangle_R$ then, $\langle z \rangle_R \subseteq \langle y \rangle_R$.*

3. Novel approximation rough sets via a grill

The main aim of this section is to develop some notions and characterizations of the rough sets theory in terms of the grill. The new definitions of the lower approximation and upper approximation aim at increasing the lower approximation of any set $B \subseteq Y$ and reducing the boundary region. Also, we extend the results of new definitions to compare with the definitions of [26, 27], [39, 40] and [3, 2].

Definition 3.1. *If R is a reflexive relation on a GTS., $B \subseteq Y$ and G is a grill on Y , then the R^G -upper and R_G -lower approximations of B are defined respectively by:*

$$R^G(B) = \{y \in Y : \langle y \rangle_R \cap B \in G\};$$

$$R_G(B) = \{y \in Y : \langle y \rangle_R \cap B^c \notin G\}.$$

Theorem 3.1. Let $G = P(Y) \setminus \{\Phi\}$ be a grill,

- (1) If R is an equivalence relation, then Definition 3.1 coincides with Pawlak's definition.
- (2) If R is a pre-order relation, then Definition 3.1 corresponds with Yao's definition.
- (3) If R is a reflexive relation, then Definition 3.1 coincides with Allam's definition.

Proof. (1) $\underline{R}(B) = \{y \in Y : [Y]_R \cap B \subseteq B\}$, that is $\langle y \rangle_R \cap B^c = \Phi$ and $\langle y \rangle_R \cap B^c \notin G$, i.e. $R_G(B) = \{y \in Y : \langle y \rangle_R \cap B^c \notin G\}$. $\overline{R}(B) = \{y \in Y : [Y]_R \cap B \neq \Phi\}$, that is $\langle y \rangle_R \cap B \in G$, i.e. $R^G(B) = \{y \in Y : \langle y \rangle_R \cap B \in G\}$.

(2) It is evident.

(3) Similarly proof(1). $\square$

Lemma 3.1. If R is a binary relation on a miss-null set Y then the following condition holds:

- (1) If $G = \{Y\}$ then $R^G(B) = \Phi$,
- (2) If $G = P(Y) \setminus \{\Phi\}$ then $R^G(B) = \overline{R}(B)$.

Proof. (1) If $G = \{Y\}$ hence $\langle y \rangle_R \cap B \neq Y$ and so $R^G(B) = \{y \in Y : \langle y \rangle_R \cap B \notin G\}$ and $R^G(B) = \Phi$.

(2) If $G = P(Y) \setminus \{\Phi\}$ then $R^G(B) = \overline{R}(B)$ for $\overline{R}^G(B)$, Pawlak's, Yao's and Allam's definition. $\square$

Theorem 3.2. Let $G = P(X) \setminus \{\Phi\}$ and R be a reflexive relation on ζ_R^G . Then, the R^G -upper approximation in Definition 3.1 satisfies the following properties:

1. $R^G(Y) = Y$,
2. $B \subseteq C \Rightarrow R^G(B) \subseteq R^G(C)$,
3. $R^G(B \cup C) = R^G(B) \cup R^G(C)$,
4. $R^G(B \cap C) \subseteq R^G(B) \cap R^G(C)$,
5. $R^G(R^g(B)) \subseteq R^G(B)$,
6. $R_G(B) = [R^G(B^c)]^c$,
7. $B \notin G \Rightarrow R^G(B) = \phi$.

Proof. 1. It is clear from Definition 3.1.

2. Let $y \in R^G(B)$, then $\langle y \rangle_R \cap B \in G$. Since $\langle y \rangle_R \cap B \subseteq \langle y \rangle_R \cap C$, so $y \in R^G(C)$.

3. $R^G(B \cup C) = \{y \in Y : \langle y \rangle_R \cap (B \cup C) \in G\} = \{y : (\langle y \rangle_R \cap B) \cup (\langle y \rangle_R \cap C) \in G\} = \{y : \langle y \rangle_R \cap B \in G\} \cup \{y : \langle y \rangle_R \cap C \in G\} = R^G(B) \cup R^G(C)$.

4. Let $y \in R^G(B \cap C)$ that is $\langle y \rangle_R \cap (B \cap C) \in G$, i.e. $(\langle y \rangle_R \cap B) \cap (\langle y \rangle_R \cap C) \in G$, and $y \in R^G(B) \cap R^G(C)$.

5. Let $Y \in R^G(R^g(B))$, hence $\langle y \rangle_R \cap R^g(B) \neq \Phi$. Then, there exists $z \in \langle y \rangle_R \cap R^g(B)$, i.e. $\langle z \rangle_R \subseteq \langle y \rangle_R$ and $\langle z \rangle_R \cap R^g(B) \in G$. It follows that $y \in R^G(B)$.

6. $[R^G(B^c)]^c = \{y \in Y : \langle y \rangle_R \cap B^c \in G\}^c = \{y \in Y : \langle y \rangle_R \cap B^c \notin G\} = R_G(B)$.

7. If $B \notin G$, then $\phi_G(B) = \Phi$ [30]. So, $\langle y \rangle_R \cap B \notin G$ and $R^G(B) = \Phi$. $\square$

The next example shows that $B \notin \overline{R}^g(B)$, in general. So, Theorem 3.2 is not valid in general, as shown in Example 3.3.

Example 3.3. Let $Y = \{1, 2, 3, 4\}$, $G = \{\{2\}, \{1, 2\}, \{2, 3\}, Y\}$ be a grill on Y and $R = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (1, 3), (2, 3), (2, 4), (3, 4), (3, 1)\}$ be a relation on Y . Hence, $1R = \{1, 2, 3\}$ and so $\langle 1 \rangle_R = \{1, 3\}$, $2R = \{2, 3, 4\}$ and so $\langle 2 \rangle_R = \{2, 3\}$, $3R = \{1, 3, 4\}$ and so $\langle 3 \rangle_R = \{3\}$, $4R = \{4\}$ and so $\langle 4 \rangle_R = \{4\}$. Let $B = \{1, 2\}$. Then, $R^G(b) = \{2\}$.

Theorem 3.4. For the binary relation R on a miss-null set Y and G_1, G_2 being two grills on a GTS Y we get:

1. $G_1 \subseteq G_2 \Rightarrow R^{G_1} \subseteq R^{G_2}$ and $R_{G_2} \subseteq R_{G_1}$;
2. $R^{G_1 \cap G_2}(B) = R^{G_1}(B) \cap R^{G_2}(B)$;
3. $R^{G_1 \cup G_2}(B) = R^{G_1}(B) \cup R^{G_2}(B)$.

Proof. 1. Let $y \in R^{G_1}$, then $\langle y \rangle_R \cap B \in G_1$. Since $G_1 \subseteq G_2$, then $\langle y \rangle_R \cap B \in G_2$ and $y \in R^{G_2}$. i.e. $R^{G_1} \subseteq R^{G_2}$.

Let $y \in R_{G_2}$, then $\langle y \rangle_R \cap B^c \notin G_2$. That, $\langle y \rangle_R \cap B^c \notin G_1$ and $y \in R_{G_1}$. i.e. $R_{G_2} \subseteq R_{G_1}$. 2. It is clear. 3. Obvious. $\square$

Theorem 3.5. Let G be a maximal grill on Y and R being a reflexive relation. Then R_G -lower approximation satisfies the following properties:

- 1- $R_G(Y) = Y$,
- 2- $B \subseteq C \Rightarrow R_G(B) \subseteq R_G(C)$,
- 3- $R_G(B \cap C) \subseteq R_G(B) \cup R_G(C)$,
- 4- $R_G(B) \cup R_G(C) \subseteq R_G(B \cap C)$,
- 5- $R_G(B) \subseteq R_G(R_G(B))$,
- 6- $R_G(B) = [R^G(B^c)]^c$,
- 7- $B^c \notin G \Rightarrow R_G(B) = Y$.

Proof. 1. From $R_G(B) = \{y \in Y : \langle y \rangle_R \cap B^c \notin G\}$. Then $R_G(Y) = \{y \in Y : \langle y \rangle_R \cap \Phi = \Phi \notin G\} = Y$.

2. If $B \subseteq C$, $y \in R_G(B)$ that $\langle y \rangle_R \cap B^c \notin G$ and $\langle y \rangle_R \cap C^c \notin G$. i.e. $y \in R_G(C)$.

3. Let $y \in R_G(B \cap C)$. Then $\langle y \rangle_R \cap (B \cap C)^c \notin G$, i.e. $y \in R_G(B) \cup R_G(C)$.

4. Let $y \in R_G(B) \cup R_G(C)$. Then $y \in \{\langle y \rangle_R \cap B^c \notin G\} \cup \{\langle y \rangle_R \cap C^c \notin G\}$ and $y \in R_G(B \cap C)$.

Proof 5, 6 and 7 are straightforward. $\square$

4. Some topological approaches for rough approximations through grill

Definition 4.1. *Supposing that G is a grill on the non-empty set Y and the relation R is reflexive on the GTS Y , $B \subseteq Y$. the rough approximations $\underline{R}$ and $\overline{R}$ of set B are assigned respectively by:*

$$\begin{aligned}\underline{R}_G(B) &= \{y \in B : < y >_R \cap B^c \notin G\}, \\ \overline{R}^G(B) &= B \cup R^G(B).\end{aligned}$$

Definition 4.2. *Let G be a grill on the GTS Y and R be a reflexive relation on the non-empty set Y . For $B \subseteq Y$ the positive, boundary, negative regions and the accuracy measure of B are assigned respectively by: $POS_{R_G}(B)$, $BND_{R_G}(B) = \frac{\overline{R}^G(B)}{\underline{R}_G(B)}$, $NEG_{R_G}(B) = X \setminus \overline{R}^G(B)$ and $\mu_{R_G}(B) = \frac{\underline{R}_G(B)}{\overline{R}^G(B)}$.*

Theorem 4.1. *If R is a binary relation on the miss-null set Y and G is any grill, then $\overline{R}(A) \subseteq \overline{R}^G(A)$.*

Proof. It is obvious. □

Theorem 4.2. *Let G be a grill and R be a reflexive relation on τ_R^g . Then, the rough approximation in Definition 4.1 satisfies the following:*

1. $\overline{R}^G(Y) = Y$,
2. $B \subseteq \overline{R}^G(B)$,
3. $B \subseteq C \Rightarrow \overline{R}^G(B) \subseteq \overline{R}^G(C)$,
4. $\overline{R}^G(B \cup C) = \overline{R}^G(B) \cup \overline{R}^G(C)$,
5. $\overline{R}^G(B \cap C) \subseteq \overline{R}^G(B) \cap \overline{R}^G(C)$,
6. $\overline{R}^G(\overline{R}^G(B)) \subseteq \overline{R}^G(B)$,
7. $\overline{R}^G(B) = [\overline{R}_G(B^c)]^c$,
8. $B \notin G \Rightarrow \overline{R}^G(B) = B$,
9. $\underline{R}_G(Y) = Y$,
10. $\underline{R}_G(B) \subseteq B$,
11. $B \subseteq C \Rightarrow \underline{R}_G(B) \subseteq \underline{R}_G(C)$,
12. $\underline{R}_G(B) \cup \underline{R}_G(C) \subseteq \underline{R}_G(B \cup C)$,
13. $\underline{R}_G(B \cap C) = \underline{R}_G(B) \cap \underline{R}_G(C)$,
14. $\underline{R}_G(\underline{R}_G(B)) = \underline{R}_G(B)$,
15. $\underline{R}_G(B) = \underline{R}_G[(B^c)]^c$,
16. *If $B^c \notin G$ then, $\underline{R}_G(B) = B$.*

Proof. The proof is clear and so it is omitted. □

Corollary 4.1. *If R is a reflexive relation on a GTS Y , then the lower approximation in Definition 4.1 satisfies Kuratowski's axioms and induces a topology on Y named τ_R^G . It is defined by $\tau_R^G = \{B \subseteq Y : \underline{R}_G(B) = B\}$.*

Theorem 4.3. Let (Y, τ_R^G) be a topological space defined in Corollary 4.1. Then,

1. $Cl_R^G(B) \subseteq Cl_R(B)$ where $Cl_R(B) = B \cup \{y \in Y : < y >_R \cap B \neq \emptyset\}$,
2. $R^G(B)$ is closed.

Proof. 1- Let $y \in Cl_R^G(B)$. Then, $y \in B$ or $< y >_R \in G$. Hence $y \in Cl_R(B)$.

2- Let $y \in Cl_R^G(R^G(B))$. It is implied that $y \in R^G(B)$ or $y \in R^G(R^G(B))$. Since $R^G(R^G(B)) \subseteq R^G(B)$, it follows the result. $\square$

Theorem 4.4. If the relation R is reflexive on the GTS Y and G_1, G_2 be two grills on Y . If $G_1 \subseteq G_2$ then $BND_{G_1} \subseteq BND_{G_2}$.

Proof. Let $y \in BND_{G_1}(B)$. Then, $y \in \overline{R}^{G_1}(A)$ and $y \in (\underline{R}_{G_1})^c$. It follows that $y \in \overline{R}_{G_2}(B)$ and $y \in (\underline{R}_{G_2})^c$. Hence, $y \in BND_{G_2}(B)$. $\square$

In Example 4.5, we show that the rough sets by grill decrease the boundary region compared to Allam's method [5].

Example 4.5. Let $Y = \{1, 2, 3, 4\}$ with $R = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (1, 4), (2, 3), (3, 2)\}$, $G = \{Y\}$ as Table 1.

Table 1: Comparison between Allam's method and a grill method.

B	$\underline{R}(B)$	$\underline{R}_G(B)$	$\overline{R}(B)$	$\overline{R}^G(B)$	$BNDR(B)$	$BNDR_G(B)$
$\{1\}$	$\emptyset$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	Φ
$\{2\}$	$\emptyset$	$\{2\}$	$\{1, 2, 3\}$	$\{2\}$	$\{1, 2, 3\}$	Φ
$\{3\}$	$\emptyset$	$\{3\}$	$\{2, 3\}$	$\{3\}$	$\{2, 3\}$	Φ
$\{4\}$	$\{4\}$	$\{4\}$	$\{1, 4\}$	$\{4\}$	$\{1\}$	Φ
$\{1, 2\}$	$\emptyset$	$\{1, 2\}$	$\{1, 2, 3\}$	$\{1, 2\}$	$\{1, 2, 3\}$	Φ
$\{1, 3\}$	$\emptyset$	$\{1, 3\}$	$\{1, 2, 3\}$	$\{1, 3\}$	$\{1, 2, 3\}$	Φ
$\{1, 4\}$	$\emptyset$	$\{1, 4\}$	$\{1, 4\}$	$\{1, 4\}$	$\{1, 4\}$	Φ
$\{2, 3\}$	$\{2, 3\}$	$\{2, 3\}$	$\{1, 2, 3\}$	$\{2, 3\}$	$\{1\}$	Φ
$\{2, 4\}$	$\{4\}$	$\{2, 4\}$	Y	$\{2, 4\}$	$\{1, 2, 3\}$	Φ
$\{3, 4\}$	$\{4\}$	$\{3, 4\}$	Y	$\{3, 4\}$	$\{1, 2, 3\}$	Φ
$\{1, 2, 3\}$	$\{2, 3\}$	$\{1, 2, 3\}$	$\{1, 2, 3\}$	$\{1, 2, 3\}$	$\{1\}$	Φ
$\{1, 2, 4\}$	$\{1, 4\}$	$\{1, 2, 4\}$	Y	$\{1, 2, 4\}$	$\{2, 4\}$	Φ
$\{1, 3, 4\}$	$\{4\}$	$\{1, 3, 4\}$	Y	$\{1, 3, 4\}$	$\{1, 2, 3\}$	Φ
$\{2, 3, 4\}$	$\{2, 3, 4\}$	$\{2, 3, 4\}$	Y	$\{2, 3, 4\}$	$\{1\}$	Φ
Φ	Φ	Φ	Φ	Φ	Φ	Φ
Y	Y	Y	Y	Y	Φ	Φ

Example 4.6. Let $Y = \{1, 2, 3, 4\}$, $R = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (1, 4), (2, 3), (3, 2)\}$, $G = \{\{2\}, \{1, 2\}, \{2, 3\}, \{2, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, Y\}$ be a grill as shown in Table 2.

From the previous examples, we notice that a grill method decreased the boundary in comparison to Allam's method.

Table 2: Comparison between Allam's method and a grill method.

B	$\underline{R}(B)$	$\underline{R}_G(B)$	$\overline{R}(B)$	$\overline{R}^G(B)$	$BNDR(B)$	$BNDR_G(B)$
$\{1\}$	Φ	Φ	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
$\{2\}$	Φ	$\{2\}$	$\{1, 2, 3\}$	$\{2, 3\}$	$\{1, 2, 3\}$	$\{2\}$
$\{3\}$	Φ	Φ	$\{2, 3\}$	$\{3\}$	$\{2, 3\}$	$\{3\}$
$\{4\}$	$\{4\}$	$\{4\}$	$\{1, 4\}$	$\{4\}$	$\{1\}$	Φ
$\{1, 2\}$	Φ	$\{1, 2\}$	$\{1, 2, 3\}$	$\{1, 2, 3\}$	$\{1, 2, 3\}$	$\{3\}$
$\{1, 3\}$	Φ	Φ	$\{1, 2, 3\}$	$\{1, 3\}$	$\{1, 2, 3\}$	$\{1, 3\}$
$\{1, 4\}$	Φ	$\{4\}$	$\{1, 4\}$	$\{1, 4\}$	$\{1, 4\}$	$\{1\}$
$\{2, 3\}$	$\{2, 3\}$	$\{2, 3\}$	$\{1, 2, 3\}$	$\{1, 2, 3\}$	$\{1\}$	$\{1\}$
$\{2, 4\}$	$\{4\}$	$\{2, 4\}$	Y	Y	$\{1, 2, 3\}$	$\{1, 3\}$
$\{3, 4\}$	$\{4\}$	$\{4\}$	Y	$\{3, 4\}$	$\{1, 2, 3\}$	$\{3\}$
$\{1, 2, 3\}$	$\{2, 3\}$	$\{1, 2, 3\}$	$\{1, 2, 3\}$	$\{1, 2, 3\}$	$\{1\}$	Φ
$\{1, 2, 4\}$	$\{1, 4\}$	$\{1, 2, 4\}$	Y	Y	$\{2, 3\}$	$\{3\}$
$\{1, 3, 4\}$	$\{4\}$	$\{4\}$	Y	$\{1, 3, 4\}$	$\{1, 2, 3\}$	$\{1, 3\}$
$\{2, 3, 4\}$	$\{2, 3, 4\}$	$\{2, 3, 4\}$	Y	Y	$\{1\}$	$\{1\}$
Φ	Φ	Φ	Φ	Φ	Φ	Φ
Y	Y	Y	Y	Y	Φ	Φ

Theorem 4.7. If the relation R is reflexive on Y and G be a grill on Y , then $\beta = \{< y >_R \setminus B : y \in Y, B \notin G\}$ is a basis for ζ_R^G .

Proof. We satisfied the relation $< y >_R \setminus B \subseteq \underline{R}_G(< y >_R \setminus B)$. So, if $z \in < y >_R \setminus B$. Then $< z >_R \subseteq < y >_R$, $< z >_R \cap (< z >_R - B)^c = < z >_R \cap ((< z >_R)^c \cup B) = < z >_R \cap B \subseteq B \notin G$. Thus, $< z >_R \cap (< y >_R - B)^c \notin G$. Hence $z \in \underline{R}_G(< y >_R \setminus B)$. Now, we prove the requirement. Firstly, let $< y >_R \setminus B, < z >_R \setminus C \in \beta$ and $x \in (< y >_R \setminus B) \cap (< z >_R \setminus C)$. Then, $< x >_R \subseteq < y >_R$ and $< x >_R \subseteq < z >_R$ hence, $(< x >_R \subseteq (B \cup C)) \subseteq < y >_R \setminus B$ and $(< x >_R \subseteq (B \cup C)) \subseteq < z >_R \setminus C$, i.e., $(< x >_R \setminus (B \cup C)) \in \beta$, $x \in (< x >_R \setminus (B \cup C)) \subseteq (< y >_R \setminus B) \cap (< z >_R \setminus C)$. Secondly, $\cup\{< y >_R \setminus B : y \in Y, B \notin G\} = Y$. $\square$

Theorem 4.8. If the relation R is the reflexive on Y and G_f be an grill of finite subsets of Y then the topological space (Y, ζ_R^G) is a T_1 -space.

Proof. Since $R^G(\{y\}) = \Phi$ for all $y \in Y$. It follows that $\overline{R}^G(\{y\}) = \{y\}$. $\square$

5. Rough membership functions via grill

In the fifth section, we introduce a new approximation to the membership function for any relation R that is reflexive on a miss-null finite set Y .

Definition 5.1. Let Y be a miss-null finite set, $B \subseteq Y$ and G be a grill on Y . Then a membership function can be defined as $\mu_B^G(y) = \max\{\frac{|(G^y - < y >_R) \cap B|}{|G^y - < y >_R|} : G^y \in G\}$, $y \in Y$ and G^y belongs to G containing y .

Example 5.1. Let $Y = \{1, 2, 3, 4\}$, $R = \{(1, 2), (1, 4), (4, 3), (1, 3), (3, 4)\}$, $\{(1, 1), (2, 2), (3, 3), (4, 4)\}$ and $G = \{\{1\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, Y\}$ be a grill on Y . $< 1 >_R = Y$; $< 2 >_R = \{2\}$; $< 3 >_R = \{3\}$; $< 4 >_R = \{3, 4\}$. Thus, $\{G^1 - < 1 >_R: G^1 \in G\} = \Phi$; $\{G^2 - < 2 >_R: G^2 \in G\} = \{\{1\}, \{1, 3\}, \{1, 4\}, \{1, 3, 4\}\}$; $\{G^3 - < 3 >_R: G^3 \in G\} = \{\{1\}, \{1, 2\}, \{1, 4\}, \{1, 2, 4\}\}$; $\{G^4 - < 4 >_R: G^4 \in G\} = \{\{1\}, \{1, 2\}\}$. Let $B = \{1, 2\}$, then $\mu_B^G(1) = 0$, $\mu_B^G(2) = 0.25$, $\mu_B^G(3) = 0.5$, $\mu_B^G(4) = 1$.

Remark 5.2. By Example 5.1, we note that $\mu_{B^c}^G(y) \neq 1 - \mu_B^G(y)$.

Theorem 5.3. Let Y be a miss-null set, R be a reflexive relation on Y , $B, C \subseteq Y$ and G be a grill on Y . Then the membership function, defined in, satisfies the following:

1. $\mu_B^G(y) = 0 \Rightarrow y \in \underline{R}_G(B)$,
2. $\mu_B^G(y) = 1 \Rightarrow y \in (\overline{R}_G(B))^c$,
3. If $R = \{(y, y) : y \in Y\}$, then μ_B^G is the characteristic function of B ,
4. If yRz , then $\mu_B^G(y) = \mu_B^G(z)$ provided that R is an equivalence relation,
5. $B \subseteq C \Rightarrow \mu_B^G(y) = \mu_C^G(y)$.

Proof. Straightforward. □

6. Conclusion

Pawlak's, Yao's, and Allam's rough set depend on the equivalence relation for Pawlak and general relation for Yao and Allam. General topology has many applications in different fields that were far from it before the emergence of rough set theory. In this study, we give another approach of rough sets by using a grill topological structure. We introduce new definitions of the approximations space and the rough membership function via the grill. By comparing new definitions with Pawlak's, Yao's, and Allam's, we have shown that the current definitions are more general. Also, it is shown that the present method is used to decrease the boundary region in the rough set theory.

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Modelling the impact of interventions against malaria-schistosomiasis co-infection dynamics

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Abstract. Sub-Saharan Africa is known to possess the greater part of the global burden of malaria and schistosomiasis infection. The co-endemicity of these two tropical diseases has initiated the investigation into the mechanisms of their co-infection due to the competing immunological responses associated with each disease in the recent time. It is known that malaria and schistosomiasis have similar epidemiological dispersal and cause challenges to public health and socio-economic development throughout the sub-Saharan region. There are very few works done on the application of optimal control theory to the dynamics of malaria-schistosomiasis co-infection to the best of our knowledge. Our aim here is to predict the impact of the present control interventions to provide necessary information for the policy makers against future control strategies. In this regard, we proposed a malaria-schistosomiasis co-infection model using a system of compartmental deterministic non-linear ordinary differential equations. Optimal control theory was applied to examine the best control strategies against malaria-schistosomiasis disease using insecticides treated bed nets (u_1), prevention by avoiding swimming or wading in freshwater (u_2), treatments of malaria with artemisinin combined therapy (u_3), treatment of schistosomiasis with praziquantel (u_4), treatment of malaria-schistosomiasis (u_5), biological control (u_6) and insecticide spray, destruction of stagnant water and mosquito breeding sites (u_7) as control interventions in line with World Health Organisation (WHO) suggestion. Scenarios on various control strategies were developed using combinations of the seven control interventions either one at a time, combinations of two at a time or more than two at a time to minimize the transmission of malaria-schistosomiasis disease. Finally, the proposed model predicted that there were reductions in the transmission dynamics of malaria-schistosomiasis co-

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infection in the presence of at least two or more control interventions even in areas where there are limited resources.

Keywords: basic reproduction number, malaria-schistosomiasis, co-infection dynamics, non-linear ordinary differential equation, optimal control.

1. Introduction

The process by which the body of a host is affected by multiple pathogen species or by more than one disease concurrently is called *Co-infection*. It is known that co-infection with multiple infectious agent strains can be seen to be present in various diseases [34]. Co-infection with multiple pathogen strains can be seen to be present in several other diseases [34] and co-infection with multiple disease-carrying pathogen strains may be seen to be profound. Some widely common distributed combinations are HIV and tuberculosis, HIV and hepatitis, HIV and malaria and others [34]. The need to study the co-infection between two different diseases is beneficial and of great importance in that co-infection may have adverse and destructive effects on the health of the co-infected persons and also posed some public health challenges in a population [34].

The marriage between malaria and schistosomiasis can be traced to the two diseases sharing the same epidemiological distribution and due to the countering effects the *Schistosoma mansoni* parasites have on the immunological cytokines which may alter the balance between *Th1* and *Th2* type immune responses which lowers immunological control of malaria [19, 20, 47, 49, 54]. Studies have shown that the interaction between malaria and schistosomiasis may reduce the effectiveness of malaria treatment for inhibiting malaria transmission [40]. It is not yet explicit the mechanisms responsible for the increase in the severity of malaria and its symptoms in individuals infected with schistosomiasis [40]. Hence, malaria-schistosomiasis co-infection still poses great challenge to public health and socio-economic advancement in the whole of the sub-Saharan African countries [40].

It is established that Malaria is widespread, severe, and highly endemic vector borne disease which is found in various part of sub-Saharan Africa. There are about 85% of global malaria cases and 90% of malaria deaths at present in the sub-Saharan African countries [40]. Malaria is caused by parasites that are transmitted to people through the bites of infected mosquitoes [8, 62]. At present, malaria can be prevented and cured if the infected person obtain early treatment [1]. There are several studies on the mathematical modelling of malaria that has been carried out and analysed to integrate other real life issues [11, 12, 30, 37, 41, 43, 65].

In various developing countries which includes Africa, the Middle East, Asia and South America schistosomiasis continues to be a public health challenge till date [16]. It is known by estimation that 200 million people are infected with the parasite already where 85% resides in underdeveloped countries in Africa [26]. Schistosomiasis can be transmitted when there is a human contact with con-

taminated fresh water occupied by snails carrying the parasite [16, 45]. This disease can cause liver damage and anaemia, particularly among children [16]. Schistosomiasis is established as the most deadly Neglected Tropical Disease (NTD) which kills an estimated 280,000 person in a year [63, 16]. There are several studies already carried out on the dynamics of schistosomiasis so far [3, 10, 14, 15, 16, 25, 31, 36, 42, 64]. Other useful modelling studies can be found in [5, 9, 13, 21, 24, 27, 29, 32, 38, 39, 48, 52, 55, 56, 58].

To the best of our knowledge, there are few studies carried out so far on the malaria-schistosomiasis co-infection dynamics. Recently, [40] proposed a co-epidemic model of malaria and schistosomiasis transmission dynamics. In their work, they reported major epidemiological coupling between malaria and schistosomiasis in terms of worse malaria incidence and amid persons with *Schistosoma mansoni* intense egg output [6, 7] formulated and analysed a mathematical model of the co-interaction between malaria and schistosomiasis. They carried out qualitative, bifurcation analysis and comprehensive mathematical study of the co-infected model. They further investigated the impact of schistosomiasis and its treatment on malaria. Their results suggested that an increase in the schistosomiasis infected population with treatments may result in a decrease in malaria cases. The aim of this work is to predict the impact of potential intervention packages in order to provide proper guidance to public health and policymakers.

This paper is organised as follows; we present a malaria-schistosomiasis co-infection transmission model and the description of its current malaria-schistosomiasis control interventions in Section 2. In Section 3, we introduce the optimal control problem of the malaria-schistosomiasis co-infection model, existence of optimal control is shown, and Pontryagin's Maximum Principle is applied to find the necessary conditions for the optimal control. In Sections 4, we show our numerical experiments and results while in Section 5, we discussed our conclusion and recommendations.

2. Mathematical model and the description of its current Malaria-Schistosomiasis control interventions

In line with the World Health Organisation (WHO) guidelines [63] for all people at risk or infected with Malaria/Schistosomiasis. The control interventions for the reduction and/or elimination of the two combined disease should be multiple intervention [63]. We introduced multiple control interventions into our malaria-schistosomiasis co-infection model which are insecticides treated bed nets (u_1), prevention by avoiding swimming or wading in freshwater (u_2), treatments of malaria with artemisinin combined therapy (u_3), treatment of schistosomiasis with praziquantel (u_4), treatment of malaria-schistosomiasis (u_5), biological control (u_6) and insecticide spray, destruction of stagnant water and mosquito breeding sites (u_7). Hence, the malaria-schistosomiasis co-infection model [6, 7] and the state variables (Table 1) with parameters in Table 2 yields

a non-autonomous co-infection model (1) below:

$$\begin{aligned}
\frac{dS_h}{dt} &= b_H + \gamma R_m + k R_{hs} + \varphi R_{ms} - (1 - u_1(t))\beta_1 \epsilon_h \sigma S_h I_v \\
&\quad - (1 - u_2(t))\beta_2 S_h I_s - d_h S_h, \\
\frac{dI_m}{dt} &= (1 - u_1(t))\beta_1 \epsilon_h \sigma S_h I_v - (1 - u_2(t))\beta_2 I_{hs} I_m \\
&\quad - (u_3(t)\omega + \theta + d_h + \tau_h) I_m, \\
\frac{dI_{hs}}{dt} &= (1 - u_2(t))\beta_2 S_h I_s - (1 - u_1(t))\beta_1 \epsilon_h \sigma I_v I_{hs} \\
&\quad - (u_4(t)q_h + \rho_h + d_h + \tau_h) I_{hs}, \\
\frac{dV_{ms}}{dt} &= (1 - u_1(t))\beta_1 \epsilon_h \sigma I_v I_{hs} + (1 - u_2(t))\beta_2 I_{hs} I_m \\
&\quad - (u_5(t)\alpha + r + m + d_h + \tau_h) V_{ms}, \\
\frac{dR_m}{dt} &= u_3(t)\omega I_m - (\gamma + d_h) R_m + \phi(1 - \alpha u_5(t)) V_{ms}, \\
\frac{dR_s}{dt} &= u_4(t)q_h I_{hs} - (k + d_h) R_s + (1 - \phi)(1 - \alpha u_6(t)) V_{ms}, \\
\frac{dR_{ms}}{dt} &= \alpha u_5(t) V_{ms} - (\varphi + d_h) R_{ms}, \\
\frac{dS_s}{dt} &= b_s - (1 - u_2(t))\beta_3 (I_{hs} + \eta_1 V_{ms}) S_s - d_s S_s - u_6(t) S_s, \\
\frac{dI_s}{dt} &= (1 - u_2(t))\beta_3 (I_{hs} + \eta_1 V_{ms}) S_s - (d_s + \rho_s) I_s - u_6(t) I_s, \\
\frac{dS_v}{dt} &= b_v - (1 - u_1(t))\beta_4 \epsilon_v \sigma (I_m + \eta_2 V_{ms}) S_v - d_v S_v - u_7(t) S_v, \\
\frac{dI_v}{dt} &= (1 - u_1(t))\beta_4 \epsilon_v \sigma (I_m + \eta_2 V_{ms}) S_v - d_v I_v - u_7(t) I_v
\end{aligned}$$

subject to the initial conditions (2)

$$\begin{aligned}
S_h(0) &= S_{h,0}, I_m(0) = I_{m,0}, I_{hs}(0) = I_{hs,0}, \\
V_{ms}(0) &= V_{ms}(t, 0), R_m(0) = R_{m,0}, R_{hs}(0) = R_{hs,0}, R_{ms}(0) = R_{ms,0}, \\
S_s(0) &= S_{s,0}, I_s(0) = I_{s,0}, S_v(0) = S_{v,0}, I_v(0) = I_{v,0}.
\end{aligned}$$

We provide the associated model variables and parameters in Tables 1 and 2.

3. Optimal control model of the Malaria-Schistosomiasis co-infection

In this section, we develop an optimal control model with the given dynamics and control constraint. We present from the model (1), time dependent controls such as the $0 \leq u_1 \leq 1$, $0 \leq u_2 \leq 1$, $0 \leq u_3 \leq 1$, $0 \leq u_4 \leq 1$, $0 \leq u_5 \leq 1$, $0 \leq u_6 \leq 1$ and $0 \leq u_7 \leq 1$ to inhibit the spread of malaria-schistosomiasis co-infection. The

Table 1: Variables used in the model

Variables	Description
S_H	Susceptible Human
I_m	Individuals infected with malaria only
I_{hs}	Individuals infected with schistosomiasis only
V_{ms}	Individuals infected with both malaria and schistosomiasis
R_m	Individuals who recovered from malaria only
R_{hs}	Individuals who recovered from schistosomiasis only
R_{ms}	Individuals who recovered from both malaria and schistosomiasis
S_s	Susceptible Snail
I_s	Snails infected with schistosomiasis
S_v	Susceptible Mosquito
I_v	Infected Mosquito with malaria

control variable $0 \leq u_1 \leq 1$ is the prevention by the use of insecticide treated bed nets, $0 \leq u_2 \leq 1$ is the prevention by avoiding swimming or wading in fresh water when you are in countries with high transmission of schistosomiasis, by drinking safe water, heating with water, and by vigorous towel drying after an accidental, very brief water exposure, the control $0 \leq u_3 \leq 1$ is the treatment of malaria with drug, the control $0 \leq u_4 \leq 1$ is the treatment of schistosomiasis with drug called praziquantel, $0 \leq u_5 \leq 1$ is the control on the treatment of co-infected person, $0 \leq u_6 \leq 1$ is the use of biological control by introducing a competitor into the snail natural habitat which can control the expansion of the snails population and eventually eliminate the snails from the breeding site while $0 \leq u_7 \leq 1$ is the use of insecticide and destruction of stagnant water and breeding sites. These controls are Lebesgue integrable and bounded.

3.1 Formulation of the optimal control model of Malaria-Schistosomiasis co-infection

The aim here is to find some controls $0 \leq u_1^*(t) \leq 1$, $0 \leq u_2^*(t) \leq 1$, $0 \leq u_3^*(t) \leq 1$, $0 \leq u_4^*(t) \leq 1$, $0 \leq u_5^*(t) \leq 1$, $0 \leq u_6^*(t) \leq 1$, $0 \leq u_7^*(t) \leq 1$ with $t \in [0, T]$ which minimises the cost functional. This implies that we want

$$\begin{aligned} & C(u_1^*(t), u_2^*(t), u_3^*(t), u_4^*(t), u_5^*(t), u_6^*(t), u_7^*(t)) \\ & \leq C(u_1(t), u_2(t), u_3(t), u_4(t), u_5(t), u_6(t), u_7(t)), \end{aligned}$$

for all controls $(u_1, u_2, u_3, u_4, u_5, u_6, u_7) \in U$. Such that controls $u_1^*(t), u_2^*(t), u_3^*(t), u_4^*(t), u_5^*(t), u_6^*(t), u_7^*(t)$ are called *optimal*. $C(u_1^*(t), u_2^*(t), u_3^*(t), u_4^*(t), u_5^*(t), u_6^*(t), u_7^*(t), I_m, I_{hs}, I_s, I_{ms}, I_v) = \int_0^T (w_1 I_m + w_2 I_{hs} + w_3 I_{ms} + w_4 I_s + w_5 I_v + w_6 u_1^2 + w_7 u_2^2 + w_8 u_3^2 + w_9 u_4^2 + w_{10} u_5^2 + w_{11} u_6^2 + w_{12} u_7^2) dt$ where $C : R^{n+m+1} \rightarrow R$ subject to the model equations of malaria-schistosomiasis co-infection (1) subject to the initial conditions (2). $S_H : [0, \infty) \rightarrow \mathbb{R}^n$, $I_m : [0, \infty) \rightarrow \mathbb{R}^n$, $I_{hs} : [0, \infty) \rightarrow \mathbb{R}^n$, $V_{ms} : [0, \infty) \rightarrow \mathbb{R}^n$, $R_m : [0, \infty) \rightarrow \mathbb{R}^n$, $R_{hs} : [0, \infty) \rightarrow \mathbb{R}^n$, $R_{ms} : [0, \infty) \rightarrow \mathbb{R}^n$, $S_s : [0, \infty) \rightarrow \mathbb{R}^n$, $I_s : [0, \infty) \rightarrow \mathbb{R}^n$, $S_v : [0, \infty) \rightarrow \mathbb{R}^n$, $I_v : [0, \infty) \rightarrow \mathbb{R}^n$. The

Table 2: Table showing parameters used in the malaria-schistosomiasis co-infection model

Parameter	Symbol
disease induced death rate due to malaria only	θ
probability of human getting infected with malaria	β_1
rate of acquiring schistosomiasis through contact with an infected snails	β_2
rate of acquiring schistosomiasis through contacts with infected humans	β_3
probability of mosquito getting infected by an infectious human	β_4
Malaria-schistosomiasis immunity waning rate	φ
Schistosomiasis induced death rate	r
malaria disease induced death rate	m
Co-infected who recovered from malaria only	ϕ
disease induced death rate of human due to schistosomiasis only	ρ_H
disease induced death rate of snails due to schistosomiasis only	ρ_s
human spontaneous recovery	ω
rate of loss of immunity to the schistosomiasis disease only	k
rate of loss of immunity to malaria and schistosomiasis disease	φ
human spontaneous recovery from schistosomiasis only	q_H
Recovery rate of co-infected individual	α
modification parameter	η_1
modification parameter	η_2
per capita birth rate of mosquitoes	b_v
per capita birth rate of snails	b_s
Natural death rate of humans	d_H
per capita biting rate of mosquitoes	ε_v
contact rate of vector per human per unit time	σ
per capita biting rate of humans	ε_h
Natural death rate of mosquitoes	d_v

Parameter	Symbol
Natural death rate of mosquitoes	d_s
rate of loss of immunity to the malaria disease only	γ
per capita birth rate of humans	b_h

aim of this work is to minimise the total number of infected human with malaria only, infected human with schistosomiasis only, infected human with schistosomiasis and malaria, infected mosquito and infected snails while at the same time minimising the associated cost of the use of insecticide treated bednets (u_1), use of prevention by avoiding swimming or wading in freshwater when one is in countries with high transmission of schistosomiasis (u_2), treatments of malaria with artemisinin combined therapy (u_3), treatment of schistosomiasis with praziquantel (u_4), the treatment of malaria-schistosomiasis (u_5), use of biological control (u_6), and the use of insecticide spray and destruction of stagnant water and mosquito breeding sites (u_7), of the population given the initial population sizes of all eleven compartments S_H , I_m , I_{hs} , V_{ms} , R_m , R_{hs} ,

$R_{ms}, S_s, I_s, S_v, I_v$. The first term in the cost functional $w_1 I_m(t)$ represents the total number of individuals infected with malaria only, $w_2 I_{hs}(t)$ stands for the total number of individuals infected with schistosomiasis only, $w_3 V_{ms}$ represents the total number of individuals infected with both malaria and schistosomiasis. The terms $w_4 I_s$ corresponds to the total number infected snails and $w_5 I_v$ denotes the total number of infected mosquitoes. The term $w_6 u_1^2$ denotes the cost associated with the use of insecticide treated bednets, $w_7 u_2^2$ denotes the cost associated with the use of preventive measures, while the term $w_8 u_3^2$ corresponds to the cost associated with the use of ACT drug for treatment of malaria. The term $w_9 u_4^2$ represents the cost associated with the use of treatment of malaria-schistosomiasis use of praziquantel for the treatment of schistosomiasis, $w_{10} u_5^2$ is the cost associated with the use of treatment of malaria-schistosomiasis, the term $w_{11} u_6^2$ is the cost associated with the use of biological control while the term $w_{12} u_7^2$ is the cost associated with the use of insecticide spray and destruction of stagnant water and mosquito breeding sites. The quadratic costs follows the nonlinear pattern of the cost of the controls and that was why it has been applied in this work. [59, 60, 61, 28, 23]. Therefore, the combined control functions $u_1(t), u_2(t), u_3(t), u_4(t), u_5(t), u_6(t)$, and $u_7(t)$ are assumed to remain in an admissible class of measurable functions. Hence, we seek the optimal control $u_1^*(t), u_2^*(t), u_3^*(t), u_4^*(t), u_5^*(t), u_6^*(t), u_7^*(t)$ such that

$$\begin{aligned}
 & C(u_1^*(t), u_2^*(t), u_3^*(t), u_4^*(t), u_5^*(t), u_6^*(t), u_7^*(t)) \\
 & = \min\{C(u_1(t), u_2(t), u_3(t), u_4(t), u_5(t), u_6(t), u_7(t)) : u_1, u_2, u_3, u_4, u_5, u_6, u_7 \in U\};
 \end{aligned}$$

where U is the control set defined above.

3.2 Existence of an optimal control model of malaria-schistosomiasis co-infection control model

In this section, we intend to state and proof that an optimal control pair $(S_h^*, I_m^*, I_{hs}^*, V_{ms}^*, R_m^*, R_s^*, R_{ms}^*, S_s^*, I_s^*, S_v^*, I_v^*, u_1^*, u_2^*, u_3^*, u_4^*, u_5^*, u_6^*, u_7^*)$ exist for the malaria-schistosomiasis co-infection control model (1) employing the result from Filippov-Cesari Existence Theorem (1983) ([36]).

Theorem 3.1. *The optimal control problem $C(u_1, u_2, u_3, u_4, u_5, u_6, u_7) = \min_{\{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}} \int_0^T (w_1 I_m + w_2 I_{hs} + w_3 I_{ms} + w_4 I_s + w_5 I_v + w_6 u_1^2 + w_7 u_2^2 + w_8 u_3^2 + w_9 u_4^2 + w_{10} u_5^2 + w_{11} u_6^2 + w_{12} u_7^2) dt$ where $u = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7 : u_i \text{ measurable } 0 \leq u_1(t) \leq 1, 0 \leq u_2(t) \leq 1 t \in [t_0, t_f] \in \mathbb{R}^+ \text{ for } i = 1, 2, \dots\}$ subject to the dynamic constraints of system equations (1) with initial conditions (2) has a solution.*

Proof. Suppose $B(t, S_h(t), I_m(t), I_{hs}(t), V_{ms}(t), R_m(t), R_s(t), R_{ms}(t), S_s(t), I_s(t), S_v(t), I_v(t)) = \{g(S_h(t), I_m(t), I_{hs}(t), V_{ms}(t), R_m(t), R_s(t), R_{ms}(t), S_s(t), I_s(t), S_v(t), I_v(t), u_1, u_2, u_3, u_4, u_5, u_6, u_7) + \zeta, f(S_h(t), I_m(t), I_{hs}(t), V_{ms}(t), R_m(t), R_s(t), R_{ms}(t), S_s(t), I_s(t), S_v(t), I_v(t), u_1, u_2, u_3, u_4, u_5, u_6, u_7) : \zeta \leq 0, u_1, u_2, u_3, u_4, u_5, u_6, u_7 \in U\}$.

Consider $z_1, z_2 \in B(t, S_h(t), I_m(t), I_{hs}(t), V_{ms}(t), R_m(t), R_s(t), R_{ms}(t), S_s(t), I_s(t), S_v(t), I_v(t))$. To show that $B(t, S_h(t), I_m(t), I_{hs}(t), V_{ms}(t), R_m(t), R_s(t), R_{ms}(t), S_s(t), I_s(t), S_v(t), I_v(t))$ is convex for each $(t, S_h(t), I_m(t), I_{hs}(t), V_{ms}(t), R_m(t), R_s(t), R_{ms}(t), S_s(t), I_s(t), S_v(t), I_v(t))$.

It will be shown that the line connecting z_1 and z_2 remains completely in $B(t, S_h(t), I_m(t), I_{hs}(t), V_{ms}(t), R_m(t), R_s(t), R_{ms}(t), S_s(t), I_s(t), S_v(t), I_v(t))$. Thus, it is required to show that $\lambda z_1 + (1 - \lambda)z_2 \in B(t, S_h(t), I_m(t), I_{hs}(t), V_{ms}(t), R_m(t), R_s(t), R_{ms}(t), S_s(t), I_s(t), S_v(t), I_v(t))$, $\forall \lambda \in [0, 1]$.

The fact that $z_i \in B(t, S_h(t), I_m(t), I_{hs}(t), V_{ms}(t), R_m(t), R_s(t), R_{ms}(t), S_s(t), I_s(t), S_v(t), I_v(t))$ indicates that there exist $\zeta_1 \leq 0$, $\zeta_2 \leq 0$, and control vectors $u_1, u_2 \in U$ such that $z_i = \{g(S_h(t), I_m(t), I_{hs}(t), V_{ms}(t), R_m(t), R_s(t), R_{ms}(t), S_s(t), I_s(t), S_v(t), I_v(t), u_i) + \zeta_i, f(S_h(t), I_m(t), I_{hs}(t), V_{ms}(t), R_m(t), R_s(t), R_{ms}(t), S_s(t), I_s(t), S_v(t), I_v(t), u_i)\}$ for $i = 1, 2$. Then, it yields $\lambda(g(S_h(t), I_m(t), I_{hs}(t), V_{ms}(t), R_m(t), R_s(t), R_{ms}(t), S_s(t), I_s(t), S_v(t), I_v(t), u_1) + \zeta_1) + (1 - \lambda)(g(S_h(t), I_m(t), I_{hs}(t), V_{ms}(t), R_m(t), R_s(t), R_{ms}(t), S_s(t), I_s(t), S_v(t), I_v(t), u_2) + \zeta_2)$.

$\lambda(w_1 I_m + w_2 I_{hs} + w_3 I_{ms} + w_4 I_s + w_5 I_v + w_6 u_1^2) + (1 - \lambda)(w_1 I_m + w_2 I_{hs} + w_3 I_{ms} + w_4 I_s + w_5 I_v + w_6 u_2^2) + \lambda(\zeta_1 + (1 - \lambda)\zeta_2)$.

Suppose $u_3 = \sqrt{\lambda u_1^2 + (1 - \lambda)u_2^2}$ it is observed that $u_3 \in U$. Likewise, suppose $\zeta_3 = \lambda(\zeta_1 + (1 - \lambda)\zeta_2)$ it is observed that $\zeta_3 \leq 0$. Therefore, the first component of the convex combination belongs to $B(t, S_h(t), E_h(t), I_h(t), R_h(t), S_v(t), E_v(t), I_v(t))$. Next, the second component is verified,

$\lambda(f(S_h(t), I_m(t), I_{hs}(t), V_{ms}(t), R_m(t), R_s(t), R_{ms}(t), S_s(t), I_s(t), S_v(t), I_v(t), u_1) + (1 - \lambda)f(S_h(t), I_m(t), I_{hs}(t), V_{ms}(t), R_m(t), R_s(t), R_{ms}(t), S_s(t), I_s(t), S_v(t), I_v(t), u_2) = \lambda(\beta_3(I_{hs} + \eta_1 V_{ms})S_s - (d_s + \rho_s)I_s - u_1(t)I_s) + (1 - \lambda)(\beta_3(I_{hs} + \eta_1 V_{ms})S_s - (d_s + \rho_s)I_s - u_1(t)I_s) = \lambda\beta_3(I_{hs} + \eta_1 V_{ms})S_s - \lambda(d_s + \rho_s)I_s - \lambda u_1(t)I_s + (1 - \lambda)\beta_3(I_{hs} + \eta_1 V_{ms})S_s - (1 - \lambda)u_2(t)I_s - (1 - \lambda)(d_s + \rho_s)I_s = \lambda\beta_3(I_{hs} + \eta_1 V_{ms})S_s - \lambda(d_s + \rho_s)I_s + (1 - \lambda)\beta_3(I_{hs} + \eta_1 V_{ms})S_s - (1 - \lambda)(d_s + \rho_s)I_s - (\lambda u_1(t) + (1 - \lambda)u_2(t))I_s$. Suppose $u_4 = \lambda u_1 + (1 - \lambda)u_2$, it is observed that $u_4 \in U$. In conclusion, the convex combination $\lambda(\zeta_1 + (1 - \lambda)\zeta_2)$ is in $B(t, S_h(t), E_h(t), I_h(t), R_h(t), S_v(t), E_v(t), I_v(t))$. Clearly, U is compact. Next, boundedness of the solution of (17) is shown. Indeed $\frac{dN_h}{dt} = b_H - d_h N_h - \psi I_h$, which implies $\frac{b_h}{d_h + \psi} \leq \liminf_{t \rightarrow \infty} N_h(t) \leq \limsup_{t \rightarrow \infty} N_h(t) \leq \frac{b_h}{d_h}$ and $b_v - d_v N_v(t) \leq \frac{dN_v(t)}{dt} \leq b_v - d_v N_v(t)$ denotes $\frac{b_v}{d_v} \leq \liminf_{t \rightarrow \infty} N_v(t) \leq \limsup_{t \rightarrow \infty} N_v(t) \leq \frac{b_v}{(d_v)}$ and $b_s - d_s N_s(t) \leq \frac{dN_s(t)}{dt} \leq b_s - d_s N_s(t)$ denotes $\frac{b_s}{d_s} \leq \liminf_{t \rightarrow \infty} N_s(t) \leq \limsup_{t \rightarrow \infty} N_s(t) \leq \frac{b_s}{(d_s)}$. Thus, $\sup N_h(t) \leq \frac{b_H}{d_h}$, $\sup N_v(t) \leq \frac{b_v}{d_v}$ and $\sup N_s(t) \leq \frac{b_s}{d_s}$. This complete the proof. $\square$

3.3 Solution to the optimal control problem

To solve the optimal control problem, the Hamiltonian function is formulated for the optimal control problem, which is given by: $H(S_h(t), I_{hs}(t), I_m(t), V_{ms}(t), R_m(t), R_{hs}(t), R_{ms}(t), S_s(t), I_s(t), S_v(t), I_v(t), \lambda_{S_h}(t), \lambda_{I_m}(t), \lambda_{I_{hs}}(t), \lambda_{V_{ms}}(t), \lambda_{R_m}(t), \lambda_{R_{hs}}(t), \lambda_{R_{ms}}(t), \lambda_{S_s}(t), \lambda_{I_s}(t), \lambda_{S_v}(t), \lambda_{I_v}(t), t) = w_1 I_m(t) + w_2 I_{hs}(t) +$

$w_3I_v(t) + w_4I_s(t) + w_5V_{ms}(t) + w_6u_1^2(t) + w_7u_2^2(t) + w_8u_3^2(t) + w_9u_4^2(t) + w_{10}u_5^2(t) + w_{11}u_6^2(t) + w_{12}u_7^2(t) + \lambda_{S_h}(t)\frac{dS_h}{dt} + \lambda_{I_m}(t)\frac{dI_m}{dt} + \lambda_{I_{hs}}(t)\frac{dI_{hs}}{dt} + \lambda_{V_{ms}}(t)\frac{dV_{ms}}{dt} + \lambda_{R_m}(t)\frac{dR_m}{dt} + \lambda_{R_s}(t)\frac{dR_s}{dt} + \lambda_{R_{ms}}(t)\frac{dR_{ms}}{dt} + \lambda_{S_s}(t)\frac{dS_s}{dt} + \lambda_{I_s}(t)\frac{dI_s}{dt} + \lambda_{S_v}(t)\frac{dS_v}{dt} + \lambda_{I_v}(t)\frac{dI_v}{dt} = w_1I_m(t) + w_2I_{hs}(t) + w_3I_v(t) + w_4I_s(t) + w_5V_{ms}(t) + w_6u_1^2(t) + w_7u_2^2(t) + w_8u_3^2(t) + w_9u_4^2(t) + w_{10}u_5^2(t) + w_{11}u_6^2(t) + w_{12}u_7^2(t) + \lambda_{S_h}(t)[b_H + \gamma R_m + kR_{hs} + \varphi R_{ms} - (1 - u_1)\beta_1\epsilon_h\sigma S_hI_v - (1 - u_2)\beta_2S_hI_s - d_hS_h] + \lambda_{I_m}(t)[(1 - u_1)\beta_1\epsilon_h\sigma S_hI_v - (1 - u_2)\beta_2I_sI_m - (u_3\omega + \theta + d_h + \tau_h)I_m] + \lambda_{I_{hs}}(t)[(1 - u_2)\beta_2S_hI_s - (1 - u_1)\beta_1\epsilon_h\sigma I_vI_{hs} - (u_4q_h + \rho_h + d_h + \tau_h)I_{hs}] + \lambda_{V_{ms}}(t)[(1 - u_1)\beta_1\epsilon_h\sigma I_vI_{hs} + (1 - u_2)\beta_2I_{hs}I_m - (u_5\alpha + r + m + d_H + \tau_h)V_{ms}] + \lambda_{R_m}(t)[u_3\omega I_m - (\gamma + d_h)R_m + \phi(1 - \alpha u_5)V_{ms}] + \lambda_{R_s}(t)[u_4q_hI_{hs} - (k + d_h)R_s + (1 - \phi)(1 - \alpha u_6)V_{ms}] + \lambda_{R_{ms}}(t)[\alpha u_5V_{ms} - (\varphi + d_h)R_{ms}] + \lambda_{S_s}(t)[b_s - \beta_3(I_{hs} + \eta_1V_{ms})S_s - d_sS_s] + \lambda_{I_s}(t)[\beta_3(I_{hs} + \eta_1V_{ms})S_s - (d_s + \rho_s)I_s] + \lambda_{S_v}(t)[b_v - (1 - u_1)\beta_4\epsilon_v\sigma(I_m + \eta_2V_{ms})S_v - d_vS_v - u_7S_v] + \lambda_{I_v}(t)[(1 - u_1)\beta_4\epsilon_v\sigma(I_m + \eta_2V_{ms})S_v - d_vI_v - u_7I_v], where $\lambda_{S_h}(t), \lambda_{I_m}(t), \lambda_{I_{hs}}(t), \lambda_{V_{ms}}(t), \lambda_{R_m}(t), \lambda_{R_{hs}}(t), \lambda_{R_{ms}}(t), \lambda_{S_s}(t), \lambda_{I_s}(t), \lambda_{S_v}(t), \lambda_{I_v}(t)$ indicates the adjoint function to be determined.$

The first and second order necessary condition with respect to the control are optimal conditions for $u_1^*, u_2^*, u_3^*, u_4^*, u_5^*, u_6^*, u_7^*$ if

$$\begin{aligned}
 &H(S_H(t), I_{hs}(t), I_m(t), V_{ms}(t), R_m(t), R_{hs}(t), R_{ms}(t), S_s(t), I_s(t), S_v(t), \\
 &I_v(t), \lambda_{S_h}(t), \lambda_{I_m}(t), \lambda_{I_{hs}}(t), \lambda_{V_{ms}}(t), \lambda_{R_m}(t), \lambda_{R_{hs}}(t), \lambda_{R_{ms}}(t), \\
 &\lambda_{S_s}(t), \lambda_{I_s}(t), \lambda_{S_v}(t), \lambda_{I_v}(t), t) = H.
 \end{aligned}$$

In order to shorten the notation let us denote $S_h(t), I_{hs}(t), \dots, \lambda_{S_h}(t), \dots, t$ by " $\odot$ ".

$$\nabla_{u_1}H(\odot)|_{u_1=u_1^*, u_1=u_1^*, u_2=u_2^*, u_3=u_3^*, u_4=u_4^*, u_5=u_5^*, u_6=u_6^*, u_7=u_7^*} = \frac{\partial H}{\partial u_1}|_{u_1=u_1^*},$$

$$\begin{aligned}
 &2w_6u_1^* + \beta_1\epsilon_h\sigma S_hI_v\lambda_{S_h}(t) - \beta_1\epsilon_h\sigma S_hI_v\lambda_{I_m}(t) + \beta_1\epsilon_h\sigma I_vI_{hs}\lambda_{I_{hs}}(t) - \\
 &\beta_1\epsilon_h\sigma I_vI_{hs}\lambda_{V_{ms}}(t) + \beta_4\epsilon_v\sigma(I_m + V_{ms})S_v\lambda_{S_v}(t) - \beta_4\epsilon_v\sigma(I_m + V_{ms})S_v\lambda_{I_v}(t) = 0
 \end{aligned}$$

$$u_1^* = \frac{\beta_1\epsilon_h\sigma S_hI_v(\lambda_{I_m}(t) - \lambda_{S_h}(t)) + \beta_1\epsilon_h\sigma I_vI_{hs}(\lambda_{V_{ms}}(t) - \lambda_{I_{hs}}(t)) + \beta_4\epsilon_v\sigma(I_m + V_{ms})S_v(\lambda_{I_v}(t) - \lambda_{S_v}(t))}{2w_6},$$

$$\nabla_{u_2}H(\odot)|_{u_1=u_1^*, u_1=u_1^*, u_2=u_2^*, u_3=u_3^*, u_4=u_4^*, u_5=u_5^*, u_6=u_6^*, u_7=u_7^*} = \frac{\partial H}{\partial u_2}|_{u_2=u_2^*}$$

$$\begin{aligned}
 &2w_7u_2^* + \beta_2S_hI_s\lambda_{S_h}(t) + \beta_2I_{hs}I_m\lambda_{I_m}(t) - \beta_2S_hI_s\lambda_{I_{hs}}(t) - \beta_2I_{hs}I_m\lambda_{V_{ms}}(t) + \beta_3(I_{hs} + \\
 &\eta_1V_{ms})S_s\lambda_{S_s}(t) - \beta_3(I_{hs} + \eta_1V_{ms})S_s\lambda_{S_v}(t) = 0,
 \end{aligned}$$

$$u_2^* = \frac{\beta_2S_hI_s(\lambda_{I_{hs}}(t) - \lambda_{S_h}(t)) + \beta_2I_{hs}I_m(\lambda_{V_{ms}}(t) - \lambda_{I_m}(t)) + \beta_3(I_{hs} + \eta_1V_{ms})S_s(\lambda_{S_v}(t) - \lambda_{S_s}(t))}{2w_7}.$$

$$\nabla_{u_3}H(\odot)|_{u_1=u_1^*, u_1=u_1^*, u_2=u_2^*, u_3=u_3^*, u_4=u_4^*, u_5=u_5^*, u_6=u_6^*, u_7=u_7^*} = \frac{\partial H}{\partial u_3}|_{u_3=u_3^*},$$

$$2w_3u_3^* - w\lambda_{I_m}(t) = 0, \quad u_3^* = \frac{\lambda_{I_m}(t)wI_m}{2w_3}.$$

$$\nabla_{u_4}H(\odot)|_{u_1=u_1^*, u_1=u_1^*, u_2=u_2^*, u_3=u_3^*, u_4=u_4^*, u_5=u_5^*, u_6=u_6^*, u_7=u_7^*} = \frac{\partial H}{\partial u_4}|_{u_4=u_4^*}$$

$$2w_9u_4^* - q_h I_{hs} \lambda_{I_{hs}}(t) = 0, \quad u_4^* = \frac{h I_{hs} \lambda_{I_{hs}}(t)}{2w_9}.$$

$$\nabla_{u_5}H(\odot)|_{u_1=u_1^*, u_1=u_1^*, u_2=u_2^*, u_3=u_3^*, u_4=u_4^*, u_5=u_5^*, u_6=u_6^*, u_7=u_7^*} = \frac{\partial H}{\partial u_5}|_{u_5=u_5^*}$$

$$2w_{10}u_5^* - \alpha V_{ms} \lambda_{V_{ms}}(t) - \phi \alpha V_{ms} \lambda_{R_m}(t) - \alpha V_{ms} \lambda_{R_{hs}}(t) + \phi \alpha V_{ms} \lambda_{R_{hs}}(t) \\ + \alpha V_{ms} \lambda_{R_{ms}}(t) = 0;$$

$$u_5^* = \frac{\alpha V_{ms} \lambda_{V_{ms}}(t) - \phi \alpha V_{ms} \lambda_{R_m}(t) - \alpha V_{ms} \lambda_{R_{hs}}(t) + \phi \alpha V_{ms} \lambda_{R_{hs}}(t) + \alpha V_{ms} \lambda_{R_{ms}}(t)}{2w_{10}}.$$

$$\nabla_{u_6}H(\odot)|_{u_1=u_1^*, u_1=u_1^*, u_2=u_2^*, u_3=u_3^*, u_4=u_4^*, u_5=u_5^*, u_6=u_6^*, u_7=u_7^*} = \frac{\partial H}{\partial u_6}|_{u_6=u_6^*}$$

$$2w_{11}u_6^* - \lambda_{S_s}(t)S_s - \lambda_{I_s}(t)I_s = 0, \quad u_6^* = \frac{\lambda_{S_s}(t)S_s - \lambda_{I_s}(t)I_s}{2w_{11}}$$

$$\nabla_{u_7}H(\odot)|_{u_1=u_1^*, u_1=u_1^*, u_2=u_2^*, u_3=u_3^*, u_4=u_4^*, u_5=u_5^*, u_6=u_6^*, u_7=u_7^*} = \frac{\partial H}{\partial u_7}|_{u_7=u_7^*}$$

$$2w_{12}u_7^* - \lambda_{S_v}(t)S_v - \lambda_{I_v}(t)I_v = 0, \quad u_7^* = \frac{\lambda_{S_v}(t)S_v - \lambda_{I_v}(t)I_v}{2w_{12}}.$$

$$\nabla_{u_1}H|_{u_1} = u_1^* = \frac{\partial^2 H}{\partial u_1^2} = 2w_6 > 0$$

$$\nabla_{u_2}H|_{u_2} = u_2^* = \frac{\partial^2 H}{\partial u_2^2} = 2w_7 > 0$$

$$\nabla_{u_3}H|_{u_3} = u_3^* = \frac{\partial^2 H}{\partial u_3^2} = 2w_8 > 0$$

$$\nabla_{u_4}H|_{u_4} = u_4^* = \frac{\partial^2 H}{\partial u_4^2} = 2w_9 > 0$$

$$\nabla_{u_5}H|_{u_5} = u_5^* = \frac{\partial^2 H}{\partial u_5^2} = 2w_{10} > 0$$

$$\nabla_{u_6}H|_{u_6} = u_6^* = \frac{\partial^2 H}{\partial u_6^2} = 2w_{11} > 0$$

$$\nabla_{u_7}H|_{u_7} = u_7^* = \frac{\partial^2 H}{\partial u_7^2} = 2w_{12} > 0.$$

(i) If $\frac{\partial H}{\partial u_1} < 0$ i.e. $u_1^* = 0$, if

$$\frac{\beta_1 \epsilon_h \sigma S_h I_v (\lambda_{I_m}(t) - \lambda_{S_h}(t)) + \beta_1 \epsilon_h \sigma I_v I_{hs} (\lambda_{V_{ms}}(t) - \lambda_{I_{hs}}(t)) \\ + \beta_4 \epsilon_v \sigma (I_m + V_{ms}) S_v (\lambda_{I_v}(t) - \lambda_{S_v}(t))}{2w_6} \leq 0 \leq u_1^* \leq 1 \text{ if}$$

$$\frac{\partial H}{\partial u_1} = 0 \text{ i.e. } 0 \leq u_1 \leq 1 \text{ if}$$

$$\frac{\beta_1 \epsilon_h \sigma S_h I_v (\lambda_{I_m}(t) - \lambda_{S_h}(t)) + \beta_1 \epsilon_h \sigma I_v I_{hs} (\lambda_{V_{ms}}(t) \\ - \lambda_{I_{hs}}(t)) + \beta_4 \epsilon_v \sigma (I_m + V_{ms}) S_v (\lambda_{I_v}(t) - \lambda_{S_v}(t))}{2w_6} \leq 1$$

$$u_1^*(t) = 1, \text{ if } \frac{\partial H}{\partial u_1} > 0 \text{ i.e. } u_1^* \geq 1, \text{ if}$$

$$\frac{\beta_1 \epsilon_h \sigma S_h I_v (\lambda_{I_m}(t) - \lambda_{S_h}(t)) \\ + \beta_1 \epsilon_h \sigma I_v I_{hs} (\lambda_{V_{ms}}(t) - \lambda_{I_{hs}}(t)) + \beta_4 \epsilon_v \sigma (I_m + V_{ms}) S_v (\lambda_{I_v}(t) - \lambda_{S_v}(t))}{2w_6} > 0.$$

(ii) If $\frac{\partial H}{\partial u_2} < 0$ i.e. $u_2^* = 0$, if

$$\frac{\beta_2 S_h I_s (\lambda_{I_{hs}}(t) - \lambda_{S_h}(t)) + \beta_2 I_{hs} I_m (\lambda_{V_{ms}}(t) - \lambda_{I_m}(t)) + \beta_3 (I_{hs} + \eta_1 V_{ms}) S_s (\lambda_{S_v}(t) - \lambda_{S_s}(t))}{2w_7} \leq 0$$

$$0 \leq u_2^* \leq 1 \text{ if } \frac{\partial H}{\partial u_2} = 0 \text{ i.e. } 0 \leq u_2 \leq 1 \text{ if}$$

$$0 < \frac{\beta_2 S_h I_s (\lambda_{I_{hs}}(t) - \lambda_{S_h}(t)) + \beta_2 I_{hs} I_m (\lambda_{V_{ms}}(t) - \lambda_{I_m}(t)) + \beta_3 (I_{hs} + \eta_1 V_{ms}) S_s (\lambda_{S_v}(t) - \lambda_{S_s}(t))}{2w_7} \leq 1$$

$$u_2^*(t) = 1, \text{ if } \frac{\partial H}{\partial u_2} > 0 \text{ i.e. } u_2^* \geq 1, \text{ if}$$

$$\frac{\beta_2 S_h I_s (\lambda_{I_{hs}}(t) - \lambda_{S_h}(t)) + \beta_2 I_{hs} I_m (\lambda_{V_{ms}}(t) - \lambda_{I_m}(t)) + \beta_3 (I_{hs} + \eta_1 V_{ms}) S_s (\lambda_{S_v}(t) - \lambda_{S_s}(t))}{2w_7} > 0.$$

$$(iii) \text{ If } \frac{\partial H}{\partial u_3} < 0 \text{ i.e. } u_3^* = 0, \text{ if } \frac{\lambda_{I_m}(t) w_{I_m}}{2w_3} \leq 0, 0 \leq u_3^* \leq 1 \text{ if } \frac{\partial H}{\partial u_3} = 0 \text{ i.e.}$$

$$0 \leq u_3 \leq 1 \text{ if}$$

$$0 < \frac{\lambda_{I_m}(t) w_{I_m}}{2w_3} \leq 1, u_3^*(t) = 1, \text{ if } \frac{\partial H}{\partial u_3} > 0 \text{ i.e. } u_3^* \geq 1, \text{ if } \frac{\lambda_{I_m}(t) w_{I_m}}{2w_3} > 0.$$

$$(iv) \text{ If } \frac{\partial H}{\partial u_4} < 0 \text{ i.e. } u_4^* = 0, \text{ if } \frac{q_h I_{hs} \lambda_{I_{hs}}(t)}{2w_9} \leq 0, 0 \leq u_4^* \leq 1 \text{ if } \frac{\partial H}{\partial u_4} = 0$$

$$\text{i.e. } 0 \leq u_4 \leq 1 \text{ if } 0 < \frac{q_h I_{hs} \lambda_{I_{hs}}(t)}{2w_9} \leq 1, u_4^*(t) = 1, \text{ if } \frac{\partial H}{\partial u_4} > 0 \text{ i.e. } u_4^* \geq 1, \text{ if}$$

$$\frac{q_h I_{hs} \lambda_{I_{hs}}(t)}{2w_9} > 0.$$

$$(v) \text{ If } \frac{\partial H}{\partial u_5} < 0 \text{ i.e. } u_5^* = 0, \text{ if}$$

$$\frac{\alpha V_{ms} \lambda_{V_{ms}}(t) - \phi \alpha V_{ms} \lambda_{R_m}(t) - \alpha V_{ms} \lambda_{R_{hs}}(t) + \phi \alpha V_{ms} \lambda_{R_{hs}}(t) + \alpha V_{ms} \lambda_{R_{ms}}(t)}{2w_{10}} \leq 0$$

$$0 \leq u_5^* \leq 1 \text{ if } \frac{\partial H}{\partial u_5} = 0 \text{ i.e. } 0 \leq u_5 \leq 1 \text{ if}$$

$$0 < \frac{\alpha V_{ms} \lambda_{V_{ms}}(t) - \phi \alpha V_{ms} \lambda_{R_m}(t) - \alpha V_{ms} \lambda_{R_{hs}}(t) + \phi \alpha V_{ms} \lambda_{R_{hs}}(t) + \alpha V_{ms} \lambda_{R_{ms}}(t)}{2w_{10}} \geq 1, \text{ if}$$

$$\frac{\alpha V_{ms} \lambda_{V_{ms}}(t) - \phi \alpha V_{ms} \lambda_{R_m}(t) - \alpha V_{ms} \lambda_{R_{hs}}(t) + \phi \alpha V_{ms} \lambda_{R_{hs}}(t) + \alpha V_{ms} \lambda_{R_{ms}}(t)}{2w_{10}} > 0.$$

$$(vi) \text{ If } \frac{\partial H}{\partial u_6} < 0 \text{ i.e. } u_6^* = 0, \text{ if } \frac{\lambda_{S_s}(t) S_s - \lambda_{I_s}(t) I_s}{2w_{11}} \leq 0, 0 \leq u_6^* \leq 1 \text{ if } \frac{\partial H}{\partial u_6} = 0, \text{ i.e.}$$

$$0 \leq u_6 \leq 1 \text{ if } 0 < \frac{\lambda_{S_s}(t) S_s - \lambda_{I_s}(t) I_s}{2w_{11}} \geq 1, \text{ if } \frac{\lambda_{S_s}(t) S_s - \lambda_{I_s}(t) I_s}{2w_{11}} > 0.$$

$$(vii) \text{ If } \frac{\partial H}{\partial u_7} < 0 \text{ i.e. } u_7^* = 0, \text{ if } \frac{\lambda_{S_v}(t) S_v - \lambda_{I_v}(t) I_v}{2w_{12}} \leq 0, 0 \leq u_7^* \leq 1 \text{ if } \frac{\partial H}{\partial u_7} = 0 \text{ i.e.}$$

$$0 \leq u_7 \leq 1 \text{ if } 0 < \frac{\lambda_{S_v}(t) S_v - \lambda_{I_v}(t) I_v}{2w_{12}} \geq 1, \text{ if } \frac{\lambda_{S_v}(t) S_v - \lambda_{I_v}(t) I_v}{2w_{12}} > 0.$$

The costate system and transversality conditions (3) are

$$\begin{aligned} \lambda'_{S_H} &= (\lambda_{S_H} - \lambda_{I_m})(1 - u_1)\beta_1 \epsilon_h \sigma I_v + \lambda_{S_H}((1 - u_2)\beta_2 I_s + d_h) \\ &\quad - \lambda_{I_{hs}}((1 - u_2))\beta_2 I_s \\ \lambda'_{I_m} &= -w_1 + \lambda_{I_m}((1 - u_2)\beta_2 I_s + 2u_3\omega + \theta + d_h + \tau_h)) \\ &\quad - \lambda_{V_{ms}}(1 - u_2)\beta_2 I_s - \lambda_{R_m} u_3\omega + (\lambda_{S_v} - \lambda_{I_v})(1 - u_1)\beta_4 \epsilon_v \sigma S_v \\ \lambda'_{I_{hs}} &= -w_2 + (\lambda_{I_{hs}} - \lambda_{V_{ms}})(1 - u_1)\beta_1 \epsilon_h \sigma I_v + \lambda_{I_{hs}}(u_4 q_h + d_h + \rho_h + \tau_h) - \\ &\quad \lambda_{V_{ms}}(1 - u_1)\beta_1 \epsilon_h \sigma I_v - \lambda_{R_{hs}} u_4 q_h + \lambda_{S_s}(1 - u_2)\beta_3 S_s - \lambda_{I_v}(1 - u_2)\beta_3 S_s \\ \lambda'_{V_{ms}} &= -w_5 + \lambda_{V_{ms}}(\alpha u_5 + r + m + d_h + \tau_h) + \lambda_{R_m} \phi(\alpha u_5 - 1) \\ &\quad - \lambda_{R_{hs}}(1 - \phi)(1 - \alpha u_5) - \lambda_{R_{ms}} \alpha u_5 + \lambda_{S_s}(1 - u_2)\beta_3 \eta_1 S_s \\ &\quad - \lambda_{I_s}(1 - u_2)\beta_3 \eta_1 S_s + (\lambda_{S_v} - \lambda_{I_v})(1 - u_1)\beta_4 \epsilon_v \sigma \eta_2 S_v \\ \lambda'_{R_m} &= -\lambda_{S_H} \gamma + \lambda_{R_m}(\gamma + d_h) \\ \lambda'_{R_{hs}} &= -\lambda_{S_H} k + \lambda_{R_{hs}}(k + d_h) \\ \lambda'_{R_{ms}} &= -\lambda_{S_H} \varphi + \lambda_{R_{ms}}(\varphi + d_h) \end{aligned}$$

$$\begin{aligned}
\lambda'_{S_s} &= (\lambda_{S_s} - \lambda_{I_s})(1 - u_2)\beta_3(V_{ms}\eta_1 + I_{hs}) - (d_s + u_6)\lambda_{S_s} \\
\lambda'_{I_s} &= -w_4 + (\lambda_{I_m} - \lambda_{V_{ms}})(1 - u_2)\beta_2 * I_m \\
&\quad + (\lambda_{S_H} - \lambda_{I_{hs}})(1 - u_2)\beta_2 S_H + \lambda_{I_s}(d_s + \rho_s + u_6) \\
\lambda'_{S_v} &= (\lambda_{S_v} - \lambda_{I_v})(1 - u_1)\beta_4\epsilon_v\sigma(V_{ms}\eta_2 + I_m) + \lambda_{S_v}(d_v + u_7) \\
\lambda'_{I_v} &= -w_3 + (\lambda_{S_H} - \lambda_{I_m})(1 - u_1)\beta_1\epsilon_h\sigma S_H + (\lambda_{I_{hs}} - \lambda_{V_{ms}})(1 - u_1)\beta_1\epsilon_h\sigma I_{hs} \\
&\quad + \lambda_{I_v}(d_v + u_7)
\end{aligned}$$

with transversality condition $\lambda_{S_H}(T) = \lambda_{I_m}(T) = \lambda_{I_{hs}}(T) = \lambda_{V_{ms}}(T) = \lambda_{R_m}(T) = \lambda_{R_s}(T) = \lambda_{R_{ms}}(T) = \lambda_{S_s}(T) = \lambda_{I_s}(T) = \lambda_{S_v}(T) = \lambda_{I_v}(T)$. The optimal control problem above can be analysed using the Pontryagin's maximum principle (PMP) for the necessary condition of optimality.

Theorem 3.2. Suppose $u^*(t) = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$ and a solution $X^* = \{S_h(t), I_m(t), I_{hs}(t), V_{ms}(t), R_m(t), R_s(t), R_{ms}(t), S_s(t), I_s(t), S_v(t), I_v(t)\}$ of the corresponding state system (6) then there exist adjoint variables $\lambda_{S_h}(t), \lambda_{I_m}(t), \lambda_{I_{hs}}(t), \lambda_{V_{ms}}(t), \lambda_{R_m}(t), \lambda_{R_{hs}}(t), \lambda_{R_{ms}}(t), \lambda_{S_s}(t), \lambda_{I_s}(t), \lambda_{S_v}(t), \lambda_{I_v}(t)$ which satisfy the following:

(i) All the costate equations $\lambda'_{S_h} = -\frac{\partial H}{\partial S_h}, \dots, \lambda'_{S_s} = -\frac{\partial H}{\partial S_h}, \dots, \lambda'_{I_v} = -\frac{\partial H}{\partial I_v}$ with transversality conditions $\lambda_{S_h}(T) = \lambda_{I_m}(T) = \lambda_{I_{hs}}(T) = \lambda_{V_{ms}}(T) = \lambda_{R_m}(T) = \lambda_{R_s}(T) = \lambda_{R_{ms}}(T) = \lambda_{S_s}(T) = \lambda_{I_s}(T) = \lambda_{S_v}(T) = \lambda_{I_v}(T)$. Thus, we find the optimal control $u_1^*, u_2^*, u_3^*, u_4^*, u_5^*, u_6^*, u_7^*$.

$$\begin{aligned}
u_1^* &= \max\{\min\{u_1, 1\}, 0\}, u_2^* = \max\{\min\{u_2, 1\}, 0\}, u_3^* = \max\{\min\{u_3, 1\}, 0\}, \\
u_5^* &= \max\{\min\{u_5, 1\}, 0\}, u_6^* = \max\{\min\{u_6, 1\}, 0\}, u_6^* = \max\{\min\{u_6, 1\}, 0\}, \\
u_7^* &= \max\{\min\{u_7, 1\}, 0\}.
\end{aligned}$$

The optimal point is determined by solving the equation at the given optimal $u_1^*, u_2^*, u_3^*, u_4^*, u_5^*, u_6^*, u_7^*$

$$\begin{aligned}
\frac{dS_h}{dt} &= b_H + \gamma R_m + k R_{hs} + \varphi R_{ms} - (1 - u_1^*)\beta_1\epsilon_h\sigma S_h I_v - (1 - u_2^*)\beta_2 S_h I_s \\
&\quad - d_h S_h, \\
\frac{dI_m}{dt} &= (1 - u_1^*)\beta_1\epsilon_h\sigma S_h I_v - (1 - u_2^*)\beta_2 I_s I_m - (u_3^*\omega + \theta + d_h + \tau_h)I_m, \\
\frac{dI_{hs}}{dt} &= (1 - u_2^*)\beta_2 S_h I_s - (1 - u_1)\beta_1\epsilon_h\sigma I_v I_{hs} - (u_4^*q_h + \rho_h + d_h + \tau_h)I_{hs}, \\
\frac{dV_{ms}}{dt} &= (1 - u_1^*)\beta_1\epsilon_h\sigma I_v I_{hs} + (1 - u_2^*)\beta_2 I_{hs} I_m - (u_5^*\alpha + r + m + d_h + \tau_h)V_{ms}, \\
\frac{dR_m}{dt} &= u_3^*\omega I_m - (\gamma + d_h)R_m + \phi(1 - \alpha u_5^*)V_{ms},
\end{aligned}$$

$$\begin{aligned}
\frac{dR_s}{dt} &= u_4^* q_h I_{hs} - (k + d_h) R_s + (1 - \phi)(1 - \alpha u_6^*) V_{ms}, \\
\frac{dR_{ms}}{dt} &= \alpha u_5^* V_{ms} - (\varphi + d_h) R_{ms}, \\
\frac{dS_s}{dt} &= b_s - \beta_3(I_{hs} + \eta_1 V_{ms}) S_s - d_s S_s, \\
\frac{dI_s}{dt} &= \beta_3(I_{hs} + \eta_1 V_{ms}) S_s - (d_s + \rho_s) I_s, \\
\frac{dS_v}{dt} &= b_v - (1 - u_1^*) \beta_4 \epsilon_v \sigma (I_m + \eta_2 V_{ms}) S_v - d_v S_v - u_7^* S_v, \\
\frac{dI_v}{dt} &= (1 - u_1^*) \beta_4 \epsilon_v \sigma (I_m + \eta_2 V_{ms}) S_v - d_v I_v - u_7^* I_v
\end{aligned}$$

subject to the initial conditions (2), where the Hamiltonian H is defined by $H(S_h(t), I_{hs}(t), I_m(t), V_{ms}(t), R_m(t), R_s(t), R_{ms}(t), S_s(t), I_s(t), S_v(t), I_v(t), \lambda_{S_h}(t), \lambda_{I_m}(t), \lambda_{I_{hs}}(t), \lambda_{V_{ms}}(t), \lambda_{R_m}(t), \lambda_{R_{hs}}(t), \lambda_{R_{ms}}(t), \lambda_{S_s}(t), \lambda_{I_s}(t), \lambda_{S_v}(t), \lambda_{I_v}(t), t) = w_1 I_m(t) + w_2 I_{hs}(t) + w_3 I_v(t) + w_4 I_s(t) + w_5 V_{ms}(t) + w_6 u_1^{*2}(t) + w_7 u_2^{*2}(t) + w_8 u_3^{*2}(t) + w_9 u_4^{*2}(t) + w_{10} u_5^{*2}(t) + w_{11} u_6^{*2}(t) + w_{12} u_7^{*2}(t) + \lambda_{S_h}(t) \frac{dS_h}{dt} + \lambda_{I_m}(t) \frac{dI_m}{dt} + \lambda_{I_{hs}}(t) \frac{dI_{hs}}{dt} + \lambda_{V_{ms}}(t) \frac{dV_{ms}}{dt} + \lambda_{R_m}(t) \frac{dR_m}{dt} + \lambda_{R_{hs}}(t) \frac{dR_{hs}}{dt} + \lambda_{R_{ms}}(t) \frac{dR_{ms}}{dt} + \lambda_{S_s}(t) \frac{dS_s}{dt} + \lambda_{I_s}(t) \frac{dI_s}{dt} + \lambda_{S_v}(t) \frac{dS_v}{dt} + \lambda_{I_v}(t) \frac{dI_v}{dt} = w_1 I_m(t) + w_2 I_{hs}(t) + w_3 I_v(t) + w_4 I_s(t) + w_5 V_{ms}(t) + w_6 u_1^{*2}(t) + w_7 u_2^{*2}(t) + w_8 u_3^{*2}(t) + w_9 u_4^{*2}(t) + w_{10} u_5^{*2}(t) + w_{11} u_6^{*2}(t) + w_{12} u_7^{*2}(t) + \lambda_{S_h}(t) [b_H + \gamma R_m + k R_{hs} + \varphi R_{ms} - (1 - u_1^*) \beta_1 \epsilon_h \sigma S_h I_v - (1 - u_2^*) \beta_2 S_h I_s - d_h S_h] + \lambda_{I_m}(t) [(1 - u_1^*) \beta_1 \epsilon_h \sigma S_h I_v - (1 - u_2^*) \beta_2 I_s I_m - (u_3^* \omega + \theta + d_h + \tau_h) I_m] + \lambda_{I_{hs}}(t) [(1 - u_2^*) \beta_2 S_h I_s - (1 - u_1^*) \beta_1 \epsilon_h \sigma I_v I_{hs} - (u_4^* q_h + \rho_h + d_h + \tau_h) I_{hs}] + \lambda_{V_{ms}}(t) [(1 - u_1^*) \beta_1 \epsilon_h \sigma I_v I_{hs} + (1 - u_2^*) \beta_2 I_{hs} I_m - (u_5^* \alpha + r + m + d_h + \tau_h) V_{ms}] + \lambda_{R_m}(t) [u_3^* \omega I_m - (\gamma + d_h) R_m + \phi(1 - \alpha u_5^*) V_{ms}] + \lambda_{R_{hs}}(t) [u_4 q_h I_{hs} - (k + d_h) R_s + (1 - \phi)(1 - \alpha u_6^*) V_{ms}] + \lambda_{R_{ms}}(t) [\alpha u_5^* V_{ms} - (\varphi + d_h) R_{ms}] + \lambda_{S_s}(t) [b_s - \beta_3(I_{hs} + \eta_1 V_{ms}) S_s - d_s S_s] + \lambda_{I_s}(t) [\beta_3(I_{hs} + \eta_1 V_{ms}) S_s - (d_s + \rho_s) I_s] + \lambda_{S_v}(t) [b_v - (1 - u_1^*) \beta_4 \epsilon_v \sigma (I_m + \eta_2 V_{ms}) S_v - d_v S_v - u_7^* S_v] + \lambda_{I_v}(t) [(1 - u_1^*) \beta_4 \epsilon_v \sigma (I_m + \eta_2 V_{ms}) S_v - d_v I_v - u_7^* I_v].$

Hence, the resulting optimality system is given in the next section.

3.4 Optimality system

The optimality system consists of the system with initial condition (4) coupled with the adjoint system and the transversality condition (3) together with the characterization of the optimal control $u_1^*, u_2^*, u_3^*, u_4^*, u_5^*, u_6^*, u_7^*$ which is defined as $\frac{\partial H}{\partial u_1} = \frac{\partial H}{\partial u_2} = \frac{\partial H}{\partial u_3} = \frac{\partial H}{\partial u_4} = \frac{\partial H}{\partial u_5} = \frac{\partial H}{\partial u_6} = \frac{\partial H}{\partial u_7} = 0$ at $u_1^*, u_2^*, u_3^*, u_4^*, u_5^*, u_6^*, u_7^*$

$$u_1^* = \begin{cases} 0, & \text{if } \delta_1^* \leq 0, \\ \delta_1^*, & \text{if } 0 < \delta_1^* < 1, \\ 1, & \text{if } \delta_1^* \geq 1 \end{cases}$$

where

$$\delta_1^* = \frac{\beta_1 \epsilon_h \sigma S_h I_v (\lambda_{I_m}(t) - \lambda_{S_h}(t)) + \beta_1 \epsilon_h \sigma I_v I_{hs} (\lambda_{V_{ms}}(t) - \lambda_{I_{hs}}(t)) + \beta_4 \epsilon_v \sigma (I_m + V_{ms}) S_v (\lambda_{I_v}(t) - \lambda_{S_v}(t))}{2w_6}$$

$$u_2^* = \begin{cases} 0, & \text{if } \delta_2^* \leq 0, \\ \delta_2^*, & \text{if } 0 < \delta_1^* < 1, \\ 1, & \text{if } \delta_2^* \geq 1 \end{cases}$$

where

$$\delta_2^* = \frac{\beta_2 S_h I_s (\lambda_{I_{hs}}(t) - \lambda_{S_h}(t)) + \beta_2 I_{hs} I_m (\lambda_{V_{ms}}(t) - \lambda_{I_m}(t)) + \beta_3 (I_{hs} + \eta_1 V_{ms}) S_s (\lambda_{S_v}(t) - \lambda_{S_s}(t))}{2w_7}.$$

$$u_3^* = \begin{cases} 0, & \text{if } \delta_3^* \leq 0, \\ \delta_3^*, & \text{if } 0 < \delta_3^* < 1, \\ 1, & \text{if } \delta_3^* \geq 1 \end{cases}$$

where

$$\delta_3^* = \frac{\lambda_{I_m}(t) w I_m}{2w_3}.$$

$$u_4^* = \begin{cases} 0, & \text{if } \delta_4^* \leq 0, \\ \delta_4^*, & \text{if } 0 < \delta_4^* < 1, \\ 1, & \text{if } \delta_4^* \geq 1 \end{cases}$$

where

$$\delta_4^* = \frac{h I_{hs} \lambda_{I_{hs}}(t)}{2w_9}.$$

$$u_5^* = \begin{cases} 0, & \text{if } \delta_5^* \leq 0, \\ \delta_5^*, & \text{if } 0 < \delta_5^* < 1, \\ 1, & \text{if } \delta_5^* \geq 1 \end{cases}$$

where

$$\delta_5^* = \frac{\alpha V_{ms} \lambda_{V_{ms}}(t) - \phi \alpha V_{ms} \lambda_{R_m}(t) - \alpha V_{ms} \lambda_{R_{hs}}(t) + \phi \alpha V_{ms} \lambda_{R_{hs}}(t) + \alpha V_{ms} \lambda_{R_{ms}}(t)}{2w_{10}}.$$

$$u_6^* = \begin{cases} 0, & \text{if } \delta_6^* \leq 0, \\ \delta_6^*, & \text{if } 0 < \delta_6^* < 1, \\ 1, & \text{if } \delta_6^* \geq 1 \end{cases}$$

where

$$\delta_6^* = \frac{\lambda_{S_s}(t) S_s - \lambda_{I_s}(t) I_s}{2w_{11}}.$$

$$u_7^* = \begin{cases} 0, & \text{if } \delta_7^* \leq 0, \\ \delta_7^*, & \text{if } 0 < \delta_7^* < 1, \\ 1, & \text{if } \delta_7^* \geq 1 \end{cases}$$

where $\delta_7^* = \frac{\lambda_{S_v}(t) S_v - \lambda_{I_v}(t) I_v}{2w_{12}}.$

4. Numerical simulation and graphical illustration of the malaria-schistosomiasis co-infection model

The numerical solutions of the optimality system and the associate results of the optimal control were obtained by working out the optimality system which is made up of eleven ODE's from the state and adjoint systems. We employ an iterative scheme to solve the optimality system and start to solve the state equations with a guess for the controls over the simulated time using fourth order Runge Kutta scheme. The adjoint systems are solved through a backward sweep method, using the existing iterations solution of the state equations because of the transversality conditions. Using a convex combination of the earlier controls and the value from the characterisation of the controls were updated. We repeat the same process and iterations and they are stopped if the values of the preceding iterations are very close to the ones at the existing iterations [1]. We investigate the co-infection model with multiple control measures to study the impacts of control practices and the transmission dynamics of the malaria-schistosomiasis co-infection. By applying several combinations of the seven controls, one control at a time, two controls at a time, three controls at a time, four controls at a time, five controls at a time, six controls at a time and all the seven controls at a time, we examine and compare numerical results from simulations with the following circumstances:

1. using treatments of malaria with artemisinin combined therapy (u_3) only, no use of insecticide treated bednets ($u_1 = 0$), without the use of prevention by avoiding swimming ($u_2 = 0$), no treatment of schistosomiasis with praziquantel ($u_4 = 0$), without treatment of malaria-schistosomiasis ($u_5 = 0$), no use of biological control ($u_6 = 0$), and no use of insecticide spray and destruction of stagnant water and mosquito breeding sites ($u_7 = 0$);
2. using treatment of schistosomiasis with praziquantel (u_4) only, no use treatments of malaria with artemisinin combined therapy ($u_3 = 0$), no use of insecticide treated bednets ($u_1 = 0$), without the use of prevention by avoiding swimming ($u_2 = 0$), without treatment of malaria-schistosomiasis ($u_5 = 0$), no use of biological control ($u_6 = 0$), and no use of insecticide spray and destruction of stagnant water and mosquito breeding sites ($u_7 = 0$);
3. the use of insecticide treated bednets (u_1) and the use of prevention by avoiding swimming (u_2) only, no treatments of malaria with artemisinin combined therapy (u_3), no treatment of schistosomiasis with praziquantel (u_4), without treatment of malaria-schistosomiasis (u_5), no use of biological control (u_6), and no use of insecticide spray and destruction of stagnant water and mosquito breeding sites (u_7);

4. the use of prevention by avoiding swimming(u_2) and treatments of malaria with artemisinin combined therapy(u_3) only, no use of insecticide treated bednets (u_1), no treatment of schistosomiasis with praziquantel (u_4), without treatment of malaria-schistosomiasis (u_5), no use of biological control (u_6), and no use of insecticide spray and destruction of stagnant water and mosquito breeding sites (u_7);
5. using treatments of malaria with artemisinin combined therapy (u_3) and use of insecticide treated bednets (u_1) only, without the use of prevention by avoiding swimming (u_2), no treatment of schistosomiasis with praziquantel (u_4), without treatment of malaria-schistosomiasis (u_5), no use of biological control (u_6), and no use of insecticide spray and destruction of stagnant water and mosquito breeding sites (u_7);
6. using treatment of schistosomiasis with praziquantel (u_4) and use treatments of malaria with artemisinin combined therapy (u_3) only, no use of insecticide treated bednets (u_1), without the use of prevention by avoiding swimming (u_2), without treatment of malaria-schistosomiasis (u_5), no use of biological control (u_6), and no use of insecticide spray and destruction of stagnant water and mosquito breeding sites (u_7);
7. the use of insecticide treated bednets (u_1), the use of prevention by avoiding swimming (u_2), use treatments of malaria with artemisinin combined therapy (u_3), and using treatment of schistosomiasis with praziquantel (u_4) only, without treatment of malaria-schistosomiasis (u_5), no use of biological control (u_6), and no use of insecticide spray and destruction of stagnant water and mosquito breeding sites (u_7);
8. the use of insecticide treated bednets (u_1), the use of prevention by avoiding swimming (u_2) and treatments of malaria with artemisinin combined therapy (u_3), using treatment of schistosomiasis with praziquantel (u_4), using treatment of malaria-schistosomiasis (u_5), use of biological control (u_6) and use of insecticide spray and destruction of stagnant water and mosquito breeding sites (u_7).

Table 3 below gave the list of parameter values used in the numerical simulation of the model. We made use of the following weight constants in our numerical experiment $w_1 = 210$; $w_2 = 910$; $w_3 = 800$; $w_4 = 760$; $w_5 = 600$; $w_6 = 150$; $w_7 = 30$; $w_8 = 200$; $w_9 = 250$; $w_{10} = 310$; $w_{11} = 230$; $w_{12} = 200$; $b_h = 0.00011$; $b_v = 0.071$; $b_s = 0.06$; $\gamma = 0.00134$; $\beta_1 = 0.03$; $\beta_4 = 0.09$; $\sigma = 0.6$; $\epsilon_h = 0.2$; $\epsilon_v = 0.3$; $d_h = 0.00004$; $\theta = 0.005$; $d_v = 0.1429$; $k = 0.013$; $\tau_h = 0.0005$; $\varphi = 0.0005$; $\beta_2 = 0.406$; $\beta_3 = 0.004$; $q_H = 0.56$; $\alpha = 0.7$; $\eta_1 = 0.01$; $\eta_2 = 0.02$; $r = 0.02$; $m = 0.06$; $d_s = 0.003$; $\phi = 0.12$; $\omega = 0.05$; $\rho_h = 0.0039$; $\rho_s = 0.0004012$.

4.1 Optimal treatments of malaria with artemisinin combined therapy(u_3) only

Only the control (u_3) on treatment of malaria with artemisinin combined therapy is used to optimise the cost functional C , while the other controls u_1 , u_2 , u_4 , u_5 , u_6 and u_7 were set to zero. In Fig.1, the result showed a key difference in the I_m , I_{hs} , V_{ms} , I_s , and I_v with optimal strategy compared to I_m , I_{hs} , V_{ms} , I_s , and I_v without control. Specifically, it is observed that in Fig.1a, the control strategy lead to a decrease in the number of human infected with malaria only (I_m) as against the uncontrolled case. Similarly, in Fig.1b, the uncontrolled case resulted in the increased number of human with schistosomiasis only (I_{hs}), while the control strategy led to an increase in the number of human with schistosomiasis only (I_{hs}) until it maintain a steady state. In Fig.1c, the uncontrolled case resulted in decrease number of co-infected human with both malaria-schistosomiasis (V_{ms}) slowly, while the control strategy led to faster declining in the malaria-schistosomiasis co-infected individuals. It was also observed that in Fig.1d, the population of infected mosquitoes decreases with or without control, but seems to swiftly decline in the present of the optimal control strategy. In Fig.1e, the population of infected snails increases with or without control strategies. The control profile is shown in Fig.1f, here it is observed that the optimal treatment control u_3 remains at the upper bound throughout the time period.

4.2 Optimal treatment of schistosomiasis with praziquantel (u_4) only

Only the control (u_4) on treatment of schistosomiasis with praziquantel is used to optimise the cost functional C , while the other controls u_1 , u_2 , u_3 , u_5 , u_6 and u_7 were set to zero. “In Fig.2, the result revealed major difference in the I_m , I_{hs} , V_{ms} , I_s , and I_v with optimal strategy compared to I_m , I_{hs} , V_{ms} , I_s , and I_v without control. It is observed that in Fig.2a, the control strategy reduced to zero at time $T = 90$ days in the number of human infected with malaria only (I_m) while I_m without control declined to zero at $T = 80$ days. Likewise, in Fig.2b, the uncontrolled case resulted in the increased number of human with schistosomiasis only (I_{hs}), while the control strategy led to a swift decrease in the number of human with schistosomiasis only (I_{hs}) until it maintain a steady state. In Fig.2c, the uncontrolled case resulted in decrease number of co-infected human with both malaria-schistosomiasis (V_{ms}) slowly, while the control strategy led to faster declining in the malaria-schistosomiasis co-infected individuals. “It was also observed that in Fig.2d, the population of infected mosquitoes decreases with or without control, but seems to swiftly decline in the present of the optimal control strategy. In Fig.1e, the population of infected snails increases without control strategy but declines in the presence of optimal strategy. The control profile is shown in Fig.1f, here it is observed that the optimal treatment control u_4 remains at the upper bound throughout the time period.

Table 3: Table showing numerical values of parameters used in the model.

Parameter	Symbol	Value	Source
disease induced death rate due to malaria only	θ	0.005	[50]
probability of human getting infected with malaria	β_1	0.03	[1, 18, 49]
rate of acquiring schistosomiasis through contact with inf. snails	β_2	0.406	[53]
rate of acquiring schistosomiasis through contacts with inf. hum.	β_3	0.004	[ass]
probability of mosquito getting infected by an infectious hum.	β_4	0.09	[1, 18, 49]
Malaria-schistosomiasis immunity waning rate	φ	0.0005	[ass]
Schistosomiasis induced death rate	r	0.02	[ass]
malaria disease induced death rate	m	0.06	[ass]
Co-infected who recovered from malaria only	ϕ	0.12	[ass]
disease induced death rate of human due to schistosomiasis only	ρ_H	0.0039	[17]
disease induced death rate of snails due to schistosomiasis only	ρ_s	0.0004012	[2]
human spontaneous recovery	ω	0.05	[12]
rate of loss of immunity to the schistosomiasis disease only	k	0.013	[ass]
rate of loss of immunity to malaria and schistosomiasis disease	φ	0.0005	[ass]
human spontaneous recovery from schistosomiasis only	q_H	0.56	[ass]
modification parameter	η_1	0.01	[ass]
modification parameter	η_2	0.02	[ass]
per capita birth rate of mosquitoes	b_v	0.071	[1, 2, 18]
drug induced death rate	τ_h	0.0005	[ass]

Parameter	Symbol	Value	Source
per capita birth rate of snails	b_s	0.06	[16]
Natural death rate of humans	d_H	0.00004	[1, 63]
per capita biting rate of mosquitoes	ε_v	0.3	[22, 33, 51]
contact rate of vector per human per unit time	σ	0.502	[12]
per capita biting rate of humans	ε_h	0.2	[4, 22]
natural death rate of mosquitoes	d_v	0.1429	[18]
natural death rate of snails	d_s	0.003	[16]
rate of loss of immunity to the malaria disease only	γ	0.00134	[2, 18, 44]
per capita birth rate of humans	b_h	0.00011	[1, 57]

4.3 Optimal insecticide treated bednets (u_1) and prevention by avoiding swimming (u_2) only

The controls (u_1) and (u_2) were used to optimise the cost functional C , while the other controls u_3 , u_4, u_5 , u_6 and u_7 were set to zero. It is observed that in Fig.3a, the number of human infected with malaria only (I_m) declines to zero without control strategies while it decreases to a certain point with control strategies as against the uncontrolled case. Similarly, in Fig.3b, the uncontrolled case resulted in the increased number of human with schistosomiasis only (I_{hs}), while the control strategies led to a decrease in the number of human with

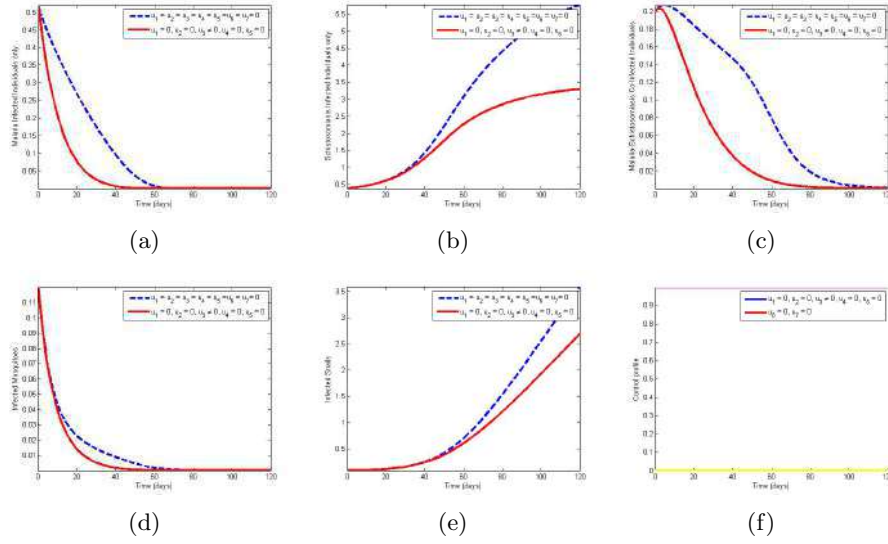


Figure 1: Simulation showing the variation of Infected Human with malaria only (I_m) in Fig.1a, Infected Human with Schistosomiasis only (I_{hs}) in Fig.1b, Humans infected with both malaria and schistosomiasis (V_{ms}) in Fig.1c, Infected mosquitoes (I_v) in Fig.1d, Infected snails (I_s) in Fig.1e and control profile in Fig.1f

schistosomiasis only (I_{hs}). In Fig.3c, the uncontrolled case resulted in decrease number of co-infected human with both malaria-schistosomiasis (V_{ms}) slowly, while the control strategy led to faster declining in the malaria-schistosomiasis co-infected individuals. It was also observed that in Fig.3d, the population of infected mosquitoes decreases with or without control, but seems to swiftly decline in the presence of the optimal control strategy. In Fig.3e, the population of infected snails increases without control strategies but decreases in the presence of the optimal strategies. The control profile is shown in Fig.3f, here it is observed that the optimal treated bednets control u_1 and prevention by avoiding swimming u_2 remains at the upper bound throughout the time period.

4.4 Optimal prevention by avoiding swimming (u_2) and treatments of malaria with artemisinin combined therapy (u_3) only

Here the controls (u_2) and (u_3) were used to optimise the cost functional C , while the other controls u_1 , u_4 , u_5 , u_6 and u_7 were set to zero. It is observed that in Fig.4a, the number of human infected with malaria only (I_m) declines to zero without control strategies while it decreases to a certain point with control strategies as against the uncontrolled case. Similarly, in Fig.4b, the uncontrolled case resulted in the increased number of human with schistosomiasis only (I_{hs}), while the control strategies led to a decrease in the number of human with

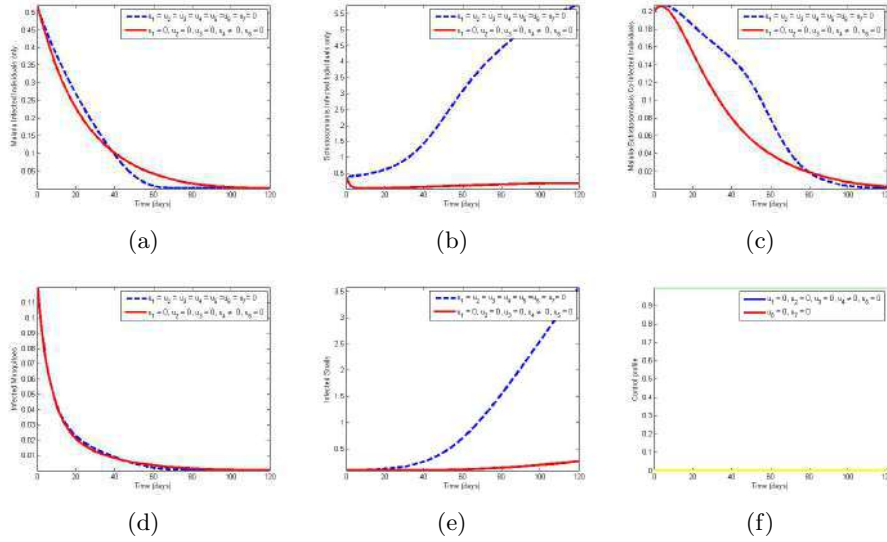


Figure 2: Simulation showing the variation of Infected Human with malaria only (I_m) in Fig.2a, Infected Human with Schistosomiasis only (I_{hs}) in Fig.2b, Humans infected with both malaria and schistosomiasis (V_{ms}) in Fig.2c, Infected mosquitoes (I_v) in Fig.2d, Infected snails (I_s) in Fig.2e and control profile in Fig.2f.

schistosomiasis only (I_{hs}). In Fig.4c, the uncontrolled case resulted in decrease number of co-infected human with both malaria-schistosomiasis (V_{ms}) slowly, while the control strategy led to faster declining in the malaria-schistosomiasis co-infected individuals. It is also observed that in Fig. 4d, the population of infected mosquitoes decreases with or without control, but seems to swiftly decline in the presence of the optimal control strategy. In Fig.4e, the population of infected snails increases without control strategies but decreases in the presence of the optimal strategies. The control profile is shown in Fig.4f, here it is observed that the optimal prevention by avoiding swimming control u_2 and treatments of malaria with artemisinin combined therapy u_3 remains at the upper bound throughout the time period.

4.5 Optimal treatments of malaria with artemisinin combined therapy (u_3) and insecticide treated bednets (u_1) only

Here the controls (u_3) on treatment of malaria with artemisinin combined therapy and insecticide treated bednets u_1 were applied to optimise the cost functional C , while the other controls u_2, u_4, u_5, u_6 and u_7 were set to zero. Specifically, it is observed that in Fig.5a, the control strategy lead to a decrease in the number of human infected with malaria only (I_m) while the uncontrolled case decreases slowly. Similarly, in Fig.5b, the uncontrolled case resulted in the increased num-

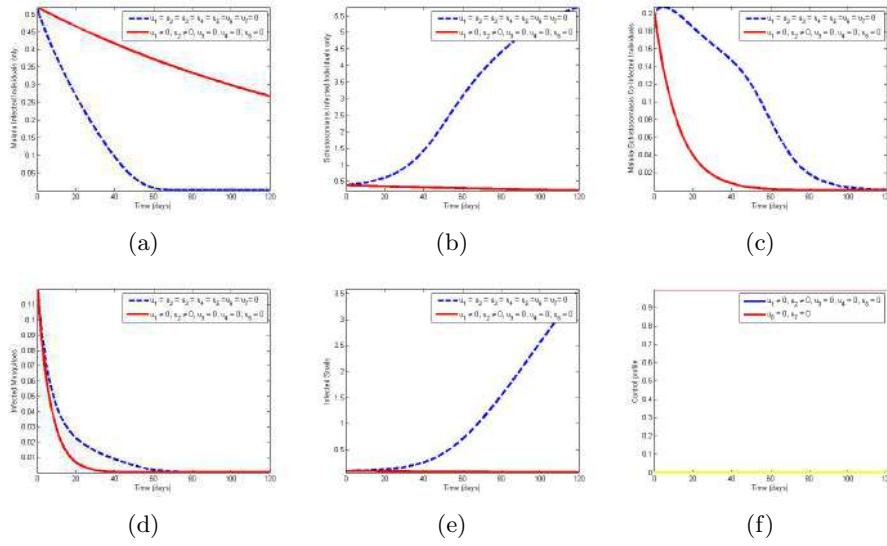


Figure 3: Simulation showing the variation of Infected Human with malaria only (I_m) in Fig.3a, Infected Human with Schistosomiasis only (I_{hs}) in Fig.3b, Humans infected with both malaria and schistosomiasis (V_{ms}) in Fig.3c, Infected mosquitoes (I_v) in Fig.3d, Infected snails (I_s) in Fig.3e and control profile in Fig.3f

ber of human with schistosomiasis only (I_{hs}), while the control strategy led to an increase in the number of human with schistosomiasis only (I_{hs}) until it maintain a steady state. In Fig.5c, the uncontrolled case resulted in decrease number of co-infected human with both malaria-schistosomiasis (V_{ms}) slowly, while the control strategy led to faster declining in the malaria-schistosomiasis co-infected individuals. It is also observed that in Fig.5d, the population of infected mosquitoes decreases with or without control, but seems to swiftly decline in the present of the optimal control strategy. In Fig.5e, the population of infected snails increases with or without control strategies. The control profile is shown in Fig.5f, here it is observed that the Optimal treatments of malaria with artemisinin combined therapy (u_3) and insecticide treated bednets (u_1) remains at the upper bound throughout the time period.

4.6 Optimal treatment of schistosomiasis with praziquantel (u_4) and use treatments of malaria with artemisinin combined therapy (u_3) only

Here the controls (u_3) on treatment of malaria with artemisinin combined therapy and treatment of schistosomiasis with praziquantel u_4 were applied to optimise the cost functional C , while the other controls u_2 , u_4 , u_5 , u_6 and u_7 were set to zero. It is observed that in Fig.6a, the control strategy lead to a decrease

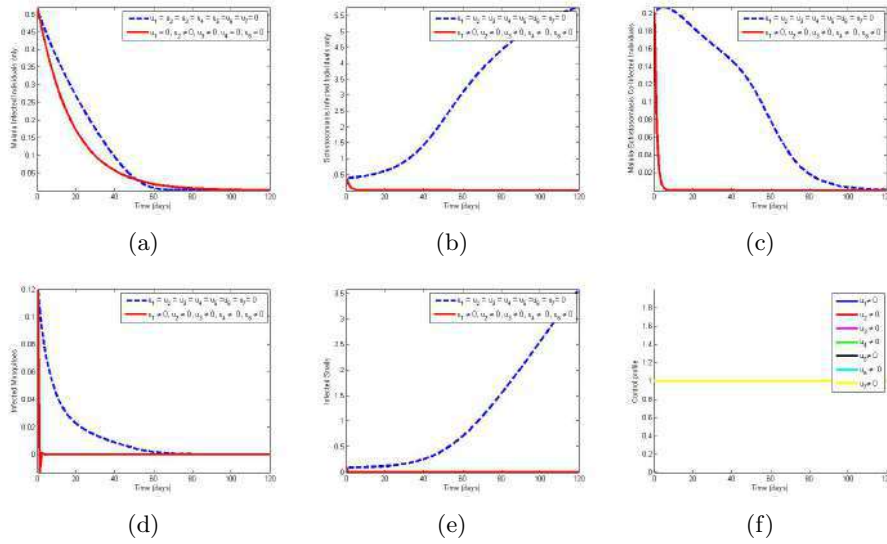


Figure 4: Simulation showing the variation of Infected Human with malaria only (I_m) in Fig.4a, Infected Human with Schistosomiasis only (I_{hs}) in Fig.4b, Humans infected with both malaria and schistosomiasis (V_{ms}) in Fig.4c, Infected mosquitoes (I_v) in Fig.4d, Infected snails (I_s) in Fig.4e and control profile in Fig.4f

in the number of human infected with malaria only (I_m) while the uncontrolled case decreases slowly. Likewise, in Fig.6b, the uncontrolled case resulted in the increased number of human with schistosomiasis only (I_{hs}), while the control strategy led to a decrease in the number of human with schistosomiasis only (I_{hs}). In Fig.6c, the uncontrolled case resulted in decrease number of co-infected human with both malaria-schistosomiasis (V_{ms}) slowly, while the control strategy led to faster declining in the malaria-schistosomiasis co-infected individuals. It was also observed that in Fig.6d, the population of infected mosquitoes decreases with or without control, but seems to fastly decline in the presence of the optimal control strategy. In Fig.6e, the population of infected snails increases without control strategies but decreases in the presence of optimal strategy. The control profile is shown in Fig.6f, here it is observed that the optimal treatments of malaria with artemisinin combined therapy(u_3) and treatment of schistosomiasis with praziquantel u_4 remains at the upper bound throughout the time period.

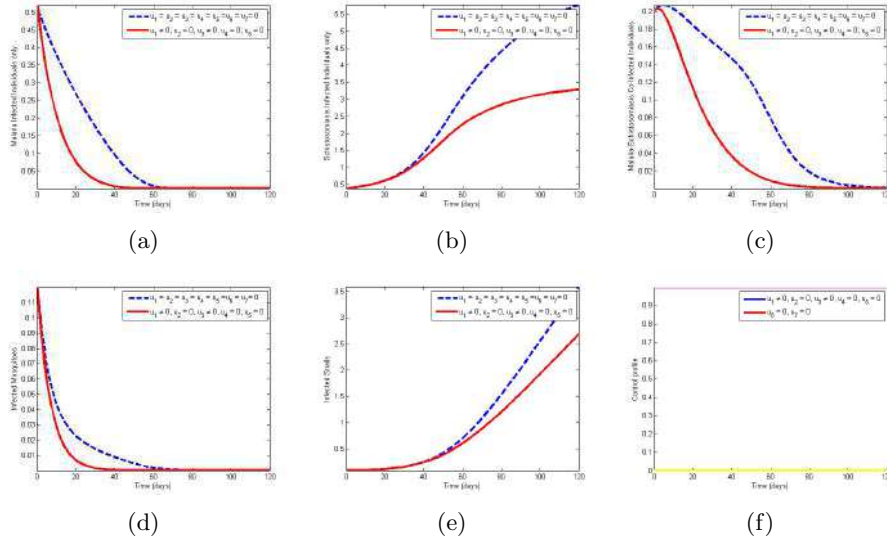


Figure 5: Simulation showing the variation of Infected Human with malaria only (I_m) in Fig.5a, Infected Human with Schistosomiasis only (I_{hs}) in Fig.5b, Humans infected with both malaria and schistosomiasis (V_{ms}) in Fig.5c, Infected mosquitoes (I_v) in Fig. 5d, Infected snails (I_s) in Fig.5e and control profile in Fig.5f

4.7 Optimal insecticide treated bednets (u_1), prevention by avoiding swimming (u_2), treatments of malaria with artemisinin combined therapy(u_3), and treatment of schistosomiasis with praziquantel (u_4) only

Here the controls (u_1) on insecticide treated bednets, prevention by avoiding swimming (u_2), treatments of malaria with artemisinin combined therapy (u_3), and treatment of schistosomiasis with praziquantel (u_4) only were applied to optimise the cost functional C , while the other controls u_5 , u_6 and u_7 were set to zero. Particularly, it is observed that in Fig.7a, the control strategy lead to a decrease in the number of human infected with malaria only (I_m) while the uncontrolled case declines slowly. Similarly, in Fig.7b, the uncontrolled case resulted in the increased number of human with schistosomiasis only (I_{hs}), while the control strategy led to a decrease in the number of human with schistosomiasis only (I_{hs}) until it maintain a steady state. In Fig.7c, the uncontrolled case resulted in decrease number of co-infected human with both malaria-schistosomiasis (V_{ms}) slowly, while the control strategy led to faster declining in the malaria-schistosomiasis co-infected individuals. It was also observed that in Fig.7d, the population of infected mosquitoes decreases with or without control, but seems to swiftly decline in the present of the optimal control strategy. In Fig.7e, the population of infected snails increases without control

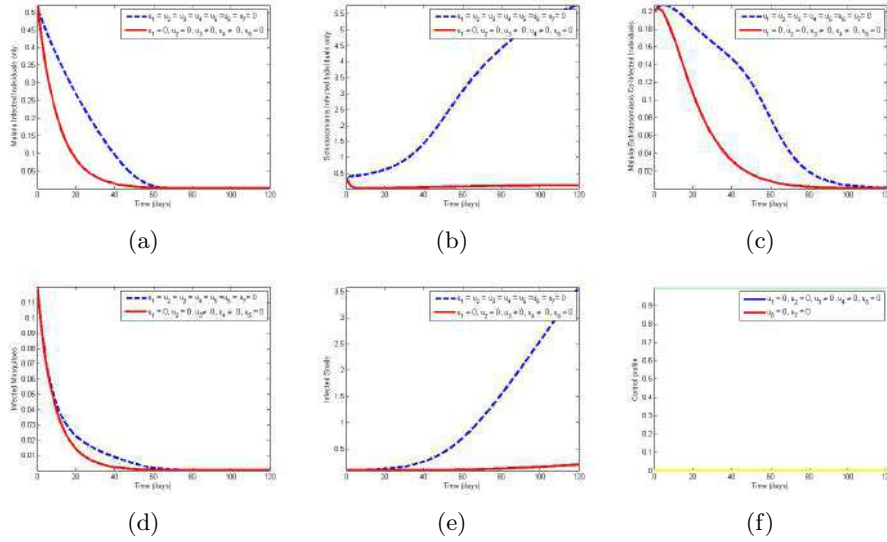


Figure 6: Simulation showing the variation of Infected Human with malaria only (I_m) in Fig.6a, Infected Human with Schistosomiasis only (I_{hs}) in Fig.6b, Humans infected with both malaria and schistosomiasis (V_{ms}) in Fig.6c, Infected mosquitoes (I_v) in Fig.6d, Infected snails (I_s) in Fig.6e and control profile in Fig.6f.

strategy while it decreases in the presence of optimal strategy. The control profile is shown in Fig.7f, here it is observed that the optimal insecticide treated bednets (u_1), prevention by avoiding swimming (u_2), treatments of malaria with artemisinin combined therapy (u_3), and treatment of schistosomiasis with praziquantel (u_4) remains at the upper bound throughout the time period.

4.8 Optimal insecticide treated bednets (u_1), prevention by avoiding swimming (u_2), treatments of malaria with artemisinin combined therapy (u_3), treatment of schistosomiasis with praziquantel (u_4), treatment of malaria-schistosomiasis (u_5), use of biological control (u_6) and insecticide spray and destruction of stagnant water and mosquito breeding sites (u_7)

Here the all the seven controls that is optimal insecticide treated bednets (u_1), prevention by avoiding swimming (u_2), treatments of malaria with artemisinin combined therapy (u_3), treatment of schistosomiasis with praziquantel (u_4), treatment of malaria-schistosomiasis (u_5), use of biological control (u_6) and insecticide spray and destruction of stagnant water and mosquito breeding sites (u_7) were applied to optimise the cost functional C . In particular, it is observed that in Fig.8a, the control strategy lead to a decrease in the number of human infected with malaria only (I_m) while the uncontrolled case decreases

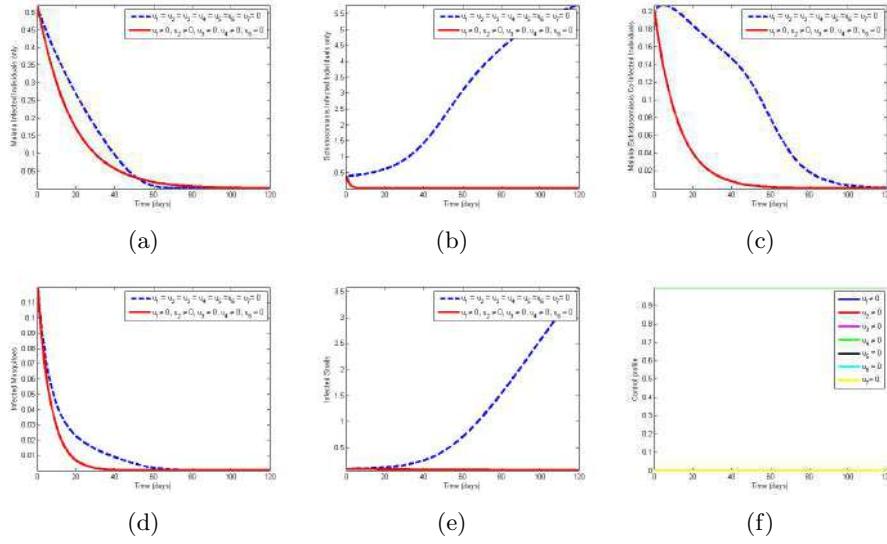


Figure 7: Simulation showing the variation of Infected Human with malaria only (I_m) in Fig.7a, Infected Human with Schistosomiasis only (I_{hs}) in Fig.7b, Humans infected with both malaria and schistosomiasis (V_{ms}) in Fig.7c, Infected mosquitoes (I_v) in Fig.7d, Infected snails (I_s) in Fig.7e and control profile in Fig.7f.

slowly. Similarly, in Fig.8b, the uncontrolled case resulted in the increased number of human with schistosomiasis only (I_{hs}), while the control strategy led to a total decrease in the number of human with schistosomiasis only (I_{hs}) until it reaches zero. In Fig.8c, the uncontrolled case resulted in decrease number of co-infected human with both malaria-schistosomiasis (V_{ms}) slowly, while the control strategy led to faster declining in the malaria-schistosomiasis co-infected individuals completely to zero. It was also observed that in Fig.8d, the population of infected mosquitoes decreases without control slowly, but seems to swiftly decline in the presence of the optimal control strategy. In Fig.8e, the population of infected snails increases without control strategies but decreases drastically to zero as time goes on. The control profile is shown in Fig.8f, here it is observed that the optimal insecticide treated bednets (u_1), prevention by avoiding swimming (u_2), treatments of malaria with artemisinin combined therapy (u_3), treatment of schistosomiasis with praziquantel (u_4), treatment of malaria-schistosomiasis (u_5), use of biological control (u_6) and insecticide spray and destruction of stagnant water and mosquito breeding sites (u_7) remains at the upper bound throughout the time period.

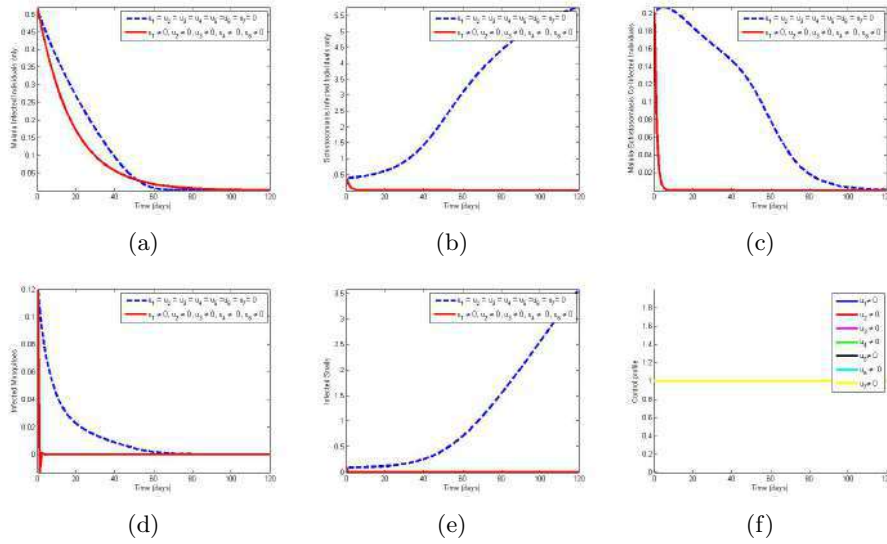


Figure 8: Simulation showing the variation of Infected Human with malaria only (I_m) in Fig.8a, Infected Human with Schistosomiasis only (I_{hs}) in Fig.8b, Humans infected with both malaria and schistosomiasis (V_{ms}) in Fig.8c, Infected mosquitoes (I_v) in Fig.8d, Infected snails (I_s) in Fig.8e and control profile in Fig.8f.

5. Conclusion

In this paper, we developed and analysed a deterministic model for the transmission of malaria-schistosomiasis co-infection with control interventions using system of non linear ordinary differential equations. The optimal control problem was formulated, where we used one control intervention at a time, combinations of two, four, and seven control interventions at a time while we set others to zero to examine and compared the influence of the control strategies on malaria-schistosomiasis elimination. Our numerical results revealed that the combinations of seven control strategies at a time had the greatest impact on the reduction or stopping the transmission chain of the co-infected disease. This is followed by the combinations of four control strategies and lastly by the combination of two i.e. u_3 and u_4 . In places where there are scarce or limited resources, it our suggestion that the combination of the two controls (u_3 and u_4) be adopted having observed their impact breaking the transmission chain of co-infection. Thus, we recommend that there should be many effective elimination programs both at local and national level on the control and elimination of malaria and schistosomiasis. The obtained results present a good framework for planning and designing cost-effective strategies for good interventions in dealing with malaria and malaria-schistosomiasis co-infection and provided guideline to

identify the most effective intervention packages by the national control bodies. This work can be extended by including the effects of the environment on the dynamics of malaria-schistosomiasis co-infection and also to incorporate some heterogeneities within and without, a community. Model validation and parameter estimation could also be essential aspect to explore by applying it to a particular malaria-schistosomiasis endemic region of the world in order to link available data with the model.

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The total game chromatic number of paths, cycles and stars

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Abstract. The total game coloring is one of the extensions of the game coloring problem in which two players alternatively color both vertices and edges of a given graph. In this paper, the total game chromatic number is determined for the classes of paths, cycles, and stars. The incidence game chromatic number for large paths is also discussed by relating the incidence and the total graphs of paths.

Keywords: game coloring, total coloring.

1. Introduction

Steven Brams defined the map-coloring game [1] in 1981, which was the introduction of graph coloring game problem. Further in 1991, Bodlaeder introduced the graph coloring game parameters [2]. The incidence game coloring and the total game coloring are some extensions of game coloring problem introduced by Andres [4], and Bartnicki and Miechowicz [3], respectively.

Let G be a graph and consider a set of colors $C = \{1, 2, 3, \dots, k\}$. Further, consider two-players, Alice and Bob who take alternative turns to color vertices of an initially uncolored graph G using the colors in the set C . They both follow proper coloring rule, i.e., each pair of adjacent vertices must receive distinct colors. Skipping a turn is not allowed and the game ends when it is not possible for a turn to carry on. Alice wins if all the vertices of the graph are properly colored using the set C ; otherwise, Bob wins. The minimum number of colors, say k , for which Alice has a winning strategy while playing on the graph G with the color set $\{1, 2, 3, \dots, k\}$ is called the *game chromatic number* of G , i.e., $\chi_g(G) = k$.

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Total graph coloring is the assignment of colors or labels to both vertices and edges of the graph G such that each pair of adjacent or incident elements, such as vertex adjacent to vertex, edge adjacent to edge, and vertex incident upon edge, must receive different colors. The *total chromatic number* of a graph G , $\chi''(G)$ is the minimum number of colors required to color the graph.

Total game coloring of a graph G is considered as follows: Let C be a set of colors and let Alice and Bob be two players who alternatively make moves on the vertices and edges of the graph G by assigning colors from C . Here, it is assumed that Alice always starts the game and skipping a move is not permitted. The game ends when it is not possible to make a move. Alice wins if the graph is completely colored with the colors in C ; otherwise, Bob wins. Moreover, it is always aimed to make Alice win so that the graph is completely colored. The minimum number of colors, say k , required for Alice to have a winning strategy on G is the *total game chromatic number* of G , i.e., $\chi_g''(G) = k$.

The *incidence graph* of G , denoted as G^I , is said to be the graph whose vertices are the incidences of G and whose edges denote adjacencies between the incidences of G . The *total graph* of a graph G , denoted as $T(G)$, has a vertex for each edge and vertex in G and an edge for every edge-edge, vertex-edge, and vertex-vertex adjacency in G .

2. The bounds of $\chi_g''(G)$

In this section, the upper and lower bounds of the function χ_g'' and also some terminologies used in the up-coming theorems are discussed.

Proposition 2.1. $\chi_g''(G) \geq \Delta(G) + 1$.

It is obvious that, $\chi_g''(G) \geq \chi''(G)$ and $\chi''(G) \geq \Delta(G) + 1$. Hence, $\chi_g''(G) \geq \Delta(G) + 1$. Moreover, it is shown in [3] that $\chi_g''(G) \leq 2\Delta(G) + 1$. Thus, the bound for $\chi_g''(G)$ can be given as: $\Delta(G) + 1 \leq \chi_g''(G) \leq 2\Delta(G) + 1$.

The following terminologies are being used in the construction of proofs:

A *configuration* or a *set-up* is an arrangement of elements in a particular formation. A *strategy* is a plan of action designed to defeat the opponent. A *trap* is a strategy followed by a player to force a win against the opponent.

3. Total game chromatic number of some graphs

In this section, the exact value of the function χ_g'' of paths, cycles, and stars are discussed. Let us consider each vertex and each edge as *positions*. Therefore, a path on n vertices will have $2n - 1$ positions.

Theorem 3.1. $\chi_g''(P_n) = 5$ for $n \geq 9$.

Proof. Let P_n be a path with $n \geq 9$. Since $\chi_g''(G) \leq 2\Delta(G) + 1$ is the upper bound, $\chi_g''(P_n) \leq 5$. Therefore, it remains to prove that Alice has a winning strategy on P_n with at least five colors, $\chi_g''(P_n) \geq 5$; in other words, Bob has a

winning strategy on P_n with four or lesser colors. It is obvious for Alice to win with five colors and for Bob to win with three, hence it suffices to show that Bob wins on P_n with four colors. Assume that the players have four colors in the set, say $C = \{1, 2, 3, 4\}$, to use. In order to win, Bob needs the following set-up:



Figure 1: Required set-up for Bob to win.

The fundamental strategy of Bob is to color five positions away from the position already colored by Alice with the same color.

Let $p : c$ denote that the position p is colored with c and $*$ denotes the position which is outside of the configuration. Consider the configuration in Figure 2, Table 1 explains Bob's strategy for winning when Alice colors anywhere in this configuration.

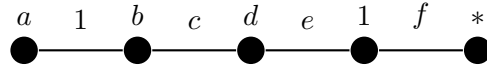


Figure 2: Configuration for trap 1.

Alice's move	Bob's move
$a : 2$	$e : 2$. This leads to Bob's win, since b , c , and d must receive three different colors.
$b : 2$	$e : 3$. This leads to Bob's win, since c and d must receive two different colors.
$c : 2$	$d : 3$. This brings a two-way winning chance for Bob in which he could either color $a : 4$ or $f : 4$ in his next move, but Alice can block only one of these.

Table 1: The above responses are considered as trap 1.

Other plays on d , e , and f follow by symmetry.

With respect to Alice's initial move, let the proof be seen in two cases.

Case 1. Suppose Alice does not start the game by coloring any of the first three positions from one end of P_n .

Without loss of generality, assume that Alice plays in the fourth position on one side of the path. Bob can always form the following set-up with the help of his fundamental strategic move:

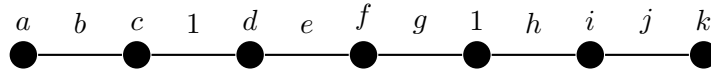


Figure 3: This formation is always possible in case 1.

The above set-up is always possible in this case. In order to win, Bob must build trap 2 given in Figure 4.

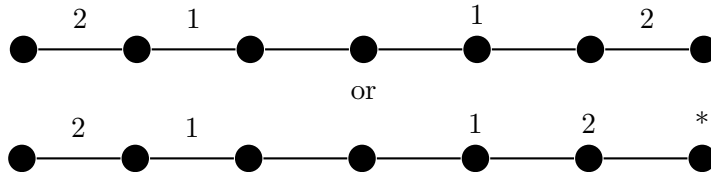


Figure 4: The above are the set-up for trap 2.

Consider Figure 3,

- If Alice colors either $i : 2$ or $j : 2$, then Bob can easily form trap 2 by coloring $b : 2$.
- Similarly, by symmetry, if Alice colors $a : 2$ or $b : 2$, then Bob can color $i : 2$.
- If Alice colors $k : 2$, then Bob has to use a different strategy by coloring $h : 3$. This forces a two-way winning for Bob, i.e., in his next move, he can color either $j : 4$ or $d : 3$. But Alice can block only one of these.
- If Alice colors anywhere away from this set-up, then Bob should color $i : 2$ (or $b : 2$). In this situation, Bob must be very careful about a (or j) because if it is already colored with 1, then this becomes a problem for Bob when Alice, in her next move, colors $b : 3$ (or $i : 3$). Therefore, the choice of i or b must be made with certainty.

The advantage of constructing trap 2 is that the usage of color 2 is limited within the segment. So, if Alice colors within it, then Bob wins with the help of trap 1. If she does not color inside the configuration, then Bob can color either $e : 2$ or $f : 2$. Now, the usage of 2 is completely ruled-out. This obviously leads to Bob's win.

Case 2. Suppose Alice starts the game by coloring any of the first three positions from one end of P_n .

As usual, Bob follows with his fundamental strategic move. In order to win, Bob should build trap 3 or trap 4.

If trap 3 (see Figure 5) is formed and if Alice colors inside the set-up, then Bob can win with the help of trap 1 and trap 2. Otherwise, Bob should color

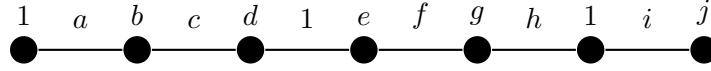


Figure 5: The above set-up is considered to be trap 3.

either d or e , which builds a two-way winning chance for him. Even if Alice, in her second move, colors inside this configuration, before the trap is formed, she cannot stop Bob from winning.

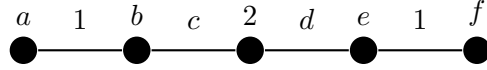


Figure 6: The above set-up is considered to be trap 4.

When trap 4 (see Figure 6) is formed, it must be made clear that either a or f or both can never acquire color 2. Assume if a cannot be colored with 2, then the usage of 1 and 2 are blocked in a, b, c, d , and e . Therefore, if Alice colors anywhere in the configuration, then Figure 1 can be easily built; otherwise, Bob can color $c : 3$ to build a two-way chance for him to win.

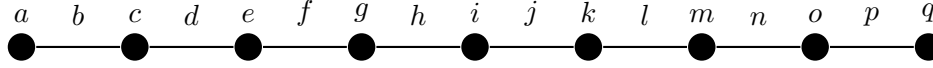


Figure 7: A set-up with 17 positions.

Consider Figure 7, assume that, Alice in her first move colors $a : 1$ and Bob colors $f : 1$. Alice is now forced to color only at $i : 1$, $j : 1$, $k : 2$, or $m : 1$; otherwise, Bob can form trap 3 or trap 4.

- If Alice in her second move colors $i : 1$ (or $j : 1$), then Bob must color at five positions away from i (or j) with 1. This set-up also serves as a trap because wherever Alice colors inside the set-up, except d , Bob can win. If Alice colors away from the set-up or colors $d : 2$, then Bob should color $g : 3$ which brings a two-way chance to win, i.e., Bob, in his next move, can color either $b : 3$ or $p : 3$ (or $q : 3$) to form trap 1 or trap 2, respectively.
- If Alice in her second move colors $k : 2$, then Bob should color $g : 3$ which again forms a two-way winning chance for him.
- If Alice colors $m : 1$, then Bob should color $i : 2$. This forces Alice to play either at $j : 1$ or at $i : 2$; otherwise, Bob colors $j : 3$. Now, Bob, at his third move, can easily form trap 4 by coloring $c : 2$. Thus Bob wins.

Similarly, when Alice colors at b or c , Bob can easily win with the above-mentioned strategies. Thus, Alice requires a minimum of five colors to win. Hence, $\chi''_g(P_n) = 5$. $\square$

The above-mentioned traps and strategies can be used to improve the incidence game chromatic number of paths. The trap 1 discussed in this paper was also discussed by Kim in [5]. He proved that the incidence game chromatic number of paths, $i_g(P_n) = 5$ for $n \geq 13$. In order to prove the following corollary, it is necessary to establish a relationship between the incidence and the total graphs of paths. The total game coloring of a graph G can also be defined as the vertex coloring game on the total graph of G , $T(G)$, i.e., $\chi_g''(G) = \chi_g(T(G))$. Since, $T(P_n)$ has $2n - 1$ vertices, it can be concluded that $\chi_g(T(P_n)) = 5$ for $|V(T(P_n))| \geq 17$. Similarly, the incidence game coloring of a graph G can be defined as the vertex coloring game on the incidence graph of the graph, G^I , i.e., $i_g(G) = \chi_g(G^I)$. Moreover, the incidence graph of a path can be obtained by deleting exactly one vertex of $T(P_n)$ with degree $\delta(= 2)$; therefore, P_n^I has $2n - 2$ vertices.

Corollary 3.1. $i_g(P_n) = 5$, for $|V(P_n)| \geq 10$.

Let C_n be a cycle on n vertices, and it has $2n$ positions. The total game chromatic number of cycles can also be found using the same way as that of paths. Moreover, the traps discussed in paths will again be recalled in the proof of cycles.

Theorem 3.2. $\chi_g''(C_n) = 5$, for $n \geq 3$.

Proof. Let C_n be a cycle with $n \geq 3$, where n denotes the number of vertices. It is trivial from the upper bound that $\chi_g''(C_n) \leq 5$. Following the proof of paths, it is enough to show that Bob has a winning strategy on C_n with four colors. Assume that the players have four colors in the color set, say $C = \{1, 2, 3, 4\}$, to use. In order to win, Bob needs to build the set-up in Figure 1.



Figure 8: The game on C_3 .

Case 1. Consider C_3 , there are six positions in C_3 , namely, p_1, p_2, p_3, p_4, p_5 , and p_6 . If Alice, in her first move, colors some position p_i with 1, then Bob should color the position not adjacent to it with a different color (see Figure 8 - left). Thus, the usage of these two colors are not allowed anywhere in C_3 , this forces Alice to use another color, and Bob should repeat the same (see Figure 8 - right). This finally helps Bob win by forming the set-up discussed in Figure 1.

Case 2. Consider C_4 and C_5 , let Alice start the game by coloring any position, say p_0 , with 1. Bob has to color five positions away from p_0 with 1. Now, the

usage of 1 is not allowed anywhere in the graph. Thus, in Alice's second move, wherever she colors, Bob can win with the help of trap 1.

Case 3. Consider C_6 , here, the players start the game as described in case 2 (see Figure 9).

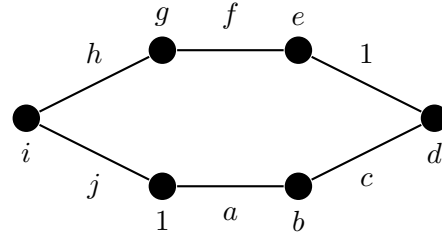


Figure 9: The situation on C_6 after the first move of the players.

In the above figure, Alice cannot color anywhere at a, b, c, d, e, f, i , and j ; otherwise, Bob wins with the help of trap 1 or trap 2. Therefore, the only places left for Alice to color are h and g . Moreover, Alice cannot use color 2 at these places, because Bob can easily win by forming trap 4. If Alice colors any of them with 1, then Bob should form trap 2 by coloring i or f with 2. Thus Bob wins.

Case 4. Consider C_7 , C_8 , and C_9 , the first move follows as already discussed. If Alice in her second move uses any color other than 1, then Bob can form trap 3 by coloring five positions away from any of the previously colored 1's. Therefore, Alice, in her second move, is forced to use 1.

In case of C_7 ,

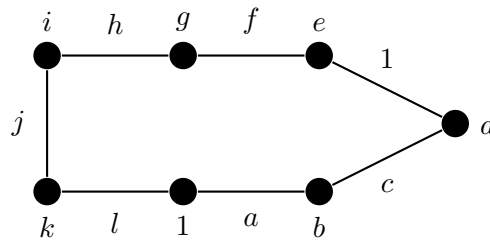


Figure 10: The situation on C_7 after the first move of the players.

Alice can only color either g or j with 1; otherwise, as mentioned, Alice will herself form trap 3. With this, Bob can easily form trap 4.

In case of C_8 ,

Alice can only color h or k with 1; otherwise, Bob can form trap 3 or Alice herself forms trap 3. Without loss of generality (see Figure 11), let Alice colors $h : 1$, then Bob should color $l : 2$. This forces Alice to color either $i : 2$ or $k : 1$, thus Bob can color $c : 2$ and win by the formation of trap 4.

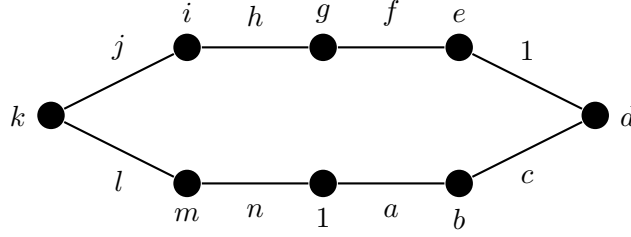


Figure 11: The situation on C_8 after the first move of the players.

In case of C_9 , with the similar construction, it can be seen that Alice can possibly color either j or k with 1. Moreover, even if Alice colors at j or k , Bob can easily form trap 4.

Case 5. For cycles with $n \geq 10$, irrespective of Alice's move, trap 3 can easily be formed by Bob in his second move.

Therefore, Alice requires at least five colors to win on cycles, $\chi_g''(C_n) \geq 5$. Hence, it can be concluded that $\chi_g''(C_n) = 5$ for cycles with $n \geq 3$. $\square$

A star, $K_{1,n}$ is a tree with a central vertex adjacent to all other outer vertices, where the degree of all the outer vertices are one. Also, $\Delta(K_{1,n}) = n$.

Theorem 3.3. $\chi_g''(K_{1,n}) = \Delta + 1$, for $n \geq 2$.

Proof. Let $K_{1,n}$ be a star with $n \geq 2$. Let Alice, always, in her first move color the central vertex, say v_0 , with 1. Since v_0 is adjacent to all other vertices and edges, 1 cannot be repeated anywhere on the graph. Thus, Bob has to opt for another color. Alice has to follow a simple strategy that for every color Bob places on a vertex (or edge), Alice should always have to use the same color on an edge (or a vertex) which is not incident to it. Therefore, with n vertices, other than v_0 , and n edges, Alice can always win with $n + 1$ colors. $\square$

The following theorem is a direct consequence of the results related to total coloring of a graph:

Theorem 3.4. Let G be a graph

1. $\chi_g''(G) = 1$ if and only if G is an empty graph.
2. If G is a connected graph, then $\chi_g''(G) \geq 3$.
3. There exists no graph G such that $\chi_g''(G) = 2$.

4. Conclusion

Certainly, game coloring is a topic with enormous ideas directly or indirectly providing several real world applications, especially while dealing with two or

more uncooperative partners. It can also be used in the fields of optimization, network security, traffic maintenance, etc. In this paper, three general classes of graphs are considered and the total game chromatic number of those graphs are obtained. These results can also be extended to other graphs.

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(p, q) - φ relative Gol'dberg order and (p, q) - φ relative Gol'dberg type oriented certain growth properties of entire functions of several complex variables

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Abstract. In this paper our concern is to study some growth properties based on different growth indicators such as (p, q) - φ relative Gol'dberg order, (p, q) - φ relative Gol'dberg type etc. of entire functions of several complex variables.

Keywords: entire functions of several complex variables, growth, (p, q) - φ relative Gol'dberg order, (p, q) - φ relative Gol'dberg type, (p, q) - φ relative Gol'dberg weak type.

1. Introduction, definitions and notations

In theory of Mathematical analysis, we generally indicate the complex and real n -spaces by the symbols $\mathbb{C}^n$ and $\mathbb{R}^n$, respectively. Considering I as the set of non-negative integers, if someone presumes that the points $(z_1, z_2, \dots, z_n)$, $(m_1, m_2, \dots, m_n)$ of $\mathbb{C}^n$ or I^n are represented by the respective unsuffixed symbols z, m then the modulus of z , denoted by $|z|$, is defined to be $|z| = (|z_1|^2 + \dots + |z_n|^2)^{\frac{1}{2}}$. Further if the coordinates of the vector m are non-negative integers $m_1, m_2, \dots, m_n$, then the expression $z_1^{m_1} \dots z_n^{m_n}$ may be presented by the symbol z^m where $\|m\| = m_1 + \dots + m_n$.

Suppose that $D \subseteq \mathbb{C}^n$ is an arbitrary bounded complex n -circular domain with the center at the origin. Then, for any entire function $f(z)$ of n complex variables and $R > 0$, the maximum modulus function $M_{f,D}(R)$ may be defined to be $M_{f,D}(R) = \sup_{z \in D_R} |f(z)|$ where a point $z \in D_R \Leftrightarrow \frac{z}{R} \in D$. It is noticeable from the way of defining the maximum modulus function that $M_{f,D}(R)$ is

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strictly increasing when $f(z)$ is non-constant and the inverse of $M_{f,D}(R)$ i.e., $M_{f,D}^{-1} : (|f(0)|, \infty) \rightarrow (0, \infty)$ exists such that $\lim_{R \rightarrow \infty} M_{f,D}^{-1}(R) = \infty$.

For $k \in [0, \infty)$ and $k \in \mathbb{N}$ where $\mathbb{N}$ be the set of all positive integers, we define iterations of the exponential and logarithmic functions as $\exp^{[k]} R = \exp(\exp^{[k-1]} R)$ and $\log^{[k]} R = \log(\log^{[k-1]} R)$, with convention that $\log^{[0]} R = R$, $\log^{[-1]} R = \exp R$, $\exp^{[0]} R = R$ and $\exp^{[-1]} R = \log R$. Further we assume that throughout the present paper p, q, m and n always denote positive integers. Further let $L(R)$ be a class of non-decreasing unbounded functions $\varphi(R) : [0, +\infty) \rightarrow (0, +\infty)$. Also, in this paper, by an entire function $f(z)$ of n -complex variables we mean an entire function $f(z)$ for any bounded complete n -circular domain D having the center at origin in $\mathbb{C}^n$. Considering this let us recall the definitions of (p, q) - φ relative Gol'dberg order and the (p, q) - φ relative Gol'dberg lower order of an entire function $f(z)$ of n -complex variables which are as follows:

Definition 1.1 ([3]). *Let $\varphi(R) \in L(R)$. Also, let $f(z)$ and $g(z)$ be any two entire functions of n -complex variables. The (p, q) - φ relative Gol'dberg order and the (p, q) - φ relative Gol'dberg lower order of $f(z)$ with respect to $g(z)$ are defined as*

$$\rho_{g,D}^{(p,q)}(f, \varphi) = \lim_{R \rightarrow \infty} \sup \frac{\log^{[p]} M_{g,D}^{-1}(M_{f,D}(R))}{\log^{[q]} \varphi(R)}.$$

In this paper, we assume that the nondecreasing unbounded function $\varphi(R) : [0, +\infty) \rightarrow (0, +\infty)$ always satisfies $\lim_{R \rightarrow +\infty} \frac{\log^{[q]} \varphi(\alpha R)}{\log^{[q]} \varphi(R)} = 1$ for all $\alpha > 0$. Since, Biswas et al. [3] have already shown that $\rho_{g,D}^{(p,q)}(f, \varphi)$ and $\lambda_{g,D}^{(p,q)}(f, \varphi)$ are independent of the choice of the domain D when $\varphi(R) : [0, +\infty) \rightarrow (0, +\infty)$ is a nondecreasing unbounded function and satisfies $\lim_{R \rightarrow +\infty} \frac{\log^{[q]} \varphi(\alpha R)}{\log^{[q]} \varphi(R)} = 1$ for all $\alpha > 0$, so here we shall always use the notations $\rho_g^{(p,q)}(f, \varphi)$ and $\lambda_g^{(p,q)}(f, \varphi)$ instead of $\rho_{g,D}^{(p,q)}(f, \varphi)$ and $\lambda_{g,D}^{(p,q)}(f, \varphi)$, respectively.

Now, for the development of such growth indicators, one may introduce (p, q) - φ relative Gol'dberg type $\sigma_{g,D}^{(p,q)}(f, \varphi)$ and (p, q) - φ relative Gol'dberg weak type $\tau_{g,D}^{(p,q)}(f, \varphi)$ in the following way:

Definition 1.2 ([6]). *Let $\varphi(R) \in L(R)$. Let $f(z)$ and $g(z)$ be any two entire functions of n -complex variables such that $0 < \rho_g^{(p,q)}(f, \varphi) < \infty$. Then, the (p, q) - φ relative Gol'dberg type $\sigma_{g,D}^{(p,q)}(f, \varphi)$ and the (p, q) - φ relative Gol'dberg lower type $\bar{\sigma}_{g,D}^{(p,q)}(f, \varphi)$ of $f(z)$ with respect to $g(z)$ are defined as:*

$$\frac{\sigma_{g,D}^{(p,q)}(f, \varphi)}{\bar{\sigma}_{g,D}^{(p,q)}(f, \varphi)} = \lim_{R \rightarrow \infty} \sup \frac{\log^{[p-1]} M_{g,D}^{-1}(M_{f,D}(R))}{\left(\log^{[q-1]} \varphi(R)\right)^{\rho_g^{(p,q)}(f, \varphi)}}.$$

Definition 1.3 ([6]). Let $\varphi(R) \in L(R)$. Let $f(z)$ and $g(z)$ be any two entire functions of n -complex variables such that $0 < \lambda_g^{(p,q)}(f, \varphi) < \infty$. Then, the (p, q) - φ relative Gol'dberg weak type $\tau_{g,D}^{(p,q)}(f, \varphi)$ and (p, q) - φ relative Gol'dberg upper weak type $\bar{\tau}_{g,D}^{(p,q)}(f, \varphi)$ of $f(z)$ with respect to $g(z)$ are defined as:

$$\frac{\bar{\tau}_{g,D}^{(p,q)}(f, \varphi)}{\tau_{g,D}^{(p,q)}(f, \varphi)} = \lim_{R \rightarrow \infty} \sup \frac{\log^{[p-1]} M_{g,D}^{-1}(M_{f,D}(R))}{\left(\log^{[q-1]} \varphi(R)\right)^{\lambda_g^{(p,q)}(f, \varphi)}}.$$

As Gol'dberg has shown that (see [8]) Gol'dberg type depends on the domain D , therefore in general all the growth indicators defined in Definition 1.2 and Definition 1.3 also depend on D .

During the past decades, several authors (see [1] to [10]) made closed investigations on the growth properties of entire functions of several complex variables using different growth indicator such as Gol'dberg order, (p, q) -th Gol'dberg order, relative Gol'dberg order, etc. In the present paper our concern is to study some growth properties based upon the discussion of several growth indicators such as (p, q) - φ relative Gol'dberg type, (p, q) - φ relative Gol'dberg weak type etc. of entire functions of several complex variables.

2. Main results

In this section we present the main results of this paper. Here we always suppose that $\varphi_1(R)$, $\varphi_2(R)$ both belong to $L(R)$ with $\lim_{R \rightarrow +\infty} \frac{\log^{[q]} \varphi_1(\alpha R)}{\log^{[q]} \varphi_1(R)} = 1$ and $\lim_{R \rightarrow +\infty} \frac{\log^{[n]} \varphi_2(\alpha R)}{\log^{[n]} \varphi_2(R)} = 1$ for all $\alpha > 0$.

Theorem 2.1. Let $f(z)$, $g(z)$, $h(z)$ and $k(z)$ be any four entire functions of n -complex variables and D be a bounded complete n -circular domain with center at origin in $\mathbb{C}^n$. Also, let $0 < \lambda_h^{(p,q)}(f, \varphi_1) \leq \rho_h^{(p,q)}(f, \varphi_1) < \infty$ and $0 < \lambda_k^{(m,n)}(g, \varphi_2) \leq \rho_k^{(m,n)}(g, \varphi_2) < \infty$. Then

$$\begin{aligned} \frac{\lambda_h^{(p,q)}(f, \varphi_1)}{\rho_k^{(m,n)}(g, \varphi_2)} &\leq \liminf_{R \rightarrow +\infty} \frac{\log^{[p]} M_{h,D}^{-1}(M_{f,D}(\varphi_1^{-1}(\exp^{[q]} R)))}{\log^{[m]} M_{k,D}^{-1}(M_{g,D}(\varphi_2^{-1}(\exp^{[n]} R)))} \\ &\leq \min \left\{ \frac{\lambda_h^{(p,q)}(f, \varphi_1)}{\lambda_k^{(m,n)}(g, \varphi_2)}, \frac{\rho_h^{(p,q)}(f, \varphi_1)}{\rho_k^{(m,n)}(g, \varphi_2)} \right\} \leq \max \left\{ \frac{\lambda_h^{(p,q)}(f, \varphi_1)}{\lambda_k^{(m,n)}(g, \varphi_2)}, \frac{\rho_h^{(p,q)}(f, \varphi_1)}{\rho_k^{(m,n)}(g, \varphi_2)} \right\} \\ &\leq \limsup_{R \rightarrow +\infty} \frac{\log^{[p]} M_{h,D}^{-1}(M_{f,D}(\varphi_1^{-1}(\exp^{[q]} R)))}{\log^{[m]} M_{k,D}^{-1}(M_{g,D}(\varphi_2^{-1}(\exp^{[n]} R)))} \leq \frac{\rho_h^{(p,q)}(f, \varphi_1)}{\lambda_k^{(m,n)}(g, \varphi_2)}. \end{aligned}$$

Proof of Theorem 2.1. From the definitions of $\lambda_k^{(m,n)}(g, \varphi_2)$, $\rho_k^{(m,n)}(g, \varphi_2)$, $\lambda_h^{(p,q)}(f, \varphi_1)$ and $\rho_h^{(p,q)}(f, \varphi_1)$ we have for arbitrary positive ε and for all sufficiently large values of R that

$$(1) \quad \log^{[p]} M_{h,D}^{-1} \left(M_{f,D} \left(\varphi_1^{-1}(\exp^{[q]} R) \right) \right) \geq (\lambda_h^{(p,q)}(f, \varphi_1) - \varepsilon) R$$

$$(2) \quad \log^{[m]} M_{k,D}^{-1} \left(M_{g,D} \left(\varphi_2^{-1}(\exp^{[n]} R) \right) \right) \leq (\rho_k^{(m,n)}(g, \varphi_2) + \varepsilon)R$$

$$(3) \quad \log^{[m]} M_{k,D}^{-1} \left(M_{g,D} \left(\varphi_2^{-1}(\exp^{[n]} R) \right) \right) \geq (\lambda_k^{(m,n)}(g, \varphi_2) - \varepsilon)R$$

and

$$(4) \quad \log^{[p]} M_{h,D}^{-1} \left(M_{f,D} \left(\varphi_1^{-1}(\exp^{[q]} R) \right) \right) \leq (\rho_h^{(p,q)}(f, \varphi_1) + \varepsilon)R.$$

Again, we get for a sequence of positive numbers of R tending to infinity that

$$(5) \quad \log^{[p]} M_{h,D}^{-1} \left(M_{f,D} \left(\varphi_1^{-1}(\exp^{[q]} R) \right) \right) \leq (\lambda_h^{(p,q)}(f, \varphi_1) + \varepsilon)R$$

$$(6) \quad \log^{[m]} M_{k,D}^{-1} \left(M_{g,D} \left(\varphi_2^{-1}(\exp^{[n]} R) \right) \right) \leq (\lambda_k^{(m,n)}(g, \varphi_2) + \varepsilon)R$$

$$(7) \quad \log^{[m]} M_{k,D}^{-1} \left(M_{g,D} \left(\varphi_2^{-1}(\exp^{[n]} R) \right) \right) \geq (\rho_k^{(m,n)}(g, \varphi_2) - \varepsilon)R$$

and

$$(8) \quad \log^{[p]} M_{h,D}^{-1} \left(M_{f,D} \left(\varphi_1^{-1}(\exp^{[q]} R) \right) \right) \geq (\rho_h^{(p,q)}(f, \varphi_1) - \varepsilon)R.$$

Now, from (1) and (2), it follows for all sufficiently large positive numbers of R that

$$\frac{\log^{[p]} M_{h,D}^{-1} (M_{f,D} (\varphi_1^{-1}(\exp^{[q]} R)))}{\log^{[m]} M_{k,D}^{-1} (M_{g,D} (\varphi_2^{-1}(\exp^{[n]} R)))} \geq \frac{(\lambda_h^{(p,q)}(f, \varphi_1) - \varepsilon)R}{(\rho_k^{(m,n)}(g, \varphi_2) + \varepsilon)R}.$$

As $\varepsilon (> 0)$ is arbitrary, we obtain that

$$(9) \quad \liminf_{R \rightarrow +\infty} \frac{\log^{[p]} M_{h,D}^{-1} (M_{f,D} (\varphi_1^{-1}(\exp^{[q]} R)))}{\log^{[m]} M_{k,D}^{-1} (M_{g,D} (\varphi_2^{-1}(\exp^{[n]} R)))} \geq \frac{\lambda_h^{(p,q)}(f, \varphi_1)}{\rho_k^{(m,n)}(g, \varphi_2)}.$$

Combining (5) and (3), we get for a sequence of positive numbers of R tending to infinity that

$$\frac{\log^{[p]} M_{h,D}^{-1} (M_{f,D} (\varphi_1^{-1}(\exp^{[q]} R)))}{\log^{[m]} M_{k,D}^{-1} (M_{g,D} (\varphi_2^{-1}(\exp^{[n]} R)))} \leq \frac{(\lambda_h^{(p,q)}(f, \varphi_1) + \varepsilon)R}{(\lambda_k^{(m,n)}(g, \varphi_2) - \varepsilon)R}.$$

Since $\varepsilon (> 0)$ is arbitrary, it follows that

$$(10) \quad \liminf_{R \rightarrow +\infty} \frac{\log^{[p]} M_{h,D}^{-1} (M_{f,D} (\varphi_1^{-1}(\exp^{[q]} R)))}{\log^{[m]} M_{k,D}^{-1} (M_{g,D} (\varphi_2^{-1}(\exp^{[n]} R)))} \leq \frac{\lambda_h^{(p,q)}(f, \varphi_1)}{\lambda_k^{(m,n)}(g, \varphi_2)}.$$

Now, from (1) and (6), we obtain for a sequence of positive numbers of R tending to infinity that

$$\frac{\log^{[p]} M_{h,D}^{-1} (M_{f,D} (\varphi_1^{-1}(\exp^{[q]} R)))}{\log^{[m]} M_{k,D}^{-1} (M_{g,D} (\varphi_2^{-1}(\exp^{[n]} R)))} \geq \frac{(\lambda_h^{(p,q)}(f, \varphi_1) - \varepsilon)R}{(\lambda_k^{(m,n)}(g, \varphi_2) + \varepsilon)R}.$$

As $\varepsilon (> 0)$ is arbitrary, we get from above that

$$(11) \quad \limsup_{R \rightarrow +\infty} \frac{\log^{[p]} M_{h,D}^{-1} (M_{f,D} (\varphi_1^{-1}(\exp^{[q]} R)))}{\log^{[m]} M_{k,D}^{-1} (M_{g,D} (\varphi_2^{-1}(\exp^{[n]} R)))} \geq \frac{\lambda_h^{(p,q)} (f, \varphi_1)}{\lambda_k^{(m,n)} (g, \varphi_2)}.$$

Now, it follows from (3) and (4) for all sufficiently large positive numbers of R that

$$\frac{\log^{[p]} M_{h,D}^{-1} (M_{f,D} (\varphi_1^{-1}(\exp^{[q]} R)))}{\log^{[m]} M_{k,D}^{-1} (M_{g,D} (\varphi_2^{-1}(\exp^{[n]} R)))} \leq \frac{(\rho_h^{(p,q)} (f, \varphi_1) + \varepsilon)R}{(\lambda_k^{(m,n)} (g, \varphi_2) - \varepsilon)R}.$$

Since $\varepsilon (> 0)$ is arbitrary, we obtain that

$$(12) \quad \limsup_{R \rightarrow +\infty} \frac{\log^{[p]} M_{h,D}^{-1} (M_{f,D} (\varphi_1^{-1}(\exp^{[q]} R)))}{\log^{[m]} M_{k,D}^{-1} (M_{g,D} (\varphi_2^{-1}(\exp^{[n]} R)))} \leq \frac{\rho_h^{(p,q)} (f, \varphi_1)}{\lambda_k^{(m,n)} (g, \varphi_2)}.$$

Now, from (4) and (7), it follows for a sequence of positive numbers of R tending to infinity that

$$\frac{\log^{[p]} M_{h,D}^{-1} (M_{f,D} (\varphi_1^{-1}(\exp^{[q]} R)))}{\log^{[m]} M_{k,D}^{-1} (M_{g,D} (\varphi_2^{-1}(\exp^{[n]} R)))} \leq \frac{(\rho_h^{(p,q)} (f, \varphi_1) + \varepsilon)R}{(\rho_k^{(m,n)} (g, \varphi_2) - \varepsilon)R}.$$

As $\varepsilon (> 0)$ is arbitrary, we obtain that

$$(13) \quad \liminf_{R \rightarrow +\infty} \frac{\log^{[p]} M_{h,D}^{-1} (M_{f,D} (\varphi_1^{-1}(\exp^{[q]} R)))}{\log^{[m]} M_{k,D}^{-1} (M_{g,D} (\varphi_2^{-1}(\exp^{[n]} R)))} \leq \frac{\rho_h^{(p,q)} (f, \varphi_1)}{\rho_k^{(m,n)} (g, \varphi_2)}.$$

Again, combining (2) and (8), we get for a sequence of positive numbers of R tending to infinity that

$$\frac{\log^{[p]} M_{h,D}^{-1} (M_{f,D} (\varphi_1^{-1}(\exp^{[q]} R)))}{\log^{[m]} M_{k,D}^{-1} (M_{g,D} (\varphi_2^{-1}(\exp^{[n]} R)))} \geq \frac{(\rho_h^{(p,q)} (f, \varphi_1) - \varepsilon)R}{(\rho_k^{(m,n)} (g, \varphi_2) + \varepsilon)R}.$$

Since $\varepsilon (> 0)$ is arbitrary, it follows that

$$(14) \quad \limsup_{R \rightarrow +\infty} \frac{\log^{[p]} M_{h,D}^{-1} (M_{f,D} (\varphi_1^{-1}(\exp^{[q]} R)))}{\log^{[m]} M_{k,D}^{-1} (M_{g,D} (\varphi_2^{-1}(\exp^{[n]} R)))} \geq \frac{\rho_h^{(p,q)} (f, \varphi_1)}{\rho_k^{(m,n)} (g, \varphi_2)}.$$

Thus, the theorem follows from (9), (10), (11), (12), (13) and (14).

The proof of the following theorem can be carried out as of the Theorem 2.1, therefore we omit the details.

Theorem 2.2. Let f, g, h, k be any four entire function such that $0 < \overline{\sigma}_{h,D}^{(p,q)}(f, \varphi_1) \leq \sigma_{h,D}^{(p,q)}(f, \varphi_1) < \infty$, $0 < \overline{\sigma}_{k,D}^{(m,n)}(g, \varphi_2) \leq \sigma_{k,D}^{(m,n)}(g, \varphi_2) < \infty$ and $\rho_h^{(p,q)}(f, \varphi_1) = \rho_k^{(m,n)}(g, \varphi_2)$. Then

$$\begin{aligned} \frac{\overline{\sigma}_{h,D}^{(p,q)}(f, \varphi_1)}{\sigma_{k,D}^{(m,n)}(g, \varphi_2)} &\leq \liminf_{R \rightarrow +\infty} \frac{\log^{[p-1]} M_{h,D}^{-1}(M_{f,D}(\varphi_1^{-1}(\exp^{[q-1]} R)))}{\log^{[m-1]} M_{k,D}^{-1}(M_{g,D}(\varphi_2^{-1}(\exp^{[n-1]} R)))} \\ &\leq \min \left\{ \frac{\overline{\sigma}_{h,D}^{(p,q)}(f, \varphi_1)}{\overline{\sigma}_{k,D}^{(m,n)}(g, \varphi_2)}, \frac{\sigma_{h,D}^{(p,q)}(f, \varphi_1)}{\sigma_{k,D}^{(m,n)}(g, \varphi_2)} \right\} \leq \max \left\{ \frac{\overline{\sigma}_{h,D}^{(p,q)}(f, \varphi_1)}{\overline{\sigma}_{k,D}^{(m,n)}(g, \varphi_2)}, \frac{\sigma_{h,D}^{(p,q)}(f, \varphi_1)}{\sigma_{k,D}^{(m,n)}(g, \varphi_2)} \right\} \\ &\leq \limsup_{R \rightarrow +\infty} \frac{\log^{[p-1]} M_{h,D}^{-1}(M_{f,D}(\varphi_1^{-1}(\exp^{[q-1]} R)))}{\log^{[m-1]} M_{k,D}^{-1}(M_{g,D}(\varphi_2^{-1}(\exp^{[n-1]} R)))} \leq \frac{\sigma_{h,D}^{(p,q)}(f, \varphi_1)}{\overline{\sigma}_{k,D}^{(m,n)}(g, \varphi_2)}. \end{aligned}$$

Remark 2.1. If we take “ $0 < \tau_{h,D}^{(p,q)}(f, \varphi_1) \leq \overline{\tau}_{h,D}^{(p,q)}(f, \varphi_1) < \infty$ ”, “ $0 < \tau_{k,D}^{(m,n)}(g, \varphi_2) \leq \overline{\tau}_{k,D}^{(m,n)}(g, \varphi_2) < \infty$ ” and “ $\lambda_h^{(p,q)}(f, \varphi_1) = \lambda_k^{(m,n)}(g, \varphi_2)$ ” instead of “ $0 < \overline{\sigma}_{h,D}^{(p,q)}(f, \varphi_1) \leq \sigma_{h,D}^{(p,q)}(f, \varphi_1) < \infty$ ”, “ $0 < \overline{\sigma}_{k,D}^{(m,n)}(g, \varphi_2) \leq \sigma_{k,D}^{(m,n)}(g, \varphi_2) < \infty$ ” and “ $\rho_h^{(p,q)}(f, \varphi_1) = \rho_k^{(m,n)}(g, \varphi_2)$ ”, respectively and other conditions remain same, the conclusion of Theorem 2.2 remains true with “ $\overline{\sigma}_{h,D}^{(p,q)}(f, \varphi_1)$ ”, “ $\sigma_{h,D}^{(p,q)}(f, \varphi_1)$ ”, “ $\overline{\sigma}_{k,D}^{(m,n)}(g, \varphi_2)$ ” and “ $\sigma_{k,D}^{(m,n)}(g, \varphi_2)$ ” replaced by “ $\tau_{h,D}^{(p,q)}(f, \varphi_1)$ ”, “ $\overline{\tau}_{h,D}^{(p,q)}(f, \varphi_1)$ ”, “ $\tau_{k,D}^{(m,n)}(g, \varphi_2)$ ” and “ $\overline{\tau}_{k,D}^{(m,n)}(g, \varphi_2)$ ”, respectively.

Remark 2.2. If we take “ $0 < \tau_{k,D}^{(m,n)}(g, \varphi_2) \leq \overline{\tau}_{k,D}^{(m,n)}(g, \varphi_2) < \infty$ ” and “ $\rho_h^{(p,q)}(f, \varphi_1) = \lambda_k^{(m,n)}(g, \varphi_2)$ ” instead of “ $0 < \overline{\sigma}_{k,D}^{(m,n)}(g, \varphi_2) \leq \sigma_{k,D}^{(m,n)}(g, \varphi_2) < \infty$ ” and “ $\rho_h^{(p,q)}(f, \varphi_1) = \rho_k^{(m,n)}(g, \varphi_2)$ ”, respectively and other conditions remain same, the conclusion of Theorem 2.2 remains true with “ $\overline{\sigma}_{k,D}^{(m,n)}(g, \varphi_2)$ ” and “ $\sigma_{k,D}^{(m,n)}(g, \varphi_2)$ ” replaced by “ $\tau_{k,D}^{(m,n)}(g, \varphi_2)$ ” and “ $\overline{\tau}_{k,D}^{(m,n)}(g, \varphi_2)$ ”, respectively.

Remark 2.3. If we take “ $0 < \tau_{h,D}^{(p,q)}(f, \varphi_1) \leq \overline{\tau}_{h,D}^{(p,q)}(f, \varphi_1) < \infty$ ” and “ $\lambda_h^{(p,q)}(f, \varphi_1) = \rho_k^{(m,n)}(g, \varphi_2)$ ” instead of “ $0 < \overline{\sigma}_{h,D}^{(p,q)}(f, \varphi_1) \leq \sigma_{h,D}^{(p,q)}(f, \varphi_1) < \infty$ ” and “ $\rho_h^{(p,q)}(f, \varphi_1) = \rho_k^{(m,n)}(g, \varphi_2)$ ”, respectively and other conditions remain same, the conclusion of Theorem 2.2 remains true with “ $\overline{\sigma}_{h,D}^{(p,q)}(f, \varphi_1)$ ” and “ $\sigma_{h,D}^{(p,q)}(f, \varphi_1)$ ” replaced by “ $\tau_{h,D}^{(p,q)}(f, \varphi_1)$ ” and “ $\overline{\tau}_{h,D}^{(p,q)}(f, \varphi_1)$ ”, respectively.

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Generalized relative order (α, β) and generalized relative type (α, β) oriented some growth analysis of composite analytic functions in the unit disc

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Abstract. In this paper we introduce the idea of generalized relative order (α, β) and generalized relative type (α, β) of an analytic function with respect to another analytic function in the unit disc where α and β are continuous non-negative on $(-\infty, +\infty)$ functions. Hence we study some growth properties relating to the composition of two analytic functions in the unit disc on the basis of generalized relative order (α, β) and generalized relative type (α, β) as compared to the growth of their corresponding left and right factors.

Keywords: growth, analytic function, composition, unit disc, generalized relative order (α, β) , generalized relative hyper order (α, β) , generalized relative logarithmic order (α, β) , generalized relative type (α, β) , generalized relative weak type (α, β) .

1. Introduction, definitions and notations

Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be analytic in the unit disc $U = \{z : |z| < 1\}$ and $M_f(r)$ be the maximum of $|f(z)|$ on $|z| = r$. In [4], Sons defined the order $\varrho(f)$ and the lower order $\lambda(f)$ as

$$\frac{\varrho(f)}{\lambda(f)} = \lim_{r \rightarrow 1} \sup \frac{\log^{[2]} M_f(r)}{-\log(1-r)}.$$

Now, let L be a class of continuous non-negative on $(-\infty, \infty)$ functions α such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ with $\alpha(x) \uparrow \infty$ as $x \rightarrow \infty$. Further we assume that throughout the present paper $\alpha, \beta \in L$. Now, considering this, we have introduced the definitions of the generalized order (α, β) and generalized

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lower order (α, β) of an analytic function f in the unit disc U which are as follows:

Definition 1.1. *The generalized order (α, β) denoted by $\varrho_{(\alpha, \beta)}[f]$ and generalized lower order (α, β) denoted by $\lambda_{(\alpha, \beta)}[f]$ of an analytic function f in the unit disc U are defined as:*

$$\frac{\varrho_{(\alpha, \beta)}[f]}{\lambda_{(\alpha, \beta)}[f]} = \lim_{r \rightarrow 1} \sup \frac{\alpha(M_f(r))}{\inf \beta\left(\frac{1}{1-r}\right)}.$$

Clearly $\varrho_{(\log \log r, \log r)}[f] = \varrho(f)$ and $\lambda_{(\log \log r, \log r)}[f] = \lambda(f)$.

Now, we can introduce the definitions of the generalized relative order (α, β) and generalized relative lower order (α, β) of an analytic function f with respect to another analytic function h in the unit disc U which are as follows:

Definition 1.2. *The generalized relative order (α, β) denoted by $\varrho_{(\alpha, \beta)}[f]_h$ and generalized relative lower order (α, β) denoted by $\lambda_{(\alpha, \beta)}[f]_h$ of an entire function f with respect to another entire function h in the unit disc U are defined as:*

$$\frac{\varrho_{(\alpha, \beta)}[f]_h}{\lambda_{(\alpha, \beta)}[f]_h} = \lim_{r \rightarrow 1} \sup \frac{\alpha(M_h^{-1}(M_f(r)))}{\inf \beta\left(\frac{1}{1-r}\right)}.$$

Clearly, if $\alpha(r) = \beta(r) = \log r$ and $h(z) = \exp z$, then $\varrho_{(\alpha, \beta)}[f]_h = \varrho(f)$ and $\lambda_{(\alpha, \beta)}[f]_h = \lambda(f)$.

Now, one may give the definitions of generalized relative hyper order (α, β) and generalized relative logarithmic order (α, β) of an analytic function f with respect to another analytic function h in the unit disc U in the following way:

Definition 1.3. *The generalized relative hyper order (α, β) denoted by $\bar{\varrho}_{(\alpha, \beta)}[f]_h$ and generalized relative hyper lower order (α, β) denoted by $\bar{\lambda}_{(\alpha, \beta)}[f]_h$ of an entire function f with respect to entire function h in the unit disc U are defined as:*

$$\frac{\bar{\varrho}_{(\alpha, \beta)}[f]_h}{\bar{\lambda}_{(\alpha, \beta)}[f]_h} = \lim_{r \rightarrow 1} \sup \frac{\alpha(\log(M_h^{-1}(M_f(r))))}{\inf \beta\left(\frac{1}{1-r}\right)}.$$

Definition 1.4. *The generalized relative logarithmic order (α, β) denoted by $\underline{\varrho}_{(\alpha, \beta)}[f]_h$ and generalized relative logarithmic lower order (α, β) denoted by $\underline{\lambda}_{(\alpha, \beta)}[f]_h$ of an entire function f with respect to entire function h in the unit disc U are defined as:*

$$\frac{\underline{\varrho}_{(\alpha, \beta)}[f]_h}{\underline{\lambda}_{(\alpha, \beta)}[f]_h} = \lim_{r \rightarrow 1} \sup \frac{\alpha(M_h^{-1}(M_f(r)))}{\inf \beta\left(\log\left(\frac{1}{1-r}\right)\right)}.$$

Now, in order to refine the growth scale namely the generalized relative order (α, β) , we introduce the definitions of another growth indicators, called generalized relative type (α, β) and generalized relative lower type (α, β) respectively of an analytic function f with respect to another analytic function h in the unit disc U which are as follows:

Definition 1.5. *The generalized relative type (α, β) and generalized relative lower type (α, β) of an entire function f with respect to another entire function h in the unit disc U having finite positive generalized relative order (α, β) ($0 < \varrho_{(\alpha, \beta)}[f]_h < \infty$) are defined as :*

$$\frac{\sigma_{(\alpha, \beta)}[f]_h}{\bar{\sigma}_{(\alpha, \beta)}[f]_h} = \lim_{r \rightarrow 1} \sup \frac{\exp(\alpha(M_h^{-1}(M_f(r))))}{\inf \left(\exp \left(\beta \left(\frac{1}{1-r} \right) \right) \right)^{\varrho_{(\alpha, \beta)}[f]_h}}.$$

It is obvious that $0 \leq \bar{\sigma}_{(\alpha, \beta)}[f]_h \leq \sigma_{(\alpha, \beta)}[f]_h \leq \infty$.

Analogously, to determine the relative growth of two analytic functions in the unit disc U having same non zero finite generalized relative lower order (α, β) , one can introduced the definitions of generalized relative weak type (α, β) and generalized relative upper weak type (α, β) of an analytic function f with respect to analytic function h in the unit disc U of finite positive generalized relative lower order (α, β) , $\lambda_{(\alpha, \beta)}[f]_h$ in the following way:

Definition 1.6. *The generalized relative upper weak type (α, β) and generalized relative weak type (α, β) of an entire function f with respect to another entire function h in the unit disc U having finite positive generalized relative lower order (α, β) ($0 < \lambda_{(\alpha, \beta)}[f]_h < \infty$) are defined as :*

$$\frac{\bar{\tau}_{(\alpha, \beta)}[f]_h}{\tau_{(\alpha, \beta)}[f]_h} = \lim_{r \rightarrow 1} \sup \frac{\exp(\alpha(M_h^{-1}(M_f(r))))}{\inf \left(\exp \left(\beta \left(\frac{1}{1-r} \right) \right) \right)^{\lambda_{(\alpha, \beta)}[f]_h}}.$$

It is obvious that $0 \leq \tau_{(\alpha, \beta)}[f]_h \leq \bar{\tau}_{(\alpha, \beta)}[f]_h \leq \infty$.

In this paper, we study some growth properties relating to the composition of two analytic function of in the unit disc on the basis of generalized relative order (α, β) , generalized relative hyper order (α, β) , generalized relative logarithmic order (α, β) , generalized relative type (α, β) and generalized relative weak type (α, β) as compared to the growth of their corresponding left and right factors. We do not explain the standard definitions and notations in the theory of entire functions as those are available in [1], [2] and [3].

2. Theorems

In this section we present the main results of the paper.

Theorem 2.1. *Let f, g, h and k be non-constant entire functions in the unit disc U such that $0 < \lambda_{(\alpha,\beta)}[f(g)]_h \leq \varrho_{(\alpha,\beta)}[f(g)]_h < \infty$ and $0 < \lambda_{(\alpha,\beta)}[f]_k \leq \varrho_{(\alpha,\beta)}[f]_k < \infty$. Then*

$$\begin{aligned} \frac{\lambda_{(\alpha,\beta)}[f(g)]_h}{\varrho_{(\alpha,\beta)}[f]_k} &\leq \liminf_{r \rightarrow 1} \frac{\alpha(M_h^{-1}(M_{f(g)}(r)))}{\alpha(M_k^{-1}(M_f(r)))} \\ &\leq \min \left\{ \frac{\lambda_{(\alpha,\beta)}[f(g)]_h}{\lambda_{(\alpha,\beta)}[f]_k}, \frac{\varrho_{(\alpha,\beta)}[f(g)]_h}{\varrho_{(\alpha,\beta)}[f]_k} \right\} \leq \max \left\{ \frac{\lambda_{(\alpha,\beta)}[f(g)]_h}{\lambda_{(\alpha,\beta)}[f]_k}, \frac{\varrho_{(\alpha,\beta)}[f(g)]_h}{\varrho_{(\alpha,\beta)}[f]_k} \right\} \\ &\leq \limsup_{r \rightarrow 1} \frac{\alpha(M_h^{-1}(M_{f(g)}(r)))}{\alpha(M_k^{-1}(M_f(r)))} \leq \frac{\varrho_{(\alpha,\beta)}[f(g)]_h}{\lambda_{(\alpha,\beta)}[f]_k}. \end{aligned}$$

Proof of Theorem 2.1. From the definitions of $\lambda_{(\alpha,\beta)}[f(g)]_h$, $\varrho_{(\alpha,\beta)}[f(g)]_h$, $\lambda_{(\alpha,\beta)}[f]_k$, $\varrho_{(\alpha,\beta)}[f]_k$ and we have for arbitrary positive ε and for all sufficiently large values of $\frac{1}{1-r}$ that

$$\begin{aligned} (1) \quad &\alpha(M_h^{-1}(M_{f(g)}(r))) \geq (\lambda_{(\alpha,\beta)}[f(g)]_h - \varepsilon)\beta((1-r)^{-1}), \\ (2) \quad &\alpha(M_h^{-1}(M_{f(g)}(r))) \leq (\varrho_{(\alpha,\beta)}[f(g)]_h + \varepsilon)\beta((1-r)^{-1}), \\ (3) \quad &\alpha(M_k^{-1}(M_f(r))) \geq (\lambda_{(\alpha,\beta)}[f]_k - \varepsilon)\beta((1-r)^{-1}), \\ (4) \quad &\alpha(M_k^{-1}(M_f(r))) \leq (\varrho_{(\alpha,\beta)}[f]_k + \varepsilon)\beta((1-r)^{-1}). \end{aligned}$$

Again for a sequence of values of $\frac{1}{1-r}$ tending to infinity,

$$\begin{aligned} (5) \quad &\alpha(M_h^{-1}(M_{f(g)}(r))) \leq (\lambda_{(\alpha,\beta)}[f(g)]_h + \varepsilon)\beta((1-r)^{-1}), \\ (6) \quad &\alpha(M_h^{-1}(M_{f(g)}(r))) \geq (\varrho_{(\alpha,\beta)}[f(g)]_h - \varepsilon)\beta((1-r)^{-1}), \\ (7) \quad &\alpha(M_k^{-1}(M_f(r))) \leq (\lambda_{(\alpha,\beta)}[f]_k + \varepsilon)\beta((1-r)^{-1}), \\ (8) \quad &\text{and } \alpha(M_k^{-1}(M_f(r))) \geq (\varrho_{(\alpha,\beta)}[f]_k - \varepsilon)\beta((1-r)^{-1}). \end{aligned}$$

Now, from (1) and (4) it follows for all sufficiently large values of $\frac{1}{1-r}$ that

$$\frac{\alpha(M_h^{-1}(M_{f(g)}(r)))}{\alpha(M_k^{-1}(M_f(r)))} \geq \frac{\lambda_{(\alpha,\beta)}[f(g)]_h - \varepsilon}{\varrho_{(\alpha,\beta)}[f]_k + \varepsilon}.$$

As $\varepsilon (> 0)$ is arbitrary, we obtain that

$$(9) \quad \liminf_{r \rightarrow 1} \frac{\alpha(M_h^{-1}(M_{f(g)}(r)))}{\alpha(M_k^{-1}(M_f(r)))} \geq \frac{\lambda_{(\alpha,\beta)}[f(g)]_h}{\varrho_{(\alpha,\beta)}[f]_k},$$

which is the first part of the theorem.

Combining (5) and (3) we get for a sequence of values of $\frac{1}{1-r}$ tending to infinity that

$$\frac{\alpha(M_h^{-1}(M_{f(g)}(r)))}{\alpha(M_k^{-1}(M_f(r)))} \leq \frac{\lambda_{(\alpha,\beta)}[f(g)]_h + \varepsilon}{\lambda_{(\alpha,\beta)}[f]_k - \varepsilon}.$$

Since $\varepsilon (> 0)$ is arbitrary it follows that

$$(10) \quad \liminf_{r \rightarrow 1} \frac{\alpha(M_h^{-1}(M_{f(g)}(r)))}{\alpha(M_k^{-1}(M_f(r)))} \leq \frac{\lambda_{(\alpha, \beta)}[f(g)]_h}{\lambda_{(\alpha, \beta)}[f]_k}.$$

Now, from (1) and (7) we obtain for a sequence of values of $\frac{1}{1-r}$ tending to infinity that

$$\frac{\alpha(M_h^{-1}(M_{f(g)}(r)))}{\alpha(M_k^{-1}(M_f(r)))} \geq \frac{\lambda_{(\alpha, \beta)}[f(g)]_h - \varepsilon}{\lambda_{(\alpha, \beta)}[f]_k + \varepsilon}.$$

As $\varepsilon (> 0)$ is arbitrary, we get from above that

$$(11) \quad \limsup_{r \rightarrow 1} \frac{\alpha(M_h^{-1}(M_{f(g)}(r)))}{\alpha(M_k^{-1}(M_f(r)))} \geq \frac{\lambda_{(\alpha, \beta)}[f(g)]_h}{\lambda_{(\alpha, \beta)}[f]_k}.$$

Now, it follows from (3) and (2), for all sufficiently large values of $\frac{1}{1-r}$ that

$$\frac{\alpha(M_h^{-1}(M_{f(g)}(r)))}{\alpha(M_k^{-1}(M_f(r)))} \leq \frac{\varrho_{(\alpha, \beta)}[f(g)]_h + \varepsilon}{\lambda_{(\alpha, \beta)}[f]_k - \varepsilon}.$$

Since $\varepsilon (> 0)$ is arbitrary, we obtain that

$$(12) \quad \limsup_{r \rightarrow 1} \frac{\alpha(M_h^{-1}(M_{f(g)}(r)))}{\alpha(M_k^{-1}(M_f(r)))} \leq \frac{\varrho_{(\alpha, \beta)}[f(g)]_h}{\lambda_{(\alpha, \beta)}[f]_k}.$$

Which is the last part of the theorem.

Now, from (2) and (8), it follows for a sequence of values of $\frac{1}{1-r}$ tending to infinity that

$$\frac{\alpha(M_h^{-1}(M_{f(g)}(r)))}{\alpha(M_k^{-1}(M_f(r)))} \leq \frac{\varrho_{(\alpha, \beta)}[f(g)]_h + \varepsilon}{\varrho_{(\alpha, \beta)}[f]_k - \varepsilon}.$$

As $\varepsilon (> 0)$ is arbitrary, we obtain that

$$(13) \quad \liminf_{r \rightarrow 1} \frac{\alpha(M_h^{-1}(M_{f(g)}(r)))}{\alpha(M_k^{-1}(M_f(r)))} \leq \frac{\varrho_{(\alpha, \beta)}[f(g)]_h}{\varrho_{(\alpha, \beta)}[f]_k}.$$

So, combining (4) and (6), we get for a sequence of values of $\frac{1}{1-r}$ tending to infinity that

$$\frac{\alpha(M_h^{-1}(M_{f(g)}(r)))}{\alpha(M_k^{-1}(M_f(r)))} \geq \frac{\varrho_{(\alpha, \beta)}[f(g)]_h - \varepsilon}{\varrho_{(\alpha, \beta)}[f]_k + \varepsilon}.$$

Since $\varepsilon (> 0)$ is arbitrary, it follows that

$$(14) \quad \limsup_{r \rightarrow 1} \frac{\alpha(M_h^{-1}(M_{f(g)}(r)))}{\alpha(M_k^{-1}(M_f(r)))} \geq \frac{\varrho_{(\alpha, \beta)}[f(g)]_h}{\varrho_{(\alpha, \beta)}[f]_k}.$$

Thus, the second part of the theorem follows from (10) and (13), the third part is trivial and fourth part follows from (11) and (14).

Thus, the theorem follows from (9), (10), (11), (12), (13) and (14).

Remark 2.1. If we take “ $0 < \lambda_{(\alpha,\beta)}[g]_k \leq \varrho_{(\alpha,\beta)}[g]_k < \infty$ ” instead of “ $0 < \lambda_{(\alpha,\beta)}[f]_k \leq \varrho_{(\alpha,\beta)}[f]_k < \infty$ ” and other conditions remain same, the conclusion of Theorem 2.1 remains true with “ $\lambda_{(\alpha,\beta)}[f]_k$ ”, “ $\varrho_{(\alpha,\beta)}[f]_k$ ” and “ $\alpha(M_k^{-1}(M_f(r)))$ ” replaced by “ $\lambda_{(\alpha,\beta)}[g]_k$ ”, “ $\varrho_{(\alpha,\beta)}[g]_k$ ” and “ $\alpha(M_k^{-1}(M_g(r)))$ ” respectively in the denominator.

Theorem 2.2. Let f, g, h and k be non-constant entire functions in the unit disc U such that $0 < \lambda_{(\alpha,\beta)}[f]_k \leq \varrho_{(\alpha,\beta)}[f]_k < \infty$ and $\lambda_{(\alpha,\beta)}[f(g)]_h = \infty$. Then

$$\lim_{r \rightarrow 1} \frac{\alpha(M_h^{-1}(M_{f(g)}(r)))}{\alpha(M_k^{-1}(M_f(r)))} = \infty.$$

Proof of Theorem 2.2. Let us suppose that the conclusion of the theorem do not hold. Then we can find a constant $\Delta > 0$ such that for a sequence of values of $\frac{1}{1-r}$ tending to infinity

$$(15) \quad \alpha(M_h^{-1}(M_{f(g)}(r))) \leq \Delta \cdot \alpha(M_k^{-1}(M_f(r))).$$

Again from the definition of $\varrho_{(\alpha,\beta)}[f]_k$, it follows for all sufficiently large values of $\frac{1}{1-r}$ that

$$(16) \quad \alpha(M_k^{-1}(M_f(r))) \leq (\varrho_{(\alpha,\beta)}[f]_k + \epsilon)\beta((1-r)^{-1}).$$

Thus, from (15) and (16), we have for a sequence of values of r tending to 1 that

$$\begin{aligned} \alpha(M_h^{-1}(M_{f(g)}(r))) &\leq \Delta(\varrho_{(\alpha,\beta)}[f]_k + \epsilon)\beta((1-r)^{-1}), \\ \text{i.e., } \frac{\alpha(M_h^{-1}(M_{f(g)}(r)))}{\beta((1-r)^{-1})} &\leq \Delta(\varrho_{(\alpha,\beta)}[f]_k + \epsilon), \\ \text{i.e., } \liminf_{r \rightarrow 1} \frac{\alpha(M_h^{-1}(M_{f(g)}(r)))}{\beta((1-r)^{-1})} &= \lambda_{(\alpha,\beta)}[f(g)]_h < \infty. \end{aligned}$$

This is a contradiction.

Thus the theorem follows.

Remark 2.2. If we take “ $0 < \lambda_{(\alpha,\beta)}[g]_k \leq \varrho_{(\alpha,\beta)}[g]_k < \infty$ ” instead of “ $0 < \lambda_{(\alpha,\beta)}[f]_k \leq \varrho_{(\alpha,\beta)}[f]_k < \infty$ ” and other conditions remain same, the conclusion of Theorem 2.2 remains true with “ $\alpha(M_k^{-1}(M_f(r)))$ ” replaced by “ $\alpha(M_k^{-1}(M_g(r)))$ ” in the denominator.

Remark 2.3. Theorem 2.2 and Remark 2.2 are also valid with “limit superior” instead of “limit” if “ $\lambda_{(\alpha,\beta)}[f(g)]_h = \infty$ ” is replaced by “ $\varrho_{(\alpha,\beta)}[f(g)]_h = \infty$ ” and the other conditions remain the same.

We may now state the following theorem without proof based on Definition 1.3.

Theorem 2.3. Let f, g, h and k be non-constant entire functions in U such that $0 < \bar{\lambda}_{(\alpha, \beta)}[f(g)]_h \leq \bar{\varrho}_{(\alpha, \beta)}[f(g)]_h < \infty$ and $0 < \bar{\lambda}_{(\alpha, \beta)}[f]_k \leq \bar{\varrho}_{(\alpha, \beta)}[f]_k < \infty$. Then

$$\begin{aligned} \frac{\bar{\lambda}_{(\alpha, \beta)}[f(g)]_h}{\bar{\varrho}_{(\alpha, \beta)}[f]_k} &\leq \liminf_{r \rightarrow 1} \frac{\alpha(\log(M_h^{-1}(M_{f(g)}(r))))}{\alpha(\log(M_k^{-1}(M_f(r))))} \\ &\leq \min \left\{ \frac{\bar{\lambda}_{(\alpha, \beta)}[f(g)]_h}{\bar{\lambda}_{(\alpha, \beta)}[f]_k}, \frac{\bar{\varrho}_{(\alpha, \beta)}[f(g)]_h}{\bar{\varrho}_{(\alpha, \beta)}[f]_k} \right\} \leq \max \left\{ \frac{\bar{\lambda}_{(\alpha, \beta)}[f(g)]_h}{\bar{\lambda}_{(\alpha, \beta)}[f]_k}, \frac{\bar{\varrho}_{(\alpha, \beta)}[f(g)]_h}{\bar{\varrho}_{(\alpha, \beta)}[f]_k} \right\} \\ &\leq \limsup_{r \rightarrow 1} \frac{\alpha(\log(M_h^{-1}(M_{f(g)}(r))))}{\alpha(\log(M_k^{-1}(M_f(r))))} \leq \frac{\bar{\varrho}_{(\alpha, \beta)}[f(g)]_h}{\bar{\lambda}_{(\alpha, \beta)}[f]_k}. \end{aligned}$$

Remark 2.4. If we take “ $0 < \bar{\lambda}_{(\alpha, \beta)}[g]_k \leq \bar{\varrho}_{(\alpha, \beta)}[g]_k < \infty$ ” instead of “ $0 < \bar{\lambda}_{(\alpha, \beta)}[f]_k \leq \bar{\varrho}_{(\alpha, \beta)}[f]_k < \infty$ ” and other conditions remain same, the conclusion of Theorem 2.3 remains true with “ $\bar{\lambda}_{(\alpha, \beta)}[f]_k$ ”, “ $\bar{\varrho}_{(\alpha, \beta)}[f]_k$ ” and “ $\alpha(\log(M_k^{-1}(M_f(r))))$ ” replaced by “ $\bar{\lambda}_{(\alpha, \beta)}[g]_k$ ”, “ $\bar{\varrho}_{(\alpha, \beta)}[g]_k$ ” and “ $\alpha(\log(M_k^{-1}(M_g(r))))$ ” respectively in the denominator.

We may now state the following theorem without proof based on Definition 1.4.

Theorem 2.4. Let f, g, h and k be non-constant entire functions in the unit disc U such that $0 < \underline{\lambda}_{(\alpha, \beta)}[f(g)]_h \leq \underline{\varrho}_{(\alpha, \beta)}[f(g)]_h < \infty$ and $0 < \underline{\lambda}_{(\alpha, \beta)}[f]_k \leq \underline{\varrho}_{(\alpha, \beta)}[f]_k < \infty$. Then

$$\begin{aligned} \frac{\underline{\lambda}_{(\alpha, \beta)}[f(g)]_h}{\underline{\varrho}_{(\alpha, \beta)}[f]_k} &\leq \liminf_{r \rightarrow 1} \frac{\alpha(M_h^{-1}(M_{f(g)}(r)))}{\alpha(M_k^{-1}(M_f(r)))} \\ &\leq \min \left\{ \frac{\underline{\lambda}_{(\alpha, \beta)}[f(g)]_h}{\underline{\lambda}_{(\alpha, \beta)}[f]_k}, \frac{\underline{\varrho}_{(\alpha, \beta)}[f(g)]_h}{\underline{\varrho}_{(\alpha, \beta)}[f]_k} \right\} \leq \max \left\{ \frac{\underline{\lambda}_{(\alpha, \beta)}[f(g)]_h}{\underline{\lambda}_{(\alpha, \beta)}[f]_k}, \frac{\underline{\varrho}_{(\alpha, \beta)}[f(g)]_h}{\underline{\varrho}_{(\alpha, \beta)}[f]_k} \right\} \\ &\leq \limsup_{r \rightarrow 1} \frac{\alpha(M_h^{-1}(M_{f(g)}(r)))}{\alpha(M_k^{-1}(M_f(r)))} \leq \frac{\underline{\varrho}_{(\alpha, \beta)}[f(g)]_h}{\underline{\lambda}_{(\alpha, \beta)}[f]_k}. \end{aligned}$$

Remark 2.5. If we take “ $0 < \underline{\lambda}_{(\alpha, \beta)}[g]_k \leq \underline{\varrho}_{(\alpha, \beta)}[g]_k < \infty$ ” instead of “ $0 < \underline{\lambda}_{(\alpha, \beta)}[f]_k \leq \underline{\varrho}_{(\alpha, \beta)}[f]_k < \infty$ ” and other conditions remain same, the conclusion of Theorem 2.4 remains true with “ $\underline{\lambda}_{(\alpha, \beta)}[f]_k$ ”, “ $\underline{\varrho}_{(\alpha, \beta)}[f]_k$ ” and “ $\alpha(M_k^{-1}(M_f(r)))$ ” replaced by “ $\underline{\lambda}_{(\alpha, \beta)}[g]_k$ ”, “ $\underline{\varrho}_{(\alpha, \beta)}[g]_k$ ” and “ $\alpha(M_k^{-1}(M_g(r)))$ ” respectively in the denominator.

In the line of Theorem 2.1, one can easily deduce the Theorem 2.5 and Theorem 2.6, so proofs are omitted.

Theorem 2.5. *Let f, g, h and k be non-constant entire functions in the unit disc U such that $0 < \bar{\sigma}_{(\alpha,\beta)}[f(g)]_h \leq \sigma_{(\alpha,\beta)}[f(g)]_h < \infty$, $0 < \bar{\sigma}_{(\alpha,\beta)}[f]_k \leq \sigma_{(\alpha,\beta)}[f]_k < \infty$ and $\varrho_{(\alpha,\beta)}[f(g)]_h = \varrho_{(\alpha,\beta)}[f]_k$. Then*

$$\begin{aligned} \frac{\bar{\sigma}_{(\alpha,\beta)}[f(g)]_h}{\sigma_{(\alpha,\beta)}[f]_k} &\leq \liminf_{r \rightarrow 1} \frac{\exp(\alpha(M_h^{-1}(M_{f(g)}(r))))}{\exp(\alpha(M_k^{-1}(M_f(r))))} \\ &\leq \min \left\{ \frac{\bar{\sigma}_{(\alpha,\beta)}[f(g)]_h}{\bar{\sigma}_{(\alpha,\beta)}[f]_k}, \frac{\sigma_{(\alpha,\beta)}[f(g)]_h}{\sigma_{(\alpha,\beta)}[f]_k} \right\} \leq \max \left\{ \frac{\bar{\sigma}_{(\alpha,\beta)}[f(g)]_h}{\bar{\sigma}_{(\alpha,\beta)}[f]_k}, \frac{\sigma_{(\alpha,\beta)}[f(g)]_h}{\sigma_{(\alpha,\beta)}[f]_k} \right\} \\ &\leq \limsup_{r \rightarrow 1} \frac{\exp(\alpha(M_h^{-1}(M_{f(g)}(r))))}{\exp(\alpha(M_k^{-1}(M_f(r))))} \leq \frac{\sigma_{(\alpha,\beta)}[f(g)]_h}{\bar{\sigma}_{(\alpha,\beta)}[f]_k}. \end{aligned}$$

Remark 2.6. If we take “ $0 < \bar{\sigma}_{(\alpha,\beta)}[g]_k \leq \sigma_{(\alpha,\beta)}[g]_k < \infty$ ” and “ $\varrho_{(\alpha,\beta)}[f(g)]_h = \varrho_{(\alpha,\beta)}[g]_k$ ” instead of “ $0 < \bar{\sigma}_{(\alpha,\beta)}[f]_k \leq \sigma_{(\alpha,\beta)}[f]_k < \infty$ ” and “ $\varrho_{(\alpha,\beta)}[f(g)]_h = \varrho_{(\alpha,\beta)}[f]_k$ ” and other conditions remain same, the conclusion of Theorem 2.5 remains true with “ $\sigma_{(\alpha,\beta)}[f]_k$ ”, “ $\bar{\sigma}_{(\alpha,\beta)}[f]_k$ ” and “ $\exp(\alpha(M_k^{-1}(M_f(r))))$ ” replaced by “ $\sigma_{(\alpha,\beta)}[g]_k$ ”, “ $\bar{\sigma}_{(\alpha,\beta)}[g]_k$ ” and “ $\exp(\alpha(M_k^{-1}(M_g(r))))$ ” respectively in the denominator.

Remark 2.7. If we take “ $0 < \tau_{(\alpha,\beta)}[f]_k \leq \bar{\tau}_{(\alpha,\beta)}[f]_k < \infty$ ” and “ $\varrho_{(\alpha,\beta)}[f(g)]_h = \lambda_{(\alpha,\beta)}[f]_k$ ” instead of “ $0 < \bar{\sigma}_{(\alpha,\beta)}[f]_k \leq \sigma_{(\alpha,\beta)}[f]_k < \infty$ ” and “ $\varrho_{(\alpha,\beta)}[f(g)]_h = \varrho_{(\alpha,\beta)}[f]_k$ ” and other conditions remain same, the conclusion of Theorem 2.5 remains true with “ $\sigma_{(\alpha,\beta)}[f]_k$ ” and “ $\bar{\sigma}_{(\alpha,\beta)}[f]_k$ ” replaced by “ $\bar{\tau}_{(\alpha,\beta)}[f]_k$ ” and “ $\tau_{(\alpha,\beta)}[f]_k$ ” respectively in the denominator.

Remark 2.8. If we take “ $0 < \tau_{(\alpha,\beta)}[g]_k \leq \bar{\tau}_{(\alpha,\beta)}[g]_k < \infty$ ” and “ $\varrho_{(\alpha,\beta)}[f(g)]_h = \lambda_{(\alpha,\beta)}[g]_k$ ” instead of “ $0 < \bar{\sigma}_{(\alpha,\beta)}[f]_k \leq \sigma_{(\alpha,\beta)}[f]_k < \infty$ ” and “ $\varrho_{(\alpha,\beta)}[f(g)]_h = \varrho_{(\alpha,\beta)}[f]_k$ ” and other conditions remain same, the conclusion of Theorem 2.5 remains true with “ $\bar{\sigma}_{(\alpha,\beta)}[f]_k$ ”, “ $\sigma_{(\alpha,\beta)}[f]_k$ ” and “ $\exp(\alpha(M_k^{-1}(M_f(r))))$ ” replaced by “ $\tau_{(\alpha,\beta)}[g]_k$ ”, “ $\bar{\tau}_{(\alpha,\beta)}[g]_k$ ” and “ $\exp(\alpha(M_k^{-1}(M_g(r))))$ ” respectively in the denominator.

Theorem 2.6. *Let f, g, h and k be non-constant entire functions in the unit disc U such that $0 < \tau_{(\alpha,\beta)}[f(g)]_h \leq \bar{\tau}_{(\alpha,\beta)}[f(g)]_h < \infty$, $0 < \tau_{(\alpha,\beta)}[f]_k \leq \bar{\tau}_{(\alpha,\beta)}[f]_k < \infty$ and $\lambda_{(\alpha,\beta)}[f(g)]_h = \lambda_{(\alpha,\beta)}[f]_k$. Then*

$$\begin{aligned} \frac{\tau_{(\alpha,\beta)}[f(g)]_h}{\bar{\tau}_{(\alpha,\beta)}[f]_k} &\leq \liminf_{r \rightarrow 1} \frac{\exp(\alpha(M_h^{-1}(M_{f(g)}(r))))}{\exp(\alpha(M_k^{-1}(M_f(r))))} \\ &\leq \min \left\{ \frac{\tau_{(\alpha,\beta)}[f(g)]_h}{\tau_{(\alpha,\beta)}[f]_k}, \frac{\bar{\tau}_{(\alpha,\beta)}[f(g)]_h}{\bar{\tau}_{(\alpha,\beta)}[f]_k} \right\} \leq \max \left\{ \frac{\tau_{(\alpha,\beta)}[f(g)]_h}{\tau_{(\alpha,\beta)}[f]_k}, \frac{\bar{\tau}_{(\alpha,\beta)}[f(g)]_h}{\bar{\tau}_{(\alpha,\beta)}[f]_k} \right\} \\ &\leq \limsup_{r \rightarrow 1} \frac{\exp(\alpha(M_h^{-1}(M_{f(g)}(r))))}{\exp(\alpha(M_k^{-1}(M_f(r))))} \leq \frac{\bar{\tau}_{(\alpha,\beta)}[f(g)]_h}{\tau_{(\alpha,\beta)}[f]_k}. \end{aligned}$$

Remark 2.9. If we take “ $0 < \tau_{(\alpha,\beta)}[g]_k \leq \bar{\tau}_{(\alpha,\beta)}[g]_k < \infty$ ” and “ $\lambda_{(\alpha,\beta)}[f(g)]_h = \lambda_{(\alpha,\beta)}[g]_k$ ” instead of “ $0 < \tau_{(\alpha,\beta)}[f]_k \leq \bar{\tau}_{(\alpha,\beta)}[f]_k < \infty$ ” and “ $\lambda_{(\alpha,\beta)}[f(g)]_h = \lambda_{(\alpha,\beta)}[f]_k$ ” and other conditions remain same, the conclusion of Theorem 2.6 remains true with “ $\tau_{(\alpha,\beta)}[f]_k$ ” and “ $\bar{\tau}_{(\alpha,\beta)}[f]_k$ ” replaced by “ $\tau_{(\alpha,\beta)}[g]_k$ ” and “ $\bar{\tau}_{(\alpha,\beta)}[g]_k$ ” respectively in the denominator.

$\lambda_{(\alpha, \beta)}[f]_k$ ” and other conditions remain same, the conclusion of Theorem 2.6 remains true with “ $\tau_{(\alpha, \beta)}[f]_k$ ”, “ $\bar{\tau}_{(\alpha, \beta)}[f]_k$ ” and “ $\exp(\alpha(M_k^{-1}(M_f(r))))$ ” replaced by “ $\tau_{(\alpha, \beta)}[g]_k$ ”, “ $\bar{\tau}_{(\alpha, \beta)}[g]_k$ ” and “ $\exp(\alpha(M_k^{-1}(M_g(r))))$ ” respectively in the denominator.

Remark 2.10. If we take “ $0 < \bar{\sigma}_{(\alpha, \beta)}[f]_k \leq \sigma_{(\alpha, \beta)}[f]_k < \infty$ ” and “ $\lambda_{(\alpha, \beta)}[f(g)]_h = \varrho_{(\alpha, \beta)}[f]_k$ ” instead of “ $0 < \tau_{(\alpha, \beta)}[f]_k \leq \bar{\tau}_{(\alpha, \beta)}[f]_k < \infty$ ” and “ $\lambda_{(\alpha, \beta)}[f(g)]_h = \lambda_{(\alpha, \beta)}[f]_k$ ” and other conditions remain same, the conclusion of Theorem 2.6 remains true with “ $\tau_{(\alpha, \beta)}[f]_k$ ” and “ $\bar{\tau}_{(\alpha, \beta)}[f]_k$ ” replaced by “ $\bar{\sigma}_{(\alpha, \beta)}[f]_k$ ” and “ $\sigma_{(\alpha, \beta)}[f]_k$ ” respectively in the denominator.

Remark 2.11. If we take “ $0 < \bar{\sigma}_{(\alpha, \beta)}[f]_k \leq \sigma_{(\alpha, \beta)}[f]_k < \infty$ ” and “ $\lambda_{(\alpha, \beta)}[f(g)]_h = \varrho_{(\alpha, \beta)}[g]_k$ ” instead of “ $0 < \tau_{(\alpha, \beta)}[f]_k \leq \bar{\tau}_{(\alpha, \beta)}[f]_k < \infty$ ” and “ $\lambda_{(\alpha, \beta)}[f(g)]_h = \lambda_{(\alpha, \beta)}[f]_k$ ” and other conditions remain same, the conclusion of Theorem 2.6 remains true with “ $\tau_{(\alpha, \beta)}[f]_k$ ”, “ $\bar{\tau}_{(\alpha, \beta)}[f]_k$ ” and “ $\exp(\alpha(M_k^{-1}(M_f(r))))$ ” replaced by “ $\bar{\sigma}_{(\alpha, \beta)}[g]_k$ ”, “ $\sigma_{(\alpha, \beta)}[g]_k$ ” and “ $\exp(\alpha(M_k^{-1}(M_g(r))))$ ” respectively in the denominator.

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Generalized relative order (α, β) and generalized relative type (α, β) oriented some growth properties of composite entire and meromorphic functions

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Abstract. In this paper we wish to prove some results relating to the growth rates of composite entire and meromorphic functions with their corresponding left and right factors on the basis of generalized relative order (α, β) and generalized relative type (α, β) where α and β are continuous non-negative functions defined on $(-\infty, +\infty)$.

Keywords: Entire function, meromorphic function, growth, generalized relative order (α, β) , generalized relative type (α, β) , generalized relative weak type (α, β) .

1. Introduction, definitions and notations

Let $g(z)$ be an entire function defined in the open complex plane $\mathbb{C}$ and let $M_g(r) = \max \{|g(z)| : |z| = r\}$. When f is meromorphic, the Nevanlinna's characteristic function $T_f(r)$ (see [6, p.4]) plays the same role as $M_f(r)$. All the standard notations of the Nevanlinna theory of meromorphic functions which are available in [6, 7, 11]. We also use the standard notations and definitions of the theory of entire functions which are available in [10] and therefore we do not explain those in details. The Nevanlinna's characteristic function of a meromorphic function f is defined as

$$T_f(r) = N_f(r) + m_f(r),$$

wherever the function $N_f(r, a)$ ($\bar{N}_f(r, a)$) known as counting function of a -points (distinct a -points) of meromorphic f is defined as follows:

$$N_f(r, a) = \int_0^r \frac{n_f(t, a) - n_f(0, a)}{t} dt + n_f(0, a) \log r$$

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$$\left(\overline{N}_f(r, a) = \int_0^r \frac{\overline{n}_f(t, a) - \overline{n}_f(0, a)}{t} dt + \overline{n}_f(0, a) \log r \right),$$

in addition we represent by $n_f(r, a)$ ($\overline{n}_f(r, a)$) the number of a -points (distinct a -points) of f in $|z| \leq r$ and an ∞ -point is a pole of f . In many occasions $N_f(r, \infty)$ and $\overline{N}_f(r, \infty)$ are symbolized by $N_f(r)$ and $\overline{N}_f(r)$ respectively.

On the other hand, the function $m_f(r, \infty)$ alternatively indicated by $m_f(r)$ known as the proximity function of f is defined as:

$$m_f(r) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f(re^{i\theta}) \right| d\theta, \quad \text{where}$$

$$\log^+ x = \max(\log x, 0) \quad \text{for all } x \geq 0.$$

Also, we may employ $m(r, \frac{1}{f-a})$ by $m_f(r, a)$.

If f is entire, then the Nevanlinna's characteristic function $T_f(r)$ of f is defined as

$$T_f(r) = m_f(r).$$

Moreover, if f is non-constant entire then $T_f(r)$ is also strictly increasing and continuous functions of r . Therefore its inverse $T_f^{-1} : (T_f(0), \infty) \rightarrow (0, \infty)$ exists and is such that $\lim_{s \rightarrow \infty} T_f^{-1}(s) = \infty$.

Now, let L be a class of continuous non-negative functions α defined on $(-\infty, +\infty)$ such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ with $\alpha(x) \uparrow +\infty$ as $x \rightarrow +\infty$. For any $\alpha \in L$, we say that $\alpha \in L_1^0$, if $\alpha((1+o(1))x) = (1+o(1))\alpha(x)$ as $x \rightarrow +\infty$ and $\alpha \in L_2^0$, if $\alpha(\exp((1+o(1))x)) = (1+o(1))\alpha(\exp(x))$ as $x \rightarrow +\infty$. Finally for any $\alpha \in L$, we also say that $\alpha \in L_1$, if $\alpha(cx) = (1+o(1))\alpha(x)$ as $x_0 \leq x \rightarrow +\infty$ for each $c \in (0, +\infty)$ and $\alpha \in L_2$, if $\alpha(\exp(cx)) = (1+o(1))\alpha(\exp(x))$ as $x_0 \leq x \rightarrow +\infty$ for each $c \in (0, +\infty)$. Clearly, $L_1 \subset L_1^0$, $L_2 \subset L_2^0$ and $L_2 \subset L_1$.

Considering this, the value

$$\varrho_{(\alpha, \beta)}[f] = \limsup_{r \rightarrow +\infty} \frac{\alpha(\log M_f(r))}{\beta(\log r)} \quad (\alpha \in L, \beta \in L)$$

is called [9] generalized order (α, β) of an entire function f .

Now, Biswas et al. [3, 4] have introduced the definitions of the generalized order (α, β) of entire and meromorphic functions in the following way after giving a minor modification to the original definition of generalized order (α, β) of an entire function (see, e.g. [9]) which considerably extend the definition of φ -order introduced by Chyzhykov et al. [5].

Definition 1.1 ([3, 4]). *The generalized order (α, β) denoted by $\rho_{(\alpha, \beta)}[f]$ and generalized lower order (α, β) denoted by $\lambda_{(\alpha, \beta)}[f]$ of an entire function f are defined as:*

$$\rho_{(\alpha, \beta)}[f] = \limsup_{r \rightarrow \infty} \frac{\alpha(M_f(r))}{\beta(r)} \quad \text{and}$$

$$\lambda_{(\alpha, \beta)}[f] = \liminf_{r \rightarrow \infty} \frac{\alpha(M_f(r))}{\beta(r)} \quad \text{where } \alpha, \beta \in L_1$$

If f is a meromorphic function, then

$$\begin{aligned}\rho_{(\alpha,\beta)}[f] &= \limsup_{r \rightarrow \infty} \frac{\alpha(\exp(T_f(r)))}{\beta(r)} \text{ and} \\ \lambda_{(\alpha,\beta)}[f] &= \liminf_{r \rightarrow \infty} \frac{\alpha(\exp(T_f(r)))}{\beta(r)}, \text{ where } \alpha \in L_2 \text{ and } \beta \in L_1\end{aligned}$$

Using the inequality $T_f(r) \leq \log M_f(r) \leq 3T_f(2r)$ {cf. [6]}, for an entire function f , one may easily verify that

$$\frac{\rho_{(\alpha,\beta)}[f]}{\lambda_{(\alpha,\beta)}[f]} = \lim_{r \rightarrow \infty} \frac{\sup}{\inf} \frac{\alpha(M_f(r))}{\beta(r)} = \lim_{r \rightarrow \infty} \frac{\sup}{\inf} \frac{\alpha(\exp(T_f(r)))}{\beta(r)},$$

when $\alpha \in L_2$ and $\beta \in L_1$.

Mainly the growth investigation of entire and meromorphic functions has usually been done through their maximum moduli or Nevanlinna's characteristic function in comparison with those of exponential function. But if one is paying attention to evaluate the growth rates of any entire and meromorphic function with respect to a new entire function, the notions of relative growth indicators (see, e.g., [1, 2, 8]) will come. Now, in order to make some progress in the study of relative order, Biswas et al. [3] have introduced the definitions of generalized relative order (α, β) and generalized relative lower order (α, β) of a meromorphic function with respect to an entire function in the following way:

Definition 1.2 ([3]). Let $\alpha, \beta \in L_1$. The generalized relative order (α, β) and generalized relative lower order (α, β) denoted by $\rho_{(\alpha,\beta)}[f]_g$ and $\lambda_{(\alpha,\beta)}[f]_g$ respectively of a meromorphic function f with respect to an entire function g are defined as:

$$\rho_{(\alpha,\beta)}[f]_g = \limsup_{r \rightarrow \infty} \frac{\alpha(T_g^{-1}(T_f(r)))}{\beta(r)} \text{ and } \lambda_{(\alpha,\beta)}[f]_g = \liminf_{r \rightarrow \infty} \frac{\alpha(T_g^{-1}(T_f(r)))}{\beta(r)}.$$

Similarly one may give the definitions of generalized relative hyper order (α, β) and generalized relative logarithmic order (α, β) of a meromorphic function f with respect to an entire function g in the following way:

Definition 1.3. Let $\alpha, \beta \in L_1$. The generalized relative hyper order (α, β) denoted by $\bar{\rho}_{(\alpha,\beta)}[f]_g$ and generalized relative hyper lower order (α, β) denoted by $\bar{\lambda}_{(\alpha,\beta)}[f]_g$ of an meromorphic function f with respect to an entire function g are defined as:

$$\begin{aligned}\bar{\rho}_{(\alpha,\beta)}[f]_g &= \limsup_{r \rightarrow \infty} \frac{\alpha(\log(T_g^{-1}(T_f(r))))}{\beta(r)}, \\ \text{and } \bar{\lambda}_{(\alpha,\beta)}[f]_g &= \liminf_{r \rightarrow \infty} \frac{\alpha(\log(T_g^{-1}(T_f(r))))}{\beta(r)}.\end{aligned}$$

Definition 1.4. Let $\alpha, \beta \in L_1$. The generalized relative logarithmic order (α, β) denoted by $\underline{\rho}_{(\alpha, \beta)}[f]_g$ and generalized relative logarithmic lower order (α, β) denoted by $\underline{\lambda}_{(\alpha, \beta)}[f]_g$ of a meromorphic function f with respect to an entire function g are defined as:

$$\begin{aligned}\underline{\rho}_{(\alpha, \beta)}[f]_g &= \limsup_{r \rightarrow \infty} \frac{\alpha(T_g^{-1}(T_f(r)))}{\beta(\log(r))}, \\ \text{and } \underline{\lambda}_{(\alpha, \beta)}[f]_g &= \liminf_{r \rightarrow \infty} \frac{\alpha(T_g^{-1}(T_f(r)))}{\beta(\log(r))}.\end{aligned}$$

To compare the relative growth of two meromorphic functions of having same non-zero finite generalized relative order (α, β) with respect to an entire function, Biswas et al. [3] have introduced the definitions of generalized relative type (α, β) and generalized relative lower type (α, β) in the following way:

Definition 1.5 ([3]). Let $\alpha, \beta \in L_1$. The generalized relative type (α, β) denoted by $\sigma_{(\alpha, \beta)}[f]_g$ and generalized relative lower type (α, β) denoted by $\bar{\sigma}_{(\alpha, \beta)}[f]_g$ of a meromorphic function f with respect to an entire function g having non-zero finite generalized relative order (α, β) are defined as:

$$\begin{aligned}\sigma_{(\alpha, \beta)}[f]_g &= \limsup_{r \rightarrow \infty} \frac{\exp(\alpha(T_g^{-1}(T_f(r))))}{(\exp(\beta(r)))^{\rho_{(\alpha, \beta)}[f]_g}}, \\ \text{and } \bar{\sigma}_{(\alpha, \beta)}[f]_g &= \liminf_{r \rightarrow \infty} \frac{\exp(\alpha(T_g^{-1}(T_f(r))))}{(\exp(\beta(r)))^{\rho_{(\alpha, \beta)}[f]_g}}.\end{aligned}$$

Analogously, to determine the relative growth of a meromorphic function f having same non-zero finite generalized relative lower order (α, β) with respect to an entire function g , Biswas et al. [3] have introduced the definitions of generalized relative upper weak type (α, β) denoted by $\tau_{(\alpha, \beta)}[f]_g$ and generalized relative weak type (α, β) denoted by $\bar{\tau}_{(\alpha, \beta)}[f]_g$ of f with respect to g of finite positive generalized relative lower order (α, β) in the following way:

Definition 1.6 ([3]). Let $\alpha, \beta \in L_1$. The generalized relative upper weak type (α, β) denoted by $\bar{\tau}_{(\alpha, \beta)}[f]_g$ and generalized relative weak type (α, β) denoted by $\tau_{(\alpha, \beta)}[f]_g$ of an meromorphic function f with respect to entire function g having finite positive generalized relative lower order (α, β) ($0 < \lambda_{(\alpha, \beta)}[f]_g < \infty$) are defined as :

$$\begin{aligned}\bar{\tau}_{(\alpha, \beta)}[f]_g &= \limsup_{r \rightarrow \infty} \frac{\exp(\alpha(T_g^{-1}(T_f(r))))}{(\exp(\beta(r)))^{\lambda_{(\alpha, \beta)}[f]_g}}, \\ \text{and } \tau_{(\alpha, \beta)}[f]_g &= \liminf_{r \rightarrow \infty} \frac{\exp(\alpha(T_g^{-1}(T_f(r))))}{(\exp(\beta(r)))^{\lambda_{(\alpha, \beta)}[f]_g}}.\end{aligned}$$

It is obvious that $0 \leq \tau_{(\alpha, \beta)}[f]_g \leq \bar{\tau}_{(\alpha, \beta)}[f]_g \leq \infty$.

In this paper we intend to establish some results relating to the growth properties of composite entire and meromorphic functions on the basis of generalized relative order (α, β) , generalized relative type (α, β) and generalized relative weak type (α, β) .

2. Main results

In this section we present the main results of the paper. Below we suppose that functions $\alpha_1, \alpha_2, \beta_1, \beta_2$ belong to the class L_1 .

Theorem 2.1. *Let f be a meromorphic function and g, h, k be an entire functions such that $0 < \lambda_{(\alpha_1, \beta_1)}[f \circ g]_h \leq \rho_{(\alpha_1, \beta_1)}[f \circ g]_h < \infty$ and $0 < \lambda_{(\alpha_2, \beta_2)}[f]_k \leq \rho_{(\alpha_2, \beta_2)}[f]_k < \infty$. Then*

$$\begin{aligned} \frac{\lambda_{(\alpha_1, \beta_1)}[f \circ g]_h}{\rho_{(\alpha_2, \beta_2)}[f]_k} &\leq \liminf_{r \rightarrow +\infty} \frac{\alpha_1(T_h^{-1}(T_{f \circ g}(r)))}{\alpha_2(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r)))))} \\ &\leq \min \left\{ \frac{\lambda_{(\alpha_1, \beta_1)}[f \circ g]_h}{\lambda_{(\alpha_2, \beta_2)}[f]_k}, \frac{\rho_{(\alpha_1, \beta_1)}[f \circ g]_h}{\rho_{(\alpha_2, \beta_2)}[f]_k} \right\} \\ &\leq \max \left\{ \frac{\lambda_{(\alpha_1, \beta_1)}[f \circ g]_h}{\lambda_{(\alpha_2, \beta_2)}[f]_k}, \frac{\rho_{(\alpha_1, \beta_1)}[f \circ g]_h}{\rho_{(\alpha_2, \beta_2)}[f]_k} \right\} \\ &\leq \limsup_{r \rightarrow +\infty} \frac{\alpha_1(T_h^{-1}(T_{f \circ g}(r)))}{\alpha_2(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r)))))} \leq \frac{\rho_{(\alpha_1, \beta_1)}[f \circ g]_h}{\lambda_{(\alpha_2, \beta_2)}[f]_k}. \end{aligned}$$

Proof. From the definition of $\lambda_{(\alpha_2, \beta_2)}[f]_k$, $\rho_{(\alpha_2, \beta_2)}[f]_k$, $\lambda_{(\alpha_1, \beta_1)}[f \circ g]_h$ and $\rho_{(\alpha_1, \beta_1)}[f \circ g]_h$ we have for arbitrary positive ε and for all sufficiently large positive numbers of r that

$$\begin{aligned} (1) \quad &\alpha_2(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r)))))) \geq (\lambda_{(\alpha_2, \beta_2)}[f]_k - \varepsilon)\beta_1(r), \\ (2) \quad &\alpha_2(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r)))))) \leq (\rho_{(\alpha_2, \beta_2)}[f]_k + \varepsilon)\beta_1(r), \\ (3) \quad &\alpha_1(T_h^{-1}(T_{f \circ g}(r))) \geq (\lambda_{(\alpha_1, \beta_1)}[f \circ g]_h - \varepsilon)\beta_1(r), \\ (4) \quad &\alpha_1(T_h^{-1}(T_{f \circ g}(r))) \leq (\rho_{(\alpha_1, \beta_1)}[f \circ g]_h + \varepsilon)\beta_1(r). \end{aligned}$$

Again, we get for a sequence of positive numbers of r tending to infinity that

$$\begin{aligned} (5) \quad &\alpha_2(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r)))))) \leq (\lambda_{(\alpha_2, \beta_2)}[f]_k + \varepsilon)\beta_1(r), \\ (6) \quad &\alpha_2(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r)))))) \geq (\rho_{(\alpha_2, \beta_2)}[f]_k - \varepsilon)\beta_1(r), \\ (7) \quad &\alpha_1(T_h^{-1}(T_{f \circ g}(r))) \leq (\lambda_{(\alpha_1, \beta_1)}[f \circ g]_h + \varepsilon)\beta_1(r), \\ (8) \quad &\alpha_1(T_h^{-1}(T_{f \circ g}(r))) \geq (\rho_{(\alpha_1, \beta_1)}[f \circ g]_h - \varepsilon)\beta_1(r). \end{aligned}$$

Now, from (2) and (3), it follows for all sufficiently large positive numbers of r that

$$\frac{\alpha_1(T_h^{-1}(T_{f \circ g}(r)))}{\alpha_2(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r)))))} \geq \frac{(\lambda_{(\alpha_1, \beta_1)}[f \circ g]_h - \varepsilon)\beta_1(r)}{(\rho_{(\alpha_2, \beta_2)}[f]_k + \varepsilon)\beta_1(r)}.$$

As $\varepsilon (> 0)$ is arbitrary, we obtain that

$$(9) \quad \liminf_{r \rightarrow +\infty} \frac{\alpha_1(T_h^{-1}(T_{f \circ g}(r)))}{\alpha_2(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r)))))} \geq \frac{\lambda_{(\alpha_1, \beta_1)}[f \circ g]_h}{\rho_{(\alpha_2, \beta_2)}[f]_k}.$$

Combining (1) and (7), we get for a sequence of positive numbers of r tending to infinity that

$$\frac{\alpha_1(T_h^{-1}(T_{f \circ g}(r)))}{\alpha_2(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r)))))} \leq \frac{(\lambda_{(\alpha_1, \beta_1)}[f \circ g]_h + \varepsilon)\beta_1(r)}{(\lambda_{(\alpha_2, \beta_2)}[f]_k - \varepsilon)\beta_1(r)}.$$

Since $\varepsilon (> 0)$ is arbitrary, it follows that

$$(10) \quad \liminf_{r \rightarrow +\infty} \frac{\alpha_1(T_h^{-1}(T_{f \circ g}(r)))}{\alpha_2(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r)))))} \leq \frac{\lambda_{(\alpha_1, \beta_1)}[f \circ g]_h}{\lambda_{(\alpha_2, \beta_2)}[f]_k}.$$

Now, from (3) and (5), we obtain for a sequence of positive numbers of r tending to infinity that

$$\frac{\alpha_1(T_h^{-1}(T_{f \circ g}(r)))}{\alpha_2(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r)))))} \geq \frac{(\lambda_{(\alpha_1, \beta_1)}[f \circ g]_h - \varepsilon)\beta_1(r)}{(\lambda_{(\alpha_2, \beta_2)}[f]_k + \varepsilon)\beta_1(r)}.$$

As $\varepsilon (> 0)$ is arbitrary, we get from above that

$$(11) \quad \limsup_{r \rightarrow +\infty} \frac{\alpha_1(T_h^{-1}(T_{f \circ g}(r)))}{\alpha_2(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r)))))} \geq \frac{\lambda_{(\alpha_1, \beta_1)}[f \circ g]_h}{\lambda_{(\alpha_2, \beta_2)}[f]_k}.$$

Now, it follows from (1) and (4) for all sufficiently large positive numbers of r that

$$\frac{\alpha_1(T_h^{-1}(T_{f \circ g}(r)))}{\alpha_2(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r)))))} \leq \frac{(\rho_{(\alpha_1, \beta_1)}[f \circ g]_h + \varepsilon)\beta_1(r)}{(\lambda_{(\alpha_2, \beta_2)}[f]_k - \varepsilon)\beta_1(r)}.$$

Since $\varepsilon (> 0)$ is arbitrary, we obtain that

$$(12) \quad \limsup_{r \rightarrow +\infty} \frac{\alpha_1(T_h^{-1}(T_{f \circ g}(r)))}{\alpha_2(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r)))))} \leq \frac{\rho_{(\alpha_1, \beta_1)}[f \circ g]_h}{\lambda_{(\alpha_2, \beta_2)}[f]_k}.$$

Now, from (4) and (6), it follows for a sequence of positive numbers of r tending to infinity that

$$\frac{\alpha_1(T_h^{-1}(T_{f \circ g}(r)))}{\alpha_2(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r)))))} \leq \frac{(\rho_{(\alpha_1, \beta_1)}[f \circ g]_h + \varepsilon)\beta_1(r)}{(\rho_{(\alpha_2, \beta_2)}[f]_k - \varepsilon)\beta_1(r)}.$$

As $\varepsilon (> 0)$ is arbitrary, we obtain that

$$(13) \quad \liminf_{r \rightarrow +\infty} \frac{\alpha_1(T_h^{-1}(T_{f \circ g}(r)))}{\alpha_2(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r)))))} \leq \frac{\rho_{(\alpha_1, \beta_1)}[f \circ g]_h}{\rho_{(\alpha_2, \beta_2)}[f]_k}.$$

Again combining (2) and (8), we get for a sequence of positive numbers of r tending to infinity that

$$\frac{\alpha_1(T_h^{-1}(T_{f \circ g}(r)))}{\alpha_2(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r)))))} \geq \frac{(\rho_{(\alpha_1, \beta_1)}[f \circ g]_h - \varepsilon)\beta_1(r)}{(\rho_{(\alpha_2, \beta_2)}[f]_k + \varepsilon)\beta_1(r)}.$$

Since $\varepsilon (> 0)$ is arbitrary, it follows that

$$(14) \quad \limsup_{r \rightarrow +\infty} \frac{\alpha_1(T_h^{-1}(T_{f \circ g}(r)))}{\alpha_2(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r)))))} \geq \frac{\rho_{(\alpha_1, \beta_1)}[f \circ g]_h}{\rho_{(\alpha_2, \beta_2)}[f]_k}.$$

Thus the theorem follows from (9), (10), (11), (12), (13) and (14). $\square$

Remark 2.1. If we take “ $0 < \lambda_{(\alpha_2, \beta_2)}[g]_k \leq \rho_{(\alpha_2, \beta_2)}[g]_k < \infty$ ” instead of “ $0 < \lambda_{(\alpha_2, \beta_2)}[f]_k \leq \rho_{(\alpha_2, \beta_2)}[f]_k < \infty$ ” and other conditions remain same, the conclusion of Theorem 2.1 remains true with “ $\lambda_{(\alpha_2, \beta_2)}[g]_k$ ”, “ $\rho_{(\alpha_2, \beta_2)}[g]_k$ ” and “ $\alpha_2(T_k^{-1}(T_g(\beta_2^{-1}(\beta_1(r))))$ ” in place of “ $\lambda_{(\alpha_2, \beta_2)}[f]_k$ ”, “ $\rho_{(\alpha_2, \beta_2)}[f]_k$ ” and “ $\alpha_2(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r))))$ ” respectively in the denominator.

We may now state the following theorem without proof based on Definition 1.3.

Theorem 2.2. Let f be a meromorphic function and g, h, k be an entire functions such that $0 < \bar{\lambda}_{(\alpha_1, \beta_1)}[f \circ g]_h \leq \bar{\rho}_{(\alpha_1, \beta_1)}[f \circ g]_h < \infty$ and $0 < \bar{\lambda}_{(\alpha_2, \beta_2)}[f]_k \leq \bar{\rho}_{(\alpha_2, \beta_2)}[f]_k < \infty$. Then

$$\begin{aligned} \frac{\bar{\lambda}_{(\alpha_1, \beta_1)}[f \circ g]_h}{\bar{\rho}_{(\alpha_2, \beta_2)}[f]_k} &\leq \liminf_{r \rightarrow +\infty} \frac{\alpha_1(\log(T_h^{-1}(T_{f \circ g}(r))))}{\alpha_2(\log(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r))))))} \\ &\leq \min \left\{ \frac{\bar{\lambda}_{(\alpha_1, \beta_1)}[f \circ g]_h}{\bar{\lambda}_{(\alpha_2, \beta_2)}[f]_k}, \frac{\bar{\rho}_{(\alpha_1, \beta_1)}[f \circ g]_h}{\bar{\rho}_{(\alpha_2, \beta_2)}[f]_k} \right\} \\ &\leq \max \left\{ \frac{\bar{\lambda}_{(\alpha_1, \beta_1)}[f \circ g]_h}{\bar{\lambda}_{(\alpha_2, \beta_2)}[f]_k}, \frac{\bar{\rho}_{(\alpha_1, \beta_1)}[f \circ g]_h}{\bar{\rho}_{(\alpha_2, \beta_2)}[f]_k} \right\} \\ &\leq \limsup_{r \rightarrow +\infty} \frac{\alpha_1(\log(T_h^{-1}(T_{f \circ g}(r))))}{\alpha_2(\log(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r))))))} \leq \frac{\bar{\rho}_{(\alpha_1, \beta_1)}[f \circ g]_h}{\bar{\lambda}_{(\alpha_2, \beta_2)}[f]_k}. \end{aligned}$$

Remark 2.2. If we take “ $0 < \bar{\lambda}_{(\alpha_2, \beta_2)}[g]_k \leq \bar{\rho}_{(\alpha_2, \beta_2)}[g]_k < \infty$ ” instead of “ $0 < \bar{\lambda}_{(\alpha_2, \beta_2)}[f]_k \leq \bar{\rho}_{(\alpha_2, \beta_2)}[f]_k < \infty$ ” and other conditions remain same, the conclusion of Theorem 2.2 remains true with “ $\bar{\lambda}_{(\alpha_2, \beta_2)}[g]_k$ ”, “ $\bar{\rho}_{(\alpha_2, \beta_2)}[g]_k$ ” and “ $\alpha_2(\log(T_k^{-1}(T_g(\beta_2^{-1}(\beta_1(r))))$ ” in place of “ $\bar{\lambda}_{(\alpha_2, \beta_2)}[f]_k$ ”, “ $\bar{\rho}_{(\alpha_2, \beta_2)}[f]_k$ ” and “ $\alpha_2(\log(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r))))$ ” respectively in the denominator.

Now, we may state the following theorem without proof based on Definition 1.4.

Theorem 2.3. Let f be a meromorphic function and g, h, k be an entire functions such that $0 < \underline{\lambda}_{(\alpha_1, \beta_1)}[f \circ g]_h \leq \underline{\rho}_{(\alpha_1, \beta_1)}[f \circ g]_h < \infty$ and $0 < \underline{\lambda}_{(\alpha_2, \beta_2)}[f]_k \leq \underline{\rho}_{(\alpha_2, \beta_2)}[f]_k < \infty$. Then

$$\begin{aligned} \frac{\underline{\lambda}_{(\alpha_1, \beta_1)}[f \circ g]_h}{\underline{\rho}_{(\alpha_2, \beta_2)}[f]_k} &\leq \liminf_{r \rightarrow +\infty} \frac{\alpha_1(T_h^{-1}(T_{f \circ g}(r)))}{\alpha_2(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r))))))} \\ &\leq \min \left\{ \frac{\underline{\lambda}_{(\alpha_1, \beta_1)}[f \circ g]_h}{\underline{\lambda}_{(\alpha_2, \beta_2)}[f]_k}, \frac{\underline{\rho}_{(\alpha_1, \beta_1)}[f \circ g]_h}{\underline{\rho}_{(\alpha_2, \beta_2)}[f]_k} \right\} \\ &\leq \max \left\{ \frac{\underline{\lambda}_{(\alpha_1, \beta_1)}[f \circ g]_h}{\underline{\lambda}_{(\alpha_2, \beta_2)}[f]_k}, \frac{\underline{\rho}_{(\alpha_1, \beta_1)}[f \circ g]_h}{\underline{\rho}_{(\alpha_2, \beta_2)}[f]_k} \right\} \\ &\leq \limsup_{r \rightarrow +\infty} \frac{\alpha_1(T_h^{-1}(T_{f \circ g}(r)))}{\alpha_2(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r))))))} \leq \frac{\underline{\rho}_{(\alpha_1, \beta_1)}[f \circ g]_h}{\underline{\lambda}_{(\alpha_2, \beta_2)}[f]_k}. \end{aligned}$$

Remark 2.3. If we take “ $0 < \underline{\lambda}_{(\alpha_2, \beta_2)}[g]_k \leq \underline{\rho}_{(\alpha_2, \beta_2)}[g]_k < \infty$ ” instead of “ $0 < \underline{\lambda}_{(\alpha_2, \beta_2)}[f]_k \leq \underline{\rho}_{(\alpha_2, \beta_2)}[f]_k < \infty$ ” and other conditions remain same, the conclusion of Theorem 2.3 remains true with “ $\underline{\lambda}_{(\alpha_2, \beta_2)}[g]_k$ ”, “ $\underline{\rho}_{(\alpha_2, \beta_2)}[g]_k$ ” and “ $\alpha_2(T_k^{-1}(T_g(\beta_2^{-1}(\beta_1(r)))))$ ” in place of “ $\underline{\lambda}_{(\alpha_2, \beta_2)}[f]_k$ ”, “ $\underline{\rho}_{(\alpha_2, \beta_2)}[f]_k$ ” and “ $\alpha_2(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r)))))$ ” respectively in the denominator.

The proof of the following theorem can be carried out as of the Theorem 2.1, therefore we omit the details.

Theorem 2.4. Let f be a meromorphic function and g, h, k be an entire functions such that $0 < \overline{\sigma}_{(\alpha_1, \beta_1)}[f \circ g]_h \leq \sigma_{(\alpha_1, \beta_1)}[f \circ g]_h < \infty$, $0 < \overline{\sigma}_{(\alpha_2, \beta_2)}[f]_k \leq \sigma_{(\alpha_2, \beta_2)}[f]_k < \infty$ and $\rho_{(\alpha_1, \beta_1)}[f \circ g]_h = \rho_{(\alpha_2, \beta_2)}[f]_k$. Then

$$\begin{aligned} \frac{\overline{\sigma}_{(\alpha_1, \beta_1)}[f \circ g]_h}{\sigma_{(\alpha_2, \beta_2)}[f]_k} &\leq \liminf_{r \rightarrow +\infty} \frac{\exp(\alpha_1(T_h^{-1}(T_{f \circ g}(r))))}{\exp(\alpha_2(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r))))))} \\ &\leq \min \left\{ \frac{\overline{\sigma}_{(\alpha_1, \beta_1)}[f \circ g]_h}{\overline{\sigma}_{(\alpha_2, \beta_2)}[f]_k}, \frac{\sigma_{(\alpha_1, \beta_1)}[f \circ g]_h}{\sigma_{(\alpha_2, \beta_2)}[f]_k} \right\} \\ &\leq \max \left\{ \frac{\overline{\sigma}_{(\alpha_1, \beta_1)}[f \circ g]_h}{\overline{\sigma}_{(\alpha_2, \beta_2)}[f]_k}, \frac{\sigma_{(\alpha_1, \beta_1)}[f \circ g]_h}{\sigma_{(\alpha_2, \beta_2)}[f]_k} \right\} \\ &\leq \limsup_{r \rightarrow +\infty} \frac{\exp(\alpha_1(T_h^{-1}(T_{f \circ g}(r))))}{\exp(\alpha_2(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r))))))} \leq \frac{\sigma_{(\alpha_1, \beta_1)}[f \circ g]_h}{\overline{\sigma}_{(\alpha_2, \beta_2)}[f]_k}. \end{aligned}$$

Remark 2.4. If we take “ $0 < \tau_{(\alpha_2, \beta_2)}[f]_k \leq \overline{\tau}_{(\alpha_2, \beta_2)}[f]_k < \infty$ ” and “ $\rho_{(\alpha_1, \beta_1)}[f \circ g]_h = \lambda_{(\alpha_2, \beta_2)}[f]_k$ ” instead of “ $0 < \overline{\sigma}_{(\alpha_2, \beta_2)}[f]_k \leq \sigma_{(\alpha_2, \beta_2)}[f]_k < \infty$ ” and “ $\rho_{(\alpha_1, \beta_1)}[f \circ g]_h = \rho_{(\alpha_2, \beta_2)}[f]_k$ ” respectively and other conditions remain same, the conclusion of Theorem 2.4 remains true with “ $\tau_{(\alpha_2, \beta_2)}[f]_k$ ” and “ $\overline{\tau}_{(\alpha_2, \beta_2)}[f]_k$ ” in place of “ $\overline{\sigma}_{(\alpha_2, \beta_2)}[f]_k$ ” and “ $\sigma_{(\alpha_2, \beta_2)}[f]_k$ ” respectively in the denominator.

Remark 2.5. If we take “ $0 < \bar{\sigma}_{(\alpha_2, \beta_2)}[g]_k \leq \sigma_{(\alpha_2, \beta_2)}[g]_k < \infty$ ” and “ $\rho_{(\alpha_1, \beta_1)}[f \circ g]_h = \rho_{(\alpha_2, \beta_2)}[g]_k$ ” instead of “ $0 < \bar{\sigma}_{(\alpha_2, \beta_2)}[f]_k \leq \sigma_{(\alpha_2, \beta_2)}[f]_k < \infty$ ” and “ $\rho_{(\alpha_1, \beta_1)}[f \circ g]_h = \rho_{(\alpha_2, \beta_2)}[f]_k$ ” respectively and other conditions remain same, the conclusion of Theorem 2.4 remains true with “ $\bar{\sigma}_{(\alpha_2, \beta_2)}[g]_k$ ”, “ $\sigma_{(\alpha_2, \beta_2)}[g]_k$ ” and “ $\exp(\alpha_2(T_k^{-1}(T_g(\beta_2^{-1}(\beta_1(r))))))$ ” in place of “ $\bar{\sigma}_{(\alpha_2, \beta_2)}[f]_k$ ”, “ $\sigma_{(\alpha_2, \beta_2)}[f]_k$ ” and “ $\exp(\alpha_2(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r))))))$ ”.

Remark 2.6. If we take “ $0 < \tau_{(\alpha_2, \beta_2)}[g]_k \leq \bar{\tau}_{(\alpha_2, \beta_2)}[g]_k < \infty$ ” and “ $\rho_{(\alpha_1, \beta_1)}[f \circ g]_h = \lambda_{(\alpha_2, \beta_2)}[g]_k$ ” instead of “ $0 < \bar{\sigma}_{(\alpha_2, \beta_2)}[f]_k \leq \sigma_{(\alpha_2, \beta_2)}[f]_k < \infty$ ” and “ $\rho_{(\alpha_1, \beta_1)}[f \circ g]_h = \rho_{(\alpha_2, \beta_2)}[f]_k$ ” and other conditions remain same, the conclusion of Theorem 2.4 remains true with “ $\tau_{(\alpha_2, \beta_2)}[g]_k$ ”, “ $\bar{\tau}_{(\alpha_2, \beta_2)}[g]_k$ ” and “ $\exp(\alpha_2(T_k^{-1}(T_g(\beta_2^{-1}(\beta_1(r))))))$ ” in place of “ $\bar{\sigma}_{(\alpha_2, \beta_2)}[f]_k$ ”, “ $\sigma_{(\alpha_2, \beta_2)}[f]_k$ ” and “ $\exp(\alpha_2(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r))))))$ ” respectively in the denominator.

Theorem 2.5. Let f be a meromorphic function and g, h, k be an entire functions such that $0 < \tau_{(\alpha_1, \beta_1)}[f \circ g]_h \leq \bar{\tau}_{(\alpha_1, \beta_1)}[f \circ g]_h < \infty$, $0 < \tau_{(\alpha_2, \beta_2)}[f]_k \leq \bar{\tau}_{(\alpha_2, \beta_2)}[f]_k < \infty$ and $\lambda_{(\alpha_1, \beta_1)}[f \circ g]_h = \lambda_{(\alpha_2, \beta_2)}[f]_k$. Then

$$\begin{aligned} \frac{\tau_{(\alpha_1, \beta_1)}[f \circ g]_h}{\bar{\tau}_{(\alpha_2, \beta_2)}[f]_k} &\leq \liminf_{r \rightarrow +\infty} \frac{\exp(\alpha_1(T_h^{-1}(T_{f \circ g}(r))))}{\exp(\alpha_2(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r))))))} \\ &\leq \min \left\{ \frac{\tau_{(\alpha_1, \beta_1)}[f \circ g]_h}{\tau_{(\alpha_2, \beta_2)}[f]_k}, \frac{\bar{\tau}_{(\alpha_1, \beta_1)}[f \circ g]_h}{\bar{\tau}_{(\alpha_2, \beta_2)}[f]_k} \right\} \\ &\leq \max \left\{ \frac{\tau_{(\alpha_1, \beta_1)}[f \circ g]_h}{\tau_{(\alpha_2, \beta_2)}[f]_k}, \frac{\bar{\tau}_{(\alpha_1, \beta_1)}[f \circ g]_h}{\bar{\tau}_{(\alpha_2, \beta_2)}[f]_k} \right\} \\ &\leq \limsup_{r \rightarrow +\infty} \frac{\exp(\alpha_1(T_h^{-1}(T_{f \circ g}(r))))}{\exp(\alpha_2(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r))))))} \leq \frac{\bar{\tau}_{(\alpha_1, \beta_1)}[f \circ g]_h}{\tau_{(\alpha_2, \beta_2)}[f]_k}. \end{aligned}$$

Remark 2.7. If we take “ $0 < \bar{\sigma}_{(\alpha_2, \beta_2)}[f]_k \leq \sigma_{(\alpha_2, \beta_2)}[f]_k < \infty$ ” and “ $\lambda_{(\alpha_1, \beta_1)}[f \circ g]_h = \rho_{(\alpha_2, \beta_2)}[f]_k$ ” instead of “ $0 < \tau_{(\alpha_2, \beta_2)}[f]_k \leq \bar{\tau}_{(\alpha_2, \beta_2)}[f]_k < \infty$ ” and “ $\lambda_{(\alpha_1, \beta_1)}[f \circ g]_h = \lambda_{(\alpha_2, \beta_2)}[f]_k$ ” respectively and other conditions remain same, the conclusion of Theorem 2.5 remains true with “ $\bar{\sigma}_{(\alpha_2, \beta_2)}[f]_k$ ” and “ $\sigma_{(\alpha_2, \beta_2)}[f]_k$ ” in place of “ $\tau_{(\alpha_2, \beta_2)}[f]_k$ ” and “ $\bar{\tau}_{(\alpha_2, \beta_2)}[f]_k$ ” respectively in the denominator.

Remark 2.8. If we take “ $0 < \tau_{(\alpha_2, \beta_2)}[g]_k \leq \bar{\tau}_{(\alpha_2, \beta_2)}[g]_k < \infty$ ” and “ $\lambda_{(\alpha_1, \beta_1)}[f \circ g]_h = \lambda_{(\alpha_2, \beta_2)}[g]_k$ ” instead of “ $0 < \tau_{(\alpha_2, \beta_2)}[f]_k \leq \bar{\tau}_{(\alpha_2, \beta_2)}[f]_k < \infty$ ” and “ $\lambda_{(\alpha_1, \beta_1)}[f \circ g]_h = \lambda_{(\alpha_2, \beta_2)}[f]_k$ ” respectively and other conditions remain same, the conclusion of Theorem 2.5 remains true with “ $\tau_{(\alpha_2, \beta_2)}[g]_k$ ”, “ $\bar{\tau}_{(\alpha_2, \beta_2)}[g]_k$ ” and “ $\exp(\alpha_2(T_k^{-1}(T_g(\beta_2^{-1}(\beta_1(r))))))$ ” in place of “ $\tau_{(\alpha_2, \beta_2)}[f]_k$ ”, “ $\bar{\tau}_{(\alpha_2, \beta_2)}[f]_k$ ” and “ $\exp(\alpha_2(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r))))))$ ” respectively in the denominator.

Remark 2.9. If we take “ $0 < \bar{\sigma}_{(\alpha_2, \beta_2)}[g]_k \leq \sigma_{(\alpha_2, \beta_2)}[g]_k < \infty$ ” and “ $\lambda_{(\alpha_1, \beta_1)}[f \circ g]_h = \rho_{(\alpha_2, \beta_2)}[g]_k$ ” instead of “ $0 < \tau_{(\alpha_2, \beta_2)}[f]_k \leq \bar{\tau}_{(\alpha_2, \beta_2)}[f]_k < \infty$ ” and “ $\lambda_{(\alpha_1, \beta_1)}[f \circ g]_h = \lambda_{(\alpha_2, \beta_2)}[f]_k$ ” respectively and other conditions remain same, the conclusion of Theorem 2.5 remains true with “ $\bar{\sigma}_{(\alpha_2, \beta_2)}[g]_k$ ”, “ $\sigma_{(\alpha_2, \beta_2)}[g]_k$ ”

and “ $\exp(\alpha_2(T_k^{-1}(T_g(\beta_2^{-1}(\beta_1(r))))))$ ” in place of “ $\tau_{(\alpha_2, \beta_2)}[f]_k$ ”, “ $\bar{\tau}_{(\alpha_2, \beta_2)}[f]_k$ ” and “ $\exp(\alpha_2(T_k^{-1}(T_f(\beta_2^{-1}(\beta_1(r))))))$ ” respectively in the denominator.

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On some class of degenerate elliptic equations with $L^1(\Omega)$ coefficients in $P(X)$ -Sobolev spaces

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Abstract. This paper deals with a class of nonlinear degenerate elliptic problem with coefficients in $L^1(\Omega)$ of the form $-\operatorname{div} a(x, u, \nabla u) = f$, where $a(x, u, \nabla u)$ is allowed to be degenerate with the unknown u . We prove the existence of weak and entropy solutions under some hypothesis on f . We also study the same problem with a lower order term.

Keywords: $L^1(\Omega)$ coefficients, weak and entropy solutions, nonlinear degenerate elliptic equations, Sobolev spaces with variable exponent.

1. Introduction

Let Ω be a bounded subset of $\mathbb{R}^N$, $N \geq 2$.

$$\begin{cases} -\operatorname{div}(g(u)|\nabla u|^{p-2}\nabla u) = h(u)|\nabla u|^p + f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

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with $1 < p < \infty$, g is a continuous positive function. Also under the additional hypothesis

$$\lim_{s \rightarrow \infty} \frac{h(s)}{g(s)} = 0,$$

they obtain L^∞ -estimate. This kind of problems has been widely studied. we refer the reader to [7, 11, 12, 13, 15, 16].

In this paper we study the existence of weak and entropy solutions for the following nonlinear degenerate elliptic problem:

$$(1) \quad \begin{cases} -\operatorname{div} a(x, u, \nabla u) = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

in the setting of the Sobolev space with variable exponent $W_0^{1,p(\cdot)}(\Omega)$, where Ω is a bounded open subset of $\mathbb{R}^N$ ($N \geq 2$), and

$$f \in L^m(\Omega), \quad \text{where } m = 1 \text{ or } m = (p^*)'.$$

Moreover we assume that $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is Carathéodory function such that for every s in $\mathbb{R}$, ξ, ξ' in $\mathbb{R}^N$ ($\xi \neq \xi'$), and for almost every $x \in \Omega$,

$$(2) \quad |a(x, s, \xi)| \leq \gamma \alpha(x)(1 + g(s))|\xi|^{p(x)-1},$$

$$(3) \quad a(x, s, \xi)\xi \geq \alpha(x)g(s)|\xi|^{p(x)},$$

$$(4) \quad [a(x, s, \xi) - a(x, s, \xi')][\xi - \xi'] > 0,$$

where γ is a positive real constant.

The function $\alpha(x)$ satisfies, for $\nu > 0$

$$(5) \quad \alpha \in L^1(\Omega), \quad \alpha(x) > \nu,$$

and

$$(6) \quad g : \mathbb{R} \mapsto \mathbb{R}^+, \text{ is a positive continuous function, } g(0) = 0 \text{ and}$$

$$(7) \quad g^{\frac{1}{p^\pm - 1}} \notin L^1([0, +\infty[) \cup L^1(]-\infty, 0]).$$

Then we study the perturbed problem

$$(8) \quad \begin{cases} -\operatorname{div} a(x, u, \nabla u) + H(x, u, \nabla u) = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $H(x, s, \xi)$ is a Carathéodory function satisfying

$$(9) \quad |H(x, s, \xi)| \leq \beta(x)h(s) |\xi|^{p(x)},$$

for almost every $x \in \Omega$, for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$, $\gamma > 0$, where the functions $\beta(\cdot)$ satisfies, for $\theta > 0$

$$(10) \quad \beta \in L^1(\Omega), \quad \beta(x) \geq 0 \quad \text{and} \quad \beta(x) \leq \theta \alpha(x) \quad \text{a.e. in } \Omega, \quad \text{and}$$

$h, g : \mathbb{R} \mapsto \mathbb{R}^+$, are positive continuous functions, $g > 0$,

$$(11) \quad \frac{h}{g} \in L^1(\mathbb{R}).$$

Furthermore, we will study the case when $f \in L^1(\Omega)$ and $f \in L^{(p^*(x))'}(\Omega)$.

To deal with this kind of problem, it is natural to work under the framework of Sobolev spaces with variable exponents. The study of differential equations with variable exponents has been a very active field in recent years, with applications in electro-rheological fluids and image processing, and so on. We refer the readers to [13] and references therein.

Some remarks concerning the difficulties in dealing with this problem are in order. First of all, the nonlinearity of the operator and the singularity of the coefficient will be solved by coupling energy estimates of type $\int_{\Omega} \alpha(x) |\nabla u|^{p(x)} < \infty$, on the one hand, we suppose no growth limitation on function g so that it can happen $a(x, u, \nabla u) \notin W^{-1, p'(\cdot)}(\Omega)$ for all $u \in W_0^{1, p(\cdot)}(\Omega)$ and, on the other hand, it is not coercive. To deal with that equation we will obtain the solution by approximation, getting a sequence of approximated solutions $(u_n)_n$ which converge to the solution u . To avoid some troubles in this convergence process, we will consider the convergence of another sequence $(A(u_n))_n$, function A being the primitive of $g^{\frac{1}{p^{\pm}-1}}$ such that $A(0) = 0$. The other remark is about the lower order term: it does not satisfy the sign condition, since h is positive. Thus, it appears the problem of getting the a priori estimates. This hindrance is overcome by considering test functions of exponential type.

With a modified version of the classical Minty Lemma (see [5]). We stress that in our treatment we avoid the use of weighted Sobolev spaces.

The content of this paper is organized as follows. We start by presenting preliminary results, some notations and definitions in Section 2. Section 3 is devoted to stating our main result.

2. Preliminaries

In this section, we define at first Lebesgue and Sobolev spaces with variable exponent and recall some of their properties. Let Ω be an open bounded set in $\mathbb{R}^N$, $N \geq 2$. The function $p(\cdot)$ satisfies the log-Hölder continuity on Ω if

$$(12) \quad |p(y) - p(x)| \leq \frac{C}{|\log|y - x||}, \quad \text{for all } x, y \in \overline{\Omega} \text{ such that } |y - x| \leq \frac{1}{2},$$

with C being a positive constant.

We denote

$$C^+(\overline{\Omega}) = \{\log\text{-H\"older continuous function } p : \overline{\Omega} \rightarrow \mathbb{R} \text{ with } 1 < p^- \leq p^+ < N\},$$

where

$$p^+ = \max_{x \in \overline{\Omega}} p(x) \quad \text{and} \quad p^- = \min_{x \in \overline{\Omega}} p(x).$$

The variable exponent Lebesgue space is defined as

$$L^{p(\cdot)}(\Omega) = \{u : u \text{ is a measurable real-valued function, } \int_{\Omega} |u(x)|^{p(x)} dx < \infty\}.$$

We can introduce the norm on $L^{p(\cdot)}(\Omega)$ by

$$\|u\|_{p(\cdot)} = \inf\{\lambda > 0 : \int_{\Omega} |\frac{u(x)}{\lambda}|^{p(x)} dx \leq 1\}.$$

The variable exponent Lebesgue spaces resemble classical Lebesgue spaces in many respects: they are Banach spaces, the Hölder inequality holds, they are reflexive if and only if $1 < p_- < p_+ < \infty$ and continuous functions are dense in $L^{p(\cdot)}(\Omega)$ if $p_+ < \infty$, (see Kováčik and Rákosník [16]).

We denote by $L^{p'(\cdot)}(\Omega)$ the dual space of $L^{p(\cdot)}(\Omega)$ where $1/p(\cdot) + 1/p'(\cdot) = 1$ (see [10],[12]). For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$, the generalized Hölder inequality

$$\left| \int_{\Omega} uv \right| \leq \left(\frac{1}{p_-} + \frac{1}{p'_-} \right) \|u\|_{p(x)} \|v\|_{p'(x)},$$

holds true.

Proposition 2.1 ([9],[19]). *if we denote*

$$\rho(u) = \int_{\Omega} |u|^{p(x)} dx, \forall u \in L^{p(\cdot)}(\Omega),$$

then the following assertions holds:

(i) $\|u\|_{p(x)} < 1$ (respectively, $= 1, > 1$) $\Leftrightarrow \rho(u) < 1$ (respectively, $= 1, > 1$);

(ii) $\begin{cases} \|u\|_{p(x)} > 1 \Rightarrow \|u\|_{p(x)}^{p_-} \leq \rho(u) \leq \|u\|_{p(x)}^{p_+}, \\ \|u\|_{p(x)} < 1 \Rightarrow \|u\|_{p(x)}^{p_+} \leq \rho(u) \leq \|u\|_{p(x)}^{p_-} \end{cases};$

(iii) $\|u\|_{p(x)} \rightarrow 0 \Leftrightarrow \rho(u) \rightarrow 0; \|u\|_{p(x)} \rightarrow \infty \Leftrightarrow \rho(u) \rightarrow \infty.$

We define the variable Sobolev space by

$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) \text{ and } |\nabla u| \in L^{p(\cdot)}(\Omega)\}.$$

normed by,

$$(13) \quad \|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)} \quad \forall u \in W^{1,p(\cdot)}(\Omega).$$

We denote by $W_0^{1,p(\cdot)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$ and

$$p^*(\cdot) = \begin{cases} \frac{Np(\cdot)}{N-p(\cdot)}, & \text{for } p(\cdot) < N, \\ \infty, & \text{for } p(\cdot) \geq N. \end{cases}$$

Proposition 2.2 ([9]). (i) Assuming $p_- > 1$, the spaces $W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$ are separable and reflexive Banach spaces.

(ii) if $q \in C_+(\bar{\Omega})$ and $q(x) < p^*(x)$ for any $x \in \bar{\Omega}$, then

$$(14) \quad W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$$

is compact and continuous (see, Theorem 8.4.2 [8]).

(iii) Let $p \in C_+(\bar{\Omega})$. Then, for $u \in W_0^{1,p(\cdot)}(\Omega)$, the $p(\cdot)$ -Poincaré inequality

$$\|u\|_{p(\cdot)} \leq C \|\nabla u\|_{p(\cdot)}$$

holds, where the positive constant C depends on $p(\cdot)$ and Ω .

Remark 2.1. By (iii) of Proposition 2.2, we know that $\|\nabla u\|_{p(\cdot)}$ and $\|u\|_{1,p(\cdot)}$ are equivalent norms on $W_0^{1,p(\cdot)}(\Omega)$.

Some technical Lemmas

Lemma 2.1 ([19]). Let $q \in C_+(\bar{\Omega})$, $g \in L^{q(\cdot)}(\Omega)$ and $(g_n)_n \in L^{q(\cdot)}(\Omega)$ with $\|g_n\|_{q(\cdot)} \leq C$, if $g_n(x) \rightarrow g(x)$ a.e in Ω , then $g_n(x) \rightarrow g(x)$ in $L^{q(x)}(\Omega)$, where C is a positive constant.

Lemma 2.2 ([19]). Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be uniformly lipshitz function, with $F(0) = 0$ and $p \in C_+(\bar{\Omega})$.

If $u \in W_0^{1,p(\cdot)}(\Omega)$, then $F(u) \in W_0^{1,p(\cdot)}(\Omega)$. Moreover, if the set D of discontinuity point of F' is finite, then

$$\frac{\partial(F \circ u)}{\partial x_i} = \begin{cases} F'(u) \frac{\partial u}{\partial x_i}, & \text{for a.e. in } \{x \in \Omega \setminus u(x) \notin D\}, \\ 0, & \text{for a.e. in } \{x \in \Omega \setminus u(x) \in D\}, \end{cases}$$

Lemma 2.3 ([9]). Let $u \in W_0^{1,p(\cdot)}(\Omega)$, then, we have $T_k(u) \in W_0^{1,p(\cdot)}(\Omega)$, with $k > 0$. Moreover, we have also $T_k(u) \rightarrow u \in W_0^{1,p(\cdot)}(\Omega)$ when $k \rightarrow \infty$.

Lemma 2.4 ([2]). Let $(u_n)_n$ be a sequence in $W_0^{1,p(\cdot)}(\Omega)$ with $u_n \rightharpoonup u$ in $W_0^{1,p(\cdot)}(\Omega)$. then, $T_k(u_n) \rightharpoonup T_k(u)$ in $W_0^{1,p(\cdot)}(\Omega)$.

Lemma 2.5 ([19]). Assume that (2)-(4) hold, with $g \geq \lambda$ in (3), $\lambda > 0$ and let $(u_n)_n$ be a sequence in $W_0^{1,p(\cdot)}(\Omega)$ such that $u_n \rightharpoonup u$ in $W_0^{1,p(\cdot)}(\Omega)$ and

$$\int_{\Omega} [a_n(x, u_n, \nabla u_n) - a_n(x, u_n, \nabla u)] \nabla(u_n - u) \rightarrow 0.$$

Then, $u_n \rightarrow u$ in $W_0^{1,p(\cdot)}(\Omega)$.

Notations and definitions

Definition 2.1. For all $k > 0$ and $s \in \mathbb{R}$, the truncation function $T_k(\cdot)$ can be defined by

$$T_k(s) = \begin{cases} s, & \text{if } |s| \leq k \\ k \cdot \text{sign}(s), & \text{if } |s| > k, \end{cases}$$

and we define $G_k(s) = s - T_k(s)$.

We define

$$(15) \quad \eta(s) = \int_0^s \frac{h(\tau)}{g(\tau)} d\tau,$$

we moreover define the space of functions

$$(16) \quad X_0^{p(\cdot)}(\Omega) = \{\varphi \in W_0^{1,p(\cdot)}(\Omega) \text{ such that } \int_{\Omega} \alpha(x) |\nabla \varphi|^{p(x)} < \infty\},$$

and

$$\mathcal{T}_0^{1,p(\cdot)}(\Omega) = \{u : \Omega \rightarrow \mathbb{R} \text{ measurable such that } T_k(u) \in W_0^{1,p(\cdot)}(\Omega)\},$$

i.e. the set of measurable functions whose truncates belong to $W_0^{1,p(\cdot)}(\Omega)$ (we refer to [4], for more details). We note

$$(17) \quad A(s) = \int_0^s g(\tau)^{1/(p^+-1)} d\tau \quad \text{if } g \text{ is unbounded,}$$

$$(18) \quad A(s) = \int_0^s g(\tau)^{1/(p^--1)} d\tau \quad \text{if } g \text{ is bounded.}$$

With C_i , $i \in \mathbb{N}$, we indicate generic positive constants that may depend on the dimension N , on the domain Ω , and on the other data of the problem, while with ε_n we indicate a generic sequence that goes to zero as n diverges.

3. Existence results

Theorem 3.1. Assume that (2)-(7) hold true, then for every $f \in L^1(\Omega)$ there exists $u \in \mathcal{T}_0^{1,p(\cdot)}(\Omega)$ entropy solution of (1) in the following sense $\forall k > 0$, $\int_{\Omega} \alpha(x) g(u)^{p'(x)} |\nabla T_k(u)|^{p(x)} < \infty$, and

$$(19) \quad \int_{\Omega} a(x, u, \nabla u) \nabla T_k(u - \phi) = \int_{\Omega} f T_k(u - \phi) \quad \forall \phi \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega).$$

If moreover $f \in L^{(p^*(\cdot))'}(\Omega)$, then $u \in W_0^{1,p(\cdot)}(\Omega)$

$$\begin{aligned} \int_{\Omega} \alpha(x) g(u)^{p'(x)} |\nabla u|^{p(x)} &< \infty, \\ \int_{\Omega} a(x, u, \nabla u) \nabla \phi &= \int_{\Omega} f \phi \quad \forall \phi \in W_0^{p(\cdot)}(\Omega). \end{aligned}$$

Theorem 3.2. Assume that (2)-(7) and (9)-(11) hold true, then for every $f \in L^1(\Omega)$ there exists $u \in \mathcal{T}_0^{1,p(\cdot)}(\Omega)$ entropy solution of (8) in the following sense

$$\begin{aligned} \forall k > 0, \quad \int_{\Omega} \alpha(x) g(u)^{p'(x)} |\nabla T_k(u)|^{p(x)} < \infty, \quad H(x, u, \nabla u) \in L^1(\Omega), \text{ and} \\ \int_{\Omega} a(x, u, \nabla u) \nabla T_k(u - \phi) + \int_{\Omega} H(x, u, \nabla u) T_k(u - \phi) = \int_{\Omega} f T_k(u - \phi), \\ \forall \phi \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega). \end{aligned}$$

If, moreover $f \in L^{(p^*(\cdot))'}(\Omega)$, then $u \in W_0^{1,p(\cdot)}(\Omega)$

$$\begin{aligned} \int_{\Omega} \alpha(x) g(u)^{p'(x)} |\nabla u|^{p(x)} < \infty, \quad H(x, u, \nabla u) \in L^1(\Omega) \text{ and} \\ \int_{\Omega} a(x, u, \nabla u) \nabla \phi + \int_{\Omega} H(x, u, \nabla u) \phi = \int_{\Omega} f \phi, \quad \forall \phi \in X_0^{p(\cdot)}(\Omega) \cap L^\infty(\Omega). \end{aligned}$$

We moreover state and prove a preliminary Lemma (see, [6] Lemma 3.3) that will be often used in the sequel.

Lemma 3.1. Let σ_n be a sequence of nonnegative bounded functions, almost everywhere convergent to some function σ , and let $\rho_n : \mathbb{R}^N \rightarrow \mathbb{R}^N$ be a sequence of functions which is weakly convergent in $(L^{q(x)}(\Omega))^N$ ($q(x) > 1$) to some function ρ . If the sequence $\sigma_n |\rho_n|^{q(x)}$ is bounded in $L^1(\Omega)$, then $\sigma |\rho|^{q(x)}$ belongs to $L^1(\Omega)$ and

$$(20) \quad \int_{\Omega} \sigma |\rho|^{q(x)} \leq \liminf_{n \rightarrow +\infty} \int_{\Omega} \sigma_n |\rho_n|^{q(x)}.$$

If moreover $\sigma_n \rightarrow \sigma$ strongly in $L^1(\Omega)$ then it holds true also that

$$(21) \quad \sigma_n^{1/q(x)} \rho_n \rightharpoonup \sigma^{1/q(x)} \rho \text{ weakly in } L^{q(x)}(\Omega)^N.$$

Proof. Since ρ_n weakly convergent to ρ in $(L^{q(x)}(\Omega))^N$, then there exists a functional $h \in (L^{q(x)}(\Omega))^N$ such that $|\rho_n| \leq h$.

Let $k > 0$ and $\psi \in (L^{q'(x)}(\Omega))^N$ we have $|T_k(\sigma_n)^{1/q(x)} \rho_n \cdot \psi| \leq k^{1/q(x)} h(x) \cdot \psi(x)$ and $T_k(\sigma_n)^{1/q(x)} \rho_n \cdot \psi \rightarrow T_k(\sigma)^{1/q(x)} \rho \cdot \psi$ a.e. in Ω , witch give

$$(22) \quad T_k(\sigma_n)^{1/q(x)} \rho_n \rightharpoonup T_k(\sigma)^{1/q(x)} \rho \text{ weakly in } (L^{q(x)}(\Omega))^N.$$

Hence by the lower semi-continuity of the $L^{q(x)}$ -norm it results

$$(23) \quad \int_{\Omega} T_k(\sigma) |\rho|^{q(x)} \leq \liminf_{n \rightarrow +\infty} \int_{\Omega} T_k(\sigma_n) |\rho_n|^{q(x)} \leq \liminf_{n \rightarrow +\infty} \int_{\Omega} \sigma_n |\rho_n|^{q(x)} \leq C.$$

Then letting k tend to infinity and using the monotone convergence theorem we obtain (20). Let now ψ be an arbitrary element of $(L^{q'(x)}(\Omega))^N$, then

$$\begin{aligned} \int_{\Omega} \sigma_n^{1/q(x)} \rho_n \psi &= \int_{\{\sigma_n \leq k\}} \sigma_n^{1/q(x)} \rho_n \psi + \int_{\{\sigma_n > k\}} \sigma_n^{1/q(x)} \rho_n \psi \\ &= \varepsilon_n + \int_{\{\sigma \leq k\}} \sigma_n^{1/q(x)} \rho_n \psi + \int_{\{\sigma_n > k\}} \sigma_n^{1/q(x)} \rho_n \psi. \end{aligned}$$

Moreover

$$\begin{aligned} \int_{\Omega} \sigma^{1/q(x)} \rho_n \psi &= \int_{\{\sigma_n \leq k\}} \sigma^{1/q(x)} \rho_n \psi + \int_{\{\sigma_n > k\}} \sigma^{1/q(x)} \rho_n \psi \\ &= \varepsilon_n + \int_{\{\sigma \leq k\}} \sigma^{1/q(x)} \rho \psi + \int_{\{\sigma_n > k\}} \sigma^{1/q(x)} \rho \psi. \end{aligned}$$

It follows that

$$\left| \int_{\Omega} [\sigma_n^{1/q(x)} \rho_n - \sigma^{1/q(x)} \rho] \cdot \psi \right| \leq \varepsilon_n + \int_{\{\sigma_n > k\}} \sigma_n^{1/q(x)} |\rho_n| |\psi| + \int_{\{\sigma_n > k\}} \sigma^{1/q(x)} |\rho| |\psi|.$$

Using Hölder inequality we have that

$$\begin{aligned} \int_{\{\sigma_n > k\}} \sigma_n^{1/q(x)} \rho_n \psi &\leq \|\sigma_n^{1/q(x)} \rho_n\|_{q(x)} \|\psi\|_{L^{q'(x)}(\sigma_n > k)}, \\ \int_{\{\sigma_n > k\}} \sigma^{1/q(x)} \rho \psi &\leq \|\sigma^{1/q(x)} \rho\|_{q(x)} \|\psi\|_{L^{q'(x)}(\sigma_n > k)}. \end{aligned} \quad \square$$

Proof of Theorem 3.1

We divide the rest of the proof into two steps; in the first step, we assume only $f \in L^1(\Omega)$, while in the second one we consider $L^{(p^*(x))'}(\Omega)$ data.

Step 1: Approximation and a priori estimates. Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of smooth function such $f_n \rightarrow f$ in $L^1(\Omega)$ and $|f_n| \leq |f|$, we consider the following problem

$$(24) \quad \int_{\Omega} a_n(x, u_n, \nabla u_n) \nabla \phi dx = \int_{\Omega} f_n \phi dx, \quad \forall \phi \in W_0^{1,p(\cdot)}(\Omega),$$

where

$$\begin{aligned} a_n(x, s, \xi) &= \frac{a(x, T_n(s), \xi)}{\alpha(x)} \alpha_n(x), \\ \alpha_n(x) &= T_n(\alpha(x)), \quad g_n(s) = g(T_n(s)). \end{aligned}$$

Note that g is continuous, so there exists $\lambda_n \geq 0$ such that $g_n(s) \geq \lambda_n$ and since $g_n^{\frac{1}{p^{\pm}-1}} \notin L^1([0, +\infty[) \cup L^1(]-\infty, 0])$. Then by definition $a_n(x, s, \xi)$ satisfies

$$\begin{aligned} |a_n(x, s, \xi)| &\leq \gamma \alpha_n(x) (1 + g_n(s)) |\xi|^{p(x)-1}, \\ a_n(x, s, \xi) \xi &\geq \alpha_n(x) g_n(s) |\xi|^{p(x)} \geq \lambda_n \alpha_n(x) |\xi|^{p(x)}, \\ [a_n(x, s, \xi) - a_n(x, s, \xi')][\xi - \xi'] &> 0, \quad ; \quad , \text{ for } \xi \neq \xi', \end{aligned}$$

we can use the theorem 4.3 of [1] to infer the existence of $u_n \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$ solution to problem (24).

Case 1. The function g is unbounded, i.e. $\lim_{s \rightarrow +\infty} g(s) = +\infty$. Then, there exists $g_0 > 0$ such that $\alpha(s) > 1$, for all $s \geq g_0$ since $\lim_{s \rightarrow +\infty} g(s) = +\infty$. For every $k > A(g_0)$, with

$$A(s) = \int_0^s g(\tau)^{1/(p^+-1)} d\tau,$$

taking $T_k(A(u_n))$ as test function in (24), there exists $C_0 > 0$, and by assumption (3), we have

(25)

$$\begin{aligned} C_0 \nu \int_{|A(u)| \leq k} |\nabla T_k(A(u_n))|^{p(x)} &\leq C_0 \int_{|A(u)| \leq k} \alpha_n(x) |\nabla T_k(A(u_n))|^{p(x)} \\ &\leq C_0 \int_{|A(u)| \leq k} \alpha_n(x) g(u_n)^{\frac{p(x)}{p^+-1}} |\nabla u_n|^{p(x)} \\ &\leq \int_{|A(u)| \leq k} \alpha_n(x) g(u_n)^{\frac{p^+}{p^+-1}} |\nabla u_n|^{p(x)} \\ &\leq \int_{|A(u)| \leq k} a_n(x, u_n, \nabla u_n) \nabla u_n g^{\frac{1}{p^+-1}}(u_n) dx \\ &\leq \int_{\Omega} |f_n| |T_k(A(u_n))| dx \\ &\leq k \|f\|_{L^1}. \end{aligned}$$

Case 2. The function g is bounded, i.e. there exists a constant $M > 0$ such that $g(s) \leq M$, for every $s \in [0, +\infty[$. Taking $T_k(A(u_n))$ as an admissible test function in (24), with

$$A(s) = \int_0^s g(\tau)^{1/(p^--1)} d\tau.$$

There exists $C_1 > 0$, and by assumption (3), we have

(26)

$$\begin{aligned} C_1 \nu \int_{|A(u)| \leq k} |\nabla T_k(A(u_n))|^{p(x)} &\leq C_1 \int_{|A(u)| \leq k} \alpha_n(x) |\nabla T_k(A(u_n))|^{p(x)} \\ &\leq C_1 \int_{|A(u)| \leq k} \alpha_n(x) g(u_n)^{\frac{p(x)}{p^--1}} |\nabla u_n|^{p(x)} \\ &\leq \int_{|A(u)| \leq k} a_n(x, u_n, \nabla u_n) \nabla u_n g^{\frac{1}{p^--1}}(u_n) dx \\ &\leq \int_{\Omega} |f_n| |T_k(A(u_n))| dx \\ &\leq k \|f\|_{L^1}. \end{aligned}$$

And consequently there exist $C_2, C_3 > 0$ such that

$$(27) \quad \int_{\Omega} |\nabla T_k(A(u_n))|^{p(x)} dx \leq C_2 \int_{\Omega} \alpha_n(x) |\nabla T_k(A(u_n))|^{p(x)} \leq C_3 k.$$

Therefore,

$$(28) \quad \|\nabla T_k(A(u_n))\|_{p(\cdot)}^r dx \leq C_3 k,$$

with

$$r = \begin{cases} p^+, & \text{if } \|\nabla T_k(A(u_n))\|_{p(\cdot)} < 1, \\ p^-, & \text{if } \|\nabla T_k(A(u_n))\|_{p(\cdot)} \geq 1. \end{cases}$$

Let $k \geq 1$, we have

$$k \operatorname{meas} \{|A(u_n)| > k\} = \int_{|A(u_n)| > k} |T_k(A(u_n))| dx \leq C_3 k^{\frac{1}{r}},$$

which implies that

$$\operatorname{meas}\{|A(u_n)| > k\} \leq C_3 \frac{1}{k^{1-\frac{1}{r}}} \rightarrow 0 \text{ as } k \rightarrow \infty.$$

By the usual method we get that $T_k(A(u_n))$ is bounded in $W_0^{1,p(\cdot)}(\Omega)$, then there exists a subsequence still denoted $(T_k(A(u_n)))_{n \in \mathbb{N}}$ such that

$$T_k(A(u_n)) \rightharpoonup \eta_k \text{ weakly in } W_0^{1,p(\cdot)}(\Omega),$$

and by the Sobolev compact embedding, we have

$$T_k(A(u_n)) \rightarrow \eta_k \text{ strongly in } L^{p(\cdot)}(\Omega), \text{ and a.e. in } \Omega.$$

Consequently, we can assume that $(T_k(A(u_n)))_{n \in \mathbb{N}}$ is a Cauchy sequence in measure. Thus,

$$\operatorname{meas}\{|T_k(A(u_n)) - T_k(A(u_m))| > \delta\} \leq \frac{\varepsilon}{2}, \text{ for all } m, n \geq n_0(\delta, \varepsilon).$$

We conclude that, for all $\delta, \varepsilon > 0$ there exists $n_0 = n_0(\delta, \varepsilon)$ such that $\operatorname{meas}\{|u_n - u_m| > \delta\} \leq \varepsilon$ for all $\delta, \varepsilon > 0$. It follows that $(T_k(A(u_n)))_{n \in \mathbb{N}}$ is a Cauchy sequence in measure, then it converges almost everywhere, for a subsequence, to some measurable function u . Consequently, we have

$$(29) \quad \begin{aligned} T_k(A(u_n)) &\rightharpoonup T_k(A(u)) \text{ weakly in } W_0^{1,p(\cdot)}(\Omega), \\ T_k(A(u_n)) &\rightarrow T_k(A(u)) \text{ strongly in } L^{p(\cdot)}(\Omega), \text{ and a.e. in } \Omega. \end{aligned}$$

Moreover by (27), (29) and Lemma 3.1 we get

$$(30) \quad \begin{aligned} \alpha_n(x)^{\frac{1}{p(x)}} \nabla T_k(A(u_n)) &\rightharpoonup \alpha(x)^{\frac{1}{p(x)}} \nabla T_k(A(u)) \text{ weakly in } L^{p(\cdot)}(\Omega)^N, \text{ and} \\ \alpha(x) |\nabla T_k(A(u))|^{p(x)} &\in L^1(\Omega). \end{aligned}$$

Step 2: Strong convergence of truncations. Let $w_n = T_{2k}(Z_n)$ be a test function in (24), where $Z_n = (A(u_n) - T_h(A(u_n)) + T_k(A(u_n)) - T_k(A(u)))$, for $h > k > 0$. Taking $M = 4k + h$ we get that,

$$\int_{\Omega} a_n(x, u_n, \nabla u_n) \nabla w_n dx \leq \int_{\Omega} |f_n| |w_n| dx.$$

We have

$$(31) \quad \int_{\Omega} a_n(x, u_n, \nabla u_n) \nabla w_n dx = \int_{\{|A(u_n)| > k\}} a_n(x, u_n, \nabla u_n) \nabla w_n dx + \int_{\{|A(u_n)| \leq k\}} a_n(x, u_n, \nabla u_n) \nabla w_n dx.$$

Concerning the first term on the right hand side of (31), since $\nabla w_n = 0$ on $\{|A(u_n)| > M\}$, we have

$$\begin{aligned} \int_{\{|A(u_n)| > k\}} a_n(x, u_n, \nabla u_n) \nabla w_n dx &= \int_{\{|A(u_n)| > k\}} a(x, T_{\widehat{M}}(u_n), \nabla T_{\widehat{M}}(u_n)) \nabla z_n \\ &\geq - \int_{\{|A(u_n)| > k\}} a_n(x, T_{\widehat{M}}(u_n), \nabla T_{\widehat{M}}(u_n)) \nabla T_k(A(u)) dx \\ &\geq -\varepsilon_0(n). \end{aligned}$$

Where $\widehat{M}$ is a positive constant depending on M such that $\{|A(u_n)| > M\}$, can be equivalently replaced by $\{|u_n| > \widehat{M}\}$.

For the second term on the right hand side of (31); we have for $\widehat{k}$ defined as $\widehat{M}$

$$\begin{aligned} \int_{\{|A(u_n)| \leq k\}} a_n(x, T_{\widehat{k}}(u_n), \nabla T_{\widehat{k}}(u_n)) (\nabla T_k(A(u_n)) - \nabla T_k(A(u))) dx \\ \leq \int_{\Omega} |f_n| |w_n| dx + \varepsilon_0(n). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} &\int_{\{|A(u_n)| \leq k\}} a_n(x, T_{\widehat{k}}(u_n), \nabla T_{\widehat{k}}(u_n)) (\nabla T_k(A(u_n)) - \nabla T_k(A(u))) dx \\ &= \int_{\{|A(u_n)| \leq k\}} (a_n(x, T_{\widehat{k}}(u_n), \nabla T_{\widehat{k}}(u_n)) - a_n(x, T_{\widehat{k}}(u_n), \nabla T_{\widehat{k}}(u))) \\ &\quad \times (\nabla T_k(A(u_n)) - \nabla T_k(A(u))) dx \\ &+ \int_{\{|A(u_n)| \leq k\}} a_n(x, T_{\widehat{k}}(u_n), \nabla T_{\widehat{k}}(u)) (\nabla T_k(A(u_n)) - \nabla T_k(A(u))) dx. \end{aligned}$$

By (30) and the definition of $a_n(x, T_{\widehat{k}}(u_n), \nabla T_{\widehat{k}}(u))$ the third terms of the right hand tend to 0, as n tend to infinity. So, we have

$$\begin{aligned} &\int_{\{|A(u_n)| \leq k\}} (a_n(x, T_{\widehat{k}}(u_n), \nabla T_{\widehat{k}}(u_n)) - a_n(x, T_{\widehat{k}}(u_n), \nabla T_{\widehat{k}}(u))) \\ &\quad \times (\nabla T_k(A(u_n)) - \nabla T_k(A(u))) dx \\ (32) \quad &= \int_{\{|A(u_n)| \leq k\}} a_n(x, T_{\widehat{k}}(u_n), \nabla T_{\widehat{k}}(u_n)) (\nabla T_k(A(u_n)) - \nabla T_k(A(u))) dx + \varepsilon_1(n) \\ &\leq \int_{\Omega} |f_n(x)| |w_n| dx + \varepsilon_1(n) + \varepsilon_0(n) \leq \varepsilon_2(n), \end{aligned}$$

then

$$\begin{aligned} \int_{\Omega} [a_n(x, T_{\widehat{k}}(u_n), \nabla T_{\widehat{k}}(u_n)) - a_n(x, T_{\widehat{k}}(u_n), \nabla T_{\widehat{k}}(u_n))] \\ \times (\nabla T_k(A(u_n)) - \nabla T_k(A(u))) dx \leq \varepsilon_2(n), \end{aligned}$$

as $f_n \rightarrow f$ strongly in $L^1(\Omega)$, and $w_n \rightharpoonup 0$ weakly* in $L^\infty(\Omega)$. So, by the Lemma 2.5, we conclude that

$$\begin{aligned} T_k(A(u_n)) &\rightarrow T_k(A(u)) \text{ in } W_0^{1,p(\cdot)}(\Omega), \\ \nabla T_k(A(u_n)) &\rightarrow \nabla T_k(A(u)) \text{ almost everywhere in } \Omega. \end{aligned}$$

If g is unbounded, we have

$$\begin{aligned} |\nabla T_k(A(u_n))|^{p(x)} &= |\nabla A(u_n)|^{p(x)} \chi_{|A(u_n)| \leq k} \\ &= g^{\frac{p(x)}{p^+-1}}(u_n) \chi_{|A(u_n)| \leq k} |\nabla u_n|^{p(x)}, \\ g^{p(x)/(p^+-1)}(u_n) \chi_{|u_n| \leq \widehat{k}} |\nabla u_n|^{p(x)} &= g^{p(x)/(p^+-1)}(T_{\widehat{k}}(u_n)) |\nabla T_{\widehat{k}}(u_n)|^{p(x)}, \\ |\nabla T_{\widehat{k}}(u_n)|^{p(x)} &= \frac{|\nabla T_k(A(u_n))|^{p(x)}}{g^{\frac{p(x)}{p^+-1}}(T_{\widehat{k}}(u_n))}, \end{aligned}$$

and if g is bounded we have

$$\begin{aligned} |gT_k(A(u_n))|^{p(x)} &= |\nabla A(u_n)|^{p(x)} \chi_{|A(u_n)| \leq k} = g^{\frac{p(x)}{p^--1}}(u_n) \chi_{|A(u_n)| \leq k} |\nabla u_n|^{p(x)}, \\ g^{\frac{p(x)}{p^--1}}(u_n) \chi_{|u_n| \leq \widehat{k}} |\nabla u_n|^{p(x)} &= g^{\frac{p(x)}{p^--1}}(T_{\widehat{k}}(u_n)) |\nabla T_m(u_n)|^{p(x)}, \\ |\nabla T_{\widehat{k}}(u_n)|^{p(x)} &= \frac{|\nabla T_k(A(u_n))|^{p(x)}}{g^{\frac{p(x)}{p^--1}}(T_{\widehat{k}}(u_n))}, \end{aligned}$$

since g continuous we have $g(T_{\widehat{k}}(u_n)) \geq \min_{[0, \widehat{k}]}(g(s)) = g_{\widehat{k}}$. Finally, we have

$$(33) \quad |\nabla T_m(u_n)|^{p(x)} \leq C |\nabla T_k(A(u_n))|^{p(x)}.$$

and on the other, by Lemma 3.1, that for every $k > 0$

$$\alpha(x) |\nabla T_k(u)|^{p(x)} \in L^1(\Omega),$$

and

$$(34) \quad \alpha_n(x) \frac{1}{p(x)} \nabla T_k(u_n) \rightharpoonup \alpha(x) \frac{1}{p(x)} \nabla T_k(u) \text{ weakly in } (L^{p(x)}(\Omega))^N.$$

Step 3: Entropy solutions.

Case 1. The function α is unbounded, taking $T_k(A(u_n))$ as test function in (24), with $A(s) = \int_0^s g(\tau)^{1/(p^- - 1)} d\tau$, we obtaining

$$\begin{aligned}
 (35) \quad C_4 \int_{|A(u)| \leq k} \alpha_n(x) g(u_n)^{\frac{p(x)}{p(x)-1}} |\nabla u_n|^{p(x)} & \\
 & \leq \int_{|A(u)| \leq k} a_n(x, u_n, \nabla u_n) \nabla T_k(u_n) g^{\frac{1}{p^- - 1}}(u_n) dx \\
 & \leq \int_{\Omega} |f_n| |T_k(A(u_n))| dx \\
 & \leq k \|f\|_{L^1}.
 \end{aligned}$$

Case 2. The function g is bounded, taking $T_k(A(u_n))$ as test function in (24), with $A(s) = \int_0^s g(\tau)^{1/(p^+ - 1)} d\tau$, we have

$$\begin{aligned}
 (36) \quad C_5 \int_{|A(u)| \leq k} \alpha_n(x) g(u_n)^{\frac{p(x)}{p(x)-1}} |\nabla u_n|^{p(x)} & \\
 & \leq \int_{|A(u)| \leq k} a_n(x, u_n, \nabla u_n) \nabla T_k(u_n) g^{\frac{1}{p^+ - 1}}(u_n) dx \\
 & \leq \int_{\Omega} |f_n| |T_k(A(u_n))| dx \\
 & \leq k \|f\|_{L^1}.
 \end{aligned}$$

By means of (35) and (36), we obtain

$$(37) \quad \int_{\Omega} \alpha_n(x) g(u_n)^{p'(x)} |\nabla T_k(u_n)|^{p(x)} \leq C_6 k \|f\|_{L^1}.$$

Let $\phi \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$, taking $T_k(u_n - \phi)$ as a test function in the approximate problem (24), we get

$$\int_{\Omega} a_n(x, u_n, \nabla u_n) \nabla T_k(u_n - \phi) dx = \int_{\Omega} f_n T_k(u_n - \phi) dx.$$

Adding and subtracting in the equation above the term

$$\int_{\Omega} a_n(x, u_n, \nabla \phi) \nabla T_k(u_n - \phi) dx,$$

and taking advantage of the monotonicity condition (4) we get

$$(38) \quad \int_{\Omega} a_n(x, u_n, \nabla \phi) \nabla T_k(u_n - \phi) dx \leq \int_{\Omega} f_n T_k(u_n - \phi) dx.$$

Noticing that $\{|u_n - \phi| \leq k\} \subset \{|u_n| \leq k + \|\phi\|_{L^\infty}\}$ and that

$$\int_{\Omega} a_n(x, u_n, \nabla \phi) \nabla T_k(u_n - \phi) dx = \int_{\Omega} \frac{a(x, T_n(u_n), \nabla \phi)}{\alpha(x)} \alpha_n(x) \nabla T_k(u_n - \phi) dx,$$

we can pass to the limit in (38) using assumption (2), (33), (34) and (37). Hence we obtain, for every $k > 0$,

$$(39) \quad \int_{\Omega} a(x, u, \nabla \phi) \nabla T_k(u - \phi) dx \leq \int_{\Omega} f T_k(u - \phi) dx.$$

To recover (19), let h and k be positive real numbers, let $t = \pm 1$, and let ψ be a function in $W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$. Choose $\phi = T_h(u) + tT_k(u - \psi)$ in the inequality in (39) we obtain

$$(40) \quad \begin{aligned} \int_{\Omega} a(x, u, \nabla T_h(u) + t \nabla T_k(u - \psi)) \nabla T_k(G_h(u) - tT_k(u - \psi)) dx \\ \leq \int_{\Omega} f T_k(G_h(u) - tT_k(u - \psi)) dx, \end{aligned}$$

and we follow the same steps as the Lemma 7 in [5] we get

$$(41) \quad -t \int_{\Omega} a(x, u, \nabla u + t \nabla T_k(u - \psi)) \nabla T_k(u - \psi) dx \leq -t \int_{\Omega} f T_k(u - \psi) dx.$$

for every $\psi \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$, and for every $k > 0$. Choosing $t > 0$, dividing by t , and then letting t tend to zero, we obtain

$$\int_{\Omega} a(x, u, \nabla u) \nabla T_k(u - \psi) dx \geq \int_{\Omega} f T_k(u - \psi) dx.$$

While the reverse inequality is obtained choosing $t < 0$, dividing by $-t$, and then letting t tend to zero.

Step 4: weak solutions. In this last part of the proof we consider the stronger assumption $f \in L^{(p^*(\cdot))'}(\Omega)$. Recalling that $u_n \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$, we can choose $A(u_n)$ as test function in (24), where $A(s) = \int_0^s g(\tau)^{1/(p^+-1)} d\tau$. We obtain

$$(42) \quad \begin{aligned} \int_{\Omega} a_n(x, u_n, \nabla u_n) \nabla u_n g^{\frac{1}{p^+-1}}(u_n) dx &\leq \int_{\Omega} |f| |T_k(A(u_n))| dx \\ &\leq \|A(u_n)\|_{L^{p^*(x)}} \|f\|_{L^{(p^*(x))'}}. \end{aligned}$$

Using Sobolev inequality, we get

$$(43) \quad \int_{\Omega} \alpha_n(x) g(u_n)^{p'(x)} |\nabla u_n|^{p(x)} \leq \frac{(C_7 \|f\|_{L^{(p^*(x))'}})^{p^+}}{\nu^{\frac{1}{1+p^+}}}.$$

Thanks to the a.e. convergence of ∇u_n to ∇u and Fatou Lemma,

$$(44) \quad \int_{\Omega} \alpha(x) g(u)^{p'(x)} |\nabla u|^{p(x)} \leq \frac{(C_7 \|f\|_{L^{(p^*(x))'}})^{p^+}}{\nu^{\frac{1}{1+p^+}}}.$$

Thus, we have that the sequence

$$\frac{a(x, T_n(u_n), \nabla u_n)}{\alpha(x)} \alpha_n(x)^{\frac{1}{p'(x)}} \text{ is bounded in } (L^{p'(\cdot)}(\Omega))^N,$$

and converges almost everywhere to $a(x, u, \nabla u) \alpha(x)^{\frac{1}{p'(x)}-1}$. Hence, it also converges weakly in $(L^{p'(\cdot)}(\Omega))^N$ to its a.e.-limit. Noticing that, for every $\phi \in X_0^{p(\cdot)}(\Omega)$

$$\alpha_n(x)^{\frac{1}{p(x)}} \nabla \phi \rightarrow \alpha(x)^{\frac{1}{p(x)}} \nabla \phi \text{ strongly in } L^{p(\cdot)}(\Omega)^N,$$

we can pass to the limit in (24) and conclude the proof.

3.1 Proof of Theorem 3.2

Let us divide the proof in five steps:

Step 1: Approximation and a priori estimates.

Step 2: Strong convergence of truncations.

Step 3: The equi-integrability of $H_n(x, u_n, \nabla u_n)$.

Step 4: Entropy solutions.

Step 5: Weak solutions case $f \in L^{p^{*'}(\cdot)}(\Omega)$.

Step 1: Approximation and a priori estimates. Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of smooth function such $f_n \rightarrow f$ in $L^1(\Omega)$ and $|f_n| \leq |f|$, we consider the following problem

$$(45) \quad \begin{cases} -\operatorname{div} a_n(x, u_n, \nabla u_n) + H_n(x, u_n, \nabla u_n) = f_n, & \text{in } \Omega, \\ u_n \in W_0^{1,p(\cdot)}(\Omega), \end{cases}$$

where

$$H_n(x, s, \xi) = \min[T_n(\beta_n(x)h_n(s)|\xi|^{p(x)}), \max(-T_n(\beta_n(x)h_n(s)|\xi|^{p(x)}), H_n(x, s, \xi))],$$

$$h_n(s) = g_n(s) \frac{h(s)}{g(s)}, \quad \beta_n(x) = T_n(\beta(x)), \text{ and } \beta_n(x) \leq \theta \alpha_n(x).$$

Note that, the function H_n is bounded and

$$(46) \quad |H_n(x, s, \xi)| \leq T_n(\beta_n(x)h_n(s)|\xi|^{p(x)-1}) \leq \beta_n(x)h_n(s)|\xi|^{p(x)-1}.$$

We also, observe that $\frac{h_n}{g_n} = \frac{h}{g} \in L^1(\mathbb{R})$ and

$$h_n \leq \max\{g(s) : |s| \leq n\} \frac{h}{g}, \text{ so that } h_n \in L^1(\mathbb{R}).$$

Let $(T_k(A(u_n)))^+ e^{\theta\eta(u_n)}$ be a test function in (45), we have

$$\begin{aligned}
 (47) \quad & \int_{\Omega} a_n(x, u_n, \nabla u_n) \nabla T_k(A(u_n)) e^{\theta\eta(u_n)} dx \\
 & + \int_{\Omega} H_n(x, u_n, \nabla u_n) (T_k(A(u_n)))^+ e^{\theta\eta(u_n)} dx \\
 & + \int_{\Omega} a_n(x, u_n, \nabla u_n) \nabla u_n \frac{\theta h(u_n)}{g(u_n)} (T_k(A(u_n)))^+ e^{\theta\eta(u_n)} dx \\
 & = \int_{\Omega} f_n(T_k(A(u_n)))^+ e^{\theta\eta(u_n)} dx.
 \end{aligned}$$

By (3) and (9), we have

$$\begin{aligned}
 & \int_{\Omega} H_n(x, u_n, \nabla u_n) (T_k(A(u_n)))^+ e^{\theta\eta(u_n)} dx \\
 & + \int_{\Omega} a_n(x, u_n, \nabla u_n) \nabla u_n \frac{\theta h(u_n)}{g(u_n)} (T_k(A(u_n)))^+ e^{\theta\eta(u_n)} dx \geq 0.
 \end{aligned}$$

It follows that

$$\int_{A(u_n) > 0} a_n(x, u_n, \nabla u_n) \nabla (T_k(A(u_n))) e^{\theta\eta(u_n)} dx \leq \int_{\Omega} |f_n| (T_k(A(u_n)))^+ e^{\theta\eta(u_n)},$$

so we get, if g unbounded,

$$\begin{aligned}
 (48) \quad & \int_{0 < A(u_n) \leq k} a_n(x, u_n, \nabla u_n) \nabla u_n g^{\frac{1}{p^+-1}}(u_n) e^{\theta\eta(u_n)} dx \\
 & \leq \int_{\Omega} |f_n| |T_k(A(u_n))| e^{\theta\eta(u_n)} dx.
 \end{aligned}$$

By the assumption (3), there exist $C_8, C_9 > 0$ such that

$$\begin{aligned}
 (49) \quad & \nu \int_{0 < A(u_n) \leq k} |\nabla T_k(A(u_n))|^{p(x)} dx \leq C_8 \int_{0 < A(u_n) \leq k} \alpha_n(x) |\nabla T_k(A(u_n))|^{p(x)} \\
 & \leq \int_{0 < A(u_n) \leq k} \alpha_n(x) g(u_n)^{\frac{p(x)}{p^+-1}} |\nabla u_n|^{p(x)} \leq C_9 k \|f\|_{L^1},
 \end{aligned}$$

and if g bounded we obtain

$$\begin{aligned}
 (50) \quad & \int_{0 < A(u_n) \leq k} a_n(x, u_n, \nabla u_n) \nabla u_n g^{\frac{1}{p^--1}}(u_n) e^{\theta\eta(u_n)} dx \\
 & \leq \int_{\Omega} |f_n| |T_k(A(u_n))| e^{\theta\eta(u_n)} dx.
 \end{aligned}$$

Using again the assumption (3), there exist $C_{10}, C_{11} > 0$ such that

$$\begin{aligned}
 (51) \quad & \nu \int_{0 < A(u_n) \leq k} |\nabla T_k(A(u_n))|^{p(x)} dx \leq C_{10} \int_{0 < A(u_n) \leq k} \alpha_n(x) |\nabla T_k(A(u_n))|^{p(x)} \\
 & \leq \int_{0 < A(u_n) \leq k} \alpha_n(x) g(u_n)^{\frac{p(x)}{p^--1}} |\nabla u_n|^{p(x)} \leq C_{11} k \|f\|_{L^1},
 \end{aligned}$$

By using the test function $-(T_k(A(u_n)))^- e^{-\eta(u_n)}$, and reasoning as before we get

$$(52) \quad \begin{aligned} & \nu \int_{-k < A(u_n) \leq 0} |\nabla T_k(A(u_n))|^{p(x)} dx \\ & \leq C_{12} \int_{-k < A(u_n) \leq 0} \alpha_n(x) |\nabla T_k(A(u_n))|^{p(x)} dx \leq C_{13} k \|f\|_{L^1}. \end{aligned}$$

Combining (51) and (52), we get

$$(53) \quad \nu \int_{\Omega} |\nabla T_k(A(u_n))|^{p(x)} dx \leq C_{14} \int_{\Omega} \alpha_n(x) |\nabla T_k(A(u_n))|^{p(x)} dx \leq C_{15} k \|f\|_{L^1}.$$

Therefore,

$$(54) \quad \|\nabla T_k(A(u_n))\|_{p(\cdot)}^{r_1} dx \leq C_{15} k,$$

$$r_1 = \begin{cases} p^+, & \text{if } \|\nabla T_k(A(u_n))\|_{p(\cdot)} < 1, \\ p^-, & \text{if } \|\nabla T_k(A(u_n))\|_{p(\cdot)} \geq 1. \end{cases}$$

That implies $(T_k(A(u_n)))_{n \in \mathbb{N}}$ converges almost everywhere, for a subsequence, to some measurable function u . Consequently, we have

$$\begin{aligned} T_k(A(u_n)) &\rightharpoonup T_k(A(u)) \text{ weakly in } W_0^{1,p(\cdot)}(\Omega), \\ T_k(A(u_n)) &\rightarrow T_k(A(u)) \text{ strongly in } L^{p(\cdot)}(\Omega), \text{ and a.e. in } \Omega, \end{aligned}$$

and on the other, by Lemma 3.1, that for every $k > 0$

$$(55) \quad \begin{aligned} & \alpha_n(x) \frac{1}{p(x)} \nabla T_k(A(u_n)) \rightharpoonup \alpha(x) \frac{1}{p(x)} \nabla T_k(A(u)) \text{ weakly in } L^{p(\cdot)}(\Omega)^N, \text{ and} \\ & \alpha(x) |\nabla T_k(A(u))|^{p(x)} \in L^1(\Omega). \end{aligned}$$

Step 2: Strong convergence of truncations. Let $w_n^+ e^{\theta \eta(u_n)}$ be a test function in (45), where $w_n = T_{2k}(Z_n)$ be a test function in (24), where $Z_n = (A(u_n) - T_h(A(u_n)) + T_k(A(u_n)) - T_k(A(u)))$, for $h > k > 0$. Taking $M = 4k + h$, we have

$$(56) \quad \begin{aligned} & \int_{w_n > 0} a_n(x, u_n, \nabla u_n) \nabla w_n e^{\theta \eta(u_n)} dx + \int_{w_n > 0} H_n(x, u_n, \nabla u_n) w_n^+ e^{\theta \eta(u_n)} dx \\ & + \theta \int_{w_n > 0} a_n(x, u_n, \nabla u_n) \nabla u_n \frac{h(u_n)}{g(u_n)} w_n^+ e^{\theta \eta(u_n)} dx = \int_{w_n > 0} f_n w_n^+ e^{\theta \eta(u_n)} dx, \end{aligned}$$

by (3) and (9), we have

$$\begin{aligned} & \int_{w_n > 0} H_n(x, u_n, \nabla u_n) w_n^+ e^{\theta \eta(u_n)} dx \\ & + \theta \int_{w_n > 0} a_n(x, u_n, \nabla u_n) \nabla u_n \frac{h(u_n)}{g(u_n)} w_n^+ e^{\theta \eta(u_n)} dx \geq 0. \end{aligned}$$

So, we get that,

$$\int_{w_n > 0} a_n(x, u_n, \nabla u_n) \nabla w_n e^{\theta \eta(u_n)} dx \leq \int_{w_n > 0} f_n w_n^+ e^{\gamma(u_n)} dx.$$

We have

$$\begin{aligned} (57) \quad & \int_{\{w_n > 0\}} a_n(x, u_n, \nabla u_n) \nabla w_n e^{\theta \eta(u_n)} dx \\ &= \int_{\{w_n > 0\} \cap \{|A(u_n)| > k\}} a_n(x, u_n, \nabla u_n) \nabla w_n e^{\theta \eta(u_n)} dx \\ &+ \int_{\{w_n > 0\} \cap \{|A(u_n)| \leq k\}} a_n(x, u_n, \nabla u_n) \nabla w_n e^{\theta \eta(u_n)} dx. \end{aligned}$$

Concerning the first term on the right hand side of (57); since $\nabla w_n = 0$ on $\{|A(u_n)| > M\}$, we have

$$\begin{aligned} & \int_{\{w_n > 0\} \cap \{|A(u_n)| > k\}} a_n(x, u_n, \nabla u_n) \nabla w_n e^{\theta \eta(u_n)} dx \\ &= \int_{\{w_n > 0\} \cap \{|A(u_n)| > k\}} a_n(x, T_{\widehat{M}}(u_n), \nabla T_{\widehat{M}}(u_n)) \nabla Z_n e^{\theta \eta(u_n)} \\ &\geq - \int_{\{w_n > 0\} \cap \{|A(u_n)| > k\}} a_n(x, T_{\widehat{M}}(u_n), \nabla T_{\widehat{M}}(u_n)) \nabla T_k(A(u)) e^{\theta \eta(u_n)} dx \\ &\geq -e^{\theta \eta(\infty)} \int_{\{|A(u_n)| > k\}} a_n(x, T_{\widehat{M}}(u_n), \nabla T_{\widehat{M}}(u_n)) \nabla T_k(A(u)) dx \\ &\geq -\varepsilon_3(n). \end{aligned}$$

For the second term on the right hand side of (57); we have for $\widehat{k} > 0$

$$\begin{aligned} & \int_{\{w_n > 0\} \cap \{|A(u_n)| \leq k\}} a(x, T_{\widehat{k}}(u_n), \nabla T_{\widehat{k}}(u_n)) (\nabla T_k(A(u_n)) - \nabla T_k(A(u))) e^{\theta \eta(u_n)} dx \\ &\leq e^{\theta \eta(\infty)} \int_{\{w_n > 0\}} |f_n| |w_n| dx + \varepsilon_3(n). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} & \int_{\{w_n > 0\} \cap \{|A(u_n)| \leq k\}} a_n(x, T_{\widehat{k}}(u_n), \nabla T_{\widehat{k}}(u_n)) [\nabla T_k(A(u_n)) - \nabla T_k(A(u))] e^{\theta \eta(u_n)} dx \\ &= \int_{\{w_n > 0\} \cap \{|A(u_n)| \leq k\}} [a_n(x, T_{\widehat{k}}(u_n), \nabla T_{\widehat{k}}(u_n)) - a_n(x, T_{\widehat{k}}(u_n), \nabla T_{\widehat{k}}(u))] \\ &\times [\nabla T_k(A(u_n)) - \nabla T_k(A(u))] e^{\theta \eta(u_n)} dx \\ &+ \int_{\{w_n > 0\} \cap \{|A(u_n)| \leq k\}} a_n(x, T_{\widehat{k}}(u_n), \nabla T_{\widehat{k}}(u)) [\nabla T_k(A(u_n)) - \nabla T_k(A(u))] e^{\theta \eta(u_n)} dx. \end{aligned}$$

The third terms of the right hand tend to 0, as n tend to infinity. Thanks to step 2 of theorem 3.1. So, we have

$$\begin{aligned}
 & \int_{\{w_n > 0\} \cap \{|A(u_n)| \leq k\}} [a_n(x, T_{\hat{k}}(u_n), \nabla T_{\hat{k}}(u_n)) \\
 & - a_n(x, T_{\hat{k}}(u_n), \nabla T_{\hat{k}}(u))] [\nabla T_k(A(u_n)) - \nabla T_k(A(u))] e^{\theta \eta(u_n)} dx \\
 (58) \quad & = \int_{\{|A(u_n)| \leq k\}} a_n(x, T_{\hat{k}}(u_n), \nabla T_{\hat{k}}(u_n)) [\nabla T_k(A(u_n)) - \nabla T_k(A(u))] e^{\theta \eta(u_n)} dx \\
 & + \varepsilon_4(n) \leq e^{\theta \eta(\infty)} \int_{\{w_n > 0\}} |f_n(x)| |w_n| dx + \varepsilon_4(n) + \varepsilon_3(n) \leq \varepsilon_5(n),
 \end{aligned}$$

as $f_n \rightarrow f$ strongly in $L^1(\Omega)$, and $w_n \rightarrow 0$ weakly* in $L^\infty(\Omega)$.

Let $-(w_n)^- \exp(-\theta \eta(u_n))$ be a test function in the problem (45), we obtain

$$\begin{aligned}
 & \int_{w_n < 0} a_n(x, u_n, \nabla u_n) \nabla w_n e^{-\theta \eta(u_n)} dx \\
 & + \int_{w_n < 0} -H_n(x, u_n, \nabla u_n) (w_n)^- e^{-\theta \eta(u_n)} dx \\
 (59) \quad & + \theta \int_{w_n < 0} a_n(x, u_n, \nabla u_n) - \nabla u_n \frac{h(u_n)}{g(u_n)} - (w_n)^- e^{-\theta \eta(u_n)} dx \\
 & = \int_{w_n < 0} -f_n(w_n)^- e^{-\theta \eta(u_n)} dx,
 \end{aligned}$$

Reasoning as before, we get that

$$\int_{\{w \leq 0\} \cap \{|A(u_n)| > k\}} a(x, u_n, \nabla u_n) \nabla w_n e^{-\theta \eta(u_n)} dx \geq -\varepsilon_6(n),$$

where $\varepsilon_6(n)$ tend to 0 as n tend to infinity.

$$\begin{aligned}
 & \int_{w \leq 0} a_n(x, T_{\hat{k}}(u_n), \nabla T_{\hat{k}}(u_n)) (\nabla T_{\hat{k}}(u_n) - \nabla T_{\hat{k}}(u)) e^{-\theta \eta(u_n)} dx \\
 & \leq \int_{w \leq 0} |f_n| |w_n| e^{-\theta \eta(u_n)} dx + \varepsilon_6(n).
 \end{aligned}$$

Since w_n tend to 0 weakly* in $L^\infty(\Omega)$, and f_n converge strongly to f in $L^1(\Omega)$ we conclude that

$$\int_{w \leq 0} a_n(x, T_{\hat{k}}(u_n), \nabla T_{\hat{k}}(u_n)) (\nabla T_k(A(u_n)) - \nabla T_k(A(u))) e^{-\theta \eta(u_n)} dx \leq \varepsilon_7(n).$$

By adding the term to the last expression, we get

$$\begin{aligned}
 (60) \quad & \int_{w \leq 0} [a_n(x, T_{\hat{k}}(u_n), \nabla T_{\hat{k}}(u_n)) - a_n(x, T_{\hat{k}}(u_n), \nabla T_{\hat{k}}(u))] \\
 & \times (\nabla T_k(A(u_n)) - \nabla T_k(A(u))) e^{-\theta \eta(u_n)} dx \leq \varepsilon_7(n).
 \end{aligned}$$

Combining (58) and (60), we get

$$\int_{\Omega} [a_n(x, T_{\widehat{k}}(u_n), \nabla T_{\widehat{k}}(u_n)) - a_n(x, T_k(u_n), \nabla T_k(u_n))] \times \\ (\nabla T_k(A(u_n)) - \nabla T_k(A(u))) e^{-\theta \eta(u_n)} dx \leq \varepsilon_8(n),$$

so by the Lemma 2.5, we conclude that

$$T_k(A(u_n)) \rightarrow T_k(A(u)) \text{ in } W_0^{1,p(\cdot)}(\Omega), \nabla A(u_n) \rightarrow \nabla A(u) \text{ a.e. in } \Omega,$$

moreover we have

$$(61) \quad |\nabla T_m(u_n)|^{p(x)} \leq c |\nabla T_k(A(u_n))|^{p(x)},$$

thanks to the continuity of g .

By Lemma 3.1, for every $k > 0$ we have

$$(62) \quad \alpha(x) |\nabla T_k(u)| \in L^1(\Omega),$$

and

$$(63) \quad \alpha_n(x)^{\frac{1}{p(x)}} \nabla T_k(u_n) \rightharpoonup \alpha(x)^{\frac{1}{p(x)}} \nabla T_k(u) \text{ weakly in } (L^{p(x)}(\Omega))^N.$$

Step 3: The equi-integrability of $H_n(x, u_n, \nabla u_n)$. In order to pass to the limit in the approximate problem, we shall show that

$$(64) \quad H_n(x, u_n, \nabla u_n) \rightarrow H(x, u, \nabla u) \text{ in } L^1(\Omega).$$

Let E a set of Ω such that $\text{meas}(E) = 0$, and $l > 0$. We have

$$\begin{aligned} \int_E |H_n(x, u_n, \nabla u_n)| dx &\leq \int_E \beta(x) h(u_n) |\nabla u_n|^{p(x)} dx \\ &= \int_{E \cap \{|u_n| > l\}} \beta(x) h(u_n) |\nabla u_n|^{p(x)} dx + \int_{E \cap \{|u_n| \leq l\}} \beta(x) h(u_n) |\nabla u_n|^{p(x)} dx \\ &= \int_{E \cap \{|u_n| > l\}} \beta(x) h(u_n) |\nabla u_n|^{p(x)} dx + \int_{E \cap \{|u_n| \leq l\}} \beta(x) h(T_l(u_n)) |\nabla T_l(u_n)|^{p(x)} dx \\ &\leq \int_{E \cap \{|u_n| > l\}} \beta(x) h(u_n) |\nabla u_n|^{p(x)} dx + \max_{[0, l]}(h(s)) \int_{E \cap \{|u_n| \leq l\}} \beta(x) |\nabla T_l(u_n)|^{p(x)} dx. \end{aligned}$$

From (62), we deduce that the second term of the right hand side of the last inequality equal to 0 as $\text{meas}(E) = 0$. Prove that

$$\int_{E \cap \{|u_n| > l\}} \beta(x) h(u_n) |\nabla T_l(u_n)|^{p(x)} dx \rightarrow 0.$$

Let $(T_1(u_n - T_l(u_n)))^+ \exp(2\theta\eta(u_n))$ be a test function in the problem (45). We have

$$\begin{aligned} & \int_{l < u_n \leq l+1} a_n(x, u_n, \nabla u_n) \nabla T_1(u_n - T_l(u_n)) \exp(2\theta\eta(u_n)) dx \\ & + \int_{l < u_n} 2\theta a_n(x, u_n, \nabla u_n) \nabla u_n \frac{h(u_n)}{g(u_n)} (T_1(u_n - T_l(u_n)))^+ \exp(2\theta\eta(u_n)) dx \\ & + \int_{l < u_n} H_n(x, u_n, \nabla u_n) T_1(u_n - T_l(u_n))^+ \exp(2\theta\eta(u_n)) dx \\ & = \int_{l < u_n} f_n(T_1(u_n - T_l(u_n)))^+ \exp(2\theta\eta(u_n)) dx. \end{aligned}$$

By assumption (3) and (9) we have

$$\begin{aligned} & \int_{l < u_n} |\nabla u_n|^{p(x)} \beta(x) h(u_n) (T_1(u_n - T_l(u_n)))^+ \exp(2\theta\eta(u_n)) dx \\ (65) \quad & \leq C_{16} \int_{l < u_n} |f| dx. \end{aligned}$$

Let $-(T_1(u_n - T_l(u_n)))^- \exp(-2\theta\eta(u_n))$ be a test function in the problem (45). Reason as above, we get

$$\begin{aligned} & \int_{u_n < -l} |\nabla u_n|^{p(x)} \beta(x) h(u_n) (T_1(u_n - T_l(u_n)))^+ \exp(2\theta\eta(u_n)) dx \\ (66) \quad & \leq C_{17} \int_{u_n < -l} |f| dx. \end{aligned}$$

From (65) and (66) we conclude that

$$(67) \quad \int_{l < |u_n|} |\nabla u_n|^{p(x)} \beta(x) h(u_n) dx \leq C_{18} \int_{l < |u_n|} |f| dx.$$

Let l tend to infinity we get

$$\int_{l < |u_n|} |\nabla u_n|^{p(x)} \beta(x) h(u_n) dx \rightarrow 0.$$

Finally, we get the equi-integrability of H .

Step 4: Entropy solutions. Let $(T_k(A(u_n)))^+ e^{\theta\eta(u_n)}$ be a test function in (45), we have

$$\begin{aligned} & \int_{\Omega} a_n(x, u_n, \nabla u_n) \nabla T_k(A(u_n)) e^{\theta\eta(u_n)} dx \\ & + \int_{\Omega} H_n(x, u_n, \nabla u_n) (T_k(A(u_n)))^+ e^{\theta\eta(u_n)} dx \\ (68) \quad & + \int_{\Omega} a_n(x, u_n, \nabla u_n) \nabla u_n \frac{\theta h(u_n)}{g(u_n)} (T_k(A(u_n)))^+ e^{\theta\eta(u_n)} dx \\ & = \int_{\Omega} f_n(T_k(A(u_n)))^+ e^{\theta\eta(u_n)} dx. \end{aligned}$$

By (3) and (9), we have

$$\begin{aligned} & \int_{\Omega} H_n(x, u_n, \nabla u_n) (T_k(A(u_n)))^+ e^{\theta \eta(u_n)} dx \\ & + \int_{\Omega} a_n(x, u_n, \nabla u_n) \nabla u_n \frac{\theta h(u_n)}{g(u_n)} (T_k(A(u_n)))^+ e^{\theta \eta(u_n)} dx \geq 0. \end{aligned}$$

It follows that

$$\int_{A(u_n) > 0} a_n(x, u_n, \nabla u_n) \nabla (T_k(A(u_n))) e^{\theta \eta(u_n)} dx \leq \int_{\Omega} |f_n| (T_k(A(u_n)))^+ e^{\theta \eta(u_n)}.$$

If the function g is unbounded, taking $A(s) = \int_0^s g(\tau)^{1/(p^- - 1)} d\tau$, and by (3) we have

$$\begin{aligned} (69) \quad & C_{19} \int_{0 < A(u) \leq k} \alpha_n(x) g_n(u_n)^{\frac{p(x)}{p(x)-1}} |\nabla u_n|^{p(x)} e^{\theta \eta(u_n)} \\ & \leq \int_{0 < A(u) \leq k} a_n(x, u_n, \nabla u_n) \nabla T_k(u_n) g^{\frac{1}{p^- - 1}}(u_n) e^{\theta \eta(u_n)} dx \\ & \leq \int_{\Omega} |f_n| |T_k(A(u_n))| e^{\theta \eta(u_n)} dx \\ & \leq C_{20} k \|f\|_{L^1}. \end{aligned}$$

Else, taking $A(s) = \int_0^s g(\tau)^{1/(p^+ - 1)} d\tau$, and by (3) we obtain

$$\begin{aligned} (70) \quad & C_{21} \int_{0 < A(u) \leq k} \alpha_n(x) g_n(u_n)^{\frac{p(x)}{p(x)-1}} |\nabla u_n|^{p(x)} e^{\theta \eta(u_n)} \\ & \leq \int_{0 < A(u) \leq k} a_n(x, u_n, \nabla u_n) \nabla T_k(u_n) g^{\frac{1}{p^+ - 1}}(u_n) e^{\theta \eta(u_n)} dx \\ & \leq \int_{\Omega} |f_n| |T_k(A(u_n))| e^{\theta \eta(u_n)} dx \\ & \leq C_{22} k \|f\|_{L^1}. \end{aligned}$$

By means of (69) and (70), we obtain

$$(71) \quad \int_{0 < A(u) \leq k} \alpha_n(x) g_n(u_n)^{p'(x)} |\nabla u_n|^{p(x)} \leq C_{23} k \|f\|_{L^1}.$$

Let $(T_k(A(u_n)))^- \exp(-\theta \eta(u_n))$ be a test function in the problem (45). Reason as above, we get

$$(72) \quad \int_{-k < A(u) \leq 0} \alpha_n(x) g_n(u_n)^{p'(x)} |\nabla u_n|^{p(x)} \leq C_{24} k \|f\|_{L^1}.$$

Combining (71) and (72), we have

$$(73) \quad \int_{\Omega} \alpha_n(x) g_n(u_n)^{p'(x)} |\nabla u_n|^{p(x)} \leq C_{25} k \|f\|_{L^1}.$$

Let $\varphi \in X_0^{p(\cdot)}(\Omega) \cap L^\infty(\Omega)$, where $X_0^{p(\cdot)}(\Omega)$ is the space defined in (16), taking $\phi = T_k(u_n - \varphi)$ as a test function in the approximate problem, we get

$$\begin{aligned} & \int_{\Omega} a_n(x, u_n, \nabla u_n) \nabla T_k(u_n - \varphi) dx + \int_{\Omega} H_n(x, u_n, \nabla u_n) T_k(u_n - \varphi) dx \\ &= \int_{\Omega} f_n T_k(u_n - \varphi) dx. \end{aligned}$$

Adding and subtracting in the equation above the term

$$\int_{\Omega} a_n(x, u_n, \nabla \varphi) \nabla T_k(u_n - \varphi) dx$$

we obtain

$$\begin{aligned} & \int_{\Omega} \left[a_n(x, u_n, \nabla u_n) - a_n(x, u_n, \nabla \varphi) \right] \nabla T_k(u_n - \varphi) dx \\ &+ \int_{\Omega} a_n(x, u_n, \nabla \varphi) \nabla T_k(u_n - \varphi) dx \\ &+ \int_{\Omega} H_n(x, u_n, \nabla u_n) T_k(u_n - \varphi) dx = \int_{\Omega} f_n T_k(u_n - \varphi) dx. \end{aligned}$$

We then have, once again by (4)

$$(74) \quad \begin{aligned} & \int_{\Omega} a_n(x, u_n, \nabla \varphi) \nabla T_k(u_n - \varphi) dx \\ &+ \int_{\Omega} H_n(x, u_n, \nabla u_n) T_k(u_n - \varphi) dx \leq \int_{\Omega} f_n T_k(u_n - \varphi) dx. \end{aligned}$$

Noticing that $\{|u_n - \varphi| \leq k\} \subset \{|u_n| \leq k + \|\varphi\|_{L^\infty}\}$ and that

$$\int_{\Omega} a_n(x, u_n, \nabla \varphi) \nabla T_k(u_n - \varphi) dx = \int_{\Omega} \frac{a(x, T_n(u_n), \nabla \varphi)}{\alpha(x)} \alpha_n(x) \nabla T_k(u_n - \varphi) dx$$

we can pass to the limit in (74) using assumption (2), the a.e. convergence of u_n and the weak convergence of $\alpha(x)^{\frac{1}{p(x)}} \nabla T_k(u_n)$ proved in Step 2. Hence, we obtain, for every $k > 0$,

$$(75) \quad \begin{aligned} & \int_{\Omega} a(x, u, \nabla \varphi) \nabla T_k(u - \varphi) dx \\ &+ \int_{\Omega} H(x, u, \nabla u) T_k(u - \varphi) dx \leq \int_{\Omega} f T_k(u - \varphi) dx. \end{aligned}$$

As in the last part of Theorem 3.1 we take advantage of Lemma 7 of [5] to infer that, for every $k > 0$,

$$\int_{\Omega} a(x, u, \nabla u) \nabla T_k(u - \varphi) + \int_{\Omega} H(x, u, \nabla u) T_k(u - \varphi) = \int_{\Omega} f T_k(u - \varphi),$$

$$\forall \varphi \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega).$$

Step 5: weak solutions. Let $f \in L^{(p^*(\cdot))'}(\Omega)$, for each $n \in \mathbb{N}$, there exists a weak solution $u_n \in W_0^{1,p(\cdot)}(\Omega)$, which is an admissible test function in the weak sense (45). By Theorem 2.10 in [1], we have $u_n \in L^\infty(\Omega)$, and so $A(u_n) \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ for all $n \in \mathbb{N}$, where $A(s) = \int_0^s g(\tau)^{1/(p^- - 1)} d\tau$.

Let $v = e^{\theta\eta(u_n)}(A(u_n))^+$ be a test function in (45), we obtain

$$\begin{aligned} & \int_{A(u_n) \geq 0} a_n(x, u_n, \nabla u_n) \nabla A(u_n) e^{\theta\eta(u_n)} dx \\ & + \int_{A(u_n) \geq 0} H_n(x, u_n, \nabla u_n) A(u_n) e^{\theta\eta(u_n)} dx \\ & + \int_{A(u_n) \geq 0} a_n(x, u_n, \nabla u_n) A(u_n) \frac{\theta h(u_n)}{g(u_n)} e^{\theta\eta(u_n)} \nabla u_n dx \\ & = \int_{A(u_n) \geq 0} f A(u_n) e^{\theta\eta(u_n)} dx. \end{aligned} \quad (76)$$

On the other hand, by (3) and (9) we have

$$\begin{aligned} & \int_{A(u_n) \geq 0} a_n(x, u_n, \nabla u_n) A(u_n) \frac{\theta h(u_n)}{g(u_n)} e^{\theta\eta(u_n)} \nabla u_n dx \\ & + \int_{A(u_n) \geq 0} H_n(x, u_n, \nabla u_n) A(u_n) e^{\theta\eta(u_n)} dx \geq 0 \end{aligned}$$

So, (76) becomes

$$(77) \quad \int_{A(u_n) \geq 0} a_n(x, u_n, \nabla u_n) \nabla A(u_n) e^{\theta\eta(u_n)} dx \leq \int_{A(u_n) \geq 0} f A(u_n) e^{\theta\eta(u_n)} dx.$$

By the assumption (3), there exists $C_{26} > 0$ such that

$$\begin{aligned} & C_{26} \int_{A(u_n) \geq 0} \alpha_n(x) g(u_n)^{\frac{p(x)}{p(x)-1}} |\nabla u_n|^{p(x)} e^{\theta\eta(u_n)} dx \\ & \leq \int_{A(u_n) \geq 0} f A(u_n) e^{\theta\eta(u_n)} dx. \end{aligned} \quad (78)$$

Let $v = -e^{-\theta\eta(u_n)}(A(u_n))^-$ be a test function in (45), and reasing as above we get

$$\begin{aligned} & C_{27} \int_{A(u_n) \leq 0} \alpha_n(x) g(u_n)^{\frac{p(x)}{p(x)-1}} |\nabla u_n|^{p(x)} e^{\theta\eta(u_n)} dx \\ & \leq \int_{A(u_n) \leq 0} f A(u_n) e^{\theta\eta(u_n)} dx \end{aligned} \quad (79)$$

Adding up (78) and (79), we conclude that there exists $C_{28} > 0$ such that

$$(80) \quad \begin{aligned} & \int_{\Omega} \alpha_n(x) g(u_n)^{\frac{p(x)}{p(x)-1}} |\nabla u_n|^{p(x)} \\ & \leq C_{28} \int_{\Omega} f A(u_n) dx \leq C_{28} \|A(u_n)\|_{L^{p^*(x)}} \|f\|_{L^{(p^*(x))'}}. \end{aligned}$$

Using Sobolev inequality and, thanks to the a.e. convergence of ∇u_n to ∇u and Fatou Lemma,

$$\int_{\Omega} \alpha_n(x) g(u_n)^{\frac{p(x)}{p(x)-1}} |\nabla u_n|^{p(x)} \leq C_{29} (\|f\|_{(p^*(\cdot))'})^{p^{+}}.$$

Thus, we have that the sequence

$$(81) \quad \frac{a(x, T_n(u_n), \nabla u_n)}{\alpha(x)} \alpha_n(x)^{\frac{1}{p'(x)}} \text{ is bounded in } (L^{p'(\cdot)}(\Omega))^N,$$

and converges almost everywhere to $a(x, u, \nabla u) g(|u|)^{\frac{1}{p'(x)}-1}$ and weakly in $L^{p'(x)}(\Omega)^N$ to its a.e.-limit. Noticing that, for every $\phi \in X_0^{p(\cdot)}(\Omega)$

$$\alpha_n(x)^{\frac{1}{p(x)}} \nabla \phi \rightarrow \alpha(x)^{\frac{1}{p(x)}} \nabla \phi \text{ strongly in } L^{p(\cdot)}(\Omega)^N.$$

Let $\varphi \in X_0^{p(\cdot)}(\Omega) \cap L^{\infty}(\Omega)$, then

$$(82) \quad \int_{\Omega} a_n(x, u_n, \nabla u_n) \varphi dx + \int_{\Omega} H_n(x, u_n, \nabla u_n) \varphi dx = \int_{\Omega} f \varphi dx,$$

it follows from (64) and (81) that we may pass to the limit in (82) and conclude the proof.

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Time optimizing of fractal image compression using the Scatter Search algorithm

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Abstract. Image compression is one of the essential requirements for the efficient use of storage space and bandwidth. A new technique based on fractal theory is proposed for encoding the image; it is known as fractal image compression. In the procedure of encoding, the mechanism of search is considered as one of the main problems of this technique. In this work, an attempt to speed up the encoding process with minimal loss of the compressed image quality is adopted based on the Scatter Search algorithm. It is a sibling of Tabu search based on similar origins. The experimental results show a significant reduction in the computation time, where the mean square error measures between blocks are decreased after comparing them to full search methods. Consequently, the decoding process evinced that the reconstructed images were of high quality.

Keywords: fractal image coding (FIC), iterated function system (IFS), optimization algorithm, metaheuristics, scatter search method.

1. Introduction

Fractal image compression (FIC) is demonstrated to deliver an excellent compression ratio, and high quality reconstructed images. These two main significant concepts attract researcher attention since this technique was discovered by Barnsley in 1988 [1]. His method is based on the theory of iterated function system (IFS) [2, 3]. He obtained around 1000:1 compression ratio, but it required manual interference. This problem motivates his student Jacquin to improve this technique to be functioned automatically through partitioning the image into blocks [4, 5]. His approach focused on how efficiently can find the appropriate domain block for each range block, which in turn, given the approximate transformation after partitioning the image into a non-overlapping range block and an overlapping domain block,. This process is known as an encoding

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method. The attractor (fractal) is constructed by the decoding process based on the fixed point theorem, starting with any initial image. However, Jacquin's approach is considered as the main core of this research direction. Despite the encouraging properties of this high compressed technique, it has a significant constraint, and this is due to its intensive computation required in searching for the matched areas in the image. This limitation motivates the scholarly to trend their efforts toward overcoming such problem and optimizing the time performance of the encoding process.

Focusing on two main categories in the FIC technique, which are; finding the IFS, and an efficient algorithm to construct the approximate corresponding IFS for the given image, makes it completely different from conventional image compression techniques. After the backbone of FIC via the IFS, which was introduced by Barnsley and Sloan [6, 7], several attempts are highlighted toward improving the search for matching blocks or the type of segmentation utilizing different approaches [8, 9, 10, 11, 12, 13, 14, 15, 16]. The optimization algorithm is proved to be efficient for such a purpose and considered as a significant objective for many authors, because it helps in increasing the abilities to match computation. Based on evolutionary computation methods, many contributions are found in recent literature, some of them are based on the natural inventor, and the others are bio-inspired [17, 18, 19, 20, 21, 22, 23, 24, 25].

Some of the literatures that based on evolutionary methods were proposed to increase the compression ratio and preserve the quality of the decoded fractal attractor to generate the compressed image. The optimization is performed by the mechanism the number of selection that achieves a high reduction in the time, which may approach 150, as well as, a sound decrease in mean square error that may approach 66%. Examples of such literatures in this context are; genetic algorithm [19, 20, 21, 22], crowding method [23], ant colony [24], particle swarm optimization [25, 26], harmony search method [27]. Also, the one that used the heuristic algorithm based on the Wolf Pack algorithm was introduced in 2017 [28]. This algorithm works by exploring the whole space looking for similar small blocks. Another improvement is achieved in [29, 30] by adopting of Cuckoo inspired algorithm which arrange the range blocks as an ordered vector by the similarity measure or coordinate distance.

In contrast to these methods and association with the evolutionary computation methods, we adopted in this work the scatter search (SS) method [31, 32] as one of the metaheuristics and a global optimization algorithm utilized to reduce the searching strategy. Scatter Search is a sibling of Tabu search [33, 34]. This searching strategy is introduced to combine decision rules in the context of scheduling techniques, and to obtain improved decision-making rules for scheduling problems [5]. The purpose of this approach is to assume that the information about the relative desire for alternative options can be achieved in different forms through different rules. This information can be used more effectively when it is integrated with the combination mechanism and processed through the standard strategy by selecting different rules at the same time. By

the SS, the bset is continually updating; its update depends on the quality and diversity of the solutions. In each updated set, the weak solutions that do not give the desired results are eliminated after calculating the least square error.

Besides all of these advantages of the scatter search method, it provides precise and fast results to exclude the redundant blocks with a shortening time. Therefore, it was used to speed up the fractal image compression technique.

Several multimedia and communication applications are based on image compression, which plays an essential role in evolving such services. Consequently, a small size image can be transmitted quickly. Fractal techniques for image compression are proved to be the best over other techniques due to their ability to remove redundancy and maintain image resolution. However, the problem with this technique is time-consuming. The main contribution of the present work is to utilize the SS algorithm; it works toward improving the solution and speeding up the matching process to exclude the redundant areas.

The rest of the paper is organized as follows: Section 2 presents the original algorithms of the scatter search method. Section 3 describes the proposed fractal image compression based on the scatter search method. Section 4 discusses and analyses the implemented, and the results in terms of some statistical measurement. Conclusions are summarized in Section 5.

2. Scatter search method

As a nature-inspired method, the scatter search method, in combination with the sibling of Tabu search, was introduced in 1977 by Glover [31]. He used it to maintain a good quality nominates solution and diversity set based on the principle that the vital information for the global optima is saved in a reference set of solutions. After that, this information is exploited by the recombining of some other samples. This procedure is performed through the iteration process, and it is highly used for reducing the cost. Glover was proposed many literature for different purposes [31, 32, 33, 34, 35]. He proved that the dispersion patterns generated by SS are useful in various application areas and applied to many optimization problems. Its mechanism emphasizes on creating of a new solution based on the basic set (bset) that represents the combinations of high-quality solutions with diverse solutions, where the bset represents a weighted of the combination to expatiate the optimization problem directly. By the SS method, it is evident that the randomness degree is lower than those appears in the meta-heuristic method that emphasized on the population, as well as, using the local search method to speed up the convergence to the optimum range is considered as another differences between the SS and other conventional methodologies that based on the population.

The main three parts of this method are:

1. Generate a collection of diverse trial solutions.
2. The local search, contain the bset.

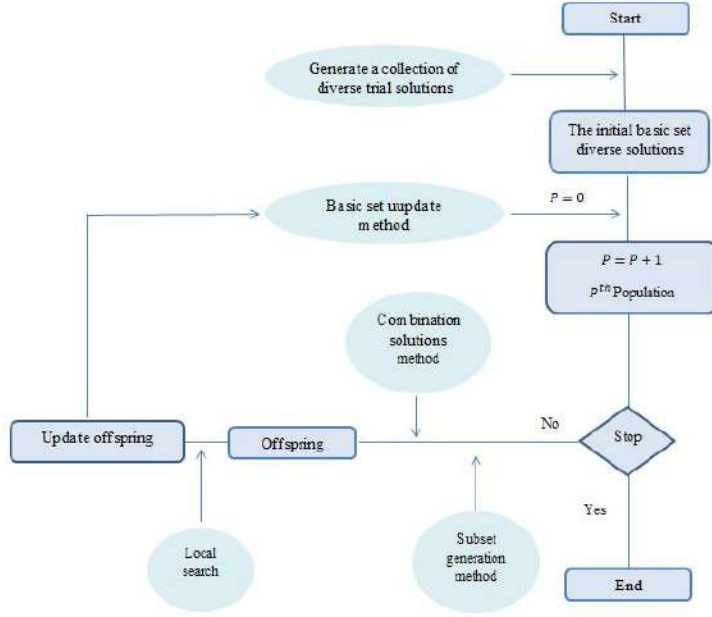


Figure 1: Scatter Search Steps

3. Update the bset to create and preserve new bset that consists the best solution.

These parts represent the initialization stage, while the other that used for search purposes are:

4. The method of configuring a subset that operated on bset is to create a subset of solutions to be used as a basis for producing collective solutions.
5. The method of combined solutions is used to convert a given subset of solutions produced by the subset creation way into one or more collected solution vectors, Glover (1999) [32]. Figure 1 illustrates the five scatter search steps.

The basic set arrangement and repetition: By this process, the elements of the bset are sorted based on their quality. When some elements are neighbors and close to each other, it should be replaced by a random solution, as can be seen in steps (8-12, **Algorithm 1**). This comparison process is applied for each pair in the bset. Besides, step 9 illustrates how two solutions are close to each other if the maximum distance between them is more significant than a given threshold. However, this step leads to increase the diversity of the bset and prevent the stagnation of the search process.

Basic set = $\{\alpha^1, \alpha^2, \dots, \alpha^m\}$, $\text{bset} \subset P, f(\alpha^i) \leq f(\alpha^j), i, j \in \{1, 2, \dots, m\}, i < j$, where P represents the population, m is the size of the population, and f is the nonlinear dynamic problem.

If $\max |\frac{\alpha^i - \alpha^j}{\alpha^j}| \leq \epsilon, i < j$ then change α^j in the *bset* by a random solution to be evaluated, such that: $\Rightarrow eval = eval + 1$ *Generating of new solutions:* This process is performed by the steps (13-23, Algorithm 1), the new solution is generated in a particular rectangles according to its position, and the distance between the elements. It is accomplished by combining all the dimension of the boundaries with a random weight and bounded by the combined solution distance, such that;

$$\beta = \beta \cup \beta^{i,j} \text{ and evaluate } \beta^{i,j} \Rightarrow eval = eval + m - 1$$

Updating of the bset: The solution created by the combination of the solution can replace the bset members, as shown in step (26, **Algorithm 2**), if they exceed the predefined threshold. Otherwise, the solution cannot replace the members (parent); it should replace the original elements in the primary group. This technique avoids combinations of identical solutions in the early stages of algorithm that can lead to premature inactivity in the search process.

Replace classified bset: The best children $\beta^{i,*}$ replaces the classified *bset* members then reset $n(i)$.

After that, two additional techniques are used to make the search more efficient (**Algorithm 2**). The first technique (go-beyond) is to use the new solution if it exceeds the original and a new solution is created in an excessive rectangle, this process is repeated as long as there are improved solutions as shown in steps (1-26, **Algorithm 2**).

Let $\beta^{i,*} = \text{best children} \in \beta^{i,j} \forall j \in \{1, 2, \dots, m\}, i \neq j$.

If $\beta^{i,*} < f(\alpha^i)$ then create a new solution α_{new} in the hyper-rectangle, such that:

$\alpha_{new} = \left(\beta^{i,*} - \frac{\alpha^i - \beta^{i,*}}{\Omega}, \beta^{i,*} \right), \Omega = 1$, where Ω depends on the number of improvements in **Algorithm 2**, and $\beta^{i,*} = \alpha_{new}$.

If $(\beta^{i,*}) < f_{best}$ then $\alpha_{best} = \beta^{i,*}$, and $f_{best} = f(\beta^{i,*})$.

The stagnation exam: The second technique includes the stagnation exam, it is considered as a local solution, and it should be replaced by a new random solution in the *bset* as illustrated in steps (27-32, **Algorithm 2**). Let $n(i)$ be the counter that replace the classified bset members by best children $\beta^{i,*}$ $n(j) = n(i) + 1$, where j is the index of the unclassified *basicset* members. If $n(i) > n_{change}$ then change $\alpha^i \in bset$ by the random solution and set $n(i) = 0$.

The evaluation process: This process is performed by testing the function of local search $LS(w_1)$, if its value is less than or equal to an evaluator tolerance, then a local search over *abset* is applied. Otherwise, the function $LS(w_2)$ is checked with the same tolerance to determine the previous local search. However, such a local search can only be applied to the best existing solutions that have been updated in the current iteration, as illustrated in the following steps.

Let $eval$ denotes the evaluator of α^i . If $eval \leq LS(w_1)$ then the local search is applied over α_{best} sol.

If $eval \geq LS(w_2)$ then create $\beta = \{\beta^1, \beta^2, \dots, \beta^m\}$ such that $f(\beta^i) \leq f(\beta^j)$ if $i < j$ $d_{\min}(i) = |\beta^i - LS|^2, i \in \{1, 2, \dots, m\} \subset \beta$ and LS refers to the local solution, where all the local optima are found. Sort β according to their diversity, creating $\beta_d = \{\beta_d^1, \beta_d^2, \dots, \beta_d^m\}$, such that $d_{\min}(i) \geq d_{\min}(j)$.

The distance between the new and the previous solutions is used to measure the diversity and the balance between their diversity and quality. The newly obtained solution is added to the set of local solutions.

In the following equation the parameter balance is used as a measure of quality or diversity by giving them a weight, score (β^k) represents the counter for this balance such that: $score(\beta^k) = (1 - balance) * i + (balance * j)$, where i is the index of $\beta^k \in \beta$ and j is the index of $\beta^k \in \beta_d$, and $i < j$. Now, apply local search over β^l , such that $score(\beta^1) = \min(score(\beta))$. If $X^* \notin LS \Rightarrow LS = LS \cup \{X^*\}$.

Algorithm 1 Classical Scatter Search

- 1: Set parameters: $eval$, Local solutions (LS), $diverse$, $bset$, dimension solutions(m), $LS(w_1)$, $LS(w_2)$, $balance$, $score$ best solution
 - 2: Initialize n , $eval$;
 - 3: $LS = \phi$;
 - 4: Generate a collection of diverse trial solutions P ;
 - 5: $eval = eval + diverse$;
 - 6: Create the initial $bset$ of dimension solutions m with the best solutions and random elements.
 - 7: **Repeat**
 - 8: Arrange $bset$ by quality, such that $bset = \{\alpha^1, \alpha^2, \dots, \alpha^m\}$, and $f(\alpha^i) \leq f(\alpha^j)$, where $i, j \in \{1, 2, \dots, m\}, i < j$.
 - 9: **if** $\max \left| \frac{a^i - a^j}{a^j} \right| \leq \epsilon, i < j$ **then**
 - 10: Put α^j in the $bset$ by a random solution and evaluate it;
 - 11: $eval = eval + 1$;
 - 12: **end if**
 - 13: $\beta = \phi$;
 - 14: **for** $i=1$ to m **do**
 - 15: **for** $j=1$ to m **do**
 - 16: **if** $i \neq j$ **then**
 - 17: Combine α^i with α^j to generate a new solution $\beta^{(i,j)}$;
 - 18: $\beta = \beta \cup \beta^{(i,j)}$;
 - 19: Evaluate $\beta^{(i,j)}$;
 - 20: **end if**
 - 21: **end for**
 - 22: $eval = eval + m - 1$;
 - 23: **end for**
-

-
- 24: Applied algorithm of additional techniques (**Algorithm2**)
 25: Until a stopping criterion is met to best found solution or set of solutions.
-

In Algorithm 2, the two methods are used to make the search for an improvement solution more efficient. It is based on two techniques. First, the enhanced solution is used instead of the original solution, which can be shown in steps (1-26). In the second, the local solution is replaced by the random solution from the solutions set; this can be seen in steps (27-32).

Algorithm 2 Additional Techniques

```

1: for  $i=1$  to  $m$  do
2:    $\beta^{(i,*)} = \text{best children in } \beta^{(i,j)} \forall j \in \{1, 2, \dots, m\}, i \neq j;$ 
3:   if  $(\beta^{(i,*)}) < f(\alpha^i)$  then
4:      $improvement = 1;$ 
5:      $\Omega = 1;$  do
6:       Create a new solution,  $\alpha_{new}$  in the hyper-rectangle.
7:        $\alpha_{new} = \left( \beta^{i,*} - \frac{\alpha^i - \beta^{L*}}{\Omega}, \beta^{i,*} \right);$ 
8:        $\beta^{(i,*)} = \alpha_{new}$ 
9:        $eval = eval + 1;$ 
10:       $\beta = \beta \cup \beta^{(i,*)};$ 
11:       $improvement = improvement + 1;$ 
12:      if  $improvement = 2$  then then
13:         $\Omega = \frac{\Omega}{2};$ 
14:         $improvement = 0;$ 
15:        if  $f(\beta^{(i,*)}) < f_{best}$  then
16:           $\alpha_{best} = \beta^{(i,*)};$ 
17:           $f_{best} = f(\beta^{(i,*)});$ 
18:        end if
19:      end if
20:    end if
21:  end for
22:  if the local search is activated then
23:    Applied algorithm of local search (3)
24:  end if
25:  Replace classified basic set members by best children  $\beta^{(i,*)}$  and reset  $n(i);$ 
26:   $n(j) = n(i) + 1;$   $j$  is the index of a not classified basic set members.
27:  for  $i = 1$  to  $m$  do
28:    if  $n(i) > n_{change}$ 
```

```

29:      Replace  $\alpha^i \in \text{basic}$  set by a random solution and set  $n(i) = 0$ ;
30:  end if
31: end for

```

Algorithm 3 The Local Search

```

1: if  $LS = \phi$  then
2:   if  $eval \geq LS(w_1)$  then
3:     Apply local search over  $\alpha$  best;
4:   end if
5: end if
6: elseif
7: if  $Localsolutions(LS) = \phi$  then
8:   if  $eval \geq LS(w_2)$  then
9:     Sort  $\beta$  by quality, create  $\beta = \{\beta^1, \beta^2, \dots, \beta^m\}$ , such that  $f(\beta^i) \leq f(\beta^j)$  if  $i < j$ .
10:    Calculating the minimum distance between each element  $i \in 1, 2, \dots, m \subset \beta$ , and all the local optima that are found.
11:     $d_{min}(i) = |\beta^i - LS|^2$ ;
12:    Sort  $\beta$  by diversity, creating  $\beta = \{\beta_d^1, \beta_d^2, \dots, \beta_d^m\}$ , such that  $d_{min}(i) \geq d_{min}(j)$ ;
13:    if  $i < j$  then
14:      for each solution  $\beta^k \in \beta$  do  $score(\beta^k) = (1 - balance) * i + balance * j$ , such that  $i$  is the index of  $\beta^k \in \beta$ , and  $j$  is the index of  $\beta^k \in \beta_d$ ;
15:    end for
16:    Apply local search over  $\beta^1$ , such that  $score(\beta^1) = \min(score(beta))$ ;
17:    end if
18:  end if
19: end if
20: Create the Local Solution  $X^*$ ;
21: if  $X^* \notin LS$  then
22:    $LS = LS \cup \{X^*\}$ ;
23: end if
24: if  $f(X^*) < f_{best}$  then
25:    $\alpha_{best} = X^*$ ;
26:    $f_{best} = f(X^*)$ ;
27: end if

```

3. The Scatter Search method to improve Fractal Image Compression technique

In this section, FIC technique is implemented using SS method improvement was accomplished using SS algorithm is one of the population-based Metaheuristics.

The classical Jacquin approach is began by partitioning the given image into non-overlapping blocks called range blocks (RBs) $R = \{R^1, R^2, \dots, R^m\}$ of size $r \times r$ and overlapping domain blocks (DBs), $D = \{D^1, D^2, \dots, D^n\}$ of size $2r \times 2r$, where $m = (\frac{M}{r})^2$, and $n = (M - 2r + 1)^2$. For each R^i , search for its similar $D^k, k = 1, \dots, n$ with an appropriate contractive transformation w^{ik} that persuaded the equation $d(R^i, w^{ik}(D^k)) = \min d(R^i, w^{ij}(D^j))$. Here, w^{ik} represents the contractive affine transformation between D^j to R^i . This can be represented by $d(R^i, w^{ij}(D^j))$, where d is the MSE between R^i and $w^{ij}(D^j)$, which in turn consist of two mappings T^j and θ^{ij} such that, $w^{ij} = \theta^{ij} * T^j$. Meanwhile, T^j represents the scaling transformation that used to contract D^j size to equal R^i size by averaging of the four pixels in the block of D^j . However, $T^j(D^j)$ is equal to R^i size as illustrated in Figure 2.

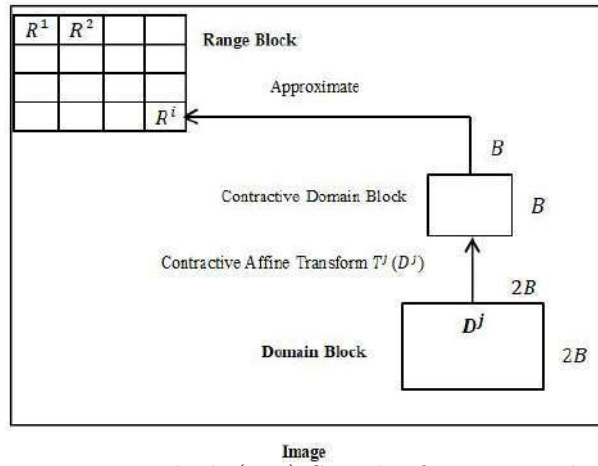


Figure 2: Domain Block (DB) Search of a Range Block (RB)

The mapping ϕ^{ij} is subjected to the eight isometries transformations given in Table 1.

Subsequently, W^{ij} , is used to transform the pixel values of the transformed block, such that:

$$(1) \quad W^{ij}(z) = s^i z + o^i.$$

The variables z, s^i , and o^i in Eq.(1) represent the pixel value, scaling, and the offset o^i respectively, where s^i , and o^i are obtained by Eq.s (2 and 3) respectively. These two values are used to calculate the MSE.

$$(2) \quad s = \frac{[n \sum_{i=1}^n a_i b_i - \sum_{i=1}^n a_i \sum_{i=1}^n b_i]}{[n \sum_{i=1}^n a_i^2 - (\sum_{i=1}^n a_i)^2]},$$

$$(3) \quad o = \frac{1}{n} \left[\sum_{i=1}^n b_i - s \sum_{i=1}^n a_i \right].$$

Table 1: The eight isometries

Transform T^j	Angle	Isometry
T^0	Identity	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
T^1	Rotation (+90°)	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
T^2	Rotation (+180°)	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$
T^3	Rotation (+270°)	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
T^4	Reflection abt x axis	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
T^5	Reflection & Rotation (+90°)	$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$
T^5	Reflection & Rotation (+180°)	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
T^5	Reflection & Rotation (+270°)	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

The MSE difference is used to measure the distances between the solutions, as computed in Eq.(4):

$$(4) \quad MSE = \frac{1}{n} \left[\sum_{i=1}^n b_i^2 - s \left(s \sum_{i=1}^n a_i^2 - 2 \sum_{i=1}^n a_i b_i + 2o \sum_{i=1}^n a_i \right) + o \left(n^* o - 2 \sum_{i=1}^n b_i \right) \right].$$

This equation is used to calculate the scaling as a geometric part and the image pixels as a massive part. By using scatter search, a random search is performed in the selected community $P = \{P_1, P_2, \dots, P_m\}$, the local search for (initial P) is then conducted. Then calculated the difference between the best children (best solution) $\beta^{i,*}$ and w^{ij}

$$(5) \quad d = \sum |\beta^{i,*} - w^{ij}|,$$

where $w^{ij} = \theta^{ij} * T^j$ and $T^j \in P$ (P is collection of diverse trial solutions). Updating the bset by choosing the best solution after tests, where bset update method equal high quality union the diversity then the best solution $\beta^{i,*}$ is chosen. The worst solution from the bset is removed and again working to get improved solutions by applying steps 7 and 8 in Algorithm 4 Finally, stop the processing when getting the best solution or the loop of find the solution is

end. Algorithm 4 describes fractal image compression based on scatter search algorithm:

Algorithm 4 The Proposed Algorithm

- 1: Initialize the scatter search;
 - 2: Split the given image M into two types of blocks, one of them is called range blocks $RB = \{R^1, R^2, \dots, R^m\}$ of size $\times r$, which are non-overlapping, where $m = (\frac{M}{r})^2$, and the other is called domain blocks $DB = \{D^1, D^2, \dots, D^n\}$ of size $2r \times 2r$ blocks of overlapped area, on which $n = (M - 2r + 1)^2$.
 - 3: For each range block R^1 , choose the domain block $D^k, k = 1, \dots, n$ and a suitable contractive affine transformation w^{ik} that satisfied $d(R^i, w^{ik}(D^k)) = \min_d(R^i, w^{ij}(D^j))$, where w^{ik} is a contractive affine transformation, $w^{ik} : D^i \rightarrow R^i$. Here d represents the MSE between R^i , and w^{ik} , where $w^{ij} = \theta^{ij} * T^j$. The function $T^j(D^j)$ is carrying out using eight isometries given in Figure 2.
 - 4: The pixel values of the output values after implementing of the first step is transformed by W^{ij} , such that, $w^{ij}(z) = s^i z + o^i$, where s^i and o^i are given in equations (2) and (3) respectively.
 - 5: This comparison resulting $T^j = [x^i, y^i, w^i, s^i, o^i]$;
 - 6: The scatter search affects the above steps as follows:
 - 7: Generate a collection of diverse trial solutions $P = (P_1, P_2, \dots, P_m)$. However, by the scatter search, a population of 10 samples is generated in each time.
 - 8: Create the initial bset of dimension solutions equals m with the best solutions and random elements.
 - 9: **Repeat**
 - 10: Arrange the bset according to its quality, such that $bset = \{T^1, T^2, \dots, T^m\}$. Hence, $w(T^i) \leq w(T^j)$, where $i, j \in \{1, 2, \dots, m\}, i < j$
 - 11: Determine the LS for (initial P)
 - 12: $d = \sum |\beta^{(i,*)} - w^{ij}|, i, j = 1, 2, \dots, m$, and $i < j$;
 - 13: The new $bset = highquality \cup diversity$;
 - 14: Apply local search iteration
 - 15: **Repeat**
 - 16: Choose the best solution $\beta^{(i,*)}$, where $\beta^{(i,*)} \in \beta$;
 - 17: Calculate the minimum distance between each element w^{ij} with the local solution,
 - 18: $d_{min}(i) = |w^{ij} - LS|^2$;
 - 19: **if** best changed **then**
 - 20: $X^* = bestsolution$
 - 21: **Else**
 - 22: **end if**
 - 23: Remove the worst solutions $\frac{X^*}{2}$ from the bset;
-

-
- 24: Combine T^i with T^j to generate a new solution $\beta^{(i,j)}$
 - 25: Replace classified bset members by best children $\beta^{(i,*)}$
 - 26: Make new population size improved solutions applying step 7 and so on
 - 27: **Until** a stopping criterion is met to the best solution or set of solutions
-

4. Implementation and analysis

Image compression plays an essential role in the evolution of several multimedia services and communication applications. Images need high storage area. Consequently, decreasing the volume of the transmitted image data leads to speed up the transportation of such data. The techniques of image compression aim to take out the redundant data with preserving of acceptable resolution of the retrieved image. As an emerging technique, FIC which is based on IFS is demonstrated as the best for image compression technology due to its high ability to remove redundancy and to maintain image resolution. The main task of the SS algorithm is to build a community for generating a complete solution based on Sipling of Tabo search and accordingly, it work toward improving the solution. We calculate the fitness for a set of images and then enter each image by 100 generating and divide into 10 generating produces an image and repeat this process for the other 10 generation and so on. The proposed model was applied on four grayscale images with a size 512 x 512, and a range block of size (2×2) , (4×4) and (8×8) . The SS algorithm was implemented in MATLAB on a PC with the specification: Intel 2.5 GHz Core i7, with 8 MB memory. Some statistical test are used to evaluate the performance of the proposed method, such as, Peak Signal to Noise Ratio (PSNR), Mean Square Error (MSE), Compression Ratio (CR), these are the crucial parameters. These parameters are evaluated for some images, they are defined as follows: Peak-Signal to Noise Ratio (PSNR): It is the metric that used to measure the quality between the original and the cipher image. It is the ratio between the possible maximum signal value and the value of the corrupted noise. The PSNR is usually expressed by the logarithmic function due to the wide dynamic range of numerous signals. It is defined by Eq. (6) as in the following [6]:

$$(6) \quad PSNR = 10 \log_{10} \left(\frac{255^2}{MSE} \right) (dB).$$

The equation for calculating mean square error (MSE), which is considered as a risk function and used to measure the square error, is given as follows:

$$(7) \quad MSE = \frac{1}{n * m} \sum_{i=1}^n \sum_{j=1}^m (x_{i,j} - y_{i,j})^2.$$

The PSNR for color images is computed at every color plane of the RGB image taking the average of the MSE's of the three colors. Therefore, the PSNR of the

RGB image is; where:

$$(8) \quad PSNR = 10 \log_{10} \left(\frac{255^2}{MSE_{RGB}} \right) (dB),$$

$$(9) \quad MSE_{RGB} = \frac{1}{3} (MSE_{Red} + MSE_{green} + MSE_{blue}).$$

The CR is defined as the ratio of the original image data size over the compressed image data size:

$$(10) \quad CR = \frac{\text{Size of original image}}{\text{Size of compressed image}}.$$

When the range block size is increased, a low compression time is obtained with a high PSNR, which in turn causes a loss in the quality of the compressed image. Whereas, by decreasing range block size, a high PSNR value is achieved, which refers to high compression ratio with good quality but vise verse time consuming. Therefore, to balance between the PSNR and reasonable computation time besides good compressed image quality, the range blocks is chosen to be of sizes (2×2) , (4×4) or (8×8) . The experiments show that the decoded image quality of a range blocks of sizes (2×2) and (4×4) are better than that of a size (8×8) , and this can be seen obviously in Figure 3.

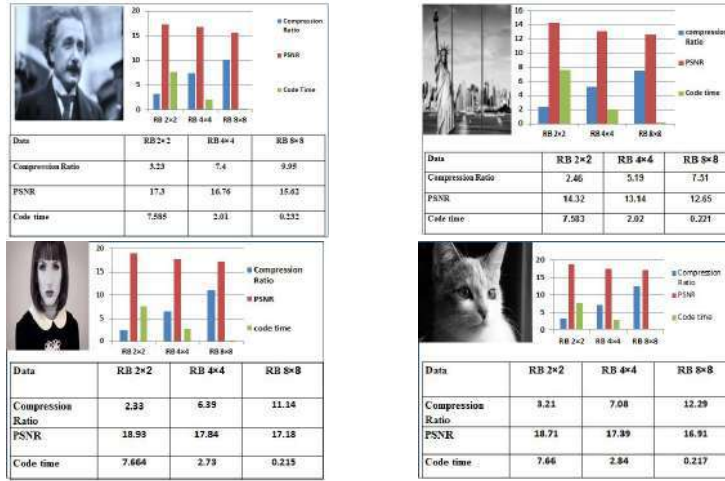


Figure 3: Compression ratio, PSNR and coding time based on the proposed method for different range size

The experimental results show the compression ability that in some images reached 98%. The time consuming of this technique as a weak point motivates us to optimize the computation time by proposing the scatter search method, where the experiments show its efficiency over the classical Jacquin method and also over the IFS based genetic algorithm as shown in Table 2 and Figure 4.

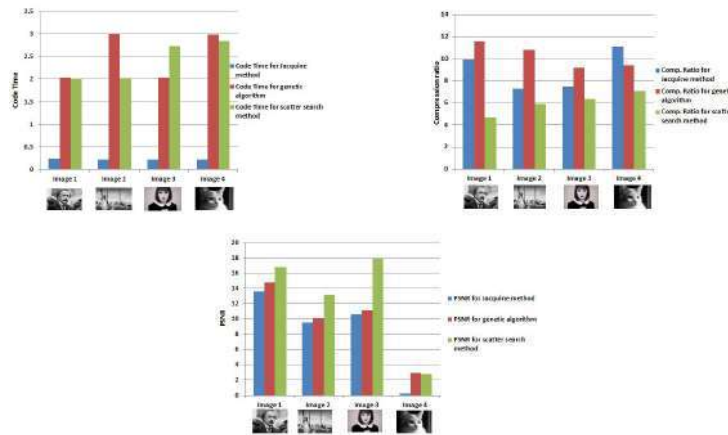


Figure 4: Comparison between Jacquin method, genetic algorithm and the proposed scatter search for range block of size 4.

Table 2: Comparison between Jacquin approach, genetic and the proposed scatter search for range block of size 4.

	Images				
Fractal image compression based on Jacquin method	Compression Ratio	9.97	7.33	7.53	11.16
	PSNR	13.62	9.52	10.65	15.18
	Code Time	0.247	0.226	0.227	0.221
Fractal image compression based on genetic algorithm	Compression Ratio	11.6	10.82	9.2	19.41
	PSNR	14.76	10.11	11.14	15.84
	Code Time	2.03	2.99	2.03	2.98
Fractal image compression based on scatter search algorithm	Compression Ratio	4.7	5.9	6.39	7.08
	PSNR	16.76	13.14	17.84	17.93
	Code Time	2.01	2.02	2.73	2.84

5. Conclusions

An improved scatter search algorithm is introduced to enhance the fractal image compression technique. What distinguishes the present algorithm from the original are the differences in the primary phases. The modification of the fractal image compression algorithm via the SS method is performed. By this method, various experimental solutions are generated to be modified. In each iteration, 10 solutions are generated for the image, followed by other 10 until getting the best solution or a 100 trial is achieved. After each 10 generated solution, they are compared to get the best and during this process, either the image resolution get improvement or still as it is, sometimes, we may not need more than 1 or 2 repetitions to get such improvement. The local search algorithm is used to calculate the minimum distance between their elements. The two best solutions

are positioned in a hyper rectangle. However, the high-quality image with high compressed ration and less error is obtained. Besides, it helps to accelerate the encoding time, which is needed to find the appropriate IFS for each image. The analysis of the SS algorithm in comparing to two fractal image compression methods, the original Jacquin method and GA, shows its high performance among them.

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Optimum solution of time fractional coupled system of partial differential equations

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Abstract. In the present article, the Optimal Homotopy Asymptotic Method (OHAM) has been extended for the first time to derive the approximate solution of coupled system of fractional order partial differential equations. The fractional Whitham-Broer-Kaup system has been solved as test example. Numerical results obtained by the proposed method are compared with that of Adomian Decomposition Method (ADM) and Variational Iteration Method (VIM). The fractional derivatives are described in the Caputo sense. Numerical results show that the proposed method is reliable and efficient for solution of fractional order coupled system partial differential equations. The accuracy of the method increases by taking higher order approximations.

Keywords: Optimal Homotopy Asymptotic Method (OHAM), time fractional Whitham-Broer-Kaup (WBK) equation, fractional order coupled system of partial differential equations.

1. Introduction

Differential equations of fractional order have been the center of attention of many studies due to their frequent applications in the areas of electromagnetic,

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electrochemistry, acoustics, material science, physics, viscoelasticity and engineering [1, 2]. These kinds of problems are more complex than the differential equations of integer order. Due to the complexities of fractional calculus, most fractional order differential equations do not have the exact solutions, and as alternative approximation methods are used extensively for solution of these types of equations. Some of the recent approximate methods for fractional order differential equations are Homotopy Perturbation Method (HPM), Homotopy Perturbation Transform Method (HPTM), Homotopy Analysis Method (HAM), Reduced Differential Transform Method (RDTM) and Adomian decomposition method (ADM) [3, 4, 5, 6, 7].

Marinca and Herisanu introduced (OHAM) for solving Non-linear Differential equations which made the perturbation methods independent of the assumption of small parameters and huge computational work [8, 9, 10, 11]. The method has been recently extended by Sarwar et al. for solution of differential equations of fractional order [12, 13, 14, 15].

In this presentation, we have extended (OHAM) for the first time for finding the approximate solution of coupled system of fractional order partial differential equations. Particularly, the extended formulation is demonstrated by illustrative examples of the following fractional version of the Whitham-Broer-Kaup (WBK) system.

The fractional order (WBK) system describes the propagation of shallow water waves with different dispersion relations. It takes the following general form:

$$(1) \quad \begin{cases} D_t^\alpha u + uu_x + v_x + bu_{xx} = 0, \\ D_t^\alpha v + (uv)_x + au_{xxx} - bv_{xx} = 0, \end{cases}$$

where $u(x, t)$ denotes the horizontal velocity, $v(x, t)$ is the height that deviates from the equilibrium position, a, b are constants that are represented in different diffusion powers, $0 < \alpha \leq 1$ for $\alpha = 1$ we get the standard (WBK) system. The fractional order modified Bouissnesq (MB) system and fractional order approximate long wave (ALW) system are the special cases of and (WBK) System for $a = 1, b = 0$ and $a = 0, b = 0.5$ respectively. These equations have become the subject of active and broad research area in recent decades [16, 17, 18].

The present paper is divided into 5 sections. In Section 2 the basic definitions and properties of fractional calculus are given. Section 3 is devoted to analysis of (OHAM) to coupled system of partial differential equations of fractional order. In Section 4, the 1st order approximate solutions of Time-fractional (WBK) and time-fractional (ALW) equation are given, in which the time fractional derivatives are described in the Caputo sense. In section 5 comparisons are made between the results of 1st order approximate solution by the proposed method with 3rd order (ADM) and (VIM) solutions [19, 20]. In all cases the proposed method yields better results.

2. Preliminaries and notations

In this section we state some definitions and results from the literature which are relevant to our work. Riemann-Liouville, Welyl, Reize, Compos, Caputo proposed many definitions.

Definition 2.1. A real function $f(r), r > 0$ is said to be in space $C_\mu, \mu \in R$ if there a real number $p > \mu$ such that $f(r) = r^p f_1(r)$, where $f_1(r) \in C(0, \infty)$, and it is said to be in the space C_μ^m if only if $f^{(m)} \in C_\mu, m \in N$.

Definition 2.2. The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ of a function $f \in C_\mu, \mu \geq -1$ is defined as

$$J_a^\alpha f(r) = \frac{1}{\Gamma(\alpha)} \int_a^r (r - \mu)^{\alpha-1} f(\mu) d\mu, \alpha r > 0, J_a^0 f(r) = f(r).$$

When we formulate the model of real world problems with fractional calculus, the Riemann-Liouville have certain disadvantages. Caputo proposed a modified fractional differential operator D_a^α in his work on the theory of viscoelasticity.

Definition 2.3. The fractional derivative of $f(r)$ in Caputo sense is defined as

$$D_a^\alpha f(r) = J_a^{m-\alpha} D^m f(r) = \frac{1}{\Gamma(m-\alpha)} \int_a^r (r - \eta)^{m-\alpha-1} f^{(m)}(\eta) d\eta,$$

$m - 1 < \alpha \leq m, m \in N, r > 0, f \in C_{-1}^m$.

Definition 2.4. If $m - 1 < \alpha \leq m, m \in N$ and $f \in C_\mu^m, \mu \geq -1$, then $D_a^\alpha J_a^\alpha f(r) = f(r)$ and $J_a^\alpha D_a^\alpha f(r) = f(r) - \sum_{k=0}^{m-1} f^{(k)}(r) \frac{(r-a)^k}{k!}, r > 0$

The operator J_a^α satisfies the following properties: for $f \in C_\mu^m, \alpha, \beta > 0, \mu \geq -1, \gamma \geq -1$,

- (a) $J_a^\alpha f(r)$ exists for almost every $r \in [a, b]$.
- (b) $J_a^\alpha J_a^\beta f(r) = J_a^{\alpha+\beta} f(r)$.
- (c) $J_a^\alpha J_a^\beta f(r) = J_a^\beta J_a^\alpha f(r)$.
- (d) $J_a^\alpha (r-a)^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} (r-a)^{\alpha+\gamma}$.

3. Analysis of (OHAM) for system of fractional order partial differential equations

Consider the general fractional order differential system:

$$(2) \quad \begin{cases} \frac{\partial^\alpha u(r, t)}{\partial t} = A(u(r, t)) + f_1(r, t), \alpha > 0, \\ \frac{\partial^\alpha v(r, t)}{\partial t} = B(v(r, t)) + f_2(r, t), \alpha > 0. \end{cases}$$

Subject to the initial conditions

$$(3) \quad \begin{cases} D_0^{\alpha-k_1} u(r, 0) = h_{k_1}(r), k_1 = 0, 1, \dots, n-1, \\ D_0^{\alpha-n} u(r, 0) = 0, n = [\alpha], \\ D_0^{k_1} u(r, 0) = g_{k_1}(r), k_1 = 0, 1, \dots, n-1, \\ D_0^n u(r, 0) = 0, n = [\alpha], \\ D_0^{\beta-k_2} v(r, 0) = h_{k_2}(r), k_2 = 0, 1, \dots, n-1, \\ D_0^{\beta-n} v(r, 0) = 0, n = [\beta], \\ D_0^{k_2} v(r, 0) = g_{k_2}(r), k_2 = 0, 1, \dots, n-1, \\ D_0^n v(r, 0) = 0, n = [\beta], \end{cases}$$

where $\frac{\partial^\alpha}{\partial t^\alpha}$, $\frac{\partial^\beta}{\partial t^\beta}$ denote the Caputo or Riemann-Liouville fraction derivative operator, A and B are differential operators, $u(r, t)$ and $v(r, t)$ are unknown functions and $f_1(r, t)$, $f_2(r, t)$ are known analytic functions, r and t denote spatial and temporal independent variables respectively.

Construct an optimal Homotopy for fractional order system of partial differential equation, $\phi(r, t; p) : \Omega \times [0, 1] \rightarrow R$ which is

$$(4) \quad \begin{cases} (1-p) \left(\frac{\partial^\alpha u(r, t)}{\partial t^\alpha} - f_1(r, t) \right) \\ \quad - H_1(r, p) \left(\frac{\partial^\alpha u(r, t)}{\partial t^\alpha} - A(u(r, t)) + f_1(r, t) \right) = 0, \\ (1-p) \left(\frac{\partial^\beta v(r, t)}{\partial t^\beta} - f_2(r, t) \right) \\ \quad - H_2(r, p) \left(\frac{\partial^\beta v(r, t)}{\partial t^\beta} - B(v(r, t)) + f_2(r, t) \right) = 0, \end{cases}$$

where $r \in \Omega$ and $p \in [0, 1]$ is an embedding parameter, $H_1(r, p)$ and $H_2(r, p)$ are non-zero auxiliary functions for $p \neq 0$ and $H_1(r, 0) = H_2(r, 0) = 0$ when p increase in the interval $[0, 1]$ the solution $\phi(r, t)$ ensures a fast convergence to the exact solution. The efficiency of (OHAM), which does not need a small parameter in the equation, is based on the construction and determination of the auxiliary function combined with a convenient way to optimally control the convergence of the solution. The auxiliary function provides us with a simple way to adjust and control the convergence also increase the accuracy of the results and effectiveness of the method.

The auxiliary functions $H_1(r, p)$ and $H_2(r, p)$ can be written in the form:

$$(5) \quad \begin{cases} H_1(r, p) = pk_1(r, C_{1i}) + p^2k_2(r, C_{1i}) + \dots + p^mk_m(r, C_{1i}), \\ H_2(r, p) = pk_1(r, C_{2i}) + p^2k_2(r, C_{2i}) + \dots + p^mk_m(r, C_{2i}), \end{cases}$$

where C_{1i} and C_{2i} , $i = 1, 2, \dots$ are auxiliary convergence control parameters and $k_i(r)$, $i = 1, 2, \dots$ can be function on the variable r .

By expanding $\phi(r, t, p, C_i)$ in Taylor's series about p , to get approximate solutions as:

$$(6) \quad \begin{cases} u(r, t, C_i) = u_0(r, t) + \sum_{k_1=1}^m u_k(r, t, C_{1i}) p^{k_1}, i = 1, 2, \dots \\ v(r, t, C_i) = v_0(r, t) + \sum_{k_2=1}^m v_k(r, t, C_{2i}) p^{k_2}, i = 1, 2, \dots \end{cases}$$

It is clear if it convergence at $p = 1$, one has:

$$(7) \quad \begin{cases} u(r, t, C_i) = u_0(r, t) + \sum_{k_1=1}^m u_k(r, t, C_{1i}) \\ v(r, t, C_i) = v_0(r, t) + \sum_{k_2=1}^m v_k(r, t, C_{2i}). \end{cases}$$

Equating the coefficients of like powers of p after substituting Eq. (6) in Eq.(4), we get zero order, first order, second order and high order problems

$$(8) \quad \begin{cases} P^0 : \left\{ \begin{aligned} \frac{\partial^\alpha u_0(r, t)}{\partial t^\alpha} - f_1 &= 0, \frac{\partial^\beta v_0(r, t)}{\partial t^\beta} - f_2 = 0 \\ P^1 : \left\{ \begin{aligned} \frac{\partial^\alpha u_1(r, t, C_{11})}{\partial t^\alpha} \\ - (1 + C_{11}) \frac{\partial^\alpha u_0(r, t)}{\partial t^\alpha} + (1 + C_{11}) f_1 + C_{11} A(u_0(r, t)) &= 0, \\ \frac{\partial^\beta v_1(r, t, C_{21})}{\partial t^\beta} \\ - (1 + C_{21}) \frac{\partial^\beta v_0(r, t)}{\partial t^\beta} + (1 + C_{21}) f_2 + C_{21} B(v_0(r, t)) &= 0, \\ \dots\dots\dots \end{aligned} \end{cases} \end{cases}$$

These problems contain the time fractional derivatives. Therefore we apply the J^α, J^β operators on the above problems, and obtain a series of solutions as follow:

$$(9) \quad \begin{cases} u_0(r, t) = J^\alpha[f_1], \quad v_0(r, t) = J^\beta[f_2], \\ u_1(r, t; C_{11}) = J^\alpha \left[(1 + C_{11}) \frac{\partial^\alpha u_0(r, t)}{\partial t^\alpha} - (1 + C_{11}) f_1 - C_{11} A(u_0(r, t)) \right], \\ v_1(r, t; C_{21}) = J^\beta \left[(1 + C_{21}) \frac{\partial^\beta v_0(r, t)}{\partial t^\beta} - (1 + C_{21}) f_2 - C_{21} B(v_0(r, t)) \right], \\ \dots\dots\dots \end{cases}$$

By putting the above solutions in Eq. (7), one can get the approximate solution $u(r, t, C_{1i})$, and $v(r, t, C_{2i})$. The auxiliary convergence control parameters

$C_{11}, C_{12}, \dots$ and $C_{21}, C_{22}, \dots$ can be found by using either: Ritz method, least square method, collocation method, or Galerkins's method. In this paper, least square method is used to find the optimal values of auxiliary convergence control parameters in which we first construct the residuals

$$R_1^2(r, t; C_{in}) = \frac{\partial^\alpha u(r, t)}{\partial t^\alpha} - A(u(r, t)) + f_1(r, t), \alpha > 0,$$

$$R_2^2(r, t; C_{ni}) = \frac{\partial^\beta v(r, t)}{\partial t^\beta} - B(v(r, t)) + f_2(r, t), \beta > 0,$$

minimize the residuals by following relation

$$(10) \quad \begin{cases} \chi(C_{in}) = \int_0^t \int_\Omega R_1^2(r, t; C_{in}) dr dt, \\ \chi(C_{ni}) = \int_0^t \int_\Omega R_2^2(r, t; C_{ni}) dr dt \end{cases}$$

and then calculate the optimum values of auxiliary constants C_{1i} and C_{2i} by solving the following system of equations.

$$(11) \quad \begin{cases} \frac{\partial \chi}{\partial C_{11}} = \frac{\partial \chi}{\partial C_{12}} = \dots = \frac{\partial \chi}{\partial C_{1m}} = 0, \\ \frac{\partial \gamma}{\partial C_{21}} = \frac{\partial \gamma}{\partial C_{22}} = \dots = \frac{\partial \gamma}{\partial C_{2m}} = 0. \end{cases}$$

The convergence of (OHAM) [10] is based on the construction and determination of auxiliary functions $H_1(r, p)$ and $H_2(r, p)$. The solution $\phi(r, t)$ ensures a fast convergence to the exact solution as p increase in the interval $[0, 1]$. The auxiliary functions are combined with a convenient way to optimally control the convergence of the solution. They provide us with a simple way to adjust and control the convergence and also increase the accuracy of the results and effectiveness of the method.

Theorem 3.1. *If the series (6) convergence to $u(x, t), v(x, t)$, where $u_k(x, t), v_k(x, t) \in L(R^+)$ is produced by zero order problem and the K -order deformation, then $u(x, t), v(x, t)$ is the exact solution of (1).*

Proof. Since the series

$$(12) \quad \left\{ \sum_{k=1}^{\infty} u_{i,k}(r, t, C_1, c_2, \dots), \sum_{k=1}^{\infty} v_{i,k}(r, t, C_1, c_2, \dots) \right\}$$

converges, it can be written as

$$(13) \quad \begin{cases} \psi_{1i}(r, t) = \sum_{k=1}^{\infty} u_{1i,k}(r, t, C_1, c_2, \dots), \\ \psi_{2i}(r, t) = \sum_{k=1}^{\infty} v_{2i,k}(r, t, C_1, c_2, \dots) \end{cases}$$

and it holds that

$$(14) \quad \begin{cases} \lim_{k \rightarrow \infty} u_{1i,k}(r, t, C_1, c_2, \dots) = 0, \\ \lim_{k \rightarrow \infty} v_{2i,k}(r, t, C_1, c_2, \dots) = 0. \end{cases}$$

In fact the following equation is satisfied

$$(15) \quad \begin{cases} u_{i,1}(r, t, C_1) + \sum k = 2^n u_{i,k}(r, t, \vec{C}_k) - \sum_{k=2}^n u_{i,k-1}(r, t, \vec{C}_{k-1}), \\ = u_{i,2}(r, t, \vec{C}_2) - u_{i,1}(r, t, C_1) + \dots + u_{i,n}(r, t, \vec{C}_n) - u_{i,n-1}(r, t, \vec{C}_{n-1}) \\ = u_{i,n}(r, t, \vec{C}_n), \\ v_{i,1}(r, t, C_1) + \sum k = 2^n v_{i,k}(r, t, \vec{C}_k) - \sum_{k=2}^n v_{i,k-1}(r, t, \vec{C}_{k-1}), \\ = v_{i,2}(r, t, \vec{C}_2) - v_{i,1}(r, t, C_1) + \dots + v_{i,n}(r, t, \vec{C}_n) - v_{i,n-1}(r, t, \vec{C}_{n-1}) \\ = v_{i,n}(r, t, \vec{C}_n). \end{cases}$$

Now, we have

$$(16) \quad \begin{cases} L_{i,1}(u_{i,1}(r, t, C_1)) + \sum_{k=2}^{\infty} L_i(u_{i,k}(r, t, \vec{C}_k)) - \sum_{k=2}^{\infty} L_i(u_{i,k-1}(r, t, \vec{C}_{k-1})) \\ = L_i(u_{i,1}(r, t, C_1)) + \sum_{k=2}^{\infty} L_i(u_{i,k}(r, t, \vec{C}_k)) - \sum_{k=2}^{\infty} L_i(u_{i,k-1}(r, t, \vec{C}_{k-1})) = 0, \\ L_{i,1}(v_{i,1}(r, t, C_1)) + \sum_{k=2}^{\infty} L_i(v_{i,k}(r, t, \vec{C}_k)) - \sum_{k=2}^{\infty} L_i(v_{i,k-1}(r, t, \vec{C}_{k-1})) \\ = L_i(v_{i,1}(r, t, C_1)) + \sum_{k=2}^{\infty} L_i(v_{i,k}(r, t, \vec{C}_k)) - \sum_{k=2}^{\infty} L_i(v_{i,k-1}(r, t, \vec{C}_{k-1})) = 0 \end{cases}$$

which satisfies

$$(17) \quad \begin{cases} L_{i,1}(u_{i,1}(r, t, C_1)) + L_i\left(\sum_{k=2}^{\infty} u_{i,k}(r, t, \vec{C}_k)\right) - L_i\left(\sum_{k=2}^{\infty} u_{i,k-1}(r, t, \vec{C}_{k-1})\right) \\ = \sum_{k=2}^{\infty} C_m [L_i(u_{i,k-m}(r, t, \vec{C}_{k-m})) + N_{i,k-m}(u_{i,k-1}(r, t, C_{k-1}))] + f_{1i}(r, t) = 0, \\ L_{i,1}(v_{i,1}(r, t, C_1)) + L_i\left(\sum_{k=2}^{\infty} v_{i,k}(r, t, \vec{C}_k)\right) - L_i\left(\sum_{k=2}^{\infty} v_{i,k-1}(r, t, \vec{C}_{k-1})\right) \\ = \sum_{k=2}^{\infty} C_m [L_i(v_{i,k-m}(r, t, \vec{C}_{k-m})) + N_{i,k-m}(v_{i,k-1}(r, t, C_{k-1}))] + f_{2i}(r, t) = 0. \end{cases}$$

Now, if $C_m, m = 1, 2, \dots$ is properly chosen, then equation leads to

$$(18) \quad \begin{cases} L_i(u_i(r, t)) + A = 0, \\ L_i(v_i(r, t)) + B = 0, \end{cases}$$

which is the exact solution. $\square$

4. Applications

4.1 Time fractional Whitam-Broker-Kaup equation

Consider the time fractional WBK system (1) subject to initial conditions

$$u(x, 0) = \lambda - 2\beta k \coth(k\zeta), v(x, 0) = -2\beta(\beta + b)k^2 \operatorname{csch}^2(k\zeta),$$

where $\beta = \sqrt{a + b^2}$, $\zeta = x + c$, and λ, c, k are arbitrary constants. For $\alpha = 1$, the exact solution of the given system is

$$(19) \quad \begin{cases} u(x, t) = \lambda - 2\beta k \coth(k(\zeta - \lambda t)), \\ v(x, t) = -2\beta(\beta + b)k^2 \operatorname{csch}^2(k(\zeta - \lambda t)). \end{cases}$$

According the formulation discussed in section 3, we have the following zero and first order problems with initial conditions are:

Zero order problem:

$$(20) \quad \begin{cases} \frac{\partial^\alpha u_0(x, t)}{\partial t^\alpha} = 0, u_0(x, 0) = \lambda - 2\beta k \coth(k\zeta), \\ \frac{\partial^\alpha v_0(x, t)}{\partial t^\alpha} = 0, v_0(x, 0) = -2\beta(\beta + b)k^2 \operatorname{csch}^2(k\zeta). \end{cases}$$

First order problem:

$$(21) \quad \begin{cases} -\frac{\partial u_1(x, t)}{\partial t} - C_{12} \frac{\partial u_0(x, t)}{\partial t} - C_{12} u_0(x, t) \frac{\partial u_0(x, t)}{\partial x} \\ + C_{12} \frac{\partial v_0(x, t)}{\partial x} - b C_{12} \frac{\partial^2 u_0(x, t)}{\partial x^2} = 0, u_1(x, 0) = 0 \\ -\frac{\partial v_1(x, t)}{\partial t} - C_{11} \frac{\partial v_0(x, t)}{\partial t} \\ - C_{11} v_0(x, t) \frac{\partial v_0(x, t)}{\partial x} - C_{11} u_0(x, t) \frac{\partial v_0(x, t)}{\partial x} \\ - b C_{11} \frac{\partial^2 v_0(x, t)}{\partial x^2} - \& C_{11} a \frac{\partial^3 u_0(x, t)}{\partial x^3} = 0, \\ v_1(x, 0) = 0. \end{cases}$$

Using J^α both sides of eq. (20-21), then we have,

$$(22) \quad u_0(x, t) = \lambda - 2\beta k \coth(k\zeta),$$

$$(23) \quad v_0(x, t) = -2\beta(\beta + b)k^2 \operatorname{csch}^2(k\zeta),$$

$$(24) \quad u_1(x, t, C_{12}) = \frac{2C_{12}\beta k^2 t^\alpha \lambda \operatorname{csch}^2(k(x+c))}{\Gamma(1+\alpha)},$$

$$(25) \quad v_1(x, t, C_{11}) = \frac{1}{\Gamma(1+\alpha)} \left(2C_{11}\beta k^3 t^\alpha \operatorname{csch}^4(k(x+c)) \left(2k(b^2 - \beta^2 + a) \right. \right. \\ \left. \left. (2 + \cosh(2k(x+c))) + (b + \beta)\lambda \sinh(2k(x+c)) \right) \right).$$

For numerical calculation, we take $k = 0.1$, $\lambda = 0.005$, $a = b = 1.5$, $c = 10$ in the above problems.

The values of C_{11} and C_{12} , are found by using least square method, which are given below. $C_{12} = -1.000316679785577005$ and $C_{11} = -0.9985613944569582$.

4.2 Time fractional (ALW) equation

Consider the time fractional (WBK)

$$(26) \quad \begin{cases} D_t^\alpha u + uu_x + \frac{1}{2}u_{xx} + v_x = 0, & D_t^\alpha v + (uv)_x - \frac{1}{2}v_{xx} = 0. \end{cases}$$

Subject to initial conditions:

$$u(x, 0) = \lambda - k \coth(k\zeta), v(x, 0) = -k^2 \operatorname{csch}^2(k\zeta),$$

where $\zeta = x + c$, and λ, c, k , are arbitrary constants. For $\alpha = 1$, the exact solution of the given system is:

$$(27) \quad \{u(x, t) = \lambda - k \coth(k(\zeta - \lambda t)), \quad v(x, t) = -k^2 \operatorname{csch}^2(k(\zeta - \lambda t)).\}$$

According the formulation discussed in section 2 we have the following zero and 1st order problems with initial conditions are given as:

Zero order problem:

$$(28) \quad \begin{cases} \frac{\partial^\alpha u_0(x, t)}{\partial t^\alpha} = 0, u_0(x, 0) = \lambda - k \coth(k\zeta), \\ \frac{\partial^\alpha v_0(x, t)}{\partial t^\alpha} = 0, v_0(x, 0) = -k^2 \operatorname{csch}^2(k\zeta) \end{cases}$$

First order problem:

$$(29) \quad \begin{cases} -\frac{\partial u_0(x, t)}{\partial t} - C_{11} \frac{\partial u_0(x, t)}{\partial t} + \frac{\partial u_1(x, t)}{\partial t} - C_{11} u_0(x, t) \frac{\partial u_0(x, t)}{\partial x} \\ - C_{11} \frac{\partial v_0(x, t)}{\partial x} - \frac{1}{2} C_{11} \frac{\partial^2 u_0(x, t)}{\partial x^2} = 0, \\ u_1(x, 0) = 0 \end{cases}$$

$$(30) \quad \begin{cases} \frac{-\partial v_0(x, t)}{\partial t} - C_{12} \frac{\partial v_0(x, t)}{\partial t} + \frac{\partial v_1(x, t)}{\partial t} - C_1 v_0(x, t) \frac{\partial u_0(x, t)}{\partial x} \\ - C_1 u_0(x, t) \frac{\partial v_0(x, t)}{\partial x} + \frac{1}{2} C_1 \frac{\partial^2 v_0(x, t)}{\partial x^2} = 0, \\ v_1(x, 0) = 0 \end{cases}$$

Using the same procedure we have zero and 1st order problems solution are

$$(31) \quad u_0(x, t) = \lambda - k \coth(k\zeta),$$

$$(32) \quad v_0(x, t) = -k^2 \operatorname{csch}^2(k\zeta),$$

$$(33) \quad u_1(x, t, C_{11}) = \frac{C_{11} k^2 t^\alpha \lambda \operatorname{csch}^2(k(x+c))}{\Gamma(1+\alpha)},$$

$$(34) \quad v_1(x, t, C_1) = \frac{-1}{\Gamma(1+\alpha)} \left(C_1 k^3 t^\alpha \operatorname{csch}^4(k(x+c)) \left(2k \right. \right. \\ \left. \left. + \left(k \cosh(2k(x+c)) - \lambda \sinh(2k(x+c)) \right) \right) \right)$$

We take $k = 0.1, \lambda = 0.005, a = b = 1.5, c = 10$. To find optimum values we used collection method. $C_1 = 0.026058512796348854, C_{11} = -1.0008938281135809$

5. Results and discussion

(OHAM) formulation is tested upon the fractional (WBK) and (ALW) equations. We have used mathematica 7 for most of our computational work.

Tables 1-2 shows comparison of absolute errors of 1st order (OHAM) solution with (ADM) and (VIM) solutions of $u(x, t)$ and $v(x, t)$ for (WBK) system at $\alpha = 1$. In Table 3 and 4 the results obtained by 1st order approximation of proposed method for the (ALW) equation are compared with (ADM) and (VIM).

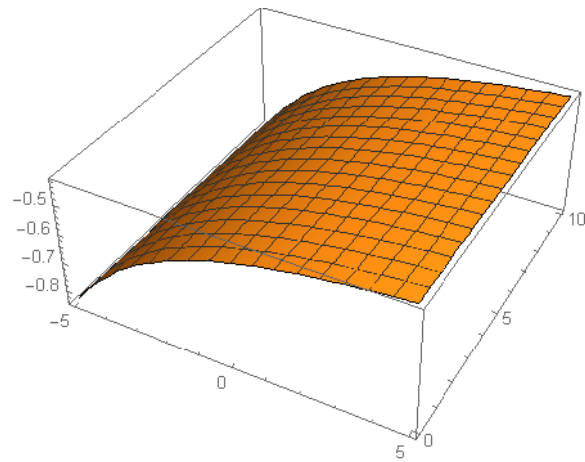
Figures 1-2 show the 3D plots of exact verses approximate solutions of $u(x, t)$ and $v(x, t)$ for (WBK) equation. Figures 3 and 4 show the 2D plots of 1st order approximate solution by proposed method for $u(x, t)$ and $v(x, t)$ of (WBK) equation. Figures 5-6 show the 3D plots of exact verses approximate solutions of $u(x, t)$ and $v(x, t)$ for (ALW) equation. Figures 7 and 8 show the 2D plots of 1st order approximate solution by proposed method for $u(x, t)$ and $v(x, t)$ of (ALW) equation.

Table 1: Comparison of 1st order (OHAM) solution of $u(x, t)$ at $\alpha = 1$, with (ADM) and (VIM)

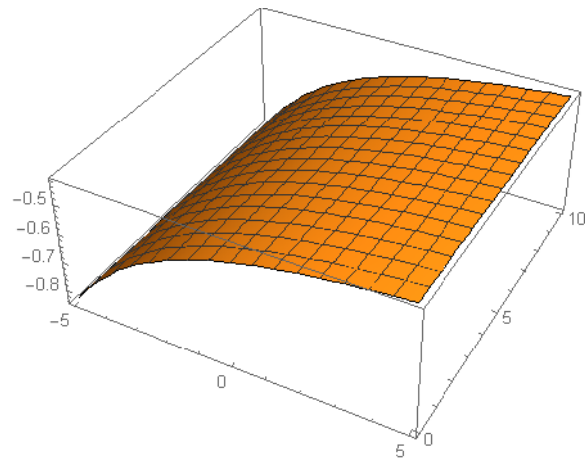
(x, t)	Absolute Error (ADM) [1]	Absolute Error (VIM) [2]	Absolute Error (OHAM)
(0.1,0.1)	1.04892×10^{-4}	1.23033×10^{-4}	3.43361×10^{-9}
(0.1,0.3)	9.64474×10^{-5}	3.69597×10^{-4}	4.9484×10^{-9}
(0.1,0.5)	8.88312×10^{-5}	6.16873×10^{-4}	6.7427×10^{-10}
(0.2,0.1)	4.25408×10^{-4}	1.19869×10^{-4}	3.34996×10^{-9}
(0.2,0.3)	3.91098×10^{-4}	3.60098×10^{-4}	4.86307×10^{-9}
(0.2,0.5)	3.60161×10^{-4}	6.01006×10^{-4}	5.41391×10^{-10}
(0.3,0.1)	9.71922×10^{-4}	1.16789×10^{-4}	3.26868×10^{-9}
(0.3,0.3)	8.93309×10^{-4}	3.50866×10^{-4}	4.77828×10^{-9}
(0.3,0.5)	8.22452×10^{-4}	5.85610×10^{-4}	4.17512×10^{-10}
(0.4,0.1)	1.75596×10^{-3}	1.13829×10^{-4}	3.18968×10^{-9}
(0.4,0.3)	1.61430×10^{-3}	3.41948×10^{-4}	4.6943×10^{-9}
(0.4,0.5)	1.48578×10^{-3}	5.70710×10^{-4}	3.02393×10^{-10}
(0.5,0.1)	2.79519×10^{-3}	1.10936×10^{-4}	3.11288×10^{-9}
(0.5,0.3)	2.56714×10^{-3}	3.33274×10^{-4}	4.61117×10^{-9}
(0.5,0.5)	2.36148×10^{-3}	5.56235×10^{-4}	1.95473×10^{-10}

Table 2: Comparison of 1st order (OHAM) solution of $v(x, t)$ at $\alpha = 1$, with (ADM) and (VIM)

(x, t)	Absolute Error (ADM) [1]	Absolute Error (VIM) [2]	Absolute Error (OHAM)
(0.1,0.1)	6.41419×10^{-3}	1.10430×10^{-4}	1.75426×10^{-8}
(0.1,0.3)	5.99783×10^{-3}	3.31865×10^{-4}	4.68301×10^{-8}
(0.1,0.5)	5.61507×10^{-3}	5.54071×10^{-4}	6.8385×10^{-8}
(0.2,0.1)	1.33181×10^{-2}	1.07016×10^{-4}	1.70078×10^{-8}
(0.2,0.3)	1.24441×10^{-2}	3.21601×10^{-4}	4.54496×10^{-8}
(0.2,0.5)	1.16416×10^{-2}	5.36927×10^{-4}	6.64572×10^{-8}
(0.3,0.1)	2.07641×10^{-2}	1.03737×10^{-4}	1.64933×10^{-8}
(0.3,0.3)	1.93852×10^{-2}	3.11737×10^{-4}	4.41191×10^{-8}
(0.3,0.5)	1.81209×10^{-2}	5.20447×10^{-4}	6.45949×10^{-8}
(0.4,0.1)	2.88100×10^{-2}	1.00579×10^{-4}	1.5998×10^{-8}
(0.4,0.3)	2.68724×10^{-2}	3.02245×10^{-4}	4.28363×10^{-8}
(0.4,0.5)	2.50985×10^{-2}	5.04593×10^{-4}	6.27954×10^{-8}
(0.5,0.1)	3.75193×10^{-2}	9.75385×10^{-5}	1.55211×10^{-8}
(0.5,0.3)	3.49617×10^{-2}	2.93107×10^{-4}	4.15991×10^{-8}
(0.5,0.5)	3.26339×10^{-2}	4.89335×10^{-4}	6.10561×10^{-8}

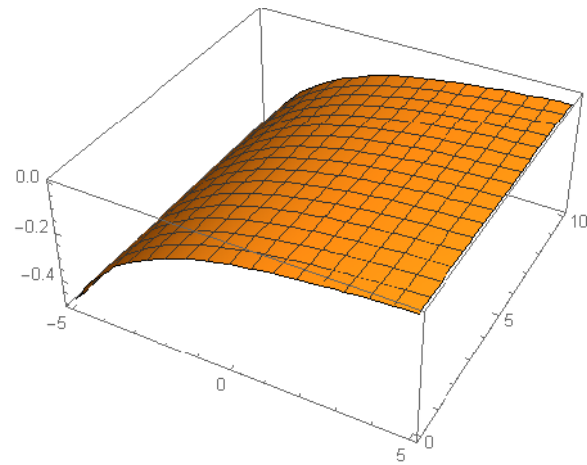


(a) 1st order approximate solution $u(x, t)$ obtained by (OHAM) for eq. (10)

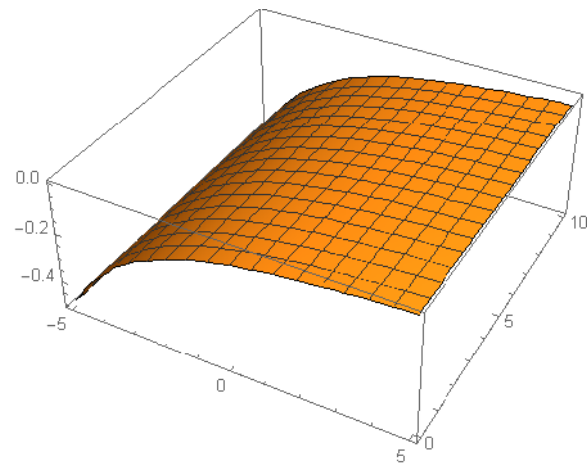


(b) Exact solution of $u(x, t)$ for eq. (10)

Figure 1: The exact and approximate solution for $u(x, t)$

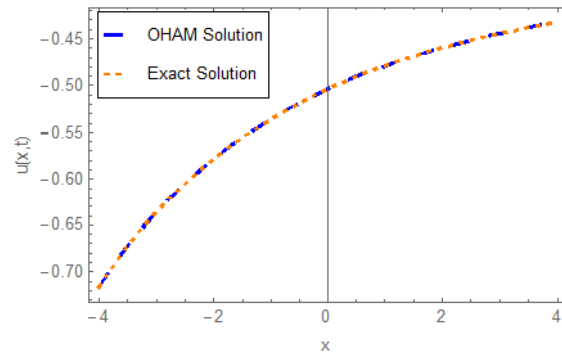


(a) 1st order approximate solution $v(x,t)$ obtained by (OHAM) for eq. (10)

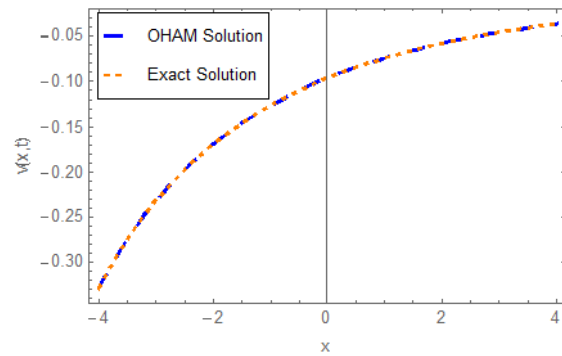


(b) Exact solution of $v(x,t)$ for eq. (10)

Figure 2: The Exact and Approximate solution for $v(x,t)$



(a) 2D plots of 1st order (OHAM) verses exact solution for $u(x, t)$ when $-4 \leq x \leq 4$ at $t = 0.1$



(b) 2D plots of 1st order (OHAM) verses exact solution for $v(x, t)$ when $-4 \leq x \leq 4$ at $t = 0.1$

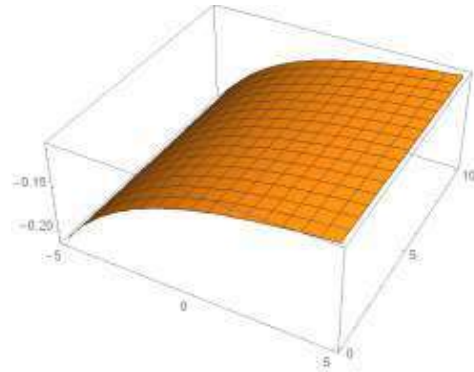
Figure 3: 2D plots of 1st order (OHAM) verses exact solution for $u(x, t), v(x, t)$

Table 3: Comparison of 1st order (OHAM) solution of $u(x, t)$ at $\alpha = 1$, with (ADM) and (VIM)

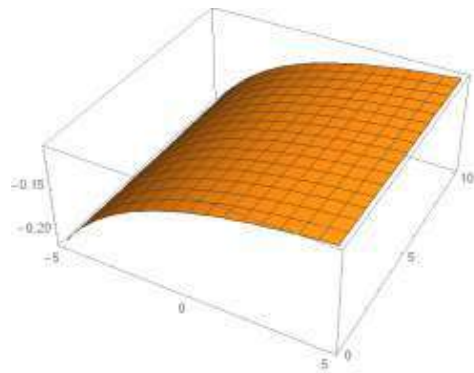
(x, t)	Absolute Error (ADM) [1]	Absolute Error (VIM) [2]	Absolute Error (OHAM)
(0.1,0.1)	8.02989×10^{-6}	3.17634×10^{-5}	3.43361×10^{-9}
(0.1,0.3)	7.38281×10^{-6}	9.54273×10^{-5}	4.9484×10^{-9}
(0.1,0.5)	6.79923×10^{-6}	1.59274×10^{-4}	6.7427×10^{-10}
(0.2,0.1)	3.23228×10^{-5}	3.09466×10^{-5}	3.34996×10^{-9}
(0.2,0.3)	2.97172×10^{-5}	9.29725×10^{-5}	4.86307×10^{-9}
(0.2,0.5)	2.73673×10^{-5}	1.55176×10^{-4}	5.41391×10^{-10}
(0.3,0.1)	7.32051×10^{-5}	3.01549×10^{-5}	3.26868×10^{-9}
(0.3,0.3)	6.73006×10^{-5}	9.05935×10^{-5}	4.77828×10^{-9}
(0.3,0.5)	6.19760×10^{-5}	1.51204×10^{-4}	4.17512×10^{-10}
(0.4,0.1)	1.31032×10^{-4}	2.93874×10^{-5}	3.18968×10^{-9}
(0.4,0.3)	1.20455×10^{-4}	8.82871×10^{-5}	4.6943×10^{-9}
(0.4,0.5)	1.10919×10^{-4}	1.47354×10^{-4}	3.02393×10^{-10}
(0.5,0.1)	2.06186×10^{-4}	2.86433×10^{-5}	3.11288×10^{-9}
(0.5,0.3)	1.89528×10^{-4}	8.60509×10^{-5}	4.61117×10^{-9}
(0.5,0.5)	1.74510×10^{-4}	1.43620×10^{-4}	1.95473×10^{-10}

Table 4: Comparison of 1st order (OHAM) solution of $v(x, t)$ at $\alpha = 1$, with (ADM) and (VIM)

(x, t)	Absolute Error (ADM) [1]	Absolute Error (VIM) [2]	Absolute Error (OHAM)
(0.1,0.1)	4.81902×10^{-4}	8.29712×10^{-6}	1.75426×10^{-8}
(0.1,0.3)	4.50818×10^{-4}	2.49346×10^{-5}	4.68301×10^{-8}
(0.1,0.5)	4.22221×10^{-4}	4.16299×10^{-5}	6.8385×10^{-8}
(0.2,0.1)	9.76644×10^{-4}	8.04063×10^{-6}	1.70078×10^{-8}
(0.2,0.3)	9.13502×10^{-4}	2.41634×10^{-5}	4.5496×10^{-8}
(0.2,0.5)	8.55426×10^{-4}	4.03419×10^{-5}	6.64572×10^{-8}
(0.3,0.1)	1.48482×10^{-3}	7.79401×10^{-6}	1.64933×10^{-8}
(0.3,0.3)	1.38858×10^{-3}	2.34220×10^{-5}	4.41191×10^{-8}
(0.3,0.5)	1.30009×10^{-3}	3.91034×10^{-5}	6.45949×10^{-8}
(0.4,0.1)	2.00705×10^{-3}	7.55675×10^{-6}	1.5998×10^{-8}
(0.4,0.3)	1.87661×10^{-3}	2.27087×10^{-5}	4.28363×10^{-8}
(0.4,0.5)	1.75670×10^{-3}	3.79121×10^{-5}	6.27954×10^{-8}
(0.5,0.1)	2.54396×10^{-3}	7.32847×10^{-6}	1.55211×10^{-8}
(0.5,0.3)	2.37815×10^{-3}	2.20224×10^{-5}	4.15991×10^{-8}
(0.5,0.5)	2.22578×10^{-3}	3.67658×10^{-5}	6.10561×10^{-8}

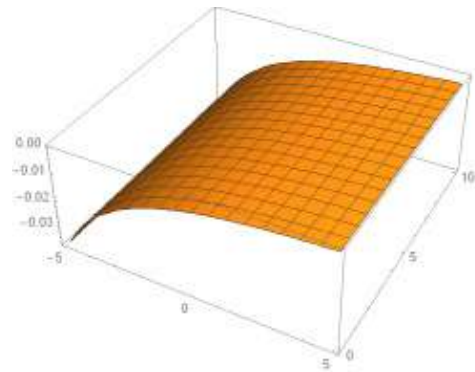


(a) 1st order approximate solution $u(x, t)$ obtained by (OHAM) for eq. (26)

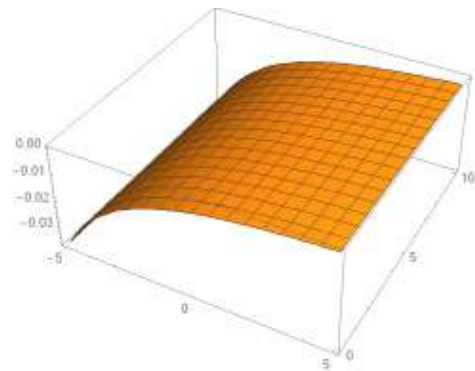


(b) Exact solution of $u(x, t)$ for eq. (26)

Figure 4: The Exact and Approximate solution for $u(x, t)$

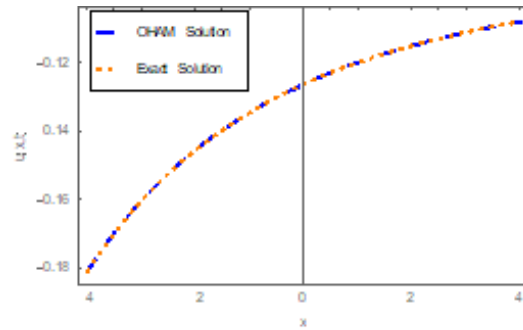


(a) 1st order approximate solution $v(x, t)$ obtained by (OHAM) for eq. (26)

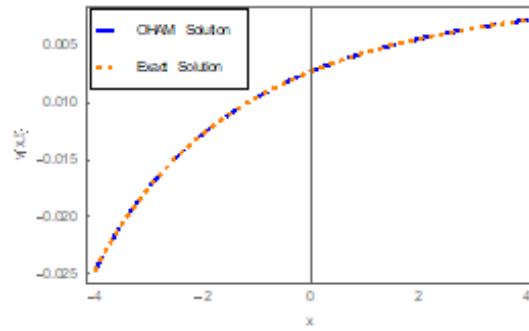


(b) Exact solution of $v(x, t)$ for eq. (26)

Figure 5: The Exact and Approximate solution for $v(x, t)$



(a) 2D plots of 1st order (OHAM) verses exact solution for $u(x, t)$ when $-4 \leq x \leq 4$ at $t = 0.1$



(b) 2D plots of 1st order (OHAM) verses exact solution for $v(x, t)$ when $-4 \leq x \leq 4$ at $t = 0.1$

Figure 6: 2D plots of 1st order (OHAM) verses exact solution for $u(x, t), v(x, t)$

6. Conclusion

(OHAM) converges rapidly to the exact solution at lower order of approximations for (WBK) system. The results obtained by proposed method are very encouraging in comparison with (ADM) and (VIM). As a result it would be more appealing for researchers to apply this method for solving systems of non-linear fractional order partial differential equations in different fields of science especially in fluid dynamics and physics. The accuracy of method can further be increased by taking higher order of approximations of the proposed method.

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Weakly ω -continuous functions in bitopological spaces

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Abstract. In this paper, as a generalization of u - ω -continuous functions, we introduce the notion of weakly ω -continuous functions in bitopological spaces and obtain several characterizations and some of its properties.

Keywords: bitopological spaces, u - ω -open sets, weakly continuous function.

1. Introduction

The concept of bitopological spaces was first introduced by Kelly [13]. After the introduction of the definition of a bitopological space by Kelly, a large number of

*. Corresponding auhtor

topologists have turned their attention to the generalization of different concepts used in topological space. Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real analysis concerns the various modified forms of continuity, separation axioms etc. by utilizing generalized open sets. Recently, as generalization of closed sets, the notion of ω -closed sets were introduced and studied by Hdeib [11]. Several characterizations and properties of ω -closed sets were provided in [4, 2, 3, 11, 12, 16]. In this paper, we introduce and study (i, j) -weakly ω -continuous functions on bitopological space and obtain several characterizations and its relation with another (i, j) - ω -continuous functions-

2. Preliminaries

Throughout this paper, (X, τ_1, τ_2) and (Y, σ_1, σ_2) always mean bitopological spaces in which no separation axioms are assumed unless explicitly stated. For a subset A of (X, τ) , $i\text{Cl}(A)$ and $i\text{Int}(A)$ denote the closure of A with respect to τ_i and the interior of A with respect to τ_i , respectively. A point $x \in X$ is called a condensation point of A if for each $U \in \tau$ with $x \in U$, the set $U \cap A$ is uncountable. A is said to be ω -closed [11] if it contains all its condensation points. The complement of an ω -closed set is said to be an ω -open set. It is well known that a subset W of a space (X, τ) is ω -open if and only if for each $x \in W$, there exists $U \in \tau$ such that $x \in U$ and $U \setminus W$ is countable. The family of all ω -open subsets of a topological space (X, τ) forms a topology on X finer than τ . The intersection of all ω -closed sets containing A is called the ω -closure [11] of A and is denoted by $\omega\text{Cl}(A)$. The family of all ω -open sets of X is denoted by $\omega(\tau)$.

Definition 2.1. A subset A of a bitopological space (X, τ_1, τ_2) is said to be:

1. (i, j) -preopen [14] if $A \subset i\text{Int}(j\text{Cl}(A))$;
2. (i, j) -regular open [8] if $A \subset j\text{Cl}(i\text{Int}(A))$;

On each definition above, $i, j = 1, 2$ and $i \neq j$.

The complement of an (i, j) -preopen (resp. (i, j) -regular open) set is called an (i, j) -preclosed (resp. (i, j) -regular closed) set. The family of all (i, j) -preopen (resp. (i, j) -regular open, (i, j) -preclosed, (i, j) -regular closed) sets of (X, τ_1, τ_2) is denote by $(i, j)\text{-}PO(X)$ (resp. $(i, j)\text{-}RO(X)$, $(i, j)\text{-}PC(X)$, $(i, j)\text{-}RC(X)$).

Definition 2.2 ([1]). Let (X, τ_1, τ_2) be a bitopological space and let $A \subset X$. Then

1. A is said to be u - ω -open in (X, τ_1, τ_2) if $A \in \omega(\tau_1) \cup \omega(\tau_2)$,

2. A is said to be u - ω -closed in (X, τ_1, τ_2) if $X - A$ is u - ω -open in (X, τ_1, τ_2) .

The family of all u - ω -open sets in (X, τ_1, τ_2) is denoted by $\omega(\tau_1, \tau_2)$, and the family of all u - ω -closed sets in (X, τ_1, τ_2) is denoted by $\omega C(\tau_1, \tau_2)$.

Definition 2.3. 1. The u - ω -closure of A in (X, τ_1, τ_2) is denoted by (τ_1, τ_2) - $\omega \text{Cl}(A)$ and defined as follows: (τ_1, τ_2) - $\omega \text{Cl}(A) = \omega \text{Cl}_{\tau_1}(A) \cap \omega \text{Cl}_{\tau_2}(A)$.

2. The u - ω -interior of A in (X, τ_1, τ_2) is denoted by (τ_1, τ_2) - $\omega \text{Int}(A)$ and defined as follows: (τ_1, τ_2) - $\omega \text{Int}(A) = \omega \text{Int}_{\tau_1}(A) \cup \omega \text{Int}_{\tau_2}(A)$.

Lemma 2.4. Let (X, τ_1, τ_2) be a bitopological space and A a subset of X . Then

1. (τ_1, τ_2) - $\omega \text{Int}(A)$ is u - ω -open;
2. (τ_1, τ_2) - $\omega \text{Cl}(A)$ is u - ω -closed;
3. A is u - ω -open if and only if $A = (\tau_1, \tau_2)$ - $\omega \text{Int}(A)$;
4. A is u - ω -closed if and only if $A = (\tau_1, \tau_2)$ - $\omega \text{Cl}(A)$;
5. (τ_1, τ_2) - $\omega \text{Int}(X \setminus A) = X \setminus (\tau_1, \tau_2)$ - $\omega \text{Cl}(A)$;
6. (τ_1, τ_2) - $\omega \text{Cl}(X \setminus A) = X \setminus (\tau_1, \tau_2)$ - $\omega \text{Int}(A)$.

Lemma 2.5. Let (X, τ_1, τ_2) be a bitopological space and $A \subset X$. A point $x \in (\tau_1, \tau_2)$ - $\omega \text{Cl}(A)$ if and only if $U \cap A \neq \emptyset$ for every u - ω -open set U of (X, τ_1, τ_2) containing x .

Definition 2.6 ([9]). A subset A of X is said to be (i, j) - θ -closed if $A = (i, j)$ - $\text{Cl}_\theta(A)$. A subset A of X is said to be (i, j) - θ -open if $X \setminus A$ is (i, j) - θ -closed. The (i, j) - θ -interior of A , denoted by (i, j) - $\text{Int}_\theta(A)$, is defined as the union of all (i, j) - θ -open sets contained in A . Hence $x \in (i, j)$ - $\text{Int}_\theta(A)$ if and only if there exists a τ_i -open set U containing x such that $x \in U \subset j \text{Cl}(U) \subset A$.

Lemma 2.7 ([9]). Let (X, τ_1, τ_2) be a bitopological space and A a subset of X . Then

1. (i, j) - $\text{Int}_\theta(X \setminus A) = X \setminus (i, j)$ - $\text{Cl}_\theta(A)$;
2. (i, j) - $\text{Cl}_\theta(X \setminus A) = X \setminus (i, j)$ - $\text{Int}_\theta(A)$.

Lemma 2.8 ([9]). Let (X, τ_1, τ_2) be a bitopological space. If U is a τ_j -open set of X , then (i, j) - $\text{Cl}_\theta(U) = i \text{Cl}(U)$.

Definition 2.9 ([1]). A function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is said to be u - ω -continuous if for each $x \in X$ and each $V \in \sigma_1 \cup \sigma_2$ containing $f(x)$, there exists an u - ω -open set U containing x such that $f(U) \subset V$.

3. Properties of (i, j) -weakly ω -continuous functions

Definition 3.1. A function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is said to be (i, j) -weakly ω -continuous if for each $x \in X$ and each σ_i -open set V of Y containing $f(x)$, there exists an u - ω -open set U containing x such that $f(U) \subset j \text{ Cl}(V)$.

Proposition 3.2. Every u - ω -continuous function is (i, j) -weakly ω -continuous.

Proof. Straightforward. $\square$

The following example shows that the converse of Proposition 3.2 is not true in general.

Example 3.3. Take $X = \mathbb{R}$ and two topologies $\tau_1 = \tau_2 = \{\emptyset, \mathbb{R}, \mathbb{R} \setminus \mathbb{Q}\}$ and $Y = \{a, b, c\}$ with topologies $\sigma_1 = \sigma_2 = \{\emptyset, Y, \{a\}, \{a, b\}\}$. Define $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ as

$$f(x) = \begin{cases} a, & \text{if } x \in \mathbb{Q}, \\ b, & \text{if } x \notin \mathbb{Q} \end{cases}$$

It is easy to see that f is (i, j) -weakly ω -continuous but does not is u - ω -continuous function.

Theorem 3.4. For a function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$, the following properties are equivalent;

1. f is (i, j) -weakly ω -continuous;
2. $(\tau_1, \tau_2)\text{-}\omega \text{ Cl}(f^{-1}(j \text{ Int}(i \text{ Cl}(B)))) \subset f^{-1}(i \text{ Cl}(B))$ for every subset B of Y ;
3. $(\tau_1, \tau_2)\text{-}\omega \text{ Cl}(f^{-1}(j \text{ Int}(F))) \subset f^{-1}(F)$ for every (i, j) -regular closed set F of Y ;
4. $(\tau_1, \tau_2)\text{-}\omega \text{ Cl}(f^{-1}(V)) \subset f^{-1}(i \text{ Cl}(V))$ for every σ_j -open set V of Y ;
5. $f^{-1}(V) \subset (\tau_1, \tau_2)\text{-}\omega \text{ Int}(f^{-1}(j \text{ Cl}(V)))$ for every σ_i -open set V of Y .

Proof. (1) $\Rightarrow$ (2): Let B be any subset of Y and $x \in X \setminus f^{-1}(i \text{ Cl}(B))$. Then, $f(x) \in Y \setminus i \text{ Cl}(B)$ and there exists a σ_i -open set V of Y containing $f(x)$ such that $V \cap B = \emptyset$. Therefore, $V \cap j \text{ Int}(i \text{ Cl}(B)) = \emptyset$ and hence $j \text{ Cl}(V) \cap j \text{ Int}(i \text{ Cl}(B)) = \emptyset$. Therefore, there exists an u - ω -open set U containing x such that $f(U) \subset j \text{ Cl}(V)$. Hence $U \cap f^{-1}(j \text{ Int}(i \text{ Cl}(B))) = \emptyset$ and $x \in X \setminus (\tau_1, \tau_2)\text{-}\omega \text{ Cl}(f^{-1}(j \text{ Int}(i \text{ Cl}(B))))$. Thus, $(\tau_1, \tau_2)\text{-}\omega \text{ Cl}(f^{-1}(j \text{ Int}(i \text{ Cl}(B)))) \subset f^{-1}(i \text{ Cl}(B))$.

(2) $\Rightarrow$ (3): Let F be an (i, j) -regular closed set of Y .

Then, we have $F = i \text{ Cl}(j \text{ Int}(F))$ and $(\tau_1, \tau_2)\text{-}\omega \text{ Cl}(f^{-1}(j \text{ Int}(F))) = (\tau_1, \tau_2)\text{-}\omega \text{ Cl}(f^{-1}(j \text{ Int}(i \text{ Cl}(j \text{ Int}(F)))) \subset f^{-1}(i \text{ Cl}(j \text{ Int}(F))) = f^{-1}(F)$.

(3) $\Rightarrow$ (4): Let V be a σ_j -open set of Y . Then $i \text{ Cl}(V)$ is (i, j) -regular closed. Then $(\tau_1, \tau_2)\text{-}\omega \text{ Cl}(f^{-1}(V)) \subset (\tau_1, \tau_2)\text{-}\omega \text{ Cl}(f^{-1}(j \text{ Int}(i \text{ Cl}(V)))) \subset f^{-1}(i \text{ Cl}(V))$.

(4) $\Rightarrow$ (5): Let V be a σ_i -open set of Y . Then $Y \setminus j\text{Cl}(V)$ is σ_j -open and we have $(\tau_1, \tau_2)\text{-}\omega\text{Cl}(f^{-1}(Y \setminus j\text{Cl}(V))) \subset f^{-1}(i\text{Cl}(Y \setminus j\text{Cl}(V)))$ and $X \setminus (\tau_1, \tau_2)\text{-}\omega\text{Int}(f^{-1}(j\text{Cl}(V))) \subset X \setminus f^{-1}(i\text{Int}(j\text{Cl}(V))) \subset X \setminus f^{-1}(V)$. Therefore, we obtain $f^{-1}(V) \subset (\tau_1, \tau_2)\text{-}\omega\text{Int}(f^{-1}(j\text{Cl}(V)))$.

(5) $\Rightarrow$ (1): Let $x \in X$ and V be a σ_i -open set containing $f(x)$. We have $x \in f^{-1}(V) \subset (\tau_1, \tau_2)\text{-}\omega\text{Int}(f^{-1}(j\text{Cl}(V)))$. Put $U = (\tau_1, \tau_2)\text{-}\omega\text{Int}(f^{-1}(j\text{Cl}(V)))$. Then U is an u - ω -open set containing x and $f(U) \subset j\text{Cl}(V)$. This shows that f is (i, j) -weakly ω -continuous. $\square$

Theorem 3.5. *For a function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

1. f is (i, j) -weakly ω -continuous;
2. $f((\tau_1, \tau_2)\text{-}\omega\text{Cl}(A)) \subset (i, j)\text{-Cl}_\theta(f(A))$ for every subset A of X ;
3. $(\tau_1, \tau_2)\text{-}\omega\text{Cl}(f^{-1}(B)) \subset (f^{-1}(i, j)\text{-Cl}_\theta(B))$ for every subset B of Y ;
4. $(\tau_1, \tau_2)\text{-}\omega\text{Cl}(f^{-1}(j\text{Int}((i, j)\text{-Cl}_\theta(B)))) \subset f^{-1}((i, j)\text{-Cl}_\theta(B))$ for every subset B of Y .

Proof. (1) $\Rightarrow$ (2): Assume that f is (i, j) -weakly ω -continuous. Let A be any subset of X , $x \in (\tau_1, \tau_2)\text{-}\omega\text{Cl}(A)$ and V be a σ_i -open set of Y containing $f(x)$. Then, there exists an u - ω -open set U containing x such that $f(U) \subset j\text{Cl}(V)$. Since $x \in (\tau_1, \tau_2)\text{-}\omega\text{Cl}(A)$, $U \cap A \neq \emptyset$ and hence $\emptyset \neq f(U) \cap f(A) \subset j\text{Cl}(V) \cap f(A)$. Therefore, we obtain $f(x) \in (i, j)\text{-Cl}_\theta(f(A))$.

(2) $\Rightarrow$ (3): Let B be any subset of Y . Then $f((\tau_1, \tau_2)\text{-}\omega\text{Cl}(f^{-1}(B))) \subset (i, j)\text{-Cl}_\theta(f(f^{-1}(B))) \subset (i, j)\text{-Cl}_\theta(B)$ and hence $(\tau_1, \tau_2)\text{-}\omega\text{Cl}(f^{-1}(B)) \subset f^{-1}((i, j)\text{-Cl}_\theta(B))$.

(3) $\Rightarrow$ (4): Let B be any subset of Y . Since $(i, j)\text{-Cl}_\theta(B)$ is σ_i -closed in Y , by Lemma 2.8, $(\tau_1, \tau_2)\text{-}\omega\text{Cl}(f^{-1}(j\text{Int}((i, j)\text{-Cl}_\theta(B)))) \subset f^{-1}((i, j)\text{-Cl}_\theta(j\text{Int}((i, j)\text{-Cl}_\theta(B)))) = f^{-1}(i\text{Cl}(j\text{Int}((i, j)\text{-Cl}_\theta(B)))) \subset f^{-1}(i\text{Cl}(i, j)\text{-Cl}_\theta(B)) = f^{-1}((i, j)\text{-Cl}_\theta(B))$.

(4) $\Rightarrow$ (1): Let V be any σ_j -open set of Y . Then by Lemma 2.8, $V \subset j\text{Int}(i\text{Cl}(V)) = j\text{Int}((i, j)\text{-Cl}_\theta(V))$ and we have $(\tau_1, \tau_2)\text{-}\omega\text{Cl}(f^{-1}(V)) \subset (\tau_1, \tau_2)\text{-}\omega\text{Cl}(f^{-1}(j\text{Int}((i, j)\text{-Cl}_\theta(V)))) \subset f^{-1}((i, j)\text{-Cl}_\theta(V)) = f^{-1}(i\text{Cl}(V))$. Thus we obtain $(\tau_1, \tau_2)\text{-}\omega\text{Cl}(f^{-1}(V)) \subset f^{-1}(i\text{Cl}(V))$. It follows from Theorem 3.4, that f is (i, j) -weakly ω -continuous. $\square$

Theorem 3.6. *For a function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

1. f is (i, j) -weakly ω -continuous;
2. $(\tau_1, \tau_2)\text{-}\omega\text{Cl}(f^{-1}(V)) \subset f^{-1}(i\text{Cl}(V))$ for every (j, i) -preopen set V of Y ;
3. $f^{-1}(V) \subset (\tau_1, \tau_2)\text{-}\omega\text{Int}(f^{-1}(j\text{Cl}(V)))$ for every (i, j) -preopen set V of Y .

Proof. (1) $\Rightarrow$ (2): Let V be any (j, i) -preopen set of Y . Suppose that $x \notin f^{-1}(i \text{Cl}(V))$. Then there exists a σ_i -open set W containing $f(x)$ such that $W \cap V = \emptyset$. Hence we have $i \text{Cl}(W \cap V) = \emptyset$. Since V is (j, i) -preopen, we have $V \cap j \text{Cl}(W) \subset j \text{Int}(i \text{Cl}(V)) \cap j \text{Cl}(W) \subset j \text{Cl}(j \text{Int}(i \text{Cl}(V)) \cap W) \subset j \text{Cl}(i \text{Cl}(V)) \cap W \subset j \text{Cl}(i \text{Cl}(V \cap W)) = \emptyset$. Since f is (i, j) -weakly ω -continuous and W is a σ_i -open set containing $f(x)$, there exists an u - ω -open set U of X containing x such that $f(U) \subset j \text{Cl}(W)$. Then $f(U) \cap V = \emptyset$ and hence $U \cap f^{-1}(V) = \emptyset$. This shows that $x \notin (\tau_1, \tau_2)\text{-}\omega \text{Cl}(f^{-1}(V))$. Therefore, we obtain $(\tau_1, \tau_2)\text{-}\omega \text{Cl}(f^{-1}(V)) \subset f^{-1}(i \text{Cl}(V))$.

(2) $\Rightarrow$ (3): Let V be any (i, j) -preopen set of Y . By (2), we have $f^{-1}(V) \subset f^{-1}(i \text{Int}(j \text{Cl}(V))) = X \setminus f^{-1}(i \text{Cl}(Y \setminus j \text{Cl}(V))) \subset X \setminus ((\tau_1, \tau_2)\text{-}\omega \text{Cl}(f^{-1}(Y \setminus j \text{Cl}(V)))) = (\tau_1, \tau_2)\text{-}\omega \text{Int}(f^{-1}(j \text{Cl}(V)))$.

(3) $\Rightarrow$ (1): Let V be any σ_i -open set of Y . Then V is (i, j) -preopen set in Y and $f^{-1}(V) \subset (\tau_1, \tau_2)\text{-}\omega \text{Int}(f^{-1}(j \text{Cl}(V)))$. By Theorem 3.4, f is (i, j) -weakly ω -continuous. $\square$

Lemma 3.7. *If $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is (i, j) -weakly ω -continuous and $g : (Y, \sigma_1, \sigma_2) \rightarrow (Z, \eta_1, \eta_2)$ is pairwise continuous, then the composition $g \circ f : (X, \tau_1, \tau_2) \rightarrow (Z, \eta_1, \eta_2)$ is (i, j) -weakly ω -continuous.*

Proof. Let $x \in X$ and W be an η_i -open set of Z containing $g(f(x))$. Then, $g^{-1}(W)$ is a σ_i -open set of Y containing $f(x)$ and there exists an u - ω -open set U of X containing x such that $f(U) \subset j \text{Cl}(g^{-1}(W))$. Since g is pairwise continuous, we obtain $(g \circ f)(U) \subset g(j \text{Cl}(g^{-1}(W))) \subset g(g^{-1}(j \text{Cl}(W))) \subset j \text{Cl}(W)$. $\square$

Definition 3.8. *A bitopological space (X, τ_1, τ_2) is said to be (i, j) -regular [13] if for each $x \in X$ and each τ_i -open set U containing x , there exists a τ_i -open set V such that $x \in V \subset j \text{Cl}(V) \subset U$.*

Lemma 3.9 ([15]). *If a bitopological space (X, τ_1, τ_2) is (i, j) -regular, then $(i, j)\text{-Cl}_\theta(F) = F$ for every τ_i -closed set F .*

Theorem 3.10. *Let (Y, σ_1, σ_2) be an (i, j) -regular bitopological space. For a function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

1. f is u - ω -continuous;
2. $f^{-1}((i, j)\text{-Cl}_\theta(B))$ is u - ω -closed in X for every subset B of Y ;
3. f is (i, j) -weakly ω -continuous;
4. $f^{-1}(F)$ is u - ω -closed in X for every (i, j) - θ -closed set F of Y ;
5. $f^{-1}(V)$ is u - ω -open in X for every (i, j) - θ -closed set V of Y .

Proof. (1) $\Rightarrow$ (2): Let B be any subset of Y . Since $(i, j)\text{-Cl}_\theta(B)$ is σ_i -closed in Y , $f^{-1}((i, j)\text{-Cl}_\theta(B))$ is u - ω -closed in X .

(2) $\Rightarrow$ (3): Let B be any subset of Y . Then we have $(\tau_1, \tau_2)\text{-}\omega \text{Cl}(f^{-1}(B)) \subset (\tau_1, \tau_2)\text{-}\omega \text{Cl}(f^{-1}((i, j)\text{-Cl}_\theta(B))) = f^{-1}((i, j)\text{-Cl}_\theta(B))$. By Theorem 3.5, f is (i, j) -weakly ω -continuous.

(3) $\Rightarrow$ (4): Let F be any (i, j) - θ -closed set of Y . Then by Theorem 3.5, $(\tau_1, \tau_2)\text{-}\omega \text{Cl}(f^{-1}(F)) \subset f^{-1}((i, j)\text{-Cl}_\theta(F)) = f^{-1}(F)$. Therefore, by Lemma 2.4, $f^{-1}(F)$ is u - ω -closed in X .

(4) $\Rightarrow$ (5): Let V be any (i, j) - θ -open set of Y . By (4) $f^{-1}(Y - V) = X \setminus f^{-1}(V)$ is u - ω -closed in X and hence $f^{-1}(V)$ is u - ω -open in X .

(5) $\Rightarrow$ (1): Since Y is (i, j) -regular, by Lemma 3.9, $(i, j)\text{-Cl}_\theta(B) = B$ for every σ_i -closed set B of Y and hence every σ_i -open set is (i, j) - θ -open. Therefore, $f^{-1}(V)$ is u - ω -open for every σ_i -open set V of Y . Hence f is u - ω -continuous. $\square$

Definition 3.11. A function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is said to be weakly (i, j) -* quasicontinuous [15] if for every σ_i -open set V of Y , $f^{-1}(j \text{Cl}(V) \setminus V)$ is bi-closed in X .

Theorem 3.12. If a function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is (i, j) -weakly ω -continuous and weakly (i, j) -* quasicontinuous, then f is u - ω -continuous.

Proof. Let $x \in X$ and V be any σ_i -open set of Y containing $f(x)$. Since f is (i, j) -weakly ω -continuous, there exists an u - ω -open set U of X containing x such that $f(U) \subset j \text{Cl}(V)$. Hence $x \notin f^{-1}(j \text{Cl}(V) \setminus V)$. Then $x \in U \setminus f^{-1}(j \text{Cl}(V) \setminus V) = U \cap (X \setminus f^{-1}(j \text{Cl}(V) \setminus V))$.

Since U is u - ω -open and $X \setminus f^{-1}(j \text{Cl}(V) \setminus V)$ is biopen,

$$G = U \cap (X \setminus f^{-1}(j \text{Cl}(V) \setminus V))$$

is u - ω -open. Then $x \in G$ and $f(G) \subset V$. For, if $y \in G$, then $f(y) \notin j \text{Cl}(V) \setminus V$ and hence $f(y) \in V$. Therefore, f is u - ω -continuous. $\square$

Definition 3.13. A function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is said to have u - ω -interiority condition if $(\tau_1, \tau_2)\text{-}\omega \text{Int}(f^{-1}(j \text{Cl}(V))) \subset f^{-1}(V)$ for every σ_i -open set V of Y .

Example 3.14. Take $X = \mathbb{R}$ and two topologies $\tau_1 = \tau_2 = \{\emptyset, \mathbb{R}, \mathbb{Q}\}$ and $Y = \{a, b, c\}$ with topologies $\sigma_1 = \sigma_2 = \{\emptyset, Y, \{a\}, \{a, b\}\}$. Define $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ as

$$f(x) = \begin{cases} a, & \text{if } x \in \mathbb{Q}, \\ b, & \text{if } x \notin \mathbb{Q} \end{cases}$$

It is easy to see that the function f , satisfies the u - ω -interiority condition.

Example 3.15. in Example 3.3, we can see easily that f does not satisfies the u - ω -interiority condition.

Theorem 3.16. If a function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is (i, j) -weakly ω -continuous and satisfies the u - ω -interiority condition, then f is u - ω -continuous.

Proof. Let V be any σ_i -open set of Y . Since f is (i, j) -weakly ω -continuous, by Theorem 3.4, $f^{-1}(V) \subset (\tau_1, \tau_2)\text{-}\omega \text{Int}(f^{-1}(j \text{Cl}(V)))$. By the u - ω -interiority condition of f , we have $(\tau_1, \tau_2)\text{-}\omega \text{Int}(f^{-1}(j \text{Cl}(V))) \subset f^{-1}(V)$ and hence $f^{-1}(V) = (\tau_1, \tau_2)\text{-}\omega \text{Int}(f^{-1}(j \text{Cl}(V)))$. By Lemma 2.4, $f^{-1}(V)$ is u - ω -open in X and so f is u - ω -continuous. $\square$

Definition 3.17. Let (X, τ_1, τ_2) be a bitopological space and A be a subset of X . The (τ_1, τ_2) - ω -frontier of A is defined as $(\tau_1, \tau_2)\text{-}\omega Fr(A) = (\tau_1, \tau_2)\text{-}\omega \text{Cl}(A) \cup (\tau_1, \tau_2)\text{-}\omega \text{Cl}(X \setminus A) = (\tau_1, \tau_2)\text{-}\omega \text{Cl}(A) \setminus (\tau_1, \tau_2)\text{-}\omega \text{Int}(A)$.

Theorem 3.18. The set of all points x of X at which a function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is not (i, j) -weakly ω -continuous is identical with the union of the (τ_1, τ_2) - ω -frontiers of the inverse images of the σ_i -closure of σ_i -open sets of Y containing $f(x)$.

Proof. Let x be a point of X at which $f(x)$ is not (i, j) -weakly ω -continuous.

Then, there exists a σ_i -open set V of Y containing $f(x)$ such that $U \cap (X \setminus f^{-1}(j \text{Cl}(V))) \neq \emptyset$ for every u - ω -open set U of X containing x . By Lemma 2.5, $x \in (\tau_1, \tau_2)\text{-}\omega \text{Cl}(X \setminus f^{-1}(j \text{Cl}(V)))$. Since $x \in f^{-1}(j \text{Cl}(V))$, we have $x \in (\tau_1, \tau_2)\text{-}\omega \text{Cl}(f^{-1}(j \text{Cl}(V)))$ and hence $x \in (\tau_1, \tau_2)\text{-}\omega Fr(f^{-1}(j \text{Cl}(V)))$. Conversely, if f is (i, j) -weakly ω -continuous at x , then for each σ_i -open set V of Y containing $f(x)$, there exists an u - ω -open set U containing x such that $f(U) \subset j \text{Cl}(V)$ and hence $x \in U \subset f^{-1}(j \text{Cl}(V))$. Hence $x \in (\tau_1, \tau_2)\text{-}\omega \text{Int}(f^{-1}(j \text{Cl}(V)))$. This contradicts that $x \in (\tau_1, \tau_2)\text{-}\omega Fr(f^{-1}(j \text{Cl}(V)))$. $\square$

Definition 3.19. A function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is said to be (i, j) -almost ω -continuous [6] if for each $x \in X$ and each σ_i -open set V containing $f(x)$, there exists an u - ω -open set U of X containing x such that $f(U) \subset i \text{Int}(j \text{Cl}(V))$.

Lemma 3.20. A function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is almost u - ω -continuous if and only if $f^{-1}(V)$ is u - ω -open for each (i, j) -regular open set V of Y .

Definition 3.21. A bitopological space (X, τ_1, τ_2) is said to be (i, j) -almost regular [17] if for each $x \in X$ and each (i, j) -regular open set U containing x , there exists an (i, j) -regular open set V of X such that $x \in V \subset j \text{Cl}(V) \subset U$.

Theorem 3.22. Let a bitopological space (Y, σ_1, σ_2) be (i, j) -almost regular. Then a function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is (i, j) -almost ω -continuous if and only if it is (i, j) -weakly ω -continuous.

Proof. *Necessity.* This is obvious. *Sufficiency.* Suppose that f is (i, j) -weakly ω -continuous. Let V be any (i, j) -regular open set of Y and $x \in f^{-1}(V)$. Then we have $f(x) \in V$. By the almost (i, j) -regularity of Y , there exists an (i, j) -regular open set V_0 of Y such that $f(x) \in V_0 \subset j \text{Cl}(V_0) \subset V$. Since f is (i, j) -weakly ω -continuous, there exists an u - ω -open set U of X containing x such that $f(U) \subset j \text{Cl}(V_0) \subset V$. This implies that $x \in U \subset f^{-1}(V)$. Therefore, we have

$f^{-1}(V) \subset (\tau_1, \tau_2)\text{-}\omega\text{Int}(f^{-1}(V))$ and hence $f^{-1}(V) = (\tau_1, \tau_2)\text{-}\omega\text{Int}(f^{-1}(V))$. By Lemma 2.4, $f^{-1}(V)$ is $u\text{-}\omega$ -open and by Lemma 3.20, f is (i, j) -almost ω -continuous. $\square$

Definition 3.23. A subset K of a bitopological space (X, τ_1, τ_2) is said to be (i, j) -quasi H -closed relative to X [7] if for each cover $\{U_\alpha : \alpha \in \Omega\}$ of K by τ_i -open sets of X , there exists a finite subset Ω_0 of Ω such that $K \subset \cup\{j\text{Cl}(U_\alpha) : \alpha \in \Omega_0\}$.

Definition 3.24. A subset K of a bitopological space (X, τ_1, τ_2) is said to be $u\text{-}\omega$ -compact relative to X if every cover of K by $u\text{-}\omega$ -open sets of X has a finite subcover.

Theorem 3.25. If $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is (i, j) -weakly ω -continuous and K is $u\text{-}\omega$ -compact relative to X , then $f(K)$ is (i, j) -quasi H -closed relative to Y .

Proof. Let K be $u\text{-}\omega$ -compact relative to X and $\{V_\alpha : \alpha \in \Omega\}$ any cover of $f(K)$ by σ_i -open sets of (Y, σ_1, σ_2) . Then $f(K) \subset \cup\{V_\alpha : \alpha \in \Omega\}$ and so $K \subset \cup\{f^{-1}(V_\alpha) : \alpha \in \Omega\}$. Since f is (i, j) -weakly ω -continuous, by Theorem 3.4 we have $f^{-1}(V_\alpha) \subset (\tau_1, \tau_2)\text{-}\omega\text{Int}(f^{-1}(j\text{Cl}(V_\alpha)))$ for each $\alpha \in \Omega$. Therefore, $K \subset \cup\{(\tau_1, \tau_2)\text{-}\omega\text{Int}(f^{-1}(j\text{Cl}(V_\alpha))) : \alpha \in \Omega\}$. Since K is $u\text{-}\omega$ -compact relative to X and $(\tau_1, \tau_2)\text{-}\omega\text{Int}(f^{-1}(j\text{Cl}(V_\alpha)))$ is $u\text{-}\omega$ -open for each $\alpha \in \Omega$, there exists a finite subset Ω_0 of Ω such that $K \subset \cup\{(\tau_1, \tau_2)\text{-}\omega\text{Int}(f^{-1}(j\text{Cl}(V_\alpha))) : \alpha \in \Omega_0\}$. This implies that $f(K) \subset \cup\{f((\tau_1, \tau_2)\text{-}\omega\text{Int}(f^{-1}(j\text{Cl}(V_\alpha)))) : \alpha \in \Omega_0\} \subset \cup\{f(f^{-1}(j\text{Cl}(V_\alpha))) : \alpha \in \Omega_0\} \subset \cup\{j\text{Cl}(V_\alpha) : \alpha \in \Omega_0\}$. Hence $f(K)$ is (i, j) -quasi H -closed relative to Y . $\square$

For a function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$, we define $D_{w\omega}(f)$ as follows: $D_{w\omega}(f) = \{x \in X : f \text{ is not } (i, j)\text{-weakly } \omega\text{-continuous at } x\}$.

Theorem 3.26. For a function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$, the following properties hold:

$$\begin{aligned}
 D_{w\omega}(f) &= \bigcup_{B \in P(Y)} \{(\tau_1, \tau_2) - \omega\text{Cl}(f^{-1}(j\text{Int}(i\text{Cl}(B)))) \setminus f^{-1}(i\text{Cl}(B))\} \\
 &= \bigcup_{F \in (i, j)\text{-RC}(Y)} \{(\tau_1, \tau_2) - \omega\text{Cl}(f^{-1}(j\text{Int}(F))) \setminus f^{-1}(F)\} \\
 &= \bigcup_{V \in \sigma_j} \{(\tau_1, \tau_2) - \omega\text{Cl}(f^{-1}(V)) \setminus f^{-1}(i\text{Cl}(V))\} \\
 &= \bigcup_{V \in \sigma_i} \{f^{-1}(V) \setminus ((\tau_1, \tau_2) - \omega\text{Int}(f^{-1}(j\text{Cl}(V))))\} \\
 &= \bigcup_{A \in P(X)} \{f((\tau_1, \tau_2) - \omega\text{Cl}(A)) \setminus (i, j) - \text{Cl}_\theta(f(A))\} \\
 &= \bigcup_{B \in P(Y)} \{(\tau_1, \tau_2) - \omega\text{Cl}(f^{-1}(B)) \setminus (f^{-1}(i, j) - \text{Cl}_\theta(B))\}
 \end{aligned}$$

$$\begin{aligned}
&= \bigcup_{B \in P(Y)} \{(\tau_1, \tau_2) - \omega \text{Cl}(f^{-1}(j \text{Int}(i, j) - \text{Cl}_\theta(B))) \setminus f^{-1}((i, j) - \text{Cl}_\theta(B))\} \\
&= \bigcup_{V \in (j, i) - PO(Y)} \{(\tau_1, \tau_2) - \omega \text{Cl}(f^{-1}(V)) \setminus f^{-1}(i \text{Cl}(V))\} \\
&= \bigcup_{V \in (i, j) - PO(Y)} \{f^{-1}(V) \setminus ((\tau_1, \tau_2) - \omega \text{Int}(f^{-1}(j \text{Cl}(V))))\}.
\end{aligned}$$

Proof. The proof follows from Theorems 3.4, 3.5 and 3.6. $\square$

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ω^* -topology

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Abstract. For any space (X, τ) , we introduced the concept of ω^* -open sets, then we found and discussed a new topology on X , namely τ_{ω^*} which is strictly closed between τ_{θ} (θ -topology) and τ_{ω} (ω -topology) on X and it is independent with the original topology τ . Furthermore, the relationship of ω^* -open sets with some other types of sets are given.

Keywords: topology, ω -topology, θ -topology, ω^* -topology and ω^* -open sets.

1. Introduction

In the twentieth century until now, many generalizations of open sets were introduced, such as in (Stone [18], Levien [12], Njastad [15], Velićo [19], Mashhour et. al. [14], Abd El-Monsef [1], Hdeib [11], Andrijević [5], Park et. al. [16], Al-Hawary et. al. [3], Ekici et. al. [10], Darwesh [6], Darwesh et.al. [9], Darwesh [7], Darwesh [8], Al Ghour et. al. [2]) introduced the notion of (regular open, semi-open, α -open, θ -open and δ -open, preopen, β -open, ω -open, b -open, δ -semiopen, ω^o -open, ω_{θ} -open, δsc -open, $k\omega$ -open, ω_{δ} -open, ω_p -open, θ_{ω} -open) sets in topological spaces. Throughout the present paper, R (resp., Q , I and N) denote the set of all real (resp., rational, irrational and natural) numbers and (X, τ) denotes a topological space (briefly, a space) on which no separation axioms are assumed unless otherwise explicitly stated. For any space (X, τ) , the family of all the above types of sets denote by $(RO(X), SO(X), \tau_{\alpha}, \tau_{\theta}, \tau_{\delta}, PO(X), \beta O(X), \tau_{\omega}, bO(X), \delta SO(X), \tau_{\omega^o}, \tau_{\omega_{\theta}}, \delta SCO(X), k\omega O(X), \tau_{\omega_{\delta}}, \omega_p O(X), \tau_{\theta_{\omega}})$, respectively. Stone [18] showed $RO(X)$ forms a base of some topology on X , named semi-regularization topology on X and its denoted by

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τ_s . Each of the families $(\tau_\alpha, \tau_\theta, \tau_\delta = \tau_s, \tau_\omega, \tau_{\omega^o}, \tau_{\omega_\theta}, \tau_{\omega_\delta} \text{ and } \tau_{\theta_\omega})$ forms a topology on X . While, each of $SO(X)$, $PO(X)$, $\beta O(X)$, $bO(X)$, $\delta SO(X)$, $\delta SCO(X)$, $\omega_p O(X)$ forms a supra topology on X , but $RO(X)$ and $k\omega O(X)$ form neither a topology nor a supra topology on X .

In the present paper, we construct a new topology τ_{ω^*} on a space (X, τ) , by using a new notion of ω^* -open sets. Then, we show that τ_{ω^*} is strictly finer than τ_θ and strictly coarser than τ_ω . Also, we obtain some basic properties of the space (X, τ_{ω^*}) . For any subset A of a space X , we denote by ClA (resp., $Cl_\omega A$, $Cl_\delta A$, $Cl_\theta A$, $IntA$, $Int_\delta A$ and $Int_\theta A$) the closure (resp., ω -closure, δ -closure, θ -closure, interior, δ -interior and θ -interior) of A .

Definition 1.1. A subset A of a space (X, τ) is said to be regular open [18] (resp., semi-open [12], preopen [14], α -open [15], β -open [1], b -open [5], δ -semiopen [16], ω_p -open [8]) if $A = IntClA$ (resp., $A \subseteq ClIntA$, $A \subseteq IntClA$, $A \subseteq IntClIntA$, $A \subseteq ClIntClA$, $A \subseteq IntClA \cup ClIntA$, $A \subseteq Clint_\delta A$, $A \subseteq IntCl_\omega A$).

Definition 1.2. A subset A of a space (X, τ) is said to be θ -open [19] (resp., δ -open [19], θ_ω -open), if for each $x \in A$, there exists an open set G such that $x \in G \subseteq ClG \subseteq A$ (resp., $x \in G \subseteq IntClG \subseteq A$, $x \in G \subseteq Cl_\omega G \subseteq A$).

Theorem 1.1 ([13]). A space (X, τ) is regular if and only if $\tau_\theta = \tau$.

Definition 1.3 ([9]). A subset A of a space (X, τ) is said to be $k\omega$ -open, if its both open and ω -compact subset of (X, τ) .

Definition 1.4 ([6]). A δ -semiopen subset A of a space (X, τ) is said to be δsc -open if, for each point $x \in A$ there is a closed set F in X such that $x \in F \subseteq A$.

Definition 1.5. A subset A of a space (X, τ) is said to be ω -open [11] (resp., ω^o -open [3], ω_θ -open [10], and ω_δ -open [7]), if for each $x \in A$, there exists an open set G containing x such that $G \setminus A$ (resp., $G \setminus IntA$, $G \setminus Int_\theta A$, $G \setminus Int_\delta A$) is a countable subset of X .

Definition 1.6 ([4]). A space (X, τ) is said to be:

1. locally countable, if for each point $x \in X$ there exists a countable open subset G of X containing x ,
2. anti-locally countable, if each nonempty open subset of X is uncountable.

Theorem 1.2 ([11, 4]). A space (X, τ) is lindelof if and only if (X, τ_ω) is lindelof.

2. ω^* -open sets and ω^* -topology

We begin this section with the definition of ω^* -open sets, and then we use it to define a new topology on the given space. Also, we give some of its basic properties.

Definition 2.1. A subset A of a space (X, τ) is said to be an ω^* -open set, if for each $x \in A$, there exists an open set G such that $x \in G$ and $ClG \setminus A$ is a countable set. Also, A is said to be ω^* -closed, if $X \setminus A$ is ω^* -open.

Theorem 2.1. A subset A of a space (X, τ) is ω^* -open if and only if for each $x \in A$, there exist an open set G_x containing x and a countable set C_x such that $ClG_x \setminus C_x \subseteq A$.

Proof. Let A be an ω^* -open subset of X and $x \in A$. There exists an open set G_x such that $x \in G$ and $ClG_x \setminus A$ is a countable set. Hence, the set $C_x = ClG_x \setminus A$ is the required countable set.

Conversely, suppose $x \in A$, so by our hypothesis, there exist an open set G_x containing x and a countable set C_x such that $ClG_x \setminus C_x \subseteq A$. This implies that, $ClG_x \setminus A \subseteq C_x$. Thus, A is an ω^* -open subset of X . $\square$

Theorem 2.2. If A is an ω^* -closed subset of (X, τ) , then $A \subseteq IntF \cup C$ for some closed set F and a countable set C .

Proof. If $A = X$, we put $F = A$ and $C = \emptyset$, to get $A \subseteq IntF \cup C$. Otherwise, we choose a point $x \in X$ which is not in A , and since $X \setminus A$ is ω^* -open, then by Theorem 2.1, there is an open set G which contains x and a countable set C such that $ClG \setminus C \subseteq X \setminus A$. Hence, $F = X \setminus G$ and C are the required sets. $\square$

Theorem 2.3. A subset A of a space (X, τ) is ω^* -closed if and only if $A = X$ or for each point x not in A , there is a closed set F not containing x and a countable set C such that $A \subseteq IntF \cup C$.

Proof. Let A be an ω^* -closed subset of X . Then, either $A = X$ or A is a proper subset of X . If $A = X$, then there is nothing to prove, otherwise A is a proper ω^* -closed subset of X , then by Theorem 2.2 there exists a closed set F not containing x and a countable set C such that $A \subseteq IntF \cup C$.

Conversely, if $A = X$, then its ω^* -closed. Otherwise, suppose that for each $x \in X \setminus A$, there is a closed set F not contain x and a countable set C such that $A \subseteq IntF \cup C$. The set $G = X \setminus F$ is an open subset of X which contains x and $ClG \setminus C = X \setminus (IntF \cup C) \subseteq X \setminus A$. By Theorem 2.1, we obtaining that $X \setminus A$ is ω^* -open, whence A is ω^* -closed. $\square$

Theorem 2.4. The intersection (union) of two ω^* -open (ω^* -closed) sets is ω^* -open (ω^* -closed).

Proof. Let A and B be two ω^* -open sets. If $A \cap B = \emptyset$, there is nothing to prove. Otherwise, for $x \in A \cap B$, there are two open sets G and U containing x such that $ClG \setminus A$ and $ClU \setminus B$ are countable sets. Since, $Cl(G \cap U) \setminus (A \cap B) \subseteq (ClG \setminus A) \cup (ClU \setminus B)$, so $A \cap B$ is ω^* -open. $\square$

Theorem 2.5. The union (intersection) of any family of ω^* -open sets is ω^* -open (ω^* -closed).

Proof. Let $\{A_\lambda; \lambda \in \Lambda\}$ be any family of ω^* -open sets and $x \in \bigcup_{\lambda \in \Lambda} A_\lambda$. Then, there is $\lambda_0 \in \Lambda$ and an open set G such that $x \in G \cap A_{\lambda_0}$ and $ClG \setminus A_{\lambda_0}$ is a countable set. Since, $ClG \setminus (\bigcup_{\lambda \in \Lambda} A_\lambda) \subseteq ClG \setminus A_{\lambda_0}$. Hence, $\bigcup_{\lambda \in \Lambda} A_\lambda$ is ω^* -open. $\square$

Theorem 2.6. *For any space (X, τ) . The family τ_{ω^*} of all ω^* -open subsets of X form a topology on X .*

Proof. Since $\emptyset$ has no any point, so $\emptyset \in \tau_{\omega^*}$. Since $ClX \setminus X = \emptyset$, so $X \in \tau_{\omega^*}$ and hence, the result is followed by using Theorem 2.4 and Throrem2.5. $\square$

we call the topology of Theorem 2.6 is the ω^* -topology.

Definition 2.2. *A point $x \in X$ is said to be a $*$ -condensation point of a subst A of a space X , if $ClG \cap A$ is an uncountable set. The set of all $*$ -condensation points of a set A is denoted by $cond^*(A)$.*

Theorem 2.7. *A subset A of a space X is ω^* -closed if and only if $cond^*(A) \subseteq A$.*

Proof. Let A be an ω^* -closed subset of X and $x \in cond^*(A)$. On contrary, we suppose that $x \notin A$. there exists an open set G containing x such that $ClG \setminus (X \setminus A)$ is countable. This implies that, $ClG \cap A$ is countable. So, $x \notin cond^*(A)$ which is a contradiction. Hence, $cond^*(A) \subseteq A$.

Conversely, suppose that $cond^*(A) \subseteq A$ and $x \in X \setminus A$. There exists an open set G containing x such that $ClG \cap A$ is countable. Hence, $ClG \setminus (X \setminus A)$ is countable, so $X \setminus A$ is ω^* -open. Therefore, A is ω^* -closed. $\square$

Corollary 2.1. *Each countable subset of any space is ω^* -closed.*

Proof. If A is countable, then $cond^*(A) = \emptyset$. So, by Theorem 2.7 A is ω^* -closed. $\square$

Corollary 2.2. *If (X, τ) is any space, then the co countable topology T_{coc} on X is coarser than τ_{ω^*} .*

Proof. If $A \in T_{coc}$, then $X \setminus A$ is a countable subset of X . By Corollary 2.1 $X \setminus A$ is ω^* -closed. Hence, $A \in \tau_{\omega^*}$. $\square$

Corollary 2.3. *For any space (X, τ) , the space (X, τ_{ω^*}) is a T_1 -space.*

Proof. It follows from Corollary 2.2. $\square$

Corollary 2.4. *If (X, τ) is a countable space, then (X, τ_{ω^*}) is a discrete space.*

Proof. It follows from Corollary 2.2. $\square$

Recalling that a space (X, τ) is said be a hyperconnected space [17], if each nonempty open subsets of X is dense in X .

Theorem 2.8. *If (X, τ) is a hyperconnected space, then τ_{ω^*} is the co countable topology T_{coc} on X .*

Proof. Let $A \in T_{\omega^*}$. If $A = \emptyset$, then $A \in \tau_{coc}$. Otherwise, we choose a fixed point x in A , and an open sets G containing x such that $ClG \setminus A = C$, for some countable set C . Since X is hyperconnected, so $ClG = R$ and $A = R \setminus C$. Hence, $A \in T_{coc}$. Thus, $\tau_{\omega^*} \subseteq T_{coc}$. The incusion $T_{coc} \subseteq \tau_{\omega^*}$ is followed from Corollary 2.2. Therefore, $\tau_{\omega^*} = T_{coc}$. $\square$

3. Some properties and relationships of τ_{ω^*}

In this section, we give more properties of the topology τ_{ω^*} and investigate its relationships with other classes of sets.

Theorem 3.1. *If τ and $\mathfrak{S}$ are two topologies on X such that $\tau \subseteq \mathfrak{S}$, then $\tau_{\omega^*} \subseteq \mathfrak{S}_{\omega^*}$.*

Proof. Let $A \in \tau_{\omega^*}$. If $A = \emptyset$, then $A \in \mathfrak{S}_{\omega^*}$. Let $A \neq \emptyset$. Then by Theorem 2.1, for each $x \in A$, there is $G \in \tau$ (hence $G \in \mathfrak{S}$) and a countable subset C of X such that $Cl_{\mathfrak{S}}G \setminus C \subseteq Cl_{\tau}G \setminus C \subseteq A$. Thus, by Theorem 2.1 we obtaining that $A \in \mathfrak{S}_{\omega^*}$. $\square$

For a space (X, τ) , we denote by $(\tau_{\theta})_{\omega}$ the ω -topology of (X, τ_{θ}) , so we have:

Theorem 3.2. $\tau_{\theta} \subseteq (\tau_{\theta})_{\omega} \subseteq \tau_{\omega^*} \subseteq \tau_{\omega}$, for any space (X, τ) .

Proof. By [4, Theorem 2.5], we have $\tau_{\theta} \subseteq (\tau_{\theta})_{\omega}$. Let A be any member of $(\tau_{\theta})_{\omega}$. If $A = \emptyset$, then $A \in \tau_{\omega^*}$. Otherwise, for an arbitrary point x in A , there exists $V \in \tau_{\theta}$ such that $x \in V$ and $V \setminus A$ is a countable set. Since $V \in \tau_{\theta}$, there is $G \in \tau$ in which $x \in G \subseteq ClG \subseteq V$. Therefore, $ClG \setminus A$ is countable. Hence, $A \in \tau_{\omega^*}$, so $(\tau_{\theta})_{\omega} \subseteq \tau_{\omega^*}$.

Now, since $G \setminus A \subseteq ClG \setminus A$ for any open subset G of X , this implies that for any $A \in \tau_{\omega^*}$, we have $A \in \tau_{\omega}$. Hence, $\tau_{\omega^*} \subseteq \tau_{\omega}$. $\square$

The following examples show that the inclusions in the above result cannot replace with equality in general:

Example 3.1. Consider the set $A = \{1\}$ and $B = R \setminus N$ in the one point excluded topological space (R, τ) [17, Example 13, P. 47], where $\tau = \{G \subseteq R; 0 \notin G\} \cup \{R\}$. Then, $A = \{1\} \in \tau_{\omega^*}$ but $A \notin (\tau_{\theta})_{\omega}$. However, the set $B \in (\tau_{\theta})_{\omega}$ but $B \notin \tau_{\theta}$.

Example 3.2. Consider the set $A = \{0, 1\}$ in the one point included topological space (R, τ) [17, Example 8, P. 44], where $\tau = \{G \subseteq R; 0 \in G\} \cup \{\emptyset\}$. Then, $A = \{0, 1\} \in \tau$ (and hence, $A \in \tau_{\omega}$) but $A \notin \tau_{\omega^*}$.

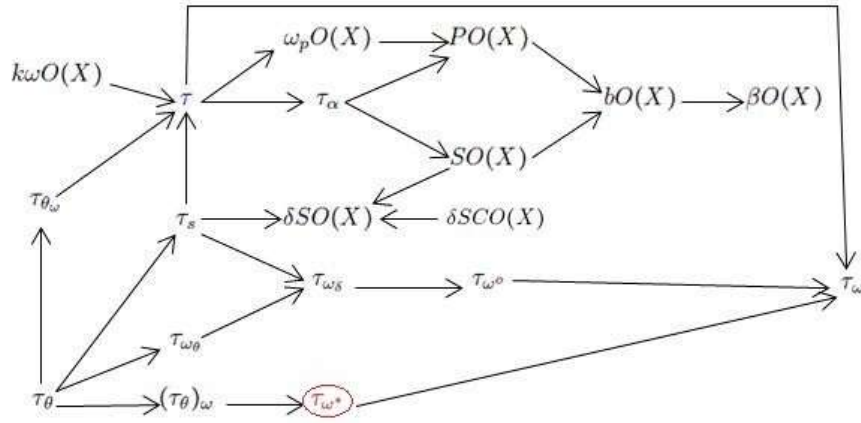
Proposition 3.1. *If (X, τ_{θ}) is a locally countable space, then (X, τ_{ω^*}) is a discrete space.*

Proof. Let (X, τ_θ) be a locally countable space. By [4, Theorem 3.2], $(X, (\tau_\theta)_\omega)$ is a discrete space. Then, by Theorem 3.2 (X, τ_{ω^*}) is a discrete space. $\square$

Remark 3.1. .

1. Since the space of Example 3.2 is locally countable but (R, τ_{ω^*}) is neither a T_2 -space nor a discrete space, this means that; the countability of X in Corollary 2.4 unable to replace with locally countable, and the T_1 in Corollary 2.3 can not replace with any higher separation axioms.
2. If $\tau = \{G \subseteq R; 0 \in G\} \cup \{\emptyset\}$ and $\mathfrak{S} = \{G \subseteq R; 1 \in G\} \cup \{\emptyset\}$ are two different one point included topologies on R . Then, neither $\tau \subseteq \mathfrak{S}$ nor $\mathfrak{S} \subseteq \tau$, but $\tau_{\omega^*} = T_{coc} = \mathfrak{S}_{\omega^*}$. This means that, the converse of Theorem 3.1 is not true in general.
3. The set $A = \{0, 1\}$ in Example 3.2 is open, α -open, preopen, semi-open, b -open, β -open, ω_p -open and $k\omega$ -open but not ω^* -open. While, the set $B = \{2\}$ in the sierpinski space (X, τ) [17, Example 11, P. 44], where $X = \{1, 2\}$ and $\tau = \{\emptyset, \{1\}, X\}$ is ω^* -open but it is neither open, α -open, preopen, semi-open, b -open, β -open, ω_p -open nor $k\omega$ -open. This means that:
 - (a) τ_{ω^*} is independent with each of the topologies τ and τ_α .
 - (b) τ_{ω^*} is independent with each of the classes $PO(X)$, $SO(X)$, $bo(X)$, $\beta O(X)$, $\omega_p O(X)$ and $k\omega O(X)$.
4. The set $A = Q$ in the space (R, τ) , where $\tau = \{\emptyset, Q, R\}$ is ω_θ -open, ω_δ -open, ω^o -open and θ_ω -open but not ω^* -open. However, the set $B = R \setminus \{0\}$ in the indiscrete space (R, T_{ind}) is ω^* -open but it is neither ω_θ -open, ω_δ -open, ω^o -open nor θ_ω -open. This means that; τ_{ω^*} is independent with each of the topologies τ_{ω_θ} , τ_{ω_δ} , τ_{ω^o} and τ_{θ_ω} .
5. Since the set $A = R \setminus \{0\}$ in the indiscrete space (R, T_{ind}) is ω^* -open but it is neither δ -open, δ -semiopen nor δsc -open, while the set $A = Q$ in the space (R, τ) , where $\tau = \{\emptyset, Q, Ir \cap (0, 1), Q \cup (0, 1), R\}$ is δ -open and δ -semiopen but not ω^* -open. However, the set $A = Q$ in the space (R, σ) , where $\sigma = \tau \cap T_{coc}$, is δsc -open but not ω^* -open. Then, we obtain:
 - (a) τ_{ω^*} is independent with the semi-regularization topology τ_s .
 - (b) τ_{ω^*} is independent with each of the classes $\delta SO(X)$ and $\delta SCO(X)$.
6. Thus, we have the following diagram:

Theorem 3.3. If (X, τ) is a regular space, then $\tau \subseteq \tau_{\omega^*}$ and $\tau_\omega = \tau_{\omega^*}$.



Proof. It follows from Theorem 1.1 and Theorem 3.2 that $\tau \subseteq \tau_{\omega^*}$ and $\tau_{\omega^*} \subseteq \tau_\omega$. So, it remains only to show $\tau_\omega \subseteq \tau_{\omega^*}$. Suppose that $A \in \tau_\omega$, there is an open set G such that $x \in G$ and $G \setminus A$ is countable. Since X is regular space and $x \in G$, so there exists an open set U such that $x \in U \subseteq ClU \subseteq G$. Hence, $ClU \setminus A$ is countable, so $A \in \tau_{\omega^*}$. Thus, $\tau_\omega = \tau_{\omega^*}$. $\square$

Since the sierpinski space (X, τ) in part 3 of Remark 3.1 (also, see [17, Example 8, P. 44]) is not regular but $\tau \subseteq \tau_{\omega^*} = \tau_\omega$. This means that, the converse of Theorem 3.3 is not true in general.

Theorem 3.4. $(\tau_{\omega^*})_{\omega^*} \subseteq \tau_{\omega^*}$, for any space (X, τ) .

Proof. Let $x \in A \in (\tau_{\omega^*})_{\omega^*}$. By Theorem 2.1, there is $U_x \in \tau_{\omega^*}$ and a countable set K_x such that $Cl_{\tau_{\omega^*}} U_x \setminus K_x \subseteq A$. Again, by Theorem 2.1 there is $G_x \in \tau$ and a countable set L_x in which $Cl_\tau G_x \setminus L_x \subseteq U_x$. Since $K_x \cup L_x$ is countable and $Cl_\tau G_x \setminus (K_x \cup L_x) \subseteq U_x \setminus K_x \subseteq Cl_{\tau_{\omega^*}} U_x \setminus K_x \subseteq A$. Hence, by Theorem 2.1 $A \in \tau_{\omega^*}$. $\square$

Corollary 3.1. $(\tau_{\omega^*})_{\omega^*} = \tau_{\omega^*}$, if (X, τ) is a regular space.

Proof. It follows from Theorem 3.1, Theorem 3.3 and Theorem 3.4 $\square$

Since the sierpinski space (X, τ) in part 3 of Remark 3.1 is not regular but $(\tau_{\omega^*})_{\omega^*} = \tau_{\omega^*}$. This means that, the converse of Corollary 3.1 is not true in general.

Theorem 3.5. If Y is a subset of a space (X, τ) , then $(\tau_{\omega^*})_Y \subseteq (\tau_Y)_{\omega^*}$.

Proof. Let $A \in (\tau_{\omega^*})_Y$. Then, there is $G \in \tau_{\omega^*}$ such that $A = G \cap Y$. For $x \in A$, there is $V \in \tau$ such that $x \in V$ and $ClV \setminus G$ is countable. Since $U = V \cap Y \in \tau_Y$, $x \in U$ and $Cl_Y U \subseteq ClV$, then $Cl_Y U \setminus A = Cl_Y U \setminus (G \cap Y) = Cl_Y U \setminus G \subseteq ClV \setminus G$ which is countable. Hence, $A \in (\tau_Y)_{\omega^*}$. Thus, $(\tau_{\omega^*})_Y \subseteq (\tau_Y)_{\omega^*}$. $\square$

The following examples show that the converse inclusion in Theorem 3.5 is not true even Y is open (or Y is closed).

Example 3.3. Consider the open subset $Y = Ir$ in the space (R, τ) , where $\tau = \{\emptyset, R\} \cup \{A; A \subseteq Ir\}$. Then τ_Y is the discrete topology on Y , therefore $(\tau_Y)_{\omega^*}$ is also the discrete topology on Y . But, Since (R, τ) is hyperconnected space, so by Theorem 2.8, we get $\tau_{\omega^*} = T_{coc}$, so that $(\tau_{\omega^*})_Y = T_{coc}$. Hence, $(\tau_Y)_{\omega^*} \not\subseteq (\tau_{\omega^*})_Y$.

Example 3.4. Consider the closed subset $Y = R \setminus \{0\}$ in the space of Example 3.2. Then, τ_Y is the discrete topology on Y , and so $(\tau_Y)_{\omega^*}$ is the discrete topology on Y . Now, Since (R, τ) is hyperconnected space, then Theorem 2.8, we get $\tau_{\omega^*} = T_{coc}$. Thus, $(\tau_{\omega^*})_Y = T_{coc}$. This means that $(\tau_Y)_{\omega^*} \not\subseteq (\tau_{\omega^*})_Y$.

Theorem 3.6. *If Y is a clopen subset of (X, τ) , then $(\tau_Y)_{\omega^*} \subseteq \tau_{\omega^*}$ and $(\tau_Y)_{\omega^*} = (\tau_{\omega^*})_Y$.*

Proof. Let $A \in (\tau_Y)_{\omega^*}$. Then for each $x \in A$, there is $G \in \tau_Y$ such that $x \in G$ and $Cl_Y G \setminus A$ is countable. Since Y is clopen, so $G \in \tau$ and $Cl_Y G = Cl G$. Hence, $Cl G \setminus A$ is countable. This means that $A \in \tau_{\omega^*}$. Thus, $(\tau_Y)_{\omega^*} \subseteq \tau_{\omega^*}$. Since, $A = A \cap Y \in (\tau_{\omega^*})_Y$, then $(\tau_Y)_{\omega^*} \subseteq (\tau_{\omega^*})_Y$, and from by Theorem 3.5 $(\tau_{\omega^*})_Y \subseteq (\tau_Y)_{\omega^*}$, then $(\tau_Y)_{\omega^*} = (\tau_{\omega^*})_Y$. $\square$

We say that a space (X, τ) is ω^* -lindelof if (X, τ_{ω^*}) is a lindelof space.

Theorem 3.7. *If (X, τ) is lindelof, then (X, τ_{ω^*}) is lindelof.*

Proof. Let (X, τ) be a lindelof sapce. By Theorem 1.2, we obtaining that (X, τ_{ω}) is lindelof. Since $\tau_{\omega^*} \subseteq \tau_{\omega}$, then (X, τ_{ω^*}) is lindelof. $\square$

The following example shows that the converse of above result is not true in general:

Example 3.5. Consider the left ray topological sapce (R, τ) , $\tau = \{(-\infty, r); r \in R\} \cup \{\emptyset, R\}$. It is clear that, this space a non-lindelof hyperconnected space, then by Theorem 2.8, we get $\tau_{\omega^*} = T_{coc}$, whence (R, τ_{ω^*}) is lindelof.

Corollary 3.2. *If (X, τ_{ω^*}) is lindelof, then (X, τ_{θ}) is lindelof.*

Proof. It follows from Theorem 3.4. $\square$

Theorem 3.8. *If (X, τ) is almost lindelof space, then (X, τ_{ω^*}) is lindelof.*

Proof. Let $\{V_{\lambda}; \lambda \in \Lambda\}$ be an ω^* -open cover of X . Then, for each $x \in X$, there is $\lambda_x \in \Lambda$ and an open set G_{λ_x} containing x such that $Cl G_{\lambda_x} \setminus V_{\lambda_x} = C_{\lambda_x}$ for some countable set C_{λ_x} . Since $\{G_{\lambda_x}; x \in X\}$ is an open cover of X , so there exists a countable subset X_0 of X such that $X = \bigcup_{x \in X_0} Cl G_{\lambda_x} = (\bigcup_{x \in X_0} Cl G_{\lambda_x} \setminus V_{\lambda_x}) \cup \bigcup_{x \in X_0} V_{\lambda_x} = \bigcup_{x \in X_0} C_{\lambda_x} \cup \bigcup_{x \in X_0} V_{\lambda_x}$. Since $\bigcup_{x \in X_0} C_{\lambda_x}$ is countable, then there exists a countable subset Λ_0 of Λ such that $\bigcup_{x \in X_0} C_{\lambda_x} \subseteq \bigcup_{\lambda \in \Lambda_0} V_{\lambda}$. Thus, $X = \bigcup_{\lambda \in \Lambda_0} V_{\lambda} \cap \bigcup_{x \in X_0} V_{\lambda_x}$. Hence, (X, τ_{ω^*}) is a lindelof space. $\square$

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Analysis of chaos dynamics of emotions in employment relation model

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Abstract. This paper studies the mathematical modelling and analysis of chaos for employment relation system in which the interactions between tendencies of employer, employee and intermediate employee are being studied. In the modelling the social frame work of an urban class office system is being considered where employee and employers have their own set of disappointments and grief in their mutual relation. Stable phase, critical phase and chaotic phase have been observed in the simulation of the dynamics of the office system for different parameter values. Through bifurcation plot for variation in the level of disappointment the critical value of disappointment is observed to be in accordance with the critical value evaluated through stability analysis. It is concluded that the tendencies of both employer and employee remain stable till they have their disappointment level below the critical value. The moment the critical value is crossed the employment relation transits from stable to critical phase where the tendencies of both the partners start oscillating which later becomes chaotic as disappointment level further increases.

Keywords: intermediate model, stability analysis, employment relations, individualistic interactions, chaos, bifurcation.

1. Introduction

An office is a functional unit of a trading society where different individuals work together in an institutional frame work and hierarchal system. In this hierarchal system the employer is at the topmost position being the governing authority of the corporate enterprise while employees under his/her supervision

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work for the institution. The employment relation is the connection between the employer and employee through which the exchange of labour take place in the economy. All employees and employers have fundamental interests which they pursue through the employment relationship. Such a relationship in corporate world are mediated and governed strictly by rule and regulations documented in the form of a contract.

Evolution of dynamics between the employee and employer in an office is governed by set of various types of interactions that take place between the employer and employee often mediated by intermediating colleagues. Every interaction is different with its own interaction parameter accounting for direct and indirect dependence of the couple on each other's evolution of emotions. Baker [1] examined and extended Noer's theoretical model of the new employment relationship. Relationships among ID (interpersonal deviance), OD (organizational deviance), and their common correlates were meta-analyzed by Berry et al. [3]. It was observed ID and OD were highly correlated and had differential relationships with five key variables and organizational citizenship behaviours, lending support to the separability of ID and OD. Bodankin and Tziner [4] further showed that neuroticism and agreeableness were related to both types of constructive deviance, whereas conscientiousness was associated with both types of destructive deviance. The relationship between pay satisfaction and outcomes was investigated by Currall et al. [6] using multi-level and multi-method data, at the organizational level of analysis.

The main advantage of developing such models is to study the effect of various factors which affect an employment relation between the employee and employer who live in an urban working class social set up. They are educated, mature and thus hold sufficient self-esteem to be confident in presenting their opinion with firmness. Nikolski et al. [11] discussed the distinction between industrial and employment relations. They concluded that employment relations unlike industrial relations are more focussed on individual relationship between employee and employer than the collective ones. The study of interaction of different individuals coupled in the employment system and mathematically modelling them is an important aspect to study the evolution of emotions and growth of different individuals in the corporate sector driven by professionalism and work ethics in principle. The level of authority is limited as observed in the corporate arena but available in adequate amount to be uniformly envisaged between the employee and employer. With passage of time situation arise where both assert their authorities and opinion in accordance to their self-discretion and start affecting each other's emotions depending on reciprocations.

Every argument followed by unsettled grievances intolerance grows which leads to faster exhaustion of faith and respect for each-other's status of equality generating disappointment. Thus decrease in mutual status of equality leads to increase in encounters between both the employer and employee which is used by the intermediates who utilize it to make their position stronger taking advantage. Both the employees and employer may interact through linear or non-linear

interactions in employment relation depending on which the dynamics can be simple or complex respectively. The interaction parameters of the employee-employer relation govern the direction in which the dynamics between them evolves.

These parameters can be controlled in preventing separation of the couple or suppression of any one partner's individual authority at cost of others' dominance and thus conserving the mutual tolerance in the relation by promotion of status of equality in the couple. In direction of understanding the employment relation [17] studied the underlying factors that gives rise to job idiosyncrasies in jobs which required workers to hold non trivial job and task specific skills during the course of employment. It was shown by him that implied demands on the rationality demands limit human actors severely and costs associated with adapting to changing market circumstances and job are considerable jobs of idiosyncratic kind. The significance of implication of conceptualization of company on employee via three models was explained in [12]. Studying the linear stability of nonlinear couple model with two delays analysing characteristic equation [21] observed Hopf bifurcation when sum of delays crossed a critical value. The effect of employment relation and working condition on health inequalities through several psychological, behavioural and physio pathological path ways discussed in [2]. It was observed that management-employee relation in small firms were more satisfactory than large firms due to lower job satisfaction in large firms highlighting the relation of management-employee relation, firm size and job satisfaction [14]. The key factors of employee-employer relationship, its benefits and key role played by job satisfaction in development good employee-employer relation for growth of business was outlined in [18].

After the establishment of mathematical models based on human relation following an interdisciplinary approach in [15] the deviant behaviour observed in employees and employer's corresponding reaction to it was modelled. In this study the deviant behaviour theory and practices were observed by examining the behaviour of employees proposing a series of application to illustrate the usefulness of the mathematical model. In [13] using logistic regression and odd ratios analysed the conjunction of several indicators on employment of citizens in Republic of Macedonia to see the effect of gender, age, ethnicity, education, area of residence and number of family members on employment. At the same time various mathematical models used for studying problem of labour potential particularly describing the stages of process of creation of mathematical model for economic and social systems as whole were discussed in [19]. Recently in [17] it was explored that the organizational structure, efficiency and evolution in perspective of its relationship to bureaucracy are essential in order to construct a mathematical model. The approach used is reminiscent of non-mathematical treatment of organizational system complexity [20]. From the simulation it is revealed that through five stages of development organizations evolve. Marsden [9] explained the diffusion of different employment systems within national economies applying theory of evolutionary games studying their

interaction with established institutions at both national and sectoral levels. In [22] the dynamics of interaction between partners in a marital relation mediated by an intermediate was discussed and studied.

The interactions between the employee and employer mediated by intermediate employee has been modelled in this paper to study how the dynamics evolve between them and growth of mutual intolerance can push the employee-employer relation towards a chaotic phase. Using Lyapunov spectra Marshall and Sprott [10] studied the second-order autonomous and simple conservative chaotic complex variable systems. The significance of using computer simulation for building theories of organization was demonstrated by Lomi and Larsen in [8]. Using different techniques of nonlinear analysis different phases of employee-employer interaction dynamics considering presence of an intermediary have been analysed. The study is significant as in most office interaction system the interaction between the employee and employer are intermediated by some other colleague. It is interesting to see the effect of disappointment level on dynamics of the employee-employer relation in presence of an intermediating colleague.

2. Mathematical model

Let us consider x_e , x_E and x_i represent the emotions of employee, employer and the intermediate colleague in the employee-employer relation. The model is described as follows:

$$\begin{aligned} (1) \quad \dot{x}_e &= c_2x_e - d_2x_E, \\ (2) \quad \dot{x}_E &= c_1x_E - d_1x_e + E_1x_ex_i, \\ (3) \quad \dot{x}_i &= c_3x_i - E_2x_ex_E, \end{aligned}$$

where c_1 =coefficient of compromise of employee

c_2 =coefficient of compromise of employer

c_3 =coefficient of compromise of intermediate colleague

d_1 =coefficient of disappointment of employee

d_2 =coefficient of disappointment of employer

E_1 =coefficient of encounter between employee and intermediate colleague

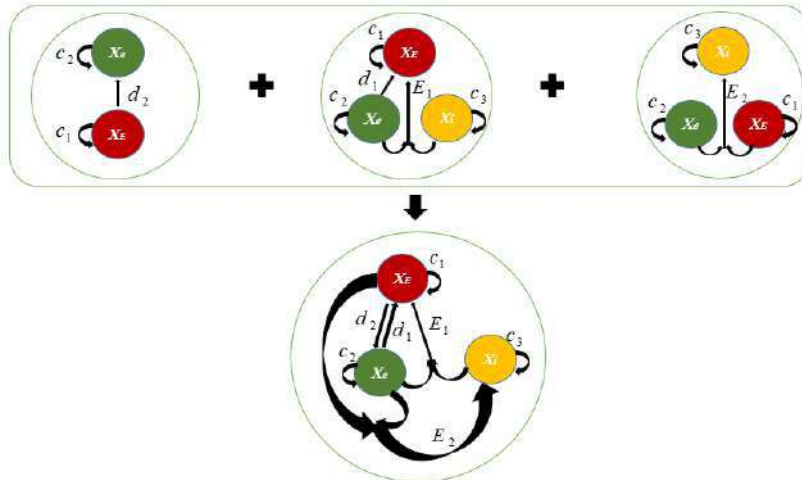
E_2 =coefficient of encounter between intermediate colleague and employer

The employer and employees belonging to an urban educated working class background in corporate sector are being considered. The employment relation is Pluralist in nature as employment relation relations in corporate sector companies is being discussed where employee is both psychologically and economically complex being with equity and voice as stated by Budd (2004) [5]. Both employer and employee are educated and mature individuals with equal rights to voice their opinion following the official protocols. As individuals they are assumed to have had good brought up and moral upbringing. Ideally education brings self-awareness, realisation, maturity and sensitivity thus employer and employees considered in the system are assumed to possess these qualities however to what extent these quality effect these individuals may vary.

Being self-aware and holding self-esteem both employer and employee are encouraged by each other's emotion while they are more cautious about their own respective emotions. Education, maturity, compatibility and sensitivity restrain them from engaging in disputes. It is the emotions of both individuals which cast effect on each other's state while disputes lose their intensity and negligibly affect them by virtue of the compatibility they share as mature and cautious people. The self-esteem make critic of his or her opinions during compromise affect his/her emotional state negatively. Tolerance however casts positive impact on emotional state of the individual as it makes it easier to understand the emotions of other person. The emotional state of intermediary however is affected by the encounter between the employee and employer positively because it strengthens the discretion of intermediary to make his position stronger.

As every individual is different both the employer and employee have different or same level of tendency to compromise and tolerance for each other. The employer with lower tolerance or realisation is bothered and more casual in their consideration of employee's opinion. Due to this the employer acts as a dominant person while employee serves with commitment and respect. The schematic diagram of the system under study is shown in Figure 1.

Sub-interaction systems of Employee-Employer-Intermediate relations within combined Employment System



Combined Employment System with Employee, Employer and Intermediate

Figure 1: Schematic Diagram of the Employment Relation System under Study

3. Fixed point analysis

For the system $eq^n(1) - (3)$ three fixed points exists which are given as follows:

- 1) $(x_e, x_E, x_i) = (0, 0, 0),$
- 2) $(x_e, x_E, x_i) = \left(\sqrt{\frac{(d_1 d_2 - c_1 c_2) c_3}{c_2 E_2 E_1}}, \frac{c_2}{d_2} \sqrt{\frac{(d_1 d_2 - c_1 c_2) c_3}{c_2 E_2 E_1}}, \frac{(d_1 d_2 - c_1 c_2)}{d_2 E_1} \right),$
- 3) $(x_e, x_E, x_i) = \left(-\sqrt{\frac{(d_1 d_2 - c_1 c_2) c_3}{c_2 E_2 E_1}}, -\frac{c_2}{d_2} \sqrt{\frac{(d_1 d_2 - c_1 c_2) c_3}{c_2 E_2 E_1}}, \frac{(d_1 d_2 - c_1 c_2)}{d_2 E_1} \right).$

Jacobian of the system is as follows:

$$J = \begin{bmatrix} c_2 & -d_2 & 0 \\ -d_1 + E_1 x_i & c_1 & E_1 x_e \\ -E_2 x_E & -E_2 x_e & c_3 \end{bmatrix}.$$

For $(x_e, x_E, x_i) = (0, 0, 0)$ the Jacobian is as follows:

$$J = \begin{bmatrix} c_2 & -d_2 & 0 \\ -d_1 & c_1 & 0 \\ 0 & 0 & c_3 \end{bmatrix}.$$

For this the characteristic equation is as follows:

$$\lambda^3 + e_1 \lambda^2 + e_2 \lambda + e_3 = 0,$$

where

$$e_1 = -(c_1 + c_2 + c_3), e_2 = (c_1 c_2 + c_2 c_3 + c_1 c_3 - d_1 d_2), e_3 = -c_3 (d_1 d_2 - c_1 c_2).$$

From Routh-Hurwitz criteria for stability $e_1 > 0, e_3 > 0$ and $e_1 e_2 - e_3 > 0$. Thus, for stability of $(x_e, x_E, x_i) = (0, 0, 0)$ it is required that $d_1 < (\frac{c_1 c_2}{d_2}) = d_0$. For fixed points

$$(x_e, x_E, x_i) = \left(\sqrt{\frac{(d_1 d_2 - c_1 c_2) c_3}{c_2 E_2 E_1}}, \frac{c_2}{d_2} \sqrt{\frac{(d_1 d_2 - c_1 c_2) c_3}{c_2 E_2 E_1}}, \frac{(d_1 d_2 - c_1 c_2)}{d_2 E_1} \right)$$

and

$$(x_e, x_E, x_i) = \left(-\sqrt{\frac{(d_1 d_2 - c_1 c_2) c_3}{c_2 E_2 E_1}}, -\frac{c_2}{d_2} \sqrt{\frac{(d_1 d_2 - c_1 c_2) c_3}{c_2 E_2 E_1}}, \frac{(d_1 d_2 - c_1 c_2)}{d_2 E_1} \right),$$

both show invariance and have same behaviour and conditions of stability. For them the Jacobian is as follows:

$$J = \begin{bmatrix} c_2 & -d_2 & 0 \\ -d_1 + E_1 \left(\frac{(d_1 d_2 - c_1 c_2)}{d_2 E_1} \right) & c_1 & E_1 \left(\sqrt{\frac{(d_1 d_2 - c_1 c_2) c_3}{c_2 E_2 E_1}} \right) \\ -E_2 \left(\frac{c_2}{d_2} \sqrt{\frac{(d_1 d_2 - c_1 c_2) c_3}{c_2 E_2 E_1}} \right) & -E_2 \left(\sqrt{\frac{(d_1 d_2 - c_1 c_2) c_3}{c_2 E_2 E_1}} \right) & c_3 \end{bmatrix}.$$

For this the characteristic equation is as follows:

$$\lambda^3 + e_1\lambda^2 + e_2\lambda + e_3 = 0,$$

where

$$e_1 = -(c_1 + c_2 + c_3), e_2 = \left(c_2c_3 + \frac{d_1d_2c_3}{c_2} \right), e_3 = -2c_3(d_1d_2 - c_1c_2).$$

Thus, for their stability it is required that

$$\left(\frac{c_1c_2}{d_2} \right) < d_1 < \left(\frac{2c_1c_2^2 + c_2^2(c_1 + c_2 + c_3)}{d_2(2c_2 - (c_1 + c_2 + c_3))} \right).$$

4. Results and discussion

Numerical simulation of the above system was carried out for different values of the parameters. The factors on which the parameter values depend upon and the relation through which they have been obtained are as follows:

1. Coefficient of compromise depends on ego and adjustability. The value of coefficient of compromise is determined by following relation:

$$c_i = ego \times W_{ego} - adjustability \times W_{adjustability}$$

2. Coefficient of disappointment depends on self-esteem, maturity and submissiveness of individual. The value of coefficient of disappointment is determined by following relation:

$$d_i = self - esteem \times W_{self-esteem} + maturity \times W_{maturity} \\ + intolerance \times W_{intolerance}$$

3. Coefficient of encounter depend on encounter intensity and given by following relation:

$$E_i = encounter\ intensity \times W_{encounter\ intensity}$$

The different weights (W) attain value as per the following Table1 where the levels of impact of the factors and their weight values are mentioned.

The coefficient of compromise for employee, employer and intermediary for the considered model are determined as follows:

1. The employee is benevolent and loyal to employer he/she has no ego. Employee's tendency to adjust affects his/her discretion up to level of consideration but self-respect restrains it from reaching the level of action.

$$c_2 = ego \times W_{ego} - adjustability \times W_{adjustability} \\ = 1 \times 0 - 1 \times 3 \\ = -3$$

Table 1: Levels of impact of the factors and their weight values

Level 1: Effect	Level 2: Influence	Level 3: Motivating Consideration	Level 4: Motivating Action	Weight Value
No	No	No	No	0
Yes	No	No	No	1
Yes	Yes	No	No	2
Yes	Yes	Yes	No	3
Yes	Yes	Yes	Yes	4

2. The employer being at the top position has ego up to the level of affecting his/her discretion and has lower adjustability as compared to employee.

$$\begin{aligned}
c_1 &= ego \times W_{ego} - adjustibility \times W_{adjustibility} \\
&= 1 \times 1 - 1 \times 2 \\
&= -1
\end{aligned}$$

3. The intermediate colleague has no ego as he or she is not involved in the relation directly but mediate externally by advising. But being an intermediate employee he/she has adjustability up to the level of effecting his/her emotions.

$$\begin{aligned}
c_3 &= ego \times W_{ego} - adjustibility \times W_{adjustibility} \\
&= 1 \times 0 - 1 \times 1 \\
&= -1
\end{aligned}$$

The employee and employer both has education and so their self-esteem motivate actions. Three cases may arise depending on how maturity and intolerance affects employee mentioned as follows:

Case 1. Both employee and employer hold self-esteem and their maturity affects their emotion but none is intolerant at the time of argument and give equal weightage to each other's opinion.

$$\begin{aligned}
d_{i=1,2} = d &= self - esteem \times W_{self-esteem} + maturity \times W_{maturity} \\
&\quad + intolerance \times W_{intolerance} \\
&= 1 \times 4 + 1 \times 1 + 1 \times 0 \\
&= 5
\end{aligned}$$

Case 2. Both employee and employer are submissive but their intolerance start affecting their emotions. Thus despite of giving space to each other's opinion

but they present their outlook firmly without accepting much change.

$$\begin{aligned}
 d_{i=1,2} = d &= self - esteem \times W_{self - esteem} + maturity \times W_{maturity} \\
 &\quad + intolerance \times W_{intolerance} \\
 &= 1 \times 4 + 1 \times 1 + 1 \times 1 \\
 &= 6
 \end{aligned}$$

Case 3. The intolerance of both is at peak affecting his/her action and thus there are disputes, misunderstanding and grievance.

$$\begin{aligned}
 d_{i=1,2} = d &= self - esteem \times W_{self - esteem} + maturity \times W_{maturity} \\
 &\quad + intolerance \times W_{intolerance} \\
 &= 1 \times 4 + 1 \times 1 + 1 \times 4 \\
 &= 9
 \end{aligned}$$

The value of coefficient of interaction are as follows:

1. The encounter between both the employee and employer positively influence the intermediate colleague's emotion to make his position stronger in institution.

$$\begin{aligned}
 E_2 &= encounter\ intensity \times W_{encounter\ intensity} \\
 &= 1 \times 1 \\
 &= 1
 \end{aligned}$$

2. The encounter between the intermediate colleague and employee negatively affects the emotions of employer who desires his/her employees to work with efficiency and not involve in disputes wasting time.

$$\begin{aligned}
 E_1 &= encounter\ intensity \times W_{encounter\ intensity} \\
 &= 1 \times 2 \\
 &= 2
 \end{aligned}$$

The phase portrait, time series and plot of Lyapunov exponent for:

- Case 1 ($c_1 = -1, c_2 = -3, c_3 = -1, d_1 = 5, d_2 = 5, E_1 = 2, E_2 = 1$):
- Case 2 ($c_1 = -1, c_2 = -3, c_3 = -1, d_1 = 6, d_2 = 6, E_1 = 2, E_2 = 1$);
- Case 3 ($c_1 = -1, c_2 = -3, c_3 = -1, d_1 = 9, d_2 = 9, E_1 = 2, E_2 = 1$)

are shown in Figure 2, 3 and 4 respectively. In Figure 5 and 6 the bifurcation diagrams for variation in and have been shown.

Case 1. ($c_1 = -1, c_2 = -3, c_3 = -1, d_1 = 5, d_2 = 5, E_1 = 2, E_2 = 1$). When both employee and employer display zero level of intolerance by virtue of education the stable phase of employee-employer relation exists. This is the starting phase of the dynamics between the employer and employee where the education plays significant role in determining the level of understanding they display to one another. As it is the starting era maturity affect both the employer and employee. Thus none of them is intolerant to others view instead they give equal weightage to each other's opinion. Both share equity and have positive emotion levels despite of encounters and mediation by intermediate colleague which doesn't affect the dynamics much.

Case 2. ($c_1 = -1, c_2 = -3, c_3 = -1, d_1 = 6, d_2 = 6, E_1 = 2, E_2 = 1$). When both the employer and employee begin to get affected by intolerance towards each other's opinion their relation enter critical phase. The employer being dominant and at the top most position no more give weightage to the opinion of employee even if he/she allows employee to present their opinion without hesitation. The compromising employee on the other hand starts getting affected by this treatment from the employer and thus in return begins disregarding the employer's decision. This leads to continuous oscillations in the emotions of the employer and employee. Development of grudge or grievance between the employee and employer begin to occur which doesn't allow restoration of stability. This is the middle phase of the employee-employer relation dynamics where both start getting affected by intolerance towards each other's opinion leading to misunderstanding and thus the stability in the relation doesn't tend to restore. While the employee and employer undergo oscillations in their emotions in negative domain the intermediate continue to have emotions in positive domains. The intermediate thus continues to maintain his/her position with aspirations to make it stronger with the weakening of employee's stature.

Case 3. ($c_1 = -1, c_2 = -3, c_3 = -1, d_1 = 9, d_2 = 9, E_1 = 2, E_2 = 1$). When the mutual intolerance of employee and employer further extend to the level of motivating their action then disputes and misunderstanding occurs causing grievance to both of them. Thus the oscillations in emotions become chaotic for employer and employer with their relation enter chaotic phase. The emotions of the intermediate continue to be positive and he/she now holds stronger position than employee with the employer and employee turning completely intolerant towards each other. From the bifurcation plot in Figure 5 for variation in $d_1 = d_2 = d$ parameter all the three stages of employment dynamics are clearly observed. The critical value observed through bifurcation diagram for variation in $d_1 = d_2 = d$ is 5.26 while the value obtained from fixed point analysis is also 5.26 as per the relation

$$d_1 < \left(\frac{2c_1c_2^2 + c_2^2(c_1 + c_2 + c_3)}{d_2(2c_2 - (c_1 + c_2 + c_3))} \right).$$

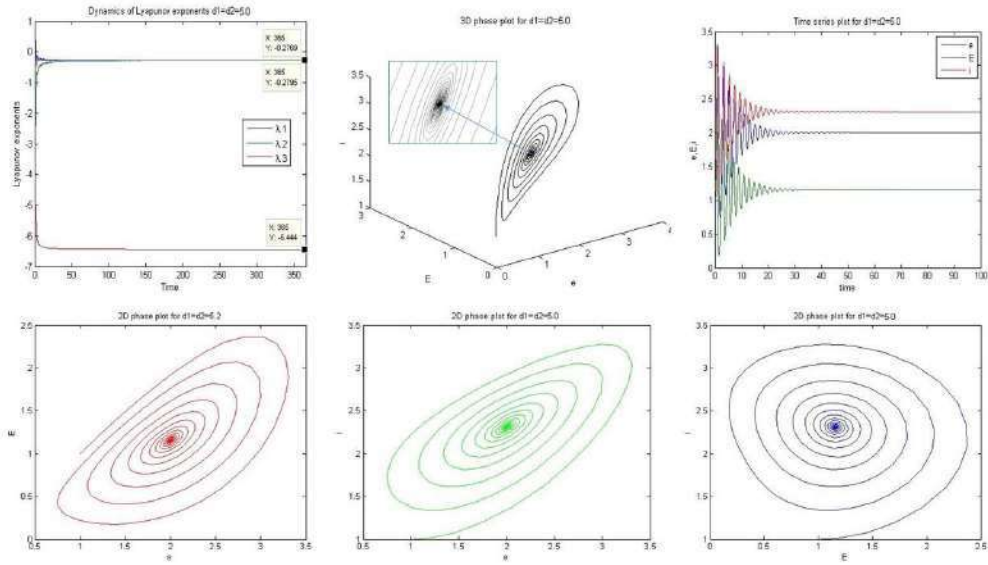


Figure 2: 3D and 2D Phase Plots, Time series Plot, Lyapunov Exponent Plot for Case 1 ($c_1 = -1, c_2 = -3, c_3 = -1, d_1 = 5, d_2 = 5, E_1 = 2, E_2 = 1$), Stable Phase

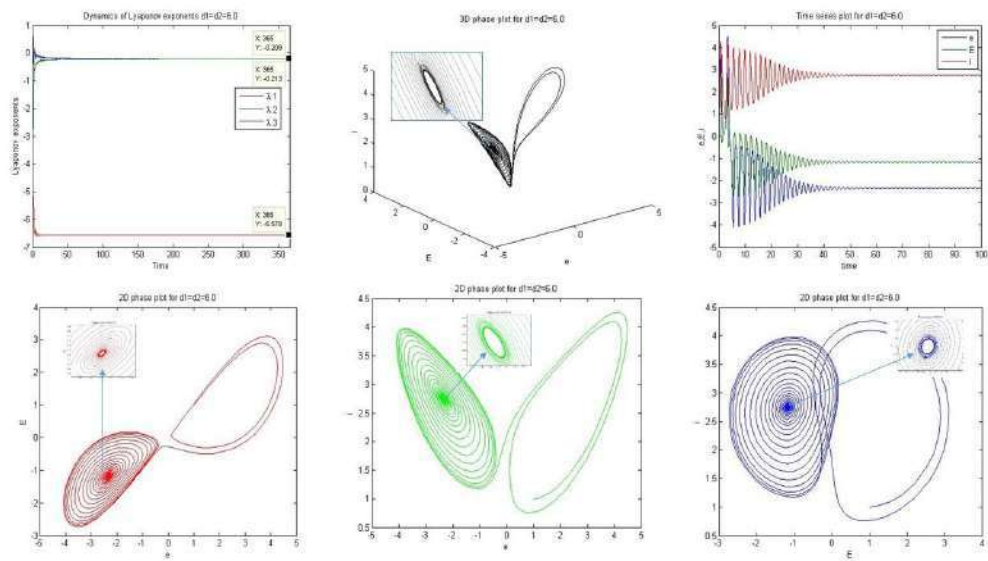


Figure 3: 3D and 2D Phase Plots, Time series Plot, Lyapunov Exponent Plot for Case 1 ($c_1 = -1, c_2 = -3, c_3 = -1, d_1 = 6, d_2 = 6, E_1 = 2, E_2 = 1$), Critical Phase

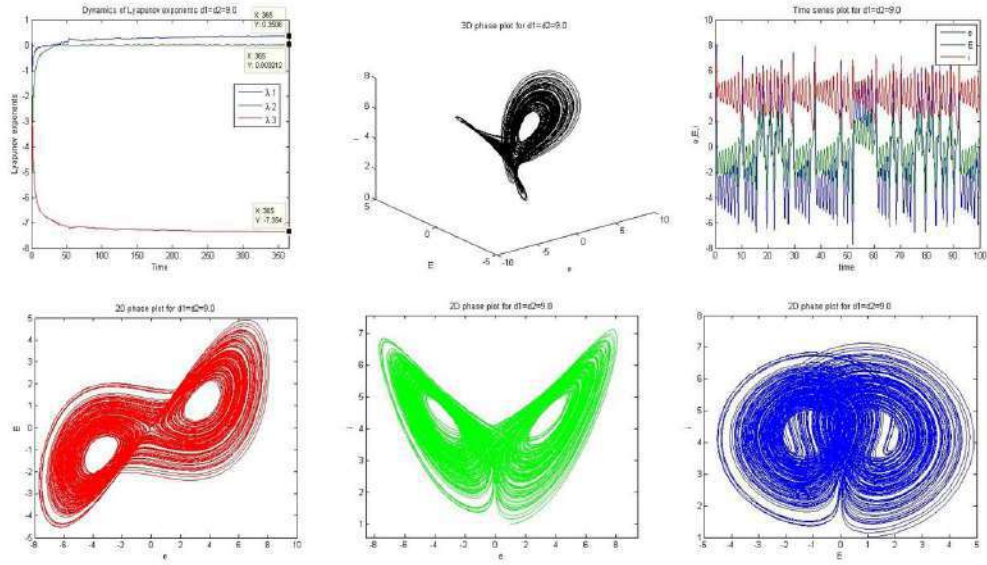


Figure 4: 3D and 2D Phase Plots, Time series Plot, Lyapunov Exponent Plot for Case 1($c_1 = -1, c_2 = -3, c_3 = -1, d_1 = 9, d_2 = 9, E_1 = 2, E_2 = 1$),, Chaotic Phase

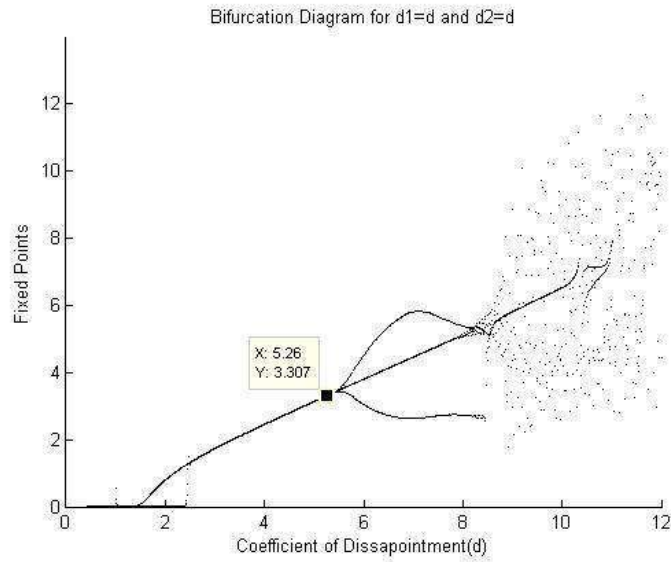


Figure 5: Bifurcation diagram for variation in $d_1=d_2=d$ parameter

In case of employee and employer having unequal level of disappointment and intolerance then it is observed that if employee has more intolerance then system enters chaotic phase faster as evident from bifurcation plot in Figure 6.

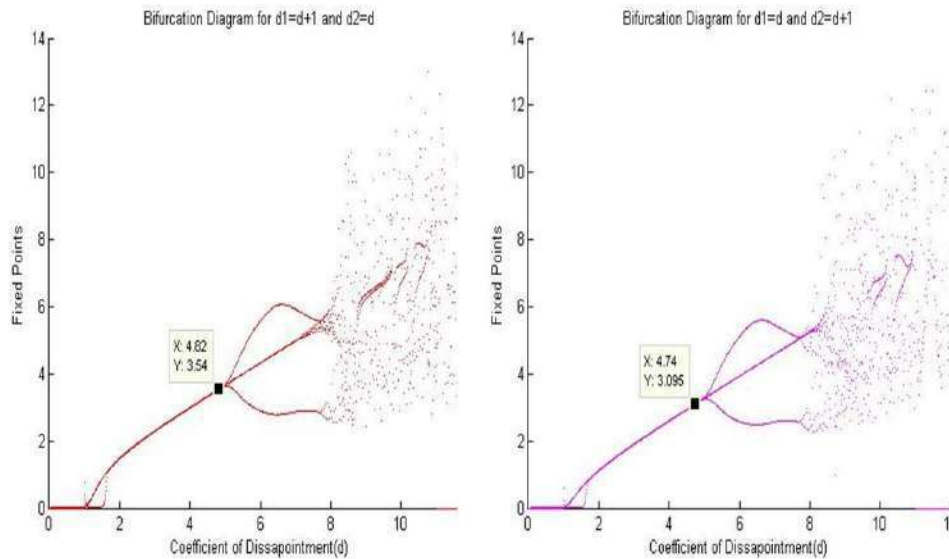


Figure 6: Bifurcation diagram for variation in unequal level of coefficient of disappointment

5. Conclusion

In this paper the pluralist employment relation in a corporate system has been modelled. From stability analysis the conditions between the parameters for the stability of system has been determined. The validity of the condition has been verified from Phase plots, Lyapunov plots and Bifurcation plots obtained from simulation of system at different parameter values. The stable, critical and chaotic stages has been observed with variation in levels of disappointment level of both the employee and employer. When employee and employer hold unequal amount of disappointment then the system transits to chaos faster if employee has more disappointment than employer. It is concluded that if employee holds more tolerance then the relation between employee and employer sustain despite of intermediate colleague mediating their interactions.

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CANOPY, COBWEB, DBSCAN clustering data mining techniques: study and analysis

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Abstract. Clustering is a process of grouping objects belonging to similar class or kind of objects which are collected and put together in same cluster or else grouped in other cluster based on similarity. The study focuses on the CANOPY, COBWEB and DBSCAN methods of clustering to cluster weather parameters in order to gain insight of the pattern being followed. Clustering technique as an unsupervised form of learning in Data mining helps in providing insight into the distribution of data to visualise and analyze characteristic of each cluster. The data considered in the study is of daily weather parameters for Delhi region from 1st January, 2017 to 31st October, 2018. The clustering of data is carried out to study the nature using different methods of clustering and the efficiency of these methods are then compared to assess the best suited method based on time taken to form clusters.

Keywords: Canopy, Cobweb, DBSCAN, data mining, weather parameters.

1. Introduction

Clustering is the process of making group of abstract objects into classes of same objects. The objects are grouped together due to similarity in them, either on the basis of pattern or nature. Cluster of the data values are taken as one group. The algorithm for calculating maximum likelihood from incomplete data at different levels has been done. The study was done on the basis of many examples that are sketched, including missing value situations, applications to grouped, censored or truncated data, finite mixture models, variance component estimation, hyperparameter estimation, iteratively re-weighted least squares and

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factor analysis [1]. AI with wavelet decomposition conjunction model was applied for analyzing the river water quality [2]. Hybrid soft computing techniques were also used for studying the behavior of Covid-19 using the data [3]. The machine learning bases assessment of HIV epidemiology was carried out for the Asian region in [4]. Also for nonlinear systems a lot of studies have been carried out using stability analysis in different fields of ecology [6] and physics [5].

For instance different methods such as consensus algorithm, developing a system using Blockchain for prevention of Diabetes, numerical simulation for analyzing the COVID-19 data to understand the situation and analyze, optimizing techniques, evolutionary techniques, model development for forecasting using neural networks, study of effects of magnetic and temperature variation on aluminum oxide was done [7-10, 13-15]. Under time series analysis autoregressive methods have been also used to study the time series of rainfall on daily basis and further the best suited model on the basis of AIC was selected to forecast rainfall [11].

Different parameters of weather were considered to study the pattern of the weather parameters and the classification and clustering techniques were used to study the parameters and the best model among classification and clustering were chosen on the basis of least error in classification techniques and the model taking less time was selected for the clustering methods [12]. The fractal and wavelet to study air and water pollutants behavior were studied by calculating fractal dimension, Hurst exponent and predictability index; further concluding time series showing Brownian behavior [16]. Another study of environmental pollutants was done using nonlinear techniques was done [17] while analysis of spread of COVID-19 for China region was done [18]. Soft computing methods have been used to study water quality in [19]. On the other hand EM clustering algorithm was studied in order to cluster 2 stochastic versions [20]. In [21] Forecasting of weather has been done using K-Means clustering was done while fuzzy randomness was studied in [22]. In article [23] derivation and uses of measures of similarity among 2 hierarchical clusters was studied.

The Density-based clusters are separated from each other by contiguous regions of low density of objects in the article [24]. A general algorithm is studied which includes the application of James-Stein type adjustment further in which the James-Stein shrinkage estimators act as the new centroids in the next clustering iteration until convergence, further the testing of accuracy has been done by using real data example [25]. The clustering methods have been examined in high dimension, EM algorithm significantly outperforms the other methods, and proceed to investigate the effect of various initialization schemes on the final solution produced by the EM algorithm [26]. It is a density-based clustering algorithm because it finds a number of clusters starting from the estimated density distribution of corresponding nodes. DBSCAN is one of the most common clustering algorithms and also most cited in scientific literature [27].

In this study a new clustering technique is introduced which will not just make clusters of data and will also create proper order of data base showing density based clustering structures [28]. The ranking and clustering method is used for aggregating inconsistent information [29]. The study showed that the hierarchical techniques show better results than the K means method in terms of connectivity [30]. The study is about effective methods for spatial data mining [31]. SLINK algorithm carries out single-link (nearest-neighbor) cluster analysis on an arbitrary dissimilarity coefficient and provides a representation of the resultant dendrogram which can readily be converted into the usual tree-diagram [32]. The new hybrid algorithm were compared to existing clustering algorithms on the basis of different measures and it was concluded that hybrid clustering algorithm was more accurate [33].

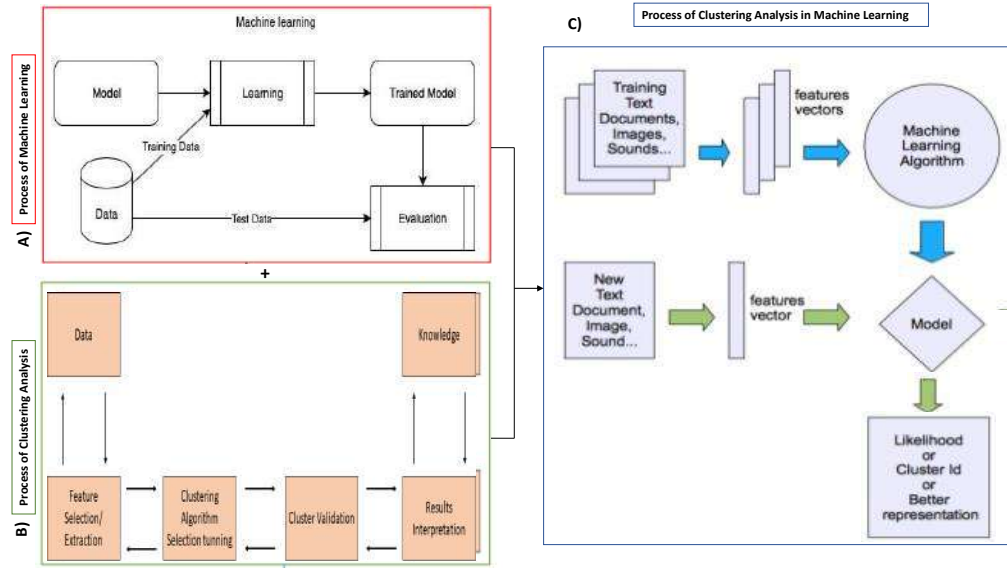
Hierarchical clustering method was studied and its performance was evaluated in [34]. Time series analysis of COVID-19 in Saudi Arabia was carried out [35]. The study introduced new concepts of density threshold and adaptive density using the p-DBSCAN algorithm [36]. In this study the method analyzes source-specific clustering and identifies a consensus set-partition which is as close as possible to all of them [37]. The study includes application of K-Means algorithm and K-prototypes algorithm with application on real world data and both the algorithms showed good results on their application [38].

The study includes application of different methods for modeling the uncertainty of NWP forecasts, various clustering algorithms are applied to group the performance records as the first step, second step was a range of methods are used to fit appropriate probability distributions to errors of each cluster, Results show that incorporating trained uncertainty model outputs into the NWP point predictions provides PI forecasts with higher reliability and skill. This can lead to improvement of decision processes for many applications that rely on these forecasts [39].

In this study different clustering techniques were used to cluster the weather data on the basis of the already provided training data in order to for clusters of different weather parameters on the basis of understanding the nature of different weather parameters from training set, further the best clustering method will be chosen on the basis of least time taking method for both training full set and building model. The clusters were later compared to the actual data of the temperature Maximum and Minimum and rainfall in order to see where the clusters formed were appropriate or not.

Data mining is a procedure of extracting information from existing set of data. Extraction of knowledge about the data in terms of pattern and nature is termed as data mining. Machine learning is divided into 3 main categories such as: Supervised Learning, Unsupervised Learning, and Reinforcement Learning. The basic steps of machine learning are shown in Figure 1 a). Unsupervised learning is a type of learning that helps in learning and understanding which helps us in understanding, discovering patterns in our data related to various features. Two of the main methods used in unsupervised learning are: Principle

Figure 1: Flow Chart Diagram of Analysis Technique in the study



component Analysis and Cluster Analysis. Clustering analysis is a part of data mining which helps in looking into the distribution of data and analyse the characteristics of every cluster formed.

A flowchart of the process of clustering analysis is shown in Figure 1 b). The process of clustering analysis in Machine Learning is as shown in Figure 3 c). The clustering techniques discussed in the present study are namely CANOPY, COBWEB, DBSCAN under the cluster analysis in order to analyze that whether these tools serve the purpose of clustering data in the correct cluster and how much time does each tool take to cluster the data values so that the study of the patterns can be done. The time taken by the tools is taken is considered in two ways namely Time taken to train full data(T1) and Time taken to make model(T2).

2. Methodology

(i) CANOPY

This algorithm is unsupervised learning algorithm. Objective of the algorithm is to fasten up clustering operations on huge data set, but in terms of handling the large data sets a few may not work or may not provide desired results due to the size. The algorithm of canopy clustering works as follows, where X and Y are two thresholds and X is greater than Y:

- Extract the data set to be used.
- Exclude one value from the set, start new 'canopy' having the value.

- For every value remaining, mark it to a new ‘canopy’ further if distance to first value of canopy is lesser than loose distance X.
- Now, if value is at a distance which is relatively lesser than Y, exclude it from original set.
- Repeat second step till the time no more data values are left to cluster.
- The comparatively poorly clustered canopies can further be sub-clustered using expansive but more precise algorithm.

The canopy algorithm is useful as the number of training data which can be compared at every step is decreasing and also the resulting clusters are improved.

(ii) COBWEB

The cobweb clustering algorithm is a simple method of incremental conceptual learning. The algorithm forms a hierarchical group in the form of a classification tree. Every node is a concept and has a probabilistic depiction of the concept. At a given level the sibling nodes do form a partition and for mentioned as follows, the rise in the number of values which can be correctly guessed refers to category utility.

$$(1) \quad \frac{\sum_{k=1}^n P(C_k) \left[\sum_i \sum_j P(A_i = v_{ij} | C_k)^2 - \sum_i \sum_j P(A_i = v_{ij})^2 \right]}{n}.$$

There are four operations that Cobweb uses while making tree. The application of the operations is based on category utility. The operations are as follows:

- Meeting 2 nodes: meeting of two nodes refers to substitute them with a node, sub node of which is the collection of original nodes’ set of sub nodes and objects classified under them are summarizes.
- Ripping a node: subnodes substitute the node ripped.
- Adding a new node: When an object is inserted into the tree a node is created correspondingly.
- Shifting values in ladder: objective is to applying COBWEB algo on values and the subtree rooted in the node.

(iii) DBSCAN

DBSCAN algo is a clustering algo which is a technique in data mining and machine learning. On the basis of the set of points, DBSCAN algorithm groups values which are near to each other on the basis of measured distance that is EuclideanDistance formula & a least number of points. DBSCAN algorithm needs two parameters:

- Eps: eps determines the distance between the points for them to be considered as part of cluster. This means that the distance in between 2 values less than or equals to this value (eps), these values are taken as neighbours.
- Min points: least number of values to make a dense region. Eg: mark min points parameter to 15, further minimum 15 values are needed to form the dense region.

The estimation of parameter is important in data mining. The basic knowledge about the data is needed to choose the correct parameters. If the value of eps is very small, then a huge portion of the values will be grouped. If the number of values required to form dense region is not satisfied then the values will be removed. Taking too large values will also not be good as maximum values will come in one group and the study will get affected. Choose eps on the basis of distance of data values, but generally eps value is taken small. In general terms, least min points can be calculated from the value of dimensions (d) in the data set, as min points are greater or equals to $d+1$. More values are considered to be appropriate for data set which has noise and makes better clusters. The least value for the min points should be 3, but the larger the number of points in the data set, larger the min points value that must be chosen. The DBSCAN algorithm is applied to look for associations and structures in data which are hard to allocate manually but it can be easy and quite of use to see if there exists pattern and forecast trends.

3. Results and discussion

The daily weather data for different parameters is taken into consideration from 1st January, 2017 to 31st October, 2018 for Delhi region. The parameters taken are Maximum Temperature, Wind Speed, Minimum Temperature, Evaporation, Rainfall and Bright Sunshine whose time series are plotted in Figure 2 a)-f). The study includes clustering of data into clusters to study the nature and also to analyse which method serves as the better method amongst others based the time taken to form clusters. The time taken by the tools is taken is considered in two ways namely Time taken to train full data (T1) and Time taken to make model (T2).

The Table 1 shows the time taken to train the full data set and to build the model. It is seen very clearly that Canopy took lesser time to make clusters that will help in better and faster study of the clusters. Canopy in total formed twenty one clusters in the lesser time. Figure 2 g)-h) shows that Canopy clustering method takes lesser time in both situations and hence, can be chosen for making clusters. The Table 1 shows the time taken to train the full data set and to build the model. It is seen very clearly that Canopy took lesser time to make clusters that will help in better and faster study of the clusters. Canopy in total formed twenty one clusters in the lesser time.

Figure 2: a)-f) Weather Data Time Series Plot and g)-h) Clustering Technique Performance Comparison Plot

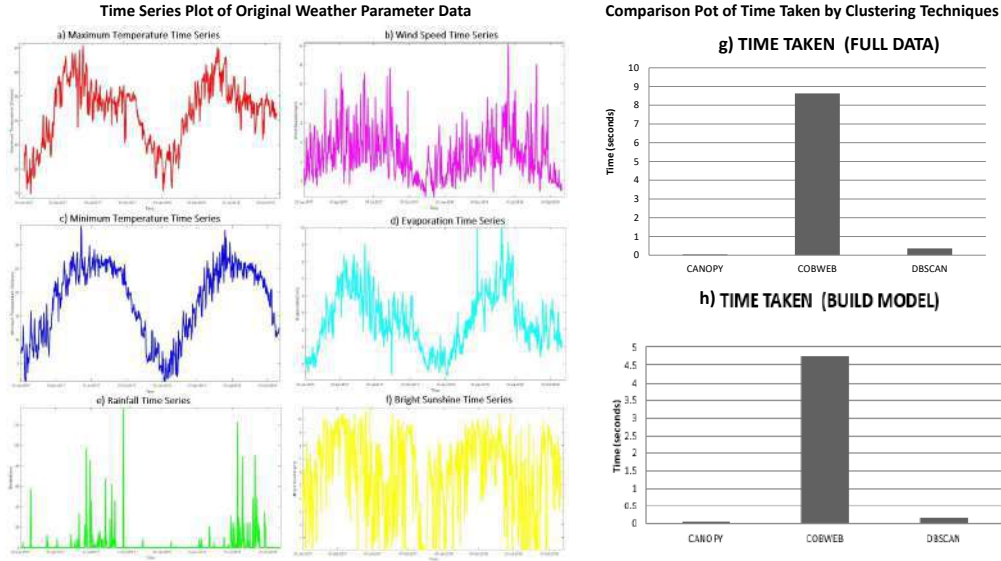


Table 1: Comparison of T1 and T2 for different clustering tools

Clustering Tools	Time Taken (Full Data: T1)	Time Taken (Build Model: T2)
CANOPY	0.04	0.01
COBWEB	8.63	4.76
DBSCAN	0.36	0.16

4. Conclusion

In the field of data analysis Clustering of data is an important problem. Clustering analysis is done in order to study the pattern and nature of the data. The objective of the study was to analyse different clustering methods and to see which among them is best suited for clustering of the weather time series data of different weather parameters. For the study the daily data of different weather parameters for Delhi region from 1st January 2017 to 31st October 2018 has been considered. Different clustering techniques such as CANOPY, COBWEB and DBSCAN were applied and compared for their efficiency to cluster data based on how much time was taken to build the model and process the full testing data to the assigned clusters. From the comparison of the performance efficiency for all the three methods it was observed that the least processing time was taken by CANOPY method. This result can be further analysed and understood in terms of the faster algorithm of CANOPY method and as the

resulting clusters are improved so better efficiency is observed on using it for clustering of the considered data. Clustering technique extracts knowledge from the data by grouping the data values of same pattern or nature in one group and hence provided insight about the data and hence helps in studying the nature of data. Thus selection of an effective and efficient clustering technique in terms of processing time is essential. It can be concluded from the analysis that CANOPY technique is faster and better in clustering the daily weather data in comparison to other methods compared.

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Coupled fixed point theorem in quasi metric space

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Abstract. In the present paper a unique common coupled fixed point theorem has been proven for quasi metric space with modified - ω distance function. This result is improvement, modification and extension in the study of quasi metric space. An example has been given to illustrate the work.

Keywords: fixed point, quasi-metric space, coupled fixed point, modified- ω distance.

1. Introduction and preliminaries

The fixed point theory is one of the important topics of functional analysis. The whole metric fixed point theory based on a very powerful theorem Banach Contraction principle [5]. Since then many researchers worked on it and develop the results in different directions.

Bhaskar and Lakshmikantham [6] initiated the concept of coupled fixed point in the following way:

Definition 1.1 ([6]). An element $(a, b) \in X \times X$ is called a coupled fixed point of mapping $J : X \times X \rightarrow X$ if $J(a, b) = a, J(b, a) = b$.

Further several researchers Lakshmikantham and Ćirić [10], Choudhary and Kundu [8], Luong et al. [11], Razani and Parvaneh [12], Alsulami [3], Samet et al. [13], Alotaibi [2] extended the coupled fixed point theorems in partial order metric space under different constraints but it is not found in quasi metric space, so the authors are motivated towards the present work.

Remark 1. Every metric space is quasi metric space but the converse is not true.

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Remark 2. The results which are true for quasi metric space need not be true for metric space.

Wilson [16] introduced the concept of quasi metric space as here under:

Definition 1.2 ([16]). The function $q : X \times X \rightarrow [0, \infty)$ is a quasi-metric if it satisfies

- (i) $q(a, b) = 0 \Leftrightarrow a = b$;
- (ii) $q(a, b) \leq q(a, c) + q(c, b)$, for all $a, b, c \in X$.

The pair (X, q) is called quasi metric space.

The study of fixed point theorems on quasi metric space further continued by Aydi, et al. [4], Bilgili, et al. [7], Shatanawi, et al. [14], Shatanawi, et al. [15], Alegre, et al. [1].

Definition 1.3 ([4,9]). A sequence $\{a_l\}$ converges to $a \in X$ if $\lim_{l \rightarrow \infty} q(a_l, a) = \lim_{l \rightarrow \infty} q(a, a_l) = 0$.

Definition 1.4 ([9]). Let $\{a_l\}$ be a sequence in X . Then:

- (i) $\{a_l\}$ is called left Cauchy if for any $\delta > 0, \exists N_0 \in N$, such that $q(a_l, a_m) < \delta \forall l \geq m > N_0$
- (ii) $\{a_l\}$ is called right Cauchy if for any $\delta > 0, \exists N_0 \in N$, such that $q(a_l, a_m) < \delta \forall m \geq l > N_0$.

Definition 1.5 ([4,9]). $\{a_l\}$ is called Cauchy sequence if for any $\delta > 0, \exists N_0 \in N$ such that $q(a_l, a_m) \leq \delta \forall l, m > N_0$ or $\{a_l\}$ is right and left Cauchy.

Definition 1.6 ([1]). The modified $-\omega$ distance on (X, q) is a function $p : X \times X \rightarrow [0, \infty)$ which satisfies the following

- (Q1) $p(a_1, a_2) \leq p(a_1, a_3) + p(a_3, a_2), \forall a_1, a_2, a_3 \in X$.
- (Q2) $p : X \rightarrow [0, \infty)$ is lower semi continuous $\forall a \in X$.
- (Q3) $\forall \varepsilon > 0 \exists \delta > 0$ such that $p(a_1, a_2) \leq \delta, p(a_2, a_3) \leq \delta \Rightarrow q(a_1, a_3) \leq \varepsilon \forall a_1, a_2, a_3 \in X$.

2. Main result

The aim of this paper is to establish coupled fixed point theorems in quasi metric space.

Theorem 2.1. *Let (X, q) be a complete quasi metric space equipped with an mw distance mapping p , also J and $K: X \times X \rightarrow X$ be two continuous functions such that the pair (J, K) satisfy*

$$(2.1) \quad p\{J(a, b), K(c, d)\} \leq h \max\{p(a, J(a, b)), p(c, K(c, d))\},$$

$$(2.2) \quad p\{K(a, b), J(c, d)\} \leq h \max\{p(a, K(a, b)), p(c, J(c, d))\}.$$

here $h \in [0, 1)$. Then J and K have unique common coupled fixed point.

Proof. Let $a_{2l+1} = J(a_{2l}, b_{2l}), b_{2l+1} = J(b_{2l}, a_{2l})$,

$$a_{2l+2} = K(a_{2l+1}, b_{2l+1}), b_{2l+2} = K(b_{2l+1}, a_{2l+1}).$$

Now, to prove that $p(a_l, a_{l+1}) = 0$ or $p(a_{l+1}, a_l) = 0$ and $p(b_l, b_{l+1}) = 0$ or $p(b_{l+1}, b_l) = 0$.

Case I. Consider $p(a_l, a_{l+1}) = 0$, for $l = 2n$,

$$\begin{aligned} p(a_{2n+1}, a_{2n+2}) &= p\{J(a_{2n}, b_{2n}), K(a_{2n+1}, b_{2n+1})\}, \\ &\leq h \max\{p(a_{2n}, J(a_{2n}, b_{2n})), p(a_{2n+1}, K(a_{2n+1}, b_{2n+1}))\} \\ &= h \max\{p(a_{2n}, a_{2n+1}), p(a_{2n+1}, a_{2n+2})\}. \end{aligned}$$

(i) If $p(a_{2n}, a_{2n+1})$ is maximum, then $p(a_{2n+1}, a_{2n+2}) = 0$.

(ii) If $p(a_{2n+1}, a_{2n+2})$ is maximum, then $p(a_{2n+1}, a_{2n+2}) = 0$,

$$(2.3) \quad p(a_{2n+1}, a_{2n+2}) = 0.$$

Now,

$$\begin{aligned} p(b_{2n+1}, b_{2n+2}) &= p\{J(b_{2n}, a_{2n}), K(b_{2n+1}, a_{2n+1})\} \\ &\leq h \max\{p(b_{2n}, J(b_{2n}, a_{2n})), p(b_{2n+1}, K(b_{2n+1}, a_{2n+1}))\} \\ &= h \max\{p(b_{2n}, b_{2n+1}), p(b_{2n+1}, b_{2n+2})\}. \end{aligned}$$

(i) If $p(b_{2n}, b_{2n+1})$ is maximum then $p(b_{2n+1}, b_{2n+2}) = 0$.

(ii) If $p(b_{2n+1}, b_{2n+2})$ is maximum then $p(b_{2n+1}, b_{2n+2}) = 0$,

$$(2.4) \quad p(b_{2n+1}, b_{2n+2}) = 0.$$

Now,

$$\begin{aligned} p(a_{2n+2}, a_{2n+1}) &= p\{K(a_{2n+1}, b_{2n+1}), J(a_{2n}, b_{2n})\} \\ &\leq h \max\{p(a_{2n+1}, K(a_{2n+1}, b_{2n+1})), p(a_{2n}, J(a_{2n}, b_{2n}))\} \\ &= h \max\{p(a_{2n+1}, a_{2n+2}), p(a_{2n}, a_{2n+1})\}. \end{aligned}$$

- (i) $p(a_{2n+1}, a_{2n+2})$ is maximum then $p(a_{2n+2}, a_{2n+1}) = 0$.
(ii) If $p(a_{2n}, a_{2n+1})$ is maximum then $p(a_{2n+2}, a_{2n+1}) = 0$,

$$(2.5) \quad p(a_{2n+2}, a_{2n+1}) = 0.$$

Similarly one can show that

$$(2.6) \quad p(a_{2n}, a_{2n+2}) = 0, p(b_{2n}, b_{2n+2}) = 0$$

using Q1. From (2.1.5), (2.1.6) and Q3,

$$(2.7) \quad q(a_{2n}, a_{2n+1}) = 0 \begin{cases} \because q(a_l, a_{l+1}) = 0 \Rightarrow a_l = a_{l+1} = J(a_l, b_l), \\ q(b_l, b_{l+1}) = 0 \Rightarrow b_l = b_{l+1} = K(b_l, a_l). \end{cases}$$

(a_l, b_l) is coupled fixed point.

Also, (2.1.8) $p(a_{2n}, a_{2n+1}) \leq p(a_{2n}, a_{2n+2}) + p(a_{2n+2}, a_{2n+1}) = 0$. From (2.1.3), (2.1.8) and Q3

$$(2.8) \quad \begin{aligned} q(a_{2n}, a_{2n+2}) &= 0, \\ q(a_{2n+1}, a_{2n+2}) &\leq q(a_{2n+1}, a_{2n}) + q(a_{2n}, a_{2n+2}) = 0. \end{aligned}$$

Similarly, $q(b_{2n+1}, b_{2n+2}) \leq q(b_{2n+1}, b_{2n}) + q(b_{2n}, b_{2n+2}) = 0$. Hence,

$$\begin{aligned} a_{2n} = a_{2n+1} = a_{2n+2} &\Rightarrow a_l = a_{l+1} = a_{l+2} \\ &\Rightarrow a_l = J(a_l, b_l) = K(a_{l+1}, b_{l+1}), \end{aligned}$$

when $l \rightarrow \infty$, as J and K are continuous $a = J(a, b) = K(a, b)$.

Also, $b_{2n} = b_{2n+1} = b_{2n+2} \Rightarrow b_l = b_{l+1} = b_{l+2} \Rightarrow b_l = J(b_l, a_l) = K(b_{l+1}, a_{l+1})$, when $l \rightarrow \infty$, as J and K are continuous $b = J(b, a) = K(b, a)$. Thus, (a, b) is common coupled fixed point of J and K .

Case II: When l is odd, $l = 2n + 1$,

$$(2.9) \quad \begin{aligned} p(a_{2n+1}, a_{2n+2}) &= 0, \\ p(a_{2n+2}, a_{2n+3}) &= p\{K(a_{2n+1}, b_{2n+1}), J(a_{2n+2}, b_{2n+2})\} \\ &\leq h \max\{p(a_{2n+1}, K(a_{2n+1}, b_{2n+1})), p(a_{2n+2}, J(a_{2n+2}, b_{2n+2}))\} \\ &= h \max\{p(a_{2n+1}, a_{2n+2}), p(a_{2n+2}, a_{2n+3})\}. \end{aligned}$$

This implies

$$(2.10) \quad p(a_{2n+2}, a_{2n+3}) = 0.$$

Also, it can be proved that

$$(2.11) \quad p(a_{2n+3}, a_{2n+2}) = 0, p(b_{2n+2}, b_{2n+3}) = 0, p(b_{2n+3}, b_{2n+2}) = 0.$$

Now, by Q1

$$(2.12) \quad p(a_{2n+1}, a_{2n+3}) \leq p(a_{2n+1}, a_{2n+2}) + p(a_{2n+2}, a_{2n+3}) = 0.$$

From (2.1.12), (2.1.13) and Q₃

$$(2.13) \quad q(a_{2n+1}, a_{2n+2}) = 0.$$

Also, $q(a_{2n+1}, a_{2n+3}) = 0$. Similarly, $q(b_{2n+1}, b_{2n+2}) = 0$ and $q(b_{2n+1}, b_{2n+3}) = 0$. From (2.1.10), (2.1.11) and Q₃

$$(2.14) \quad \begin{aligned} q(a_{2n+1}, a_{2n+3}) = 0, a_{2n+1} = a_{2n+2} = a_{2n+3} &\Rightarrow a_l = a_{l+1} = a_{l+2} \\ &\Rightarrow a_l = J(a_l, b_l) = K(a_{l+1}, b_{l+1}). \end{aligned}$$

Similarly, one can prove that $\Rightarrow b = J(b, a) = K(b, a)$. Thus, (a, b) are common coupled fixed point of J and K .

Similarly, one can show that if $p(a_{l+1}, a_l) = 0$, (a, b) are common coupled fixed point of J and K . Now, assume that $p(a_l, a_{l+1}) \neq 0, p(a_{l+1}, a_l) \neq 0$. Then,

$$(2.15) \quad \begin{aligned} p(a_{2n+1}, a_{2n}) &= p((Ja_{2n}, b_{2n}), K(a_{2n+1}, b_{2n+1})) \\ &\leq h \max\{p(a_{2n}, J(a_{2n}, b_{2n})), p(a_{2n+1}, K(a_{2n+1}, b_{2n+1}))\} \\ &= h \max\{p(a_{2n}, a_{2n+1}), p(a_{2n+1}, a_{2n+2})\}. \end{aligned}$$

(i) If $p(a_{2n}, a_{2n+1})$ is maximum, then $p(a_{2n+1}, a_{2n+2}) \leq hp(a_{2n}, a_{2n+1})$.

(ii) If $p(a_{2n+1}, a_{2n+2})$ is maximum, then $p(a_{2n+1}, a_{2n+2}) = 0$, which is contradiction.

Hence, $p(a_{2n+1}, a_{2n+2}) \leq hp(a_{2n}, a_{2n+1})$. By similar process we can prove that

$$p(a_{2n}, a_{2n+1}) \leq hp(a_{2n-1}, a_{2n}).$$

Also, we can prove

$$p(b_{2n+1}, b_{2n+2}) \leq hp(b_{2n}, b_{2n+1})$$

and

$$p(b_{2n}, b_{2n+1}) \leq hp(b_{2n-1}, b_{2n}).$$

Thus,

$$(2.16) \quad p(a_l, a_{l+1}) \leq hp(a_{l-1}, a_l) \text{ and } p(b_l, b_{l+1}) \leq hp(b_{l-1}, b_l).$$

Now,

$$(2.17) \quad \begin{aligned} p(a_{l+1}, a_l) &= p(J(a_l, b_l), K(a_{l-1}, b_{l-1})) \\ &\leq h \max\{p(a_l, J(a_l, b_l)), p(a_{l-1}, K(a_{l-1}, b_{l-1}))\} \\ &= h \max\{p(a_l, a_{l+1}), p(a_{l-1}, a_l)\}, \\ p(a_{l+1}, a_l) &\leq hp(a_{l-1}, a_l). \end{aligned}$$

Repeating l times

$$(2.18) \quad p(a_{l+1}, a_l) \leq h^l p(a_1, a_0).$$

Also, one can prove

$$(2.19) \quad p(a_l, a_{l+1}) \leq h^l p(a_0, a_1).$$

Similarly, we can prove that $p(b_{l+1}, b_l) \leq h^l p(b_1, b_0)$ and $p(b_l, b_{l+1}) \leq h^l p(b_0, b_1)$. Thus, as $l \rightarrow \infty$, $p(a_l, a_{l+1}) = 0$, $p(a_{l+1}, a_l) = 0$, $p(b_l, b_{l+1}) = 0$, $p(b_{l+1}, b_l) = 0$. Now, to prove $\{a_l\}$ and $\{b_l\}$ are Cauchy sequences, we need to prove $\lim_{s,t \rightarrow \infty} p(a_s, a_t) = 0$, for each $s, t \in N$.

Case I: If s is odd and t is even with $s < t$, then

$$\begin{aligned} p(a_s, a_t) &= p(Ja_{s-1}, b_{s-1}), K(a_{t-1}, b_{t-1}) \\ &\leq h \max\{p(a_{s-1}, J(a_{s-1}, a_{s-1})), p(a_{t-1}, K(a_{t-1}, a_{t-1}))\} \\ &= h \max\{p(a_{s-1}, a_s), p(a_{t-1}, a_t)\} \\ &= hp(a_{s-1}, a_s). \end{aligned}$$

Thus, we have $p(a_s, a_t) \leq h^s p(a_0, a_1)$.

Let $s, t \rightarrow \infty$. Then,

$$(2.20) \quad \lim_{s,t \rightarrow \infty} p(a_s, a_t) = 0,$$

with $s < t$. Similarly $\lim_{s,t \rightarrow \infty} p(b_s, b_t) = 0$, with $s < t$.

Case II: If s is odd and t is even with $s > t$

$$\begin{aligned} p(a_s, a_t) &= p(J(a_{s-1}, b_{s-1}), K(a_{t-1}, b_{t-1})) \\ &\leq h \max\{p(a_{s-1}, J(a_{s-1}, b_{s-1})), p(a_{t-1}, K(a_{t-1}, b_{t-1}))\} \\ &= h \max\{p(a_{s-1}, a_s), p(a_{t-1}, a_t)\} \\ &= hp(a_{t-1}, a_t). \end{aligned}$$

Thus, $p(a_s, a_t) \leq h^t p(a_0, a_1)$.

Let $s, t \rightarrow \infty$. Then

$$(2.21) \quad \lim_{s,t \rightarrow \infty} p(a_s, a_t) = 0,$$

with $s > t$. Similarly $\lim_{s,t \rightarrow \infty} p(b_s, b_t) = 0$ with $s > t$.

By similar argument, one can show for s even and t odd, for $s < t$ and $s > t$ $\lim_{s,t \rightarrow \infty} p(a_s, a_t) = 0$ and $\lim_{s,t \rightarrow \infty} p(b_s, b_t) = 0$.

Now, we show that $\{x_l\}$ is right Cauchy sequence. Consider the following cases:

(i) If $l, r \in N$ such that l is odd and r is even with $r > l$, then we have $\lim_{l,r \rightarrow \infty} p(a_l, a_r) = 0$ from (2.1.19).

(ii) If $l, r \in N$ such that l is even and r is odd with $r > l$, then we have $\lim_{l,r \rightarrow \infty} p(a_l, a_r) = 0$ from (2.1.21).

(iii) $l, r \in N$ such that l and r both are even with $r > l$, then we have $p(a_l, a_r) \leq p(a_l, a_{l+1}) + p(a_{l+1}, a_r)$. Thus, $\lim_{l,r \rightarrow \infty} p(a_l, a_r) = 0, l > r$.

(iv) If $l, r \in N$ such that l and r both are odd with $r > l$ then, we have $p(a_l, a_r) \leq p(a_l, a_{r+1}) + p(a_{r+1}, a_r)$. Thus, $\lim_{l,r \rightarrow \infty} p(a_l, a_r) = 0, l > r$.

Hence, $\{a_l\}$ is right Cauchy sequence. With similar argument $\{b_l\}$ is right Cauchy sequence and also $\{a_l\}, \{b_l\}$ are left Cauchy sequences. Hence $\{a_l\}$ and $\{b_l\}$ both are Cauchy sequences.

Since (X, q) is complete quasi metric space, therefore $\lim_{l \rightarrow \infty} p(a_{2l}, a) = 0 = \lim_{l \rightarrow \infty} p(a, a_{2l})$

Now, $a = \lim_{l \rightarrow \infty} a_{2l+1} = \lim_{l \rightarrow \infty} J(a_{2l}, b_{2l}) = J(a, b)$ and $b = \lim_{l \rightarrow \infty} b_{2l+1} = \lim_{l \rightarrow \infty} J(b_{2l}, a_{2l}) = J(b, a)$. Thus, (a, b) is coupled fixed point of J and with similar argument it can be shown that (a, b) is coupled fixed point of K .

Hence, (a, b) is common coupled fixed point of J and K . To prove uniqueness first we have to prove $p(v, v) = 0$,

$$p(v, v) = p\{J(v, v), K(v, v)\} \leq h \max\{p(v, J(v, v)), p(v, K(v, v))\} = hp(v, v).$$

Thus, $p(v, v) = 0$.

Let (a, b) and (a', b') be two coupled fixed points of J and K , $J(a, b) = K(a, b) = a$ and $J(b, a) = K(b, a) = b$, $J(a', b') = K(a', b') = a'$ and $J(b', a') = K(b', a') = b'$.

Now,

$$\begin{aligned} p(a, a') &= p\{J(a, b), K(a', b')\} \leq h \max\{p(a, J(a, b)), p(a', K(a', b'))\} \\ &= h \max\{p(a, a), p(a', a')\} = 0. \end{aligned}$$

Thus, $a = a'$, similarly we can show that $b = b'$. This implies $(a, b) \equiv (a', b')$. $\square$

Hence, (a, b) is unique common coupled fixed point of J and K .

Corollary 2.2. Let (X, q) be a complete quasi metric space equipped with an m -distance mapping p and $J: X \times X \rightarrow X$ be continuous function such that:

(i) $p\{J(a, b), J(c, d)\} \leq h \max\{p(a, J(a, b)), p(c, J(c, d))\}$, where $h \in [0, 1)$. Then J has unique common coupled fixed point.

Proof. Consider $K = I$ (Identity map), we get the result. The following example validate our result. $\square$

Example 2.3. Let $X = (0, 3]$, $p(a, b) = \frac{a+2b}{8}$, $q(a, b) = \frac{a+3b-2}{4}$, $J(a, b) = 4a + 3b - 15$, $K(a, b) = 3a + 2b - 10$. Here, q is quasi metric space, p is modified- ω distance function.

L.H.S. $= p\{J(a, b), K(c, d)\} = p(4a + 3b - 15, 3c + 2d - 10) = \frac{1}{8}(4a + 3b + 6c + 4d - 35)$

R.H.S. $= h \max\{p(a, J(a, b)), p(c, K(c, d))\} = h, \max\{p(a, 4a+3b-15), p(c, 3c+2d-10)\} = h \max\{\frac{9a+6b-30}{8}, \frac{7c+4d-20}{8}\} = \frac{h}{8} \max\{9a+6b-30, 7c+4d-20\}$. Condition (i) and (ii) of theorem are satisfied. Also, $J(2, 3) = 2 = K(2, 3)$, $J(3, 2) = 3 = K(3, 2)$. Thus, $(2, 3)$ is coupled common fixed point of J and K .

Conclusion. A unique common coupled fixed point theorem proved for quasi metric space.

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Hybrid of shifted Legendre and rational Legendre spectral methods on a semi-infinite interval

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Abstract. In this study, spectral and pseudospectral methods on the half-line are studied using orthogonal systems of shifted Legendre polynomials on a subinterval $[0, S]$ and rational Legendre functions on $[S, \infty)$. Based on this idea a hybrid orthogonal system is introduced. We establish primary results on hybrid approximations of interpolations and some orthogonal projections. The obtained results organize the theory of developing spectral and pseudospectral methods for solving differential equations on a semi-infinite interval. Error analysis for a model problem are established. Numerical results are included to support the theoretical results and shows the effectiveness and performance of this method.

Keywords: hybrid legendre function, semi-infinite interval, spectral and pseudospectral.

1. Introduction

In recent years, considerable growths in the fields of physics, engineering, chemistry and other sciences have been made that can be explained successfully with models that use mathematical tools on infinite intervals. A direct method is to use the spectral method dependent on orthogonal systems on infinite intervals (see, Hermite and Laguerre methods [2, 4, 9, 12, 14, 28, 29]). In these methods, main problems with infinite intervals are transformed to the problems with finite intervals. Jacobi's polynomials are used to approximate the results. For instance, Maleki [19], Boyd [13, 14] and Christov [8] used mutually orthogonal systems of hybrid functions for developing several spectral methods with infinite

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intervals. Some of the recent methods include hybrid functions [17, 20, 26, 27], Chebyshev methods on semi-infinite intervals [6, 13, 21, 22, 23] and Legendre methods [20, 24] and rational methods [10, 18].

In this study, we examine the spectral and pseudospectral methods on a combination of finite and semi-infinite intervals. We use a new hybrid mutual orthogonal system of shifted Legendre polynomials and rational Legendre functions with the weight function $(x - S + 1)^{-2}$. We also provide a detailed theoretical analysis of the above hybrid approximation in weighted Sobolev space and assess some error estimates for a model problem.

This method has the following advantages. First, the proposed domain decomposition scheme enables us to approximate problems with oscillatory behaviour with more accuracy compared to the usual rational approximations. Second, this hybrid approach is flexible in that the value of S could be tuned according to the structure of the problem. Third, we do not need variable transformations to solve differential equations on a semi-infinite interval and the problem is solved directly.

The system of hybrid functions is obtained by the Legendre polynomials and its primary properties are introduced in the next Section. In Section 3, we denote several results for various orthogonal projections. In Section 4, we use a Legendre hybrid interpolation approximation. In Section 5, we discuss a model problem and prove error estimates for the Legendre hybrid spectral and pseudospectral methods. In Section 6, the numerical results are compared with rational method [5]. Finally, Section 7 includes the conclusion of the study.

2. Properties of hybrid Legendre functions

Let $S > 0$ and $I = [0, \infty)$ is partitioned into $(n + 1)$ sub-intervals as following

$$\Lambda_{S,n} = \prod_{i=1}^{n+1} \Lambda_i$$

with

$$\Lambda_i = \begin{cases} \left[\frac{S(i-1)}{n}, \frac{Si}{n} \right), & i = 1, 2, \dots, n \\ [S, \infty), & i = n + 1. \end{cases}$$

For $1 \leq p < \infty$, let

$$L_w^p(\Lambda_{S,n}) = \left\{ v \mid v \text{ is measurable and } \|v\|_{L_{w_i}^p(\Lambda_i)} < \infty, \quad (i = 1, \dots, n, n + 1) \right\}$$

where w_i 's are positive weight functions

$$\begin{aligned} w(x) &> 0, \quad w|_{\Lambda_i} = w_i, \quad \forall x \in \Lambda_i, \quad w_i(x) > 0, \\ w_i(x) &= \begin{cases} 1, & i = 1, 2, \dots, n \\ (x - S + 1)^{-2}, & i = n + 1 \end{cases} \end{aligned}$$

and

$$(2.1) \quad \|v\|_{L_w^p} = \begin{cases} \left(\sum_{i=1}^{n+1} \int_{\Lambda_i} |v(x)|^p w_i(x) dx \right)^{\frac{1}{p}}, & 1 \leq p < \infty \\ \operatorname{ess\,sup}_{x \in \Lambda_{S,n}} |v(x)|, & p = \infty. \end{cases}$$

Inner product of the space $L_w^2(\Lambda_{S,n})$ is defined as follows:

$$(f, g)_w = \sum_{i=1}^{n+1} \int_{\Lambda_i} f(x)g(x)w_i(x)dx.$$

For any integer $m \geq 0$, we set

$$H_w^m(\Lambda_{S,n}) = \left\{ u \mid \partial_x^k u = \frac{d^k u}{dx^k} \in L_w^2(\Lambda_{S,n}), \quad 0 \leq k \leq m, m > 0 \right\}.$$

In this space, the semi-norm, the norm and the inner product are as follows:

$$(f, g)_{m,w}(\Lambda_{S,n}) = \sum_{k=0}^m (\partial_x^k f, \partial_x^k g)_w, \quad |f|_{m,w} = \|\partial_x^m f\|_w, \quad \|f\|_{m,w} = (f, f)_{m,w}^{\frac{1}{2}}.$$

We consider the space $H_w^r(\Lambda_{S,n})$ with the norm $\|f\|_{r,w}$ by space interpolation as in Adams [25] for any real number $r > 0$. As usual w will be omitted from the relations if $w_i(x) = 1$.

As we know, the Legendre polynomial of degree z , is the eigenfunction of the singular Sturm-Liouville problem

$$(2.2) \quad \partial_x \left((1 - x^2) \partial_x L_z(x) \right) + z(z+1)L_z(x) = 0, \quad z = 0, 1, 2, \dots$$

We consider the Legendre hybrid functions of degree z in sub-intervals $[0, S]$ and in interval $[S, \infty)$, as:

$$F_{z,i}(x) = \begin{cases} L_z \left(\frac{2n}{S}x - 2i + 1 \right), & x \in \Lambda_i, i = 1, 2, \dots, n \\ L_z \left(\frac{x - S - 1}{x - S + 1} \right), & x \in \Lambda_{n+1}, i = n + 1 \\ 0, & \text{otherwise.} \end{cases}$$

The functions $F_{z,i}(x), i = 1, \dots, n + 1$ are the eigenfunctions of the following singular Sturm-Liouville problems

$$(2.3) \quad \partial_x \left(\left(\frac{(2i-1)Sx}{n} - x^2 + (i-i^2) \frac{S^2}{n^2} \right) \partial_x v \right) + z(z+1)v = 0, x \in \Lambda_i, i = 1, \dots, n, \\ (x - S + 1)^2 \partial_x((x - S) \partial_x v) + z(z + 1)v = 0, x \in \Lambda_{n+1}.$$

Clearly, we have

$$(2.4) \quad \lim_{x \rightarrow \infty} F_{z,n+1}(x) = 1, \quad \lim_{x \rightarrow \infty} (x - S) \partial_x F_{z,n+1}(x) = 0.$$

From Legendre's polynomials recursive formulas, the following relationships are obtained.

$$\begin{cases} F_{0,i}(x) = 1, & x \in \Lambda_i, (i = 1, \dots, n), \\ F_{1,i}(x) = x, \\ F_{z+1,i}(x) = \frac{2z-1}{z} \left(\frac{2n}{S} x - 2i + 1 \right) F_{z,i}(x) - \frac{z-1}{z} F_{z-1,i}(x), & z \geq 1, \\ F_{z,i}(x) = \frac{S}{2n} \left[\frac{\partial_x F_{z+1,i}(x) - \partial_x F_{z-1,i}(x)}{2z+1} \right], \end{cases}$$

and

$$\begin{cases} F_{0,n+1}(x) = 1, & x \in \Lambda_{n+1} \\ F_{1,n+1}(x) = \frac{x - T - 1}{x - T + 1}, \\ F_{z+1,n+1}(x) = \frac{2z-1}{z} \frac{x - S - 1}{x - S + 1} F_{z,n+1}(x) - \frac{z-1}{z} F_{z-1,n+1}(x), & z \geq 1, \\ F_{z,n+1}(x) = \left[\frac{\partial_x F_{z+1,n+1}(x) - \partial_x F_{z-1,n+1}(x)}{2z+1} \right]^z \left(\frac{(x - S + 1)^2}{2} \right). \end{cases}$$

By using orthogonal relations of Legendre polynomials, we get

$$(2.5) \quad \int_{\Lambda_{S,n}} F_{z,i}(x) F_{m,i}(x) w_i(x) dx = (z + \frac{1}{2})^{-1} \delta_{z,m} \delta_{i,j},$$

where $\delta_{z,m}$ and $\delta_{i,j}$ are the Kronecker functions.

According to the above definitions, the functions $\{F_{z,i}(x) \mid i=1, \dots, n+1; z=0, 1, 2, \dots\}$ are orthogonal. For $v \in L_w^2(\Lambda_{S,n})$, we have

$$v(x) = \sum_{z=0}^{\infty} \sum_{i=1}^{n+1} \hat{v}_{z,i} F_{z,i}(x)$$

with

$$\hat{v}_{z,i} = \left(z + \frac{1}{2} \right) \int_{\Lambda_i} v(x) F_{z,i}(x) w_i(x) dx.$$

Let

$$l_i(x) = \begin{cases} \frac{(2i-1)Sx}{n} - x^2 + \frac{(i-i^2)S^2}{n^2}, & i = 1, \dots, n \\ x - S, & i = n+1. \end{cases}$$

By virtue of Eqs. (2.3) and (2.4), we know that $\{\partial_x F_{z,i}(x), i = 1, \dots, n+1\}$ are mutually orthogonal in $L_{l_i}^2(\Lambda_i)$ ($i = 1, \dots, n+1$), namely

$$\begin{aligned} & \int_{\Lambda_i} \partial_x F_{z,i}(x) \partial_x F_{m,i}(x) l_i(x) dx \\ (2.6) \quad & = z(z+1)(z + \frac{1}{2})^{-1} \delta_{z,m}, (i = 1, \dots, n+1), (x \in \Lambda_i). \end{aligned}$$

For any integer $N > 0$, we consider

$$\mathcal{H}_{N,n} = \text{span} \{F_{z,i} : z = 0, 1, \dots, N, i = 1, 2, \dots, n+1\}.$$

We set $\tilde{\Lambda} = (-1, 1)$ and derive some useful inequalities and embedding inequalities in the following theorems.

Theorem 2.1. *Let $\phi \in \mathcal{H}_{N,n}$ and $1 \leq p \leq q < \infty$, then*

$$\|\phi\|_{L_w^q} \leq \left(\frac{S+1}{2}\right)^{\frac{p-q}{pq}} [(p+1)N^2]^{\frac{1}{p}-\frac{1}{q}} \|\phi\|_{L_w^p}.$$

Proof. Let $y \in \tilde{\Lambda}$, $x_{n+1} = \frac{y(S-1)-S-1}{y-1}$, $x_i = \frac{S}{2n}(y+2i-1)$ ($i = 1, \dots, n$). For any $\phi \in \mathcal{H}_{N,n}$, we set

$$\psi(y) = \begin{cases} \phi(x_i) = \phi\left(\frac{S}{2n}(y+2i-1)\right), & i = 1, \dots, n \\ \phi(x_{n+1}) = \phi\left(\frac{y(S-1)-S-1}{y-1}\right). \end{cases}$$

By definition of $\mathcal{H}_{N,n}$, we have $\psi(y) \in \mathcal{P}_N$, which is the space of polynomials of degree not exceeding N . By using $\mathcal{P}_N$ [1], we have

$$\left(\int_{\tilde{\Lambda}} |\psi(y)|^q dy\right)^{\frac{1}{q}} \leq ((p+1)N^2)^{\frac{1}{p}-\frac{1}{q}} \left(\int_{\tilde{\Lambda}} |\psi(y)|^p dy\right)^{\frac{1}{p}}.$$

Therefore,

$$\begin{aligned} \|\phi\|_{L_w^q}^q &= \sum_{i=1}^{n+1} \int_{\Lambda_i} |\phi(x)|^q \omega_i(x) dx = \left(\frac{S+1}{2}\right) \int_{\tilde{\Lambda}} |\psi(y)|^q dy \\ &\leq \left(\frac{S+1}{2}\right) [(p+1)N^2]^{\frac{q}{p}-1} \left(\int_{\tilde{\Lambda}} |\psi(y)|^p dy\right)^{\frac{q}{p}} \\ &= \left(\frac{S+1}{2}\right)^{\frac{p-q}{p}} [(p+1)N^2]^{\frac{q}{p}-1} \|\phi\|_{L_w^p}^q. \quad \square \end{aligned}$$

Theorem 2.2. *For any integer $m \geq 0$, $2 \leq p < \infty$, $\phi \in \mathcal{H}_{N,n}$ and $r > 0$, we have*

$$\|\partial_x^m \phi\|_{L_w^p} \leq cN^{2m} \|\phi\|_{L_w^p},$$

and

$$\|\phi\|_{r,w} \leq cN^{2r} \|\phi\|_w.$$

Proof. Let $y \in \tilde{\Lambda}$, $x_{n+1} = \frac{y(S-1)-S-1}{y-1}$, $x_i = \frac{S}{2n}(y+2i-1)$ ($i = 1, \dots, n$). For any $\phi \in \mathcal{H}_{N,n}$, we set

$$\psi(y) = \begin{cases} \phi(x_i) = \phi\left(\frac{S}{2n}(y+2i-1)\right), & i = 1, \dots, n \\ \phi(x_{n+1}) = \phi\left(\frac{y(S-1)-S-1}{y-1}\right). \end{cases}$$

$$\begin{aligned}
\|\partial_x \phi\|_{L_w^p}^p &= \sum_{i=1}^{n+1} \int_{\Lambda_i} |\partial_x \phi|^p w_i(x) dx \\
&= n \left(\frac{2n}{S} \right)^{p-1} \int_{\tilde{\Lambda}} |\partial_y \psi|^p dy + \frac{1}{2^{p+1}} \int_{\tilde{\Lambda}} |\partial_y \psi|^p (y-1)^{2p} dy.
\end{aligned}$$

According to Theorem 2.2 in [5], we have

$$\begin{aligned}
\|\partial_x \phi\|_{L_w^p}^p &\leq n \left(\frac{2n}{S} \right)^{p-1} \int_{\tilde{\Lambda}} |\partial_y \psi|^p dy + 2^p \int_{\tilde{\Lambda}} |\partial_y \psi|^p dy \\
&\leq \left[\left(\frac{2^{(p-1)} n^p}{S^{p-1}} \right) + 2^p \right] \|\partial_y \psi\|_{L^p(\tilde{\Lambda})}^p.
\end{aligned}$$

Since $\psi(y) \in \mathcal{P}_N$ and using Theorem 2.8 of [1] we have

$$\|\partial_y^m \psi\|_{L^p(\tilde{\Lambda})} \leq c N^{2m} \|\psi\|_{L^p(\tilde{\Lambda})}.$$

By setting $m = 1$, we obtain

$$\|\partial_x \phi\|_{L_w^p}^p \leq \left[\left(\frac{2^{p-1} n^p}{S^{p-1}} \right) + 2^p \right] c^p N^{2p} \|\psi\|_{L^p(\tilde{\Lambda})}^p.$$

Thus

$$\|\partial_x \phi\|_{L_w^p}^p \leq \left[\left(\frac{2^{p-1} n^p}{S^{p-1}} \right) + 2^p \right] c^p N^{2p} \left(\frac{2}{S+1} \right) \|\phi\|_{L_w^p}^p.$$

By setting

$$c = \left[\left[\frac{2^{p-1} n^p}{S^{p-1}} + 2^p \right] \left(\frac{2c^p}{S+1} \right) \right]^{\frac{1}{p}},$$

we get

$$\|\partial_x \phi\|_{L_w^p}^p \leq c N^2 \|\phi\|_{L_w^p}^p.$$

If above procedure is repeated, we obtain

$$\|\partial_x^m \phi\|_{L_w^p} \leq c N^{2m} \|\phi\|_{L_w^p} (m \geq 0).$$

By setting $p = 2$, we have

$$\|\phi\|_{r,w} \leq c N^{2r} \|\phi\|_w.$$

□

Theorem 2.3. If $v \in L_{w^2}^2(\Lambda_{S,n})$, $\partial_x v \in L_w^2(\Lambda_{S,n})$ and $v(\frac{S}{n}(i-1)) = 0$, ($i = 1, \dots, n+1$), $c = \frac{16n^2}{9S^2}$, then

$$\|v\|_{w^2} \leq \frac{2}{3} |v|_{1,w}.$$

Proof. Let $y \in \tilde{\Lambda}$,

$$\psi(y) = \begin{cases} v(x_i) = v\left(\frac{S}{2n}(y + 2i - 1)\right), & i = 1, \dots, n \\ v(x_{n+1}) = v\left(\frac{y(S-1) - S - 1}{y-1}\right). \end{cases}$$

$$\|v\|_{w^2}^2 = \sum_{i=1}^{n+1} \int_{\Lambda_i} v^2(x) w_i^2(x) dx = \frac{S}{2} \int_{\tilde{\Lambda}} \psi^2(y) dy + \frac{1}{8} \int_{\tilde{\Lambda}} \psi^2(y) (y-1)^2 dy.$$

According to Poincare inequality, since $\psi(-1) = 0$, we can conclude

$$\|v\|_{w^2}^2 \leq \frac{cS}{2} \int_{\tilde{\Lambda}} (\partial_y \psi)^2 dy + \frac{1}{8} \int_{\tilde{\Lambda}} (\psi(y))^2 (y-1)^2 dy.$$

From Theorem 2.3 in [5], we have

$$\begin{aligned} \|v\|_{w^2}^2 &\leq \frac{cS}{2} \int_{\tilde{\Lambda}} (\partial_y \psi)^2 dy + \frac{1}{8} \times \frac{4}{9} \int_{\tilde{\Lambda}} (\partial_y \psi)^2 (1-y)^4 dy \\ &\leq \frac{cS}{2n} \int_{\tilde{\Lambda}} (\partial_y \psi)^2 dy + \dots + \frac{cS}{2n} \int_{\tilde{\Lambda}} (\partial_y \psi)^2 dy + \frac{1}{8} \times \frac{4}{9} \int_{\tilde{\Lambda}} (\partial_y \psi)^2 (1-y)^4 dy \\ &\leq \frac{cS^2}{4n^2} \sum_{i=1}^n \int_{\Lambda_i} (\partial_x v)^2 w_i(x) dx + \frac{4}{9} \int_{\Lambda_{n+1}} (\partial_x v)^2 w_{n+1}(x) dx. \end{aligned}$$

By letting $c = \frac{16n^2}{9S^2}$, we obtain

$$\|v\|_{w^2}^2 \leq \frac{4}{9} \sum_{i=1}^{n+1} \int_{\Lambda_i} (\partial_x v(x))^2 w_i(x) dx \leq \frac{4}{9} |v|_{1,w}^2. \quad \square$$

3. Legendre hybrid function approximations

In this section, we present some orthogonal projections for error estimates.

Definition 3.1. We define the $L_w^2(\Lambda_{S,n})$ -orthogonal projection $P_N : L_w^2(\Lambda_{S,n}) \rightarrow \mathcal{H}_{N,n}$ by

$$(P_N u - u, \phi)_w = 0, \quad \forall \phi \in \mathcal{H}_{N,n}.$$

The space $H_{w,A}^r(\Lambda_{S,n})$ which is considered by space interpolation is introduced to estimate $\|P_N u - u\|_w$, where

$$H_{w,A}^r(\Lambda_{S,n}) = \{u \mid u \text{ is measurable and } \|u\|_{r,w,A} < \infty\} \quad (r \geq 0)$$

and

$$\|u\|_{r,w,A} = \left(\sum_{k=0}^r \|(x - S + 1)^{\frac{r}{2}+k} \partial_x^k u\|_{w_{n+1}}^2 + n \|\partial_x^r u\|^2 \right)^{\frac{1}{2}}, \quad (i = 1, \dots, n).$$

By using (2.3), we have

$$Au(x) = \begin{cases} -\partial x \left(\left(\frac{2i-1}{n} Sx - x^2 + \frac{i-i^2}{n^2} S^2 \right) \partial_x u(x) \right), & x \in \Lambda_i, \\ -(x-S+1)^2 \partial x ((x-S) \partial_x u(x)), & x \in \Lambda_{n+1}. \end{cases}$$

where A is Sturm-Liouville operator. Now, by induction, we obtain

$$(3.1) \quad A^m u(x) = \begin{cases} \sum_{k=1}^{2m} p_k(x) \partial_x^k u(x), & (x \in \Lambda_i), \\ \sum_{k=1}^{2m} (x-S+1)^{m+k} q_k(x) \partial_x^k u(x), & (x \in \Lambda_{n+1}), \end{cases}$$

where $p_k(x)$ and $q_k(x)$ are some Legendre hybrid function.

Theorem 3.1. *If $r \geq 0$, for any $u \in H_{w,A}^r(\Lambda_{S,n})$, we have*

$$\|P_N u - u\|_w \leq cN^{-r} \|u\|_{r,w,A}$$

Proof. At first, we set $r = 2m$. By placing

$$F_{z,i}(x) = \begin{cases} -\frac{\partial x \left(\left(\frac{2i-1}{n} Sx - x^2 + \frac{i-i^2}{n^2} S^2 \right) \partial_x F_{z,i}(x) \right)}{z(z+1)}, & i = 1, \dots, n, \\ -\frac{(x-S+1)^2 \partial x ((x-S) \partial_x F_{z,i}(x))}{z(z+1)}, & i = n+1 \end{cases}$$

and using Eqs. (2.3) and (2.5) and integration by parts,

$$\begin{aligned} \hat{u}_{z,i} &= (z + \frac{1}{2}) \int_{\Lambda_i} u(x) F_{z,i}(x) w_i(x) dx \\ &= -\frac{2z+1}{2z(z+1)} \int_{\Lambda_i} u(x) \partial_x \left(\left(\frac{2i-1}{n} Sx - x^2 + \frac{i-i^2}{n^2} S^2 \right) \partial_x F_{z,i}(x) \right) dx \\ &= \frac{2z+1}{2z(z+1)} \int_{\Lambda_i} \left(\frac{2i-1}{n} Sx - x^2 + \frac{i-i^2}{n^2} S^2 \right) (\partial_x u(x)) (\partial_x F_{z,i}(x)) dx \\ &= \frac{2z+1}{2z(z+1)} \int_{\Lambda_i} Au(x) F_{z,i}(x) dx = \dots \\ (3.2) \quad &= \frac{2z+1}{2z^m(z+1)^m} \int_{\Lambda_i} A^m u(x) F_{z,i}(x) dx, (i = 1, \dots, n). \end{aligned}$$

$$\begin{aligned}
\hat{u}_{z,n+1} &= (z + \frac{1}{2}) \int_{\Lambda_{n+1}} u(x) F_{z,n+1}(x) w_{n+1}(x) dx \\
&= -\frac{2z+1}{2z(z+1)} \int_{\Lambda_{n+1}} u(x) (x-S+1)^2 \partial_x((x-S) \partial_x F_{z,n+1}(x)) w_{n+1}(x) dx \\
&= -\frac{2z+1}{2z(z+1)} \int_{\Lambda_{n+1}} u(x) \partial_x((x-S) \partial_x F_{z,n+1}(x)) dx \\
&= \frac{2z+1}{2z(z+1)} \int_{\Lambda_{n+1}} (x-S) \partial_x u(x) \partial_x F_{z,n+1}(x) dx \\
&= -\frac{2z+1}{2z(z+1)} \int_{\Lambda_{n+1}} \partial_x((x-S) \partial_x u(x)) F_{z,n+1}(x) dx \\
&= \frac{2z+1}{2z(z+1)} \int_{\Lambda_{n+1}} A u(x) F_{z,n+1}(x) w_{n+1}(x) dx = \dots \\
(3.3) &= \frac{2z+1}{2z^m(z+1)^m} \int_{\Lambda_{n+1}} A^m u(x) F_{z,n+1}(x) w_{n+1}(x) dx.
\end{aligned}$$

Therefore, we derive from (3.1) and (3.2) and (3.3) and the definition of $H_{w,A}^r(\Lambda_{S,n})$ that

$$\begin{aligned}
\|P_N u - u\|_w^2 &= \sum_{z=N+1}^{\infty} \sum_{i=1}^{n+1} \hat{u}_{z,i}^2 \|F_{z,i}\|_{w_i}^2 \\
&= \sum_{z=N+1}^{\infty} \frac{1}{z^{2m}(z+1)^{2m}} \left[\frac{\int_{\Lambda_1} A^m u(x) F_{z,1}(x) dx}{\|F_{z,1}\|_{w_1}^2} \right]^2 \|F_{z,1}\|_{w_1}^2 + \dots \\
&\quad + \frac{1}{z^{2m}(z+1)^{2m}} \left[\frac{\int_{\Lambda_n} A^m u(x) F_{z,n}(x) dx}{\|F_{z,n}\|_{w_n}^2} \right]^2 \|F_{z,n}\|_{w_n}^2 \\
&\quad + \frac{1}{z^{2m}(z+1)^{2m}} \left[\frac{\int_{\Lambda_{n+1}} A^m u(x) F_{z,n+1}(x) w_{n+1}(x) dx}{\|F_{z,n+1}\|_{w_{n+1}}^2} \right]^2 \|F_{z,n+1}\|_{w_{n+1}}^2 \\
&\leq cN^{-4m} (n \|A^m u\|^2 + \|A^m u\|_{w_{n+1}}^2).
\end{aligned}$$

Next, we set $r = 2m + 1$. By using (2.3) and (2.5) and integration by parts,

$$\begin{aligned}
\hat{u}_{z,i} &= \frac{2z+1}{2z^m(z+1)^m} \int_{\Lambda_i} A^m u(x) F_{z,i}(x) dx \\
(3.4) &= -\frac{2z+1}{2z^{m+1}(z+1)^{m+1}} \int_{\Lambda_i} A^m u(x) \partial_x \left(\left(\frac{2i-1}{n} Sx - x^2 + \frac{i-i^2}{n^2} S^2 \right) \partial_x F_{z,i}(x) \right) dx \\
&= \frac{2z+1}{2z^{m+1}(z+1)^{m+1}} \int_{\Lambda_i} \partial_x (A^m u(x)) \partial_x F_{z,i}(x) l_i(x) dx, \quad (i = 1, \dots, n).
\end{aligned}$$

and

$$\begin{aligned}
 \hat{u}_{z,n+1} &= \frac{2z+1}{2z^m(z+1)^m} \int_{\Lambda_{n+1}} A^m u(x) F_{z,n+1}(x) w_{n+1}(x) dx \\
 &= -\frac{2z+1}{2z^{m+1}(z+1)^{m+1}} \int_{\Lambda_{n+1}} A^m u(x) \partial_x((x-S)\partial_x F_{z,n+1}(x)) dx \\
 (3.5) \quad &= \frac{2z+1}{2z^{m+1}(z+1)^{m+1}} \int_{\Lambda_{n+1}} \partial_x(A^m u(x)) \partial_x F_{z,n+1}(x) l_{n+1}(x) dx.
 \end{aligned}$$

By using (2.6) and (3.4) and (3.5), we have

$$\begin{aligned}
 \|P_N u - u\|_w^2 &= \sum_{z=N+1}^{\infty} \sum_{i=1}^{n+1} \hat{u}_{z,i}^2 \|F_{z,i}\|_{w_i}^2 \\
 &\leq cN^{-2(2m+1)} \sum_{z=N+1}^{\infty} \left(\frac{\int_{\Lambda_1} \partial_x(A^m u) \partial_x F_{z,1}(x) l_1(x) dx}{\|\partial_x F_{z,1}(x)\|_{l_1}^2} \right)^2 \|\partial_x F_{z,1}\|_{l_1}^2 + \dots \\
 &\quad + \left(\frac{\int_{\Lambda_n} \partial_x(A^m u) \partial_x F_{z,n}(x) l_n(x) dx}{\|\partial_x F_{z,n}(x)\|_{l_n}^2} \right)^2 \|\partial_x F_{z,n}\|_{l_n}^2 \\
 &\quad + \left(\frac{\int_{\Lambda_{n+1}} \partial_x(A^m u) \partial_x F_{z,n+1}(x) l_{n+1}(x) dx}{\|\partial_x F_{z,n+1}(x)\|_{l_{n+1}}^2} \right)^2 \|\partial_x F_{z,n+1}\|_{l_{n+1}}^2 \\
 &\leq cN^{-2(2m+1)} (\|\partial_x A^m u\|_{l_1}^2 + \dots + \|\partial_x A^m u\|_{l_n}^2 + \|\partial_x A^m u\|_{l_{n+1}}^2) \\
 &\leq cN^{-2(2m+1)} (n\|A^m u\|^2 + \|A^m u\|_{w_{n+1}}^2) \\
 &\leq cN^{-2m(2m+1)} \|u\|_{r,w,A}^2.
 \end{aligned}$$

Which completes the proof. $\square$

Definition 3.2. We define $H_w^1(\Lambda_{S,n})$ -orthogonal projection $P_N^1 : H_w^1(\Lambda_{S,n}) \rightarrow \mathcal{H}_{N,n}$ by

$$(P_N^1 u - u, \phi)_{1,w} = 0, \quad \forall \phi \in \mathcal{H}_{N,n}, \quad \forall u \in H_w^1(\Lambda_{S,n}).$$

For estimating $\|P_N^1 u - u\|_{1,w}$, we introduce

$$H_{w,B}^r(\Lambda_{S,n}) = \{u \mid u \text{ is measurable on } \Lambda_{S,n} \text{ and } \|u\|_{r,w,B} < +\infty\} \quad (r \geq 0)$$

where

$$(3.6) \quad \|u\|_{r,w,B} = \left(\sum_{k=1}^r \|(x-S+1)^{\frac{r}{2}+k-\frac{1}{2}} \partial_x^k u\|_{w_{n+1}}^2 + n^2 \|\partial_x^k u\|^2 \right)^{\frac{1}{2}}.$$

Theorem 3.2. For any $u \in H_{w,B}^r(\Lambda_{S,n})$, with $r \geq 1$

$$\|P_N^1 u - u\|_{1,w} \leq cN^{1-r} \|u\|_{r,w,B}.$$

Proof. We have $\|P_N^1 u - u\|_{1,w} \leq \|\phi - u\|_{1,w}$ for any $\phi \in \mathcal{H}_{N,n}$. Let $y \in \tilde{\Lambda} = (-1, 1)$,

$$\psi(y) = \begin{cases} u(x_i) = u(\frac{S}{2n}(y + 2i - 1)) & i = 1, \dots, n \\ u(x_{n+1}) = u(\frac{y(S-1) - S - 1}{y-1}). \end{cases}$$

By taking

$$\phi(x) = \begin{cases} \phi(x_i) = \tilde{P}_{N,0,0,0,0}^1 \psi(y) & i = 1, \dots, n \\ \phi(x_{n+1}) = \tilde{P}_{N,4,0,0,0}^1 \psi(y). \end{cases}$$

By using Theorem 2.5 in [3], we obtain

$$\begin{aligned} \|\phi - u\|_{1,w}^2 &= \sum_{k=0}^1 \left\{ \sum_{i=1}^{n+1} \int_{\Lambda_i} \partial_x^k (\phi - u) \partial_x^k (\phi - u) w_i(x) dx \right\} \\ &= \frac{S}{2n} \int_{\tilde{\Lambda}} (\tilde{P}_{N,0,0,0,0}^1 \psi(y) - \psi(y))^2 dy + \frac{2n}{S} \int_{\tilde{\Lambda}} (\partial_y \tilde{P}_{N,0,0,0,0}^1 \psi(y) - \partial_y \psi(y))^2 dy + \dots \\ &+ \frac{S}{2n} \int_{\tilde{\Lambda}} (\tilde{P}_{N,0,0,0,0}^1 \psi(y) - \psi(y))^2 dy + \frac{2n}{S} \int_{\tilde{\Lambda}} (\partial_y \tilde{P}_{N,0,0,0,0}^1 \psi(y) - \partial_y \psi(y))^2 dy \\ &+ \frac{1}{2} \int_{\tilde{\Lambda}} (\tilde{P}_{N,4,0,0,0}^1 \psi(y) - \psi(y))^2 dy + \frac{1}{8} \int_{\tilde{\Lambda}} (\partial_y \tilde{P}_{N,4,0,0,0}^1 \psi(y) - \partial_y \psi(y))^2 (y-1)^4 dy \\ &\leq \frac{2n}{S} \|\tilde{P}_{N,0,0,0,0}^1 \psi - \psi\|_{1,0,0,0,0}^2 + \dots + \frac{2n}{S} \|\tilde{P}_{N,0,0,0,0}^1 \psi - \psi\|_{1,0,0,0,0}^2 \\ &+ \|\tilde{P}_{N,4,0,0,0}^1 \psi - \psi\|_{1,4,0,0,0}^2 \\ &\leq \frac{2c_1 n^2}{S} N^{2-2r} \|\psi\|_{r,0,0,*} + c_2 N^{2-2r} \|\psi\|_{r,4,0,*}^2 \\ &\leq c N^{2-2r} \|\psi\|_{r,0,0,*} + c N^{2-2r} \|\psi\|_{r,4,0,*}^2, \quad (c = \max(c_2, \frac{2c_1 n^2}{S})). \end{aligned}$$

Note that

$$1 - y^2 = \begin{cases} \frac{-4x^2}{S^2} x^2 - 4i^2 + \frac{8ni}{S} + 4i - \frac{4n}{S} x, & x \in \Lambda_i, i = 1, \dots, n, \\ \frac{4x}{(x - S + 1)^2}, & x \in \Lambda_{n+1}, \end{cases}$$

$$1 - y = \begin{cases} 2i - \frac{2n}{S}, & i = 1, \dots, n, \\ \frac{2}{x - S + 1}, & x \in \Lambda_{n+1} \end{cases}$$

and one can show easily by induction that

$$\partial_y^k \psi(y) = \begin{cases} (\frac{S}{2n})^k \partial_x^k u(x), & x \in \Lambda_i, i = 1, \dots, n \\ \sum_{j=1}^k q_j(x) (x - S + 1)^{k+j} \partial_x^j u(x), & x \in \Lambda_{n+1}. \end{cases}$$

where $q_j(x)$ are some hybrid polynomials with degree j which are uniformly bounded on $\Lambda_{S,n}$. According to [3], we obtain the following relations for $r \geq 0$

$$\begin{aligned} A_{r,0,0}^{(1)}(\psi) &\leq \sum_{k=r-\lfloor \frac{r}{2} \rfloor + 1}^r \left(\frac{S}{2n}\right)^{2k-1} \|\partial_x^k u\|^2, \\ A_{r,0,0}^{(2)}(\psi) &\leq \sum_{k=1}^{\lfloor \frac{r+1}{2} \rfloor} \left(\frac{S}{2n}\right)^{2k-1} \|\partial_x^k u\|^2, \\ A_{r,4,0}^{(1)}(\psi) &\leq c_3 \sum_{k=r-\lfloor \frac{r}{2} \rfloor + 1}^r \sum_{j=1}^k \|(x-S+1)^{\frac{r}{2}+j-\frac{1}{2}} \partial_x^j u\|_{w_{n+1}}^2, \\ A_{r,4,0}^{(2)}(\psi) &\leq c_4 \sum_{k=r}^{\lfloor r+\frac{1}{2} \rfloor} \sum_j^k \|(x-S+1)^{k+j-1} \partial_x^j u\|_{w_{n+1}}^2. \end{aligned}$$

Now, by setting $(c = \max(c_2, \frac{2c_1 n^2}{S}, c_3, c_4))$, we obtain

$$\begin{aligned} \|\phi - u\|_{1,w}^2 &\leq cN^{2-2r} (A_{r,0,0}^{(1)}(\psi) + A_{r,0,0}^{(2)}(\psi)) + cN^{2-2r} (A_{r,4,0}^{(1)}(\psi) + A_{r,4,0}^{(2)}(\psi)) \\ &\leq cN^{2-2r} (A_{r,4,0}^{(1)}(\psi) + n^2 A_{r,0,0}^{(1)}(\psi) + A_{r,4,0}^{(2)}(\psi) + n^2 A_{r,0,0}^{(2)}(\psi)) \\ &\leq cN^{2-2r} \left(\sum_{k=1}^r \|(x-T+1)^{\frac{r}{2}+k-\frac{1}{2}} \partial_x^k u\|_{w_{n+1}}^2 + n^2 \|\partial_x^k u\|^2 \right) \\ &\leq cN^{2-2r} \|u\|_{r,w,B}^2. \end{aligned}$$

Which completes the proof. $\square$

We define another orthogonal projection for Dirichlet boundary conditions at $x = 0$, in partial differential equations.

Definition 3.3. Let us denote

$$\begin{aligned} H_{0,w}^1(\Lambda_{S,n}) &= \left\{ v \mid v \in H_w^1(\Lambda_{S,n}), \ v(S(i-1)/n) = 0, i = 1, \dots, n+1, \right. \\ &\quad \left. v(x)(x-S+1)^{-\frac{3}{2}} \rightarrow 0 \text{ as } x \rightarrow \infty \right\} \\ \mathcal{H}_{N,n}^0 &= \{ \phi \in \mathcal{H}_{N,n} \mid \phi(S(i-1)/n) = 0, i = 1, \dots, n+1 \}, \end{aligned}$$

$$(3.7) \quad a_w^v(f, v) = (\partial_x f, \partial_x(vw)) + v(f, v)_w.$$

We consider the $H_{0,w}^1(\Lambda_{S,n})$ -orthogonal projection $P_N^{1,0} : H_{0,w}^1(\Lambda_{S,n}) \rightarrow \mathcal{H}_{N,n}^0$ by

$$a_w^v(P_N^{1,0} v - v, \phi) = 0, \quad \forall \phi \in \mathcal{H}_{N,n}^0, \quad \forall v \in H_{0,w}^1(\Lambda_{S,n})$$

Lemma 3.1. For any $f, v \in H_{0,w}^1(\Lambda_{S,n})$ and $v > \frac{3}{4}$,

$$\begin{aligned} a_w^v(v, v) &\geq c\|v\|_{1,w}^2, \\ |a_w^v(f, v)| &\leq c\|f\|_{1,w}\|v\|_{1,w} \end{aligned}$$

Proof. Let $v = \frac{3}{4} + \varepsilon$ and $\varepsilon > 0$. By using Theorem 2.3, integrating by parts and Lemma 3.1 in [5], we find that

$$\begin{aligned} a_w^v(v, v) &= (\partial_x v, \partial_x(vw)) + v(v, v)w \\ &= \sum_{i=1}^{n+1} \int_{\Lambda_i} \partial_x v \partial_x(vw) w_i(x) dx + v \sum_{i=1}^{n+1} \int_{\Lambda_i} v(x) v(x) w_i(x) dx, \\ &= |v|_{1,w}^2 + v\|v\|_w^2 + \frac{1}{2} \int_{\Lambda_{n+1}} \partial_x(v^2(x)) \partial_x w_{n+1}(x) dx, \\ &= |v|_{1,w}^2 + v\|v\|_w^2 - 3 \int_{\Lambda_{n+1}} v^2(x) (x - S + 1)^{-4} dx, \\ &\geq |v|_{1,w}^2 + v\|v\|_w^2 - 3\|v\|_{w^2}^2, \\ &\geq |v|_{1,w}^2 - \left(\frac{9}{4} - \frac{\varepsilon}{2}\right)\|v\|_{w^2}^2 + \left(\frac{3}{4} + \varepsilon\right)\|v\|_w^2 - \left(\frac{3}{4} + \frac{\varepsilon}{2}\right)\|v\|_{w^2}^2, \\ (3.8) \quad &\geq \frac{2}{9}\varepsilon|v|_{1,w}^2 + \frac{\varepsilon}{2}\|v\|_w^2. \end{aligned}$$

The first result is obtained. Next, by using Theorem 2.3 and Cauchy-Schwartz inequality, we obtain

$$\begin{aligned} a_w^v(f, v) &= |(\partial_x f, \partial_x(vw)) + v(f, v)_w| \\ &\leq \|\partial_x f\|_w \|\partial_x v\|_w + 2\|\partial_x f\|_w \|v\|_{w^2} + v\|f\|_w \|v\|_w, \\ &\leq \|f\|_{1,w} \|v\|_{1,w} + \frac{4}{3}\|f\|_{1,w} \|v\|_{1,w} + v\|f\|_{1,w} \|v\|_{1,w}, \\ &\leq c\|f\|_{1,w} \|v\|_{1,w}. \end{aligned}$$

Which indicates the second result. $\square$

Theorem 3.3. If $r \geq 1$, for any $v \in H_{w,B}^r(\Lambda_{S,n}) \cap H_{0,w}^1(\Lambda_{S,n})$, $v > \frac{3}{4}$, we can conclude

$$\|P_N^{1,0}v - v\|_{1,w} \leq cN^{1-r}\|v\|_{r,w,B}.$$

Proof. By using Lemma 3.1, for any $\phi \in \mathcal{H}_{N,n}^0$, we set

$$\begin{aligned} \|P_N^{1,0}v - v\|_{1,w}^2 &\leq ca_w^v(P_N^{1,0}v - v, P_N^{1,0}v - v) \leq ca_w^v(P_N^{1,0}v - v, \phi - v) \\ &\leq c\|P_N^{1,0}v - v\|_{1,w}\|\phi - v\|_{1,w}. \end{aligned}$$

Therefore,

$$(3.9) \quad \|P_N^{1,0}v - v\|_{1,w} \leq c \inf_{\phi \in \mathcal{H}_{N,n}^0} \|\phi - v\|_{1,w}.$$

Next, let $y \in \tilde{\Lambda} = (-1, 1)$,

$$\psi(y) = \begin{cases} v(x_i) = v(\frac{S}{2n}(y + 2i - 1)), & i = 1, \dots, n \\ v(x_{n+1}) = v(\frac{y(S-1) - S - 1}{y-1}), & \end{cases}$$

and take

$$\phi(x) = \begin{cases} \phi(x_i) = \tilde{P}_{N,0,0,0,0}^{1,0} \psi(y), & i = 1, \dots, n \\ \phi(x_{n+1}) = \tilde{P}_{N,4,0,0,0}^{1,0} \psi(y), & \end{cases}$$

in (3.9).

By using Theorem 2.6 in [3] and similar arguments in the proof of Theorem 3.2, the desired result is obtained. $\square$

Definition 3.4. Let $\hat{a}_w(f, v) = \frac{1}{2}(\partial_x f, w \frac{-3}{2}(x)\partial_x v)_w + (f, v)_w$. We consider $H_{w,A}^1(\Lambda_{S,n})$ -orthogonal projection $\hat{P}_N^1 : H_{w,A}^1(\Lambda_{S,n}) \rightarrow \mathcal{H}_{N,n}$ by

$$(3.10) \quad \hat{a}_w(\hat{P}_N^1 v - v, \phi) = 0, \quad \forall \phi \in \mathcal{H}_{N,n}, \forall v \in H_{w,A}^1(\Lambda_{S,n}).$$

Theorem 3.4. For any $v \in H_{w,A}^r(\Lambda_{S,n})$ and $r \geq 1$,

$$\|\hat{P}_N^1 v - v\|_w \leq cN^{-r} \|v\|_{r,w,A}$$

and

$$\|w \frac{-3}{4}(x)\partial_x(\hat{P}_N^1 v - v)\|_w \leq cN^{1-r} \|v\|_{r,w,A}.$$

Proof. For $y \in \tilde{\Lambda} = (-1, 1)$, let us denote

$$\psi(y) = \begin{cases} v(x_i) = v(\frac{S}{2n}(y + 2i - 1)), & i = 1, \dots, n \\ v(x_{n+1}) = v(\frac{y(S-1) - S - 1}{y-1}), & \end{cases}$$

$$\psi_N^*(y) = \hat{P}_N^1 v(x) = \begin{cases} \hat{P}_N^1 v(x_i) = \tilde{P}_{N,0,0,0,0}^1 \psi(y), & i = 1, \dots, n \\ \hat{P}_N^1 v(x_{n+1}) = \tilde{P}_{N,1,0,0,0}^1 \psi(y), & \end{cases}$$

$$\begin{aligned} \|\hat{P}_N^1 v - v\|_w^2 &= \sum_{i=1}^{n+1} \int_{\Lambda_i} (\hat{P}_N^1 v - v)^2 w_i(x) dx \\ &= \frac{S}{2n} n \int_{\tilde{\Lambda}} (\psi_N^* - \psi) dy + \frac{1}{2} \int_{\tilde{\Lambda}} (\psi_N^* - \psi) dy \\ &= \frac{S}{2n} n \|\tilde{P}_{N,0,0,0,0}^1 \psi - \psi\|_{0,0,0,0,0}^2 + \frac{1}{2} \|\tilde{P}_{N,1,0,0,0}^1 \psi - \psi\|_{0,1,0,0,0}^2 \\ (3.11) \quad &\leq cN^{-2r} (n \|\psi\|_{r,0,0,*}^2 + \|\psi\|_{r,1,0,*}^2). \end{aligned}$$

For the second formula, we set

$$\begin{aligned}
 & \|w^{\frac{-3}{4}}(x)\partial x(\hat{P}_N^1 v - v)\|_w^2 \\
 &= \sum_{i=1}^{n+1} \int_{\Lambda_i} (w_i^{\frac{-3}{4}}(x)\partial_x \hat{P}_N^1 v - \partial_x v)^2 w_i(x) dx \\
 (3.12) \quad &= \frac{2n}{S} \int_{\tilde{\Lambda}} [\partial_y(\psi_N^* - \psi)]^2 dy + \dots + \frac{2n}{S} \int_{\tilde{\Lambda}} [\partial_y(\psi_N^* - \psi)]^2 dy \\
 &+ \frac{1}{4} \int_{\tilde{\Lambda}} [\partial_y(\psi_N^* - \psi)]^2 (1-y) dy \\
 &\leq \frac{2n}{S} n \|\tilde{P}_{N,0,0,0,0}^1 \psi - \psi\|_{1,0,0,0,0}^2 + \frac{1}{4} \|\tilde{P}_{N,1,0,0,0}^1 \psi - \psi\|_{1,1,0,0,0}^2 \\
 &\leq cN^{2-2r} \left(\frac{2n}{S} n \|\psi\|_{r,0,0,*}^2 + \|\psi\|_{r,1,0,*}^2 \right).
 \end{aligned}$$

According to [3], we obtain the following relations for $r \geq 0$

$$\begin{aligned}
 A_{r,0,0}^{(1)}(\psi) &\leq \left(\frac{S}{2n}\right)^{2r-1} \sum_{k=r-\lfloor \frac{r}{2} \rfloor + 1}^r \|\partial_x^k v\|^2, \\
 A_{r,0,0}^{(2)}(\psi) &\leq \left(\frac{S}{2n}\right)^{2r-1} \sum_{k=r}^{\lfloor \frac{r+1}{2} \rfloor} \|\partial_x^k v\|^2, \\
 A_{r,1,0}^{(1)}(\psi) &\leq c \sum_{k=r-\lfloor \frac{r}{2} \rfloor + 1}^r \sum_{j=1}^k \|(x-S+1)^{\frac{r}{2}+j-\frac{1}{2}} \partial_x^j v\|_{w_{n+1}}^2, \\
 A_{r,1,0}^{(2)}(\psi) &\leq c \sum_{k=1}^{\lfloor \frac{r+1}{2} \rfloor} \sum_{j=1}^k \|(x-S+1)^{k+j-1} \partial_x^j v\|_{w_{n+1}}^2.
 \end{aligned}$$

Now, we have

$$\begin{aligned}
 \|\hat{P}_N^1 v - v\|_w^2 &\leq cN^{-2r} (n \|\psi\|_{r,0,0,*}^2 + \|\psi\|_{r,1,0,*}^2) \\
 &\leq cN^{-2r} (nA_{r,0,0}^{(1)}(\psi) + A_{r,1,0}^{(1)}(\psi) + nA_{r,0,0}^{(2)}(\psi) + A_{r,1,0}^{(2)}(\psi)) \\
 &\leq cN^{-2r} \|v\|_{r,w,A}^2
 \end{aligned}$$

and

$$\begin{aligned}
 \|w^{\frac{-3}{4}}(x)\partial x(\hat{P}_N^1 v - v)\|_w^2 &\leq cN^{2-2r} \left(\frac{2n}{S} n \|\psi\|_{r,0,0,*}^2 + \|\psi\|_{r,1,0,*}^2 \right) \\
 &\leq cN^{2-2r} \|v\|_{r,w,A}^2.
 \end{aligned}$$

This proves the first and second result. $\square$

4. Legendre hybrid interpolation approximation

4.1 Legendre-Gauss-Radau hybrid interpolation approximation

To get the high accuracy of the approximated solutions, we use Legendre-Gauss-Radau hybrid interpolation. Let

$$(4.1) \quad \int_{\Lambda_{S,n}} \phi(x)w(x)dx = \sum_{i=1}^{n+1} \sum_{j=0}^N \phi(\alpha_{N,i,j})w_{N,i,j} \quad \forall \phi \in \mathcal{H}_{2N,n}$$

such that for $j = 0, \dots, N-1$

$$(4.2) \quad \alpha_{N,i,j} = \begin{cases} \frac{S}{2n}(\delta_{N,i,j} + 2i - 1), & i = 1, \dots, n, \\ \frac{\delta_{N,i,j}(S-1) - S-1}{\delta_{N,i,j} - 1}, & i = n+1, \end{cases}$$

$$\alpha_{N,n+1,N} = S, \quad \alpha_{N,i,N} = \frac{S(i-1)}{n} \quad (i = 1, \dots, n),$$

where $\alpha_{N,i,j}$ are the zeros of $F_{N,i}(x) + F_{N+1,i}(x)$, $\delta_{N,i,j}$ are the zeros of $L_N(x) + L_{N+1}(x)$ and $w_{N,i,j}$ are the corresponding Christoffel numbers. By using the weights of the Legendre-Gauss-Radau quadrature for $(j = 0, \dots, N-1)$, we obtain

$$(4.3) \quad w_{N,i,j} = \begin{cases} \frac{1}{(N+1)^2} \left(\frac{2i - \frac{2n}{S}\alpha_{N,i,j}}{[F_{N,i}(\alpha_{N,i,j})]^2} \right), & i = 1, \dots, n, \\ \frac{1}{(N+1)^2} \left(\frac{2}{(\alpha_{N,i,j} - S + 1)(F_{N,i}(\alpha_{N,i,j}))^2} \right), & i = n+1, \end{cases}$$

and

$$w_{N,i,N} = \frac{1}{(N+1)^2} \quad (i = 1, \dots, n+1).$$

According to (15.3.10) in Szegő [11],

$$(4.4) \quad w_{N,i,j} = \begin{cases} \frac{2\pi}{N} \sqrt{\frac{8ni}{S}\alpha_{N,i,j} + 4i - \frac{4n}{S}\alpha_{N,i,j} - \frac{4n^2}{S^2}\alpha_{N,i,j}^2 - 4i^2}, & i = 1, \dots, n, j = 0, \dots, N-1, \\ \frac{4\pi(\alpha_{N,i,j} - S)^{1/2}}{N(\alpha_{N,i,j} - S + 1)}, & i = n+1. \end{cases}$$

The discrete norm and discrete inner product related to the Legendre-Gauss-Radau hybrid interpolation points from continuous functions f, g are,

$$(f, g)_{w,N} = \sum_{i=1}^{n+1} \sum_{j=0}^N f(\alpha_{N,i,j})g(\alpha_{N,i,j})w_{N,i,j},$$

$$\|f\|_{w,N} = (f, f)_{w,N}^{\frac{1}{2}}.$$

Thanks to (4.1), we have

$$(4.5) \quad (\phi, \psi)_{w,N} = (\phi, \psi)_w \quad \forall \phi, \psi \in \mathcal{H}_{2N,n}.$$

For any $f \in C(\Lambda_{S,n})$, the Legendre-Gauss-Radau hybrid interpolant $I_N f(x) \in \mathcal{H}_{N,n}$, satisfying

$$\begin{aligned} I_N f(\alpha_{N,i,j}) &= f(\alpha_{N,i,j}) \quad (j = 0, \dots, N; \quad i = 1, \dots, n+1), \\ (I_N f - f, \phi)_{w,N} &= 0, \quad \forall \phi \in \mathcal{H}_{N,n}. \end{aligned}$$

Theorem 4.1. *The following relation is obtained for any $v \in H_{w,A}^1(\Lambda_{S,n})$,*

$$\|I_N v\|_w \leq c(\|v\|_w + N^{-1}\|w(x)^{-\frac{1}{4}} \partial x v\|)$$

Proof. By using (4.4) and (4.5), we obtain

$$\begin{aligned} \|I_N v\|_w^2 &= \|I_N v\|_{w,N}^2 = \sum_{i=1}^{n+1} \sum_{j=0}^N v^2(\xi_{N,i,j}) w_{N,i,j} \\ &= \sum_{i=1}^{n+1} \sum_{j=0}^{N-1} v^2(\alpha_{N,i,j}) w_{N,i,j} + \sum_{i=1}^{n+1} v^2(\alpha_{N,i,N}) w_{N,i,N} \\ &= \sum_{i=1}^n \sum_{j=0}^{N-1} v^2(\alpha_{N,i,j}) w_{N,i,j} + \sum_{j=0}^{N-1} v^2(\alpha_{N,n+1,j}) w_{N,n+1,j} \\ &\quad + \sum_{i=1}^n v^2(\alpha_{N,i,N}) w_{N,i,N} + v^2(\alpha_{N,n+1,N}) w_{N,n+1,N} \\ &\leq cN^{-1} \left\{ \sum_{i=1}^n \sum_{j=0}^{N-1} v^2(\alpha_{N,i,j}) \left[\frac{8ni}{S} \alpha_{N,i,j} + 4i - \frac{4n}{S} \alpha_{N,i,j} - \frac{4n^2}{S^2} \alpha_{N,i,j} - 4i^2 \right]^{\frac{1}{2}} \right. \\ &\quad \left. + \sum_{j=0}^{N-1} v^2(\alpha_{N,n+1,j}) (\alpha_{N,n+1,j} - S)^{\frac{1}{2}} (\alpha_{N,n+1,j} - S + 1)^{-1} \right\} \\ &\quad + \sum_{i=1}^n v^2\left(\frac{S(i-1)}{n}\right) (N+1)^{-2} + v^2(S) (N+1)^{-2}. \end{aligned}$$

According to trace theorem, we have

$$\begin{aligned} \|v(S)\|^2 &\leq c\|v\|_{1,w}^2, \\ \|v\left(\frac{S(i-1)}{n}\right)\|^2 &\leq c\|v\|_{1,w}^2 \quad (i = 1, \dots, n). \end{aligned}$$

Let

$$\alpha_{N,i,j} = \begin{cases} \frac{S}{2n} (\cos \theta_{N,i,j} + 2i - 1), & i = 1, \dots, n \\ \frac{(S-1) \cos \theta_{N,i,j} - S - 1}{\cos \theta_{N,i,j} - 1}, & i = n+1, \end{cases}$$

and

$$\hat{v}(\theta_{N,i,j}) = \begin{cases} v\left(\frac{S}{2n}(\cos \theta_{N,i,j} + 2i - 1)\right) & i = 1, \dots, n \\ v\left(\frac{(S-1)\cos \theta_{N,i,j} - S - 1}{\cos \theta_{N,i,j} - 1}\right) & i = n + 1. \end{cases}$$

By using (4.2) and Theorem 8.9.1 [11],

$$\theta_{N,i,j} = \frac{1}{N}(j\pi + O(1)); \quad j = 0, \dots, N-1.$$

where $O(1)$ is bounded uniformly.

Now, let $a_0 = \frac{O(1)}{N+1}$ and $a_1 = \frac{N\pi + O(1)}{N+1}$. Then $\theta_{N,i,j} \in K_j \subset [a_0, a_1]$, K_j being of size $\frac{c}{N+1}$. Consequently,

$$\begin{aligned} \|I_N v\|_w^2 &\leq cN^{-1} \left\{ \sum_{i=1}^n \sum_{j=0}^{N-1} \hat{v}^2(\theta_{N,i,j}) \left[\frac{8ni}{S} \left(\frac{S}{2n}(\cos \theta_{N,i,j} + 2i - 1) \right) \right. \right. \\ &\quad \left. \left. - \frac{4n}{S} \left(\frac{S}{2n}(\cos \theta_{N,i,j} + 2i - 1) \right) - \frac{4n^2}{S^2} \left(\frac{S}{2n}(\cos \theta_{N,i,j} + 2i - 1) \right)^2 - 4i^2 + 4i \right]^{\frac{1}{2}} \right\} \\ &\quad + cN^{-1} \left\{ \sum_{j=0}^{N-1} \hat{v}^2(\theta_{N,n+1,j}) \left(\frac{-\cos \theta_{N,n+1,j} - 1}{\cos \theta_{N,n+1,j} - 1} \right)^{\frac{1}{2}} \left(\frac{-2}{\cos \theta_{N,n+1,j} - 1} \right)^{-1} \right\} \\ &\quad + N^{-2} \sum_{i=1}^n \hat{v}^2\left(\frac{S(i-1)}{n}\right) + \hat{v}^2(S) \\ &\leq cN^{-1} \left\{ \sum_{i=1}^n \sum_{j=0}^N \hat{v}^2(\hat{\theta}_{N,i,j}) \sin \hat{\theta}_{N,i,j} + \sum_{j=0}^N \hat{v}^2(\hat{\theta}_{N,n+1,j}) \sin \hat{\theta}_{N,n+1,j} \right\} \\ &\quad + cN^{-2}(n+1)\|v(x)\|_{1,w}^2. \end{aligned}$$

We apply inequality (13.7) in [7] for interval K_j . Thus,

$$\begin{aligned} \|I_N v\|_w^2 &\leq c \sum_{i=1}^{n+1} \sum_{j=0}^{N-1} \left(\|\hat{v}(\theta) \sin^{\frac{1}{2}} \theta\|_{L^2(K_j)}^2 + N^{-2} \|\partial_\theta \hat{v}(\theta) \sin^{\frac{1}{2}} \theta\|_{L^2(K_j)}^2 \right) \\ &\quad + c(n+1)N^{-2}\|v(x)\|_{1,w}^2 \\ &\leq c(\|v(x)\|_{L_w^2(\Lambda_{a,n})}^2 + N^{-2}\|w(x)^{\frac{-1}{4}} \partial_x v\|_{L_w^2(\Lambda_{a,n})}^2 + N^{-2}\|\partial_x v\|_{L_w^2(\Lambda_{a,n})}^2). \end{aligned}$$

Which implies the desired result. $\square$

Theorem 4.2. For any $v \in H_{w,A}^r(\Lambda_{S,n})$ and $0 \leq \beta \leq 1 \leq r$,

$$\|I_N v - v\|_{\beta,w} \leq cN^{2\beta-r}\|v\|_{r,w,A}.$$

Proof. By using Theorems 3.4 and 4.1 and Cauchy-Schwartz inequality and $I_N(\hat{P}_N^1 v) = \hat{P}_N^1 v$, we can conclude

$$(4.6) \quad \begin{aligned} \|I_N v - \hat{P}_N^1 v\|_w &\leq c(\|\hat{P}_N^1 v - v\|_w + N^{-1}\|w(x)^{\frac{-1}{4}}\partial_x(\hat{P}_N^1 v - v)\|), \\ \|I_N v - \hat{P}_N^1 v\|_w &\leq cN^{-r}\|v\|_{r,w,A} \end{aligned}$$

and

$$(4.7) \quad \|\hat{P}_N^1 v - v\|_w \leq cN^{-r}\|v\|_{r,w,A}.$$

Moreover, by (4.6) and (4.7) and Theorem 2.2, we get

$$(4.8) \quad \begin{aligned} \|I_N v - v\|_{1,w} &\leq \|\hat{P}_N^1 v - v\|_{1,w} + \|I_N v - \hat{P}_N^1 v\|_{1,w} \\ &\leq cN^2\|\hat{P}_N^1 v - v\|_w + cN^2\|I_N v - \hat{P}_N^1 v\|_w \\ &\leq cN^{2-r}\|v\|_{r,w,A}. \end{aligned}$$

Finally, the result follows from (4.6), (4.7), (4.8) and space interpolation. $\square$

5. Error estimation

We consider the following problem ($v > 0$)

$$(5.1) \quad \begin{cases} -\partial_x^2 V(x) + vV(x) = g(x), & 0 < x < \infty \\ V(0) = 0, & i = 1, \dots, n, n+1 \\ (x-S+1)^{-\frac{3}{2}}V(x) \rightarrow 0, & \text{as } x \rightarrow \infty. \end{cases}$$

To solve this problem with $v > \frac{3}{4}$, we find $V \in H_{0,w}^1(\Lambda_{S,n})$ such that

$$(5.2) \quad a_w^v(V, v) = (g, v)_w, \quad \forall v \in H_{0,w}^1(\Lambda_{S,n}).$$

By considering $g \in (H_{0,w}^1(\Lambda_{S,n}))'$ and the Lax-Milgram Lemma and Lemma 3.1, (5.2) has a unique solution in $H_{0,w}^1(\Lambda_{S,n})'$.

The Legendre hybrid spectral method for (5.1) is to obtain $v_N \in \mathcal{H}_{N,n}^0$ such that

$$(5.3) \quad a_w^v(v_N, \phi) = (g, \phi)_w, \quad \forall \phi \in \mathcal{H}_{N,n}^0.$$

Theorem 5.1. *If $V \in H_{w,B}^r(\Lambda_{S,n}) \cap H_{0,w}^1(\Lambda_{S,n})$, $v > \frac{3}{4}$ and $r \geq 1$, then $\|v_N - V\|_{1,w} \leq cN^{1-r}\|V\|_{r,w,B}$.*

Proof. Let $V_N = P_N^{1,0}V$ and $\hat{V}_N = v_N - V_N$. By using equations (5.2) and (5.3), we have

$$(5.4) \quad a_w^v(V_N, \phi) = (g, \phi)_w, \quad \forall \phi \in \mathcal{H}_{N,n}^0$$

and

$$(5.5) \quad a_w^v(\hat{V}_N, \phi) = 0, \quad \forall \phi \in \mathcal{H}_{N,n}^0.$$

Therefore, $v_N = V_N$ and the result is obtained from Theorem 3.3. $\square$

We use the Legendre-Gauss-Radau hybrid pseudospectral method for (5.1). Thus, we have $a_{w,N}^v(u, v) = n(\partial_x u, \partial_x v)_{w_i,N} + (\partial_x u, \partial_x v - 2v(x-S+1)^{-1})_{w_{n+1},N} + v(u, v)_{w,N}, i=1, \dots, n$. We know from (4.5) that for any $\phi, \psi \in \mathcal{H}_{N-1,n}$, we have

$$(5.6) \quad a_{w,N}^v(\phi, \psi) = a_w^v(\phi, \psi).$$

A Legendre hybrid pseudospectral method for (5.1) is to obtain $v_N \in \mathcal{H}_{N-1,n}^0$ such that

$$(5.7) \quad a_{w,N}^v(v_N, \phi) = (g, \phi)_{w,N}, \forall \phi \in \mathcal{H}_{N-1,n}^0.$$

Theorem 5.2. *If $V \in H_{w,B}^r(\Lambda_{S,n}) \cap H_{0,w}^1(\Lambda_{S,n}), g \in H_{w,A}^{r-1}(\Lambda_{S,n}), v > \frac{3}{4}$ and $r \geq 1$, then*

$$(5.8) \quad \|v_N - V_N\|_{1,w} \leq cN^{1-r}(\|V\|_{r,w,B} + \|g\|_{r-1,w,A})$$

Proof. By using (4.5) and Theorem 2.3, we obtain

$$(5.9) \quad \begin{aligned} |(g, \phi)_{w,N}| &= |(I_N(w^{\frac{-1}{2}}(x)g), w^{\frac{1}{2}}(x)\phi(x))_{w,N}| = |(I_N(w^{\frac{-1}{2}}(x)g), w^{\frac{1}{2}}(x)\phi(x))_w| \\ &\leq \|I_N(w^{\frac{-1}{2}}(x)g)\|_{w^2} \|\phi\|_{w^2} \\ &\leq \|I_N(w^{\frac{-1}{2}}(x)g)\|_{1,w} \|\phi\|_{1,w} (\forall \phi \in \mathcal{H}_{N-1,n}). \end{aligned}$$

Hence, by Lemma 3.1 and the Lax-Milgram Lemma, (5.7) has a unique solution such that $\|v_N\|_{1,w} \leq c\|I_N(w^{\frac{-1}{2}}(x)g)\|_{1,w}$. Let $V_N = P_{N-1}^{1,0}V$. Then by (5.6) and (5.7), we have for any $\phi \in \mathcal{H}_{N-1,n}^0$,

$$(5.10) \quad a_w^v(V_N, \phi) = (g, \phi)_w, \quad a_w^v(v_N, \phi) = (I_N g, \phi)_w.$$

Therefore, $a_w^v(V_N - v_n, \phi) = (g - I_N g, \phi)_w, \phi \in \mathcal{H}_{N-1,n}^0$. Let $\varepsilon = v - \frac{3}{4} > 0$. By taking $\phi = V_N - v_N$ and using (3.8), we obtain

$$(5.11) \quad \begin{aligned} \frac{\varepsilon}{2}\|v_N - V_N\|_w^2 + \frac{2\varepsilon}{9}|v_N - V_N|_{1,w}^2 &\leq a_w^v(v_N - V_N, v_N - V_N) \\ &\leq (g - I_N g, V_N - v_N)_w \leq \|g - I_N g\|_w \|V_N - v_N\|_w \\ &\leq \varepsilon\|g - I_N g\|_w^2 + \frac{\varepsilon}{2}\|V_N - v_N\|_w^2. \end{aligned}$$

Therefore, by using Theorems 3.3 and 4.2, we arrive at $\|v_N - V\|_{1,w} \leq \|V_N - V\|_{1,w} + \|v_N - V_N\|_{1,w} \leq cN^{1-r}(\|V\|_{r,w,B} + \|g\|_{r-1,w,A})$. $\square$

6. Numerical results

We present an efficient algorithm for solving (5.1) and compare and report some numerical experiments by using the hybrid pseudospectral scheme (5.7). The

combinations of hybrid Legendre polynomials should be used as basis functions like [16] (see [15]). Indeed, setting

$$\psi_j(x) = \begin{cases} F_{j,i}(x) + F_{j+1,i}(x), & x \in \Lambda_i, i = 1, \dots, n \\ F_{j,n+1}(x) + F_{j+1,n+1}(x), & x \in \Lambda_{n+1}, \end{cases}$$

we have

$$(6.1) \quad \begin{aligned} \mathcal{H}_{N,n}^0 &= \{ \psi \in \mathcal{H}_{N,n} : \psi(0) = 0, \psi(Si/n) = 0, i = 1, \dots, n \}, \\ \mathcal{H}_{N-1,n}^0 &= \text{span} \{ \psi_j : j = 0, 1, \dots, N-2 \}. \end{aligned}$$

Hence, setting

$$(6.2) \quad \begin{aligned} b_{kj} &= (\psi_j, \psi_k)_{w,N} = (\psi_j, \psi_k)_w, a_{kj} = a_{w,N}^v(\psi_j, \psi_k) = -(\psi_j'', \psi_k)_w, \\ v_N &= \sum_{j=0}^{N-2} x_j \psi_j(x), \bar{x} = (x_0, x_1, \dots, x_{N-2})^t, \\ \bar{g} &= (g_0, g_1, \dots, g_{N-2})^t \text{ with } g_j = (g, \psi_j)_w, \end{aligned}$$

the Legendre hybrid pseudospectral relation (5.7) is changed to

$$(6.3) \quad (vB + A)\bar{x} = \bar{g}.$$

Now, with replacing hybrid functions, we have

$$\begin{aligned} b_{kj} &= \sum_{i=1}^{n+1} \int_{\Lambda_i} \psi_j(x) \psi_k(x) w_i(x) dx \\ &= \sum_{i=1}^n \int_{\Lambda_i} (F_{j,i}(x) + F_{j+1,i}(x))(F_{k,i}(x) + F_{k+1,i}(x)) w_i(x) dx \\ &\quad + \int_{\Lambda_{n+1}} (F_{j,n+1}(x) + F_{j+1,n+1}(x))(F_{k,n+1}(x) + F_{k+1,n+1}(x)) w_{n+1}(x) dx \end{aligned}$$

and

$$\begin{aligned} a_{kj} &= - \sum_{i=1}^{n+1} \int_{\Lambda_i} \partial_x^2 \psi_j(x) \psi_k(x) w_i(x) dx \\ &= - \sum_{i=1}^n \int_{\Lambda_i} (\partial_x^2 F_{j,i}(x) + \partial_x^2 F_{j+1,i}(x))(F_{k,i}(x) + F_{k+1,i}(x)) w_i(x) dx \\ &\quad - \int_{\Lambda_{n+1}} (\partial_x^2 F_{j,n+1}(x) + \partial_x^2 F_{j+1,n+1}(x))(F_{k,n+1}(x) + F_{k+1,n+1}(x)) w_{n+1}(x) dx. \end{aligned}$$

Setting

$$x = \begin{cases} \frac{S}{2n}(\gamma + 2i - 1), & (i = 1, \dots, n) \\ \frac{\gamma(S-1) - S - 1}{\gamma - 1}, & \end{cases}$$

one verify that

$$b_{kj} = \frac{S}{2n} \left(\sum_{i=1}^n \int_{-1}^1 (L_j(\gamma) + L_{j+1}(\gamma))(L_k(\gamma) + L_{k+1}(\gamma)) d\gamma \right) \\ - \frac{1}{4} \int_{-1}^1 (L_j(\gamma) + L_{j+1}(\gamma))(L_k(\gamma) + L_{k+1}(\gamma)) d\gamma$$

and

$$a_{kj} = -\frac{2n}{S} \left(\sum_{i=1}^n \int_{-1}^1 (\partial_\gamma^2 L_j(\gamma) + \partial_\gamma^2 L_{j+1}(\gamma))(L_k(\gamma) + L_{k+1}(\gamma)) d\gamma \right) \\ - \int_{-1}^1 \left(\frac{2}{(1-\gamma)^2} (\partial_\gamma L_j(\gamma) + \partial_\gamma L_{j+1}(\gamma)) + \partial_\gamma^2 L_j(\gamma) \right. \\ \left. + \partial_\gamma^2 L_{j+1}(\gamma) \right) (L_k(\gamma) + L_{k+1}(\gamma)) (1-\gamma) d\gamma.$$

By using Orthogonality of Legendre polynomials, the equation is solved. We test the proposed method with numerical examples for solving (5.1) with $v=2$ and different values of S . We solve them and compare the results with rational method [5].

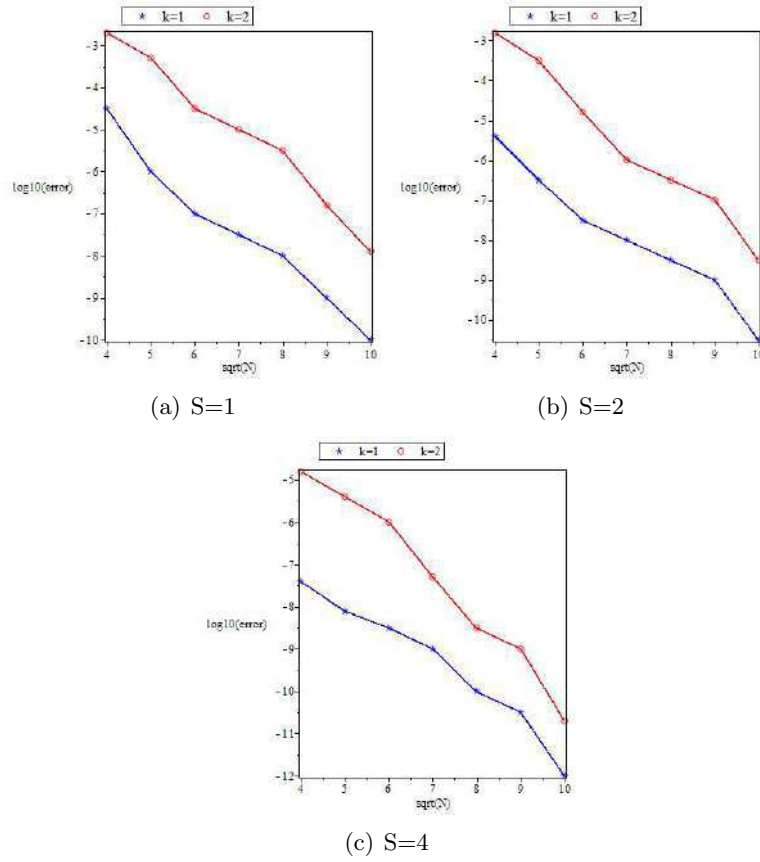
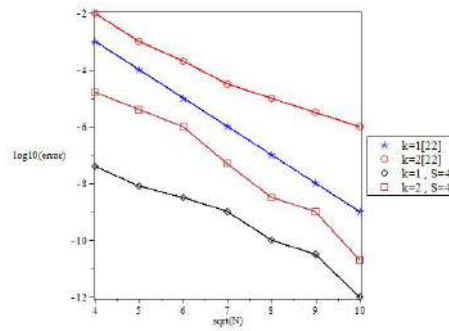
Example 1. $V(x) = \sin kxe^{-x}$.

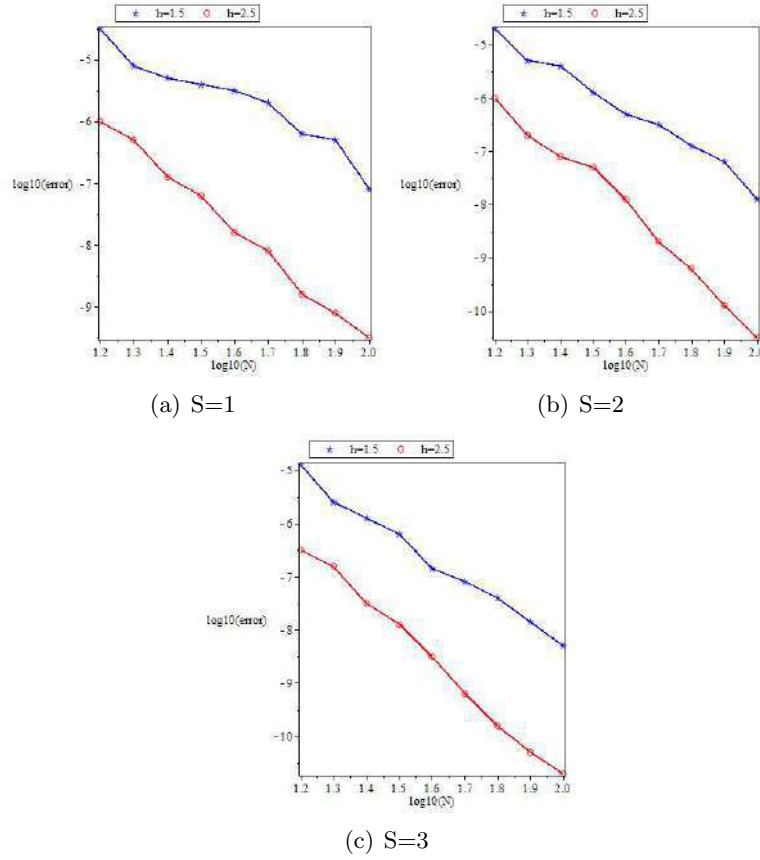
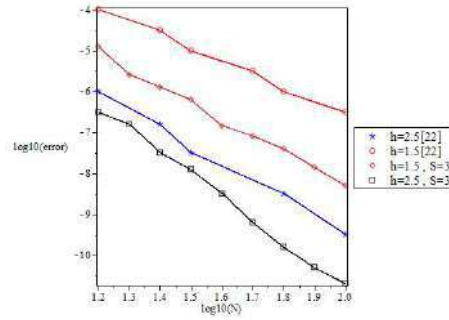
This function decreases exponentially at infinity. Theorem 5.2 predicts that errors will decrease faster than any algebraic rate. We solve (5.1) for this function by using the proposed method with $v=2$ and $S = 1, 2, 4$. In Fig. 1, we plot the $\log_{10}$ of errors for $k = 1, 2$. The results show that errors have been further reduced with ($S=4$). In Fig. 2, we plot the $\log_{10}$ of errors and compare the present method($S=4$) with rational method [5].

Example 2. $V(x) = x/(1+x)^h$. The second example decays algebraically at infinity without essential singularity. We solve (5.1) for this function by using the proposed method with $v=2$ and $S = 1, 2, 3$. In Fig. 3, we plot the $\log_{10}$ of errors for $h = 1.5, 2.5$. The results show that errors have been further reduced with ($S=3$). In Fig. 4, we plot the $\log_{10}$ of errors and compare the present method($S=3$) with rational method [5].

Example 3. $V(x) = (\sin 2x)/(1+x)^h$.

This function decays algebraically at infinity but it also has an essential singularity at infinity. We solve (5.1) for this function by using the proposed method with $v = 2$ and $S = 2, 4, 6$. In Fig. 5, we plot the $\log_{10}$ of errors for $h = 3.5, 5$. The results show that errors have been further reduced with ($S=6$). In Fig. 6, we plot the $\log_{10}$ of errors and compare the present method ($S = 6$) with rational method [5]. The observed convergence rate plotted in Fig. 6 agrees well with the theoretical result.

Figure 1: Log-errors with different values of S and k Figure 2: Log-errors and Comparison of the present method ($S=4$) with rational method [5].

Figure 3: Log-errors with different values of S and h Figure 4: Log-errors and Comparison of the present method ($S = 3$) with rational method [5].

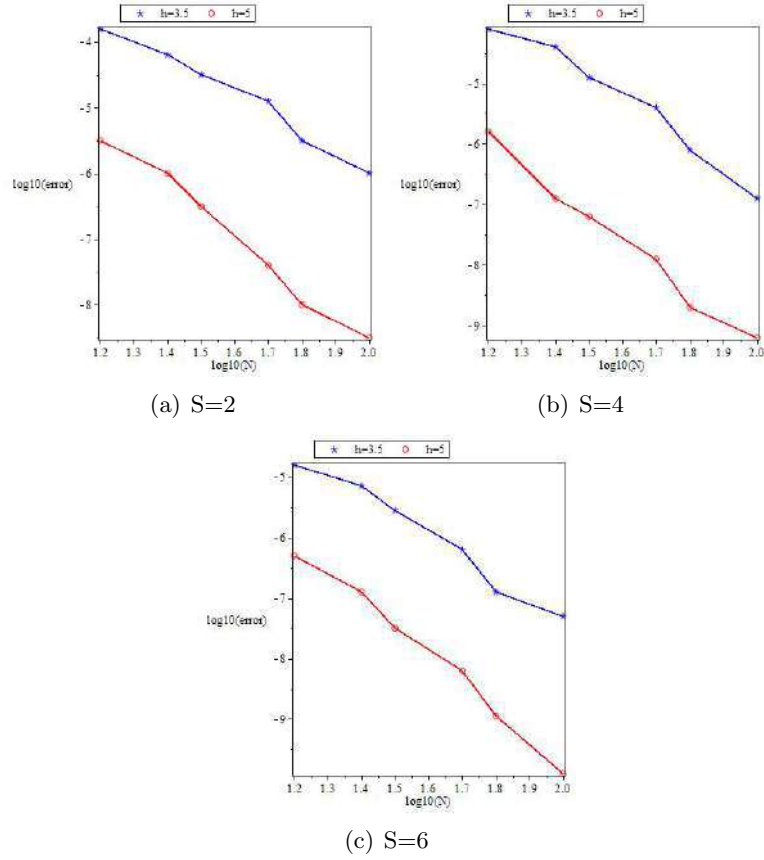
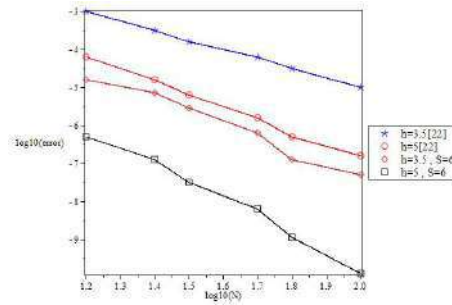


Figure 5: Log-errors with different values of S and h

Figure 6: Log-errors and Comparison of the present method ($S = 6$) with rational method [5].

7. Conclusion

This paper presented spectral method and pseudospectral method on semi-infinite interval $[0, \infty)$ by using hybrid Legendre functions and decomposition of the half-line into $[0, S]$ and $[S, \infty)$ depending on the problem. Basic results on the proposed hybrid approximation of several projections and interpolation were established. The obtained results indicate that the errors decay more rapidly as compared to usual rational approximation. Moreover, it was shown that the proposed hybrid scheme is more suited for problems with oscillatory behaviour.

8. Suggestion

The results of this research can be successfully used in engineering, physics and mechanics and other sciences in infinite domains. In solving the differential equations in semi-infinite interval $[0, \infty)$, where the desired interval is decomposed into $[0, S]$ and $[S, \infty)$, it is suggested that the best value for S would be chosen in future research. In physical models for functions, there is damping from a point onwards and the best value for S can be chosen. To select the value of S , the difference between the values of the function at consecutive points would be obtained. Whenever this difference is less than a certain value of ϵ , the desired point is the best value for S .

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A study on the fuzzy dynamical systems by using of dynamical relations

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Abstract. In this paper, some dynamical relations on fuzzy dynamical systems are introduced and their properties are studied. This relations are special subsets of $X \times X$, where X denotes the base space of the fuzzy dynamical systems $(X, f, M, *)$. By using this relations (i.e. $O_F f, R_F f, N_F f$) a new method for studying fuzzy dynamical systems is introduced.

Keywords: fuzzy dynamical system, orbit relation, recurrent relation, prolongation relation.

1. Introduction

The theory of fuzzy sets was introduced by L. Zadeh in 1965 [8]. After the pioneering work of Zadeh, there has been a great effort to obtain fuzzy analogues of classical theories [4, 5, 8]. Among other fields, a progressive developments is made in the field of fuzzy dynamical systems. The theory of fuzzy dynamical systems has been investigated by many authors from different points of view

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[2, 7]. In analogy to the classical case, this paper is an attempt to present a new method for studying fuzzy dynamical systems. To do this, we introduce the concept of three new dynamical relations Of , Rf and Nf . By using this relations the properties of fuzzy dynamical systems are investigated.

2. Preliminary facts

This section is devoted to provide the prerequisites that are necessary for the next section. Let us to recall the definition of fuzzy metric space (see, [2]).

Definition 2.1. A binary operation $*$: $[0, 1] \times [0, 1] \longrightarrow [0, 1]$ is a continuous t -norm if it satisfies the following condition:

1. $*$ is an associative and commutative operation,
2. $a * 1 = a$, for all $a \in (0, 1]$,
3. $a * b \leq c * d$, whenever $a \leq c, b \leq d$ where $a, b, c, d \in (0, 1]$,
4. if $a * b = a * c$ then $b = c$,
5. $*$ is continuous.

Definition 2.2. A fuzzy metric space is a triple $(X, M, *)$ where X is a non-empty set, $*$ is a continuous t -norm and $M : X^2 \times [0, \infty) \longrightarrow [0, 1]$ is a mapping which has the following properties:

For every $x, y, z \in X$ and $t, s > 0$,

1. $M(x, y, t) > 0$,
2. $M(x, y, t) = 1$ if and only if $x = y$,
3. $M(x, y, t) = M(y, x, t)$,
4. $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$,
5. $M(x, y, \cdot) : (0, \infty) \longrightarrow [0, 1]$ is a continuous map.

Let $(X, M, *)$ be a fuzzy metric space. A set $A \subset X$ is called a fuzzy open set if for any $x \in A$ there exist $0 < r < 1$ and $T_0 \in (0, \infty)$ so that if $M(x, y, t) > 1 - r$ then $y \in A$ for all $t > T_0$. Let $(X, M, *)$ be a fuzzy metric space. An open ball $B(x, r, t)$ with center $x \in X$ and radius $r, 0 < r < 1, t > 0$ is defined by $B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}$.

Let $(X, M, *)$ be a fuzzy metric space. Let $\tau_M = \{A \subset X : x \in A \iff \text{there exist } t > 0 \text{ and } r \in (0, 1) \text{ s.t. } B(x, r, t) \subset A\}$. Then, τ_M is a topology on X . A fuzzy metric space is called a compact fuzzy metric space if (X, τ_M) is a compact space [2]. A fuzzy map $f : X \rightarrow X$ is said to be fuzzy continuous at x_0 , if for each $\epsilon \in (0, 1)$ and each $t > 0$ there is $\delta \in (0, 1)$ so that for each x with $M(x, x_0, t) > 1 - \delta$, we deduce $M(f(x), f(x_0), t) > 1 - \epsilon$. Also f is called

uniformly fuzzy continuous if for any $\epsilon \in (0, 1)$ there is $\delta \in (0, 1)$ so that for each x and y with $M(x, y, t) > 1 - \delta$, we deduce $M(f(x), f(y), t) > 1 - \delta$. A fuzzy map $f : X \rightarrow X$ is said to be fuzzy homeomorphism, if $f : X \rightarrow X$ be a fuzzy continuous bijection map with fuzzy continuous inverse. Let $(X, M, *)$ be a fuzzy metric space and $f : X \rightarrow X$ be a fuzzy homeomorphism then $(X, f, M, *)$ is called a fuzzy dynamical system [2].

In the remaining of the paper, we suppose that X is a compact metric space and $f : X \rightarrow X$ is a fuzzy homeomorphism.

Definition 2.3. A relation from X to Y , written $F : X \rightarrow Y$, is an arbitrary subset of the product $X \times Y$. Also, we define $F(x) = \{y : (x, y) \in F\}$. If $X = Y$, we call F a relation on X . Also for any relation F on X we define the cyclic set

$$|F| := \{x \in X : (x, x) \in F\}.$$

In this paper, we will use the following notations:

$$F^0 = 1_X = \{(x, x) : x \in X\}, \quad F^n = F \circ F \circ \dots \circ F, \\ F^n \circ F^m = F^{n+m}.$$

Definition 2.4. Suppose that $F : X \rightarrow Y$ and $G : Y \rightarrow Z$ are two relations. If $A \subseteq X$, $B \subseteq Y$, then

1. We define the inverse relation of F by $F^{-1} := \{(y, x) : (x, y) \in F\}$,
2. $F(A) := \bigcup_{x \in A} F(x)$,
3. $F^{-1}(B) := \{x : F(x) \cap B \neq \emptyset\}$,
4. The composition $G \circ F : X \rightarrow Z$ is the relation which is defined by $G \circ F := \{(x, z) : \text{there exists } y \in Y, (x, y) \in F \text{ and } (y, z) \in G\}$.

Definition 2.5. Let F be a relation on X . We call F a closed relation when it is a closed subset of $X \times X$ (see, [3]).

Notice that, the composition of closed relations is closed. Also, if F is a closed relation then $|F|$, is a closed subset of X .

Definition 2.6. A relation F on X is reflexive if $1_X \subseteq F$, symmetric if $F^{-1} = F$ and transitive if $F \circ F \subseteq F$.

3. Fuzzy dynamical relations

Definition 3.1. Suppose that $(X, f, M, *)$ be a fuzzy dynamical system, then the orbit relation Of is defined by

$$Of = \bigcup_{n=1}^{\infty} f^n$$

If $x \in X$,

$$Of(x) = \{f^n(x), n \in \mathbb{Z}^+\}.$$

We recall that for relation F ,

$$|F| = \{x; (x, x) \in F\}.$$

So,

$$|Of| = \{x; (x, x) \in Of\} = \{x; \exists n; x = f^n(x)\}.$$

Thus, $|Of|$ is a set, consisting of the periodic points.

Definition 3.2. The point $y \in X$ is called ω -limit for $x \in X$ if there exist sequences $\{n_k\}$ of natural numbers, such that if $k \rightarrow \infty$ then

$$M(f^{n_k}(x), y, t) \rightarrow 1.$$

The set of all limit points of an orbit $Of(x)$ is denoted by

$$\omega f(x) = \limsup_n \{f^n(x)\} = \{y, \exists n_i \rightarrow \infty, M(f^{n_i}(x), y, t) \rightarrow 1\}.$$

ω -limit sets give fundamental information about asymptotic behaviour of a fuzzy dynamical system.

In fact, in this case $\omega f(x)$ is the set of ω -limit points of an orbit $x \in X$ that

$$\omega f(x) = \bigcap_{n=1}^{\infty} \overline{\bigcup_{k \geq n} f^k(x)}.$$

And the set of α -limit points is defined as follows:

$$\alpha f(x) = \bigcap_{n=1}^{\infty} \overline{\bigcup_{k \geq n} f^{-k}(x)}.$$

Definition 3.3. Let $(X, f, M, *)$ be a fuzzy dynamical system, the relation Rf is defined by $Rf = Of \cup \omega f$. If $x \in X$, we have

$$Rf(x) = Of(x) \cup \omega f(x).$$

The relation Rf is called the recurrent relation.

Notice that the cyclic set $|Rf| = \{x; (x, x) \in Rf\}$, consisting of the recurrent points.

Theorem 3.1. Let $(X, f, M, *)$ be a fuzzy dynamical system then for each $x \in X$,

- 1) $\omega f(x) \neq \emptyset$ and $Rf(x)$, $\omega f(x)$ are fuzzy closed relations.
- 2) Of is a transitive relation.

3) Rf is a transitive relation.

Proof. 1) We know that, $\omega f(x) = \bigcap_{n=1}^{\infty} \overline{\bigcup_{k \geq n} f^k(x)}$, and the proof is clear.

2) Suppose $(x, y) \in (Of) \circ (Of)$, then there is $z \in X$ such that $(x, z) \in Of$ and $(z, y) \in Of$. By definition Of

$$\exists n \in \mathbb{Z}^+; z = f^n(x) \wedge \exists m \in \mathbb{Z}^+; y = f^m(z).$$

So,

$$y = f^m(f^n(x)) = f^{m+n}(x),$$

therefore $(x, y) \in Of$.

3) Suppose that $(x, z) \in (Rf) \circ (Rf)$, then

$$\exists y; (x, y) \in Rf \wedge (y, z) \in Rf,$$

from definition Rf we have

$$\forall \epsilon_1, t_1 > 0, \exists n; M(f^n(x), y, t_1) > 1 - \epsilon_1$$

and

$$\forall \epsilon_2, t_2 > 0, \exists m; M(f^m(y), z, t_2) > 1 - \epsilon_2.$$

Since, f is a fuzzy homeomorphism on the compact fuzzy metric space $(X, M, *)$, so f is uniformly fuzzy continuous and therefore,

$$\forall \epsilon_3, t_3 > 0, ; M(f^{n+m}(x), f^m(y), t_3) > 1 - \epsilon_3.$$

Now, by triangle inequality we have,

$$\begin{aligned} M(f^{n+m}(x), z, t_1 + t_2) &\geq M(f^{n+m}(x), f^m(y), t_1) * M(f^m(y), z, t_2) \\ &> (1 - \epsilon_3) * (1 - \epsilon_2) > 1 - \epsilon. (\epsilon \in (0, 1)). \end{aligned}$$

Thus, $(x, z) \in Rf$. Therefore, Rf is a transitive relation. $\square$

Theorem 3.2. Let $(X, f, M, *)$ be a fuzzy dynamical system, then

$$1) Of = f \cup ((Of) \circ f) = f \cup (f \circ (Of)).$$

$$2) Of^{-1} = (Of)^{-1}.$$

$$3) Rf = f \cup ((Rf) \circ f) = f \cup (f \circ (Rf)).$$

Proof. 1) Since Of is a transitive relation and $f \subseteq Of$, we have

$$f \circ (Of) \subseteq (Of) \circ (Of) \subseteq Of.$$

So,

$$(1) \quad f \cup (f \circ (Of)) \subseteq Of.$$

Conversely, for every $(x, y) \in Of$, we have

$$\exists n \in \mathbb{Z}^+; y = f^n(x) = f(f^{n-1}(x)).$$

Let $z = f^{n-1}(x)$, then $(x, z) \in Of$ and $(z, y) \in f$ therefore $(x, y) \in f \circ (Of)$, and we have

$$(2) \quad Of \subseteq f \cup (f \circ (Of)).$$

Hence, one conclude $Of = f \cup (f \circ (Of))$.

2) Since $Of = \{(x, f^n(x)); x \in X, n \in \mathbb{Z}^+\}$. So,

$$(Of)^{-1} = \{(f^n(x), x); x \in X, n \in \mathbb{Z}^+\}$$

let, $y = f^n(x)$, then

$$\begin{aligned} (Of)^{-1} &= \{(y, f^{-n}(y)); y \in X\} \\ &= \{(y, f^{-n}(y)); y \in X, n \in \mathbb{Z}^+\} \\ &= O(f^{-1}). \end{aligned}$$

3) Since, $f \subseteq R_F f$ and R_F is a transitive relation we have $f \circ (R_F f) \subseteq R_F \circ R_F \subseteq R_F f$. So, $f \cup (f \circ (R_F f)) \subseteq R_F f$. Conversely, let $(x, y) \in R_F f$ and $y \neq f(x)$. Then, the subsequence $\{n_k\}$ of natural numbers is exist such that if $k \rightarrow \infty$, $M(f^{n_k}(x), y, t) \rightarrow 1$.

So, $M(f^{n_k-1}(f(x)), y, t) \rightarrow 1$. Let $z = f(x)$ then, we have $(x, z) \in f$ and $(z, y) \in R_F f$. Therefore, $(x, y) \in (R_F f) \circ f$. which completes the proof.

The proof of the other properties is similar. $\square$

Definition 3.4. Suppose that $(X, f, M, *)$ be a fuzzy dynamical system and $A \subseteq X$ then,

- 1) The set A is called forward invariant if $f(A) \subseteq A$.
- 2) The set A is called invariant respect to f if $f(A) = A$.

Theorem 3.3. Let $(X, f, M, *)$ be a fuzzy dynamical system and suppose $x \in X$, then

- 1) If $A = O, R$, then $Af(x)$ respect to f is forward invariant.
- 2) If $A = \alpha, \omega$, then $f(Af(x)) = Af(x)$.
- 3) If $A = O, R$, then $f(| Af |) = | Af |$.

Proof. 1) According to Theorem 3.2, $Af = f \cup (f \circ (Af))$ hence $f \circ Af \subseteq Af$. That is mean for each $x \in X$, $f(Af(x)) \subseteq Af(x)$.

2) According to Theorem 3.2, $Af = f \circ Af = f^{-1} \circ Af = Af \circ f^{-1}$, then for each $x \in X$, $f(Af(x)) = Af(x)$.

3) In the case $A = O$, the proof is obvious. For case $A = R$, according to Definition 3.3 and cyclic set be result $f(|Rf|) = \{f(x); x \in |Rf|\} = \{f(x); (x, x) \in Rf\}$. If $x \in |Rf|$ then $f(x) \in |Rf|$. Thus, $f(|Rf|) \subseteq \{f(x); (f(x), f(x)) \in Rf\}$. Set $z = f(x)$, thus $f(|Rf|) \subseteq \{z; (z, z) \in Rf\} = |Rf|$. On the other hand, similar to process the above we have $f^{-1}(|Rf|) \subseteq |Rf|$. Then, $f(|Rf|) = Rf$. $\square$

Theorem 3.4. *Let the set A respect to fuzzy dynamical system f be forward invariant, if A be closed then $Rf(A) \subseteq A$.*

Proof. Suppose A be closed, according to Rf we have $Rf(A) = \overline{Of(A)} \subseteq \overline{A} \subseteq A$, then $Rf(A) \subseteq A$. $\square$

Definition 3.5. *Suppose that $(X, f, M, *)$ be a fuzzy dynamical system, then Ωf is the relation of all $(x, y) \in \Omega f$ if and only if sequence $\{x_n\}$ and subsequence $\{n_k\}$ of sequence $\{n\}$ of natural numbers be exist that as $n \rightarrow \infty$*

$$M(x_n, x, t) \rightarrow 1, n_k \rightarrow \infty, M(f^{n_k}(x_k), y, t) \rightarrow 1.$$

Theorem 3.5. *Suppose that $(X, f, M, *)$ be a fuzzy dynamical system, $x, y, z \in X$ then:*

- (1) *if $z, y \in \omega f(x)$ then $z \in \omega f(y)$;*
- (2) *if $y \in \omega f(x)$ then $y \in |\Omega f|$.*

Proof. (1) According definition ω -limit relation, sequence $\{n_k\}$ of natural numbers is exist such that for $k \rightarrow \infty$

$$n_k \rightarrow \infty, M(f^{n_k}(x), y, t) \rightarrow 1.$$

Also sequence $\{n'_k\}$ of natural numbers is exist such that if

$$n'_k \rightarrow \infty, M(f^{n'_k}(x), z, t) \rightarrow 1.$$

Set $y_k = f^{n_k}(x)$ and $\{m_k\}$ is increasing sequence of natural numbers such that for each $k \rightarrow \infty$

$$M(y_k, y, t) \rightarrow 1, m_k \rightarrow \infty, M(f^{m_k}(y_k), z, t) \rightarrow 1.$$

Therefore, $z \in \omega f(y)$.

- (2) Set in (1) $z = y$. $\square$

Theorem 3.6. *Suppose that $(X, f, M, *)$ be a fuzzy dynamical system, then*

$$\Omega f = f \circ (\Omega f) = f^{-1} \circ (\Omega f) = (\Omega f) \circ f^{-1} = (\Omega f) \circ f.$$

Proof. Let $(x, y) \in \Omega f$. According definition of relation Ωf , sequence $\{x_n\}$ and subsequence $\{n_k\}$ of sequence $\{n\}$ of natural numbers be exist that as $n \rightarrow \infty$

$$M(x_n, x, t) \rightarrow 1, n_k \rightarrow \infty, M(f^{n_k}(x), y, t) \rightarrow 1.$$

Since X is compact metric fuzzy space, sequence $\{f^{n_k-1}(x)\}$ has convergent subsequence, that is mean,

$$\exists \{n_{i_j}\} \subseteq \{n_k\}, M(f^{n_{i_j}-1}(x), z, t) \rightarrow 1 \implies M(f^{n_{i_j}}(x), f(z), t) \rightarrow 1.$$

Due to uniqueness of limit we have $z = f(y)$, so $(x, z) \in (\Omega f) \wedge (z, y) \in f$, hence $(x, y) \in f \circ (\omega f)$, so

$$(3) \quad \Omega f \subseteq f \circ \Omega f,$$

On the other hand

$$(4) \quad f \circ (\Omega f) \subseteq \Omega f,$$

From (3) and (4) be result $f \circ \Omega f = \Omega f$. Similar to process the above $\Omega f = f^{-1} \circ \Omega f = (\Omega f) \circ f^{-1}$. $\square$

Theorem 3.7. Suppose that $(X, f, M, *)$ be a fuzzy dynamical system and $x \in X$. Then $\Omega f(x)$ is invariant with respect to f .

Proof. According Theorem 3.6, $f \circ \Omega f = \Omega f$. Hence the proof is complete. $\square$

In the following example, we examine the difference between two relations of ωf and ωf .

Example 3.1. Let $X = [0, 1]$, $a * b = ab$ and d be Euclidean meter on $[0, 1]$, then (X, d) is compact metric space and $(X, M, *)$ that $M(x, y, t) = \frac{t}{t+d(x,y)}$ is compact fuzzy metric space. Consider homeomorphism $f(x) = x^2$ on X , for fuzzy dynamical system $(X, f, M, *)$, $\omega f(0) = \{0\}$, $\omega f(1) = \{1\}$. If $x \in (0, 1)$, $\omega f(x) = (0, 1]$. hence $\omega f = ((0, 1] \times \{1\}) \cup \{(0, 0)\}$, while $\omega f = ((0, 1] \times \{1\}) \cup (\{0\} \times \{[0, 1]\})$.

Definition 3.6. Suppose that $(X, f, M, *)$ be a fuzzy dynamical system, then relation Nf is defined as the follows:

$$Nf = Of \cup \Omega f.$$

The relation Nf is called prolongation of f .

Remark 3.1. From definition easily, one can result that Nf is a closed relation. Also

$$|Nf| = \{x, (x, x) \in Nf\}.$$

The points of set $|Nf|$ are called non-wandering points.

Theorem 3.8. Suppose that $(X, f, M, *)$ be a dynamical system, then

$$\{x \in X, \omega f(x) = \omega f(x)\} = \{x \in X, Rf(x) = Nf(x)\}.$$

Proof. For all $x \in X$, $Rf(x) = Of(x) \cup \omega f(x) = Nf(x) = Of(x) \cup \Omega f(x) \iff \omega f(x) = \Omega f(x)$, that proof is complete. $\square$

Example 3.2. Let $X = S^1$, $a * b = ab$ and $d(x, y)$ be a distance between of x and y that

$$d(x = (\cos 2\pi t, \sin 2\pi t), y = (\cos 2\pi s, \sin 2\pi s)) = |t - s|, \quad t, s \in [0, 1)$$

then (X, d) is compact metric space and define

$$M(x, y, t) = \frac{t}{t + d(x, y)}$$

that $(X, M, *)$ is compact fuzzy metric space. Consider homeomorphism $f(x) = x$ on X , then

$$Of = \bigcup_{n=1}^{\infty} f^n = f = \{(x, x), x \in X\},$$

for all $x \in X$ $\omega f(x) = \omega f(x) = Rf(x) = Nf(x) = \{x\}$ hence $|Nf| = |Rf| = |Of| = |f|$. In fact fuzzy fixed points, recurrence points and non-wandering points are similar.

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Dynamically defined topological entropy of co-compact open covers

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Abstract. In this paper, the concept of dynamically defined topological entropy of co-compact open sets for iteration of a continuous system on non-compact Hausdorff space is introduced. Dynamically defined topological entropy is introduced as an invariant of topological conjugation for perfect mappings defined on any Hausdorff space (compactness and metrizability are not necessarily required). This is achieved through the consideration of a topological dynamic system and co-compact covers of the space. An example in the genetic space is presented.

Keywords: co-compact open set, dynamically defined co-compact entropy, topological dynamical system, genetic space.

1. Introduction

Entropy was first introduced into the theory of dynamical systems by Kolmogorov and Sinai [7] in 1958. Later, on the basis of this work, in 1965, Adler, Konheim and McAndrew [1] defined topological entropy for a continuous map on a compact topological space, using open covers. From then on, entropy soon

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became the center of attention. The importance of entropy arises from its invariance under conjugacy. Therefore, systems with different entropies cannot be conjugate. This article is an attempt to present an approach to the entropy for iteration of a continuous system $f : X \rightarrow X$ on a non-compact Hausdorff space X .

Instead of using all open covers of the space to define entropy, we consider the open covers consisting of the co-compact open sets. The advantages of dynamically defined topological entropy include:

- (i) It does not require the space to be metrizable, compared to Bowen's entropy, which is metric dependent.
- (ii) It does not require the space to be compact.
- (iii) It is an invariant of topological conjugation.

2. Basic notions

Let (X, f) be a topological dynamical system, where X is a Hausdorff and $f : X \rightarrow X$ is a continuous mapping.

Definition 2.1. Let X be a Hausdorff space. For an open subset, U of X , if $X \setminus U$ is a compact subset of X , then U is called a co-compact open subset. If every element of an open cover α of X is co-compact, then α is called a co-compact open cover of X (see [13]).

Theorem 2.1. *The intersection of finitely many co-compact open subsets is co-compact, and the union of any collection of co-compact open sets is co-compact open.*

Proof. See [13]. □

Theorem 2.2. *Let X be Hausdorff. Then, any co-compact open cover has a finite subcover.*

Proof. See [13]. □

Definition 2.2. Let X and Y be Hausdorff spaces and let $f : X \rightarrow Y$ be a continuous mapping. If f is a closed mapping and all fibers, $f^{-1}(x), x \in Y$, are co-compact, then f is called a perfect mapping.

In particular, if X is compact Hausdorff and Y is Hausdorff, every continuous mapping from X into Y is perfect. If $f : X \rightarrow Y$ is perfect, then $f^{-1}(F)$ is compact for each compact subset, $F \subset Y$ (see [6]).

Theorem 2.3. *Let X, Y be two Hausdorff spaces and let $f : X \rightarrow Y$ be a perfect mapping. If U is co-compact open in Y , then $f^{-1}(U)$ is co-compact open in X . Moreover if α is a co-compact open cover of Y , then $f^{-1}(\alpha)$ is a co-compact open cover of X .*

Proof. See [13]. □

3. Dynamically defined topological entropy of co-compact open covers

For compact topological systems, Adler, Konheim and McAndrew introduced the concept of topological entropy and studied their properties [1]. But in many physical systems the state space is not compact. For this goal we would like to present another approach to topological entropy for non-compact Hausdorff spaces. Therefore, in the remainder of the paper, a space is assumed to be Hausdorff and a mapping is assumed to be perfect.

Let X be Hausdorff and α be an open co-compact cover of X . Moreover let $N(f^n(X), \alpha)$ be the minimum cardinal number of subsets of α which are a cover for $f^n(X)$, where n is a natural number. By Theorem 2.3, when α is a co-compact open cover of X , α has a finite subcover. Hence, $N(f^n(X), \alpha)$ is a finite positive integer. We define $H(f^n(X), \alpha)$ as the number $\log N(f^n(X), \alpha)$.

Let α, β be two co-compact open covers of X . Define

$$\alpha \vee \beta = \{U \cap V; U \in \alpha \text{ and } V \in \beta\}.$$

If for any $U \in \alpha$, there exists $V \in \beta$, such that $U \subset V$, then α is said to be a refinement of β and is denoted by $\beta \prec \alpha$.

The following are some obvious facts:

- (i) For any open covers α, β of X , $\alpha \prec \alpha \vee \beta$.
- (ii) For any open covers, α and β of X , if β is a subcover of α , then $\alpha \prec \beta$.
- (iii) $H(f^n(X), \alpha) \geq 0$.
- (iv) For any co-compact open cover α of X and $n \in \mathbb{N}$, $H(f^n(X), \alpha) = 0 \iff N(f^n(X), \alpha) = 1 \iff f^n(X) \in \alpha$.

Theorem 3.1. *Let α, β be two co-compact open covers of X and n be a natural number then,*

- (i) *If $\alpha \prec \beta$ then $H(f^n(X), \alpha) \leq H(f^n(X), \beta)$.*
- (ii) $H(f^n(X), \alpha \vee \beta) \leq H(f^n(X), \alpha) + H(f^n(X), \beta)$.
- (iii) *If $f : X \rightarrow X$ is a continuous map then $H(f^{n-1}(X), f^{-1}(\alpha)) \leq H(f^n(X), \alpha)$.*

Proof. One can deduce (i) by straightforward calculations based on the definition. For the proof of (ii), let $\{A_1, A_2, \dots, A_l\} \subset \alpha$ and $\{B_1, B_2, \dots, B_m\} \subset \beta$ be two minimal subcovers for $f^n(X)$. Then, $\{B_i \cap A_j\}$ is an open subcover for $f^n(X)$. So,

$$H(f^n(X), \alpha \vee \beta) \leq \log(lm) = \log l + \log m = H(f^n(X), \alpha) + H(f^n(X), \beta).$$

- (iii) Let $\{A_1, \dots, A_m\} \subset \alpha$ be a minimal subcover for $f^n(X)$. Then

$$\{f^{-1}(A_1), f^{-1}(A_2), \dots, f^{-1}(A_m)\} \subset f^{-1}(\alpha)$$

is an open cover for $f^{n-1}(X)$. So, $H(f^{n-1}(X), f^{-1}(\alpha)) \leq H(f^n(X), \alpha)$. $\square$

Definition 3.1. Let X be Hausdorff space, $f : X \rightarrow X$ be a perfect mapping, and α be a co-compact open cover of X . The non-negative number, $h(f^n(X), \alpha) = \limsup_{m \rightarrow \infty} \frac{1}{m} H(f^n(X), \bigvee_{i=0}^{m-1} f^{-i}(\alpha))$, is said to be the dynamically defined topological entropy of f^n relative to α .

In particular when X is compact Hausdorff, any open set of X is co-compact, and any continuous mapping $f : X \rightarrow X$ is perfect. Note that when X is Hausdorff and $f : X \rightarrow X$ is perfect, $f^m : X \rightarrow X$ is also a perfect mapping [6]. It should be made aware that the new entropy is well defined for perfect mappings on non-compact spaces, e.g., $\mathbb{R}^n$. But Adler, Konheim and McAndrew's topological entropy requires that the space be compact.

4. An example of dynamically defined topological entropy in the genetic space

Each genetic code consists an alphabet of four letters [12]. DNA's alphabet is the set $\{T, C, A, G\}$ (corresponding to the bases thymine, cytosine, adenine, and guanine), while RNA's is $\{U, C, A, G\}$ (corresponding to the bases uracil, cytosine, adenine, and guanine). To model each genetic code we consider n letters as a mapping from the set of integer numbers to $\{0, 1, 2, 3\}$, where 0, 1, 2 are identification numbers for adenine, guanine, and cytosine respectively, which 3 is the identification number for either thymine or uracil. If we denote $\omega(i)$ by ω_i , then a genetic code can be denoted by a sequence (ω_n) where $(\omega_n) \in \{0, 1, 2, 3\}$.

So, the set of genetic codes is the set $G = \{(\omega_n) : \omega_n \in \{0, 1, 2, 3\}\}$.

We know that there is a one to one correspondence between G and the set of real numbers. So, in this case G is not a compact metric space. The mapping $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = 2x$ creates a dynamics on the genetic space. f is clearly a perfect mapping. For this linear system, $(\mathbb{R}, f)$, defined by $f(x) = 2x$, the dynamically defined topological entropy is zero.

5. Dynamically defined topological entropy and conjugacy

Theorem 5.1. Let X be Hausdorff and $f : X \rightarrow X$ be perfect. Then, $h(f^n(X), \alpha) = h(f^n(X), \bigvee_{j=0}^k f^{-j}(\alpha))$.

Proof. We obtain immediately

$$\begin{aligned} h(f^n(X), \bigvee_{j=0}^k f^{-j}(\alpha)) &= \limsup_{m \rightarrow \infty} \frac{1}{m} H(f^n(X), \bigvee_{i=0}^{m-1} f^{-i}(\bigvee_{j=0}^k f^{-j}(\alpha))) \\ &= \limsup_{m \rightarrow \infty} \frac{1}{m} H(f^n(X), \bigvee_{t=0}^{m+k-1} f^{-t}(\alpha)) \end{aligned}$$

$$\begin{aligned}
&= \limsup_{p \rightarrow \infty} \frac{p}{p-k} \frac{1}{p} H(f^n(X), \bigvee_{t=0}^{p-1} f^{-t}(\alpha)) \\
&= h(f^n(X), \alpha). \quad \square
\end{aligned}$$

Theorem 5.2. *Let X be Hausdorff and $f : X \rightarrow X$ be perfect. Then for each natural numbers n, k such that $k < n$, we have, $h(f^{n-k}(X), f^{-k}(\alpha)) \leq h(f^n(X), \alpha)$.*

Proof. It suffices to show that $h(f^{n-1}(X), f^{-1}(\alpha)) \leq h(f^n(X), \alpha)$.

By Theorem 3.1 (iii), we have

$$\begin{aligned}
h(f^{n-1}(X), f^{-1}(\alpha)) &= \limsup_{m \rightarrow \infty} \frac{1}{m} H(f^{n-1}(X), \bigvee_{i=0}^{m-1} f^{-i}(f^{-1}(\alpha))) \\
&\leq \limsup_{m \rightarrow \infty} \frac{1}{m} H(f^n(X), \bigvee_{i=0}^{m-1} f^{-i}(\alpha)) \\
&= h(f^n(X), \alpha).
\end{aligned}$$

□

Definition 5.1. Let X be Hausdorff space, $f : X \rightarrow X$ be a perfect mapping. The non-negative number, $h(f^n(X)) = \sup_{\alpha} h(f^n(X), \alpha)$, is said to be the dynamically defined topological entropy of f^n . Where the supremum is taken over all co-compact open covers of X .

Theorem 5.3. *Let X be Hausdorff and $id : X \rightarrow X$ be the identity mapping. Then $h(id(X)) = 0$.*

Proof. Obvious. □

Theorem 5.4. *Let X be Hausdorff space, $f : X \rightarrow X$ be a perfect mapping. If Ω is a closed subset of X and invariant under f , i.e., $f(\Omega) \subseteq \Omega$, then $h(f^n(\Omega)) \leq h(f^n(X))$.*

Proof. Obvious. □

Theorem 5.5. *Let (X, f) and (Y, g) be two topological dynamical systems, where X and Y are Hausdorff, $f : X \rightarrow X$ and $g : Y \rightarrow Y$ are perfect mappings. If there exists a topological conjugation, $\varphi : X \rightarrow Y$, where φ is also perfect, then $h(f^n(X)) = h(g^n(Y))$.*

Proof. Let α be any co-compact open cover of Y . As φ is perfect and α is co-compact open cover of Y , $\varphi^{-1}(\alpha)$ is co-compact open cover of X by applying

Theorem 2.3. Hence, we have:

$$\begin{aligned}
 h(g^n(Y), \alpha) &= \limsup_{m \rightarrow \infty} \frac{1}{m} H(g^n(Y), \bigvee_{i=0}^{m-1} g^{-i}(\alpha)) \\
 &= \limsup_{m \rightarrow \infty} \frac{1}{m} H(\varphi^{-1}(g^n(Y)), \bigvee_{i=0}^{m-1} \varphi^{-1}(g^{-i}(\alpha))) \\
 &= \limsup_{m \rightarrow \infty} \frac{1}{m} H(f^n(\varphi^{-1}(Y)), \bigvee_{i=0}^{m-1} f^{-i}(\varphi^{-1}(\alpha))) \\
 &= \limsup_{m \rightarrow \infty} \frac{1}{m} H(f^n(X), \bigvee_{i=0}^{m-1} f^{-i}(\varphi^{-1}(\alpha))) \\
 &= h(f^n(X), \varphi^{-1}(\alpha)).
 \end{aligned}$$

Therefore, $h(f^n(X)) \geq h(g^n(Y))$. When φ is a topological conjugation, it is, of course, perfect, too. Hence, we have both $h(f^n(X)) \geq h(g^n(Y))$ and $h(g^n(Y)) \geq h(f^n(X))$ from the above proof, implying $h(f^n(X)) = h(g^n(Y))$. $\square$

6. Conclusions

In this paper an extension of the notion of topological entropy for non-compact Hausdorff spaces (compactness and metrizability are not necessarily required) is introduced. The dynamically defined topological entropy is defined based on the co-compact open covers. Interestingly, for non-compact systems this new entropy retains invariant under topological conjugation. The computation of dynamically defined topological entropy of locally compact manifolds for well known examples can be a topic for future research.

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CLR property in Menger spaces and related common fixed point theorems

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Abstract. In this paper, we have proved some common fixed point theorems for weakly compatible mappings in Menger space using the notion of CLR and JCLR property and control functions. Some illustrative examples are also given to show the usability of the presented results.

Keywords: Menger space, compatible mappings, weakly compatible mappings, CLR property and JCLR property.

1. Introduction

The notion of probabilistic metric space as a generalization of metric space was introduced by Menger [16]. In Menger theory, the notion of probabilistic metric space corresponds to situations when we do not know exactly the distance between two points, but we know probabilities of possible values of this distance. In this note he explained how to replace the numerical distance between two points p and q by a function $F(p, q, t)$ whose value $F(p, q, t)$ at the real number t is interpreted as the probability that the distance between p and q is less than

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t . In fact the study of such spaces received an impetus with the pioneering work of Schweizer and Sklar [20]. The theory of probabilistic metric space is of paramount importance in Probabilistic Functional Analysis especially due to its extensive applications in random differential as well as random integral equations.

Now, we give preliminaries and basic definitions in Menger space which are useful in this paper.

Definition 1.1 ([20]). *A mapping $F : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is called distribution function if it is non decreasing and left continuous with $\inf\{F(t) : t \in \mathbb{R}^+\} = 0$ and $\sup\{F(t) : t \in \mathbb{R}^+\} = 1$. We will denote the set of all distribution functions by $\mathcal{L}$.*

Let $\mathcal{L}$ be the set of all distribution functions whereas H be the set of specific distribution functions (also known as Heaviside functions) defined by

$$H(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 1, & \text{if } t > 0. \end{cases}$$

Definition 1.2 ([16]). *A probabilistic metric space is a pair (K, F) , where K is a nonempty set and $F : K \times K \times [0, \infty) \rightarrow \mathcal{L}$ is a mapping satisfying the following conditions for all $p, q, r \in K$ and $t, s \geq 0$,*

$$(p_1) \quad F(p, q, t) = 1 \text{ if and only if } p = q;$$

$$(p_2) \quad F(p, q, 0) = 0;$$

$$(p_3) \quad F(p, q, t) = F(q, p, t);$$

$$(p_4) \quad F(p, q, t) = 1 \text{ and } F(q, r, s) = 1, \text{ then } F(p, r, (t + s)) = 1.$$

Every metric space (K, d) can always be realized as a Probabilistic metric space by $F(p, q, t) = H(t - d(p, q))$, for all $p, q \in K$, where H be the set of specific distribution functions (also known as Heaviside functions) defined in [20, Definition 1.1].

Probabilistic metric space offers a wider framework than that of the metric space and cover even wider statistical situations.

Definition 1.3 ([20]). *A mapping $\Delta : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is called a t -norm if for all $a_1, b_1, c_1, d_1 \in [0, 1]$,*

$$(1) \quad \Delta(a_1, 1) = a_1, \quad \Delta(0, 0) = 0;$$

$$(2) \quad \Delta(a_1, b_1) = \Delta(b_1, a_1);$$

$$(3) \quad \Delta(c_1, d_1) \geq \Delta(a_1, b_1) \text{ for } c_1 \geq a_1, \quad d_1 \geq b_1;$$

$$(4) \quad \Delta(\Delta(a_1, b_1), c_1) = \Delta(a_1, \Delta(b_1, c_1)).$$

Example 1.1. Following are the four basic t -norms:

- (i) The minimum t -norm: $\Delta_M(a_1, b_1) = \min\{a_1, b_1\}$.
- (ii) The product t -norm: $\Delta_P(a_1, b_1) = a_1 b_1$.
- (iii) The Lukasiewicz t -norm: $\Delta_L(a_1, b_1) = \max\{a_1 + b_1 - 1, 0\}$.
- (iv) The weakest t -norm, the drastic product:

$$\Delta_D(a_1, b_1) = \begin{cases} \min\{a_1, b_1\} & \text{if } \max\{a_1, b_1\} = 1, \\ 0, & \text{otherwise.} \end{cases}$$

We have the following ordering in the above stated norms:

$$\Delta_D < \Delta_L < \Delta_P < \Delta_M.$$

Definition 1.4 ([16]). A Menger space is a triplet (K, F, Δ) , where (K, F) is a probabilistic metric space and Δ is a t -norm with the following condition:

For all $p, q, r \in K$ and $t, s \geq 0$,

$$(p_5) \quad F(p, r, (t + s)) \geq \Delta(F(p, q, t), F(q, r, s)).$$

Example 1.2. Let $K = \mathbb{R}$, $\Delta(a_1, b_1) = \min(a_1, b_1)$, for all a_1, b_1 in $[0, 1]$ and

$$F(p, q, t) = \begin{cases} H(t), & \text{if } p \neq q \\ 1, & \text{if } p = q \end{cases}; \quad \text{where} \quad H(t) = \begin{cases} 0, & \text{if } t \leq 0 \\ 1, & \text{if } t > 0. \end{cases}$$

Then (K, F, Δ) is a Menger space.

Definition 1.5. A sequence $\{p_n\}$ in Menger space (K, F, Δ) is said to be:

- (i) Convergent at a point $p \in K$ if for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer $N_{\epsilon, \lambda}$ such that $F(p_n, p, \epsilon) > 1$ for all $n \geq N_{\epsilon, \lambda}$.
- (ii) Cauchy sequence in K if for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer $N_{\epsilon, \lambda}$ such that $F(p_n, p_m, \epsilon) > 1$ for all $n, m \geq N_{\epsilon, \lambda}$.
- (iii) Complete if every Cauchy sequence in K is convergent in K .

In 1996, Jungck [12] introduce the notion of weakly commuting mappings.

Definition 1.6 ([12]). Two self-mapping f_1 and g_1 of a Menger space (K, F, Δ) are said to be weakly commuting if $F(f_1 g_1 p, g_1 f_1 p, t) \geq F(f_1 p, g_1 p, t)$, for each $p \in K$ and for each $t > 0$.

In 1982, Sessa [21] weakened the concept of commutativity to weakly commuting mappings. Afterwards, Jungck [13] enlarged the concept of weakly commuting mappings to compatible mappings. In 1991, Mishra [17] introduced the notion of compatible mappings in the setting of probabilistic metric space.

Definition 1.7 ([17]). Let (K, F, Δ) be a Menger space such that the t -norm Δ is continuous and f_1, g_1 be mappings from K into itself. Then f_1 and g_1 are said to be compatible if $\lim_{n \rightarrow \infty} F(f_1 g_1 p_n, g_1 f_1 p_n, t) = 1$, whenever $\{p_n\}$ is a sequence in K such that $\lim_{n \rightarrow \infty} f_1 p_n = \lim_{n \rightarrow \infty} g_1 p_n = u_1$, for some $u_1 \in K$ and for all $t > 0$.

Definition 1.8. Two self mappings f_1 and g_1 on a Menger space (K, F, Δ) are said to be non-compatible if either $\lim_{n \rightarrow \infty} F(f_1 g_1 p_n, g_1 f_1 p_n, t)$ is non-existent or $\lim_{n \rightarrow \infty} F(f_1 g_1 p_n, g_1 f_1 p_n, t) \neq 1$, whenever $\{p_n\}$ is a sequence in K such that $\lim_{n \rightarrow \infty} f_1 p_n = \lim_{n \rightarrow \infty} g_1 p_n = u_1$, for some $u_1 \in K$ and for all $t > 0$.

Further, Singh and Jain [22] proved some fixed point theorems for weakly compatible maps in the setting of Menger space.

Definition 1.9 ([22]). Two maps f_1 and g_1 are said to be weakly compatible if they commute at their coincidence points i.e., if $f_1 p = g_1 p$, for some $p \in K$, then $f_1 g_1 p = g_1 f_1 p$.

Definition 1.10 ([1]). A pair of self-mappings (f_1, g_1) on a Menger space (K, F, Δ) are said to satisfy the property (E.A) if there exists a sequence $\{p_n\}$ in K such that

$$\lim_{n \rightarrow \infty} F(f_1 p_n, u_1, t) = \lim_{n \rightarrow \infty} F(g_1 p_n, u_1, t) = 1, \text{ for some } u_1 \in K \text{ and for all } t > 0.$$

Remark 1.1. It is easy to see that two non-compatible self-mappings of a Menger space satisfy the property (E.A), but the converse is not true in general.

Definition 1.11. The pairs (A, S) and (B, T) on a Menger space (K, F, Δ) are said to satisfy the common property (E.A) if there exists two sequence $\{p_n\}$ and $\{q_n\}$ in K such that

$$\lim_{n \rightarrow \infty} A p_n = \lim_{n \rightarrow \infty} S p_n = \lim_{n \rightarrow \infty} B q_n = \lim_{n \rightarrow \infty} T q_n = u_1, \text{ for some } u_1 \in K.$$

If $B = A$ and $T = S$ in the definition we get the definition of the property (E.A).

Definition 1.12. A pair of self-mappings A and S of a Menger space (K, F, Δ) is said to satisfy the common limit range property with respect to the mapping S (briefly CLR_S property) if there exists a sequence $\{p_n\}$ in K such that

$$\lim_{n \rightarrow \infty} F(A p_n, u_1, t) = \lim_{n \rightarrow \infty} F(S p_n, u_1, t) = 1, \text{ for some } u_1 \in S(K) \text{ and for all } t > 0.$$

Now, we give an example of self-mappings A and S satisfying the CLR_S property (see [2])

Example 1.3. Let (K, F, Δ) be a Menger space with $K = [0, \infty)$ and for all $p, q \in K$ by $F(p, q, t) = H(t - |p - q|)$, $t > 0$ and $F(p, q, 0) = 0$ where $\Delta(a_1, b_1) = \min\{a_1, b_1\}$ for all $a_1, b_1 \in [0, 1]$. Define self-mappings A and S on K by $A p = p + 3$, $S p = 4p$. Let a sequence $\{p_n = 1 + \frac{1}{n}\}$, $n \in N$ in K .

Since $\lim_{n \rightarrow \infty} Ap_n = \lim_{n \rightarrow \infty} Sp_n = 4$, then

$$\lim_{n \rightarrow \infty} F(Ap_n, 4, t) = \lim_{n \rightarrow \infty} F(Sp_n, 4, t) = 1,$$

where $4 \in K$. Therefore, the mappings A and S satisfy the CLR property.

Remark 1.2. From Example 1.3, it is clear that a pair (A, S) satisfying the property (E.A) property with the closeness of the subspace $S(K)$ always verify the CLR property.

Definition 1.13. The pairs (A, S) and (B, T) on a Menger space (K, F, Δ) are said to satisfy common limit range property with respect to mappings S and T (briefly CLR_{ST} property) if there exists two sequence $\{p_n\}$ and $\{q_n\}$ in K such that for all $t > 0$

$$\begin{aligned} \lim_{n \rightarrow \infty} F(Ap_n, u_1, t) &= \lim_{n \rightarrow \infty} F(Sp_n, u_1, t) = \lim_{n \rightarrow \infty} F(Bq_n, u_1, t) \\ &= \lim_{n \rightarrow \infty} F(Tq_n, u_1, t) = 1, \text{ where } u_1 \in S(K) \cap T(K). \end{aligned}$$

Remark 1.3. If $B = A$ and $T = S$ in the definition we get the definition of CLR property.

Remark 1.4. The CLR_{ST} property implies the common property (E.A), but the converse is not true in general (see [5]).

Proposition 1.1 ([15]). If the pairs (A, S) and (B, T) satisfy the common property (E.A) and $S(K)$ and $T(K)$ are closed subsets of K , then the pairs satisfy also the CLR_{ST} property.

In 2012, Imdad et al. [9] introduce the CLR_{ST} property in Menger space as follows:

Definition 1.14 ([9]). The pairs (A, S) and (B, T) on a Menger space (K, F, Δ) are said to satisfy common limit range property with respect to mappings S and T (briefly CLR_{ST} property) if there exists two sequence $\{p_n\}$ and $\{q_n\}$ in K such that for all $t > 0$

$$\begin{aligned} \lim_{n \rightarrow \infty} F(Ap_n, u_1, t) &= \lim_{n \rightarrow \infty} F(Sp_n, u_1, t) = \lim_{n \rightarrow \infty} F(Bq_n, u_1, t) \\ &= \lim_{n \rightarrow \infty} F(Tq_n, u_1, t) = 1, \end{aligned}$$

where $u_1 = Sr = Tr$, for some $r \in K$.

Remark 1.5. If $B = A$ and $T = S$ in the above definition we get the definition of CLR property.

Definition 1.15 ([8]). Two families of self-mappings $\{A_i\}$ and $\{S_j\}$ are said to be pair wise commuting if

- (1) $A_i A_j = A_j A_i$, $i, j \in \{1, 2, \dots, m\}$;
- (2) $S_k S_l = S_l S_k$, $k, l \in \{1, 2, \dots, n\}$;
- (3) $A_i S_k = S_k A_i$, $i \in \{1, 2, \dots, m\}$, $k \in \{1, 2, \dots, n\}$.

Lemma 1.1 ([22]). *Let $\{p_n\}$ be a sequence in a Menger space (K, F, Δ) with continuous t -norm Δ and $\Delta(t, t) \geq t$. If there exists a constant $k \in (0, 1)$ such that*

$$F(p_n, p_{n+1}, kt) \geq F(p_{n-1}, p_n, t),$$

for all $t > 0$ and $n = 1, 2, 3, \dots$, then $\{p_n\}$ is a Cauchy sequence in K .

Lemma 1.2 ([22]). *Let (K, F, Δ) be a Menger space. If there exists $k \in (0, 1)$ such that*

$$F(p, q, kt) \geq F(p, q, t), \text{ for all } p, q \in K \text{ and } t > 0, \text{ then } p = q.$$

2. Main results

Now, we prove the main results as follows.

Lemma 2.1. *Let A, B, S and T be mappings of a complete Menger space (K, F, Δ) into itself satisfying the following conditions:*

- (2.1) *the pair (A, S) satisfies the CLR_S property or the pair (B, T) satisfies the CLR_T property,*
- (2.2) $A(K) \subseteq T(K)$, $B(K) \subseteq S(K)$,
- (2.3) $T(K)$ or $S(K)$ is a closed subset of K ,
- (2.4) $B(q_n)$ converges for every sequence $\{q_n\}$ in K whenever $T(q_n)$ converges or $A(p_n)$ converges for every sequence $\{p_n\}$ in K whenever $S(p_n)$ converges
- (2.5) $(1 + \alpha F(Sp, Tq, t))F(Ap, Bq, t) > \alpha \min\{F(Ap, Sp, t)F(Bq, Tq, t), F(Sp, Bq, t)F(Ap, Tq, t)\} + \min\{F(Sp, Tq, t), \sup_{t_1+t_2=\frac{2t}{k}} \min\{F(Ap, Sp, t_1), F(Bq, Tq, t_2), \sup_{t_3+t_4=2t} \min\{F(Sp, Bq, t_3), F(Ap, Tq, t_4)\}\}, \text{ for all } p, q \in K, t > 0, \text{ for some } \alpha \geq 0 \text{ and } 1 \leq k < 2. \text{ Then the pairs } (A, S) \text{ and } (B, T) \text{ satisfy the } CLR_{ST} \text{ property.}$

Proof. Suppose that the pair (A, S) satisfies the CLR_S property and $T(K)$ is closed subset of K . Then there exists a sequence $\{p_n\}$ in K such that

$$\lim_{n \rightarrow \infty} Ap_n = \lim_{n \rightarrow \infty} Sp_n = r, \text{ where } r \in S(K).$$

Since $A(K) \subseteq T(K)$, there exists a sequence $\{q_n\}$ in K such that $A(p_n) = T(q_n)$. So,

$$\lim_{n \rightarrow \infty} Tq_n = \lim_{n \rightarrow \infty} Ap_n = r, \text{ where } r \in S(K) \cap T(K).$$

Thus, $Ap_n \rightarrow r$, $Sp_n \rightarrow r$ and $Tq_n \rightarrow r$. Now, we show the $Bq_n \rightarrow r$.

Let $\lim_{n \rightarrow \infty} F(Bq_n, l, t_0) = 1$. Now, we show that $l = r$. Assume that $l \neq r$. We prove that there exists $t_0 > 0$ such that

$$(2.6) \quad F\left(r, l, \frac{2}{k}t_0\right) > F(r, l, t_0).$$

Suppose the contrary. Therefore for all $t > 0$, we have

$$(2.7) \quad F\left(r, l, \frac{2}{k}t\right) \leq F(r, l, t).$$

Using repeatedly (2.7), we obtain

$$F(r, l, t) \geq F\left(r, l, \frac{2}{k}t\right) \geq \cdots \geq F\left(r, l, \left(\frac{2}{k}\right)^n t\right) \rightarrow 1 \text{ as } n \rightarrow \infty,$$

this shows that $F(r, l, t) = 1$ for all $t > 0$, which contradicts $l \neq r$.

Without loss of generality, we may assume that t_0 in (2.6) is a continuous point of $F(r, l)$. Since every distance distribution function is left-continuous, (2.6) implies that there exists $\epsilon > 0$ such that (2.6) holds for all $t \in (t_0 - \epsilon, t_0)$. Since $F(r, l)$ is non-decreasing, the set of all discontinuous points of $F(r, l)$ is a countable set at most. Thus, when t_0 is a discontinuous point of $F(r, l)$, we can choose a continuous point t_1 of $F(r, l)$ and $F(r, l)$ in $(t_0 - \epsilon, t_0)$ to replace t_0 .

Using the inequality (2.5) with $p = p_n$, $q = q_n$, we get for some $t_0 > 0$

$$\begin{aligned} & (1 + \alpha F(Sp_n, Tq_n, t_0))F(Ap_n, Bq_n, t_0) \\ & > \alpha \min\{F(Ap_n, Sp_n, t_0)F(Bq_n, Tq_n, t_0), F(Sp_n, Bq_n, t_0)F(Ap_n, Tq_n, t_0)\} \\ & \quad + \min\left\{F(Sp_n, Tq_n, t_0), \min\left\{F(Ap_n, Sp_n, \epsilon), F\left(Bq_n, Tq_n, \left(\frac{2}{k}t_0 - \epsilon\right)\right)\right\}, \right. \\ & \quad \left. \min\{F(Sp_n, Bq_n, (2t_0 - \epsilon)), F(Ap_n, Tq_n, \epsilon)\}\right\}. \end{aligned}$$

For all $\epsilon \in (0, \frac{2}{k}t_0)$. Letting $n \rightarrow \infty$, we have

$$\begin{aligned} & F(r, l, t_0) + \alpha F(r, l, t_0) \\ & \geq \alpha F(r, l, t_0) + \min\left\{F\left(r, l, \left(\frac{2}{k}t_0 - \epsilon\right)\right), F(r, l, (2t_0 - \epsilon))\right\}. \end{aligned}$$

As $\epsilon \rightarrow 0$, we obtain $F(r, l, t_0) \geq F\left(r, l, \frac{2}{k}t_0\right)$, which contradicts (2.6). Thus, the pairs (A, S) and (B, T) satisfy the CLR_{ST} property. $\square$

Remark 2.1. The converse of Lemma 2.1 is not true.

Theorem 2.1. Let A, B, S and T be mappings of a complete Menger space (K, F, Δ) into itself satisfying the inequality (2.5) of Lemma 2.1. If the pairs (A, S) and (B, T) satisfy the CLR_{ST} property, the pairs (A, S) and (B, T) have coincidence points. Moreover, if (A, S) and (B, T) are weakly compatible, then A, B, S and T have a unique common fixed point in K .

Proof. Suppose that the pairs (A, S) and (B, T) satisfies the CLR_{ST} property, there exists two sequences $\{p_n\}$ and $\{q_n\}$ in K such that

$$\lim_{n \rightarrow \infty} Ap_n = \lim_{n \rightarrow \infty} Sp_n = \lim_{n \rightarrow \infty} Tq_n = \lim_{n \rightarrow \infty} Bq_n = r,$$

where $r \in S(K) \cap T(K)$.

Hence, there exists $u_1, v_1 \in K$ such that $Su_1 = Tv_1 = r$.

Now, we show that $Au_1 = Su_1 = r$.

If $Au_1 \neq Su_1$, putting $p = u_1$ and $q = q_n$ in inequality (2.5), we get for some $t_0 > 0$,

$$\begin{aligned} & (1 + \alpha F(Su_1, Tq_n, t_0))F(Au_1, Bq_n, t_0) \\ & > \alpha \min\{F(Au_1, Su_1, t_0)F(Bq_n, Tq_n, t_0), F(Su_1, Bq_n, t_0)F(Au_1, Tq_n, t_0)\} \\ & \quad + \min\left\{F(Su_1, Tq_n, t_0), \min\left\{F\left(Au_1, Su_1, \left(\frac{2}{k}t_0 - \epsilon\right)\right), F(Bq_n, Tq_n, \epsilon), \right. \right. \\ & \quad \left. \left. \min\{F(Su_1, Bq_n, \epsilon), F(Au_1, Tq_n, (2t_0 - \epsilon))\}\right\}, \right. \\ & (1 + \alpha F(r, r, t_0))F(Au_1, r, t_0) \\ & > \alpha \min\{F(Au_1, r, t_0)F(r, r, t_0), F(r, r, t_0)F(Au_1, r, t_0)\} \\ & \quad + \min\left\{F(r, r, t_0), \min\left\{F\left(Au_1, r, \left(\frac{2}{k}t_0 - \epsilon\right)\right), F(r, r, \epsilon), \right. \right. \\ & \quad \left. \left. \min\{F(r, r, \epsilon), F(Au_1, r, (2t_0 - \epsilon))\}\right\}, \right. \\ & F(Au_1, r, t_0) + \alpha F(Au_1, r, t_0) \\ & \geq \alpha F(Au_1, r, t_0) + \min\left\{F\left(Au_1, r, \left(\frac{2}{k}t_0 - \epsilon\right)\right), F(Au_1, r, (2t_0 - \epsilon))\right\}, \end{aligned}$$

for all $\epsilon \in (0, \frac{2}{k}t_0)$. Letting $n \rightarrow \infty$, we have as $\epsilon \rightarrow 0$, $F(Au_1, l, t_0) \geq F(Au_1, l, \frac{2}{k}t_0)$, which contradicts (2.6) and so $Au_1 = Su_1 = r$. Therefore, u_1 is a coincidence point of the pair (A, S) .

Now, we assert that $Bv_1 = Tv_1 = r$.

If $r \neq Bv_1$, putting $p = u_1$ and $q = v_1$ in the inequality (2.5), we get for some $t_0 > 0$,

$$\begin{aligned} & (1 + \alpha F(Su_1, Tv_1, t_0))F(Au_1, Bv_1, t_0) \\ & > \alpha \min\{F(Au_1, Su_1, t_0)F(Bv_1, Tv_1, t_0), F(Su_1, Bv_1, t_0)F(Au_1, Tv_1, t_0)\} \\ & \quad + \min\left\{F(Su_1, Tv_1, t_0), \min\left\{F(Au_1, Su_1, \epsilon), F\left(Bv_1, Tv_1, \left(\frac{2}{k}t_0 - \epsilon\right)\right), \right. \right. \\ & \quad \left. \left. \min\{F(Su_1, Bv_1, (2t_0 - \epsilon)), F(Au_1, Tv_1, \epsilon)\}\right\} \right\} \end{aligned}$$

$$\begin{aligned}
& (1 + \alpha F(r, r, t_0))F(r, Bv_1, t_0) \\
& > \alpha \min\{F(r, r, t_0)F(Bv_1, r, t_0), F(r, Bv_1, t_0)F(r, r, t_0)\} \\
& \quad + \min\left\{F(r, r, t_0), \min\left\{F(r, r, \epsilon), F\left(Bv_1, r, \left(\frac{2}{k}t_0 - \epsilon\right)\right), \right. \right. \\
& \quad \left. \left. \min\{F(Bv_1, r, (2t_0 - \epsilon)), F(r, r, \epsilon)\}\right\} \right. \\
& F(r, Bv_1, t_0) + \alpha F(r, Bv_1, t_0) \\
& \geq \alpha F(r, Bv_1, t_0) + \min\left\{F\left(Bv_1, r, \left(\frac{2}{k}t_0 - \epsilon\right)\right), F(Bv_1, r, (2t_0 - \epsilon))\right\},
\end{aligned}$$

for all $\epsilon \in (0, \frac{2}{k}t_0)$. As $\epsilon \rightarrow 0$, $F(Bv_1, r, t_0) \geq F(Bv_1, r, \frac{2}{k}t_0)$, which contradicts (2.6) and so $Bv_1 = Tv_1 = r$. Therefore v_1 is a coincidence point of the pair (B, T) .

Since the pair (A, S) is weakly compatible and $Au_1 = Su_1$ we obtain $Ar = Sr$.

Now, we prove that r is a common fixed point of A and S . If $r \neq Ar$, applying the inequality (2.5) with $p = r$ and $q = v_1$, we get for some $t_0 > 0$

$$\begin{aligned}
& t(1 + \alpha F(Sr, Tv_1, t_0))F(Ar, Bv_1, t_0) \\
& > \alpha \min\{F(Ar, Sr, t_0)F(Bv_1, Tv_1, t_0), F(Sr, Bv_1, t_0)F(Ar, Tv_1, t_0)\} \\
& \quad + \min\left\{F(Ar, r, t_0), \min\left\{F(Ar, Sr, \epsilon), F\left(Bv_1, Tv_1, \left(\frac{2}{k}t_0 - \epsilon\right)\right), \right. \right. \\
& \quad \left. \left. \min\{F(Sr, Bv_1, (2t_0 - \epsilon)), F(Ar, Tv_1, \epsilon)\}\right\} \right. \\
& (1 + \alpha F(Ar, r, t_0))F(Ar, r, t_0) \\
& > \alpha \min\{F(Ar, Ar, t_0)F(r, r, t_0), F(Ar, r, t_0)F(Ar, r, t_0)\} \\
& \quad + \min\left\{F(Ar, r, t_0), \min\left\{F(Ar, r, \epsilon), F\left(r, r, \left(\frac{2}{k}t_0 - \epsilon\right)\right), \right. \right. \\
& \quad \left. \left. \min\{F(Ar, r, (2t_0 - \epsilon)), F(r, r, \epsilon)\}\right\} \right. \\
& F(Ar, r, t_0) + \alpha(F(Ar, r, t_0))^2 \geq \alpha(F(Ar, r, t_0))^2 + F(Ar, r, t_0),
\end{aligned}$$

for all $\epsilon \in (0, \frac{2}{k}t_0)$. $F(Ar, r, t_0) \geq F(Ar, r, t_0)$, which contradicts (2.6) and so $Ar = Sr = r$, which shows that r is a common fixed point of A and S . Since the pair (B, T) is weakly compatible, we get $Br = Tr$. Similarly, we can prove that r is a common fixed point of B and T . Hence r is a common fixed point of A, B, S and T . $\square$

We now give an example to illustrate the above theorem see for details [2].

Example 2.1. Let (K, F, Δ) be a Menger space with $K = [3, 11]$ and for all $p, q \in K$ by $F(p, q, t) = H(t - |p - q|)$, $t > 0$ and $F(p, q, 0) = 0$, where $\Delta(a_1, b_1) =$

$\min\{a_1, b_1\}$ for all $a_1, b_1 \in [0, 1]$. Define self-mappings A and S on K by

$$\begin{aligned} Ap &= \begin{cases} 3, & \text{if } p \in 3 \cup (5, 11) \\ 10, & \text{if } p \in (3, 5) \end{cases}, & Bp &= \begin{cases} 3, & \text{if } p \in 3 \cup (5, 11) \\ 9, & \text{if } p \in (3, 5) \end{cases}, \\ Sp &= \begin{cases} 3, & \text{if } p = 3 \\ 7, & \text{if } p \in (3, 5) \\ \frac{p+1}{3}, & \text{if } p \in (5, 11) \end{cases}, & Tp &= \begin{cases} 3, & \text{if } p = 3 \\ p+4, & \text{if } p \in (3, 5) \\ p-2, & \text{if } p \in (5, 11). \end{cases} \end{aligned}$$

We take $\{p_n = 3\}$, $\{q_n = 5 + \frac{1}{n}\}$ or $\{p_n = 5 + \frac{1}{n}\}$, $\{q_n = 3\}$, since

$$\lim_{n \rightarrow \infty} Ap_n = \lim_{n \rightarrow \infty} Sp_n = \lim_{n \rightarrow \infty} Bq_n = \lim_{n \rightarrow \infty} Tq_n = 3 \in S(K) \cap T(K).$$

Then, the pairs (A, S) and (B, T) satisfy the CLR_{ST} property. Thus all the conditions of Theorem 2.1 are satisfied and 3 is the unique common fixed point of the pairs (A, S) and (B, T) . Remark that all the mappings are even discontinuous at their unique common fixed point 3. In this example $S(K)$ and $T(K)$ are not closed subsets of K .

Lemma 2.2. *Let A, B, S and T be mappings of a complete Menger space (K, F, Δ) into itself satisfying the conditions (2.1)-(2.4) of Lemma 2.1 and*

$$\begin{aligned} (2.8) \quad & (1 + \alpha F(Sp, Tq, t))F(Ap, Bq, t) \\ & > \alpha \min\{F(Ap, Sp, t)F(Bq, Tq, t), F(Sp, Bq, t)F(Ap, Tq, t)\} \\ & + \min \left\{ F(Sp, Tq, t) \sup_{t_1+t_2=\frac{2t}{k}} \min\{F(Ap, Sp, t_1), F(Sp, Bq, t_2)\} \right. \\ & \quad \left. \sup_{t_3+t_4=\frac{2t}{k}} \min\{F(Bq, Tq, t_3), F(Ap, Tq, t_4)\} \right\}, \end{aligned}$$

for all $p, q \in K$, $t > 0$, for some $\alpha \geq 0$ and $1 \leq k < 2$. Then the pairs (A, S) and (B, T) satisfy the CLR_{ST} property.

Proof. As in the proof of Lemma 2.1, there exists $t_0 > 0$ such that (2.6) holds. Using the inequality (2.8) with $p = p_n$, $q = q_n$, we have

$$\begin{aligned} & (1 + \alpha F(Sp_n, Tq_n, t_0))F(Ap_n, Bq_n, t_0) \\ & > \alpha \min\{F(Ap_n, Sp_n, t_0)F(Bq_n, Tq_n, t_0), F(Sp_n, Bq_n, t_0)F(Ap_n, Tq_n, t_0)\} \\ & + \min \left\{ F(Sp_n, Tq_n, t_0) \min \left\{ F(Ap_n, Sp_n, \epsilon), F \left(Sp_n, Bq_n, \left(\frac{2}{k}t_0 - \epsilon \right) \right) \right. \right. \\ & \quad \left. \left. \min \left\{ F \left(Bq_n, Tq_n, \left(\frac{2}{k}t_0 - \epsilon \right) \right), F(Ap_n, Tq_n, \epsilon) \right\} \right\} \right\}, \end{aligned}$$

for all $\epsilon \in (0, \frac{2}{k}t_0)$. Letting $n \rightarrow \infty$, we have

$$\begin{aligned}
& (1 + \alpha F(r, r, t_0))F(r, l, t_0) \\
& > \alpha \min\{F(r, r, t_0)F(l, r, t_0), F(r, l, t_0)F(r, r, t_0)\} \\
& \quad + \min\left\{F(r, r, t_0) \min\left\{F(r, r, \epsilon), F\left(r, l, \left(\frac{2}{k}t_0 - \epsilon\right)\right)\right.\right. \\
& \quad \left.\left.\min\left\{F\left(l, r, \left(\frac{2}{k}t_0 - \epsilon\right)\right), F(r, r, \epsilon)\right\}\right.\right. \\
& F(r, l, t_0) + \alpha F(r, l, t_0) \\
& > \alpha F(r, l, t_0) + \min\left\{F(r, r, t_0) \min\left\{F(r, r, (\epsilon)), F\left(r, l, \left(\left(\frac{2}{k}t_0 - \epsilon\right)\right)\right)\right.\right. \\
& \quad \left.\left.\min\left\{F\left(l, r, \left(\left(\frac{2}{k}t_0 - \epsilon\right)\right)\right), F(r, r, (\epsilon))\right\}\right.\right. \\
& F(r, l, t_0) > \min\left\{1, F\left(r, l, \left(\left(\frac{2}{k}t_0 - \epsilon\right)\right)\right)\right\}.
\end{aligned}$$

As $\epsilon \rightarrow 0$, we obtain, $F(r, l, t_0) \geq F(r, l, \frac{2}{k}t_0)$, which contradicts (2.6) and so we have $r = l$. Then the pairs (A, S) and (B, T) satisfy the CLR_{ST} property. $\square$

Theorem 2.2. *Let A, B, S and T be mappings of a complete Menger space (K, F, Δ) into itself satisfying the inequality (2.8) of Lemma 2.2. If the pairs (A, S) and (B, T) satisfy the CLR_{ST} property, the pairs (A, S) and (B, T) have coincidence points. Moreover, if (A, S) and (B, T) are weakly compatible, then A, B, S and T have a unique common fixed point in K .*

Proof. Suppose that the pairs (A, S) and (B, T) satisfies the CLR_{ST} property, there exists two sequences $\{p_n\}$ and $\{q_n\}$ in K such that

$$\lim_{n \rightarrow \infty} Ap_n = \lim_{n \rightarrow \infty} Sp_n = \lim_{n \rightarrow \infty} Tq_n = \lim_{n \rightarrow \infty} Bq_n = r,$$

where $r \in S(K) \cap T(K)$.

Hence, there exists $u, v \in K$ such that $Su = Tv = r$. From Theorems 2.1, we can easily prove that r is a common fixed point of A, B, S and T .

If $B = A$ and $T = S$ in Theorems 2.1 and 2.2, we obtain a common fixed point for a set of self-mappings. $\square$

In the proof of the following lemma, we do not need to prove the inequality (2.6).

Lemma 2.3. *Let A, B, S and T be mappings of a complete Menger space (K, F, Δ) into itself satisfying the conditions (2.1)-(2.4) of Lemma 2.1 and*

$$(2.9) \quad (1 + \alpha F(Sp, Tq, t))F(Ap, Bq, t) \\ > \alpha \min\{F(Ap, Sp, t)F(Bq, Tq, t), F(Sp, Bq, t)F(Ap, Tq, t)\} \\ + \min\left\{F(Sp, Tq, t) \sup_{t_1+t_2=\frac{2t}{k}} \max\{F(Ap, Sp, t_1), F(Bq, Tq, t_2)\} \right. \\ \left. \sup_{t_3+t_4=2t} \max\{F(Sp, Bq, t_3), F(Ap, Tq, t_4)\}\right\},$$

for all $p, q \in K$, $t > 0$, for some $\alpha \geq 0$ and $1 \leq k < 2$. Then the pairs (A, S) and (B, T) satisfy the CLR_{ST} property.

Proof. Suppose that the pair (A, S) satisfies the CLR_S property and $T(K)$ is closed subset of K . Then there exists a sequence $\{p_n\}$ in K such that

$$\lim_{n \rightarrow \infty} Ap_n = \lim_{n \rightarrow \infty} Sp_n = r,$$

where $r \in S(K)$.

Since $A(K) \subseteq T(K)$, there exists a sequence $\{q_n\}$ in K such that $A(p_n) = T(q_n)$. So,

$$\lim_{n \rightarrow \infty} Tq_n = \lim_{n \rightarrow \infty} Ap_n = r,$$

where $r \in S(K) \cap T(K)$. Thus, $Ap_n \rightarrow r$, $Sp_n \rightarrow r$ and $Tq_n \rightarrow r$. Now, we show the $Bq_n \rightarrow r$. Let $\lim_{n \rightarrow \infty} F(Bq_n, l, t_0) = 1$. Now, we show that $l = r$. Assume that $l \neq r$.

Using the inequality (2.9) with $p = p_n$, $q = q_n$, we get for some $t_0 > 0$,

$$(1 + \alpha F(Sp_n, Tq_n, t_0))F(Ap_n, Bq_n, t_0) \\ > \alpha \min\{F(Ap_n, Sp_n, t_0)F(Bq_n, Tq_n, t_0), F(Sp_n, Bq_n, t_0)F(Ap_n, Tq_n, t_0)\} \\ + \min\left\{F(Sp_n, Tq_n, t_0) \max\left\{F(Ap_n, Sp_n, \epsilon), F\left(Bq_n, Tq_n, \left(\frac{2}{k}t_0 - \epsilon\right)\right)\right\} \right. \\ \left. \min\{F(Sp_n, Bq_n, (2t_0 - \epsilon)), F(Ap_n, Tq_n, \epsilon)\}\right\},$$

for all $\epsilon \in (0, \frac{2}{k}t_0)$. Letting $n \rightarrow \infty$, we have

$$(1 + \alpha F(r, r, t_0))F(r, l, t_0) \\ > \alpha \min\{F(r, r, t_0)F(l, r, t_0), F(r, l, t_0)F(r, r, t_0)\} \\ + \min\left\{F(r, r, t_0) \max\left\{F(r, r, \epsilon), F\left(r, l, \left(\frac{2}{k}t_0 - \epsilon\right)\right)\right\} \right. \\ \left. \max\{F(r, l, (2t_0 - \epsilon)), F(r, r, \epsilon)\}\right\}$$

$$\begin{aligned}
& F(r, l, t_0) + \alpha F(r, r, t_0) \\
& > \alpha F(r, r, t_0) + \min \left\{ F(r, r, t_0) \max \left\{ F(r, r, (\epsilon)), F\left(r, l, \left(\left(\frac{2}{k}t_0 - \epsilon\right)\right)\right) \right\} \right. \\
& \quad \left. \max\{F(l, r, ((2t_0 - \epsilon))), F(r, r, (\epsilon))\} \right\} \\
& F(r, l, t_0) > 1,
\end{aligned}$$

for some $t_0 > 0$ and so we have $r = l$. Then the pairs (A, S) and (B, T) satisfy the CLR_{ST} property. $\square$

Theorem 2.3. *Let A, B, S and T be mappings of a complete Menger space (K, F, Δ) into itself satisfying the inequality (2.9) of Lemma 2.3. If the pairs (A, S) and (B, T) satisfy the CLR_{ST} property, the pairs (A, S) and (B, T) have coincidence points. Moreover, if (A, S) and (B, T) are weakly compatible, then A, B, S and T have a unique common fixed point in K .*

Proof. Suppose that the pairs (A, S) and (B, T) satisfies the CLR_{ST} property, there exists two sequences $\{p_n\}$ and $\{q_n\}$ in K such that

$$\lim_{n \rightarrow \infty} Ap_n = \lim_{n \rightarrow \infty} Sp_n = \lim_{n \rightarrow \infty} Tq_n = \lim_{n \rightarrow \infty} Bq_n = r,$$

where $r \in S(K) \cap T(K)$.

Hence, there exists $u, v \in K$ such that $Su = Tv = r$. Now, we show that $Au = Su = r$.

If $Au \neq Su$, putting $p = u$ and $q = q_n$ in inequality (2.9), we get for some $t_0 > 0$,

$$\begin{aligned}
& (1 + \alpha F(Su, Tq_n, t_0))F(Au, Bq_n, t_0) \\
& > \alpha \min\{F(Au, Su, t_0)F(Bq_n, Tq_n, t_0), F(Su, Bq_n, t_0)F(Au, Tq_n, t_0)\} \\
& \quad + \min \left\{ F(Su, Tq_n, t_0), \max \left\{ F\left(Au, Su, \left(\frac{2}{k}t_0 - \epsilon\right)\right), F(Bq_n, Tq_n, \epsilon) \right\}, \right. \\
& \quad \left. \max\{F(Su, Bq_n, \epsilon), F(Au, Tq_n, (2t_0 - \epsilon))\} \right\} \\
& (1 + \alpha F(r, r, t_0))F(Au, r, t_0) \\
& > \alpha \min\{F(Au, r, t_0)F(r, r, t_0), F(r, r, t_0)F(Au, r, t_0)\} \\
& \quad + \min \left\{ F(r, r, t_0), \max \left\{ F\left(Au, r, \left(\frac{2}{k}t_0 - \epsilon\right)\right), F(r, r, \epsilon), \right. \right. \\
& \quad \left. \left. \max\{F(r, r, \epsilon), F(Au, r, (2t_0 - \epsilon))\} \right\} \right\} \\
& F(Au, r, t_0) > 1,
\end{aligned}$$

and so $Au = Su = r$. Therefore, u is a coincidence point of the pair (A, S) .

Now, we assert that $Bv = Tv = r$. If $r \neq Bv$, putting $p = u$ and $q = v$ in the inequality (2.9), we get for some $t_0 > 0$

$$\begin{aligned}
& (1 + \alpha F(Su, Tv, t_0))F(Au, Bv, t_0) \\
& > \alpha \min\{F(Au, Su, t_0)F(Bv, Tv, t_0), F(Su, Bv, t_0)F(Au, Tv, t_0)\} \\
& \quad + \min\left\{F(Su, Tv, t_0), \max\left\{F(Au, Su, \epsilon), F\left(Bv, Tv, \left(\frac{2}{k}t_0 - \epsilon\right)\right)\right\}\right\} \\
& \quad \max\{F(Su, Bv, (2t_0 - \epsilon)), F(Au, Tv, \epsilon)\} \\
& (1 + \alpha F(r, r, t_0))F(r, Bv, t_0) \\
& > \alpha \min\{F(r, r, t_0)F(Bv, r, t_0), F(r, Bv, t_0)F(r, r, t_0)\} \\
& \quad + \min\left\{F(r, r, t_0), \max\left\{F(r, r, \epsilon), F\left(Bv, r, \left(\frac{2}{k}t_0 - \epsilon\right)\right)\right\}\right\} \\
& \quad \max\{F(Bv, r, (2t_0 - \epsilon)), F(r, r, \epsilon)\} \\
& F(Bv, r, t_0) > 1
\end{aligned}$$

and so $Bv = Tv = r$. Therefore, v is a coincidence point of the pair (B, T) .

Since the pair (A, S) is weakly compatible and $Au = Su$ we obtain $Ar = Sr$. Now, we prove that r is a common fixed point of A and S . If $r \neq Ar$, applying the inequality (2.9) with $p = r$ and $q = v$, we get for some $t_0 > 0$,

$$\begin{aligned}
& (1 + \alpha F(Sr, Tv, t_0))F(Ar, Bv, t_0) \\
& > \alpha \min\{F(Ar, Sr, t_0)F(Bv, Tv, t_0), F(Sr, Bv, t_0)F(Ar, Tv, t_0)\} \\
& \quad + \min\left\{F(Ar, r, t_0), \max\left\{F(Ar, Sr, \epsilon), F\left(Bv, Tv, \left(\frac{2}{k}t_0 - \epsilon\right)\right)\right\}\right\} \\
& \quad \max\{F(Sr, Bv, \epsilon), F(Ar, Tv, (2t_0 - \epsilon))\} \\
& (1 + \alpha F(Ar, r, t_0))F(Ar, r, t_0) \\
& > \alpha \min\{F(Ar, Ar, t_0)F(r, r, t_0), F(Ar, r, t_0)F(Ar, r, t_0)\} \\
& \quad + \min\left\{F(Ar, r, t_0), \max\left\{F(Ar, Ar, \epsilon), F\left(r, r, \left(\frac{2}{k}t_0 - \epsilon\right)\right)\right\}\right\} \\
& \quad \max\{F(Ar, r, \epsilon), F(Ar, r, (2t_0 - \epsilon))\} \\
& F(Ar, r, t_0) + \alpha(F(Ar, r, t_0))^2 \geq \alpha(F(Ar, r, t_0))^2 + F(Ar, r, t_0) \\
& F(Ar, r, t_0) > F(Ar, r, t_0)
\end{aligned}$$

which is impossible and so $Ar = Sr = r$, which shows that r is a common fixed point of A and S . Since the pair (B, T) is weakly compatible, we get $Br = Tr$. Similarly, we can prove that r is a common fixed point of B and T . Hence r is a common fixed point of A, B, S and T . $\square$

Let φ be the set of all non-decreasing and continuous functions $\varphi : (0, 1] \rightarrow (0, 1]$ such that $\varphi(t) > t$ for all $t \in (0, 1]$.

Lemma 2.4. *Let A, B, S and T be mappings of a complete Menger space (K, F, Δ) into itself satisfying the conditions (2.1)-(2.4) of Lemma 2.1 and*

$$(2.10) \quad (1 + \alpha F(Sp, Tq, t))F(Ap, Bq, t) \\ > \alpha \min\{F(Ap, Sp, t)F(Bq, Tq, t), F(Sp, Bq, t)F(Ap, Tq, t)\} \\ + \varphi \left\{ F(Sp, Tq, tt) \sup_{t_1+t_3=\frac{2t}{k}} \min\{F(Ap, Sp, t_1), F(Bq, Tq, t_3)\} \right. \\ \left. \sup_{t_2+t_4=\frac{2t}{k}} \min\{F(Sp, Bq, t_2), F(Ap, Tq, t_4)\} \right\},$$

for all $p, q \in K$, $t > 0$, for some $\alpha \geq 0$ and $1 \leq k < 2$, $\varphi \in \varphi$. Then the pairs (A, S) and (B, T) satisfy the CLR_{ST} property.

Proof. From Lemma 2.3 we have $Ap_n \rightarrow r$, $Sp_n \rightarrow r$ and $Tp_n \rightarrow r$. Now we show the $Bq_n \rightarrow r$.

Let $\lim_{n \rightarrow \infty} F(Bq_n, l, t_0) = 1$. Now we show that $l = r$. Assume that $l \neq r$. Using the inequality (2.10) with $p = p_n$, $q = q_n$, we get for some $t_0 > 0$

$$(1 + \alpha F(Sp_n, Tq_n, t_0))F(Ap_n, Bq_n, t_0) \\ > \alpha \min\{F(Ap_n, Sp_n, t_0)F(Bq_n, Tq_n, t_0), F(Sp_n, Bq_n, t_0)F(Ap_n, Tq_n, t_0)\} \\ + \varphi \left\{ F(Sp_n, Tq_n, t_0) \min \left\{ F(Ap_n, Sp_n, \epsilon), F\left(Bq_n, Tq_n, \left(\frac{2}{k}t_0 - \epsilon\right)\right) \right. \right. \\ \left. \left. \min \left\{ F\left(Sp_n, Bq_n, \left(\frac{2}{k}t_0 - \epsilon\right)\right), F(Ap_n, Tq_n, \epsilon) \right\} \right\} \right\},$$

for all $\epsilon \in (0, \frac{2}{k}t_0)$. Letting $n \rightarrow \infty$, we have

$$(1 + \alpha F(r, r, t_0))F(r, l, t_0) \\ > \alpha \min\{F(r, r, t_0)F(l, r, t_0), F(r, l, t_0)F(r, r, t_0)\} \\ + \varphi \left\{ F(r, r, t_0) \min \left\{ F(r, r, \epsilon), F\left(l, r, \left(\frac{2}{k}t_0 - \epsilon\right)\right) \right. \right. \\ \left. \left. \min \left\{ F\left(r, l, \left(\frac{2}{k}t_0 - \epsilon\right)\right), F(r, r, \epsilon) \right\} \right\} \right\} \\ F(r, l, t_0) + \alpha F(r, l, t_0) \\ > \alpha F(r, l, t_0) + \varphi \left\{ F(r, r, t_0) \min \left\{ F(r, r, (\epsilon)), F\left(r, l, \left(\left(\frac{2}{k}t_0 - \epsilon\right)\right)\right) \right. \right. \\ \left. \left. \min \left\{ F\left(l, r, \left(\left(\frac{2}{k}t_0 - \epsilon\right)\right)\right), F(r, r, (\epsilon)) \right\} \right\} \right\} \\ F(r, l, t_0) > \varphi \left(F\left(l, r, \left(\left(\frac{2}{k}t_0 - \epsilon\right)\right)\right) \right),$$

for all $\epsilon \in (0, \frac{2}{k}t_0)$. As $\epsilon \rightarrow 0$, $F(r, l, t_0) > F(l, r, ((\frac{2}{k}t_0)))$, for some $t_0 > 0$.

By (2.6) so we have $r = l$. Then the pairs (A, S) and (B, T) satisfy the CLR_{ST} property. $\square$

Remark 2.2. Lemmas 2.1, 2.2, 2.3 and 2.4 remain true if we assume that the pair (B, T) satisfies the CLR_T property, $B(K) \subseteq S(K)$ and $S(K)$ is a closed subset of K .

Theorem 2.4. Let A, B, S and T be mappings of a complete Menger space (K, F, Δ) into itself satisfying the inequality (2.10) of Lemma 2.4. If the pairs (A, S) and (B, T) satisfy the CLR_{ST} property, the pairs (A, S) and (B, T) have coincidence points. Moreover, if (A, S) and (B, T) are weakly compatible, then A, B, S and T have a unique common fixed point in K .

Proof. Suppose that the pairs (A, S) and (B, T) satisfies the CLR_{ST} property, there exists two sequences $\{p_n\}$ and $\{q_n\}$ in K such that

$$\lim_{n \rightarrow \infty} Ap_n = \lim_{n \rightarrow \infty} Sp_n = \lim_{n \rightarrow \infty} Tq_n = \lim_{n \rightarrow \infty} Bq_n = r,$$

where $r \in S(K) \cap T(K)$.

From Theorem 2.3 and using function φ , we can easily prove the theorems. $\square$

Remark 2.3. In Theorems 2.1, 2.2, 2.3 and 2.4 by a similar manner, we can prove that A, B, S and T have a unique common fixed point in K if we assume that the pairs (A, S) and (B, T) verify $JCLR_{ST}$ property or CLR_{AB} property instead of CLR_{ST} property.

Theorem 2.5. Let A, B, S and T be mappings of a complete Menger space (K, F, Δ) into itself satisfying the conditions of Lemma 2.1, or Lemma 2.2, Lemma 2.3, Lemma 2.4. If the pairs (A, S) and (B, T) are weakly compatible, then A, B, S and T have a unique common fixed point in K .

Proof. In view of Lemma 2.1, Lemma 2.2, Lemma 2.3 and Lemma 2.4, the pairs (A, S) and (B, T) satisfies the CLR_{ST} property, there exists two sequences $\{p_n\}$ and $\{q_n\}$ in K such that

$$\lim_{n \rightarrow \infty} Ap_n = \lim_{n \rightarrow \infty} Sp_n = \lim_{n \rightarrow \infty} Tq_n = \lim_{n \rightarrow \infty} Bq_n = r,$$

where $r \in S(K) \cap T(K)$.

The rest of the proof follows the proof of Theorems 2.1, 2.2, 2.3 and 2.4. $\square$

Example 2.2. We retain A and B and replace S and T in Example 2.1 by the following mappings

$$Sp = \begin{cases} 3, & \text{if } p = 3 \\ 6, & \text{if } p \in (3, 5) \\ \frac{p+1}{3}, & \text{if } p \in [5, 11], \end{cases} \quad Tp = \begin{cases} 3, & \text{if } p = 3 \\ 9, & \text{if } p \in (3, 5) \\ p - 2, & \text{if } p \in [5, 11]. \end{cases}$$

Therefore, $A(K) = \{3, 4\} \subset [3, 9] = T(K)$ and $B(K) = \{3, 5\} \subset [3, 6] = S(K)$.

Thus, all the conditions of Theorem 2.3 are satisfied and 3 is the unique common fixed point of the pairs (A, S) and (B, T) . Also, it is noted that Theorem 2.1 cannot be used in the context of this example as $S(K)$ and $T(K)$ are closed subsets of K .

Applying Theorems 2.1, 2.2, 2.3 and 2.4, we deduce a common fixed point for four finite families of self-mappings given by the following corollary.

Corollary 2.1. *Let $\{A_i\}_{i=1}^m$, $\{B_r\}_{r=1}^n$, $\{S_k\}_{k=1}^p$ and $\{T_h\}_{h=1}^q$ be four finite families of self-mappings of Menger space (K, F, Δ) , where Δ is a continuous t -norm with $A = A_1 A_2 \dots A_m$, $B = B_1 B_2 \dots B_n$, $S = S_1 S_2 \dots S_p$, $T = T_1 T_2 \dots T_q$ satisfies the inequality (2.5) of Lemma 2.1 or the inequality (2.8) of Lemma 2.2. Suppose that the pairs (A, S) and (B, T) verify the CLR_{ST} property. Then $\{A_i\}_{i=1}^m$, $\{B_r\}_{r=1}^n$, $\{S_k\}_{k=1}^p$ and $\{T_h\}_{h=1}^q$ a unique common fixed point in K , provided that the pairs of families $(\{A_i\}_{i=1}^m, \{S_k\}_{k=1}^p)$, $(\{B_r\}_{r=1}^n, \{T_h\}_{h=1}^q)$ commute pair wise.*

By setting $A = A_1 = A_2 = \dots = A_m$, $B = B_1 = B_2 = \dots = B_n$, $S = S_1 = S_2 = \dots = S_p$, $T = T_1 = T_2 = \dots = T_q$ in Corollary 2.1, we get that A, B, S and T have a unique common fixed point in K , provided that the pairs (A^m, S^p) and (B^n, T^q) commute pair wise.

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Fractional Bessel differential equation and fractional Bessel functions

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Abstract. Using a new simple and well-behaved definition of the fractional derivative which is different from the Caputo and Riemann-Liouville fractional derivative and recently introduced by Khalil and others, we reformulate the second order Bessel differential equation in this new setting. In this article by the use of power series, one of the solution of the fractional differential equation is obtained. Moreover, we find the generating function and use it to prove some nice standard results and recurrence relations. Finally, we present some application and integral representations of Bessel functions of fractional type including sines and cosines.

Keywords: Bessel equation, fractional derivative, Bessel functions.

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1. Introduction

Special functions [10, 12, 14] are of extreme importance in the study of mathematics and physics. At some level in quantum mechanics spherical harmonic are important in problems with spherical symmetry in electricity and magnetism, where they play the role of cosines and sines in the Fourier expanding of functions. Bessel functions and integrals involving them are in constant demand in applied mathematics and physics. They arise in problems with spherical symmetry. A typical example from Fourier analysis is the Fourier transform of radial functions in the Euclidean space $\mathbb{R}^n$.

Bessel functions which form the standard solution of the Bessel differential equation

$$x^2 y'' + xy' + (x^2 - p^2)y = 0,$$

where p is the order of the differential equation are named for the great mathematician Wilhelm Bessel (1784 – 1846). However Daniel Bernoulli (1732), is credited to be the first mathematician to introduce the concept of Bessel functions. It is known [11], that these functions can be obtained by separating the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

in spherical coordinates. That's why Bessel functions plays a significant role in spherical harmonic and Fourier analysis. Bessel equation has two linearly independent solutions $J_n(x)$ and $Y_n(x)$, being a second order differential equation. For more on Bessel functions we refer the reader to the excellent books by Luke [10] and Watson [14].

Fractional calculus owes its origin to the question raised by L'Hospital in 1695 of whether the derivative of an integer n could be extended and still be valid when n is not an integer. In the last decades fractional calculus caught attention of great mathematicians like Euler, Laplace, Fourier, Riemann, Liouville and others who directly or indirectly contributed to its development, [9, 11, 13]. Recently, [8] a new definition has been simply formulated and called conformable fractional derivative depending on the basic limit definition of derivative. It is defined as: for a function $f : (0, \infty) \rightarrow \mathbb{R}$, the conformable fractional derivative of order $\alpha \in (0, 1)$ of f at $t > 0$ is given by

$$D^\alpha(f)(t) = \lim_{\epsilon \rightarrow 0} \frac{f(t + \epsilon t^{1-\alpha}) - f(t)}{\epsilon}.$$

The fractional derivative of f at $t = 0$ is defined as $D^\alpha f(0) = \lim_{t \rightarrow 0} D^\alpha f(t)$. However the definition of fractional derivative either of Caputo or of Riemann-Liouville, both used an expression involving Gamma function and integral form.

The new fractional derivative (conformable fractional derivative) satisfies the product rule, the quotient rule and have results which are natural extension of the corresponding Roll's Theorem and Mean Value Theorem, but unfortunately it does not obey the natural chain rule, see [3, 8].

Note that if a function f is differentiable, then $D_\alpha(f)(t) = (t)^{1-\alpha} f'(t)$. It is clear that the conformable fractional derivative of the constant function is zero. On letting $\alpha = 1$ in these derivatives, we get the corresponding ordinary derivatives. One should notice that a function could be α -conformable differentiable at a point but not differentiable, for example, take $f(t) = 2\sqrt{t}$. Then $D_{\frac{1}{2}}(f)(t) = 1$. Hence $D_{\frac{1}{2}}(f)(0) = 1$. But $D_1(f)(0)$ does not exist. This is not the case for the known classical fractional derivatives. For more on conformable fractional derivative and its applications we refer to [1]–[8] and references therein.

Using this new derivative we reformulate the second order Bessel differential equation namely, conformable Bessel fractional differential equation,

$$(1.1) \quad x^{2\alpha} D^\alpha D^\alpha y + \alpha x^\alpha D^\alpha y + (x^{2\alpha} - \alpha^2 p^2) y = 0.$$

With the use of power series, the first solution of the fractional differential equation is obtained. Then we find the generating function and use it to prove some nice standard results and recurrence relations. Finally we present some application and integral representations of Bessel functions of fractional type including sines and cosines.

2. Conformable fractional Bessel's differential equation

Analogous to second order homogeneous linear differential equation

$$P(x)y'' + Q(x)y' + R(x)y = 0,$$

we formulate the second order homogeneous linear fractional differential equation as

$$(2.1) \quad P(x)D^\alpha D^\alpha y + Q(x)D^\alpha y + R(x)y = 0.$$

Definition 2.1. A point $x = x_0$ is called α -**regular** singular point of equation (2.1) if:

$$\lim_{x \rightarrow x_0} (x - x_0)^\alpha Q(x) \text{ exists and } \lim_{x \rightarrow x_0} (x - x_0)^{2\alpha} R(x) \text{ exists.}$$

If P, Q and R are polynomials with no common factors, then the singular points of the differential equation are those for which $P(x) = 0$. The point $x = 0$ is an α -regular singular point for equation (1.1).

Fractional Bessel functions form the standard solution of the fractional Bessel differential equation

$$x^{2\alpha} D^\alpha D^\alpha y + \alpha x^\alpha D^\alpha y + (x^{2\alpha} - \alpha^2 p^2) y = 0,$$

where p is the order of the differential equation. To find the first solution for equation (1.1), we begin by taking the fractional power series

$$(2.2) \quad y = \sum_{m=0}^{\infty} a_m x^{\alpha m + \beta},$$

where β is any real number which will be plugged into the Bessel equation (1.1) and solve for necessary component. For convenience the first and second fractional derivatives for equations (2.2) are

$$\begin{aligned} D^\alpha y &= \sum_{m=0}^{\infty} (\alpha m + \beta) a_m x^{\alpha m + \beta - \alpha}, \\ D^\alpha D^\alpha y &= \sum_{m=0}^{\infty} (\alpha m + \beta)(\alpha m + \beta - \alpha) a_m x^{\alpha m + \beta - 2\alpha}. \end{aligned}$$

Substituting into the differential equation (1.1), define a_{-1} and a_{-2} to be zero and collecting terms, we get

$$\sum_{m=0}^{\infty} (((\alpha m + \beta)(\alpha m + \beta - \alpha) + \alpha(\alpha m + \beta) - \alpha^2 p^2) a_m + a_{m-2}) x^{\alpha m + \beta} = 0.$$

Compare the coefficients of $x^{\alpha m + \beta}$ to yield

$$\begin{aligned} ((\alpha m + \beta)(\alpha m + \beta - \alpha) + \alpha(\alpha m + \beta) - \alpha^2 p^2) a_m + a_{m-2} &= 0, \\ ((\alpha m + \beta)^2 - \alpha(\alpha m + \beta) + \alpha(\alpha m + \beta) - \alpha^2 p^2) a_m + a_{m-2} &= 0, \end{aligned}$$

$$(2.3) \quad ((\alpha m + \beta)^2 - \alpha^2 p^2) a_m + a_{m-2} = 0.$$

Letting $m = 0$ in (2.3) and assuming $a_{m-2} = 0$, we obtain the indicial equation

$$(\beta^2 - \alpha^2 p^2) a_0 = 0.$$

Since we have no condition on a_0 , we can choose a convenient value as needed. Assuming $a_0 \neq 0$, we have $\beta^2 - \alpha^2 p^2 = 0$ or $\beta = \pm \alpha p$ and so we have two cases:

Case 1. $\beta = \alpha p$. In this case (2.3) becomes

$$\begin{aligned} ((\alpha m + \alpha p)^2 - \alpha^2 p^2) a_m + a_{m-2} &= 0, \\ (\alpha^2 m^2 + 2\alpha^2 m p + \alpha^2 p^2 - \alpha^2 p^2) a_m + a_{m-2} &= 0, \end{aligned}$$

$$(2.4) \quad \alpha^2 m (m + 2p) a_m + a_{m-2} = 0.$$

Putting $m = 1, 2$ successively in (2.4), we get

$$\alpha^2(1 + 2p)a_1 + a_{-1} = 0, \quad 2\alpha^2(2 + 2p)a_2 + a_0 = 0.$$

So, $a_1 = 0$ and

$$a_2 = \frac{-a_0}{2\alpha^2(2 + 2p)} = \frac{-a_0}{(2\alpha)^2(1 + p)}.$$

Since $a_1 = 0$, we can infer that $a_3 = 0$. Continuing in this manner to obtain $a_{2k-1} = 0$, for all positive integers k . For $m = 4$, we have

$$4\alpha^2(4+2p)a_4 + a_2 = 0,$$

$$a_4 = \frac{-a_2}{4\alpha^2(2)(2+p)} = \frac{a_0}{(2\alpha)^4(2)(1+p)(2+p)}.$$

Continue the process for $m = 2r$ (even), a general formula for the coefficients a_m is obtained,

$$a_m = a_{2r} = \frac{(-1)^r a_0}{(2\alpha)^{2r} r! (1+p)(2+p)(3+p) \dots (r+p)}.$$

Substituting these coefficients into (2.2), gives one of the solution to the conformable fractional Bessel equation:

$$(2.5) \quad y = a_0 \sum_{r=0}^{\infty} \frac{(-1)^r}{(2\alpha)^{2r} r! (1+p)(2+p)(3+p) \dots (r+p)} x^{2\alpha r + \alpha p},$$

where $a_0 \neq 0$ is arbitrary. With the aid of the gamma function giving a_0 the value

$$a_0 = \frac{1}{(2\alpha)^p \Gamma(p+1)},$$

this solution may be written in a nice way and is denoted by $J_{\alpha p}$ called the conformable fractional Bessel function of order αp ,

$$(2.6) \quad J_{\alpha p}(x) = \sum_{r=0}^{\infty} (-1)^r \frac{x^{2\alpha r + \alpha p}}{(2\alpha)^{2r+p} r! \Gamma(r+p+1)}.$$

Case 2. $\beta = -\alpha p$. Replacing p by $-p$ in case 1, we find

$$(2.7) \quad J_{-\alpha p}(x) = \sum_{r=0}^{\infty} (-1)^r \frac{x^{2\alpha r - \alpha p}}{(2\alpha)^{2r-p} r! \Gamma(r-p+1)}.$$

For $p = 0$ in (2.6), the fractional Bessel function $J_0(x)$ has the form

$$J_0(x) = 1 - \frac{x^{2\alpha}}{\alpha^2(2)^2} + \frac{x^{4\alpha}}{\alpha^4(2)^2(4)^2} - \frac{x^{6\alpha}}{\alpha^6(2)^2(4)^2(6)^2} + \frac{x^{8\alpha}}{\alpha^8(2)^2(4)^2(6)^2(8)^2} + \dots$$

The functions $J_{\alpha p}(x)$ and $J_{-\alpha p}(x)$ in (2.6) and (2.7) are called conformable fractional Bessel functions. If $\alpha p = n$, for some integer n , then $J_n(x)$ is just the classical Bessel functions.

3. Fractional Bessel functions and generating function

Most of the properties of the fractional Bessel functions can be proved using their generating functions.

A generating function of a function J_n is another function $G(J_n, t)$ whose power series expansion has the function J_n as the coefficient of t^n . That means

$$G(J_n, t) = \sum t^n J_n(x).$$

In this section we find the generating function of the fractional Bessel functions of the first kind $J_{\alpha p}$.

Theorem 3.1. *Let p be an integer. Then the generating function of the fractional Bessel functions of the first kind is*

$$G(x, t) = \sum_{p=-\infty}^{\infty} J_{\alpha p}(x) t^p = e^{\frac{x^\alpha}{2\alpha}(t-\frac{1}{t})}.$$

Proof. Recall that the power series representation $e^t = \sum_{r=0}^{\infty} \frac{t^r}{r!}$. So, we have

$$\begin{aligned} e^{\frac{x^\alpha}{2\alpha}(t-\frac{1}{t})} &= e^{\frac{x^\alpha t}{2\alpha}} \cdot e^{-\frac{x^\alpha}{2\alpha}t^{-1}} \\ &= \sum_{r=0}^{\infty} \frac{x^{\alpha r} t^r}{(2\alpha)^r r!} \sum_{k=0}^{\infty} \frac{(-1)^k x^{\alpha k} t^{-k}}{(2\alpha)^k k!}. \end{aligned}$$

Since the multiplication of two infinite power series can be written as a Cauchy Product, we have

$$e^{\frac{x^\alpha}{2\alpha}(t-\frac{1}{t})} = \sum_{r=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k x^{\alpha r + \alpha k} t^{r-k}}{(2\alpha)^{r+k} k! r!}.$$

Letting $p = r - k$ so that p varies from $-\infty$ to ∞ , the power series expansion can be written as

$$\begin{aligned} e^{\frac{x^\alpha}{2\alpha}(t-\frac{1}{t})} &= \sum_{p=-\infty}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k x^{\alpha(p+k)+\alpha k} t^p}{(2\alpha)^{p+2k} k! (p+k)!} \\ &= \sum_{p=-\infty}^{\infty} \left(\sum_{k=0}^{\infty} \frac{(-1)^k x^{2\alpha k + \alpha p}}{(2\alpha)^{p+2k} k! \Gamma(p+k+1)} \right) t^p. \end{aligned}$$

Using (2.6) and (2.7) we get

$$G(x, t) = e^{\frac{x^\alpha}{2\alpha}(t-\frac{1}{t})} = \sum_{p=-\infty}^{\infty} J_{\alpha p}(x) t^p.$$

□

As an application of the generating function we have the following nice recurrence relations:

Theorem 3.2. *The following results are valid for all values of αp :*

- (i) $J_{-\alpha p}(x) = (-1)^p J_{\alpha p}(x)$;
- (ii) $\frac{d^\alpha}{dx^\alpha} [x^{\alpha p} J_{\alpha p}(x)] = x^{\alpha p} J_{\alpha(p-1)}$;
- (iii) $\frac{d^\alpha}{dx^\alpha} [x^{-\alpha p} J_{\alpha p}(x)] = -x^{-\alpha p} J_{\alpha p} + \alpha$.

Proof. (i)

$$\begin{aligned} J_{-\alpha p}(x) &= \sum_{r=0}^{\infty} (-1)^r \frac{x^{2\alpha r - \alpha p}}{(2\alpha)^{2r-p} r! \Gamma(r-p+1)} \\ &= \sum_{r=0}^{p-1} (-1)^r \frac{x^{2\alpha r - \alpha p}}{(2\alpha)^{2r-p} r! \Gamma(r-p+1)} \\ &\quad + \sum_{r=p}^{\infty} (-1)^r \frac{x^{2\alpha r - \alpha p}}{(2\alpha)^{2r-p} r! \Gamma(r-p+1)}. \end{aligned}$$

Regarding that $\Gamma(-p+r+1)$ is infinite for $r = 0, 1, 2, 3, \dots, p$, the finite sum is zero. Letting $r = p+k$, the infinite series can be written as

$$\begin{aligned} J_{-\alpha p}(x) &= \sum_{k=0}^{\infty} (-1)^{k+p} \frac{x^{2\alpha(k+p) - \alpha p}}{(2\alpha)^{2(k+p)-p} (k+p)! \Gamma(k+1)} \\ &= (-1)^p \sum_{k=0}^{\infty} (-1)^k \frac{x^{2\alpha k + \alpha p}}{(2\alpha)^{2k+p} k! \Gamma(k+p+1)} = (-1)^p J_{\alpha p}(x). \end{aligned}$$

To prove (ii), we calculate the α -fractional derivative of $x^{\alpha p} J_{\alpha p}(x)$ with respect to x to get

$$\begin{aligned} \frac{d^\alpha}{dx^\alpha} [x^{\alpha p} J_{\alpha p}(x)] &= \frac{d^\alpha}{dx^\alpha} \left(\sum_{r=0}^{\infty} (-1)^r \frac{x^{2\alpha r + 2\alpha p}}{(2\alpha)^{2r+p} r! \Gamma(r+p+1)} \right) \\ &= \sum_{r=0}^{\infty} (-1)^r \frac{2\alpha(r+p) x^{2\alpha r + 2\alpha p - \alpha}}{(2\alpha)^{2r+p} r! (r+p) \Gamma(r+p)} \\ &= \sum_{r=0}^{\infty} (-1)^r \frac{x^{\alpha p} x^{2\alpha r + \alpha(p-1)}}{(2\alpha)^{2r+p-1} r! \Gamma(r+p-1+1)} = x^{\alpha p} J_{\alpha(p-1)}. \end{aligned}$$

Similarly (iii) can be proved by taking the α -fractional derivative of $x^{-\alpha p} J_{\alpha p}(x)$,

$$\begin{aligned} \frac{d^\alpha}{dx^\alpha} [x^{-\alpha p} J_{\alpha p}(x)] &= \sum_{r=1}^{\infty} (-1)^r \frac{(2\alpha r) x^{2\alpha r - \alpha}}{(2\alpha)^{2r+p} r! \Gamma(r+p+1)} \\ &= \sum_{r=1}^{\infty} (-1)^r \frac{x^{2\alpha r - \alpha}}{(2\alpha)^{2r+p-1} (r-1)! \Gamma(r+p+1)} \end{aligned}$$

$$\begin{aligned}
&= - \sum_{r=0}^{\infty} (-1)^r \frac{x^{2\alpha r + \alpha(p+1)}}{(2\alpha)^{2r + \alpha(p+1)} r! \Gamma((p+1) + r + 1)} x^{-\alpha p} \\
&= -x^{-\alpha p} J_{\alpha(p+1)}(x).
\end{aligned}$$

□

4. Some applications

In this section some classical nice formulas of fractional type has been deduced and presented.

Lemma 4.1.

$$\int x^{\alpha p} J_{\alpha p - \alpha}(x) d^{\alpha} x = x^{\alpha p} J_{\alpha p}(x).$$

Proof. This follows immediately from Theorem 3.2(ii). In fact since

$$\frac{d^{\alpha}}{dx^{\alpha}} [x^{\alpha p} J_{\alpha p}(x)] = x^{\alpha p} J_{\alpha p - \alpha},$$

it follows that

$$\int x^{\alpha p} J_{\alpha p - \alpha}(x) d^{\alpha} x = x^{\alpha p} J_{\alpha p}(x).$$

□

Lemma 4.2. (1) $\int x^3 J_{\frac{1}{2}}(x) dx^{\frac{1}{2}} = x^3 J_1(x) - 2x^{\frac{5}{2}} J_{\frac{3}{2}}(x) + x^2 J_2(x).$

(2) $\int x^2 J_{\frac{5}{2}}(x) d^{\frac{1}{2}} x = -x^2 J_2(x) - 3x^{\frac{3}{2}} J_{\frac{3}{2}}(x) - 12x J_1(x) - 24x^{\frac{1}{2}} J_{\frac{1}{2}}(x) + 24J_0(x).$

Proof. (1) Using Theorem 3.2(ii) and Integration by parts and

$$u = x^2 \implies d^{\frac{1}{2}} u = 2x^{\frac{3}{2}} d^{\frac{1}{2}} x, \quad d^{\frac{1}{2}} v = x J_{\frac{1}{2}} \implies v = x J_1$$

we get

$$\begin{aligned}
\int x^3 J_{\frac{1}{2}}(x) d^{\frac{1}{2}} x &= \int x^2 x J_{\frac{1}{2}}(x) d^{\frac{1}{2}} x \\
&= x^3 J_1(x) - 2 \int x^{\frac{5}{2}} J_1(x) d^{\frac{1}{2}} x.
\end{aligned}$$

Repeating the process for the second integral

$$u = x \implies d^{\frac{1}{2}} u = x^{\frac{1}{2}} d^{\frac{1}{2}} x, \quad d^{\frac{1}{2}} v = x^{\frac{3}{2}} J_1(x) d^{\frac{1}{2}} x \implies v = x^{\frac{3}{2}} J_{\frac{3}{2}}(x),$$

we get

$$\begin{aligned}
\int x^3 J_{\frac{1}{2}}(x) d^{\frac{1}{2}} x &= x^3 J_1(x) - 2 \left(x^{\frac{5}{2}} J_{\frac{3}{2}}(x) - \int x^2 J_{\frac{3}{2}}(x) d^{\frac{1}{2}} x \right) \\
&= x^3 J_1(x) - 2 x^{\frac{5}{2}} J_{\frac{3}{2}}(x) + x^2 J_2(x).
\end{aligned}$$

For part (2), the proof is similar to that of (1), using integration by parts

$$u = x^4 \implies d^{\frac{1}{2}}u = 4x^{\frac{7}{2}}d^{\frac{1}{2}}x, d^{\frac{1}{2}}v = x^{-2}J_{\frac{5}{2}}(x)d^{\frac{1}{2}}x \implies v = -x^{-2}J_2(x),$$

we have

$$\begin{aligned} \int x^2 J_{\frac{5}{2}}(x) d^{\frac{1}{2}}x &= \int x^4 x^{-2} J_{\frac{5}{2}}(x) d^{\frac{1}{2}}x \\ &= -x^2 J_2(x) + 4 \int x^{\frac{3}{2}} J_2(x) d^{\frac{1}{2}}x \\ &= -x^2 J_2(x) + 4 \int x^3 x^{-\frac{3}{2}} J_2(x) d^{\frac{1}{2}}x. \end{aligned}$$

Continue repeating the process for the second integral

$$\begin{aligned} u &= x^3 \implies d^{\frac{1}{2}}u = 3x^{\frac{5}{2}}d^{\frac{1}{2}}x, \\ d^{\frac{1}{2}}v &= x^{-\frac{3}{2}}J_2(x)d^{\frac{1}{2}}x \implies v = -x^{-\frac{3}{2}}J_{\frac{3}{2}}(x) \end{aligned}$$

we have

$$\begin{aligned} \int x^2 J_{\frac{5}{2}}(x) d^{\frac{1}{2}}x &= -x^2 J_2(x) - 4x^{\frac{1}{2}}J_{\frac{3}{2}}(x) + 12 \int x J_{\frac{3}{2}}(x) d^{\frac{1}{2}}x \\ &= -x^2 J_2(x) - 3x^{\frac{1}{2}}J_{\frac{3}{2}}(x) + 12 \int x^2 x^{-1} J_{\frac{3}{2}}(x) d^{\frac{1}{2}}x \\ &= -x^2 J_2(x) - 3x^{\frac{3}{2}}J_{\frac{3}{2}}(x) - 12xJ_1(x) + 24 \int x^{\frac{1}{2}}J_1(x) d^{\frac{1}{2}}x \\ &= -x^2 J_2(x) - 3x^{\frac{3}{2}}J_{\frac{3}{2}}(x) - 12xJ_1(x) + 24 \int x x^{-\frac{1}{2}}J_1(x) d^{\frac{1}{2}}x \\ &= -x^2 J_2(x) - 3x^{\frac{3}{2}}J_{\frac{3}{2}}(x) - 12xJ_1(x) - 24x^{\frac{1}{2}}J_{\frac{1}{2}}(x) \\ &\quad + 24 \int J_{\frac{1}{2}}(x) d^{\frac{1}{2}}x \\ &= -x^2 J_2(x) - 3x^{\frac{3}{2}}J_{\frac{3}{2}}(x) - 12xJ_1(x) - 24x^{\frac{1}{2}}J_{\frac{1}{2}}(x) + 24J_0(x). \end{aligned}$$

□

Lemma 4.3. (1) $\cos(\frac{x^\alpha}{\alpha} \sin \theta) = J_0(x) + 2J_{2\alpha}(x) \cos 2\theta + 2J_{4\alpha}(x) \cos 4\theta + \dots$
 (2) $\sin(\frac{x^\alpha}{\alpha} \sin \theta) = 2J_\alpha(x) \sin \theta + 2J_{3\alpha}(x) \sin 3\theta + \dots$

Proof. Using the generating function, $e^{\frac{x^\alpha}{2\alpha}(t-\frac{1}{t})}$, where $t = e^{i\theta}$, we get:

$$\begin{aligned} e^{\frac{x^\alpha}{2\alpha}(e^{i\theta} - e^{-i\theta})} &= e^{\frac{ix^\alpha}{\alpha} \sin \theta} \\ &= \sum_{-\infty}^{\infty} J_{\alpha p}(x) t^p \end{aligned}$$

$$\begin{aligned}
&= \sum_{-\infty}^{\infty} J_{\alpha p}(x) (\cos p\theta + i \sin p\theta) \\
&= \left\{ \begin{aligned} &J_0(x) + (J_{-\alpha}(x) + J_{\alpha}(x)) \cos \theta \\ &+ [J_{-2\alpha}(x) + J_{2\alpha}(x)] \cos 2\theta + \dots \end{aligned} \right\} \\
&\quad + i \left\{ \begin{aligned} &(J_{\alpha}(x) - J_{-\alpha}(x)) \sin \theta \\ &+ (J_{2\alpha}(x) - J_{-2\alpha}(x)) \sin 2\theta + \dots \end{aligned} \right\} \\
&= \left\{ \begin{aligned} &J_0(x) + (J_{-\alpha}(x) + J_{\alpha}(x)) \cos \theta \\ &+ (J_{-2\alpha}(x) + J_{2\alpha}(x)) \cos 2\theta + \dots \end{aligned} \right\} \\
&\quad + i \left\{ \begin{aligned} &(J_{\alpha}(x) - J_{-\alpha}(x)) \sin \theta \\ &+ (J_{2\alpha}(x) - J_{-2\alpha}(x)) \sin 2\theta + \dots \end{aligned} \right\} \\
&= \{J_0(x) + 2J_{2\alpha}(x) \cos 2\theta + \dots\} \\
&\quad + i\{2J_{\alpha}(x) \sin \theta + 3J_{3\alpha}(x) \sin 2\theta + \dots\}.
\end{aligned}$$

The proof of (2) is similar. □

Finally, let us solve:

$$x D^{\frac{1}{2}} D^{\frac{1}{2}} y + \frac{1}{2} x^{\frac{1}{2}} D^{\frac{1}{2}} y + (x^2 - 1)y = 0.$$

Here $\alpha = \frac{1}{2}$ and $p = 2$. Using equation (2.6) we get the solution to be.

$$J_1(x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^{r+1}}{r! \Gamma(r+3)}.$$

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On a Diophantine inequality with different powers of primes

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Abstract. Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6$ be non-zero real numbers, not all negative. Assume that $\frac{\lambda_1}{\lambda_2}$ is irrational and algebraic. Let $\mathcal{V}$ be a well-spaced sequence, and $\delta > 0$. In this paper, we proved that, for any $\varepsilon > 0$, the number of $v \in \mathcal{V}$ with $1 \leq v \leq X$ such that the inequality

$$|\lambda_1 p_1^2 + \lambda_2 p_2^2 + \lambda_3 p_3^3 + \lambda_4 p_4^3 + \lambda_5 p_5^4 + \lambda_6 p_6^4 - v| < v^{-\delta}$$

has no solution in primes $p_1, p_2, p_3, p_4, p_5, p_6$ does not exceed $O(X^{1-\frac{1}{8}+2\delta+\varepsilon})$.

Keywords: Waring-Goldbach problem, Diophantine inequality, primes.

1. Introduction

In 1970, Vaughan [14] proved that every large integer N can be represented as the sum of two squares, two cubes and two biquadrates of natural numbers by showing that

$$\sum_{N=x_1^2+x_2^2+x_3^3+x_4^3+x_5^4+x_6^4} 1 = \frac{\Gamma^2(\frac{3}{2})\Gamma^2(\frac{4}{3})\Gamma^2(\frac{5}{4})}{\Gamma(\frac{13}{6})} \mathfrak{S}'(N) N^{\frac{7}{6}} + O(N^{\frac{7}{6}-\frac{1}{96}+\varepsilon}),$$

where $x_1, x_2, x_3, x_4, x_5, x_6$ are natural numbers and

$$\mathfrak{S}'(N) = \sum_{q=1}^{\infty} \frac{1}{q^6} \sum_{\substack{a=1 \\ (a,q)=1}}^q \prod_{i=1}^3 \left(\sum_{x_i=1}^q e\left(\frac{ax_i^{i+1}}{q}\right) \right)^2 e\left(-\frac{aN}{q}\right).$$

In 2015, Lü [12] proved that every large even integer N can be represented in the form

$$N = x^2 + p_2^2 + p_3^3 + p_4^3 + p_5^4 + p_6^4,$$

where x is an almost-prime $\mathcal{P}_6$ and p_2, p_3, p_4, p_5, p_6 are primes. Very recently, Liu [11] obtained a stronger result with $\mathcal{P}_6$ replaced by $\mathcal{P}_4$. It seems reasonable to conjecture that every sufficiently large even integer n can be expressed in the form

$$(1.1) \quad n = p_1^2 + p_2^2 + p_3^3 + p_4^3 + p_5^4 + p_6^4,$$

where $p_1, p_2, p_3, p_4, p_5, p_6$ are primes. Recently, Zhang and Li [16] considered the set of possible exceptions of the representation (1.1). Let $E(N)$ denote the number of positive even integers $n \leq N$, which can not be represented in the form (1.1). Zhang and Li [16] showed that

$$E(N) \ll N^{\frac{13}{16}+\varepsilon}.$$

Later, Zhang [18] improved this result to

$$E(N) \ll N^{\frac{17}{192}+\varepsilon}.$$

Zhang and Li [17] also considered the exceptional problem of the form (1.1) in short intervals.

Typically, when one can handle a representation problem, it is of interest to study the analogous forms with real coefficients (see e.g. [1, 3, 4, 6, 7, 8, 9, 10]). We call a set of positive real numbers $\mathcal{V}$ a *well-spaced set* if there exists a $c > 0$ such that

$$u, v \in \mathcal{V}, u \neq v \Rightarrow |u - v| > c.$$

We further assume that, for any $\varepsilon > 0$,

$$|\{v \in \mathcal{V} : 0 \leq v \leq X\}| \gg X^{1-\varepsilon}.$$

In this paper, we consider a Diophantine inequality with two squares, two cubes and two biquadrates of primes. Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6$ be non-zero real numbers, not all negative and $\mathcal{V}$ be a well-spaced sequence. Let $\delta > 0$. Denote by $E(\mathcal{V}, X, \delta)$ the number of $v \in \mathcal{V}$ with $1 \leq v \leq X$ such that the inequality

$$(1.2) \quad |\lambda_1 p_1^2 + \lambda_2 p_2^2 + \lambda_3 p_3^3 + \lambda_4 p_4^3 + \lambda_5 p_5^4 + \lambda_6 p_6^4 - v| < v^{-\delta}$$

has no solutions in primes $p_1, p_2, p_3, p_4, p_5, p_6$. Our results are as follows:

Theorem 1.1. *Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6$ be non-zero real numbers, not all negative. Suppose that $\frac{\lambda_1}{\lambda_2}$ is irrational and algebraic. Let $\mathcal{V}$ be a well-spaced sequence and $\delta > 0$. Then, we have, for any $\varepsilon > 0$,*

$$E(\mathcal{V}, X, \delta) \ll X^{1-\frac{1}{8}+2\delta+\varepsilon}.$$

Theorem 1.2. *Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6$ be non-zero real numbers, not all negative. Suppose that $\frac{\lambda_1}{\lambda_2}$ is irrational. Let $\mathcal{V}$ be a well-spaced sequence and $\delta > 0$. Then, there is a sequence $X_j \rightarrow \infty$ such that*

$$(1.3) \quad E(\mathcal{V}, X_j, \delta) \ll X_j^{1-\frac{1}{8}+2\delta+\varepsilon}$$

for any $\varepsilon > 0$. Moreover, if the convergent denominators q_j for λ_1/λ_2 satisfy

$$(1.4) \quad q_{j+1}^{1-w} \ll q_j \text{ for some } w \in [0, 1),$$

then we have, for all $X \geq 1$ and any $\varepsilon > 0$,

$$(1.5) \quad E(\mathcal{V}, X, \delta) \ll X^{2\chi+2\delta+\varepsilon}$$

with

$$(1.6) \quad \chi = \max\left(\frac{2-w}{6-4w}, \frac{7}{16}\right).$$

Theorem 1.1 follows immediately from Theorem 1.2, since, in the case of λ_1/λ_2 algebraic, we can take $w = \varepsilon$ and then $\chi = \frac{7}{16}$. Thus, we focus on proving Theorem 1.2 in the following.

Notation. Throughout this paper, the letter p , with or without a subscript, always denotes a prime. The letter ε denotes a sufficiently small positive number, and the value of ε may change from statement to statement. Constants, both explicit and implicit, in Vinogradov symbols may depend on $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6$. We abbreviate $e^{2\pi i x}$ to $e(x)$.

2. Outline of the method and preliminary lemmas

We follow the adaption of the Hardy-Littlewood method first introduced by Davenport and Heilbronn [2]. Let X be some large positive quantity to be chosen later (we shall take $X = q^{\frac{4}{3}}$ in Lemma 4.2) and η be a fixed sufficiently small positive integer. Let $0 < \tau < 1$ (we shall chose $\tau = X^{-\delta}$ in Section 6). Let

$$K_\tau(\alpha) = \left(\frac{\sin(\pi\tau\alpha)}{\tau\alpha}\right)^2$$

for $\tau > 0$ and $\alpha \neq 0$. By continuity, we define $K_\tau(0) = \tau^2$. Then we have, by [15],

$$(2.1) \quad K_\tau(\alpha) \ll \min(\tau^2, |\alpha|^{-2}),$$

and

$$(2.2) \quad A(x) := \int_{-\infty}^{+\infty} e(x\alpha) K_\tau(\alpha) d\alpha = \max(0, \tau - |x|).$$

We define, for $i = 2, 3, 4$,

$$(2.3) \quad S_i(\alpha) = \sum_{p \in I_i} (\log p) e(p^i \alpha), \quad U_i(\alpha) = \sum_{n \in I_i} e(n^i \alpha),$$

where

$$I_i = \left[(\eta X)^{\frac{1}{i}}, X^{\frac{1}{i}} \right].$$

If we write

$$N_v = \frac{1}{\tau} \sum_{\substack{p_1, p_2 \in I_2 \\ p_3, p_4 \in I_3 \\ p_5, p_6 \in I_4}} \left(\prod_{j=1}^6 \log p_j \right) A(\lambda_1 p_1^2 + \lambda_2 p_2^2 + \lambda_3 p_3^3 + \lambda_4 p_4^3 + \lambda_5 p_5^4 + \lambda_6 p_6^4 - v),$$

then $0 \leq N_v \leq \Psi(v)$, where $\Psi(v)$ counts the number of solutions to

$$|\lambda_1 p_1^2 + \lambda_2 p_2^2 + \lambda_3 p_3^3 + \lambda_4 p_4^3 + \lambda_5 p_5^4 + \lambda_6 p_6^4 - v| < \tau,$$

weighted by a term $\prod_{j=1}^6 \log p_j$. We shall restrict our attention to those v satisfying $X/2 \leq v \leq X$. In general, one can consider $2^{-j}X \leq v \leq 2^{1-j}X$, $j = 1, 2, \dots$, and obtain a satisfactory bound for the exceptional set. Then, from (2.2), we have

$$(2.4) \quad N_v = \frac{1}{\tau} \int_{-\infty}^{+\infty} S_2(\lambda_1 \alpha) S_2(\lambda_2 \alpha) S_3(\lambda_3 \alpha) S_3(\lambda_4 \alpha) S_4(\lambda_5 \alpha) S_4(\lambda_6 \alpha) \cdot K_\tau(\alpha) e(-v\alpha) d\alpha.$$

To estimate the integral in (2.4), we divide the real line into the major arc $\mathfrak{M}$, the minor arc $\mathfrak{m}$ and the trivial arc $\mathfrak{t}$, which are defined by

$$(2.5) \quad \mathfrak{M} = \{\alpha : |\alpha| \leq \phi\}, \quad \mathfrak{m} = \{\alpha : \phi < |\alpha| < \xi\}, \quad \mathfrak{t} = \{\alpha : |\alpha| \geq \xi\},$$

with $\phi = X^{-\frac{3}{4}}$ and $\xi = \tau^{-2} X^{\frac{1}{6} + 2\varepsilon}$. The main contribution to the integral (2.4) comes from the major arc. In order to handle integrals on these arcs, we need the following lemmas.

Lemma 2.1 ([5, Theorem 3]). *Let $k \geq 2$ and define*

$$\rho(k) = \begin{cases} \frac{1}{8}, & \text{if } k = 2, \\ \frac{1}{14}, & \text{if } k = 3, \\ \frac{2}{3} \times 2^{-k}, & \text{if } k \geq 4. \end{cases}$$

Suppose that α is a real number, and there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfying

$$(2.6) \quad 1 \leq q \leq Q, \quad (a, q) = 1, \quad |q\alpha - a| < Q^{-1}$$

with

$$Q = \begin{cases} X^{\frac{3}{4}}, & \text{if } k = 2, \\ X^{\frac{k-2\rho(k)}{2k-1}}, & \text{if } k \geq 3. \end{cases}$$

Then, for any $\varepsilon > 0$, we have

$$(2.7) \quad \sum_{1 \leq p^k \leq X} (\log p) e(p^k \alpha) \ll X^{\frac{1}{2}(1-\rho(k))+\varepsilon} + \frac{q^\varepsilon X^{\frac{1}{2}} (\log X)^c}{(q + X|q\alpha - a|)^{\frac{1}{2}}},$$

where $c > 0$ is an absolute constant and the constant implied by $\ll$ depends at most on k and ε .

The following lemma shows an improvement of [3, Corollary 2.2].

Lemma 2.2. *Suppose that $X^{\frac{1}{2}} \geq Z \geq X^{\frac{1}{2}-\frac{1}{16}+\varepsilon}$ and $|S_2(\lambda\alpha)| > Z$. Then there two coprime integers a, q satisfying*

$$1 \leq q \ll \left(\frac{X^{\frac{1}{2}+\varepsilon}}{Z} \right)^2, \quad |q\lambda\alpha - a| \ll X^{-1} \left(\frac{X^{\frac{1}{2}+\varepsilon}}{Z} \right)^2.$$

Proof. We prove this lemma by following a similar argument of [1, Corollary 3]. By Dirichlet's theorem in Diophantine approximation, there exist coprime integers a, q satisfying

$$1 \leq q \leq X \left(\frac{X^{\frac{1}{2}+\varepsilon}}{Z} \right)^{-2}, \quad |q\lambda\alpha - a| \ll X^{-1} \left(\frac{X^{\frac{1}{2}+\varepsilon}}{Z} \right)^2.$$

Then from (2.7), we have

$$S_2(\lambda\alpha) \ll X^{\frac{1}{2}-\frac{1}{16}+\varepsilon} + \frac{X^{\frac{1}{2}+\varepsilon}}{(q + X|q\lambda\alpha - a|)^{\frac{1}{2}}}.$$

Since, we have $|S_2(\lambda\alpha)| > Z \geq X^{\frac{1}{2}-\frac{1}{16}+\varepsilon}$, then

$$Z < |S_2(\lambda\alpha)| \ll \frac{X^{\frac{1}{2}+\varepsilon}}{(q + X|q\lambda\alpha - a|)^{\frac{1}{2}}} \ll X^{\frac{1}{2}+\varepsilon} q^{-\frac{1}{2}},$$

which means that

$$1 \leq q \ll \left(\frac{X^{\frac{1}{2}+\varepsilon}}{Z} \right)^2.$$

Therefore, we complete the proof of this lemma. $\square$

Lemma 2.3. *Suppose that $X^{\frac{5}{24}-1-\varepsilon} \leq |\alpha| \leq X^{-\frac{3}{4}}$. Then we have*

$$|S_4(\lambda\alpha)| \ll X^{\frac{1}{4}-\frac{1}{2 \cdot 4^4}+\varepsilon}.$$

Proof. This lemma is [3, Corollary 2.3] with $k = 4$. $\square$

Lemma 2.4. *Suppose that $F \in \{S_2^4, S_3^8, S_2^2 S_3^4, S_2^2 S_4^4\}$. Then we have*

$$\int_{-1}^1 |F(\alpha)| d\alpha \ll X^{-1} (F(0))^{1+\varepsilon}, \quad \int_{-\infty}^{\infty} |F(\alpha)| K_{\tau}(\alpha) d\alpha \ll \tau X^{-1} (F(0))^{1+\varepsilon}.$$

Proof. All of these results follow from [14] by using Hua's Lemma. The similar proof can be found in in [3]. $\square$

For $r \geq 1$, we set

$$J_r(X, h) = \int_{\varepsilon X}^X \left(\theta((x+h)^{\frac{1}{r}}) - \theta(x^{\frac{1}{r}}) - ((x+h)^{\frac{1}{r}} - x^{\frac{1}{r}}) \right)^2 dx,$$

where $\theta(x) = \sum_{p \leq x} \log p$ is the Chebyshev function. The following Lemmas 2.5 and 2.6 are Theorems 3.1 and 3.2 in [6], respectively.

Lemma 2.5. *Let $r \geq 1$ be a real number. For $0 < Y \leq 1/2$ we have*

$$\int_{-Y}^Y |S_r(\alpha) - U_r(\alpha)|^2 d\alpha \ll \frac{X^{\frac{2}{r}-2} \log^2 X}{Y} + Y^2 X + Y^2 J_r \left(X, \frac{1}{2Y} \right).$$

Lemma 2.6. *Let $r \geq 1$ be a real number. There exists a positive constant $c_1 = c_1(\varepsilon)$, which does not depend on r , such that*

$$J_r(X, h) \ll h^2 X^{\frac{2}{r}-1} \exp \left(-c_1 \left(\frac{\log X}{\log \log X} \right)^{\frac{1}{3}} \right)$$

uniformly for $X^{1-\frac{5}{6r}+\varepsilon} \leq h \leq X$.

Combining Lemmas 2.5 and 2.6, we can deduce the following lemma easily.

Lemma 2.7. *Let $r \geq 1$ be a real number. For any fixed real number $A \geq 6$, we have*

$$\int_{|\alpha| \leq X^{\frac{5}{6r}-1-\varepsilon}} |S_r(\alpha) - U_r(\alpha)|^2 d\alpha \ll X^{\frac{2}{r}-1} (\log X)^{-A}.$$

3. The major arc

In this section, we handle the contribution from the major arc and show a positive lower bound. To do this, we divide the major arc into two regions: $\mathfrak{M}_1$ and $\mathfrak{M}_2$, which are defined by

$$\mathfrak{M}_1 = \{\alpha : |\alpha| \leq X^{\frac{5}{24}-1-\varepsilon}\}, \quad \mathfrak{M}_2 = \{\alpha : X^{\frac{5}{24}-1-\varepsilon} < |\alpha| \leq X^{-\frac{3}{4}}\}.$$

In the region $\mathfrak{M}_1$, which can be seen as the standard major arc, we use the idea of Languasco and Zaccagnini [6]. While in the region $\mathfrak{M}_2$, we follow the idea of Ge and Wang [3] (see also Harman [4]), which can enlarge the standard major arc.

3.1 The region $\mathfrak{M}_1$

In this subsection, we give the lower bound of the integral on the region $\mathfrak{M}_1$. For $i = 2, 3, 4$, we set

$$T_i(\alpha) = \int_{I_i} e(t^i \alpha) dt.$$

Then, by the first derivative estimate for trigonometric integrals (see Titchmarsh [13]), one has

$$(3.1) \quad T_i(\alpha) \ll X^{\frac{1}{i}-1} \min(X, |\alpha|^{-1}).$$

We have the following lemma.

Lemma 3.1. *We have*

$$\int_{\mathfrak{M}_1} S_2(\lambda_1\alpha)S_2(\lambda_2\alpha)S_3(\lambda_3\alpha)S_3(\lambda_4\alpha)S_4(\lambda_5\alpha)S_4(\lambda_6\alpha)K_\tau(\alpha)e(-v\alpha)d\alpha \gg \tau^2 X^{\frac{7}{6}}.$$

Proof. It is easy to show that

$$\begin{aligned} & \int_{\mathfrak{M}_1} S_2(\lambda_1\alpha)S_2(\lambda_2\alpha)S_3(\lambda_3\alpha)S_3(\lambda_4\alpha)S_4(\lambda_5\alpha)S_4(\lambda_6\alpha)K_\tau(\alpha)e(-v\alpha)d\alpha \\ &= \int_{\mathfrak{M}_1} T_2(\lambda_1\alpha)T_2(\lambda_2\alpha)T_3(\lambda_3\alpha)T_3(\lambda_4\alpha)T_4(\lambda_5\alpha)T_4(\lambda_6\alpha)K_\tau(\alpha)e(-v\alpha)d\alpha \\ &+ \int_{\mathfrak{M}_1} \left(S_2(\lambda_1\alpha) - T_2(\lambda_1\alpha) \right) T_2(\lambda_2\alpha)T_3(\lambda_3\alpha)T_3(\lambda_4\alpha)T_4(\lambda_5\alpha)T_4(\lambda_6\alpha)K_\tau(\alpha)e(-v\alpha)d\alpha \\ &+ \int_{\mathfrak{M}_1} S_2(\lambda_1\alpha) \left(S_2(\lambda_2\alpha) - T_2(\lambda_2\alpha) \right) T_3(\lambda_3\alpha)T_3(\lambda_4\alpha)T_4(\lambda_5\alpha)T_4(\lambda_6\alpha)K_\tau(\alpha)e(-v\alpha)d\alpha \\ &+ \int_{\mathfrak{M}_1} S_2(\lambda_1\alpha)S_2(\lambda_2\alpha) \left(S_3(\lambda_3\alpha) - T_3(\lambda_3\alpha) \right) T_3(\lambda_4\alpha)T_4(\lambda_5\alpha)T_4(\lambda_6\alpha)K_\tau(\alpha)e(-v\alpha)d\alpha \\ &+ \int_{\mathfrak{M}_1} S_2(\lambda_1\alpha)S_2(\lambda_2\alpha)S_3(\lambda_3\alpha) \left(S_3(\lambda_4\alpha) - T_3(\lambda_4\alpha) \right) T_4(\lambda_5\alpha)T_4(\lambda_6\alpha)K_\tau(\alpha)e(-v\alpha)d\alpha \\ &+ \int_{\mathfrak{M}_1} S_2(\lambda_1\alpha)S_2(\lambda_2\alpha)S_3(\lambda_3\alpha)S_3(\lambda_4\alpha) \left(S_4(\lambda_5\alpha) - T_4(\lambda_5\alpha) \right) T_4(\lambda_6\alpha)K_\tau(\alpha)e(-v\alpha)d\alpha \\ &+ \int_{\mathfrak{M}_1} S_2(\lambda_1\alpha)S_2(\lambda_2\alpha)S_3(\lambda_3\alpha)S_3(\lambda_4\alpha)S_4(\lambda_5\alpha) \left(S_4(\lambda_6\alpha) - T_4(\lambda_6\alpha) \right) K_\tau(\alpha)e(-v\alpha)d\alpha \\ &:= J_1 + J_2 + J_3 + J_4 + J_5 + J_6 + J_7. \end{aligned}$$

In the following, we will show that $J_1 \gg \tau^2 X^{\frac{7}{6}}$ and $J_i = o(\tau^2 X^{\frac{7}{6}})$, for $i = 2, 3, 4, 5, 6, 7$. Thus, this lemma follows. $\square$

Since the estimates for J_3, J_5 and J_7 are similar to J_2, J_4 and J_6 , respectively, we restrict our attention to estimate J_1, J_2, J_4 and J_6 ,

3.1.1 Lower bound for J_1

We first establish the lower bound for J_1 . Note that

$$\begin{aligned} J_1 &= \int_{\mathbb{R}} T_2(\lambda_1\alpha)T_2(\lambda_2\alpha)T_3(\lambda_3\alpha)T_3(\lambda_4\alpha)T_4(\lambda_5\alpha)T_4(\lambda_6\alpha)K_\tau(\alpha)e(-v\alpha)d\alpha \\ (3.2)+ &O\left(\int_{|\alpha|>\phi} |T_2(\lambda_1\alpha)T_2(\lambda_2\alpha)T_3(\lambda_3\alpha)T_3(\lambda_4\alpha)T_4(\lambda_5\alpha)T_4(\lambda_6\alpha)|K_\tau(\alpha)d\alpha\right). \end{aligned}$$

Together with (2.1) and (3.1), we obtain that the error term in (3.2) is

$$(3.3) \quad \ll \tau^2 X^{-\frac{23}{6}} \int_{|\alpha| > X^{\frac{5}{24}-1-\varepsilon}} \frac{d\alpha}{|\alpha|^6} \ll \tau^2 X^{\frac{95}{24}-\frac{23}{6}+5\varepsilon} = o(\tau^2 X^{\frac{7}{6}}).$$

For convenience, we write $\mathbf{y} = (y_1, y_2, \dots, y_6)$. Changing the order of integration and substituting variables, we have

$$\begin{aligned} & \int_{\mathbb{R}} T_2(\lambda_1 \alpha) T_2(\lambda_2 \alpha) T_3(\lambda_3 \alpha) T_3(\lambda_4 \alpha) T_4(\lambda_5 \alpha) T_4(\lambda_6 \alpha) K_\tau(\alpha) e(-v\alpha) d\alpha \\ &= \frac{1}{576(\lambda_1 \lambda_2)^{\frac{1}{2}} (\lambda_3 \lambda_4)^{\frac{2}{3}} (\lambda_5 \lambda_6)^{\frac{3}{4}}} \int (y_1 y_2)^{-\frac{1}{2}} (y_3 y_4)^{-\frac{2}{3}} (y_5 y_6)^{-\frac{3}{4}} \\ & \times \int_{\mathbb{R}} e\left(\left(\sum_{j=1}^6 y_j - v\right)\alpha\right) K_\tau(\alpha) e(-v\alpha) d\alpha d\mathbf{y} \\ & \gg \int (y_1 y_2)^{-\frac{1}{2}} (y_3 y_4)^{-\frac{2}{3}} (y_5 y_6)^{-\frac{3}{4}} A\left(\sum_{j=1}^6 y_j - v\right) d\mathbf{y}, \end{aligned}$$

where the domain of integration for $\mathbf{y} = (y_1, y_2, \dots, y_6)$ are $[\lambda_i \eta X, \lambda_i X]$, respectively, and the integral satisfies $|\sum_{j=1}^6 y_j - v| < \tau$. Taking $|\sum_{j=1}^6 y_j - v| < \frac{\tau}{2}$, and noting that $\lambda_6 \eta X \leq y_6 \leq \lambda_6 X$, then we can get a lower bound for the last integral:

$$\begin{aligned} & \gg \tau X^{-\frac{3}{4}} \int_{\lambda_1 \eta X}^{\lambda_1 X} \cdots \int_{\lambda_5 \eta X}^{\lambda_5 X} \int_{\sum_{j=1}^5 y_j - v - \frac{\tau}{2}}^{\sum_{j=1}^5 y_j - v + \frac{\tau}{2}} (y_1 y_2)^{-\frac{1}{2}} (y_3 y_4)^{-\frac{2}{3}} (y_5)^{-\frac{3}{4}} \\ & \times A\left(\sum_{j=1}^6 y_j - v\right) dy_1 \cdots dy_6 \\ & \gg \tau^2 X^{\frac{7}{6}}, \end{aligned}$$

which means that

$$(3.4) \quad \gg \tau^2 X^{\frac{7}{6}}.$$

Thus, combining (3.2), (3.3) and (3.4), we have $J_1 \gg \tau^2 X^{\frac{7}{6}}$.

3.1.2 Upper bounds for J_2 and J_3

By Euler's summation formula, we have

$$(3.5) \quad U_j(\lambda\alpha) - T_j(\lambda\alpha) \ll 1 + |\alpha|X, \quad j = 2, 3, 4.$$

Using (2.1), we have

$$J_2 \ll \tau^2 \int_{\mathfrak{M}_1} |S_2(\lambda_1 \alpha) - T_2(\lambda_1 \alpha)|$$

$$\begin{aligned}
& \cdot |T_2(\lambda_2\alpha)T_3(\lambda_3\alpha)T_3(\lambda_4\alpha)T_4(\lambda_5\alpha)T_4(\lambda_6\alpha)|d\alpha \\
& \ll \tau^2 \int_{\mathfrak{M}_1} |S_2(\lambda_1\alpha) - U_2(\lambda_1\alpha)| \\
(3.6) \quad & \cdot |T_2(\lambda_2\alpha)T_3(\lambda_3\alpha)T_3(\lambda_4\alpha)T_4(\lambda_5\alpha)T_4(\lambda_6\alpha)|d\alpha \\
& + \tau^2 \int_{\mathfrak{M}_1} |U_2(\lambda_1\alpha) - T_2(\lambda_1\alpha)| \\
& \cdot |T_2(\lambda_2\alpha)T_3(\lambda_3\alpha)T_3(\lambda_4\alpha)T_4(\lambda_5\alpha)T_4(\lambda_6\alpha)|d\alpha \\
& := \tau^2(A_2 + B_2).
\end{aligned}$$

By Cauchy's inequality and Lemma 2.7, we obtain

$$\begin{aligned}
(3.7) \quad A_2 & \ll X^{\frac{7}{6}} \left(\int_{\mathfrak{M}_1} |S_2(\lambda_1\alpha) - U_2(\lambda_1\alpha)|^2 d\alpha \right)^{\frac{1}{2}} \left(\int_{\mathfrak{M}_1} |T_2(\lambda_2\alpha)|^2 d\alpha \right)^{\frac{1}{2}} \\
& \ll X^{\frac{7}{6}} (\log X)^{-\frac{A}{2}} \left(\int_0^{\frac{1}{X}} X d\alpha + \int_{\frac{1}{X}}^{X^{\frac{5}{24}-1-\varepsilon}} X^{-1} |\alpha|^{-2} d\alpha \right)^{\frac{1}{2}} \\
& \ll X^{\frac{7}{6}} (\log X)^{-\frac{A}{2}},
\end{aligned}$$

where we have used (3.1). It follows from (3.1) and (3.5) that

$$\begin{aligned}
(3.8) \quad B_2 & \ll \int_0^{\frac{1}{X}} |T_2(\lambda_2\alpha)T_3(\lambda_3\alpha)T_3(\lambda_4\alpha)T_4(\lambda_5\alpha)T_4(\lambda_6\alpha)|d\alpha \\
& + X \int_{\frac{1}{X}}^{X^{\frac{5}{24}-1-\varepsilon}} |\alpha| |T_2(\lambda_2\alpha)T_3(\lambda_3\alpha)T_3(\lambda_4\alpha)T_4(\lambda_5\alpha)T_4(\lambda_6\alpha)|d\alpha \\
& \ll X^{\frac{2}{3}} + X \int_{\frac{1}{X}}^{X^{\frac{5}{24}-1-\varepsilon}} |\alpha| X^{-\frac{10}{3}} |\alpha|^{-5} d\alpha \\
& \ll X^{\frac{2}{3}}.
\end{aligned}$$

Then, from (3.6)-(3.8), we have $J_2 = o(\tau^2 X^{\frac{7}{6}})$.

Arguing similarly, we can also get $J_3 = o(\tau^2 X^{\frac{7}{6}})$.

3.1.3 Upper bounds for J_4 and J_5

By (2.1), we have

$$\begin{aligned}
J_4 & \ll \tau^2 \int_{\mathfrak{M}_1} |S_2(\lambda_1\alpha)S_2(\lambda_2\alpha)| |S_3(\lambda_3\alpha) - T_3(\lambda_3\alpha)| \\
& \cdot |T_3(\lambda_4\alpha)T_4(\lambda_5\alpha)T_4(\lambda_6\alpha)|d\alpha \\
& \ll \tau^2 \int_{\mathfrak{M}_1} |S_2(\lambda_1\alpha)S_2(\lambda_2\alpha)| |S_3(\lambda_3\alpha) - U_3(\lambda_3\alpha)| \\
(3.9) \quad & \cdot |T_3(\lambda_4\alpha)T_4(\lambda_5\alpha)T_4(\lambda_6\alpha)|d\alpha
\end{aligned}$$

$$\begin{aligned}
& + \tau^2 \int_{\mathfrak{M}_1} |S_2(\lambda_1 \alpha) S_2(\lambda_2 \alpha)| |U_3(\lambda_3 \alpha) - T_3(\lambda_3 \alpha)| \\
& \cdot |T_3(\lambda_4 \alpha) T_4(\lambda_5 \alpha) T_4(\lambda_6 \alpha)| d\alpha \\
& := \tau^2 (A_3 + B_3).
\end{aligned}$$

Applying Hölder's inequality, by Lemmas 2.4 and 2.7, we obtain

$$\begin{aligned}
(3.10) \quad A_3 & \ll X^{\frac{5}{6}} \left(\int_{-1}^1 |S_2(\lambda_1 \alpha)|^4 d\alpha \right)^{\frac{1}{4}} \left(\int_{-1}^1 |S_2(\lambda_2 \alpha)|^4 d\alpha \right)^{\frac{1}{4}} \\
& \times \left(\int_{\mathfrak{M}_1} |S_3(\lambda_3 \alpha) - U_3(\lambda_3 \alpha)|^2 d\alpha \right)^{\frac{1}{2}} \\
& \ll X^{\frac{5}{6}} X^{\frac{1}{2}} (\log X) X^{\frac{1}{3} - \frac{1}{2}} (\log X)^{-\frac{A}{2}} \\
& \ll X^{\frac{7}{6}} (\log X)^{-2}.
\end{aligned}$$

It follows from (3.1) and (3.5) that

$$\begin{aligned}
(3.11) \quad B_3 & \ll \int_0^{\frac{1}{X}} |S_2(\lambda_1 \alpha) S_2(\lambda_2 \alpha) T_3(\lambda_4 \alpha) T_4(\lambda_5 \alpha) T_4(\lambda_6 \alpha)| d\alpha \\
& + X \int_{\frac{1}{X}}^{X^{\frac{5}{24} - 1 - \varepsilon}} |\alpha| |S_2(\lambda_1 \alpha) S_2(\lambda_2 \alpha) T_3(\lambda_4 \alpha) T_4(\lambda_5 \alpha) T_4(\lambda_6 \alpha)| d\alpha \\
& \ll X^{\frac{5}{6}} + X^{-\frac{7}{6}} \int_{\frac{1}{X}}^{X^{\frac{5}{24} - 1 - \varepsilon}} |\alpha|^{-2} |S_2(\lambda_1 \alpha) S_2(\lambda_2 \alpha)| d\alpha \\
& \ll X^{\frac{5}{6}} + X^{-\frac{7}{6}} \left(\int_{\frac{1}{X}}^{X^{\frac{5}{24} - 1 - \varepsilon}} |\alpha|^{-4} d\alpha \right)^{\frac{1}{2}} \left(\int_{-1}^1 |S_2(\lambda_1 \alpha)|^4 d\alpha \right)^{\frac{1}{4}} \\
& \times \left(\int_{-1}^1 |S_2(\lambda_2 \alpha)|^4 d\alpha \right)^{\frac{1}{4}} \\
& \ll X^{\frac{5}{6}} \log X,
\end{aligned}$$

Then, from (3.9)-(3.11), we have $J_4 = o(\tau^2 X^{\frac{7}{6}})$.

Arguing similarly, we can also get $J_5 = o(\tau^2 X^{\frac{7}{6}})$.

3.1.4 Upper bounds for J_6 and J_7

The computation for J_6 and J_7 are similar to that for J_4 and J_5 . Also by (2.1), we have

$$\begin{aligned}
J_6 & \ll \tau^2 \int_{\mathfrak{M}_1} |S_2(\lambda_1 \alpha) S_2(\lambda_2 \alpha) S_3(\lambda_3 \alpha) S_3(\lambda_4 \alpha)| \\
& \cdot |S_4(\lambda_5 \alpha) - T_4(\lambda_5 \alpha)| |T_4(\lambda_6 \alpha)| d\alpha
\end{aligned}$$

$$\begin{aligned}
(3.12) \quad & \ll \tau^2 \int_{\mathfrak{M}_1} |S_2(\lambda_1 \alpha) S_2(\lambda_2 \alpha) S_3(\lambda_3 \alpha) S_3(\lambda_4 \alpha)| \\
& \cdot |S_4(\lambda_5 \alpha) - U_4(\lambda_5 \alpha)| |T_4(\lambda_6 \alpha)| d\alpha \\
& + \tau^2 \int_{\mathfrak{M}_1} |S_2(\lambda_1 \alpha) S_2(\lambda_2 \alpha) S_3(\lambda_3 \alpha) S_3(\lambda_4 \alpha)| \\
& \cdot |U_4(\lambda_5 \alpha) - T_4(\lambda_5 \alpha)| |T_4(\lambda_6 \alpha)| d\alpha \\
& := \tau^2 (A_4 + B_4).
\end{aligned}$$

Applying Hölder's inequality, by Lemmas 2.4 and 2.7, we obtain

$$\begin{aligned}
(3.13) \quad A_4 & \ll X^{\frac{11}{12}} \left(\int_{-1}^1 |S_2(\lambda_1 \alpha)|^4 d\alpha \right)^{\frac{1}{4}} \left(\int_{-1}^1 |S_2(\lambda_2 \alpha)|^4 d\alpha \right)^{\frac{1}{4}} \\
& \times \left(\int_{\mathfrak{M}_1} |S_4(\lambda_5 \alpha) - U_4(\lambda_5 \alpha)|^2 d\alpha \right)^{\frac{1}{2}} \\
& \ll X^{\frac{11}{12}} X^{\frac{1}{2}} (\log X) X^{\frac{1}{4} - \frac{1}{2}} (\log X)^{-\frac{A}{2}} \\
& \ll X^{\frac{7}{6}} (\log X)^{-2}.
\end{aligned}$$

From (3.1) and (3.5), we have

$$\begin{aligned}
(3.14) \quad B_4 & \ll \int_0^{\frac{1}{X}} |S_2(\lambda_1 \alpha) S_2(\lambda_2 \alpha) S_3(\lambda_3 \alpha) S_3(\lambda_4 \alpha) T_4(\lambda_6 \alpha)| d\alpha \\
& + X \int_{\frac{1}{X}}^{X^{\frac{5}{24} - 1 - \varepsilon}} |\alpha| |S_2(\lambda_1 \alpha) S_2(\lambda_2 \alpha) S_3(\lambda_3 \alpha) S_3(\lambda_4 \alpha) T_4(\lambda_6 \alpha)| d\alpha \\
& \ll X^{\frac{11}{12}} + X^{\frac{1}{4}} \int_{\frac{1}{X}}^{X^{\frac{5}{24} - 1 - \varepsilon}} |S_2(\lambda_1 \alpha) S_2(\lambda_2 \alpha) S_3(\lambda_3 \alpha) S_3(\lambda_4 \alpha)| d\alpha \\
& := X^{\frac{11}{12}} + B_{41}.
\end{aligned}$$

Using Hölder's inequality and Lemma 2.4, we have

$$\begin{aligned}
(3.15) \quad B_{41} & \ll X^{\frac{1}{4}} \left(\int_{-1}^1 |S_2(\lambda_1 \alpha)|^4 d\alpha \right)^{\frac{1}{4}} \left(\int_{-1}^1 |S_2(\lambda_2 \alpha)|^4 d\alpha \right)^{\frac{1}{4}} \\
& \times \left(\int_{-1}^1 |S_3(\lambda_3 \alpha)|^8 d\alpha \right)^{\frac{1}{8}} \left(\int_{-1}^1 |S_3(\lambda_4 \alpha)|^8 d\alpha \right)^{\frac{1}{8}} \left(\int_{\frac{1}{X}}^{X^{\frac{5}{24} - 1 - \varepsilon}} 1 d\alpha \right)^{\frac{1}{4}} \\
& \ll X^{\frac{93}{96} + \varepsilon}.
\end{aligned}$$

Then, from (3.12)-(3.15), we have $J_6 = o(\tau^2 X^{\frac{7}{6}})$.

Arguing similarly, we can also get $J_7 = o(\tau^2 X^{\frac{7}{6}})$.

3.2 The region $\mathfrak{M}_2$

In this subsection, we give the upper bound for the integral on the region $\mathfrak{M}_2$.

Lemma 3.2 ([3, Lemma 3.2]). *We have*

$$\int_{|\alpha| \leq X^{-\frac{2}{3}}} |S_2(\lambda\alpha)|^2 d\alpha \ll 1.$$

Then, we have the following lemma:

Lemma 3.3. *We have*

$$\int_{\mathfrak{M}_2} |S_2(\lambda_1\alpha)S_2(\lambda_2\alpha)S_3(\lambda_3\alpha)S_3(\lambda_4\alpha)S_4(\lambda_5\alpha)S_4(\lambda_6\alpha)|K_\tau(\alpha)d\alpha = o(\tau^2 X^{\frac{7}{6}}).$$

Proof. By Lemmas 2.3, 2.4 and 3.2, we have

$$\begin{aligned} & \int_{\mathfrak{M}_2} |S_2(\lambda_1\alpha)S_2(\lambda_2\alpha)S_3(\lambda_3\alpha)S_3(\lambda_4\alpha)S_4(\lambda_5\alpha)S_4(\lambda_6\alpha)|K_\tau(\alpha)d\alpha \\ & \ll \tau^2 X^{2(\frac{1}{4}-\frac{1}{2\cdot 4^4})+\varepsilon} \left(\int_{\mathfrak{M}} |S_2(\lambda_1\alpha)|^2 d\alpha \right)^{\frac{1}{2}} \left(\int_{-1}^1 S_2^2(\lambda_2\alpha)S_3^4(\lambda_3\alpha)d\alpha \right)^{\frac{1}{4}} \\ & \quad \times \left(\int_{-1}^1 S_2^2(\lambda_2\alpha)S_3^4(\lambda_4\alpha)d\alpha \right)^{\frac{1}{4}} \\ & \ll \tau^2 X^{\frac{7}{6}-\frac{1}{256}+\varepsilon} = o(\tau^2 X^{\frac{7}{6}}). \end{aligned}$$

□

Therefore, combining Lemmas 3.1 and 3.3, we complete the estimate of the integral from the major arc. We have the following lemma:

Lemma 3.4. *We have*

$$\int_{\mathfrak{M}} S_2(\lambda_1\alpha)S_2(\lambda_2\alpha)S_3(\lambda_3\alpha)S_3(\lambda_4\alpha)S_4(\lambda_5\alpha)S_4(\lambda_6\alpha)K_\tau(\alpha)e(-v\alpha)d\alpha \gg \tau^2 X^{\frac{7}{6}}.$$

4. The minor arc

In this section, we give the estimate of the integral on the minor arc. Let $\tilde{\mathfrak{m}} = \mathfrak{m}_1 \cup \mathfrak{m}_2$, where

$$\mathfrak{m}_j = \{\alpha \in \mathfrak{m} : |S_2(\lambda_j\alpha)| \leq X^{\frac{7}{16}+2\varepsilon}\}, \quad \text{for } j = 1, 2.$$

Lemma 4.1. *We have*

$$\int_{\tilde{\mathfrak{m}}} |S_2(\lambda_1\alpha)S_2(\lambda_2\alpha)S_3(\lambda_3\alpha)S_3(\lambda_4\alpha)S_4(\lambda_5\alpha)S_4(\lambda_6\alpha)|^2 K_\tau(\alpha)d\alpha \ll \tau X^{1+\frac{7}{3}-\frac{1}{8}+\varepsilon}.$$

Proof. Using Hölder's inequality and Lemma 2.4, we can obtain

$$\begin{aligned}
& \int_{\mathfrak{m}_1} |S_2(\lambda_1\alpha)S_2(\lambda_2\alpha)S_3(\lambda_3\alpha)S_3(\lambda_4\alpha)S_4(\lambda_5\alpha)S_4(\lambda_6\alpha)|^2 K_\tau(\alpha) d\alpha \\
& \ll \left(\sup_{\alpha \in \mathfrak{m}_1} |S_2(\lambda_1\alpha)| \right)^2 X^{\frac{4}{3}} \left(\int_{\mathbb{R}} |S_2^2(\lambda_2\alpha)S_4^4(\lambda_5\alpha)| K_\tau(\alpha) d\alpha \right)^{\frac{1}{2}} \\
& \times \left(\int_{\mathbb{R}} |S_2^2(\lambda_2\alpha)S_4^4(\lambda_6\alpha)| K_\tau(\alpha) d\alpha \right)^{\frac{1}{2}} \\
& \ll \tau X^{1+\frac{7}{3}-\frac{1}{8}+\varepsilon}.
\end{aligned}$$

By symmetry we can get the same bound for the integral on $\mathfrak{m}_2$. Thus, we have

$$\int_{\tilde{\mathfrak{m}}} |S_2(\lambda_1\alpha)S_2(\lambda_2\alpha)S_3(\lambda_3\alpha)S_3(\lambda_4\alpha)S_4(\lambda_5\alpha)S_4(\lambda_6\alpha)|^2 K_\tau(\alpha) d\alpha \ll \tau X^{1+\frac{7}{3}-\frac{1}{8}+\varepsilon}.$$

□

Now, we turn to handle the range $\mathfrak{m}^* = \mathfrak{m} \setminus \tilde{\mathfrak{m}}$. We have the following lemma:

Lemma 4.2. *We have*

$$\int_{\mathfrak{m}^*} |S_2(\lambda_1\alpha)S_2(\lambda_2\alpha)S_3(\lambda_3\alpha)S_3(\lambda_4\alpha)S_4(\lambda_5\alpha)S_4(\lambda_6\alpha)|^2 K_\tau(\alpha) d\alpha \ll \tau X^{1+\frac{7}{3}-\frac{3}{4}+\varepsilon}.$$

Proof. Note that, for any $\alpha \in \mathfrak{m}^*$, we have

$$|S_1(\lambda_1\alpha)| > X^{\frac{7}{16}+2\varepsilon} \text{ and } |S_2(\lambda_2\alpha)| > X^{\frac{7}{16}+2\varepsilon}.$$

We divide $\mathfrak{m}^*$ into disjoint sets $S(Z_1, Z_2, y)$, such that for $\alpha \in S(Z_1, Z_2, y)$, we have

$$Z_1 < |S_1(\lambda_1\alpha)| \leq 2Z_1, \quad Z_2 < |S_2(\lambda_2\alpha)| \leq 2Z_2, \quad y < |\alpha| \leq 2y,$$

where $Z_1 = 2^{k_1} X^{\frac{7}{16}+2\varepsilon}$, $Z_2 = 2^{k_2} X^{\frac{7}{16}+2\varepsilon}$ and $y = 2^{k_3} \phi$ for some non-negative integers k_1, k_2, k_3 . Then, from Lemma 2.2, there exist two pairs of coprime integers (a_1, q_1) and (a_2, q_2) satisfying

$$(4.1) \quad a_1 a_2 \neq 0, 1 \leq q_i \ll \left(\frac{X^{\frac{1}{2}+\varepsilon}}{Z_i} \right)^2, |q_i \lambda_i \alpha - a_i| \ll X^{-1} \left(\frac{X^{\frac{1}{2}+\varepsilon}}{Z_i} \right)^2, i = 1, 2.$$

Furthermore, we subdivide $S(Z_1, Z_2, y)$ into sets $S(Z_1, Z_2, y, Q_1, Q_2)$, where $Q_j < q_j \leq 2Q_j$ on each set. Then, we have

$$\begin{aligned}
\left| a_2 q_1 \frac{\lambda_1}{\lambda_2} - a_1 q_2 \right| &= \left| (q_1 \lambda_1 \alpha - a_1) \frac{a_2}{\lambda_2 \alpha} - (q_2 \lambda_2 \alpha - a_2) \frac{a_1}{\lambda_2 \alpha} \right| \\
&\ll Q_2 X^{-1} \left(\frac{X^{\frac{1}{2}+\varepsilon}}{Z_1} \right)^2 + Q_1 X^{-1} \left(\frac{X^{\frac{1}{2}+\varepsilon}}{Z_2} \right)^2 \\
&\ll X^{1+4\varepsilon} Z_1^{-2} Z_2^{-2} \\
&\ll X^{-\frac{3}{4}-4\varepsilon}.
\end{aligned}$$

We take $X = q^{\frac{4}{3}}$. Thus,

$$\left| a_2 q_1 \frac{\lambda_1}{\lambda_2} - a_1 q_2 \right| = o(q^{-1}).$$

We also have $|a_2 q_1| \ll y Q_1 Q_2$. Thus, if $|a_2 q_1|$ took R distinct values, we could apply the pigeon-hole principle to deduce the existence of n satisfying

$$\left\| n \frac{\lambda_1}{\lambda_2} \right\| \ll X^{-\frac{3}{4}-4\varepsilon}, \quad n \ll \frac{y Q_1 Q_2}{R}.$$

This would contradict a/q being a convergent to λ_1/λ_2 if q is sufficiently large, unless

$$R \ll \frac{y Q_1 Q_2}{q}.$$

From the well-known bound for the divisor function, each value of $|a_2 q_1|$ corresponds to $O(X^\varepsilon)$ values of a_2, q_1 . Then, we conclude that each set of $S(Z_1, Z_2, y, Q_1, Q_2)$ is made up of $O(RX^\varepsilon)$ intervals of length

$$\begin{aligned} &\ll \min \left(Q_1^{-1} X^{-1} \left(\frac{X^{\frac{1}{2}+\varepsilon}}{Z_1} \right)^2, Q_2^{-1} X^{-1} \left(\frac{X^{\frac{1}{2}+\varepsilon}}{Z_2} \right)^2 \right) \\ &\ll \frac{X^{2\varepsilon}}{Z_1 Z_2 Q_1^{1/2} Q_2^{1/2}}. \end{aligned}$$

Let $\mathcal{A}$ denote such a set $S(Z_1, Z_2, y, Q_1, Q_2)$. Then, integrating over $\mathcal{A}$ gives

$$\begin{aligned} (4.2) \quad &\int_{\mathcal{A}} |S_2(\lambda_1 \alpha) S_2(\lambda_2 \alpha) S_3(\lambda_3 \alpha) S_3(\lambda_4 \alpha) S_4(\lambda_5 \alpha) S_4(\lambda_6 \alpha)|^2 K_\tau(\alpha) d\alpha \\ &\ll \min(\tau^2, y^{-2}) Z_1^2 Z_2^2 X^{\frac{4}{3}+1} \frac{X^{2\varepsilon}}{Z_1 Z_2 Q_1^{1/2} Q_2^{1/2}} \frac{X^\varepsilon y Q_1 Q_2}{q} \\ &\ll \tau X^{\frac{4}{3}+2+5\varepsilon} q^{-1} \\ &\ll \tau X^{\frac{31}{12}+5\varepsilon}. \end{aligned}$$

Then, summing over all possible values of Z_1, Z_2, y, Q_1, Q_2 , we conclude that

$$\int_{\mathfrak{m}^*} |S_2(\lambda_1 \alpha) S_2(\lambda_2 \alpha) S_3(\lambda_3 \alpha) S_3(\lambda_4 \alpha) S_4(\lambda_5 \alpha) S_4(\lambda_6 \alpha)|^2 K_\tau(\alpha) d\alpha \ll \tau X^{1+\frac{7}{3}-\frac{3}{4}+\varepsilon}.$$

□

Combining Lemmas 4.1 and 4.2 we can deduce the following lemma immediately.

Lemma 4.3. *We have*

$$\int_{\mathfrak{m}} |S_2(\lambda_1 \alpha) S_2(\lambda_2 \alpha) S_3(\lambda_3 \alpha) S_3(\lambda_4 \alpha) S_4(\lambda_5 \alpha) S_4(\lambda_6 \alpha)|^2 K_\tau(\alpha) d\alpha \ll \tau X^{1+\frac{7}{3}-\frac{1}{8}+\varepsilon}.$$

5. The trivial arc

Applying Hölder's inequality, we have

$$\begin{aligned}
 & \int_t^\infty |S_2(\lambda_1\alpha)S_2(\lambda_2\alpha)S_3(\lambda_3\alpha)S_3(\lambda_4\alpha)S_4(\lambda_5\alpha)S_4(\lambda_6\alpha)|K_\tau(\alpha)d\alpha \\
 (5.1) \quad & \ll \left(\int_\xi^\infty |S_2^2(\lambda_1\alpha)S_3^4(\lambda_3\alpha)|K_\tau(\alpha)d\alpha \right)^{\frac{1}{4}} \left(\int_\xi^\infty |S_2^2(\lambda_1\alpha)S_3^4(\lambda_4\alpha)|K_\tau(\alpha)d\alpha \right)^{\frac{1}{4}} \\
 & \times \left(\int_\xi^\infty |S_2^2(\lambda_2\alpha)S_4^4(\lambda_5\alpha)|K_\tau(\alpha)d\alpha \right)^{\frac{1}{4}} \left(\int_\xi^\infty |S_2^2(\lambda_2\alpha)S_4^4(\lambda_6\alpha)|K_\tau(\alpha)d\alpha \right)^{\frac{1}{4}}.
 \end{aligned}$$

These four integrals at the right side of (5.1) can be handled in a similar argument. Using (2.1) and Lemma 2.4, we have, for $j = 3, 4$,

$$\begin{aligned}
 & \int_\xi^\infty |S_2^2(\lambda_1\alpha)S_3^4(\lambda_j\alpha)|K_\tau(\alpha)d\alpha \\
 & \ll \sum_{n=[\xi]}^\infty \int_n^{n+1} S_2^2(\lambda_1\alpha)S_3^4(\lambda_j\alpha) \frac{1}{\alpha^2} d\alpha \\
 & \ll \sum_{n=[\xi]}^\infty \frac{1}{n^2} \int_0^1 S_2^2(\lambda_1\alpha)S_3^4(\lambda_j\alpha) d\alpha \\
 & \ll \xi^{-1} X^{\frac{4}{3}+\varepsilon}.
 \end{aligned}$$

Similarly, we can get, for $j = 5, 6$,

$$\int_\xi^\infty |S_2^2(\lambda_1\alpha)S_4^4(\lambda_j\alpha)|K_\tau(\alpha)d\alpha \ll \xi^{-1} X^{1+\varepsilon}.$$

Thus, inserting these bounds into (5.1) we have

$$\begin{aligned}
 & \int_t^\infty |S_2(\lambda_1\alpha)S_2(\lambda_2\alpha)S_3(\lambda_3\alpha)S_3(\lambda_4\alpha)S_4(\lambda_5\alpha)S_4(\lambda_6\alpha)|K_\tau(\alpha)d\alpha \\
 (5.2) \quad & \ll \xi^{-1} X^{\frac{4}{3}+\varepsilon} \ll \tau^2 X^{\frac{7}{6}-\varepsilon},
 \end{aligned}$$

where $\xi = \tau^{-2} X^{\frac{1}{6}+2\varepsilon}$ is used.

6. Proof of Theorem 1.2

In this section, we complete the proof of Theorem 1.2. We take $\tau = X^{-\delta}$. Let $\mathcal{E} = \mathcal{E}(\mathcal{V}, X, \delta)$ denote the set of $v \in [X/2, X] \cap \mathcal{V}$ such that the inequality (1.2) has no solution in primes $p_1, p_2, p_3, p_4, p_5, p_6$, and $E = E(\mathcal{V}, X, \delta) = |\mathcal{E}(\mathcal{V}, X, \delta)|$. Then from Lemma 3.4 and (5.2), we have

$$\begin{aligned}
 & \left| \sum_{v \in \mathcal{E}^{\mathbf{m}}} \int S_2(\lambda_1\alpha)S_2(\lambda_2\alpha)S_3(\lambda_3\alpha)S_3(\lambda_4\alpha)S_4(\lambda_5\alpha)S_4(\lambda_6\alpha)e(-v\alpha)K_\tau(\alpha)d\alpha \right| \\
 (6.1) \quad & \gg \tau^2 X^{\frac{7}{6}} E.
 \end{aligned}$$

Applying Cauchy's inequality and Lemma 4.3, we have

$$\begin{aligned}
 & \left| \sum_{v \in \mathcal{E}} \int_{\mathfrak{m}} S_2(\lambda_1 \alpha) S_2(\lambda_2 \alpha) S_3(\lambda_3 \alpha) S_3(\lambda_4 \alpha) S_4(\lambda_5 \alpha) S_4(\lambda_6 \alpha) e(-v\alpha) K_\tau(\alpha) d\alpha \right| \\
 & \ll \left(\int_{-\infty}^{+\infty} \left| \sum_{v \in \mathcal{E}} e(-v\alpha) \right|^2 K_\tau(\alpha) d\alpha \right)^{\frac{1}{2}} \\
 (6.2) \quad & \times \left(\int_{\mathfrak{m}} |S_2(\lambda_1 \alpha) S_2(\lambda_2 \alpha) S_3(\lambda_3 \alpha) S_3(\lambda_4 \alpha) S_4(\lambda_5 \alpha) S_4(\lambda_6 \alpha)|^2 K_\tau(\alpha) d\alpha \right)^{\frac{1}{2}} \\
 & \ll \left(\tau X^{1+\frac{7}{3}-\frac{1}{8}+\varepsilon} \right)^{\frac{1}{2}} \left(\sum_{v_1, v_2 \in \mathcal{E}} \max(0, \tau - |v_1 - v_2|) \right)^{\frac{1}{2}} \\
 & \ll \tau E^{\frac{1}{2}} \left(X^{1+\frac{7}{3}-\frac{1}{8}+\varepsilon} \right)^{\frac{1}{2}}.
 \end{aligned}$$

Then combining (6.1) and (6.2), we have

$$(6.3) \quad E(\mathcal{V}, X_j, \delta) \ll X_j^{1-\frac{1}{8}+2\delta+\varepsilon}.$$

Since λ_1/λ_2 is irrational, there are infinitely many q we could have taken, and this gives the sequence $X_j \rightarrow +\infty$.

Now, if the convergent denominators for λ_1/λ_2 satisfy (1.4), then we can modify our works in Lemmas 4.1 and 4.2. We now assume that

$$\min(Z_1, Z_2) > X^{\chi+2\varepsilon}$$

with χ given by (1.6). We then obtain

$$\left| a_2 q_1 \frac{\lambda_1}{\lambda_2} - a_1 q_2 \right| \ll X^{1-4\chi-4\varepsilon}.$$

However, we know from (1.4) that there is a convergent a/q to λ_1/λ_2 with

$$X^{(1-w)(4\chi-1)} \ll q \ll X^{4\chi-1}.$$

The expression corresponding to (4.2) is now

$$\begin{aligned}
 & \int_{\mathcal{A}} |S_2(\lambda_1 \alpha) S_2(\lambda_2 \alpha) S_3(\lambda_3 \alpha) S_3(\lambda_4 \alpha) S_4(\lambda_5 \alpha) S_4(\lambda_6 \alpha)|^2 K_\tau(\alpha) d\alpha \\
 & \ll \tau X^{\frac{4}{3}+2+\varepsilon} q^{-1} \ll \tau X^{\frac{10}{3}-(1-w)(4\chi-1)+\varepsilon} \\
 & \ll \tau X^{\frac{7}{3}+2\chi+\varepsilon}
 \end{aligned}$$

by our choice of χ . Thus,

$$\int_{\mathfrak{m}} |S_2(\lambda_1 \alpha) S_2(\lambda_2 \alpha) S_3(\lambda_3 \alpha) S_3(\lambda_4 \alpha) S_4(\lambda_5 \alpha) S_4(\lambda_6 \alpha)|^2 K_\tau(\alpha) d\alpha \ll \tau X^{\frac{7}{3}+2\chi+\varepsilon}.$$

Working as (6.1)-(6.3), we can complete the proof of Theorem 1.2 easily.

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On the edge version of topological indices for certain networks

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Abstract. In QSPR/QSAR study, topological indices such as Shultz index, generalized Randić index, Zagreb index, general sum-connectivity index, atom-bond connectivity (ABC) index and geometric-arithmetic (GA) index are utilized to guess the physico-chemical and bioactivity of chemical compounds. They can be classified based on the structural properties of graphs used for calculation. There is numerous applications of graph theory in this field of research. The aims of this paper is to investigate the generalize on the edge version Randić, GA, ABC, multiplicative ABC, Zagreb and inverse sum indices for certain graph networks by using the concept of line graphs.

Keywords: line graphs, topological indices, networks.

1. Introduction

Chemical graph theory is a branch of mathematical chemistry which has an important effect on the development of the chemical sciences. Every individual number used to characterize some properties of a graph is known as a topological index of a related (molecular) graph. Topological indices are the numerical quantities which represent the structure of any simple finite graph. It's apparent that the numbers of vertices and edges are topological variants. The first graph invariant reported as (distance based) topological index, Wiener index, is denoted as half of distances between all the pairs of vertices in a graph [1]. Meanwhile, on the basis of the distance between edges, the edge version of Wiener index was proposed by Iranmanesh et al. [6]. Several recent results on various types of Wiener indices (such as Wiener polarity index) can be referred to Ma et al. [7,8]. The meaningful of topological indices is generally

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related with quantitative structure activity (QSAR) and quantitative structures property relationship (QSPR), see [2, 4, 5].

In this article, we will consider only simple graphs, as an unweighted, undirected graph who has no loop and multiple edges attached. As the concept in theoretical and applies areas, a graph is represented by a collection points and lines, we can be separately called vertices and edges. Suppose e is an edge of G , which connects the u and v , then we denote $e = uv$ and state that u and v are adjacent. The degree d_u of a vertex u is the number of edges that are incident to it.

The first and oldest degree based topological index was introduced by Milan Randić in 1975 named Randić index [15]. The Randić index of graph G is defined as

$$(1) \quad R(G) = \sum_{u,v \in G} (d_u d_v)^{-1/2},$$

where d_u and d_v represent the degree of the vertices u and v , respectively.

In 1998, Bollobas and Erdos [16] generalized this Randić index for any real number α , and known as the generalized Randić index $R_\alpha(G)$:

$$(2) \quad R_\alpha(G) = \sum_{uv \in E(G)} (d_u d_v)^\alpha.$$

For further study of Randić index of various graph families, see [2, 13, 18, 32]. Referring to the end vertex degree d_e and d_f of the edges e and f in a line graph of G , Gao et al. [22] proposed the edge version Randić index. This idea is expressed as follows:

$$(3) \quad {}_e R_\alpha(G) = \sum_{ef \in E(L(G))} (d_e \bullet d_f)^{-1/2}.$$

Furthermore, Randić [14] raised one of the most important topological indices as branching index, that is, the sum of certain bond contributions calculated from the vertex degree of the hydrogen suppressed molecular graphs.

With the reference to Randić connectivity index, Vukicevic and Furtula [33] put forward the geometric-arithmetic index (briefly, GA), a topological index, and it is designed on the ground of the end-vertex degrees of edges in a graph connected G with the vertex set $V(G)$ and the edge set $E(G)$. To be specific, this index can be described as

$$(4) \quad GA(G) = \sum_{uv \in E(G)} \frac{2\sqrt{d_u \bullet d_v}}{d_u + d_v},$$

where d_u states the degree of the vertex u in G . All the information about geometric-arithmetic index is wanted, please refer to Fath-Tabar et al. [19] and Yuan et al. [34]. If more results on topological index computation are in need,

please refer to Gao and Farahani [25], Vukicevic and Furtula [29], Gao and Wang [26] and [27], Asadpour [35] and [14, 33, 37-41].

Iranmanesh et al. [36] proposed the edge version of geometric arithmetic index, referring to the end-vertex degrees of edges in a line graph of G . This idea is described in the following:

$$(5) \quad GA_e(G) = \sum_{ef \in E(L(G))} \frac{2\sqrt{d_e \bullet d_f}}{d_e + d_f}$$

where d_e represents the degree of the edge e in G . Recently, Nadeem et al. [42] also studied GA_e index of nanocones.

Estrada et al. [9] put forward a topological index with the name of the atom-bond connectivity index (briefly, ABC) as

$$(6) \quad ABC(G) = \sum_{uv \in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u \bullet d_v}},$$

where d_u and d_v represent the degrees of the vertices u and v , respectively.

Recent advances on ABC index can be referred to Das et al. [10], Lin et al. [11], Gao and Shao [23], and Bianchi et al. [12]. Farahani [17] proposed the edge version of atom-bond connectivity index. This idea is expressed as follows: This idea is expressed as follows:

$$(7) \quad ABC_e(G) = \sum_{ef \in E(L(G))} \sqrt{\frac{d_e + d_f - 2}{d_e \bullet d_f}},$$

where d_e is the degree of the edge e in $L(G)$.

The multiplicative atom-bond connectivity index was introduced by Kulli in 2016 [20]. Later, the edge version of multiplicative atom-bond connectivity index [21] of a graph G was introduced and it is defined as

$$(8) \quad ABCII_e(G) = \prod_{ef \in E(L(G))} \sqrt{\frac{d_e + d_f - 2}{d_e \bullet d_f}},$$

where d_e is the degree of the edge e in $L(G)$.

Zagreb index is a degree based molecular descriptor that was introduced by Gutman and Trinajstić [12] more than 3 decades ago. Gao et al. [22] proposed the edge version Zagreb index. This idea is expressed as follows:

$$(9) \quad {}_eM_1(G) = \sum_{ef \in E(L(G))} (d_e + d_f),$$

where d_e is the degree of the edge e in $L(G)$.

In 2010, Vukicevic and Gasperov [28] introduced the inverse sum indeg index of a graph G . The edge version of inverse sum indeg index of graph G was found by Bhanumathi et al. [30] and is defined as

$$(10) \quad ISI_e(G) = \sum_{ef \in E(L(G))} \frac{d_e \bullet d_f}{d_e + d_f},$$

where d_e is the degree of the edge e in $L(G)$. For further study of edge version of indices of various graph families, see [20, 30, 32-36].

Now, we define some notations of the graph theory. In graph G , if the corresponding edges share a vertex in G , the line graph $L(G)$ of a graph G is considered as a graph with vertices of the edges in G , and it possesses two adjacent vertices. Similarly, the degree of an edge $e \in E(G)$ is represented by the number of its adjacent vertices in $V(L(G))$.

2. Topological indices of $L(G)$

In 2012, Farahani [17] computed ABC_e index of the circumcoronene series of benzenoid. Recently, Gao et al. [24] studies on the ABC_e and GA_e indices of certain graph operations. They also studied the edge version of Randić, Zagreb, ABC_e and GA_e of $HAC_5C_6C_7[p, q]$ in [22]. In 2018, Kulli [20] computed the edge version of multiplicative ABC index of certain nanotubes and nanotorus. Also, Bhanumathi et al. [30] studied the edge inverse sum indeg index connected graph.

Motivated by the result of [17, 20, 22, 24, 30], we computed close formulae of the edge version of Randić, GA , ABC , multiplicative ABC , Zagreb and inverse sum indices by using the concept of line graphs for chain silicate (CS_n), hexagonal (HX_n) and oxide (OX_n) networks. For more information about these networks see [31].

2.1 The edge version of topological indices for $L(CS_n)$

Lemma 2.1. *Let G be the line graph CS_n with dimension of $n = 1, 2$ and 3.*

1. *For $n=1$. Then the edge version of topological indices is given by*

- (a) ${}_eR_{-1/2}(G) = 3;$
- (b) $GA_e(G) = 12;$
- (c) $ABC_e(G) = 3\sqrt{6};$
- (d) $ABCII_e(G) = 3\sqrt{6};$
- (e) ${}_eM_1(G) = 96;$
- (f) $ISI_e(G) = 24.$

2. *For $n = 2$. Then the edge version of topological indices is given by*

- (a) ${}_eR_{-1/2}(G) = \frac{3}{2} + \frac{6}{\sqrt{7}} + \frac{15}{7};$
 (b) $GA_e(G) = 27 + \frac{48\sqrt{7}}{11};$
 (c) $ABC_e(G) = \frac{3\sqrt{6}}{2} + \frac{18\sqrt{7}+30\sqrt{3}}{7};$
 (d) $ABCI_e(G) = \frac{2430\sqrt{14}}{49};$
 (e) ${}_eM_1(G) = 390;$
 (f) $ISI_e(G) = \frac{2091}{22}.$

3. For $n=3$. Then the edge version of topological indices is given by

- (a) ${}_eR_{-1/2}(G) = \sqrt{\frac{10}{7}} + \frac{66}{14} + \frac{8\sqrt{7}}{7};$
 (b) $GA_e(G) = 28 + \frac{64\sqrt{7}}{11} + \frac{20\sqrt{70}}{17};$
 (c) $ABC_e(G) = \frac{3\sqrt{6}}{2} + \frac{24\sqrt{7}+44\sqrt{3}}{7} + 10\sqrt{\frac{3}{14}};$
 (d) $ABCI_e(G) = \frac{47520\sqrt{3}}{49};$
 (e) ${}_eM_1(G) = 720;$
 (f) $ISI_e(G) = \frac{31959}{187}.$

Proof. First consider the line graph $L(CS_n)$ with $n = 1$ as shown in Fig. 1(b). In this graph there are total 6 vertices, consisting 6 vertices of degree 4. Hence, we get the edge partition, based on the degree of an edge e and f as shown in Table 1. We apply Formulas (3), (5), (7) – (10) to Table 1 and obtain the required results. By similar arguments we can obtain the expression of the edge version of topological indices for $n = 2$ and 3 by viewing edge partition in Tables 2 and 3. $\square$



Figure 1: (a) A chain silicate network CS_1 ; (b) the line graph $L(CS_1)$.

Table 1: The edge partition of $L(CS_n)$ for $n = 1$

(d_e, d_f) where $ef \in E(L(G))$	Number of edges
$(4, 4)$	12

Table 2: The edge partition of $L(CS_n)$ for $n = 2$

(d_e, d_f) where $ef \in E(L(G))$	Number of edges
(4, 4)	6
(4, 7)	12
(7, 7)	15

Table 3: The edge partition of $L(CS_n)$ for $n = 3$

(d_e, d_f) where $ef \in E(L(G))$	Number of edges
(4, 4)	6
(4, 7)	16
(7, 7)	22
(7, 10)	10

Theorem 2.1. Let G be the line graph CS_n with dimension of $n \geq 4$, then

1. ${}_eR_{-1/2}(G) = \left(28\sqrt{\frac{2}{5}} + \frac{16}{\sqrt{7}} + 8\sqrt{7} + \sqrt{10}\right)n + 12 - 49\sqrt{\frac{2}{5}} + \frac{16}{\sqrt{7}} - 2\sqrt{7} - 3\sqrt{10};$
2. $GA_e(G) = \left(9 + \frac{16\sqrt{7}}{11} + \frac{16\sqrt{70}}{17}\right)n + 1 + \frac{16\sqrt{7}}{11} - \frac{28\sqrt{70}}{17};$
3. $ABC_e(G) = \left(4\sqrt{\frac{6}{7}} + \frac{3}{5\sqrt{2}} + \frac{16\sqrt{3}}{7} + \frac{6}{\sqrt{7}}\right)n + 3\sqrt{\frac{3}{2}} - \frac{9}{5\sqrt{2}} - \frac{4\sqrt{3}}{7} + \frac{6}{\sqrt{7}} - \sqrt{42};$
4. $ABCI_e(G) = \sqrt{6} \left(\frac{5184n^4 - 20736n^3 + 7452n^2 + 26568n}{245} - \frac{972}{35} \right);$
5. ${}_eM_1(G) = 312n - 234;$
6. $ISI_e(G) = \frac{14235}{187}n - \frac{10746}{187}.$

Proof. The graph $L(CS_n)$ for $n \geq 4$ as shown in Fig. 2(b) and contain the number vertices and edges in $L(G)$ are $|V(L(G))| = 6n$ and $|E(L(G))| = 21n - 9$. It also consists of five partitions as given in Table 4. Apply Formulas (3), (5), (7) – (10) to the edge partition shown in Table 4 to get the required results. $\square$

Table 4: The edge partition of $L(CS_n)$ for $n = 4$

(d_e, d_f) where $ef \in E(L(G))$	Number of edges
(4, 4)	6
(4, 7)	$4n + 4$
(7, 7)	$8n - 2$
(7, 10)	$8n - 14$
(10, 10)	$n - 3$

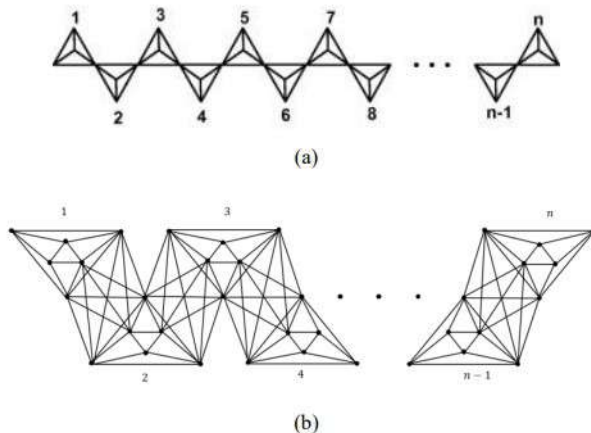


Figure 2: (a) A chain silicate network CS_n ; (b) the line graph $L(CS_n)$.

2.2 The edge version of topological indices for $L(HX_n)$

Lemma 2.2. *Let H be the line graph HX_n with dimension of $n = 2, 3$ and 4.*

1. For $n=2$. Then the edge version of topological indices is given by

- (a) ${}_eR_{-1/2}(H) = \frac{3}{2} + \frac{6}{\sqrt{7}} + \frac{15}{7}$;
- (b) $GA_e(H) = 27 + \frac{48\sqrt{7}}{11}$;
- (c) $ABC_e(H) = \frac{3\sqrt{6}}{2} + \frac{18\sqrt{7}+30\sqrt{3}}{7}$;
- (d) $ABCII_e(H) = \frac{2430\sqrt{14}}{49}$;
- (e) ${}_eM_1(H) = 390$;
- (f) $ISI_e(H) = \frac{2091}{22}$.

2. For $n=3$. Then the edge version of topological indices is given by

- (a) ${}_eR_{-1/2}(H) = \frac{36}{5} + \frac{15\sqrt{14}+63\sqrt{5}+42\sqrt{10}+12\sqrt{35}+9\sqrt{70}}{35}$;
- (b) $GA_e(H) = 57 + 2\sqrt{35} + 16\sqrt{5} + \frac{16\sqrt{14}}{5} + \frac{96\sqrt{10}}{13} + \frac{36\sqrt{70}}{17}$;
- (c) $ABC_e(H) = 12\sqrt{\frac{2}{7}} + 18\sqrt{\frac{3}{14}} + 35\sqrt{\frac{1}{5}} + \frac{3(343\sqrt{2}+28\sqrt{100}+10\sqrt{182}+35\sqrt{14})}{70}$;
- (d) $ABCII_e(H) = \frac{997691904\sqrt{858}}{875}$;
- (e) ${}_eM_1(H) = 2562$;
- (f) $ISI_e(H) = \frac{696994}{1105}$.

3. For $n=4$. Then the edge version of topological indices is given by

- (a) ${}_eR_{-1/2}(H) = 2\sqrt{3} + \frac{201}{10} + \frac{147\sqrt{5}+14\sqrt{30}+42\sqrt{10}+12\sqrt{35}+9\sqrt{70}+15\sqrt{14}}{35}$;

$$\begin{aligned}
(b) \quad GA_e(H) &= 189 + 2\sqrt{35} + \frac{96\sqrt{3}}{7} + \frac{16\sqrt{14}}{5} + \frac{112\sqrt{5}}{3} + \frac{24\sqrt{30}}{11} + \frac{96\sqrt{10}}{13} + \frac{36\sqrt{70}}{17}; \\
(c) \quad ABC_e(H) &= 12\sqrt{\frac{3}{10}} + 12\sqrt{\frac{2}{7}} + 18\sqrt{\frac{3}{14}} + 3\sqrt{14} \\
&\quad + 84\sqrt{\frac{1}{5}} + \frac{3(1169\sqrt{2}+28\sqrt{110}+10\sqrt{182})}{70} + 12; \\
(d) \quad ABCII_e(H) &= \frac{692216782848\sqrt{\frac{143}{5}}}{125}; \\
(e) \quad {}_eM_1(H) &= 6546; \\
(f) \quad ISI_e(H) &= \frac{414051719}{255255}.
\end{aligned}$$

Proof. The graph $L(HX_n)$ for $n = 2$ as shown in Fig. 3(b). It contains 12 vertices among which 6 vertices are of degree 4 and remaining all are of degree 7. Hence, we get the edge partition, based on the degree of an edge e and f as shown in Table 5. We apply Formulas (3), (5), (7) – (10) to Table 5 and obtain the required results. By similar arguments we can obtain the expression of the edge version of topological indices for $n = 3$ and 4 (see Fig. 4(b) and 5(b)) by viewing edge partition in Tables 6 and 7. $\square$



Figure 3: (a) A hexagonal network HX_2 ; (b) the line graph $L(HX_2)$.

Table 5: The edge partition of $L(HX_2)$

(d_e, d_f) where $ef \in E(L(G))$	Number of edges
(4, 4)	6
(4, 7)	12
(7, 7)	15

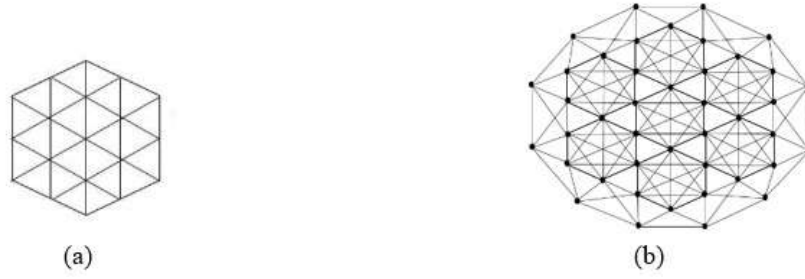


Figure 4: (a) A hexagonal network HX_3 ; (b) the line graph $L(HX_3)$.

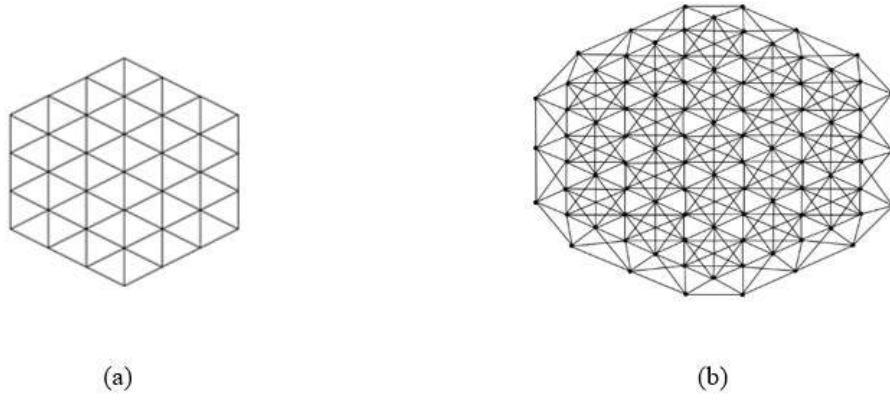


Figure 5: (a) A hexagonal network HX_4 ; (b) the line graph $L(HX_4)$.

Table 6: The edge partition of $L(HX_3)$

(d_e, d_f) where $ef \in E(L(G))$	Number of edges
(5, 5)	12
(5, 7)	12
(5, 8)	24
(7, 8)	12
(7, 10)	18
(8, 8)	12
(8, 10)	36
(10, 10)	33

Table 7: The edge partition of $L(HX_4)$

(d_e, d_f) where $ef \in E(L(G))$	Number of edges
(5, 5)	6
(5, 6)	12
(5, 7)	12
(5, 8)	24
(6, 8)	24
(7, 8)	12
(7, 10)	18
(8, 8)	24
(8, 10)	84
(10, 10)	159

Theorem 2.2. Let H be the line graph HX_n with dimension of $n \geq 5$, then

1. ${}^eR_{-1/2}(H) = \frac{9}{2}n^2 + \left(2\sqrt{3} + \frac{12}{\sqrt{5}} - \frac{82}{5}\right)n + \frac{137}{10} + 9\sqrt{\frac{2}{35}} + 3\sqrt{\frac{2}{7}} + 6\sqrt{\frac{2}{5}} + 2\sqrt{\frac{6}{5}} - 6\sqrt{3} - \frac{27}{\sqrt{5}} + \frac{12}{\sqrt{35}};$
2. $GA_e(H) = 45^2 + \left(\frac{96\sqrt{3}}{7} + \frac{64\sqrt{5}}{3} - 171\right)n + 153 - \frac{288\sqrt{3}}{7} - 48\sqrt{5} + \frac{96\sqrt{10}}{13} + \frac{16\sqrt{14}}{5} + \frac{24\sqrt{30}}{11} + 2\sqrt{35} + \frac{36\sqrt{70}}{17};$
3. $ABC_e(H) = \frac{27}{\sqrt{2}}n^2 + \left(12 - \frac{567}{5\sqrt{2}} + 3\sqrt{\frac{7}{2}} + \frac{48}{\sqrt{5}} + \sqrt{10}\right)n - 36 - 12\sqrt{\frac{2}{7}} + 9\sqrt{\frac{6}{7}} + 6\sqrt{\frac{6}{5}} + \frac{117}{\sqrt{2}} + \frac{12\sqrt{2}}{5} + 3\sqrt{\frac{26}{7}} + 6\sqrt{\frac{22}{5}} - \frac{108}{\sqrt{5}} - 4\sqrt{10} - 3\sqrt{14};$
4. $ABCI_e(H) = \frac{104976}{875}\sqrt{286}(6n-24)(12n-24)(24n-72)(48n-108)(45n^2 - 189n + 195);$
5. ${}^eM_1(H) = 900n^2 - 1500n - 1038;$
6. $ISI_e(H) = 225n^2 - \frac{12251}{21}n + \frac{30259113}{85085}.$

Proof. Let H be the line graph $L(HX_n)$ for $n \geq 5$, contains the number of vertices and edges in $L(H)$ are $|V(L(H))| = 9n^2 - 15n + 6$ and $|E(L(H))| = 45n^2 - 99n + 51$. It consists of eleven partitions as given in Table 8. Apply Formulas (3), (5), (7) – (10) to the edge partition shown in Table 8 to get the required results. $\square$

Table 8: The edge partition of $L(HX_n)$ for $n \geq 5$

(d_e, d_f) where $ef \in E(L(G))$	Number of edges
(5, 5)	6
(5, 6)	12
(5, 7)	12
(5, 8)	24
(6, 6)	$6n - 24$
(6, 8)	$24n - 72$
(7, 8)	12
(7, 10)	18
(8, 8)	$12n - 24$
(8, 10)	$48n - 108$
(10, 10)	$45n^2 - 189n + 195$

2.3 The edge version of topological indices for $L(OX_n)$

Theorem 2.3. Let K be the line graph HX_n with dimension of $n \geq 1$, then

1. ${}_eR_{-1/2}(K) = 9n^2 + (3\sqrt{6} - \frac{15}{2})n + \frac{5}{2} - \sqrt{6}$;
2. $GA_e(K) = 54n^2 + (\frac{72\sqrt{6}}{5} - 48)n + 12 - \frac{24\sqrt{6}}{5}$;
3. $ABC_e(K) = 9\sqrt{10}n^2 - (3\sqrt{\frac{3}{2}} + 12\sqrt{3} - 9\sqrt{10})n + 3\sqrt{\frac{3}{2}} - 4\sqrt{3} + \sqrt{10}$;
4. $ABCI_e(K) = \sqrt{5}(972n^4 - 324n^3 - 864n^2 + 396n - 36)$;
5. ${}_eM_1(K) = 648n^2 - 240n$;
6. $ISI_e(K) = 162n^2 - \frac{318}{5}n + \frac{6}{5}$.

Proof. Let K be the line graph $L(OX_n)$ for $n \geq 1$ and the graph K in Fig. 6(b) and 7(b). It contain the number of vertices and edges in $L(K)$ are $|V(L(K))| = 18n^2$ and $|E(L(K))| = 54n^2 - 12n$. Also, it consists of three partitions as given in Table 9. Apply Formulas (3), (5), (7) – (10) to the edge partition shown in Table 9 to get the required results. $\square$

Table 9: The edge partition of $L(OX_n)$ for $n \geq 1$

(d_e, d_f) where $ef \in E(L(G))$	Number of edges
(4, 4)	$6n + 6$
(4, 6)	$36n - 12$
(6, 6)	$54n^2 - 54n + 6$



Figure 6: (a) An oxide network OX_1 ; (b) the line graph $L(OX_1)$.



Figure 7: (a) An oxide network OX_2 ; (b) the line graph $L(OX_2)$.

3. Conclusion

In this paper, certain degree based topological indices, namely the edge version of Randić, GA, ABC, multiplicative ABC, Zagreb and inverse sum indices for the line graphs of chains silicate, hexagonal and oxide networks were studied. In future, we will pay attention to some new classes of the line graph and study their topological indices which will be reasonably useful to recognize their underlying topologies.

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Some results on uniqueness of certain types of difference polynomials

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Abstract. In this article we investigate the uniqueness of certain type of difference polynomials that share a small function and obtain some results which improve and extend some recent results of Sujoy Majumder ([12]).

Keywords: weighted sharing, meromorphic functions, difference-differential, uniqueness.

1. Introduction

Let $\mathbb{C}$ be a open complex plane and two functions f and g are non-constant and meromorphic in $\mathbb{C}$. In this article standard notations like $T(r, f)$, $m(r, f)$, $N(r, f)$ of Value Distribution Theory are used see ([1], [2], [3]). In this paper, we denote $\mathbb{C} \setminus \{0\}$ as $\mathbb{C}'$. For $a \in \mathbb{C} \cup \{\infty\}$, if f and g have the same point a with same multiplicities then f and g share a CM. If we do not take the multiplicities into account then f and g share the value a IM. A meromorphic

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function a is said to be a small function of f provided $T(r, a) = S(r, f)$ i.e., $T(r, a) = O\{T(r, f)\}$ as $r \rightarrow \infty, r \notin E$.

Definition 1.1 ([14]). Let $k \in \mathbb{N} \cup \{0\} \cup \{\infty\}$. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a -points of f where an a -point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k .

We write f, g share (a, k) to mean that f, g share the value a with weight k . Clearly, if f, g share (a, k) then f, g share (a, p) for any integer $p, 0 \leq p < k$. We note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

Definition 1.2 ([8]). Let $a \in \mathbb{C} \cup \{\infty\}$. For $p \in \mathbb{N}$ we denote by $N(r, a; f| \leq p)$ the counting function of those a -points of f (counted with multiplicities) whose multiplicities are not greater than p . By $\overline{N}(r, a; f| \leq p)$ we denote the corresponding reduced counting function.

Similarly we can define $N(r, a; f| \geq p)$ and $\overline{N}(r, a; f| \geq p)$.

Definition 1.3 ([14]). Let $p \in \mathbb{N} \cup \{\infty\}$. We denote by $N_p(r, a; f)$ the counting function of a -points of f , where an a -point of multiplicity m is counted m times if $m \leq p$ and p times if $m > p$. Then

$$N_p(r, a; f) = \overline{N}(r, a; f) + \overline{N}(r, a; f| \geq 2) + \dots + \overline{N}(r, a; f| \geq p).$$

Clearly $N_1(r, a; f) = \overline{N}(r, a; f)$.

In 2011, K. Liu, X. L Liu and T. B Cao [10] studied the uniqueness of the difference monomials and obtained the following results

Theorem 1 ([10]). Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions with finite order. Suppose that $c \in \mathbb{C}'$ and $n \in \mathbb{N}$. If $n \geq 14$, $f^n(z)f(z+c)$ and $g^n(z)g(z+c)$ share 1 CM, then $f(z) \equiv tg(z)$ or $f(z)g(z) \equiv t$, where $t^{n+1} = 1$.

Theorem 2 ([10]). Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions with finite order. Suppose that $c \in \mathbb{C}'$ and $n \in \mathbb{N}$. If $n \geq 26$, $f^n(z)f(z+c)$ and $g^n(z)g(z+c)$ share 1 IM, then $f(z) \equiv tg(z)$ or $f(z)g(z) \equiv t$, where $t^{n+1} = 1$.

In 2015, Y. Liu, J. P Wang and F. H Liu [13] improved Theorems 1 and 2 and obtained the following results.

Theorem 3 ([13]). Let $c \in \mathbb{C}'$ and let $f(z)$ and $g(z)$ be two transcendental meromorphic functions with finite order, and $n(\geq 14), k(\geq 3)$ be two positive integers. If $E_k(1, f^n(z)f(z+c)) = E_k(1, g^n(z)g(z+c))$, then $f(z) \equiv t_1g(z)$ or $f(z)g(z) \equiv t_2$ for some constants t_1 and t_2 satisfying $t_1^{n+1} = 1$ and $t_2^{n+1} = 1$.

Theorem 4 ([13]). *Let $c \in \mathbb{C}'$ and let $f(z)$ and $g(z)$ be two transcendental meromorphic functions with finite order, and $n(\geq 16)$ be a positive integers. If $E_2(1, f^n f(z+c)) = E_2(1, g^n g(z+c))$, then $f(z) \equiv t_1 g(z)$ or $f(z)g(z) \equiv t_2$ for some constants t_1 and t_2 satisfying $t_1^{n+1} = 1$ and $t_2^{n+1} = 1$.*

Theorem 5 ([13]). *Let $c \in \mathbb{C}'$ and let $f(z)$ and $g(z)$ be two transcendental meromorphic functions with finite order, and $n(\geq 22)$ be a positive integers. If $E_1(1, f^n f(z+c)) = E_1(1, g^n g(z+c))$, then $f(z) \equiv t_1 g(z)$ or $f(z)g(z) \equiv t_2$ for some constants t_1 and t_2 satisfying $t_1^{n+1} = 1$ and $t_2^{n+1} = 1$.*

In 2017, S. Majumder [12] proved following theorem by taking non-zero polynomial in place of sharing value 1 CM and came out with the following results.

Theorem 6 ([12]). *Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions of finite order, $c \in \mathbb{C}'$ and $n \in \mathbb{N}$ be such that $n \geq 14$. Let $p(z) (\neq 0)$ be a polynomial such that $\deg(p) < \frac{n-1}{2}$. Suppose $f^n(z)f(z+c) - p(z)$ and $g^n(z)g(z+c) - p(z)$ share $(0, 2)$, then one of the following two cases holds:*

- (i) $f(z) \equiv tg(z)$ for some constant t such that $t^{n+1} = 1$;
- (ii) $f(z)g(z) \equiv t$, where $p(z)$ reduces to a non-zero constant c and t is a constant such that $t^{n+1} = c^2$.

Theorem 7 ([12]). *Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions of finite order, $c \in \mathbb{C}'$ and $n \in \mathbb{N}$ be such that $n \geq 16$. Let $p(z) (\neq 0)$ be a polynomial such that $\deg(p) < \frac{n-1}{2}$. Suppose $f^n(z)f(z+c) - p(z)$ and $g^n(z)g(z+c) - p(z)$ share $(0, 1)$. Then conclusion of Theorem 6 holds.*

Theorem 8 ([12]). *Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions of finite order, $c \in \mathbb{C}'$ and $n \in \mathbb{N}$ be such that $n \geq 26$. Let $p(z) (\neq 0)$ be a polynomial such that $\deg(p) < \frac{n-1}{2}$. Suppose $f^n(z)f(z+c) - p(z)$ and $g^n(z)g(z+c) - p(z)$ share $(0, 0)$. Then conclusion of Theorem 6 holds.*

Regarding Theorems 6-8, one may ask the following question.

Question 1. What happens if $f^n(z)f(z+c)$ is replaced by $f^n(z)P(f) \prod_{j=1}^d f(z+c_j)^{v_j}$ in Theorems 6-8?

To answer the above question affirmatively, we prove the following results which is the main results of this article.

Theorem 9. *Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions of finite order, $c \in \mathbb{C}'$ and $n \in \mathbb{N}$ be such that $n > 7 + 6m + 7\sigma$. Let $p(z) (\neq 0)$ be a polynomial such that $\deg(p) < \frac{n+m-\sigma}{2}$. Suppose $f^n(z)P(f) \prod_{j=1}^d f(z+c_j)^{v_j} - p(z)$ and $g^n(z)P(g) \prod_{j=1}^d g(z+c_j)^{v_j} - p(z)$ share $(0, 2)$, then one of the following two cases holds:*

- (i) $f(z) \equiv tg(z)$ for some constant t such that $t^{n+m+\sigma} = 1$;
- (ii) $f(z)g(z) \equiv t$, where $p(z)$ reduces to a non-zero constant c and t is a constant such that $t^{n+m+\sigma} = c^2$.

Theorem 10. Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions of finite order, $c \in \mathbb{C}'$ and $n \in \mathbb{N}$ be such that $n > 8 + 6m + 8\sigma$. Let $p(z) (\neq 0)$ be a polynomial such that $\deg(p) < \frac{n+m-\sigma}{2}$. Suppose $f^n(z)P(f) \prod_{j=1}^d f(z + c_j)^{v_j} - p(z)$ and $g^n(z)P(g) \prod_{j=1}^d g(z + c_j)^{v_j} - p(z)$ share $(0, 1)$. Then conclusion of Theorem 9 holds.

Theorem 11. Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions of finite order, $c \in \mathbb{C}'$ and $n \in \mathbb{N}$ be such that $n > 13 + 6m + 13\sigma$. Let $p(z) (\neq 0)$ be a polynomial such that $\deg(p) < \frac{n+m-\sigma}{2}$. Suppose $f^n(z)P(f) \prod_{j=1}^d f(z + c_j)^{v_j} - p(z)$ and $g^n(z)P(g) \prod_{j=1}^d g(z + c_j)^{v_j} - p(z)$ share $(0, 0)$. Then conclusion of Theorem 9 holds.

Remark 1. If $m = 0$ and $\sigma = 1$ in Theorems 9-11, then Theorems 9-11 reduces to Theorems 6-8.

2. Some lemmas

In this segment, we present few Lemmas which will be used further.

Let F, G be two non-constant meromorphic functions. From now onwards let us use H to denote the following functions.

$$(1) \quad H = \left(\frac{F''}{F'} - \frac{2F'}{F-1} \right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1} \right).$$

Lemma 2.1 ([11]). Suppose $f(z)$ is a meromorphic function in the complex plane $\mathbb{C}$ and the polynomial is defined by $P(z) = a_m f^m + a_{m-1} f^{m-1} + \dots + a_1 f + a_0$, where $a_m (\neq 0)$, $a_0, a_1, \dots, a_{m-1}$ are small functions of $f(z)$. Then

$$T(r, P(f)) = mT(r, f) + S(r, f).$$

Lemma 2.2 ([5]). Let $f(z)$ be a meromorphic function of finite order ρ , and let $c \in \mathbb{C}'$ be fixed. Then for each $\varepsilon > 0$, we have

$$m \left(r, \frac{f(z+c)}{f(z)} \right) + m \left(r, \frac{f(z)}{f(z+c)} \right) = O(r^{\rho-1+\varepsilon}).$$

Lemma 2.3 ([5]). Let $f(z)$ be a transcendental meromorphic function of finite order, $c \in \mathbb{C}'$ be fixed. Then

$$T(r, f(z+c)) = T(r, f) + S(r, f).$$

Lemma 2.4 ([6]). Let $f(z)$ be non-constant meromorphic function of finite order and $c \in \mathbb{C}$. Then:

$$\begin{aligned} N(r, 0; f(z+c)) &\leq N(r, 0; f(z)) + S(r, f), \\ N(r, \infty; f(z+c)) &\leq N(r, \infty; f(z)) + S(r, f), \\ \overline{N}(r, 0; f(z+c)) &\leq \overline{N}(r, 0; f(z)) + S(r, f), \\ \overline{N}(r, \infty; f(z+c)) &\leq \overline{N}(r, \infty; f(z)) + S(r, f). \end{aligned}$$

Lemma 2.5 ([16]). *Let $f(z)$ be a transcendental meromorphic function of hyper order, $\rho_2 < 1$ and $F(z) = f^n(z)P(f) \prod_{j=1}^d f(z + c_j)^{v_j}$, where $m, n \in \mathbb{N}$. Then we have,*

$$(n + m - \sigma)T(r, f) \leq T(r, F) + S(r, f) \leq (n + m + \sigma)T(r, f).$$

Lemma 2.6. *Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions of finite order, $c \in \mathbb{C}'$ and $n \in \mathbb{N}$ such that $n \geq 2$. Let $p(z)$ be a non-zero polynomial such that $\deg(p) < \frac{n+m-\sigma}{2}$. Then:*

- (i) *if $\deg(p) \geq 1$, then $f^n(z)P(f) \prod_{j=1}^d f(z + c_j)^{v_j} g^n(z)P(g) \prod_{j=1}^d g(z + c_j)^{v_j} \not\equiv p^2(z)$;*
- (ii) *if $p(z) = c \in \mathbb{C}'$, then the relation*

$$f^n(z)P(f) \prod_{j=1}^d f(z + c_j)^{v_j} g^n(z)P(g) \prod_{j=1}^d g(z + c_j)^{v_j} \equiv p^2(z)$$

always implies that $fg = t$ where t is a constant such that $t^{n+m+\sigma} = c^2$.

Proof. Suppose

$$(2) \quad f^n(z)P(f) \prod_{j=1}^d f(z + c_j)^{v_j} g^n(z)P(g) \prod_{j=1}^d g(z + c_j)^{v_j} \equiv p^2(z).$$

Let $h = fg$ then from equation (2) becomes,

$$(3) \quad \begin{aligned} & h^{n+m} \left(a_m \frac{1}{g^m} + \dots + \frac{a_0}{h^m} \right) (a_m g^m + a_{m-1} g^{m-1} + \dots + a_0) \\ &= \frac{p^2(z)}{\prod_{j=1}^d h(z + c_j)^{v_j}}. \end{aligned}$$

Let us discuss following two cases.

Case 1. Suppose h is a transcendental meromorphic function. Now, by Lemmas 2.1, 2.2 and 2.4, we get

$$\begin{aligned} (n + m)T(r, h) &= T(r, h^{n+m}) + S(r, h) \\ &= T\left(r, \frac{p^2(z)}{\prod_{j=1}^d h(z + c_j)^{v_j}}\right) + S(r, h) \\ &\leq \sigma T(r, h) + S(r, h) \end{aligned}$$

which is a contradiction.

Case 2. Suppose h is rational function. Let

$$(4) \quad h = \frac{h_1}{h_2}.$$

where h_1 and h_2 are two non-zero relatively prime polynomials. By equation (4), we have

$$(5) \quad T(r, h) = \max\{\deg(h_1), \deg(h_2)\} \log r + O(1).$$

Now, by equations (3) - (5), we have

$$(6) \quad \begin{aligned} & (n+m) \max\{\deg(h_1), \deg(h_2)\} \log r \\ & = T(r, h^{n+m}) + O(1) \\ & \leq T(r, h^\sigma) + T(r, p^2(z)) + S(r, f) \\ & \leq \sigma T(r, h) + 2T(r, p) + S(r, f) \\ & = \max\{\deg(h_1), \deg(h_2)\} \log r + 2\deg(p) \\ & \quad + \log r + O(1). \end{aligned}$$

This implies,

$$(n+m-\sigma)T(r, h) \leq 2T(r, p) + S(r, f).$$

Hence,

$$(n+m-\sigma) \leq 2\deg(p).$$

We see that $\max\{\deg(h_1), \deg(h_2)\} \geq 1$. Now by see (6), we deduce that $(n+m-\sigma)/2 \leq \deg(p)$, which contradicts our assumption that $\deg(p) < \frac{n+m-\sigma}{2}$. Hence h must be a non-zero constant. Let

$$(7) \quad h = t \in \mathbb{C}'.$$

Now, when $\deg(p) \geq 1$. By see (3) and (7) we arrive at a contradiction. Therefore $f^n(z)P(f) \prod_{j=1}^d f(z+c_j)^{v_j} g^n(z)P(g) \prod_{j=1}^d g(z+c_j)^{v_j} \not\equiv p^2(z)$. Suppose $p(z) = c \in \mathbb{C}'$. So by 3 we see that $h^{n+m+\sigma} \equiv c^2$. By (7) we get $t^{n+m+\sigma} \equiv c^2$. This completes the proof. $\square$

Lemma 2.7 ([14]). *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions sharing (1,2). Then one of the following holds:*

- (i) $T(r, f) \leq N_2(r, 0; f) + N_2(r, 0; g) + N_2(r, \infty; f) + N_2(r, \infty; g) + S(r, f) + S(r, g);$
- (ii) $fg \equiv 1;$
- (iii) $f \equiv g.$

Lemma 2.8 ([4]). *Let F and G be two non-constant meromorphic functions sharing (1,1) and $H \not\equiv 0$. Then*

$$\begin{aligned} T(r, F) & \leq N_2(r, 0; F) + N_2(r, 0; G) + N_2(r, \infty; F) + N_2(r, \infty; G) + \frac{1}{2}\overline{N}(r, 0; F) \\ & \quad + \frac{1}{2}\overline{N}(r, \infty; F) + S(r, F) + S(r, G). \end{aligned}$$

Lemma 2.9 ([4]). *Let F and G be two non-constant meromorphic functions sharing $(1, 0)$ and $H \not\equiv 0$. Then*

$$T(r, F) \leq N_2(r, 0; F) + N_2(r, 0; G) + N_2(r, \infty; F) + N_2(r, \infty; G) + 2\overline{N}(r, 0; F) \\ + \overline{N}(r, 0; G) + 2\overline{N}(r, \infty; F) + \overline{N}(r, \infty; G) + S(r, F) + S(r, G).$$

Lemma 2.10 ([15]). *Let H be defined as in (1). If $H \equiv 0$ and*

$$\lim_{r \rightarrow \infty} \frac{\overline{N}(r, 0; F) + \overline{N}(r, 0; G) + \overline{N}(r, \infty; F) + \overline{N}(r, \infty; G)}{T(r)} < 1, \quad r \in I,$$

where I is a set of infinite linear measure, then $F \equiv G$ or $F.G \equiv 1$.

3. Proof of main results

Proof of Theorem 9.

Proof. Let $F(z) = \frac{f^n(z)P(f)\prod_{j=1}^d f(z+c_j)^{v_j}}{p(z)}$ and $G(z) = \frac{g^n(z)P(g)\prod_{j=1}^d g(z+c_j)^{v_j}}{p(z)}$. Then, F and G share $(1, 2)$ except for the zeros of $p(z)$. Now by Lemma 2.7, we see that one of the following three cases holds:

Case 1. Suppose

$$T(r, F) \leq N_2(r, 0; F) + N_2(r, 0; G) + N_2(r, \infty; F) + N_2(r, \infty; G) + S(r, F) + S(r, G).$$

Now, by applying Lemmas 2.1 and 2.4, we have

$$\overline{N}(r, 0; F) \leq (1 + m + \sigma)T(r, f) + S(r, f).$$

Similarly,

$$\begin{aligned} \overline{N}(r, 0; G) &\leq (1 + m + \sigma)T(r, g) + S(r, f), \\ \overline{N}(r, \infty; F) &\leq (1 + m + \sigma)T(r, f) + S(r, f), \\ \overline{N}(r, \infty; G) &\leq (1 + m + \sigma)T(r, g) + S(r, f). \end{aligned}$$

Therefore,

$$\begin{aligned} T(r, F) &\leq N_2(r, 0; F) + N_2(r, 0; G) + N_2(r, \infty; F) \\ &\quad + N_2(r, \infty; G) + S(r, F) + S(r, G) \\ &\leq 2\overline{N}(r, 0; F) + 2\overline{N}(r, 0; G) \\ &\quad + \overline{N}(r, \infty; F) + \overline{N}(r, \infty; G) + S(r, f) + S(r, g) \\ &\leq 2(1 + m + \sigma)T(r, f) + 2(1 + m + \sigma)T(r, g) + (1 + m + \sigma)T(r, f) \\ &\quad + (1 + m + \sigma)T(r, g) + S(r, f) + S(r, g) \\ &\leq 3(1 + m + \sigma)\{T(r, f) + T(r, g)\} + S(r, f) + S(r, g). \end{aligned}$$

By Lemma 2.5,

$$(8) \quad (n + m - \sigma)T(r, f) \leq 3(1 + m + \sigma)\{T(r, f) + T(r, g)\} + S(r, f) + S(r, g).$$

Similarly,

$$(9) \quad (n + m - \sigma)T(r, g) \leq 3(1 + m + \sigma)\{T(r, f) + T(r, g)\} + S(r, f) + S(r, g).$$

Combining equation (8), (9)

$$(n + m - \sigma)T(r) \leq 6(1 + m + \sigma)T(r) + S(r).$$

where $T(r) = \max\{T(r, f), T(r, g)\}$ and $S(r)$ denotes any quantity satisfying $S(r) = o(T(r))$ as $r \rightarrow \infty$, outside of a possible exceptional set of finite logarithmic measure. Therefore $n \leq 6 + 5m + 7\sigma$ which contradicts with $n > 7 + 5m + 7\sigma$.

Case 2. $F \equiv G$. Then we have

$$(10) \quad f^n(z)P(f) \prod_{j=1}^d f(z + c_j)^{v_j} \equiv g^n(z)P(g) \prod_{j=1}^d g(z + c_j)^{v_j}$$

Let $h = \frac{f}{g}$. Then $f = gh$, with these equation (10) becomes,

$$(gh)^n(z)P(gh) \prod_{j=1}^d (gh)(z + c_j)^{v_j} \equiv g^n(z)P(g) \prod_{j=1}^d g(z + c_j)^{v_j}.$$

$$h^n(z)[a_m g^m h^m + \dots + a_0] \prod_{j=1}^d h(z + c_j)^{v_j} \equiv P(g).$$

$$[a_m g^m h^{n+m} + \dots + a_0 h^n] \equiv \frac{P(g)}{\prod_{j=1}^d h(z + c_j)^{v_j}}.$$

$$\begin{aligned} (n + m)T(r, h) &\leq T(r, h^{n+m}) + S(r, h) \\ &\leq N \left(r, \frac{1}{\prod_{j=1}^d h(z + c_j)^{v_j}} \right) + m \left(r, \frac{1}{\prod_{j=1}^d h(z + c_j)^{v_j}} \right) \\ &\leq T \left(r, \frac{1}{\prod_{j=1}^d h(z + c_j)^{v_j}} \right) + S(r, h) \\ &\leq T(r, h^\sigma) + S(r, h) \\ &\leq \sigma T(r, h) + S(r, h). \end{aligned}$$

Since $n \geq 2$, we see that h is a constant. Then, we have $h^{n+m+\sigma} = 1$. Therefore, $f(z) = tg(z)$, where $t^{n+m+\sigma} = 1$.

Case 3. $F.G \equiv 1$. Then we have $f^n(z)P(f) \prod_{j=1}^d f(z + c_j)^{v_j} g^n(z)P(g) \prod_{j=1}^d g(z + c_j)^{v_j} \equiv p^2(z)$. Hence Theorem 9 follows by Lemma 2.6. This completes the proof. $\square$

Proof of Theorem 10. Let

$$F = \frac{f^n(z)P(f) \prod_{j=1}^d f(z + c_j)^{v_j}}{p(z)}$$

and

$$G = \frac{g^n(z)P(g) \prod_{j=1}^d g(z + c_j)^{v_j}}{p(z)}.$$

Then, F and G share $(1, 1)$ except for the zeros of $p(z)$. We now consider the following two cases.

Case 1. $H \not\equiv 0$. By Lemma 2.3, we have

$$\overline{N}(r, 0; F) \leq (1 + m + \sigma)T(r, f) + S(r, f)$$

Similarly,

$$\overline{N}(r, 0; G) \leq (1 + m + \sigma)T(r, g) + S(r, f)$$

$$\overline{N}(r, \infty; F) \leq (1 + m + \sigma)T(r, f) + S(r, f)$$

$$\overline{N}(r, \infty; G) \leq (1 + m + \sigma)T(r, g) + S(r, f).$$

Now by applying Lemmas 2.1, 2.4 and 2.8 we have,

$$\begin{aligned} T(r, F) &\leq N_2(r, 0; F) + N_2(r, 0; G) + N_2(r, \infty; F) + N_2(r, \infty; G) + \frac{1}{2}\overline{N}(r, 0; F) \\ &\quad + \frac{1}{2}\overline{N}(r, \infty; F) + S(r, f) + S(r, g) \\ &\leq 2\overline{N}(r, 0; F) + 2\overline{N}(r, 0; G) + \overline{N}(r, \infty; F) + \overline{N}(r, \infty; G) + \frac{1}{2}\overline{N}(r, 0; F) \\ &\quad + \frac{1}{2}\overline{N}(r, \infty; F) + S(r, f) + S(r, g) \\ &\leq \{2(1 + m + \sigma) + (1 + m + \sigma) + \frac{1}{2}(1 + m + \sigma) + \frac{1}{2}(1 + m + \sigma)\}T(r, f) \\ &\quad + \{2(1 + m + \sigma) + (1 + m + \sigma)\}T(r, g) + S(r, f) + S(r, g) \\ &\leq 4(1 + m + \sigma)T(r, f) + 3(1 + m + \sigma)T(r, g) + S(r, f) + S(r, g). \end{aligned}$$

By Lemma 2.5, we have

$$(11) \quad \begin{aligned} (n + m - \sigma)T(r, f) &\leq 4(1 + m + \sigma)T(r, f) + 3(1 + m + \sigma)T(r, g) \\ &\quad + S(r, f) + S(r, g). \end{aligned}$$

Similarly,

$$(12) \quad \begin{aligned} (n + m - \sigma)T(r, g) &\leq 4(1 + m + \sigma)T(r, g) + 3(1 + m + \sigma)T(r, f) \\ &\quad + S(r, f) + S(r, g). \end{aligned}$$

Adding inequalities (11) and (12) we get,

$$(n + m - \sigma)T(r) \leq 7(1 + m + \sigma)T(r) + S(r).$$

where $T(r) = \max\{T(r, F), T(r, G)\}$, $S(r) = \max\{S(r, f), S(r, g)\}$. Which implies that $n \leq 7 + 6m + 8\sigma$ which contradicts with $n > 8 + 6m + 8\sigma$.

Case 2. $H \equiv 0$. In view of Lemmas 2.4 and 2.5 we get

$$\begin{aligned} \overline{N}(r, 0; F) + \overline{N}(r, 0; G) + \overline{N}(r, \infty; F) + \overline{N}(r, \infty; G) \\ \leq 2(1 + m + \sigma)T(r, f) + 2(1 + m + \sigma)T(r, g) + S(r, f) + S(r, g) \\ \leq \frac{2(1 + m + \sigma)}{n + m - \sigma}T(r, F) + \frac{2(1 + m + \sigma)}{n + m - \sigma}T(r, G) + S(r, f) + S(r, g) \\ \leq \frac{4(1 + m + \sigma)}{n + m - \sigma}T(r) + S(r), \end{aligned}$$

where $T(r) = \max\{T(r, F), T(r, G)\}$, $S(r) = \max\{S(r, f), S(r, g)\}$. Since $n > 8 + 6m + 8\sigma$, we have

$$\lim_{r \rightarrow \infty} \frac{\overline{N}(r, 0; F) + \overline{N}(r, 0; G) + \overline{N}(r, \infty; F) + \overline{N}(r, \infty; G)}{T(r)} < 1$$

and so by Lemma 2.10, we have either $F \equiv G$ or $F.G \equiv 1$. Hence, Theorem 10 follows by the proof of Theorem 9. This completes the proof.

Proof of Theorem 11. Let

$$F(z) = \frac{f^n(z)P(f) \prod_{j=1}^d f(z + c_j)^{v_j}}{p(z)}$$

and

$$G(z) = \frac{g^n(z)P(g) \prod_{j=1}^d g(z + c_j)^{v_j}}{p(z)}.$$

Then, F and G share $(1, 0)$ except for the zeros of $p(z)$. We now consider the following two cases.

Case 1. $H \not\equiv 0$. By Lemma 2.3, we have

$$\overline{N}(r, 0; F) \leq (1 + m + \sigma)T(r, f) + S(r, f).$$

Similarly,

$$\overline{N}(r, 0; G) \leq (1 + m + \sigma)T(r, g) + S(r, f),$$

$$\overline{N}(r, \infty; F) \leq (1 + m + \sigma)T(r, f) + S(r, f),$$

$$\overline{N}(r, \infty; G) \leq (1 + m + \sigma)T(r, g) + S(r, f).$$

Now, by Lemmas 2.1, 2.4 and 2.9 we have

$$\begin{aligned}
 T(r, F) &\leq N_2(r, 0; F) + N_2(r, 0; G) + N_2(r, \infty; F) + N_2(r, \infty; G) + 2\overline{N}(r, 0; F) \\
 &\quad + \overline{N}(r, 0; G) + 2\overline{N}(r, \infty; F) + \overline{N}(r, \infty; G) + S(r, f) + S(r, g) \\
 &\leq 2\overline{N}(r, 0; F) + 2\overline{N}(r, 0; G) + \overline{N}(r, \infty; F) + \overline{N}(r, \infty; G) + 2\overline{N}(r, 0; F) \\
 &\quad + \overline{N}(r, 0; G) + 2\overline{N}(r, \infty; F) + \overline{N}(r, \infty; G) + S(r, f) + S(r, g) \\
 &\leq \{2(1+m+\sigma) + (1+m+\sigma) + 2(1+m+\sigma)\}T(r, f) \\
 &\quad + \{2(1+m+\sigma) + (1+m+\sigma) + (1+m+\sigma) + (1+m+\sigma)\}T(r, g) \\
 &\leq 7(1+m+\sigma)T(r, f) + 5(1+m+\sigma)T(r, g) + S(r, f) + S(r, g).
 \end{aligned}$$

By Lemma 2.5, we have

$$\begin{aligned}
 (n+m-\sigma)T(r, f) &\leq 7(1+m+\sigma)T(r, f) + 5(1+m+\sigma)T(r, g) \\
 (13) \qquad \qquad \qquad &\quad + S(r, f) + S(r, g).
 \end{aligned}$$

Similarly

$$\begin{aligned}
 (n+m-\sigma)T(r, g) &\leq 7(1+m+\sigma)T(r, g) + 5(1+m+\sigma)T(r, f) \\
 (14) \qquad \qquad \qquad &\quad + S(r, f) + S(r, g).
 \end{aligned}$$

Adding inequalities (13) and (14) we get

$$\begin{aligned}
 (n+m-\sigma)\{T(r, g) + T(r, f)\} &\leq 12(1+m+\sigma)\{T(r, g) + T(r, f)\} \\
 &\quad + S(r, f) + S(r, g).
 \end{aligned}$$

i.e., $n \leq 12 + 11m + 13\sigma$, which contradicts with $n > 13 + 11m + 13\sigma$.

Case 3. $H \equiv 0$. Hence Theorem 11 follows from the proof of Theorems 9 and 10. This completes the proof.

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Solving oscillation problems using optimized integrator method

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Abstract. In this paper, an explicit optimized integrator method of order four is developed for solving second order ordinary differential equations with oscillatory solutions. The new optimized integrator method (NOIM) depend on the existing hybrid methods with dissipative of order infinity. The constant coefficients of new method is found after using the phase-lag, the amplification error (dissipative error) and the first derivative of the phase-lag. Numerical results are presented to illustrate the robustness and competency of the proposed integrator method compared with the existing methods in the scientific literature for solving oscillatory problems.

Keywords: oscillatory problem, explicit hybrid method, dissipative error, phase-lag, numerical solution.

1. Introduction

This paper devotes to the initial value problems (IVPs) of the form

$$(1) \quad z''(t) = g(t, z), \quad z(t_0) = z_0, \quad z'(t_0) = z'_0.$$

Where the solution of (1) shows an obvious oscillatory behaviour. This problems arise in a several of sciences fields like theoretical physics, quantum mechanics, electronics, celestial mechanics (see [1, 2]). Many numerical methods are derived to approximate the solutions for (1) such as Runge-Kutta method [3] after reducing the problem to an equivalent system of the first order (ODEs) and then solved it. But several authors are proposed direct integration method, for instance Runge-Kutta Nyström method and multistep methods for second order ODEs to obtain the best approximation for the solution of second order ODEs (1), we can find such work in Simos ([4, 5, 6]), Ming et al. [7], and Tsitouras [8]. Franco [10] proposed a class up to six order hybrid methods using the order conditions constructed by Coleman [9]. Searching for the dispersion and dissipation errors of numerical methods for solving oscillatory or periodic problems is extremely significant. Several authors have proposed hybrid methods with the objective to make the dispersion of the method smaller (see [11, 12, 13, 14]) In

this paper, we construct a new optimized integrator method based on existing fourth order hybrid method which is dispersive of order six and zero dissipative proposed by Franco [10] and using the technique suggested by Kosti et al. [15]. The paper is organized as follows: In section 2, we present the dispersion and dissipation analysis of the integrator method. In section 3 we describe the derivation of the method. In section 4, we give numerical results that prove the efficiency of our new optimized integrator method by the well known oscillating problems. In Section 5 we observe some conclusions.

2. Phase-Lag analysis of NOIM method

The hybrid method for solving second order ODEs (1) can be written in following form

$$(2) \quad z_{n+1} = 2z_n - z_{n-1} + h^2 \sum_{i=1}^s b_i g(t_n + c_i h),$$

$$(3) \quad z_i = (1 + c_i)z_n - c_i z_{n-1} + h^2 \sum_{j=1}^s a_{ij} g(t_n + c_j h). \quad i = 1, \dots, s.$$

where the parameters b_i, a_{ij} , and c_i of the hybrid methods are expressed using Butcher notation as follows we can defined the methods of the form (2)–(3) as

$$\begin{array}{c|c} \mathbf{c} & \mathbf{A} \\ \hline & \mathbf{b}^T \end{array} = \begin{array}{c|cc} c_1 & a_{11} & \dots & a_{1s} \\ \vdots & \vdots & \ddots & \vdots \\ c_s & a_{s1} & \dots & a_{ss} \\ \hline & b_1 & \dots & b_s \end{array}$$

follows

$$(4) \quad Z_1 = z_{n-1}, \quad Z_2 = z_n,$$

$$(5) \quad z_{n+1} = 2z_n - z_{n-1} + h^2 \left[b_1 g_{n-1} + b_2 g_n + \sum_{i=3}^s b_i g(t_n + c_i h, Z_i) \right],$$

$$(6) \quad Z_i = (1 + c_i)z_n - c_i z_{n-1} + h^2 \sum_{j=3}^{i-1} a_{ij} g(t_n + c_j h, Z_j). \quad i = 3, \dots, s.$$

where $g_n = g(t_n, z_n)$ and $g_{n-1} = g(t_{n-1}, z_{n-1})$ and the two first nodes are $c_1 = -1$ and $c_2 = 0$.

The method need only the evaluation of $g(t_n, z_n), g(t_n + c_3 h, Z_3), \dots, g(t_n + c_s h, Z_s)$ for every step after the proceeding start. Thus, hybrid method is considered with $s-1$ stages for each step which can express in the following Butcher notation The construction of NOIM method is based on the using simple test equation

$$(7) \quad z'' = -\omega^2 z, \quad \omega > 0$$

$$\begin{array}{c|c} \mathbf{c} & \mathbf{A} \\ \hline & \mathbf{b}^T \end{array} = \begin{array}{c|cccccc} -1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ c_3 & a_{31} & a_{32} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ c_s & a_{s1} & a_{s2} & \dots & a_{s,s-1} & 0 \\ \hline & b_1 & b_2 & \dots & b_{s-1} & b_s \end{array}$$

Applying the two step hybrid method (2)–(3) to the test equation (7) we obtain

$$(8) \quad z_{n+1} = 2z_n - z_{n-1} - h^2 \sum_{i=1}^s b_i \omega^2 z_i,$$

$$(9) \quad z_i = (1 + c_i)z_n - c_i z_{n-1} - h^2 \sum_{j=1}^s a_{ij} \omega^2 z_j, \quad i = 1, \dots, s.$$

The hybrid method (8)–(9) can be expressed in the following vector form

$$(10) \quad z_{n+1} = 2z_n - z_{n-1} - v^2 \mathbf{b}^T \mathbf{Z}, \quad v = h\omega,$$

$$(11) \quad \mathbf{Z} = (\mathbf{c} + \mathbf{e})z_n - \mathbf{c} z_{n-1} + v^2 \mathbf{A} \mathbf{Z},$$

where $\mathbf{e} = (1, \dots, 1)^T$, $\mathbf{c} = (c_1, \dots, c_s)^T$, $\mathbf{Z} = (Z_1, \dots, Z_s)^T$, $\mathbf{b} = (b_1, \dots, b_s)$ and

$$\mathbf{A} = \begin{bmatrix} a_{11} & \dots & a_{1s} \\ \vdots & \ddots & \vdots \\ a_{s1} & \dots & a_{ss} \end{bmatrix}.$$

Solving equation (11) we get

$$(12) \quad \mathbf{Z} = (\mathbf{1} + v^2 \mathbf{A})^{-1} (\mathbf{c} + \mathbf{e})z_n - (\mathbf{1} + v^2 \mathbf{A})^{-1} \mathbf{c} z_{n-1}.$$

Substituting equation (12) into equation (10) we get the following difference equation

$$(13) \quad z_{n+1} - P(v^2) z_n + D(v^2) z_{n-1} = 0,$$

where

$$(14) \quad P(v^2) = 2 - v^2 \mathbf{b}^T (\mathbf{1} + v^2 \mathbf{A})^{-1} (\mathbf{c} + \mathbf{e}), \quad D(v^2) = 1 - v^2 \mathbf{b}^T (\mathbf{1} + v^2 \mathbf{A})^{-1} \mathbf{c}.$$

Definition. The dispersion (or phase-lag error) and the dissipation (or amplification error) are defined respectively as follows (see [10, 15])

$$(15) \quad \alpha(v) = 1 - \sqrt{D(v^2)}, \quad \Psi(v) = v - \cos^{-1} \left(\frac{P(v^2)}{2\sqrt{D(v^2)}} \right),$$

If $\alpha(v) = O(v^{r+1})$ and $\Psi(v) = O(v^{p+1})$ therefore the hybrid method is said to have dispersion of order p and dissipative of order r respectively.

3. Derivation of the NOIM method

New optimized integrator method (NOIM) of order four with four stage will be derived based on existing hybrid method proposed by Franco [10] (see Table 1). Alternatively (14) can be written in polynomial form as follows:

$$(16) \quad P(v^2) = 2 - \beta_1 v^2 + \beta_2 v^4 - \beta_3 v^6 + \beta_4 v^8 + \dots + \beta_n v^{2i},$$

$$(17) \quad D(v^2) = 1 - \gamma_1 v^2 + \gamma_2 v^4 - \gamma_3 v^6 + \gamma_4 v^8 + \dots + \gamma_n v^{2i}.$$

To construct the (NOIM) method of order four with four stage, the equations (16)-(17) should be satisfied the order conditions given in [9] up to fourth order. Consequently, we obtain

$$(18) \quad \begin{aligned} P(v^2) = & 2 + \left(-b_1 - b_2 - b_3 - b_4 - b_1 c_1 - b_2 c_2 - b_3 c_3 - b_4 c_4 \right) v^2 + \left(b_3 a_{31} + \right. \\ & b_3 a_{32} + b_4 a_{41} + b_4 a_{42} + b_4 a_{43} + b_3 a_{31} c_1 + b_3 a_{32} c_2 + \\ & \left. b_4 a_{41} c_1 + b_4 a_{42} c_2 + b_4 a_{43} c_3 \right) v^4 + \\ & \left(-b_4 a_{43} a_{31} - b_4 a_{43} a_{32} - b_4 a_{43} a_{31} c_1 - b_4 a_{43} a_{32} c_2 \right) v^6, \end{aligned}$$

$$(19) \quad \begin{aligned} D(v^2) = & 1 + \left(-b_1 c_1 - b_2 c_2 - b_3 c_3 - b_4 c_4 \right) v^2 + \left(b_3 a_{31} c_1 + b_3 a_{32} c_2 + \right. \\ & \left. b_4 a_{41} c_1 + b_4 a_{42} c_2 + b_4 a_{43} c_3 \right) v^4 + \left(-b_4 a_{43} a_{31} c_1 - b_4 a_{43} a_{32} c_2 \right) v^6. \end{aligned}$$

Table 1: Franco's hybrid method

-1	0	0	0	0
0	0	0	0	0
$\frac{33}{50}$	0	$\frac{2739}{5000}$	0	0
$-\frac{13}{17}$	$\frac{314860}{20796729}$	$-\frac{1058746}{8268579}$	$\frac{15743000}{686292057}$	0
	$-\frac{89}{1992}$	$\frac{545}{858}$	$\frac{625000}{3316929}$	$\frac{83521}{377832}$

To achieve our goal which is to optimize the Franco's hybrid method given in Table 1, The following conditions should be met

$$(20) \quad 1 - \eta(v) = v - \cos^{-1} \left(\frac{P(v^2)}{2\sqrt{D(v^2)}} \right) = 0,$$

$$(21) \quad 2 - \alpha(v) = 1 - \sqrt{D(v^2)} = 0,$$

$$(22) \quad 3 - \eta(v)' = 0.$$

Substituting the parameters of Franco's hybrid method in Table 1 into equations (20)-(22) and the coefficients a_{41} , a_{42} and a_{43} are chosen as free parameters. Solving the system simultaneously we get the new optimized integrator method (NOIM) with the parameters a_{41} , a_{42} and a_{43} which give by

$$a_{41} = \frac{800}{1199278039 v^6} \left(\sqrt{47229} v \sqrt{1413516741 - 1413516741 (\cos(v))^2} + 32682468 \cos(v) - 32682468 + 8170617 v^2 \right),$$

$$a_{42} = \frac{8}{476821389 v^6} \left(1617782166 \cos(v) v^2 - 2434843866 v^2 + 539260722 v^4 - 27831375 v^6 - 100 \sqrt{47229} v \sqrt{1413516741 - 1413516741 (\cos(v))^2} - 3268246800 \cos(v) + 3268246800 + 33 \sqrt{47229} v^3 \sqrt{1413516741 - 1413516741 (\cos(v))^2} \right),$$

$$a_{43} = \frac{40000}{39576175287 v^6} \left(\sqrt{47229} v \sqrt{1413516741 - 1413516741 (\cos(v))^2} + 32682468 \cos(v) - 32682468 + 8170617 v^2 \right).$$

Consequently, the new optimized integrator method (NOIM) of four-order with four stage as follows Taylor series expansion of the free parameters a_{41} , a_{42} and

-1	0	0	0	0
0	0	0	0	0
$\frac{33}{50}$	0	$\frac{2739}{5000}$	0	0
$-\frac{13}{17}$	a_{41}	a_{42}	a_{43}	0
	$-\frac{89}{1992}$	$\frac{545}{858}$	$\frac{625000}{3316929}$	$\frac{83521}{377832}$

a_{43} is practical and is given as follows

$$a_{41} = \frac{314860}{20796729} - \frac{11245}{20796729} v^2 + \frac{2249}{249560748} v^4 - \frac{2249}{24706514052} v^6 + \frac{865}{1383564786912} v^8 - \frac{173}{55342591476480} v^{10} + O(v^{12}),$$

$$a_{42} = -\frac{1058746}{8268579} + \frac{11245}{8268579} v^2 - \frac{245141}{992229480} v^4 + \frac{150683}{28891387800} v^6 - \frac{1601807}{27504601185600} v^8 + \frac{46537}{110018404742400} v^{10} + O(v^{12}),$$

$$a_{43} = \frac{15743000}{686292057} - \frac{562250}{686292057} v^2 + \frac{56225}{4117752342} v^4 - \frac{56225}{407657481858} v^6 \\ + \frac{21625}{22828818984048} v^8 - \frac{865}{182630551872384} v^{10} + O(v^{12}).$$

4. Problems tested

Some oscillatory problems are solved to show the accuracy of NOIM method in this section. We are compared the numerical results with the effective methods in the interval from 0 to 1000. In the comparison we have chosen the following acronyms:

- NOIM4: New optimized integrator method of four order developed in this paper.
- PFRKN4: Existing Phase fitted Runge-Kutta Nyström of four order method presented in [16].
- RKN5: Existing five order Runge-Kutta Nyström method in [2].

The measurement of the efficiency is tested by using the following absolute error

$$\text{Absolute error} = \max \{|z(t_n) - z_n|\}$$

where z_n is the numerical solution and $z(t_n)$ is the analytic solution. It can be observed from the figures (1)-(3) that the curve of the NOIM method locates below every other curve. Consequently, we conclude that optimized integrator method is more efficiency and effectiveness than the existing methods.

Problem 1 ([17]). $z''(t) = -100z(t)$, $z(0) = 1$, $z'(0) = -2$. The fitted frequency, $v = 10$, and the analytic solution is: $z(t) = -(1/5) \sin(10t) + \cos(10t)$.

Problem 2 ([16]). $z''(t) = -v^2 z(t) + (v^2 - 1) \sin(t)$, $z(0) = 1$, $z'(0) = v + 1$, $v = 10$. the analytic solution is: $z(t) = \cos(vt) + \sin(vt) + \sin(t)$.

Problem 3 ([18]).

$$z_1''(t) + z_1(t) = 0.001 \cos(t), \quad z_1(0) = 1, \quad z_1'(0) = 0, \\ z_2''(t) + z_2(t) = 0.001 \sin(t), \quad z_2(0) = 0, \quad z_2'(0) = 0.009995,$$

The fitted frequency, $v = 1$, and the analytic solution is:

$$z_1 = \cos(t) + 0.0005t \sin(t), \\ z_2 = \sin(t) - 0.0005t \cos(t),$$

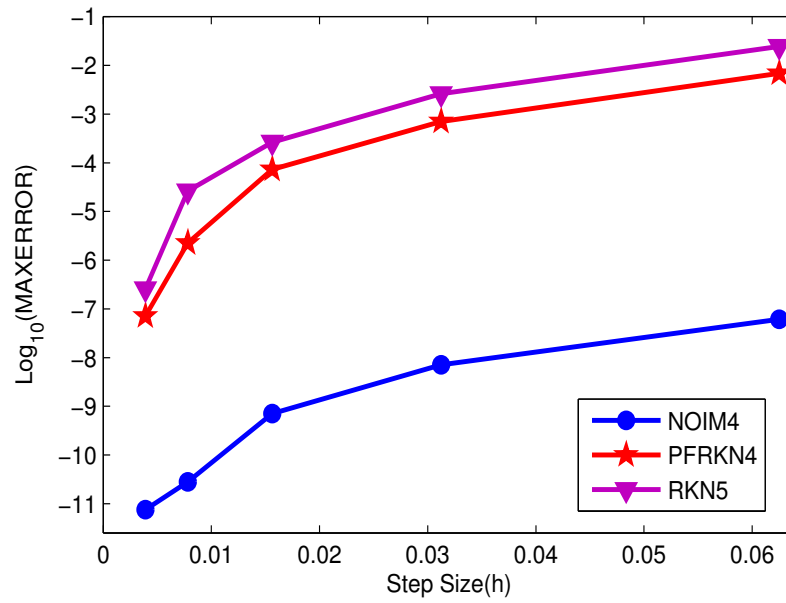


Figure 1: Efficiency Results for Problem 1

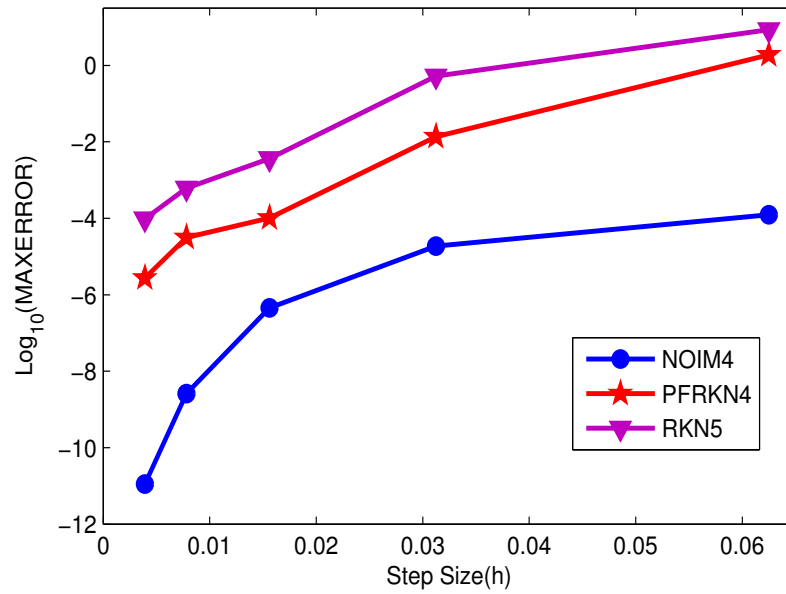


Figure 2: Efficiency Results for Problem 2

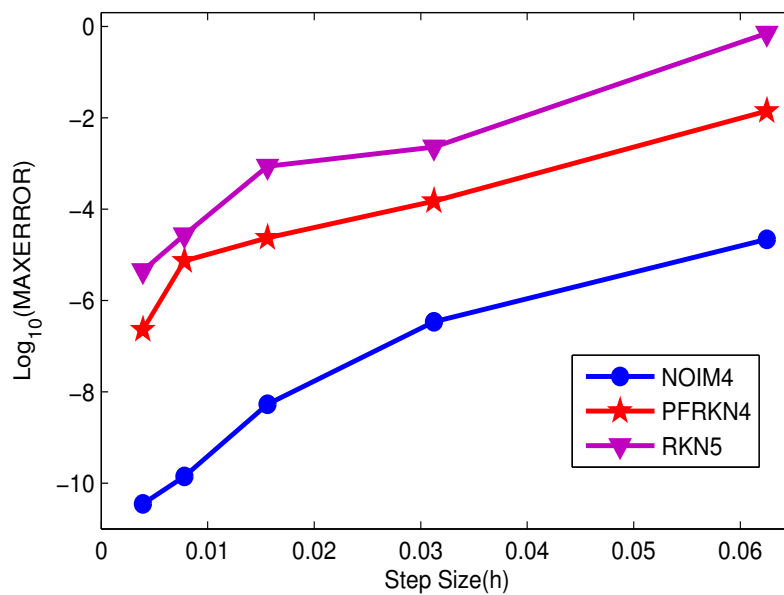


Figure 3: Efficiency Results for Problem 3

5. Conclusion

In this study, optimized integrator method of order four is derived based on the existing hybrid methods. The new method has dissipative of order infinity and dispersive of order six. Numerical results detect that the NOIM method is more competence than the existing methods.

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On supra $\widehat{\omega}_p$ -Lindelöf spaces

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Abstract. We submit in this research a new kind of supra compact and supra Lindelöf spaces via new kind of open sets in supra topological space, also, to introduce new type of supra continuous functions. Our results supported by examples, remarks and some facts.

Keywords: supra $\widehat{\omega}_p$ -open set, supra $\widehat{\omega}_p$ -limit point supra $\widehat{\omega}_p$ -compact space, supra $\widehat{\omega}_p$ -Lindelöf space, supra $\widehat{\omega}_p$ -continuous function.

1. Introduction

Mashhour was the first researcher whose defined the supra topological space in (1983) and he introduced the notion of supra continuous functions [1], in which every topological space is supra topological space [2]. After him many researchers dealt with this space such as in [3]. In [4] Sayed, submitted the definition of supra closure and supra interior for any set in a supra space. Also, he posed the notion of supra pre-open set in [5] and he defined it as (any set $\mathfrak{L}$

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of a supra space is named supra pre-open in case $\mathfrak{L} \subseteq \text{su.int}(\text{su.cl}(\mathfrak{L}))$. In [6], Humadi and Ali defined new set in supra spaces named it supra $\widehat{\omega}$ -open set [6], where they defined it as (Any set $\mathfrak{L}$ in a supra space is named supra $\widehat{\omega}$ -open set whenever for each $x \in \mathfrak{L}$ we can find supra open set $\mathfrak{g}$ in X with $x \in \mathfrak{g}$ and $\mathfrak{g} - \mathfrak{L}$ is a countable set). In the other hand, Al-Shami in [7] defined the supra compact space and supra Lindelöf space. And in [8] the researchers defined supra $\widehat{\omega}$ -compact space. In this our research we introduce the definition of new set named it supra $\widehat{\omega}$ -pre-open set (shortly *su. $\widehat{\omega}_p$ -open*) also we defined *su. $\widehat{\omega}_p$ -compact* and *su. $\widehat{\omega}_p$ -Lindelöf* spaces as well as, we defined *su. $^*\widehat{\omega}_p$ -continuous* function in the same manner of definition for *su. $^*\widehat{\omega}$ -continuous* function in [9], [10].

2. Supra $\widehat{\omega}_p$ -compact and supra $\widehat{\omega}_p$ -Lindelöf spaces

We will define the notions of *su. $\widehat{\omega}_p$ -compact* and *su. $\widehat{\omega}_p$ -Lindelöf* spaces. Also we will show the relation between them. We will use the abbreviation "su" instead of the word "supra".

Definition 2.1. A subset $\mathfrak{L}$ of a *su. space* is said to be *su. $\widehat{\omega}_p$ -open* set if for each $x \in \mathfrak{L}$ we can find a *su. pre-open* set $\mathfrak{g}$ in X with $x \in \mathfrak{g}$ and $\mathfrak{g} - \mathfrak{L}$ is a countable set.

Remark 2.2. Each *su. open* set is *su. $\widehat{\omega}_p$ -open*, this remark is irreversible as in the next example.

Example 2.3. Take indiscrete *su. space* $(\mathcal{R}, \mu_{ind})$, for any point x in $\mathcal{R} - 0$, we can find a unique *su. pre-open* set $\mathcal{R}$ containing x and $\mathcal{R} - (\mathcal{R} - 0)$ is finite, so it is countable. This implies that $\mathcal{R} - 0$ is *su. $\widehat{\omega}_p$ -open* but not *su. open* set.

Example 2.4. The set Z^+ in the indiscrete *su. space* (Z, μ_{ind}) is *su. $\widehat{\omega}_p$ -open* set.

Lemma 2.5. Suppose (X, μ) is a *su. space*. Then the union of any family of *su. $\widehat{\omega}_p$ -open* sets is *su. $\widehat{\omega}_p$ -open*.

Proof. Give $\{U_\beta : \beta \in \Lambda\}$ as a family of *su. $\widehat{\omega}_p$ -open* sets in X and $x \in \bigcup_{\beta \in \Lambda} U_\beta$. Then $x \in U_\alpha$ for some $\alpha \in \Lambda$. Hence there is *su. pre-open* subset $\mathcal{H}$ of X with $x \in \mathcal{H}$ in which $\mathcal{H} - U_\alpha$ is countable. We know that $\mathcal{H} - \bigcup_{\beta \in \Lambda} U_\beta \subseteq \mathcal{H} - U_\alpha$, then $\mathcal{H} - \bigcup_{\beta \in \Lambda} U_\beta$ is countable. Therefore, $\bigcup_{\beta \in \Lambda} U_\beta$ is *su. $\widehat{\omega}_p$ -open*. $\square$

Definition 2.6. A *su. $\widehat{\omega}_p$ -open* cover for any set $\mathcal{M}$ in a *su. space* X is a family $\{\mathfrak{g}_\alpha : \alpha \in \Lambda\}$ of *su. $\widehat{\omega}_p$ -open* sets in which $\mathcal{M} \subseteq \bigcup\{\mathfrak{g}_\alpha : \alpha \in \Lambda\}$.

Definition 2.7. A *su. space* X is named *su. $\widehat{\omega}_p$ -compact* (*su. $\widehat{\omega}_p$ -Lindelöf*) whenever each *su. $\widehat{\omega}_p$ -open* cover of X owns a finite (countable) *su. sub* cover.

Remark 2.8. If X is *su. $\widehat{\omega}_p$ -compact* (*su. $\widehat{\omega}_p$ -Lindelöf*) space, so it is *su. compact* (*su. Lindelöf*) but the convers is not true.

Example 2.9. The su. space $(\mathcal{R}, \mu_{ind})$ is su. compact (su. Lindelöf) but not su. $\widehat{\omega}_p$ -compact, since $\{x\}_{x \in \mathcal{R}}$ is su. $\widehat{\omega}_p$ -open cover to $\mathcal{R}$ which has no finite (countable) su. sub cover, where is su. $\widehat{\omega}_p$ -open (Since $\mu Int(\mu cl\{x\}) = \mathcal{R}$, so $\{x\} \subseteq \mu Int(\mu cl\{x\}) = \mathcal{R}$).

Proposition 2.10. *If X is finite su. space, then it is su. $\widehat{\omega}_p$ -compact.*

Proof. Since X is finite, so any su. $\widehat{\omega}_p$ -open cover to it is finite. Also X is countable then any su. $\widehat{\omega}_p$ -open cover to it is countable. $\square$

Proposition 2.11. *Every su. $\widehat{\omega}_p$ -compact space is su. $\widehat{\omega}_p$ -Lindelöf.*

Proof. Give $\mathcal{C} = \{G_\alpha, \alpha \in \Lambda\}$ as a su. $\widehat{\omega}_p$ -open cover to a su. $\widehat{\omega}_p$ -compact space X , so $\mathcal{C}$ owns a finite sub cover to X , then it is countable, therefore X is su. $\widehat{\omega}_p$ -Lindelöf. $\square$

Example 2.12. Suppose $\mu_{\mathcal{R}} = \{\phi, \mathcal{R}, G_j \subseteq \mathcal{R} : G_j = \{j, j+1, j+2, \dots : j \in N\}\}$ is su. topology on $\mathcal{R}$, so $(\mathcal{R}, \mu_{\mathcal{R}})$ is su. compact space, but not su. $\widehat{\omega}_p$ -compact, since $\mathcal{C} = \{Q^c \cup \{x\} : x \in Q\}$ is su. $\widehat{\omega}_p$ -open cover to $\mathcal{R}$. But it has no finite su. sub cover.

Remark 2.13. In case X is su. $\widehat{\omega}_p$ -compact space, hence it is su. $\widehat{\omega}_p$ -Lindelöf, this remark is irreversible as in above example.

Proposition 2.14. *Each su. $\widehat{\omega}_p$ -closed subset of su. $\widehat{\omega}_p$ -compact space is su. $\widehat{\omega}_p$ -compact.*

Proof. Take $\mathfrak{L}$ as a su. $\widehat{\omega}_p$ -closed subset of a su. $\widehat{\omega}_p$ -compact space X and $\mathcal{C} = \{V_\alpha : \alpha \in \Lambda\}$ be su. $\widehat{\omega}_p$ -open cover to $\mathfrak{L}$, thus $\mathfrak{L} \subseteq \bigcup_{\alpha \in \Lambda} V_\alpha$. But $\mathfrak{L}$ is a su. $\widehat{\omega}_p$ -closed, so $\mathfrak{L}^c$ is $\widehat{\omega}_p$ -open set. Therefore, $X \subseteq \bigcup_{\alpha \in \Lambda} V_\alpha \cup \mathfrak{L}^c$ which is $\widehat{\omega}_p$ -compact, so $X \subseteq \bigcup_{i=1}^n V_{\alpha_i} \cup \mathfrak{L}^c$, then $\mathfrak{L} \subseteq \bigcup_{i=1}^n V_{\alpha_i}$ which implies $\mathfrak{L}$ is su. $\widehat{\omega}_p$ -compact. The case of su. $\widehat{\omega}_p$ -Lindelöf, the proof is similar. $\square$

Proposition 2.15. *A su. space X is su. $\widehat{\omega}_p$ -compact if and only if each family of su. $\widehat{\omega}_p$ -closed sets in X , satisfies the finite intersection property (shortly FIP), has itself a non-null intersection necessity.*

Proof. Necessity give $\mathcal{C} = \{V_\alpha : \alpha \in \Lambda\}$ as a family of su. $\widehat{\omega}_p$ -closed sets in X and has FIP, suppose $\bigcap_{\alpha \in \Lambda} V_\alpha = \phi$, then $\bigcup_{\alpha \in \Lambda} V_\alpha^c = X$. But X is su. $\widehat{\omega}_p$ -compact, thus $X = \bigcup_{i=1}^n V_{\alpha_i}^c$, so $\bigcap_{i=1}^n V_{\alpha_i} = \phi$ which this contradicts with fact that $\mathcal{C}$ has FIP. Therefore, $\mathcal{C}$ owns a non-null intersection. Sufficiency: Give $\mathcal{M} = \{\mathcal{K}_i : i \in I\}$ as a su. $\widehat{\omega}_p$ -open cover to X and it owns no finite su. sub cover. Thus $X - \bigcup_{i=1}^n \mathcal{K}_i \neq \phi$, so $\bigcap_{i=1}^n \mathcal{K}_i^c \neq \phi$, which implies the collection $\{\mathcal{K}_i^c : i \in I\}$ of su. $\widehat{\omega}_p$ -closed subsets of X which has FIP, then $\bigcap_{i \in I} \mathcal{K}_i^c \neq \phi$ by hypothesis. So $\bigcup_{i \in I} \mathcal{K}_i \neq X$ which is a contradiction. So $\mathcal{M}$ owns a finite sub cover to X . $\square$

Proposition 2.16. *If $\mathcal{H}$ is a su. $\widehat{\omega}_p$ -compact set in a su. space X and $\mathcal{K}$ is su. $\widehat{\omega}_p$ -closed set in X , then $\mathcal{H} \cap \mathcal{K}$ is su. $\widehat{\omega}_p$ -compact.*

Proof. If $\mathcal{C} = \{U_\alpha : \alpha \in \Lambda\}$ is a su. $\widehat{\omega}_p$ -open cover to $\mathcal{H} \cap \mathcal{K}$, then $\mathcal{H} \subseteq (\bigcup_{\alpha \in \Lambda} U_\alpha) \cup \mathcal{K}^c$. Since $\mathcal{H}$ is su. $\widehat{\omega}_p$ -compact, then $\mathcal{H} \subseteq (\bigcup_{i=1}^n U_{\alpha_i}) \cup \mathcal{K}^c$, which implies $\mathcal{H} \cap \mathcal{K} \subseteq (\bigcup_{i=1}^n U_{\alpha_i}) \cup \mathcal{K}^c \cap \mathcal{K}$. Hence, $\mathcal{H} \cap \mathcal{K}$ is su. $\widehat{\omega}_p$ -compact. $\square$

Corollary 2.17. *If $\mathcal{H}$ is a su. $\widehat{\omega}_p$ -Lindelöf and $\mathcal{K}$ is a su. $\widehat{\omega}_p$ -closed set in a su. space X , thus $\mathcal{H} \cap \mathcal{K}$ is su. $\widehat{\omega}_p$ -Lindelöf.*

Proof. Is similar to Proposition 2.16 $\square$

Definition 2.18. *A function h from a su. space X into a su. space Y is named su. $^*\widehat{\omega}_p$ -continuous whenever $h^{-1}(\mathfrak{L})$ is su. $\widehat{\omega}_p$ -open in X for any su. open set $\mathfrak{L}$ in Y .*

Proposition 2.19. *If a function h from a su. space X into a su. space Y is su. $^*\widehat{\omega}_p$ -continuous and $\mathfrak{L}$ is su. $\widehat{\omega}_p$ -compact set in X , then $h(\mathfrak{L})$ is su. compact.*

Proof. Give $\mathfrak{L}$ is a su. $\widehat{\omega}_p$ -compact set in a su. space X and $\mathcal{C} = \{\bigcup_\alpha : \alpha \in \Lambda\}$ is a su. open cover for $h(\mathfrak{L})$, which means $h(\mathfrak{L}) \subseteq (\bigcup_{\alpha \in \Lambda} U_\alpha)$, then $\mathfrak{L} \subseteq h^{-1}(f(\mathfrak{L})) \subseteq h^{-1}(\bigcup_{\alpha \in \Lambda} U_\alpha)$. But $\mathfrak{L}$ is su. $\widehat{\omega}_p$ -compact and $h^{-1}(U_\alpha)$ is su. $\widehat{\omega}_p$ -open set, since h is su. $^*\widehat{\omega}_p$ -continuous, so $\mathfrak{L} \subseteq \bigcup_{i=1}^n h^{-1}(U_{\alpha_i})$, then $h(\mathfrak{L}) \subseteq h(\bigcup_{i=1}^n h^{-1}(U_{\alpha_i})) \subseteq \bigcup_{i=1}^n h(h^{-1}(U_{\alpha_i})) \subseteq \bigcup_{i=1}^n U_{\alpha_i}$. Therefore, $h(\mathfrak{L})$ is su. compact. $\square$

Proposition 2.20. *If f from a su. space X onto a su. space Y and if X is su. $\widehat{\omega}_p$ -Lindelöf, then Y is a su. $\widehat{\omega}_p$ -Lindelöf, whenever f is su. $^*\widehat{\omega}_p$ -continuous.*

Proof. Is similar to above proposition 2.19 $\square$

Definition 2.21. *A function h from a su. space X into a su. space Y is called su. $^*\widehat{\omega}_p^*(\widehat{\omega}_p^{**})$ -continuous, whenever $h^{-1}(\mathfrak{L})$ is su. open (su. $\widehat{\omega}_p$ -open) in X for any su. $\widehat{\omega}_p$ -open set $\mathfrak{L}$ in Y .*

Proposition 2.22. *If h from a su. space X onto a su. space Y , and X is su. Lindelöf (su. $\widehat{\omega}_p$ -Lindelöf), then Y is su. $\widehat{\omega}_p$ -Lindelöf, whenever h is su. $^*\widehat{\omega}_p^*(\widehat{\omega}_p^{**})$ -continuous.*

Proof. Let $\mathcal{C} = \{G_\alpha : \alpha \in \Lambda\}$ be a su. $\widehat{\omega}_p$ -open cover to Y , thus $Y = \bigcup_{\alpha \in \Lambda} G_\alpha$. Thus $h^{-1}(Y) = h^{-1}(\bigcup_{\alpha \in \Lambda} G_\alpha)$, hence $X = \bigcup_{\alpha \in \Lambda} (h^{-1}(G_\alpha))$. But h is su. $^*\widehat{\omega}_p^*(\widehat{\omega}_p^{**})$ -continuous, so $h^{-1}(G_\alpha)$ is su. $\widehat{\omega}_p$ -open set in X for all, but X is su. Lindelöf (su. $\widehat{\omega}_p$ -Lindelöf). Then $X \subseteq \bigcup_{i=1}^n (h^{-1}(G_{\alpha_i}))$, so $Y = h(X) \subseteq h(\bigcup_{i=1}^n (h^{-1}(G_{\alpha_i}))) = \bigcup_{i=1}^n (h(h^{-1}(G_{\alpha_i}))) = \bigcup_{i=1}^n (G_{\alpha_i})$. Since h is onto, then Y is su. $\widehat{\omega}_p$ -Lindelöf. $\square$

Definition 2.23. *Any point $y \in X$ is named su. $\widehat{\omega}_p$ -limit point to a subset $\mathfrak{L}$ of a su. space X , if every su. $\widehat{\omega}_p$ -open set $\mathcal{K}$ where $y \in \mathcal{K}$ has non-empty intersection with $\mathfrak{L}$ whenever y has been select. The set of all su. $\widehat{\omega}_p$ -limit points to $\mathfrak{L}$ is denoted by $\mu\widehat{\omega}_p - d(\mathfrak{L})$.*

Proposition 2.24. *Any set $\mathcal{D}$ of su. space X is su. $\widehat{\omega}_p$ -closed if and only if $\text{su. } \mu\widehat{\omega}_p - d(\mathcal{D}) \subseteq \mathcal{D}$.*

Proof. Necessity: Let $\mathcal{D}$ be a su. $\widehat{\omega}_p$ -closed set, and let $x \in \mu\widehat{\omega}_p - d(\mathcal{D})$. Suppose $x \notin \mathcal{D}$ then $x \in \mathcal{D}^c$ which is su. $\widehat{\omega}_p$ -open set and $\mathcal{D} \cap \mathcal{D}^c = \phi$, then x is not su. $\widehat{\omega}_p$ -limit point to $\mathcal{D}$, so it contradicts with $x \in \mu\widehat{\omega}_p - d(\mathcal{D})$, then $\mu\widehat{\omega}_p - d(\mathcal{D}) \subseteq \mathcal{D}$. Sufficiency: To show that $\mathcal{D}$ is su. $\widehat{\omega}_p$ -closed, if $x \in \mathcal{D}^c$ and we have $\mu\widehat{\omega}_p - d(\mathcal{D}) \subseteq \mathcal{D}$, then x is not su. $\widehat{\omega}_p$ -limit point to $\mathcal{D}$, so there is a su. $\widehat{\omega}_p$ -open U with $x \in U$ and $(U - \{x\}) \cap \mathcal{D} = \phi$, but $x \notin \mathcal{D}$, then $U \cap \mathcal{D} = \phi$ and $U \subseteq \mathcal{D}^c$, so $\mathcal{D}^c = \cup\{U_x : x \in \mathcal{D}^c\}$. Thus $\mathcal{D}^c$ is su. $\widehat{\omega}_p$ -open (by Lemma 2.5), therefor $\mathcal{D}$ is su. $\widehat{\omega}_p$ -closed. $\square$

Proposition 2.25. *Any set G in a su. space X is su. $\widehat{\omega}_p$ -open set if and only if all its points are su. $\widehat{\omega}_p$ -interior points to G .*

Proof. Suppose G is su. $\widehat{\omega}_p$ -open subset of X and $x \in G$, since each set is a subset of itself, so x is su. $\widehat{\omega}_p$ -interior point to G , but x is arbitrary point in G , hence all the points in G are su. $\widehat{\omega}_p$ -interior points to G . Conversely, since $G = \bigcup_{x \in G} \{x\}$, and every point in G is su. $\widehat{\omega}_p$ -interior point to it, thus for all $x \in G$ we can find su. $\widehat{\omega}_p$ -open set U_x in X with $x \in U_x \subseteq G$, in order that, G is a union of the su. $\widehat{\omega}_p$ -open sets U_x for any $x \in G$, hence G is su. $\widehat{\omega}_p$ -open set (by Lemma 2.5). $\square$

Proposition 2.26. *Whenever H is an infinite set in a su. $\widehat{\omega}_p$ -compact space X , hence H has a su. $\widehat{\omega}_p$ -limit point.*

Proof. Suppose H is an infinite subset of su. $\widehat{\omega}_p$ -compact space X . Let H has no su. $\widehat{\omega}_p$ -limit point, then there exist a su. $\widehat{\omega}_p$ -open set $U_{(x_i)}$ for each $x_i \in H$ such that $(U_{x_i} - \{x_i\}) \cap H = \phi$. But the whole space is a union of its points, so $X = \cup\{U_{x_i} : x_i \in H\}$ and $\{U_{x_i} : x_i \in H\}$ form an su. $\widehat{\omega}_p$ -open cover to X , which is su. $\widehat{\omega}_p$ -compact space, then $X = \bigcup_{i=1}^n U_{x_i}$. So X is finite, then H is also finite which is contradicted with the fact that H is infinite, then H has an su. $\widehat{\omega}_p$ -limit point. $\square$

Definition 2.27. *A function h from a su. space X into a su. space Y is called su. $\widehat{\omega}_p$ -open (su. $\widehat{\omega}_p^*$ -open) if $h(\mathfrak{L})$ is su. $\widehat{\omega}_p$ -open (su. open) in Y for any su. open (su. $\widehat{\omega}_p$ -open) set $\mathfrak{L}$ in X .*

Proposition 2.28. *If a bijective function h from a su. space X into a su. space Y is su. $\widehat{\omega}_p^*$ -open function, then X is su. $\widehat{\omega}_p$ -Lindelöf (su. $\widehat{\omega}_p$ -compact) whenever Y is su. Lindelöf (su. compact).*

Proof. Let $\{V_\alpha : \alpha \in \Lambda\}$ be a su. $\widehat{\omega}_p$ -open cover to X , thus $X = \bigcup_{\alpha \in \Lambda} V_\alpha$, so $h(X) = h(\bigcup_{\alpha \in \Lambda} V_\alpha) = \bigcup_{\alpha \in \Lambda} h(V_\alpha)$. But Y is su. Lindelöf, so $Y = \bigcup_{\alpha \in \delta} h(V_\alpha)$ for some countable set $\delta \in \Lambda$, then

$$X = h^{-1}(Y) = h^{-1} \bigcup_{\alpha \in \delta} h(V_\alpha) = \bigcup_{\alpha \in \delta} h^{-1}(h(V_\alpha)) = \bigcup_{\alpha \in \delta} V_\alpha,$$

so X is su. $\widehat{\omega}_p$ -Lindelöf.

In case of compactness the proof is similar. $\square$

Proposition 2.29. *Every countable space is su. $\widehat{\omega}_p$ -Lindelöf.*

Proof. Since every su. $\widehat{\omega}_p$ -open cover to it must be countable. $\square$

Lemma 2.30. *If (X, μ) is a su. space and $\mu' \subseteq \mu$, and if G is su. pre-open set in (X, μ') , thus it is su. pre-open in (X, μ) .*

Proof. Take G as su. pre-open in (X, μ') , then $G \subseteq \mu' - \text{int}^\mu(\mu' - \text{cl}^\mu(G))$. But $\mu' \subseteq \mu$, then $G \subseteq \mu - \text{int}^\mu(\mu - \text{cl}^\mu(G))$. So G is su. pre-open set in (X, μ) . $\square$

Lemma 2.31. *Suppose (X, μ) is a su. space and $\mu' \subseteq \mu$, whenever $\mathfrak{L}$ is a su. $\widehat{\omega}_p$ -open set in (X, μ') , hence it is su. $\widehat{\omega}_p$ -open in (X, μ) .*

Proof. If $\mathfrak{L}$ is su. $\widehat{\omega}_p$ -open set in (X, μ') and let $x \in \mathfrak{L}$, then there is a su. pre-open set $\mathcal{K}$ in (X, μ') where $x \in \mathcal{K}$ and $\mathcal{K} - \mathfrak{L}$ = countable. But $\mathcal{K}$ is also su. pre-open in (X, μ) , so $\mathfrak{L}$ is su. $\widehat{\omega}_p$ -open in (X, μ) . $\square$

Proposition 2.32. *Suppose (X, μ) is a su. $\widehat{\omega}_p$ -Lindelöf space, whenever μ' is coarser than μ , hence (X, μ') is su. $\widehat{\omega}_p$ -Lindelöf.*

Proof. Give $\mathcal{C} = \{U_\alpha : \alpha \in \Lambda\}$ as a su. $\widehat{\omega}_p$ -open cover to (X, μ') , so by Lemma 2.31 $\mathcal{C}$ is su. ω_p -open cover to (X, μ) , which is su. $\widehat{\omega}_p$ -Lindelöf, then $\mathcal{C}$ owns a countable sub cover, that is (X, μ') is su. $\widehat{\omega}_p$ -Lindelöf. $\square$

Corollary 2.33. *Suppose (X, μ) is a su. $\widehat{\omega}_p$ -Lindelöf space and if μ' is coarser than μ , hence (X, μ') is Lindelöf.*

Proof. Take $\mathcal{C} = \{U_{\alpha \in \Lambda} : \alpha \in \Lambda\}$ as an μ' -open cover to (X, μ') , but $\mu' \subseteq \mu$ then $\mathcal{C}$ is μ -open cover to (X, μ) , so $\mathcal{C}$ is a su. $\widehat{\omega}_p$ -open cover to (X, μ) (Each open set is su. $\widehat{\omega}_p$ -open). But (X, μ) is su. $\widehat{\omega}_p$ -Lindelöf, then $\mathcal{C}$ owns a countable sub cover, that is (X, μ') is su. Lindelöf space. $\square$

Remark 2.34. Not necessary that any su. sub space of su. $\widehat{\omega}_p$ -Lindelöf is su. $\widehat{\omega}_p$ -Lindelöf space too.

Example 2.35. Take X as an uncountable set and μ consist of X and the subsets of X which not contain x_0 belong to X . If $\mathcal{M}$ is a su. $\widehat{\omega}_p$ -open cover to X , also the only su. ω_p -open set which contain x_0 is just X . So $X \in \mathcal{M}$, thus $\{X\}$ is countable sub cover of X . It follows that X is su. $\widehat{\omega}_p$ -Lindelöf. But $Y = X - \{x_0\}$ is not su. $\widehat{\omega}_p$ -Lindelöf subspace, since $\{y : y \in Y\} = \mathcal{M}^*$ is a su. $\widehat{\omega}_p$ -open cover to Y which has no countable sub cover (since $\{y\}$ is su. $\widehat{\omega}_p$ -open set in X and $\{y\} = Y \cap \{y\}$, so $\{y\}$ is su. $\widehat{\omega}_p$ -open in Y , then μ_Y is the discrete topology. But $\mathcal{M}^*$ is uncountable su. $\widehat{\omega}_p$ -open cover to Y and $\mathcal{M}^*$ has no proper sub cover, where if we select one member of $\mathcal{M}^*$ the reminder is not cover to Y . That is $\mathcal{M}^*$ has no a countable sub cover to Y , therefore Y is not su. $\widehat{\omega}_p$ -Lindelöf.

Proposition 2.36. *Any su. $\widehat{\omega}_p$ -closed subspace of su. $\widehat{\omega}_p$ -Lindelöf space is su. $\widehat{\omega}_p$ -Lindelöf.*

Proof. Let Y be su. $\widehat{\omega}_p$ -closed subset of a su. $\widehat{\omega}_p$ -Lindelöf space X , and $\mathcal{V} = \{V_\alpha : \alpha \in \Lambda\}$ be an $\mu_Y - \widehat{\omega}_p$ -open cover to Y . Then there exist $\mu_X - \widehat{\omega}_p$ -open sets U_α such that $V_\alpha = U_\alpha \cap Y$ that is, $\{U_\alpha : \alpha \in \Lambda\}$ be a $\mu_X - \widehat{\omega}_p$ -open cover to Y . Then $X = \{U_\alpha : \alpha \in \Lambda\} \cup Y^c$ is $\mu_X - \widehat{\omega}_p$ -open cover to X , but it is su. ω_p -Lindelöf, thus we can find a countable set $j \subseteq \Lambda$ in which $X = \{U_\alpha : \alpha \in j\} \cup Y^c$, then $Y = \{U_\alpha : \alpha \in j\}$. Therefore, Y is su. $\widehat{\omega}_p$ -Lindelöf. $\square$

Definition 2.37. *A su. topological space X is said to be su. locally $\widehat{\omega}_p$ -compact (for short $\mu\widehat{\omega}_p$ -compact) if every $x \in X$ owns a su. neighbourhood its su. closure is su. $\widehat{\omega}_p$ -compact.*

Proposition 2.38. *Whenever X is su. $\widehat{\omega}_p$ -compact space, then it is $\mu\widehat{\omega}_p$ -compact.*

Proof. Take X as a su. $\widehat{\omega}_p$ -compact space. If we take any point x in hole su. space X , we have X su. clopen, so X a su. neighbourhood and it is equal to its su. closure, which is su. $\widehat{\omega}_p$ -compact, then it is $\mu\widehat{\omega}_p$ -compact. $\square$

This proposition is irreversible.

Example 2.39. The su. discrete space $(\mathcal{R}, \mu_D)$ is $\mu\widehat{\omega}_p$ -compact space but not su. $\widehat{\omega}_p$ -compact.

Proposition 2.40. *Every su. closed subspace of $\mu\widehat{\omega}_p$ -compact space is $\mu\widehat{\omega}_p$ -compact.*

Proof. Take Y as su. $\widehat{\omega}_p$ -closed subspace of $\mu\widehat{\omega}_p$ -compact space X . Also $P \in Y \subseteq X$, then $P \in X$. So, there is a su. open neighbourhood G to P in X with su. $cl(G)$ is su. $\widehat{\omega}_p$ -compact. Since $G \cap Y$ is su. open neighbourhood of P in Y and $G \cap Y \subset G$ and su. $cl(G \cap Y) \subset cl(G)$. But su. $cl(G \cap Y)$ is su. closed, so it is su. $\widehat{\omega}_p$ -closed subset of su. $\widehat{\omega}_p$ -compact su. $cl(G)$, that is su. $cl(G \cap Y)$ is su. $\widehat{\omega}_p$ -compact since su. $cl(G \cap Y)$ is su. closed in X , and a subset of Y , then $cl_{inX}(G \cap Y) = cl_{inY}(G \cap Y)$ so it is su. $\widehat{\omega}_p$ -compact in Y . So we have every point in Y has a su. neighbourhood its su. closure is su. $\widehat{\omega}_p$ -compact. $\square$

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Inverse one-dimensional wave equation problem under upper-base as additional information

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Abstract. Final displacement measurement 'upper-base' is used in this research for finding force in non-homogeneous hyperbolic equation. Same as [7, 8] finite difference method (FDM) with separation variable are applied, but different additional condition is investigated. That means additional data were given by the left end displacement measurement in [7], and the right end displacement measurement in [8].

Furthermore, the wave equation with force split in two parts, part one is direct homogenous problem and numerically using FDM and second one is inverse non-homogeneous ill-posed problem, solved separation variable has been applied. Sequentially, for solving inverse force problem require initial and the boundary condition with additional data, in this study extra data is upper-base. Moreover, second part is unstable due to ill-posedness of the problem where a small error in addition data causes major error in out force data [6]. A stable solution for direct problem and reasonable for inverse problem has been obtained as shown on figures and presented numerical result. Compare with the [7, 8] different shapes of figures and different condition numbers are provided.

Keywords: final displacement measurement, direct problem, finite difference method, separation variables method, regularization, inverse force problem.

1. Introduction

The target of this research is to solve inverse force problem with initial, boundary and additional condition, which is different to the one had used in [7, 8]. Furthermore, dealing with end displacement boundary in [7, 8] and upper based in this research are considered. An application of wave equation in science especially physics and engineering is dominant [2, 3, 6]. Added small amount of error to the extra data 'noisy data' to see instability after that using zeroth order Tikhonov regularization in order to stabilize the solution. For choosing regularization parameter the minimum error is explored. Numerically finite difference method [7, 8, 9] is used instead boundary element method (BEM) [1, 4, 6] with separation variable and different condition are studied.

The innovation of this study is to develop difference additional data and to see how far is working compare to the [4, 6, 7, 8] for that reason same numerically example have been applied.

The mathematical formulation is in Section 2, numerical results and discussed are in Section 3 and conclusions are in Section 4.

2. Mathematical formulation

The governing equation for a boundary $[0, L]$ with force $f(x)$ is given by the hyperbolic equation [4, 5, 6, 7, 8]

$$\begin{aligned} (1) \quad & u_{tt}(x, t) = \nabla^2 u(x, t) + f(x), \quad (x, t) \in (0, L) \times (0, T), \quad T > 0 \\ (2) \quad & u(x, 0) = u_0(x), \quad u_t(x, 0) = v_0(x), \quad x \in [0, L], \\ (3) \quad & u_x(0, t) = q_0(t), \quad u(L, t) = p_L(t), \quad t \in (0, T), \\ (4) \quad & u(0, t) = p_0(t), \quad u_x(L, t) = q_L(t), \quad t \in (0, T), \\ (5) \quad & u(x, T) = u_T(x), \quad x \in (0, L), \quad T > 0, \end{aligned}$$

where (5) is called the final displacement measurement, when deal with inverse problem part for reaching force $f(x)$ is used. In addition, $\{u(x, t), u_0, v_0\}$ variables represent $\{\text{displacement, initial displacement, velocity}\}$, respectively.

One of the equation (3) or (4) using with (1) and (2) in case solving hyperbolic equation.

Split equation (1) into $u = v + w$. In direct problem: $v_{tt} = v_{xx}$ with initial condition (2) and boundary condition (3) (or (4)), and inverse problem: $w_{tt} = w_{xx} + f(x)$ with initial condition $u(x, 0) = u_t(x, 0) = 0$, boundary condition $u_x(0, t) = u(L, t) = 0$ (or $u(0, t) = u_x(L, t) = 0$) and $w(x, T) = u_T(x) - v_T(x)$.

2.1 Direct problem

v in part one is well-posed direct problem without force/source ($v_{tt} = v_{xx}$). Numerically FDM applying to get $v(x, T) = v_T(x)$ (i.e $v_T(x_i)$), with initial condition (2) and boundary condition (3) (or (4))

$$\begin{aligned} (6) \quad & v_{i,j+1} = r^2 v_{i+1,j} + 2(1-r^2)v_{i,j} + r^2 v_{i-1,j} - v_{i,j-1}, \\ & i = \overline{1, (M-1)}, \quad j = \overline{1, (N-1)}, \\ & v_{i,1} = \frac{1}{2}r^2 u_0(x_{i+1}) + (1-r^2)u_0(x_i) + \frac{1}{2}r^2 u_0(x_{i-1}) + (\Delta t)v_0(x_i), \\ (7) \quad & j = 0, i = \overline{1, (M-1)}. \\ (8) \quad & v_{i,0} = u_0(x_i), \quad i = \overline{0, M}, \quad \frac{v_{i,1} - v_{i,-1}}{2\Delta t} = v_0(x_i), \quad i = \overline{1, (M-1)}, \\ (9) \quad & -\frac{\partial v}{\partial x}(0, t_j) = -\frac{4v_{1,j} - v_{2,j} - 3v_{0,j}}{2\Delta x}, \quad j = \overline{1, N}, \quad v_{M,j} = P_L(t_j), \quad j = \overline{0, N}, \\ (10) \quad & v_{0,j} = P_0(t_j), \quad j = \overline{0, N}, \quad \frac{\partial v}{\partial x}(L, t_j) = \frac{3v_{M,j} - 4v_{M-1,j} + v_{M-2,j}}{2\Delta x}, \quad j = \overline{1, N}, \end{aligned}$$

where $v_{i,j} := v(x_i, t_j)$, $x_i = i\Delta x$, $t_j = j\Delta t$ and $r = c\Delta t/\Delta x$, for $i = \overline{0, M}$, $j = \overline{0, N}$, in addition $\Delta x = L/M$ and $\Delta t = T/N$ such that divide the domain $(0, L) \times (0, T)$ into M and N [9, 4, 5, 6, 7, 8].

After that using $v_T(x)$ in the inverse problem part to get (w, f) .

2.2 Inverse problem

Separation variable has studied to solve $w_{tt} = w_{xx} + f(x)$ with 'zero' initial condition $u(x, 0) = u_t(x, 0) = 0$ and 'zero' boundary condition $u_x(0, t) = u(L, t) = 0$ get equations (11) and (12), or with 'zero' boundary condition $u(0, t) = u_x(L, t) = 0$ reach equations (13) and (14) [1, 9, 4, 5, 6, 7, 8], as following:

$$(11) \quad w_K(x, t; \underline{b}) = \frac{\sqrt{2}}{c^2} \sum_{k=1}^K \frac{b_k}{\lambda_k^2} (1 - \cos(c\lambda_k t)) \cos(\lambda_k x),$$

$$(12) \quad f_K(x) = \sqrt{2} \sum_{k=1}^K b_k \cos(\lambda_k x), \quad x \in (0, L),$$

$$(13) \quad w_K(x, t; \underline{b}) = \frac{\sqrt{2}}{c^2} \sum_{k=1}^K \frac{b_k}{\lambda_k^2} (1 - \cos(c\lambda_k t)) \sin(\lambda_k x),$$

$$(14) \quad f_K(x) = \sqrt{2} \sum_{k=1}^K b_k \sin(\lambda_k x), \quad x \in (0, L),$$

where K is a truncation number and $\lambda_k = \frac{(k-\frac{1}{2})\pi}{L}$ for $k = \overline{1, K}$. As it is clear from equations (11)-(14) $\underline{b} = (b_k)_{k=\overline{1, K}}$ is unknown, which can calculate from $w(x, T) = u_T(x) - v_T(x)$, i.e

$$(15) \quad \begin{aligned} w(x, T; \underline{b}) &= u_T(x) - v_T(x) =: g(x)|h(x) \\ &= \frac{\sqrt{2}}{c^2} \sum_{k=1}^K \frac{b_k}{\lambda_k^2} (1 - \cos(c\lambda_k T)) \cos(\lambda_k x) | \sin(\lambda_k x), \quad x \in [0, L], \end{aligned}$$

the left side of the equation (15) is known and b is unknown on the right side. Moreover, the least squares solution is given $\underline{b}_\lambda = (Q^{\text{tr}}Q)^{-1}Q^{\text{tr}}g|h$ where the superscript 'tr' denotes the transpose and $Q_{mk} = \frac{\sqrt{2}(1-\cos(c\lambda_k T)) \cos(\lambda_k x) | \sin(\lambda_k x)}{c^2 \lambda_k^2}$.

The existence and uniqueness had provided in [1, 4, 6], and for stability adding some noise to the $u_T(x)$ as

$$(16) \quad u_T^\epsilon(x_m) = u_T(x_m) + \epsilon, \quad m = \overline{1, M}.$$

For knowing the value of ϵ : studying a Gaussian normal distribution as mean zero and standard deviation σ are given by the following [4, 5, 6, 7] and $p\%$ represents the percentage of noise

$$(17) \quad \sigma = p\% \times \max_{x \in [0, L]} |u_T(x)|.$$

The noisy data (16) also causes noise in $g|h$ term as

$$(18) \quad g^\epsilon(x_m)|h^\epsilon(x_m) = u_T^\epsilon(x_m) - v_T(x_m) = g(x_m)|h(x_m) + \epsilon, \quad m = \overline{1, M}.$$

As mentioned above this noise makes unstable for solution as appear in example section, then to overcome this obstacle zeroth-order Tikhonov regularization is applied to find the solutions (for more detail can see [4, 6, 7]) as:

$$(19) \quad \underline{b}_\lambda = (Q^{\text{tr}}Q + \lambda I)^{-1}Q^{\text{tr}}\underline{g}^\epsilon|\underline{h}^\epsilon.$$

Note that in equations (15), (18) and (19) the "—" means "or". Furthermore, this 'or' has been used when (3) or (4) was applied.

3. Numerical results and discussion

With an observation of understanding how changing additional condition is modified solution for the problem, the same example in [4, 6, 7, 8] is chosen with analytic solution for $(u(x, t), f(x))$

$$(20) \quad u(x, t) = \sin(\pi x) + t + \frac{t^2}{2}, \quad f(x) = 1 + \pi^2 \sin(\pi x), \quad x \in [0, 1],$$

$$(21) \quad u(x, 0) = u_0(x) = \sin(\pi x), \quad u_t(x, 0) = v_0(x) = 1, \quad x \in [0, 1],$$

$$(22) \quad u_x(0, t) = q_0(t) = \pi, \quad u(1, t) = p_L(t) = t + \frac{t^2}{2}, \quad t \in (0, 1].$$

Instead of (22) can be used boundary condition

$$(23) \quad u(0, t) = p_0(t) = t + \frac{t^2}{2}, \quad u_x(L, t) = q_L(t) = -\pi, \quad t \in (0, 1].$$

An additional overdetermined boundary condition is upper-base

$$(24) \quad u(x, T = 1) = u_T(x) = \frac{3}{2} + \sin(\pi x), \quad x \in [0, 1].$$

First, find $v_T(x)$ by using finite difference method (FDM), the discrete form of v problem for more details see [5, 7, 8], as shown in Figure 1 $M = N \in \{20, 40, 80\}$ increase the result quite close to each other (this means solution good convergence), when using boundary condition (22) and (23) appear in Figure 1(a) and 1(b), respectively. After that, is obtained $\underline{b}$ from equation (13) after substituting $v_T(x)$ in it. Moreover, condition number for matrix Q 'cond(Q)' is calculating in Table 1, the left side of the Table 1 is getting with boundary condition (22), and the right side from boundary condition (23). However, when M and K are both increased, the condition number of Q increases too, they are not effected to the numerically result for $\underline{b}$. Figure 2 showing ill-condition for inverse problem, the figure comes from singular values for matrix Q .

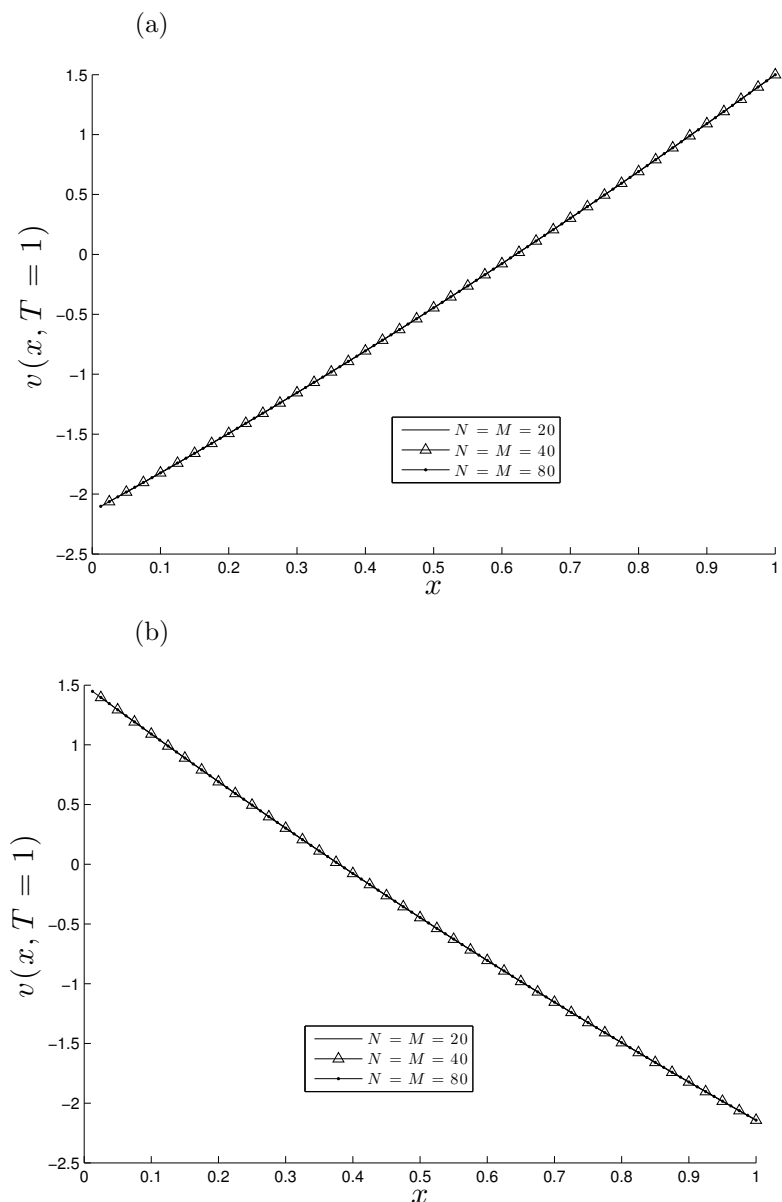


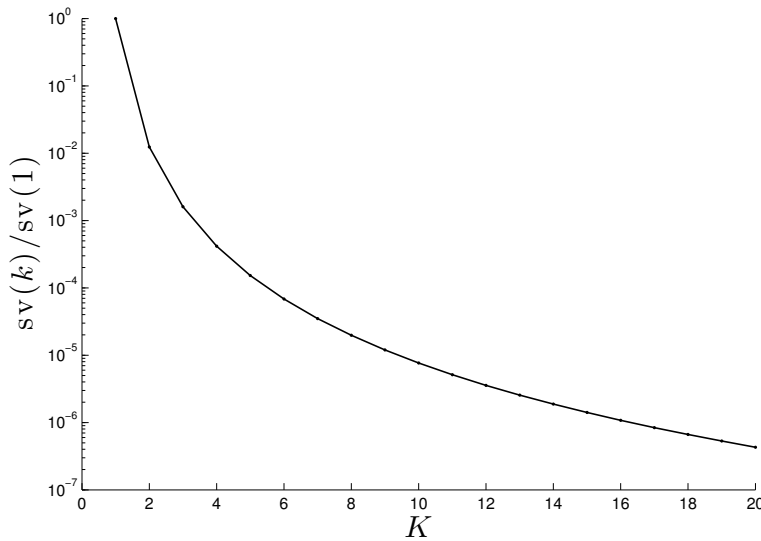
Figure 1: The numerical values of $v_T(x)$ using the boundary condition (22) are shown in (a), whereas the corresponding $v_T(x)$ values when boundary condition (23) is applied, are shown in (b), obtained using the FDM (see [5, 7, 8]) with $M = N \in \{20, 40, 80\}$.

Next step is finding a numerical solution for $\underline{b}$ from equation (15) and for $f(x)$ then compare them with exact one.

In Figure 4(a) the exact solution is $b_k = \sqrt{2} \int_0^1 f(x) \cos((k - \frac{1}{2})\pi x) dx$. In Figure 4(b) is $b_k = \sqrt{2} \int_0^1 f(x) \sin((k - \frac{1}{2})\pi x) dx$ (see [4, 6, 7, 8]). From these

Table 1: Condition number of matrix Q .

BC (22) K	$M = 20$	$M = 40$	$M = 80$
5	6666.95	6583.35	6566.18
10	139798.90	131715.67	130607.63
20	$1.28E + 21$	2432128.57	2328937.11
BC (23) K	$M = 20$	$M = 40$	$M = 80$
5	6621.60	6577.98	6565.51
10	132709.88	131061.46	130531.14
20	2380475.12	2337248.57	2320963.73

Figure 2: $sv(k)/sv(1)$ normalised singular values for $k = \overline{1, 20}$ and $M = 80$

figures fixed $K = 20, N = M = 80$, whereas different shapes obtain where boundary condition has changed, but still can observe a good approach exist between exact and numerical one.

The value of force $f(x)$ has obtained after substituted numerical values b in equation (12)—or (14) where $k \in \{5, 10, 20\}$ and $N = M = 80$. Furthermore, Figure 6 shows comparative between numerical (12)—or (14) and exact values (20) of $f(x)$, where applying boundary condition (22) for finding force appear in Figure 6(a), similarly, for boundary condion (23) shown in Figure 6(b). Although, there is unstable from left of the Figure 6(a) and right Figure 6(b), still get quite good approximation where K increase between exact and numerical values. Add noise to the input data (16) to see how far out put data "force" stay stable which clear in Figure 8 oscillations appear high as K increase and amount

of noise $p\% = 1\%$. In Figure 8(a) right side and Figure 8(b) left side are more unstable. Consequently, boundary conditions (22) and (23) have researched for achieving Figures 8(a) and 8(b), respectively.

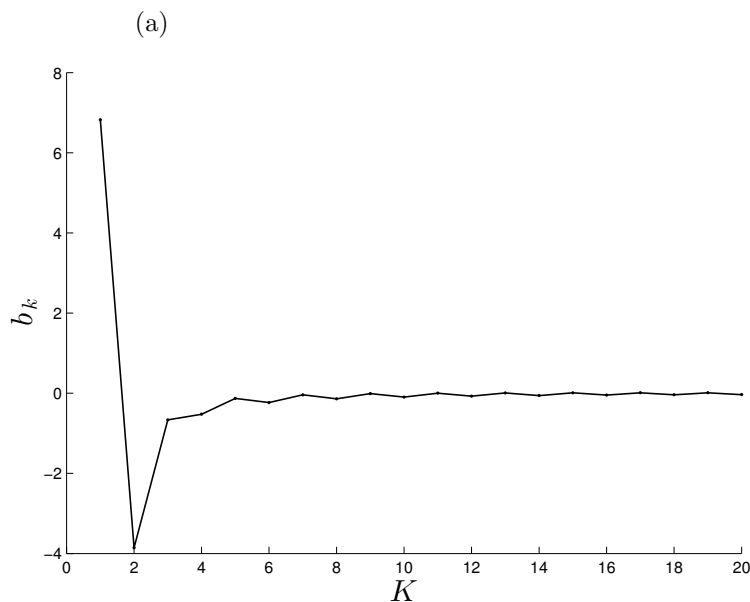
Final step, for stable solution zeroth-order Tikhonov regularization has been applied (19). In order to find good parameter, several parameters λ are tested such as

$$\lambda \in \{10^{-4}, 5 \times 10^{-4}, 10^{-3}, 5 \times 10^{-3}, \dots, 10^0\}$$

and minimum norm error

$$\|f_{\text{numerical}} - f_{\text{exact}}\| = \sqrt{\sum_{n=1}^N (f_{\text{numerical}}(t_n) - f_{\text{exact}}(t_n))^2}$$

[4, 6, 7, 8] have applied to choose the best one, as shown in Figure 10. In both Figures 10(a) and 10(b) receive $\lambda = 10^{-3}$ best approach $f(x)$ reach with exact solution which is showing in Figure 12. This result is derived from the fact that there are two figures in Figures 10 and 12, using boundary condition (22)|or (23).



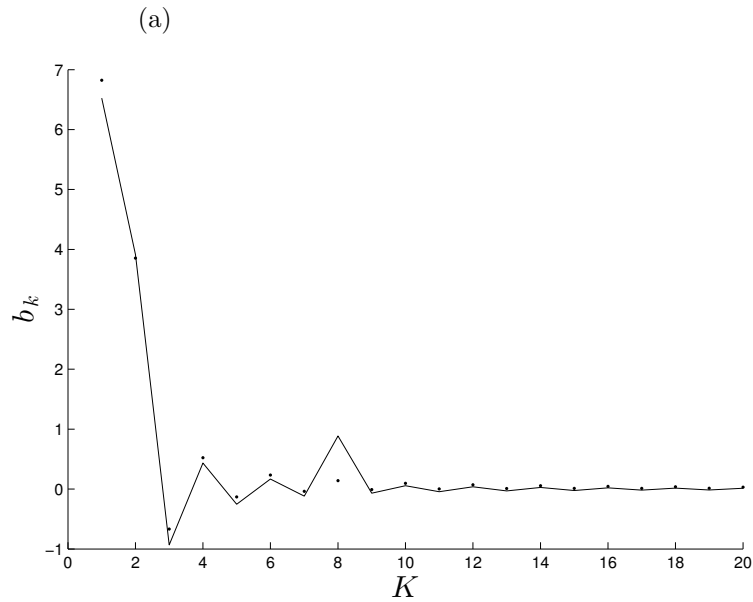
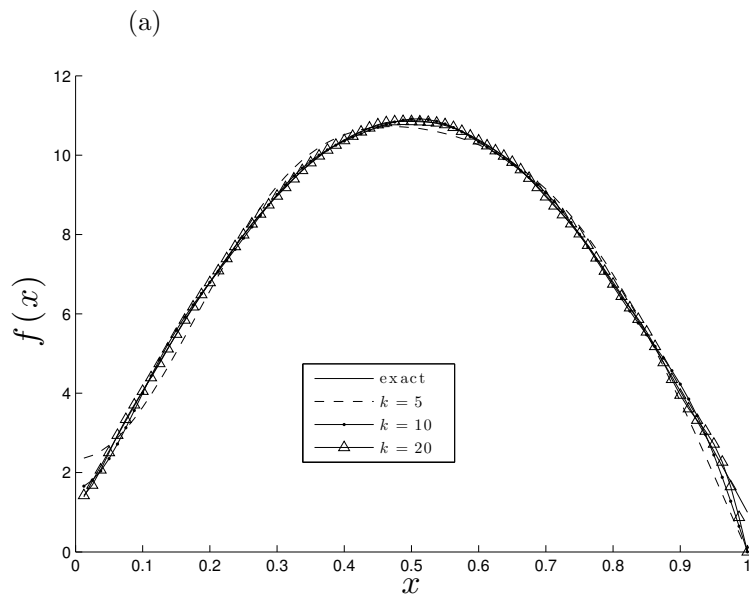


Figure 4: The numerical solution (...) for $\underline{b}$ for $K = 20, N = 80$, obtained from equation (15) in comparison with the exact solution (a) $b_k = \sqrt{2} \int_0^1 f(x) \cos((k - \frac{1}{2}) \pi x) dx$ (-), (b) $b_k = \sqrt{2} \int_0^1 f(x) \sin((k - \frac{1}{2}) \pi x) dx$ (-)



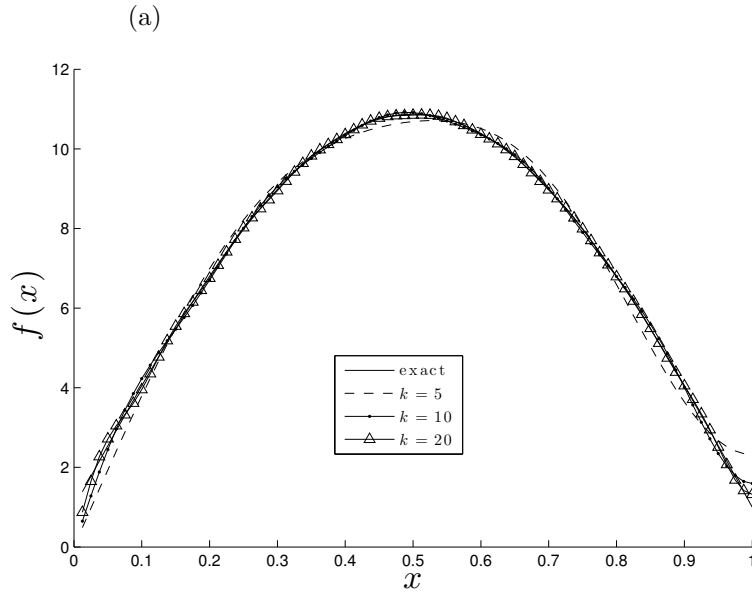
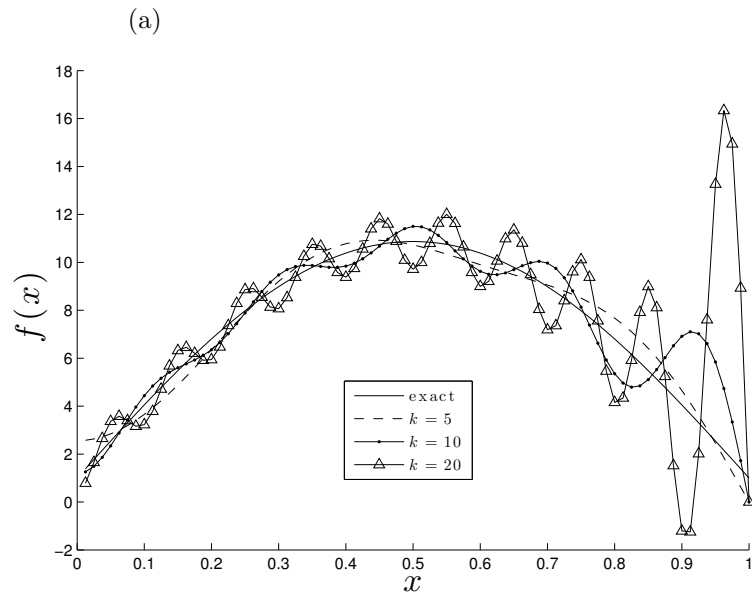


Figure 6: The exact solution (20) for $f(x)$ in comparison with the numerical solution (12)—or(14) (a) with condition (22) (b) with boundary condition (23) are used for various $K \in \{5, 10, 20\}$.



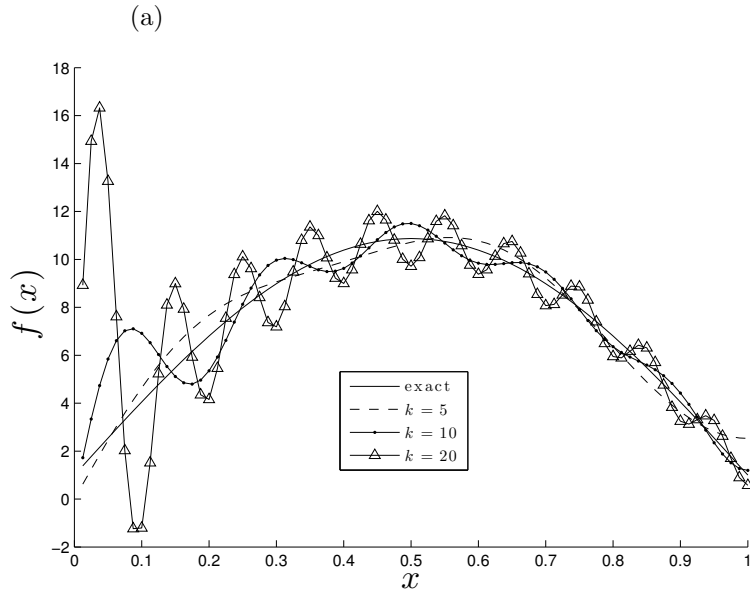
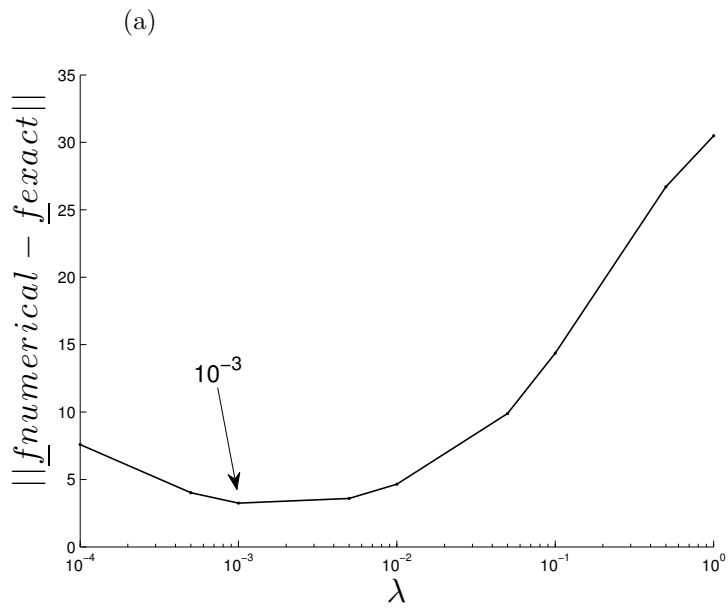


Figure 8: The exact solution (20) for $f(x)$ in comparison with the numerical solution (12)—or(14), for $p\% = 1\%$ and $K \in \{5, 10, 20\}$ using (a) boundary condition (22) (b) boundary condition (23), in noisy data.



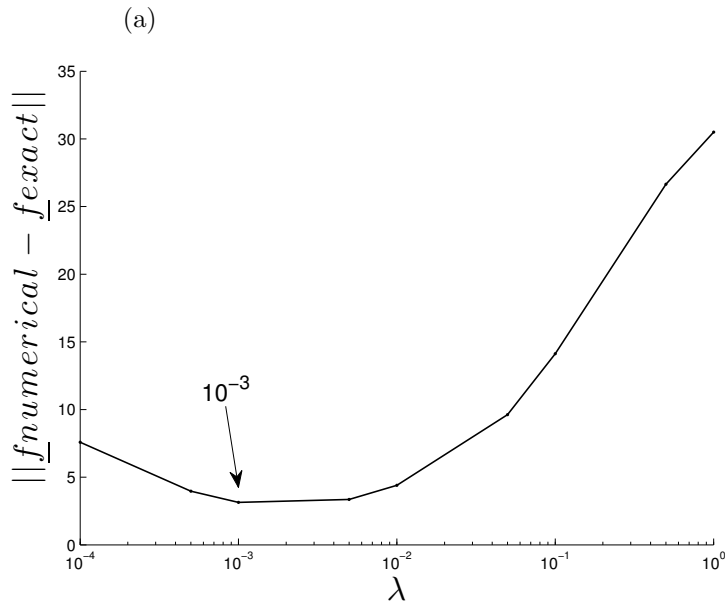
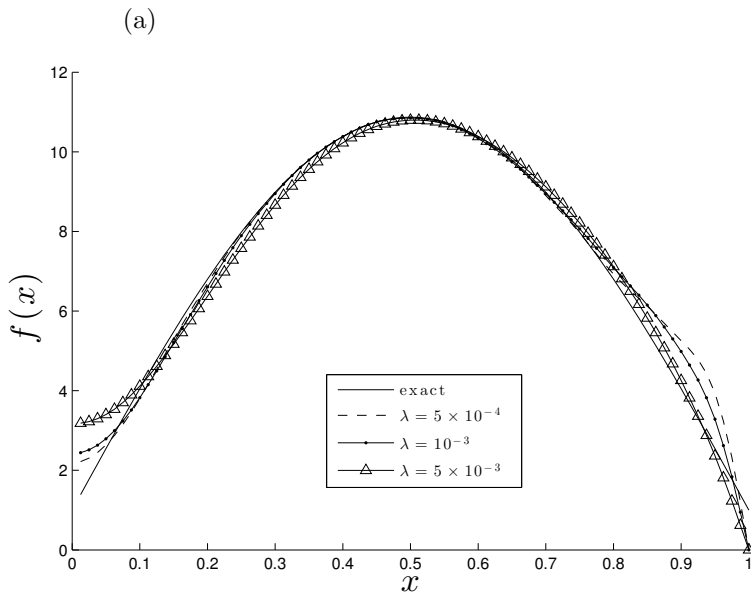


Figure 10: The accuracy error $\|f_{\text{numerical}} - f_{\text{exact}}\|$ after applying Tikhonov regularization technique, as a function of λ , for $K = 20$ and $p\% = 1\%$ noise, boundary condition are applied (a) (22) (b) (23).



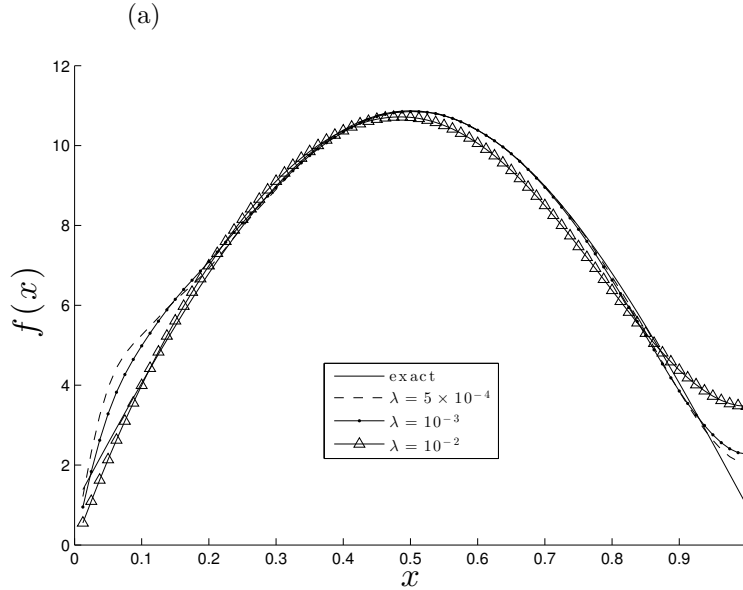


Figure 12: The exact solution (20) for $f(x)$ in comparison with the numerical solution (12)—or (14), for $K = 20$, $p\% = 1\%$ noise, and regularization parameters (a) $\lambda \in \{5 \times 10^{-4}, 10^{-3}, 5 \times 10^{-3}\}$ boundary condition are applied (22) (b) $\lambda \in \{5 \times 10^{-4}, 10^{-3}, 10^{-2}\}$ boundary condition (23).

4. Conclusions

Applying upper-base as additional data has been studying in this research. The non-homogeneous hyperbolic problem has solved by splitting in two parts, part of direct problem and part of inverse problem. Furthermore, the direct problem is well-posed and inverse problem is ill-posed which is unstable small error in upper-base $u_T(x)$ case error in force $f(x)$. The difference from this work in comparison with the other two parts [7, 8] is additional data depended on x "space" whereas the other on t "time". As I said in [7, 8], zeroth-order Tikhonov regularization and minimum norm errors have been used to regularized the solution which is clear from the presented figures.

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Properties of weakly 2-absorbing primal ideals

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Abstract. Let R be a commutative ring with unity ($1 \neq 0$). In this paper we introduce the concept of weakly 2-absorbing primal ideal which is a generalization of weakly primal ideal. Let I be a proper ideal of R . An element $a \in R$ is defined to be a weakly 2-absorbing prime to I if for any $r, s, t \in R$ with $0 \neq rsta \in I$, then $rs \in I$ or $rt \in I$ or $st \in I$. An element $a \in R$ is not a weakly 2-absorbing prime to I if there exist $r, s, t \in R$, with $0 \neq rsta \in I$, such that $rs, rt, st \in R \setminus I$. We denote by $\nu_0(I)$ the set of all elements in R that are not weakly 2-absorbing prime to I . We define a proper ideal I of R to be a weakly 2-absorbing primal if the set $\nu_0(I) \cup \{0\}$ forms an ideal of R . Many results concerning weakly 2-absorbing primal ideals and examples of weakly 2-absorbing primal ideals are given.

Keywords: weakly 2-absorbing ideal, weakly 2-absorbing primal ideal.

1. Introduction

We study in this paper weakly 2-absorbing primal ideals in commutative rings with unity, which are generalization of weakly primal ideals. Ebrahimi Atani and Ahmad Darani gave a generalization of weakly primal ideals (see [3]). Recall from [3] that if R is a commutative ring with unity and I is a proper ideal of R , then $a \in R$ is a weakly prime to I if $0 \neq ra \in I$, for some $r \in R$, then $r \in I$. Also recall from [3] that $a \in R$ is not a weakly prime to I if there exists $r \in R \setminus I$ such that $0 \neq ra \in I$. In [3] Ebrahimi Atani defined $S_0(I)$ by the set of all elements a in R that are not weakly prime to I . In [3] Ebrahimi Atani defined I to be a weakly primal ideal in R if $S_0(I) \cup \{0\}$ forms an ideal in R .

The idea of a weakly 2-absorbing primal ideal of a given ring R which is a generalization of a weakly primal ideal of R is come from the concept of a weakly 2-absorbing ideal of R which is a generalization of a weakly prime ideal of R . Recall that weakly 2-absorbing ideals, which are generalizations of weakly prime ideals, were introduced in [5]. Let I be a proper ideal of R , in [5] A. Badawi and A. Y. Darani defined I to be a weakly 2-absorbing ideal of R if whenever $a, b, c \in R$ and $0 \neq abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$.

In this paper, we define $a \in R$ to be a weakly 2-absorbing prime to a proper ideal I of R , if $0 \neq rsta \in I$, where $r, s, t \in R$, then $rs \in I$ or $rt \in I$ or $st \in I$. Also we define $a \in R$ to be not a weakly 2-absorbing prime to a proper ideal I of

R , if there exist $r, s, t \in R$ with rs, rt and st are in $R \setminus I$ such that $0 \neq rsta \in I$. Let I be a proper ideal of R , and let $\nu_0(I)$ be the set of all $a \in R$ such that a is not a weakly 2-absorbing prime to I . In this paper I is defined to be a weakly 2-absorbing primal ideal of R if $\nu_0(I) \cup \{0\}$ forms an ideal in R . Let I be a proper ideal of R , in [9] we define $a \in R$ to be a 2-absorbing prime to I if for any $r, s, t \in R$ such that $rsta \in I$, then $rs \in I$ or $rt \in I$ or $st \in I$. In [9] we define I to be a 2-absorbing primal ideal of R if $\nu(I)$ forms an ideal of R , where $\nu(I) = \{a \in R : a \text{ is not } 2\text{-absorbing prime to } I\}$. In this paper some basic properties of weakly 2-absorbing primal ideals are studied and classified, and some examples are given. Also the relations between 2-absorbing primal ideals and weakly 2-absorbing primal ideals are studied.

2. Basic properties of weakly 2-absorbing primal ideals

Let R be a commutative ring with unity and let I be a proper ideal of R . An element $a \in R$ is a weakly 2-absorbing prime to I if for any $r, s, t \in R$ with $0 \neq rsta \in I$, then rs or rt or st is in I . An element $a \in R$ is not a weakly 2-absorbing prime to I if there exist $r, s, t \in R$, with $0 \neq rsta \in I$, such that rs, rt and st are in $R \setminus I$. We denote by $\nu_0(I)$ the set of all elements in R that are not weakly 2-absorbing prime to I .

It is clear that every weakly primal ideal of a ring R is weakly 2-absorbing primal ideal of R . If $R = \mathbb{Z}_{16}$ and $I = \{0, 8\}$, then one can easily see that $\nu_0(I) \cup \{0\} = \mathbb{Z}_{16}$ since $0 \neq 2 \cdot 2 \cdot 2 \in I$ and $4 \notin I$, so I is a weakly 2-absorbing primal ideal of $\mathbb{Z}_{16}$ with $\nu_0(I) \cup \{0\} = \mathbb{Z}_{16}$. Also one can easily see that $S_0(I) \cup \{0\} = 2\mathbb{Z}_{16} \neq \nu_0(I) \cup \{0\}$. Therefore, $I = \{0, 8\}$ is a weakly primal and weakly 2-absorbing primal ideal of $\mathbb{Z}_{16}$ with $S_0(I) \neq \nu_0(I)$. Also if $I = \{0\}$ is an ideal in $\mathbb{Z}_6$, then, from the definition of weakly 2-absorbing primal ideals, I is a weakly 2-absorbing primal ideal of $\mathbb{Z}_6$ with $\nu_0(I) \cup \{0\} = \{0\}$, but easy computations implies that I is not a 2-absorbing primal ideal of $\mathbb{Z}_6$. The following are two examples of nonzero weakly 2-absorbing primal ideals that are not weakly primal ideals.

Example 2.1. (1) Let $R = \mathbb{Z}$ and let $I = 30\mathbb{Z}$. Then I is a weakly 2-absorbing primal ideal of $\mathbb{Z}$ with $\nu_0(I) \cup \{0\} = \mathbb{Z}$, since $(2)(3)(5) = 30 \in I$ and $(2)(3) = 6 \notin I$, $(2)(5) = 10 \notin I$ and $(3)(5) = 15 \notin I$. On the other hand I is not a weakly primal ideal in $\mathbb{Z}$, because $2, 3 \in S_0(I)$ but $1 \notin S_0(I)$. Note that if $1 \in S_0(I)$, then there exists $r \notin I$ with $1 \cdot r = r \in I$, a contradiction.

(2) Let $R = \mathbb{Z}[x, y, z]$ and let $I = xyzR$. Then I is a proper ideal of R and since $0 \neq xyz \in I$ with xy, xz , and yz are in $R \setminus I$ we get that $\nu_0(I) \cup \{0\} = R$. Therefore, I is a weakly 2-absorbing primal ideal of R .

On the other hand, since $0 \neq xyz \in I$ and $yz \in R \setminus I$ we get that $x \in S_0(I)$. Similarly, $y \in S_0(I)$. Now we show that $x + y$ can't be in $S_0(I)$. If there exists $f(x, y, z) \in \mathbb{Z}[x, y, z]$ with $0 \neq (x + y)f(x, y, z) \in I$, then xyz divides $(x + y)f(x, y, z)$ and since x divides xyz we get that x divides $(x + y)f(x, y, z)$ but x does not divide $x + y$, so x must divide $f(x, y, z)$. Similarly, y divides

$f(x, y, z)$ and z divides $f(x, y, z)$. Therefore, xyz divides $f(x, y, z)$ which implies that $f(x, y, z) \in I$, so $x + y \notin S_0(I)$ and hence I is not a weakly primal ideal of R .

Now, we have the following easy result.

Lemma 2.1. *Every weakly primal ideal of R is a weakly 2-absorbing primal.*

Theorem 2.1. *Let I be a proper ideal of R such that I is a weakly 2-absorbing primal ideal of R with $\nu_0(I) \cup \{0\} \neq R$. Then $\nu_0(I) \cup \{0\}$ is a weakly prime ideal of R .*

Proof of Theorem 2.1. Let $a, b \in R$ such that $0 \neq ab \in \nu_0(I)$. Then there exist $r, s, t \in R$ with $0 \neq rst(ab) \in I$ such that $rs, rt, st \in R \setminus I$. Assume that $a \notin \nu_0(I)$. We must show that $b \in \nu_0(I)$. Since $0 \neq r(sb)ta \in I$ and $a \notin \nu_0(I)$, $0 \neq rsb \in I$ or $0 \neq rt \in I$ or $0 \neq sbt \in I$. But $rt \in R \setminus I$ implies $rsb \in I$ or $sbt \in I$. If $rsb \in I$, then $b \in \nu_0(I)$ since $r, s, rs \in R \setminus I$. Similarly, if $stb \in I$, then $b \in \nu_0(I)$. Therefore, $\nu_0(I) \cup \{0\}$ is a weakly prime ideal of R . $\square$

For example $I = 4\mathbb{Z}$ is a proper ideal of $\mathbb{Z}$ with $\nu_0(I) \cup \{0\} = 2\mathbb{Z}$. So I is a weakly 2-absorbing primal ideal of $\mathbb{Z}$ and $\nu_0(I) \cup \{0\} = 2\mathbb{Z}$ is a weakly prime ideal of $\mathbb{Z}$. But if $I = 6\mathbb{Z}$, then I is not a weakly 2-absorbing primal ideal of $\mathbb{Z}$, since $0 \neq (2)(3) \in I$ and $2, 3 \notin I$ so $2, 3 \in \nu_0(I)$, therefore, if $\nu_0(I) \cup \{0\}$ is an ideal of $\mathbb{Z}$, then $1 \in \nu_0(I)$ which implies that there exist $r, s, t \in \mathbb{Z} \setminus 6\mathbb{Z}$ such that $0 \neq rst \in 6\mathbb{Z}$ with $rs, rt, st \notin 6\mathbb{Z}$, but since 6 divides rst we have that 2 must divide r or s or t and 3 must divide r or s or t so 6 must divide rs or st or rt , a contradiction.

Definition 2.1. Let R be a commutative ring with unity ($1 \neq 0$) and let I be a proper ideal of R such that I is a weakly 2-absorbing primal ideal of R . Let $r, s, t \in R$, then (r, s, t) is called a triple zero of I if $rst = 0$ with $rs, rt, st \in R \setminus I$.

The following five results on weakly 2-absorbing primal ideals over R and the results on weakly 2-absorbing ideals of R are the same and proved in [5] with another approach.

Theorem 2.2. Let R be a commutative ring with unity ($1 \neq 0$) and let I be a proper ideal of R . Suppose that I is a weakly 2-absorbing primal ideal of R with $1 \notin \nu_0(I)$. If (r, s, t) is a triple zero of I , then:

- (1) $rsI = rtI = stI = 0$;
- (2) $rI^2 = sI^2 = tI^2 = 0$.

Proof of Theorem 2.2. (1) If $rsI \neq 0$, then there exists $a \in I$ such that $0 \neq rsa \in I$. So, $0 \neq rs(t + a) = rsa \in I$ with $rs, r(t + a), s(t + a) \in R \setminus I$ implies that $1 \in \nu_0(I)$, a contradiction. Therefore, $rsI = 0$. Similarly, $rtI = 0$ and $stI = 0$.

(2) Suppose $rI^2 \neq 0$, then there exist $a, b \in I$ such that $rab \neq 0$. So, $0 \neq r(s + a)(t + b) = ab \in I$ with $r(s + a), r(t + b), (s + a)(t + b) \in R \setminus I$ implies

that $1 \in \nu_0(I)$, a contradiction. Therefore, $rI^2 = 0$. Similarly, $sI^2 = 0$ and $tI^2 = 0$. $\square$

Let I be a proper ideal of R such that I is a weakly 2-absorbing primal ideal of R with $1 \notin \nu_0(I)$. If I is a 2-absorbing primal ideal of R with $\nu(I) = R$. Then, by using Theorem 2.2, we have the following result.

Theorem 2.3. *Let R be a commutative ring with unity ($1 \neq 0$) and let I be a proper ideal of R . Suppose that I is a weakly 2-absorbing primal ideal of R with $1 \notin \nu_0(I)$ such that $\nu(I) = R$, then $I^3 = 0$.*

Proof of Theorem 2.3. Since $\nu(I) = R$ we have that $1 \in \nu(I)$, and hence there exist $r, s, t \in R$ with $rst = 0$ such that $rs, rt, st \in R \setminus I$. Thus, (r, s, t) is a triple zero of I , since if $0 \neq rst \in I$, then $1 \in \nu_0(I)$, a contradiction. Suppose that $I^3 \neq 0$, then there exist $a, b, c \in I$ such that $0 \neq abc \in I$. So by Theorem 2.2 we have $0 \neq (r+a)(s+b)(t+c) = abc \in I$ and since $1 \notin \nu_0(I)$ we must have that $(r+a)(s+b) \in I$ or $(r+a)(t+c) \in I$ or $(s+b)(t+c) \in I$ and hence we have either $rs \in I$ or $rt \in I$ or $st \in I$, a contradiction. Hence $I^3 = 0$. $\square$

Let R be a commutative ring with unity ($1 \neq 0$). Then $\text{Nil}(R)$ denotes the ideal of nilpotent elements of R .

Corollary 2.1. *Let R be a commutative ring with unity ($1 \neq 0$) and let I be a proper ideal of R . Suppose that I is a weakly 2-absorbing primal ideal of R with $1 \notin \nu_0(I)$. If $\nu(I) = R$, then $I \subseteq \text{Nil}(R)$.*

Theorem 2.4. *Let I be a proper ideal of R . Suppose that I is a weakly 2-absorbing primal ideal of R with $1 \notin \nu_0(I)$ such that $\nu(I) = R$. Then:*

- (1) *If $a \in \text{Nil}(R)$, then either $a^2 \in I$ or $a^2I = aI^2 = 0$;*
- (2) *$(\text{Nil}(R))^2I^2 = 0$.*

Proof of Theorem 2.4. (1) Let $a \in \text{Nil}(R)$. First, we show that if $a^2I \neq 0$, then $a^2 \in I$. Now assume that $a^2I \neq 0$. Let $i \in I$ such that $0 \neq a^2i \in I$ and suppose that n is the smallest positive integer such that $a^n = 0$. Then $n \geq 3$ and we have $0 \neq a^2(i + a^{n-2}) \in I$, since $1 \notin \nu_0(I)$ we must have $a^2 \in I$ or $a^{n-1} \in I$. If $a^2 \in I$, then we are done. If $a^{n-1} \in I$, then $0 \neq a^2a^{n-3} \in I$ again since $1 \notin \nu_0(I)$ we have $a^{n-2} \in I$. Continuing this procedure, then we arrive at $a^2 \in I$. Therefore for each $a \in \text{Nil}(R)$ we have either $a^2 \in I$ or $a^2I = 0$. Now, assume that $b^2 \notin I$ for some $b \in \text{Nil}(R)$, then $b^2I = 0$. We show that $bI^2 = 0$. If $bI^2 \neq 0$, then there exist $i_1, i_2 \in I$ such that $bi_1i_2 \neq 0$. Let m be the smallest positive integer such that $b^m = 0$, then $m \geq 3$ since $b^2 \notin I$. Hence $0 \neq b(b + i_1)(b^{m-2} + i_2) = bi_1i_2 \in I$ and since $1 \notin \nu_0(I)$ we have $b(b + i_1) \in I$ which implies that $b^2 \in I$ (a contradiction) or $b(b^{m-2} + i_2) \in I$ which implies that $b^{m-1} \in I$ (a contradiction). Therefore, $bI^2 = 0$.

(2) Let $r, s \in \text{Nil}(R)$. If $r^2 \notin I$ or $s^2 \notin I$, then, by (1), $(rs)I^2 = 0$. Therefore we may assume that $r^2 \in I$ and $s^2 \in I$. So, $rs(r + s) \in I$. If $(r, s, r + s)$ is a triple zero of I , then, by Theorem 2.2(1), $(rs)I = 0$ and hence $(rs)I^2 = 0$.

If $0 \neq rs(r+s) \in I$, then $rs \in I$ since $1 \notin \nu_0(I)$. So, by Theorem 2.3, $(rs)I^2 \subseteq I^3 = 0$. $\square$

Corollary 2.2. *Let R be a commutative ring with unity ($1 \neq 0$) and let A, B, C be proper ideals of R . Suppose that A, B, C are weakly 2-absorbing primal ideals of R with $1 \notin \nu_0(A) \cup \nu_0(B) \cup \nu_0(C)$ such that $\nu(A) = \nu(B) = \nu(C) = R$. Then, $A^2BC = AB^2C = ABC^2 = A^2B^2 = A^2C^2 = B^2C^2 = 0$.*

Proof of Corollary 2.2. By Corollary 2.1, $B \subseteq \text{Nil}(R)$ and $C \subseteq \text{Nil}(R)$, so $A^2BC \subseteq A^2(\text{Nil}(R))^2$ and by Theorem 2.4(2), $A^2(\text{Nil}(R))^2 = 0$. Similarly, $AB^2C = ABC^2 = 0$. Also, by Corollary 2.1, $A^2B^2 \subseteq (\text{Nil}(R))^2B^2 = 0$. Similarly, $A^2C^2 = B^2C^2 = 0$. $\square$

In the next result we give a condition on a weakly 2-absorbing primal ideal of R to be a 2-absorbing primal ideal of R .

Theorem 2.5. *Let R be a commutative ring with unity ($1 \neq 0$) and let I be a proper ideal of R . If I is a weakly 2-absorbing primal ideal of R with $I^2 \neq 0$, then I is a 2-absorbing primal ideal of R .*

Proof of Theorem 2.5. If $1 \in \nu(I)$, then $\nu(I) = R$ which implies that I is a 2-absorbing primal ideal of R . Therefore we may assume that $1 \notin \nu(I)$. To prove that I is a 2-absorbing primal ideal of R we must show that $\nu(I) = \nu_0(I) \cup \{0\}$. It is clear that $\nu_0(I) \cup \{0\} \subseteq \nu(I)$. Conversely, let $a \in \nu(I)$, then there exist $r, s, t \in R$ with $rs, rt, st \in R \setminus I$ such that $(rst)a \in I$. If $(rst)a \neq 0$, then $a \in \nu_0(I)$. So we may assume that $rsta = 0$. If $(rst)I \neq 0$, then there exists $c \in I$ such that $rstc \neq 0$. Therefore, $0 \neq (rst)(a+c) \in I$ which implies that $a+c \in \nu_0(I)$ and hence $a \in \nu_0(I)$, since $c \in \nu_0(I)$. Therefore we may assume that $(rst)I = 0$. If $rst \in I$, then $1 \in \nu(I)$ which is a contradiction. Therefore we may assume that $rst \notin I$. Since $I^2 \neq 0$, there exist $x, y \in I$ such that $xy \neq 0$. Hence, $(a+y)(trs+x) = 0 + ax + 0 + xy \in I$. If $0 \neq ax + xy \in I$ and since $trs+x \notin I$, then $a+y \in \nu_0(I)$ which implies that $a \in \nu_0(I)$, since $y \in \nu_0(I)$. But, if $ax + xy = 0$, then $0 \neq ax \in I$ which implies that $0 \neq a(x+trs) = ax \in I$, so $a \in \nu_0(I)$. Thus, $\nu(I) = \nu_0(I) \cup \{0\}$. $\square$

We have to remark that if a proper ideal I of R is a weakly 2-absorbing primal ideal of R with $I^2 \neq 0$, and $1 \notin \nu(I)$, then $\nu_0(I) \cup \{0\}$ is a prime ideal of R since by Theorem 2.5, $\nu(I) = \nu_0(I) \cup \{0\}$.

We recall that if R and S are commutative rings with unity and P, Q are weakly prime ideals in R, S (respectively), then $P \times S$ and $R \times Q$ are weakly prime ideals of $R \times S$.

Theorem 2.6. *Let $R \times S$ be a commutative ring with unity, where R, S are commutative rings with unities, let I be a proper ideal of R with $I \times S \not\subseteq \text{Nil}(R \times S)$. Then, the following statements are equivalent.*

- (1) $I \times S$ is a weakly 2-absorbing primal ideal of $R \times S$;
- (2) $I \times S$ is a 2-absorbing primal ideal of $R \times S$;
- (3) I is a 2-absorbing primal ideal of R .

Proof of Theorem 2.6. $(1 \rightarrow 2)$. Since $I \times S \not\subseteq \text{Nil}(R \times S)$, by Corollary 2.1 we have that $\nu(I \times S) \neq R \times S$. To prove that $I \times S$ is a 2-absorbing primal ideal of $R \times S$ we must show that $\nu(I) = \nu_0(I) \cup I$. Let $a \in \nu(I)$ and let $(rst)a = 0$ for some $r, s, t \in R$ with $rs, rt, st \in R \setminus I$. Since $1 \notin \nu(I)$ and $rs \notin I$ we must have $rta \in I$ or $sta \in I$. If $rta \in I$, then $ra \in I$ or $ta \in I$ since $1 \notin \nu(I)$ and $rt \notin I$. If $ra \in I$, then $a \in I$ since $r \notin I$ and $1 \notin \nu(I)$, similarly, if $ta \in I$, then $a \in I$. Similarly, if $sta \in I$, then $a \in I$. Therefore, $\nu(I) = \nu_0(I) \cup I$ and hence $\nu(I \times S) = \nu(I) \times S$ is an ideal of $R \times S$ which implies that $I \times S$ is a 2-absorbing primal ideal of $R \times S$.

$(2 \rightarrow 3)$. Since $\nu(I \times S) = \nu(I) \times S$ is a prime ideal of $R \times S$ we have that $\nu(I)$ is a prime ideal of R , so I is a 2-absorbing primal ideal of R .

$(3 \rightarrow 1)$. Since I is a 2-absorbing primal ideal of R we have that $I \times S$ is a 2-absorbing primal ideal of $R \times S$, hence $I \times S$ is a weakly 2-absorbing primal ideal of $R \times S$. $\square$

Theorem 2.7. *Let $R \times S$ be a commutative ring with unity, where R, S are commutative rings with unities, let I be a nonzero proper ideal of R and J a nonzero ideal of S with $I \times J \not\subseteq \text{Nil}(R \times S)$. Then, the following statements are equivalent.*

- (1) $I \times J$ is a weakly 2-absorbing primal ideal of $R \times S$;
- (2) $J = S$ and I is a 2-absorbing primal ideal of R ;
- (3) $I \times J$ is a 2-absorbing primal ideal of $R \times S$.

Proof of Theorem 2.7. $(1 \rightarrow 2)$. Suppose $I \times J$ is a weakly 2-absorbing primal ideal of $R \times S$. Since $I \times J \not\subseteq \text{Nil}(R \times S)$ and if $J = S$, then I is a 2-absorbing primal ideal of R by Theorem 2.6. We show that the case $J \neq S$ can not be happened. Suppose $J \neq S$, we show that J is a prime ideal in S and I is a prime ideal of R . Since $I \times J \not\subseteq \text{Nil}(R \times S)$, by Corollary 2.1 we have that $\nu(I \times J) \neq R \times S$. Let $a, b \in S$ such that $ab \in J$ and let $0 \neq i \in I$. Then, $(i, 1)(1, a)(1, b) = (i, ab) \neq (0, 0) \in I \times J$ since $(1, ab) \notin I \times J$ and since $(1, 1) \notin \nu_0(I \times J)$ we must have $(i, a) \in I \times J$ or $(i, b) \in I \times J$ so $a \in J$ or $b \in J$. Thus J is a prime ideal of S . Similarly, let $c, d \in R$ such that $cd \in I$, and let $0 \neq j \in J$. Then, $(c, 1)(d, 1)(1, j) = (cd, j) \neq (0, 0) \in I \times J$, since $(cd, 1) \notin I \times J$ and since $(1, 1) \notin \nu_0(I \times J)$ we must have $(c, j) \in I \times J$ or $(d, j) \in I \times J$ so $c \in I$ or $d \in I$. Hence I is a prime ideal of R . In this case we show that $(1, 1) \in \nu(I \times J)$. Now, $(1, 0)(0, 1) \in I \times J$ and $(1, 0) \notin I \times J$, $(0, 1) \notin I \times J$, so $(1, 0), (0, 1) \in \nu(I \times J)$. Therefore, if $\nu(I \times J)$ is an ideal in $R \times S$, then $(1, 1) = (1, 0) + (0, 1) \in \nu(I \times J)$, a contradiction to Corollary 2.1. Therefore the only case of part (2) is that $J = S$ and I is a 2-absorbing primal ideal of R .

$(2 \rightarrow 3)$. If $J = S$ and I is a 2-absorbing primal ideal of R , then $I \times J$ is a 2-absorbing primal ideal of $R \times S$ by Theorem 2.6(2).

$(3 \rightarrow 1)$. Clearly from Theorem 2.6 $\square$

Theorem 2.8. *Let $R \times S$ be a commutative ring with unity, where each R and S are commutative rings with unities, let $I = I_1 \times I_2$ be a proper ideal of $R \times S$.*

If I is a weakly 2-absorbing primal ideal of $R \times S$ with $\nu_0(I) \neq \phi$. Then, either $I = (0, 0)$ or I is a 2-absorbing primal ideal of $R \times S$.

Proof of Theorem 2.8. Suppose $I \neq (0, 0)$. Since $\nu_0(I) \neq \phi$, by Theorem 2.1, $\nu_0(I) \cup \{(0, 0)\}$ is a weakly prime ideal of $R \times S$. Therefore, $\nu_0(I) \cup \{(0, 0)\}$ has one of the following cases:

(1) $\nu_0(I) \cup \{(0, 0)\} = P_1 \times P_2$, where each $P_i = \{0\}$. In this case $\nu_0(I) \cup \{(0, 0)\} = (0, 0)$, a contradiction.

(2) $\nu_0(I) \cup \{(0, 0)\} = P_1 \times S$, where P_1 is a weakly prime ideal in R and since $S \neq 0$, P_1 is a prime ideal of R , so $\nu_0(I) \cup \{(0, 0)\}$ is a prime ideal of $R \times S$. Now, we show that $I_2 = S$. Since $S \neq 0$, there exists $(a, b) \neq (0, 0) \in I$ and so $(0, 0) \neq (a, b) = (1, 1)(a, 1)(1, b)$. If $(a, 1) \notin I$, then $(1, b) \in \nu_0(I)$ and hence $1 \in P_1$, a contradiction. Therefore, $(a, 1) \in I$ which implies that $1 \in I_2$, hence $I_2 = S$. Now, we show that I_1 is a 2-absorbing primal ideal of R with $\nu(I_1) = P_1$. Let $a_1 \in P_1$, then $(a_1, 0) \in \nu_0(I)$. If $a_1 = 0$, then $a_1 \in I_1 \subseteq \nu(I_1)$, so we may assume that $a_1 \neq 0$. Thus, there exist $(r_1, r_2), (s_1, s_2), (t_1, t_2) \in R \times S$ with $(r_1s_1, r_2s_2), (r_1t_1, r_2t_2), (t_1s_1, t_2s_2) \in R \times S \setminus I = (R \setminus I_1) \times S$ such that $(0, 0) \neq (r_1s_1t_1, r_2s_2t_2)(a_1, 0) \in I$, so $0 \neq r_1s_1t_1a_1 \in I_1$ with $r_1s_1, r_1t_1, s_1t_1 \notin I_1$ which implies that a_1 is not a weakly 2-absorbing prime to I_1 , hence a_1 is not a 2-absorbing prime to I_1 and so $a_1 \in \nu(I_1)$. Conversely, let $a_1 \in \nu(I_1)$. Then, there exist $r, s, t \in R$ with $rs, rt, st \in R \setminus I_1$ such that $rsta_1 \in I_1$. Since $S \neq 0$, we have $(0, 0) \neq (rsta_1, 1) = (r, 1)(s, 1)(t, 1)(a_1, 1) \in I_1 \times S$ with $(rs, 1), (rt, 1), (st, 1) \in (R \setminus I_1) \times S = (R \times S) \setminus I$ which implies that $(a_1, 1)$ is not a weakly 2-absorbing prime to I , hence $(a_1, 1) \in \nu_0(I) \cup \{(0, 0)\} = P_1 \times S$, so $a_1 \in P_1$. Therefore, $\nu(I_1) = P_1$. Thus, $\nu(I) = P_1 \times S$. Hence $I = I_1 \times S$ is a 2-absorbing primal ideal of $R \times S$.

(3) $\nu_0(I) \cup \{(0, 0)\} = R \times P_2$, where P_2 is a weakly prime ideal in S . By using the same approach in part 2 we conclude that $I = R \times I_2$ and $\nu(I_2) = P_2$, and so $\nu(I) = R \times P_2$. Hence $I = R \times I_2$ is a 2-absorbing primal ideal of $R \times S$. $\square$

3. More properties of weakly 2-absorbing primal ideals

For a commutative ring R , let $\mathcal{J}(R)$ denotes the intersection of all maximal ideals of R .

Lemma 3.1. *Let R be a commutative ring and $a, b \in \mathcal{J}(R)$. Then the ideal $I = abR$, where $1 \notin \nu_0(I)$, is a weakly 2-absorbing primal ideal of R if and only if $ab = 0$.*

Proof of Lemma 3.1. If $ab = 0$, then $I = 0$ is a weakly 2-absorbing primal ideal of R by definition. If $0 \neq ab \in I$ with $a, b \notin I$, then $1 \in \nu_0(I)$, contradiction. Therefore, a or b is in $I = abR$. If $a \in I$, then $a = abk$ for some $k \in R$. So, $a(1 - bk) = 0$ and since $bk \in \mathcal{J}(R)$ we have that $1 - bk$ is a unit in R . Thus, $a(1 - bk) = 0$ implies that $a = 0$ and hence $ab = 0$, a contradiction. Therefore, $I = 0$. $\square$

We recall that R is defined to be quasi-local ring if R has a unique maximal ideal. If (R, M) is a quasi-local ring, where M is the unique maximal ideal of R , then we have the following two results about a weakly 2-absorbing primal ideal I of R with $1 \notin \nu_0(I)$.

Theorem 3.1. *Let (R, M) be a quasi-local ring with $\nu_0(I) \neq R$, for all proper ideals I of R . Then every proper ideal of R is a weakly 2-absorbing primal ideal if and only if $M^2 = 0$.*

Proof of Theorem 3.1. Let $a, b \in M$, then $I = abR$ is a weakly 2-absorbing primal ideal of R with $1 \notin \nu_0(I)$, hence, by Lemma 3.1, $M^2 = 0$. Conversely, suppose $M^2 = 0$. Let I be a proper ideal of R and let $a \in \nu_0(I)$. If a is a unit in R , then $1 \in \nu_0(I)$, a contradiction. So we may assume that a is not a unit in R . Let $r, s, t \in R$ with $0 \neq rsta \in I$ such that $rs, rt, st \notin I$. If $0 \neq rst \in I$, then $M^2 = 0$ and $a \in M$ implies $rst \notin M$. So rst is a unit in R which implies that $a \in I$. So $\nu_0(I) \cup \{0\} = I$ which implies that I is weakly 2-absorbing primal ideal of R . $\square$

Corollary 3.1. *Let (R, M) be a quasi-local ring with $M^2 = 0$ and with $\nu_0(I) \neq R$, for all proper ideals I of R . Then every proper ideal of R is a 2-absorbing primal ideal of R .*

Proof of Corollary 3.1. Let I be a proper ideal of R , then, by Theorem 3.1, I is a weakly 2-absorbing primal ideal of R since $M^2 = 0$. We show that $\nu(I)$ is an ideal in R . Let a, b be nonzero elements in $\nu(I)$. Then, there exist $r, s, t \in R$ with $rs, rt, st \in R \setminus I$ such that $rsta \in I$. If $0 \neq rsta \in I$, then, by Theorem 3.1, $a \in I \subseteq M$. Since $rs \notin I$ we have r or s is a unit in R . Therefore, if $rsta = 0$, then $(st)a = 0$ or $(rt)a = 0$. Say $(st)a = 0$ again since $st \notin I$ we have s or t is a unit in R which implies that $sa = 0$ or $ta = 0$. Say $ta = 0$, hence t is not a unit in R , since $0 \neq a \in I$. Therefore if $ta = 0 \in I \subseteq M$ and a is not a unit in R (if a is a unit in R , then $t = 0$ a contradiction), then a must be in M since M is a prime ideal. Similarly, $b \in M$, so $a + b \in M$. If $t(a + b) \neq 0$, then t is a unit in R since $a + b \in M$, a contradiction. Therefore, $t(a + b) = 0 \in I$ which implies that $a + b \in \nu(I)$ since $t \notin I$. Hence $\nu(I)$ is an ideal of R . $\square$

In the next result we give the condition on a proper ideal I of R such that I/J is a weakly 2-absorbing primal ideal of R/J where J is a proper ideal of R contained in I .

Theorem 3.2. *Let I, J be proper ideals of R with $J \subseteq I$. If I is a weakly 2-absorbing primal ideal of R with $\nu_0(J) \subseteq I$. Then I/J is a weakly 2-absorbing primal ideal of R/J .*

Proof of Theorem 3.2. To prove that I/J is a weakly 2-absorbing primal ideal of R/J , we must show that $\nu_0(I/J) \cup \{0\} = [\nu_0(I) \cup J]/J$. Let $a + J \in \nu_0(I/J)$, then there exist $r + J, s + J, t + J \in R/J$ with $0 \neq rsta + J \in I/J$ such that $rs + J, rt + J, st + J \notin I/J$. So $0 \neq rsta \in I$, since $rsta \notin J$, with

$rs, rt, st \notin I$ hence $a \in \nu_0(I)$, therefore, $a + J \in [\nu_0(I) \cup J]/J$. Conversely, let $0 \neq a + J \in [\nu_0(I) \cup J]/J$, then $a \in \nu_0(I) - J$. If $a \in I$, then $a + J \in \nu_0(I/J)$. So we may assume that $a \notin I$. Then, there exist $r, s, t \in R$ with $0 \neq rsta \in I$ such that $rs, rt, st \notin I$. If $0 \neq rsta \in J$, then $a \in \nu_0(J)$, a contradiction, since $\nu_0(J) \subseteq I$ and $a \notin I$. Therefore, $r + J, s + J, t + J \in R/J$ with $0 \neq rsta + J = (rst + J)(a + J) \in I/J$ such that $rs + J, rt + J, st + J \notin I/J$, so $a + J \in \nu_0(I/J)$. Hence $\nu_0(I/J) \cup \{0\} = [\nu_0(I) \cup J]/J$ which implies that I/J is a weakly 2-absorbing primal ideal of R/J . $\square$

Corollary 3.2. *Let R_0 be a subring of R with unity. If I is a weakly 2-absorbing primal ideal of R , then $I \cap R_0$ is a weakly 2-absorbing primal ideal of R_0 .*

Proof of Corollary 3.2. The easy proof is left to the reader. $\square$

Let R be a commutative ring with unity ($1 \neq 0$) and let S be a multiplicatively closed proper subset of R with $1 \in S$. We recall that if R is a commutative ring with unity, then $R_S = \{\frac{a}{s} : a \in R, s \in S\}$ is a commutative ring with unity. Also if I is an ideal in R , then I_S is an ideal of R_S , where $I_S = \{\frac{a}{s} : a \in I, s \in S\}$. Moreover, if J is an ideal of R_S , then $J \cap R$ is an ideal R .

Consider the canonical homomorphism $\rho : R \rightarrow R_S$ which is defined by $r \mapsto \frac{r}{1}$, for all $r \in R$. Then ρ is a homogenous homomorphism of degree 0. Note that if P is a prime ideal of R with $P \cap S = \emptyset$, then $\rho^{-1}(P_S) = P$.

Theorem 3.3. *Let I be a weakly 2-absorbing-primal ideal of R with $\nu_0(I) \cap S = \emptyset$. If $0 \neq \frac{a}{s} \in I_S$, then $a \in I$.*

Proof of Theorem 3.3. Let $0 \neq \frac{a}{s} \in I_S$, then $\frac{a}{s} = \frac{b}{t}$ for some $b \in I$. So there exists $u \in S$ such that $uta = usb \in I$, since $b \in I$. If $uta = 0$, then $\frac{a}{s} = 0$, a contradiction, so $0 \neq uta \in I$. If $a \notin I$, then ut is not a weakly 2-absorbing prime to I which implies that $ut \in \nu_0(I)$, a contradiction. Therefore, $a \in I$. $\square$

We recall that $a \in R$ is a regular element in R , if $(0 : a) = 0$. In the rest of this section we assume that S is multiplicatively closed subset of R with $1 \in S$ such that all elements of S are regular.

Suppose that J is a 2-absorbing-primal ideal of R_S , we define the set of all elements in R_S that are not weakly 2-absorbing prime to J by $\nu_0(J)$. Let I be a proper ideal of R , under the condition $\nu_0(I) \cap S = \emptyset$ we have the following two results.

Lemma 3.2. *let I be a weakly 2-absorbing-primal ideal of R with $\nu_0(I) \cap S = \emptyset$. Then:*

- (1) I_S is a 2-absorbing-primal ideal of R_S ;
- (2) $I = I_S \cap R$.

Proof of Lemma 3.2. (1) It is enough to show that $\nu_0(I_S) = (\nu_0(I))_S$. Let $\frac{a}{u} \in \nu_0(I_S)$, then there exist $\frac{r}{t_1}, \frac{s}{t_2}, \frac{v}{t_3} \in R_S$ with $0 \neq (\frac{r}{t_1} \frac{s}{t_2} \frac{v}{t_3})(\frac{a}{u}) \in I_S$ such that

$\frac{rs}{t_1t_2}, \frac{rv}{t_1t_3}, \frac{sv}{t_2t_3} \notin I_S$. Therefore, $rs, rv, sv \notin I$ with $0 \neq (rsva)w \in I$ for some $w \in S$, since $\frac{rsva}{ut_1t_2t_3} \neq 0$. So $wa \in \nu_0(I)$ and hence $\frac{a}{u} \in (\nu_0(I))_S$. Conversely, let $\frac{a}{u} \in (\nu_0(I))_S$, then there exists $w \in S$ such that $aw \in \nu_0(I)$. So, let $r, s, t \in R$ with $rs, rt, st \notin I$ such that $0 \neq (rst)wa \in I$. If $rsta \notin I$, then $w \in \nu_0(I)$, a contradiction. Therefore, $0 \neq rsta \in I$ and since S consists of regular elements we have that $0 \neq \frac{rsta}{u} \in I_S$. If $\frac{rs}{1} \in I_S$, then $0 \neq rsv \in I$ for some $v \in S$ and so $v \in \nu_0(I)$, a contradiction. Therefore, $\frac{rs}{1} \notin I_S$, similarly, $\frac{rt}{1} \notin I_S$ and $\frac{st}{1} \notin I_S$. Thus, $\frac{a}{u} \in \nu_0(I_S)$ and hence I_S is a 2-absorbing-primal ideal of R_S .

(2) It is clear that $I \subseteq I_S \cap R$. Conversely, let $\frac{a}{1} \in I_S \cap R$, then there exists $w \in S$ such that $aw \in I$. If $aw = 0$, then $a = 0 \in I$. If $0 \neq aw \in I$ and $a \notin I$, then $w \in \nu_0(I)$, a contradiction. Thus, $a \in I$. So, $I = I_S \cap R$. $\square$

Proposition 3.1. *If J is a weakly 2-absorbing primal ideal of R_S with $\nu_0(J) \neq R_S$, then $J \cap R$ is a weakly 2-absorbing primal ideal of R with $\nu_0(J \cap R) \cap S = \phi$.*

Proof of Proposition 3.1. To prove that $J \cap R$ is a weakly 2-absorbing primal ideal of R we must show that $\nu_0(J \cap R) = \nu_0(J) \cap R$. Let $a \in \nu_0(J) \cap R$, then $\frac{a}{1} \in \nu_0(J)$. So there exist $\frac{r}{t_1}, \frac{s}{t_2}, \frac{u}{t_3} \in R_S$ with $0 \neq (\frac{r}{t_1} \frac{s}{t_2} \frac{u}{t_3})(\frac{a}{1}) \in I_S$ such that $\frac{rs}{t_1t_2}, \frac{ru}{t_1t_3}, \frac{su}{t_2t_3} \notin I_S$. Therefore, $0 \neq rsua \in J \cap R$ with $rs, ru, su \notin J \cap R$, hence $a \in \nu_0(J \cap R)$. Conversely, let $a \in \nu_0(J \cap R)$, then there exist $r, s, t \in R$ such that $rs, rt, st \notin J \cap R$ with $0 \neq (rst)a \in J \cap R$. Therefore, $\frac{rs}{1}, \frac{rt}{1}, \frac{st}{1} \notin J$ with $0 \neq \frac{(rst)a}{1} \in J$, since S consists of regular elements, thus, $\frac{a}{1} \in \nu_0(J)$. So $J \cap R$ is a weakly 2-absorbing primal ideal of R . Now, we show that $\nu_0(J \cap R) \cap S = \phi$. Let $w \in \nu_0(J \cap R) \cap S$, then there exist $r, s, t \in R$ such that $rs, rt, st \notin J \cap R$ with $0 \neq (rst)w \in J \cap R$. Therefore, $0 \neq \frac{rst}{1} \in J$ with $\frac{rs}{1}, \frac{rt}{1}, \frac{st}{1} \notin J$ implies that $1 \in \nu_0(J)$, a contradiction. So, $\nu_0(J \cap R) \cap S = \phi$. $\square$

It is well known that if S is multiplicatively closed subset of R with $1 \in S$ such that S consists of regular elements, then there is a one-to-one correspondence between weakly prime ideals P of R with $P \cap S = \phi$ and weakly prime ideals of R_S . In the next result we also prove that there is a one-to-one correspondence between weakly 2-absorbing primal ideals I of R with $\nu_0(I) \cap S = \phi$ and weakly 2-absorbing primal ideals of R_S .

Theorem 3.4. *There is a one-to-one correspondence between weakly 2-absorbing primal ideals I of R with $\nu_0(I) \cap S = \phi$ and weakly 2-absorbing primal ideals of R_S .*

Proof of Theorem 3.4. This follows directly from Lemma 3.2 and Proposition 3.1 and the fact that there is a one-to-one correspondence between weakly prime ideals P of R with $P \cap S = \phi$ and weakly prime ideals of R_S . $\square$

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Relative relation matrix based approaches for updating approximations in neighborhood multigranulation rough sets

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Abstract. With the revolution of computing and biology technology, data sets containing information could be huge and complex that sometimes are difficult to handle. Dynamic computing is an efficient approach to solve problems. Since neighborhood multigranulation rough sets (NMGRS) have been proposed, few papers focused on how to calculate approximations in NMGRS and how to update approximations in NMGRS dynamically. The purpose of this study is try to propose relative relation matrix based approaches for computing approximations in NMGRS and updating them dynamically. First, static approaches for computing approximations in NMGRS were proposed. Second, relative relation matrix based approaches for updating approximations in NMGRS while decreasing and increasing neighborhood classes were proposed. Third, incremental algorithms for updating approximations in NMGRS while decreasing and increasing neighborhood classes were designed. Finally, the efficiency and the validity of the designed algorithms were verified by experiments.

Keywords: approximation computation, multigranulation rough set, knowledge acquisition, decision making.

Since rough set has been proposed by Pawlak in 1982, it has been widely used in various fields such as pattern recognition [9, 2, 7, 11, 21, 20, 25, 29, 12], machine learning [24, 6, 13, 14, 3, 18, 30, 34, 15], image precessing[16, 17], data mining and other relevant areas. After rough set theory has been proposed, many models were proposed to extend its application, including covering based rough sets [31], variable precision rough sets[8], probabilistic rough sets [30], fuzzy rough sets [26, 23, 25, 6], fuzzy variable precision rough sets [33], and etc.

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Lin et al. proposed NMGRS [10] in 2012. It is a powerful mathematical tool in representing some real life situations and extend the application field of multigranulation rough sets [19](MGRS) into a broader area. When we apply NMGRS in real life situations, approximation computing is an essential part. Approximations must be computed before decision making. Positive region must be calculated so that attributes can be selected in attribute reduction process. Calculating approximations is necessary when applying rough set theory, thus how to compute the approximations is a significant issue in NMGRS theory.

In the information explosion era, approximation computing becomes more and more difficult: sizes of data sets sometimes are too huge to handled, the structure of data sets are changing to be more and more complex and granular structures often increase or decrease. It is of great importance to calculate approximations of NMGRS by dynamic approaches [28, 5]. In this paper, we attempt to propose relative relation matrix based approaches for computing and updating approximations in NMGRS. In real life applications, neighborhood classes often decreasing and increasing, so it is important to update approximations in NMGRS based on approximations we have computed. We try to design algorithms for updating approximations while decreasing and increasing neighborhood classes. First, we propose relative relation matrix based static approach. Second, we design incremental relative relation matrix based approach for updating approximations in NMGRS need to be proposed.

The rest of this paper is organized as follows. Several basic concepts of NMGRS are introduced in Section 2, and so is relative relation matrix based static algorithm to calculate approximation in NMGRS. In Section 3, dynamic approaches for updating approximations in NMGRS while decreasing and increasing neighborhood classes are proposed. Several algorithms are designed according to the approaches we proposed in Section 4. Experimental evaluations are conducted in Section 5 to verify the efficiency and validity of algorithms we designed. Finally, some conclusions and future work are given in Section 6.

1. Preliminaries

In this section, we review several basic concepts of NMGRS, and then, we propose relative relation matrix based approaches for computing approximations in NMGRS.

1.1 Neighborhood multigranulation rough sets

In this subsection, several basic concepts of NMGRS are reviewed.

Definition 1 ([10]). Let $NIS = (U, AT, \delta)$ be a neighborhood information system, where $A_k \in AT$ for any $k \in \{1, 2, \dots, m\}$, δ is a nonnegative number. $\forall k \in \{1, 2, \dots, m\}$, $A_k^C, A_k^N \subseteq A_k$ are categorical and numeric attributes, respectively. the neighborhood granules of objects x induced by A_k^C, A_k^N and $A_k^C \cup A_k^N$ are defined as

- $n_{A_k^C}(x) = \{x_i \in U | d_{A_k^C}(x, x_i) = 0\};$
- $n_{A_k^N}(x) = \{x_i \in U | d_{A_k^N}(x, x_i) \leq \delta\};$
- $n_{A_k^C \cup A_k^N}(x) = \{x_i \in U | d_{A_k^C}(x, x_i) = 0 \wedge d_{A_k^N}(x, x_i) \leq \delta\};$

where d is a distance [22] between two samples, δ is a nonnegative number. Accordingly, we say (U, δ) is a neighborhood approximation space, if there is an attribute subset in the system generating a neighborhood relation on the universe, we can regard this system as a neighborhood information system. Denote by $NIS = (U, AT, \delta)$, where U is a nonempty finite set and AT is a group of attribute set.

In this paper, we consider distance between two objects as Euclidean distance, Euclidean distance is defined as follows:

Definition 2 ([1]). *The Euclidean distance between points p and q is the length of the line segment connecting them in Cartesian coordinates, if $p = (p_1, p_2, \dots, p_n)$ and $q = (q_1, q_2, \dots, q_n)$ are two points in Euclidean n -space, then the distance (d) from p to q , or from q to p is given by the Pythagorean formula:*

$$d(p, q) = d(q, p) = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2 + \dots + (q_n - p_n)^2}.$$

Definition 3 ([10]). *(ONMGRS) Let $NIS = (U, AT, \delta)$ be a neighborhood information system, where $A_k \in AT$ for any $k \in \{1, 2, \dots, m\}$, δ is a nonnegative number. For any $X \subseteq U$, the optimistic neighborhood multigranulation lower and upper approximations of X are denoted by $\sum_{k=1}^m \delta_{A_k}^O(X)$ and $\sum_{k=1}^m \delta_{A_k}^O(X)$, respectively,*

$$\begin{aligned} \sum_{k=1}^m \delta_{A_k}^O(X) &= \left\{ x \in U | n_{A_1^C \cup A_1^N}(x) \subseteq X \vee n_{A_2^C \cup A_2^N}(x) \subseteq X \right. \\ &\quad \left. \vee \dots \vee n_{A_m^C \cup A_m^N}(x) \subseteq X \right\} \\ \sum_{k=1}^m \delta_{A_k}^O(X) &= \sim \sum_{k=1}^m \delta_{A_k}^O(\sim X) \end{aligned}$$

where $n_{A_k^C \cup A_k^N}(x)$ is the neighborhood class of x in terms of the attribute set A_k and neighborhood radius δ , $\sim X$ is the complement of the set X .

Theorem 1 ([10]). *Let $NIS = (U, AT, \delta)$ be a neighborhood information system, where $A_k \in AT$ for any $k \in \{1, 2, \dots, m\}$, δ is a nonnegative number. For any $X \subseteq U$, the optimistic neighborhood multigranulation upper approximation of X*

is denoted by $\overline{\sum_{k=1}^m \delta_{A_k}^O}(X)$,

$$\overline{\sum_{k=1}^m \delta_{A_k}^O}(X) = \left\{ x \in U \mid n_{A_1^C \cup A_1^N}(x) \cap X \neq \emptyset \wedge n_{A_2^C \cup A_2^N}(x) \cap X \neq \emptyset \right. \\ \left. \wedge \dots \wedge n_{A_m^C \cup A_m^N}(x) \cap X \neq \emptyset \right\}$$

where $n_{A_k^C \cup A_k^N}(x)$ is the neighborhood class of x in terms of the attribute set A_k and neighborhood radius δ , $\sim X$ is the complement of the set X .

Definition 4 ([10]). (PNMGRS) Let $NIS = (U, AT, \delta)$ be a neighborhood information system, where $A_k \in AT$ for any $k \in \{1, 2, \dots, m\}$, δ is a nonnegative number. For any $X \subseteq U$, the pessimistic neighborhood multigranulation lower and upper approximation of X are denoted by $\overline{\sum_{k=1}^m \delta_{A_k}^P}(X)$ and $\overline{\sum_{k=1}^m \delta_{A_k}^P}(X)$, respectively,

$$\overline{\sum_{k=1}^m \delta_{A_k}^P}(X) = \left\{ x \in U \mid n_{A_1^C \cup A_1^N}(x) \subseteq X \wedge n_{A_2^C \cup A_2^N}(x) \subseteq X \right. \\ \left. \wedge \dots \wedge n_{A_m^C \cup A_m^N}(x) \subseteq X \right\}$$

$$\overline{\sum_{k=1}^m \delta_{A_k}^P}(X) = \sim \overline{\sum_{k=1}^m \delta_{A_k}^P}(\sim X)$$

where $n_{A_k^C \cup A_k^N}(x)$ is the neighborhood class of x in terms of the attribute set A_k and neighborhood radius δ , $\sim X$ is the complement of the set X .

Theorem 2 ([10]). Let $NIS = (U, AT, \delta)$ be a neighborhood information system, where $A_k \in AT$ for any $k \in \{1, 2, \dots, m\}$, δ is a nonnegative number and for any $X \subseteq U$. The pessimistic neighborhood multigranulation upper approximation of X is denoted by $\overline{\sum_{k=1}^m \delta_{A_k}^P}(X)$,

$$\overline{\sum_{k=1}^m \delta_{A_k}^P}(X) \\ = \left\{ x \in U \mid n_{A_1^C \cup A_1^N}(x) \cap X \neq \emptyset \vee n_{A_2^C \cup A_2^N}(x) \cap X \neq \emptyset \vee \dots \vee n_{A_m^C \cup A_m^N}(x) \cap X \neq \emptyset \right\},$$

where $n_{A_k^C \cup A_k^N}(x)$ is the neighborhood class of x in terms of the attribute set A_k and neighborhood radius δ , $\sim X$ is the complement of the set X .

1.2 Relative relation matrix-based algorithm for computing approximations in multigranulation rough set

Definition 5 ([4]). Let U be an universe and $|U| = n$, the matrix representation of $X \subseteq U$ is denoted as $G^U(X) = [g_1^U(X), \dots, g_{|U|}^U(X)]^T$, where

$$g_i^U(X) = \begin{cases} 1, & x_i \in X \\ 0, & x_i \notin X \end{cases}, \quad i \in \{1, 2, \dots, |U|\}$$

and "T" denotes the transpose operation.

Definition 6. Let U be an universe and $|U| = n$, the set representation of column matrix $P = [p_1, p_2, \dots, p_n]^T$ is denoted as $G_{-1}^U(P) = \{x_{r_1}, x_{r_2}, \dots, x_{r_t}\}$, where

$$x_{r_s} = \begin{cases} x_i, & p_i = 1 \\ \text{None}, & p_i = 0 \end{cases}, \quad i \in \{1, 2, \dots, |U|\}.$$

Definition 7. Let $NIS = (U, AT, \delta)$ be a neighborhood information system, where $A_k \in AT$ for any $k \in \{1, 2, \dots, m\}$, δ is a nonnegative number and for any $X \subseteq U$, R_k denotes the neighborhood relation of the granular structure A_k on U , the relative approximation space which is relative to X is denoted by $V^k = \{y_1^k, y_2^k, \dots, y_r^k\}$, where for any $x_i, x_j \in U$, if $(x_i, x_j) \in R_k$ and $x_i \in X$, exists $y_p^k, y_q^k \in V^k$ satisfied that $y_p^k = x_i, y_q^k = x_j$. Further more, we view $RNIS = (VT, AT, \delta)$ as a relative neighborhood information system where $VT = \{V^1, V^2, \dots, V^m\}$.

Definition 8. Let $NIS = (U, AT, \delta)$ be a neighborhood information system, where $A_k \in AT$ for any $k \in \{1, 2, \dots, m\}$, δ is a nonnegative number and for any $X \subseteq U$, suppose $RNIS = (VT, AT, \delta)$ is a relative neighborhood information system, R_k denotes the neighborhood relation of the granular structure A_k on U and V^k for any $k \in \{1, 2, \dots, m\}$, if $R_k(X)\{x_{s_1}, x_{s_2}, \dots, x_{s_{r_k}}\}$ satisfied that $(x_{s_p}, x_{s_q}) \in R_k$ and $x_{s_p} \in X$, then $y_1 = x_{s_1}, \dots, y_2 = x_{s_2}, y_r = x_{s_{r_k}}$, where $s_1 < s_2 < \dots < s_{r_k}$, then exists bijective mapping $f : U \rightarrow V^k$ and its inverse mapping f^{-1} satisfied that

$$f(x_{s_p}) = y_p \wedge f^{-1}(y_p) = x_{s_p},$$

for all $p \in \{1, 2, \dots, r_k\}$.

Definition 9. Let $NIS = (U, AT, \delta)$ be a neighborhood information system, where $A_k \in AT$ for any $k \in \{1, 2, \dots, m\}$, δ is a nonnegative number and for any $X \subseteq U$, suppose $RNIS = (VT, AT, \delta)$ is a relative neighborhood information system, R_k denotes the neighborhood relation of the granular structure A_k on U and V^k for any $k \in \{1, 2, \dots, m\}$. For any $Y \subseteq V^k$, if the boolean column matrix of Y in U and V is denoted by $G^{V^k}(Y)$ and $G^U(f^{-1}(Y))$, then exists bijective mapping F and its inverse mapping F^{-1} satisfied that

$$F(G^U(f^{-1}(Y))) = G^{V^k}(Y) \wedge F^{-1}(G^{V^k}(Y)) = G^U(f^{-1}(Y)).$$

Definition 10 ([27]). Let $NIS = (U, AT, \delta)$ be a neighborhood information system, where $A_k \in AT$ for any $k \in \{1, 2, \dots, m\}$, δ is a nonnegative number and for any $X \subseteq U$, suppose $RNIS = (VT, AT, \delta)$ is a relative neighborhood information system, R_k denotes the neighborhood relation of the granular structure

Table 1: A neighborhood information system.

U	a_1	a_2	a_3	a_4
x_1	1	3	1.5	0.5
x_2	2	2	0.5	1
x_3	3	1	2	2.5
x_4	2	3	1.5	1
x_5	1	1	2	0.5
x_6	3	2	0.5	2

A_k on U and V^k for any $k \in \{1, 2, \dots, m\}$, $MX_{A_k} = (mx_{ij}^{A_k})_{n \times n}$ denote the relative neighborhood relation matrix of the granular structure A_k which is relative to X , where

$$(1) \quad mx_{ij}^{A_k} = \begin{cases} 1, & (y_p, y_q) \in R_k \\ 0, & (y_p, y_q) \notin R_k \end{cases}, \quad p, q \in \{1, 2, \dots, r\}$$

and the elements that relative relation matrix perform their relations are $R_k(X) = \{x_{s_1}, x_{s_2}, \dots, x_{s_r}\} = \{y_1^k, y_2^k, \dots, y_r^k\}$, $|R(X)| = \max_{k=1}^m |R_{A_k}(X)|$.

Example 1. Let $NIS = (U, AT, \delta)$ be a neighborhood information system, as shown in Table 1, where $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, $A_1 = A_1^C \cup A_1^N$, $A_1^C = \{a_1\}$ and $A_1^N = \{a_3\}$, $A_2 = A_2^C \cup A_2^N$, $A_2^C = \{a_2\}$ and $A_2^N = \{a_4\}$. Let $X = \{x_2, x_4, x_5\}$ and $\delta = 1$. According to Definition 10, we have that for any $k \in \{1, 2\}$, the relative neighborhood relation matrix can be calculate as follows,

$$M_{A_1} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, M_{A_2} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

and

$$R_{A_1}(X) = \{x_1, x_2, x_4, x_5\} = \{y_1, y_2, y_3, y_4\} - V^1, \\ R_{A_2}(X) = \{x_1, x_2, x_4, x_5, x_6\} = \{y_1, y_2, y_3, y_4, y_5\} = V^2.$$

then, we have $f(R_{A_1}(X)) = V^1$, $f(R_{A_2}(X)) = V^2$, $f^{-1}(V^1) = R_{A_1}(X)$, $f^{-1}(V^2) = R_{A_2}(X)$. $G^U(X) = [0, 1, 0, 1, 1, 0]^T$, $G^{V^1}(X) = [0, 1, 1, 1]^T$, $G^{V^2}(X) = [0, 1, 1, 1, 0]^T$, $F(G^U(X)) = G^{V^1}(X)$, $F^{-1}(G^{V^1}(X)) = G^U(X)$.
 $F(G^U(X)) = G^{V^2}(X)$, $F^{-1}(G^{V^2}(X)) = G^U(X)$.

Definition 11. Let $NIS = (U, AT, \delta)$ be a neighborhood information system, where $A_k \in AT$ for any $k \in \{1, 2, \dots, m\}$, δ is a nonnegative number and for any $X \subseteq U$, suppose $RNIS = (VT, AT, \delta)$ is a relative neighborhood information

system, R_k denotes the neighborhood relation of the granular structure A_k on U and V^k for any $k \in \{1, 2, \dots, m\}$, Λ_{A_k} denotes the diagonal matrix of granular structure A_k induced by the relative neighborhood relation matrix,

$$\Lambda_{A_k} = \text{diag}\left[\frac{1}{\lambda_{A_k}^1}, \frac{1}{\lambda_{A_k}^2}, \dots, \frac{1}{\lambda_{A_k}^{r_k}}\right],$$

where $\lambda_{A_k}^q = \sum_{i=1}^{r_k} mx_{ij}^{A_k}$ for any $i \in \{1, 2, \dots, r_k\}$

Example 2 (Continuation of Example 1). by Definition 11, for any $k \in \{1, 2, 3\}$, we can calculate the diagonal matrix of A_k as follows.

$$(2) \quad \begin{aligned} \Lambda_{A_1} &= \text{diag}\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right], \\ \Lambda_{A_2} &= \text{diag}\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}\right]. \end{aligned}$$

Theorem 3. Let $NIS = (U, AT, \delta)$ be a neighborhood information system, where $A_k \in AT$ for any $k \in \{1, 2, \dots, m\}$, δ is a nonnegative number and for any $X \subseteq U$, suppose $RNIS = (VT, AT, \delta)$ is a relative neighborhood information system, R_k denotes the neighborhood relation of the granular structure A_k on U and V^k for any $k \in \{1, 2, \dots, m\}$, Λ_{A_k} denotes the diagonal matrix of granular structure A_k induced by the relative neighborhood relation matrix, then

$$\Lambda_{A_k} = \text{diag}\left[\frac{1}{|n_{A_1^C \cup A_k^N}(y_1^k)|}, \frac{1}{|n_{A_1^C \cup A_k^N}(y_2^k)|}, \dots, \frac{1}{|n_{A_1^C \cup A_k^N}(y_r^k)|}\right],$$

where $|n_{A_1^C \cup A_k^N}(y_s)|$ denote the cardinality of the neighborhood class of y_s with respect to granular structure A_k , $d \in 1, 2, \dots, r_k$.

Definition 12. Let $NIS = (U, AT, \delta)$ be a neighborhood information system, where $A_k \in AT$ for any $k \in \{1, 2, \dots, m\}$, δ is a nonnegative number and for any $X \subseteq U$, suppose $RNIS = (VT, AT, \delta)$ is a relative neighborhood information system, R_k denotes the neighborhood relation of the granular structure A_k on U and V^k for any $k \in \{1, 2, \dots, m\}$, Λ_{A_k} denotes the diagonal matrix of granular structure A_k induced by the relative neighborhood relation matrix, then the column matrix $H_{A_k}(X)$ of the granular structure A_k can be calculated as follows

$$H_{A_k}(X) = \Lambda_{A_k} \cdot (MX_{A_k} \cdot G^{V^k}(X)),$$

where " $\cdot$ " denote dot product of matrix.

Example 3 (Continuation of Example 1). According to Definition 12, the column matrix $H_{A_1}(X)$ and $H_{A_2}(X)$ can be calculated as follows.

$$\begin{aligned} H_{A_1}(X) &= \Lambda_{A_1} \cdot (MX_{A_1} \cdot G^{V^1}(X)) \\ &= \text{diag}\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right] \cdot \left[\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right] = \begin{bmatrix} 1/2 \\ 1 \\ 1 \\ 1/2 \end{bmatrix} \end{aligned}$$

$$\begin{aligned}
H_{A_2}(X) &= \Lambda_{A_2} \cdot (MX_{A_2} \cdot G^{V^2}(X)) \\
&= \text{diag} \left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2} \right] \left[\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right] = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1 \\ 1/2 \end{bmatrix}
\end{aligned}$$

Definition 13 ([4]). Let $NIS = (U, AT, \delta)$ be a neighborhood information system, where $A_k \in AT$ for any $k \in \{1, 2, \dots, m\}$, δ is a nonnegative number and for any $X \subseteq U$, suppose $RNIS = (VT, AT, \delta)$ is a relative neighborhood information system, R_k denotes the neighborhood relation of the granular structure A_k on U and V^k for any $k \in \{1, 2, \dots, m\}$, Λ_{A_k} denotes the diagonal matrix of granular structure A_k induced by the relative neighborhood relation matrix, $H_{A_k}(X)$ is the column matrix of granular structure A_k , then two cut matrices of A_k can be denoted as follows,

- $H_{A_k}^{[\alpha, \beta]}(X) = (h_{A_k}^{i\downarrow})_{r \times 1}$, where

$$(3) \quad h_{A_k}^{i\downarrow} = \begin{cases} 1, & \alpha \leq h_{A_k}^i \leq \beta \\ 0, & \text{otherwise} \end{cases}, \quad i \in \{1, 2, \dots, r_k\}$$

- $H_{A_k}^{(\alpha, \beta]}(X) = (h_{A_k}^{i\uparrow})_{r \times 1}$, where

$$(4) \quad h_{A_k}^{i\uparrow} = \begin{cases} 1, & \alpha < h_{A_k}^i \leq \beta \\ 0, & \text{otherwise} \end{cases}, \quad i \in \{1, 2, \dots, r_k\}$$

Example 4 (Continuation of Example 3). Assume that $\alpha = 1$ and $\beta = 1$. According to Definition 13. the cut matrix of H_{A_1} and H_{A_2} can be calculated as follows,

$$\begin{aligned}
H_{A_1}^{[1,1]}(X) &= [0 \ 1 \ 1 \ 0 \ 0]^T, \\
H_{A_2}^{[1,1]}(X) &= [0 \ 0 \ 0 \ 1 \ 0]^T.
\end{aligned}$$

Assume that $\alpha = 0$ and $\beta = 1$. the cut matrix of H_{A_1} and H_{A_2} can be calculated as follows,

$$\begin{aligned}
H_{A_1}^{(0,1]}(X) &= [1 \ 1 \ 1 \ 1 \ 1]^T, \\
H_{A_2}^{(0,1]}(X) &= [1 \ 1 \ 1 \ 1 \ 1]^T.
\end{aligned}$$

Theorem 4. Let $NIS = (U, AT, \delta)$ be a neighborhood information system, where $A_k \in AT$ for any $k \in \{1, 2, \dots, m\}$, δ is a nonnegative number and

for any $X \subseteq U$, suppose $RNIS = (VT, AT, \delta)$ is a relative neighborhood information system, R_k denotes the neighborhood relation of the granular structure A_k on U and V^k for any $k \in \{1, 2, \dots, m\}$, Λ_{A_k} denotes the diagonal matrix of granular structure A_k induced by the relative neighborhood relation matrix. If $H_{A_k}(X)$ is the column matrix of granular structure A_k , the lower and upper approximations in ONMGRS are denoted by $\underline{\sum_{k=1}^m \delta_{A_k}^O(X)}$ and $\overline{\sum_{k=1}^m \delta_{A_k}^O(X)}$, respectively, the following results hold,

1. If $G^U(\underline{\sum_{k=1}^m \delta_{A_k}^O(X)})$ denotes the boolean column matrix of lower approximation in ONMGRS, then we have

$$G^U(\underline{\sum_{k=1}^m \delta_{A_k}^O(X)}) = \bigvee_{k=1}^m F^{-1}(H_{A_k}^{[1,1]}(X)),$$

2. If $G^U(\overline{\sum_{k=1}^m \delta_{A_k}^O(X)})$ denotes the boolean column matrix of upper approximation in ONMGRS, then we have

$$G^U(\overline{\sum_{k=1}^m \delta_{A_k}^O(X)}) = \bigwedge_{k=1}^m F^{-1}(H_{A_k}^{(0,1]}(X)).$$

Proof. 1. $\forall s \in \{1, 2, \dots, r_k\}$, $g_s^{V_k}(\underline{\sum_{k=1}^m \delta_{A_k}^O(X)}) = 1 \Leftrightarrow y_s \in \underline{\sum_{k=1}^m \delta_{A_k}^O(X)} \Leftrightarrow \exists k \in \{1, 2, \dots, m\}$, $n_{A_k^C \cup A_k^N}(y_s^{A_k}) \subseteq X$. Since $(y_t, y_s) \in R_k$ and $y_s \in X$, we have $mx_{st}^{A_k} = 1$ and $g_s^{V_k}(X) = 1$, thus we have $g_s^{V_k}(\underline{\sum_{k=1}^m \delta_{A_k}^O(X)}) = 1 \Leftrightarrow \exists k \in \{1, 2, \dots, m\}$, s.t. $h_{A_k}^{s\downarrow} = \frac{\sum_{t=1}^{r_k} mx_{st}^{A_k} \times g_t^{V_k}(X)}{\sum_{t=1}^{r_k} mx_{st}^{A_k}} = 1$ holds. Then we have $g_s^{V_k}(\underline{\sum_{k=1}^m \delta_{A_k}^O(X)}) = 1 \Leftrightarrow \bigvee_{k=1}^m h_{A_k}^{s\downarrow} = 1$, $G^{V_k}(\underline{\sum_{k=1}^m \delta_{A_k}^O(X)}) = \bigvee_{k=1}^m H_{A_k}^{[1,1]}(X)$, in other words, $G^U(\underline{\sum_{k=1}^m \delta_{A_k}^O(X)}) = \bigvee_{k=1}^m F^{-1}(H_{A_k}^{[1,1]}(X))$.

2. The proof is similar to that of 1. □

Theorem 5. Let $NIS = (U, AT, \delta)$ be a neighborhood information system, where $A_k \in AT$ for any $k \in \{1, 2, \dots, m\}$, δ is a nonnegative number and for any $X \subseteq U$, suppose $RNIS = (VT, AT, \delta)$ is a relative neighborhood information system, R_k denotes the neighborhood relation of the granular structure A_k on U and V^k for any $k \in \{1, 2, \dots, m\}$, Λ_{A_k} denotes the diagonal matrix of granular structure A_k induced by the relative neighborhood relation matrix. If $H_{A_k}(X)$ is the column matrix of granular structure A_k , the lower and upper approximations in ONMGRS are denoted by $\underline{\sum_{k=1}^m \delta_{A_k}^O(X)}$ and $\overline{\sum_{k=1}^m \delta_{A_k}^O(X)}$, respectively, the following results hold,

1. If $G^U(\underline{\sum_{k=1}^m \delta_{A_k}^P(X)})$ denote the boolean column matrix of lower approximation in ONMGRS, then we have

$$G^U(\underline{\sum_{k=1}^m \delta_{A_k}^P(X)}) = \bigwedge_{k=1}^m F^{-1}(H_{A_k}^{[1,1]}(X)),$$

2. If $G^U(\overline{\sum_{k=1}^m \delta_{A_k}^P}(X))$ denotes the boolean column matrix of upper approximation in ONMGRS, then we have

$$G^U(\overline{\sum_{k=1}^m \delta_{A_k}^P}(X)) = \bigvee_{k=1}^m F^{-1}(H_{A_k}^{(0,1]}(X)).$$

Proof. The proof is similar to that of Theorem 4. $\square$

Example 5 (Continuation of Example 3). The boolean column matrix of lower and upper approximations in PNMGRS can be computed as follows,

$$\begin{aligned} G^U(\overline{\sum_{k=1}^m \delta_{A_k}^O}(X)) &= \bigvee_{k=1}^2 F^{-1}(H_{A_k}^{[1,1]}(X)) \\ &= [0 \ 1 \ 0 \ 1 \ 0 \ 0]^T \vee [0 \ 0 \ 0 \ 0 \ 1 \ 0]^T \\ &= [0 \ 1 \ 0 \ 1 \ 1 \ 0]^T. \\ G^U(\overline{\sum_{k=1}^m \delta_{A_k}^O}(X)) &= \bigwedge_{k=1}^2 F^{-1}(H_{A_k}^{(0,1]}(X)) \\ &= [1 \ 1 \ 0 \ 1 \ 1 \ 0]^T \wedge [1 \ 1 \ 1 \ 1 \ 1 \ 0]^T \\ &= [1 \ 1 \ 0 \ 1 \ 1 \ 0]^T. \end{aligned}$$

The boolean column matrix of lower and upper approximations in PNMGRS can be computed as follows,

$$\begin{aligned} G^U(\overline{\sum_{k=1}^m \delta_{A_k}^P}(X)) &= \bigwedge_{k=1}^2 F^{-1}(H_{A_k}^{[1,1]}(X)) \\ &= [0 \ 1 \ 0 \ 1 \ 0 \ 0]^T \wedge [0 \ 0 \ 0 \ 0 \ 1 \ 0]^T \\ &= [0 \ 0 \ 0 \ 0 \ 0 \ 0]^T. \\ G^U(\overline{\sum_{k=1}^m \delta_{A_k}^P}(X)) &= \bigvee_{k=1}^2 F^{-1}(H_{A_k}^{(0,1]}(X)) \\ &= [1 \ 1 \ 0 \ 1 \ 1 \ 0]^T \vee [1 \ 1 \ 1 \ 1 \ 1 \ 0]^T \\ &= [1 \ 1 \ 1 \ 1 \ 1 \ 0]^T. \end{aligned}$$

Algorithm 1 is a relative matrix based algorithm for computing approximations in NMGRS. If $N_X = \max_{k=1}^m \{|R_{A_k}(X)|\}$, and then the total time complexity of the algorithm is $O(|AT|N_X^2)$. Steps 4-9 are to calculate $R_k(X)$ whose time complexity is $O(|AT||X| \sim X)$, Steps 12-17 are to calculate MX_{A_k} whose time complexity is $O(|AT|N_X^2)$. Steps 18-22 are to calculate H_{A_k} whose time complexity is $O(|AT|N_X^2)$, 24-27 are to compute the approximations of NMGRS whose time complexity is $O(|AT||U|)$.

Algorithm 1 Relative relation matrix based algorithm for computing approximations in NMGRS

Require: (1) A neighborhood information system $NIS = (U, AT, \delta)$, the granular structure is $A_1, A_2, \dots, A_m$. (2) A target concept $X \subseteq U$

Ensure: Approximations in NMGRS.

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1:  $m \leftarrow |AT^t|$ 
2:  $n \leftarrow |U|$ 
3: for  $k = 1 \rightarrow m$  do
4:   for  $i = 1 \rightarrow n$  do
5:     for  $j = 1 \rightarrow n$  do
6:       if  $g_i^U(X) = 1 \wedge g_j^U(n_{A_1^C \cup A_1^N}(x_i)) = 1$  then  $g_i^U(R_k(X)) = 1$ 
7:       end if
8:     end for
9:   end for
10:   $V \leftarrow f^{-1}(R_k(X))$ 
11:   $r \leftarrow |V^k|$ 
12:  for  $i = 1 \rightarrow r_k$  do
13:    for  $j = 1 \rightarrow r_k$  do
14:      if  $(y_i, y_j) \in R_k = 1$  then  $mx_{ij}^{A_k} = 1$ 
15:      end if
16:    end for
17:  end for
18:  for  $i = 1 \rightarrow r_k$  do
19:     $g_i^{V^k}(\Lambda_{A_k}) \leftarrow \sum_{j=1}^{r_k} mx_{ij}^{A_k}$ 
20:  end for
21:   $G^{V^k}(X) \leftarrow F^{-1}(G^U(X))$ 
22:   $H_{A_k} \leftarrow \text{diag}(\Lambda_{A_k}) \cdot (MX_{A_k} \cdot G^{V^k}(X))$ 
23: end for
24:  $G^U(\sum_{k=1}^m \delta_{A_k}^O(X)) \leftarrow \bigvee_{i=1}^m F^{-1}(H_{A_k})^{[1,1]}$ 
25:  $G^U(\sum_{k=1}^m \overline{\delta_{A_k}^O}(X)) \leftarrow \bigwedge_{i=1}^m F^{-1}(H_{A_k})^{(0,1]}$ 
26:  $G^U(\sum_{k=1}^m \delta_{A_k}^P(X)) \leftarrow \bigwedge_{i=1}^m F^{-1}(H_{A_k})^{[1,1]}$ 
27:  $G^U(\sum_{k=1}^m \overline{\delta_{A_k}^P}(X)) \leftarrow \bigvee_{i=1}^m F^{-1}(H_{A_k})^{(0,1]}$ 
28: Return  $\sum_{k=1}^m \delta_{A_k}^O(X)$ ,  $\sum_{k=1}^m \overline{\delta_{A_k}^O}(X)$ ,  $\sum_{k=1}^m \delta_{A_k}^P(X)$  and  $\sum_{k=1}^m \overline{\delta_{A_k}^P}(X)$ .

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2. Relative relation matrix based dynamic approaches for updating approximations in NMGRS while neighborhood classes are decreasing and increasing

2.1 Relative relation matrix based approaches for updating approximations while neighborhood classes are decreasing

In this subsection, we present relative relation matrix based dynamic approaches for updating approximations in NMGRS while neighborhood classes are decreasing. We denote $NIS^t = (U, AT^t, \Delta)$ be a neighborhood information system at time t , $NIS^{t+1} = (U, AT^{t+1}, \delta)$ be a neighborhood information system at time $t + 1$, $RNIS^{t+1} = (VT^{t+1}, AT^{t+1}, \delta)$ be a relative neighborhood information system at time $t + 1$ where δ and Δ are two nonnegative numbers and $\delta \leq \Delta$. For all $A_k^t \in AT^t (k \leq m)$, exists $A_k^{t+1} \in AT^{t+1}$, s.t. $(A_k^C)^t \subseteq (A_k^C)^{t+1}$, $(A_k^N)^t \subseteq (A_k^N)^{t+1}$. Also, for all $x \in U$, we denote neighborhood class of x at time t by $n_{A_k^C \cup A_k^N}^t(x)$. Denote neighborhood class of x at time $t + 1$ by $n_{A_k^C \cup A_k^N}^{t+1}(x)$. For all $X \subseteq U$, we denote approximations in PNMGRS by $\overline{\sum_{k=1}^m \Delta_{A_k}^t}^P(X)$ and $\overline{\sum_{k=1}^m \Delta_{A_k}^t}^P(X)$ at time t . Denote approximations in PNMGRS $\overline{\sum_{k=1}^m \delta_{A_k}^{t+1}}^P(X)$ and $\overline{\sum_{k=1}^m \delta_{A_k}^{t+1}}^P(X)$ at time $t + 1$. Denote approximations in ONMGRS by $\overline{\sum_{k=1}^m \Delta_{A_k}^t}^O(X)$ and $\overline{\sum_{k=1}^m \Delta_{A_k}^t}^O(X)$ at time t . Denote approximations in ONMGRS by $\overline{\sum_{k=1}^m \delta_{A_k}^{t+1}}^O(X)$ and $\overline{\sum_{k=1}^m \delta_{A_k}^{t+1}}^O(X)$ at time $t + 1$.

Theorem 6 ([32]). *Let $NIS^t = (U, AT^t, \Delta)$ be a neighborhood information system at time t , $NIS^{t+1} = (U, AT^{t+1}, \delta)$ be a neighborhood information system at time $t + 1$. For any $X \subseteq U$, the following result holds,*

$$n_{A_k^C \cup A_k^N}^{t+1}(x) \subseteq n_{A_k^C \cup A_k^N}^t(x).$$

Theorem 7 ([32]). *Let $NIS^t = (U, AT^t, \Delta)$ be a neighborhood information system at time t , $NIS^{t+1} = (U, AT^{t+1}, \delta)$ be a neighborhood information system at time $t + 1$. For any $X \subseteq U$, the following results hold,*

- (1) $\overline{\sum_{k=1}^m \Delta_{A_k}^t}^O(X) \subseteq \overline{\sum_{k=1}^m \delta_{A_k}^{t+1}}^O(X)$;
- (2) $\overline{\sum_{k=1}^m \Delta_{A_k}^t}^O(X) \supseteq \overline{\sum_{k=1}^m \delta_{A_k}^{t+1}}^O(X)$.

Theorem 8 ([32]). *Let $NIS^t = (U, AT^t, \Delta)$ be a neighborhood information system at time t , $NIS^{t+1} = (U, AT^{t+1}, \delta)$ be a neighborhood information system at time $t + 1$. For any $X \subseteq U$, the following results hold,*

- (1) $\overline{\sum_{k=1}^m \Delta_{A_k}^t}^P(X) \subseteq \overline{\sum_{k=1}^m \delta_{A_k}^{t+1}}^P(X)$;
- (2) $\overline{\sum_{k=1}^m \Delta_{A_k}^t}^P(X) \supseteq \overline{\sum_{k=1}^m \delta_{A_k}^{t+1}}^P(X)$.

Theorem 9. *Let $NIS^t = (U, AT^t, \Delta)$ be a neighborhood information system at time t , $RNIS^t = (VT^t, AT^t, \Delta)$ be a relative neighborhood information system*

at time t ; $NIS^{t+1} = (U, AT^{t+1}, \delta)$ be a neighborhood information system at time $t+1$, $RNIS^{t+1} = (VT^{t+1}, AT^{t+1}, \delta)$ be a relative neighborhood information system at time $t+1$. For any $X \in U$, the upper approximations in NMGRS at time $t+1$ are denoted by $\overline{\sum_{k=1}^m \delta_{A_k}^{t+1} O}(X)$ and $\overline{\sum_{k=1}^m \delta_{A_k}^{t+1} P}(X)$, respectively. If $BN^P(X) = \overline{\sum_{k=1}^m \Delta_{A_k}^t P}(X) - \underline{\sum_{k=1}^m \Delta_{A_k}^t P}(X)$, the following results hold,

1. If $G^U(\overline{\sum_{k=1}^m \delta_{A_k}^{t+1} O}(X))$ denote the boolean column matrix of lower approximation in ONMGRS, then we have

$$G^U(\overline{\sum_{k=1}^m \delta_{A_k}^{t+1} O}(X)) = G^U(\overline{\sum_{k=1}^m \Delta_{A_k}^t O}(X)) \wedge \sim (\bigvee_{k=1}^m F^{-1}(H_{A_k}^{[0,0]}(BN^P(X)))) ,$$

2. If $G^U(\overline{\sum_{k=1}^m \delta_{A_k}^{t+1} P}(X))$ denote the boolean column matrix of upper approximation in ONMGRS, then we have

$$G^U(\overline{\sum_{k=1}^m \delta_{A_k}^{t+1} P}(X)) = G^U(\overline{\sum_{k=1}^m \Delta_{A_k}^t P}(X)) \wedge \sim (\bigwedge_{k=1}^m F^{-1}(H_{A_k}^{[0,0]}(BN^P(X)))) .$$

Proof. 1. $\forall i \in \{1, 2, \dots, n\}, g_i^U(\overline{\sum_{k=1}^m \delta_{A_k}^{t+1} O}(X)) = 1 \Leftrightarrow \forall k \in \{1, 2, \dots, m\}, n_{A_k^C \cup A_k^N}(x_i) \subseteq X \wedge \forall x_j \in BN^P(X), \exists k \in \{1, 2, \dots, m\}, n_{A_k^C \cup A_k^N}(x_j) \not\subseteq X \Leftrightarrow g_i^U(\overline{\sum_{k=1}^m \delta_{A_k}^{t+1} O}(X)) = 1 \wedge g_j^U(\bigvee_{k=1}^m F^{-1}(H_{A_k}^{[0,0]}(BN^P(X)))) = 0$.

2. The proof is similar to that of 1. □

Theorem 10. Let $NIS^t = (U, AT^t, \Delta)$ be a neighborhood information system at time t , $RNIS^t = (VT^t, AT^t, \Delta)$ be a relative neighborhood information system at time t ; $NIS^{t+1} = (U, AT^{t+1}, \delta)$ be a neighborhood information system at time $t+1$, $RNIS^{t+1} = (VT^{t+1}, AT^{t+1}, \delta)$ be a relative neighborhood information system at time $t+1$. For any $X \in U$, the lower approximations in NMGRS at time $t+1$ are denoted by $\underline{\sum_{k=1}^m \delta_{A_k}^{t+1} O}(X)$ and $\underline{\sum_{k=1}^m \delta_{A_k}^{t+1} P}(X)$, respectively. If $BN^P(X) = \overline{\sum_{k=1}^m \Delta_{A_k}^t P}(X) - \underline{\sum_{k=1}^m \Delta_{A_k}^t P}(X)$, the following results hold,

1. If $G^U(\underline{\sum_{k=1}^m \delta_{A_k}^{t+1} O}(X))$ denote the boolean column matrix of lower approximation in ONMGRS, then we have

$$G^U(\underline{\sum_{k=1}^m \delta_{A_k}^{t+1} O}(X)) = G^U(\underline{\sum_{k=1}^m \delta_{A_k}^t O}(X)) \vee (\bigvee_{k=1}^m F^{-1}(H_{A_k}^{[1,1]}(BN^P(X)))) ,$$

2. If $G^U(\sum_{k=1}^m \delta_{A_k}^{t+1P}(X))$ denote the boolean column matrix of lower approximation in ONMGRS, then we have

$$G^U(\sum_{k=1}^m \delta_{A_k}^{t+1P}(X)) = G^U(\sum_{k=1}^m \delta_{A_k}^t(X)) \vee (\bigwedge_{k=1}^m F^{-1}(H_{A_k}^{[1,1]}(BN^P(X)))).$$

Proof. The proof is similar to that of Theorem 9. $\square$

Example 6. (Continuation of Example 1) Supposed that $AT^t = \{A_1^t, A_2^t\}$, $A_1^t = (A_1^C)^t \cup (A_1^N)^t$, $(A_1^C)^t = \{a_1\}$ and $(A_1^N)^t = \emptyset$, $A_2^t = (A_2^C)^t \cup (A_2^N)^t$, $(A_2^C)^t = \emptyset$ and $(A_2^N)^t = \{a_3\}$, $AT^{t+1} = \{A_1^{t+1}, A_2^{t+1}\}$, $A_1^{t+1} = (A_1^C)^{t+1} \cup (A_1^N)^{t+1}$, $(A_1^C)^{t+1} = \{a_1\}$ and $(A_1^N)^{t+1} = \{a_4\}$, $A_2^{t+1} = (A_2^C)^{t+1} \cup (A_2^N)^{t+1}$, $(A_2^C)^{t+1} = \{a_1\}$ and $(A_2^N)^{t+1} = \{a_3\}$. Let $\Delta = 0.5, \delta = 0.5$. According to Definition 10, we have that for any $k \in \{1, 2\}$, the relative neighborhood relation matrix can be calculate as follows. According to Theorem 9 and Theorem 10, $BN^P(X) = \{x_1, x_2, x_3, x_5\}$, $R_{A_2}(BN^P(X)) = \{x_1, x_2, x_3, x_5\}$, $R_{A_1}(BN^P(X)) = U$, the column matrix $H_{A_1}(BN^P(X))$ and $H_{A_2}(BN^P(X))$ can be calculated as follows.

$$\begin{aligned} H_{A_1}(BN^P(X)) &= \Lambda_{A_1} \cdot (M_{A_1} \cdot G^{V^1}(X)) \\ &= \text{diag} \left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right] \cdot \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1 \\ 0 \\ 1 \\ 1/2 \\ 0 \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} H_{A_2}(BN^P(X)) &= \Lambda_{A_2} \cdot (M_{A_2} \cdot G^{V^2}(X)) \\ &= \text{diag} \left[\frac{1}{2}, 1, 1, \frac{1}{2} \right] \cdot \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \\ 1/2 \end{bmatrix}. \end{aligned}$$

According to Theorem 9 and Theorem 10 we have

$$\begin{aligned} \overline{\sum_{k=1}^2 \delta_{A_k}^{t+1}}^O(X) &= \overline{\sum_{k=1}^2 \Delta_{A_k}^t}^O(X) \wedge \sim (\bigvee_{k=1}^2 F^{-1}(H_{A_k}^{[0,0]}(BN^P(X)))) \\ &= [1 \ 1 \ 0 \ 1 \ 1 \ 0]^T \wedge \sim \\ &\quad \left[[0 \ 0 \ 1 \ 0 \ 0 \ 1]^T \vee [0 \ 0 \ 1 \ 0 \ 0 \ 0]^T \right] \\ &= [1 \ 1 \ 0 \ 1 \ 1 \ 0]^T \wedge [1 \ 1 \ 0 \ 1 \ 1 \ 0]^T \\ &= [1 \ 1 \ 0 \ 1 \ 1 \ 0]^T. \end{aligned}$$

$$\begin{aligned}
\overline{\sum_{k=1}^2 \delta_{A_k}^{t+1}}^P (X) &= \overline{\sum_{k=1}^2 \Delta_{A_k}^t}^P (X) \wedge \sim \left(\bigwedge_{k=1}^2 F^{-1}(H_{A_k}^{[0,0]}(BN^P(X))) \right) \\
&= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}^T \wedge \sim \\
&\quad \left[\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}^T \wedge \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}^T \right] \\
&= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}^T \wedge \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}^T \\
&= \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}^T. \\
\\
\overline{\sum_{k=1}^2 \delta_{A_k}^{t+1}}^O (X) &= \overline{\sum_{k=1}^2 \delta_{A_k}^t}^O (X) \vee \left(\bigvee_{k=1}^m F^{-1}(H_{A_k}^{[1,1]}(BN^P(X))) \right) \\
&= \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}^T \vee \\
&\quad \left[\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}^T \vee \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T \right] \\
&= \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}^T \vee \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}^T \\
&= \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}^T. \\
\\
\overline{\sum_{k=1}^2 \delta_{A_k}^{t+1}}^P (X) &= \overline{\sum_{k=1}^2 \delta_{A_k}^t}^P (X) \vee \left(\bigwedge_{k=1}^m F^{-1}(H_{A_k}^{[1,1]}(BN^P(X))) \right) \\
&= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}^T \vee \\
&\quad \left[\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}^T \wedge \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T \right] \\
&= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}^T \vee \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T \\
&= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}^T.
\end{aligned}$$

2.2 Matrix-based approaches for updating approximations while neighborhood classes are increasing

In this subsection, we present matrix based dynamic approaches for updating approximations in neighborhood multigranulation rough set while neighborhood classes are increasing. We denote $NIS^t = (U, AT^t, \delta)$ be an neighborhood information system at time t , $NIS^{t+1} = (U, AT^{t+1}, \Delta)$ be an neighborhood information system at time $t+1$, $RNIS^{t+1} = (VT^{t+1}, AT^{t+1}, \Delta)$ be an relative neighborhood information system at time $t+1$ where δ and Δ are two nonnegative numbers and $\delta \leq \Delta$. For all $A_k^t \in AT^t (k \leq m)$, exists $A_k^{t+1} \in AT^{t+1}$, s.t. $(A_k^C)^{t+1} \subseteq (A_k^C)^t$ and $(A_k^N)^{t+1} \subseteq (A_k^N)^t$. Also, for all $x \in U$, we denote neighborhood class of x at time t by $n_{A_k^C \cup A_k^N}^t(x)$. Denote neighborhood class of x at time $t+1$ by $n_{A_k^C \cup A_k^N}^{t+1}(x)$. For all $X \in U$, we denote approximations in PNMGRS by $\overline{\sum_{k=1}^m \delta_{A_k}^t}^P(X)$ and $\overline{\sum_{k=1}^m \delta_{A_k}^t}^P(X)$ at time t . Denote approxima-

tions in PNMGRS $\underline{\sum_{k=1}^m \Delta_{A_k}^{t+1P}}(X)$ and $\overline{\sum_{k=1}^m \Delta_{A_k}^{t+1P}}(X)$ at time $t + 1$. Denote approximations in ONMGRS by $\underline{\sum_{k=1}^m \delta_{A_k}^t O}(X)$ and $\overline{\sum_{k=1}^m \delta_{A_k}^t O}(X)$ at time t . Denote approximations in ONMGRS by $\underline{\sum_{k=1}^m \Delta_{A_k}^{t+1O}}(X)$ and $\overline{\sum_{k=1}^m \Delta_{A_k}^{t+1O}}(X)$ at time $t + 1$.

Theorem 11 ([32]). *Let $NIS^t = (U, AT^t, \delta)$ be a neighborhood information system at time t , $NIS^{t+1} = (U, AT^{t+1}, \Delta)$ be a neighborhood information system at time $t + 1$, . For any $X \subseteq U$, the following result holds,*

$$n_{A_k^C \cup A_k^N}^{t+1}(x) \supseteq n_{A_k^C \cup A_k^N}^t(x).$$

Theorem 12 ([32]). *Let $NIS^t = (U, AT^t, \delta)$ be a neighborhood information system at time t , $NIS^{t+1} = (U, AT^{t+1}, \Delta)$ be a neighborhood information system at time $t + 1$, . For any $X \subseteq U$, the following results hold:*

- (1) $\underline{\sum_{k=1}^m \delta_{A_k}^t O}(X) \supseteq \underline{\sum_{k=1}^m \Delta_{A_k}^{t+1O}}(X)$;
- (2) $\overline{\sum_{k=1}^m \delta_{A_k}^t O}(X) \subseteq \overline{\sum_{k=1}^m \Delta_{A_k}^{t+1O}}(X)$.

Theorem 13 ([32]). *Let $NIS^t = (U, AT^t, \delta)$ be a neighborhood information system at time t , $NIS^{t+1} = (U, AT^{t+1}, \Delta)$ be a neighborhood information system at time $t + 1$, . For any $X \subseteq U$, the following results hold:*

- (1) $\underline{\sum_{k=1}^m \delta_{A_k}^t P}(X) \supseteq \underline{\sum_{k=1}^m \Delta_{A_k}^{t+1P}}(X)$;
- (2) $\overline{\sum_{k=1}^m \delta_{A_k}^t P}(X) \subseteq \overline{\sum_{k=1}^m \Delta_{A_k}^{t+1P}}(X)$.

Theorem 14. *Let $NIS^t = (U, AT^t, \delta)$ be an neighborhood information system at time t , $RNIS^t = (VT^t, AT^t, \delta)$ be an relative neighborhood information system at time t , $NIS^{t+1} = (U, AT^{t+1}, \Delta)$ be an neighborhood information system at time $t + 1$, $RNIS^{t+1} = (VT^{t+1}, AT^{t+1}, \Delta)$ be an relative neighborhood information system at time $t + 1$. For all $X \in U$, the upper approximations in NMGRS are denoted by $\underline{\sum_{k=1}^m \delta_{A_k}^t O}(X)$ and $\overline{\sum_{k=1}^m \delta_{A_k}^t O}(X)$, respectively. If $BN^O(X) = \underline{\sum_{k=1}^m \delta_{A_k}^t O}(X) \cup (U - \overline{\sum_{k=1}^m \delta_{A_k}^t O}(X))$, the following results hold,*

1. *If $G^U(\overline{\sum_{k=1}^m \Delta_{A_k}^{t+1O}}(X))$ denotes the boolean column matrix of upper approximation in PNMGRS, then we have*

$$G^U(\overline{\sum_{k=1}^m \Delta_{A_k}^{t+1O}}(X)) = \overline{\sum_{k=1}^m \delta_{A_k}^t O}(X) \vee (\bigwedge_{k=1}^m F^{-1}(H_{A_k}^{(0,1]}(BN^P(X)))).$$

2. *If $G^U(\underline{\sum_{k=1}^m \delta_{A_k}^{t+1P}}(X))$ denotes the boolean column matrix of upper approximation in ONMGRS, then we have*

$$G^U(\underline{\sum_{k=1}^m \Delta_{A_k}^{t+1P}}(X)) = \underline{\sum_{k=1}^m \delta_{A_k}^t P}(X) \vee (\bigvee_{k=1}^m F^{-1}(H_{A_k}^{(0,1]}(BN^P(X)))).$$

Proof. 1. $\forall i \in \{1, 2, \dots, n\}, g_i^U(\overline{\sum_{k=1}^m \delta_{A_k}^{t+1}}^O(X)) = 1 \Leftrightarrow \forall k \in \{1, 2, \dots, m\}, n_{A_k^C \cup A_k^N}(x_i) \subseteq X \wedge \forall x_j \in BN^O(X), \exists k \in \{1, 2, \dots, m\}, n_{A_k^C \cup A_k^N}(x_j) \subseteq X \Leftrightarrow g_i^U(\overline{\sum_{k=1}^m \delta_{A_k}^{t+1}}^O(X)) = 1 \vee g_j^U(\bigvee_{k=1}^m F^{-1}(H_{A_k}^{[0,1]}(BN^O(X)))) = 1$.

2. The proof is similar to that of 1. $\square$

Definition 14. Let $NIS = (U, AT, \delta)$ be a neighborhood information system, where $A_k \in AT$ for any $k \in \{1, 2, \dots, m\}$, δ is a nonnegative number and for any $X \subseteq U$, suppose $RNIS = (VT, AT, \delta)$ is a relative neighborhood information system, R_k denotes the neighborhood relation of the granular structure A_k on U and V^k for any $k \in \{1, 2, \dots, m\}$, Λ_{A_k} denotes the diagonal matrix of granular structure A_k induced by the relative neighborhood relation matrix, then the cut matrix of A_k can be denoted as $H_{A_k}^{[\alpha, \beta]}(X) = (h_{A_k}^{i\downarrow})_{r \times 1}$, where

$$h_{A_k}^{i\downarrow} = \begin{cases} 1, & \alpha \leq h_{A_k}^i < \beta \\ 0, & \text{otherwise} \end{cases}, \quad i \in \{1, 2, \dots, r\}.$$

Theorem 15. Let $NIS^t = (U, AT^t, \delta)$ be a neighborhood information system at time t , $RNIS^t = (VT^t, AT^t, \delta)$ be a relative neighborhood information system at time t , $NIS^{t+1} = (U, AT^{t+1}, \Delta)$ be a neighborhood information system at time $t+1$, $RNIS^{t+1} = (VT^{t+1}, AT^{t+1}, \Delta)$ be a relative neighborhood information system at time $t+1$. For all $X \in U$, the lower approximations in NMGRS are denoted by $\underline{\sum_{k=1}^m \delta_{A_k}^P(X)}$ and $\underline{\sum_{k=1}^m \delta_{A_k}^O(X)}$, respectively. If $BN^O(X) = \underline{\sum_{k=1}^m \delta_{A_k}^t}^O(X) \cup (U - \underline{\sum_{k=1}^m \delta_{A_k}^t}^O(X))$, the following results hold,

1. If $G^U(\underline{\sum_{k=1}^m \delta_{A_k}^{t+1}}^O(X))$ denote the boolean column matrix of lower approximation in ONMGRS, then we have

$$G^U(\underline{\sum_{k=1}^m \Delta_{A_k}^{t+1}}^O(X)) = \underline{\sum_{k=1}^m \delta_{A_k}^t}^O(X) \wedge \sim (\bigwedge_{k=1}^m F^{-1}(H_{A_k}^{[0,1]}(BN^P(X)))) ,$$

2. If $G^U(\underline{\sum_{k=1}^m \delta_{A_k}^{t+1}}^P(X))$ denote the boolean column matrix of lower approximation in ONMGRS, then we have

$$G^U(\underline{\sum_{k=1}^m \Delta_{A_k}^{t+1}}^P(X)) = \underline{\sum_{k=1}^m \delta_{A_k}^t}^P(X) \wedge \sim (\bigvee_{k=1}^m F^{-1}(H_{A_k}^{[0,1]}(BN^P(X)))) .$$

Proof. The proof is similar to that of Theorem 9. $\square$

Example 7. (Continuation of Example 1) Supposed that $AT^t = \{A_1^t, A_2^t\}$, $A_1^t = (A_1^C)^t \cup (A_1^N)^t$, $(A_1^C)^t = \{a_1, a_2\}$ and $(A_1^N)^t = \emptyset$, $A_2^t = (A_2^C)^t \cup (A_2^N)^t$, $(A_2^C)^t = \emptyset$ and $(A_2^N)^t = \{a_3, a_4\}$, $AT^{t+1} = \{A_1^{t+1}, A_2^{t+1}\}$, $A_1^{t+1} = (A_1^C)^{t+1} \cup (A_1^N)^{t+1}$,

$(A_1^C)^{t+1} = \{a_2\}$ and $(A_1^N)^{t+1} = \emptyset$, $A_2^{t+1} = (A_2^C)^{t+1} \cup (A_2^N)^{t+1}$, $(A_2^C)^{t+1} = \emptyset$ and $(A_2^N)^{t+1} = \{a_4\}$. Let $\delta = 0.5$, $\Delta = 0.5$. According to Definition 10, we have that for any $k \in \{1, 2\}$, the relative neighborhood relation matrix can be calculate as follows, According to Theorem 14 and Theorem 15, $BN^O(X) = \{x_2, x_3, x_4, x_6\}$, $R_{A_2}(BN^P(X)) = U$, $R_{A_1}(BN^P(X)) = U$, the column matrix $H_{A_1}(BN^P(X))$ and $H_{A_2}(BN^P(X))$ can be calculated as follows.

$$H_{A_1}(BN^P(X)) = \Lambda_{A_1} \cdot (M_{A_1} \cdot G^{V^1}(X))$$

$$= \text{diag} \left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right] \cdot \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix},$$

$$H_{A_2}(BN^P(X)) = \Lambda_{A_2} \cdot (M_{A_2} \cdot G^{V^2}(X))$$

$$= \text{diag} \left[\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{2} \right] \cdot \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3/4 \\ 3/4 \\ 0 \\ 3/4 \\ 3/4 \\ 0 \end{bmatrix}.$$

According to Theorem 9) and Theorem 10) we have

$$\begin{aligned} \overline{\sum_{k=1}^2 \delta_{A_k}^{t+1}}^O(X) &= G^U(\sum_{k=1}^2 \Delta_{A_k}^t(X)) \vee (\bigwedge_{k=1}^2 F^{-1}(H_{A_k}^{[0,1]}(BN^P(X)))) \\ &= [1 \ 1 \ 0 \ 1 \ 1 \ 0]^T \vee \\ &\quad \left[[1 \ 1 \ 1 \ 1 \ 1 \ 1]^T \wedge [1 \ 1 \ 0 \ 1 \ 1 \ 0]^T \right] \\ &= [1 \ 1 \ 0 \ 1 \ 1 \ 0]^T \wedge [1 \ 1 \ 0 \ 1 \ 1 \ 0]^T \\ &= [1 \ 1 \ 0 \ 1 \ 1 \ 0]^T. \end{aligned}$$

$$\begin{aligned} \overline{\sum_{k=1}^2 \delta_{A_k}^{t+1}}^P(X) &= G^U(\sum_{k=1}^2 \Delta_{A_k}^t(X)) \wedge \sim (\bigwedge_{k=1}^2 F^{-1}(H_{A_k}^{[0,0]}(BN^P(X)))) \\ &= [1 \ 1 \ 1 \ 1 \ 1 \ 0]^T \vee \\ &\quad \left[[1 \ 1 \ 1 \ 1 \ 1 \ 1]^T \vee [1 \ 1 \ 0 \ 1 \ 1 \ 0]^T \right] \\ &= [1 \ 1 \ 1 \ 1 \ 1 \ 0]^T \wedge [1 \ 1 \ 1 \ 1 \ 1 \ 1]^T \\ &= [1 \ 1 \ 1 \ 1 \ 1 \ 0]^T. \end{aligned}$$

$$\begin{aligned}
G^U(\underbrace{\sum_{k=1}^2 \delta_{A_k}^{t+1}}^O(X)) &= G^U(\underbrace{\sum_{k=1}^2 \delta_{A_k}^t}_{\sim}^O(X)) \wedge \sim (\bigwedge_{k=1}^m F^{-1}(H_{A_k}^{[0,1]}(BN^P(X)))) \\
&= [0 \ 1 \ 0 \ 1 \ 0 \ 0]^T \wedge \sim \\
&\quad \left[[1 \ 1 \ 1 \ 1 \ 1 \ 1]^T \vee [1 \ 1 \ 1 \ 1 \ 1 \ 1]^T \right] \\
&= [1 \ 1 \ 1 \ 1 \ 1 \ 1]^T \wedge [0 \ 0 \ 0 \ 0 \ 0 \ 0]^T \\
&= [0 \ 0 \ 0 \ 0 \ 0 \ 0]^T. \\
G^U(\underbrace{\sum_{k=1}^2 \delta_{A_k}^{t+1}}^P(X)) &= G^U(\underbrace{\sum_{k=1}^2 \delta_{A_k}^t}_{\sim}^P(X)) \vee (\bigvee_{k=1}^m F^{-1}(H_{A_k}^{[0,1]}(BN^P(X)))) \\
&= [0 \ 0 \ 0 \ 1 \ 0 \ 0]^T \wedge \sim \\
&\quad \left[[1 \ 1 \ 1 \ 1 \ 1 \ 1]^T \vee [1 \ 1 \ 1 \ 1 \ 1 \ 1]^T \right] \\
&= [0 \ 0 \ 0 \ 1 \ 0 \ 0]^T \wedge [0 \ 0 \ 0 \ 0 \ 0 \ 0]^T \\
&= [0 \ 0 \ 0 \ 0 \ 0 \ 0]^T.
\end{aligned}$$

3. Matrix-based dynamic algorithms for updating approximations while neighborhood classes decreasing and increasing

Based on Theorem 9 and 10, we propose a relative matrix based algorithm for updating approximations in NMGRS while neighborhood classes are decreasing. Algorithm 2 is a relative relation matrix based algorithm for updating the approximations of NMGRS while neighborhood classes are decreasing. Suppose that $N_{BN^P(X)} = \max_{k=1}^{|AT^t|} \{|R_{A_k}(BN^P(X))|\}$, the total time complexity of Algorithm 2 is $O(|AT^t|N_{BN^P(X)}^2)$. Steps 4-12 are to calculate $R_{A_k}(BN^P(X))$ ($k \in \{1, 2, \dots, m\}$) whose time complexity is $O(|AT^t||X| \sim |X|)$. Steps 13-20 are to compute MX_{A_k} ($k \in \{1, 2, \dots, m\}$) whose time complexity is $O(|AT^t|R(BN^P(X))^2)$. Steps 21-25 are to compute $H_{A_k}(BN^P(X))$ whose time complexity is $O(|AT^t|N_{BN^P(X)}^2)$. Steps 27-29 are to compute $\sum_{k=1}^m \Delta_{A_k}^{t+1O}(X)$, $\overline{\sum_{k=1}^m \Delta_{A_k}^{t+1O}(X)}$, $\sum_{k=1}^m \Delta_{A_k}^{t+1P}(X)$ and $\overline{\sum_{k=1}^m \Delta_{A_k}^{t+1P}(X)}$ whose time complexity is $O(|AT^t||U|)$. Steps 24-27 are to update the approximations of NMGRS while neighborhood classes are decreasing whose time complexity is $O(|AT^t||U|)$.

Since the time complexity of Algorithm 1 is $O(|AT^t|N_X^2)$, and in general, $O(|AT^t|N_{BN^P(X)}^2) \leq O(|AT^t|N_X^2)$ is not hold, Algorithm 3 was proposed to make sure the total time complexity is no more than $O(|AT^t|N_X^2)$, in other words, when $N_{BN^P(X)} \leq N_X$ we call Algorithm 2, otherwise, we call Algorithm 1.

Algorithm 2 Matrix-based algorithm for computing approximations in NM-GRS

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1: Require (1) Let  $NIS^t = (U, AT^t, \Delta)$  (2)  $NIS^{t+1} = (U, AT^{t+1}, \delta)$  (3) A target concept  $X \subseteq U$  (4)  $\sum_{k=1}^m \Delta_{A_k}^t O(X)$ ,  $\overline{\sum_{k=1}^m \Delta_{A_k}^t O(X)}$ ,  $\sum_{k=1}^m \Delta_{A_k}^t P(X)$  and  $\overline{\sum_{k=1}^m \Delta_{A_k}^t P(X)}$ . (4) neighborhood classes  $n_{A_k^C \cup A_k^N}^{t+1}(x), \forall x \in U, \forall k \in \{1, 2, \dots, m\}$ .
Ensure:  $\sum_{k=1}^m \Delta_{A_k}^{t+1 O}(X)$ ,  $\overline{\sum_{k=1}^m \Delta_{A_k}^{t+1 O}(X)}$ ,  $\sum_{k=1}^m \Delta_{A_k}^{t+1 P}(X)$  and  $\overline{\sum_{k=1}^m \Delta_{A_k}^{t+1 P}(X)}$ .
2:  $m \leftarrow |AT^t|$ 
3:  $n \leftarrow |U|$ 
4:  $BN^P(X) \leftarrow \overline{\sum_{k=1}^m \Delta_{A_k}^t P(X)} - \sum_{k=1}^m \Delta_{A_k}^t P(X)$ 
5: for  $k = 1 \rightarrow m$  do
6:   for  $i = 1 \rightarrow n$  do
7:     for  $j = 1 \rightarrow n$  do
8:       if  $g_i^U(BN^P(X)) = 1 \wedge g_j^U(n_{A_1^C \cup A_1^N}(x_i)) = 1$  then
9:          $g_i^U(R_k(BN^P(X))) = 1$ 
10:       end if
11:     end for
12:   end for
13:    $V \leftarrow f^{-1}(R_k(BN^P(X)))$ 
14:    $r_k \leftarrow |V^k|$ 
15:   for  $i = 1 \rightarrow r_k$  do
16:     for  $j = 1 \rightarrow r_k$  do
17:       if  $(y_i, y_j) \in R_k = 1$  then  $m_{ij}^{A_k} = 1$ 
18:     end if
19:   end for
20:   end for
21:   for  $i = 1 \rightarrow r_k$  do
22:      $g_i^{V^k}(\Lambda_{A_k}) \leftarrow \sum_{j=1}^r m_{ij}^{A_k}$ 
23:   end for
24:    $G^{V^k}(BN^P(X)) \leftarrow F^{-1}(G^U(X))$ 
25:    $H_{A_k}(BN^P(X)) \leftarrow \text{diag}(\Lambda_{A_k}) \cdot (M_{A_k} \cdot G^{V^k}(X))$ 
26: end for
27:  $G^U(\sum_{k=1}^m \delta_{A_k}^O(X)) \leftarrow G^U(\sum_{k=1}^m \Delta_{A_k}^t O(X)) \vee (\bigvee_{k=1}^m F^{-1}(H_{A_k}^{[1,1]}(BN^P(X))))$ 
28:  $G^U(\overline{\sum_{k=1}^m \delta_{A_k}^O(X)}) \leftarrow G^U(\overline{\sum_{k=1}^m \Delta_{A_k}^t O(X)}) \wedge (\bigvee_{k=1}^m F^{-1}(H_{A_k}^{[0,0]}(BN^P(X))))$ 
29:  $G^U(\sum_{k=1}^m \delta_{A_k}^P(X)) \leftarrow G^U(\sum_{k=1}^m \Delta_{A_k}^t P(X)) \vee (\bigwedge_{k=1}^m F^{-1}(H_{A_k}^{[1,1]}(BN^P(X))))$ 
30:  $G^U(\overline{\sum_{k=1}^m \delta_{A_k}^P(X)}) \leftarrow G^U(\overline{\sum_{k=1}^m \Delta_{A_k}^t P(X)}) \wedge (\bigwedge_{k=1}^m F^{-1}(H_{A_k}^{[0,0]}(BN^P(X))))$ 
31: Return  $\sum_{k=1}^m \delta_{A_k}^O(X)$ ,  $\overline{\sum_{k=1}^m \delta_{A_k}^O(X)}$ ,  $\sum_{k=1}^m \delta_{A_k}^P(X)$  and  $\overline{\sum_{k=1}^m \delta_{A_k}^P(X)}$ 

```

Algorithm 3 Ensure total time complexity of updating approximations in NMGRS while increasing granular structures no more than $O(|AT^t|R(X)^2)$

1: Require (1) Let $NIS^t = (U, AT^t, \Delta)$ (2) $NIS^{t+1} = (U, AT^{t+1}, \delta)$ (3) A target concept $X \subseteq U$ (4) $\sum_{k=1}^m \Delta_{A_k}^t O(X)$, $\overline{\sum_{k=1}^m \Delta_{A_k}^t O(X)}$, $\sum_{k=1}^m \Delta_{A_k}^t P(X)$ and $\overline{\sum_{k=1}^m \Delta_{A_k}^t P(X)}$. (4) neighborhood classes $n_{A_k^C \cup A_k^N}^{t+1}(x), \forall x \in U, \forall k \in \{1, 2, \dots, m\}$.

Ensure: $\sum_{k=1}^m \delta_{A_k}^{t+1} O(X)$, $\overline{\sum_{k=1}^m \delta_{A_k}^{t+1} O(X)}$, $\sum_{k=1}^m \delta_{A_k}^{t+1} P(X)$ and $\overline{\sum_{k=1}^m \delta_{A_k}^{t+1} P(X)}$.

2: **if** $N_{BN^P(X)} \leq N_X$ **then** Call Algorithm 3

3: **end if**

4: **if** $N_X \leq N_{BN^P(X)}$ **then** Call Algorithm 2

5: **end if**

6: **Return** $\sum_{k=1}^m \delta_{A_k}^{t+1} O(X)$, $\overline{\sum_{k=1}^m \delta_{A_k}^{t+1} O(X)}$, $\sum_{k=1}^m \delta_{A_k}^{t+1} P(X)$ and $\overline{\sum_{k=1}^m \delta_{A_k}^{t+1} P(X)}$.

Based on Theorem 14 and Theorem 15, we proposed a relative matrix based algorithm for updating approximations in NMGRS while neighborhood classes are increasing. Algorithm 3 is a relative matrix based algorithm for updating the approximations of NMGRS while neighborhood classes are increasing. Suppose that $N_{BN^O(X)} = \max_{k=1}^{|AT^t|} \{|R_{A_k}(BN^O(X))|\}$, the total time complexity of Algorithm 2 is $O(|AT^t|R(X)^2)$. Steps 5-12 are to calculate $R_{A_k}(BN^O(X))$ whose time complexity is $O(|X| \sim X)$. Steps 13-20 are to compute $MX^{A_k} (k \in \{1, 2, \dots, m\})$ whose time complexity is $O(|AT^t|N_{BN^O(X)}^2)$. Steps 21-25 are to compute $H_{A_k}(BN^O(X))$ whose time complexity is $O(|AT^t|N_{BN^O(X)}^2)$.

Steps 27-29 are to update approximations in NMGTs whose time complexity is $O(|AT^t||U|)$. Since the time complexity of Algorithm 1 is $O(|AT^t|N_X^2)$, and In general, $O(|AT^t|N_{BN^O(X)}^2) \leq O(|AT^t|N_X^2)$ is not hold, Algorithm 5 was proposed to make sure the total time complexity is no more than $O(|AT^t|N_X^2)$, in other words, when $N_{BN^O(X)} \leq N_X$ we call Algorithm 4, otherwise, we call Algorithm 1.

4. Experimental evaluations

In this section, several experiments were conducted to evaluate effectiveness and efficiency of the algorithms that we devised. In order to verify the effectiveness and efficiency of Algorithm 3 and Algorithm 5, six data sets have been chosen from UCI machine learning repository to perform tests in the experiments. The details of data sets is listed in Table 2. All the experiments were carried out

Algorithm 4 Matrix-based algorithm for computing approximations in NM-GRS

```

1: Require (1) Let  $NIS^t = (U, AT^t, \delta)$  (2)  $NIS^{t+1} = (U, AT^{t+1}, \Delta)$  (3) A target concept  $X \subseteq U$  (4)  $\sum_{k=1}^m \delta_{A_k}^t O(X)$ ,  $\overline{\sum_{k=1}^m \delta_{A_k}^t O(X)}$ ,  $\sum_{k=1}^m \delta_{A_k}^t P(X)$  and  $\overline{\sum_{k=1}^m \delta_{A_k}^t P(X)}$ . (4) neighborhood classes  $n_{A_k^C \cup A_k^N}^{t+1}(x), \forall x \in U, \forall k \in \{1, 2, \dots, m\}$ .
Ensure:  $\sum_{k=1}^m \Delta_{A_k}^{t+1 O}(X)$ ,  $\overline{\sum_{k=1}^m \Delta_{A_k}^{t+1 O}(X)}$ ,  $\sum_{k=1}^m \Delta_{A_k}^{t+1 P}(X)$  and  $\overline{\sum_{k=1}^m \Delta_{A_k}^{t+1 P}(X)}$ .
2:  $m \leftarrow |AT^t|$ 
3:  $n \leftarrow |U|$ 
4:  $BN^O(X) \leftarrow \sum_{k=1}^m \delta_{A_k}^t O(X) \cup (U - \overline{\sum_{k=1}^m \delta_{A_k}^t O(X)})$ 
5: for  $k = 1 \rightarrow m$  do
6:   for  $i = 1 \rightarrow n$  do
7:     for  $j = 1 \rightarrow n$  do
8:       if  $g_i^U(BN^O(X)) = 1 \wedge g_j^U(n_{A_1^C \cup A_1^N}(x_i)) = 1$  then
9:          $g_i^U(R_k(BN^O(X))) = 1$ 
10:       end if
11:     end for
12:   end for
13:    $V^k \leftarrow f^{-1}(R_k(BN^O(X)))$ 
14:    $r_k \leftarrow |V^k|$ 
15:   for  $i = 1 \rightarrow r_k$  do
16:     for  $j = 1 \rightarrow r_k$  do
17:       if  $(y_i, y_j) \in R_k = 1$  then  $m_{ij}^{A_k} = 1$ 
18:     end if
19:   end for
20:   end for
21:   for  $i = 1 \rightarrow r_k$  do
22:      $g_i^{V^k}(\Lambda_{A_k}) \leftarrow \sum_{j=1}^r m_{ij}^{A_k}$ 
23:   end for
24:    $G^{V^k}(X) \leftarrow F^{-1}(G^U(X))$ 
25:    $H_{A_k}(BN^O(X) \leftarrow \text{diag}(\Lambda_{A_k}) \cdot (M_{A_k} \cdot G^{V^k}(X))$ 
26: end for
27:  $G^U(\sum_{k=1}^m \Delta_{A_k}^O(X)) \leftarrow G^U(\sum_{k=1}^m \delta_{A_k}^t O(X)) \wedge (\bigwedge_{k=1}^m F^{-1}(H_{A_k}^{(0,1)}(BN^O(X)))) \sim$ 
28:  $G^U(\overline{\sum_{k=1}^m \Delta_{A_k}^O(X)}) \leftarrow G^U(\overline{\sum_{k=1}^m \delta_{A_k}^t O(X)}) \vee (\bigwedge_{k=1}^m F^{-1}(H_{A_k}^{(0,1]}(BN^O(X))))$ 
29:  $G^U(\sum_{k=1}^m \Delta_{A_k}^P(X)) \leftarrow G^U(\sum_{k=1}^m \delta_{A_k}^t P(X)) \wedge (\bigvee_{k=1}^m F^{-1}(H_{A_k}^{(0,1)}(BN^O(X)))) \sim$ 
30:  $G^U(\overline{\sum_{k=1}^m \Delta_{A_k}^P(X)}) \leftarrow G^U(\overline{\sum_{k=1}^m \delta_{A_k}^t P(X)}) \vee (\bigvee_{k=1}^m F^{-1}(H_{A_k}^{(0,1]}(BN^O(X))))$ 
31: Return  $\sum_{k=1}^m \Delta_{A_k}^O(X)$ ,  $\overline{\sum_{k=1}^m \Delta_{A_k}^O(X)}$ ,  $\sum_{k=1}^m \Delta_{A_k}^P(X)$  and  $\overline{\sum_{k=1}^m \Delta_{A_k}^P(X)}$ 

```

Algorithm 5 Ensure total time complexity of updating approximations in NMGRS while increasing granular structures no more than $O(|AT^t|R(X)^2)$

1: Require (1) Let $NIS^t = (U, AT^t, \delta)$ (2) $NIS^{t+1} = (U, AT^{t+1}, \Delta)$ (3) A target concept $X \subseteq U$ (4) $\sum_{k=1}^m \delta_{A_k}^t O(X)$, $\overline{\sum_{k=1}^m \delta_{A_k}^t O(X)}$, $\sum_{k=1}^m \delta_{A_k}^t P(X)$ and $\overline{\sum_{k=1}^m \delta_{A_k}^t P(X)}$. (4) neighborhood classes $n_{A_k^C \cup A_k^N}^{t+1}(x), \forall x \in U, \forall k \in \{1, 2, \dots, m\}$.

Ensure: $\sum_{k=1}^m \Delta_{A_k}^{t+1 O}(X)$, $\overline{\sum_{k=1}^m \Delta_{A_k}^{t+1 O}(X)}$, $\sum_{k=1}^m \Delta_{A_k}^{t+1 P}(X)$ and $\overline{\sum_{k=1}^m \Delta_{A_k}^{t+1 P}(X)}$.

2: **if** $N_{BNO}(X) \leq N_X$ **then** Call Algorithm 4

3: **end if**

4: **if** $N_X \leq N_{BNO}(X)$ **then** Call Algorithm 1

5: **end if**

6: **Return** $\sum_{k=1}^m \Delta_{A_k}^{t+1 O}(X)$, $\overline{\sum_{k=1}^m \Delta_{A_k}^{t+1 O}(X)}$, $\sum_{k=1}^m \Delta_{A_k}^{t+1 P}(X)$ and $\overline{\sum_{k=1}^m \Delta_{A_k}^{t+1 P}(X)}$.

Table 2: Details of data sets

No.	Data sets	Samples	Attributes
1	Cylinder Bands Data Set	541	40
2	Dermatology	366	20
3	Flags	194	30
4	LasVegas Trip Advisor Reviews Data set	504	20
5	Soybean Large	307	36
6	Student	396	33

on a personal computer with 64-bit windows 10, Inter(R) Core(TM) i7 6700HQ CPU @2.60GHz, and 8GB memory. The program language is Matlab r2015b.

4.1 The comparison of computational time with different size of data sets

The computational time were compared between static and incremental algorithms in NMGRS when the size of data increases. While neighborhood classes are decreasing and increasing, first of all, we made $\delta = 1, \Delta = 1$ and we randomly chose the categorical and numeric attributes $\widehat{a_C}$ and $\widehat{a_N}$ in the data set and divided the rest of the categorical and numeric attributes into three parts randomly respectively to contribute to three granular structures. While neighborhood classes decreasing, we added the categorical and numeric attributes $\widehat{a_C}$ and $\widehat{a_N}$ into the three granular structures at the same time. While neighborhood classes increasing, we combined the categorical and numeric attributes $\widehat{a_C}$

and $\widehat{a}_N$ with each granular structure and deleted the categorical and numeric attributes $\widehat{a}_C$ and $\widehat{a}_N$ from the three granular structures at the same time. We randomly divided every data set U into ten subsets which is denoted by $\{U_1, U_2, \dots, U_{10}\}$. Then U_1 was chosen as the first temporary data set. After that, some samples of temporary data set was randomly selected to contribute to the target concept X , the size of target concept X was about 0.2 times the size of the each temporary data set. We calculated the four approximations in NMGRS by static and incremental algorithms ten times and compared the averages. Then we made $U_1 \cup U_2$ the second temporary data set and repeat the whole process.

When the size of data sets increases, Experimental results of the static and the incremental while neighborhood classes are decreasing and increasing in NMGRS are shown in Figure 1 and Figure 2, by comparing the computational time of the static and the incremental algorithms in Figure 1 and Figure 2, we have that the computational time of the incremental algorithms is less than or equal to the static algorithm in most situations of these experiments. Sometimes the computational time of the incremental algorithms was a little more than the static algorithm while deleting attributes, which is due to additional computation in the incremental algorithms and it is within the expected range. According to Figure 1 and Figure 2 we can observe that the incremental algorithms for updating approximations while neighborhood classes are decreasing and increasing are more efficient than the static algorithm in most of the situations.

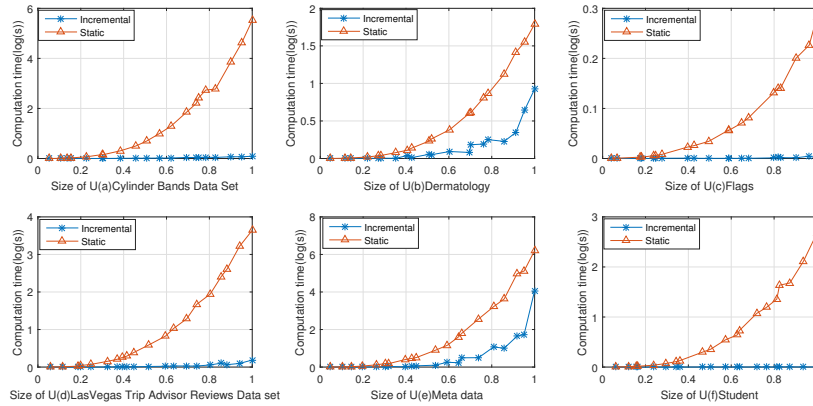


Figure 1: Computational time of Algorithm 3 when the size of U increasing gradually

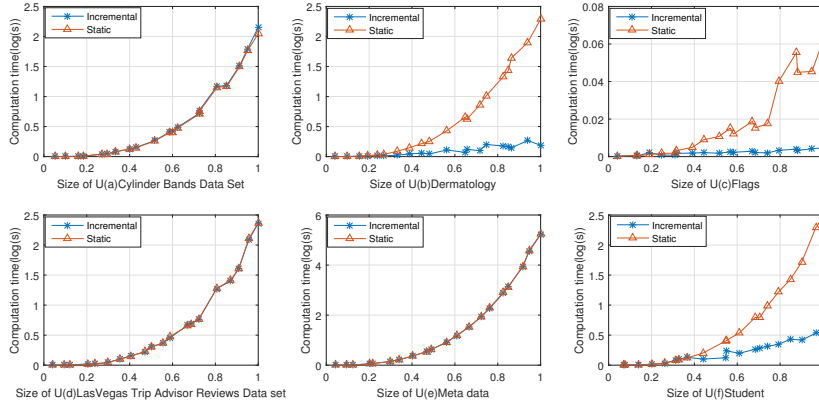


Figure 2: Computational time of Algorithm 5 when the size of U increasing gradually

4.2 The comparison of computational time with different size of target concept

The computational time are compared between the static and the incremental algorithms in NMGRS when the size of target concept increases. While neighborhood classes are decreasing and increasing, the process of setting parameters and constructing granular structures is similar to subsection 5.1. We randomly divided every data set U into ten subsets which is denoted by $\{U_1, U_2, \dots, U_{10}\}$. And then U_1 was chosen as the first temporary target concept X . Finally, we calculated the four approximations in NMGRS by the static and the incremental algorithm ten times and compared the averages. We made $U_1 \cup U_2$ the second temporary target concept and repeat the whole process.

When the size of target concept increases, Experimental results of the static and the incremental while neighborhood classes decreasing and increasing in NMGRS are shown in Figure 3 and Figure 4, by comparing the computational time of the static and the incremental algorithms in Figure 3 and Figure 4, we can also get the same result that incremental methods are more efficient than static method. According to Figure 3 and Figure 4 we can observe that the incremental algorithms for updating approximations while neighborhood classes decreasing and increasing are more efficient than the static algorithm in most of the situations.

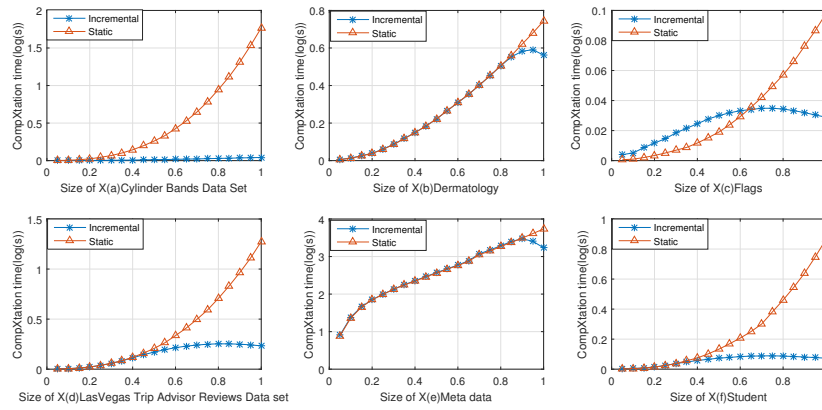


Figure 3: Computational time of Algorithm 3 when the size of X increasing gradually

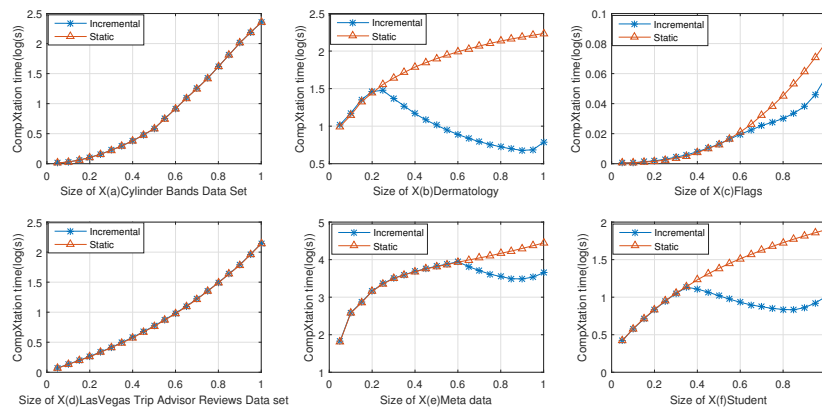


Figure 4: Computational time of Algorithm 5 when the size of X increasing gradually

5. Conclusion

Data sets in real life applications sometimes are complex and huge, i.e, it is difficult to use and granular structures often increase and decrease in some data sets. It is a significant issue to design algorithms to update approximation in NMGRS. In this paper, four algorithms have been proposed to ensure time complexity of incremental algorithms is less than static algorithm. Experimental results showed that the time complexity of the incremental algorithms is no more than the static algorithm in most of situations. Approximation computation is a basic process of attribute reduction, in the future, we will further investigate attribute reduction algorithm using approaches we proposed.

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Symmetry classification and solutions for Shigesada-Kawasaki-Teramoto system

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Abstract. In this paper, the symmetry classification of the Shigesada-Kawasaki-Teramoto system is presented, then the symmetry reduction and solutions of the classified system are constructed by the generalized conditional symmetry method.

Keywords: symmetry classification, symmetry reduction, solution.

1. Introduction

The reaction-diffusion system

$$\begin{aligned} (1) \quad u_t &= [(d_1 + d_2u + d_3v)u]_{xx} + u(a_{11} - b_{11}u - c_1v), \\ v_t &= [(d_4 + d_5u + d_6v)v]_{xx} + v(a_{21} - b_{12}u - c_2v) \end{aligned}$$

is the well-known Shigesada-Kawasaki-Teramoto system which was first proposed by Shigesada et al [1]. Where $u = u(x, t)$, $v = v(x, t)$ denote the unknown functions of the space variable x and time t , a_{ij} , b_{ij} , c_i and d_i are arbitrary constants. The generalized forms of system (1) play a significant role in mathematical biology, and some results are shown in [2]-[8] (and references cited therein). In the following sections, we will use the form of generalized conditional symmetry (GCS) [9]-[15]

$$\begin{aligned} (2) \quad \eta_1 &= u_{l_1x} + a_{(l_1-1)}u_{(l_1-1)x} + \cdots + a_1u, \\ \eta_2 &= v_{l_2x} + b_{(l_2-1)}v_{(l_2-1)x} + \cdots + b_1v \end{aligned}$$

to classify system (1) and seek for symmetry reduction of the obtained system, here $l_1 > 2, l_2 \geq 2$.

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The remainder of this paper is organized as follows. In section 2, we perform symmetry classification for system (1), which admits the GCS are presented. In section 3, the symmetry reduction and exact solutions to the classified equations are constructed by the GCS method.

2. Symmetry classification of system (1)

In this section, we display the symmetry classification results of system (1). We will consider the following eight cases.

Case 1. When $l_1 = 3$, $l_2 = 2$.

According to the definition of GCS method and calculation procedure, we obtain the following determining system, then it takes the form

$$\begin{aligned}
 F_1 = & 14a_2d_2u_{xx}^2 + [(2a_2^2d_2 + 24a_1d_2 - 6b_{11})u_x + (a_2^2d_3 + 4a_2b_1d_3 + 10b_1^2d_3 \\
 & + 2a_1d_3 + 10b_0d_3 - 3c_1)v_x + 20a_0d_2u + (4a_2b_0d_3 + 10b_0b_1d_3)v]u_{xx} + (2a_1a_2d_2 \\
 & + 8a_0d_2 + 2a_2b_{11})u_x^2 + [(-4a_2b_1^2d_3 + 5b_1^3d_3 + a_1a_2d_3 + 7a_1b_1d_3 - 4a_2b_0d_3 \\
 & + 10b_0b_1d_3 + 3a_0d_3 + 2a_2c_1 - 3b_1c_1)v_x + 2a_0a_2d_2u + (-4a_2b_0b_1d_3 + 5b_0b_1^2d_3 \\
 & + 7a_1b_0d_3 + 5b_0^2d_3 - 3b_0c_1)v]u_x + (-a_2b_1^3d_3 + b_1^4d_3 - a_1b_1^2d_3 - 2a_2b_0b_1d_3 \\
 & + 3b_0b_1^2d_3 + a_0a_2d_3 + 9a_0b_1d_3 - a_1b_0d_3 + a_2b_1c_1 + b_0^2d_3 - b_1^2c_1 + a_1c_1 - b_0c_1)uv_x \\
 & - u^2a_0b_{11} + (-a_2b_0b_1^2d_3 + b_0b_1^3d_3 - a_1b_0b_1d_3 - a_2b_0^2d_3 + 2b_0^2b_1d_3 \\
 & + 9a_0b_0d_3 + a_2b_0c_1 - b_0b_1c_1)vu = 0, \\
 F_2 = & [(4a_2d_5 + 3b_1d_5)v_x + (a_2^2d_5 - a_2b_1d_5 + a_1d_5 + 5b_0d_5 - b_{12})v]u_{xx} \\
 & + [(b_1^2d_5 + 4a_1d_5 + 2b_0d_5 - 2b_{12})v_x + (a_1a_2d_5 - a_1b_1d_5 + b_0b_1d_5 + a_0d_5 + b_1b_{12})v]u_x \\
 & + (8b_1^2d_6 + 6b_0d_6 - 2c_2)v_x^2 + (14b_0b_1d_6v + 4a_0d_5u)v_x + (a_0a_2d_5 - a_0b_1d_5)uv \\
 & + (6b_0^2d_6 - b_0c_2)v^2 = 0.
 \end{aligned}$$

The coefficients of $u, v, v_x, u_x, u_{xx}, v_{xx}$ are equal to zero, and the classification results are shown in the following forms.

(i)

$$\begin{aligned}
 u_t &= [(d_1 + d_2u)u]_{xx} + u(a_{11} - 4a_1d_2u), \\
 v_t &= [(d_4 + d_6v)v]_{xx} + v(a_{21} - 4b_1^2d_6v), \\
 \eta_1 &= u_{xxx} - a_1u_x, \\
 \eta_2 &= v_{xx} - b_1v_x;
 \end{aligned}$$

(ii)

$$\begin{aligned}
 u_t &= [(d_1 + d_2u + d_3v)u]_{xx} + u(a_{11} - 4b_1^2d_2u - 4b_1^2d_3v), \\
 v_t &= [(d_4 + d_6v)v]_{xx} + v(a_{21} - 4b_1^2d_6v), \\
 \eta_1 &= u_{xxx} - b_1^2u_x, \\
 \eta_2 &= v_{xx} - b_1v_x;
 \end{aligned}$$

(iii)

$$\begin{aligned}
u_t &= [(d_1 + d_2u + d_3v)u]_{xx} + a_{11}u, \\
v_t &= [(d_4 + d_5u + d_6v)v]_{xx} + a_{21}v, \\
\eta_1 &= u_{xxx}, \\
\eta_2 &= v_{xx};
\end{aligned}$$

(iv)

$$\begin{aligned}
u_t &= [(d_1 + d_3v)u]_{xx} + a_{11}u, \\
v_t &= [(d_4 + d_5u + d_6v)v]_{xx} + a_{21}v, \\
\eta_1 &= u_{xxx}, \\
\eta_2 &= v_{xx};
\end{aligned}$$

(v)

$$\begin{aligned}
u_t &= [(d_1 + d_2u)u]_{xx} + u(a_{11} - 4a_1d_2u), \\
v_t &= d_4v_{xx} + a_{21}v, \\
\eta_1 &= u_{xxx} - a_1u_x, \\
\eta_2 &= v_{xx} - b_1v_x - b_0v;
\end{aligned}$$

(vi)

$$\begin{aligned}
u_t &= [(d_1 + d_2u)u]_{xx} + u(a_{11} - 4a_1d_2u), \\
v_t &= d_4v_{xx} + a_{21}v, \\
\eta_1 &= u_{xxx} - a_1u_x, \\
\eta_2 &= v_{xx} - b_0v;
\end{aligned}$$

(vii)

$$\begin{aligned}
u_t &= [(d_1 + d_2u + d_3v)u]_{xx} + u(a_{11} - 4b_0d_2u - 4b_0d_3v), \\
v_t &= d_4v_{xx} + a_{21}v, \\
\eta_1 &= u_{xxx} - b_0u_x, \\
\eta_2 &= v_{xx} - b_0v;
\end{aligned}$$

(viii)

$$\begin{aligned}
u_t &= [(d_1 + d_2u + d_3v)u]_{xx} + a_{11}u, \\
v_t &= [(d_4 + d_5u)v]_{xx} + a_{21}v, \\
\eta_1 &= u_{xxx}, \\
\eta_2 &= v_{xx};
\end{aligned}$$

(ix)

$$\begin{aligned}
u_t &= [(d_1 + d_2 u)u]_{xx} + u(a_{11} - 16b_0 d_2 u), \\
v_t &= [(d_4 + d_5 u)v]_{xx} + v(a_{21} - 9b_0 d_5 u), \\
\eta_1 &= u_{xxx} - 4b_0 u_x, \\
\eta_2 &= v_{xx} - b_0 v;
\end{aligned}$$

(x)

$$\begin{aligned}
u_t &= d_1 u_{xx} + a_{11} u, \\
v_t &= [(d_4 + d_6 v)v]_{xx} + v(a_{21} - 4b_1^2 d_6 v), \\
\eta_1 &= u_{xxx} - a_2 u_{xx} - a_1 u_x - a_0 u, \\
\eta_2 &= v_{xx} - b_1 v_x;
\end{aligned}$$

(xi)

$$\begin{aligned}
u_t &= [(d_1 + d_3 v)u]_{xx} + u \left[a_{11} - \left(\frac{1}{9} a_2^2 + \frac{4}{3} a_2 b_1 + 4b_1^2 \right) d_3 v \right], \\
v_t &= [(d_4 + d_6 v)v]_{xx} + v(a_{21} - 4b_1^2 d_6 v), \\
\eta_1 &= u_{xxx} - a_2 u_{xx} + \left(\frac{1}{3} a_2^2 - b_1^2 \right) u_x - \left(\frac{1}{27} a_2^3 - \frac{1}{3} a_2 b_1^2 \right) u, \\
\eta_2 &= v_{xx} - b_1 v_x;
\end{aligned}$$

(xii)

$$\begin{aligned}
u_t &= d_1 u_{xx} + a_{11} u, \\
v_t &= d_4 v_{xx} + a_{21} v, \\
\eta_1 &= u_{xxx} - a_2 u_{xx} - a_1 u_x - a_0 u, \\
\eta_2 &= v_{xx} - b_1 v_x - b_0 v;
\end{aligned}$$

(xiii)

$$\begin{aligned}
u_t &= d_1 u_{xx} + a_{11} u, \\
v_t &= d_4 v_{xx} + a_{21} v, \\
\eta_1 &= u_{xxx} + \frac{5}{2} b_1 u_{xx} - a_1 u_x - \left(\frac{91}{216} b_1^2 + \frac{10}{27} a_1 + \frac{43}{54} b_0 \right) b_1 u, \\
\eta_2 &= v_{xx} - b_1 v_x - b_0 v;
\end{aligned}$$

(xiv)

$$\begin{aligned}
u_t &= d_1 u_{xx} + a_{11} u, \\
v_t &= d_4 v_{xx} + a_{21} v, \\
\eta_1 &= u_{xxx} - a_1 u_x, \\
\eta_2 &= v_{xx} - b_0 v;
\end{aligned}$$

(xv)

$$\begin{aligned}
u_t &= [(d_1 + d_3 v)u]_{xx} + u(a_{11} - 4b_0 d_3 v), \\
v_t &= d_4 v_{xx} + a_{21} v, \\
\eta_1 &= u_{xxx} - b_0 u_x, \\
\eta_2 &= v_{xx} - b_0 v;
\end{aligned}$$

(xvi)

$$\begin{aligned}
u_t &= d_1 u_{xx} + a_{11} u, \\
v_t &= d_4 v_{xx} + a_{21} v, \\
\eta_1 &= u_{xxx} + \frac{5}{2} b_1 u_{xx} - \left(\frac{29}{20} b_1^2 + \frac{77}{5} b_0 \right) u_x - \left(\frac{23}{24} b_1^3 + \frac{351}{54} b_0 b_1 \right) u, \\
\eta_2 &= v_{xx} - b_1 v_x - b_0 v;
\end{aligned}$$

(xvii)

$$\begin{aligned}
u_t &= d_1 u_{xx} + a_{11} u, \\
v_t &= d_4 v_{xx} + a_{21} v, \\
\eta_1 &= u_{xxx} + \frac{5}{2} b_1 u_{xx} + \frac{71}{36} b_1^2 u_x + \frac{35}{72} b_1^3 u, \\
\eta_2 &= v_{xx} - b_1 v_x + \frac{2}{9} b_1^2 v;
\end{aligned}$$

(xviii)

$$\begin{aligned}
u_t &= [(d_1 + d_3 v)u]_{xx} + a_{11} u, \\
v_t &= d_4 v_{xx} + a_{21} v, \\
\eta_1 &= u_{xxx}, \\
\eta_2 &= v_{xx};
\end{aligned}$$

(xix)

$$\begin{aligned}
u_t &= d_1 u_{xx} + a_{11} u, \\
v_t &= [(d_4 + d_5 u)v]_{xx} + v \left(a_{21} - \frac{1}{64} b_1^2 d_5 u \right), \\
\eta_1 &= u_{xxx} + \frac{3}{4} b_1 u_{xx} + \frac{1}{8} b_1^2 u_x, \\
\eta_2 &= v_{xx} - b_1 v_x + \frac{15}{64} b_1^2 v.
\end{aligned}$$

Case 2. When $l_1 = 4$, $l_2 = 2$.

The system (1) is equivalent to one of the following cases.

(i)

$$\begin{aligned}
u_t &= d_1 u_{xx} + a_{11} u, \\
v_t &= [(d_4 + d_6 v)v]_{xx} + v(a_{21} - 4b_1^2 d_6 v), \\
\eta_1 &= u_{xxxx} - a_3 u_{xxx} - a_2 u_{xx} - a_1 u_x - a_0 u, \\
\eta_2 &= v_{xx} - b_1 v_x;
\end{aligned}$$

(ii)

$$\begin{aligned}
u_t &= d_1 u_{xx} + a_{11} u, \\
v_t &= d_4 v_{xx} + a_{21} v, \\
\eta_1 &= u_{xxxx} - a_3 u_{xxx} - a_2 u_{xx} - a_1 u_x - a_0 u, \\
\eta_2 &= v_{xx} - b_1 v_x - b_0 v;
\end{aligned}$$

(iii)

$$\begin{aligned}
u_t &= d_1 u_{xx} + a_{11} u, \\
v_t &= [(d_4 + d_6 v)v]_{xx} + v(a_{21} - 4b_1^2 d_6 v), \\
\eta_1 &= u_{xxxx} - a_3 u_{xxx} - a_2 u_{xx} \\
&\quad - \left(5b_1^3 - \frac{1}{4}a_3^3 - \frac{3}{4}a_3^2 b_1 + \frac{5}{6}a_3 b_1^2 - \frac{5}{6}a_2 a_3 - 2a_2 b_1 \right) u_x - a_0 u, \\
\eta_2 &= v_{xx} - b_1 v_x;
\end{aligned}$$

(iv)

$$\begin{aligned}
u_t &= [(d_1 + d_3 v)u]_{xx} + u \left[a_{11} - \left(\frac{13}{2}b_1^2 d_3 + \frac{1}{2}a_2 d_3 \right) v \right], \\
v_t &= [(d_4 + d_6 v)v]_{xx} + v(a_{21} - 4b_1^2 d_6 v), \\
\eta_1 &= u_{xxxx} + 6b_1 u_{xxx} - a_2 u_{xx} - (27b_1^3 + 3a_2 b_1) u_x + \left(\frac{55}{4}b_1^4 + 4a_2 b_1^2 + \frac{1}{4}a_2^2 \right) u, \\
\eta_2 &= v_{xx} - b_1 v_x;
\end{aligned}$$

(v)

$$\begin{aligned}
u_t &= d_1 u_{xx} + a_{11} u, \\
v_t &= d_4 v_{xx} + a_{21} v, \\
\eta_1 &= u_{xxxx} - a_3 u_{xxx} - a_2 u_{xx} + \left(\frac{17}{192}a_3^3 + \frac{1}{3}a_2 a_3 + \frac{5}{3}a_3 b_0 \right) u_x - a_0 u, \\
\eta_2 &= v_{xx} + \frac{1}{4}a_3 v_x - b_0 v;
\end{aligned}$$

(vi)

$$\begin{aligned}
u_t &= [(d_1 + d_3v)u]_{xx} + u(a_{11} - \frac{9}{4}b_1^2d_3v), \\
v_t &= d_4v_{xx} + a_{21}v, \\
\eta_1 &= u_{xxxx} + 4b_1u_{xxx} + \frac{7}{2}b_1^2u_{xx} - b_1^3u_x - \frac{15}{16}b_1^4u, \\
\eta_2 &= v_{xx} - b_1v_x;
\end{aligned}$$

(vii)

$$\begin{aligned}
u_t &= d_1u_{xx} + a_{11}u, \\
v_t &= d_4v_{xx} + a_{21}v, \\
\eta_1 &= u_{xxxx} + \frac{116}{7}b_1u_{xxx} + \frac{841}{14}b_1^2u_{xx} - \frac{24389}{343}b_1^3u_x - \frac{10609215}{38416}b_1^4u, \\
\eta_2 &= v_{xx} - \frac{29}{7}b_1v_x;
\end{aligned}$$

(viii)

$$\begin{aligned}
u_t &= [(d_1 + d_3v)u]_{xx} + a_{11}u, \\
v_t &= d_4v_{xx} + a_{21}v, \\
\eta_1 &= u_{xxxx}, \\
\eta_2 &= v_{xx};
\end{aligned}$$

Case 3. When $l_1 = 3$, $l_2 = 3$, the system (1) is equivalent to one of the following cases.

(i)

$$\begin{aligned}
u_t &= [(d_1 + d_2u)u]_{xx} + u(a_{11} - 4a_1d_2u), \\
v_t &= d_4v_{xx} + a_{21}v, \\
\eta_1 &= u_{xxx} - a_1u_x, \\
\eta_2 &= v_{xxx} - b_2v_{xx} - b_1v_x - b_0v;
\end{aligned}$$

(ii)

$$\begin{aligned}
u_t &= [(d_1 + d_2u + d_3v)u]_{xx} + u(a_{11} - 4b_1d_2u - 4b_1d_3v), \\
v_t &= [(d_4 + d_5u + d_6v)v]_{xx} + v(a_{21} - 4b_1d_5u - 4b_1d_6v), \\
\eta_1 &= u_{xxx} - b_1u_x, \\
\eta_2 &= v_{xxx} - b_1v_x;
\end{aligned}$$

(iii)

$$\begin{aligned}
u_t &= [(d_1 + d_2u)u]_{xx} + u(a_{11} - 4a_1d_2u), \\
v_t &= [(d_4 + d_6v)v]_{xx} + v(a_{21} - 4b_1d_6v), \\
\eta_1 &= u_{xxx} - a_1u_x, \\
\eta_2 &= v_{xxx} - b_1v_x;
\end{aligned}$$

(iv)

$$\begin{aligned}
u_t &= d_1 u_{xx} + a_{11} u, \\
v_t &= [(d_4 + d_6 v)v]_{xx} + v(a_{21} - 4b_1 d_6 v), \\
\eta_1 &= u_{xxx} - a_2 u_{xx} - a_1 u_x - a_0 u, \\
\eta_2 &= v_{xxx} - b_1 v_x;
\end{aligned}$$

(v)

$$\begin{aligned}
u_t &= d_1 u_{xx} + a_{11} u, \\
v_t &= d_4 v_{xx} + a_{21} v, \\
\eta_1 &= u_{xxx} - a_2 u_{xx} - a_1 u_x - a_0 u, \\
\eta_2 &= v_{xxx} - b_2 v_{xx} - b_1 v_x - b_0 v;
\end{aligned}$$

(vi)

$$\begin{aligned}
u_t &= d_1 u_{xx} + a_{11} u, \\
v_t &= [(d_4 + d_5 u)v]_{xx} + \left(a_{21} - \frac{1}{225} b_2^2 d_5 u\right) v, \\
\eta_1 &= u_{xxx} + \frac{2}{5} b_2 u_{xx} + \frac{8}{225} b_2^2 u_x, \\
\eta_2 &= v_{xxx} - b_2 v_{xx} + \frac{71}{225} b_2^2 v_x - \frac{7}{225} b_2^3 v;
\end{aligned}$$

(vii)

$$\begin{aligned}
u_t &= [(d_1 + d_3 v)u]_{xx} + u(a_{11} - \frac{a_2^2 d_3}{225} v), \\
v_t &= d_4 v_{xx} + a_{21} v, \\
\eta_1 &= u_{xxx} - a_2 u_{xx} + \frac{71}{225} a_2^2 u_x - \frac{7}{225} a_2^3 u, \\
\eta_2 &= v_{xxx} + \frac{2}{5} a_2 v_{xx} + \frac{8}{225} a_2^2 v_x.
\end{aligned}$$

Case 4. When $l_1 = 4$, $l_2 = 3$, the system (1) is equivalent to one of the following cases.

(i)

$$\begin{aligned}
u_t &= d_1 u_{xx} + a_{11} u, \\
v_t &= [(d_4 + d_6 v)v]_{xx} + v(a_{21} - 4b_1 d_6 v), \\
\eta_1 &= u_{xxxx} - a_3 u_{xxx} - a_2 u_{xx} - a_1 u_x - a_0 u, \\
\eta_2 &= v_{xxx} - b_1 v_x;
\end{aligned}$$

(ii)

$$\begin{aligned}
u_t &= [(d_1 + d_3v)u]_{xx} + u \left(a_{11} - \frac{25}{4}b_1d_3v \right), \\
v_t &= [(d_4 + d_6v)v]_{xx} + v(a_{21} - 4b_1d_6v), \\
\eta_1 &= u_{xxxx} - \frac{5}{2}b_1u_{xx} + \frac{9}{16}b_1^2u, \\
\eta_2 &= v_{xxx} - b_1v_x;
\end{aligned}$$

(iii)

$$\begin{aligned}
u_t &= d_1u_{xx} + a_{11}u, \\
v_t &= d_4v_{xx} + a_{21}v, \\
\eta_1 &= u_{xxxx} - a_3u_{xxx} - a_2u_{xx} - a_1u_x - a_0u, \\
\eta_2 &= v_{xxx} - b_2v_{xx} - b_1v_x - b_0v;
\end{aligned}$$

(iv)

$$\begin{aligned}
u_t &= [(d_1 + d_3v)u]_{xx} + u \left(a_{11} - \frac{1}{576}a_3^2d_3v \right), \\
v_t &= d_4v_{xx} + a_{21}v, \\
\eta_1 &= u_{xxxx} - a_3u_{xxx} + \frac{103}{288}a_3^2u_{xx} - \frac{31}{576}a_3^3u_x + \frac{35}{12288}a_3^4u, \\
\eta_2 &= v_{xxx} + \frac{1}{4}a_3v_{xx} + \frac{1}{72}a_3^2v_x.
\end{aligned}$$

Case 5. When $l_1 = 5$, $l_2 = 3$, the system (1) is equivalent to one of the following cases.

(i)

$$\begin{aligned}
u_t &= d_1u_{xx} + a_{11}u, \\
v_t &= d_4v_{xx} + a_{21}v, \\
\eta_1 &= u_{xxxxx} - a_4u_{xxxx} - a_3u_{xxx} - a_2u_{xx} - a_1u_x - a_0u, \\
\eta_2 &= v_{xxx} - b_2v_{xx} - b_1v_x - b_0v;
\end{aligned}$$

(ii)

$$\begin{aligned}
u_t &= [(d_1 + d_3v)u]_{xx} + u(a_{11} - 9b_1d_3v), \\
v_t &= d_4v_{xx} + a_{21}v, \\
\eta_1 &= u_{xxxxx} - 5b_1u_{xxx} + 4b_1^2u_x, \\
\eta_2 &= v_{xxx} - b_1v_x;
\end{aligned}$$

(iii)

$$\begin{aligned}
u_t &= d_1 u_{xx} + a_{11} u, \\
v_t &= d_4 v_{xx} + a_{21} v, \\
\eta_1 &= u_{xxxxx} - a_4 u_{xxxx} + \left(\frac{86}{245} a_4^2 - 5b_1\right) u_{xxx} + \left(5a_4 b_1 - \frac{46}{1225} a_4^3\right) u_{xx} - a_1 u_x - a_0 u, \\
\eta_2 &= v_{xxx} + \frac{6}{35} a_4 v_{xx} - b_1 v_x;
\end{aligned}$$

(iv)

$$\begin{aligned}
u_t &= d_1 u_{xx} + a_{11} u, \\
v_t &= [(d_4 + d_6 v)v]_{xx} + (a_{21} - 4b_1 d_6 v)v, \\
\eta_1 &= u_{xxxxx} - a_4 u_{xxxx} - a_3 u_{xxx} - a_2 u_{xx} - a_1 u_x - a_0 u, \\
\eta_2 &= v_{xxx} - b_1 v_x;
\end{aligned}$$

(v)

$$\begin{aligned}
u_t &= d_1 u_{xx} + a_{11} u, \\
v_t &= [(d_4 + d_6 v)v]_{xx} + (a_{21} - 4b_1 d_6 v)v, \\
\eta_1 &= u_{xxxxx} - 5b_1 u_{xxx} - a_1 u_x, \\
\eta_2 &= v_{xxx} - b_1 v_x;
\end{aligned}$$

(vi)

$$\begin{aligned}
u_t &= [(d_1 + d_3 v)u]_{xx} + (a_{11} - 9b_1 d_3 v)u, \\
v_t &= [(d_4 + d_6 v)v]_{xx} + (a_{21} - 4b_1 d_6 v)v, \\
\eta_1 &= u_{xxxxx} - 5b_1 u_{xxx} + 4b_1^2 u_x, \\
\eta_2 &= v_{xxx} - b_1 v_x.
\end{aligned}$$

Case 6. When $l_1 = 4$, $l_2 = 4$, the system (1) is equivalent to the following case.

(i)

$$\begin{aligned}
u_t &= d_1 u_{xx} + a_{11} u, \\
v_t &= d_4 v_{xx} + a_{21} v, \\
\eta_1 &= u_{xxxxx} - a_3 u_{xxx} - a_2 u_{xx} - a_1 u_x - a_0 u, \\
\eta_2 &= v_{xxxxx} - b_3 v_{xxx} - b_2 v_{xx} - b_1 v_x - b_0 v.
\end{aligned}$$

Case 7. When $l_1 = 5$, $l_2 = 4$, the system (1) is equivalent to the following case.

(i)

$$\begin{aligned}
u_t &= d_1 u_{xx} + a_{11} u, \\
v_t &= d_4 v_{xx} + a_{21} v, \\
\eta_1 &= u_{xxxxx} - a_4 u_{xxxx} - a_3 u_{xxx} - a_2 u_{xx} - a_1 u_x - a_0 u, \\
\eta_2 &= v_{xxxxx} - b_3 v_{xxx} - b_2 v_{xx} - b_1 v_x - b_0 v.
\end{aligned}$$

Case 8. When $l_1 = 5$, $l_2 = 5$, the system (1) is equivalent to one of the following cases.

(i)

$$\begin{aligned} u_t &= d_1 u_{xx} + (a_{21} - d_1 a_4^2 + d_4 a_4^2) u, \\ v_t &= d_4 v_{xx} + a_{21} v, \\ \eta_1 &= u_{xxxxx} - a_4 u_{xxxx}, \\ \eta_2 &= v_{xxxxx} - b_4 v_{xxxx} - b_3 v_{xxx} - b_2 v_{xx} - b_1 v_x - b_0 v; \end{aligned}$$

(ii)

$$\begin{aligned} u_t &= d_1 u_{xx} + (a_{21} - d_1 a_3 + d_4 a_3) u, \\ v_t &= d_4 v_{xx} + a_{21} v, \\ \eta_1 &= u_{xxxxx} - a_3 u_{xxx}, \\ \eta_2 &= v_{xxxxx} - b_4 v_{xxxx} - b_3 v_{xxx} - b_2 v_{xx} - b_1 v_x - b_0 v; \end{aligned}$$

(iii)

$$\begin{aligned} u_t &= d_1 u_{xx} + a_{11} u, \\ v_t &= d_4 v_{xx} + a_{21} v, \\ \eta_1 &= u_{xxxxx} - a_4 u_{xxxx} - a_3 u_{xxx} - a_2 u_{xx} - a_1 u_x - a_0 u, \\ \eta_2 &= v_{xxxxx} - b_3 v_{xxx} - b_2 v_{xx} - b_1 v_x - b_0 v; \end{aligned}$$

(iv)

$$\begin{aligned} u_t &= d_1 u_{xx} + a_{11} u, \\ v_t &= d_1 v_{xx} + a_{11} v, \\ \eta_1 &= u_{xxxxx} - a_4 u_{xxxx} - a_3 u_{xxx} - a_2 u_{xx} - a_1 u_x - a_0 u, \\ \eta_2 &= v_{xxxxx} - b_4 v_{xxxx} - b_3 v_{xxx} - b_2 v_{xx} - b_1 v_x - b_0 v. \end{aligned}$$

3. Example

In this section, we show two examples to illustrate the main feature of the symmetry reduction procedure and solutions.

Example 1. The system

$$\begin{aligned} (3) \quad u_t &= [(d_1 + d_2 u + d_3 v)u]_{xx} + a_{11} u, \\ v_t &= [(d_4 + d_5 u + d_6 v)v]_{xx} + a_{21} v, \end{aligned}$$

admits the GCS

$$\begin{aligned} (4) \quad \eta_1 &= u_{xxx}, \\ \eta_2 &= v_{xx}. \end{aligned}$$

By integrating the system (4), the corresponding solutions are given as below.

$$(5) \quad \begin{aligned} u &= \phi_1(t)x^2 + \phi_2(t)x + \phi_3(t), \\ v &= \psi_1(t)x + \psi_2(t). \end{aligned}$$

Inserting (5) into system (3) yields the following ordinary differential equations(ODEs).

$$(6) \quad \begin{aligned} \frac{d\phi_1}{dt} &= 12\phi_1(t)^2d_2 + \phi_1(t)a_{11}, \\ \frac{d\phi_2}{dt} &= (6d_3\psi_1(t) + 12\phi_2(t)d_2)\phi_1(t) + \phi_2(t)a_{11}, \\ \frac{d\phi_3}{dt} &= (4d_2\phi_3(t) + 2d_3\psi_2(t) + 2d_1)\phi_1(t) + 2\psi_1(t)\phi_2(t)d_3 + 2\phi_2(t)^2d_2 \\ &\quad + \phi_3(t)a_{11}, \\ \frac{d\psi_1}{dt} &= \psi_1(t)(6d_5\phi_1(t) + a_{21}), \\ \frac{d\psi_2}{dt} &= 2\psi_1(t)^2d_6 + 2\psi_1(t)\phi_2(t)d_5 + (2d_5\phi_1(t) + a_{21})\psi_2(t). \end{aligned}$$

By solving the above system, we obtain

$$\begin{aligned} \phi_1(t) &= \frac{a_{11}}{a_{11}N_5e^{-a_{11}t} - 12d_2}, \\ \phi_2(t) &= \left[\int 6d_3\phi_1(t)\psi_1(t)e^{-\int(12d_2\phi_1(t)+a_{11})dt}dt + N_3 \right] e^{\int(12d_2\phi_1(t)+a_{11})dt}, \\ \phi_3(t) &= \left[\int 2(d_3\phi_1(t)\psi_2(t) + d_3\psi_1(t)\phi_2(t) + d_2\phi_2(t)^2 \right. \\ &\quad \left. + d_1\phi_1(t))e^{-\int(4d_2\phi_1(t)+a_{11})dt}dt + N_1 \right] \\ &\quad e^{\int(4d_2\phi_1(t)+a_{11})dt}, \\ \psi_1(t) &= N_4e^{\int(6d_5\phi_1(t)+a_{21})dt}, \\ \psi_2(t) &= \left[\int 2\psi_1(t)(d_6\psi_1(t) + d_5\phi_2(t))e^{-\int(2d_5\phi_1(t)+a_{21})dt}dt + N_2 \right] e^{\int(2d_5\phi_1(t)+a_{21})dt}. \end{aligned}$$

The solution is obtained by substituting the above system into (5).

Example 2. The system

$$(7) \quad \begin{aligned} u_t &= [(d_1 + d_2u + d_3v)u]_{xx} + u(a_{11} - 4d_2b_1u - 4d_3b_1v), \\ v_t &= [(d_4 + d_5u + d_6v)v]_{xx} + v(a_{21} - 4d_5b_1u - 4d_6b_1v) \end{aligned}$$

admits GCS

$$(8) \quad \begin{aligned} \eta_1 &= u_{xxx} - b_1u_x, \\ \eta_2 &= v_{xxx} - b_1v_x. \end{aligned}$$

The solutions of (8) depend on the sign of the parameter b_1 , so we can give three cases, then the solutions are

$$(9) \quad \begin{aligned} u &= \phi_1(t)x^2 + \phi_2(t)x + \phi_3(t), b_1 = 0; \\ v &= \psi_1(t)x^2 + \psi_2(t)x + \psi_3(t); \end{aligned}$$

$$(10) \quad \begin{aligned} u &= \phi_1(t) + \phi_2(t)e^{bx} + \phi_3(t)e^{-bx}, b_1 = b^2 > 0; \\ v &= \psi_1(t) + \psi_2(t)e^{bx} + \psi_3(t)e^{-bx}; \end{aligned}$$

$$(11) \quad \begin{aligned} u &= \phi_1(t) + \phi_2(t) \sin bx + \phi_3(t) \cos bx, b_1 = -b^2 < 0; \\ v &= \psi_1(t) + \psi_2(t) \sin bx + \psi_3(t) \cos bx; \end{aligned}$$

- For $b_1 = 0$, the system (7) can be reduced to the following ODEs

$$\begin{aligned} \frac{d\phi_1}{dt} &= 12\phi_1(t)^2d_2 + (12\psi_1(t)d_3 + a_{11})\phi_1(t), \\ \frac{d\phi_2}{dt} &= (6\psi_2(t)d_3 + 12\phi_2(t)d_2)\phi_1(t) + (6\psi_1(t)d_3 + a_{11})\phi_2(t), \\ \frac{d\phi_3}{dt} &= (4\phi_3(t)d_2 + 2\psi_3(t)d_3 + 2d_1)\phi_1(t) + 2\phi_2(t)^2d_2 + 2\psi_2(t)\phi_2(t)d_3 \\ &\quad + (2\psi_1(t)d_3 + a_{11})\phi_3(t) \\ \frac{d\psi_1}{dt} &= 12\psi_1(t)^2d_6 + (12\phi_1(t)d_5 + a_{21})\psi_1(t), \\ \frac{d\psi_2}{dt} &= (12\psi_2(t)d_6 + 6\phi_2(t)d_5)\psi_1(t) + (6\phi_1(t)d_5 + a_{21})\psi_2(t), \\ \frac{d\psi_3}{dt} &= (2\phi_3(t)d_5 + 4\psi_3(t)d_6 + 2d_4)\psi_1(t) + 2\psi_2(t)^2d_6 + 2\psi_2(t)\phi_2(t)d_5 \\ &\quad + (2\phi_1(t)d_5 + a_{21})\psi_3(t). \end{aligned}$$

- For $b_1 = b^2 > 0$, the system (7) can be reduced to the following ODEs

$$\begin{aligned} \frac{d\phi_1}{dt} &= -4\phi_1(t)^2b^2d_2 + (-4\psi_1(t)b^2d_3 + a_{11})\phi_1(t) + (-8\phi_3(t)b^2d_2 - 4\psi_3(t)b^2d_3)\phi_2(t) \\ &\quad - 4\phi_3(t)\psi_2(t)b^2d_3, \\ \frac{d\phi_2}{dt} &= (-6\phi_2(t)b^2d_2 - 3\psi_2(t)b^2d_3)\phi_1(t) + (-3\psi_1(t)b^2d_3 + b^2d_1 + a_{11})\phi_2(t), \\ \frac{d\phi_3}{dt} &= (-6\phi_3(t)b^2d_2 - 3\psi_3(t)b^2d_3)\phi_1(t) + (-3\psi_1(t)b^2d_3 + b^2d_1 + a_{11})\phi_3(t), \\ \frac{d\psi_1}{dt} &= -4\psi_1(t)^2b^2d_6 + (-4\phi_1(t)b^2d_5 + a_{21})\psi_1(t) + (-4\phi_3(t)b^2d_5 - 8\psi_3(t)b^2d_6)\psi_2(t) \\ &\quad - 4\phi_2(t)\psi_3(t)b^2d_5, \\ \frac{d\psi_2}{dt} &= (-3\phi_2(t)b^2d_5 - 6\psi_2(t)b^2d_6)\psi_1(t) + (-3\phi_1(t)b^2d_5 + b^2d_4 + a_{21})\psi_2(t), \\ \frac{d\psi_3}{dt} &= (-3\phi_3(t)b^2d_5 - 6\psi_3(t)b^2d_6)\psi_1(t) + (-3\phi_1(t)b^2d_5 + b^2d_4 + a_{21})\psi_3(t). \end{aligned}$$

- For $b_1 = -b^2 < 0$, the system (7) can be reduced to the following ODEs

$$\begin{aligned}\frac{d\phi_1}{dt} &= 4\phi_1(t)^2b^2d_2 + (4\psi_1(t)b^2d_3 + a_{11})\phi_1(t) + 2\phi_2(t)^2b^2d_2 + 2\phi_2(t)\psi_2(t)b^2d_3 \\ &\quad + 2\phi_3(t)^2b^2d_2 + 2\phi_3(t)\psi_3(t)b^2d_3, \\ \frac{d\phi_2}{dt} &= (6\phi_2(t)b^2d_2 + 3\psi_2(t)b^2d_3)\phi_1(t) + (3\psi_1(t)b^2d_3 - b^2d_1 + a_{11})\phi_2(t), \\ \frac{d\phi_3}{dt} &= (6\phi_3(t)b^2d_2 + 3\psi_3(t)b^2d_3)\phi_1(t) + (3\psi_1(t)b^2d_3 - b^2d_1 + a_{11})\phi_3(t), \\ \frac{d\psi_1}{dt} &= 4\psi_1(t)^2b^2d_6 + (4\phi_1(t)b^2d_5 + a_{21})\psi_1(t) + 2\phi_2(t)\psi_2(t)b^2d_5 + 2\phi_3(t)\psi_3(t)b^2d_5, \\ &\quad + 2\psi_2(t)^2b^2d_6 + 2\psi_3(t)^2b^2d_6, \\ \frac{d\psi_2}{dt} &= (3\phi_2(t)b^2d_5 + 6\psi_2(t)b^2d_6)\psi_1(t) + (3\phi_1(t)b^2d_5 - b^2d_4 + a_{21})\psi_2(t), \\ \frac{d\psi_3}{dt} &= (3\phi_3(t)b^2d_5 + 6\psi_3(t)b^2d_6)\psi_1(t) + (3\phi_1(t)b^2d_5 - b^2d_4 + a_{21})\psi_3(t).\end{aligned}$$

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A unified inequality of differential polynomials related to small functions

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Abstract. In this paper, we consider the value distribution of the differential polynomials $\varphi f^n f' - 1$ ($n \geq 2$), where f is a transcendental meromorphic function and φ is a small function, and obtain a precise unified inequality by the reduced counting function.

Keywords: meromorphic function, differential polynomials, Nevanlinna theory, value distribution, small function.

1. Introduction and results

Let $f(z)$ be a meromorphic function in the complex plane, we say $a(z)$ is a small function if $a(z)$ is a non-vanishing meromorphic function such that $T(r, a) = S(r, f)$ and $S(r, f)$ denotes $o(T(r, f))(r \rightarrow \infty)$, possibly outside a set of r of finite linear measure. We assumed that the reader is familiar with the notations of Nevanlinna theory (see, e.g., [6, 9, 18, 20]).

Definition 1.1. Let k be a positive integer, for any constant a in the complex plane. We denote by $N_k(r, 1/(f-a))$ the counting function of a -points of f with multiplicity $\leq k$, by $N_{(k)}(r, 1/(f-a))$ the counting function of a -points of f with multiplicity $\geq k$, by $N_k(r, 1/(f-a))$ the counting function of a -points of f with multiplicity of k . and denote the reduced counting function by $\overline{N}_k(r, 1/(f-a))$, $\overline{N}_{(k)}(r, 1/(f-a))$ and $\overline{N}_k(r, 1/(f-a))$, respectively.

In 1979, E. Mues [11] proved that for a transcendental meromorphic function f in the open plane, $f^2 f' - 1$ has infinitely many zeros. This is a qualitative result. In 1992, Q. Zhang [22] has obtained a quantitative result and proved the following theorem.

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Theorem A. *Let f be transcendental meromorphic in the complex plane, then*

$$(1) \quad T(r, f) \leq 6N(r, \frac{1}{f^2 f' - 1}) + S(r, f).$$

In [14], Xu, Yi and Zhang improved Theorem A by the reduced counting function and proved the following:

Theorem B. *Let f be a transcendental meromorphic function. Then*

$$(2) \quad T(r, f) \leq 6\overline{N}(r, \frac{1}{f^2 f' - 1}) + S(r, f).$$

In the value distribution theory, it is a very important problem whether we can use the small function to instead of the constant in the counting function (or the reduced counting function)? For example, K. Yamanoi proved the second Nevanlinna main theorem for small functions in [17]. There were some works on the value distribution differential polynomials related to small functions. (see, [1, 2, 13, 14, 15, 16, 19, 21])

In 1993, Q. Zhang [23] studied the zeros of $f^2(z)f'(z) - a(z)$, where $a(z) \not\equiv 0$ is a small function, and improved Theorem A.

Theorem C. *Let $f(z)$ be transcendental meromorphic in the complex plane and $\varphi(z) (\not\equiv 0)$ be a small function, then*

$$(3) \quad T(r, f) \leq 6N(r, \frac{1}{\varphi f^2 f' - 1}) + S(r, f).$$

Corresponding Theorem B, it is naturally to consider the value distribution of $\varphi f^2 f' - 1$ by the reduced counting function. Recently, Xu and Yi proved the following inequality in [16], which generalized Theorem B.

Theorem D. *Let $f(z)$ be a transcendental meromorphic function and $\varphi(z) (\not\equiv 0)$ be a small function. Then*

$$(4) \quad T(r, f) \leq 6\overline{N}(r, \frac{1}{\varphi f^2 f' - 1}) + S(r, f).$$

Moreover, noting that from the proof of Theorem 2 in [5], we can obtain the following result.

Theorem E. *Let $f(z)$ be a transcendental meromorphic function and $\varphi(z) (\not\equiv 0)$ be a small function. Then for $n \geq 3$,*

$$(5) \quad T(r, f) \leq 2\overline{N}(r, \frac{1}{\varphi f^n f' - 1}) + S(r, f).$$

Naturally, the following question arises: Is it possible to give a unified inequality for $f^n f' - a$ for $n \geq 2$. In fact, we prove

Theorem 1.1. *Let $f(z)$ be a transcendental meromorphic function and $\varphi(z) (\not\equiv 0)$ be a small function. If $n \geq 2$, then*

$$(6) \quad T(r, f) \leq \left(\frac{6}{2n-3} \right) \bar{N}(r, \frac{1}{\varphi f^n f' - 1}) + S(r, f).$$

Remark 1.1. Obviously, our result gives an affirmative answer. The result improves Theorem E when $n \geq 4$, and generalizes Theorem 1.1 when $k = 1$ and φ is a constant in [8].

2. Lemmas

In order to prove our result, we need the following lemmas.

Lemma 2.1. *Let f be a transcendental meromorphic function, and let $\varphi(z) (\not\equiv 0)$ be a small function. Then*

$$(7) \quad \begin{aligned} (n+1)T(r, f) &\leq \bar{N}(r, f) + \bar{N}(r, \frac{1}{f}) + N_1(r, \frac{1}{f}) + 2N_2(r, \frac{1}{f}) \\ &\quad + \bar{N}(r, \frac{1}{\varphi f^n f' - 1}) - N_0(r, \frac{1}{(\varphi f^n f')'}) + S(r, f), \end{aligned}$$

where $N_0(r, \frac{1}{(\varphi f^n f')'})$ denotes the counting function of the zeros of $(\varphi f^n f')'$, not of $f(\varphi f^n f' - 1)$.

Proof. We know that $\varphi f^n f' \not\equiv \text{constant}$ by Lemma 3.4 in [3].

Let

$$\frac{1}{f^{n+1}} \equiv \frac{\varphi f^n f'}{f^{n+1}} - \frac{(\varphi f^n f')' \varphi f^n f' - 1}{f^{n+1} (\varphi f^n f')'},$$

we have

$$(8) \quad \begin{aligned} (n+1)m(r, \frac{1}{f}) &= m(r, \frac{1}{f^{n+1}}) \\ &\leq m(r, \frac{\varphi f^n f' - 1}{(\varphi f^n f')'}) + m(r, \varphi \frac{f'}{f}) + m(r, \frac{(\varphi f^n f')'}{f^{n+1}}) + O(1) \\ &\leq N(r, \frac{(\varphi f^n f')'}{\varphi f^n f' - 1}) - N(r, \frac{\varphi f^n f' - 1}{(\varphi f^n f')'}) + S(r, f) \\ &= N(r, (\varphi f^n f')') + N(r, \frac{1}{\varphi f^n f' - 1}) \\ &\quad - N(r, \frac{1}{(\varphi f^n f')'}) - N(r, \varphi f^n f') + S(r, f) \\ &= \bar{N}(r, f) + N(r, \frac{1}{\varphi f^n f' - 1}) - N(r, \frac{1}{(\varphi f^n f')'}) + S(r, f). \end{aligned}$$

Hence

$$\begin{aligned}
 (n+1)T(r, f) &= (n+1)m(r, \frac{1}{f}) + (n+1)N(r, \frac{1}{f}) + O(1) \\
 (9) \quad &= \overline{N}(r, f) + (n+1)N(r, \frac{1}{f}) \\
 &\quad + N(r, \frac{1}{\varphi f^n f' - 1}) - N(r, \frac{1}{(\varphi f^n f')'}) + S(r, f).
 \end{aligned}$$

Let

$$(10) \quad N(r, \frac{1}{(\varphi f^n f')'}) = N_{000}(r, \frac{1}{(\varphi f^n f')'}) + N_{00}(r, \frac{1}{(\varphi f^n f')'}) + N_0(r, \frac{1}{(\varphi f^n f')'}),$$

where $N_{000}(r, \frac{1}{(\varphi f^n f')'})$ denotes the counting function of the zeros of $(\varphi f^n f')'$, which come from the zeros of $\varphi f^n f' - 1$, $N_{00}(r, \frac{1}{(\varphi f^n f')'})$ denotes the counting function of the zeros of $(\varphi f^n f')'$, which come from the zeros of f . Hence we have

$$(11) \quad N(r, \frac{1}{\varphi f^n f' - 1}) - N_{000}(r, \frac{1}{(\varphi f^n f')'}) = \overline{N}(r, \frac{1}{\varphi f^n f' - 1}).$$

Supposed that z_0 is a zero of f with multiplicity q and the pole of φ with multiplicity of t .

Case I. Supposed that $t \leq nq - 1$. If $q = 1$, then z_0 is a zero of $(\varphi f^n f')'$ with multiplicity at least $nq - 1 - t$; if $q \geq 2$, then z_0 is a zero of $(\varphi f^n f')'$ with multiplicity at least $(n+1)q - 2 - t$.

Case II. Supposed that $t \geq 2q$, z_0 is at most the pole of φ^2 . Hence, we have

$$\begin{aligned}
 (n+1)N(r, \frac{1}{f}) - N_{00}(r, \frac{1}{(\varphi f^n f')'}) \\
 \leq nN(r, \frac{1}{f}) + \overline{N}_1(r, \frac{1}{f}) + 2\overline{N}_2(r, \frac{1}{f}) \\
 (12) \quad + N(r, \varphi^2) - N_{00}(r, \frac{1}{(\varphi f^n f')'}) \\
 = N_1(r, \frac{1}{f}) + \overline{N}(r, \frac{1}{f}) + \overline{N}_2(r, \frac{1}{f}) + S(r, f).
 \end{aligned}$$

Combining (9)-(12), we have

$$\begin{aligned}
 (n+1)T(r, f) &\leq \overline{N}(r, f) + \overline{N}(r, \frac{1}{f}) + N_1(r, \frac{1}{f}) + 2\overline{N}_2(r, \frac{1}{f}) \\
 &\quad + \overline{N}(r, \frac{1}{\varphi f^n f' - 1}) - N_0(r, \frac{1}{(\varphi f^n f')'}) + S(r, f).
 \end{aligned}$$

This completes the proof of the lemma. $\square$

If the coefficients of differential polynomials $M[f]$ are $a_j, j = 0, 1, \dots, n$, which satisfy $m(r, a_j) = S(r, f)$, then differential polynomials $M[f]$ is called a quasi-differential polynomials in f . The following lemma is nothing but an easy variant of standard Clunie lemma [4], Lemma 1.

Lemma 2.2 ([14]). *Let f be a non-constant meromorphic in the complex plane, $Q_1[f], Q_2[f]$ are quasi-differential polynomials in f , satisfy $f^n Q_1[f] = Q_2[f]$, if the total degree of $Q_2 \leq n$, then $m(r, Q_1[f]) = S(r, f)$.*

Lemma 2.3. *Let f be a transcendental meromorphic function and $n \geq 3$, $F(z) = \varphi(z)f^n(z)f'(z) - 1$ and*

$$h(z) = \frac{F'(z)}{f^{n-1}(z)} = \varphi'(z)f(z)f'(z) + \varphi(z)f(z)f''(z) + n\varphi(z)f'^2(z), \phi(z) = \frac{h(z)}{F(z)}.$$

Also, let

$$\begin{aligned} G(z) = & a_1 \left(\frac{F'(z)}{F(z)} \right)^2 + a_2 \left(\frac{F'(z)}{F(z)} \right)' + a_3 \left(\frac{F'(z)}{F(z)} \frac{h'(z)}{h(z)} \right) + a_4 \left(\frac{h'(z)}{h(z)} \right)^2 + a_5 \left(\frac{h'(z)}{h(z)} \right)' \\ & + a_6 \left(\frac{\varphi'(z)}{\varphi(z)} \frac{F'(z)}{F(z)} \right) + a_7 \left(\frac{\varphi'(z)}{\varphi(z)} \frac{h'(z)}{h(z)} \right) + a_8 \left(\frac{\varphi'(z)}{\varphi(z)} \right)' + a_9 \left(\frac{\varphi'(z)}{\varphi(z)} \right)^2. \end{aligned}$$

Where

$$\begin{cases} a_1 = 2(4n^2 + 8n + 7), \\ a_2 = 2(n+2)(2n+1)(2n-1), \\ a_3 = -2(n+2)(2n^2 + 3n + 4), \\ a_4 = (n+1)(n+2)^2, \\ a_5 = -(n+2)^2(2n-1), \\ a_6 = 2(n+2)(n+1)(2n-5), \\ a_7 = 3(n+2)^2, \\ a_8 = -(n+2)^2(2n-1), \\ a_9 = -(n+2)^2(n-2). \end{cases}$$

Then, $G(z) \not\equiv 0$ and the simple pole of $f(z)$ is the zero of $G(z)$.

Proof. Suppose that $G(z) \equiv 0$. We first give the following inequalities in a similar way to Lemma 4 and 5 in [13]. For any $z_0 \in \mathbb{C}$, we have

$$(13) \quad (n-1)\omega\left(\frac{1}{f}, z_0\right) + \omega\left(\frac{1}{h}, z_0\right) \leq \omega\left(\frac{1}{f^{n-1}h}, z_0\right) + \omega(\varphi, z_0) + \omega\left(\frac{1}{\varphi}, z_0\right),$$

$$(14) \quad \omega(\phi, z_0) \leq 2\omega(\varphi, z_0) + \omega\left(\frac{1}{\varphi}, z_0\right),$$

$$(15) \quad \omega\left(\frac{1}{F}, z_0\right) \leq \omega\left(\frac{1}{h}, z_0\right) + 2\omega(\varphi, z_0) + \omega\left(\frac{1}{\varphi}, z_0\right).$$

We omit the proof of (13) and (14) here. We just prove (15). One can claim that the zeros of $F(z)$ must be the zeros or poles of $\varphi(z)$. This is

$$(16) \quad \bar{\omega}\left(\frac{1}{F}, z_0\right) \leq \omega(\varphi, z_0) + \omega\left(\frac{1}{\varphi}, z_0\right).$$

Let z_1 be a zero of $F(z)$ of multiplicity l ($l \geq 1$) and $\varphi(z_1) \neq 0, \infty$. Then $F(z_1) = 0$ and $\varphi(z_1)f^n(z_1)f'(z_1) - 1 = 0$ such that $f(z_1) \neq 0, \infty$. From $h = \frac{F'}{f^{n-1}}$, we get that z_1 is a zero of h with multiplicity $l - 1$. Using the Laurent series of G at the point z_1 , we obtain the coefficient of $(z - z_1)^{-2}$:

$$A(l) = (a_1 + a_3 + a_4)l^2 - (a_2 + a_3 + 2a_4 + a_5)l + (a_4 + a_5).$$

Therefore,

$$A(l) = -(3n^2 + n^2 - 4n - 2)l^2 - (n + 2)(4n^2 - 3n - 4)l - (n - 2)(n + 2)^2.$$

Obviously, $A(l) < 0$ for all positive integers l . Hence z_1 is a pole of $G(z)$ which contradicts $G(z) \equiv 0$. Hence the zeros of $F(z)$ must be the zeros or poles of $\varphi(z)$. Therefore, (16) holds. In the following we begin to prove (15).

If $F(z_0) \neq 0$, then the inequality (15) obviously holds. If $F(z_0) = 0$, then from (16) we obtain

$$\begin{aligned} \omega\left(\frac{1}{F}, z_0\right) - \omega\left(\frac{1}{h}, z_0\right) &= [\omega\left(\frac{1}{F}, z_0\right) - \omega\left(\frac{1}{F'}, z_0\right)] + [\omega\left(\frac{1}{f^{n-1}h}, z_0\right) - \omega\left(\frac{1}{h}, z_0\right)] \\ &\leq \bar{\omega}\left(\frac{1}{F}, z_0\right) + (n - 1)\omega\left(\frac{1}{f}, z_0\right) \\ &\leq (n - 1)\omega\left(\frac{1}{f}, z_0\right) + \omega(\varphi, z_0) + \omega\left(\frac{1}{\varphi}, z_0\right). \end{aligned}$$

If $f(z_0) \neq 0$, then we have $\omega\left(\frac{1}{f}, z_0\right) = 0$.

If $f(z_0) = 0$, then from $\varphi(z_0)f^n(z_0)f'(z_0) = F(z_0) + 1 = 1$, we have $n\omega\left(\frac{1}{f}, z_0\right) \leq \omega(\varphi, z_0)$. Hence

$$\omega\left(\frac{1}{F}, z_0\right) - \omega\left(\frac{1}{h}, z_0\right) \leq 2\omega(\varphi, z_0) + \omega\left(\frac{1}{\varphi}, z_0\right).$$

Thus, the inequality (15) holds. From (14) and (15), we have

$$(17) \quad N(r, \phi) \leq 2N(r, \varphi) + N\left(r, \frac{1}{\varphi}\right) = S(r, f),$$

and

$$(18) \quad N\left(r, \frac{1}{F}\right) - N\left(r, \frac{1}{h}\right) \leq 2N(r, \varphi) + N\left(r, \frac{1}{\varphi}\right) = S(r, f).$$

By (8), we have

$$(19) \quad (n + 1)m\left(r, \frac{1}{f}\right) \leq \bar{N}(r, f) + N\left(r, \frac{1}{F}\right) - N\left(r, \frac{1}{f^{n-1}h}\right) + S(r, f).$$

By (13), we have

$$(20) \quad (n-1)N(r, \frac{1}{f}) + N(r, \frac{1}{h}) \leq N(r, \frac{1}{f^{n-1}h}) + N(r, \varphi) + N(r, \frac{1}{\varphi}).$$

From (19) and (20), we have

$$(21) \quad (n+1)m(r, \frac{1}{f}) \leq \overline{N}(r, f) + N(r, \frac{1}{F}) - (n-1)N(r, \frac{1}{f}) - N(r, \frac{1}{h}) + S(r, f).$$

From (18) and (21), we have

$$(22) \quad nm(r, \frac{1}{f}) \leq N(r, \frac{1}{F}) - N(r, \frac{1}{h}) + S(r, f) = S(r, f).$$

From (17) and (22), we have

$$(23) \quad \begin{aligned} T(r, \phi) &= m(r, \phi) + N(r, \phi) = m(r, \frac{1}{f^{n-1}} \cdot \frac{F'}{F}) + N(r, \phi) \\ &\leq (n-1)m(r, \frac{1}{f}) + m(r, \frac{F'}{F}) + N(r, \phi) = S(r, f). \end{aligned}$$

Note that $\frac{F'}{F} = f^{n-1}\phi$ and

$$\frac{h'}{h} = \frac{F'}{F} + \frac{\phi'}{\phi} = f^{n-1}\phi + \frac{\phi'}{\phi}.$$

Substituting the above expressions into $G(z)$, we have

$$(24) \quad \begin{aligned} &(a_1 + a_3 + a_4)f^{2n-2}\phi^2 + [(a_2 + a_3 + 2a_4 + a_5)\frac{\phi'}{\phi} + (a_6 + a_7)\frac{\varphi'}{\varphi}]\phi f^{n-1} \\ &+ (n-1)(a_2 + a_5)f^{n-2}f'\phi + [a_4(\frac{\phi'}{\phi})^2 \\ &+ a_5(\frac{\phi'}{\phi})' + a_7\frac{\varphi'}{\varphi}\frac{\phi'}{\phi} + a_8(\frac{\varphi'}{\varphi})' + a_9(\frac{\varphi'}{\varphi})^2] = 0. \end{aligned}$$

Note that $(n-1)(a_2 + a_5) = 3n(2n^3 + n^2 - 5n + 2) \neq 0$. By (24), we have

$$(25) \quad f' \equiv \frac{1}{\phi f^{n-2}}c_1(z) + f c_2(z) + f^2 \phi c_3(z),$$

where $c_1(z) = [a_4(\frac{\phi'}{\phi})^2 + a_5(\frac{\phi'}{\phi})' + a_7\frac{\varphi'}{\varphi}\frac{\phi'}{\phi} + a_8(\frac{\varphi'}{\varphi})' + a_9(\frac{\varphi'}{\varphi})^2]/(n-1)(a_2 + a_5)$,
 $c_2(z) = [(a_2 + a_3 + 2a_4 + a_5)\frac{\phi'}{\phi} + (a_6 + a_7)\frac{\varphi'}{\varphi}]/(n-1)(a_2 + a_5)$, $c_3(z) = -[(a_1 + a_3 + a_4)]/(n-1)(a_2 + a_5)$.

From (24), we can obtain

$$(26) \quad F = \varphi f^n f' - 1 = \frac{\varphi}{\phi} c_1 f^2 + c_2 \varphi f^{n+1} + c_3 \phi \varphi f^{n+2} - 1.$$

$$\begin{aligned}
F' &= (n+2)c_3^2\varphi\phi^2f^{n+3} + [c_3\phi\varphi' + c_3\phi'\varphi + (2n+3)c_2c_3\phi\varphi]f^{n+2} \\
&\quad + [c_2'\phi + c_2\phi' + 2\phi c_1c_3 + (n+1)c_2^2\phi]f^{n+1} + [(\frac{\phi}{\varphi})'c_1 + \frac{\phi}{\varphi}c_1' \\
(27) \quad &\quad + (n+3)\frac{\phi}{\varphi}c_1c_2]f^2 + 2\frac{\phi}{\varphi^2}c_1^2f^{3-n}.
\end{aligned}$$

Substituting (26) and (27) into $F' = f^{n-1}\varphi F$, we have

$$\begin{aligned}
&c_3\phi\varphi^2f^{3n+1} + c_2\varphi f^{3n} - (n+2)c_3^2\phi^2\varphi f^{2n+3} \\
&\quad - [c_3\phi\varphi' + c_3\phi'\varphi + (2n+3)c_2c_3\phi\varphi] \\
&\quad f^{2n+2} + [\frac{\varphi^2}{\phi}c_1 - c_2'\phi - c_2\phi' - (n+1)c_2^2\phi]f^{2n+1} - \varphi f^{2n-1} - (n+4)c_1c_3\varphi \\
&\quad f^{3+n} - [(\frac{\varphi}{\phi})'c_1 + \frac{\varphi}{\phi}c_1' + (n+3)\frac{\phi}{\varphi}c_1c_2]f^{2+n} - 2\frac{\phi}{\varphi^2}c_1^2f^3 = 0.
\end{aligned}$$

By (23) and Lemma 3 in [7], we have $c_3\phi\varphi^2 \equiv 0$, note that $c_3 \neq 0$, therefore $\phi \equiv 0$ or $\varphi \equiv 0$. It is impossible. Hence $G(z) \not\equiv 0$.

Suppose that, z_0 is a simple pole of $f(z)$. Then

$$\begin{aligned}
\varphi(z) &= A\{1 + x(z - z_0) + y(z - z_0)^2 + O[(z - z_0)^3]\} \quad (A \neq 0), \\
f(z) &= \frac{a}{(z - z_0)} \{1 + b_0(z - z_0) + b_1(z - z_0)^2 + O[(z - z_0)^3]\} \quad (a \neq 0),
\end{aligned}$$

and

$$\begin{aligned}
h(z) &= \frac{Aa^2}{(z - z_0)^4} \left\{ (n+2) + [(n+1)x + 2b_0](z - z_0) \right. \\
&\quad \left. + [b_0x + ny - 2(n-1)b_1](z - z_0)^2 + O[(z - z_0)^3] \right\},
\end{aligned}$$

where $a \neq 0, b_0, b_1, b_2$ are constants. Therefore, we have

$$\begin{aligned}
F(z) &= \frac{-Aa^{n+1}}{(z - z_0)^{n+2}} \left\{ 1 + (x + nb_0)(z - z_0) + [y + nxb_0 + \frac{1}{2}n(n-1)b_0^2 \right. \\
&\quad \left. + (n-1)b_1](z - z_0)^2 + O[(z - z_0)^3] \right\}, \\
\frac{F'(z)}{F(z)} &= -\frac{n+2}{(z - z_0)} + (x + nb_0) + [2y - x^2 - nb_0^2 + 2(n-1)b_1](z - z_0) \\
&\quad + O[(z - z_0)^2], \\
\frac{\varphi'(z)}{\varphi(z)} &= x + (2y - x^2)(z - z_0) + O[(z - z_0)^2], \\
\frac{h'(z)}{h(z)} &= \frac{-4}{(z - z_0)} + [\frac{(n+1)}{(n+2)}x + \frac{2}{(n+2)}b_0] + [\frac{2n}{(n+2)}y - \frac{2n}{(n+2)^2}xb_0 \\
&\quad - \frac{4(n-1)}{(n+2)}b_1 - \frac{(n+1)^2}{(n+2)^2}x^2 - \frac{4}{(n+2)^2}b_0^2](z - z_0) + O[(z - z_0)^2],
\end{aligned}$$

$$(28) \quad \left(\frac{F'(z)}{F(z)} \right)^2 = \frac{(n+2)^2}{(z-z_0)^2} + \frac{-2(n+2)x - 2(n+2)nb_0}{(z-z_0)} + [-4(n+2)y + (2n+5)x^2 + 2nxb_0 + (3n^2+4n)b_0^2 - 4(n-1)(n+2)b_1] + O(z-z_0),$$

$$(29) \quad \left(\frac{F'(z)}{F(z)} \right)' = \frac{(n+2)}{(z-z_0)^2} + [2y - x^2 - nb_0^2 + 2(n-1)b_1] + O(z-z_0),$$

$$(30) \quad \frac{F'(z)}{F(z)} \frac{h'(z)}{h(z)} = \frac{4(n+2)}{(z-z_0)^2} + \frac{-(n+5)x - (4n+2)b_0}{(z-z_0)} + [-(2n+8)y + (n+5)x^2 + (n+1)xb_0 + (4n+2)b_0^2 + (4-4n)b_1] + O(z-z_0),$$

$$(31) \quad \left(\frac{h'(z)}{h(z)} \right)^2 = \frac{16}{(z-z_0)^2} + \left[\frac{-8(n+1)}{(n+2)}x + \frac{-16}{(n+2)}b_0 \right] \frac{1}{(z-z_0)} + \left[\frac{-16n}{(n+2)}y + \frac{9(n+1)^2}{(n+2)^2}x^2 + \frac{(20n+4)}{(n+2)^2}xb_0 + \frac{36}{(n+2)^2}b_0^2 + \frac{32(n-1)}{(n+2)}b_1 \right] + O(z-z_0),$$

$$(32) \quad \left(\frac{h'(z)}{h(z)} \right)' = \frac{4}{(z-z_0)^2} + \left[\frac{2n}{(n+2)}y - \frac{(n+1)^2}{(n+2)^2}x^2 - \frac{2n}{(n+2)^2}xb_0 - \frac{4}{(n+2)^2}b_0^2 - \frac{4(n-1)}{(n+2)}b_1 \right] + O(z-z_0),$$

$$(33) \quad \frac{\varphi'(z)}{\varphi(z)} \frac{F'(z)}{F(z)} = \frac{-(n+2)x}{(z-z_0)} + [-2(n+2)y + (n+3)x^2 + nxb_0] + O(z-z_0),$$

$$(34) \quad \frac{\varphi'(z)}{\varphi(z)} \frac{h'(z)}{h(z)} = \frac{-4x}{(z-z_0)} + (-8y + \frac{5n+9}{n+2}x^2 + \frac{2}{n+2}xb_0) + O(z-z_0),$$

$$(35) \quad \left(\frac{\varphi'(z)}{\varphi(z)} \right)' = (2y - x^2) + O(z-z_0),$$

$$(36) \quad \left(\frac{\varphi'(z)}{\varphi(z)} \right)^2 = x^2 + 2x(2y - x^2)(z-z_0) + O(z-z_0)^2.$$

By substituting (28)-(36) in the expression of $G(z)$ and performing some simple calculations, it can be easily seen that $G(z) = O[(z-z_0)]$. Hence the simple pole of $f(z)$ is the zero of $G(z)$. $\square$

Remark 2.1. Obviously, the lemma does not hold for $n = 2$. In fact, Lemma 2.6 also holds just for $n \geq 3$ in [8]. If $n = 2$, the result has been proved by Zhang in [23], and the similar method can be found in [13].

3. The proof of the Theorem 1.1

If $n = 2$, we can obtain the conclusion by Theorem D. In the following, we only proved the case of $n \geq 3$.

By differentiating the equation $F = \varphi f^n f' - 1$, we get

$$f^{n-1}\beta = -\frac{F'}{F},$$

where

$$\beta = \varphi' f f' + n\varphi f'^2 + \varphi f f'' - \varphi f f' \frac{F'}{F}, \quad h = -\beta F.$$

Note that the poles of $G(z)$ whose multiplicity are at most two come from the multiple poles of $f(z)$, $F(z)$ or the zeros of $h(z)$.

We consider the poles of $\beta^2 G$. We can see the zeros of h either are the zeros of F , or the zeros of β . From the above we know that the multiple poles of f with the multiplicity $q(\geq 2)$ are the zeros of β with the multiplicity of $(n-1)q-1$. Hence the poles of $\beta^2 G$ only come from the zeros of F , and the multiplicity is at most 4. Hence,

$$N(r, \beta^2 G) \leq 4\overline{N}(r, 1/F).$$

By $m(r, G) = S(r, f)$ and Lemma 2.2, we have $m(r, \beta^2 G) = S(r, f)$. Hence

$$T(r, \beta^2 G) \leq 4\overline{N}(r, 1/F).$$

Since the multiple zeros of f with the multiplicity $p(\geq 2)$ are the multiple zeros of β with multiplicity at least $2p-2$, therefore, are at least the zeros of $\beta^2 G$ with the multiplicity $2(2p-2)-2 = 4p-6$. Also note that the simple poles of f are the zeros of $\beta^2 G$. Hence we have

$$(37) \quad N(r, f) + 2N(r, \frac{1}{f}) - 2\overline{N}(r, \frac{1}{f}) \leq N(r, \frac{1}{\beta^2 G}) \leq T(r, \beta^2 G) \leq 4\overline{N}(r, \frac{1}{F}).$$

From (7) and (37), we have

$$\begin{aligned} & 2(n+1)T(r, f) + N_1(r, f) + 2N(r, \frac{1}{f}) - 2\overline{N}(r, \frac{1}{f}) \\ & \leq 2\overline{N}(r, f) + 4\overline{N}(r, \frac{1}{f}) + 6\overline{N}(r, \frac{1}{\varphi f^n f' - 1}) - 2N_0(r, \frac{1}{(\varphi f^n f')'}) + S(r, f). \end{aligned}$$

i.e.,

$$\begin{aligned} & 2(n+1)T(r, f) + N_1(r, f) + 2N(r, \frac{1}{f}) - 6\overline{N}(r, \frac{1}{f}) \\ & \leq 2N_1(r, f) + 2\overline{N}(r, \frac{1}{f}) + 6\overline{N}(r, \frac{1}{\varphi f^n f' - 1}) - 2N_0(r, \frac{1}{(\varphi f^n f')'}) + S(r, f), \end{aligned}$$

which gives

$$(2n-3)T(r, f) + m(r, f) + N(r, f) + 4m(r, \frac{1}{f}) + 6N(r, \frac{1}{f}) \\ - 2\overline{N}_{(2)}(r, f) - 6\overline{N}(r, \frac{1}{f}) \leq N_1(r, f) + 6\overline{N}(r, \frac{1}{\varphi f^n f' - 1}) + S(r, f).$$

Therefore,

$$(2n-3)T(r, f) + N_{(2)}(r, f) - 2\overline{N}_{(2)}(r, f) + m(r, f) + 4m(r, \frac{1}{f}) + 6N_1(r, \frac{1}{f}) \\ \leq 6\overline{N}(r, \frac{1}{f}) + S(r, f),$$

Hence, we have

$$T(r, f) < \left(\frac{6}{2n-3}\right)\overline{N}(r, \frac{1}{\varphi f^n f' - 1}) + S(r, f).$$

4. Conclusions

In our present investigation, we applied several methods which appeared in [8, 15, 23] and gave a unified inequality for $\varphi f^n f' - 1$, where $n \geq 2$. The result also improved the coefficient in Theorem E. The Lemma 2.3 plays a key role in the study. Obviously, the lemma does not hold for $n = 2$. In fact, Lemma 2.6 also does not hold for $n = 2$ in [8]. When $n = 2$, the result has been proved by Zhang in [23]. The reader can find a similar method used in [13]. For further study of related problems, we would like to pose the following question.

Question 4.1. *If f' is replaced by $f^{(k)}$ in Theorem 1.1, whether the inequality is still true?*

Remark 4.1. *There are some complete generalizations of the Hayman result about the zeros of $f^n f' - 1$ where f is a transcendental meromorphic function. For example, in 1981 N. Steinmetz [12] proved that for a transcendental meromorphic function f , $\Psi = f^{n_0}(f')^{n_1} \dots (f^{(k)})^{n_k} - 1$ has infinitely many zeros, where $n_0 \geq 2$, $n_1 + n_2 + \dots + n_k \geq 1$. In 1982, W. Doeringer [5] proved that for a transcendental meromorphic function f , let $Q(f), P(f)$ be differential polynomials in f satisfying $P(f) \not\equiv 0$, $Q(f) \not\equiv 0$. Then $\Psi = f^n P(f) + Q(f)$ has infinitely many zeros, where $n \geq 3 + \Gamma_p$. For Question 4.1, if φ is a constant, there are some results obtained in [3] and [8].*

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On refined Young's inequality

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Abstract. In this paper, we obtain new refinements of the classical Young's inequality for positive real numbers, and by using these results, we establish the corresponding inequalities for Hilbert space operators.

Keywords: operator inequality, Young's inequality, positive operator, operator mean.

1. Introduction

Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the C^* -algebra of all bounded linear operators on $\mathcal{H}$ with the operator norm $\|\cdot\|$ and the identity $I_{\mathcal{H}}$. If $\dim \mathcal{H} = n$, then we identify $\mathcal{B}(\mathcal{H})$ with the space $\mathcal{M}_n$ of all $n \times n$ complex matrices and denote the identity matrix by I_n . A norm $\|\cdot\|$ on $\mathcal{M}_n$ is said to be unitarily invariant if $\|UAV\| = \|A\|$ for all $A \in \mathcal{M}_n$ and all unitary matrices $U, V \in \mathcal{M}_n$. For an operator $A \in \mathcal{B}(\mathcal{H})$, we write $A \geq 0$ if A is positive (positive semidefinite for matrices), and $A > 0$ if A is positive and invertible (positive definite for matrices). For $A, B \in \mathcal{B}(\mathcal{H})$, we say $A \geq B$ if $A - B \geq 0$. For $A = (a_{ij}) \in \mathcal{M}_n$, the Hilbert-Schmidt (or Frobenius) norm is defined by

$\|A\|_2 = \left(\sum_{i,j=1}^n |a_{ij}|^2 \right)^{\frac{1}{2}}$. $\|\cdot\|_2$ has the unitarily invariant property.

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Let $A, B \in \mathcal{B}(\mathcal{H})$ be two positive operators and $\nu \in [0, 1]$. The ν -weighted arithmetic mean of A and B , denoted by $A\nabla_\nu B$, is defined as

$$A\nabla_\nu B = (1 - \nu)A + \nu B.$$

If A is invertible, ν -geometric mean of A and B , denoted by $A\sharp_\nu B$, is defined by

$$A\sharp_\nu B = A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^\nu A^{\frac{1}{2}}.$$

When $\nu = \frac{1}{2}$, we write $A\nabla B$ and $A\sharp B$ for brevity, respectively.

The well-known Young's inequality is a classical result attributed to the English mathematician William Henry Young (1863-1942) stating that

$$(1) \quad a^\nu b^{1-\nu} \leq \nu a + (1 - \nu)b,$$

where a and b are distinct positive real numbers and $\nu \in [0, 1]$.

The inequality (1) was refined by Kittaneh and Mansarah [3] in the following form

$$(2) \quad (a^\nu b^{1-\nu})^2 + r^2(a - b)^2 \leq (\nu a + (1 - \nu)b)^2,$$

where $r = \min\{\nu, 1 - \nu\}$.

The authors of [6] obtained another refinement of the Young inequality as follows:

$$(3) \quad (a^\nu b^{1-\nu})^2 + r(a - b)^2 \leq \nu a^2 + (1 - \nu)b^2.$$

More recently, Hu in [4] gave the following Young type inequalities:

$$(4) \quad \begin{cases} ((\nu a)^\nu b^{1-\nu})^2 + \nu^2(a - b)^2 \leq \nu^2 a^2 + (1 - \nu)^2 b^2, & 0 \leq \nu \leq \frac{1}{2} \\ \left\{ (a^\nu (1 - \nu)b)^{1-\nu} \right\}^2 + (1 - \nu)^2(a - b)^2 \leq \nu^2 a^2 + (1 - \nu)^2 b^2, & \frac{1}{2} \leq \nu \leq 1 \end{cases}.$$

When comparing inequalities (4) with the inequalities (2) and (3), it is easy to observe that both the left-hand and the right-hand sides of inequalities (4) are greater than or equal to the corresponding sides in (2) and (3), respectively. It should be noticed that neither inequalities (4) nor (2) and (3) is uniformly better than the other. These inequalities are extended to matrices in various contexts. The original Young's inequality was first extended to $\mathcal{M}_n$ in [1] as follows:

$$|||A^\nu B^{1-\nu}||| \leq |||\nu A + (1 - \nu)B|||,$$

where $A, B \in \mathcal{M}_n$ are positive semidefinite and $\nu \in [0, 1]$.

The Frobenius norm version of Young's inequality was established by Kosaki [7] and also by Bhatia and Parthasarathy [2] as

$$\|A^\nu X B^{1-\nu}\|_2 \leq \|\nu AX + (1 - \nu)XB\|_2,$$

for $A, B, X \in \mathcal{M}_n(\mathbb{C})$, with A and B positive semidefinite and $\nu \in [0, 1]$. Then, each refinement of the scalar Young's inequality accompanies a corresponding refinement of the matrix inequality. For example, let $A, B, X \in \mathcal{M}_n$, such that A and B are positive semidefinite matrices. Then, the matrix versions of (4) are (see [4])

$$(5) \quad \begin{aligned} & \nu^2 \|AX - XB\|_2^2 + \nu^{2\nu} \|A^\nu XB^{1-\nu}\|_2^2 + 2\nu(1-\nu) \left\| A^{\frac{1}{2}}XB^{\frac{1}{2}} \right\|_2^2 \\ & \leq \|\nu AX + (1-\nu)XB\|_2^2 \end{aligned}$$

for $0 \leq \nu \leq \frac{1}{2}$, and

$$(6) \quad \begin{aligned} & (1-\nu)^2 \|AX - XB\|_2^2 + (1-\nu)^{2\nu} \|A^\nu XB^{1-\nu}\|_2^2 + 2\nu(1-\nu) \left\| A^{\frac{1}{2}}XB^{\frac{1}{2}} \right\|_2^2 \\ & \leq \|\nu AX + (1-\nu)XB\|_2^2, \end{aligned}$$

for $\frac{1}{2} \leq \nu \leq 1$.

The primary objective of this paper is to present new inequalities of Young's-type. Here we employ the analysis, which is utilized in [5]. In section 2, we first propose a refinement of the inequalities in (4). Furthermore, we establish the operator analogue of the new scalar inequalities.

2. Main results

2.1 Scalar inequalities

The main aim in this section is the improvement of Young's inequality established in [4].

Throughout the paper we will use the following notations:

$$S_N(\nu; a, b) = \sum_{j=1}^N s_j(\nu) \left(\sqrt[2^j]{b^{2^{j-1}-k_j(\nu)} a^{k_j(\nu)}} - \sqrt[2^j]{a^{k_j(\nu)+1} b^{2^{j-1}-k_j(\nu)-1}} \right)^2$$

with

$$k_j(\nu) = \left\lfloor 2^{j-1}\nu \right\rfloor, \quad r_j(\nu) = \left\lfloor 2^j\nu \right\rfloor \quad \text{and} \quad s_j(\nu) = (-1)^{r_j(\nu)} 2^{j-1}(\nu) + (-1)^{r_j(\nu)+1} \left\lfloor \frac{r_j(\nu)+1}{2} \right\rfloor,$$

for $N \in \mathbb{N}$ and $j = 1, 2, \dots, N$. Notice that $[x]$ is the greatest integer less than or equal to x .

Recall that, in [8], it has been shown that:

$$(7) \quad a^\nu b^{1-\nu} + S_N(\nu; a, b) \leq \nu a + (1-\nu)b.$$

Now, we start with some numerical results.

Theorem 2.1. *Let $a, b > 0$ and $N \in \mathbb{N}$:*

(i) If $0 \leq \nu \leq \frac{1}{2}$, then

$$(8) \quad bS_N(2\nu; \nu a, b) + ((\nu a)^\nu b^{1-\nu})^2 + \nu^2(a-b)^2 \leq \nu^2 a^2 + (1-\nu)^2 b^2.$$

(ii) If $\frac{1}{2} \leq \nu \leq 1$, then

$$(9) \quad aS_N(2\nu-1; a, (1-\nu)b) + \left\{a^\nu((1-\nu)b)^{1-\nu}\right\}^2 + (1-\nu)^2(a-b)^2 \leq \nu^2 a^2 + (1-\nu)^2 b^2.$$

Proof of Theorem 2.1. We consider the case $0 \leq \nu \leq \frac{1}{2}$ at first. By (7), we have

$$(10) \quad \begin{aligned} & \nu^2 a^2 + (1-\nu)^2 b^2 - \nu^2(a-b)^2 \\ &= b[2\nu(\nu a) + (1-2\nu)b] \\ &\geq b[(\nu a)^{2\nu} b^{1-2\nu} + S_N(2\nu; \nu a, b)] \\ &= ((\nu a)^\nu b^{1-\nu})^2 + bS_N(2\nu; \nu a, b). \end{aligned}$$

Similarly, if $\frac{1}{2} \leq \nu \leq 1$, then we have

$$(11) \quad \begin{aligned} & \nu^2 a^2 + (1-\nu)^2 b^2 - (1-\nu)^2(a-b)^2 \\ &= a[(2\nu-1)a + 2(1-\nu)b] \\ &\geq a[a^{2\nu-1}[(1-\nu)b]^{2-2\nu} + S_N(2\nu-1; a, (1-\nu)b)] \\ &= \left\{a^\nu((1-\nu)b)^{1-\nu}\right\}^2 + aS_N(2\nu-1; a, (1-\nu)b). \end{aligned}$$

This completes the proof.

Remark 2.1. Since $bS_N(2\nu; \nu a, b)$, $aS_N(2\nu-1; a, (1-\nu)b) \geq 0$, so Theorem 2.1 improves the inequalities in (4).

As a direct consequence of Theorem 2.1, we have:

Corollary 2.1. Let $a, b > 0$ and $N \in \mathbb{N}$:

(i) If $0 \leq \nu \leq \frac{1}{2}$, then

$$(12) \quad \sqrt{b}S_N(2\nu; \nu\sqrt{a}, \sqrt{b}) + \nu^{2\nu}(a^\nu b^{1-\nu}) + \nu^2(\sqrt{a} - \sqrt{b})^2 \leq \nu^2 a + (1-\nu)^2 b.$$

(ii) If $\frac{1}{2} \leq \nu \leq 1$, then

$$(13) \quad \begin{aligned} & \sqrt{a}S_N(2\nu-1; \sqrt{a}, (1-\nu)\sqrt{b}) + (1-\nu)^{2(1-\nu)}(a^\nu b^{1-\nu}) \\ &+ (1-\nu)^2(\sqrt{a} - \sqrt{b})^2 \leq \nu^2 a + (1-\nu)^2 b. \end{aligned}$$

Corollary 2.2. Assume that $a, b \geq 1$:

- (i) If $0 \leq \nu \leq \frac{1}{2}$, then $S_N(2\nu; \nu a, b) + ((\nu a)^\nu b^{1-\nu})^2 + \nu^2(a-b)^2 \leq \nu^2 a^2 + (1-\nu)^2 b^2$.
- (ii) If $\frac{1}{2} \leq \nu \leq 1$, then $S_N(2\nu-1; a, (1-\nu)b) + (a^\nu((1-\nu)b)^{1-\nu})^2 + (1-\nu)^2(a-b)^2 \leq \nu^2 a^2 + (1-\nu)^2 b^2$.

2.2 Operator versions

Here, the operator versions of the inequalities proved in the previous section are established.

Theorem 2.2. Let $A, B \in \mathcal{B}(\mathcal{H})$ be two positive and invertible operators, and $N \in \mathbb{N}$. If $0 \leq \nu \leq \frac{1}{2}$ and $\alpha_j(2\nu) = \frac{k_j(2\nu)}{2^j}$, then

$$\begin{aligned}
 (14) \quad & \sum_{j=1}^N s_j(2\nu) \left(\nu^{2\alpha_j(2\nu)} A \sharp_{\alpha_j(2\nu)} B + \nu^{2\alpha_j(2\nu)+2^{1-j}} A \sharp_{\alpha_j(2\nu)+2^{-j}} B \right. \\
 & \left. - 2\nu^{2\alpha_j(2\nu)+2^{-j}} A \sharp_{\alpha_j(2\nu)+2^{-(j+1)}} B \right) \\
 & + \nu^{2\nu} A \sharp_\nu B + 2\nu^2 (A \nabla B - A \sharp B) \\
 & \leq ((1-\nu)A) \nabla_\nu (\nu B).
 \end{aligned}$$

On the other hand, if $\frac{1}{2} \leq \nu \leq 1$ and $\beta_j(2\nu-1) = \frac{k_j(2\nu-1)}{2^j}$, then

$$\begin{aligned}
 (15) \quad & \sum_{j=1}^N s_j(2\nu-1) \left((1-\nu)^{1-2\beta_j(2\nu-1)} B \sharp_{\frac{1}{2}-\beta_j(2\nu-1)} A \right. \\
 & + (1-\nu)^{1-2\beta_j(2\nu-1)-2^{1-j}} B \sharp_{\frac{1}{2}-\beta_j(2\nu-1)-2^{-j}} A \\
 & \left. - 2(1-\nu)^{1-2\beta_j(2\nu-1)-2^{-j}} B \sharp_{\frac{1}{2}-\beta_j(2\nu-1)-2^{-(j+1)}} A \right) \\
 & + (1-\nu)^{2(1-\nu)} A \sharp_\nu B + 2(1-\nu)^2 (A \nabla B - A \sharp B) \\
 & \leq ((1-\nu)A) \nabla_\nu (\nu B).
 \end{aligned}$$

Proof of Theorem 2.2. Let $b = 1$ in (12) and expand the summand to get

$$S_N(2\nu; \nu\sqrt{a}, 1) + \nu^{2\nu} a^\nu + \nu^2(a+1-2\sqrt{a}) \leq \nu^2 a + (1-\nu)^2,$$

where

$$\begin{aligned}
 S_N(2\nu; \nu\sqrt{a}, 1) = & \sum_{j=1}^N s_j(2\nu) \left(\sqrt[2^j]{\nu^{2k_j(2\nu)}} a^{\frac{k_j(2\nu)}{2^j}} + \sqrt[2^j]{\nu^{2k_j(2\nu)+2}} a^{\frac{k_j(2\nu)+1}{2^j}} \right. \\
 & \left. - 2 \sqrt[2^j]{\nu^{2k_j(2\nu)+1}} a^{\frac{2k_j(2\nu)+1}{2^{j+1}}} \right).
 \end{aligned}$$

Now, replacing a with the positive operator $X = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$, we get

$$S_N\left(2\nu; \nu X^{\frac{1}{2}}, 1\right) + \nu^{2\nu} X^\nu + \nu^2 \left(X + I_{\mathcal{H}} - 2X^{\frac{1}{2}}\right) \leq \nu^2 X + (1 - \nu)^2 I_{\mathcal{H}},$$

where

$$S_N\left(2\nu; \nu X^{\frac{1}{2}}, 1\right) = \sum_{j=1}^N s_j(2\nu) \left(\sqrt[2^j]{\nu^{2k_j(2\nu)}} X^{\frac{k_j(2\nu)}{2^j}} + \sqrt[2^j]{\nu^{2k_j(2\nu)+2}} X^{\frac{k_j(2\nu)+1}{2^j}} - 2 \sqrt[2^j]{\nu^{2k_j(2\nu)+1}} X^{\frac{2k_j(2\nu)+1}{2^{j+1}}} \right).$$

Multiplying both sides by $A^{\frac{1}{2}}$, we deduce the desired inequality (14).

Likewise, if $\frac{1}{2} \leq \nu \leq 1$, by (13), we can write

$$S_N\left(2\nu - 1; 1, (1 - \nu) \sqrt{b}\right) + (1 - \nu)^{2(1-\nu)} b^{1-\nu} + (1 - \nu)^2 \left(1 + b - 2\sqrt{b}\right) \leq \nu^2 + (1 - \nu)^2 b,$$

where

$$\begin{aligned} & S_N\left(2\nu - 1; 1, (1 - \nu) \sqrt{b}\right) \\ &= \sum_{j=1}^N s_j(2\nu - 1) \left((1 - \nu)^{\frac{2(2^{j-1} - k_j(2\nu-1))}{2^j}} b^{\frac{2^{j-1} - k_j(2\nu-1)}{2^j}} \right. \\ & \quad \left. + (1 - \nu)^{\frac{2(2^{j-1} - k_j(2\nu-1)-1)}{2^j}} b^{\frac{2^{j-1} - k_j(2\nu-1)-1}{2^j}} - 2(1 - \nu)^{\frac{2^j - 2k_j(2\nu-1)-1}{2^j}} b^{\frac{2^j - 2k_j(2\nu-1)-1}{2^{j+1}}} \right). \end{aligned}$$

Replacing b with the positive operator $Y = B^{-\frac{1}{2}}AB^{-\frac{1}{2}}$, and then multiplying both sides of the inequality by $B^{\frac{1}{2}}$, we get (15). This completes the proof.

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On classes of Janowski functions associated with a conic domain

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Abstract. The purpose of this paper is to define a new class of close-to-convex function which are analytic, normalized functions in the open unit disk subordinating with a conic region and to derive initial coefficient estimates a_2 , a_3 for the function class. We also investigate interesting characteristic properties like sufficient conditions, inclusion relationship, and radius of convexity. Further by appropriate choice of parameters, we provided some new and well known results of our main result.

Keywords: Janowski starlike functions, uniformly convex, close-to-convex function, subordination, coefficient inequalities, q -calculus, differential operator.

1. Introduction

Denote by $\mathcal{A}$ the class of functions having a Taylor series expansion of the form

$$(1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in \mathbb{U} = \{z : |z| < 1\}).$$

We let $\mathcal{S}$ to denote the class of functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$. We let $\mathcal{S}^*$, $\mathcal{C}$ and $\mathcal{K}$ to denote the well known classes of starlike, convex and close-

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to-convex (normalized) function respectively. For $0 \leq \alpha < 1$, $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$ symbolize the classes of starlike functions of order α and convex functions of order α respectively. Also let $\mathcal{P}$ denote the class of functions of the form $p(z) = 1 + p_1z + p_2z^2 + p_3z^3 + \cdots$ that are analytic in $\mathbb{U}$ and such that $\operatorname{Re}(p(z)) > 0$ for all z in $\mathbb{U}$.

Let $f(z)$ and $g(z)$ be analytic in $\mathbb{U}$. Then we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$, if there exists an Schwarz function $w(z)$ in $\mathbb{U}$ such that $|w(z)| < |z|$ and $f(z) = g(w(z))$, denoted by $f(z) \prec g(z)$. If $g(z)$ is univalent in $\mathbb{U}$, then the subordination is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

Using the concept of subordination of analytic functions, Ma and Minda [13] introduced the class $\mathcal{S}^*(\phi)$ of functions in $\mathcal{A}$ satisfying $\frac{zf'(z)}{f(z)} \prec \phi$, where $\phi \in \mathcal{P}$ with $\phi'(0) > 0$ maps $\mathbb{U}$ onto a region starlike with respect to 1 and symmetric with respect to real axis. This class specializes to several classes of univalent functions for suitable choice of ϕ . For instance, the class $\mathcal{S}^*(\frac{1+Az}{1+Bz}) =: \mathcal{S}^*[A; B]$ where $-1 \leq B < A \leq 1$, is the class of the Janowski starlike functions (see [7]).

In 1993, Goodman [4] introduced the geometrically defined class of uniformly convex functions (UCV), for which Rønning [20] and Ma and Minda [14] independently gave the following one variable characterization for the class UCV , as

$$(2) \quad \Re \left(1 + \frac{zf''(z)}{f'(z)} \right) > \left| \frac{zf''(z)}{f'(z)} \right|, \quad z \in \mathbb{U}.$$

Further, Bharati et al., [2] defined the subclass $k-UCV$, of order α as below: A function $f(z) \in \mathcal{A}$ is said to be k -uniformly convex of order α , if

$$(3) \quad \Re \left(1 + \frac{zf''(z)}{f'(z)} \right) > k \left| \frac{zf''(z)}{f'(z)} \right| + \alpha, \quad z \in \mathbb{U}.$$

Special cases of $k-UCV(\alpha)$ namely $k-UCV(0) = k-UCV$ were introduced by Kanas and Wiśniowska [8, 9]. Geometrically, a function f is in $k-UCV(\alpha)$ if and only if $1 + \frac{zf''(z)}{f'(z)}$ takes all values in the convex domain $\mathcal{D}_{k,\alpha}$ which is included in right half plane. Moreover, $\mathcal{D}_{k,\alpha}$ is an elliptic region for $k > 1$, a parabolic region for $k = 1$, a hyperbolic region for $0 < k < 1$, the half plane $u > \alpha$ for $k = 0$.

The function $p_{k,\alpha}(z)$ plays the role of an extremal functions those related to these conic domain $\mathcal{D}_{k,\alpha}$ and is given by

$$(4) \quad \hat{p}_{k,\alpha}(z) = \begin{cases} \frac{1+(1-2\alpha)z}{1-z}, & \text{if } k = 0 \\ 1 + \frac{2(1-\alpha)}{\pi^2} \left(\log \frac{1+\sqrt{z}}{1-\sqrt{z}} \right)^2, & \text{if } k = 1 \\ 1 + \frac{2(1-\alpha)}{1-k^2} \sinh^2 \left[\left(\frac{2}{\pi} \arccos k \right) \operatorname{arctanh} \sqrt{z} \right], & \text{if } 0 < k < 1 \\ 1 + \frac{2(1-\alpha)}{1-k^2} \sin \left(\frac{\pi}{2R(t)} \int_0^{\frac{u(z)}{t}} \frac{1}{\sqrt{1-x^2}\sqrt{1-(tx)^2}} dx \right) + \frac{1}{k^2-1}, & \text{if } k > 1 \end{cases}$$

where $u(z) = \frac{z - \sqrt{t}}{1 - \sqrt{tz}}$, $t \in (0, 1)$ and t is chosen such that $k = \cosh(\frac{\pi R'(t)}{4R(t)})$, with $R(t)$ is Legendre's complete elliptic integral of the first kind and $R'(t)$ is complementary integral of $R(t)$. Clearly, $\hat{p}_{k,\alpha}(z)$ is in $\mathcal{P}$ with the expansion of the form

$$(5) \quad \hat{p}_{k,\alpha}(z) = 1 + \delta_1 z + \delta_2 z^2 + \cdots, \quad (\delta_j = p_j(k, \alpha), j = 1, 2, 3, \dots),$$

we get

$$(6) \quad \delta_1 = \begin{cases} \frac{8(1-\alpha)(\arccos k)^2}{\pi^2(1-k^2)}, & \text{if } 0 \leq k < 1, \\ \frac{8(1-\alpha)}{\pi^2}, & \text{if } k = 1 \\ \frac{\pi^2(1-\alpha)}{4\sqrt{t}(k^2-1)R^2(t)(1+t)}, & \text{if } k > 1. \end{cases}$$

Now we briefly recall the q -calculus and the notations which are required for our study. Quantum calculus (q -calculus and h -calculus) is common classical calculus without the notion of limits. Here, h represents Planck's constant, while q represents quantum. Due to its application in a variety of branches such as physics, mathematics, the area of q -calculus has gained great importance for researchers. The first study on q -calculus was systematically established by Jackson [6] as q -derivative is merely a ratio which is given by

$$D_q f(z) = \frac{f(qz) - f(z)}{(q-1)z}, \quad (0 < q < 1).$$

Note that $\lim_{q \rightarrow 1^-} D_q f(z) = f'(z)$. Notations and symbols play an very important role in the study of q -calculus. Throughout this paper, we let

$$[n]_q = \sum_{k=1}^n q^{k-1}, \quad [0]_q = 0, \quad (q \in \mathbb{C})$$

and the q -shifted factorial by

$$(a; q)_n = \begin{cases} 1, & n = 0 \\ (1-a)(1-aq) \cdots (1-aq^{n-1}), & n = 1, 2, \dots \end{cases}$$

In [5], Ismail et. al. introduced the class $\mathcal{S}_q^*$ to be the class of functions which satisfy the condition

$$\left| \frac{z D_q f(z)}{f(z)} - \frac{1}{1-q} \right| \leq \frac{1}{1-q}, \quad (f \in \mathcal{S}).$$

Equivalently, a function $f \in \mathcal{S}_q^*$ if and only if the following subordination condition (see [17, 24])

$$\frac{z D_q f(z)}{f(z)} \prec \frac{1+z}{1-qz}$$

holds. Recently in [19], the authors defined the following q -differential operator

$$\mathcal{J}_\lambda^m(a_1, b_1; q, z)f : \mathbb{U} \rightarrow \mathbb{U}$$

given by

$$(7) \quad \mathcal{J}_\lambda^m(a_1, b_1; q, z)f = z + \sum_{n=2}^{\infty} [1 - \lambda + \lambda[n]_q]^m \Gamma_n a_n z^n, \\ (m \in N_0 = N \cup \{0\} \text{ and } \lambda \geq 0),$$

where

$$\Gamma_n = \frac{(a_1; q)_{n-1} (a_2; q)_{n-1} \cdots (a_r; q)_{n-1}}{(q; q)_{n-1} (b_1; q)_{n-1} \cdots (b_s; q)_{n-1}}, \quad (|q| < 1).$$

For detailed study and applications of the operator $\mathcal{J}_\lambda^m(a_1, b_1; q, z)f$, refer to [19] and the references provided therein.

Throughout our present discussion, to avoid repetition, we lay down once for all that $-1 \leq B < A \leq 1$, $z \in \mathbb{U}$ and Γ_n is real.

Definition 1.1. For $p_{k,\alpha}(z)$ defined as in (4), a function $f \in k\text{-UCV}_{q,t}^{\lambda,m}(a_1, b_1; A, B)$ if and only if

$$(8) \quad p(z) \prec \frac{(A+1)\hat{p}_{k,\alpha}(z) - (A-1)}{(B+1)\hat{p}_{k,\alpha}(z) - (B-1)}, \quad (f \in \mathcal{A}; |t| \leq 1, t \neq 0)$$

when $p(z) = \frac{tz^2[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]'}{[\mathcal{J}_\lambda^m(a_1, b_1; q, z)g(z)][\mathcal{J}_\lambda^m(a_1, b_1; q, z)g(tz)]}$ and $\mathcal{J}_\lambda^m(a_1, b_1; q, z)g(z) \in \mathcal{S}^*\left(\frac{1}{2}\right)$.

Analogous to the class $k\text{-UCV}_{q,t}^{\lambda,m}(a_1, b_1; A, B)$, we define the following.

Definition 1.2. Let $\mathcal{V}_{q,t}^{\lambda,m}(a_1, b_1; A, B)$, $|t| \leq 1$, $t \neq 0$ denote the class of functions $f(z) \in \mathcal{A}$ satisfying the conditions

$$(9) \quad \frac{tz^2[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]'}{[\mathcal{J}_\lambda^m(a_1, b_1; q, z)g(z)][\mathcal{J}_\lambda^m(a_1, b_1; q, z)g(tz)]} \prec \frac{(A+1)\hat{p}(z) - (A-1)}{(B+1)\hat{p}(z) - (B-1)},$$

where $\mathcal{J}_\lambda^m(a_1, b_1; q, z)g(z) \in \mathcal{S}^*\left(\frac{1}{2}\right)$ and $\hat{p}(z) = \frac{1+z}{1-qz}$, $q \in (0, 1)$.

By definition of subordination, a function $f \in \mathcal{A}$ is said to be in $\mathcal{V}_{q,t}^{\lambda,m}(a_1, b_1; A, B)$ if and only if

$$(10) \quad \frac{tz^2[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]'}{[\mathcal{J}_\lambda^m(a_1, b_1; q, z)g(z)][\mathcal{J}_\lambda^m(a_1, b_1; q, z)g(tz)]} = \frac{(A+1)w(z) + 2 + (A-1)qw(z)}{(B+1)w(z) + 2 + (B-1)qw(z)},$$

where $q \in (0, 1)$; $z \in \mathbb{U}$; $w(z)$ is analytic in $\mathbb{U}$ and $w(0) = 0$, $|w(z)| < 1$.

Remark 1.1. The classes $\mathcal{V}_{q,t}^{\lambda,m}(a_1, b_1; A, B)$ and $k\text{-UCV}_{q,t}^{\lambda,m}(a_1, b_1; A, B)$ were motivated by study of [3, 12]. For study of classes associated with Janowski functions on conic domains, refer to [1, 15, 16, 24].

In the present work, we obtain the coefficient estimates, sufficient conditions, inclusion relation and radius of convexity for the functions in the classes $\mathcal{V}_{q,t}^{\lambda,m}(a_1, b_1; A, B)$ and $k\text{-UCV}_{q,t}^{\lambda,m}(a_1, b_1; A, B)$. Also, we obtain the Fekete-Szegő problem for the functions in the class $\mathcal{V}_{q,t}^{\lambda,m}(a_1, b_1; A, B)$.

2. Coefficient estimates and Fekete-Szegő inequality

To prove our results, we make use of the following Lemmas.

Lemma 2.1 ([18]). *Let $f(z) = \sum_{n=1}^{\infty} a_n z^n$ be analytic in $\mathbb{U}$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$ is analytic and convex in $\mathbb{U}$. If $f(z) \prec g(z)$, then $|a_n| \leq |b_1|$, for $n = 1, 2, \dots$*

Lemma 2.2 ([16]). *Let $p(z) \in \mathcal{P}$ satisfy the subordination condition*

$$p(z) \prec \frac{(A+1)p_{k,\alpha}(z) - (A-1)}{(B+1)p_{k,\alpha}(z) - (B-1)},$$

then

$$(11) \quad |p_n| \leq \frac{(A-B)\delta_1}{2},$$

where δ_1 is given by (6).

Lemma 2.3 ([23]). *Let $g(z) \in \mathcal{S}^*(\frac{1}{2})$ and $0 < |t| \leq 1$, then $\frac{g(z)g(tz)}{tz} \in \mathcal{S}^*$.*

Using Lemma 2.3 and proceeding on lines similar to Theorem 2.3 of [23], we get the following result.

Lemma 2.4 ([23]). *Let $\mathcal{J}_\lambda^m(a_1, b_1; q, z)g \in \mathcal{S}^*(\frac{1}{2})$, then*

$$(12) \quad \begin{aligned} G(z) &= \frac{[\mathcal{J}_\lambda^m(a_1, b_1; q, z)g(z)][\mathcal{J}_\lambda^m(a_1, b_1; q, z)g(tz)]}{tz} \\ &= z + \sum_{n=2}^{\infty} c_n z^n \in \mathcal{S}^*, \quad \text{and} \quad |c_n| \leq n, \end{aligned}$$

where

$$c_n = [\Psi_n b_n + \Psi_{n-1} \Psi_2 b_{n-1} b_2 t + \dots + \Psi_n b_n t^{n-1}], \quad \Psi_n = [1 - \lambda + [n]_q \lambda]^m \Gamma_n.$$

To start with we prove the following result :

Theorem 2.1. *If $f(z) \in k - \mathcal{UCV}_{q,t}^{\lambda,m}(a_1, b_1; A, B)$, then*

$$(13) \quad |a_n| \leq \frac{1}{|\Gamma_n| [1 - \lambda + [n]_q \lambda]^m} \left(1 + \frac{(A-B)\delta_1(n-1)}{4} \right).$$

Proof. Let $f(z) \in k - \mathcal{UCV}_{q,t}^{\lambda,m}(a_1, b_1; A, B)$, then there exist a function $p(z) \in \mathcal{P}$ analytic in $\mathbb{U}$ such that

$$(14) \quad p(z) = \frac{z[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]'}{G(z)} \prec \frac{(A+1)p_{k,\alpha}(z) - (A-1)}{(B+1)p_{k,\alpha}(z) - (B-1)},$$

where $p_{k,\alpha}(z)$ is defined by the equation (4). Clearly $p_{k,\alpha}(z)$ is analytic and maps $\mathbb{U}$ on to a convex domain. From (14), we have

$$(15) \quad \frac{z[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]'}{G(z)} = p(z), \quad (p(z) \in \mathcal{P}).$$

Using Lemma 2.2 and Lemma 2.4 in (15), we get

$$(16) \quad z + \sum_{n=2}^{\infty} [1 - \lambda + [n]_q \lambda]^m n \Gamma_n a_n z^n = (1 + \sum_{n=1}^{\infty} p_n z^n) (z + \sum_{n=2}^{\infty} c_n z^n).$$

Equating the coefficients of z^n in (16), we have

$$(17) \quad n \Gamma_n a_n [1 - \lambda + [n]_q \lambda]^m = c_n + p_1 c_{n-1} + p_2 c_{n-2} + \cdots + p_{n-1}.$$

Therefore using (11) and Lemma 2.4, we get

$$(18) \quad n \Gamma_n [1 - \lambda + [n]_q \lambda]^m |a_n| \leq n \left(1 + \frac{(A-B)\delta_1(n-1)}{4} \right).$$

From (18), we get the desired result in (13). $\square$

Theorem 2.2. *If $f(z) \in \mathcal{V}_{q,t}^{\lambda,m}(a_1, b_1; A, B)$, then*

$$(19) \quad |a_n| \leq \frac{1}{|\Gamma_n| [1 - \lambda + [n]_q \lambda]^m} \left(1 + \frac{(A-B)(1+q)(n-1)}{4} \right).$$

Proof. It can be easily seen that

$$p(z) = \frac{(A+1)w(z) + 2 + (A-1)qw(z)}{(B+1)w(z) + 2 + (B-1)qw(z)} = 1 + (A-B)(1+q)z + \cdots$$

is analytic and maps $\mathbb{U}$ on to a convex domain. By Lemma 2.1, we obtain

$$|p_n| \leq \frac{(A-B)(1+q)}{2}, \quad (n \geq 1).$$

Now, following the steps as in Theorem 2.1, we get the desired result. $\square$

If we let $k = m = 0$, $r = 2$, $s = 1$, $a_1 = b_1$, $a_2 = q$ and $q \rightarrow 1^-$ in Theorem 2.2, we get the following result

Corollary 2.1 ([22]). *If $X_t(A, B)$, then $|a_n| \leq 1 + \frac{(n-1)(A-B)}{2}$.*

Letting $A = 1 - 2\alpha$, $B = -1$ in Corollary 2.1, the following result due to Prajapat [21] becomes obvious.

Corollary 2.2. *If $f \in X_t(\alpha)$ then $|a_n| \leq 1 + (n-1)(1-\alpha)$.*

2.1 Fekete-Szegő problem

We recall the following lemmas to prove our results in this subsection:

Lemma 2.5 ([10]). *If $p(z) = 1 + p_1z + p_2z^2 + \dots$ is a function with positive real part, then for each complex number μ*

$$(20) \quad |p_2 - \mu p_1^2| \leq 2 \max(1, |2\mu - 1|)$$

and the result is sharp for the functions given by $p(z) = \frac{1+z^2}{1-z^2}$, $p(z) = \frac{1+z}{1-z}$.

Lemma 2.6 ([11]). *If*

$$G(z) = z + \sum_{n=2}^{\infty} c_n z^n \in \mathcal{S}^*,$$

then for each complex number λ we have $|c_3 - \lambda c_2^2| \leq \max(1, |3 - 4\lambda|)$ and the result is sharp for the Koebe function

$$k(z) = \frac{z}{(1-z)^2} \quad \text{if} \quad \left| \lambda - \frac{3}{4} \right| \geq \frac{1}{4}$$

and for

$$k^{\frac{1}{2}}(z^2) = \frac{z}{1-z^2} \quad \text{if} \quad \left| \lambda - \frac{3}{4} \right| \leq \frac{1}{4}.$$

Theorem 2.3. *If $f(z) \in k - \mathcal{UCV}_{q,t}^{\lambda,m}(a_1, b_1; A, B)$, then for $\mu \in \mathbb{C}$ we have*

$$(21) \quad |a_3 - \mu a_2^2| \leq \frac{(A-B)\delta_1}{6\Gamma_3[1-\lambda+[3]_q\lambda]^m} \max(1, |2\vartheta - 1|) \\ + \frac{1}{3} \max(1, |3 - 4\nu_1|) + \frac{(A-B)\delta_1}{\Gamma_3[1-\lambda+[3]_q\lambda]^m} \left| \frac{1}{3} - \frac{\mu}{2} \right|,$$

where

$$\vartheta = \frac{(B+1)\delta_1 + 2\left(1 - \frac{\delta_2}{\delta_1}\right)}{4} + \frac{3(A-B)\delta_1\mu\Gamma_3[1-\lambda+[3]_q\lambda]^m}{16\Gamma_2^2[1-\lambda+[2]_q\lambda]^{2m}}$$

and $\nu_1 = \frac{3\mu}{4}$.

Proof. Let $h(z) \in \mathcal{P}$ be of the form $h(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$, we consider

$$h(z) = \frac{1+w(z)}{1-w(z)},$$

where $w(z)$ is such that $w(0) = 0$ and $|w(z)| < 1$. On simple computation, we have

$$(22) \quad w(z) = \frac{h(z) - 1}{h(z) + 1} \\ = \frac{p_1z + p_2z^2 + p_3z^3 + \dots}{2 + p_1z + p_2z^2 + p_3z^3 + \dots} \\ = \frac{1}{2}p_1z + \frac{1}{2}\left(p_2 - \frac{1}{2}p_1^2\right)z^2 + \frac{1}{2}\left(p_3 - p_1p_2 + \frac{1}{4}p_1^3\right)z^3 + \dots.$$

Using (22) in (5), we have

$$\begin{aligned}\hat{p}_{k,\alpha}(w(z)) &= 1 + \delta_1 w(z) + \delta_2 [w(z)]^2 + \delta_3 [w(z)]^3 + \dots \\ &= 1 + \frac{\delta_1 p_1}{2} z + \frac{\delta_1}{2} \left[p_2 - \frac{1}{2} \left(1 - \frac{\delta_2}{\delta_1} \right) p_1^2 \right] z^2 + \dots\end{aligned}$$

As $f(z) \in k - \mathcal{UCV}_{q,t}^{\lambda,m}(a_1, b_1; A, B)$, by (14) we have

$$(23) \quad \frac{z[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]'}{G(z)} = \frac{(A+1)p_{k,\alpha}(w(z)) - (A-1)}{(B+1)p_{k,\alpha}(w(z)) - (B-1)}.$$

From (23), we obtain

$$\begin{aligned}&1 + \Gamma_2[1 - \lambda + [2]_q \lambda]^m (2a_2 - c_2)z \\ &+ \Gamma_3[1 - \lambda + [3]_q \lambda]^m (3a_3 - 2a_2 c_2 - c_3 + c_2^2)z^2 + \dots \\ &= 1 + \frac{\delta_1 p_1 (A-B)}{4} z + \frac{(A-B)\delta_1}{4} \left[p_2 - p_1^2 \left(\frac{(B+1)\delta_1 + 2 \left(1 - \frac{\delta_2}{\delta_1} \right)}{4} \right) \right] z^2 + \dots\end{aligned}$$

Equating the coefficients at z and z^2 on both sides of the above equation, we get

$$a_2 = \frac{p_1(A-B)\delta_1 + 4c_2\Gamma_2[1 - \lambda + [2]_q \lambda]^m}{8\Gamma_2[1 - \lambda + [2]_q \lambda]^m}$$

and

$$a_3 = \frac{1}{3} \left[c_3 + \frac{(A-B)\delta_1}{4\Gamma_3[1 - \lambda + [3]_q \lambda]^m} \left(p_1 c_2 + p_2 - p_1^2 \frac{(B+1)\delta_1 + 2 \left(1 - \frac{\delta_2}{\delta_1} \right)}{4} \right) \right].$$

Therefore, we have

$$\begin{aligned}(24) \quad |a_3 - \mu a_2^2| &\leq \frac{(A-B)\delta_1}{12\Gamma_3[1 - \lambda + [3]_q \lambda]^m} |p_2 - \vartheta p_1^2| + \frac{1}{3} |c_3 - \nu_1 c_2^2| \\ &+ \frac{(A-B)\delta_1}{4\Gamma_3[1 - \lambda + [3]_q \lambda]^m} |c_2| \left| \left(\frac{1}{3} - \frac{\mu}{2} \right) \right| |p_1|.\end{aligned}$$

Using Lemma 2.5 and Lemma 2.6, we complete the proof. $\square$

Theorem 2.4. If $f(z) \in \mathcal{S}_{q,t}^{\lambda,m}(a_1, b_1; A, B)$, then for $\mu \in \mathbb{C}$ we have

$$\begin{aligned}(25) \quad |a_3 - \mu a_2^2| &\leq \frac{(A-B)(1+q)}{6\Gamma_3[1 - \lambda + [3]_q \lambda]^m} \max(1, |2\eta - 1|) \\ &+ \frac{1}{3} \max(1, |3 - 4\mu_1|) + \frac{(A-B)(1+q)}{\Gamma_3[1 - \lambda + [3]_q \lambda]^m} \left| \frac{1}{3} - \frac{\mu}{2} \right|,\end{aligned}$$

where

$$\eta = \frac{2 + B(1+q) + (1-q)}{4} + \frac{3(A-B)(1+q)\mu \Gamma_3[1 - \lambda + [3]_q \lambda]^m}{16\Gamma_2^2[1 - \lambda + [2]_q \lambda]^{2m}}$$

and $\mu_1 = \frac{3\mu}{4}$.

Proof. As $f(z) \in \mathcal{V}_{q,t}^{\lambda,m}(a_1, b_1; A, B)$, by (10) we have

$$\frac{z[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]'}{G(z)} = \frac{(A+1)w(z) + 2 + (A-1)qw(z)}{(B+1)w(z) + 2 + (B-1)qw(z)}.$$

Let

$$h(z) = \frac{1+w(z)}{1-w(z)} = 1 + p_1z + p_2z^2 + \dots,$$

then $Re(h(z)) > 0$ and $h(0) = 1$. Hence,

$$(26) \quad \begin{aligned} & \frac{z[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]'}{G(z)} \\ &= \frac{(A+1)(1+q)[h(z)-1] + 2[h(z)+1-q(h(z)-1)]}{(B+1)(1+q)[h(z)-1] + 2[h(z)+1-q(h(z)-1)]}. \end{aligned}$$

From (26), we obtain

$$\begin{aligned} & 1 + \Gamma_2[1 - \lambda + [2]_q\lambda]^m(2a_2 - c_2)z \\ & + \Gamma_3[1 - \lambda + [3]_q\lambda]^m(3a_3 - 2a_2c_2 - c_3 + c_2^2)z^2 + \dots \\ &= 1 + \frac{p_1(A-B)(1+q)}{4}z \\ & + \frac{(A-B)(1+q)}{4} \left[p_2 - p_1^2 \left(\frac{2+B(1+q)+(1-q)}{4} \right) \right] z^2 + \dots. \end{aligned}$$

Equating the corresponding coefficients of the above equation, we get

$$a_2 = \frac{p_1(A-B)(1+q) + 4c_2\Gamma_2[1 - \lambda + [2]_q\lambda]^m}{8\Gamma_2[1 - \lambda + [2]_q\lambda]^m}$$

and

$$a_3 = \frac{1}{3} \left[c_3 + \frac{(A-B)(1+q)}{4\Gamma_3[1 - \lambda + [3]_q\lambda]^m} \left(p_1c_2 + p_2 - \frac{p_1^2[2+B(1+q)+(1-q)]}{4} \right) \right].$$

Therefore, we have

$$(27) \quad \begin{aligned} |a_3 - \mu a_2^2| &\leq \frac{(A-B)(1+q)}{12\Gamma_3[1 - \lambda + [3]_q\lambda]^m} |p_2 - \eta p_1^2| \\ &+ \frac{|c_3 - \mu_1 c_2^2|}{3} + \frac{(A-B)(1+q)}{4\Gamma_3[1 - \lambda + [3]_q\lambda]^m} |c_2| \left| \left(\frac{1}{3} - \frac{\mu}{2} \right) \right| |p_1|. \end{aligned}$$

Using Lemma 2.5 and Lemma 2.6, we complete the proof. $\square$

If we let $m = 0$, $r = 2$, $s = 1$, $a_1 = b_1$, $a_2 = q$ and $q \rightarrow 1^-$ in Theorem 2.4, we get the following result.

Corollary 2.3 ([22]). *If $f(z) \in X_t(A, B)$, then for $\mu \in \mathbb{C}$ we have*

$$(28) \quad \begin{aligned} |a_3 - \mu a_2^2| &\leq \frac{A-B}{3} \max(1, |2\beta - 1|) + \frac{1}{3} \max(1, |3 - 4\mu_1|) \\ &\quad + 2(A-B) \left| \frac{1}{3} - \frac{\mu}{2} \right|, \end{aligned}$$

where

$$\beta = \frac{1+B}{2} + \frac{3(A-B)\mu}{8}, \quad \mu_1 = \frac{3\mu}{4}.$$

For $A = 1 - 2\alpha, B = -1$, Theorem 2.4 gives the following result which is analogous to one obtained by Prajapat in [21].

Corollary 2.4. *If $\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z) \in X_t(\alpha)$, then for $\mu \in \mathbb{C}$,*

$$(29) \quad \begin{aligned} |a_3 - \mu a_2^2| &\leq \frac{2(1-\alpha)}{3\Gamma_3[1-\lambda + [3]_q\lambda]^m} \max(1, |2\beta - 1|) \\ &\quad + \frac{1}{3} \max(1, |3 - 4\mu_1|) + \frac{4(1-\alpha)}{\Gamma_3[1-\lambda + [3]_q\lambda]^m} \left| \frac{1}{3} - \frac{\mu}{2} \right| \end{aligned}$$

where

$$\beta = \frac{3(1-\alpha)\mu\Gamma_3[1-\lambda + [3]_q\lambda]^m}{4\Gamma_2^2[1-\lambda + [2]_q\lambda]^{2m}} \quad \text{and} \quad \mu_1 = \frac{3\mu}{4}.$$

3. Characterization properties of $k - \mathcal{UCV}_{q,t}^{\lambda,m}(a_1, b_1; A, B)$ and $\mathcal{V}_{q,t}^{\lambda,m}(a_1, b_1; A, B)$

Theorem 3.1. *Let the function $f(z)$ defined by (1) and let*

$$(30) \quad \sum_{n=2}^{\infty} \frac{2(k+1)}{1-\alpha} |na_n - b_n| - |n(B+1)a_n - (A+1)b_n| < (A-B)$$

holds, then $f(z)$ belongs to $k - \mathcal{UCV}_{q,t}^{\lambda,m}(a_1, b_1; A, B)$.

Proof. Assume that (30) holds. It is sufficient to show that

$$\begin{aligned} &k \left| \frac{(B-1) \frac{z[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]'}{G(z)} - (A-1)}{(B+1) \frac{z[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]'}{G(z)} - (B-1)} - 1 \right| \\ &- \Re \left(\frac{(B-1) \frac{z[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]'}{G(z)} - (A-1)}{(B+1) \frac{z[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]'}{G(z)} - (B-1)} - 1 \right) < 1 - \alpha. \end{aligned}$$

Now, consider

$$\begin{aligned}
& k \left| \frac{(B-1) \frac{z[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]'}{G(z)} - (A-1)}{(B+1) \frac{z[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]'}{G(z)} - (B-1)} - 1 \right| \\
& - \Re \left(\frac{(B-1) \frac{z[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]'}{G(z)} - (A-1)}{(B+1) \frac{z[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]'}{G(z)} - (B-1)} - 1 \right) \\
& \leq (k+1) \left| \frac{(B-1) \frac{z[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]'}{G(z)} - (A-1)}{(B+1) \frac{z[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]'}{G(z)} - (B-1)} - 1 \right| \\
& \leq \frac{2(k+1) \sum_{n=2}^{\infty} |na_n - b_n|}{(A-B) - \sum_{n=2}^{\infty} |n(B+1)a_n - (A+1)b_n|}
\end{aligned}$$

The last inequality is bounded by $1 - \alpha$ if

$$\sum_{n=2}^{\infty} \frac{2(k+1)}{1-\alpha} |na_n - b_n| - |n(B+1)a_n - (A+1)b_n| < (A-B). \quad \square$$

Theorem 3.2. Let the function $f(z)$ defined by (1) and let

$$\begin{aligned}
(31) \quad & \left(1 + \frac{|B(1+q) + (1-q)|}{2} \right) \sum_{n=2}^{\infty} [1 - \lambda + [n]_q \lambda]^m n \Gamma_n |a_n| \\
& + \left(1 + \frac{|A(1+q) + (1-q)|}{2} \right) \sum_{n=2}^{\infty} |c_n| < \frac{(A-B)(1+q)}{2}
\end{aligned}$$

holds, then $f(z)$ belongs to $\mathcal{V}_{q,t}^{\lambda,m}(a_1, b_1; A, B)$.

Proof. Let us assume that (31) holds. Now, consider

$$\begin{aligned}
& |z[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]' - G(z)| \\
& - \left| \frac{A(1+q) + (1-q)}{2} G(z) - \frac{B(1+q) + (1-q)}{2} z[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]' \right| \\
& = \left| \sum_{n=2}^{\infty} [1 - \lambda + [n]_q \lambda]^m n \Gamma_n a_n z^n - \sum_{n=2}^{\infty} c_n z^n \right| \\
& - \left| \frac{A(1+q) + (1-q)}{2} \left(z + \sum_{n=2}^{\infty} c_n z^n \right) \right. \\
& \quad \left. - \frac{B(1+q) + (1-q)}{2} \left(z + \sum_{n=2}^{\infty} [1 - \lambda + [n]_q \lambda]^m n \Gamma_n a_n z^n \right) \right| \\
& \leq \sum_{n=2}^{\infty} [1 - \lambda + [n]_q \lambda]^m n \Gamma_n |a_n| |z|^n + \sum_{n=2}^{\infty} |c_n| |z|^n - \left(\frac{(A-B)(1+q)}{2} |z| \right)
\end{aligned}$$

$$\begin{aligned}
& - \frac{A(1+q) + (1-q)}{2} \sum_{n=2}^{\infty} |c_n| |z|^n \\
& - \frac{B(1+q) + (1-q)}{2} \sum_{n=2}^{\infty} [1 - \lambda + [n]_q \lambda]^m n \Gamma_n |a_n| |z^n| \Big) \\
& = - \frac{(A-B)(1+q)}{2} |z| \\
& + \left(1 + \frac{B(1+q) + (1-q)}{2} \right) \sum_{n=2}^{\infty} [1 - \lambda + [n]_q \lambda]^m n \Gamma_n |a_n| |z|^n \\
& + \left(1 + \frac{A(1+q) + (1-q)}{2} \right) \sum_{n=2}^{\infty} |c_n| |z|^n \leq 0.
\end{aligned}$$

From the above inequality, we have established that

$$\begin{aligned}
& |z[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]' - G(z)| \\
& < \left| \frac{A(1+q) + (1-q)}{2} G(z) - \frac{B(1+q) + (1-q)}{2} z[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]' \right|,
\end{aligned}$$

which is equivalent to (10). Thus $f(z) \in \mathcal{V}_{q,t}^{\lambda,m}(a_1, b_1; A, B)$. $\square$

3.1 Inclusion relation

The following lemma is useful in the proof of the main result in this subsection.

Lemma 3.1 ([23]). *If $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1$, then*

$$\frac{1 + A_1 z}{1 + B_1 z} \prec \frac{1 + A_2 z}{1 + B_2 z}.$$

Theorem 3.3. *Let $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1$, then*

$$\mathcal{V}_{q,t}^{\lambda,m}(a_1, b_1; A_1, B_1) \subset \mathcal{V}_{q,t}^{\lambda,m}(a_1, b_1; A_2, B_2).$$

Proof. As $f(z) \in \mathcal{V}_{q,t}^{\lambda,m}(a_1, b_1; A_1, B_1)$, therefore

$$(32) \quad p(z) = \frac{tz^2[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]'}{[\mathcal{J}_\lambda^m(a_1, b_1; q, z)g(z)]g(tz)} \prec \frac{(A_1 + 1)z + 2 + (A_1 - 1)qz}{(B_1 + 1)z + 2 + (B_1 - 1)qz}.$$

Since $-1 \leq B_2 \leq B_1 < A_1 < A_2 \leq 1$, by Lemma 3.1, we have

$$(33) \quad p(z) \prec \frac{(A_1 + 1)z + 2 + (A_1 - 1)qz}{(B_1 + 1)z + 2 + (B_1 - 1)qz} \prec \frac{(A_2 + 1)z + 2 + (A_2 - 1)qz}{(B_2 + 1)z + 2 + (B_2 - 1)qz}.$$

This yields that $f(z) \in \mathcal{V}_{q,t}^{\lambda,m}(a_1, b_1; A_2, B_2)$ and this proves the inclusion relation. $\square$

For completeness, we just state the following.

Theorem 3.4. *Let $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1$, then*

$$k - \mathcal{UCV}_{q,t}^{\lambda,m}(a_1, b_1; A_1, B_1) \subset k - \mathcal{UCV}_{q,t}^{\lambda,m}(a_1, b_1; A_2, B_2).$$

3.2 Radius of convexity

Theorem 3.5. *If $f(z) \in \mathcal{V}_{q,t}^{\lambda,m}(a_1, b_1; A_1, B_1)$, then $\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)$ is convex in $|z| = r_1$, where r_1 is the smallest positive root in $(0, 1)$ of the equation*

$$(34) \quad Qr^3 - Rr^2 - 4Sr - 4 = 0,$$

where $Q = AB(1+q)^2 + (A+B)(1-q^2) + (1-q)^2$, $R = [A(1+q) + (1-q)][B(1+q) - (3+q)]$ and $S = [B(1+q) - q]$.

Proof. As $f(z) \in \mathcal{V}_{q,t}^{\lambda,m}(a_1, b_1; A_1, B_1)$, we have

$$(35) \quad z[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]' = G(z)p(z).$$

By applying logarithmic differentiation in (35), we get

$$(36) \quad 1 + \frac{z[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]''}{[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]'} = \frac{zG'(z)}{G(z)} + \frac{zp'(z)}{p(z)}.$$

Now, for $G(z) \in \mathcal{S}^*$, we have

$$\Re\left(\frac{zG'(z)}{G(z)}\right) \geq \frac{1-r}{1+r}.$$

Therefore, (36) yields that

$$\Re\left(1 + \frac{z[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]''}{[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]'}\right) \geq \frac{1-r}{1+r} - \left|\frac{zp'(z)}{p(z)}\right|.$$

Further, we have

$$\begin{aligned} \Re\left(1 + \frac{z[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]''}{[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]'}\right) &\geq \frac{1-r}{1+r} \\ &\quad - \frac{2r(A-B)(1+q)}{(2+A(1+q)r + (1-q)r)(2+B(1+q)r + (1-q)r)}. \end{aligned}$$

By a straightforward computation, we have

$$\begin{aligned} &\Re\left(1 + \frac{z[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]''}{[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]'}\right) \\ &\geq \frac{-Qr^3 + Rr^2 + 4Sr + 4}{4(1+r)(2+A(1+q)r + (1-q)r)(2+B(1+q)r + (1-q)r)}. \end{aligned}$$

Hence, the function $\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)$ is convex in $|z| < r_1$, where r_1 is the smallest positive root in $(0, 1)$ of the equation $Qr^3 - Rr^2 - 4Sr - 4 = 0$. $\square$

If we let $A = 1 - 2\alpha$, $B = -1$, $m = 0$, $r = 2$, $s = 1$, $a_1 = b_1$, $a_2 = q$ and $q \rightarrow 1^-$ in Theorem 3.5 gives the following result obtained by Prajapat [21].

Corollary 3.1. *If $f(z) \in X_t(\alpha)$, then $f(z)$ is convex in $|z| < r_0 = 2 - \sqrt{3}$.*

Conclusion

The motivation of the classes $\mathcal{V}_{q,t}^{\lambda,m}(a_1, b_1; A, B)$ and $k - \mathcal{UCV}_{q,t}^{\lambda,m}(a_1, b_1; A, B)$, was from the study of univalent functions with respect to symmetric points. Many authors introduced and defined various classes of analytic functions with respect to symmetric points. For appropriate choice of parameters involved in the classes $\mathcal{V}_{q,t}^{\lambda,m}(a_1, b_1; A, B)$ and $k - \mathcal{UCV}_{q,t}^{\lambda,m}(a_1, b_1; A, B)$, several well-known and new results can be obtained as special case of our results.

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2D MA parameters identification using higher-order spatial cumulants

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Abstract. In this paper, the almost-sure convergence properties of sample estimates of higher-order spatial statistics are derived. As a practical framework, we address the problem of identification of 2D moving average (MA) models with non-Gaussian errors based on cumulants alone under a nonminimum phase assumption first and on a generalized method of moments approach after. A simulation study verifies the performance of the proposed methods.

Keywords: higher-order spatial statistics, nonminimum phase, 2D FIR, almost sure convergence.

1. Introduction

Most of natural (or real life) signals are signal processing for which the inputs are complex spatial data that exhibit non-Gaussian and / or non-linear behavior or characteristics. They need higher order correlation methods for their analyzing. Higher-order 2D statistics are able to measure periodicity between multiple points separated by given distances. They better reflect geometry than the two-point correlation function (i.e., second-order statistics) (see for example; [6], [10], [29], [16]). Also, the extraction of new properties about the spatial data leads to a significant improvement of the predictive modeling performances. So in several cases, there is necessity to use higher-order spatial cumulants.

However, to take full advantage of the several virtues of experimental spatial cumulants, we need to be able to mistake the theoretical cumulants and the experimental (or sample) ones when dealing with a finite spatial data set, and

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establish a strong convergence between them. For this purpose, we study in this work:

- the almost sure convergence of sample third- and forth-order spatial cumulants. Furthermore, for a spatial linear process we discuss the asymptotic normality of the sample estimates.
- as a practical framework of the proposed theoretical results, some methods for identification of one dimensional Moving Average systems of finite order (FIR MA systems) are extended to the 2-D case i.e., to the 2-D FIR MA system.

Our focus will be on higher-order cumulants based methods under non invertibility assumption. Indeed, recent studies in many science and engineering applications have shown that parameter values are in the nonminimum phase range.

A key finding in these studies is that higher order cumulants are necessary for the identification of nonminimum phase systems, implying that the assumption of Gaussian errors must be abandoned. In the 1-D case, theory and computational techniques of linear prediction have received extensive attention in the system theory and signal processing. However, despite the efforts of many researchers, most of this theory has not been extended to the 2-D case. 2D MA representation for random fields is relevant due to the great popularity of their Wold-type decomposition in the regular homogeneous case. Applications of the Wold-type decomposition and 2D MA modeling include modeling natural textures [12], image segmentation and restoration problems [22], image modeling and retrieval [24] and adaptive processing radar systems [13].

The identification problem is often recasted as one of finding a unique solution to a system of non-linear equations. Higher order cumulants are useful for identification of possibly nonminimum phase systems because the Fourier transform of the k -th order joint cumulants of a k -th order stationary random field X is the k -th order polyspectrum of X : Indeed, provided that the k -th order cumulant of the errors exists and is non-zero for $k \geq 3$, one can recover the phase function from any k -th order spectrum, see [23] Lemma 1, [15], [14] for 1D case and [32] and [33] for the multidimensional one. Hence, two approaches for the identification of non-Gaussian 2D MA processes parameters are proposed. The first approach is based on experimental cumulants alone and without imposing minimum phase. While a generalized method of moments (GMM) is used in the second approach.

Throughout, we take $(X_t)_{t \in \mathbb{Z}^d}$ to be a real-valued k -th-order weakly stationary random field having mean zero. Its k -th-order joint moment is defined for $t, v^1, v^2, \dots, v^{k-1}$ in $\mathbb{Z}^d$ by $r_k(v^1; v^2; \dots; v^{k-1}) = EX_t X_{t+v^1} \dots X_{t+v^{k-1}}$, and its k -th-order joint cumulant is defined for $t, v^1, v^2, \dots, v^{k-1}$ in $\mathbb{Z}^d$ by $C_{kX}(v^1; v^2; \dots; v^{k-1}) = cum \{X_t X_{t+v^1} \dots X_{t+v^{k-1}}, \}$.

Suppose that the stationary random field X_t is observed over the cube of lattice points $t_i = 1, \dots, N$, $i = 1, \dots, d$. Finally, define the lexicographic order $\prec$ as: For any $s = (s_1, s_2), t = (t_1, t_2) \in \mathbb{Z}^2$, we write $s \prec t$ if and only if $[(s_1 < t_1) \vee (s_1 = t_1 \text{ and } s_2 < t_2)]$.

The paper is organized as follows. In Section 2, Almost sure convergence of sample third and fourth spatial cumulants is discussed as well as the asymptotic normality of the sample estimates for a spatial linear process. Section 3 presents two approaches for the identification of 2D nonminimum Finite Impulse Response Moving Average systems (FIR MA systems). A Monte Carlo study is presented in section 4.

2. Asymptotic properties of spatial cumulants

We consider the sample third-order moment (cumulant) $\hat{r}_3(\nu^1; \nu^2)$ of X_t , namely,

$$(2.1) \quad \hat{r}_{3X}(\nu^1; \nu^2) = \frac{1}{N^d} \sum_{\substack{t_i=1 \\ i=1, \dots, d}}^N X_t X_{t+\nu^1} X_{t+\nu^2}.$$

Our basic assumption on X_t is

(A) $C_{4X}(\nu^1; \nu^2; \nu^3) \in L_1(\mathbb{R}^3)$ and $C_{6X}(\nu^1; \nu^2; \nu^3; \nu^4; \nu^5) \in L_1(\mathbb{R}^5)$. Before presenting the almost sure convergence of $\hat{r}_3(\nu^1; \nu^2)$, we prove the following lemma

Lemma 1. *If $(X_t)_{t \in \mathbb{Z}^d}$, $EX_t = 0$, is a real 6-th-order weakly stationary process, and $r_{2X}(\nu)$, $r_{3X}(\nu^1; \nu^2)$, $C_{4X}(\nu^1; \nu^2; \nu^3)$ and $C_{6X}(\nu^1; \nu^2; \nu^3; \nu^4; \nu^5) \in L_1$, then for $t = (t_1, \dots, t_d)$, $\nu^1 = (\nu_1^1, \nu_2^1, \dots, \nu_d^1)$ and $\nu^2 = (\nu_1^2, \nu_2^2, \dots, \nu_d^2)$*

$$Y_t^{(\nu^1, \nu^2)} = X_t X_{t+\nu^1} X_{t+\nu^2} - r_{3X}(\nu^1; \nu^2)$$

is a stationary process, and the spectral density function $f_{Y^{(\nu^1, \nu^2)}}(\lambda)$ satisfies

$$\sup_{\lambda \in \mathbb{Z}^d} f_{Y^{(\nu^1, \nu^2)}}(\lambda) < \infty.$$

Proof. See Appendix A. □

The first important result of this section is the almost sure convergence of sample third-order moment (cumulant).

Theorem 1. *If $(X_t)_{t \in \mathbb{Z}^d}$, $EX_t = 0$, is a real 6-th-order weakly stationary process, and $r_{2X}(\nu)$, $r_{3X}(\nu^1; \nu^2)$, $C_{4X}(\nu^1; \nu^2; \nu^3)$ and $C_{6X}(\nu^1; \nu^2; \nu^3; \nu^4; \nu^5) \in L_1$, then for $t = (t_1, \dots, t_d)$, $\nu^1 = (\nu_1^1, \nu_2^1, \dots, \nu_d^1)$ and $\nu^2 = (\nu_1^2, \nu_2^2, \dots, \nu_d^2)$*
(1)

$$(2.2) \quad E \left\{ \frac{1}{N^{2d}} \left| \sum_{\substack{t_i=1 \\ i=1, \dots, d}}^N X_t X_{t+\nu^1} X_{t+\nu^2} - r_3(\nu^1; \nu^2) \right|^2 \right\} = O\left(\frac{1}{N^d}\right)$$

(2) for $k > \frac{3}{4}$,

$$(2.3) \quad \frac{1}{N^d} \left| \sum_{\substack{t_i=1 \\ i=1,\dots,d}}^N X_{\mathbf{t}} X_{\mathbf{t}+\nu^1} X_{\mathbf{t}+\nu^2} - r_3(\nu^1; \nu^2) \right| \stackrel{a.s.}{=} o\left(\frac{1}{N^{d(1-k)}}\right),$$

where a.s. represents almost sure convergence.

Proof. See Appendix B. $\square$

Remark 1. It is not difficult to generalize Theorem 1 to k -th-order moment and cumulant estimates, provided that we take $Y_{\mathbf{t}}^{(\nu^1; \dots; \nu^{k-1})} = X_{\mathbf{t}} X_{\mathbf{t}+\nu^1} \dots X_{\mathbf{t}+\nu^{k-1}} - r_{kX}(\nu^1; \dots; \nu^{k-1})$, and that the cumulants up to $2k$ -th-order are absolutely summable.

Now, we show the asymptotic normality of $\hat{r}_{3X}(\nu^1, \nu^2)$ for spatial linear processes.

Theorem 2. Let $X_{\mathbf{t}} = \sum_{\mathbf{j} \in \mathbb{Z}^d} \psi_{\mathbf{j}} \varepsilon_{\mathbf{t}-\mathbf{j}}$, where $\varepsilon_{\mathbf{j}}$ are independent and identically distributed with $E(\varepsilon_{\mathbf{j}}) = 0$, $E(\varepsilon_{\mathbf{j}}^2) = 1$, $E(\varepsilon_{\mathbf{j}}^3) = \gamma_3$, $E(\varepsilon_{\mathbf{j}}^6) < \infty$ and $\sum_{\mathbf{j} \in \mathbb{Z}^d} |\psi_{\mathbf{j}}| < \infty$, then

$$(2.4) \quad N^{\frac{d}{2}} (\hat{r}_{3X}(\nu^1; \nu^2) - r_{3X}(\nu^1; \nu^2)) \rightarrow \mathcal{N}(0, \sigma_X^2(\nu^1; \nu^2)),$$

where $\sigma_X^2(\nu^1; \nu^2) = \sum_{\substack{\tau_i=-\infty \\ i=1,\dots,d}}^{+\infty} \{C_{6X}(\nu^1; \nu^2; \tau; \tau + \nu^1; \tau + \nu^2) + \{r_{2X}(\nu^1)C_{4X}(\tau - \nu^2; \tau + \nu^1 - \nu^2; \nu^2)\}_{15} + \{r_{2X}(\nu^1)r_{2X}(\tau - \nu^2)r_{2X}(\nu^2 - \nu^1)\}_{15}\}.$

Proof. See Appendix C. $\square$

3. 2D FIR moving average identification

In the sequel, we propose two approaches for the identification of non-Gaussian 2D MA processes parameters. The first approach is based on experimental cumulants alone and without imposing minimum phase. While a generalized method of moments (GMM) is used in the second approach.

3.1 2D FIR MA identification using third order cumulants alone

In this subsection, we address the problem of identification of 2D moving average (MA) models with non-Gaussian errors based on experimental cumulants alone and without imposing minimum phase. The proposed algorithm is an extension to the 2D case of the one proposed by authors in [40] in the 1D case.

Consider the 2-D finite impulse response (FIR) signal process given by the following equation :

$$(3.1) \quad x(n, m) = \sum_{i=0}^{q_1} \sum_{j=0}^{q_2} b(i, j) w(n - i, m - j),$$

where $b(i, j)$, $i = 1, \dots, q_1$, $j = 1, \dots, q_2$ are unknown MA coefficients, and the signal $x(n, m)$ is observed in additive noise :

$$(3.2) \quad y(n, m) = x(n, m) + v(n, m).$$

The following conditions are assumed to hold:

- i The driving noise sequence $w(n, m)$ is unobservable and is a zero-mean i.i.d. non-Gaussian process with at least one cumulant $0 < |\gamma_{m,w}| < \infty$, $m > 2$.
- ii The system is of nonminimum phase (i.e. $x(n, m)$ is not invertible).
- iii The additive noise $v(n, m)$ is a Gaussian ARMA process with unknown power spectrum and is independent of the input $w(n, m)$ and hence of the output $x(n, m)$.

To obtain conditions of minimum phase for model (3.1), we need to write it in a polynomial form as follows:

Let $\mathbf{B} = (B_1, B_2)$ the back shift operator, for any $\mathbf{i} = (i_1, i_2) \in \mathbb{Z}^2$, $\mathbf{z} = (z_1, z_2) \in \mathbb{C}^2$, write $B^{\mathbf{i}}X(t_1, t_2) = B_1^{i_1}B_2^{i_2}X(t_1, t_2) = X(t_1 - i_1, t_2 - i_2)$ and $\mathbf{z}^{\mathbf{i}} = z_1^{i_1}z_2^{i_2}$. Define $\Theta(\mathbf{z}) = \sum_{i=0}^{q_1} \sum_{j=0}^{q_2} b(i, j) z_1^i z_2^j$, then the Model (3.1) can be written in a shorthand notation as:

$$x(n, m) = \Theta(\mathbf{B})w(n, m).$$

(3.1) is of minimum phase (or invertible) if the following condition is satisfied:

$\mathbf{C}_0$. none of the roots of $\Theta(z_1, z_2)$ lie within the closed unit polydisc ($|z_1| \leq 1, |z_2| \leq 1$).

If we consider a semi-infinite prediction support from (3.1) with known order (q_1, q_2) , we can derive a relationship between the MA parameters $b(i, j)$, $i = 0, \dots, q_1$, $j = 0, \dots, q_2$, by arranging its terms by the lexicographic order, and some functions of the statistics. Indeed, it's not difficult to see that under condition (iii),

$$(3.3) \quad C_{ky}(\nu^1; \dots; \nu^{k-1}) = C_{kx}(\nu^1; \dots; \nu^{k-1}),$$

and that the k th order-cumulants are related to the impulse response by

$$(3.4) \quad \begin{aligned} C_{ky}(\nu^1; \dots; \nu^{k-1}) &= \gamma_{k,w} \sum_{i=0}^{q_1} \sum_{j=0}^{q_2} h(i, j) \times h[(i, j) + \nu^1] \\ &\times \dots \times h[(i, j) + \nu^{k-1}], \end{aligned}$$

with $\nu^1 = (t_1, t_2)$, $\nu^2 = \dots = \nu^{k-1} = \mathbf{0}$ (where $\mathbf{0} = (0, 0)$) and $k = 3$ in (3.4) we have,

$$(3.5) \quad C_{3y}((t_1, t_2); \mathbf{0}) = \gamma_{3,w} \sum_{i=0}^{q_1} \sum_{j=0}^{q_2} h^2(i, j) h(i + t_1, j + t_2).$$

We substitute $h(i + t_1, j + t_2) = b(i + t_1, j + t_2)$ in (3.5) to get a new normal equation that is,

$$(3.6) \quad C_{3y}((t_1, t_2), \mathbf{0}) = \gamma_{3,w} \sum_{i=0}^{q_1} \sum_{j=0}^{q_2} b(i, j) \times \\ h^2(i - t_1, j - t_2), \text{ for any } (t_1, t_2),$$

where we have used $h(i, j) = b(i, j) = 0$ for $i < 0$ or $j < 0$ or $i > q_1$ or $j > q_2$. Recall [32] that

$$(3.7) \quad h(i, j) = b(i, j) = \frac{C_{3y}((q_1, q_2); (i, j))}{C_{3y}((q_1, q_2); \mathbf{0})}.$$

Substituting (3.7) into (3.6) yields

$$(3.8) \quad \gamma_{3,w} \sum_{i=0}^{q_1} \sum_{j=0}^{q_2} b(i, j) C_{3y}^2((q_1, q_2); (i - t_1, j - t_2)) \\ = C_{3y}((t_1, t_2); \mathbf{0}) \times C_{3y}^2((q_1, q_2); \mathbf{0}), \text{ for any } (t_1, t_2).$$

We refer to (3.8) as the spatial cumulant-based normal equation for pure 2D MA model. To remove $\gamma_{3,w}$ from equation (3.8) we use again (3.5) and (3.7) to get

$$(3.9) \quad \gamma_{3,w} = \frac{C_{3y}^2((q_1, q_2); \mathbf{0})}{C_{3y}((q_1, q_2); (q_1, q_2))},$$

where we have used $b(0, 0) = 1$ and $C_{3y}((q_1, q_2); (t_1, t_2)) = 0$ for $t_1 < 0$ or $t_2 < 0$ or $t_1 > q_1$ or $t_2 > q_2$, now substituting (3.9) into (3.8), we obtain

$$(3.10) \quad \sum_{i=0}^{q_1} \sum_{j=0}^{q_2} b(i, j) C_{3y}^2((q_1, q_2); (i - t_1, j - t_2)) \\ = C_{3y}((t_1, t_2); \mathbf{0}) \times C_{3y}((q_1, q_2); (q_1, q_2)).$$

If we arrange vectors and matrices in the lexicographic order, concatenating the normal equation (3.10) for $-q_1 \leq t_1 \leq q_1$ and $-q_2 \leq t_2 \leq q_2$ we obtain the following single matricial equation

$$(3.11) \quad \begin{bmatrix} \mathbf{C}_1 \\ \mathbf{C}_2 \end{bmatrix} \mathbf{b} = \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \end{bmatrix},$$

(see Appendix D for the expressions of $\mathbf{C}_1$, $\mathbf{C}_2$, $\mathbf{c}_1$ and $\mathbf{c}_2$)

The following theorem guarantees the identification for 2D FIR MA parameters (without imposing minimum phase) by solving (3.11).

Theorem 3. Under the conditions i)-iii), given the true output cumulants $C_{3y}(\mathbf{q}; \mathbf{t})$ and $C_{3y}(\mathbf{r}; \mathbf{0})$ of model (3.1) and (3.2) for lags $(0 \leq t_1 \leq q_1 \text{ and } 0 \leq t_2 \leq q_2)$ and $(-q_1 \leq \tau_1 \leq q_1 \text{ and } -q_2 \leq \tau_2 \leq q_2)$ respectively, the $(q_1+1)(q_2+1)$ unknowns $b(i, j), i = 0, \dots, q_1, j = 0, \dots, q_2$ can be recovered uniquely by the solution to (3.11).

Proof. The proof is quite similar to the proof of Theorem 1 of [40] in the 1D FIR MA case and hence omitted. $\square$

In practice, only sample estimates of $C_{3y}((q_1, q_2); (t_1, t_2))$ and $C_{3y}((\tau_1, \tau_2); \mathbf{0})$ are available from the finite sample $\{y(n, m), 1 \leq n \leq N_1, 1 \leq m \leq N_2\}$. However, it may be possible to replace the theoretical cumulants by experimental ones when solving (3.11) for $\mathbf{b}$ which would help to identify it. So, the algorithm for 2D MA parameter identification is summarized in Algorithm I as follows.

Algorithm I.

Step 1. From the output $y(n, m), 1 \leq n \leq N_1, 1 \leq m \leq N_2$, form the estimates

$$\hat{C}_{3y}((q_1, q_2); (t_1, t_2)) = \frac{1}{N_1 N_2} \sum_{i=1}^{N_1-q_1} \sum_{j=1}^{N_2-q_2} y(i, j) \times y(i+q_1, j+q_2) y(i+t_1, j+t_2);$$

for $0 \leq t_1 \leq q_1$ and $0 \leq t_2 \leq q_2$,

and

$$\hat{C}_{3y}((\tau_1, \tau_2); \mathbf{0}) = \frac{1}{N_1 N_2} \sum_{i=\max(1, -\tau_1)}^{\min(N_1, N_1-\tau_1)} \sum_{j=\max(1, -\tau_2)}^{\min(N_2, N_2-\tau_2)} y^2(i, j) \times y(i+\tau_1, j+\tau_2),$$

for $-q_1 \leq \tau_1 \leq q_1$ and $-q_2 \leq \tau_2 \leq q_2$.

Step 2. Using the estimates $\hat{C}_{3y}((q_1, q_2); (t_1, t_2))$ and $\hat{C}_{3y}((\tau_1, \tau_2); \mathbf{0})$, iteratively solve the system (3.11) to obtain the estimates $\hat{b}(i, j)$ of $b(i, j)$ $i = 0, \dots, q_1, j = 0, \dots, q_2$.

Step 3. Finally to satisfy the normalized condition $b(0, 0) = 1$, we take $\hat{b}_1(i, j) = \frac{\hat{b}(i, j)}{\hat{b}(0, 0)}$ as the final estimate of $b(i, j)$ $0 \leq i, j \leq 1, (i, j) \neq (0, 0)$.

Remark 2. Theorem 1 and Theorem 2 established the almost sure convergence and the asymptotic normality of the estimates $\hat{C}_{3y}((q_1, q_2); (t_1, t_2))$ and $\hat{C}_{3y}((\tau_1, \tau_2); \mathbf{0})$.

This strong consistency guarantees that estimating the parameters $b(i, j), i = 0, \dots, q_1, j = 0, \dots, q_2$ results in strong consistent estimators provided that the output process $y(i, j)$ satisfies the stationary conditions of Theorem 1.

Indeed, both 2D MA parameters are identified as measurable functions of $C_{3y}((q_1, q_2), (t_1, t_2))$ and $C_{3y}((\tau_1, \tau_2); \mathbf{0})$, and $\hat{C}_{3y}((q_1, q_2); (t_1, t_2)), \hat{C}_{3y}((\tau_1, \tau_2); \mathbf{0})$ are consistent estimators of $C_{3y}((q_1, q_2); (t_1, t_2)), C_{3y}((\tau_1, \tau_2); \mathbf{0})$, respectively. Hence the 2D MA parameters will themselves be consistent. Also, the asymptotic distribution of our parameters estimators will be Gaussian, because they

are functions of the asymptotically Gaussian $\hat{C}_{3y}((q_1, q_2); (t_1, t_2))$, $\hat{C}_{3y}((\tau_1, \tau_2); \mathbf{0})$ estimators.

Remark 3. In practical applications, true order (q_1, q_2) is not known a priori. In the proof of Theorem 3 it is shown that the rank of the cumulant matrix $\begin{bmatrix} \mathbf{C}_1 \\ \mathbf{C}_2 \end{bmatrix}$ is $R = (q_1 + 1)(q_2 + 1)$. Hence, (q_1, q_2) can be determined using the singular value decomposition (SVD) by testing for

$$(3.12) \quad \hat{C}_{3y}^R((q_1, q_2); \mathbf{0}) \neq 0, \hat{C}_{3y}^{\kappa+1}((k_1, k_2); \mathbf{0}) \approx 0,$$

where $\kappa = (k_1 + 1)(k_2 + 1)$. See [39], for details, of the SVD based on (3.12).

3.2 2D FIR MA identification using GMM estimators

The results in Subsection 2.2 suggest a second estimation within the method of moments framework. In the case of 2D FIR MA(1) model given by (3.1) and (3.2), we can derive orthogonality conditions to estimate

$$\gamma = \left(\frac{b^2(0,0)}{b(1,1)}, \frac{b^2(0,1)}{b(1,1)}, \frac{b^2(1,0)}{b(1,1)}, b(1,1) \right)'$$

from equations (3.4), (3.7) and (3.9). Indeed, for $k = 3$, $\nu^1 = (-t_1, -t_2)$ and $\nu^2 = \mathbf{0}$ equation (3.4) gives,

$$C_{3y}((-t_1, -t_2); \mathbf{0}) = \gamma_{3,w} \sum_{i=0}^1 \sum_{j=0}^1 h^2(i, j) h(i - t_1, j - t_2),$$

since $C_{3y}((-t_1, -t_2); \mathbf{0}) = C_{3y}((t_1, t_2); (t_1, t_2))$, and using (3.7) and (3.9) we get,

$$\begin{aligned} C_{3y}((t_1, t_2); (t_1, t_2)) &= \frac{b^2(0,0)}{b(1,1)} C_{3y}((q_1, q_2); (-t_1, -t_2)) \\ &\quad + \frac{b^2(0,1)}{b(1,1)} C_{3y}((q_1, q_2); (-t_1, 1 - t_2)) \\ &\quad + \frac{b^2(1,0)}{b(1,1)} C_{3y}((q_1, q_2); (1 - t_1, -t_2)) \\ &\quad + b(1,1) C_{3y}((q_1, q_2); (1 - t_1, 1 - t_2)), \end{aligned}$$

for $0 \leq t_1 \leq 1$ and $0 \leq t_2 \leq 1$. This represents a system of four orthogonality conditions given by

$$\begin{cases} h(\gamma, (i, j)) = A(i, j) \times \gamma - H(i, j), \\ E(h(\gamma, (i, j))) = 0 \end{cases}$$

where

$$A(i, j) = y(i, j)y(i+1, j+1) \\ \times \begin{pmatrix} y(i, j) & y(i, j+1) & y(i+1, j) & y(i+1, j+1) \\ y(i, j-1) & y(i, j) & y(i+1, j-1) & y(i+1, j) \\ y(i-1, j) & y(i-1, j+1) & y(i, j) & y(i, j+1) \\ y(i-1, j-1) & y(i-1, j) & y(i, j-1) & y(i, j) \end{pmatrix},$$

and

$$H(i, j) = y(i, j) \times \begin{pmatrix} y^2(i, j) \\ y^2(i, j+1) \\ y^2(i+1, j) \\ y^2(i+1, j+1) \end{pmatrix}.$$

For given data $\{y(n, m), 1 \leq n \leq N_1, 1 \leq m \leq N_2\}$, define $g_{\mathbf{N}}(\gamma, (i, j)) = \frac{1}{N_1 N_2} \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} h(\gamma, (i, j))$, where $\mathbf{N} = (N_1, N_2)$, and assume that γ_0 , the true value of γ , is in the interior of a compact parameter space Γ . The GMM estimator of γ is then defined by solving the minimization problem,

$$(3.13) \quad \hat{\gamma}_{\mathbf{N}}^{GMM} = \arg \min_{\gamma \in \Omega} [g_{\mathbf{N}}(\gamma, (i, j))]^T W_{\mathbf{N}} [g_{\mathbf{N}}(\gamma, (i, j))],$$

where the weight matrix $W_{\mathbf{N}}$ is positive definite. Since there are four parameters for four conditions of orthogonality, the system is exactly identified and the GMM estimator $\hat{\gamma}_{\mathbf{N}}^{GMM}$ of γ based on the $N_1 N_2$ observations is defined by the equation $g_{\mathbf{N}}(\hat{\gamma}_{\mathbf{N}}^{GMM}, (i, j)) = 0$.

Hence, we just solve the system $\frac{1}{N_1 N_2} \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} h(\hat{\gamma}_{\mathbf{N}}^{GMM}, (i, j)) = 0$, that can be rewritten as $\left[\sum_{i=1}^{N_1} \sum_{j=1}^{N_2} A(i, j) \right] \times \hat{\gamma}_{\mathbf{N}}^{GMM} = \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} H(i, j)$, consequently

$$(3.14) \quad \hat{\gamma}_{\mathbf{N}}^{GMM} = \left[\sum_{i=1}^{N_1} \sum_{j=1}^{N_2} A(i, j) \right]^{-1} \times \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} H(i, j).$$

The consistency and asymptotic normality of $\hat{\gamma}_{\mathbf{N}}^{GMM}$ can be obtained after the check of some regularity conditions. First, a good way for the weighting matrix $W_{\mathbf{N}}$ to solve the minimization problem (3.13) is by putting $W_{\mathbf{N}} \equiv \tilde{W}_{\mathbf{N}} = \left[\frac{1}{N_1 N_2} \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} h(\gamma_{\mathbf{N}}^{(1)}, (i, j)) h'(\gamma_{\mathbf{N}}^{(1)}, (i, j)) \right]^{-1}$ where

$$\gamma_{\mathbf{N}}^{(1)} = \left(\frac{\hat{b}_1^2(0, 0)}{\hat{b}_1(1, 1)}, \frac{\hat{b}_1^2(0, 1)}{\hat{b}_1(1, 1)}, \frac{\hat{b}_1^2(1, 0)}{\hat{b}_1(1, 1)}, \hat{b}_1(1, 1) \right)'$$

with $\hat{b}_1(0, 0)$, $\hat{b}_1(0, 1)$, $\hat{b}_1(1, 0)$, $\hat{b}_1(1, 1)$ being the consistent estimates of $b(0, 0)$, $b(0, 1)$, $b(1, 0)$, $b(1, 1)$, respectively, derived in Algorithm I. Since $h(\gamma, (i, j))$ is a measurable function of $X(i, j)$, $0 \leq i, j \leq 1$ and γ is assumed to be such that conditions in Theorem 3 hold, we get $\tilde{W}_{\mathbf{N}} \xrightarrow{P} \tilde{W} = [Eh(\gamma_0, (i, j)) h'(\gamma_0, (i, j))]^{-1}$, it is clear that $\tilde{W}_{\mathbf{N}}$ and $\tilde{W}$ are both definite and positive. Hence, the consistency and asymptotic normality of $\hat{\gamma}_{\mathbf{N}}^{GMM}$ can be established in the following theorem:

Theorem 4. Consider a 2D FIR MA(1) model and suppose that the conditions in Theorem 3 hold. Then,

$$\sqrt{N_1 N_2} (\gamma_{\mathbf{N}}^{GMM} - \gamma_0) \overset{\mathcal{L}}{\rightsquigarrow} \mathcal{N}(\mathbf{0}, \Sigma),$$

where $\Sigma = V^{-1} \Omega V^{-1}$ with $V = E(A(i, j))$ and $\Omega = \tilde{W}^{-1}$.

Proof. In view of (3.14) we can write,

$$(3.15) \quad \begin{aligned} (\gamma_{\mathbf{N}}^{GMM} - \gamma_0) &= \left[\frac{1}{\sqrt{N_1 N_2}} \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} A(i, j) \right]^{-1} \\ &\times \frac{1}{\sqrt{N_1 N_2}} \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} h(\gamma_0, (i, j)), \end{aligned}$$

so it is not difficult to show that, under the conditions of Theorem 3, $h(\gamma_0, (i, j))$ is a 2D martingale difference with finite variance $\Omega = \tilde{W}^{-1}$. Hence, by Huang martingale difference central limit theorem $\frac{1}{\sqrt{N_1 N_2}} \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} h(\gamma_0, (i, j))$ converges to the multivariate normal distribution with mean zero and covariance matrix Ω . From (3.15) and by Cramer-Wold device, the result follows. $\square$

Now, to derive the estimate of the parameter of interest $\mathbf{b}$ from $\gamma_{\mathbf{N}}^{GMM}$, notice first that $\gamma_{\mathbf{N}}^{GMM}(i)$ and $\gamma_{\mathbf{N}}^{GMM}(4)$ $i = 1, 2, 3$ have the same sign so we simply compute directly $\hat{b}_{GMM}(1, 1) = \gamma_{\mathbf{N}}^{GMM}(4)$; $|\hat{b}_{GMM}(0, 0)| = [\gamma_{\mathbf{N}}^{GMM}(1)\gamma_{\mathbf{N}}^{GMM}(4)]^{\frac{1}{2}}$; $|\hat{b}_{GMM}(0, 1)| = [\gamma_{\mathbf{N}}^{GMM}(2)\gamma_{\mathbf{N}}^{GMM}(4)]^{\frac{1}{2}}$ and $|\hat{b}_{GMM}(1, 0)| = [\gamma_{\mathbf{N}}^{GMM}(3)\gamma_{\mathbf{N}}^{GMM}(4)]^{\frac{1}{2}}$ by keeping the sign of the estimates of $b(0, 0)$, $b(0, 1)$ and $b(1, 0)$ given in Algorithm I. Finally to satisfy the normalized condition $b(0, 0) = 1$ we take $\frac{\hat{b}_{GMM}(i, j)}{\hat{b}_{GMM}(0, 0)}$ as the final estimate of $b(i, j)$ $0 \leq i, j \leq 1$, $(i, j) \neq (0, 0)$.

3.3 Simulation results

In this section, we evaluate the proximity of the parameter estimators of $\mathbf{b}$ to their expected values by a Monte Carlo study and compare the performances of the two proposed estimators. This study allows to check and to corroborate, first the efficiency of the previous identification algorithm and second the validity of the proposed theoretical results presented in the previous subsections. Consider the 2D MA model of first order,

$$(4.1) \quad \begin{aligned} x(n, m) &= b_1 w(n, m-1) + b_2 w(n-1, m) \\ &\quad + b_3 w(n, m-2) + w(n, m), \end{aligned}$$

The additive noise $v(n, m)$ is a Gaussian ARMA process given by

$$\begin{aligned} v(n, m) &= 0.3v(n, m-1) - 0.08v(n-2, m-1) \\ &\quad + 0.54 + e(n, m) - 1.25e(n, m-1), \end{aligned}$$

the signal $x(n, m)$ is observed in additive noise:

$$y(n, m) = x(n, m) + v(n, m),$$

where $e(n, m) \sim \mathcal{N}(0, 0.5)$ and $w(n, m)$ are i.i.d $(0, 1)$ and follows a generalized lambda distribution (GLD) with a skewness $\kappa_3 = 0$ and a kurtosis $\kappa_4 = 3$. The true values of the parameters are and $(b_1 = 1, b_2 = 0.25, b_3 = -1, 65)$ (nonminimum phase case) and $(b_1 = 0, 25, b_2 = 0, 35, b_3 = 0, 15)$ (minimum phase case), Proposition 1 in [2] gives a convenient set of conditions on the parameters b_1, b_2 and b_3 to satisfy $\mathbf{C}_0$). Notice that in the proposed identification algorithm the minimum phase is not imposed.

Furthermore, errors with nonGaussian features are the key to identification. It is why we choose a GLD distribution errors considered in [28] that can be simulated in a flexible fashion and can accommodate a wide range of values for skewness and excess kurtosis parameters. Table 1 and 2 present the averages mean columns, mean square errors columns, kurtosis columns and skewness columns of b_1, b_2 and b_3 estimators over 500 Monte Carlo replications for the sample sizes $(100, 100)$, $(500, 500)$ and $(1000, 1000)$ in both the minimum and nonminimum phase respectively. The average values of the estimator given by Algorithm I ($\hat{b}_i$) and the GMM estimator ($\hat{b}_i^{gmm}$), $i = 1, 2, 3$, in the two cases, over the 500 replications were always close to the true values for all grid sizes. Furthermore and as expected, the increase in the sample size substantially improves the results in terms of bias reduction. Also, the averages of the coefficients of kurtosis and skweness over the 500 replications tends to be closer to the reference values 3 and 0 respectively when the sample size is increasing for the two estimators. The observations from Table 1 and 2 confirm the consistency and the asymptotic normality of the proposed estimators as indicated in Remark 2 and Theorem 4. They also suggest that the GMM estimator produces estimates of $\mathbf{b}$ that are somewhat more precise than those given by Algorithm I. However, despite the reasonable computational burden of the GMM estimator of $\mathbf{b}$, a major advantage of the estimator given by Algorithm I is that it doesn't require a preliminary consistent estimator.

Table 1: Sample distribution of the estimators $\hat{b}_1$, $\hat{b}_2$ and $\hat{b}_3$ and $\hat{b}_1^{gmm}$, $\hat{b}_2^{gmm}$ and $\hat{b}_3^{gmm}$ (between brackets) in the minimum phase case with $\kappa_3 = 0$ and $\kappa_4 = 3$.

	$N = (100, 100)$		
$true\ b_i$	0.25	0.35	0.15
	$\hat{b}_1$ ($\hat{b}_1^{gmm}$)	$\hat{b}_2$ ($\hat{b}_2^{gmm}$)	$\hat{b}_3$ ($\hat{b}_3^{gmm}$)
<i>mean</i>	0.2501 (0.2498)	0.3502 (0.3501)	0.150 (0.153)
<i>rmse</i>	0.18×10^{-4} (0.12×10^{-4})	0.0029 (0.002)	0.0038 (0.0025)
<i>kurt</i>	2.5910 (3.2985)	3.1432 (3.176)	2.879 (2.764)
<i>skew</i>	0.0151 (0.0099)	0.0092 (0.0111)	0.0098 (0.0089)

	$N =$ (500, 500)		
<i>true b_i</i>	0.25	0.35	0.15
	$\hat{b}_1$ ($\hat{b}_1^{gmm}$)	$\hat{b}_2$ ($\hat{b}_2^{gmm}$)	$\hat{b}_3$ ($\hat{b}_3^{gmm}$)
<i>mean</i>	0.25009 (0.2500)	0.3501 (0.3497)	0.15006 (0.1502)
<i>rmse</i>	0.119×10^{-4} (0.099×10^{-4})	0.0015 (0.0014)	0.0029 (0.0025)
<i>kurt</i>	2.7910 (3.321)	2.9143 (3.156)	3.1787 (2.8691)
<i>skew</i>	0.0105 (0.0098)	0.0100 (0.0113)	0.0091 (0.0079)

	$N =$ (1000, 1000)		
<i>true b_i</i>	0.25	0.35	0.15
	$\hat{b}_1$ ($\hat{b}_1^{gmm}$)	$\hat{b}_2$ ($\hat{b}_2^{gmm}$)	$\hat{b}_3$ ($\hat{b}_3^{gmm}$)
<i>mean</i>	0.25001 (0.2489)	0.35005 (0.35005)	0.150 (0.1499)
<i>rmse</i>	0.119×10^{-4} (0.074×10^{-4})	0.0015 (0.0013)	0.0029 (0.0025)
<i>kurt</i>	2.7910 (3.146)	2.9143 (3.09)	3.1787 (3.234)
<i>skew</i>	0.0105 (0.0098)	0.0100 (0.0054)	0.0091 (0.0123)

Table 2: Sample distribution of the estimators $\hat{b}_1$, $\hat{b}_2$ and $\hat{b}_3$ and $\hat{b}_1^{gmm}$, $\hat{b}_2^{gmm}$ and $\hat{b}_3^{gmm}$ (between brackets) in the nonminimum case with $\kappa_3 = 0$ and $\kappa_4 = 3$.

	$N =$ (100, 100)		
<i>true b_i</i>	1	0.25	-1.65
	$\hat{b}_1$ ($\hat{b}_1^{gmm}$)	$\hat{b}_2$ ($\hat{b}_2^{gmm}$)	$\hat{b}_3$ ($\hat{b}_3^{gmm}$)
<i>mean</i>	1.0003 (1.0005)	0.2499 (0.2503)	-1.6501 (-1.6498)
<i>rmse</i>	0.9229×10^{-4} (0.9×10^{-4})	0.00218 (0.002)	0.00079 (0.00062)
<i>kurt</i>	3.2110 (2.763)	2.8942 (3.245)	3.1787 (3.0985)
<i>skew</i>	0.0099 (0.0134)	0.0210 (0.0125)	0.0081 (0.0099)

	$N =$ (500, 500)		
<i>true b_i</i>	1	0.25	-1.65
	$\hat{b}_1$ ($\hat{b}_1^{gmm}$)	$\hat{b}_2$ ($\hat{b}_2^{gmm}$)	$\hat{b}_3$ ($\hat{b}_3^{gmm}$)
<i>mean</i>	1 (0.999)	0.2499 (0.2508)	-1.6502 (-1.65)
<i>rmse</i>	0.8589×10^{-4} (0.812×10^{-4})	0.00191 (0.0019)	0.00059 (0.0005)
<i>kurt</i>	2.9902 (3.146)	2.6942 (2.885)	3.0840 (3.1328)
<i>skew</i>	0.0089 (0.0121)	0.0110 (0.0085)	0.0071 (0.0099)

	$N =$ (1000, 1000)		
<i>true b_i</i>	1	0.25	-1.65
	$\hat{b}_1$ ($\hat{b}_1^{gmm}$)	$\hat{b}_2$ ($\hat{b}_2^{gmm}$)	$\hat{b}_3$ ($\hat{b}_3^{gmm}$)
<i>mean</i>	1 (1)	0.25005 (0.2481)	-1.65004 (-1.65)
<i>rmse</i>	0.8143×10^{-4} (0.835×10^{-4})	0.00132 (0.0011)	0.00055 (0.00043)
<i>kurt</i>	3.002 (3.0904)	2.9941 (3.281)	3.1840 (2.871)
<i>skew</i>	0.0009 (0.0012)	0.0120 (0.0088)	0.0069 (0.005)

4. Conclusion

This paper presented new approaches to take into account complex non-linear and /or non-Gaussian signal processing data. The originality of this work was the investigation of asymptotic properties of high-order statistics in a spatial context. Convergence properties of third- and fourth-order moments and cumulants estimates have been described and proven. The study also presented new ways to identify a linear 2-D nonminimum phase system using higher order cumulants alone and with GMM estimators. A Monte Carlo study has been conducted.

Appendix A (proof of Lemma 1)

Proof. It is obvious that $Y_{\mathbf{t}}^{(\nu^1, \nu^2)}$ is a stationary process. Put for all $\tau \in \mathbb{Z}^d$, $R_{Y^{(\nu^1, \nu^2)}}(\tau) = EY_{\mathbf{t}}^{(\nu^1, \nu^2)}Y_{\mathbf{t}+\tau}^{(\nu^1, \nu^2)} = r_{6X}(\nu^1, \nu^2, \tau, \tau + \nu^1, \tau + \nu^2) - [r_{3X}(\nu^1, \nu^2)]^2$. Using the conversion relationship between higher-order moments and cumulants we have (see proof of Lemma1 of [18])

$$\begin{aligned} R_{Y^{(\nu^1, \nu^2)}}(\tau) &= C_{6X}(\nu^1, \nu^2, \tau, \tau + \nu^1, \tau + \nu^2) \\ &\quad + \{r_{2X}(\nu^1) C_{4X}(\tau - \nu^2, \tau + \nu^1 - \nu^2, \nu^2)\}_{15} \\ &\quad + \{r_{2X}(\nu^1) r_{2X}(\tau - \nu^2) r_{2X}(\nu^2 - \nu^1)\}_{15}, \end{aligned}$$

where the notation $\{.\}_j$ denotes the sum of all j different terms obtained by interchanging the arguments of the terms in brackets, and by the absolutely summable properties of $r_{2X}(\nu)$, $r_{3X}(\nu^1, \nu^2)$, $C_{4X}(\nu^1, \nu^2, \nu^3)$ and $C_{6X}(\nu^1, \nu^2, \nu^3, \nu^4, \nu^5)$, we have

$$\sum_{\tau \in \mathbb{Z}^d} |R_{Y^{(\nu^1, \nu^2)}}(\tau)| < +\infty.$$

If $f_{Y^{(\nu^1, \nu^2)}}(\cdot)$ is the spectral density function of $Y_{\mathbf{t}}^{(\nu^1, \nu^2)}$, by spectral theory (see [17]), $f_{Y^{(\nu^1, \nu^2)}}(\lambda)$ has the representation

$$f_{Y^{(\nu^1, \nu^2)}}(\lambda) = \frac{1}{(2\pi)^d} \sum_{\mathbf{n} \in \mathbb{Z}^d} R_{Y^{(\nu^1, \nu^2)}}(\mathbf{n}) e^{-i\langle \lambda, \mathbf{n} \rangle},$$

where $\langle \lambda, \mathbf{n} \rangle = \sum_{i=1}^d \lambda_i n_i$, hence

$$\sup_{\lambda \in \mathbb{Z}^d} f_{Y^{(\nu^1, \nu^2)}}(\lambda) \leq \frac{1}{(2\pi)^d} \sum_{\mathbf{n} \in \mathbb{Z}^d} |R_{Y^{(\nu^1, \nu^2)}}(\mathbf{n})| < +\infty. \quad \square$$

Appendix B (proof of Theorem 1)

Proof. By spectral theory (see [17]), we have

$$R_{Y^{(\nu^1, \nu^2)}}(\tau) = \int_{[0, 2\pi]^d} e^{i\langle \lambda, \tau \rangle} f_{Y^{(\nu^1, \nu^2)}}(\lambda) d\lambda,$$

furthermore

$$\begin{aligned}
& E \left\{ \left| \sum_{\substack{t_i=1 \\ i=1,\dots,d}}^N X_{\mathbf{t}} X_{\mathbf{t}+\nu^1} X_{\mathbf{t}+\nu^2} - r_3(\nu^1, \nu^2) \right|^2 \right\} \\
&= E \left\{ \left| \sum_{\substack{t_i=1 \\ i=1,\dots,d}}^N Y_{\mathbf{t}}^{(\nu^1, \nu^2)} \right|^2 \right\} \\
&= E \left\{ \sum_{\substack{t_i, s_i=1 \\ i=1,\dots,d}}^N Y_{\mathbf{t}}^{(\nu^1, \nu^2)} Y_{\mathbf{s}}^{(\nu^1, \nu^2)} \right\} \\
&= \sum_{\substack{t_i, s_i=1 \\ i=1,\dots,d}}^N R_{Y^{(\nu^1, \nu^2)}}(\mathbf{t} - \mathbf{s}),
\end{aligned}$$

hence

$$\begin{aligned}
& E \left\{ \left| \sum_{\substack{t_i=1 \\ i=1,\dots,d}}^N X_{\mathbf{t}} X_{\mathbf{t}+\nu^1} X_{\mathbf{t}+\nu^2} - r_3(\nu^1, \nu^2) \right|^2 \right\} \\
&= \sum_{\substack{t_i, s_i=1 \\ i=1,\dots,d}}^N \int_{[0, 2\pi]^d} e^{i\langle \lambda, \mathbf{t}-\mathbf{s} \rangle} f_{Y^{(\nu^1, \nu^2)}}(\lambda) d\lambda \\
&\leq \sup_{\lambda \in \mathbb{Z}^d} f_{Y^{(\nu^1, \nu^2)}}(\lambda) \sum_{\substack{t_i, s_i=1 \\ i=1,\dots,d}}^N \int_{[0, 2\pi]^d} e^{i\langle \lambda, \mathbf{t}-\mathbf{s} \rangle} d\lambda \leq C \times N^d,
\end{aligned}$$

where C represents a constant which is independent of N , then

$$E \left\{ \frac{1}{N^{2d}} \left| \sum_{\substack{t_i=1 \\ i=1,\dots,d}}^N X_{\mathbf{t}} X_{\mathbf{t}+\nu^1} X_{\mathbf{t}+\nu^2} - r_3(\nu^1, \nu^2) \right|^2 \right\} \leq C \times \frac{N^d}{N^{2d}} = \frac{C}{N^d},$$

which is (2.2). Following [18], consider (2.3) on the basis of (2.2) and let

$$S(N) = \frac{1}{N^{kd}} \left| \sum_{\substack{t_i=1 \\ i=1,\dots,d}}^N Y_{\mathbf{t}}^{(\nu^1, \nu^2)} \right|.$$

For $k > \frac{3}{4}$, there is a β which satisfies $\beta > 1$ and,

$$k - \frac{1}{2} > \frac{1}{2\beta} > \frac{1}{4}.$$

Now, consider the two case where $N = M^\beta$ and for general N .

(I) When $N = M^\beta$, we have

$$\begin{aligned} E |S(N)|^2 &= E \frac{1}{N^{2kd}} \left| \sum_{\substack{t_i=1 \\ i=1,\dots,d}}^N Y_{\mathbf{t}}^{(\nu^1, \nu^2)} \right|^2 \\ &= \frac{1}{N^{2kd-2d}} E \frac{1}{N^{2d}} \left| \sum_{\substack{t_i=1 \\ i=1,\dots,d}}^N Y_{\mathbf{t}}^{(\nu^1, \nu^2)} \right|^2 \leq \frac{1}{N^{2kd-2d}} \times \frac{C}{N^d} = CM^{\beta d(1-2k)}, \end{aligned}$$

since $\beta d(1-2k) < -d \leq -1$, hence $\sum_{M=1}^{+\infty} M^{\beta d(1-2k)}$ is a convergent Riemman series, consequently

$$\sum_{M=1}^{+\infty} E |S(M^\beta)|^2 < +\infty.$$

It follows from the Chebychev inequality and the Borel-Cantelli lemma, that

$$(2.5) \quad \lim_{M \rightarrow \infty} S(M^\beta) \stackrel{a.s.}{=} 0.$$

(II) For general N , and $N_M \leq N \leq N_{M+1}$, where $N_M = M^\beta$, we have

$$\left| S(N) - N^{-kd} M^{kd\beta} S(M^\beta) \right| \leq M^{-\beta kd} \left| \sum_{\substack{t_i=M^\beta+1 \\ i=1,\dots,d}}^N Y_{\mathbf{t}}^{(\nu^1, \nu^2)} \right|,$$

and

$$\begin{aligned} &E \left\{ \sup_{N_M \leq N \leq N_{M+1}} \left| S(N) - N^{-kd} M^{kd\beta} S(M^\beta) \right|^2 \right\} \\ &\leq M^{-2\beta kd} \sup_{N_M \leq N \leq N_{M+1}} E \left| \sum_{\substack{t_i=M^\beta+1 \\ i=1,\dots,d}}^N Y_{\mathbf{t}}^{(\nu^1, \nu^2)} \right|^2 \\ &\leq CM^{-2\beta kd} \sup_{N_M \leq N \leq N_{M+1}} \left[N - M^\beta \right]^d \leq CM^{-2\beta kd} \left[(M+1)^\beta - M^\beta \right]^d, \end{aligned}$$

noting that $(M+1)^\beta - M^\beta \leq (2M)^\beta$, we deduce that

$$E \sup_{N_M \leq N \leq N_{M+1}} \left| S(N) - N^{-kd} M^{kd\beta} S(M^\beta) \right|^2 \leq C_1 M^{\beta d(1-2k)},$$

where C_1 represents a constant which is independent of M . Since $\beta d(1-2k) < -d \leq -1$, we obtain

$$\sum_{M=1}^{+\infty} E \sup_{N_M \leq N \leq N_{M+1}} \left| S(N) - N^{-kd} M^{kd\beta} S(M^\beta) \right|^2 < \infty.$$

It follows from the Chebychev inequality and the Borel-Cantelli lemma, that

$$(2.6) \quad \lim_{M \rightarrow \infty} \sup_{N_M \leq N \leq N_{M+1}} \left| S(N) - N^{-kd} M^{kd\beta} S(M^\beta) \right|^2 \stackrel{a.s.}{=} 0.$$

By (2.5), (2.6) and

$$S(N) \leq \sup_{N_M \leq N \leq N_{M+1}} |S(N) - N^{-kd} M^{kd\beta} S(M^\beta)| + N^{-kd} M^{kd\beta} S(M^\beta),$$

we have, for $k > \frac{3}{4}$,

$$S(N) = \frac{1}{N^{kd}} \left| \sum_{\substack{t_i=1 \\ i=1, \dots, d}}^N Y_{\mathbf{t}}^{(\nu^1, \nu^2)} \right| \xrightarrow{a.s.} 0,$$

which completes the proof. $\square$

Appendix C: Proof of Theorem 2

Proof. For $\mathbf{k} = (k, \dots, k) \in \mathbb{Z}^d$ such that $k > 0$, let

$$\begin{aligned} X_{\mathbf{t}, \mathbf{k}} &= \sum_{\substack{s_i = -k \\ i=1, \dots, d}}^{+k} \psi_{\mathbf{s}} \varepsilon_{\mathbf{t}-\mathbf{s}}, \\ \hat{r}_{3, \mathbf{k}\mathbf{X}}(\nu^1, \nu^2) &= \frac{1}{N^d} \sum_{\substack{t_i=1 \\ i=1, \dots, d}}^N X_{\mathbf{t}, \mathbf{k}} X_{\mathbf{t}+\nu^1, \mathbf{k}} X_{\mathbf{t}+\nu^2, \mathbf{k}}, \\ E(X_{\mathbf{t}, \mathbf{k}} X_{\mathbf{t}+\nu^1, \mathbf{k}} X_{\mathbf{t}+\nu^2, \mathbf{k}}) &= r_{3, \mathbf{k}\mathbf{X}}(\nu^1, \nu^2). \end{aligned}$$

By the symmetries of $r_{3X}(\nu^1, \nu^2)$, we can assume $0 \leq \nu_i^1 \leq \nu_i^2$ for all $i = 1, \dots, d$, then

$$E(X_{\mathbf{t}, \mathbf{k}} X_{\mathbf{t}+\nu^1, \mathbf{k}} X_{\mathbf{t}+\nu^2, \mathbf{k}}) = \begin{cases} \gamma_3 \sum_{\substack{s_i = -k \\ i=1, \dots, d}}^{k-\nu_i^2} \psi_{\mathbf{s}} \psi_{\nu^1+\mathbf{s}} \psi_{\nu^2+\mathbf{s}}, & \text{for at least one } i \\ \nu_i^2 \leq 2k, & \\ 0, & \text{if not} \end{cases}.$$

For $\nu_i^2 \leq 2k$ for at least one $i = 1, \dots, d$, we have

$$r_{3,\mathbf{k}\mathbf{X}}(\nu^1, \nu^2) = E\hat{r}_{3,\mathbf{k}\mathbf{X}}(\nu^1, \nu^2).$$

We deduce that $X_{\mathbf{t},\mathbf{k}}X_{\mathbf{t}+\nu^1,\mathbf{k}}X_{\mathbf{t}+\nu^2,\mathbf{k}}$ is independent of $X_{\mathbf{s},\mathbf{k}}X_{\mathbf{s}+\nu^1,\mathbf{k}}X_{\mathbf{s}+\nu^2,\mathbf{k}}$ provided that $|s_i - t_i| > 2k + \nu_i^2$ for at least one $i = 1, \dots, d$ via the independence of $\varepsilon_{\mathbf{t}}$. Hence, if we put $\mathbf{m} = 2\mathbf{k} + \nu^2$, $X_{\mathbf{t},\mathbf{k}}X_{\mathbf{t}+\nu^1,\mathbf{k}}X_{\mathbf{t}+\nu^2,\mathbf{k}} - r_{3,\mathbf{k}\mathbf{X}}(\nu^1, \nu^2)$ is a $\mathbf{m}$ -dependent stationary random field with mean zero, it follows that

$$(2.7) \quad \begin{aligned} & N^{-\frac{d}{2}} \left(\sum_{\substack{t_i=1 \\ i=1,\dots,d}}^N \{X_{\mathbf{t},\mathbf{k}}X_{\mathbf{t}+\nu^1,\mathbf{k}}X_{\mathbf{t}+\nu^2,\mathbf{k}} - r_{3,\mathbf{k}\mathbf{X}}(\nu^1, \nu^2)\} \right) \\ & \rightarrow \mathcal{N}(0, \sigma_{X,\mathbf{k}}^2(\nu^1, \nu^2)), \end{aligned}$$

where

$$\sigma_{X,\mathbf{k}}^2(\nu^1, \nu^2) = \lim_{N \rightarrow \infty} N^d \times \text{Var} \left(N^{-d} \sum_{\substack{t_i=1 \\ i=1,\dots,d}}^N \{X_{\mathbf{t},\mathbf{k}}X_{\mathbf{t}+\nu^1,\mathbf{k}}X_{\mathbf{t}+\nu^2,\mathbf{k}} - r_{3,\mathbf{k}\mathbf{X}}(\nu^1, \nu^2)\} \right),$$

it is easy to show that

$$\begin{aligned} \sigma_{X,\mathbf{k}}^2(\nu^1, \nu^2) &= \sum_{\substack{\tau_i=-\infty \\ i=1,\dots,d}}^{+\infty} \{C_{6,\mathbf{k}\mathbf{X}}(\nu^1, \nu^2, \tau, \tau + \nu^1, \tau + \nu^2) \\ &\quad + \{r_{2,\mathbf{k}\mathbf{X}}(\nu^1)C_{4,\mathbf{k}\mathbf{X}}(\tau - \nu^2, \tau + \nu^1 - \nu^2, \nu^2)\}_{15} \\ &\quad + \{r_{2,\mathbf{k}\mathbf{X}}(\nu^1)r_{2,\mathbf{k}\mathbf{X}}(\tau - \nu^2)r_{2,\mathbf{k}\mathbf{X}}(\nu^2 - \nu^1)\}_{15}\}. \end{aligned}$$

Now, to prove (2.4) we use Proposition 6.3.9 of [7]. For this, observe that;

$$\begin{aligned} N^{\frac{d}{2}}[\hat{r}_{3X}(\nu^1, \nu^2) - r_{3X}(\nu^1, \nu^2)] &= N^{\frac{d}{2}}\{\hat{r}_{3X}(\nu^1, \nu^2) - \hat{r}_{3,\mathbf{k}\mathbf{X}}(\nu^1, \nu^2) \\ &\quad - E[\hat{r}_{3X}(\nu^1, \nu^2) - \hat{r}_{3,\mathbf{k}\mathbf{X}}(\nu^1, \nu^2)]\} + N^{\frac{d}{2}}[\hat{r}_{3,\mathbf{k}\mathbf{X}}(\nu^1, \nu^2) - r_{3,\mathbf{k}\mathbf{X}}(\nu^1, \nu^2)] \\ &= T_{k,N} + Z_{k,N}, \end{aligned}$$

where $T_{k,N} = N^{-\frac{d}{2}}\{\hat{r}_{3X}(\nu^1, \nu^2) - \hat{r}_{3,\mathbf{k}\mathbf{X}}(\nu^1, \nu^2) - E[\hat{r}_{3X}(\nu^1, \nu^2) - \hat{r}_{3,\mathbf{k}\mathbf{X}}(\nu^1, \nu^2)]\}$ and $Z_{k,N} = N^{-\frac{d}{2}}[\hat{r}_{3,\mathbf{k}\mathbf{X}}(\nu^1, \nu^2) - r_{3,\mathbf{k}\mathbf{X}}(\nu^1, \nu^2)]$. But according to Proposition 6.3.9 of [7], it remains to prove that:

- (i) $Z_{k,N} \Rightarrow Z_k$ as $N \rightarrow \infty$ for each $k = 1, 2, \dots$
- (ii) $Z_k \Rightarrow Z$ as $k \rightarrow \infty$,
- (iii) $\lim_{k \rightarrow \infty} \lim_{N \rightarrow \infty} \sup P(|T_{k,N}| > \epsilon) = 0$ for every $\epsilon > 0$,

after which we can immediately deduce (2.4). First, (2.7) establish (i). Now let's prove (iii). According to Chebychev's inequality, to show (iii) it suffices to show that

$$(2.8) \quad \lim_{k \rightarrow \infty} \lim_{N \rightarrow \infty} \sup \text{Var}(T_{k,N}) = 0.$$

By some calculations we obtain

$$\begin{aligned} & N^{\frac{d}{2}} \{ \hat{r}_{3X}(\nu^1, \nu^2) - \hat{r}_{3,\mathbf{k}\mathbf{X}}(\nu^1, \nu^2) \} \\ &= N^{-\frac{d}{2}} \sum_{\substack{t_i=1 \\ i=1, \dots, d}}^N \left\{ \underbrace{X_{\mathbf{t},\mathbf{k}} X_{\mathbf{t}+\nu^1, \mathbf{k}} u_{\mathbf{t}+\nu^2, \mathbf{k}}}_{S_1} + \underbrace{X_{\mathbf{t},\mathbf{k}} u_{\mathbf{t}+\nu^1, \mathbf{k}} X_{\mathbf{t}+\nu^2, \mathbf{k}}}_{S_2} \right. \\ & \quad \left. + \underbrace{u_{\mathbf{t}, \mathbf{k}} X_{\mathbf{t}+\nu^1, \mathbf{k}} X_{\mathbf{t}+\nu^2, \mathbf{k}}}_{S_3} \right\} = S_1 + S_2 + S_3, \end{aligned}$$

where

$$u_{\mathbf{t}, \mathbf{k}} = X_{\mathbf{t}} - X_{\mathbf{t}, \mathbf{k}} = \sum_{\substack{|s_i| > k \\ i=1, \dots, d}} \psi_{\mathbf{s}} \varepsilon_{\mathbf{t}-\mathbf{s}}.$$

It is easy to show that as $k \rightarrow \infty$,

$$\text{Var}(S_1 + S_2 + S_3) \leq 3 \{ \text{Var} S_1 + \text{Var} S_2 + \text{Var} S_3 \} \rightarrow 0,$$

which gives (2.8) and hence gives (iii).

It remains to show (ii). By (2.5), $Z_k \sim \mathcal{N}(0, \sigma_{X,\mathbf{k}}^2(\nu^1, \nu^2))$. Observe that if we put $\sigma_X^2(\nu^1, \nu^2) = \lim_{k \rightarrow \infty} \sigma_{X,\mathbf{k}}^2(\nu^1, \nu^2)$, it is not difficult to show that

$$\begin{aligned} \sigma_X^2(\nu^1, \nu^2) &= \sum_{\substack{\tau_i = -\infty \\ i=1, \dots, d}}^{+\infty} \{ C_{6X}(\nu^1, \nu^2, \tau, \tau + \nu^1, \tau + \nu^2) \\ & \quad + \{ r_{2X}(\nu^1) r_{2X}(\tau - \nu^2) r_{2X}(\nu^2 - \nu^1) \}_{15} \} < \infty, \end{aligned}$$

then, using the result of Problem 6.16 of [7] $Z_k \Rightarrow Z$, where $Z \sim \mathcal{N}(0, \sigma_X^2(\nu^1, \nu^2))$. From (i), (ii) and (iii) $N^{\frac{d}{2}}(\hat{r}_{3X}(\nu^1, \nu^2) - r_{3X}(\nu^1, \nu^2)) \rightarrow \mathcal{N}(0, \sigma_X^2(\nu^1, \nu^2))$. $\square$

Appendix D: Chapter head

Setting $\mathbf{q} = (q_1, q_2)$ and $R = (q_1 + 1)(q_2 + 1)$, the expressions of $\mathbf{C}_1$, $\mathbf{C}_2$, $\mathbf{c}_1$ and $\mathbf{c}_2$ are:

$$\mathbf{c}_1 = \begin{pmatrix} C_{3y}((-q_1, -q_2), \mathbf{0}) \times C_{3y}(\mathbf{q}, \mathbf{q}) \\ C_{3y}((-q_1, -q_2 + 1), \mathbf{0}) \times C_{3y}(\mathbf{q}, \mathbf{q}) \\ \vdots \\ C_{3y}((0, -1), \mathbf{0}) \times C_{3y}(\mathbf{q}, \mathbf{q}) \end{pmatrix}_{(R-1) \times 1}$$

$$\begin{aligned}
\mathbf{c}_2 &= \begin{pmatrix} C_{3y}((0,0), \mathbf{0}) \times C_{3y}(\mathbf{q}, \mathbf{q}) \\ C_{3y}((0,1), \mathbf{0}) \times C_{3y}(\mathbf{q}, \mathbf{q}) \\ \vdots \\ C_{3y}(\mathbf{q}, \mathbf{0}) \times C_{3y}(\mathbf{q}, \mathbf{q}) \end{pmatrix}_{R \times 1} \\
\mathbf{C}_1 &= \begin{pmatrix} C_{3y}^2(\mathbf{q}, \mathbf{q}) & 0 & \dots & 0 & 0 \\ \vdots & C_{3y}^2(\mathbf{q}, \mathbf{q}) & & & \\ \vdots & & \ddots & & \\ C_{3y}^2(\mathbf{q}, (0,1)) & \dots & \dots & C_{3y}^2(\mathbf{q}, \mathbf{q}) & 0 \end{pmatrix}_{(R-1) \times R} \\
\mathbf{C}_2 &= \begin{pmatrix} C_{3y}^2(\mathbf{q}, \mathbf{0}) & C_{3y}^2(\mathbf{q}, (0,1)) & \dots & C_{3y}^2(\mathbf{q}, \mathbf{q}) \\ 0 & C_{3y}^2(\mathbf{q}, \mathbf{0}) & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & C_{3y}^2(\mathbf{q}, \mathbf{0}) \end{pmatrix}_{R \times R}.
\end{aligned}$$

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Study on Kenmotsu manifolds admitting generalized Tanaka-Webster connection

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Abstract. In this study, we use the generalized Tanaka-Webster connection to observe curvature properties and Ricci soliton in Kenmotsu manifold. Here we look at projective curvature tensor, conharmonic curvature tensor, Weyl projective curvature tensor and recurrent conditions of Kenmotsu manifold with generalized Tanaka-Webster connection. Likewise, we demonstrate identical conditions for a Ricci soliton in a Kenmotsu manifold with generalized Tanaka-Webster connection.

Keywords: Kenmotsu manifolds, generalized Tanaka-Webster connection, Ricci solitons, shrinking, expanding, steady.

1. Introduction

The Tanaka-Webster connection is canonical affine connection defined on a non-degenerate pseudo-Hermitian CR-manifold [19, 22]. The generalized Tanaka-

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Webster connection (GTWC) for contact metric manifolds by using the canonical connection turned into first studied by Tanno [20]. This connection coincides with the Tanaka-Webster connection if the associated CR-structure is integrable. Using the GTWC, few geometers have studied some characterizations of real hypersurfaces in complex space forms [18]. Kenmotsu manifolds introduced by Kenmotsu in 1971[9]. Recently many authors[15, 7, 16] have been studied GTWC connection in Kenmotsu manifolds.

Ricci soliton, introduced by Hamilton [8] are natural generalizations of the Einstein metrics and is defined on a Riemannian manifold (M, g) . A Ricci soliton (g, V, λ) defined on (M, g) as

$$(1.1) \quad (L_X g)(U, V) + 2S(U, V) + 2\lambda g(U, V) = 0,$$

where L_X denotes the Lie-derivative of Riemannian metric g along a vector field X , λ is a constant and U, V are arbitrary vector fields on M . A Ricci soliton is said to shrinking or steady or expanding to the extent that λ is negative, zero or positive respectively. Ricci solitons have been studied extensively in the context of contact geometry; we may refer to [17, 4, 14, 11, 10, 12, 13]) and references therein.

2. Preliminaries

A smooth manifold M of $(2n+1)$ -dimension is said to be an almost contact metric manifold if it admits an almost contact metric structure (ϕ, ξ, η, g) consisting of a tensor field ϕ of type $(1, 1)$, a vector field ξ , a 1-form η and a Riemannian metric g compatible with (ϕ, ξ, η) satisfying

$$(2.1) \quad \phi^2 U = -U + \eta(U)\xi, \phi\xi = 0, g(U, \xi) = \eta(U), \eta(\xi) = 1, \eta \circ \phi = 0,$$

and

$$(2.2) \quad g(\phi U, \phi V) = g(U, V) - \eta(U)\eta(V).$$

An almost contact metric manifold is said to be a Kenmotsu manifold [1] if

$$(2.3) \quad (\nabla_U \phi)V = -g(U, \phi V)\xi - \eta(V)\phi U,$$

where ∇ denotes the Riemannian connection of g .

In a Kenmotsu manifold the following relations hold [6].

$$(2.4) \quad \nabla_U \xi = U - \eta(U)\xi,$$

$$(2.5) \quad (\nabla_U \eta)V = g(\nabla_U \xi, V),$$

$$(2.6) \quad R(U, V)\xi = \eta(U)V - \eta(V)U,$$

$$\begin{aligned}
(2.7) \quad & R(\xi, U)V = \eta(V)U - g(U, V)\xi, \\
(2.8) \quad & \eta(R(U, V)W) = g(U, W)\eta(V) - g(V, W)\eta(U), \\
(2.9) \quad & S(U, \xi) = -2n\eta(U), \\
(2.10) \quad & Q\xi = -2n\xi, \\
(2.11) \quad & S(\phi U, \phi V) = S(U, V) + 2n\eta(U)\eta(V),
\end{aligned}$$

for any vector fields U, V, W on M , where R denote the curvature tensor of type $(1, 3)$ on M .

3. Curvature properties of Kenmotsu manifolds admitting generalized Tanaka-Webster connection

Throughtout this paper we associate $*$ with the quantities with respect to generalized Tanaka-Webster connection. The generalized Tanaka-Webster connection ∇^* associated to the Levi-Civita connection ∇ is given by [21, 7]

$$(3.1) \quad \nabla_U^* V = \nabla_U V - \eta(V)\nabla_U \xi + (\nabla_U \eta)(V)\xi - \eta(U)\phi V,$$

for any vector fields U, V on M .

Using (2.4) and (2.5), the above equation yields,

$$(3.2) \quad \nabla_U^* V = \nabla_U V + g(U, V)\xi - \eta(V)U - \eta(U)\phi V.$$

By taking $V = \xi$ in (3.2) and using (2.4) we obtain

$$(3.3) \quad \nabla_U^* \xi = 0.$$

We now calculate the Riemann curvature tensor R^* using (3.2) as follows:

$$(3.4) \quad R^*(U, V)W = R(U, V)W + g(V, W)U - g(U, W)V.$$

Using (2.6) and taking $W = \xi$ in (3.4) we get

$$(3.5) \quad R^*(U, V)\xi = 0.$$

On contracting (3.4), we obtain the Ricci tensor S^* of a Kenmotsu manifold with respect to the generalized Tanaka-Webster connection ∇^* as

$$(3.6) \quad S^*(V, W) = S(V, W) + 2ng(V, W).$$

This gives

$$(3.7) \quad Q^*V = QV + 2nV.$$

Contracting with respect to V and W in (3.6), we get

$$(3.8) \quad r^* = r + 2n(2n + 1),$$

where r^* and r are the scalar curvatures with respect to the generalized Tanaka-Webster connection ∇^* and the Levi-Civita connection ∇ respectively.

Now, the projective curvature tensor [23] P^* with respect to the generalized Tanaka-Webster connection ∇^* is defined by

$$(3.9) \quad P^*(U, V)W = R^*(U, V)W - \frac{1}{2n}\{S^*(V, W)U - S^*(U, W)V\}.$$

If projective curvature tensor P^* with respect to the generalized Tanaka-Webster connection ∇^* vanishes, then from (3.9), we have

$$(3.10) \quad R^*(U, V)W = \frac{1}{2n}\{S^*(V, W)U - S^*(U, W)V\}.$$

Now, in view of (3.4) and (3.6), (3.10) takes the form

$$(3.11) \quad \begin{aligned} &g(R(U, V)W, Y) + g(V, W)g(U, Y) - g(U, W)g(V, Y) \\ &= \frac{1}{2n}[\{S(V, W) + 2ng(V, W)\}g(U, Y) \\ &- \{S(U, W) + 2ng(U, W)\}g(V, Y)]. \end{aligned}$$

Now, taking $Y = \xi$ in (3.11), we obtain

$$(3.12) \quad S(V, W)\eta(U) - S(U, W)\eta(V) = 2n\{g(U, W)\eta(U) - g(V, W)\eta(U)\}.$$

Again setting $U = \xi$ in (3.12), we get

$$(3.13) \quad S(V, W) = -2ng(V, W).$$

Contracting the above equation (3.13), we get

$$(3.14) \quad r = -2n(2n + 1).$$

Using (3.13) in (3.10), we have $R^* = 0$.

Thus, we state the following:

Theorem I. *Let M be a Kenmotsu manifold with generalized Tanaka-Webster connection. In M , vanishing of projective curvature tensor with respect to generalized Tanaka-Webster connection leads to vanishing of curvature tensor with respect to generalized Tanaka-Webster connection.*

By interchanging U and V in (3.9), we have

$$(3.15) \quad P^*(V, U)W = R^*(V, U)W - \frac{1}{2n}\{S^*(U, W)V - S^*(V, W)U\}.$$

On adding (3.9) and (3.15) and using the fact that $R(U, V)W + R(V, U)W = 0$, we get

$$(3.16) \quad P^*(U, V)W + P^*(V, U)W = 0.$$

From (3.4), (3.9) and first Bianchi identity $R(U, V)W + R(V, W)U + R(W, U)V = 0$ with respect to ∇ , we obtain

$$(3.17) \quad P^*(U, V)W + P^*(V, W)U + P^*(W, U)V = 0.$$

Hence, from (3.16) and (3.17), shows that projective curvature tensor with respect to generalized Tanaka-Webster connection in a Kenmotsu manifold is skew-symmetric and cyclic.

Now, by taking $W = \xi$ in (3.9), using (3.5) and (3.6), we get

$$(3.18) \quad P^*(U, V)\xi = 0.$$

From (3.18), we state the following.

Theorem II. *Let M be a Kenmotsu manifold with generalized Tanaka-Webster connection is ξ -projectively flat.*

Definition 3.1. *A Kenmotsu manifold is said to be ϕ -projectively semisymmetric with respect to the generalized Tanaka-Webster connection ∇^* if*

$$(3.19) \quad P^*(U, V).\phi = 0.$$

for any vector fields U, V on M .

Now, (3.19) turns into

$$(3.20) \quad (P^*(U, V).\phi)W = P^*(U, V)\phi W - \phi P^*(U, V)W = 0.$$

Making use of (3.9), (3.4) and (3.6) in (3.20), we get

$$(3.21) \quad R(U, V)\phi W - \phi R(U, V)W - \frac{1}{2n}\{S(V, \phi W)U - S(U, \phi W)V \\ + S(U, W)\phi V - S(V, W)\phi U\} = 0.$$

Taking V by ξ in (3.21), using (2.6) and (2.9), we get

$$(3.22) \quad S(U, \phi W)\xi = -2ng(U, \phi W)\xi.$$

Taking an inner product with ξ and replacing U by ϕU , using (2.2) and (2.11) in (3.22), we get

$$(3.23) \quad S(U, W) = -2ng(U, W).$$

and

$$(3.24) \quad r = -2n(2n + 1).$$

Again by substituting (3.24) in (3.9), we obtain

$$(3.25) \quad P^*(U, V)W = R(U, V)W + \{g(V, W)U - g(U, W)V\}.$$

Thus, we can state the following:

Theorem III. *Let M be a Kenmotsu manifold with generalized Tanaka-Webster connection be ϕ -projectively semisymmetric if and only if $S(U, W) = -2ng(U, W)$.*

Further if $P^* = 0$ then M is isomorphic to the hyperbolic space $H^{2n+1}(-1)$.

Suppose $(P^*(U, V).S^*)(W, Y) = 0$ holds in a Kenmotsu manifold M . Then, we have

$$(3.26) \quad S^*(P^*(U, V)W, Y) + S^*(W, P^*(U, V)Y) = 0.$$

Taking $U = \xi$ in the equation (3.26), we get

$$(3.27) \quad S^*(P^*(\xi, V)W, Y) + S^*(W, P^*(\xi, V)Y) = 0.$$

By using (3.9), equation (3.27) turns into

$$(3.28) \quad S^*(V, W)\eta(Y) + S^*(V, Y)\eta(W) = 0.$$

In view of equation (3.6), (3.28) becomes

$$(3.29) \quad S(V, W)\eta(Y) + S(V, Y)\eta(W) + 2n\{g(V, W)\eta(Y) + g(V, Y)\eta(W)\} = 0.$$

In (3.29), taking $Y = \xi$ and contracting with respect to V and W , we get

$$S(V, W) = -2ng(V, W).$$

and

$$(3.30) \quad r = -2n(2n + 1).$$

Again by substituting (3.30) in (3.9), we obtain

$$(3.31) \quad P^*(U, V)W = R(U, V)W + \{g(V, W)U - g(U, W)V\}.$$

Thus, we can state that

Theorem IV. *In a Kenmotsu manifold M with generalized Tanaka-Webster connection, $P^*.S^* = 0$ if and only if $S(V, W) = -2ng(V, W)$.*

Further if $P^* = 0$ then M is isomorphic to the hyperbolic space $H^{2n+1}(-1)$.

The conharmonic curvature tensor [5] K^* with respect to the generalized Tanaka-Webster connection ∇^* is defined by

$$(3.32) \quad \begin{aligned} K^*(U, V)W &= R^*(U, V)W - \frac{1}{2n-1}\{S^*(V, W)U - S^*(U, W)V \\ &+ g(V, W)Q^*U - g(U, W)Q^*V\}. \end{aligned}$$

If conharmonic curvature tensor K^* with respect to the generalized Tanaka-Webster connection ∇^* vanishes, then from (3.32), we have

$$(3.33) \quad \begin{aligned} R^*(U, V)W &= \frac{1}{2n-1}\{S^*(V, W)U - S^*(U, W)V \\ &+ g(V, W)Q^*U - g(U, W)Q^*V\}. \end{aligned}$$

By using (3.4), (3.6) and (3.7) in (3.33), we get

$$\begin{aligned} & g(R(U, V)W, Y) + g(V, W)g(U, Y) - g(U, W)g(V, Y) \\ &= \frac{1}{2n-1}[\{S(V, W) + 4ng(V, W)\}g(U, Y) - \{S(U, W) + 4ng(U, W)\} \\ (3.34) \quad & g(V, Y) + S(U, Y)g(V, W) - S(V, Y)g(U, W)]. \end{aligned}$$

Taking $Y = \xi$ in (3.34), we obtain

$$(3.35) \quad S(V, W)\eta(U) - S(U, W)\eta(V) - 2n\{g(U, W)\eta(V) - g(V, W)\eta(U)\} = 0.$$

Taking $U = \xi$ in (3.35), we get

$$(3.36) \quad S(V, W) = -2ng(V, W).$$

Contracting the equation (3.36), we get

$$(3.37) \quad r = -2n(2n+1).$$

Using (3.36) in (3.33), we have $R^* = 0$.

Theorem V. *Let M be a Kenmotsu manifold with generalized Tanaka-Webster connection. In M , vanishing of conharmonic curvature tensor with respect to generalized Tanaka-Webster connection leads to vanishing of curvature tensor with respect to generalized Tanaka-Webster connection.*

In a Riemannian manifold Weyl conformal curvature tensor C^* with respect to the generalized Tanaka-Webster connection ∇^* is defined as

$$\begin{aligned} C^*(U, V)W &= R^*(U, V)W \\ &- \frac{1}{2n-1}\{S^*(V, W)U - S^*(U, W)V + g(V, W)Q^*U \\ (3.38) \quad &- g(U, W)Q^*V\} + \frac{r^*}{2n(2n-1)}\{g(V, W)U - g(U, W)V\}. \end{aligned}$$

By making use of (3.4), (3.6), (3.7), (3.8) in (3.38) yields

$$(3.39) \quad C^*(U, V)W = C(U, V)W.$$

for all U, V, W . Thus, we state the following:

Theorem VI. *The Weyl conformal curvature tensor of Kenmotsu manifold with respect to the Levi-Civita connection and the generalized Tanaka-Webster connection are equivalent.*

Definition 3.2. *A Kenmotsu manifold with respect to the generalized Tanaka-Webster connection ∇^* is called recurrent, if its curvature tensor R^* satisfies the condition*

$$(3.40) \quad (\nabla_Y^* R^*)(U, V)W = A(Y)R^*(U, V)W,$$

where R^* is the curvature tensor with respect to the connection ∇^* and A is 1-form.

Using (3.40), we can write

$$(3.41) \quad \begin{aligned} & \nabla_Y^* R^*(U, V)W - R^*(\nabla_Y^* U, V)W - R^*(U, \nabla_Y^* V)W - R^*(U, V)\nabla_Y^* W \\ &= A(Y)R^*(U, V)W. \end{aligned}$$

Making use of (3.2), (3.4) and (3.6) in (3.41), we get

$$(3.42) \quad \begin{aligned} & g(Y, R(U, V)W)\xi - g(Y, U)R(\xi, V)W \\ & - g(Y, V)R(U, \xi)W - g(Y, W)R(U, V)\xi \\ & - \eta(R(U, V)W)Y + \eta(U)R(Y, V)W + \eta(V)R(U, Y)W + \eta(W)R(U, V)Y \\ & - \eta(Y)\{\phi R(U, V)W - R(\phi U, V)W - R(U, \phi V)W - R(U, V)\phi W\} \\ &= A(Y)\{g(V, W)U - g(U, W)V\}. \end{aligned}$$

Replacing W by ξ and using (2.1), (2.6), (2.7) and (2.8), we get

$$(3.43) \quad A(Y)\{\eta(V)U - \eta(U)V\} = g(Y, V)U - g(Y, U)V + R(U, V)Y.$$

Taking an inner product with Z in (3.43), we have

$$(3.44) \quad \begin{aligned} & A(Y)\{\eta(V)g(U, Z) - \eta(U)g(V, Z)\} \\ &= g(Y, V)g(U, Z) - g(Y, U)g(V, Z) + R(U, V, Y, Z). \end{aligned}$$

Let $\{e_1, e_2, e_3, \dots, e_{2n+1}\}$ be a local orthonormal basis of vector fields in M . Then by putting $U = Z = e_i$ in (3.44) and summing up with respect to i , $1 \leq i \leq 2n+1$, we obtain

$$(3.45) \quad S(V, Y) = -2n\{g(V, Y) + \eta(V)A(Y)\}.$$

Suppose the associated 1-form A is equal to the associated 1-form η , then from (3.45), we get

$$(3.46) \quad S(V, Y) = -2n\{g(V, Y) + \eta(V)\eta(Y)\}.$$

Thus, we state the following:

Theorem VII. *If a Kenmotsu manifold whose curvature tensor of manifold is covariant constant with respect to the generalized Tanaka-Webster connection, the manifold is recurrent and the associated 1-form A is equal to the associated 1-form η , then the manifold is an η -Einstein manifold.*

4. Ricci solitons in Kenmotsu manifold with generalized Tanaka-Webster connection

Suppose Kenmotsu manifold M admits a Ricci soliton with respect to the generalized Tanaka-Webster connection ∇^* . Then

$$(4.1) \quad (\overset{*}{L}_X g)(U, V) + 2S^*(U, V) + 2\lambda g(U, V) = 0.$$

If the potential vector field X is the structure vector field ξ , then since ξ is a parallel vector field with respect to the generalized Tanaka-Webster connection (from (3.3)), the first term in equation (4.1) becomes zero, hence M reduces to an Einstein manifold. In this case the results in theorem (3.4) and (3.6) hold.

If X is pointwise collinear with the structure vector field ξ , i.e. $X = b\xi$, where b is a function on M , then equation (1.1) implies that

$$(4.2) \quad \begin{aligned} &bg(\nabla_U^* \xi, V) + (Ub)\eta(V) + bg(U, \nabla_V^* \xi) + (Vb)\eta(U) \\ &+ 2S^*(U, V) + 2\lambda g(U, V) = 0. \end{aligned}$$

Using (3.3) and (3.6) in (4.2), it follows that

$$(4.3) \quad (Ub)\eta(V) + (Vb)\eta(U) + 2S(U, V) + 2\{2n + \lambda\}g(U, V) = 0.$$

By setting $V = \xi$ in (4.3) and using (2.9), we obtain

$$(4.4) \quad (Ub) = -\{2\lambda + \xi b\}\eta(U).$$

Again, replacing U by ξ in (4.4), we get

$$(4.5) \quad (\xi b) = -\lambda.$$

Substituting this in (4.4), we have

$$(4.6) \quad (Ub) = -\lambda\eta(U).$$

By applying d on (4.6), we get

$$(4.7) \quad \lambda d\eta = 0.$$

Since $d\eta \neq 0$ from (4.7), we have

$$(4.8) \quad \lambda = 0.$$

Substituting (4.8) in (4.6), we conclude that b is a constant. Hence it is verified from (4.3) that

$$(4.9) \quad S(U, V) = -(2n + \lambda)g(U, V) + \lambda\eta(U)\eta(V).$$

This leads to the following.

Theorem VIII. *If a Kenmotsu manifold with respect to the generalized Tanaka-Webster connection admits a Ricci soliton (g, X, λ) with X , pointwise collinear with ξ , then the manifold is an η -Einstein manifold and the Ricci soliton is steady.*

5. Example of a 5-dimensional Kenmotsu manifold with respect to the generalized Tanaka-Webster connection

We consider the five-dimensional manifold $M = \{(u, v, w, x, y) \in R^5\}$, where (u, v, w, x, y) are the standard coordinates in R^5 . The vector fields

$$E_1 = e^{-y} \frac{\partial}{\partial u}, \quad E_2 = e^{-y} \frac{\partial}{\partial v}, \quad E_3 = e^{-y} \frac{\partial}{\partial w}, \quad E_4 = e^{-y} \frac{\partial}{\partial x}, \quad E_5 = e^{-y} \frac{\partial}{\partial y}$$

are linearly independent at each point of M . Let g be the Riemannian metric defined by

$$g_{ij} = \begin{cases} 1, & \text{for } i = j, \\ 0, & \text{for } i \neq j. \end{cases}$$

Let η be the 1-form defined by $\eta(W) = g(W, E_5)$ for any $W \in \chi(M)$. Let ϕ be the $(1, 1)$ tensor field defined by $\phi E_1 = E_3, \phi E_2 = E_4, \phi E_3 = -E_1, \phi E_4 = -E_2, \phi E_5 = 0$. Then using the linearity of ϕ and g we have

$$\eta(E_5) = 1, \quad \phi^2(W) = -W + \eta(W)E_5, \quad g(\phi W, \phi X) = g(W, X) - \eta(W)\eta(X),$$

for any $W, X \in \chi(M)$. Thus for $E_5 = \xi, (\phi, \xi, \eta, g)$ defines an almost contact metric structure on M . Let ∇ be the Levi-Civita connection with respect to the metric g . Then, we have

$$\begin{aligned} [E_1, E_2] &= [E_1, E_3] = [E_1, E_4] = [E_2, E_3] = 0, & [E_1, E_5] &= E_1, \\ [E_4, E_5] &= E_4, & [E_2, E_4] &= [E_3, E_4] = 0, & [E_2, E_5] &= E_2, & [E_3, E_5] &= E_3. \end{aligned}$$

The Riemannian connection ∇ of the metric g is given by the Koszul's formula

$$\begin{aligned} 2g(\nabla_U V, W) &= U(g(V, W)) + V(g(W, U)) - W(g(U, V)) \\ &\quad - g(U, [V, W]) - g(V, [U, W]) + g(W, [U, V]). \end{aligned}$$

By Koszul's formula, we get

$$\begin{aligned} \nabla_{E_1} E_1 &= -E_5, & \nabla_{E_1} E_2 &= 0, & \nabla_{E_1} E_3 &= 0, & \nabla_{E_1} E_4 &= 0, & \nabla_{E_1} E_5 &= E_1, \\ \nabla_{E_2} E_1 &= 0, & \nabla_{E_2} E_2 &= -E_5, & \nabla_{E_2} E_3 &= 0, & \nabla_{E_2} E_4 &= 0, & \nabla_{E_2} E_5 &= E_2, \\ \nabla_{E_3} E_1 &= 0, & \nabla_{E_3} E_2 &= 0, & \nabla_{E_3} E_3 &= -E_5, & \nabla_{E_3} E_4 &= 0, & \nabla_{E_3} E_5 &= E_3, \\ \nabla_{E_4} E_1 &= 0, & \nabla_{E_4} E_2 &= 0, & \nabla_{E_4} E_3 &= 0, & \nabla_{E_4} E_4 &= -E_5, & \nabla_{E_4} E_5 &= E_4, \\ \nabla_{E_5} E_1 &= 0, & \nabla_{E_5} E_2 &= 0, & \nabla_{E_5} E_3 &= 0, & \nabla_{E_5} E_4 &= 0, & \nabla_{E_5} E_5 &= 0. \end{aligned}$$

Further we obtain the following:

$$\nabla_{E_i}^* E_j = 0, \quad i, j = 1, 2, 3, 4, 5.$$

and hence

$$(\nabla_{E_i}^* \phi) E_j = 0, \quad i, j = 1, 2, 3, 4, 5.$$

From the above expressions it follows that the manifold satisfies (2.2), (2.3) and (2.4) for $\xi = E_5$. Hence the manifold is a Kenmotsu manifold. With the help of the above results we can verify the following results.

$$\begin{aligned} R(E_1, E_2)E_2 &= R(E_1, E_3)E_3 = R(E_1, E_4)E_4 = R(E_1, E_5)E_5 = -E_1, \\ R(E_1, E_2)E_1 &= E_2, \quad R(E_1, E_3)E_1 = R(E_5, E_3)E_5 = R(E_2, E_3)E_5 = E_3, \\ R(E_2, E_3)E_3 &= R(E_2, E_4)E_4 = R(E_2, E_5)E_5 = -E_2, \quad R(E_3, E_4)E_4 = -E_3, \\ R(E_2, E_5)E_2 &= R(E_1, E_5)E_1 = R(E_4, E_5)E_4 = R(E_3, E_5)E_3 = E_5, \\ R(E_1, E_4)E_1 &= R(E_2, E_4)E_2 = R(E_3, E_4)E_3 = R(E_5, E_4)E_5 = E_4 \end{aligned}$$

and

$$R^*(E_i, E_j)E_k = 0, \quad i, j, k = 1, 2, 3, 4, 5.$$

From the above expressions of the curvature tensor of the Kenmotsu manifold it can be easily seen that the manifold has a constant sectional curvature -1 .

Making use of the above results we obtain the Ricci tensors as follows:

$$\begin{aligned} S(E_1, E_1) &= g(R(E_1, E_2)E_2, E_1) + g(R(E_1, E_3)E_3, E_1) + g(R(E_1, E_4)E_4, E_1) \\ &\quad + g(R(E_1, E_5)E_5, E_1) = -4. \end{aligned}$$

Similarly, we have

$$S(E_2, E_2) = S(E_3, E_3) = S(E_3, E_3) = S(E_4, E_4) = S(E_5, E_5) = -4$$

and

$$S^*(E_1, E_1) = S^*(E_2, E_2) = S^*(E_3, E_3) = S^*(E_4, E_4) = S^*(E_5, E_5) = 0.$$

Therefore, it can be easily verified that the manifold is an Einstein manifold with respect to Levi-Civita connection.

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The basis of knowledge space via Boolean matrix

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Abstract. Knowledge space theory as a mathematical theory in learning evaluation is a crucial component in mathematical psychology. In real world, basis of knowledge space can be significantly much smaller than the knowledge space, which may be essential for some purposes. In this paper, a new mechanism drawn lessons from Boolean matrix is formulated to search for the basis. In this process, according to the link between knowledge space and its projections, the judgement theorems of atoms (or irreducible elements) are put forward to suit for different situations. Furthermore, an algorithm for basis is presented. And numerical experiments are conducted to evaluate the effectiveness of the proposed approach.

Keywords: algorithm, basis, boolean matrix, knowledge space.

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1. Introduction

Knowledge space theory furnishes a mathematical cognitive science for knowledge assessment with existing software. It was introduced by Falmagne, Koppen, Villano, Doignon and Johannesen [11]. It has been developed into the most effective language of knowledge representation in the self-adaptive learning and testing system [1, 3, 7, 13, 16]. On the other hand, knowledge space has become an important role of mathematical psychology.

From Doignon and Falmagne [4], the core concept of knowledge space theory is knowledge state, which gathers particular elements of knowledge and denotes by K the subset of the whole considered problems Q . A knowledge state delineates the items answered correctly in ideal condition by an individual. In other words, elements in a knowledge state are interpreted as items mastered by an individual without lucky guess and careless. Thereby the standard answers for each item are clearly classified as “correct” and “wrong”. In this case, knowledge space theory takes careless about answer process. Usually, the construction of knowledge spaces has three main approaches: the query to experts [6, 9, 15], skill map construction [2, 14] and data-driven [10, 19].

A knowledge structure is a pair $(Q, \mathcal{K})$ in which Q is a nonempty set of examined problems and $\mathcal{K}$ is a collection of subsets of Q , including at least Q and $\emptyset$. Subsets in $\mathcal{K}$ are referred to as knowledge states. It is conceivable that $\cup \mathcal{K} = Q$, $\mathcal{K}$ is occasionally identified with a knowledge structure on Q . Furthermore, a knowledge structure $(Q, \mathcal{K})$ is said to be knowledge space when $\mathcal{K}$ is closed under union. Another terminology is quasi ordinal space, a knowledge space closed under intersection. As well known, knowledge spaces are structures for surmise relations. A compressed way of storing surmise relations is to store the basis of a knowledge space. A basis of knowledge space is the smallest subset of states from which the raw space can be reconstructed by closure under union.

It should be pointed out that the knowledge space often too large for an efficient adaptive assessment even though the domain of problems is small. The extreme case is the power set of a domain with n items. In this case, the cardinal number of knowledge states is 2^n . The basis of a knowledge space is usually much smaller than the complete space which results in a less storage in computer memory. On the other hand, as we know, it is easy to determine a quasi ordinal space corresponding to a surmise relation as well as a granular knowledge space corresponding to a surmise function. In conclusion, there is correspondence between the basis and surmise relation or surmise function. Thereby basis play a crucial role in searching surmise relation and surmise function. Wild [20] proposed a wildcard-based compression for knowledge space and learning space. Dowling [6] applied the basis of a knowledge space for controlling the questioning of an expert. Subsequently, Dowling et al. [8] suggested a procedure for computing the intersection of knowledge spaces using only their basis. Matayoshi [17] has extended several results on the properties of well-graded, from which the author has used the base of knowledge space to present theorems of well-grade. In addi-

tion, Dowling [5] proposed a simple algorithm for building the base. Rusch and Wille [18] searched for the basis by transfer a knowledge space into a knowledge context according to formal concept analysis. In this paper, our results provide a better understanding of method for searching base from Boolean matrix point of view and will allow for designing better, specialized algorithm.

The paper proceeds as follows: In Section 2, we briefly review basic notions of knowledge space. In Section 3, we first investigate the properties of atoms with the perspective of covering rough sets. Then incorporating with matrix viewpoint, the connection of bases between the complete space and its projection is investigated. In this case, judgement theorems of irreducible state are developed, which becomes a deriving force for formulating a fast approach for the basis. Furthermore, experimental analysis to further illustrate the effectiveness of our method is presented in Section 4. The paper is concluded with a summary at the final part.

2. Preliminaries

Knowledge spaces can be summarized by a subfamily of their states faithfully. In other terms, any state of the knowledge space can be interpreted as forming the union of some states in the subfamily. Therefore, from reference [12], we have the following basic notions.

Definition 2.1. *The span of a family of sets $\mathcal{G}$ is the family $S(\mathcal{G})$ containing any set which is the union of some subfamily of $\mathcal{G}$.*

In this fashion, $S(\mathcal{G})$ is closed under union. Basis of a knowledge space is the collection of small sets of knowledge states which determine knowledge space by means of a span. Consequently, a knowledge space admits at least one base. And it has a unique base especially in case $\mathcal{K}$ is finite.

By definition, a minimal knowledge state is called an atom in the knowledge space. Obviously, the base of a knowledge space is formed by the collection of all the atoms. From reference [12], another characterization of the atom in a knowledge space is given as follows.

Theorem 2.1. *A state K is a knowledge space $(Q, \mathcal{K})$ is an atom if and only if $K \in \mathcal{F}$ for any states $\mathcal{F}$ such that $K = \cup \mathcal{F}$.*

Substructure and subspace in the setting have been investigated in [12]. It defined a restriction of knowledge state of a nonempty subset of domain.

Definition 2.2. *With a knowledge space $(Q, \mathcal{K})$, take any nonempty proper subset Q' of Q , the family*

$$(1) \quad \mathcal{K}_{|Q'} = \{W \subseteq Q' \mid W = K \cap Q' \text{ for some } K \in \mathcal{K}\}$$

is called the projection of $\mathcal{K}$ on Q' .

Depending on the context, it can also refer to $\mathcal{K}_{|Q'}$ as a substructure of $\mathcal{K}$. Every set $W = K \cap Q'$ is said to be the trace of the state K on Q' .

Definition 2.3. Suppose that $(Q, \mathcal{K})$ is a knowledge structure with $\cup \mathcal{K} = Q$ and $|Q| \geq 2$, and take that Q' be any proper nonempty set of Q . Define a relation $\sim_{Q'}$ on $\mathcal{K}$ by

$$(2) \quad K \sim_{Q'} L \Leftrightarrow K \cap Q' = L \cap Q'.$$

Thus, $\sim_{Q'}$ is an equivalence relation on $\mathcal{K}$. We denote by $[K]$ the equivalence class of $\sim_{Q'}$ containing K .

For any state K in $\mathcal{K}$ and with $[K]$ as in Definition 2.3, we define the family

$$(3) \quad \mathcal{K}_{[K]} = \{M \subseteq Q \mid M = L \cap Q' \text{ for some } L \sim_{Q'} K\}.$$

3. Basis of knowledge space

3.1 Perspective of covering rough sets

From the perspective of topological theory, we know that any knowledge space $(Q, \mathcal{K})$ is in fact a cover approximate space and $\mathcal{K}$ is a cover on Q . According to covering rough sets theory [21], and the base is actually the reduction of cover. Then we shall have the following definition from reference.

Definition 3.1. Let $(Q, \mathcal{K})$ be a knowledge space and $K \in \mathcal{K}$. If K is represented as the join of several states of $\mathcal{K} - \{K\}$, then K is called an reducible state of $\mathcal{K}$. Otherwise, K is referred to as irreducible state of $\mathcal{K}$.

Thus, all the atoms of $\mathcal{K}$ are the irreducible states. Then we will present the judgement approach of reducible state to trim the proof process.

Theorem 3.1. Take that K is a reducible state in knowledge space $\mathcal{K}$ and $L \in \mathcal{K}$. L is reducible in $\mathcal{K}$ if and only if it is a reducible state in $\mathcal{K} - \{K\}$.

Proof. If L is reducible in $\mathcal{K} - \{K\}$, then it is the joint of some knowledge sates in $\mathcal{K} - \{K, L\}$. Namely, L may be interpreted as the join of several states of $\mathcal{K} - \{L\}$. That is to say, L is a reducible state of $\mathcal{K}$.

Conversely, take that L is reducible in $\mathcal{K}$, then there is a sequence of states $K_1, K_2, \dots, K_t$ such that $L = \bigcup_{i=1}^t K_i$. For any $0 < i \leq t$, it is easy verified that $K_i \subset L$. In this case, if $K_i \neq K$, then we have proved L is a reducible state of $\mathcal{K} - \{K\}$. Otherwise, suppose that $K_1 = K$ without losing generality. As K is a reducible state of $\mathcal{K}$, there exists a sequence of states $L_1, L_2, \dots, L_p$ such that $K_1 = \bigcup_{i=1}^p L_i$. Since $K_1 \subset L$, we have any L_i , $0 < i \leq p$, must be unequal to L . In other terms, $L = (\bigcup_{i=1}^p L_i) \cup (\bigcup_{i=2}^t K_i)$. In such case, we conclude that L is a reducible state of $\mathcal{K} - \{K\}$. $\square$

This theorem ensures that a new reducible state will not be produced when deleting a reducible state on $\mathcal{K}$. With perspective of this, we can reduct knowledge space by removing all reducible states or one by one at every time. Ultimately, we will obtain a family of all irreducible states, which is called the reduct of $\mathcal{K}$ [21]. Namely, the base of knowledge space.

Lemma 3.1. *Let $(Q, \mathcal{K})$ is a knowledge space. Then $\mathcal{B}$ is the base of $\mathcal{K}$ if and only if for any $B \in \mathcal{B}$ and $\mathcal{F} \subset \mathcal{B}$, $B \in \mathcal{F}$ implies $B \neq \bigcup \mathcal{F}$.*

Proof. Suppose existing $B \in \mathcal{B}$ and $\mathcal{F} \subset \mathcal{B}$ such that $B \notin \mathcal{F}$ and $B = \bigcup \mathcal{F}$. Then the span $S(\mathcal{B} - \{B\}) = \mathcal{K}$, which is contradicted with the definition of base. Conversely, if $\mathcal{B}$ is not the base of $\mathcal{K}$, there is $\mathcal{B}_1 \subset \mathcal{B}$ such that $\mathcal{B}_1$ is the base. Take $B \in \mathcal{B} - \mathcal{B}_1$, there is $\mathcal{F} \subseteq \mathcal{B}_1$ satisfying $B = \bigcup \mathcal{F}$ but $B \notin \mathcal{F}$, contradiction. $\square$

In addition, the following results will be easily caught. Firstly, for any $q \in Q$, we denote

$$\sigma(q) = \{K \in \mathcal{K} | q \in K \wedge (\forall L \in \mathcal{K} \wedge q \in L \wedge L \subseteq K \Rightarrow K = L)\},$$

which is said to be the minimal description state set with respect to item q in $\mathcal{K}$. It is easily checked that any $K \in \sigma(q)$ is an atom at q . For convenience, for $\mathcal{K}' \subseteq \mathcal{K}$, the minimal description state set with respect to item q in $\mathcal{K}'$ is denoted $\sigma_{\mathcal{K}'}(q)$.

Example 1. Let $Q = \{a, b, c, d\}$ and $\mathcal{K} = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, d\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, \{a, b, c, d\}\}$. Then the minimal description with respect to every item is $\sigma(a) = \{\{a\}\}$, $\sigma(b) = \{\{b\}\}$, $\sigma(c) = \{\{b, c\}\}$, $\sigma(d) = \{\{a, d\}, \{b, c, d\}\}$.

Theorem 3.2. *Let $(Q, \mathcal{K})$ be a knowledge space and $K \in \mathcal{K}$ be reducible. Then $\sigma(q) = \sigma_{\mathcal{K} - \{K\}}(q)$ for each $q \in Q$.*

Proof. Actually, if K is reducible in $\mathcal{K}$, then $K \notin \sigma(q)$ for any $q \in Q$. However, for any $L \in \mathcal{K}$, there is K' belonging to $\mathcal{K} - \{K\}$ such that $K' \subseteq L$ when $K \subseteq L$. Thus $\sigma(q) = \sigma_{\mathcal{K} - \{K\}}(q)$ for each $q \in Q$. $\square$

For instance, $\{a, b, d\}$ is reducible in $\mathcal{K}$. It is easy to see that the minimal descriptions of all items stay the same even when removing such state from $\mathcal{K}$.

Corollary 3.1. *For a knowledge space $\mathcal{K}$ on Q and any $q \in Q$, $\sigma_{\mathcal{B}}(q) = \sigma(q)$, where $\mathcal{B}$ is the base of $\mathcal{K}$.*

3.2 Boolean matrix viewpoint

Rusch and Wille [18] transferred a knowledge space into knowledge context and used double arrows to reduce the knowledge context without changing the basis of knowledge space. In this fashion, the dual of the reduced knowledge context is

actually the base of knowledge space. This section we shall use a novel method to search for the base efficiently.

Described verbally, a subset A in Q can be regarded as a mapping $A : Q \rightarrow \{0, 1\}$, $A(q)$ being interpreted as “the degree which q is contained to A ”. That is, the characteristic mapping from Q to $\{0, 1\}$ with

$$A(q) = \begin{cases} 1, & q \in A, \\ 0, & q \notin A. \end{cases}$$

Let $Q = \{q_1, q_2, \dots, q_n\}$, then A can be denoted by a $1 \times n$ row vector, of which each coordinate is either 0 or 1. For instance, suppose that $Q = \{q_1, q_2, q_3, q_4\}$, then $A = (1, 0, 0, 1)$ means a subset $\{q_1, q_4\}$ of Q .

As inspired from Rusch and Wille [18], a knowledge space may be converted to no matter a knowledge context but also a Boolean matrix called knowledge matrix and denoted by M_K . From the perspective of calculation, we adopt Boolean matrix to find the basis of knowledge space. In addition, it should be noted that a knowledge matrix is consisted of the collection of all knowledge states with row vectors forming with a presented ordering.

Example 2. Let $K = \{\emptyset, \{q_1\}, \{q_1, q_2\}, \{q_2, q_3\}, \{q_1, q_2, q_3\}\}$ is a knowledge space with $Q = \{q_1, q_2, q_3\}$. Then the corresponding knowledge matrix is

$$M_K = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Actually, in the search of base, it is not needed to consider the two states $\emptyset$ and Q if Q is not an irreducible state. Thus the above knowledge matrix can briefly denoted by

$$M_K = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

However, if a knowledge space is trivial, then its base $\mathcal{B} = \{Q\}$. Excluding this fashion, if no otherwise specified, the knowledge matrix corresponded to a non-trivial knowledge space neglects the states $\emptyset$ as well as Q when Q is reducible. That is to say, all zero vector and all one vector are excluded by the knowledge matrix. Furthermore, for a knowledge matrix M_K , let $M_K(i, :)$ and $M_K(:, j)$ denote the i -th row and the j -th column vectors of M_K , respectively. And take $M_K(i, j)$ be the crossed member of i -th row and the j -th column vectors.

Corollary 3.2. Let $K = \{\emptyset, K_1, \dots, K_m, Q\}$ be a knowledge space with $Q = \{q_1, q_2, \dots, q_n\}$ and its basis $\mathcal{B}$. Assume that M_K is the corresponding knowledge matrix with a preseted ordering. If $|M_K(i, :)| = 1$, $0 < i \leq m$, then the corresponding state K_i belongs to $\mathcal{B}$.

Proof. Straightforward. $\square$

Corollary 3.3. *If $|M_{\mathcal{K}}(:, j)| = 1$, $0 < j \leq n$, then the corresponding state K belongs to the basis $\mathcal{B}$.*

Proof. $|M_{\mathcal{K}}(:, j)| = 1$ indicates the corresponding state K is not the joint of some other sates. That is to say, K is irreducible in $\mathcal{K}$, then $K \in \mathcal{B}$. $\square$

Summing up above corollaries, a thing they have in common is that such knowledge state belonging to the base is an atom at only one item. A bit more technically, if the state K is an atom at item q , then $K - \{q\}$ is not atom any more. Furthermore, if K and q are chopped away from $(Q, \mathcal{K})$, it is conceivable that the following conclusion can be formulated.

Theorem 3.3. *Take K as an atom at item $q \in Q$ uniquely from knowledge space $(Q, \mathcal{K})$. Then $(\mathcal{K} - \{K\})_{|Q - \{q\}}$ is a projection of $\mathcal{K} - \{K\}$ as well as a knowledge space on $Q - \{q\}$.*

Proof. With hypothesis, we have $\sigma(q) = \{K\}$. Assume that $M_{\mathcal{K}}$ is the knowledge matrix, $K = K_i \in \mathcal{K}$ and $q = q_j \in Q$. If $|M_{\mathcal{K}}(:, j)| = 1$, then any states in $\mathcal{K}$ do not contain q_j except for K_i . In other terms, any state $L \in \mathcal{K} - \{K_i\}$ can be interpreted as a join of some states in $\mathcal{B} - \{K_i\}$, where $\mathcal{B}$ is the base of $\mathcal{K}$. Namely, $(\mathcal{K} - \{K_i\})_{|Q - \{q_j\}}$ equaling to $\mathcal{K} - \{K_i\}$ is a knowledge space on $Q - \{q_j\}$. For another case with $|M_{\mathcal{K}}(i, :)| = 1$, it is easy to verify that $(\mathcal{K} - \{K_i\})_{|Q - \{q_j\}} = \mathcal{K}_{|Q - \{q_j\}}$. And thus $(\mathcal{K} - \{K_i\})_{|Q - \{q_j\}}$ is a knowledge space on $Q - \{q_j\}$ obviously. Stated thus, $(\mathcal{K} - \{K\})_{|Q - \{q\}}$ is a projection of $\mathcal{K} - \{K\}$ as well as a knowledge space on $Q - \{q\}$. $\square$

Then the following corollary will be easily got.

Corollary 3.4. *Take that $Q' = \{q \in Q \mid |\sigma(q)| = 1\}$, where $|\sigma(q)|$ is the cardinality of $\sigma(q)$. Then $(\mathcal{K} - \bigcup_{q \in Q'} \sigma(q))_{|Q - Q'}$ is a projection of $\mathcal{K} - \bigcup_{q \in Q'} \sigma(q)$ and a knowledge space on $Q - Q'$ as well.*

We examine an example of these conclusions.

Example 3. Consider the knowledge space $\mathcal{K} = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{b, c, e\}, \{a, b, c, e\}, \{a, b, c, d\}, \{a, b, c, d, e\}\}$. By definition, it is easy to obtain the knowledge matrix

$$M_{\mathcal{K}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

At the same time, we find that $Q' = \{a, b, d\}$ and $(\mathcal{K} - \bigcup_{q \in Q'} \sigma(q))|_{\{c, e\}} = \{\emptyset, \{c\}, \{e\}, \{c, e\}\}$ is obviously a knowledge space on $\{c, e\}$. Then the new knowledge matrix is

$$M_{(\mathcal{K} - \bigcup_{q \in Q'} \sigma(q))|_{\{c, e\}}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Nevertheless, expecting a learning space, many knowledge spaces may exist no knowledge space with only one item. On the other hand, atom at more than one item can be found everywhere. In such cases, we are interested in searching for the irreducible states. As a matter of fact, it is easy to formulate the following corollary explicitly incorporating with knowledge matrix.

Theorem 3.4. *Let $(Q, \mathcal{K})$ be a knowledge space without atoms at only one item and $M_{\mathcal{K}}$ is the corresponding knowledge matrix. Take that $\mathcal{G} = \{K_j \in \mathcal{K} \mid |A(K_j)| = \min_{t=1}^{|\mathcal{K}|} \{|M_{\mathcal{K}}(t, :)|\}\}$. Then any $K \in \mathcal{G}$ is an irreducible state of $\mathcal{K}$.*

Proof. By hypothesis, if K is a reducible state of $\mathcal{K}$, then K is a join of some other knowledge states, which is in contradiction with $|A(K)|$ is the minimum. $\square$

Another interesting thing coming into our eyes is that the link between the base of knowledge space and the base of its projection. Firstly, we define an equivalence relation on $\mathcal{K}$.

Definition 3.2. *Suppose that $(Q, \mathcal{K})$ is a knowledge structure with $\bigcup \mathcal{K} = Q$ and $|Q| \geq 2$, and take that Q' be any proper nonempty set of Q . Define a relation $\approx_{Q'}$ on $\mathcal{K}$ by*

$$(4) \quad K \approx_{Q'} L \Leftrightarrow K - Q' = L - Q'.$$

Thus, $\approx_{Q'}$ is an equivalence relation on $\mathcal{K}$. We denote by $[K]_{\approx}$ the equivalence class of $\approx_{Q'}$ containing K .

Lemma 3.2. *Let $\mathcal{K}$ be any $\cup$ -closed (close under union) family. Take any $Q' \subset Q$. Then the following statements are true.*

1. *If $\mathcal{K}$ is closed under intersection, then $K \approx_{Q'} \cap [K]_{\approx}$.*
2. *$[K]_{\approx}$ is $\cup$ -closed.*
3. *The function $h : [K]_{\approx} \mapsto K - Q'$ is a well defined bijection of $\mathcal{K}_{\approx}$ onto $\mathcal{K}_{|Q-Q'}$, where $\mathcal{K}_{\approx} = \{[K]_{\approx} \mid K \in \mathcal{K}\}$.*
4. *$\approx_{Q'} = \sim_{Q-Q'}$.*

Proof. (1) By hypothesis, $\cap[K]_{\approx} \in \mathcal{K}$, and $\cap[K]_{\approx} - Q' = \bigcap_{L \in [K]_{\approx}} (L - Q') = K - Q'$. Thus, $K \approx_{Q'} \cap[K]_{\approx}$.

(2) For any $L, H \in [K]_{\approx}$, we have $L - Q' = H - Q'$ and $L \cup H \in \mathcal{K}$, then $(L \cup H) - Q' = (L - Q') \cup (H - Q') = L - Q'$, which means that $L \cup H \approx_{Q'} K$. Hence, $L \cup H \in [K]_{\approx}$.

(3) That h is a well define function is due to Eq.(4). Clearly, $h(\mathcal{K}_{\approx}) = \mathcal{K}_{|Q-Q'}$ by the definitions of h and $\mathcal{K}_{|Q-Q'}$. Assume that $h([K]_{\approx}) = K - Q' = h([L]_{\approx}) = L - Q' = X$ for some $[K]_{\approx}$ and $[L]_{\approx}$ in $\mathcal{K}_{\approx}$. Whether or not $X = \emptyset$, this entails $K \approx_{Q'} L$ and thus $[K]_{\approx} = [L]_{\approx}$.

(4) Straightforward. $\square$

Example 4. Consider the knowledge space $\mathcal{K} = \{\emptyset, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, \{b, c, e\}, \{b, d, f\}, \{a, b, c, d\}, \{a, b, c, e\}, \{b, c, d, e\}, \{b, c, d, f\}, \{b, c, e, f\}, \{a, b, d, f\}, \{a, b, c, d, e\}, \{a, b, c, d, f\}, \{a, b, c, e, f\}, \{b, c, d, e, f\}, \{a, b, c, d, e, f\}, \{a, b, d, c, e, f, g\}\}$. Take $Q' = \{b, c, e, g\}$. Then

1. $[\{a, b, d, f\}]_{\approx} = \{\{a, b, c, d, e, f, g\}, \{a, b, c, d, e, f\}, \{a, b, c, d, f\}, \{a, b, d, f\}\},$
2. $[\{b, d, f\}]_{\approx} = \{\{b, d, e, f\}, \{b, c, d, f\}, \{b, d, f\}\},$
3. $[\{a, b, d\}]_{\approx} = \{\{a, b, c, d, e\}, \{a, b, c, d\}, \{a, b, d\}\},$
4. $[\{b, d\}]_{\approx} = \{\{b, c, d, e\}, \{b, c, d\}, \{b, d\}\},$
5. $[\{a, c\}]_{\approx} = \{\{a, b, c, e\}, \{a, b, c\}, \{a, b\}, \{a, c\}\},$
6. $[\{c\}]_{\approx} = \{\emptyset, \{b\}, \{c\}, \{b, c\}, \{b, c, e\}\},$
7. $[\{a, b, c, e, f\}]_{\approx} = \{\{a, b, c, e, f\}\},$
8. $[\{b, c, e, f\}]_{\approx} = \{\{b, c, e, f\}\}.$

As illustrated before, a quasi ordinal space is a knowledge space closed under intersection. In the first place, we consider the base of quasi ordinal space in the following.

Theorem 3.5. *Let $(Q, \mathcal{K})$ be a quasi ordinal space and $\mathcal{B}$ is the base of $\mathcal{K}$. Denote $\mathcal{L} = \{L \in \mathcal{K} : |L| = \min\{|K| \mid K \in \mathcal{K} - \{\emptyset\}\}\}$ and $Q' = \bigcup \mathcal{L}$. If $Q - Q' \neq \emptyset$, then $\mathcal{L} \subset \mathcal{B}$ and $\mathcal{B}_{|Q-Q'} = \{B - Q' \mid B \in \mathcal{B}, B - Q' \neq \emptyset\}$ is the base of $\mathcal{K}_{|Q-Q'}$.*

Proof. For any $L \in \mathcal{L}$, by definition, L is obviously an irreducible state in $\mathcal{K}$. Then $\mathcal{L} \subseteq \mathcal{B}$. Next, if $\mathcal{B}_{|Q-Q'}$ is not the base of $\mathcal{K}_{|Q-Q'}$, from Lemma 1, there exists $W \in \mathcal{B}_{|Q-Q'}$ and $\mathcal{F} \subset \mathcal{B}_{|Q-Q'}$ such that $W = \bigcup \mathcal{F}$ while $W \notin \mathcal{F}$. By definition of $\mathcal{B}_{|Q-Q'}$, a state B in $\mathcal{B}$ satisfying $W = B - Q'$ as well as $K \in \mathcal{B}$ satisfying $F = K - Q'$ for $F \in \mathcal{F}$ can be found. Take $H_F = B \cap K$, then $H_F \in \mathcal{K}$ as $\mathcal{K}$ is a quasi ordinal space. Thus there is $\mathcal{H}_F \subset \mathcal{B}$ such that $H_F = \bigcup \mathcal{H}_F$ for any $F \in \mathcal{F}$. Let $A = B - \bigcup \{H_F \mid F \in \mathcal{F}\}$, then $A \subset Q - Q'$. Since $\mathcal{K}$

is a quasi ordinal space, we have $A \in \mathcal{K}$. Thus there exists $\mathcal{G} \subset \mathcal{B}$ such that $A = \bigcup \mathcal{G}$. In this fashion, it is easy to see that $\bigcup(\{\mathcal{H}_F | F \in \mathcal{F}\} \cup \{\mathcal{G}\}) \subset \mathcal{B}$ and $B = \bigcup(\{\mathcal{H}_F | F \in \mathcal{F}\} \cup \{\mathcal{G}\})$. This contradicts to the definition of $\mathcal{B}$. $\square$

More normally, if Q' is any nonempty proper subset of Q , then we shall have the following verdict.

Corollary 3.5. *Let $(Q, \mathcal{K})$ be a quasi ordinal space and $\emptyset \neq Q' \subset Q$. Then $\mathcal{B}_{|Q-Q'} = \{B - Q' | B \in \mathcal{B}, B - Q' \neq \emptyset\}$ is the base of $\mathcal{K}_{|Q-Q'}$.*

Proof. Straightforward. $\square$

Example 5. Let $\mathcal{K} = \{\emptyset, \{a, b\}, \{d, e\}, \{a, b, f\}, \{c, d, e\}, \{a, b, d, e\}, \{a, b, c, d, e\}, \{a, b, d, e, f\}, Q\}$ be a quasi ordinal space with $Q = \{a, b, c, d, e, f\}$. From Theorem 3.5, $\mathcal{L} = \{\{a, b\}, \{d, e\}\}$. Take $Q' = \bigcup \mathcal{L} = \{a, b, d, e\}$. Obviously, $\{a, b\}, \{d, e\} \in \mathcal{B}$ and $\mathcal{B}_{|Q-Q'} = \{\{c\}, \{f\}\}$ is the basis of $\mathcal{K}_{|Q-Q'}$. From knowledge matrix, the process as follows:

$$M_{\mathcal{K}} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} \Rightarrow M_{\mathcal{K}_{|Q-Q'}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

However, if $\mathcal{K}$ is a knowledge space without closing under intersection, it is worth pointing out that $B - Q'$ is not necessarily in $\mathcal{B}_{|Q-Q'}$ for some $B \in \mathcal{B}$, where $\mathcal{B}$ and $\mathcal{B}_{|Q-Q'}$ are the bases of $\mathcal{K}$ and $\mathcal{K}_{|Q-Q'}$, respectively. For instance, from Example 4, we know $\mathcal{K}_{|Q-Q'} = \{\emptyset, \{a\}, \{d\}, \{f\}, \{a, d\}, \{a, f\}, \{d, f\}, \{a, d, f\}\}$ and the state $\{a, b, c, d, e, f, g\}$ is an irreducible state of $\mathcal{K}$.

Nevertheless, $\{a, b, c, d, e, f, g\} - Q' = \{a, d, f\}$ is a reducible state of $\mathcal{K}_{|Q-Q'}$.

Fortunately, considering Q' a knowledge state in the given knowledge space, excellent results will be came out as follows.

Theorem 3.6. *Let $(Q, \mathcal{K})$ be a knowledge space with knowledge matrix $M_{\mathcal{K}}$. Take Q' a nonempty state in $\mathcal{K}$ such that $|Q'| < \max_{i=1}^{|\mathcal{K}|} \{|M_{\mathcal{K}}(i, :)|\}$. Assume that $\mathcal{B}$ and $\mathcal{B}_{|Q-Q'}$ are the bases of $\mathcal{K}$ and $\mathcal{K}_{|Q-Q'}$, respectively. Then the following conclusions hold.*

1. $\mathcal{B}_{|Q-Q'} = \{B - Q' | B \in \mathcal{B} \wedge (B - Q') \neq \emptyset\}$.
2. If $L \in [K]_{\approx}$ is a reducible state such that $L - Q' \in \mathcal{B}_{|Q-Q'}$, then $\min\{|B| | B \in [K]_{\approx}\} < |L|$.
3. If $L - Q' \in \mathcal{B}_{|Q-Q'}$ and $\min\{|B| | B \in [K]_{\approx}\} = |L|$ for $L \in [K]_{\approx}$, then $L \in \mathcal{B}$.

Proof. (1) Suppose $B \in \mathcal{K}$ with $B - Q' \neq \emptyset$. So B is not included in Q' . If $B \cap Q' = \emptyset$, then $B - Q'$ is obviously in $\mathcal{B}_{|Q-Q'}$. Conversely, if $B \cap Q' \neq \emptyset$, assume that $B - Q'$ be a reducible state in $\mathcal{K}_{|Q-Q'}$. Then there exists $\mathcal{W} \subseteq \mathcal{B}_{|Q-Q'}$ such that $B - Q' = \bigcup \mathcal{W}$. Actually, there are some $L \in \mathcal{K}$ satisfying $W = L - Q'$ for each $W \in \mathcal{W}$. Without loss of generality, take $\mathcal{L} \subseteq \mathcal{K}$ such that any state in it is one-to-one correspondence with that in $\mathcal{W}$. So it is easy to see that $B - Q' = \bigcup \mathcal{L} - Q'$. Furthermore, there is $\mathcal{H} \subseteq \mathcal{B}$ such that $\bigcup \mathcal{L} = \mathcal{H}$ as $\bigcup \mathcal{L}$ is a state in $\mathcal{K}$. That is to say, $B - Q' = \bigcup \mathcal{H} - Q'$. Moreover, $B = B \cup (B - Q') = (\bigcup \mathcal{H} \cup B) - B \cap Q'$. In other terms, $B \cap Q' \in \bigcup \mathcal{H}$. Since $B - Q' = \bigcup \mathcal{H} - Q'$, it must be $B \subseteq \bigcup \mathcal{H}$. Therefore, $B = (\bigcup \mathcal{H} \cup B) - B \cap Q' = \bigcup \mathcal{H}$. If $|\mathcal{H}| > 1$, then it is contradicted to B is an irreducible state. Otherwise, $\mathcal{H} = \{B\}$. This implies $B - Q'$ is an irreducible state in $\mathcal{B}_{|Q-Q'}$.

Next we prove that any state in $\mathcal{B}_{|Q-Q'}$ corresponds to a state in $\mathcal{B}$. Conversely, suppose this condition is not fulfilled by $W \in \mathcal{B}_{|Q-Q'}$. By definition, there is $\mathcal{L} \subseteq \mathcal{K}$ such that $W = L - Q'$ for any $L \in \mathcal{L}$. Then such state L is not in $\mathcal{B}$. Furthermore, we can find $\mathcal{F} \subseteq \mathcal{B}$ with $L = \bigcup \mathcal{F}$. In other word, $W = \bigcup_{F \in \mathcal{F}} (F - Q')$ and $F - Q' \neq \emptyset$. This implies W is a reducible state since $F - Q' \in \mathcal{B}_{|Q-Q'}$ for any $F \in \mathcal{F}$, a contradiction.

(2) Since $L - Q' \in \mathcal{B}_{|Q-Q'}$, there exists $B \in \mathcal{B}$ satisfying $L - Q' = B - Q'$. Now we only need to present $|B| < |L|$. If not, suppose that there is an index set T of $\mathcal{B}$ such that $L = \bigcup_{i \in T} B_i$ as L is a reducible state, then $B \notin \{B_i | i \in T\}$. However, $B - Q' = (\bigcup_{i \in T} B_i) - Q' = \bigcup_{i \in T} (B_i - Q')$, which contradicts to $B - Q'$ belonging to $\mathcal{B}_{|Q-Q'}$ in the light of item (1). In conclusion, $\min\{|B| | B \in [K]_{\approx}\} < |L|$.

(3) It follows from item (2). $\square$

Corollary 3.6. Let $\mathcal{K}_{|Q-Q'}$ be a projection of knowledge space $\mathcal{K}$ on a nonempty proper subset Q' of Q . Assume that $\mathcal{B}$ and $\mathcal{B}_{|Q-Q'}$ are the base of $\mathcal{K}$ and $\mathcal{K}_{|Q-Q'}$, respectively. For any $K \in \mathcal{K}$, K is an irreducible state if the following conditions hold

1. if $K - Q' \neq \emptyset$, then $K - Q' \in \mathcal{B}_{|Q-Q'}$;
2. $|K - Q'| \leq |W|$ for any $W \in [K]_{\approx}$.

Proof. It follows from Theorem 3.6. $\square$

However, there is no evidence to see that the opposite is true.

Example 6. Consider the knowledge space $\mathcal{K}$ from Example 4, take $Q' = \{a, b, d, f\}$. By definition of projection, $\mathcal{K}_{|Q-Q'} = \{\emptyset, \{c\}, \{c, e\}, \{c, e, g\}\}$. It is easily checked that $\{\{b, c, e\}, \{a, b, c, e\}, \{b, c, e, f\}, \{a, b, c, d, e\}, \{a, b, c, e, f\}, \{b, c, d, e, f\}, \{a, b, c, d, e, f\}\}$ is a equivalence class corresponding the state $\{c, e\}$ in $\mathcal{K}_{|Q-Q'}$. Since $\{c, e\}$ is an irreducible state and $|\{b, c, e\}| = 3$ is the minimum in its class, then $\{b, c, e\}$ is an irreducible state evidently in $\mathcal{K}$. In addition, $\{b, c, e, f\}$ is an irreducible state in $\mathcal{K}$ while it does not meet the condition (2) of above corollary.

Similar to Theorem 3.5, we shall have the following corollary

Corollary 3.7. *Take that $\mathcal{L} = \{L \in \mathcal{K} \mid |L| = \min\{|K| \mid K \in \mathcal{K} - \{\emptyset\}\}\}$ from knowledge space $(Q, \mathcal{K})$. Let $Q' = \bigcup \mathcal{L}$. If $Q - Q' \neq \emptyset$, then $\mathcal{L} \subset \mathcal{B}$ and $\mathcal{B}_{|Q-Q'} = \{B - Q' \mid B \in \mathcal{B}, B - Q' \neq \emptyset\}$.*

Proof. It follows immediately from the item (1) of Theorem 3.6. $\square$

Conversely, we shall have the following result to search for irreducible states from projection to the raw knowledge space. We first present a fact to trim some proofs. Actually, for any $Q' \in \mathcal{K} - \{\emptyset, Q\}$, there is a projection function $\delta : 2^{Q-Q'} \rightarrow 2^Q$ defined by

$$(5) \quad \delta(L) = \{K \in \mathcal{K} \mid K - Q' = L\}, \quad L \in \mathcal{K}_{Q-Q'}.$$

Obviously, δ is a surjection. In addition, it is easily verified that $\delta(L) = [K]_{\approx}$.

Theorem 3.7. *Let $(Q, \mathcal{K})$ be a knowledge space and $Q' \in \mathcal{K} - \{\emptyset, Q\}$. Take that*

$$(6) \quad \mathcal{L} = \{L \in \mathcal{K}_{Q-Q'} \mid |L| = \min\{|K| \mid K \in \mathcal{K}_{Q-Q'} - \{\emptyset\}\}\}.$$

For $L \in \mathcal{L}$, if $K \in \delta(L)$ such that $|K| = \min\{|H| \mid H \in \delta(L)\}$, then $K \in \mathcal{B}$.

Proof. By hypothesis, any $L \in \mathcal{L}$ is irreducible. Otherwise, it will be a reducible state. Then L can be generated by forming the union of some knowledge states in $\mathcal{K}_{Q-Q'}$, which contradicts to $|L|$ is the minimum. Then according to the item (3) of Theorem 3.6, we have $K \in \mathcal{B}$. $\square$

The next chief problem attaches our attention is the connection of bases between $\mathcal{K}$ and its sub-subspace, that is, the projection of the projection of $\mathcal{K}$.

Theorem 3.8. *Let $(Q, \mathcal{K})$ be a knowledge space and $Q' \in \mathcal{K} - \{\emptyset, Q\}$. Denote by $\mathcal{K}' = \mathcal{K}_{|Q-Q'}$ and take $Q'' \in \mathcal{K}' - \{\emptyset, Q'\}$. Suppose that*

$$(7) \quad \mathcal{H} = \{H \in \mathcal{K}'_{Q'-Q''} \mid |H| = \min\{|K| \mid K \in \mathcal{K}'_{Q'-Q''} - \{\emptyset\}\}\}.$$

For any $H \in \mathcal{H}$, there exists $K \in \mathcal{K}$ with $H = K - Q' \cup Q''$ such that $K \in \mathcal{B}$.

Proof. It follows from Theorem 3.7. $\square$

It is worth mentioning that the inspiration of these results is came from Boolean matrix. That is to say, all of them can be realised by computer. As discussed by these results, we shall develop a fast algorithm to search the base of knowledge space in the following.

It is easy to see that the algorithm is correct and has execution time in $O(|Q||\mathcal{K}|)$. However, the algorithm (SA) for constructing the base from [12] needs $O(|Q||\mathcal{K}|^2)$, and the complexity of the Rusch-Wille approach [18] from formal concept point of view takes $O(|Q||\mathcal{K}|^2 + |Q|^2|\mathcal{K}|)$.

Algorithm 1 Construction of basis based on covering rough set (CBCR)

Input: A knowledge space $\mathcal{K} = \{K_1, K_2, \dots, K_m\}$ with $Q = \{q_1, q_2, \dots, q_n\}$.

Output: The basis $\mathcal{B}$ of $\mathcal{K}$.

- 1: Generate knowledge matrix $M_{\mathcal{K}}$ and set $\mathcal{B} \leftarrow \emptyset$.
 - 2: Select state K_i such that $|M_{\mathcal{K}}(i, :)| = 1$ or there exists $j \leq n$ satisfying $|M_{\mathcal{K}}(i, j)| = 1$ and $|M_{\mathcal{K}}(:, j)| = 1$.
 - 3: $\mathcal{B} \leftarrow \{K_i\}$ and then update $M_{\mathcal{K}}$ to $M_{\mathcal{K}_{Q-Q'}}$ based on Theorem 3.3.
 - 4: **While** $M_{\mathcal{K}_{Q-Q'}} \neq \emptyset$ **do**
 - 5: Perform Step 2 and 3, and if such conditions of Step 2 do not satisfy, go to Step 6.
 - 6: $\mathcal{B} \leftarrow \mathcal{B} \cup \{K_t\}$, where K_t such that item (3) of Theorem 3.7.
 - 7: Update $M_{\mathcal{K}_{Q-Q'}}$
 - 8: **End while**
 - 9: Output $\mathcal{B}$.
-

Example 7. Let $(Q, \mathcal{K})$ be a knowledge space with $Q = \{q_1, \dots, q_7\}$ and $\mathcal{K} = \{K_0, K_1, \dots, K_{15}\}$, where $K_0 = \emptyset$ and $K_{15} = Q$. The knowledge matrix by above proper order is presented in the following:

$$M_{\mathcal{K}} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Firstly, from Step 2 and 3, we have $\{K_7\} \in \mathcal{B}$ since $M_{\mathcal{K}}(7, :) = (0, 0, 0, 0, 0, 0, 1)$, and then

$$M_{\mathcal{K}_{Q-\{q_7\}}} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

From Eq.(6), $\mathcal{L} = \{\{q_2, q_3\}, \{q_1, q_5\}, \{q_4, q_6\}\}$. By Eq.(5), We can easily check that $\delta(\{q_2, q_4\}) = \{K_1, K_9\}$, $\delta(\{q_1, q_5\}) = \{K_2, K_{10}\}$, and $\delta(\{q_4, q_6\}) = \{K_4\}$. Comparing with knowledge matrix $M_{\mathcal{K}}$, we have $|K_1| < |K_9|$, $|K_2| < |K_{10}|$.

Hence, $K_1, K_2, K_4 \in \mathcal{B}$. Update $M_{Q-\{q_7\}}$ from Step 7, the knowledge matrix of the projection becomes empty. Therefore $\mathcal{B} = \{K_1, K_2, K_4, K_7\}$.

4. Experimental analysis

In this section, to further illustrate the effectiveness of our proposed algorithm, we compare it with some existing algorithms for the bases of knowledge spaces. The experiments were performed on a personal computer with Windows 10 and Intel (R) Core (TM) i7-6700 CPU @ 3.41GHz with 8GB of memory. The algorithms are implemented by Matlab 9.3.

To achieve this task, we need some knowledge spaces. However, it is difficult to search for knowledge spaces in real life, we design a generation algorithm to construct knowledge spaces randomly. All knowledge spaces used in experimental analysis are randomly generated, which are shown in Table 1. It should be point out that every knowledge space depicted in Table 1 is closed under intersection.

Table 1: Data description of the 15 knowledge spaces

Knowledge spaces	Items	States
K1	14	5376
K2	19	207
K3	14	1036
K4	15	2576
K5	18	9216
K6	18	384
K7	17	2400
K8	17	6016
K9	22	73
K10	19	66
K11	15	4068
K12	21	928
K13	16	1388
K14	18	520
K15	14	2736

The algorithms Dowling and Wille used to numerical experiments are from References [5] and [18], respectively. And the comparison results is depicted in Table 2. From Table 2, we see that our method CBCR was the fastest on the most knowledge spaces except K5. In addition, Dowling was the most time-consuming. In K5, our approach was only 0.02 seconds longer than Wille while much faster than Dowling. To sum it up, we can conclude that the superior computational efficiency of our proposed method over the others.

Table 2: Comparison of the running time(s)

Knowledge spaces	CBCR	Dowling	Wille
K1	0.09	66.2	0.18
K2	0.01	0.11	0.18
K3	0.04	5.61	0.18
K4	0.05	15.16	0.2
K5	0.22	193.46	0.2
K6	0.01	0.36	0.2
K7	0.05	12.92	0.21
K8	0.12	75.47	0.23
K9	0.01	0.02	0.32
K10	0.08	62.41	0.32
K11	0.08	45.45	0.32
K12	0.06	1.95	0.32
K13	0.06	4.12	0.33
K14	0.02	0.65	0.33
K15	0.05	16.14	0.33

5. Conclusion and future work

Base plays an important role in knowledge space. In this article, we mainly focused on the base incorporating with the projection of knowledge space from the viewpoints of covering rough set theory and Boolean matrix. In this case, the connections of bases between raw knowledge space and its different projections have been formulated. With this mechanism, a fast approach for base has been designed subsequently. The experiment results have demonstrated that our proposed method has a superior performance on the running time by comparing with the existing algorithms. In summary, the proposed model can effectively preprocess the base of knowledge spaces.

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Coincidence and fixed point results for (Ψ, L) - M -weak contraction mapping on Mb -metric spaces

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Abstract. In this paper, we introduce the $(\Psi, L) - M - weak$ contraction and we prove some common fixed point results for self-mappings T and S and some fixed point results for a single mapping T by using a $(c) - comparison$ function and a comparison function in the sense of $Mb - metric$ space.

Keywords: D -metric space, b -metric space, common fixed point.

1. Introduction

In 2020, Malkawi et al [1], defined $Mb - metric$ space, the idea of this definition has weaker than the tetrahedral inequality axiom for $D - metric$ space [2] and generalization of b -metric [3,4].

Also, many authors gives many fixed point theorems in a $b - metric$ space and Quasi Metric Spaces (see, [6-22]).

We begin by presenting the definition of $Mb - metric$ space.

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Definition 1.1 ([1]). Let X be a non empty set and $R > 1$ be a real number. A function $M : X \times X \times X \rightarrow [0, \infty)$ is called Mb – metric space, if it satisfies the following properties for each $\zeta, \eta, \gamma \in X$.

$$(M1) : M(\zeta, \eta, \gamma) \geq 0.$$

$$(M2) : M(\zeta, \eta, \gamma) = 0 \text{ iff } \zeta = \eta = \gamma.$$

$$(M3) : M(\zeta, \eta, \gamma) = M(p(\zeta, \eta, \gamma)); \text{ for any permutation } p(\zeta, \eta, \gamma) \text{ of } \zeta, \eta, \gamma.$$

$$(M4) : M(\zeta, \eta, \gamma) \leq R[M(\zeta, \nu, \gamma) + M(\zeta, \nu, \eta) + M(\nu, \eta, \gamma)].$$

and a pair (X, M) is called Mb – metric space.

The following Mb – Cauchy, Mb – convergent and Mb – complete spaces are defined by Malkawi et al. [1]

Definition 1.2 ([1]). Let (X, M) be an Mb – metric space. A sequence $\{\zeta_n\}$ on X is said to be

(I) Mb – Cauchy iff for given $\epsilon > 0$, there exists a positive integer N such that $M(\zeta_n, \zeta_m, \zeta_p) < \epsilon$ for all $m, n, p \geq N$.

(II) Mb – convergent iff there exists an element ζ in X such that for given $\epsilon > 0$, there exists a positive integer N such that $M(\zeta_n, \zeta_m, \zeta) < \epsilon$ for all $m, n \geq N$. So we called $\{\zeta_n\}$ Mb – converges to ζ and ζ is a limit of $\{\zeta_n\}$.

(III) The Mb – metric space (X, M) is Mb – complete if every Mb – Cauchy sequence is Mb – convergent.

In the following definition, Shatanawi [5] is defined (c)-comparison function with base R .

Definition 1.3 ([5]). Let R be a constant $R \geq 1$. A map $\Psi : [0, +\infty) \rightarrow [0, +\infty)$ is called a (c) – comparison function with base R if Ψ satisfies the following:

(i) Ψ is monotone increasing,

(ii) $\sum_{n=0}^{\infty} R^n \Psi^n(Rt)$ converges for all $t \geq 0$.

Now, we present some important definitions and theories.

Definition 1.4. A single-valued mapping $f : X \rightarrow X$ is called a strong almost contraction if there a constant $\delta \in [0, 1]$ some $L \geq 0$ and for $R \geq 1$ such that

$$M(f\zeta, f\eta, f\gamma) \leq \frac{\delta}{R} \max\{RM(\zeta, \eta, \gamma), RM(\zeta, \zeta, f\zeta), RM(\eta, \eta, f\eta), RM(\gamma, \gamma, f\gamma), \\ \frac{1}{2}[M(\zeta, \zeta, f\eta) + M(\eta, \eta, f\gamma) + M(\gamma, \gamma, f\zeta)]\} + LM(\eta, \eta, f\gamma),$$

for all $\zeta, \eta, \gamma \in X$.

Definition 1.5. Let (X, M) be an Mb – metric space. Let $T : X \rightarrow X$ be mapping, a map T is called a (δ, L) – weak contraction if there exist a constant $\delta \in [0, 1]$ some $L \geq 0$ and for $R \geq 1$ such that

$$(1.1) \quad M(T\zeta, T\eta, T\gamma) \leq \frac{\delta}{R} M(\zeta, \eta, \gamma) + LM(\eta, \eta, T\gamma).$$

Because of the symmetry of the distance, the (δ, L) -weak contraction condition implicitly includes the following dual one:

$$(1.2) \quad M(T\zeta, T\eta, T\gamma) \leq \frac{\delta}{R}M(\zeta, \eta, \gamma) + LM(\zeta, \zeta, T\eta).$$

Thus by (1.1) and (1.2), the of (δ, L) -weak contraction condition can be replaced by the following condition:

$$M(T\zeta, T\eta, T\gamma) \leq \frac{\delta}{R}M(\zeta, \eta, \gamma) + L\min\{M(\eta, \eta, T\gamma), M(\zeta, \zeta, T\eta)\}.$$

Definition 1.6. Let (X, M) be an Mb -metric space. A map T is called (Ψ, L) -weak contraction if there exist a comparison function Ψ , some $L \geq 0$ and for all $R \geq 1$ such that

$$(1.3) \quad M(T\zeta, T\eta, T\gamma) \leq \frac{1}{R}\Psi(RM(\zeta, \eta, \gamma)) + LM(\eta, \eta, T\gamma).$$

Because of the symmetry of the distance, the (Ψ, L) -weak contraction condition implicitly includes the following dual one:

$$(1.4) \quad M(T\zeta, T\eta, T\gamma) \leq \frac{1}{R}\Psi(RM(\zeta, \eta, \gamma)) + LM(\zeta, \zeta, T\eta).$$

Thus by (1.3) and (1.4), the (Ψ, L) -weak contraction condition can be replaced by the following condition:

$$M(T\zeta, T\eta, T\gamma) \leq \frac{1}{R}\Psi(RM(\zeta, \eta, \gamma)) + L\min\{M(\eta, \eta, T\gamma), M(\zeta, \zeta, T\eta)\}.$$

Theorem 1.1. Assume that $\zeta_n \rightarrow \gamma$ as $n \rightarrow +\infty$ in an Mb -metric space (X, M) such that $M(\gamma, \gamma, \gamma) = 0$. Then $\lim_{n \rightarrow +\infty} M(\zeta_n, \zeta_n, \eta) = M(\gamma, \gamma, \eta)$ for every $\eta \in X$.

Proof. The proof is clear. □

Theorem 1.2. Let (X, M) be an Mb -complete metric space and $T : X \rightarrow X$ be a (Ψ, L) -weak contraction mapping with a (c) -comparison function, with M are satisfies

$$(1.5) \quad M(\zeta, \eta, \eta) \leq R[M(\zeta, \gamma, \gamma) + M(\gamma, \eta, \eta)]$$

and

$$(1.6) \quad M(\zeta, \zeta, \eta) = M(\zeta, \eta, \eta),$$

then T has a fixed point.

Proof. Let $\zeta_0 \in X$ and $\zeta_n = T\zeta_{n+1}$ for all $n \in \mathbb{N}$. Now T is a (Ψ, L) - weak contraction, we get

$$\begin{aligned} M(T\zeta_{n-1}, T\zeta_{n-1}, T\zeta_n) &\leq \frac{1}{R} \Psi(RM(\zeta_{n-1}, \zeta_{n-1}, \zeta_n) + LM(\zeta_n, \zeta_n, T\zeta_{n-1})) \\ &= \frac{1}{R} \Psi(RM(\zeta_{n-1}, \zeta_{n-1}, \zeta_n)) \end{aligned}$$

thus

$$M(\zeta_n, \zeta_n, \zeta_n) \leq \frac{1}{R} \Psi(RM(\zeta_{n-1}, \zeta_{n-1}, \zeta_n))$$

by induction, we attain

$$M(\zeta_n, \zeta_n, \zeta_{n+1}) \leq \frac{1}{R} \Psi^n(RM(\zeta_0, \zeta_0, \zeta_1))$$

for all $n \in \mathbb{N}$. By (1.5), (1.6), for $m > n$, we have

$$\begin{aligned} M(\zeta_n, \zeta_n, \zeta_m) &\leq R[M(\zeta_n, \zeta_n, \zeta_{n+1}) + M(\zeta_{n+1}, \zeta_m, \zeta_m)] \\ &= R[M(\zeta_n, \zeta_n, \zeta_{n+1}) + M(\zeta_{n+1}, \zeta_{n+1}, \zeta_m)] \\ &\leq RM(\zeta_n, \zeta_n, \zeta_{n+1}) + R^2[M(\zeta_{n+1}, \zeta_{n+1}, \zeta_{n+2}) + M(\zeta_{n+2}, \zeta_{n+2}, \zeta_m)] \\ &\leq \dots \leq \sum_{k=n}^{m-1} R^k M(\zeta_k, \zeta_k, \zeta_{k+1}) \\ &\leq \sum_{k=n}^{\infty} R^k M(\zeta_k, \zeta_k, \zeta_{k+1}) \\ &\leq \sum_{k=n}^{\infty} \frac{1}{R} \Psi^k(RM(\zeta_0, \zeta_0, \zeta_1)). \end{aligned}$$

Then $\sum_{k=n}^{\infty} R^k \Psi^k(RM(\zeta_0, \zeta_0, \zeta_1))$ is Mb - convergent because Ψ is a (c) - comparison function, and so $\{\zeta_n\}$ is an Mb - Cauchy sequence in X . Since X is Mb - complete and then $\{\zeta_n\}$ is Mb - converges with respect to τ_M . to a point $\gamma \in X$ such that $\lim_{n \rightarrow \infty} M(\zeta_n, \zeta_n, \gamma) = M(\gamma, \gamma, \gamma) = 0$.

We claim that $M(\gamma, \gamma, T\gamma) = 0$. By contradiction suppose that $M(\gamma, \gamma, T\gamma) > 0$. Since Ψ is (c) - comparison function, then $\Psi(t) < t$ for $t > 0$. Also since $\lim_{n \rightarrow \infty} M(\zeta_n, \zeta_n, \gamma) = M(\gamma, \gamma, \gamma) = 0$, then there exist $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $M(\zeta_n, \zeta_n, \gamma) < \frac{M(\gamma, \gamma, T\gamma)}{2R}$.

Consequently, we get

$$\begin{aligned} M(\gamma, \gamma, T\gamma) &\leq R[M(\gamma, \gamma, \zeta_{n+1}) + M(\zeta_{n+1}, \zeta_{n+1}, T\gamma)] \text{ by (1.5) and (1.6)} \\ &= R[M(\gamma, \gamma, \zeta_{n+1}) + M(T\zeta_n, T\zeta_n, T\gamma)] \\ &\leq R \left[M(\gamma, \gamma, \zeta_{n+1}) + \frac{1}{R} \Psi(RM(\zeta_n, \zeta_n, \gamma)) + LM(\gamma, \gamma, \zeta) \right] \\ &< R \left[M(\gamma, \gamma, \zeta_{n+1}) + \frac{M(\gamma, \gamma, T\gamma)}{2R} + LM(\gamma, \gamma, \zeta_{n+1}) \right]. \end{aligned}$$

By (1.6) and letting $n \rightarrow \infty$, we attain

$$0 < M(\gamma, \gamma, T\gamma) < \frac{M(\gamma, \gamma, T\gamma)}{2}$$

which is a contradiction. Therefore, $M(T\gamma, \gamma, \gamma) = 0$ and $\gamma = T\gamma$. $\square$

Theorem 1.3. *Let (X, M) be an Mb -complete metric space and $T : X \rightarrow X$ be a (Ψ, L) -weak contraction mapping, suppose T satisfies the following condition: There exist a comparison function Ψ_1 and some $L_1 > 0$, such that*

$$(1.7) \quad M(T\zeta, T\zeta, T\eta) \leq \frac{1}{R}\Psi_1(RM(\zeta, \zeta, \eta)) + L_1M(\zeta, \zeta, T\eta), \text{ for } R > 1$$

for all $\zeta, \eta, \gamma \in X$. Then T has a unique fixed point.

Proof. Suppose that, there are two fixed point γ and ν of T . If $M(\gamma, \gamma, \nu) = 0$, it is obvious that $\gamma = \nu$. Assume that $M(\gamma, \gamma, \nu) > 0$. For (1.7) with $\zeta = \gamma$ and $\eta = \nu$, we have

$$\begin{aligned} 0 < M(\gamma, \gamma, \nu) &= M(T\gamma, T\gamma, T\nu) \\ &\leq \frac{1}{R}\Psi_1(RM(\gamma, \gamma, \nu)) + L_1M(\gamma, \gamma, T\gamma) \\ &= \frac{1}{R}\Psi_1(RM(\gamma, \gamma, \nu)) \\ &< M(\gamma, \gamma, \nu) \end{aligned}$$

which is a contradiction. Therefore T has a unique fixed point. $\square$

2. The main result

In this section, we present some of the main findings related to the fixed point theories and $(\Psi, L) - M - weak$ contraction mapping, and we start with the following two definitions.

Definition 2.1. *Let (X, M) be an Mb -metric space and $T, S : X \rightarrow X$ be two mappings. The pair (T, S) is called a $(R, L) - M - weak$ contraction if for $R \geq 1$ and some $L \geq 0$. such that*

$$\begin{aligned} M(T\zeta, T\zeta, S\eta) &\leq \frac{1}{R} \max\{RM(\zeta, \zeta, \eta), RM(\zeta, \zeta, T\zeta), RM(\eta, \eta, T\eta), \\ (2.1) \quad &\frac{1}{2}(M(\zeta, \zeta, S\eta) + M(T\zeta, \eta, \eta))\} \\ &+ L \min\{M(\zeta, \zeta, S\eta), M(T\zeta, \eta, \eta)\}. \end{aligned}$$

for all $\zeta, \eta \in X$.

Definition 2.2. Let (X, M) be an Mb -metric space and $T, S : X \rightarrow X$ be two mappings. The pair (T, S) is called a $(\Psi, L) - M$ -weak contraction if there exist a control function, some $L \geq 0$ and for $R \geq 1$. such that

$$(2.2) \quad \begin{aligned} M(T\zeta, T\zeta, S\eta) &\leq \frac{1}{R} \Psi(\max\{RM(\zeta, \zeta, \eta), RM(\zeta, \zeta, T\zeta), RM(\eta, \eta, T\eta), \\ &\frac{1}{2}M(\zeta, \zeta, S\eta) + M(T\zeta, \zeta, \zeta)\}) \\ &+ L \min\{M(\zeta, \zeta, S\eta), M(T\zeta, \eta, \eta)\}. \end{aligned}$$

for all $\zeta, \eta \in X$.

Now, we present the first result using an Mb -complete metric space and $(\Psi, L) - M$ -weak contraction to give a common fixed point for two maps T and S .

Theorem 2.1. Let (X, M) be an Mb -complete metric space and $T, S : X \rightarrow X$ be two mappings such that the pair (T, S) is a $(\Psi, L) - M$ -weak contraction with

$$(2.3) \quad M(\zeta, \eta, \eta) \leq R[M(\zeta, \gamma, \gamma) + M(\gamma, \eta, \eta)]$$

and

$$(2.4) \quad M(\zeta, \zeta, \eta) = M(\zeta, \eta, \eta).$$

If Ψ is a (c) - comparison function, then T and S have a common fixed point.

Proof. Pick $\zeta_0 \in X$. Let $\zeta_1 = T\zeta_0$. Also, put $\zeta_2 = S\zeta_1$. Keep going this process, (ζ_n) is constructed sequence in X such that $\zeta_{2n+1} = T\zeta_{2n}$ and $\zeta_{2n+2} = S\zeta_{2n+1}$. Suppose $M(\zeta_n, \zeta_n, \zeta_{n+1}) = 0$ for some $n \in \mathbb{N}$. Without loss of generality, assume $n = 2k$ for some $k \in \mathbb{N}$. Thus $M(\zeta_{2k}, \zeta_{2k}, \zeta_{2k+1}) = 0$. Now, by (2.2), we have

$$\begin{aligned} M(\zeta_{2k+1}, \zeta_{2k+1}, \zeta_{2k+2}) &= M(T\zeta_{2k}, T\zeta_{2k}, S\zeta_{2k+1}) \\ &\leq \frac{1}{R} \Psi(\max\{RM(\zeta_{2k}, \zeta_{2k}, \zeta_{2k+1}), RM(\zeta_{2k}, \zeta_{2k}, T\zeta_{2k}), RM(\zeta_{2k+1}, \zeta_{2k+1}, S\zeta_{2k+1}), \\ &\frac{1}{2}[M(\zeta_{2k}, \zeta_{2k}, S\zeta_{2k+1}) + M(T\zeta_{2k}, T\zeta_{2k}, \zeta_{2k+1})]\}) \\ &+ L \min\{M(T\zeta_{2k}, T\zeta_{2k}, \zeta_{2k+1}), M(\zeta_{2k}, \zeta_{2k}, S\zeta_{2k+1})\} \\ &= \frac{1}{R} \Psi(\max\{RM(\zeta_{2k+1}, \zeta_{2k+1}, \zeta_{2k+2}), \\ &\frac{1}{2}[M(\zeta_{2k}, \zeta_{2k}, \zeta_{2k+2}) + M(\zeta_{2k+1}, \zeta_{2k+1}, \zeta_{2k+2})]\}) \\ &+ L \min\{M(\zeta_{2k+1}, \zeta_{2k+1}, \zeta_{2k+2}), M(\zeta_{2k}, \zeta_{2k}, \zeta_{2k+2})\} \\ &\leq \frac{1}{R} \Psi(\max\{RM(\zeta_{2k+1}, \zeta_{2k+1}, \zeta_{2k+2}), \\ &\frac{1}{2}[M(\zeta_{2k}, \zeta_{2k}, \zeta_{2k+1}) + M(\zeta_{2k+1}, \zeta_{2k+1}, \zeta_{2k+2})]\}) \\ &\leq \frac{1}{R} \Psi(RM(\zeta_{2k+1}, \zeta_{2k+1}, \zeta_{2k+2})). \end{aligned}$$

Since $\Psi(t) < t$ for all $t > 0$, we conclude that $M(\zeta_{2k+1}, \zeta_{2k+1}, \zeta_{2k+2}) = 0$. By (M2) and (M3) of the definition of Mb -metric spaces, we have $\zeta_{2k+1} = \zeta_{2k+2}$. So, $\zeta_{2k} = \zeta_{2k+1} = \zeta_{2k+2}$. Therefore, $\zeta_{2k} = T\zeta_{2k} = S\zeta_{2k}$ and hence ζ_k is a fixed point of T and S . Thus, we may assume that $M(\zeta_n, \zeta_n, \zeta_{n+1}) \neq 0$, for all $n \in \mathbb{N}$. Given $n \in \mathbb{N}$. If n is even $n = 2t$ for some $t \in \mathbb{N}$. By (2.2), we have

$$\begin{aligned}
 M(\zeta_{2t}, \zeta_{2t}, \zeta_{2t+1}) &= M(\zeta_{2t+1}, \zeta_{2t+1}, \zeta_{2t}) \\
 &= M(T\zeta_{2t}, T\zeta_{2t}, S\zeta_{2t+1}) \\
 &\leq \frac{1}{R} \Psi(\max\{RM(\zeta_{2t}, \zeta_{2t}, \zeta_{2t-1}), RM(\zeta_{2t}, \zeta_{2t}, T\zeta_{2t}), \\
 &\quad RM(\zeta_{2t-1}, \zeta_{2t-1}, S\zeta_{2t-1}), \\
 &\quad \frac{1}{2} [M(\zeta_{2t}, \zeta_{2t}, S\zeta_{2t-1}) + M(T\zeta_{2t}, T\zeta_{2t}, \zeta_{2t-1})]\}) \\
 &\quad + L \min\{M(\zeta_{2t}, \zeta_{2t}, S\zeta_{2t-1}), M(T\zeta_{2t}, T\zeta_{2t}, \zeta_{2t-1})\} \\
 &= \frac{1}{R} \Psi(\max\{RM(\zeta_{2t}, \zeta_{2t}, \zeta_{2t-1}), RM(\zeta_{2t}, \zeta_{2t}, \zeta_{2t+1}), \\
 &\quad \frac{1}{2} [M(\zeta_{2t}, \zeta_{2t}, \zeta_{2t}) + M(\zeta_{2t+1}, \zeta_{2t+1}, \zeta_{2t-1})]\}) \\
 &\quad + L \min\{M(\zeta_{2t}, \zeta_{2t}, \zeta_{2t}), M(\zeta_{2t+1}, \zeta_{2t+1}, \zeta_{2t-1})\}.
 \end{aligned}$$

From (2.3) and (2.4), we conclude

$$\begin{aligned}
 M(\zeta_{2t}, \zeta_{2t}, \zeta_{2t+1}) &\leq \frac{1}{R} \Psi(\max\{RM(\zeta_{2t}, \zeta_{2t}, \zeta_{2t-1}), RM(\zeta_{2t}, \zeta_{2t}, \zeta_{2t+1}), \\
 &\quad \frac{1}{2} [M(\zeta_{2t-1}, \zeta_{2t-1}, \zeta_{2t}) + M(\zeta_{2t}, \zeta_{2t}, \zeta_{2t+1})]\}) \\
 (2.5) \quad &\leq \frac{1}{R} \Psi(\max\{RM(\zeta_{2t}, \zeta_{2t}, \zeta_{2t-1}), RM(\zeta_{2t}, \zeta_{2t}, \zeta_{2t+1})\}).
 \end{aligned}$$

If $\max\{RM(\zeta_{2t}, \zeta_{2t}, \zeta_{2t-1}), RM(\zeta_{2t}, \zeta_{2t}, \zeta_{2t+1})\} = RM(\zeta_{2t}, \zeta_{2t}, \zeta_{2t+1})$, then (2.5) yields a contradiction. Thus, $\max\{RM(\zeta_{2t}, \zeta_{2t}, \zeta_{2t-1}), RM(\zeta_{2t}, \zeta_{2t}, \zeta_{2t+1})\} = RM(\zeta_{2t}, \zeta_{2t}, \zeta_{2t-1})$ and hence

$$(2.6) \quad M(\zeta_{2t}, \zeta_{2t}, \zeta_{2t+1}) \leq \frac{1}{R} \Psi(RM(\zeta_{2t}, \zeta_{2t}, \zeta_{2t-1})).$$

If n is odd, then $n = 2t + 1$ for some $t \in \mathbb{N} \cup \{0\}$. By similar arguments as above, we will show that

$$(2.7) \quad M(\zeta_{2t+1}, \zeta_{2t+1}, \zeta_{2t+2}) \leq \frac{1}{R} \Psi(RM(\zeta_{2t}, \zeta_{2t}, \zeta_{2t+1})).$$

From (2.6) and (2.7), we get

$$(2.8) \quad M(\zeta_n, \zeta_n, \zeta_{n+1}) \leq \frac{1}{R} \Psi(RM(\zeta_{n-1}, \zeta_{n-1}, \zeta_n)).$$

Repeat (2.8) n -times, we have $M(\zeta_n, \zeta_n, \zeta_{n+1}) \leq \frac{1}{R} \Psi^n(RM(\zeta_0, \zeta_0, \zeta_1))$. For $n, m \in \mathbb{N}$ with $m > n$, we have

$$\begin{aligned} M(\zeta_n, \zeta_n, \zeta_m) &\leq \sum_{i=n}^{m-1} R^i M(\zeta_i, \zeta_i, \zeta_{i+1}) \\ &\leq \sum_{i=n}^{m-1} R^i \Psi^i(RM(\zeta_0, \zeta_0, \zeta_1)) \\ &\leq \sum_{i=n}^{\infty} R^i \Psi^i(RM(\zeta_0, \zeta_0, \zeta_1)). \end{aligned}$$

We have $\sum_{i=n}^{\infty} R^i \Psi^i(RM(\zeta_0, \zeta_0, \zeta_1))$ Mb -converges because Ψ is (c)-comparison, and so $\lim_{n \rightarrow +\infty} \sum_{i=n}^{\infty} R^i \Psi^i(RM(\zeta_0, \zeta_0, \zeta_1)) = 0$.

Thus, $\lim_{n, m \rightarrow +\infty} M(\zeta_n, \zeta_n, \zeta_m) = 0$. Thus (ζ_n) is an Mb -Cauchy sequence in X . Now X is Mb -complete, so there exists $\gamma \in X$ such that $\zeta_n \rightarrow \gamma$ with $M(\gamma, \gamma, \gamma) = 0$. So,

$$(2.9) \quad \lim_{n, m \rightarrow +\infty} M(\zeta_n, \zeta_n, \zeta_m) = \lim_{n \rightarrow \infty} M(\zeta_n, \zeta_n, \gamma) = M(\gamma, \gamma, \gamma) = 0.$$

Now, we prove that $S\gamma = \gamma$ and $T\gamma = \gamma$. Since $M(\zeta_{2n+1}, \zeta_{2n+1}, \gamma) \rightarrow M(\gamma, \gamma, \gamma) = 0$ and $M(\zeta_{2n+2}, \zeta_{2n+2}, \gamma) \rightarrow M(\gamma, \gamma, \gamma) = 0$, By using *Theorem 1.1*, we get

$$(2.10) \quad \lim_{n \rightarrow +\infty} M(\zeta_{2n+1}, \zeta_{2n+1}, S\gamma) = M(\gamma, \gamma, S\gamma)$$

and

$$(2.11) \quad \lim_{n \rightarrow +\infty} M(\zeta_{2n+2}, \zeta_{2n+2}, S\gamma) = M(\gamma, \gamma, T\gamma).$$

By using (2.2), we have

$$\begin{aligned} M(\zeta_{2n+1}, \zeta_{2n+1}, S\gamma) &= M(T\zeta_{2n}, T\zeta_{2n}, S\gamma) \\ &\leq \frac{1}{R} \Psi(\max\{RM(\zeta_{2n}, \zeta_{2n}, \gamma), RM(\zeta_{2n}, \zeta_{2n}, T\zeta_{2n}), RM(\gamma, \gamma, S\gamma), \\ &\quad \frac{1}{2} [M(T\zeta_{2n}, T\zeta_{2n}, \gamma) + M(\zeta_{2n}, \zeta_{2n}, S\gamma)]\}) \\ &\quad + L \min\{M(T\zeta_{2n}, T\zeta_{2n}, \gamma), M(\zeta_{2n}, \zeta_{2n}, S\gamma)\} \\ &\leq \frac{1}{R} \Psi(\max\{RM(\zeta_{2n}, \zeta_{2n}, \gamma), RM(\zeta_{2n}, \zeta_{2n}, \zeta_{2n+1}), RM(\gamma, \gamma, S\gamma), \\ &\quad \frac{1}{2} [M(\zeta_{2n+1}, \zeta_{2n+1}, \gamma) + M(\zeta_{2n}, \zeta_{2n}, S\gamma)]\}) \\ &\quad + L \min\{M(\zeta_{2n+1}, \zeta_{2n+1}, \gamma), M(\zeta_{2n}, \zeta_{2n}, S\gamma)\}. \end{aligned}$$

In the above inequality, let $n \rightarrow +\infty$ and using (2.9), (2.10), we get that $M(\gamma, \gamma, S\gamma) \leq \frac{1}{R} \Psi(RM(\gamma, \gamma, S\gamma))$. Since $\Psi(t) < t$ for all $t > 0$, we conclude that $M(\gamma, \gamma, S\gamma) = 0$. By using (M2) and (M3) of the definition of Mb -metric space, we get that $S\gamma = \gamma$. Similarity, we may show that $T\gamma = \gamma$. $\square$

In the following theorem, we give the 2nd result.

Theorem 2.2. *Let (X, M) be an Mb -complete metric space and $T, S : X \rightarrow X$ be two mapping such that*

$$(2.12) \quad \begin{aligned} M(T\zeta, T\zeta, S\eta) &\leq \frac{1}{R} \Psi \left(\max \{RM(\zeta, \zeta, \eta), RM(\zeta, \zeta, T\zeta), RM(\eta, \eta, S\eta), \right. \\ &\quad \left. \frac{1}{2} [M(T\zeta, T\zeta, \eta) + M(\zeta, \zeta, S\eta)] \right\} \Bigg) \\ &\quad + L \min \{M(\zeta, \zeta, T\zeta), M(\zeta, \zeta, S\eta), M(T\zeta, T\zeta, \eta)\} \end{aligned}$$

with

$$M(\zeta, \eta, \eta) \leq R [M(\zeta, \gamma, \gamma) + M(\gamma, \eta, \eta)]$$

and

$$M(\zeta, \zeta, \eta) = M(\zeta, \eta, \eta),$$

for all $\zeta, \eta \in X$. If Ψ is a (c) -comparison function, then the common fixed point of T and S is unique.

Proof. From Theorem 2.1, the existence of the common fixed point of T and S followed. To prove the uniqueness of the common fixed point of T and S , assume there exist two fixed points ν, κ of T and S . So $T\nu = S\nu = \nu$ and $T\kappa = S\kappa = \kappa$. Now, we have to show that $\nu = \kappa$. By (2.12), we have

$$\begin{aligned} M(\nu, \nu, \kappa) &= M(T\nu, T\nu, S\kappa) \\ &\leq \frac{1}{R} \Psi \left(\max \{RM(\nu, \nu, \kappa), RM(\nu, \nu, T\nu), RM(\kappa, \kappa, S\kappa), \right. \\ &\quad \left. \frac{1}{2} [M(T\nu, T\nu, \kappa) + M(\kappa, \kappa, T\nu)] \right\} \Bigg) \\ &\quad + L \min \{M(\nu, \nu, T\nu), M(T\nu, T\nu, \kappa), M(\kappa, \kappa, T\nu)\} \\ &\leq \frac{1}{R} \Psi \left(\max \{RM(\nu, \nu, \kappa), RM(\nu, \nu, T\nu), RM(\kappa, \kappa, \kappa), \right. \\ &\quad \left. \frac{1}{2} [M(T\nu, T\nu, \kappa) + M(\kappa, \kappa, \nu)] \right\} \Bigg) \\ &\quad + L \min \{M(\nu, \nu, \nu), M(\nu, \nu, \kappa), M(\kappa, \kappa, \nu)\} \\ &= \frac{1}{R} \Psi(RM(\nu, \nu, \kappa)). \end{aligned}$$

Since $\Phi(t) < t$ for all $t > 0$, we conclude that $M(\nu, \nu, \kappa) = 0$. By (M2) and (M3) of the definition of Mb -metric spaces, we get that $\nu = \kappa$. $\square$

Taking $T = S$ in Theorems 2.1 and 2.2, we have the following results

Corollary 2.1. *Let (X, M) be an Mb-complete metric space and $T : X \rightarrow X$ be a mapping such that*

$$\begin{aligned} M(T\zeta, T\zeta, T\eta) \leq & \frac{1}{R} \Psi(\max\{RM(\zeta, \zeta, \eta), RM(\zeta, \zeta, T\zeta), RM(\eta, \eta, T\eta), \\ & \frac{1}{2}[M(T\zeta, T\zeta, \eta) + M(\zeta, \zeta, S\eta)]\}) \\ & + L \min\{M(\zeta, \zeta, T\eta), M(T\zeta, T\zeta, \eta)\} \end{aligned}$$

with

$$M(\zeta, \eta, \eta) \leq R[M(\zeta, \gamma, \gamma) + M(\gamma, \eta, \eta)]$$

and

$$M(\zeta, \zeta, \eta) = M(\zeta, \eta, \eta),$$

for all $\zeta, \eta \in X$. If Ψ is a (c) – comparison function, then T has a fixed point.

Corollary 2.2. *Let (X, M) be an Mb-complete metric space and $T : X \rightarrow X$ be a mapping such that*

$$\begin{aligned} M(T\zeta, T\zeta, T\eta) \leq & \frac{1}{R} \Psi(\max\{RM(\zeta, \zeta, \eta), RM(\zeta, \zeta, T\zeta), RM(\eta, \eta, T\eta), \\ & \frac{1}{2}[M(T\zeta, T\zeta, \eta) + M(\zeta, \zeta, S\eta)]\}) \\ & + L \min\{M(\zeta, \zeta, T\zeta), M(\zeta, \zeta, T\eta), M(T\zeta, T\zeta, \eta)\} \end{aligned}$$

with

$$M(\zeta, \eta, \eta) \leq R[M(\zeta, \gamma, \gamma) + M(\gamma, \eta, \eta)]$$

and

$$M(\zeta, \zeta, \eta) = M(\zeta, \eta, \eta),$$

for all $\zeta, \eta \in X$. If Ψ is a (c) – comparison function, then T has a unique fixed point.

Corollary 2.3. *Let (X, M) be an Mb-metric space and $T, S : X \rightarrow X$ be two mappings such that*

$$\begin{aligned} M(T\zeta, T\zeta, T\eta) \leq & \frac{1}{R} \Psi(\max\{RM(S\zeta, S\zeta, S\eta), RM(S\zeta, S\zeta, T\zeta), RM(S\eta, S\eta, T\eta), \\ & \frac{1}{2}[M(T\zeta, T\zeta, S\eta) + M(S\zeta, S\zeta, T\eta)]\}) \\ & + L \min\{M(S\zeta, S\zeta, T\eta), M(S\eta, S\eta, T\zeta)\} \end{aligned}$$

with

$$M(\zeta, \eta, \eta) \leq R[M(\zeta, \gamma, \gamma) + M(\gamma, \eta, \eta)]$$

and

$$M(\zeta, \zeta, \eta) = M(\zeta, \eta, \eta),$$

for all $\zeta, \eta \in X$. Also, suppose that

(1) $TX \subseteq SX$

(2) SX is an Mb – complete subspace of the Mb – metric space X .

If Ψ a (c) – comparison function, then T and S have a coincidence point. Moreover, the point of coincidence of T and S is unique.

Corollary 2.4. Let (X, M) be an Mb – metric space and $T, S : X \rightarrow X$ be two mappings such that

$$\begin{aligned} M(T\zeta, T\zeta, T\eta) &\leq \frac{1}{R} \Psi(\max\{RM(S\zeta, S\zeta, S\eta), RM(S\zeta, S\zeta, T\zeta), RM(S\eta, S\eta, T\eta), \\ &\quad \frac{1}{2} [M(T\zeta, T\zeta, S\eta) + M(S\zeta, S\zeta, T\eta)]\}) \\ &\quad + L \min\{M(T\zeta, T\zeta, S\zeta), M(S\zeta, S\zeta, T\eta), M(S\eta, S\eta, T\zeta)\} \end{aligned}$$

with

$$M(\zeta, \eta, \eta) \leq R[M(\zeta, \gamma, \gamma) + M(\gamma, \eta, \eta)]$$

and

$$M(\zeta, \zeta, \eta) = M(\zeta, \eta, \eta),$$

for all $\zeta, \eta \in X$. Also, suppose that

(1) $TX \subseteq SX$

(2) SX is an Mb – complete subspace of the Mb – metric space X .

If Ψ a (c) – comparison function, then the point of coincidence of T and S is unique; that is, if $T\nu = S\nu$ and $T\kappa = S\kappa$, then $T\nu = T\kappa = S\kappa = S\nu$.

In Theorems 2.1 and 2.2, the (c) – comparison function can be replaced by a comparison function if we formulated the counteractive condition to a suitable form. For example, we have the following result.

Theorem 2.3. Let (X, M) be an Mb – complete metric space and $T : X \rightarrow X$ be a mapping, such that

$$\begin{aligned} M(T\zeta, T\zeta, T\eta) &\leq \frac{1}{R} \Psi(\max\{RM(\zeta, \zeta, \eta), RM(\zeta, \zeta, T\zeta), RM(\eta, \eta, T\eta)\}) \\ (2.13) \quad &+ L \min\{M(\zeta, \zeta, T\zeta), M(\zeta, \zeta, T\eta), M(\eta, \eta, T\zeta)\} \end{aligned}$$

for all $\zeta, \eta \in X$,

$$(2.14) \quad M(\zeta, \eta, \eta) \leq R[M(\zeta, \gamma, \gamma) + M(\gamma, \eta, \eta)]$$

and

$$(2.15) \quad M(\zeta, \zeta, \eta) = M(\zeta, \eta, \eta),$$

If Ψ is a comparison function, then T has a unique fixed point.

Proof. Choose $\zeta_0 \in X$. Put $\zeta_1 = T\zeta_0$. Also, set $\zeta_2 \in X$ such that $\zeta_2 = T\zeta_1$. Continuing the same process, we can construct a sequence (ζ_n) in X such that $\zeta_{n+1} = T\zeta_n$. If $M(\zeta_k, \zeta_k, \zeta_{k+1}) = 0$ for some $k \in \mathbb{N}$, then by the definition of Mb -metric spaces, we have $\zeta_k = \zeta_{k+1} = T\zeta_k$, that is, ζ_k is a fixed point of T . Thus, we assume that $M(\zeta_n, \zeta_n, \zeta_{n+1}) \neq 0$ for all $n \in \mathbb{N}$. By (2.13), we have

$$\begin{aligned} M(\zeta_n, \zeta_n, \zeta_{n+1}) &= M(T\zeta_{n-1}, T\zeta_{n-1}, T\zeta_n) \\ &\leq \frac{1}{R} \Psi(\max \{RM(\zeta_{n-1}, \zeta_{n-1}, \zeta_n), RM(\zeta_{n-1}, \zeta_{n-1}, T\zeta_{n-1}), RM(\zeta_n, \zeta_n, T\zeta_n)\}) \\ &\quad + L \min \{M(\zeta_{n-1}, \zeta_{n-1}, T\zeta_n), M(\zeta_{n-1}, \zeta_{n-1}, T\zeta_n), M(\zeta_n, \zeta_n, T\zeta_{n-1})\} \\ &= \frac{1}{R} \Psi(\max \{RM(\zeta_{n-1}, \zeta_{n-1}, \zeta_n), RM(\zeta_n, \zeta_n, T\zeta_{n+1})\}) \\ &\quad + L \min \{M(\zeta_{n-1}, \zeta_{n-1}, T\zeta_{n+1}), M(\zeta_n, \zeta_n, \zeta_n)\} \\ &= \frac{1}{R} \Psi(\max \{RM(\zeta_{n-1}, \zeta_{n-1}, \zeta_n), RM(\zeta_n, \zeta_n, T\zeta_{n+1})\}). \end{aligned}$$

If

$$\max \{RM(\zeta_{n-1}, \zeta_{n-1}, \zeta_n), RM(\zeta_n, \zeta_n, \zeta_{n+1})\} = RM(\zeta_n, \zeta_n, \zeta_{n+1}),$$

then

$$M(\zeta_n, \zeta_n, \zeta_{n+1}) \leq \frac{1}{R} \Psi(RM(\zeta_n, \zeta_n, \zeta_{n+1})) < M(\zeta_n, \zeta_n, \zeta_{n+1}),$$

a contradiction. Thus,

$$\max \{RM(\zeta_{n-1}, \zeta_{n-1}, \zeta_n), RM(\zeta_n, \zeta_n, \zeta_{n+1})\} = RM(\zeta_{n-1}, \zeta_{n-1}, \zeta_n)$$

and hence

$$(2.16) \quad M(\zeta_n, \zeta_n, \zeta_{n+1}) \leq \frac{1}{R} \Psi(RM(\zeta_{n-1}, \zeta_{n-1}, \zeta_n)), \quad \text{for all } n \in \mathbb{N}.$$

Repeating (2.14) n -times, we get that

$$M(\zeta_n, \zeta_n, \zeta_{n+1}) \leq \frac{1}{R} \Psi^n(RM(\zeta_0, \zeta_0, \zeta_1)).$$

Now, we will prove that (ζ_n) is an Mb -Cauchy sequence in the Mb -metric space (X, M) . For this, given $\epsilon > 0$, since $\frac{1}{R(2+L)}(\epsilon - \Phi(\epsilon)) > 0$ and $\lim_{n \rightarrow +\infty} R^n \Psi^n(M(\zeta_0, \zeta_0, \zeta_1)) = 0$, there exists $k \in \mathbb{N}$ such that $M(\zeta_n, \zeta_n, \zeta_{n+1}) < \frac{1}{R(2+L)}(\epsilon - \Phi(\epsilon))$ for all $n \geq k$. Now, given $m, n \in \mathbb{N}$ with $m > n$. Claim: $M(\zeta_n, \zeta_n, \zeta_m) < \epsilon$ for all $m > n > k$. We prove our claim by induction on m . Since $k+1 > k$, then

$$M(\zeta_k, \zeta_k, \zeta_{k+1}) \leq \frac{1}{R(2+L)}(\epsilon - \Phi(\epsilon)) < \epsilon.$$

The last inequality proves our claim for $m = k+1$. Assume that our claim holds for $m = k$.

To prove our claim for $m = k + 1$, we have

$$\begin{aligned}
 M(\zeta_n, \zeta_n, \zeta_{k+1}) &\leq R[M(\zeta_n, \zeta_n, \zeta_{n+1}) + M(\zeta_{n+1}, \zeta_{n+1}, \zeta_{k+1}) - M(\zeta_{n+1}, \zeta_{n+1}, \zeta_{n+1})] \\
 &\leq R[M(\zeta_n, \zeta_n, \zeta_{n+1}) + M(\zeta_n, \zeta_n, \zeta_{k+1})] \\
 (2.17) \quad &= R[M(\zeta_n, \zeta_n, \zeta_{n+1}) + M(T\zeta_n, T\zeta_n, T\zeta_k)].
 \end{aligned}$$

By (2.13), we have

$$\begin{aligned}
 M(T\zeta_n, T\zeta_n, T\zeta_k) &\leq \frac{1}{R}\Psi(\max\{RM(\zeta_n, \zeta_n, \zeta_k), RM(\zeta_n, \zeta_n, T\zeta_n), RM(\zeta_k, \zeta_k, T\zeta_k)\}) \\
 &\quad + L \min\{M(\zeta_n, \zeta_n, T\zeta_n), M(\zeta_n, \zeta_n, T\zeta_k), M(\zeta_k, \zeta_k, T\zeta_n)\} \\
 &= \frac{1}{R}\Psi(\max\{RM(\zeta_n, \zeta_n, \zeta_k), RM(\zeta_n, \zeta_n, \zeta_{n+1}), RM(\zeta_k, \zeta_k, \zeta_{k+1})\}) \\
 &\quad + L \min\{M(\zeta_n, \zeta_n, \zeta_{n+1}), M(\zeta_n, \zeta_n, \zeta_{k+1}), M(\zeta_k, \zeta_k, \zeta_{n+1})\} \\
 &\leq \frac{1}{R}\Psi(\max\{RM(\zeta_n, \zeta_n, \zeta_k), RM(\zeta_n, \zeta_n, \zeta_{n+1}), RM(\zeta_k, \zeta_k, \zeta_{k+1})\}) \\
 &\quad + LM(\zeta_n, \zeta_n, \zeta_{n+1}).
 \end{aligned}$$

If $\max\{RM(\zeta_n, \zeta_n, \zeta_k), RM(\zeta_n, \zeta_n, \zeta_{n+1}), RM(\zeta_k, \zeta_k, \zeta_{k+1})\} = RM(\zeta_n, \zeta_n, \zeta_k)$, then by (2.17) we have

$$\begin{aligned}
 M(\zeta_n, \zeta_n, \zeta_{k+1}) &\leq R\left[M(\zeta_n, \zeta_n, \zeta_{n+1}) + \frac{1}{R}\Psi(RM(\zeta_n, \zeta_n, \zeta_k)) + LM(\zeta_n, \zeta_n, \zeta_{n+1})\right] \\
 &< \left[\frac{1+L}{R(2+L)}(\epsilon - \Phi(\epsilon)) + \frac{1}{R}\Phi(\epsilon)\right]R \\
 &< \epsilon.
 \end{aligned}$$

If $\max\{RM(\zeta_n, \zeta_n, \zeta_k), RM(\zeta_n, \zeta_n, \zeta_{n+1}), RM(\zeta_k, \zeta_k, \zeta_{k+1})\} = RM(\zeta_n, \zeta_n, \zeta_{n+1})$, then by (2.17) we have

$$\begin{aligned}
 M(\zeta_n, \zeta_n, \zeta_{k+1}) &\leq R\left[M(\zeta_n, \zeta_n, \zeta_{n+1}) + \frac{1}{R}\Psi(RM(\zeta_n, \zeta_n, \zeta_{n+1})) + LM(\zeta_n, \zeta_n, \zeta_{n+1})\right] \\
 &< (2+L)RM(\zeta_n, \zeta_n, \zeta_{n+1}) \\
 &< \frac{\epsilon - \Phi(\epsilon)}{\epsilon} \\
 &< \epsilon.
 \end{aligned}$$

If $\max\{RM(\zeta_n, \zeta_n, \zeta_k), RM(\zeta_n, \zeta_n, \zeta_{n+1}), RM(\zeta_k, \zeta_k, \zeta_{k+1})\} = RM(\zeta_k, \zeta_k, \zeta_{k+1})$, then by (2.17) we have

$$\begin{aligned}
 M(\zeta_n, \zeta_n, \zeta_{k+1}) &\leq R\left[M(\zeta_n, \zeta_n, \zeta_{n+1}) + \frac{1}{R}\Psi(RM(\zeta_k, \zeta_k, \zeta_{k+1})) + LM(\zeta_n, \zeta_n, \zeta_{n+1})\right] \\
 &< (R+L)M(\zeta_n, \zeta_n, \zeta_{n+1}) + RM(\zeta_k, \zeta_k, \zeta_{k+1}) \\
 &< \frac{R+L}{R(2+L)}(\epsilon - \Phi(\epsilon)) + \frac{R}{R(2+L)}(\epsilon - \Phi(\epsilon)) \\
 &< \epsilon.
 \end{aligned}$$

Thus (ζ_n) is an Mb – *Cauchy* sequence in X . Since X is Mb – *complete*, then (ζ_n) Mb – *converges*, with respect to τ_M , to a point γ for some $\gamma \in X$ such that

$$(2.16) \quad \lim_{n,m \rightarrow +\infty} M(\zeta_n, \zeta_n, \zeta_m) = \lim_{n \rightarrow +\infty} M(\zeta_n, \zeta_n, \gamma) = M(\gamma, \gamma, \gamma) = 0.$$

Now, assume that $M(\gamma, \gamma, T\gamma) > 0$. By using (2.14) and (2.15) of the definition of Mb – *metric* spaces and (2.13), we have

$$\begin{aligned} M(\gamma, \gamma, T\gamma) &\leq R[M(\gamma, \gamma, \zeta_{n+1}) + M(\zeta_{n+1}, \zeta_{n+1}, T\gamma)] \quad \text{by (2.14)} \\ &= R[M(\gamma, \gamma, \zeta_{n+1}) + M(T\zeta_n, T\zeta_n, T\gamma)] \\ &\leq R[M(\gamma, \gamma, \zeta_{n+1}) + \frac{1}{R}\Psi(\max\{RM(\zeta_n, \zeta_n, \gamma), \\ &\quad RM(\zeta_n, \zeta_n, T\zeta_n), RM(\gamma, \gamma, T\gamma)\}) \\ &\quad + L \min\{M(\zeta_n, \zeta_n, T\zeta_n), M(\zeta_n, \zeta_n, T\gamma), M(\zeta_n, \zeta_n, T\gamma)\})] \\ (2.19) \quad &= R[M(\gamma, \gamma, \zeta_{n+1}) + \frac{1}{R}\Psi(\max\{RM(\zeta_n, \zeta_n, \gamma), \\ &\quad RM(\zeta_n, \zeta_n, \zeta_{n+1}), RM(\gamma, \gamma, T\gamma)\}) \\ &\quad + L \min\{M(\zeta_n, \zeta_n, \zeta_{n+1}), M(\zeta_n, \zeta_n, T\gamma), M(\zeta_{n+1}, \zeta_{n+1}, S\gamma)\})]. \end{aligned}$$

Since

$$\lim_{n,m \rightarrow +\infty} M(\zeta_n, \zeta_n, \zeta_{n+1}) = \lim_{n \rightarrow +\infty} M(\zeta_n, \zeta_n, \gamma) = 0$$

and $M(\gamma, \gamma, T\gamma) > 0$, we can choose $n_0 \in \mathbb{N}$ such that

$$\max\{RM(\zeta_n, \zeta_n, \gamma), RM(\zeta_n, \zeta_n, \zeta_{n+1}), RM(\gamma, \gamma, T\gamma)\} = RM(\gamma, \gamma, T\gamma)$$

for all $n \geq n_0$. Thus (2.19) becomes

$$\begin{aligned} M(\gamma, \gamma, T\gamma) &\leq R[M(\gamma, \gamma, \zeta_{n+1}) + \frac{1}{R}\Psi(RM(\gamma, \gamma, T\gamma)) + \\ &\quad L \min\{M(\zeta_n, \zeta_n, \zeta_{n+1}), M(\zeta_n, \zeta_n, T\gamma), M(\zeta_{n+1}, \zeta_{n+1}, T\gamma)\}], \end{aligned}$$

for all $n \geq n_0$. On letting $n \rightarrow +\infty$ in the above inequality and using (2.18), we get that $M(\gamma, \gamma, T\gamma) \leq \frac{1}{R}\Psi(RM(\gamma, \gamma, T\gamma)) < M(\gamma, \gamma, T\gamma)$, a contradiction. Thus $M(\gamma, \gamma, T\gamma) = 0$. By using (M2) and (M3) of the definition of an Mb – *metric* space, we get that $\gamma = T\gamma$; that is, γ is a fixed point of T . To prove that the fixed point of T is unique, we assume that ν and κ are fixed points of T .

Thus, we have $T\nu = \nu$ and $T\kappa = \kappa$. By (2.13), we have

Since $\Psi(t) < t$ for all $t \in \mathbb{N}$, we have $M(\nu, \nu, \kappa) = 0$. By (M2) and (M3), we have $\nu = \kappa$. $\square$

Corollary 2.5. *Let (X, M) be an Mb – *metric* space and $T, S : X \rightarrow X$ be two mapping such that for some $L \geq 0$, we have*

$$\begin{aligned} M(T\zeta, T\zeta, T\eta) &\leq \frac{1}{R}\Psi(\max\{RM(S\zeta, S\zeta, S\eta), Rd(S\zeta, S\zeta, T\zeta), Rd(S\eta, S\eta, T\eta)\}) \\ &\quad + L \min\{M(S\zeta, S\zeta, T\zeta), M(S\zeta, S\zeta, T\eta), M(S\eta, S\eta, T\zeta)\} \end{aligned}$$

with

$$M(\zeta, \eta, \eta) \leq R[M(\zeta, \gamma, \gamma) + M(\gamma, \eta, \eta)]$$

and

$$M(\zeta, \zeta, \eta) = M(\zeta, \eta, \eta),$$

for all $\zeta, \eta \in X$. Also, suppose that

(1) $TX \subseteq SX$.

(2) SX is a complete subspace of the Mb -metric space X .

If Ψ is a comparison of T and S have a coincidence point. Moreover, the point of coincidence of T and S is unique.

Example 2.1. Let $X = [0, +\infty)$. Consider the complete Mb -metric space $d : X \times X \rightarrow [0, +\infty)$, $M(\zeta, \zeta, \eta) = (\zeta - \eta)^2$ with constant $R = 2$. Define the mappings $T, S : X \rightarrow X$ by $T\zeta = \frac{1}{3}\zeta$ and $S\zeta = \frac{1}{6}\zeta$. and define $\Psi : [0, +\infty) \rightarrow [0, +\infty)$ by $\Psi(t) = \frac{1}{4}\zeta$. Then:

(1) Ψ is a continuous (c) -comparison function.

(2) T, S and Ψ satisfy the following inequality:

$$\begin{aligned} M(T\zeta, T\zeta, S\eta) &\leq \frac{1}{R}\Psi(\max\{RM(\zeta, \zeta, \eta), RM(\zeta, \zeta, T\zeta), RM(\eta, \eta, S\eta), \\ &\quad \frac{1}{2}[M(T\zeta, T\zeta, \eta) + M(\zeta, \zeta, S\eta)]\}) \\ &\quad + L \min\{M(\zeta, \zeta, T\zeta), M(\zeta, \zeta, \eta), M(T\zeta, T\zeta, \eta)\} \end{aligned}$$

Proof. Clearly, to show that Ψ is a non-decreasing continuous function and clearly M satisfies (2.14), (2.15). Now, let $t \in [0, +\infty)$. Then,

$$\Psi^n(Rt) = \Psi^n(2t) = \frac{1}{4^n}(2t).$$

Thus

$$\sum_{n=0}^{\infty} R^n \Psi^n(Rt) = \sum_{n=0}^{\infty} \frac{2^n}{4^n}(2t) = 2t \sum_{n=0}^{\infty} \frac{1}{2^n} < +\infty.$$

So, Ψ is a continuous (c) -comparison function. the part (1) as required. To show (2), let $\zeta, \eta \in X$. Then

$$M(T\zeta, T\zeta, S\eta) = M\left(\frac{1}{3}\zeta, \frac{1}{3}\zeta, \frac{1}{6}\eta\right) = \left(\frac{1}{3}\zeta - \frac{1}{6}\eta\right)^2 = \frac{1}{9}\left(\zeta - \frac{1}{2}\eta\right)^2.$$

Now, we have 3 cases:

Case I. $\zeta = \frac{1}{2}\eta$. Here, we have

$$\begin{aligned} M(T\zeta, T\zeta, S\eta) &= 0 \leq \frac{1}{R}\Psi(\max\{RM(\zeta, \zeta, \eta), RM(\zeta, \zeta, T\zeta), RM(\eta, \eta, S\eta), \\ &\quad \frac{1}{2}[M(T\zeta, T\zeta, \eta) + M(\zeta, \zeta, S\eta)]\}) \\ &\quad + L \min\{M(\zeta, \zeta, T\zeta), M(\zeta, \zeta, S\eta), M(T\zeta, T\zeta, \eta)\} \end{aligned}$$

Case II: $\zeta > \frac{1}{2}\eta$. Here, we have

$$\begin{aligned} M(T\zeta, T\zeta, S\eta) &= \frac{1}{9}(\zeta - \frac{1}{2}\eta)^2 \leq \frac{\zeta^2}{6} \\ &= \frac{1}{2}(2)(\frac{2}{3}\zeta)^2(\frac{1}{4}) = \frac{1}{2}\Psi(2(\zeta - \frac{1}{3}\zeta)^2) = \frac{1}{2}\Psi(2M(\zeta, \zeta, \frac{1}{3}\zeta)) \\ &= \frac{1}{R}\Psi(RM(\zeta, \zeta, T\zeta)) \\ &\leq \frac{1}{R}\Psi(\max\{RM(\zeta, \zeta, \eta), RM(\zeta, \zeta, T\zeta), RM(\eta, \eta, S\eta), \\ &\quad \frac{1}{2}[M(T\zeta, T\zeta, \eta) + M(\zeta, \zeta, S\eta)]\}) \\ &\quad + L \min\{M(\zeta, \zeta, T\zeta), M(\zeta, \zeta, S\eta), M(T\zeta, T\zeta, \eta)\}. \end{aligned}$$

Case III: $\zeta < \frac{1}{2}\eta$. Here, we have

$$\begin{aligned} M(T\zeta, T\zeta, S\eta) &= \frac{1}{9}\left(\zeta - \frac{1}{2}\eta\right)^2 \leq \frac{y^2}{36} \\ &\leq \left(\frac{25}{36}\right)\left(\frac{\eta^2}{4}\right) = \frac{1}{2}\Psi\left(2\left(\frac{25}{36}\right)\eta^2\right) = \frac{1}{2}\Psi\left(2\left(\eta - \frac{1}{6}\eta\right)^2\right) \\ &= \frac{1}{2}\Psi\left(2M\left(\eta, \eta, \frac{1}{6}\eta\right)\right) = \frac{1}{R}\Psi(RM(\eta, \eta, S\zeta)) \\ &= \frac{1}{R}\Psi(\max\{RM(\zeta, \zeta, \eta), RM(\zeta, \zeta, T\zeta), RM(\eta, \eta, S\eta), \\ &\quad \frac{1}{2}[M(T\zeta, T\zeta, \eta) + M(\zeta, \zeta, S\eta)]\}) \\ &\quad + L \min\{M(\zeta, \zeta, T\zeta), M(\zeta, \zeta, S\eta), M(T\zeta, T\zeta, \eta)\}. \end{aligned}$$

Note that, T, S and Ψ satisfy all hypotheses of Theorem 2.2. So, T and S have a unique common fixed point. $\square$

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Ricci soliton on Sasakian manifolds admitting Zamkovoy connection

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Abstract. Object of this paper is to study Ricci soliton on concircularly flat, W_2 -flat, W_3 -flat, W_4 -flat Sasakian manifolds with respect to Zamkovoy connection. Besides these, we discuss Ricci soliton on a Sasakian manifold satisfying $W_2^*(\xi, Y) \cdot R^* = 0$, where R^* denotes Riemannian curvature tensor with respect to Zamkovoy connection and W_2^* -denotes the W_2 -curvature tensor with respect to Zamkovoy connection.

Keywords: Sasakian manifold, Zamkovoy connection, Ricci soliton, concircular curvature tensor, W_2 -curvature tensor, W_3 -curvature tensor, W_4 -curvature tensor.

1. Introduction

The notion of Sasakian structure [16] was introduced by Japanese mathematician S. Sasaki in the year 1960. If a contact metric structure is normal then the structure is said to have a normal contact metric structure or a Sasakian structure. Thus a manifold with Sasakian structure is a normal contact metric manifold. In some respect Sasakian manifold may be viewed as an odd dimensional analogous of Kähler manifold. Sasakian manifold was further studied by many authors. For details, we refer ([4], [9], [11], [5]) and the references therein.

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The concircular curvature tensor on Riemannian manifold was introduced and defined by K. Yano [20]. The concircular curvature tensor of rank 3 is given by

$$(1) \quad W(X, Y)Z = R(X, Y)Z - \frac{r}{n(n-1)}[g(Y, Z)X - g(X, Z)Y].$$

The W_2 -curvature tensor [12], W_3 -curvature tensor [13] and W_4 -curvature tensor [13] are, respectively, given by

$$(2) \quad W_2(X, Y)Z = R(X, Y)Z - \frac{1}{n-1}[g(Y, Z)QX - g(X, Z)QY],$$

$$(3) \quad W_3(X, Y)Z = R(X, Y)Z - \frac{1}{n-1}[S(X, Z)Y - g(Y, Z)QX],$$

$$(4) \quad W_4(X, Y)Z = R(X, Y)Z + \frac{1}{n-1}[g(X, Z)QY - g(X, Y)QZ],$$

for all $X, Y, Z \in \chi(M)$, where $\chi(M)$ is set of all vector fields of the manifold M and $R(X, Y)Z$ is Riemannian curvature tensor and r, S, Q are the scalar curvature, Ricci tensor, Ricci operator, respectively.

The notion of Ricci flow was first introduced by R. S. Hamilton in the early 1980s. R. S. Hamilton [6] observed that the Ricci flow is an excellent tool for simplifying the structure of a manifold. It is the process which deforms the metric of a Riemannian manifold by smoothing out the irregularities. The Ricci flow equation is given by

$$(5) \quad \frac{\partial g}{\partial t} = -2S,$$

where g is Riemannian metric, S is Ricci tensor and t is time. Solitons for the Ricci flow is the solutions of the above equation, where the metrics at different times differ by a diffeomorphism of the manifold. A Ricci soliton is represented by a triple (g, V, λ) , where g is Riemannian metric, V is a vector field and λ is a scalar, which satisfies the equation:

$$(6) \quad L_V g + 2S + 2\lambda g = 0,$$

where S is Ricci curvature tensor, $L_V g$ denotes the Lie derivative of g along the vector field V . A Ricci soliton is said to be shrinking, steady, expanding according as $\lambda < 0, \lambda = 0, \lambda > 0$ respectively. If the vector field V is gradient of a smooth function, then the Ricci soliton (g, V, λ) is called a gradient Ricci soliton and the associated function is called the potential function. Ricci soliton, was further studied by many researcher. For instance, we see ([10], [14], [17], [18]) and their references.

In 2008, S. Zamkovoy [21] introduced the notion of a new canonical connection named as Zamkovoy connection for para-contact manifolds. And this connection was defined as a canonical paracontact connection whose torsion is the

obstruction of paracontact manifold to be a para Sasakian manifold. Zamkovoy connection was further studied by many authors et al. ([7], [8], [1], [2], [3]). The Zamkovoy connection ∇^* for an n -dimensional almost contact metric manifold M equipped with an almost contact metric structure (ϕ, ξ, η, g) consisting of a $(1, 1)$ tensor field ϕ , a vector field ξ , a 1-form η and a Riemannian metric g , is defined as

$$(7) \quad \nabla_X^* Y = \nabla_X Y + (\nabla_X \eta)(Y) \xi - \eta(Y) \nabla_X \xi + \eta(X) \phi Y,$$

for all $X, Y \in \chi(M)$.

Definition 1.1. *An n -dimensional Sasakian manifold M is said to be concircularly flat with respect to Zamkovoy connection if $W^*(X, Y)Z = 0$, for all $X, Y, Z \in \chi(M)$.*

Definition 1.2. *An n -dimensional Sasakian manifold M is said to be W_2 -flat with respect to Zamkovoy connection if $W_2^*(X, Y)Z = 0$, for all $X, Y, Z \in \chi(M)$.*

Definition 1.3. *An n -dimensional Sasakian manifold M is said to be W_3 -flat with respect to Zamkovoy connection if $W_3^*(X, Y)Z = 0$, for all $X, Y, Z \in \chi(M)$.*

Definition 1.4. *An n -dimensional Sasakian manifold M is said to be W_4 -flat with respect to Zamkovoy connection if $W_4^*(X, Y)Z = 0$, for all $X, Y, Z \in \chi(M)$.*

Definition 1.5. *An n -dimensional Sasakian manifold M is said to be η -Einstein manifold if its Ricci tensor is of the form*

$$S(X, Y) = k_1 g(X, Y) + k_2 \eta(X) \eta(Y),$$

for all $X, Y \in \chi(M)$, where k_1 and k_2 are scalars.

This paper is organized as follows:

After introduction a short description of Sasakian manifold is given in section (2). In section (3), we have discussed some properties of Sasakian manifold with respect to Zamkovoy connection ∇^* and obtained Riemannian curvature tensor R^* , Ricci tensor S^* , scalar curvature tensor r^* , Ricci operator Q^* with respect to ∇^* . Section (4), section (5), section (6) and section (7) contain Ricci soliton on concircularly flat, W_2 -flat, W_3 -flat and W_4 -flat Sasakian manifolds, respectively, with respect to the Zamkovoy connection. In section (8), we have discussed Ricci soliton on Sasakian manifold satisfying $W_2^*(\xi, Y).R^* = 0$.

2. Preliminaries

Let M be an n -dimensional almost contact metric manifold equipped with an almost contact metric structure (ϕ, ξ, η, g) consisting of a $(1, 1)$ tensor field ϕ , a

vector field ξ , a 1-form η and a Riemannian metric g satisfying

$$(8) \quad \phi^2 Y = -Y + \eta(Y)\xi, \eta(\xi) = 1, \eta(\phi X) = 0, \phi\xi = 0,$$

$$(9) \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

$$(10) \quad g(X, \phi Y) = -g(\phi X, Y), \eta(Y) = g(Y, \xi),$$

for all $X, Y \in \chi(M)$.

An almost contact metric manifold M is said to be Sasakian manifold if the following relations hold in M

$$(11) \quad g(X, \phi Y) = \nabla \eta(X, Y),$$

$$(12) \quad \nabla_X \xi = -\phi X,$$

$$(13) \quad (\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X,$$

$$(14) \quad R(X, Y)\xi = \eta(Y)X - \eta(X)Y,$$

for all $X, Y \in \chi(M)$, where ∇_X denotes the covariant differentiation with respect to X .

In a Sasakian manifold equipped with the structure (ϕ, ξ, η, g) the following relations also hold ([19], [15]):

$$(15) \quad (\nabla_X \eta)Y = g(X, \phi Y),$$

$$(16) \quad R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X,$$

$$(17) \quad S(X, \xi) = (n-1)\eta(X),$$

$$(18) \quad R(X, \xi)Y = \eta(Y)X - g(X, Y)\xi,$$

$$(19) \quad Q\xi = (n-1)\xi,$$

$$(20) \quad S(X, Y) = g(QX, Y), S^2(X, Y) = S(QX, Y).$$

In Sasakian manifold, using (12) and (15) equation (7) reduces to

$$(21) \quad \nabla_X^* Y = \nabla_X Y + g(X, \phi Y)\xi + \eta(Y)\phi X + \eta(X)\phi Y,$$

with torsion tensor $T^*(X, Y) = 2g(X, \phi Y)\xi$.

3. Some properties of Sasakian manifold with respect to Zamkovoy connection

By the the help of (21), (12) and (13) we get the following results:

$$(22) \quad \nabla_X^* \eta(Y) = \eta(\nabla_X Y) + g(X, \phi Y),$$

$$\nabla_X^* (\phi Y) = \nabla_X (\phi Y) - g(\phi X, \phi Y)\xi$$

$$(23) \quad -\eta(X)Y + \eta(X)\eta(Y)\xi,$$

$$\nabla_X^* g(Y, \phi Z) = g(\nabla_X Y, \phi Z) + \eta(X)g(\phi Y, \phi Z) + g(Y, \nabla_X (\phi Z))$$

$$(24) \quad -\eta(X)g(Y, Z) + \eta(X)\eta(Y)\eta(Z)$$

Let R^* be the Riemannian curvature tensor with respect to Zamkovoy connection which is defined as

$$(25) \quad R^*(X, Y)Z = \nabla_X^* \nabla_Y^* Z - \nabla_Y^* \nabla_X^* Z - \nabla_{[X, Y]}^* Z.$$

By the help of (21), (22), (23) and (24) we obtain the expressions for $\nabla_X^* \nabla_Y^* Z$, $\nabla_Y^* \nabla_X^* Z$ and $\nabla_{[X, Y]}^* Z$ to get the Riemannian curvature tensor with respect to Zamkovoy connection as

$$(26) \quad \begin{aligned} R^*(X, Y)Z &= R(X, Y)Z - g(Z, \phi X)\phi Y - g(Y, \phi Z)\phi X \\ &\quad - 2g(Y, \phi X)\phi Z + g(X, Z)\eta(Y)\xi - \eta(X)g(Y, Z)\xi \\ &\quad + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X, \end{aligned}$$

Consequently one can easily bring out the followings:

$$(27) \quad S^*(Y, Z) = S(Y, Z) + 2g(Y, Z) - (1 + n)\eta(Y)\eta(Z),$$

$$(28) \quad S^*(Y, \xi) = 0 = S^*(\xi, Z),$$

$$(29) \quad Q^*Y = QY + 2Y - (1 + n)\eta(Y)\xi,$$

$$(30) \quad Q^*\xi = 0$$

$$(31) \quad r^* = r + n - 1,$$

$$(32) \quad R^*(X, Y)\xi = 0,$$

$$(33) \quad R^*(\xi, Y)Z = 0,$$

$$(34) \quad R^*(X, \xi)Z = 0.$$

Let M be an n -dimensional Sasakian manifold admitting Zamkovoy connection ∇^* , then (i) The curvature tensor R^* of ∇^* is given by (26), (ii) The Ricci tensor S^* of ∇^* is given by (27), (iii) The scalar curvature r^* of ∇^* is given by (31), (iv) The Ricci tensor S^* of ∇^* is symmetric.

Consider a Ricci soliton (g, ξ, λ) on M , then from (6) we have

$$(35) \quad \begin{aligned} 0 &= (L_\xi g)(Y, Z) + 2S(Y, Z) + 2\lambda g(Y, Z) \\ &= -g(\phi Y, Z) + g(\phi Y, Z) + 2S(Y, Z) + 2\lambda g(Y, Z) \\ &= S(Y, Z) + \lambda g(Y, Z). \end{aligned}$$

Setting $Z = \xi$ in (35), we get

$$(36) \quad S(Y, \xi) = -\lambda \eta(Y).$$

Replacing Y by QY in (36), we get

$$(37) \quad S^2(Y, \xi) = \lambda^2 \eta(Y).$$

4. Ricci soliton on concircularly flat Sasakian manifold with respect to Zamkovoy connection

Theorem 4.1. *If a concircularly flat Sasakian manifold M admits a Ricci soliton (g, ξ, λ) with respect to Zamkovoy connection, then the Ricci soliton is steady, shrinking or expanding according as $r = (n^2 + 1)$, $r > (n^2 + 1)$, or $r < (n^2 + 1)$.*

Proof 4.1. Let M be a concircularly flat Sasakian manifold with respect to Zamkovoy connection, i.e., $W^*(X, Y)Z = 0$, where W^* is the concircular curvature tensor with respect to Zamkovoy connection and $X, Y, Z \in \chi(M)$.

For a concircularly flat Sasakian manifold, it follows from (1) that

$$(38) \quad R^*(X, Y)Z = \frac{r^*}{n(n-1)} [g(Y, Z)X - g(X, Z)Y]$$

where R^* is the Riemannian curvature tensor with respect to Zamkovoy connection.

Taking inner product of (38) with a vector field V we get

$$(39) \quad g(R^*(X, Y)Z, V) = \frac{r+n-1}{n(n-1)} [g(Y, Z)g(X, V) - g(X, Z)g(Y, V)].$$

Taking an orthonormal frame field and contracting (39) over X and V we get

$$(40) \quad S^*(Y, Z) = \frac{r+n-1}{n} g(Y, Z).$$

Using (27) in (40) and setting $Z = \xi$, we get

$$(41) \quad S(Y, \xi) = \frac{r-n-1}{n} \eta(Y) + (1+n) \eta(Y).$$

By the help of (36) and (41), we obtain $\lambda = -\frac{r-(n^2+1)}{n}$.

Therefore, the Ricci soliton (g, ξ, λ) with respect to Zamkovoy connection is steady, shrinking or expanding according as $r = (n^2 + 1)$, $r > (n^2 + 1)$, or $r < (n^2 + 1)$.

5. Ricci soliton on W_2 -flat Sasakian manifold with respect to Zamkovoy connection

Theorem 5.1. *If a W_2 -flat Sasakian manifold M admits a Ricci soliton (g, ξ, λ) with respect to Zamkovoy connection, then the Ricci soliton is steady, shrinking or expanding according as $r = (n^2 + 1)$, $r > (n^2 + 1)$, or $r < (n^2 + 1)$.*

Proof 5.1. Let M be a W_2 -flat Sasakian manifold with respect to Zamkovoy connection, i.e., $W_2^*(X, Y)Z = 0$, where W_2^* is the W_2 -curvature tensor with respect to Zamkovoy connection and $X, Y, Z \in \chi(M)$.

For a W_2^* -flat Sasakian manifold, it follows from (2) that

$$(42) \quad R^*(X, Y)Z = \frac{1}{n-1} [g(Y, Z)Q^*X - g(X, Z)Q^*Y],$$

where R^* is the Riemannian curvature tensor with respect to Zamkovoy connection.

Taking inner product of (42) with a vector field V we get

$$(43) \quad g(R^*(X, Y)Z, V) = \frac{1}{n-1} [g(Y, Z)g(Q^*X, V) - g(X, Z)g(Q^*Y, V)].$$

Taking a frame field and contracting (43) over X and V we get

$$(44) \quad S^*(Y, Z) = \frac{1}{n-1} [r^*g(Y, Z) - S^*(Y, Z)].$$

Using (27), (31) in (44), we get

$$(45) \quad nS(Y, Z) = (r - n - 1)g(Y, Z) + n(n+1)\eta(Y)\eta(Z).$$

Setting $Z = \xi$ in (45), we have

$$(46) \quad nS(Y, \xi) = (r - n - 1)\eta(Y) + n(n+1)\eta(Y)$$

By the help of (36) and (46), we obtain

$$\lambda = -\frac{r - (n^2 + 1)}{n}.$$

This gives the theorem.

6. Ricci soliton on W_3 -flat Sasakian manifold with respect to Zamkovoy connection

Theorem 6.1. *If a W_3 -flat Sasakian manifold M of dimension n (> 3) admits a Ricci soliton (g, ξ, λ) with respect to Zamkovoy connection, then the Ricci soliton is steady, shrinking or expanding according as $r = -(n-1)(n-3)$, $r < (n-1)(n-3)$ or $r > (n-1)(n-3)$.*

Proof 6.1. Let M be a W_3 -flat Sasakian manifold with respect to Zamkovoy connection, i.e., $W_3^*(X, Y)Z = 0$, where W_3^* is the W_3 -curvature tensor with respect to Zamkovoy connection and $X, Y, Z \in \chi(M)$.

For a W_3^* -flat Sasakian manifold, it follows from (3) that

$$(47) \quad R^*(X, Y)Z = \frac{1}{n-1} [S^*(X, Z)Y - g(Y, Z)Q^*X],$$

where R^* is the Riemannian curvature tensor with respect to Zamkovoy connection.

Taking inner product of (47) with a vector field V we get

$$(48) \quad g(R^*(X, Y)Z, V) = \frac{1}{n-1} [S^*(X, Z)g(Y, V) - g(Y, Z)g(Q^*X, V)].$$

Taking an orthonormal frame field and contracting (48) over X and V we get

$$(49) \quad S^*(Y, Z) = \frac{1}{n-1} [S^*(Y, Z) - r^*g(Y, Z)].$$

Using (27), (31) in (49), we get

$$(50) \quad S(Y, Z) = \frac{(-r-3n+5)}{(n-2)}g(Y, Z) + (n+1)\eta(Y)\eta(Z).$$

Setting $Z = \xi$ in (50), we have

$$(51) \quad S(Y, \xi) = \frac{(-r-3n+5)}{(n-2)}\eta(Y) + (n+1)\eta(Y).$$

By the help of (36) and (51), we obtain

$$\lambda = \frac{r - (n-1)(n-3)}{n-2}.$$

Therefore, the Ricci soliton (g, ξ, λ) with respect to Zamkovoy connection is steady, shrinking or expanding according as $r = -(n-1)(n-3)$, $r < -(n-1)(n-3)$ or $r > -(n-1)(n-3)$.

Corollary 6.1. *If (g, ξ, λ) be a Ricci soliton with respect to Zamkovoy connection on a 3-dimensional Sasakian manifold, then the Ricci soliton is steady, shrinking or expanding according as $r = 0$, $r < 0$ or $r > 0$.*

7. Ricci soliton on W_4 -flat Sasakian manifold with respect to the Zamkovoy connection

Theorem 7.1. *If a W_4 -flat Sasakian manifold M admits a Ricci soliton (g, ξ, λ) with respect to Zamkovoy connection, then the Ricci soliton is always shrinking.*

Proof 7.1. Let M be a W_4 -flat Sasakian manifold with respect to Zamkovoy connection, i.e., $W_4^*(X, Y)Z = 0$, where W_4^* is the W_4 -curvature tensor with respect to Zamkovoy connection and $X, Y, Z \in \chi(M)$. For a W_4^* -flat Sasakian manifold, it follows from equation (4) that

$$(52) \quad R^*(X, Y)Z = -\frac{1}{n-1} [g(X, Z)Q^*Y - g(X, Y)Q^*Z],$$

where R^* is the Riemannian curvature tensor with respect to Zamkovoy connection.

Taking inner product with a vector field V in (52), we get

$$(53) \quad g(R^*(X, Y)Z, V) = -\frac{1}{n-1} [g(X, Z)g(Q^*Y, V) - g(X, Y)g(Q^*Z, V)].$$

Let $\{e_i\}$ ($1 \leq i \leq n$) be an orthonormal basis of the tangent space at any point of the manifold M . Setting $X = V = e_i$ in the equation (53) and taking summation over i ($1 \leq i \leq n$) and using (27), we get

$$(54) \quad S(Y, Z) = -2g(Y, Z) + (1+n)\eta(Y)\eta(Z).$$

Setting $Y = \xi$ in (54), we get

$$(55) \quad S(\xi, Z) = -2\eta(Z) + (1+n)\eta(Z).$$

Using (36) in (55), we get

$$\lambda = -(n-1) < 0.$$

Therefore, the Ricci soliton (g, ξ, λ) with respect to Zamkovoy connection is always shrinking.

8. Ricci soliton on Sasakian manifold satisfying $W_2^*(\xi, Y).R^* = 0$

Theorem 8.1. *If a Sasakian manifold M satisfying $W_2^*(\xi, Y).R^* = 0$ admits a Ricci soliton (g, ξ, λ) with respect to Zamkovoy connection, then the Ricci soliton cannot be steady.*

Proof 8.1. Let us consider a Sasakian manifold admitting Zamkovoy connection and satisfying

$$(W_2^*(\xi, Y).R^*)(X, U)Z = 0,$$

where $X, Y, Z, U \in \chi(M)$.

Then we have

$$(56) \quad \begin{aligned} & (W_2^*(\xi, Y).R^*)(X, U)Z \\ &= R^*(W_2^*(\xi, Y)X, U)Z + R^*(X, W_2^*(\xi, Y)U)Z \\ &+ R^*(X, U)W_2^*(\xi, Y)Z. \end{aligned}$$

Setting $U = \xi$ in (56), we have

$$(57) \quad \begin{aligned} & (W_2^*(\xi, Y).R^*)(X, \xi)Z \\ &= R^*(W_2^*(\xi, Y)X, \xi)Z + R^*(X, W_2^*(\xi, Y)\xi)Z \\ &+ R^*(X, \xi)W_2^*(\xi, Y)Z. \end{aligned}$$

By the help of (34) and (57), we get

$$(58) \quad 0 = R^*(X, Q^*Y)Z.$$

Taking inner product of (58) with a vector field V , we get

$$(59) \quad g(R^*(X, Q^*Y)Z, V) = 0.$$

Taking a frame field and contracting (59) over X and V and using (20), we get

$$(60) \quad 0 = S^2(Y, Z) + 4S(Y, Z) + 4g(Y, Z) - n(1+n)\eta(Y)\eta(Z)$$

Setting $Z = \xi$ in (60), we have

$$(61) \quad 0 = S^2(Y, \xi) + 4S(Y, \xi) + 4\eta(Y) - n(1+n)\eta(Y)$$

By the help of (37) and (61), we obtain

$$\lambda = 2 \pm \sqrt{n(n+1)} \neq 0.$$

Therefore, the Ricci soliton (g, ξ, λ) with respect to Zamkovoy connection cannot be steady.

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Comparative numerical study of time fractional coupled Korteweg-de Vries equation

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Abstract. In this paper, we construct an analytical solution for a system of the Fractional coupled Korteweg-de Vries differential equations in the sense of Caputo definition. We give the basic properties of the fractional differential equation and a coupled Korteweg-de Vries equation. We applied the homotopy analysis transform method to obtain the analytic solution for this equation, and we compared the result with other obtained solutions by different numerical methods. A comparison study observed the efficacy and accuracy of the present algorithm.

Keywords: fractional coupled Korteweg de Vries, Caputo fractional derivative, homotopy analysis transform method.

1. Introduction

Fractional differential equations are one of the important branches in the field differential equations, because of various phenomena in physics and chemistry, electronic and electrical, medicine and epidemiology spread, engineering, can be formulated into a fractional differential equation such as the coupled Korteweg-de Vries equation [1]:

$$\begin{aligned} \frac{\partial u(x, t)}{\partial t} &= \eta \frac{\partial^3 u(x, t)}{\partial x^3} + \gamma u(x, t) \frac{\partial u(x, t)}{\partial x} + \mu v(x, t) \frac{\partial v(x, t)}{\partial x}, \\ (1) \quad \frac{\partial v(x, t)}{\partial t} &= \lambda \frac{\partial^3 v(x, t)}{\partial x^3} - \nu v(x, t) \frac{\partial v(x, t)}{\partial x}. \end{aligned}$$

The solution of coupled Korteweg-de Vries is introduced using different numerical methods, to obtain the accurate approximations and their properties, such as

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the Spectral collocation method used by Khader [4], and Hussain [5]. Bashan [6] used the Finite difference method and differential quadrature method. Bakodah applied the decomposition method [7], homotopy perturbation method investigated by Bothayna S. Kashkari [8], homotopy analysis method (HAM) in [11] used by Abbsbandy and homotopy analysis transform method [12] by Saad.

The homotopy analysis method is one of the most important approximate methods for solving the linear and nonlinear differential equations, as partial differential equations [13], delay differential equations [14], fuzzy differential equations [15], including the fractional differential equations [16]. HAM proposed in Ph.D. dissertation in 1992 of Shijun Liao [17],[18], and [19]. The method has many applications in various classes of famous differential equations. For fractional differential equations, the scientists used the coupled Laplace transform with the homotopy analysis method (HATM), to write a simple algorithm corresponding to this type of equations, easily solved by Mathematica and Maple. The method introduced by Sunil Kumar [21], [22], Devendra Kumar [23], Majid Khan [24].

In this paper, we investigate the HATM for the following fractional differential equation [1]

$$(2) \quad \begin{aligned} \frac{\partial^\alpha u(x, t)}{\partial t^\alpha} &= \eta \frac{\partial^3 u(x, t)}{\partial x^3} + \gamma u(x, t) \frac{\partial u(x, t)}{\partial x} + \mu v(x, t) \frac{\partial v(x, t)}{\partial x}, \\ \frac{\partial^\beta v(x, t)}{\partial t^\beta} &= \lambda \frac{\partial^3 v(x, t)}{\partial x^3} - \nu u(x, t) \frac{\partial v(x, t)}{\partial x}, \end{aligned}$$

where ${}^C D^\alpha$ is the fractional derivative operator of order α in the sense of Caputo. We compare the numerical solutions with the result obtained by Mohamed Elbadri [1], who used the Naturel decomposition method (NDM), and spectral collection method [2].

2. Basic definitions of fractional calculus

In this section, we give some basic definitions of fractional calculus theory used in this paper:

Definition 2.1. A real function $h(t)$ is said C_μ , $\mu \in \mathbb{R}$ if there exists a real number $p > \mu$, such that $h(t) = t^p h_1(t)$ where $h_1(t) \in C(0, \infty)$ and it is said to be in space C_n if and only if $h^{(n)} \in C_\mu$, $n \in \mathbb{N}$.

Definition 2.2. The Riemann-Liouville fractional operator of order $\alpha \geq 0$, of a function $h \in C_\alpha$, $\alpha \geq -1$ is defined as [1]

$$\begin{aligned} I^\alpha h(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t - \alpha)^{(\alpha-1)} h(\tau) d\tau, \mu > 0, t > 0, \\ I^0 h(t) &= h(t), \end{aligned}$$

where $\Gamma(\cdot)$ is well-known Gamma function.

Definition 2.3. The Caputo fractional derivative of h , $h \in C_{-1}^m$ is defined as

$$D_C^\alpha h(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\varsigma)^{(m-\alpha-1)} h^{(m)}(\varsigma) d\varsigma,$$

where $m-1 < \alpha < m, m \in \mathbb{N}$

Definition 2.4. The Laplace transform for the function $h(t)$ of the Caputo fractional derivative given by [22]:

$$(3) \quad L[D_t^\alpha h(t)] = s^\alpha L[h(t)] - \sum_{k=0}^{n-1} s^{(\alpha-k-1)} h^{(k)}(0), \quad n-1 < \alpha \leq n.$$

3. The homotopy analysis transform method

Firstly, we give some basic idea of the homotopy analysis transform method, so we consider the following fractional differential equation to illustrate the application of the HATM for the fractional differential equation:

$$(4) \quad {}^C D^\alpha h(x, t) + \mathfrak{R}h(x, t) + \mathfrak{N}h(x, t) = f(x, t), \quad 0 < \alpha \leq 1,$$

where ${}^C D_t^\alpha h(x, t)$ is the Caputo derivative of $h(x, t)$, $\mathfrak{R}$ and $\mathfrak{N}$ are the linear and nonlinear operator respectively, $f(x, t)$ is the source term.

Firstly, by applying the Laplace transform L on the (4), we obtain,

$$(5) \quad s^\alpha L[h(x, t)] - s^{\alpha-1} h(x, 0) + L[\mathfrak{R}h(x, t)] + L[\mathfrak{N}h(x, t)] = L[f(x, t)], \quad 0 < \alpha \leq 1$$

after simplification,

$$(6) \quad L[h(x, t)] = \frac{1}{s} h(x, 0) + \frac{1}{s^\alpha} (L[f(x, t)] - L[\mathfrak{R}h(x, t)] - L[\mathfrak{N}h(x, t)]), \quad 0 < \alpha \leq 1.$$

By applying the homotopy analysis method shown in [10], [11] and [12], we define the non-linear operator

$$(7) \quad \begin{aligned} N[\phi(x, t, q)] &= L[\phi(x, t, q)] - \frac{1}{s} h(x, 0) - \frac{1}{s^\alpha} (L[f(x, t)] \\ &+ L[\mathfrak{R}\phi(x, t, q)] + L[\mathfrak{N}\phi(x, t, q)]). \end{aligned}$$

where ϕ is a real-valued function of x, t , and $q \in [0, 1]$. The zeroth-order deformation constructed by Liao [10], [11] is

$$(8) \quad (1-q)\mathcal{L}[\phi(x, t, q) - h_0(x, t)] = \hbar q N[\phi(x, t, q)],$$

where $\hbar \neq 0$ is nonzero convergent control parameter, $h_0(x, t)$ is the initial guess, N is the nonlinear operator and $\mathcal{L}$ is an injective linear operator. Here, we choose

the linear operator as a Laplace operator $\mathcal{L} = L$. Obviously $\phi(x, t, 0) = u_0(x, t)$ and $\phi(x, t, 1) = h(x, t)$. Expanding $\phi(x, t, q)$ in Taylor series with respect q ,

$$\phi(x, t, q) = \sum_{i=0}^n h_i(x, t) q^i,$$

where

$$(9) \quad h_i(x, t) = \frac{1}{m!} \frac{\partial^m \phi(x, t, q)}{\partial q^m} \Big|_{q=0},$$

by differentiating (8) m times with respect q , and taking $q = 0$, we obtain the m th order deformation equation

$$(10) \quad L[h_m(x, t) - \ell_m h_{m-1}(x, t)] = \hbar H(x, t) R_m(h_{m-1}(x, t)).$$

Applying L^{-1} in (10) we get:

$$(11) \quad h_m(x, t) = \ell_m h_{m-1}(x, t) + \hbar L^{-1}[R_m(h_{m-1}(x, t))],$$

where

$$\ell_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases}$$

Now, for the system (2), we define a system of nonlinear operator as:

$$(12) \quad \begin{aligned} N_1[\phi(x, t, q), \psi(x, t, q)] &= L[\phi] - \frac{1}{s} u(x, 0) - \frac{1}{s^\alpha} \left(\eta \frac{\partial^3 \phi}{\partial x^3} + \gamma \phi \frac{\partial \phi}{\partial x} + \mu \psi \frac{\partial \psi}{\partial x} \right), \\ N_2[\phi(x, t, q), \psi(x, t, q)] &= L[\psi] - \frac{1}{s} v(x, 0) - \frac{1}{s^\beta} \left(\lambda \frac{\partial^3 \psi}{\partial x^3} - \nu \phi \frac{\partial \psi}{\partial x} \right) \end{aligned}$$

the m th order deformation equation is

$$(13) \quad \begin{aligned} L[u_m(x, t) - \ell_m u_{m-1}(x, t)] &= \hbar H(x, t) K_m[\vec{u}_{m-1}(x, t), \vec{v}_{m-1}(x, t)], \\ L[v_m(x, t) - \ell_m v_{m-1}(x, t)] &= \hbar H(x, t) R_m[\vec{u}_{m-1}(x, t), \vec{v}_{m-1}(x, t)], \end{aligned}$$

where

$$\begin{aligned} K_n &= u_{n-1}(x, t) - \frac{(1 - l_n)}{s} u_0(x, t) \\ &\quad + \frac{1}{s^\alpha} \left(L \left[\eta \frac{\partial^3 u_{n-1}}{\partial x^3} + \gamma \sum_{i=0}^{n-1} u_i(x, t) \frac{\partial u_{n-1-i}}{\partial x} + \mu \sum_{i=0}^{n-1-i} v_i \frac{\partial v_{n-1-i}}{\partial x} \right] \right), \\ R_n &= v_{n-1}(x, t) - \frac{(1 - l_n)}{s} v_0(x, t) \\ &\quad + \frac{1}{s^\beta} \left(L \left[\lambda \frac{\partial^3 v_{n-1}}{\partial x^3} - \nu \sum_{i=0}^{n-1} u_i(x, t) \frac{\partial v_{n-1-i}}{\partial x} \right] \right). \end{aligned}$$

By applying the inverse Laplace operator for (13), we obtain:

$$(14) \quad \begin{aligned} u_m(x, t) &= \ell_m u_{m-1}(x, t) + \hbar L^{-1} K_m[\vec{u}_m(x, t), \vec{v}_m(x, t)], \\ v_m(x, t) &= \ell_m v_{m-1}(x, t) + \hbar L^{-1} R_m[\vec{u}_m(x, t), \vec{v}_m(x, t)], \end{aligned}$$

where L^{-1} is the inverse Laplace transform operator, and in this case, we mention that $u(x, t)$, and $v(x, t)$ is written as a series

$$(15) \quad \begin{aligned} u(x, t) &= u_0(x, t) + \sum_{i=1}^{\infty} u_i(x, t), \\ v(x, t) &= v_0(x, t) + \sum_{i=1}^{\infty} v_i(x, t), \end{aligned}$$

with the initial conditions,

$$u(x, 0) = u_0(x, t), \quad v(x, 0) = v_0(x, t).$$

4. Application

In this section, we evaluate the coupled Korteweg-de Vries (2) using HATM with two different initial conditions.

Example 4.1. We consider (2) with $\eta = -a, \gamma = -6a, \mu = 2b, \lambda = -r$ and $\nu = 3r$:

$$(16) \quad u(x, 0) = \frac{\zeta}{a} \left(\operatorname{sech}\left(\frac{1}{2}\sqrt{\frac{\zeta}{a}}x\right) \right)^2, \quad v(x, 0) = \frac{\zeta}{\sqrt{2a}} \left(\operatorname{sech}\left(\frac{1}{2}\sqrt{\frac{\zeta}{a}}x\right) \right)^2.$$

for $\eta = -1, \gamma = -6, \mu = 6, \lambda = -1$ and $\nu = 3$, we obtained the first-order approximation :

$$\begin{aligned} u_0(x, t) &= \left(\operatorname{sech}\left(\frac{x}{2}\right) \right)^2, \\ u_1(x, t) &= -\frac{\hbar t^\alpha (\operatorname{sech}[\frac{x}{2}]^4 \tanh[\frac{x}{2}]^2 + \operatorname{sech}[\frac{x}{2}]^2 \tanh[\frac{x}{2}]^3)}{\Gamma[\alpha + 1]}, \\ u_2(x, t) &= -\frac{\hbar t^\alpha (\operatorname{sech}[\frac{x}{2}]^4 \tanh[\frac{x}{2}]^2 + \operatorname{sech}[\frac{x}{2}]^2 \tanh[\frac{x}{2}]^3)}{\Gamma[\alpha + 1]} \\ &\quad + \hbar \left(\frac{\hbar t^\alpha \operatorname{sech}^6(\frac{x}{2}) (t^\alpha (22 \cosh(x) + \cosh(2x) - 39) \Gamma(\alpha + \beta + 1) - 12 \Gamma(2\alpha + 1) t^\beta (2 \cosh(x) - 3))}{8 \Gamma(2\alpha + 1) \Gamma(\alpha + \beta + 1)} \right. \\ &\quad \left. - \frac{\hbar t^\alpha (\tanh(\frac{x}{2}) \operatorname{sech}^4(\frac{x}{2}) + \tanh^3(\frac{x}{2}) \operatorname{sech}^2(\frac{x}{2}))}{\Gamma(\alpha + 1)} \right), \end{aligned}$$

for $v(x, t)$, we have:

$$\begin{aligned}
 v_0(x, t) &= \frac{\operatorname{sech}^2\left(\frac{x}{2}\right)}{\sqrt{2}}, \\
 v_1(x, t) &= \frac{\sqrt{2}ht^\beta \tanh\left(\frac{x}{2}\right) \operatorname{sech}^4\left(\frac{x}{2}\right)}{\Gamma(\beta+1)} - \frac{3ht^\beta \tanh\left(\frac{x}{2}\right) \operatorname{sech}^4\left(\frac{x}{2}\right)}{\sqrt{2}\Gamma(\beta+1)} \\
 &\quad - \frac{ht^\beta \tanh^3\left(\frac{x}{2}\right) \operatorname{sech}^2\left(\frac{x}{2}\right)}{\sqrt{2}\Gamma(\beta+1)}, \\
 v_2(x, t) &= -\frac{3h^2 \operatorname{sech}^6\left(\frac{x}{2}\right) t^{\alpha+\beta}}{2\sqrt{2}\Gamma(\alpha+\beta+1)} + \frac{3h^2 \cosh(x) \operatorname{sech}^6\left(\frac{x}{2}\right) t^{\alpha+\beta}}{2\sqrt{2}\Gamma(\alpha+\beta+1)} \\
 &\quad + \frac{9h^2 t^{2\beta} \operatorname{sech}^6\left(\frac{x}{2}\right)}{8\sqrt{2}\Gamma(2\beta+1)} - \frac{7h^2 t^{2\beta} \cosh(x) \operatorname{sech}^6\left(\frac{x}{2}\right)}{4\sqrt{2}\Gamma(2\beta+1)} \\
 &\quad + \frac{h^2 t^{2\beta} \cosh(2x) \operatorname{sech}^6\left(\frac{x}{2}\right)}{8\sqrt{2}\Gamma(2\beta+1)} - \frac{3ht^\beta \tanh\left(\frac{x}{2}\right) \operatorname{sech}^4\left(\frac{x}{2}\right)}{\sqrt{2}\Gamma(\beta+1)} \\
 &\quad + \frac{\sqrt{2}ht^\beta \tanh\left(\frac{x}{2}\right) \operatorname{sech}^4\left(\frac{x}{2}\right)}{\Gamma(\beta+1)} + \dots
 \end{aligned}$$

Example 4.2. For the second test example, we consider (2), with the new initial conditions

$$(17) \quad u(x, 0) = \frac{4c^2 e^{cx}}{(1 + e^{cx})^2}, \quad v(x, 0) = \frac{4c^2 e^{cx}}{(1 + e^{cx})^2}.$$

Applying the HATM, with $\eta = -1$, $\gamma = -6$, $\mu = 3$, $\lambda = -1$, and $\nu = 3$, we obtain

$$\begin{aligned}
 u_0(x, t) &= \frac{4c^2 e^{cx}}{(e^{cx} + 1)^2}, \\
 u_1(x, t) &= -\frac{1}{\Gamma(\alpha+1)} \left(-\frac{12c^2 e^{cx} \left(\frac{4c^3 e^{cx}}{(e^{cx}+1)^2} - \frac{8c^3 e^{2cx}}{(e^{cx}+1)^3} \right)}{(e^{cx} + 1)^2} \right. \\
 &\quad - 4c^2 \left(\frac{c^3 e^{cx}}{(e^{cx} + 1)^2} - \frac{6c^3 e^{2cx}}{(e^{cx} + 1)^3} + 3ce^{cx} \left(\frac{6c^2 e^{2cx}}{(e^{cx} + 1)^4} - \frac{2c^2 e^{cx}}{(e^{cx} + 1)^3} \right) \right. \\
 &\quad \left. \left. + \hbar t^\alpha e^{cx} \left(-\frac{2c^3 e^{cx}}{(e^{cx} + 1)^3} + \frac{18c^3 e^{2cx}}{(e^{cx} + 1)^4} - \frac{24c^3 e^{3cx}}{(e^{cx} + 1)^5} \right) \right) \right), \\
 u_2(x, t) &= -\frac{\hbar t^\alpha}{\Gamma(\alpha+1)} \left(-\frac{12c^2 e^{cx} \left(\frac{4c^3 e^{cx}}{(e^{cx}+1)^2} - \frac{8c^3 e^{2cx}}{(e^{cx}+1)^3} \right)}{(e^{cx} + 1)^2} \right. \\
 &\quad - 4c^2 \left(\frac{c^3 e^{cx}}{(e^{cx} + 1)^2} - \frac{6c^3 e^{2cx}}{(e^{cx} + 1)^3} \right. \\
 &\quad \left. + e^{cx} \left(-\frac{2c^3 e^{cx}}{(e^{cx} + 1)^3} + \frac{18c^3 e^{2cx}}{(e^{cx} + 1)^4} - \frac{24c^3 e^{3cx}}{(e^{cx} + 1)^5} \right) \right. \\
 &\quad \left. \left. + 3ce^{cx} \left(\frac{6c^2 e^{2cx}}{(e^{cx} + 1)^4} - \frac{2c^2 e^{cx}}{(e^{cx} + 1)^3} \right) \right) \right) + \dots
 \end{aligned}$$

and for $v(x, t)$, we obtain

$$\begin{aligned}
 v_0(x, t) &= \frac{4c^2 e^{cx}}{(e^{cx} + 1)^2}, \\
 v_1(x, t) &= \frac{4c^5 \hbar e^{cx} t^\beta}{\Gamma(\beta + 1) (e^{cx} + 1)^2} - \frac{56c^5 \hbar e^{2cx} t^\beta}{\Gamma(\beta + 1) (e^{cx} + 1)^3} \\
 &\quad + \frac{48c^5 \hbar e^{2cx} t^\beta}{\Gamma(\beta + 1) (e^{cx} + 1)^4} + \frac{144c^5 \hbar e^{3cx} t^\beta}{\Gamma(\beta + 1) (e^{cx} + 1)^4} \\
 &\quad - \frac{96c^5 \hbar e^{3cx} t^\beta}{\Gamma(\beta + 1) (e^{cx} + 1)^5} - \frac{96c^5 \hbar e^{4cx} t^\beta}{\Gamma(\beta + 1) (e^{cx} + 1)^5}, \\
 v_2(x, t) &= \frac{48c^5 \hbar e^{2cx} t^\beta}{\Gamma(\beta + 1) (e^{cx} + 1)^4} + \frac{144c^5 \hbar e^{3cx} t^\beta}{\Gamma(\beta + 1) (e^{cx} + 1)^4} \\
 &\quad - \frac{96c^5 \hbar e^{3cx} t^\beta}{\Gamma(\beta + 1) (e^{cx} + 1)^5} - \frac{96c^5 \hbar e^{4cx} t^\beta}{\Gamma(\beta + 1) (e^{cx} + 1)^5} \\
 &\quad + \frac{48c^8 \hbar^2 e^{4cx} t^{\alpha+\beta}}{(e^{cx} + 1)^6 \Gamma(\alpha + \beta + 1)} + \frac{4c^8 \hbar^2 e^{cx} t^{2\beta}}{\Gamma(2\beta + 1) (e^{cx} + 1)^6} \\
 &\quad - \frac{56c^8 \hbar^2 e^{2cx} t^{2\beta}}{\Gamma(2\beta + 1) (e^{cx} + 1)^6} + \frac{72c^8 \hbar^2 e^{3cx} t^{2\beta}}{\Gamma(2\beta + 1) (e^{cx} + 1)^6} + \dots
 \end{aligned}$$

5. Convergence analysis

Theorem 5.1. *If the series solutions $\sum_{i=0}^n u_i(x, t)$, and $\sum_{i=0}^n v_i(x, t)$ are convergent, where u_m and v_m are governed by equation (9), then must be solutions of (11).*

Proof of Theorem 5.1. *Suppose that $\sum_{i=0}^n u_i(x, t)$ and $\sum_{i=0}^n v_i(x, t)$ are convergent, i.e $\lim_{n \rightarrow \infty} u_n(x, t) = \lim_{n \rightarrow \infty} v_n(x, t) = 0$. From the equation (13), we get:*

$$\begin{aligned}
 \hbar H \sum_{m=1}^{\infty} K_m &= \lim_{k \rightarrow \infty} \sum_{m=0}^k L[u_m - \ell_m u_{m-1}] \\
 &= L[\lim_{k \rightarrow \infty} \sum_{m=0}^k [u_m - \ell_m u_{m-1}]] = L[\lim_{k \rightarrow \infty} u_k],
 \end{aligned}$$

where L is a Laplace operator. Since $\lim_{k \rightarrow \infty} u_k = 0$, $H \neq 0$ and $\hbar \neq 0$, this implies $\sum_{m=1}^{\infty} K_m = 0$. Now, expand $N_1[\Phi(x, t, q), \psi(x, t, q)]$ about $q = 0$, then set $q = 1$,

$$N_1[\Phi(x, t, 1), \psi(x, t, q)] = 0,$$

we can see that $u(x, t) = \Phi(x, t, 1) = \sum_{i=0}^{\infty} u_i(x, t)$, and $v(x, t) = \psi(x, t, 1) = \sum_{i=0}^{\infty} v_i(x, t)$ solve equation (2).

Theorem 5.2 ([20]). *Let the solution component $h_0(x, t), h_1(x, t), h_2(x, t), \dots$ be defined as (11). The series solution $\sum_{m=0}^{\infty} h_m(x, t)$ defined in (15) converge if there exist $0 < \gamma < 1$ such that $\|h_{m+1}(x, t)\| \leq \gamma \|h_m(x, t)\|, \forall m > m_0$, for some $m_0 \in \mathbb{N}$.*

Theorem 5.3 ([20]). Suppose that the series solution $\sum_{m=0}^{\infty} h_m(x, t)$ is convergent to the solution $h(x, t)$. If the truncated series $\sum_{m=0}^k h_m(x, t)$ is used as approximation to the solution $h(x, t)$, then the truncated error satisfies

$$\| h(x, t) - \sum_{m=0}^k h_m(x, t) \| \leq \frac{1}{1 - \gamma} \gamma^{k+1} \| h_0(x, t) \| .$$

6. Numerical result and discussion

In Fig. 2 and 3, we plot the 5-th order approximation solutions and the exact solution of (2), we can see that the graphs coincide with the graphs of the exact solution.

Now, to control the convergence of the series in the frame of HATM, by the auxiliary convergence parameter $\hbar$, we plot the so-called $\hbar$ -curve for $u_{tt}(0, 0)(\hbar)$ and $v_{tt}(0, 0)(\hbar)$ In Fig1,4,6 and 8. we can see from those figures that the best value of $\hbar$ for the first example is in the convergence region $-1.2 \leq \hbar \leq -0.6$, and for the second example is $-1.3 \leq \hbar \leq -0.7$ so we have a freedom to choice $\hbar$ in this region to get a convergence series. In table 6.1, and 6.3 we compare our results with other result obtained using the NDM solutions [1], and the spectral collection method [2]. In Table 6.2, the approximation solutions for different value of α , and β are given.

Table 6.1. Comparison of our results of $u(x, t)$, and $v(x, t)$ with other result obtained by other numerical methods for different values of $\alpha = \beta = 1$.

x	t	$Exact(u(x, t))$	$HATM$	$Error (HATM)$	$Error(NDM)[1]$	$Error[2]$
-10	0.1	0.000164304	0.000164304	2.47239×10^{-13}	2.95039×10^{-8}	2.99039×10^{-8}
-10	0.2	0.000148670	0.000148670	1.56036×10^{-11}	2.30335×10^{-7}	2.33335×10^{-7}
-5	0.1	0.024092321	0.024092321	8.91166×10^{-12}	3.93592×10^{-6}	3.96592×10^{-6}
-5	0.2	0.021824797	0.021824797	5.83638×10^{-10}	3.08049×10^{-5}	3.38049×10^{-5}
5	0.1	0.029347625	0.029347625	8.43378×10^{-12}	2.02966×10^{-4}	3.97592×10^{-6}
5	0.2	0.032383774	0.032383773	5.22271×10^{-10}	5.15558×10^{-4}	3.78049×10^{-5}
10	0.1	0.000200678	0.000200678	2.54363×10^{-13}	2.11254×10^{-8}	2.96039×10^{-8}
10	0.2	0.000221781	0.000221781	1.65158×10^{-11}	2.28173×10^{-7}	2.37335×10^{-7}
x	t	$Exact(v(x, t))$	$HATM$	$Error (HATM)$	$Error(NDM)[1]$	$Error[2]$
-10	0.1	0.000116180	0.000116180	1.74825×10^{-13}	2.08624×10^{-8}	2.18624×10^{-8}
-10	0.2	0.000105125	0.000105125	1.10334×10^{-11}	1.62872×10^{-7}	1.64872×10^{-7}
-5	0.1	0.017035843	0.017035843	6.30149×10^{-12}	2.78312×10^{-6}	2.88312×10^{-6}
-5	0.2	0.015432462	0.015432462	4.12694×10^{-10}	2.21782×10^{-5}	2.87824×10^{-5}
5	0.1	0.020751905	0.020751905	5.96358×10^{-12}	2.9094×10^{-6}	2.98312×10^{-6}
5	0.2	0.022898786	0.022898786	3.69301×10^{-10}	2.238036×10^{-5}	2.47824×10^{-5}
10	0.1	0.000141901	0.000141901	1.79862×10^{-13}	2.19313×10^{-8}	2.09624×10^{-8}
10	0.2	0.000156823	0.000156823	1.16785×10^{-11}	1.79991×10^{-7}	1.72872×10^{-7}

In the above tables, we can see that the approximation solution obtained by HATM, is agreed and appropriate with the exact solution more than the NDM and spectral collection methods.

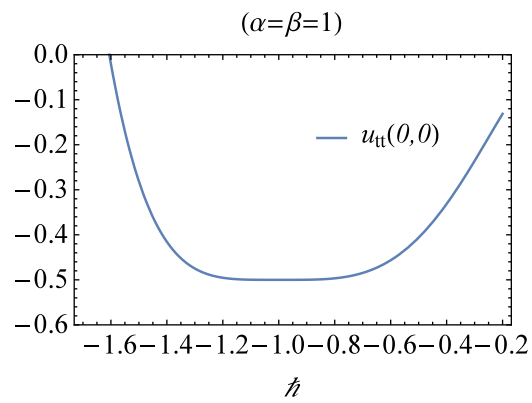
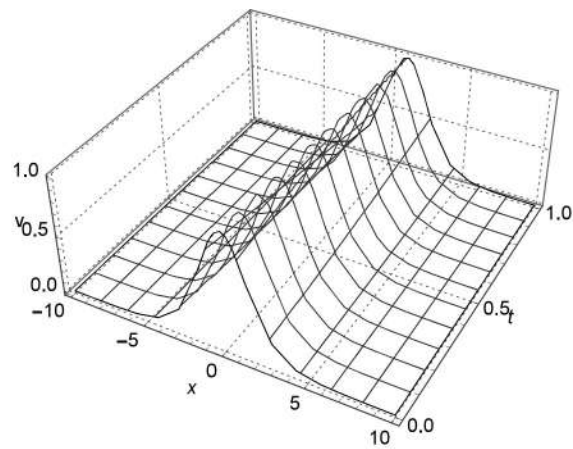
Table 6.2. Numerical solutions of $u(x, t)$, and $v(x, t)$ for different values of α, β . Numerical solutions of $u(x, t)$, and $v(x, t)$ for different values of α, β

x	t	$Exact\ u(x, t)$	$\alpha = \beta = 0.9$	$\alpha = \beta = 0.8$	$\alpha = \beta = 0.6$
-10	0.1	0.000164304	0.000159449	0.000153648	0.000139462
-10	0.2	0.000148670	0.000142868	0.000136314	0.000123400
-5	0.1	0.024092321	0.023388290	0.022546135	0.020472920
-5	0.2	0.021824797	0.020954214	0.020024033	0.018074633
5	0.1	0.029347625	0.030919078	0.031537740	0.035657141
5	0.2	0.032383774	0.033919078	0.035900711	0.041940955
10	0.1	0.000200678	0.000207156	0.000215926	0.000244977
10	0.2	0.000221781	0.000232514	0.000246460	0.000290077
x	t	$Exact\ v(x, t)$	$\alpha = \beta = 0.9$	$\alpha = \beta = 0.8$	$\alpha = \beta = 0.6$
-10	0.1	0.000116180	0.000112747	0.000108646	0.000098614
-10	0.2	0.000105125	0.000100894	0.000096389	0.000087257
-5	0.1	0.017035843	0.016538018	0.015942525	0.014476541
-5	0.2	0.015432462	0.014816867	0.014159129	0.012780695
5	0.1	0.020751905	0.021410870	0.022300550	0.025213406
5	0.2	0.022898786	0.023984410	0.025385636	0.029656733
10	0.1	0.000141901	0.000146481	0.000152682	0.000173225
10	0.2	0.000156823	0.000164412	0.000174273	0.000205115

For the second example, we compare with the exact solution, the approximate solutions for [3] in Table 5.3. In Fig. 4 the 10-th order of solution with the exact solution $u(x, t) = \frac{\zeta}{a} (\text{sech}(\frac{1}{2\sqrt{\frac{\zeta}{a}(x-\zeta t)}}))^2, u(x, t) = \frac{\zeta}{\sqrt{2a}} (\text{sech}(\frac{1}{2\sqrt{\frac{\zeta}{a}(x-\zeta t)}}))^2$ for example1 are presented, and $u(x, t) = v(x, t) = \frac{4c^2 e^{c(x-c^2 t)}}{(1+e^{c(x-c^2 t)})^2}$ for the second example. In Fig 6, the graphs traced for different values of α, β .

Table 6.3. Comparison of numerical solutions of $u(x, t)$, and $v(x, t)$ with other obtained by other numerical methods for different value of $\alpha = \beta = 1$.

x	t	$Exact$	$HATM(\alpha = \beta = 1)$	$Error\ (HATM)$	$Error(NDM)[1]$	$Error[3]$
-10	0.1	0.000164304	0.000164304	2.47240×10^{-13}	2.95039×10^{-8}	2.95039×10^{-8}
-10	0.2	0.000148670	0.000148670	1.56036×10^{-11}	2.30335×10^{-7}	2.30335×10^{-7}
-5	0.1	0.024092321	0.024092321	8.91164×10^{-12}	3.93592×10^{-6}	3.93592×10^{-6}
-5	0.2	0.021824797	0.021824797	5.83638×10^{-10}	3.08049×10^{-5}	3.08049×10^{-5}
5	0.1	0.029347625	0.029347625	8.43377×10^{-12}	4.11452×10^{-6}	4.11452×10^{-6}
5	0.2	0.032383774	0.032383773	5.22271×10^{-10}	3.36633×10^{-5}	3.36633×10^{-5}
10	0.1	0.000200678	0.000200678	2.54363×10^{-13}	3.10155×10^{-8}	3.10155×10^{-8}
10	0.2	0.000221781	0.000221781	1.65158×10^{-11}	2.54546×10^{-7}	2.54546×10^{-7}
x	t	$Exact$	$HATM(\alpha = \beta = 1)$	$Error\ (HATM)$	$Error(NDM)[1]$	$Error[3]$
-10	0.1	0.000164304	0.000164304	2.47240×10^{-13}	2.95039×10^{-8}	2.95039×10^{-8}
-10	0.2	0.000148670	0.000148670	1.56036×10^{-11}	2.30335×10^{-7}	2.30335×10^{-7}
-5	0.1	0.024092321	0.024092321	8.91164×10^{-12}	3.93592×10^{-6}	3.93592×10^{-6}
-5	0.2	0.021824797	0.021824797	5.83638×10^{-10}	3.08049×10^{-5}	3.08049×10^{-5}
5	0.1	0.029347625	0.029347625	8.43375×10^{-12}	4.11452×10^{-6}	4.11452×10^{-6}
5	0.2	0.032383774	0.032383773	5.22277×10^{-10}	3.36633×10^{-5}	3.36633×10^{-5}
10	0.1	0.000200678	0.000200678	2.54363×10^{-13}	3.10155×10^{-8}	3.10155×10^{-8}
10	0.2	0.000221781	0.000221781	1.65158×10^{-11}	2.54546×10^{-7}	2.54546×10^{-7}

Figure 1: The h -curve for the example 1Figure 2: The approximate solution $u(x,t)$ for example 1

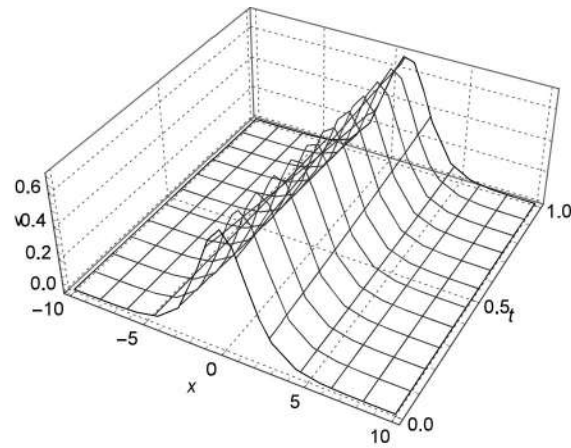


Figure 3: The approximate solution $v(x, t)$ for example 1

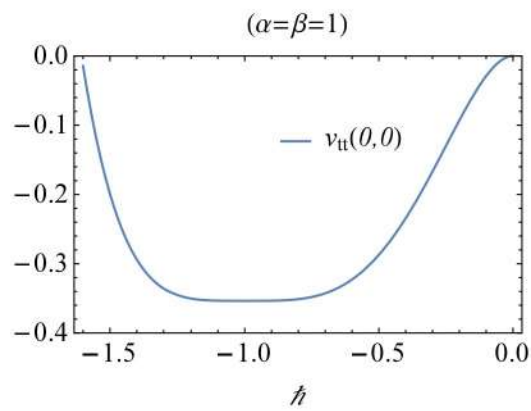


Figure 4: The h -curve for the example 1

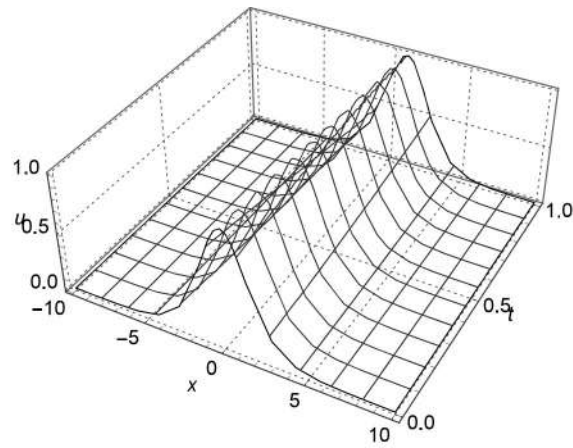


Figure 5: The approximate solution $u(x, t)$ for example 2

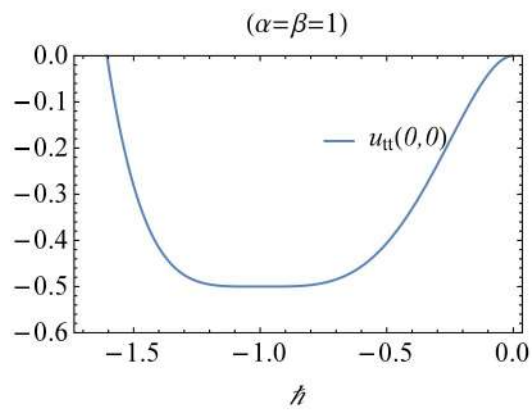


Figure 6: The h -curve for the example 1

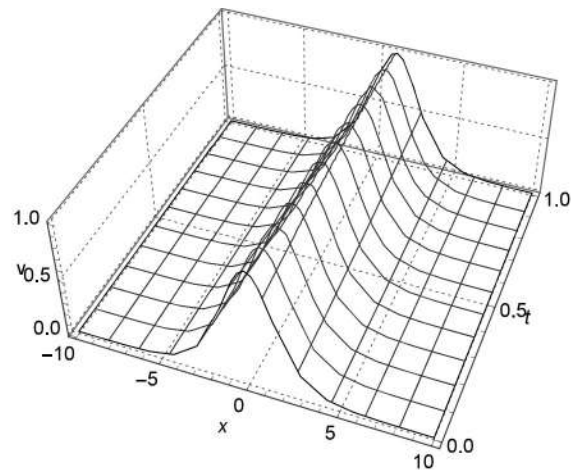


Figure 7: The approximate solution $v(x, t)$ for example 2

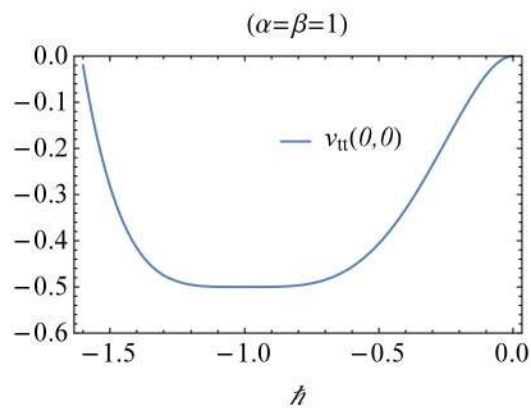


Figure 8: The h -curve for the example 1

Concluding remarks

In this work, we successfully applied the HATM for a coupled Korteweg-de Vries equation, and we show that this method is efficient and applicable for this type of fractional differential equations, we compared our results with others obtained by different numerical method. The algorithm is powerfully for solving this kinds of equations.

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On properties of composites and monoid domains

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Abstract. In this paper we consider all possible properties from commutative algebra for polynomial composites and monoid domains. The aim is full characterization of these structures. We start with the examination of group, ring, modules properties, graded, but also study of invertible elements, irreducible elements, ideals, etc. in these structures. In the second part of this paper we give examples of the use of composites and monoid domains in cryptology. Each such polynomial is the sum of the products of the variable and the coefficient. And what if subsequent coefficient sets are appropriate cryptographic systems? Similarly, monoid domains can be a very good tool between encrypting and decrypting messages.

Keywords: cryptology, domain, field, irreducible element, monoid, polynomial.

1. Introduction

By a ring we mean a commutative ring with unity. Let R be a ring. Denote by R^* the group of all invertible elements of R . The set of all irreducible elements in R will be denoted by $\text{Irr } R$. Recall that a GCD-domain is an integral domain R with the property that any two elements have a greatest common divisor (GCD); i.e., there is a unique minimal principal ideal containing the ideal generated by given two elements. We can find much informations about GCD-domains. I refer to [6], [13], [14].

The main motivation of this paper is description of some algebraic objects in the language of commutative algebra. D.D. Anderson, D.F. Anderson, M. Zafrullah in [12] called object $A + XB[X]$ as a composite for $A \subset B$ fields. If B is a domain and M is an additive cancellative monoid we can define a monoid domain $B[M] = \{a_0X^{m_0} + \dots + a_nX^{m_n} : a_0, \dots, a_n \in B, m_1, \dots, m_n \in M\}$. Monoid domains appear in many works such that [15], [16].

There are a lot of works where composites are used as examples to show some properties. But the most important works are presented below.

In 1976 [3] authors considered the structures in the form $D + M$, where D is a domain and M is a maximal ideal of ring R , where $D \subset R$. Later (2.1), we could prove that in composite in the form $D + XK[X]$, where D is a domain, K is a field with $D \subset K$, that $XK[X]$ is a maximal ideal of

$K[X]$. Next, Costa, Mott and Zafrullah ([4], 1978) considered composites in the form $D + XD_S[X]$, where D is a domain and D_S is a localization of D relative to the multiplicative subset S . In 1988 [8] Anderson and Ryckaert studied classes groups $D + M$. Zafrullah in [9] continued research on structure $D + XD_S[X]$ but he showed that if D is a GCD-domain, then the behaviour of $D^{(S)} = \{a_0 + \sum a_i X^i \mid a_0 \in D, a_i \in D_S\} = D + XD_S[X]$ depends upon the relationship between S and the prime ideals P of D such that D_P is a valuation domain (Theorem 1, [9]). Fontana and Kabbaj in 1990 ([11]) studied the Krull and valuative dimensions of composite $D + XD_S[X]$. In 1991 there was an article ([12]) that collected all previous composites and the authors began to create a theory about composites creating results. In this paper, the structures under consideration were officially called as composites. After this article, various minor results appeared. But the most important thing is that composites have been used in many theories as examples. That is why we decided to examine all possible properties of composites for commutative algebra. We put the first results in [18], and the next results we put in this article.

In the second chapter we will present a more general concept of a composite defining two types and $I(B, A)$, and monoid domain $F[X; M]$, with different configurations of rings, domains, fields, etc. We will show some simple properties from commutative algebra for these structures. In the Lemma 2.2 it will turn out that one of the types of composites together with the multiplicative action can not be a semigroup. In the Theorem 2.1 we show that every nonzero prime ideal in the composite is the maximal ideal. We will also see what are the irreducible elements of such composites. In the Theorem 2.2 we will look at what elements are irreducible in the considered structures. At the end of the chapter, we will look at the multiplicative systems of these structures.

In the third chapter we present simple observations by building on R -modules from the considered structures (Proposition 3.1). We also asked a question about exact sequence for composites. The next chapter is devoted to locations in composites. In the fifth chapter, we mention graded rings and modules.

The main motivation of sixth chapter is description composites and monoid domains in language for the monoids/groups (for composites) and for rings (for monoid rings).

The structures described in this paper often appear in the form of examples in many works. The aim of this work will be to examine as much as possible properties and applications in commutative algebra.

In last chapters of this paper I give an example of the use of composites and monoid domains in cryptology.

2. Basic properties

The aim of this chapter will be to examine the simplest structural properties of the considered structures.

Consider A and B as rings such that $A \subset B$. Put $T = A + XB[X]$. The structure defined in this way is called a composite. (The definition comes from [12]). Later results will require more assumptions.

Let us generalize the concept of a composite in two different directions.

Consider $A_0, A_1, \dots, A_{n-1}$ and B be rings for any $n \geq 0$ such that $A_0 \subset A_1 \subset \dots \subset A_{n-1} \subset B$. Put $T_n = A_0 + A_1X + \dots + A_{n-1}X^{n-1} + X^nB[X]$.

And let other $A_0, A_1, \dots, A_{n-1}$ and B be rings for any $n \geq 0$ such that there exists $i \in \{0, 1, \dots, n-1\}$, where $A_i \not\subset A_{i+1}$ and for every $j \in \{0, 1, \dots, n-1\}$ we have $A_j \subset B$. Put $T'_n = A_0 + A_1X + \dots + A_{n-1}X^{n-1} + X^nB[X]$. Also, consider $I(B, A) = \{f \in B[X], f(A) \subseteq A\}$, where $A \subset B$ are rings and monoid domain $B[M]$, where B be a domain and M be a submonoid of $\mathbb{Q}_+$.

Throughout the article, we will use the above denotation. If the above structures require stronger assumptions, then of course this will be included.

It is easy to check the following Lemmas.

Lemma 2.1. T_n is a ring, $T_n \subset T$ and $T'_n \subset T$.

Lemma 2.2. If $f, g \in T'_n$, where $n > 0$, then $fg \in A_0 + XB[X]$.

Corollary 2.1. $(T'_n, \cdot)$ is not a semigroup.

Lemma 2.3. $I(B, A)$ is a ring.

Now, let's look at invertible and nilpotent elements.

Proposition 2.1. Let $f = a_0 + a_1X + \dots + a_nX^n \in T$ for any $n \geq 0$, then $f \in T^*$ if and only if $a_0 \in A^*$ and $a_1, a_2, \dots, a_n$ are nilpotents.

Proof. We know that if R is a ring then $f = a_0 + a_1X + \dots + a_nX^n \in R[X]^*$ if and only if $a_0 \in R^*$ and $a_1, a_2, \dots, a_n$ are nilpotents. In our Proposition we have $a_1, a_2, \dots, a_n$ are nilpotents. Of course we get $a_0 \in A^*$. $\square$

Proposition 2.2. Let $f = a_0 + a_1X + \dots + a_{n-1}X^{n-1} + a_nX^n + \dots + a_mX^m \in T_n$, $(T'_n, \text{ respectively})$, where $0 \leq n \leq m$ and $a_i \in A_i$ for $i = 0, 1, \dots, n$ and $a_j \in B$ for $j = n, n+1, \dots, m$.

(i) $f \in T_n^*$, $(T_n'^*, \text{ respectively})$ if and only if $a_0 \in A_0^*$ and $a_1, a_2, \dots, a_m$ are nilpotents.

(ii) f is a nilpotent if and only if $a_0, a_1, \dots, a_m$ are nilpotents.

Proof. Analogous proof like in Proposition 2.1. $\square$

Proposition 2.3. Let $f = a_0 + a_1X + \dots + a_nX^n \in I(B, A)$, where $A \subset B$ are domains.

(i) $f \in I(B, A)^*$ if and only if $a_0 \in A^*$ and $a_1, a_2, \dots, a_n$ are nilpotents.

(ii) f is nilpotent if and only if $a_0, a_1, \dots, a_n$ are nilpotents.

Proposition 2.4. *Let B be a domain and $f = a_{m_1}X^{m_1} + a_{m_2}X^{m_2} + \cdots + a_{m_n}X^{m_n} \in B[M]$, where $m_1, m_2, \dots, m_n \in M$ and $a_{m_1}, a_{m_2}, \dots, a_{m_n} \in B$.*

(i) *$f \in B[M]^*$ if and only if there exist $m_i \in M$ such that $a_{m_i} \in B^*$ and $m_i = 0$ and for every $m_k \neq m_i$ we have a_{m_k} be nilpotents.*

(ii) *f be a nilpotent if and only if $a_{m_1}, a_{m_2}, \dots, a_{m_n}$ are nilpotents.*

Proof. (i) Assume $f \in B[M]^*$, then there exists $g = b_{m'_1}X^{m'_1} + b_{m'_2}X^{m'_2} + \cdots + b_{m'_n}X^{m'_n}$, where $m'_1, m'_2, \dots, m'_n \in M$ and $b_{m'_1}, b_{m'_2}, \dots, b_{m'_n} \in B$ such that $fg = 1$. Hence there exist $m_i, m_j \in M$ such that $a_{m_i}b_{m_j}X^{m_i+m_j} = 1$. We have $a_i \in B^*$ and $m_i, m_j = 0$. The rest of coefficients are nilpotents. On the other side of the proof it is easy.

(ii) Obvious. □

Let's recall Theorem from [12] (Theorem 2.9) in a different form.

Proposition 2.5. *Let A be a subfield of B . Consider $D = A + XB[X]$, then $\text{Irr } D = \{aX, a \in B\} \cup \{a(1 + Xf(X)), a \in A, f \in B[X], 1 + Xf(X) \in \text{Irr } B[X]\}$.*

Theorem 2.1. *Consider $T = A + XB[X]$, where A be a subfield of B ; $T_n = A_0 + A_1X + A_2X^2 + \cdots + A_{n-1}X^{n-1} + X^nB[X]$, where $A_0 \subset A_1 \subset A_2 \subset \cdots \subset A_{n-1} \subset B$ be fields. Then*

- (i) *every nonzero prime ideal of T (T_n , respectively) is maximal;*
- (ii) *every prime ideal P different from $XB[X]$ (in T) is principal;*
- (iii) *every prime ideal P different from $A_1X + A_2X^2 + \cdots + A_{n-1}X^{n-1} + X^nB[X]$ (in T_n) is principal;*
- (iv) *T is atomic, i. e., every nonzero nonunit of T is a finite product of irreducible elements (atoms);*
- (v) *T_n is atomic.*

Proof. (i) For T we have in [12], Theorem 2.9 (i). We proof for T_n .

First note that $A_1X + A_2X^2 + \cdots + A_{n-1}X^{n-1} + X^nB[X]$ is maximal since $T_n/A_1X + A_2X^2 + \cdots + A_{n-1}X^{n-1} + X^nB[X] \cong A_0$. Let P be a nonzero prime ideal of T_n . Now, $X \in P$ implies $(T_n/A_1X + A_2X^2 + \cdots + A_{n-1}X^{n-1} + X^nB[X])^2 \subseteq P$ and hence $A_1X + A_2X^2 + \cdots + A_{n-1}X^{n-1} + X^nB[X] \subseteq P$ so $P = A_1X + A_2X^2 + \cdots + A_{n-1}X^{n-1} + X^nB[X]$. So suppose that $X \notin P$, then for $N = \{1, X, X^2, \dots\}$, P_N is a prime ideal in the PID $B[X, X^{-1}] = T_{n,N}$. (In fact, $B[X, X^{-1}] \subseteq R_P$ and R_P is a DVR (discrete valuation ring).) So P is minimal and is also maximal unless $P \subsetneq A_1X + A_2X^2 + \cdots + A_{n-1}X^{n-1} + X^nB[X]$. But let $k_nX^n + \cdots + k_sX^s \in P$ with $k_n \neq 0$, where $k_n, \dots, k_s \in B$ for any n, s . Then $X^{n+1} + k_n^{-1}k_{n+1}X^{n+2} + \cdots + k_n^{-1}k_sX^s \in P$, so $X \notin P$ implies that

$1 + k_n^{-1}k_{n+1}X + \cdots + k_n^{-1}k_sX^{s-n} \in P$, a contradiction. So every nonzero prime ideal is maximal.

(ii) [12], Theorem 2.9 (ii).

(iii) If P is different from $A_1X + A_2X^2 + \cdots + A_{n-1}X^{n-1} + X^nB[X]$, then it contains an element of the form $1 + a_1X + a_2X^2 + \cdots + a_{n-1}X^{n-1} + X^n f(X)$, where $a_i \in A_i$ for $i = 1, 2, \dots, n-1$ and $f(X) \in B[X]$. Now, if $1 + a_1X + a_2X^2 + \cdots + a_{n-1}X^{n-1} + X^n f(X)$ can be factored in $A_1X + A_2X^2 + \cdots + A_{n-1}X^{n-1} + X^nB[X]$ it can be written as $(1 + b_1X + b_2X^2 + \cdots + b_{n-1}X^{n-1} + X^n g(X))(1 + c_1X + c_2X^2 + \cdots + c_{n-1}X^{n-1} + X^n h(X))$, where $b_i, c_i \in A_i$ for $i = 1, 2, \dots, n-1$ and $g(X), h(X) \in B[X]$. Hence $1 + a_1X + a_2X^2 + \cdots + a_{n-1}X^{n-1} + X^n f(X)$ is irreducible in T_n if and only if it is irreducible in $A_1X + A_2X^2 + \cdots + A_{n-1}X^{n-1} + X^nB[X]$.

Now, let $1 + a_1X + a_2X^2 + \cdots + a_{n-1}X^{n-1} + X^n f(X)$ be irreducible in T_n and suppose that $1 + a_1X + a_2X^2 + \cdots + a_{n-1}X^{n-1} + X^n f(X) \mid k(X)l(X)$ in T_n . Then $1 + a_1X + a_2X^2 + \cdots + a_{n-1}X^{n-1} + X^n f(X) \mid k(X)l(X)$ in $A_1X + A_2X^2 + \cdots + A_{n-1}X^{n-1} + X^nB[X]$, and so in $A_1X + A_2X^2 + \cdots + A_{n-1}X^{n-1} + X^nB[X]$ we have, say $1 + a_1X + a_2X^2 + \cdots + a_{n-1}X^{n-1} + X^n f(X) \mid k(X)$. Then, in $A_1X + A_2X^2 + \cdots + A_{n-1}X^{n-1} + X^nB[X]$, $k(X) = (1 + a_1X + a_2X^2 + \cdots + a_{n-1}X^{n-1} + X^n f(X))d(X)$. Now, $d(X)$ can be written as $d(X) = aX^r(1 + a_1X + a_2X^2 + \cdots + a_{n-1}X^{n-1} + X^n p(X))$. If $r > 0$, $d(X) \in T_n$, while if $r = 0$, $k(X) = (1 + a_1X + a_2X^2 + \cdots + a_{n-1}X^{n-1} + X^n f(X))(b(1 + b_1X + b_2X^2 + \cdots + b_{n-1}X^{n-1} + X^n p(X)))$ and $b \in A_0$ because $k(X) \in T_n$. In either case, $d(X) \in T_n$ and so $1 + a_1X + a_2X^2 + \cdots + a_{n-1}X^{n-1} + X^n f(X) \mid k(X)$ in T_n . Consequently, in T_n every irreducible element of the type $1 + a_1X + a_2X^2 + \cdots + a_{n-1}X^{n-1} + X^n f(X)$ is prime.

Now, since every element of the form $1 + a_1X + a_2X^2 + \cdots + a_{n-1}X^{n-1} + X^n f(X)$ is a product of irreducible elements of the same form and hence is a product of prime elements, it follows that every prime ideal of P different from $A_1X + A_2X^2 + \cdots + A_{n-1}X^{n-1} + X^nB[X]$ contains a principal prime and hence is actually principal.

(iv) [12], Theorem 2.9 (iii).

(v) From (iii) a general element of T_n can be written as $aX^r(1 + a_1X + a_2X^2 + \cdots + a_{n-1}X^{n-1} + X^n f(X))$, where $a \in B$ (with $a \in A_0$ if $r = 0$) and $1 + a_1X + a_2X^2 + \cdots + a_{n-1}X^{n-1} + X^n f(X)$ is a product of primes. $\square$

Now, We give some basic information related to ideals.

Corollary 2.2. (i) If A be a field, then $XB[X]$ be an maximal ideal in T .

(ii) If A be an integral domain, then $XB[X]$ be an prime ideal in T .

(iii) $T/(X) \cong A$.

(iv) $T/B \cong \{0\}$.

(v) Let $A \subset B$ be fields in T . $T/(aX)$ be a field for any $a \in B$.

- (vi) Let $A \subset B$ be fields in T . $T/(a(1 + Xf(X)))$ be a field for any $a \in A, f \in B[X]$ such that $1 + Xf(X) \in \text{Irr } B[X]$.

Proof. (i) Let A be a field. The proof follows from $T/XB[X] \cong A$. We have $XB[X]$ is a maximal ideal in T .

(ii) – (iv) Obvious.

(v), (vi) From Theorem 2.9 in [12] aX for any $a \in B$ is an irreducible element. We get $T/(aX)$ be a field. We also have $a(1 + Xf(X))$ for any $a \in A, f \in B[X]$ such that $1 + Xf(X) \in \text{Irr } B[X]$ is a irreducible element. We have $T/(a(1 + Xf(X)))$ be a field. $\square$

Corollary 2.3. (i) If $A_0 + A_1X + \cdots + A_{n-1}X^{n-1}$ be a field (where $A_0 \subset A_1 \subset A_2 \subset \cdots \subset A_{n-1} \subset B$), then $X^nB[X]$ be an maximal ideal in T_n .

(ii) If $A_0 + A_1X + \cdots + A_{n-1}X^{n-1}$ be a domain, then $X^nB[X]$ be an prime ideal in T_n .

(iii) $T_n/(X) \cong A_0$.

(iv) $T_n/B \cong \{0\}$.

(v) Let $A_0 \subset A_1 \subset \cdots \subset B$ be fields in T_n . $T_n/(aX)$ be a field for any $a \in B$.

(vi) Let $A_0 \subset A_1 \subset \cdots \subset B$ be fields in T_n . $T_n/(a(1 + a_1X + a_2X^2 + \cdots + a_{n-1}X^{n-1} + X^n f(X)))$ be a field for any $a \in B, a_i \in A_i (i = 1, 2, \dots, n-1), f \in B[X]$ such that $1 + Xf(X) \in \text{Irr } B[X]$.

Proof. This proof is similarly to proof of Proposition 2.2. $\square$

The Proposition 2.3 holds for $T'_n = A_0 + A_1X + A_2X^2 + \cdots + A_{n-1}X^{n-1} + X^nB[X]$ where $A_0, A_1, A_2, \dots, A_{n-1} \subset B, A_i \not\subset A_j$ for $i \neq j$ be fields.

Lemma 2.4. If $A \subset B$ be fields and $B[X]$ be a GCD-domain then $I(B, A)$ be a GCD-domain.

Example 2.1. T, T_n are no GCD-domains. Let $f = a_1 + b_1X, g = a_2 + b_2X$, where $a_1, a_2 \in A, b_1, b_2 \in B$ with $A + XB[X]$, then $\text{gcd}(f, g) = \frac{a_1b_2 - a_2b_1}{b_2}$. We see that $\text{gcd}(f, g) \in B \setminus A$ and is not symmetrical.

Recall that a domain R is a pre-Schreier domain if every element $a \in R$ is a primal, i.e. for every elements $b, c \in H$ if $a \mid bc$ then there exist $a_1, a_2 \in R$ such that $a_1 \mid b, a_2 \mid c, a = a_1a_2$.

More information about Schreier and pre-Schreier domains we can see in many works, e.g. in [1], [5], [7], [14], [18], respectively.

Lemma 2.5. If $A \subset B$ be fields, then T be a pre-Schreier domain. If $A_0 \subset A_1 \subset \dots \subset A_{n-1} \subset B$ be fields, then T_n is also pre-Schreier domain. But T'_n doesn't satisfies a pre-Schreier conditions.

The following lemma states the uniqueness of two polynomials in the composites T_n and T'_n .

Lemma 2.6. *Let $f = a_0 + a_1X + a_2X^2 + \cdots + a_nX^n + \cdots + a_rX^r$, $g = b_0 + b_1X + b_2X^2 + \cdots + b_nX^n + \cdots + b_sX^s$, where $r \geq n \geq 0$, $s \geq n \geq 0$ and $a_i, b_i \in A_i$ for $i = 0, 1, 2, \dots, n$ and $a_j \in B$ for $j = n+1, n+2, \dots, r$ and $b_j \in B$ for $j = n+1, n+2, \dots, s$. If $f = g$ then $r = s$ and $a_i = b_i$ for $i = 0, 1, 2, \dots, r$.*

Proof. Let $f = a_0 + a_1X + a_2X^2 + \cdots + a_nX^n + \cdots + a_rX^r$, $g = b_0 + b_1X + b_2X^2 + \cdots + b_nX^n + \cdots + b_sX^s$, where $s \geq r \geq n \geq 0$ and $a_i, b_i \in A_i$ for $i = 0, 1, 2, \dots, n$ and $a_j \in B$ for $j = n+1, n+2, \dots, r$ and $b_j \in B$ for $j = n+1, n+2, \dots, s$. Consider $a_0 + a_1X + a_2X^2 + \cdots + a_nX^n + \cdots + a_rX^r = b_0 + b_1X + b_2X^2 + \cdots + b_nX^n + \cdots + b_sX^s$, then $(b_0 - a_0) + (b_1 - a_1)X + (b_2 - a_2)X^2 + \cdots + (b_n - a_n)X^n + \cdots + (b_r - a_r)X^r + b_{r-1} + \cdots + b_sX^s = 0$. Hence $r = s$ and $a_i = b_i$ for $i = 0, 1, 2, \dots, r$. $\square$

Theorem 2.2. *Consider $T = A + XB[X]$, where A be a subfield of B ; $T_n = A_0 + A_1X + A_2X^2 + \cdots + A_{n-1}X^{n-1} + X^nB[X]$, where $A_0 \subset A_1 \subset A_2 \subset \cdots \subset A_{n-1} \subset B$ be fields; $I(B, A)$, where $A \subset B$ be domains. Then*

- (i) $f \in \text{Irr } T$ if and only if $f \in \text{Irr } B[X]$, $f(0) \in A$.
- (ii) $f \in \text{Irr } T_n$ if and only if $f \in \text{Irr } B[X]$, $a_i \in A_i$, where $f = a_0 + a_1X + \cdots + a_{n-1}X^{n-1} + a_nX^n + \cdots + a_mX^m$ with $a_i \in A_i$ for $i = 0, 1, \dots, n-1$ and $a_n, a_{n+1}, \dots, a_m \in B(n < m)$.
- (iii) $f \in \text{Irr } I(B, A)$ if and only if $f \in \text{Irr } B[X]$ and $f(A) \subseteq A$.

Proof. (i) Suppose that $f \notin \text{Irr } B[X]$ or $f(0) \notin A$. If $f(0) \notin A$, then $f \notin T$, so $f \notin \text{Irr } B[X]$. Now, assume that $f \notin \text{Irr } B[X]$, then $f = gh$, where $g, h \in B[x] \setminus B$. Let $g = a_0 + a_1X + \cdots + a^nX^n$, $h = b_0 + b_1X + \cdots + b_mX^m$. We have $f = (a_0 + a_1X + \cdots + a^nX^n)(b_0 + b_1X + \cdots + b_mX^m)$, then $f = (1 + \frac{a_1}{a_0}X + \cdots + \frac{a_n}{a_0}X^n)(a_0b_0 + a_0b_1X + \cdots + a_0b_mX^m)$, where $a_0b_0 = f(0) \in A$. Now, suppose that $f \notin \text{Irr } T$. If $f \notin T$, then $f(0) \notin A$. Now, assume that $f \in T$, then we have $f = gh$, where $g, h \in T \setminus A$. This implies $g, h \in B[x] \setminus B$.

(ii) Occur in the same way as in (i).

(iii) From definition $I(B, A)$ we have $f \in \text{Irr } B[X]$ and $f(A) \subseteq A$. $\square$

In [18], Proposition 8.6 with Jędrzejewicz, Marciniak and Zieliński we received the characterization of square-free elements in composite $A + XB[X]$, for $A \subset B$ fields, i.e.

Proposition 2.6. *Let L and F be fields such that $L \subset F$ and let $T = L + XF[X]$, then $\text{Sqf } T = (\text{Sqf } F[X] \cap T) \cup \{X^2h; h \in \text{Sqf } F[X], h(0) \notin \{a^2b; a \in F, b \in L\}\}$*

Corollary 2.4. *Let $A_0 \subset A_1 \subset \cdots \subset A_{n-1} \subset B$ be fields, then $\text{Sqf } T_n = (\text{Sqf } B[X] \cap T_n) \cup \{X^2h; h \in \text{Sqf } B[X], h(0) \notin \{a^2b; a \in B, b \in A\}\}$.*

In [15], Lemma 6.4 we have informations about irreducible element in monoid domain $D[S]$, where D be a domain, and S be a submonoid of $\mathbb{Q}_+$. We present a generalized Proposition.

Proposition 2.7. *Let B be an integral domain with quotient field K and M a monoid with quotient group $G \neq M$. Assume that B contains prime elements $p_1, p_2, \dots, p_{r-1}$. Assume that M is integrally closed and each nonzero element of G is type $(0, 0, \dots)$ (G satisfies the ascending chain condition on cyclic subgroups). Consider $m_1, m_2, \dots, m_r \in M$ such that $m_1 \in \text{Irr } M$ and $m_2, m_3, \dots, m_r \notin m_1 + M$, then $p_{r-1}X^{m_r} - \dots - p_2X^{m_3} - p_1X^{m_2} - X^{m_1} \in \text{Irr } B[M]$.*

Proof. Let $\leq$ be a total order on G . We may assume that $m_r < m_{r-1} < \dots < m_2 < m_1$. Suppose that $p_{r-1}X^{m_r} - \dots - p_2X^{m_3} - p_1X^{m_2} - X^{m_1} = fg$ with $f, g \in B[M]$. Write $f = a_1X^{t_1} + \dots + a_mX^{t_m}$ and $g = b_1X^{k_1} + \dots + b_nX^{k_n}$ in canonical form, where $t_1 < \dots < t_m$ and $k_1 < \dots < k_n$. First assume that either f or g is a monomial, say $f = aX^t$. Then $a \in B^*$, $m_1 = t + k_n$, $m_2 = t + k_1$, $m_3 = t + k_2, \dots, m_r = t + k_{r-1}$. Since $m_1 \in \text{Irr } M$, either t or k_n is invertible in M . If k_n is invertible, then $m_2 = t + k_1 = (m_1 - k_n) + k_1 \in m_1 + M$, $m_3 = t + k_2 = (m_1 - k_n) + k_2 \in m_1 + M, \dots, m_r \in m_1 + M$, a contradiction. Thus t is invertible in M , and hence f is a unit in $B[M]$. Thus we may assume that f and g are not monomials. Now, consider the reduction of $p_{r-1}X^{m_r} - \dots - p_2X^{m_3} - p_1X^{m_2} - X^{m_1} = fg$ modulo the ideal $(p_1, p_2, \dots, p_{r-1})$. Then $(-1 + (p_1, p_2, \dots, p_{r-1})) = ((a_m + (p_1, p_2, \dots, p_{r-1}))X^{t_m})((b_n + (p_1, p_2, \dots, p_{r-1}))X^{k_n})$. This means that $a_1 + (p_1, p_2, \dots, p_{r-1}) = b_1 + (p_1, p_2, \dots, p_{r-1}) = (p_1, p_2, \dots, p_{r-1})$. In this case $c_1p_1 + \dots + c_{r-1}p_{r-1} - 1 = a_1b_1 \in (p_1, \dots, p_{r-1})^2$, a contradiction. Thus $p_{r-1}X^{m_r} - \dots - p_2X^{m_3} - p_1X^{m_2} - X^{m_1} \in \text{Irr } B[M]$. $\square$

Proposition 2.8. *$B[M]/(p_{r-1}X^{m_r} - \dots - p_1X^{m_2} - X^{m_1})$ be a field, where B be a domain, $p_1, p_2, \dots, p_{m_r} \in B$, $m_1, m_2, \dots, m_r \in M$ with $m_1 \in \text{Irr } M$, $m_2, m_3, \dots, m_r \notin m_1 + M$.*

Proof. It follows from Proposition 2.7. $\square$

The next statements give a description of some multiplicative systems. Proofs of these statements are easy.

Proposition 2.9. *If A be an integral domain then $S = A + XB[X] \setminus XB[X]$ is a saturated multiplicative system.*

Proposition 2.10. *If $A_0 + A_1X + \dots + A_{n-1}X^{n-1}$ be an integral domain, then $S = T_n \setminus X^nB[X]$ (with appropriate assumptions) is saturated multiplicative system.*

3. Modules

In this chapter we will discuss the basic information related to modules.

Proposition 3.1. *Let $R \in \{T, B[X], I(B, A)\}$, then $M \in \{T, T_n, T'_n, I(B, A)\}$ be R -modules.*

Proof. Easy to check. $\square$

Question: What are the smallest assumptions for a sequence below to be an exact sequence in $A + XB[X]$?

$$\cdots \rightarrow A_0 + A_1X + \cdots + A_{n-1}X^{n-1} + X^nB[X] \rightarrow A + XB[X] \rightarrow B[X] \rightarrow \cdots$$

Consider functions $f: A_0 + A_1X + \cdots + A_{n-1}X^{n-1} + X^nB[X] \rightarrow A + XB[X]$ and $g: A + XB[X] \rightarrow B[X]$, where $A_0, A_1, \dots, A_{n-1}, A, B$ be a domains such that $A_0 \not\subset A_i$ for some $i \in \{1, 2, \dots, n-1\}$ and $A, A_0A_1, A_2, \dots, A_{n-1} \subset B$. Consider $y \in \ker g$, then $g(y) = 0$. This implies $y = 0 = f(0)$ and then $y \in \Im f$.

We do not know, whether we can prove $\Im f \subset \ker g$ with the same assumptions.

Lemma 3.1. (i) *If C be a subgroup of A , then $C + XB[X]$ be a submodule of $B[X]$ -module $A + XB[X]$.*

(ii) *If C_i be a subgroup of A_i for $i = 0, 1, \dots, n-1$, then $C_0 + C_1X + \cdots + C_{n-1}X^{n-1} + X^nB[X]$ be a submodule of $B[X]$ -module $A_0 + A_1X + \cdots + A_{n-1}X^{n-1} + X^nB[X]$.*

(iii) *If C be a subdomain of A and D be a subdomain of B , then $I(D, C)$ be a submodule of $B[X]$ -module $I(B, A)$.*

(iv) *If N be a submonoid of M , then $B[N]$ be a submodule of B -module $B[M]$, where B be a field.*

Proof. Easy to proof. $\square$

Proposition 3.2. (i) *If A be a simple R -module, then $XB[X]$ and T be unique submodules of T .*

(ii) *For some $i \in \{0, 1, 2, \dots, n-1\}$ consider A_i which be a simple R -module. Then $A_0 + \cdots + A_{i-1}X^{i-1} + C_iX^i + A_{i+1}X^{i+1} + \cdots + A_{n-1}X^{n-1} + X^nB[X]$ be unique submodules of T_n or T'_n , where $C_i \in \{0, A_i\}$.*

(iii) *Consider $A_0, A_1, \dots, A_{n-1} \subset B$, where for finitely $i \in \{0, 1, \dots, n-1\}$ assume A_i be simple R -modules. Then $C_0 + C_1X + \cdots + C_{n-1}X^{n-1} + X^nB[X]$ be unique submodule of T_n or T'_n , where for simple R -modules $A_{i_1}, A_{i_2}, \dots, A_{i_k}$ ($i_1, i_2, \dots, i_k \in \{0, 1, \dots, n-1\}$) we have $C_{i_1} \in \{0, A_{i_1}\}$, $C_{i_2} \in \{0, A_{i_2}\}, \dots, C_{i_k} \in \{0, A_{i_k}\}$.*

(iv) Consider a simple R -module M and a field B , then $0, B[M]$ be unique submodules of R -module $B[M]$.

Proof. (i) If A be a simple module then 0 and A be a unique submodules of A . Hence we get $C + XB[X]$, where $C \in \{0, A\}$. Proof of (ii), (iii), (iv) hold similarly to (i). $\square$

4. Localizations

This chapter will tell about building fractions. The following statements are very well known, but have been included in the structures we are considering.

Theorem 4.1. Let S be a multiplicative subset of T_n . There exists a ring $T_{n,S}$ and a homomorphism $w: T_n \rightarrow T_{n,S}$ which satisfy the following conditions:

- (i) for all $s \in S$, the elements $w(s)$ are invertible in $T_{n,S}$,
- (ii) given a homomorphism of composites $f: T_n \rightarrow T_m$ such that, for every $s \in S$, the elements $f(s)$ are invertible in T_m , there exists a unique homomorphism $F: T_{n,S} \rightarrow T_m$ such that $Fw = f$.

The ring $T_{n,S}$ is determined by the above conditions up to isomorphism.

Proof. Let us consider the set $T_n \times S$ of all pairs (g, s) , where $g \in T_n, s \in S$, and define the operations $+$, $\cdot$ in it by formulae

$$(g, s) + (g_1, s_1) = (gs_1 + g_1s, ss_1), (g, s) \cdot (g_1, s_1) = (gg_1, ss_1).$$

The pairs $(g, s), (g_1, s_1) \in T_n \times S$ are called equivalent ($=$) if and only if there exists $s_2 \in S$ such that $s_2(gs_1 - g_1s) = 0$. The relation thus defined is a relation of equivalence. The equivalence class of a pair $(g, s) \in T_n \times S$ with respect to the relation $=$ is denoted by gs^{-1} or g/s and called fraction. The set of all fractions g/s with $g \in T_n, s \in S$ forms a ring denoted $T_{n,S}$. Fractions $g/s, g'/s' \in T_{n,S}$ are equal if and only if there exists an element $s'' \in S$ such that $s''(rs' - r's) = 0$. We define the ring homomorphism $w: T_n \rightarrow T_{n,S}$ by the formula $w(g) = g/1$ for $g \in T_n$. The ring $T_{n,S}$ and the homomorphism w satisfy conditions (i) and (ii). $\square$

Corollary 4.1. $T_{n,S} = A_{0,S} + A_{1,S}X + \cdots + A_{n-1,S}X^{n-1} + X^n B_S[X]$.

Similarly, we can construct a fraction module.

Lemma 4.1. Let S be a multiplicative subset of $I(B, A)$, then $I(B, A)_S = I(B_S, A)$ with B be a field.

Proposition 4.1. Let $M \subset \mathbb{Q}_+$ be a subgroup and let B be a ring, then $B[M]_0 = B_0[M]$.

Proof. Consider $f \in B[M], g \in B[M] \setminus \{0\}$. When we divided f by g we get a sum of elements form $\frac{a}{b}X^{m-n}$, where $a \in B, b \in B \setminus \{0\}, m, n \in M$. We see that $m - n \in M$ and $\frac{a}{b} \in B_0$. $\square$

5. Graded rings and modules

This chapter shows the considered structures as graded rings and modules. Later, we can use it to study different properties.

Proposition 5.1. (i) T be a graded ring.

(ii) T_n be a graded ring.

(iii) $I(B, A)$ be a graded ring.

(iv) If M be a countable monoid and B be a domain, then $B[M]$ be a graded ring.

Proof. Obvious. □

Example 5.1. T'_n is not a graded ring. Consider $T'_3 = A_0 + A_1X + A_2X^2 + X^3B[X]$, then $R_1R_3 = A_0 + A_0A_1X + (A_0A_2 + A_1)X^2 + (B + A_1A_2)X^3 + BX^4 \in A_0 + XB[X] \neq R_4$.

Proposition 5.2. (i) T be a graded T -module.

(ii) T_n be a graded T_n -module.

(iii) $I(B, A)$ be a graded R -module, where $R \in \{I(B, A), T\}$.

(iv) $B[M]$ be a graded R -module, where $R \in \{I(B, A), B[M]\}$ where B be a field and M be a monoid.

6. Composites with monoid and group coefficients. Monoid rings

The main motivation of this paper is description of composites and monoid domains in the language of commutative algebra. Previous chapters were described but these chapters have been described for the rings (for composites) and for domains (for monoid domains). Then, we decided to try described for monoids/groups (for composites) and for rings (for monoid rings).

Lemma 6.1. T_n is a monoid, if components $(A_0, \dots, A_{n-1}, B)$ are monoids, $T_n \subset T$ and $T'_n \subset T$.

Of course Lemma 2.2 and Corollary 2.1 hold, but in monoid version. The same holds true for invertible and nilpotent elements in polynomial composites and monoid rings (Proposition 2.1 and Proposition 2.2, proofs are similar). Proposition 2.4 can also be used for monoid rings.

Unfortunately, it is not known whether the Theorem 2.1 can occur in the monoid or group version.

Similar claims are in the case of irreducible elements (Theorem 2.2, Proposition 2.7). We often come across the concept of atom more in groups or in monoids.

7. Some examples

Recall that a ring R satisfies ACCP if each chain of principal ideals of R is stabilize.

Example 7.1. In [12] Example 5.1 showed an example of an integral domain R which satisfies ACCP, but whose integral closure does not satisfy ACCP. It mean $R = \mathbb{Z} + X\overline{\mathbb{Z}}[X]$, where $\overline{\mathbb{Z}}$ be the ring of all algebraic integers. R satisfies ACCP. For if not, then there is an infinite properly ascending chain of principal ideals of R . Since the degrees of the polynomials generating these principal ideals are nonincreasing, the degrees eventually stabilize. The principal ideals in $\overline{\mathbb{Z}}$ generated by the leading coefficients of these polynomials gives an infinite ascending chain $a_1\overline{\mathbb{Z}} \subsetneq a_2\overline{\mathbb{Z}} \subsetneq \dots$ where each $a_n/a_{n+1} \in \mathbb{Z}$. Thus all $a_n \in \mathbb{Q}[a_1]$. Let $A = \overline{\mathbb{Z}} \cap \mathbb{Q}[a_1]$, then $a_1A \subsetneq a_2A \subsetneq \dots \subsetneq A$, a contradiction since A is Dedekind.

Example 7.2. Recall that a domain R is called a half-factorial domain (HFD) if R is atomic and for each nonzero nonunit $x \in R$, $x = x_1 \dots x_m = y_1 \dots y_n$ where x_i, y_j are all irreducible for $i = 1, \dots, m, j = 1, \dots, n$, implies that $m = n$. A HFD domain satisfies ACCP.

Example 7.3. If $K_1 \subsetneq K_2$ is algebraic extension of fields, then $R = K_1 + XK_2[X]$ has integral closure $K_2[X]$, a Euclidean domain. If $[K_2:K_1] < \infty$, then R is a Noetherian HFD that is not integrally closed. If K_1 is algebraically closed in K_2 , then R is an integrally close non-Noetherian HFD. Of course, R satisfies ACCP.

Example 7.4. Let $R = \mathbb{R} + X\mathbb{C}[X]$. So R is a HFD, so has ACCP, then atomic.

Example 7.5. ([2]) Let F be a field and T the additive submonoid of $\mathbb{Q}^+$ generated by $\{1/3, 1/(2 \cdot 5), \dots, 1/(2^j p_j), \dots\}$, where $p_0 = 3, p_1 = 5, \dots$ is the sequence of odd primes. Let R be the monoid domain $F[X; T] = F[T]$ and $N = \{f \in R \mid f \text{ has nonzero constant term}\}$. Then $F[T]_N$ is an atomic domain which does not satisfy ACCP.

Example 7.6. Let K be a field and T the additive submonoid of $\mathbb{Q}^+$ generated by $\{1/2, 1/3, 1/5, \dots, 1/p_j, \dots\}$, where p_j is the j th prime. Then the monoid domain $R = K[T]$ satisfies ACCP.

For a $0 \neq f = b_1X^{a_1} + \dots b_nX^{a_n} \in R$ with $a_1 < \dots < a_n$ and $b_n \neq 0$, write $\beta(f) = a_n$. If ACCP fails, then there is a strictly increasing chain $(f_1) \subset (f_2) \subset \dots$ of principal ideals in R . Then each $f_n = f_{n+1}g_{n+1}$ for some nonunit $g_{n+1} \in R$. Hence each $\beta(f_n) = \beta(f_{n+1}) + \beta(g_{n+1})$, and each term is positive. Then in T , we have $\beta(f_1) > \beta(f_2) > \dots$ with each $\beta(f_n) - \beta(f_{n+1}) \in T$, but this is impossible by the above-mentioned unique representation of each nonzero $a \in T$.

Example 7.7. ([10]) Let K be a field, $T = \{q \in \mathbb{Q} \mid q \geq 1\} \cup \{0\}$ an additive submonoid of $\mathbb{Q}^+$, and $R = K[T]$ the monoid domain. Then $R_S = K[\mathbb{Q}]$, where

$S = \{X^t \mid t \in T\}$, is not atomic since R_S is a GCD-domain, but R_S does not satisfy ACCP.

8. Applications of polynomial composites in cryptology

Each such polynomial is the sum of the products of the variable and the coefficient. And what if subsequent coefficient sets are appropriate cryptographic systems? Instead of encrypting with one system, we can create one system composed of many systems. Such a cipher is very difficult to break. If the spy detects encryption systems (composite coefficients), then the problem will be to find the right sum and product of such systems.

Assume that we have two people: Alice und Bob. Alice wants to send a message to Bob. Alice has one composite type T'_n and Bob has another one composite type T'_n .

We can build such composite by various encryption systems (even known ones). Let see note Lemma:

Lemma 8.1. *Let $f = a_0 + a_1X + \dots + a_{n-1}X^{n-1} + \sum_{j=n}^m a_jX^j$, $g = b_0 + b_1X + \dots + b_{n-1}X^{n-1} + \sum_{j=n}^m b_jX^j \in T'_n$, where $a_i, b_i \in A_i$ for $i = 1, 2, \dots, n-1$ and $a_j, b_j \in B$ for $j = n, n+1, \dots, m$. Then $fg \in A_0 + XB[X]$.*

Put A_i, B_j ($i, j = 0, 1, \dots, n-1$) be different encryption systems, then we have f and g are composition of encryption systems. No consider B . To improve security, let's fix that $\deg f = n-1$, $\deg g = n-k$, where $k \in \{2, \dots, n-1\}$. And such f, g Alice and Bob agree before the message is sent.

Alice and Bob multiply these composites to form one. We have $fg = (A_0 + A_1X + \dots + A_kX^k)(B_0 + B_1X + \dots + B_lX^l) = A_0B_0 + (A_0B_1 + A_1B_0)X + \dots + A_kB_lX^{k+l}$.

Note that the sum and product of the encryption systems must be defined in the formula above. Definitions we leave Alice and Bob. But in this section we can put $S_iS_j : x \rightarrow (x)_{S_i}(x)_{S_j}$ and $S_i + S_j : x \rightarrow ((x)_{S_i})_{S_j}$.

So in the product we encrypt the letter as two letters, the first in the first system and the second in the second system. And in the sum we encrypt the letter using the first system and then the second system. Of course, we can define completely different, at our discretion.

Assume that degree of fg is m and text to encrypt consists of more letters then $m+1$. Then we divide the text into blocks of length $m+1$. We can assume that $fg(0)$ encrypts the first letter of each block. Expression at X of fg encrypts the second letter of each block, and expression at X^2 of fg encrypts the third letter and so on.

Now, let's see how to decrypt in this idea.

Assume that we have an encrypted message $M_0M_1 \dots M_n$. If our key is degree m , then we divide message on $m+1$ partition. And every partition divide to two. Every two letters are one letter of message.

Earlier we define $S_i S_j : x \rightarrow (x)_{S_i}(x)_{S_j}$ and $S_i + S_j : x \rightarrow ((x)_{S_i})_{S_j}$. Then decryption of two letters $M_l M_{l+1}$ ($l = 0, 2, 4, \dots$) are $M_l M_{l+1} = (M_l)_{S_i}(M_{l+1})_{S_j} = N_{l,l+1}$ (one letter) and $M_l = ((M_l)_{S_i})_{S_j} = (N_l)_{ij}$ (one letter).

Example 8.1. Alice and Bob agree different encryption systems in the center: A_0, A_1, A_2, B_0, B_1 . Next, Alice has gone far from Bob.

We have two compositions: $f = A_0 + A_1 X + A_2 X^2$, $g = B_0 + B_1 X$. Their key is one composition in the form fg i.e. $A_0 B_0 + (A_0 B_1 + A_1 B_0)X + (A_2 B_0 + A_1 B_1)X^2 + A_2 B_1 X^3$.

The established systems are as follows:

A_0 is a Caesar cipher, where the letter is shifted one letter forward;

A_1 is a Caesar cipher, where the letter is shifted two letter forward;

A_2 is a Caesar cipher, where the letter is shifted three letter forward;

B_0 is a Caesar cipher, where the letter is shifted one letter back;

B_1 is a Caesar cipher, where the letter is shifted two letter back.

Suppose Alice wants to send a message saying

0 2 4 6 8 9 6 5

The degree of fg is 3. Hence message divide to $3+1$ partition. So the fourth letter is the same encrypted.

Letters 0 and 8 encrypt by $A_0 B_0$, then, from definition of A_0, B_0 , 0 will be 1 9 (two letters). 8 will be 9 7.

Letters 2 and 9 encrypt by $A_0 B_1 + A_1 B_0$, then 2 will be 5 9 and 9 will be 2 6.

Letters 4 and 6 encrypt by $A_2 B_0 + A_1 B_1$, then 4 will be 9 1 and 6 will be 1 3.

Letters 6 and 5 encrypt by $A_2 B_1$, then 6 will be 9 4 and 5 will be 8 3.

Bob receives a message from Alice:

1 9 5 9 9 1 9 4 9 7 2 6 1 3 8 3.

Now, Bob would like to read the message. Bob sees that message has 16 letters, so the original text has 8 letters, because the composition fg has degree 3 (i.e. $(3+1)2$ letters of original message). Divide message by 8 letters.

We take the first pairs from each section, i.e. 1 9 and 9 7. Bob uses decryption $(A_0 B_0)^{-1}$. So, 1 will be 0 by A_0^{-1} and 9 will be 0 by B_0^{-1} . Hence 1 9 will be 0. Similarly, 9 7 will be 8.

Next, we take the second pairs from each section, i.e. 5 9 and 2 6. Bob uses decryption $(A_0 B_1 + A_1 B_0)^{-1}$. So, 5 9 will be 2 and 2 6 will be 9.

We take next pair, i.e. 9 1 and 1 3. Bob uses decryption $(A_2 B_0 + A_1 B_1)^{-1}$. So, 9 1 will be 4 and 1 3 will be 6.

Similarly, the last pairs decrypt by $(A_2 B_1)^{-1}$. The pair 9 4 will be 6 and 8 3 will be 5.

After decrypting, Bob received the message:

0 2 4 6 8 9 6 5.

9. The concept of using monoid domains in cryptology

Recall that if F be a field and M be a submonoid of $\mathbb{Q}_+$ then we can construct a monoid domain:

$$F[M] = F[X; M] = \{a_0X^{m_0} + \dots + a_nX^{m_n} \mid a_i \in F, m_i \in M\}.$$

Any alphabet of characters creates a finite set. Most ciphers are based on finite sets. But we can have the idea of using the infinite alphabet $\mathbb{A}$, although in reality they can be cyclical sets with an index that would mean a given cycle. For example, $A_0 - 0, B_0 - 1, \dots, Z_0 - 25, A_1 - 0, B_1 - 1, \dots$, where $A_i = A, \dots, Z_i = Z$ for $i = 0, 1, \dots$. We see that this is isomorphic to a monoid $\mathbb{N}_0$ non-negative integers by a formula

$$f: \mathbb{A} \rightarrow \mathbb{N}, f(m_i) = i.$$

Then, we can use a monoid domain by a map

$$\varphi: \mathbb{A} \rightarrow F[\mathbb{A}], \varphi(m_0, m_1, \dots, m_n) = a_0X^{m_0} + \dots a_nX^{m_n}.$$

Here, one should think carefully about what a field F should be and think about additional mappings. In contrast, monoid domains can be excellent carriers of characters in the alphabet for monoids. This will make it harder to break any ciphers based on monoids for one simple reason, namely, we don't have inverse properties in a monoid.

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Some new Hermite-Hadamard Fejér type inequalities for functions whose second-order mixed derivatives are coordinated preinvex

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Abstract. In this paper, we present some new Hermite-Hadamard-Fejér type inequalities for those functions whose second order derivative is coordinated preinvex. We have given some new error bounds for the weighted trapezoidal and weighted mid-point rules for coordinated preinvex functions. The results presented here are extensions of earlier works.

Keywords: Hermite-Hadamard, Fejér, coordinated preinvexity, Hölder inequality, power mean inequality.

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1. Introduction

In recent years, a great deal of attention has been laid by many researchers to the theory of convexity because of its great importance in various fields of pure and applied sciences. The theories of convex function and inequalities are closely related to each other. A very interesting and well known inequality that establishes this connection is Hermite-Hadamard inequality, given as:

Let $g : H \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex function and $a, b \in H$ with $a < b$, then

$$(1) \quad g\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b g(v)dv \leq \frac{g(a)+g(b)}{2}.$$

Weighted generalization of the inequality (1) was presented by Fejér [8] as follows:

$$(2) \quad g\left(\frac{a+b}{2}\right) \int_a^b w(v)dv \leq \int_a^b w(v)g(v)dv \leq \frac{g(a)+g(b)}{2} \int_a^b w(v)dv,$$

where $w : [a, b] \rightarrow \mathbb{R}$ is nonnegative, integrable and symmetric about $v = \frac{a+b}{2}$.

The concept of invex sets was given by T. Antczak [2].

Definition 1. A set $H \subseteq \mathbb{R}^n$ is invex with respect to the map $\zeta : H \times H \rightarrow \mathbb{R}^n$ if for every $a, b \in H$ and $t \in [0, 1]$, $b + t\zeta(a, b) \in H$. The invex set H is also called an ζ -connected set.

Remark 1. Every convex set is an invex set but its converse is not true.

In 1998, Weir and Mond [18], defined preinvex functions as generalization of convex functions.

Definition 2. Let $H \subseteq \mathbb{R}^n$ be an invex set and a function $g : H \rightarrow \mathbb{R}$ is said to be preinvex w.r.t. ζ if $\forall a, b \in H$,

$$g(a + t\zeta(a, b)) \leq tg(b) + (1-t)g(a).$$

If $\zeta(a, b) = a - b$, then in the classical sense, the preinvex function becomes a convex functions. A function g is called preincave iff its negative is preinvex.

Definition 3 ([13]). Let $K_1 \times K_2$ be invex set with respect to $\eta_1 : K_1 \times K_1 \rightarrow \mathbb{R}^n$ and $\eta_2 : K_2 \times K_2 \rightarrow \mathbb{R}^n$. A function $f : K_1 \times K_2 \rightarrow \mathbb{R}$ is said to be preinvex in coordinates if for every $(x, y), (u, v) \in K_1 \times K_2$ and $t \in [0, 1]$, we have

$$f(u + t\eta_1(x, u), v + t\eta_2(y, v)) \leq (1-t)f(u, v) + tf(x, y).$$

Definition 4 ([11]). Let $K_1 \times K_2$ be invex set with respect to $\eta_1 : K_1 \times K_1 \rightarrow \mathbb{R}^n$ and $\eta_2 : K_2 \times K_2 \rightarrow \mathbb{R}^n$. A function $f : K_1 \times K_2 \rightarrow \mathbb{R}$ is said to be preinvex in coordinates if for every $(x, y), (x, u), (u, y), (u, v) \in K_1 \times K_2$ and $\lambda, t \in [0, 1]$, we have

$$f(u + \lambda\eta_1(x, u), v + t\eta_2(y, v)) \leq (1-\lambda)(1-t)f(u, v) + (1-\lambda)tf(u, y) \\ + (1-t)\lambda f(x, v) + \lambda tf(x, y).$$

S. S. Dragomir [6] established the following result.

Theorem 1. *suppose that $f : \Delta = [a, b] \times [c, d] \rightarrow \mathbb{R}$ is a coordinated convex function on Δ . Then one has the following inequalities:*

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\leq \frac{1}{2} \left[\frac{1}{b-a} \int_a^b f\left(x, \frac{c+d}{2}\right) dx \right. \\ &\quad \left. + \frac{1}{d-c} \int_c^d f\left(\frac{a+b}{2}, y\right) dy \right] \\ &\leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy dx \\ &\leq \frac{1}{4} \left[\frac{1}{b-a} \left(\int_a^b f(x, c) dx + \int_a^b f(x, d) dx \right) \right. \\ &\quad \left. + \frac{1}{d-c} \left(\int_c^d f(a, y) dy + \int_c^d f(b, y) dy \right) \right] \\ &\leq \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4}. \end{aligned}$$

M. A. Latif and S. S. Dragomir proposed the following Hermite-Hadamard equalities for preinvex functions on coordinates.

Theorem 2 ([11]). *Let $K_1 \times K_2 \rightarrow \mathbb{R}$ be an open convex subset of $\mathbb{R}^2$ with respect to the mappings $\eta_1 : K_1 \times K_1 \rightarrow \mathbb{R}$ and $\eta_2 : K_2 \times K_2 \rightarrow \mathbb{R}$. Suppose $f : K_1 \times K_2 \rightarrow \mathbb{R}$ be twice partial differentiable mapping such that*

$$\frac{\partial^2 f}{\partial \lambda \partial t} \in L_1([a, a + \eta_1(b, a)] \times [c, c + \eta_2(d, c)]),$$

where $\eta_1(b, a) \neq 0, \eta_2(d, c) \neq 0$, where $a, b \in K_1$ and $c, d \in K_2$. Then the following equality holds:

$$\begin{aligned} &\frac{1}{4} [f(a, c) + f(a, c + \eta_2(d, c)) + f(a + \eta_1(b, a), c) + f(a + \eta_1(b, a), c + \eta_2(d, c))] \\ &\quad + \frac{1}{\eta_1(b, a)\eta_2(d, c)} \int_a^{a+\eta_1(b, a)} \int_c^{c+\eta_2(d, c)} f(x, y) dx dy - A \\ &= \frac{\eta_1(b, a)\eta_2(d, c)}{4} \int_0^1 \int_0^1 (1-2\lambda)(1-2t) \frac{\partial^2 f}{\partial t \partial \lambda} (a + \lambda\eta_1(b, a), c + t\eta_2(d, c)) d\lambda dt, \end{aligned}$$

where

$$\begin{aligned} A &= \frac{1}{2\eta_1(b, a)} \int_a^{a+\eta_1(b, a)} [f(x, c) + f(x, c + \eta_2(d, c))] dx \\ &\quad + \frac{1}{2\eta_2(d, c)} \int_c^{c+\eta_2(d, c)} [f(a, y) + f(a + \eta_1(b, a), y)] dy. \end{aligned}$$

Fejér-type inequality for coordinated convex mappings is established in [1] as follows.

Theorem 3. *suppose that $f : \Delta = [a, b] \times [c, d] \rightarrow \mathbb{R}$ is a coordinated convex function on Δ . Then one has the following inequalities:*

$$\begin{aligned} & f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \int_a^b \int_c^d p(x, y) dy dx \\ & \leq \int_a^b \int_c^d f(x, y) p(x, y) dy dx \\ & \leq \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} \\ & \quad \times \int_a^b \int_c^d p(x, y) dy dx, \end{aligned}$$

where $p : [a, b] \times [c, d] \rightarrow \mathbb{R}$ is positive, integrable and symmetric about $x = \frac{a+b}{2}$ and $y = \frac{c+d}{2}$

In this paper, we present two new Hermite-Hadamard-Fejér identities for functions second-order mixed derivatives are coordinated preinvex. Using the new identities, we obtain some new weighted estimates connected with the left and right hand side of the Hermite-Hadamard-Fejér type inequalities for coordinated preinvex functions.

2. Main results

Throughout this section, we will let $\sup_{(s,t) \in (0,1) \times (0,1)} |w(a + s\eta(b, a), c + t\xi(d, c))| = \|w\|_\infty$.

Lemma 1. *Let $K_1 \times K_2 \subseteq \mathbb{R}^2$ be an open invex subset with respect to $\eta : K_1 \times K_1 \rightarrow \mathbb{R}^2$ and $\xi : K_2 \times K_2 \rightarrow \mathbb{R}^2$. Suppose $f : K_1 \times K_2 \rightarrow \mathbb{R}^2$ is a twice differentiable mapping on $K_1 \times K_2$ such that $f_{st} \in L([a, a + \eta(b, a)] \times ([c, c + \xi(d, c)]))$ where $\eta(b, a) > 0$ and $\xi(d, c) > 0$. If $w : ([a, a + \eta(b, a)] \times ([c, c + \xi(d, c)])) \rightarrow [0, \infty) \times [0, \infty)$ is an integrable mapping, then for every $a, b \in K_1$ and $c, d \in K_2$, the following equality holds:*

$$\begin{aligned} & f\left(a + \frac{1}{2}\eta(b, a), c + \frac{1}{2}\xi(d, c)\right) \int_0^1 \int_0^1 w(a + s\eta(b, a), c + t\xi(d, c)) ds dt \\ & - \int_0^1 \int_0^1 \left[f\left(a + s\eta(b, a), c + \frac{1}{2}\xi(d, c)\right) \right. \\ (3) \quad & \left. + f\left(a + \frac{1}{2}\eta(b, a), c + t\xi(d, c)\right) \right] \\ & \quad \times w(a + s\eta(b, a), c + t\xi(d, c)) ds dt \\ & + \int_0^1 \int_0^1 f(a + s\eta(b, a), c + t\xi(d, c)) w(a + s\eta(b, a), c + t\xi(d, c)) ds dt \\ & = \eta(b, a) \xi(d, c) \int_0^1 \left(\int_0^1 K(s, t) f_{st}(a + s\eta(b, a), c + t\xi(d, c)) ds \right) dt, \end{aligned}$$

where

$$K(s, t) = \begin{cases} \int_0^t \left(\int_0^s w(a + s\eta(b, a), c + t\xi(d, c)) ds \right) dt, & (s, t) \in [0, \frac{1}{2}] \times [0, \frac{1}{2}] \\ - \int_0^t \left(\int_s^1 w(a + s\eta(b, a), c + t\xi(d, c)) ds \right) dt, & (s, t) \in [0, \frac{1}{2}] \times (\frac{1}{2}, 1] \\ - \int_t^1 \left(\int_0^s w(a + s\eta(b, a), c + t\xi(d, c)) ds \right) dt, & (s, t) \in (\frac{1}{2}, 1] \times [0, \frac{1}{2}] \\ \int_t^1 \left(\int_s^1 w(a + s\eta(b, a), c + t\xi(d, c)) ds \right) dt, & (s, t) \in (\frac{1}{2}, 1] \times (\frac{1}{2}, 1]. \end{cases}$$

Proof. Consider

$$\begin{aligned} & \int_0^{\frac{1}{2}} \left(\int_0^t \left(\int_0^s w(a + s\eta(b, a), c + t\xi(d, c)) ds \right) dt f_{st}(a + s\eta(b, a), c + t\xi(d, c)) \right) dt \\ &= \frac{1}{\xi(d, c)} [f_s(a + s\eta(b, a), c + t\xi(d, c)) \int_0^t \left(\int_0^s w(a + s\eta(b, a), c + t\xi(d, c)) ds \right) dt]_0^{\frac{1}{2}} \\ & - \frac{1}{\xi(d, c)} \int_0^{\frac{1}{2}} \left(\left(\int_0^s w(a + s\eta(b, a), c + t\xi(d, c)) ds \right) f_s(a + s\eta(b, a), c + t\xi(d, c)) \right) dt \\ &= \frac{1}{\xi(d, c)} f_s(a + s\eta(b, a), c + \frac{1}{2}\xi(d, c)) \int_0^{\frac{1}{2}} \left(\int_0^s w(a + s\eta(b, a), c + t\xi(d, c)) ds \right) dt \\ & - \frac{1}{\xi(d, c)} \int_0^{\frac{1}{2}} \left(\left(\int_0^s w(a + s\eta(b, a), c + t\xi(d, c)) ds \right) f_s(a + s\eta(b, a), c + t\xi(d, c)) \right) dt. \end{aligned}$$

Now,

$$\begin{aligned} I_1 &= \int_0^{\frac{1}{2}} \left(\int_0^{\frac{1}{2}} \left(\int_0^t \left(\int_0^s w(a + s\eta(b, a), \right. \right. \right. \\ & \quad \left. \left. \left. c + t\xi(d, c)) ds \right) dt f_{st}(a + s\eta(b, a), c + t\xi(d, c)) \right) dt \right) ds \\ &= \frac{1}{\xi(d, c)} \int_0^{\frac{1}{2}} \left(f_s(a + s\eta(b, a), \right. \\ & \quad \left. c + \frac{1}{2}\xi(d, c)) \left(\int_0^{\frac{1}{2}} \left(\int_0^s w(a + s\eta(b, a), c + t\xi(d, c)) ds \right) dt \right) \right) ds \\ & - \frac{1}{\xi(d, c)} \int_0^{\frac{1}{2}} \left(\int_0^{\frac{1}{2}} \left(\left(\int_0^s w(a + s\eta(b, a), \right. \right. \right. \\ & \quad \left. \left. \left. c + t\xi(d, c)) ds \right) f_s(a + s\eta(b, a), c + t\xi(d, c)) \right) dt \right) ds. \end{aligned}$$

Again integrating, we get

$$\begin{aligned}
 I_1 &= \frac{1}{\eta(b, a)\xi(d, c)} [f(a + s\eta(b, a), c \\
 &+ \frac{1}{2}\xi(d, c)) \int_0^{\frac{1}{2}} (\int_0^s w(a + s\eta(b, a), c + t\xi(d, c))ds)dt]_0^{\frac{1}{2}} \\
 &- \frac{1}{\eta(b, a)\xi(d, c)} \int_0^{\frac{1}{2}} (f(a + s\eta(b, a), c \\
 &+ \frac{1}{2}\xi(d, c)) (\int_0^{\frac{1}{2}} w(a + s\eta(b, a), c + t\xi(d, c))dt))ds \\
 &- \frac{1}{\eta(b, a)\xi(d, c)} \int_0^{\frac{1}{2}} [(\int_0^s w(a + s\eta(b, a), c \\
 &+ t\xi(d, c))ds) f(a + s\eta(b, a), c + t\xi(d, c))]_0^{\frac{1}{2}} dt \\
 &+ \frac{1}{\eta(b, a)\xi(d, c)} \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} w(a + s\eta(b, a), c \\
 &+ t\xi(d, c)) f(a + s\eta(b, a), c + t\xi(d, c))dsdt.
 \end{aligned}$$

After simplification

$$\begin{aligned}
 I_1 &= \frac{1}{\eta(b, a)\xi(d, c)} f(a + \frac{1}{2}\eta(b, a), c \\
 &+ \frac{1}{2}\xi(d, c)) \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} w(a + s\eta(b, a), c + t\xi(d, c))dsdt \\
 &- \frac{1}{\eta(b, a)\xi(d, c)} \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} f(a + s\eta(b, a), c \\
 &+ \frac{1}{2}\xi(d, c)) w(a + s\eta(b, a), c + t\xi(d, c))dsdt \\
 (4) \quad &- \frac{1}{\eta(b, a)\xi(d, c)} \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} f(a + \frac{1}{2}\eta(b, a), c \\
 &+ t\xi(d, c)) w(a + s\eta(b, a), c + t\xi(d, c))dsdt \\
 &+ \frac{1}{\eta(b, a)\xi(d, c)} \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} f(a + s\eta(b, a), c \\
 &+ t\xi(d, c)) w(a + s\eta(b, a), c + t\xi(d, c))dsdt.
 \end{aligned}$$

Again

$$\begin{aligned}
 &\int_0^{\frac{1}{2}} (\int_0^t (\int_s^1 w(a + s\eta(b, a), c + t\xi(d, c))ds)dt f_{st}(a + s\eta(b, a), c + t\xi(d, c)))dt \\
 &= \frac{1}{\xi(d, c)} [f_s(a + s\eta(b, a), c + t\xi(d, c)) \int_0^t (\int_s^1 w(a + s\eta(b, a), c + t\xi(d, c))ds)]_0^{\frac{1}{2}}
 \end{aligned}$$

$$\begin{aligned}
& - \frac{1}{\xi(d, c)} \int_0^{\frac{1}{2}} \left(\left(\int_s^1 w(a + s\eta(b, a), c + t\xi(d, c)) ds \right) f_s(a + s\eta(b, a), c + t\xi(d, c)) \right) dt \\
& = \frac{1}{\xi(d, c)} f_s(a + s\eta(b, a), c + \frac{1}{2}\xi(d, c)) \int_0^{\frac{1}{2}} \left(\int_s^1 w(a + s\eta(b, a), c + t\xi(d, c)) ds \right) dt \\
& - \frac{1}{\xi(d, c)} \int_0^{\frac{1}{2}} \left(\left(\int_s^1 w(a + s\eta(b, a), c + t\xi(d, c)) ds \right) f_s(a + s\eta(b, a), c + t\xi(d, c)) \right) dt.
\end{aligned}$$

Now,

$$\begin{aligned}
I_2 & = \int_{\frac{1}{2}}^1 \left(\int_0^{\frac{1}{2}} \left(\int_0^t \left(\int_s^1 w(a + s\eta(b, a), c \right. \right. \right. \\
& \quad \left. \left. \left. + t\xi(d, c) \right) ds \right) dt f_{st}(a + s\eta(b, a), c + t\xi(d, c)) \right) dt ds \\
& = \frac{1}{\xi(d, c)} \int_{\frac{1}{2}}^1 \left(f_s(a + s\eta(b, a), c \right. \\
& \quad \left. + \frac{1}{2}\xi(d, c) \right) \left(\int_0^{\frac{1}{2}} \left(\int_s^1 w(a + s\eta(b, a), c + t\xi(d, c)) ds \right) dt \right) ds \\
& - \frac{1}{\xi(d, c)} \int_{\frac{1}{2}}^1 \left(\int_0^{\frac{1}{2}} \left(\left(\int_s^1 w(a + s\eta(b, a), c \right. \right. \right. \\
& \quad \left. \left. \left. + t\xi(d, c) \right) ds \right) f_s(a + s\eta(b, a), c + t\xi(d, c)) \right) dt ds.
\end{aligned}$$

Again integrating, we get

$$\begin{aligned}
I_2 & = \frac{1}{\eta(b, a)\xi(d, c)} [f(a + s\eta(b, a), c \\
& \quad + \frac{1}{2}\xi(d, c)) \int_0^{\frac{1}{2}} \left(\int_s^1 w(a + s\eta(b, a), c + t\xi(d, c)) ds \right) dt]_{\frac{1}{2}}^1 \\
& + \frac{1}{\eta(b, a)\xi(d, c)} \int_{\frac{1}{2}}^1 \left(f(a + s\eta(b, a), c \right. \\
& \quad \left. + \frac{1}{2}\xi(d, c) \right) \left(\int_0^{\frac{1}{2}} w(a + s\eta(b, a), c + t\xi(d, c)) dt \right) ds \\
& - \frac{1}{\eta(b, a)\xi(d, c)} \int_0^{\frac{1}{2}} \left[\left(\int_s^1 w(a + s\eta(b, a), c \right. \right. \right. \\
& \quad \left. \left. \left. + t\xi(d, c) \right) ds \right) f(a + s\eta(b, a), c + t\xi(d, c)) \right]_{\frac{1}{2}}^1 dt \\
& - \frac{1}{\eta(b, a)\xi(d, c)} \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 w(a + s\eta(b, a), c \\
& \quad + t\xi(d, c)) f(a + s\eta(b, a), c + t\xi(d, c)) ds dt.
\end{aligned}$$

After simplifying

$$\begin{aligned}
 I_2 = & -\frac{1}{\eta(b, a)\xi(d, c)}f(a + \frac{1}{2}\eta(b, a), \\
 & c + \frac{1}{2}\xi(d, c)) \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 w(a + s\eta(b, a), c + t\xi(d, c))dsdt \\
 & + \frac{1}{\eta(b, a)\xi(d, c)} \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 f(a + s\eta(b, a), \\
 & c + \frac{1}{2}\xi(d, c))w(a + s\eta(b, a), c + t\xi(d, c))dsdt \\
 (5) \quad & + \frac{1}{\eta(b, a)\xi(d, c)} \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 f(a + \frac{1}{2}\eta(b, a), c \\
 & + t\xi(d, c))w(a + s\eta(b, a), c + t\xi(d, c))dsdt \\
 & - \frac{1}{\eta(b, a)\xi(d, c)} \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 f(a + s\eta(b, a), c \\
 & + t\xi(d, c))w(a + s\eta(b, a), c + t\xi(d, c))dsdt.
 \end{aligned}$$

Again taking

$$\begin{aligned}
 & \int_{\frac{1}{2}}^1 (\int_t^1 (\int_0^s w(a + s\eta(b, a), c + t\xi(d, c))ds)dt \ f_{st}(a + s\eta(b, a), c + t\xi(d, c)))dt \\
 & = \frac{1}{\xi(d, c)}[f_s(a + s\eta(b, a), c + t\xi(d, c)) \int_t^1 (\int_0^s w(a + s\eta(b, a), c + t\xi(d, c))ds)dt]_{\frac{1}{2}}^1 \\
 & + \frac{1}{\xi(d, c)} \int_{\frac{1}{2}}^1 ((\int_0^s w(a + s\eta(b, a), c + t\xi(d, c))ds)f_s(a + s\eta(b, a), c + t\xi(d, c)))dt \\
 & = -\frac{1}{\xi(d, c)}f_s(a + s\eta(b, a), c + \frac{1}{2}\xi(d, c)) \int_{\frac{1}{2}}^1 (\int_0^s w(a + s\eta(b, a), c + t\xi(d, c))ds)dt \\
 & + \frac{1}{\xi(d, c)} \int_{\frac{1}{2}}^1 ((\int_0^s w(a + s\eta(b, a), c + t\xi(d, c))ds)f_s(a + s\eta(b, a), c + t\xi(d, c)))dt.
 \end{aligned}$$

Now,

$$\begin{aligned}
 I_3 = & \int_0^{\frac{1}{2}} (\int_{\frac{1}{2}}^1 (\int_t^t (\int_0^s w(a + s\eta(b, a), c \\
 & + t\xi(d, c))ds)dt \ f_{st}(a + s\eta(b, a), c + t\xi(d, c)))dt)ds \\
 & = -\frac{1}{\xi(d, c)} \int_0^{\frac{1}{2}} (f_s(a + s\eta(b, a), c \\
 & + \frac{1}{2}\xi(d, c))(\int_{\frac{1}{2}}^1 (\int_0^s w(a + s\eta(b, a), c + t\xi(d, c))ds)dt))ds
 \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{\xi(d, c)} \int_0^{\frac{1}{2}} \left(\int_{\frac{1}{2}}^1 \left(\int_0^s w(a + s\eta(b, a), c \right. \right. \\
& \left. \left. + t\xi(d, c)) ds \right) f_s(a + s\eta(b, a), c + t\xi(d, c)) dt \right) ds.
\end{aligned}$$

Again integrating, we get

$$\begin{aligned}
I_3 = & -\frac{1}{\eta(b, a)\xi(d, c)} \left[f(a + s\eta(b, a), c + \frac{1}{2}\xi(d, c)) \int_{\frac{1}{2}}^1 \left(\int_0^s w(a + s\eta(b, a), c \right. \right. \\
& \left. \left. + t\xi(d, c)) ds \right) dt \right]_0^{\frac{1}{2}} \\
& + \frac{1}{\eta(b, a)\xi(d, c)} \int_0^{\frac{1}{2}} \left(f(a + s\eta(b, a), c \right. \\
& \left. + \frac{1}{2}\xi(d, c)) \left(\int_{\frac{1}{2}}^1 w(a + s\eta(b, a), c + t\xi(d, c)) dt \right) \right) ds \\
& + \frac{1}{\eta(b, a)\xi(d, c)} \int_{\frac{1}{2}}^1 \left[\left(\int_0^s w(a + s\eta(b, a), c \right. \right. \\
& \left. \left. + t\xi(d, c)) ds \right) f(a + s\eta(b, a), c + t\xi(d, c)) \right]_0^{\frac{1}{2}} dt \\
& - \frac{1}{\eta(b, a)\xi(d, c)} \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} w(a + s\eta(b, a), c \\
& + t\xi(d, c)) f(a + s\eta(b, a), c + t\xi(d, c)) ds dt.
\end{aligned}$$

After simplifying

$$\begin{aligned}
I_3 = & -\frac{1}{\eta(b, a)\xi(d, c)} f(a + \frac{1}{2}\eta(b, a), c \\
& + \frac{1}{2}\xi(d, c)) \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} w(a + s\eta(b, a), c + t\xi(d, c)) ds dt \\
& + \frac{1}{\eta(b, a)\xi(d, c)} \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} f(a + s\eta(b, a), c \\
& + \frac{1}{2}\xi(d, c)) w(a + s\eta(b, a), c + t\xi(d, c)) dt ds \\
(6) \quad & + \frac{1}{\eta(b, a)\xi(d, c)} \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} f(a + \frac{1}{2}\eta(b, a), c \\
& + t\xi(d, c)) w(a + s\eta(b, a), c + t\xi(d, c)) ds dt \\
& - \frac{1}{\eta(b, a)\xi(d, c)} \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 f(a + s\eta(b, a), c \\
& + t\xi(d, c)) w(a + s\eta(b, a), c + t\xi(d, c)) ds dt.
\end{aligned}$$

Again taking

$$\begin{aligned}
& \int_{\frac{1}{2}}^1 \left(\int_t^1 \left(\int_s^1 w(a + s\eta(b, a), c + t\xi(d, c)) ds \right) dt f_{st}(a + s\eta(b, a), c + t\xi(d, c)) \right) dt \\
&= \frac{1}{\xi(d, c)} [f_s(a + s\eta(b, a), c + t\xi(d, c)) \int_t^1 \left(\int_s^1 w(a + s\eta(b, a), c + t\xi(d, c)) ds \right) dt]_{\frac{1}{2}}^1 \\
&+ \frac{1}{\xi(d, c)} \int_{\frac{1}{2}}^1 \left(\left(\int_s^1 w(a + s\eta(b, a), c + t\xi(d, c)) ds \right) f_s(a + s\eta(b, a), c + t\xi(d, c)) \right) dt \\
&= -\frac{1}{\xi(d, c)} f_s(a + s\eta(b, a), c + \frac{1}{2}\xi(d, c)) \int_{\frac{1}{2}}^1 \left(\int_s^1 w(a + s\eta(b, a), c + t\xi(d, c)) ds \right) dt \\
&+ \frac{1}{\xi(d, c)} \int_{\frac{1}{2}}^1 \left(\left(\int_s^1 w(a + s\eta(b, a), c + t\xi(d, c)) ds \right) f_s(a + s\eta(b, a), c + t\xi(d, c)) \right) dt.
\end{aligned}$$

Now,

$$\begin{aligned}
I_4 &= \int_{\frac{1}{2}}^1 \left(\int_{\frac{1}{2}}^1 \left(\int_t^1 \left(\int_s^1 w(a + s\eta(b, a), c \right. \right. \right. \\
&\quad \left. \left. \left. + t\xi(d, c)) ds \right) dt f_{st}(a + s\eta(b, a), c + t\xi(d, c)) \right) dt \right) ds \\
&= -\frac{1}{\xi(d, c)} \int_{\frac{1}{2}}^1 (f_s(a + s\eta(b, a), c \\
&\quad + \frac{1}{2}\xi(d, c)) \left(\int_{\frac{1}{2}}^1 \left(\int_s^1 w(a + s\eta(b, a), c + t\xi(d, c)) ds \right) dt \right) ds \\
&\quad + \frac{1}{\xi(d, c)} \int_{\frac{1}{2}}^1 \left(\int_{\frac{1}{2}}^1 \left(\int_s^1 w(a + s\eta(b, a), c \right. \right. \right. \\
&\quad \left. \left. \left. + t\xi(d, c)) ds \right) f_s(a + s\eta(b, a), c + t\xi(d, c)) \right) dt \right) ds.
\end{aligned}$$

Again integrating, we get

$$\begin{aligned}
I_4 &= -\frac{1}{\eta(b, a)\xi(d, c)} [f(a + s\eta(b, a), c \\
&\quad + \frac{1}{2}\xi(d, c)) \int_{\frac{1}{2}}^1 \left(\int_s^1 w(a + s\eta(b, a), c + t\xi(d, c)) ds \right) dt]_{\frac{1}{2}}^1 \\
&\quad - \frac{1}{\eta(b, a)\xi(d, c)} \int_{\frac{1}{2}}^1 (f(a + s\eta(b, a), c \\
&\quad + \frac{1}{2}\xi(d, c)) \left(\int_{\frac{1}{2}}^1 w(a + s\eta(b, a), c + t\xi(d, c)) dt \right) ds \\
&\quad + \frac{1}{\eta(b, a)\xi(d, c)} \int_{\frac{1}{2}}^1 \left[\left(\int_s^1 w(a + s\eta(b, a), c \right. \right. \right. \\
&\quad \left. \left. \left. + t\xi(d, c)) ds \right) f(a + s\eta(b, a), c + t\xi(d, c)) \right]_{\frac{1}{2}}^1 dt
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{\eta(b, a)\xi(d, c)} \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 w(a + s\eta(b, a), c \\
& + t\xi(d, c)) f(a + s\eta(b, a), c + t\xi(d, c)) ds dt.
\end{aligned}$$

After simplifying

$$\begin{aligned}
(7) \quad I_4 &= \frac{1}{\eta(b, a)\xi(d, c)} f(a + \frac{1}{2}\eta(b, a), c \\
& + \frac{1}{2}\xi(d, c)) \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 w(a + s\eta(b, a), c + t\xi(d, c)) ds dt \\
& - \frac{1}{\eta(b, a)\xi(d, c)} \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 f(a + s\eta(b, a), c \\
& + \frac{1}{2}\xi(d, c)) w(a + s\eta(b, a), c + t\xi(d, c)) dt ds \\
& - \frac{1}{\eta(b, a)\xi(d, c)} \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 f(a + \frac{1}{2}\eta(b, a), c \\
& + t\xi(d, c)) w(a + s\eta(b, a), c + t\xi(d, c)) ds dt \\
& + \frac{1}{\eta(b, a)\xi(d, c)} \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 f(a + s\eta(b, a), c \\
& + t\xi(d, c)) w(a + s\eta(b, a), c + t\xi(d, c)) ds dt.
\end{aligned}$$

Upon combining (4), (5), (6) and (7), we get required result. $\square$

Remark 2. If $w = 1$, $\eta(b, a) = b - a$ and $\xi(d, c) = d - c$, we get Lemma 2.1 in [10].

Theorem 4. Let $K_1 \times K_2 \subseteq \mathbb{R}^2$ be an open invex subset with respect to $\eta : K_1 \times K_1 \rightarrow \mathbb{R}^2$ and $\xi : K_2 \times K_2 \rightarrow \mathbb{R}^2$. Suppose $f : K_1 \times K_2 \rightarrow \mathbb{R}^2$ is a twice differentiable mapping on $K_1 \times K_2$ such that $f_{st} \in L([a, a + \eta(b, a)] \times [c, c + \xi(d, c)])$ where $\eta(b, a) > 0$ and $\xi(d, c) > 0$. If $w : ([a, a + \eta(b, a)] \times [c, c + \xi(d, c)]) \rightarrow [0, \infty) \times [0, \infty)$ is an integrable mapping and $|f_{st}|$ is preinvex function then for every $a, b \in K_1$ and $c, d \in K_2$, the following inequality holds:

$$\begin{aligned}
(8) \quad & |f(a + \frac{1}{2}\eta(b, a), c + \frac{1}{2}\xi(d, c)) \int_0^1 \int_0^1 w(a + s\eta(b, a), c + t\xi(d, c)) ds dt \\
& - \int_0^1 \int_0^1 [f(a + s\eta(b, a), c + \frac{1}{2}\xi(d, c)) + f(a + \frac{1}{2}\eta(b, a), c + t\xi(d, c))] \\
& \times w(a + s\eta(b, a), c + t\xi(d, c)) ds dt \\
& + \int_0^1 \int_0^1 f(a + s\eta(b, a), c + t\xi(d, c)) w(a + s\eta(b, a), c + t\xi(d, c)) ds dt| \\
& \leq \frac{\eta(b, a)\xi(d, c)}{16} \|w\|_\infty \frac{[|f_{st}(a, c)| + |f_{st}(a, d)| + |f_{st}(b, c)| + |f_{st}(b, d)|]}{4}.
\end{aligned}$$

Proof. Taking modulus of both sides of (3), we get

$$\begin{aligned} J &= |f(a + \frac{1}{2}\eta(b, a), c + \frac{1}{2}\xi(d, c)) \int_0^1 \int_0^1 w(a + s\eta(b, a), c + t\xi(d, c)) ds dt \\ &\quad - \int_0^1 \int_0^1 [f(a + s\eta(b, a), c + \frac{1}{2}\xi(d, c)) + f(a + \frac{1}{2}\eta(b, a), c + t\xi(d, c))] \\ &\quad \times w(a + s\eta(b, a), c + t\xi(d, c)) ds dt \\ &\quad + \int_0^1 \int_0^1 f(a + s\eta(b, a), c + t\xi(d, c)) w(a + s\eta(b, a), c + t\xi(d, c)) ds dt| \\ &= |\eta(b, a)\xi(d, c) \int_0^1 (\int_0^1 K(s, t) f_{st}(a + s\eta(b, a), c + t\xi(d, c)) ds) dt|. \end{aligned}$$

f_{st} is coordinated preinvex function, i.e

$$\begin{aligned} f_{st}(a + s\eta(b, a), c + t\xi(d, c)) &\leq (1-s)(1-t)f_{st}(a, c) + (1-s)t f_{st}(a, d) \\ &\quad + (1-t)s f_{st}(b, c) + st f_{st}(b, d). \end{aligned}$$

From preinvexity of $|f_{st}|$, we have

$$\begin{aligned} J &\leq \eta(b, a)\xi(d, c) \int_0^1 (\int_0^1 |K(s, t)| [(1-s)(1-t)|f_{st}(a, c)| + (1-s)t|f_{st}(a, d)| \\ (9) \quad &+ (1-t)s|f_{st}(b, c)| + st|f_{st}(b, d)|] dt). \end{aligned}$$

After a straightforward computation, we obtain

$$\begin{aligned} \|w\|_\infty \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} (\int_0^t (\int_0^s ds) dt [(1-s)(1-t)|f_{st}(a, c)| + (1-s)t|f_{st}(a, d)| \\ + (1-t)s|f_{st}(b, c)| + st|f_{st}(b, d)|]) ds dt \\ (10) \quad = \|w\|_\infty [\frac{|f_{st}(a, c)|}{144} + \frac{|f_{st}(a, d)|}{288} + \frac{|f_{st}(b, c)|}{288} + \frac{|f_{st}(b, d)|}{576}] \end{aligned}$$

$$\begin{aligned} \|w\|_\infty \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 (\int_0^t (\int_s^1 ds) dt [(1-s)(1-t)|f_{st}(a, c)| + (1-s)t|f_{st}(a, d)| \\ + (1-t)s|f_{st}(b, c)| + st|f_{st}(b, d)|]) ds dt \\ (11) \quad = \|w\|_\infty [\frac{|f_{st}(a, c)|}{288} + \frac{|f_{st}(a, d)|}{576} + \frac{|f_{st}(b, c)|}{144} + \frac{|f_{st}(b, d)|}{288}] \end{aligned}$$

$$\begin{aligned} \|w\|_\infty \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} (\int_t^1 (\int_0^s ds) dt [(1-s)(1-t)|f_{st}(a, c)| + (1-s)t|f_{st}(a, d)| \\ + (1-t)s|f_{st}(b, c)| + st|f_{st}(b, d)|]) ds dt \\ (12) \quad = \|w\|_\infty [\frac{|f_{st}(a, c)|}{288} + \frac{|f_{st}(a, d)|}{144} + \frac{|f_{st}(b, c)|}{576} + \frac{|f_{st}(b, d)|}{288}] \end{aligned}$$

$$\begin{aligned}
& \|w\|_\infty \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 \left(\int_t^1 \left(\int_s^1 ds \right) dt [(1-s)(1-t)|f_{st}(a, c)| + (1-s)tf|_{st}(a, d)| \right. \\
& \quad \left. + (1-t)s|f_{st}(b, c)| + st|f_{st}(b, d)|) ds dt \right. \\
(13) \quad & = \|w\|_\infty \left[\frac{|f_{st}(a, c)|}{576} + \frac{|f_{st}(a, d)|}{288} + \frac{|f_{st}(b, c)|}{288} + \frac{|f_{st}(b, d)|}{144} \right].
\end{aligned}$$

Using the results of (10), (11), (12) and (13) in (9), we get (8). $\square$

Corollary 1. *If we take $\|w\|_\infty = 1$, $\eta(b, a) = b - a$ and $\xi(d, c) = d - c$ in (6), we get Theorem 2 in [10].*

Theorem 5. *Let $K_1 \times K_2 \subseteq \mathbb{R}^2$ be an open invex subset with respect to $\eta : K_1 \times K_1 \rightarrow \mathbb{R}^2$ and $\xi : K_2 \times K_2 \rightarrow \mathbb{R}^2$. Suppose $f : K_1 \times K_2 \rightarrow \mathbb{R}^2$ is a twice differentiable mapping on $K_1 \times K_2$ such that $f_{st} \in L([a, a + \eta(b, a)] \times [c, c + \xi(d, c)])$ where $\eta(b, a) > 0$ and $\xi(d, c) > 0$. If $w : ([a, a + \eta(b, a)] \times [c, c + \xi(d, c)]) \rightarrow [0, \infty) \times [0, \infty)$ is an integrable mapping and $|f_{st}|^q$ is preinvex function then for every $a, b \in K_1$ and $c, d \in K_2$, where $q \in (1, \infty)$ and $\frac{1}{p} + \frac{1}{q} = 1$, then following inequality holds:*

$$\begin{aligned}
& |f(a + \frac{1}{2}\eta(b, a), c + \frac{1}{2}\xi(d, c)) \int_0^1 \int_0^1 w(a + s\eta(b, a), c + t\xi(d, c)) ds dt \\
& - \int_0^1 \int_0^1 [f(a + s\eta(b, a), c + \frac{1}{2}\xi(d, c)) + f(a + \frac{1}{2}\eta(b, a), c + t\xi(d, c))] \\
(14) \quad & \times w(a + s\eta(b, a), c + t\xi(d, c)) ds dt \\
& + \int_0^1 \int_0^1 f(a + s\eta(b, a), c + t\xi(d, c)) w(a + s\eta(b, a), c + t\xi(d, c)) ds dt| \\
& \leq \frac{\|w\|_\infty \eta(b, a) \xi(d, c)}{4^{1+\frac{1}{p}}(p+1)^{\frac{2}{p}}} (|f_{st}(a, c)|^q + |f_{st}(a, d)|^q + |f_{st}(b, c)|^q + |f_{st}(b, d)|^q)^{\frac{1}{q}}.
\end{aligned}$$

Proof. From Lemma 2.1 and Hölder's integral inequality, we get

$$\begin{aligned}
J & = |f(a + \frac{1}{2}\eta(b, a), c + \frac{1}{2}\xi(d, c)) \int_0^1 \int_0^1 w(a + s\eta(b, a), c + t\xi(d, c)) ds dt \\
& - \int_0^1 \int_0^1 [f(a + s\eta(b, a), c + \frac{1}{2}\xi(d, c)) + f(a + \frac{1}{2}\eta(b, a), c + t\xi(d, c))] \\
& \times w(a + s\eta(b, a), c + t\xi(d, c)) ds dt \\
& + \int_0^1 \int_0^1 f(a + s\eta(b, a), c + t\xi(d, c)) w(a + s\eta(b, a), c + t\xi(d, c)) ds dt| \\
& \leq \eta(b, a) \xi(d, c) \left(\int_0^1 \int_0^1 |K(s, t)|^p ds dt \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 \left(\int_0^1 |f_{st}(a + s\eta(b, a), c + t\xi(d, c))|^q ds \right) dt \right)^{\frac{1}{q}}.
\end{aligned}$$

From preinvexity of $|f_{st}|^q$ in coordinates, we have

$$\begin{aligned}
 J &\leq \eta(b, a)\xi(d, c)\left(\int_0^1 \int_0^1 |K(s, t)|^p ds dt\right)^{\frac{1}{p}} \\
 &\quad \times \left(\int_0^1 \left(\int_0^1 (1-s)(1-t)|f_{st}(a, c)|^q + (1-s)t|f_{st}(a, d)|^q \right. \right. \\
 (15) \quad &\quad \left. \left. + (1-t)s|f_{st}(b, c)|^q + st|f_{st}(b, d)|^q\right) dt\right)^{\frac{1}{q}}.
 \end{aligned}$$

After a straightforward computation, we obtain

$$\begin{aligned}
 &\int_0^1 \int_0^1 [(1-s)(1-t)|f_{st}(a, c)|^q + (1-s)t|f_{st}(a, d)|^q \\
 &\quad + (1-t)s|f_{st}(b, c)|^q + st|f_{st}(b, d)|^q] ds dt \\
 (16) \quad &= \frac{1}{4} [|f_{st}(a, c)|^q + |f_{st}(a, d)|^q + |f_{st}(b, c)|^q + |f_{st}(b, d)|^q]
 \end{aligned}$$

and

$$\begin{aligned}
 &\int_0^1 \int_0^1 |K(s, t)|^p ds dt \\
 &= \|w\|_\infty^p \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} t^p s^p ds dt + \|w\|_\infty^p \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 t^p (1-s)^p ds dt \\
 &\quad + \|w\|_\infty^p \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} (1-t)^p s^p ds dt + \|w\|_\infty^p \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 (1-t)^p (1-s)^p ds dt \\
 (17) \quad &= \|w\|_\infty^p \frac{4}{(p+1)^2} \left(\frac{1}{2}\right)^{2(p+1)}.
 \end{aligned}$$

Using the results of (16) and (17) in (15), we get required result (14). $\square$

Remark 3. For $p > 1$, $\frac{1}{16} < \frac{1}{4(p+1)^{\frac{2}{p}}}$, we have improvement of constant in Theorem 2.3.

Corollary 2. If we take $\|w\|_\infty = 1$, $\eta(b, a) = b - a$ and $\xi(d, c) = d - c$ in (12), we get Theorem 3 in [10].

Theorem 6. Let $K_1 \times K_2 \subseteq \mathbb{R}^2$ be an open invex subset with respect to $\eta : K_1 \times K_1 \rightarrow \mathbb{R}^2$ and $\xi : K_2 \times K_2 \rightarrow \mathbb{R}^2$. Suppose $f : K_1 \times K_2 \rightarrow \mathbb{R}^2$ is a twice differentiable mapping on $K_1 \times K_2$ such that $f_{st} \in L([a, a + \eta(b, a)] \times ([c, c + \xi(d, c)]))$, where $\eta(b, a) > 0$ and $\xi(d, c) > 0$. If $w : ([a, a + \eta(b, a)] \times ([c, c + \xi(d, c)])) \rightarrow [0, \infty) \times [0, \infty)$ is an integrable mapping and $|f_{st}|^q$ is preinvex function then for every $a, b \in K_1$ and $c, d \in K_2$, where $q \geq 1$, then following

inequality holds:

$$\begin{aligned}
& |f(a + \frac{1}{2}\eta(b, a), c + \frac{1}{2}\xi(d, c)) \int_0^1 \int_0^1 w(a + s\eta(b, a), c + t\xi(d, c)) ds dt \\
& - \int_0^1 \int_0^1 [f(a + s\eta(b, a), c + \frac{1}{2}\xi(d, c)) + f(a + \frac{1}{2}\eta(b, a), c + t\xi(d, c))] \\
& \times w(a + s\eta(b, a), c + t\xi(d, c)) ds dt \\
& + \int_0^1 \int_0^1 f(a + s\eta(b, a), c + t\xi(d, c)) w(a + s\eta(b, a), c + t\xi(d, c)) ds dt| \\
& \leq \eta(b, a)\xi(d, c) \left(\frac{\|w\|_\infty}{16} \right) \\
& \times \left(\frac{\|w\|_\infty}{4} [|f_{st}(a, c)|^q + |f_{st}(a, d)|^q + |f_{st}(b, c)|^q + |f_{st}(b, d)|^q] \right)^{\frac{1}{q}}.
\end{aligned}
\tag{18}$$

Proof. From Lemma 2.1 and Power mean inequality, we get

$$\begin{aligned}
J &= |f(a + \frac{1}{2}\eta(b, a), c + \frac{1}{2}\xi(d, c)) \int_0^1 \int_0^1 w(a + s\eta(b, a), c + t\xi(d, c)) ds dt \\
& - \int_0^1 \int_0^1 [f(a + s\eta(b, a), c + \frac{1}{2}\xi(d, c)) + f(a + \frac{1}{2}\eta(b, a), c + t\xi(d, c))] \\
& \times w(a + s\eta(b, a), c + t\xi(d, c)) ds dt \\
& + \int_0^1 \int_0^1 f(a + s\eta(b, a), c + t\xi(d, c)) w(a + s\eta(b, a), c + t\xi(d, c)) ds dt| \\
& \leq \eta(b, a)\xi(d, c) \left(\int_0^1 \int_0^1 |K(s, t)| ds dt \right)^{1-\frac{1}{q}} \\
& \times \left(\int_0^1 \left(\int_0^1 K(s, t) |f_{st}(a + s\eta(b, a), c + t\xi(d, c))|^q ds \right) dt \right)^{\frac{1}{q}}.
\end{aligned}$$

From preinvexity of $|f_{st}|^q$ in coordinates, we have

$$\begin{aligned}
J &\leq \eta(b, a)\xi(d, c) \left(\int_0^1 \int_0^1 |K(s, t)| ds dt \right)^{1-\frac{1}{q}} \\
&\times \left(\int_0^1 \left(\int_0^1 |K(s, t)| (1-s)(1-t) |f_{st}(a, c)|^q + (1-s)t |f_{st}(a, d)|^q \right. \right. \\
&\left. \left. + (1-t)s |f_{st}(b, c)|^q + st |f_{st}(b, d)|^q \right) dt \right)^{\frac{1}{q}}.
\end{aligned}
\tag{19}$$

After a straightforward computation, we obtain

$$\begin{aligned}
& \|w\|_\infty \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \left(\int_0^t \left(\int_0^s ds \right) dt [(1-s)(1-t) |f_{st}(a, c)|^q + (1-s)t |f_{st}(a, d)|^q \right. \\
& \left. + (1-t)s |f_{st}(b, c)|^q + st |f_{st}(b, d)|^q] ds dt \right) \\
& = \|w\|_\infty \frac{1}{64} \left[\frac{|f_{st}(a, c)|^q}{144} + \frac{|f_{st}(a, d)|^q}{288} + \frac{|f_{st}(b, c)|^q}{288} + \frac{|f_{st}(b, d)|^q}{576} \right]
\end{aligned}
\tag{20}$$

$$\begin{aligned}
& \|w\|_{\infty} \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 \left(\int_0^t \left(\int_s^1 ds \right) dt [(1-s)(1-t)|f_{st}(a, c)|^q + (1-s)tf|_{st}(a, d)|^q \right. \\
& \quad \left. + (1-t)s|f_{st}(b, c)|^q + st|f_{st}(b, d)|^q) ds dt \right. \\
(21) \quad & = \|w\|_{\infty} \left[\frac{|f_{st}(a, c)|^q}{288} + \frac{|f_{st}(a, d)|^q}{576} + \frac{|f_{st}(b, c)|^q}{144} + \frac{|f_{st}(b, d)|^q}{288} \right]
\end{aligned}$$

$$\begin{aligned}
& \|w\|_{\infty} \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} \left(\int_t^1 \left(\int_0^s ds \right) dt [(1-s)(1-t)|f_{st}(a, c)|^q + (1-s)tf|_{st}(a, d)|^q \right. \\
& \quad \left. + (1-t)s|f_{st}(b, c)|^q + st|f_{st}(b, d)|^q) ds dt \right. \\
(22) \quad & = \|w\|_{\infty} \left[\frac{|f_{st}(a, c)|^q}{288} + \frac{|f_{st}(a, d)|^q}{144} + \frac{|f_{st}(b, c)|^q}{576} + \frac{|f_{st}(b, d)|^q}{288} \right]
\end{aligned}$$

$$\begin{aligned}
& \|w\|_{\infty} \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 \left(\int_t^1 \left(\int_s^1 ds \right) dt [(1-s)(1-t)|f_{st}(a, c)|^q + (1-s)tf|_{st}(a, d)|^q \right. \\
& \quad \left. + (1-t)s|f_{st}(b, c)|^q + st|f_{st}(b, d)|^q) ds dt \right. \\
(23) \quad & = \|w\|_{\infty} \left[\frac{|f_{st}(a, c)|^q}{576} + \frac{|f_{st}(a, d)|^q}{288} + \frac{|f_{st}(b, c)|^q}{288} + \frac{|f_{st}(b, d)|^q}{144} \right]
\end{aligned}$$

and

$$\begin{aligned}
& \int_0^1 \int_0^1 |K(s, t)| ds dt \\
& = \|w\|_{\infty} \left(\int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \left(\int_0^t \left(\int_0^s ds \right) dt \right) ds dt + \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 \left(\int_0^t \left(\int_s^1 ds \right) dt \right) ds dt \right. \\
& \quad \left. + \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} \left(\int_t^1 \left(\int_0^s ds \right) dt \right) ds dt + \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 \left(\int_t^1 \left(\int_s^1 ds \right) dt \right) ds dt \right) \\
(24) \quad & = \frac{\|w\|_{\infty}}{16}.
\end{aligned}$$

Using the results of (20), (21), (22), (23) and (24) in (19), we get required result (18). $\square$

Corollary 3. *If we take $\|w\|_{\infty} = 1$, $\eta(b, a) = b - a$ and $\xi(d, c) = d - c$ in (6), we get Theorem 4 in [10].*

Lemma 2. *Let $K_1 \times K_2 \subseteq \mathbb{R}^2$ be an open invex subset with respect to $\eta : K_1 \times K_1 \rightarrow \mathbb{R}^2$ and $\xi : K_2 \times K_2 \rightarrow \mathbb{R}^2$. Suppose $f : K_1 \times K_2 \rightarrow \mathbb{R}^2$ is a twice differentiable mapping on $K_1 \times K_2$ such that $f_{st} \in L([a, a + \eta(b, a)] \times ([c, c + \xi(d, c)]))$ where $\eta(b, a) > 0$ and $\xi(d, c) > 0$. If $w : ([a, a + \eta(b, a)] \times ([c, c + \xi(d, c)])) \rightarrow [0, \infty) \times [0, \infty)$ is an integrable mapping, then for every $a, b \in K_1$*

and $c, d \in K_2$, the following equality holds:

$$\begin{aligned}
 & [f(a, c + \xi(d, c)) + f(a + \eta(b, a), c + \xi(d, c)) + f(a + \eta(b, a), c) + f(a, c)] \\
 & \times \int_0^1 \left(\int_0^1 w(a + s\eta(b, a), c + t\xi(d, c)) ds \right) dt \\
 & - 2 \int_0^1 \int_0^1 [f(a + s\eta(b, a), c + \xi(d, c)) + f(a + \eta(b, a), c + t\xi(d, c)) \\
 & + (f(a, c + t\xi(d, c)) + f(a + s\eta(b, a), c))] \\
 & \times (w(a + s\eta(b, a), c + t\xi(d, c))) ds dt \\
 & + 4 \int_0^1 \int_0^1 w(a + s\eta(b, a), c + t\xi(d, c)) f(a + s\eta(b, a), c + t\xi(d, c)) ds dt \\
 (25) \quad & = \eta(b, a) \xi(d, c) \int_0^1 \int_0^1 P(s, t) f_{ts}((a + s\eta(b, a), c + t\xi(d, c))) ds dt,
 \end{aligned}$$

where

$$\begin{aligned}
 P(s, t) = & \int_0^t \int_0^s w(a + s\eta(b, a), c + t\xi(d, c)) ds dt \\
 & - \int_0^t \int_s^1 w(a + s\eta(b, a), c + t\xi(d, c)) ds dt \\
 & - \int_t^1 \int_0^s w(a + s\eta(b, a), c + t\xi(d, c)) ds dt \\
 & + \int_t^1 \int_s^1 w(a + s\eta(b, a), c + t\xi(d, c)) ds dt,
 \end{aligned}$$

and $(t, s) \in [0, 1]$.

Proof. Consider

$$\begin{aligned}
 & \int_0^1 \left(\int_0^t \left(\int_0^s w(a + s\eta(b, a), c + t\xi(d, c)) ds \right) dt \right) f_{st}(a + s\eta(b, a), c + t\xi(d, c)) dt \\
 & = \frac{1}{\xi(d, c)} [f_s(a + s\eta(b, a), c + t\xi(d, c)) \int_0^t \left(\int_0^s w(a + s\eta(b, a), c + t\xi(d, c)) ds \right) dt]_0^1 \\
 & - \frac{1}{\xi(d, c)} \int_0^1 \left(\left(\int_0^s w(a + s\eta(b, a), c + t\xi(d, c)) ds \right) f_s(a + s\eta(b, a), c + t\xi(d, c)) \right) dt \\
 & = \frac{1}{\xi(d, c)} f_s(a + s\eta(b, a), c + \xi(d, c)) \int_0^1 \left(\int_0^s w(a + s\eta(b, a), c + t\xi(d, c)) ds \right) dt \\
 & - \frac{1}{\xi(d, c)} \int_0^1 \left(\left(\int_0^s w(a + s\eta(b, a), c + t\xi(d, c)) ds \right) f_s(a + s\eta(b, a), c + t\xi(d, c)) \right) dt.
 \end{aligned}$$

Now,

$$\begin{aligned}
 J_1 &= \int_0^1 \left(\int_0^1 \left(\int_0^t \left(\int_0^s w(a + s\eta(b, a), c \right. \right. \right. \\
 &\quad \left. \left. \left. + t\xi(d, c) \right) ds \right) dt f_{st}(a + s\eta(b, a), c + t\xi(d, c)) \right) dt ds \\
 &= \frac{1}{\xi(d, c)} \int_0^1 (f_s(a + s\eta(b, a), c + \xi(d, c)) \left(\int_0^1 \left(\int_0^s w(a \right. \right. \\
 &\quad \left. \left. + s\eta(b, a), c + t\xi(d, c) \right) ds \right) dt) ds \\
 &\quad - \frac{1}{\xi(d, c)} \int_0^1 \left(\int_0^1 \left(\int_0^s w(a + s\eta(b, a), c \right. \right. \right. \\
 &\quad \left. \left. \left. + t\xi(d, c) \right) ds \right) f_s(a + s\eta(b, a), c + t\xi(d, c)) \right) dt ds.
 \end{aligned}$$

Again integrating, we get

$$\begin{aligned}
 J_1 &= \frac{1}{\eta(b, a)\xi(d, c)} [f(a + s\eta(b, a), c + \xi(d, c)) \int_0^1 \left(\int_0^s w(a \right. \\
 &\quad \left. + s\eta(b, a), c + t\xi(d, c) \right) ds \right]_0^1 \\
 &\quad - \frac{1}{\eta(b, a)\xi(d, c)} \int_0^1 (f(a + s\eta(b, a), c \\
 &\quad + \xi(d, c)) \left(\int_0^1 w(a + s\eta(b, a), c + t\xi(d, c)) dt \right) ds \\
 &\quad - \frac{1}{\eta(b, a)\xi(d, c)} \int_0^1 \left[\left(\int_0^s w(a + s\eta(b, a), c \right. \right. \\
 &\quad \left. \left. + t\xi(d, c) \right) ds \right) f(a + s\eta(b, a), c + t\xi(d, c)) \right]_0^1 dt \\
 &\quad + \frac{1}{\eta(b, a)\xi(d, c)} \int_0^1 \int_0^1 w(a + s\eta(b, a), c \\
 &\quad + t\xi(d, c)) f(a + s\eta(b, a), c + t\xi(d, c)) ds dt.
 \end{aligned}$$

After simplification

$$\begin{aligned}
 J_1 &= \frac{1}{\eta(b, a)\xi(d, c)} f(a + \eta(b, a), c + \xi(d, c)) \int_0^1 \int_0^1 w(a \\
 &\quad + s\eta(b, a), c + t\xi(d, c)) ds dt \\
 &\quad - \frac{1}{\eta(b, a)\xi(d, c)} \int_0^1 \int_0^1 f(a + s\eta(b, a), c \\
 (26) \quad &\quad + \xi(d, c)) w(a + s\eta(b, a), c + t\xi(d, c)) dt ds \\
 &\quad - \frac{1}{\eta(b, a)\xi(d, c)} \int_0^1 \int_0^1 f(a + \eta(b, a), c \\
 &\quad + t\xi(d, c)) w(a + s\eta(b, a), c + t\xi(d, c)) ds dt
 \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{\eta(b, a)\xi(d, c)} \int_0^1 \int_0^1 f(a + s\eta(b, a), c \\
& + t\xi(d, c))w(a + s\eta(b, a), c + t\xi(d, c))dsdt.
\end{aligned}$$

Again taking

$$\begin{aligned}
& \int_0^1 \left(\int_0^t \left(\int_s^1 w(a + s\eta(b, a), c + t\xi(d, c))ds \right) dt f_{st}(a + s\eta(b, a), c + t\xi(d, c)) \right) dt \\
& = \frac{1}{\xi(d, c)} [f_s(a + s\eta(b, a), c + t\xi(d, c)) \int_0^t \left(\int_s^1 w(a + s\eta(b, a), c + t\xi(d, c))ds \right)]_0^1 \\
& - \frac{1}{\xi(d, c)} \int_0^1 \left(\left(\int_s^1 w(a + s\eta(b, a), c + t\xi(d, c))ds \right) f_s(a + s\eta(b, a), c + t\xi(d, c)) \right) dt \\
& = \frac{1}{\xi(d, c)} f_s(a + s\eta(b, a), c + \xi(d, c)) \int_0^1 \left(\int_s^1 w(a + s\eta(b, a), c + t\xi(d, c))ds \right) dt \\
& - \frac{1}{\xi(d, c)} \int_0^1 \left(\left(\int_s^1 w(a + s\eta(b, a), c + t\xi(d, c))ds \right) f_s(a + s\eta(b, a), c + t\xi(d, c)) \right) dt.
\end{aligned}$$

Now,

$$\begin{aligned}
J_2 & = \int_0^1 \left(\int_0^1 \left(\int_0^t \left(\int_s^1 w(a + s\eta(b, a), c \right. \right. \right. \\
& \quad \left. \left. + t\xi(d, c))ds \right) dt f_{st}(a + s\eta(b, a), c + t\xi(d, c)) \right) dt ds \\
& = \frac{1}{\xi(d, c)} \int_0^1 (f_s(a + s\eta(b, a), c + \xi(d, c)) \left(\int_0^1 \left(\int_s^1 w(a + s\eta(b, a), c \right. \right. \right. \\
& \quad \left. \left. + t\xi(d, c))ds \right) dt \right) ds \\
& - \frac{1}{\xi(d, c)} \int_0^1 \left(\int_0^1 \left(\int_s^1 w(a + s\eta(b, a), c \right. \right. \right. \\
& \quad \left. \left. + t\xi(d, c))ds \right) f_s(a + s\eta(b, a), c + t\xi(d, c)) \right) dt ds.
\end{aligned}$$

Again integrating, we get

$$\begin{aligned}
J_2 & = \frac{1}{\eta(b, a)\xi(d, c)} [f(a + s\eta(b, a), c + \xi(d, c)) \int_0^1 \left(\int_s^1 w(a \right. \\
& \quad \left. + s\eta(b, a), c + t\xi(d, c))ds \right) dt]_0^1 \\
& + \frac{1}{\eta(b, a)\xi(d, c)} \int_0^1 (f(a + s\eta(b, a), c \\
& + \xi(d, c)) \left(\int_0^1 w(a + s\eta(b, a), c + t\xi(d, c))dt \right) ds \\
& - \frac{1}{\eta(b, a)\xi(d, c)} \int_0^1 \left[\left(\int_s^1 w(a + s\eta(b, a), c \right. \right. \right. \\
& \quad \left. \left. + t\xi(d, c))ds \right) f(a + s\eta(b, a), c + t\xi(d, c)) \right]_0^1 dt
\end{aligned}$$

$$- \frac{1}{\eta(b, a)\xi(d, c)} \int_0^1 \int_0^1 w(a + s\eta(b, a), c + t\xi(d, c)) f(a + s\eta(b, a), c + t\xi(d, c)) ds dt.$$

After simplifying

$$\begin{aligned}
 J_2 = & -\frac{1}{\eta(b, a)\xi(d, c)} f(a, c + \xi(d, c)) \int_0^1 \left(\int_0^1 w(a + s\eta(b, a), c + t\xi(d, c)) ds \right) dt \\
 & + \frac{1}{\eta(b, a)\xi(d, c)} \int_0^1 \int_0^1 f(a + s\eta(b, a), c + \xi(d, c)) w(a + s\eta(b, a), c + t\xi(d, c)) ds dt \\
 & + \frac{1}{\eta(b, a)\xi(d, c)} \int_0^1 \left(\int_0^1 w(a + s\eta(b, a), c + t\xi(d, c)) ds \right) f(a, c + \xi(d, c)) dt \\
 & - \frac{1}{\eta(b, a)\xi(d, c)} \int_0^1 \int_0^1 w(a + s\eta(b, a), c + t\xi(d, c)) f(a + s\eta(b, a), c + t\xi(d, c)) ds dt
 \end{aligned}
 \tag{27}$$

Again taking

$$\begin{aligned}
 & \int_0^1 \left(\int_t^1 \left(\int_0^s w(a + s\eta(b, a), c + t\xi(d, c)) ds \right) dt f_{st}(a + s\eta(b, a), c + t\xi(d, c)) \right) dt \\
 & = \frac{1}{\xi(d, c)} [f_s(a + s\eta(b, a), c + t\xi(d, c)) \int_t^1 \left(\int_0^s w(a + s\eta(b, a), c + t\xi(d, c)) ds \right) dt]_0^1 \\
 & + \frac{1}{\xi(d, c)} \int_0^1 \left(\left(\int_0^s w(a + s\eta(b, a), c + t\xi(d, c)) ds \right) f_s(a + s\eta(b, a), c + t\xi(d, c)) \right) dt \\
 & = -\frac{1}{\xi(d, c)} f_s(a + s\eta(b, a), c) \int_0^1 \left(\int_0^s w(a + s\eta(b, a), c + t\xi(d, c)) ds \right) dt \\
 & + \frac{1}{\xi(d, c)} \int_0^1 \left(\left(\int_0^s w(a + s\eta(b, a), c + t\xi(d, c)) ds \right) f_s(a + s\eta(b, a), c + t\xi(d, c)) \right) dt.
 \end{aligned}$$

Now,

$$\begin{aligned}
 J_3 = & \int_0^1 \left(\int_0^1 \left(\int_t^1 \left(\int_0^s w(a + s\eta(b, a), c + t\xi(d, c)) ds \right) dt f_{st}(a + s\eta(b, a), c + t\xi(d, c)) \right) ds \right) dt \\
 & = -\frac{1}{\xi(d, c)} \int_0^1 (f_s(a + s\eta(b, a), c) \left(\int_0^1 \left(\int_0^s w(a + s\eta(b, a), c + t\xi(d, c)) ds \right) dt \right) ds
 \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{\xi(d, c)} \int_0^1 \left(\int_0^1 \left(\int_0^s w(a + s\eta(b, a), c \right. \right. \\
& \left. \left. + t\xi(d, c)) ds \right) f_s(a + s\eta(b, a), c + t\xi(d, c)) dt \right) ds.
\end{aligned}$$

Again integrating, we get

$$\begin{aligned}
J_3 = & -\frac{1}{\eta(b, a)\xi(d, c)} [f(a + s\eta(b, a), c) \int_0^1 \left(\int_0^s w(a \right. \\
& \left. + s\eta(b, a), c + t\xi(d, c)) ds \right) dt]_0^1 \\
& + \frac{1}{\eta(b, a)\xi(d, c)} \int_0^1 (f(a + s\eta(b, a), c) \left(\int_0^1 w(a \right. \\
& \left. + s\eta(b, a), c + t\xi(d, c)) dt \right) ds \\
& + \frac{1}{\eta(b, a)\xi(d, c)} \int_0^1 \left[\left(\int_0^s w(a + s\eta(b, a), c \right. \right. \\
& \left. \left. + t\xi(d, c)) ds \right) f(a + s\eta(b, a), c + t\xi(d, c)) \right]_0^1 dt \\
& - \frac{1}{\eta(b, a)\xi(d, c)} \int_0^1 \int_0^1 w(a + s\eta(b, a), c \\
& + t\xi(d, c)) f(a + s\eta(b, a), c + t\xi(d, c)) ds dt.
\end{aligned}$$

After simplifying

$$\begin{aligned}
J_3 = & -\frac{1}{\eta(b, a)\xi(d, c)} f(a + \eta(b, a), c) \int_0^1 \left(\int_0^1 w(a \right. \\
& \left. + s\eta(b, a), c + t\xi(d, c)) ds \right) dt \\
& + \frac{1}{\eta(b, a)\xi(d, c)} \int_0^1 \int_0^1 f(a \\
(28) \quad & + s\eta(b, a), c) w(a + s\eta(b, a), c + t\xi(d, c)) ds dt \\
& + \frac{1}{\eta(b, a)\xi(d, c)} \int_0^1 \left(\int_0^1 w(a + s\eta(b, a), c \right. \\
& \left. + t\xi(d, c)) ds \right) f(a + \eta(b, a), c + t\xi(d, c)) dt \\
& - \frac{1}{\eta(b, a)\xi(d, c)} \int_0^1 \int_0^1 w(a + s\eta(b, a), c \\
& + t\xi(d, c)) f(a + s\eta(b, a), c + t\xi(d, c)) ds dt
\end{aligned}$$

Again taking

$$\begin{aligned}
& \int_0^1 \left(\int_t^1 \left(\int_s^1 w(a + s\eta(b, a), c + t\xi(d, c)) ds \right) dt f_{st}(a + s\eta(b, a), c + t\xi(d, c)) \right) dt \\
& = \frac{1}{\xi(d, c)} [f_s(a + s\eta(b, a), c + t\xi(d, c)) \int_t^1 \left(\int_s^1 w(a + s\eta(b, a), c + t\xi(d, c)) ds \right)]_0^1
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{\xi(d, c)} \int_0^1 \left(\left(\int_s^1 w(a + s\eta(b, a), c + t\xi(d, c)) ds \right) f_s(a + s\eta(b, a), c + t\xi(d, c)) \right) dt \\
& = -\frac{1}{\xi(d, c)} f_s(a + s\eta(b, a), c) \int_0^1 \left(\int_s^1 w(a + s\eta(b, a), c + t\xi(d, c)) ds \right) dt \\
& + \frac{1}{\xi(d, c)} \int_0^1 \left(\left(\int_s^1 w(a + s\eta(b, a), c + t\xi(d, c)) ds \right) f_s(a + s\eta(b, a), c + t\xi(d, c)) \right) dt.
\end{aligned}$$

Now,

$$\begin{aligned}
J_4 & = \int_0^1 \left(\int_0^1 \left(\int_t^1 \left(\int_s^1 w(a + s\eta(b, a), c \right. \right. \right. \\
& \quad \left. \left. \left. + t\xi(d, c) \right) ds \right) dt f_{st}(a + s\eta(b, a), c + t\xi(d, c)) \right) dt ds \\
& = -\frac{1}{\xi(d, c)} \int_0^1 \left(f_s(a + s\eta(b, a), c) \left(\int_0^1 \left(\int_s^1 w(a + s\eta(b, a), c \right. \right. \right. \\
& \quad \left. \left. \left. + t\xi(d, c) \right) ds \right) dt \right) ds \\
& + \frac{1}{\xi(d, c)} \int_0^1 \left(\int_0^1 \left(\left(\int_s^1 w(a \right. \right. \right. \\
& \quad \left. \left. \left. + s\eta(b, a), c + t\xi(d, c) \right) ds \right) f_s(a + s\eta(b, a), c + t\xi(d, c)) \right) dt ds.
\end{aligned}$$

Again integrating, we get

$$\begin{aligned}
J_4 & = -\frac{1}{\eta(b, a)\xi(d, c)} [f(a + s\eta(b, a), c) \int_0^1 \left(\int_s^1 w(a \right. \\
& \quad \left. + s\eta(b, a), c + t\xi(d, c) \right) ds) dt]_0^1 \\
& - \frac{1}{\eta(b, a)\xi(d, c)} \int_0^1 \left(f(a + s\eta(b, a), c) \left(\int_0^1 w(a \right. \right. \\
& \quad \left. \left. + s\eta(b, a), c + t\xi(d, c) \right) dt \right) ds \\
& + \frac{1}{\eta(b, a)\xi(d, c)} \int_0^1 \left[\left(\int_s^1 w(a + s\eta(b, a), c \right. \right. \right. \\
& \quad \left. \left. \left. + t\xi(d, c) \right) ds \right) f(a + s\eta(b, a), c + t\xi(d, c)) \right]_0^1 dt \\
& + \frac{1}{\eta(b, a)\xi(d, c)} \int_0^1 \int_0^1 w(a \\
& \quad + s\eta(b, a), c + t\xi(d, c)) f(a + s\eta(b, a), c + t\xi(d, c)) ds dt.
\end{aligned}$$

After simplifying we get

$$\begin{aligned}
J_4 & = \frac{1}{\eta(b, a)\xi(d, c)} f(a, c) \int_0^1 \left(\int_0^1 w(a \right. \\
& \quad \left. + s\eta(b, a), c + t\xi(d, c) \right) ds) dt \\
& - \frac{1}{\eta(b, a)\xi(d, c)} \int_0^1 \left(f(a + s\eta(b, a), c) \left(\int_0^1 w(a \right. \right. \\
& \quad \left. \left. + s\eta(b, a), c + t\xi(d, c) \right) dt \right) ds.
\end{aligned}$$

$$\begin{aligned}
(29) \quad & + s\eta(b, a), c + t\xi(d, c))dt))ds \\
& - \frac{1}{\eta(b, a)\xi(d, c)} \int_0^1 \left(\int_0^1 w(a + s\eta(b, a), c \right. \\
& \left. + t\xi(d, c))ds \right) f(a, c + t\xi(d, c))dt \\
& + \frac{1}{\eta(b, a)\xi(d, c)} \int_0^1 \int_0^1 w(a + s\eta(b, a), c \\
& + t\xi(d, c))f(a + s\eta(b, a), c + t\xi(d, c))dsdt
\end{aligned}$$

Upon combining (26), (27), (28) and (29) we get (25). $\square$

Theorem 7. Let $K_1 \times K_2 \subseteq \mathbb{R}^2$ be an open invex subset with respect to $\eta : K_1 \times K_1 \rightarrow \mathbb{R}^2$ and $\xi : K_2 \times K_2 \rightarrow \mathbb{R}^2$. Suppose $f : K_1 \times K_2 \rightarrow \mathbb{R}^2$ is a twice differentiable mapping on $K_1 \times K_2$ such that $f_{st} \in L([a, a + \eta(b, a)] \times [c, c + \xi(d, c)])$ where $\eta(b, a) > 0$ and $\xi(d, c) > 0$. If $w : ([a, a + \eta(b, a)] \times [c, c + \xi(d, c)]) \rightarrow [0, \infty) \times [0, \infty)$ is an integrable mapping and $|f_{st}|$ is preinvex function then for every $a, b \in K_1$ and $c, d \in K_2$, the following inequality holds:

$$\begin{aligned}
(30) \quad & |[f(a, c + \xi(d, c)) + f(a + \eta(b, a), c + \xi(d, c)) + f(a + \eta(b, a), c) + f(a, c)] \\
& \times \int_0^1 \left(\int_0^1 w(a + s\eta(b, a), c + t\xi(d, c))ds \right) dt \\
& - 2 \int_0^1 \int_0^1 [f(a + s\eta(b, a), c + \xi(d, c)) + f(a + \eta(b, a), c + t\xi(d, c)) \\
& + (f(a, c + t\xi(d, c)) + f(a + s\eta(b, a), c))] \times (w(a + s\eta(b, a), c + t\xi(d, c)))dsdt \\
& + 4 \int_0^1 \int_0^1 w(a + s\eta(b, a), c + t\xi(d, c))f(a + s\eta(b, a), c + t\xi(d, c))dsdt| \\
& \leq \frac{\|w\|_\infty}{4} [|f_{st}(a, c)| + |f_{st}(a, d)| + |f_{st}(b, c)| + |f_{st}(b, d)|]
\end{aligned}$$

Proof. Taking modulus of both sides of (25), we get

$$\begin{aligned}
J_2 & = |[f(a, c + \xi(d, c)) + f(a + \eta(b, a), c + \xi(d, c)) + f(a + \eta(b, a), c) + f(a, c)] \\
& \times \int_0^1 \left(\int_0^1 w(a + s\eta(b, a), c + t\xi(d, c))ds \right) dt \\
& - 2 \int_0^1 \int_0^1 [f(a + s\eta(b, a), c + \xi(d, c)) + f(a + \eta(b, a), c + t\xi(d, c)) \\
& + (f(a, c + t\xi(d, c)) + f(a + s\eta(b, a), c))] \times (w(a + s\eta(b, a), c + t\xi(d, c)))dsdt \\
& + 4 \int_0^1 \int_0^1 w(a + s\eta(b, a), c + t\xi(d, c))f(a + s\eta(b, a), c + t\xi(d, c))dsdt| \\
& = |\eta(b, a)\xi(d, c) \int_0^1 \int_0^1 P(s, t)f_{ts}((a + s\eta(b, a), c + t\xi(d, c))dsdt|.
\end{aligned}$$

f_{st} is preinvex function, i.e

$$f_{st}(a + s\eta(b, a), c + t\xi(d, c)) \leq (1-s)(1-t)f_{st}(a, c) + (1-s)t f_{st}(a, d) \\ + (1-t)s f_{st}(b, c) + st f_{st}(b, d).$$

From preinvexity of $|f_{st}|$ in coordinates, we have

$$(31) \quad J_2 \leq \eta(b, a)\xi(d, c) \int_0^1 \left(\int_0^1 |P(s, t)|(1-s)(1-t)|f_{st}(a, c)| \right. \\ \left. + (1-s)t|f_{st}(a, d)| + (1-t)s|f_{st}(b, c)| + st|f_{st}(b, d)| \right) dt.$$

After a straightforward computation, we obtain

$$(32) \quad \|w\|_\infty \int_0^1 \int_0^1 [ts + t(1-s) + s(1-t) + (1-t)(1-s)] \\ \times (1-s)(1-t)|f_{st}(a, c)| ds dt = \frac{\|w\|_\infty |f_{st}(a, c)|}{4}$$

$$(33) \quad \|w\|_\infty \int_0^1 \int_0^1 [ts + t(1-s) + s(1-t) + (1-t)(1-s)] \\ \times (1-s)t|f_{st}(a, d)| ds dt = \frac{\|w\|_\infty |f_{st}(a, d)|}{4}$$

$$(34) \quad \|w\|_\infty \int_0^1 \int_0^1 [ts + t(1-s) + s(1-t) + (1-t)(1-s)] \\ \times (1-t)s|f_{st}(b, c)| ds dt = \frac{\|w\|_\infty |f_{st}(b, c)|}{4}$$

$$(35) \quad \|w\|_\infty \int_0^1 \int_0^1 [ts + t(1-s) + s(1-t) + (1-t)(1-s)] \\ \times ts|f_{st}(b, d)| ds dt = \frac{\|w\|_\infty |f_{st}(b, d)|}{4}.$$

Using the results of (32), (33), (34) and (35) in (31), we get required result (30). $\square$

Corollary 4. *If we take $\|w\|_\infty = 1$ in (30), we get similar inequality as in Theorem 2.1 of [11].*

Theorem 8. *Let $K_1 \times K_2 \subseteq \mathbb{R}^2$ be an open invex subset with respect to $\eta : K_1 \times K_1 \rightarrow \mathbb{R}^2$ and $\xi : K_2 \times K_2 \rightarrow \mathbb{R}^2$. Suppose $f : K_1 \times K_2 \rightarrow \mathbb{R}^2$ is a twice differentiable mapping on $K_1 \times K_2$ such that $f_{st} \in L([a, a + \eta(b, a)] \times [c, c + \xi(d, c)])$ where $\eta(b, a) > 0$ and $\xi(d, c) > 0$. If $w : ([a, a + \eta(b, a)] \times [c, c + \xi(d, c)]) \rightarrow [0, \infty) \times [0, \infty)$ is an integrable mapping and $|f_{st}|^q$ is preinvex function then for every $a, b \in K_1$ and $c, d \in K_2$, where $q \in (1, \infty)$ and $\frac{1}{p} + \frac{1}{q} =$*

1, then following inequality holds:

$$\begin{aligned}
 & |[f(a, c + \xi(d, c)) + f(a + \eta(b, a), c + \xi(d, c)) + f(a + \eta(b, a), c) + f(a, c)] \\
 & \times \int_0^1 (\int_0^1 w(a + s\eta(b, a), c + t\xi(d, c)) ds) dt \\
 & - 2 \int_0^1 \int_0^1 [f(a + s\eta(b, a), c + \xi(d, c)) + f(a + \eta(b, a), c + t\xi(d, c)) \\
 (36) \quad & + (f(a, c + t\xi(d, c)) + f(a + s\eta(b, a), c))] \times (w(a + s\eta(b, a), c + t\xi(d, c))) ds dt \\
 & + 4 \int_0^1 \int_0^1 w(a + s\eta(b, a), c + t\xi(d, c)) f(a + s\eta(b, a), c + t\xi(d, c)) ds dt| \\
 & \leq \eta(b, a) \xi(d, c) (\|w\|_\infty^p \frac{4}{(p+1)^2})^{\frac{1}{p}} \\
 & \times (\frac{1}{4} [|f_{st}(a, c)|^q + |f_{st}(a, d)|^q + |f_{st}(b, c)|^q + |f_{st}(b, d)|^q])^{\frac{1}{q}}.
 \end{aligned}$$

Proof. From Lemma 2.10 and Hölder's integral inequality, we get

$$\begin{aligned}
 J_2 & = |[f(a, c + \xi(d, c)) + f(a + \eta(b, a), c + \xi(d, c)) + f(a + \eta(b, a), c) + f(a, c)] \\
 & \times \int_0^1 (\int_0^1 w(a + s\eta(b, a), c + t\xi(d, c)) ds) dt \\
 & - 2 \int_0^1 \int_0^1 [f(a + s\eta(b, a), c + \xi(d, c)) + f(a + \eta(b, a), c + t\xi(d, c)) \\
 & + (f(a, c + t\xi(d, c)) + f(a + s\eta(b, a), c))] \times (w(a + s\eta(b, a), c + t\xi(d, c))) ds dt \\
 & + 4 \int_0^1 \int_0^1 w(a + s\eta(b, a), c + t\xi(d, c)) f(a + s\eta(b, a), c + t\xi(d, c)) ds dt| \\
 & = \eta(b, a) \xi(d, c) (\int_0^1 \int_0^1 |P(s, t)|^p ds dt)^{\frac{1}{p}} \\
 & \times (\int_0^1 \int_0^1 |f_{ts}((a + s\eta(b, a), c + t\xi(d, c))|^q ds dt)^{\frac{1}{q}}.
 \end{aligned}$$

From preinvexity of $|f_{st}|^q$ in coordinates, we have

$$\begin{aligned}
 J_2 & \leq \eta(b, a) \xi(d, c) (\int_0^1 \int_0^1 |P(s, t)|^p ds dt)^{\frac{1}{p}} \\
 & \times (\int_0^1 (\int_0^1 (1-s)(1-t) |f_{st}(a, c)|^q + (1-s)t |f_{st}(a, d)|^q \\
 (37) \quad & + (1-t)s |f_{st}(b, c)|^q + st |f_{st}(b, d)|^q) dt)^{\frac{1}{q}}.
 \end{aligned}$$

After a straightforward computation, we obtain

$$\int_0^1 \int_0^1 [(1-s)(1-t) |f_{st}(a, c)|^q + (1-s)t |f_{st}(a, d)|^q$$

$$\begin{aligned}
& + (1-t)s|f_{st}(b, c)|^q + st|f_{st}(b, d)|^q dsdt \\
(38) \quad & = \frac{1}{4} [|f_{st}(a, c)|^q + |f_{st}(a, d)|^q + |f_{st}(b, c)|^q + |f_{st}(b, d)|^q]
\end{aligned}$$

and

$$\begin{aligned}
& \int_0^1 \int_0^1 |P(s, t)|^p dsdt \leq (\|w\|_\infty^p \int_0^1 \int_0^1 t^p s^p dsds + \int_0^1 \int_0^1 t^p (1-s)^p dsdt \\
(39) \quad & + \int_0^1 \int_0^1 (1-t)^p s^p dsdt + \int_0^1 \int_0^1 (1-t)^p (1-s)^p dsdt) = \frac{4\|w\|_\infty^p}{(p+1)^2}.
\end{aligned}$$

Using the results of (38) and (39) in (37), we get required result (36). $\square$

Corollary 5. *If we take $\|w\|_\infty = 1$ in (36), we get similar inequality as in Theorem 2.2 of [11].*

Theorem 9. *Let $K_1 \times K_2 \subseteq \mathbb{R}^2$ be an open invex subset with respect to $\eta : K_1 \times K_1 \rightarrow \mathbb{R}^2$ and $\xi : K_2 \times K_2 \rightarrow \mathbb{R}^2$. Suppose $f : K_1 \times K_2 \rightarrow \mathbb{R}^2$ is a twice differentiable mapping on $K_1 \times K_2$ such that $f_{st} \in L([a, a + \eta(b, a)] \times [c, c + \xi(d, c)])$ where $\eta(b, a) > 0$ and $\xi(d, c) > 0$. If $w : ([a, a + \eta(b, a)] \times [c, c + \xi(d, c)]) \rightarrow [0, \infty) \times [0, \infty)$ is an integrable mapping and $|f_{st}|^q$ is preinvex function then for every $a, b \in K_1$ and $c, d \in K_2$, where $q \geq 1$, then following inequality holds:*

$$\begin{aligned}
& |[f(a, c + \xi(d, c)) + f(a + \eta(b, a), c + \xi(d, c)) + f(a + \eta(b, a), c) + f(a, c)] \\
& \times \int_0^1 (\int_0^1 w(a + s\eta(b, a), c + t\xi(d, c)) ds) dt \\
& - 2 \int_0^1 \int_0^1 [f(a + s\eta(b, a), c + \xi(d, c)) + f(a + \eta(b, a), c + t\xi(d, c)) \\
(40) \quad & + (f(a, c + t\xi(d, c)) + f(a + s\eta(b, a), c))] \times (w(a + s\eta(b, a), c + t\xi(d, c))) dsdt \\
& + 4 \int_0^1 \int_0^1 w(a + s\eta(b, a), c + t\xi(d, c)) f(a + s\eta(b, a), c + t\xi(d, c)) dsdt| \\
& \leq \eta(b, a) \xi(d, c) (\|w\|_\infty)^{1-\frac{1}{q}} \\
& \times (\|w\|_\infty \frac{[|f_{st}(a, c)|^q + |f_{st}(a, d)|^q + |f_{st}(b, c)|^q + |f_{st}(b, d)|^q]}{4})^{\frac{1}{q}}.
\end{aligned}$$

Proof. From Lemma 2.10 and Power mean inequality, we get

$$\begin{aligned}
J_2 & = |[f(a, c + \xi(d, c)) + f(a + \eta(b, a), c + \xi(d, c)) + f(a + \eta(b, a), c) + f(a, c)] \\
& \times \int_0^1 (\int_0^1 w(a + s\eta(b, a), c + t\xi(d, c)) ds) dt
\end{aligned}$$

$$\begin{aligned}
& - 2 \int_0^1 \int_0^1 [f(a + s\eta(b, a), c + \xi(d, c)) + f(a + \eta(b, a), c + t\xi(d, c)) \\
& + (f(a, c + t\xi(d, c)) + f(a + s\eta(b, a), c))] \times (w(a + s\eta(b, a), c + t\xi(d, c))) ds dt \\
& + 4 \int_0^1 \int_0^1 w(a + s\eta(b, a), c + t\xi(d, c)) f(a + s\eta(b, a), c + t\xi(d, c)) ds dt \\
& = \eta(b, a) \xi(d, c) \left(\int_0^1 \int_0^1 |P(s, t)| ds dt \right)^{1-\frac{1}{q}} \\
& \times \left(\int_0^1 \int_0^1 |P(s, t)| |f_{ts}((a + s\eta(b, a), c + t\xi(d, c)))|^q ds dt \right)^{\frac{1}{q}}.
\end{aligned}$$

From preinvexity of $|f_{st}|^q$ in coordinates, we have

$$\begin{aligned}
J_2 & \leq \eta(b, a) \xi(d, c) \left(\int_0^1 \int_0^1 |P(s, t)| ds dt \right)^{1-\frac{1}{q}} \\
& \times \left(\int_0^1 \left(\int_0^1 |P(s, t)| (1-s)(1-t) |f_{st}(a, c)|^q + (1-s)t |f_{st}(a, d)|^q \right. \right. \\
(41) \quad & \left. \left. + (1-t)s |f_{st}(b, c)|^q + st |f_{st}(b, d)|^q \right) dt \right)^{\frac{1}{q}}.
\end{aligned}$$

After a straightforward computation, we obtain

$$\begin{aligned}
& \|w\|_\infty \int_0^1 \int_0^1 [ts + t(1-s) + s(1-t) + (1-t)(1-s)] \\
(42) \quad & \times (1-s)(1-t) |f_{st}(a, c)|^q ds dt = \frac{\|w\|_\infty |f_{st}(a, c)|^q}{4}
\end{aligned}$$

$$\begin{aligned}
& \|w\|_\infty \int_0^1 \int_0^1 [ts + t(1-s) + s(1-t) + (1-t)(1-s)] \\
(43) \quad & \times (1-s)t |f_{st}(a, d)|^q ds dt = \frac{\|w\|_\infty |f_{st}(a, d)|^q}{4}
\end{aligned}$$

$$\begin{aligned}
& \|w\|_\infty \int_0^1 \int_0^1 [ts + t(1-s) + s(1-t) + (1-t)(1-s)] \\
(44) \quad & \times (1-t)s |f_{st}(b, c)|^q ds dt = \frac{\|w\|_\infty |f_{st}(b, c)|^q}{4}
\end{aligned}$$

$$\begin{aligned}
& \|w\|_\infty \int_0^1 \int_0^1 [ts + t(1-s) + s(1-t) + (1-t)(1-s)] \\
(45) \quad & \times ts |f_{st}(b, d)|^q ds dt = \frac{\|w\|_\infty |f_{st}(b, d)|^q}{4}
\end{aligned}$$

and

$$\begin{aligned}
& \int_0^1 \int_0^1 |P(s, t)| ds dt \\
& = \|w\|_\infty \left(\int_0^1 \int_0^1 \left(\int_0^t \left(\int_0^s ds \right) dt \right) ds ds + \int_0^1 \int_0^1 \left(\int_0^t \left(\int_s^1 ds \right) dt \right) ds dt \right)
\end{aligned}$$

$$\begin{aligned}
& + \int_0^1 \int_0^1 \left(\int_t^1 \left(\int_0^s ds \right) dt \right) ds dt + \int_0^1 \int_0^1 \left(\int_t^1 \left(\int_s^1 ds \right) dt \right) ds dt \\
(46) \quad & = \|w\|_\infty.
\end{aligned}$$

Using the results of (42), (43), (44), (45) and (46) in (41), we get required result (40). $\square$

Corollary 6. *If we take $\|w\|_\infty = 1$ in (40), we get similar inequality as in Theorem 2.3 of [11].*

3. Conclusion

We have obtained some new estimates for the lower and upper boundaries of Hermite-Hadamard-Fejér type inequalities for coordinated preinvex. We have also elaborated the results with special cases. Authors expect that the results of our paper will be inspiring for interested readers.

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Cocycles over top spaces

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Abstract. Evolution of a potential in a physical system can be determine by a cocycle. This essay contains the notion of cocycles on top spaces. We can associate a semigroup with an identity to each element of a manifold via a cocycle over a top space. This semigroup is a subsemigroup of a Lie group. We prove that these kind of semigroups are invariant under diffeomorphisms of manifolds. We study the concept of forward invariant set, global pullback absorbing set and global forward absorbing set for cocycles. We show that global attractors are persistence by a kind of topological equivalent relation on cocycles. We also define and study the concept of topological entropy for a sequence of cocycles.

Keywords: cocycle, physical system, attractor, entropy, top space, Schrodinger operator.

1. Introduction

Many essential physical systems can be considered by a diffeomorphism $f : M \rightarrow M$ and a smooth map $\varphi : M \rightarrow T$ where M is a smooth manifold and T is a Lie group. Potential of physical systems are the best candidates for φ . For example the map φ for a one dimensional harmonic oscillator is $\varphi(x) = \frac{m\omega^2}{2}x^2$, where m and ω are two constants. Lorenz gauge potential or Coulomb gauge potential [10] are two other examples for φ . The cocycle $\varphi_n(x) = \varphi(f^n(x))$ determines the evolution of potential on the orbit of x , where $x \in M$. For example the Schrodinger operator H [11] on the set of bi-sequences of real numbers of the form $p = \{\dots, p_{-1}, p_0, p_1, \dots\}$ with $\sum_{i \in \mathbb{Z}} |p_i|^2 < \infty$ is defined by $H(p) = q$ where $q_n = p_{n+1} + p_{n-1} + \varphi_n(p_n)$ and $\varphi_n(x) = \varphi(f^n(x))$ for a fixed $x \in M$ and a diffeomorphism $f : M \rightarrow M$.

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In this paper we replace T with a top space, and this means that we have possibility to choose new products. Top spaces are a kind of generalized lie groups which have been introduced in 2004 [5, 7, 6], generalized vector fields on top spaces create new kind of dynamics on them which are called complete semidynamical systems [8]. Top spaces are smooth manifolds which are also semigroups and each element of it has a special identity and a special inverse. Moreover identity and inverse mappings are smooth maps. In the next section we introduce cocycles [2] over top spaces and by using of them we associate a semigroup with an identity to each element of a manifold. When we work with a top space, then we have this possibility to consider generalized vector fields instead of vector fields. The image of a generalized vector field is a member of a one dimensional vector bundle on a manifold. Hence they may not be tangent to the ambient manifold. Physically when we restrict a vector field to a subset of a manifold then we deduce a generalized vector field [8]. The definition of a cocycle over a top space creates this ability to consider non-autonomous motions which their skew-products are generalized vector fields. Each top space can divide to a disjoint union of Lie groups and we deduce a class of semigroups with an identity via cocycles which each member of it, is a subset of a Lie group. One can use of the method presented in [8] to create a non-autonomous motion on M via this class of semigroups. Moreover, we show that diffeomorphic manifolds create isomorphic classes of such semigroups. In section 3, we introduce a new concept of forward invariant set, global pullback absorbing set and global forward absorbing set for cocycles. Also we introduce a new concept of global pullback attractors and global forward attractors for cocycles. We prove that global pullback attractors and global forward attractors are invariant objects under topological equivalent relation. In section 4, we study and define the notion of topological entropy for a sequence of cocycles.

2. Cocycles created by a top space

Top spaces are a kind of generalized Lie groups [7]. We recall that a smooth manifold T with a binary smooth operation

$$\begin{aligned} m : T \times T &\longrightarrow T \\ (a, b) &\mapsto ab \end{aligned}$$

is called a top space if it satisfies the following conditions.

- (i) (T, m) is a semigroup;
 - (ii) For given $a \in T$, there exists a unique $e(a) \in T$ so that $ae(a) = e(a)a = a$;
 - (iii) For given $a \in T$ there exists $a^{-1} \in T$ such that $aa^{-1} = a^{-1}a = e(a)$,
- and the inverse mapping

$$\begin{aligned} in : T &\longrightarrow T \\ a &\mapsto a^{-1} . \end{aligned}$$

is a smooth map.

One of the most important class of top spaces is the class of top spaces created by Rees matrix semigroups [12]. Let's explain it. We assume G is a Lie group, A and B are two smooth manifolds, and $p : A \times B \rightarrow G$ is a smooth mapping. It is well known that $B \times G \times A$ with the product $(n, a, s)(k, b, l) = (n, ap(s, k)b, l)$ is a semigroup, which is called Rees matrix semigroup. One can easily prove that it is also a top space [7].

We assume that M is a smooth manifold and $f : M \rightarrow M$ is a smooth diffeomorphism. We also assume that $\varphi : M \rightarrow T$ is a smooth map and the following diagram commutes.

$$\begin{array}{ccc} M & \xrightarrow{f} & M \\ \varphi \downarrow & & \downarrow \varphi \\ T & \xrightarrow{e} & T \end{array}$$

where e is the identity map of T .

Definition 2.1. A cocycle created by (φ, f) is the sequence of mappings $\{\varphi_n : M \rightarrow T \mid n \in \mathbb{Z}\}$, where φ_n is defined by

$$\varphi_n(x) = \begin{cases} \varphi(f^{n-1}(x)) \dots \varphi(f(x)) \varphi(x), & \text{if } n > 0 \\ (\varphi(f^{-n}(x)))^{-1} \dots (\varphi(f^{-1}(x)))^{-1}, & \text{if } n < 0. \\ e(\varphi(x)), & \text{if } n = 0 \end{cases}$$

We present an equivalent definition for a cocycle in the following. A smooth mapping $\alpha : \mathbb{Z} \times M \rightarrow T$ is a cocycle if $\alpha(0, x) = e(\alpha(1, x)) = e(\alpha(1, f(x)) = \dots = e(\alpha(1, f^n(x)))$, $\alpha(m + n, x) = \alpha(m, f^n(x)) \cdot \alpha(n, x)$, for all $n, m \in \mathbb{Z}$, and $x \in M$. To prove the equivalency of these two definitions it is enough to define $\varphi_n(x) = \alpha(n, x)$.

Since the tuple (φ, f) creates a cocycle, then we also use of the name cocycle for (φ, f) . (φ, f) is abelian if T is an abelian group. An abelian cocycle (φ, f) is equivalent to the notion of cocycle which is defined by Avila A., Santamaria A., Viana M., and Wilkinson A. in [2].

For given $x \in M$ we take $C_\varphi(x) = \{\prod_{i=1}^M \varphi_{n_i}(x) \mid M \in \mathbb{N}\}$, and we define the following product on it.

$$(\prod_{j=1}^K \varphi_{k_j} \varphi_{s_j}) (\prod_{v=1}^D \varphi_{l_v} \varphi_{r_v}) = \prod_{t=1}^{K+D} \varphi_{w_t} \varphi_{v_t},$$

where

$$w_t = \begin{cases} k_j, & \text{for } 1 \leq t = j \leq K \\ l_u, & \text{for } 1 \leq t = K + u \leq K + D \end{cases}$$

and

$$v_t = \begin{cases} s_j, & \text{for } 1 \leq t = j \leq K \\ r_u, & \text{for } 1 \leq t = K + u \leq K + D \end{cases}.$$

$C_\varphi(x)$ with the above product is a semigroup.

Since for given $n \in \mathbb{N}$

$$e(\varphi(x)) = e(\varphi(f(x))) = e(\varphi(f^2(x))) = \dots = e(\varphi(f^n(x))),$$

then

$$\begin{aligned} \varphi_n(x)e(\varphi(x)) &= \varphi(f^{n-1}(x))\dots\varphi(f(x))\varphi(x)e(\varphi(x)) = \varphi(f^{n-1}(x))\dots\varphi(f(x))\varphi(x) \\ &= \varphi_n(x) \\ e(\varphi(x))\varphi_n(x) &= e(\varphi(f^{n-1}(x)))\varphi_n(x) = e(\varphi(f^{n-1}(x)))\varphi(f^{n-1}(x))\dots\varphi(f(x))\varphi(x) \\ &= \varphi(f^{n-1}(x))\dots\varphi(f(x))\varphi(x) = \varphi_n(x). \end{aligned}$$

Moreover, for given $-n \in \mathbb{N}$ we have $e(\varphi(x)) = e(\varphi(x))^{-1} = e(\varphi(f(f^{-1}(x))))^{-1} = e(\varphi(f(f^{-1}(x)))) = e(\varphi(f^{-1}(x))) = e(\varphi(f^{-1}(x)))^{-1} = \dots = e(\varphi(f^n(x)))^{-1}$.

Thus

$$\begin{aligned} \varphi_n(x)e(\varphi(x)) &= (\varphi(f^n(x)))^{-1}\dots(\varphi(f^{-1}(x)))^{-1}e(\varphi(x)) \\ &= (\varphi(f^n(x)))^{-1}\dots(\varphi(f^{-1}(x)))^{-1}e(\varphi(f^{-1}(x)))^{-1} \\ &= (\varphi(f^{-n}(x)))^{-1}\dots(\varphi(f^{-1}(x)))^{-1} = \varphi_n(x)e(\varphi(x))\varphi_n(x) \\ &= e(\varphi(f^n(x)))^{-1}\varphi_n(x) \\ &= e(\varphi(f^n(x)))^{-1}(\varphi(f^n(x)))^{-1}\dots(\varphi(f^{-1}(x)))^{-1} \\ &= (\varphi(f^n(x)))^{-1}\dots(\varphi(f^{-1}(x)))^{-1} = \varphi_n(x). \end{aligned}$$

In the case $n = 0$ we have

$$\begin{aligned} \varphi_0(x)e(\varphi(x)) &= e(\varphi(x))e(\varphi(x)) = e(\varphi(x)) = \varphi_0(x), \\ e(\varphi(x))\varphi_0(x) &= e(\varphi(x))e(\varphi(x)) = e(\varphi(x)) = \varphi_0(x). \end{aligned}$$

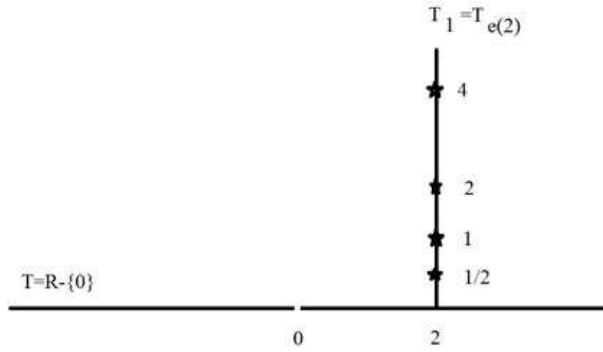
Hence, $C_\varphi(x)$ is a semigroup with the identity $e(\varphi(x))$. This implies $C_\varphi(x)$ is a subsemigroup of the Lie group $T_{e(\varphi(x))} = e^{-1}(\{e(\varphi(x))\})$. In the next example we show that $C_\varphi(x)$ may not be equal to $T_{e(\varphi(x))}$.

Example 2.1. Take $M = \mathbb{R} - \{0\}$, and $T = \mathbb{R} - \{0\}$. T with the product $a.b = a|b|$ as an open subset of the real manifold $\mathbb{R}$ is a top space. We only explain the associative property of this product. If $a, b, c \in T$ then

$$(a.b).c = (a|b|)|c| = a|bc| = a|b|c| = a|b.c| = a.(b.c).$$

If $f(x) = 2x$ and $\varphi = id$ then

$$\begin{aligned} \varphi_1(x) &= \varphi(x) = x, \\ \varphi_2(x) &= \varphi(f(x))\varphi(x) = 2x.x = 2x \mid x \mid, \\ \varphi_3(x) &= \varphi(f^2(x))\varphi_2(x) = 4x.2x \mid x \mid = 8x \mid x^2 \mid, \\ &\vdots \end{aligned}$$

Figure 1: Stars denote the points of $C_\varphi(2)$.

$$\varphi_0(x) = \begin{cases} 1, & \text{if } x > 0 \\ -1, & \text{if } x < 0 \end{cases}.$$

$$\varphi_{-1}(x) = (\varphi(f^{-1}(x)))^{-1} = \left(\frac{1}{2}x\right)^{-1} = \frac{2}{x},$$

$$\varphi_{-2}(x) = (\varphi(f^{-2}(x)))^{-1}(\varphi(f^{-1}(x)))^{-1} = \frac{4}{x} \cdot \frac{2}{x} = \frac{8}{x} \cdot \left| \frac{1}{x} \right|,$$

$$\varphi_{-3}(x) = (\varphi(f^{-3}(x)))^{-1}(\varphi_{-2}(x))^{-1} = \frac{8}{x} \cdot \frac{8}{x} \cdot \left| \frac{1}{x} \right| = \frac{64}{x} \cdot \left| \frac{1}{x^2} \right|,$$

$$\vdots$$

For $x = 2$, we have $C_\varphi(2) = \{2^m \mid m \in \mathbb{Z}\}$, and $C_\varphi(2)$ is not equal to $T_{e(\varphi(2))} = T_1 = e^{-1}\{1\} = \{x > 0 \mid x \in \mathbb{R}\}$ (see figure 1).

In the next example we show that $C_\varphi(x)$ can be sometimes equal to $T_{e(\varphi(x))}$.

Example 2.2. Let M be a manifold, and $T = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{R} \right\}$.

Then T with the product $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & r \\ g & h \end{bmatrix} = \begin{bmatrix} a & r \\ g & d \end{bmatrix}$ and the Euclidean topology is a top space. For a given smooth map $\varphi : M \longrightarrow T$, and for each diffeomorphism $f : M \longrightarrow M$ with $e(\varphi(f(x))) = e(\varphi(x))$ we have

$$C_\varphi(x) = \{e(\varphi(x))\} = \{\varphi(x)\} = T_{e(\varphi(x))}.$$

To prove this, we see that $(\varphi(f^{-1}(x)))^{-1} = e(\varphi(f^{-1}(x)))^{-1} = e(\varphi(f^{-1}(x))) = e\varphi(f(f^{-1}(x))) = e(\varphi(x)) = \varphi(x)$, and $\varphi(f(x)) = e(\varphi(f(x))) = e(\varphi(x)) = \varphi(x)$. So, $\varphi_n(x) = \varphi(x)$, for all $n \in \mathbb{Z}$.

Theorem 2.3. Suppose (φ, f) is a cocycle on T , and the mapping $\psi : M' \longrightarrow M$ is a diffeomorphism. Then the following diagram commutes and $C_\varphi(x) = C_{\varphi \circ \psi}(\psi^{-1}(x))$.

$$\begin{array}{ccc}
 M' & \xrightarrow{\psi^{-1} \circ \psi} & M' \\
 \varphi \circ \psi \downarrow & & \downarrow \varphi \circ \psi \\
 T & & T \\
 & \searrow e & \swarrow e \\
 & T &
 \end{array}$$

Proof. If $x \in M'$ then $e \circ (\varphi \circ \psi) \circ (\psi^{-1} \circ f \circ \psi)(x) = e \circ \varphi \circ f \circ \psi(x) = e \circ \varphi \circ \psi(x) = e(\varphi \circ \psi)(x)$. So, the above diagram is a commutative one. Now we prove the equality of $C_\varphi(x)$ with $C_{\varphi \circ \psi}(\psi^{-1}(x))$, where $x \in M$. For given $x \in M$ and $n \in \mathbb{Z}$ we have

$$\begin{aligned}
 & (\varphi \circ \psi)_n(\psi^{-1}(x)) \\
 = & \begin{cases} \varphi \circ \psi(\psi^{-1} \circ f \circ \psi)^{n-1}(\psi^{-1}(x)) \dots \varphi \circ \psi(\psi^{-1} \circ f \circ \psi)(\psi^{-1}(x)) \varphi \circ \psi(\psi^{-1}(x)), & \text{if } n > 0 \\ (\varphi \circ \psi(\psi^{-1} \circ f \circ \psi)^n(\psi^{-1}(x)))^{-1} \dots (\varphi \circ \psi(\psi^{-1} \circ f \circ \psi)^{-1}(\psi^{-1}(x)))^{-1}, & \text{if } n < 0 \\ e(\varphi \circ \psi)(\psi^{-1}(x)), & \text{if } n = 0 \end{cases} \\
 = & \begin{cases} \varphi(f^{n-1}(x)) \dots \varphi(f(x)) \varphi(x), & \text{if } n > 0 \\ (\varphi(f^n(x)))^{-1} \dots (\varphi(f^{-1}(x)))^{-1}, & \text{if } n < 0 \\ e(\varphi(x)), & \text{if } n = 0 \end{cases} = \varphi_n(x). \quad \square
 \end{aligned}$$

3. Attractors

In this section $Z : M \longrightarrow 2^T$ is a mapping such that $Z(x)$ is a bounded set in the top space T . For all $x \in M$, Z is called a non-autonomous set for a cocycle (φ, f) if $f^{-1}(Z^{-1}(Z(x))) = Z^{-1}(Z(f^{-1}(x)))$. We say that Z is a compact non-autonomous set if for all $x \in M$, $Z(x)$ is a compact set. Z is called a forward invariant set if $\varphi_n(Z^{-1}(Z(x))) \subseteq Z(f^n(x))$, for all $x \in M$ and $n \geq 0$.

A forward invariant set Z for a cocycle (φ, f) is called a global pullback absorbing set if for all $x \in M$, there exist $N \in \mathbb{N}$ such that $\varphi_n(f^{-n}(Z^{-1}(Z(x)))) \subseteq Z(x)$, for all $n \geq N$.

A forward invariant set Z for a cocycle (φ, f) is called a global forward absorbing set if for all $x \in M$, there is $N \in \mathbb{N}$ such that $\varphi_{-n}(f^n(Z^{-1}(Z(x)))) \subseteq Z(x)$, for all $n \geq N$.

Definition 3.1. Let Z be a compact global pullback absorbing set. Then $A : M \longrightarrow 2^T$ defined by $A(x) = \bigcap_{n=0}^{\infty} \overline{\bigcup_{m \geq n} \varphi_m(f^{-m}(Z^{-1}(Z(x))))}$ is called a global pullback attractor for a cocycle (φ, f) .

In the next theorem we prove that a global pullback attractor is an invariant object under forward motion.

Theorem 3.1. *If A is a global pullback attractor for a cocycle (φ, f) , then A is a forward invariant set .*

Proof. Assume that $\varphi_k(A^{-1}(A(x))) \not\subseteq A(f^k(x))$, then there is $y \in \varphi_k(A^{-1}(A(x)))$ such that $y \notin A(f^k(x))$. So, there exists n such that for each $m \geq n$

$$y \notin \overline{\bigcup_{m \geq n} \varphi_m(f^{-m}(Z^{-1}(Z(f^k(x)))))}.$$

Thus, $y \notin \bigcup_{m \geq n} \varphi_m(f^{-m}(Z^{-1}(Z(f^k(x)))))$. Hence, there is m so that $y \notin \varphi_m(f^{-m}(Z^{-1}(Z(f^k(x)))))$.

Therefore, there exist m such that $y \notin \varphi_m(Z^{-1}(Z(f^{k-m}(x)))) \subseteq Z(f^k(x))$ and it is a contradiction because Z is a forward invariant set. $\square$

Definition 3.2. *Let Z be a compact global forward absorbing set. Then $A : M \rightarrow 2^T$ defined by $A(x) = \bigcap_{n=0}^{\infty} \overline{\bigcup_{m \geq n} \varphi_{-m}(f^m(Z^{-1}(Z(x))))}$ is called a global forward attractor for a cocycle (φ, f) .*

By the same method of the proof of Theorem 3.2 one can prove the following theorem.

Theorem 3.2. *If A is a global forward attractor for a cocycle (φ, f) , then A is a forward absorbing set .*

Note that $A(x)$ is a compact set because it is a closed subset of the compact set $Z(x)$.

Assume (φ_1, f_1) and (φ_2, f_2) are two cocycles on T . These cocycles are equivalent if there exist a homeomorphism $h : M \rightarrow M$, a homeomorphism $L : T \rightarrow T$, and an increasing homeomorphism $S : \mathbb{Z} \rightarrow \mathbb{Z}$ with $S(0) = 0$ such that $h(f_1^n(x)) = f_2^{S(n)}(h(x))$, and $L(\varphi_{1n}(x)) = \varphi_{2S(n)}(h(x))$. We have a natural extension map $\tilde{L} : 2^T \rightarrow 2^T$ that defined by $\tilde{L}(A) = \{L(a) : a \in A\}$.

Lemma 3.1. *If (φ_1, f_1) and (φ_2, f_2) are two equivalent cocycles on T , and if $Z : M \rightarrow T$ is a global pullback absorbing set for (φ_1, f_1) , then a map $\tilde{L} \circ Z \circ h^{-1} : M \rightarrow \tilde{L}(2^T)$ is also a global pullback absorbing set for the cocycle (φ_2, f_2) .*

Proof. Since Z is a forward invariant set for a cocycle (φ_1, f_1) , then

$$\varphi_{1n}(Z^{-1}(Z(x))) \subseteq Z(f_1^n(x))$$

for all $x \in M$ and $n \geq 0$. For given $h(x) \in M$ and $S(n) > 0$ we have

$$\begin{aligned} \varphi_{2S(n)}((\tilde{L} \circ Z \circ h^{-1})^{-1}(\tilde{L} \circ Z \circ h^{-1})(h(x))) &= \varphi_{2S(n)}(h(Z^{-1}(Z(x)))) \\ &= L\varphi_{1n}(Z^{-1}(Z(x))) \subseteq L(Z(f_1^n(x))) = \tilde{L}(Z(f_1^n(x))) \\ &= (\tilde{L} \circ Z \circ h^{-1} \circ h \circ f_1^n)(x) = (\tilde{L} \circ Z \circ h^{-1})(h \circ f_1^n(x)) \\ &= (\tilde{L} \circ Z \circ h^{-1})(f_2^{S(n)}(h(x))). \end{aligned}$$

Thus, $\tilde{L} \circ Z \circ h^{-1}$ is a forward invariant set for a cocycle (φ_2, f_2) . If $h(x) \in M$, then there is $N \in \mathbb{N}$ such that $\varphi_{1n}(f_1^{-n}(Z^{-1}(Z(x)))) \subseteq Z(x)$ for all $n \geq N$. For $S(n) \geq S(N)$ we have

$$\begin{aligned} & \varphi_{2S(n)}(f_2^{-S(n)}(\tilde{L} \circ Z \circ h^{-1})^{-1}(\tilde{L} \circ Z \circ h^{-1})(h(x))) \\ &= \varphi_{2S(n)}(f_2^{-S(n)}(h \circ Z^{-1} \circ \tilde{L}^{-1}) \circ \tilde{L} \circ Z \circ h^{-1} \circ h(x)) \\ &= \varphi_{2S(n)}(f_2^{-S(n)}(h(Z^{-1}(Z(x)))) = \varphi_{2S(n)}(h(f_1^{-n}(Z^{-1}(Z(x)))) \\ &= L(\varphi_{1n}(f_1^{-n}(Z^{-1}(Z(x)))) = L(\varphi_{1n}(f_1^{-n}(Z^{-1}(Z(x)))) \subseteq L(Z(x)) = \tilde{L}Z(x) \\ &= (\tilde{L} \circ Z \circ h^{-1} \circ h)(x) = (\tilde{L} \circ Z \circ h^{-1})(h(x)). \quad \square \end{aligned}$$

Theorem 3.3. *If we have two equivalence cocycles (φ_1, f_1) and (φ_2, f_2) , and if $A^{(\varphi_1, f_1)}$ is a global pullback attractor for the cocycle (φ_1, f_1) , then $\tilde{L} \circ A^{(\varphi_1, f_1)}$ is also a global pullback attractor for the cocycle (φ_2, f_2) .*

Proof. Let $Z : M \rightarrow 2^T$ be a compact global pullback absorbing set for (φ_1, f_1) , and $A^{(\varphi_1, f_1)}(x) = \bigcap_{n=0}^{\infty} \overline{\bigcup_{m \geq n} \varphi_{1m}(f_1^{-m}(Z^{-1}(Z(x))))}$. Then Lemma 3.5 implies $\tilde{L} \circ Z \circ h^{-1}$ is a compact global pullback absorbing set for the cocycle (φ_2, f_2) . So,

$$\begin{aligned} (\tilde{L} \circ A^{(\varphi_1, f_1)})(x) &= \bigcap_{n=0}^{\infty} \overline{\bigcup_{m \geq n} \tilde{L} \circ \varphi_{1m}(f_1^{-m}(Z^{-1}(Z(x))))} \\ &= \bigcap_{n=0}^{\infty} \overline{\bigcup_{m \geq n} L \circ \varphi_{1m}(f_1^{-m}(Z^{-1}(Z(x))))} \\ &= \bigcap_{n=0}^{\infty} \overline{\bigcup_{m \geq n} \varphi_{2S(m)}(h f_1^{-m}(Z^{-1}(Z(x))))} \\ &= \bigcap_{n=0}^{\infty} \overline{\bigcup_{m \geq n} \varphi_{2S(m)}(f_2^{-S(m)}(h(Z^{-1}(Z(x))))} \\ &= \bigcap_{n=0}^{\infty} \overline{\bigcup_{m \geq n} \varphi_{2S(m)}(f_2^{-S(m)}(\tilde{L} \circ Z \circ h^{-1})^{-1}(\tilde{L} \circ Z \circ h^{-1})(h(x))))} \\ &= A^{(\varphi_2, f_2)}(h(x)). \quad \square \end{aligned}$$

One can prove the following lemma and theorem similarly

Lemma 3.2. *If (φ_1, f_1) and (φ_2, f_2) are two equivalent cocycles, and if $Z : M \rightarrow T$ is a global forward absorbing set for (φ_1, f_1) , then a map $\tilde{L} \circ Z \circ h^{-1} : M \rightarrow \tilde{L}(2^T)$ is also a global forward absorbing set for the cocycle (φ_2, f_2) .*

Theorem 3.4. *If we have two equivalent cocycles (φ_1, f_1) and (φ_2, f_2) , and if $A^{(\varphi_1, f_1)}$ is a global forward attractor for the cocycle (φ_1, f_1) , then $\tilde{L} \circ A^{(\varphi_1, f_1)}$ is a global forward attractor for the cocycle (φ_2, f_2) .*

4. Topological entropy of a cocycle

The notion of topological entropy was introduced by Konheim, Adler and McAndrew [1]. This notion has been considered by Kolmogorov [3] and then by Sinai [13] from measure theoretical point of view. Rudolf Clausius was the first scientist who introduced entropy to describe the use of dissipative energy of a thermodynamic system during a change of the state [9]. In this section we study the notion of topological entropy for a cocycle on a compact top space [4]. Assume T is a compact top space, $f : T \rightarrow T$ is a smooth diffeomorphism, $\varphi : T \rightarrow T$ is a smooth map and $e(\varphi(f(x))) = e(\varphi(x))$. We would like to define topological entropy for a cocycle created by φ and f . Let $\{\varphi_i : i \in \mathbb{Z}\}$ be the cocycle created by (φ, f) . We define $\varphi_{1,\infty} = \{\varphi_1, \varphi_2, \dots\} = \{\varphi_i\}_{i=1}^\infty$. $\varphi_i^0 = \varphi_i^{-0}$ is the identity function and for any $m \in \mathbb{N}$, we define $\varphi_i^m := \varphi_{i+m-1}$, $\varphi_i^{-1} := \varphi_{-i}$, $\varphi_i^{-m} := (\varphi_i^m)^{-1}$. Finally, we denote $\{\varphi_{in+1}^n\}_{i=0}^\infty$ by $\varphi_{1,\infty}^n$ and $\{\varphi_i^{-1}\}_{i=1}^\infty$ by $\varphi_{1,\infty}^{-1}$. The reader must pay attention to this point that, φ_i^{-1} is only a symbol and it is not the inverse of φ_i . Consider an open cover $\mathcal{U}$ of T . For two open covers $\mathcal{U}, \mathcal{V}$, a new open cover $\mathcal{U} \vee \mathcal{V}$ is defined by $\{U \cap V \mid U \in \mathcal{U}, V \in \mathcal{V}\}$. So, for open covers $\mathcal{U}_i$, $\bigvee_{i=1}^n \mathcal{U}_i = \mathcal{U}_1 \vee \mathcal{U}_2 \vee \dots \vee \mathcal{U}_n = \{U_1 \cap U_2 \cap \dots \cap U_n \mid U_1 \in \mathcal{U}_1, U_2 \in \mathcal{U}_2, \dots, U_n \in \mathcal{U}_n\}$ is an open cover of T . For an open cover $\mathcal{U}$ we denote $\varphi_i^{-m}(\mathcal{U}) = \{(\varphi_i^m)^{-1}(U) \mid U \in \mathcal{U}\}$ and $\mathcal{U}_i^m(\varphi_{1,\infty}) = \mathcal{U}_i^m = \bigvee_{j=0}^{m-1} \varphi_i^{-j}(\mathcal{U})$.

We denote by $\mathcal{N}(\mathcal{U})$ the minimal cardinality of the subcovers of $\mathcal{U}$. The topological entropy of an open cover $\mathcal{U}$ of T is denoted by $H(\mathcal{U})$ and it is defined by $\log \mathcal{N}(\mathcal{U})$. We define the entropy of a cocycle with respect to an open cover $\mathcal{U}$ by $h(\varphi_{1,\infty}, \mathcal{U}_{1,\infty}) := \limsup_{m \rightarrow \infty} \frac{1}{m} \log \mathcal{N}(\mathcal{U}_1^m)$, and the topological entropy of the cocycle by

$$h(\varphi_{1,\infty}) := \{\sup(h(\varphi_{1,\infty}, \mathcal{U}) \mid \mathcal{U} \text{ is an open cover of } T)\}.$$

$h(\varphi_{1,\infty})$ denotes the value of the complexity of $\varphi_{1,\infty}$. In fact it denotes the value of the chaotic behavior of $\varphi_{1,\infty}$.

We have the following inequalities:

- (1) $\mathcal{N}(\mathcal{U} \vee \mathcal{V}) \leq \mathcal{N}(\mathcal{U}) \cdot \mathcal{N}(\mathcal{V})$;
- (2) $\mathcal{N}(\varphi_i^{-m}(\mathcal{U})) \leq \mathcal{N}(\mathcal{U})$.

Let $\mathcal{U}$ be finer than $\mathcal{V}$ i.e. each element of $\mathcal{U}$ is contained in some element of $\mathcal{V}$. Then

- (3) $\mathcal{N}(\mathcal{U}) \geq \mathcal{N}(\mathcal{V})$.

If $\mathcal{U}$ is finer than $\mathcal{V}$ then we use of the symbol $\mathcal{U} \geq \mathcal{V}$.

Clearly if $\mathcal{U} \geq \mathcal{V}$ then $h(\varphi_{1,\infty}, \mathcal{U}) \geq h(\varphi_{1,\infty}, \mathcal{V})$.

If $\mathcal{U}$ is an open finite cover of T then the cardinality of $\mathcal{U}_i^m$ is at most $(\text{cardinal } \mathcal{U})^m$. Therefore $h(\varphi_{1,\infty}, \mathcal{U}) \leq \log \text{card } \mathcal{U}$ and so $0 \leq h(\varphi_{1,\infty}, \mathcal{U}) < \infty$.

Theorem 4.1. *Let $\varphi_{1,\infty}$ be a cocycle on T . Then*

$$h(\varphi_{1,\infty}^n) \leq n \cdot h(\varphi_{1,\infty})$$

for every $n \geq 1$.

Proof. Fix n . For any open cover $\mathcal{U}$ of T we have

$$\begin{aligned}
 h(\varphi_{1,\infty}, \mathcal{U}) &= \limsup_{k \rightarrow \infty} \frac{1}{k} \log \mathcal{N}(\mathcal{U}_1^k) \geq \limsup_{m \rightarrow \infty} \frac{1}{mn} \log \mathcal{N}(\mathcal{U}_1^{mn}) \\
 &= \limsup_{m \rightarrow \infty} \frac{1}{mn} \log \mathcal{N}(\bigvee_{j=0}^{mn-1} \varphi_1^{-j}(\mathcal{U})) \\
 &= \limsup_{m \rightarrow \infty} \frac{1}{mn} \log \mathcal{N}(\mathcal{U} \vee \varphi_1^{-1}(\mathcal{U}) \vee \varphi_1^{-2}(\mathcal{U}) \vee \dots \vee \varphi_1^{-(mn-1)}(\mathcal{U})) \\
 &\geq \limsup_{m \rightarrow \infty} \frac{1}{mn} \log \mathcal{N}(\mathcal{U} \vee \varphi_1^{-n}(\mathcal{U}) \vee \varphi_1^{-2n}(\mathcal{U}) \vee \dots \vee \varphi_1^{-(m-1)n}(\mathcal{U})) \\
 &= \frac{1}{n} \limsup_{m \rightarrow \infty} \frac{1}{m} \log \mathcal{N}(\mathcal{U} \vee (\varphi_n)^{-1}(\mathcal{U}) \vee (\varphi_{2n})^{-1}(\mathcal{U}) \vee \dots \vee (\varphi_{mn})^{-1}(\mathcal{U})) \\
 &= \frac{1}{n} h(\varphi_{1,\infty}^n, \mathcal{U}).
 \end{aligned}$$

Hence $h(\varphi_{1,\infty}^n) \leq n.h(\varphi_{1,\infty})$. □

Theorem 4.2. Let $\varphi_{1,\infty}$ be a cocycle over T . Then, for every $1 \leq i \leq j < \infty$ and every open cover $\mathcal{U}$ of T , $h(\varphi_{i,\infty}, \mathcal{U}) \leq h(\varphi_{j,\infty}, \mathcal{U})$ and $h(\varphi_{i,\infty}) \leq h(\varphi_{j,\infty})$.

Proof. For any open cover $\mathcal{U}$ of T and $i \geq 1$ we have

$$\begin{aligned}
 \mathcal{U}_i^n &= \mathcal{U} \vee \varphi_i^{-1}(\mathcal{U}) \vee \varphi_i^{-2}(\mathcal{U}) \vee \dots \vee \varphi_i^{-(n-1)}(\mathcal{U}) \\
 &= \mathcal{U} \vee \varphi_i^{-1}(\mathcal{U}) \vee \varphi_{i+1}^{-1}(\mathcal{U}) \vee \varphi_{i+2}^{-1}(\mathcal{U}) \vee \dots \vee \varphi_{i+n-2}^{-1}(\mathcal{U}) \\
 &= \varphi_i^{-1}(\mathcal{U}) \vee \mathcal{U} \vee \varphi_{i+1}^{-1}(\mathcal{U}) \vee \varphi_{i+1}^{-2}(\mathcal{U}) \vee \dots \vee \varphi_{i+1}^{-(n-2)}(\mathcal{U}) \\
 &= \varphi_i^{-1}(\mathcal{U}) \vee \mathcal{U}_{i+1}^{n-1}(\mathcal{U}).
 \end{aligned}$$

So, by monotonicity we have

$$\begin{aligned}
 h(\varphi_{i,\infty}, \mathcal{U}) &= \limsup_{n \rightarrow \infty} \frac{1}{n} \log \mathcal{N}(\mathcal{U}_i^n) \\
 &\leq \limsup_{n \rightarrow \infty} \frac{1}{n} \log(\mathcal{N}(\mathcal{U}).\mathcal{N}(\mathcal{U}_{i+1}^{n-1})) \\
 &= \limsup_{n \rightarrow \infty} \frac{1}{n-1} \log \mathcal{N}(\mathcal{U}_{i+1}^{n-1}) = h(\varphi_{i+1,\infty}, \mathcal{U}). \quad \square
 \end{aligned}$$

5. Conclusion

We consider the concept of cocycles via top spaces and we can associate an algebraic object to each element of a manifold. We also study the concept of forward invariant set, global forward absorbing set and global pullback absorbing set for cocycles. In this direction we have the following two questions.

1. What kind of invariant sets can be a candidate for being a global forward or pullback attractor?

2. If $f : M \longrightarrow M$ is a diffeomorphism is the equality $C_\varphi(x) = C_\varphi(f(x))$ true?

In section four we extend the concept of entropy for cocycles, and we prove the essential properties of it.

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The inverse exponential Rayleigh distribution and related concepts

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Abstract. The main aim of this paper is to introduce a new two-parameter distribution called the inverse exponential Rayleigh distribution (IERD) with the investigated of its statistical properties. We introduce some mathematical complexity of statistical properties such as the moment function, moment generating function, factorial moments generating function, quantity function, skewness, and Kurtosis. We utilize Taylor series expansion to obtain these statistical properties. The process establishes the new distribution relied mainly on the survival functions of both the exponential and Rayleigh distributions. For the application, real data were applied to compare the new distribution with some other distributions using the Akaike information criterion (AIC). It has been indicated that the suggested method is more efficient and high accuracy by the convergence of the real data.

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Keywords: exponential distribution, exponential Rayleigh distribution, inverse distribution, Rayleigh distribution, survival functions.

1. Introduction

It is well known that there are countless distributions in the statistical field, the development and complexity that occur in the information extracted from the statistical data. Especially in the areas that are concerned with the survival and reliability analysis necessitated the existence of new distributions to address the problems in the data that could not be handled by existing statistical distributions. This research was based on a new approach in developing a mix of distributions for use in statistical techniques, survival, and reliability studies. This approach was developed by Gauss M. Cordeiro [1], he introduced the exponential–Weibull distribution, where the mix between the exponential and Weibull distributions was made according to survival functions of both distributions. Suleman N. [2], using this technique to developed serial Weibull Rayleigh distribution. Gadde S. and Sauda. M. [3] used generalization technical to present exponentiated inverse Rayleigh distribution which is a generalized form of inverse Rayleigh distribution. Saralees N. [4] developed a new method using the generalized form to introduce the exponentiation type distributions. Muhammad S. Khan [5] studied two parameters of the modified inverse Rayleigh distribution which is a generalizing of inverse Rayleigh distribution. Pelumi E. and el at [6] used the inverse distribution technic to develop The Gompertz inverse exponential distribution. Maha A. [7] proposed a new modal called inverse Weibull inverse exponential distribution as a special case of inverse Weibull-G distribution. Mayssa J. Mohammed [8] proposed a new mixture distribution based on the same approach and developed three distributions exponential Weibull, exponential Rayleigh, and new mixture distributions.

In this paper lifetime distribution called inverse exponential Rayleigh distribution (IERD) was proposed and its new distribution system could be used in both survival and reliability analysis. Suppose that a random variable $X > 0$ distributed as an exponential Rayleigh distribution with two-scale parameters $\gamma > 0$ and $\beta > 0$, the cumulative and density functions of $X > 0$ are defined as follows:

$$F(x) = 1 - \exp\left(-\left(\gamma x + \frac{\beta}{2}x^2\right)\right)$$

and

$$f(x) = (\gamma + \beta x) \exp\left(-\left(\gamma x + \frac{\beta}{2}x^2\right)\right).$$

Generally, for any positive random variable X , the inverse distribution could be found for the random variable $Y = 1/X$, with the general form of cumulative function $G(y)$ and density function $g(y)$:

$$G(y) = P(Y \leq y) = P(X \geq 1/y) = 1 - P(X < 1/y) = 1 - F(1/y) \quad g(y) = 1/y^2 f(1/y).$$

On the possibility of available according to the above, it will define the inverse system with cumulative and probability density functions of random variable $y > 0$ as follows:

$$(1) \quad G(y) = \exp \left(-\left(\frac{\gamma}{y} + \frac{\beta}{2y^2} \right) \right), \quad y > 0$$

$$(2) \quad g(y) = \frac{1}{y^2} \left(\gamma + \frac{\beta}{y} \right) \exp \left(-\left(\frac{\gamma}{y} + \frac{\beta}{2y^2} \right) \right), \quad y > 0$$

The new distribution with two positive scale parameters γ and β called inverse exponential Rayleigh (IERD). The corresponding survival (reliability) function of IERD could be given by:

$$(3) \quad S(y) = 1 - \exp \left(-\left(\frac{\gamma}{y} + \frac{\beta}{2y^2} \right) \right), \quad y > 0.$$

The hazard function can be taken the form:

$$(4) \quad h(y) = \frac{\gamma + \frac{\beta}{y}}{y^2 \left(\exp \left(-\left(\frac{\gamma}{y} + \frac{\beta}{2y^2} \right) \right) - 1 \right)}, \quad y > 0.$$

As a result, one can obtain two special cases for the IERD when $\gamma = 0$, we have the inverse Rayleigh distribution and when $\beta = 0$, we get the inverse exponential distribution. Figs.1 and 2 discuss the shape of probability density and hazard functions both of them in two cases according to IERD scale parameters.

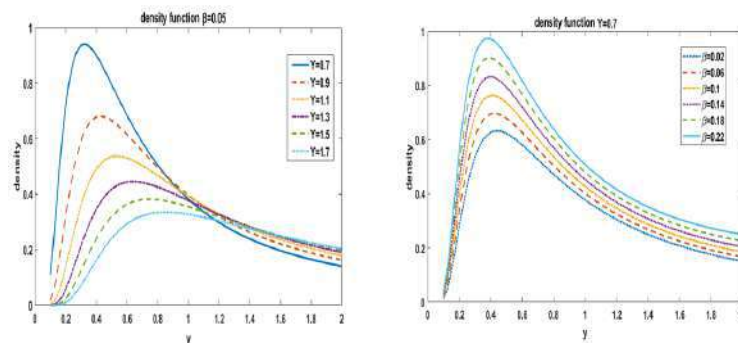


Figure 1: The first plot indicates the density function of IERD with $\beta = 0.05$ while the second represents density function of IERD with $\gamma = 4$

The rest of the paper is devoted as follows: Section 2 indicates the results of this effort, including probability density and hazard functions; Section 3 presents the application of the IERD and Section 4 concludes our efforts.

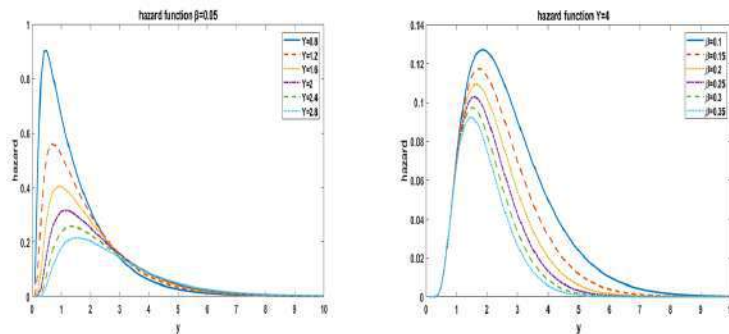


Figure 2: The first plot indicates the hazard function of IERD with $\beta = 0.05$ while the second represents the hazard function of IERD with $\gamma = 4$

2. Results

Some statistical properties of IERD are imposed in this section.

2.1 The moments

According to the r -th non-central moment general form, the r -th moment of the IERD could be written as follows:

$$\begin{aligned}
 M'_r &= E(y^r) = \int_0^\infty y^r g(y) dy \\
 &= \int_0^\infty y^{r-2} (\gamma + \beta/y) e^{-(\gamma/y + \beta/(2y^2))} dy \\
 (5) \quad &= \sum_{k=0}^\infty \frac{(-\gamma)^k}{k!} \int_0^\infty y^{r-k-2} (\gamma + \beta/y) e^{-(\beta/(2y^2))} dy \\
 &= \sum_{k=0}^\infty \frac{(-\gamma)^k}{k!} \left(\int_0^\infty \gamma y^{(r-k-2)} e^{-(\beta/(2y^2))} dy + \int_0^\infty \beta y^{(r-k-3)} e^{-(\beta/(2y^2))} dy \right).
 \end{aligned}$$

Define two integrals as follows:

$$\begin{aligned}
 I(r, \gamma, \beta) &= \sum_{k=0}^\infty \frac{(-\gamma)^k}{k!} \int_0^\infty y^{(r-k-2)} e^{-(\beta/(2y^2))} dy \\
 &= \sum_{k=0}^\infty \frac{(-\gamma)^k}{k!} \beta^{(1/2(r-k-1))} 2^{(-1/2(r-k-1))} \Gamma\left(-\frac{(r-k-1)}{2}, \frac{\beta}{2y^2}\right)
 \end{aligned}$$

and

$$I(r-1, \gamma, \beta) = \sum_{k=0}^\infty \frac{(-\gamma)^k}{k!} \int_0^\infty y^{(r-k-3)} e^{-(\beta/(2y^2))} dy$$

$$= \sum_{k=0}^{\infty} \frac{(-\gamma)^k}{k!} \beta^{(1/2(r-k-2))} 2^{(-1/2(r-k-2))} \Gamma\left(-\frac{(r-k-2)}{2}, \frac{\beta}{2y^2}\right).$$

Then, we have

$$(6) \quad M'_r = \gamma \cdot I(r, \gamma, \beta) + \beta \cdot I(r-1, \gamma, \beta).$$

2.2 The mean and the variance

The mean and the variance of the IERD could be found directly from equation (6) by specifying (r) to equal 1 and 2 respectively, as follows:

$$(7) \quad E(y) = \gamma \cdot I(1, \gamma, \beta) + \beta \cdot I(\gamma, \beta),$$

$$(8) \quad E(y^2) = \gamma \cdot I(2, \gamma, \beta) + \beta \cdot I(1, \gamma, \beta)$$

while the variance is formulated as follows :

$$(9) \quad Var(y) = (\gamma \cdot I(2, \gamma, \beta) + \beta \cdot I(1, \gamma, \beta)) - (\gamma \cdot I(1, \gamma, \beta) + \beta \cdot I(\gamma, \beta))^2$$

With the following details:

$$I(\gamma, \beta) = \sum_{k=0}^{\infty} \frac{(-\gamma)^k}{k!} \beta^{(1/2(-k-1))} 2^{(-1/2(-k-1))} \Gamma\left(\frac{(k+1)}{2}, \frac{\beta}{2y^2}\right),$$

$$I(1, \gamma, \beta) = \sum_{k=0}^{\infty} \frac{(-\gamma)^k}{k!} \beta^{(1/2(-k))} 2^{(-1/2(-k))} \Gamma\left(\frac{k}{2}, \frac{\beta}{2y^2}\right),$$

$$I(2, \gamma, \beta) = \sum_{k=0}^{\infty} \frac{-\gamma^k}{k!} \beta^{1/2(1-k)} 2^{-1/2(1-k)} \Gamma\left(\frac{k-1}{2}, \frac{\beta}{2y^2}\right).$$

2.3 Moment generating function

The moment generating function of IERD which is denoted by $M(t)$ can be expressed mathematically as follows:

$$(10) \quad \begin{aligned} M(t) &= E(e^{ty}) = \int_0^{\infty} e^{ty} g(y) dy \\ &= \int_0^{\infty} \frac{e^{ty}}{y^2} (\gamma + \beta/y) e^{-(\gamma/y + \beta/(2y^2))} dy \\ &= \sum_{k=0}^{\infty} \int_0^{\infty} \frac{(-\gamma)^k}{k!} y^{-(k+2)} (\gamma + \beta/y) e^{-(\beta/(2y^2) - ty)} dy \\ &= \sum_{n=0}^{\infty} n! \sum_{k=0}^{\infty} \frac{(-\gamma)^k}{k!} \int_0^{\infty} y^{-(k+n+2)} (\gamma + \beta/y) e^{-(\beta/(2y^2))} dy \end{aligned}$$

$$= \sum_{n=0}^{\infty} n! \sum_{k=0}^{\infty} \frac{(-\gamma)^k}{k!} \left(\int_0^{\infty} \gamma y^{-(k+n+2)} e^{(-\beta/(2y^2))} dy \right. \\ \left. + \int_0^{\infty} \beta y^{-(k+n+3)} e^{(-\beta/(2y^2))} dy \right).$$

Define the following integral equations:

$$I(-n, \gamma, \beta) = \sum_{n=0}^{\infty} n! \sum_{k=0}^{\infty} \frac{(-\gamma)^k}{k!} \left(\int_0^{\infty} y^{-(k+n+2)} e^{(-\beta/(2y^2))} dy \right) \\ = \sum_{n=0}^{\infty} n! \sum_{k=0}^{\infty} \frac{(-\gamma)^k}{k!} \beta^{(-1/2(n+k+1))} 2^{(1/2(n+k-1))} \Gamma\left(\frac{(n+k+1)}{2}, \frac{\beta}{2y^2}\right),$$

and

$$I(-(n+1), \gamma, \beta) = \sum_{n=0}^{\infty} n! \sum_{k=0}^{\infty} \frac{(-\gamma)^k}{k!} \left(\int_0^{\infty} y^{-(k+n+3)} e^{(-\beta/(2y^2))} dy \right) \\ (11) \quad = \sum_{n=0}^{\infty} n! \sum_{k=0}^{\infty} \frac{(-\gamma)^k}{k!} \beta^{(-1/2(n+k+2))} 2^{(1/2(n+k-2))} \Gamma\left(\frac{(n+k+2)}{2}, \frac{\beta}{2y^2}\right).$$

Consequently, we obtain the formula

$$(12) \quad M(t) = \Upsilon \cdot I(-n, \gamma, \beta) + \beta \cdot I(-(n+1), \gamma, \beta).$$

2.4 Factorial moments generating function

The factorial moments generating function of IERD denoted by $M_y(t)$ is given by:

$$(13) \quad M_y(t) = E(t^y) = \int_0^{\infty} t^y g(y) dy \\ = \int_0^{\infty} \frac{t^y}{y^2} (\gamma + \beta/y) e^{(-(\gamma/y + \beta/(2y^2)))} dy.$$

By using

$$I(n, \gamma, \beta) \\ = \sum_{k=0}^{\infty} \frac{(-\gamma)^k}{k!} \sum_{n=0}^{\infty} \frac{(-\ln t)^n}{n!} \beta^{(1/2(n-k-1))} 2^{(-1/2(n-k+1))} \Gamma\left(\frac{-(n-k-1)}{2}, \frac{\beta}{2y^2}\right),$$

and

$$I(n-1, \gamma, \beta) \\ = \sum_{k=0}^{\infty} \frac{(-\gamma)^k}{k!} \sum_{n=0}^{\infty} \frac{(-\ln t)^n}{n!} \beta^{(1/2(n-k-2))} 2^{(-1/2(n-k))} \Gamma\left(\frac{-(n-k-2)}{2}, \frac{\beta}{2y^2}\right),$$

we have

$$(14) \quad M_y(t) = \gamma \cdot I(n, \gamma, \beta) + \beta \cdot I(n-1, \gamma, \beta).$$

2.5 Quantifiable function (QF)

The general form of QF is

$$(15) \quad \frac{\gamma}{y} + \frac{\beta}{2y^2} = -\ln(V), \quad \ln(V)y^2 + \gamma y + 2\beta = 0.$$

Solving Eq.(15), we have $y = \frac{-\gamma \pm \sqrt{\gamma^2 - 8\beta \ln(V)}}{2\ln(V)}$. But $y > 0$ then we get the positive root only.

2.6 Skewness and Kurtosis

Skewness in statistics, which can be defined mathematically as a ratio that depends on its numerator and its denominator on the moment such that:

$$(16) \quad C.S = \frac{M'_3}{(M'_2)^{(3/2)}},$$

and for Kurtosis, we have

$$(17) \quad C.K = \frac{M'_4}{(M'_2)^4},$$

where

$$M'_r = \gamma \cdot I(r, \gamma, \beta) + \beta \cdot I(r - 1, \gamma, \beta).$$

2.7 Characteristic function

For any positive real value random variable Y the characteristic function of the inverse exponential Rayleigh distribution is denoted by φ_Y and formed as the follows:

$$(18) \quad \begin{aligned} \varphi_Y(it) &= E(e^{ity}) \\ &= \int_0^\infty e^{ity} g(y) dy \\ &= \int_0^\infty \frac{e^{ity}}{y^2} (\gamma + \beta/y) e^{-(\gamma/y + \beta/(2y^2))} dy \\ &= \sum_{k=0}^\infty \frac{(-\gamma)^k}{k!} \int_0^\infty e^{ity} y^{-2-k} (\gamma + \beta/y) e^{(-\beta/(2y^2))} dy \\ &= \sum_{k=0}^\infty \frac{(-\gamma)^k}{k!} \sum_{m=0}^\infty \frac{(it)^m}{m!} \int_0^\infty y^{m-2-k} (\gamma + \beta/y) e^{(-\beta/(2y^2))} dy. \end{aligned}$$

By letting

$$I(m, \gamma, \beta) = \sum_{k=0}^\infty \frac{(-\gamma)^k}{k!} \sum_{m=0}^\infty \frac{it^m}{m!} \frac{(-\gamma)^k}{k!} \int_0^\infty y^{m-2-k} e^{(-\beta/(2y^2))} dy$$

$$= \sum_{k=0}^{\infty} \frac{(-\gamma)^k}{k!} \sum_{m=0}^{\infty} \frac{(it)^m}{m!} 2^{-1/2(m-k+1)} \Gamma\left(\frac{-(m-k-1)}{2}, \frac{\beta}{2y^2}\right)$$

and

$$\begin{aligned} I(m-1, \gamma, \beta) &= \sum_{k=0}^{\infty} \frac{(-\gamma)^k}{k!} \sum_{m=0}^{\infty} \frac{it^m}{m!} \frac{(-\gamma)^k}{k!} \int_0^{\infty} y^{m-3-k} e^{(-\beta/(2y^2))} dy \\ &= \sum_{k=0}^{\infty} \frac{(-\gamma)^k}{k!} \sum_{m=0}^{\infty} \frac{(it)^m}{m!} 2^{-1/2(m-k)} \Gamma\left(\frac{-(m-k-3)}{2}, \frac{\beta}{2y^2}\right) \end{aligned}$$

we receive

$$(19) \quad \varphi_Y(it) = \gamma \cdot I(m, \gamma, \beta) + \beta \cdot I(m-1, \gamma, \beta).$$

2.8 MLE estimation

This section includes estimating the parameters of the inverse exponential Rayleigh distribution to estimate the two parameters γ and β . On the assumption that the size of the sample is n with the considering that $y_1, y_2, \dots, y_n$ are values of the density distribution of the inverse exponential Rayleigh distribution, then the log-likelihood is given as follows:

$$(20) \quad \ln(L(y_1, y_2, \dots, y_n; \gamma, \beta)) = \sum_{i=1}^n \ln\left(\frac{1}{y_i^2} \left(\gamma + \frac{\beta}{y_i}\right)\right) - \sum_{i=1}^n \left(\frac{\gamma}{y_i} + \frac{\beta}{2y_i^2}\right),$$

$$(21) \quad D_{\gamma} \ln(L(y_1, y_2, \dots, y_n; \gamma, \beta)) = \sum_{i=1}^n \frac{1}{((\gamma + \beta/y_i))} - \sum_{i=1}^n \frac{1}{y_i}$$

$$(22) \quad D_{\beta} \ln(L(y_1, y_2, \dots, y_n; \gamma, \beta)) = \sum_{i=1}^n \frac{1}{(y_i(\gamma + \beta/y_i))} - \sum_{i=1}^n \frac{1}{2y_i^2}.$$

To obtain the results that represent the parameters estimated by maximum likelihood estimation method (MLE), numerical methods are applied to solve equations Eq.(21) and Eq.(22) after being equal to zero.

3. Application

The application in this section relies on the use of the information criterion to compare the results from the inverse exponential Rayleigh distribution with some other distributions that are used to process data for cancer patients from the Iraqi Yarmouk Hospital. The data give information about the failure (death) times for bladder cancer patients. In view of the information provided in Table 1, inverse exponential Rayleigh distribution showed lower criterion values than other distributions through information criterion applications such as Akaike (AIC), corrected Akaike (AIC_c), and Bayesian (BIC) information criterion.

Table 1: MLE parameter estimation and information criterion results

Model	MLE	Std. Error	AIC	AIC_c	BIC
IER	$\gamma = 2.2401$ $\beta = 0.3398$	0.2411 0.6502	162.6252	164.3662	160.6252
exponential Rayleigh	$\gamma = 0.0001805$ $\beta = 0.000032$	0.00017885 0.0000232	168.6439	170.3582	166.6439
exponential Weibull	$\gamma = 0.0000979$ $\beta = 0.0007814$	0.00003901 0.00021859	203.7306	205.449	201.7306
Rayleigh Weibull	$\gamma = 0.000184$ $\beta = 0.0005473$	0.000173 0.0005274	164.7465	166.461	162.7465
Weibull	$\gamma = 35.8407$	0.1593	180.9829	181.4829	172.9829

Depending on the results in the Table 1, it can be noted that the new distribution compared to other distributions showed good fit in terms of information criterion. This would prove that the new distribution can be relied upon to process and obtain statistical information from the data. From Fig.3 it can be seen that the survival function of inverse exponential Rayleigh distribution of the real data set for bladder cancer patients provides a clear fit for the survival function estimated by the Kaplan-Meier estimate. Through this, the compatibility with the data shown by the plot, the IER distribution can be adopted for of data modeling.

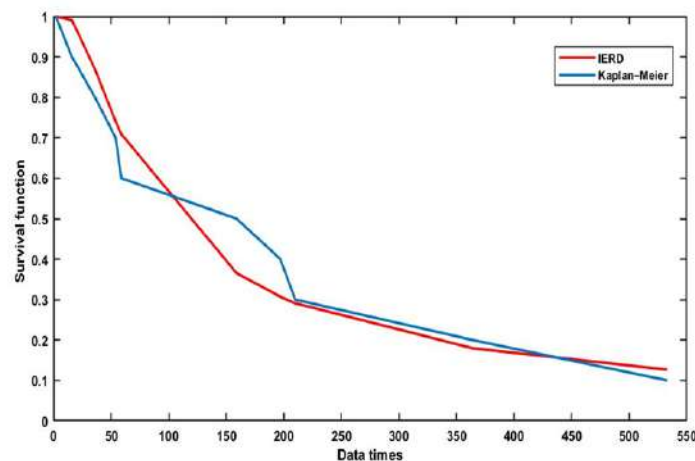


Figure 3: Estimated survival function of IRE and empirical survival function of Kaplan-Meier

4. Conclusion

This study introduced a new probability distribution called inverse exponential Rayleigh distribution, where a new approach was adopted that mixes the Exponential and Rayleigh distributions and then finds the inverted distribution of the mixing. Statistical properties derived distribution using Taylor series. In terms of application, based on information criterion, the inverse exponential Rayleigh distribution showed good fit against other distributions.

For future studies, using statistical simulation technique to generate data of random variables and using these simulation data in estimating the IER distribution parameters by applying Bayesian and classical methods and comparing the results of the estimation. For survival analysis, using real data of censoring data (type I, type II, or left and right) to study the performance of both survival (failure) and hazard functions of IER distribution.

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Resolving sets in graphs

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Abstract. Let G be a connected graph. A vertex x of a connected graph G is said to *resolve two vertices* u and v of G if $d_G(x, u) \neq d_G(x, v)$. For an ordered set $W = \{x_1, \dots, x_k\} \subseteq V(G)$ and a vertex v in G , the k -vector $r_G(v/W) = (d_G(v, x_1), d_G(v, x_2), \dots, d_G(v, x_k))$ is called the *representation* of v with respect to W . The set W is a *resolving set* for G if and only if no two vertices of G have the same representation with respect to W . The metric dimension of G , denoted by $\dim(G)$, is the minimum cardinality over all resolving sets of G . A resolving set of cardinality $\dim(G)$ is called a *basis*. In this paper, we characterize the resolving sets in the join, corona and lexicographic product of two graphs and determine the resolving number of these graphs.

Keywords: resolving set, resolving number, join, corona, lexicographic product.

1. Introduction

All graphs considered in this study are finite, simple, and undirected connected graphs, that is, without loops and multiple edges. For some basic concepts in Graph Theory, we refer readers to [2].

Let $G = (V(G), E(G))$ be a connected graph. The *open neighborhood* $N_G(v) = \{u \in V(G) : uv \in E(G)\}$. Any element u of $N_G(v)$ is called a *neighbor* of v . The *closed neighborhood* $N_G[v] = N_G(v) \cup \{v\}$. Thus, the degree of v is given by $\deg_G(v) = |N_G(v)|$. Customarily, for $S \subseteq V(G)$, $N_G(S) = \bigcup_{v \in S} N_G(v)$ and $N_G[S] = \bigcup_{v \in S} N_G[v]$.

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A vertex x of a connected graph G is said to *resolve two vertices* u and v of G if $d_G(x, u) \neq d_G(x, v)$. For an ordered set $W = \{x_1, \dots, x_k\} \subseteq V(G)$ and a vertex v in G , the k -vector $r_G(v/W) = (d_G(v, x_1), d_G(v, x_2), \dots, d_G(v, x_k))$ is called the *representation* of v with respect to W . The set W is a *resolving set* for G if and only if no two vertices of G have the same representation with respect to W . The metric dimension of G , denoted by $\dim(G)$, is the minimum cardinality over all resolving sets of G . A resolving set of cardinality $\dim(G)$ is called a *basis*.

Canoy and Malacas [1] defined a *locating set* as a set $S \subseteq V(G)$ of G if for every two distinct vertices u and v of $V(G) \setminus S$, $N_G(u) \cap S \neq N_G(v) \cap S$. The *locating number* of G , denoted by $\ln(G)$, is the smallest cardinality of a locating set of G . A locating set of G of cardinality $\ln(G)$ is referred to as an *ln-set* of G .

The problem of uniquely recognizing the possible position of an intruder such as fault in a computer network and spoiled device was the principal motivation in introducing the concept of metric dimension in graphs.

Slater [6] brought in the notion of locating sets and its minimum cardinality as locating number. The same concept was also introduced by Harary and Melter [2] but using the terms resolving sets and metric dimension to refer to locating sets and locating number, respectively. However, in recent studies, locating sets and resolving sets are defined differently. In 2013, Bailey et al. [3] defined a resolving set as a set of vertices S in a graph G such that for any two vertices u, v , there exists $x \in S$ such that the distances $d(u, x) \neq d(v, x)$. Meanwhile in the same year, Canoy and Malacas [1], defined a locating set as a subset S of $V(G)$ in a connected graph G satisfying that $N_G(u) \cap S \neq N_G(v) \cap S$, for all $u, v \in V(G) \setminus S$, $u \neq v$.

Let G be a connected graph. A set $S \subseteq V(G)$ is a *strictly locating set* of G if it is a locating set of G and $N_G(u) \cap S \neq S$, for all $u \in V(G) \setminus S$. The *strictly locating number* of G , denoted by $sln(G)$, is the smallest cardinality of a strictly locating set of G . A strictly locating set of G of cardinality $sln(G)$ is referred to as *sln-set* of G .

A connected graph G of order $n \geq 3$ is *point distinguishing* if for any two distinct vertices u and v of G , $N_G[u] \neq N_G[v]$. It is *totally point determining* if for any two distinct vertices u and v of G , $N_G(u) \neq N_G(v)$ and $N_G[u] \neq N_G[v]$.

This study aims to define and characterize the resolving sets in the join, corona and lexicographic product of two graphs and determine their corresponding resolving number.

2. Preliminary results

Remark 2.1. For any connected graph G of order n , $1 \leq \dim(G) \leq n - 1$.

Example 2.1. $\dim(P_n) = 1$ and $\dim(K_n) = n - 1$ for $n \geq 2$.

Theorem 2.1 ([5]). *Let G be a connected graph of order $n \geq 2$, then*

- (i) $\dim(G) = 1$ if and only if $G = P_n$;
- (ii) $\dim(G) = n - 1$ if and only if $G = K_n$;
- (iii) for $G = C_n, n \geq 3$, $\dim(G) = 2$.

Theorem 2.2. Let G be a connected graph of order $n = 4$, then $\dim(G) = 2$ if and only if $G \neq K_4$ and $G \neq P_4$.

Proof. Suppose that $\dim(G) = 2$, then by Theorem 2.1(i) and (ii), $G \neq K_4$ and $G \neq P_4$.

For the converse, suppose that $G \neq K_4$ and $G \neq P_4$. Since $n = 4$, $\dim(G) \geq 2$ by Theorem 2.1(i) and (ii). Choose $x, y \in V(G)$ such that $d_G(x, y) = 2$. Let $z \in N_G(x) \cap N_G(y)$ and let $p \in V(G) \setminus \{x, y, z\}$. Consider the following cases:

Case 1. Suppose that $p \notin N_G(x) \cap N_G(y)$.

Since G is connected, $p \in N_G(z)$, that is, $G = K_{1,3}$. By Theorem 2.1(iv), $\dim(G) = 2$.

Case 2. Suppose that $p \in N_G(x) \cap N_G(y)$.

Then $G = C_4$ or $G = K_1 + P_3$. Hence, by Theorem 2.1(iii), $\dim(C_4) = 2$ and $W = \{x, p\}$ is a minimum resolving set of $K_1 + P_3$. Thus, $\dim(G) = 2$.

Case 3. Suppose that $p \in N_G(x) \setminus N_G(y)$ or $p \in N_G(y) \setminus N_G(x)$, say $p \in N_G(x) \setminus N_G(y)$. Let $W = \{p, x\}$, then W is a resolving set of G . Then $G \cong K_1 + (K_1 \cup K_2)$ since $G \neq P_4$. By Theorem 2.1(iv), $\dim(G) = 2$.

Therefore, in all cases, $\dim(G) = 2$. $\square$

Theorem 2.3. Let G be a connected graph of order $n = 5$, then $\dim(G) = 2$ if and only if there exist distinct vertices x and y of G satisfying one of the following properties:

- (i) $|N_G(x) \cap N_G(y)| = 0$ and $|N_G(x) \setminus \{y\}| = |N_G(y) \setminus \{x\}| = 1$;
- (ii) $|N_G(x) \cap N_G(y)| = 1$ and $|N_G(x) \setminus \{y\}| = |N_G(y) \setminus \{x\}| = 2$ or $|N_G(x) \setminus \{y\}| = 2$ and $|N_G(y) \setminus \{x\}| = 1$ or $|N_G(y) \setminus \{x\}| = 2$ and $|N_G(x) \setminus \{y\}| = 1$.

Proof. Suppose that $\dim(G) = 2$, then there exist distinct vertices x and y of G such that $W = \{x, y\}$ is a minimum resolving set of G . Suppose $|N_G(x) \cap N_G(y)| > 1$. Then there exist distinct vertices $u, v \in N_G(x) \cap N_G(y)$. This implies that $r_G(u/W) = r_G(v/W)$, a contradiction to the assumption. Hence, $|N_G(x) \cap N_G(y)| \leq 1$. Suppose $|N_G(x) \cap N_G(y)| = 0$. Since W is a resolving set, $|N_G(x) \setminus \{y\}| \leq 1$. Suppose $|N_G(x) \setminus \{y\}| = 0$, then $|N_G(y) \setminus \{x\}| = 1$ since W is a resolving set. This implies that there exist at least two vertices say u_1, u_2 such that $u_1, u_2 \notin N_G(x) \cup N_G(y)$. Thus, $r_G(u_1/W) = r_G(u_2/W)$, contrary to our assumption that W is a resolving set. Thus, $|N_G(x) \setminus \{y\}| = 1$. Similarly, $|N_G(y) \setminus \{x\}| = 1$. Hence, (i) holds.

Suppose that $|N_G(x) \cap N_G(y)| = 1$. Let $p \in N_G(x) \cap N_G(y)$ and let $v, z \in V(G) \setminus \{x, y, p\}$, then $v, z \notin N_G(x) \cap N_G(y)$. Since the subsets of W are $\emptyset, \{x, y\}, \{x\}$, and $\{y\}$ and since $N_G(p) \cap W$ is $\{x, y\}$, the remaining two

sets $N_G(v) \cap W$ and $N_G(z) \cap W$ are $\{x\}$ and $\{y\}$ or $\{x\}$ and $\emptyset$ or $\{y\}$ and $\emptyset$, respectively. Thus, $|N_G(x) \setminus \{y\}| = |N_G(y) \setminus \{x\}| = 2$ or $|N_G(x) \setminus \{y\}| = 2$ and $|N_G(y) \setminus \{x\}| = 1$ or $|N_G(y) \setminus \{x\}| = 2$ and $|N_G(x) \setminus \{y\}| = 1$. Therefore, (ii) holds.

For the converse, suppose there exist distinct vertices $x, y \in V(G)$ satisfying (i) or (ii). Let $W = \{x, y\}$, then W is a minimum resolving set of G . Therefore, $\dim(G) = 2$. $\square$

Theorem 2.4. *If W is a locating set of G , then it is a resolving set.*

Proof. Let $W = \{w_1, w_2, \dots, w_n\}$ be a locating set of G and $x, y \in V(G) \setminus W$ with $x \neq y$, then

$$r_G(x/W) = (d_G(x, w_1), d_G(x, w_2), \dots, d_G(x, w_n)) \quad \text{and} \\ r_G(y/W) = (d_G(y, w_1), d_G(y, w_2), \dots, d_G(y, w_n)).$$

Since W is a locating set, $N_G(x) \cap W \neq N_G(y) \cap W$. Thus, there exists $w_i \in W$ such that $w_i \in N_G(x) \setminus N_G(y)$ or $w_i \in N_G(y) \setminus N_G(x)$. Hence, $d_G(x, w_i) = 1$ but $d_G(y, w_i) \neq 1$ or $d_G(y, w_i) = 1$ but $d_G(x, w_i) \neq 1$. Hence, $r_G(x/W) \neq r_G(y/W)$. Therefore, W is a resolving set of G . $\square$

Remark 2.2. (i) Let G be a connected graph and $W \subseteq V(G)$, then for any two distinct vertices $x, y \in V(G) \setminus W$ with $N_G(x) \cap W \neq N_G(y) \cap W$, we have $r_G(x/W) \neq r_G(y/W)$.

(ii) The converse of Theorem 2.4 is not true.

To see this, consider the graph G in Figure 1. Let $W = \{w_1, w_2, w_3, w_4\}$, then $r_G(x/W) = (1, 1, 1, 2)$, $r_G(y/W) = (1, 1, 1, 4)$ and $r_G(z/W) = (2, 2, 2, 1)$. Thus, W is a resolving set but it is not a locating set since $N_G(x) \cap W = N_G(y) \cap W$.

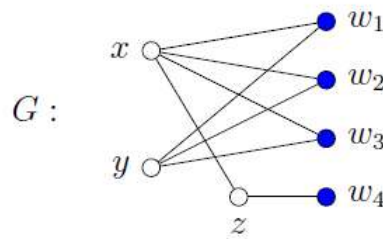


Figure 1: A resolving set $W = \{w_1, w_2, w_3, w_4\}$ of a graph G but not a locating set of G

3. Resolving sets in the join of graphs

Remark 3.1. Let G and H be non-trivial connected graphs and $W \subseteq V(G+H)$. If $W \subseteq V(G)$ or $W \subseteq V(H)$, then W is not a resolving set of $G+H$.

Theorem 3.1. *Let G and H be non-trivial connected graphs. A set $W \subseteq V(G+H)$ is a resolving set of $G+H$ if and only if W is a locating set of $G+H$.*

Proof. If W is a locating set of $G+H$, then it is a resolving set by Theorem 2.4.

Conversely, suppose W is a resolving set of $G+H$. Let $W_G = W \cap V(G)$ and $W_H = W \cap V(H)$, then $W = W_G \cup W_H$ where $W_G \subseteq V(G)$ and $W_H \subseteq V(H)$. If $W_G = \emptyset$, then $W = W_H$. Hence, $W \subseteq V(H)$. By Remark 2.2, W is not a resolving set of $G+H$, a contradiction. Thus, $W_G \neq \emptyset$. Similarly, $W_H \neq \emptyset$. Let $W_G = \{w_1, w_2, \dots, w_n\}$ and $W_H = \{w_{n+1}, w_{n+2}, \dots, w_m\}$. Let $x, y \in V(G) \setminus W_G$ with $x \neq y$. We claim that $N_G(x) \cap W_G \neq N_G(y) \cap W_G$. Suppose $N_G(x) \cap W_G = N_G(y) \cap W_G = S$. Let $S = \{w_{j_1}, w_{j_2}, \dots, w_{j_k}\}$, $j_i \in \{1, 2, \dots, n\}$, for all $i = 1, 2, \dots, k$, then $d_G(x, s) = d_G(y, s) = 1$, for all $s \in S$. Hence, $d_{G+H}(x, w) = d_{G+H}(y, w) = 1$, for all $w \in S \cup W_H$ and $d_{G+H}(x, v) = d_{G+H}(y, v) = 2$, for all $v \in W_G \setminus S$. It follows that $r_{G+H}(x/W) = r_{G+H}(y/W)$, a contradiction, since W is a resolving set of $G+H$. Thus, W_G is a locating set of G . Similarly, W_H is a locating set of H . Suppose there exists $u \in V(G) \setminus W_G$ and $v \in V(H) \setminus W_H$ such that $N_G(u) \cap W_G = W_G$ and $N_H(v) \cap W_H = W_H$. Thus, W_G or W_H is a strictly locating set. $\square$

Theorem 3.2 ([1]). *Let G and H be non-trivial connected graphs. A set $S \subseteq V(G+H)$ is a locating set in $G+H$ if and only if $S_1 = V(G) \cap S$ and $S_2 = V(H) \cap S$ are locating sets in G and H , respectively, where S_1 or S_2 is a strictly locating set.*

The next result follows immediately from Theorem 3.1 and Theorem 3.2.

Theorem 3.3. *Let G and H be non-trivial connected graphs. A set $W \subseteq V(G+H)$ is a resolving set of $G+H$ if and only if $W = W_G \cup W_H$ where $W_G \subseteq V(G)$ and $W_H \subseteq V(H)$ are locating sets of G and H , respectively, where W_G or W_H is a strictly locating set.*

Corollary 3.1 ([1]). *Let G and H be connected non-trivial graphs, then*

$$\ln(G+H) = \min \{sln(H) + \ln(G), sln(G) + \ln(H)\}.$$

Corollary 3.2 ([1]). *Let G be a non-trivial connected graph and let K_n be a complete graph of order $n \geq 2$, then $\ln(G+K_n) = sln(G) + n - 1$.*

The next results follow immediately from Corollary 3.1, Corollary 3.2 and Theorem 2.4.

Corollary 3.3. *Let G and H be connected non-trivial graphs, then*

$$\dim(G + H) = \min \{ \text{sln}(H) + \ln(G), \text{sln}(G) + \ln(H) \}.$$

Corollary 3.4. *Let G be a non-trivial connected graph and let K_n be a complete graph of order $n \geq 2$, then $\dim(G + K_n) = \text{sln}(G) + n - 1$.*

Theorem 3.4. *Let H be a non-trivial connected graph, then $W \subseteq V(H + K_1)$ is a resolving set of $H + K_1$ if and only if W is a locating set of $H + K_1$.*

Proof. Suppose that W is a locating set of $H + K_1$, then by Theorem 2.4, W is a resolving set of $H + K_1$.

For the converse, suppose that W is a resolving set of $H + K_1$ and let $K_1 = \langle v \rangle$. If $v \notin W$, then $W \subseteq V(H)$ is a locating set of H . Suppose there exists $u \in V(H) \setminus W$ such that $N_H(u) \cap W = W$, then $N_{H+K_1}(u) \cap W = W = N_{H+K_1}(v) \cap W$. Hence, W is a strictly locating set of H . Next, suppose that $W = \{v\} \cup W_H$, where $W_H = V(H) \cap W$, then $W_H \neq \emptyset$. Let $W_H = \{w_1, w_2, \dots, w_n\}$. Let $w, z \in V(H) \setminus W_H$ with $w \neq z$. We claim that

$$N_H(z) \cap W_H \neq N_H(w) \cap W_H.$$

Suppose $(N_H(z) \cap W_H) \cup \{v\} = N_{H+K_1}(z) \cap W = N_{H+K_1}(w) \cap W = (N_H(w) \cap W_H) \cup \{v\} = S$. Let $S = \{v, w_{j_1}, w_{j_2}, \dots, w_{j_k}\}$, $j_i \in \{1, 2, \dots, n\}$, for all $i = 1, 2, \dots, k$, then $d_G(w, s) = d_G(z, s) = 1$, for all $s \in S$. Hence, $d_{H+K_1}(w, y) = d_{H+K_1}(z, y) = 1$, for all $y \in S \cup W_H$ and $d_{H+K_1}(w, x) = d_{G+H}(z, x) = 2$, for all $x \in W \setminus S$. It follows that $r_{H+K_1}(w/W) = r_{H+K_1}(z/W)$, a contradiction since W is a resolving set of $H + K_1$. Hence, W_H is a locating set of H . $\square$

Theorem 3.5 ([1]). *Let H be a non-trivial connected graph and let $K_1 = \langle v \rangle$, then $S \subseteq V(H + K_1)$ is a locating set in $H + K_1$ if and only if either $v \notin S$ and S is a strictly locating set in H or $S = \{v\} \cup S_1$, where S_1 is a locating set in H .*

Corollary 3.5 ([1]). *Let H be a non-trivial connected graph, then $\ln(H + K_1) = \text{sln}(H)$.*

The next results follow immediately from Theorem 2.4, Corollary 3.5 and Theorem 3.5.

Theorem 3.6. *Let H be a non-trivial connected graph and let $K_1 = \langle v \rangle$, then $W \subseteq V(H + K_1)$ is a resolving set of $H + K_1$ if and only if either $v \notin W$ and W is a strictly locating set of H or $W = \{v\} \cup W_H$, where W_H is a locating set of H .*

Corollary 3.6. *Let H be a non-trivial connected graph, then $\dim(H + K_1) = \text{sln}(H)$.*

4. Resolving sets in the corona of graphs

Remark 4.1. Let $v \in V(G)$. For every $x, y \in V(H^v)$, $d_{G \circ H}(x, w) = d_{G \circ H}(y, w)$, and $d_{G \circ H}(x, w) \neq d_{G \circ H}(v, w)$ for every $w \in V(G \circ H) \setminus V(H^v)$.

Remark 4.2. Let G and H be non-trivial connected graphs, $W \subseteq V(G \circ H)$ and $A_v = V(H^v) \cap W$ where $v \in V(G)$. For each $x \in V(H^v) \setminus A_v$ and $z \in A_v$,

$$d_{G \circ H}(x, z) = \begin{cases} 1, & \text{if } z \in N_{H^v}(x) \\ 2, & \text{otherwise.} \end{cases}$$

Theorem 4.1. Let G and H be non-trivial connected graphs, then $W \subseteq V(G \circ H)$ is a resolving set of $G \circ H$ if and only if $W \cap V(H^v) \neq \emptyset$, for all $v \in V(G)$ and $W = A \cup B$, where $A \subseteq V(G)$ and

$$B = \cup \{B_v : v \in V(G) \text{ and } B_v \text{ is a locating set of } H^v\}.$$

Proof. Let W be a resolving set of $G \circ H$. Suppose there exists $u \in V(G)$ such that $W \cap V(H^u) = \emptyset$. By Remark 4.1, $d_{G \circ H}(p, x) = d_{G \circ H}(q, x)$ for every distinct vertices $p, q \in V(H^u)$ and for each $x \in V(G \circ H) \setminus V(H^u)$. Thus, $r_{G \circ H}(p/W) = r_{G \circ H}(q/W)$, a contradiction. Hence, $W \cap V(H^v) \neq \emptyset$ for each $v \in V(G)$. Let $A = V(G) \cap W$, $v \in V(G)$ and $B_v = W \cap V(H^v)$. Let $x, y \in V(H^v) \setminus B_v$ with $x \neq y$, then $x, y \notin W$. Suppose $N_{H^v}(x) \cap B_v = N_{H^v}(y) \cap B_v$, then $d_{G \circ H}(x, p) = 1 = d_{G \circ H}(y, p)$ for each $p \in N_{H^v}(x) \cap B_v$ and $d_{G \circ H}(x, q) = 2 = d_{G \circ H}(y, q)$ for each $q \in B_v \setminus (N_{H^v}(x) \cap B_v)$ by Remark 4.2. By Remark 4.1, it follows that $r_{G \circ H}(x/W) = r_{G \circ H}(y/W)$, a contradiction since W is a resolving set of $G \circ H$. Hence, $N_{H^v}(x) \cap B_v \neq N_{H^v}(y) \cap B_v$. Hence, B_v is a locating set of H^v . Now, let

$$B = \bigcup \{B_v : v \in V(G) \text{ and } B_v \text{ is a locating set of } H^v\},$$

then $W = A \cup B$.

For the converse, suppose $W = A \cup B$, where A and B are the sets possessing the properties described. Let $x, y \in V(G \circ H) \setminus W$ with $x \neq y$, and let $u, v \in V(G)$ such that $x \in V(u + H^u)$ and $y \in V(v + H^v)$. Suppose $u = v$. Since B_v is a locating set of H^v , $d_{G \circ H}(x, p) \neq d_{G \circ H}(y, p)$ for some $p \in B_v$ by Remark 4.2 when $x, y \in V(H^v) \setminus B_v$. If $v \notin A$ and $x = v$ or $y \notin A$, then $d_{G \circ H}(x, w) \neq d_{G \circ H}(y, w)$, for all $w \in A$ by Remark 4.1. Thus, $r_{G \circ H}(x/W) \neq r_{G \circ H}(y/W)$.

Suppose now that $u \neq v$. Since $B_u \neq \emptyset$ and $B_v \neq \emptyset$,

$$d_{G \circ H}(x, p) \neq d_{G \circ H}(y, p), \text{ for all } p \in B_u \text{ or } p \in B_v.$$

Hence, $r_{G \circ H}(x/W) \neq r_{G \circ H}(y/W)$. Therefore, W is a resolving set of $G \circ H$. $\square$

Corollary 4.1. Let G and H be non-trivial connected graphs, where $|V(G)| = m$, then $\dim(G \circ H) = m \cdot \ln(H)$.

Proof. Let W be a minimum resolving set of $G \circ H$, then $W = A \cup B$ where A and B are the sets described in Theorem 4.1. Hence,

$$\begin{aligned} \dim(G \circ H) &= |W| = |A| + |B| \\ &\geq |A| + |V(G)| \cdot \ln(H) \\ &= |A| + m \cdot \ln(H) \\ &\geq m \cdot \ln(H). \end{aligned}$$

Next, let F be a minimum locating set of H . For each $v \in V(G)$, pick $F_v \subseteq V(H^v)$ with $\langle F_v \rangle \cong \langle F \rangle$, then $S = \bigcup_{v \in V(G)} F_v$ is a resolving set of $G \circ H$ by Theorem 4.1. Hence,

$$\dim(G \circ H) \leq |S| = \left| \bigcup_{v \in V(G)} F_v \right| = m \cdot |F| = m \cdot \ln(H).$$

Therefore, $\dim(G \circ H) = m \cdot \ln(H)$. $\square$

5. Resolving sets in the lexicographic product of graphs

Remark 5.1. Let G and H be non-trivial connected graphs and $W = \bigcup_{x \in S} (\{x\} \times T_x)$ where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$.

(i) If $(x, p) \in V(G[H]) \setminus W$, then $r_{G[H]}((x, p)/W) = (p_1, p_2, \dots, p_r, c_1, c_2, \dots, c_t)$.

where $p_i = d_{G[H]}((x, p), (x, q_i))$, $q_i \in T_x$, $r = |T_x|$ and

$c_j = d_{G[H]}((x, p), (x_j, y_j))$, $x_j \in S \setminus \{x\}$, $y_j \in T_{x_j}$, $t = |S| - 1$.

Thus,

$$p_i = \begin{cases} 1, & \text{if } q_i \in N_H(p) \cap T_x \\ 2, & \text{otherwise} \end{cases}$$

and $c_j = d_G(x, x_j)$ for each $x_j \in S \setminus \{x\}$ and for any $y_j \in T_{x_j}$.

(ii) If $(x, p), (x, q) \in V(G[H]) \setminus W$, $p \neq q$ and $N_H(p) \cap T_x = N_H(q) \cap T_x$, then $r_{G[H]}((x, p)/W) = r_{G[H]}((x, q)/W)$.

(iii) If $(x, p), (y, q) \in V(G[H]) \setminus W$, where $x, y \in E(G)$ such that $N_H(p) \cap T_x = T_x$ and $N_H(q) \cap T_y = T_y$, then $r_{G[H]}((x, p)/W) = r_{G[H]}((y, q)/W)$.

Theorem 5.1 ([4]). *Let G and H be two non-trivial graphs such that G is connected, then the following assertions hold for any $a, c \in V(G)$ and $b, d \in V(H)$ such that $a \neq c$.*

(i) $N_{G[H]}(a, b) = (\{a\} \times N_H\{b\}) \cup \{N_G\{a\} \times V(H)\}$

(ii) $d_{G[H]}((a, b), (c, d)) = d_G(a, c)$

$$(iii) \ d_{G[H]}((a, b), (a, d)) = \min\{d_H(b, d), 2\}.$$

Theorem 5.2. *Let G and H be non-trivial connected graphs with $\Delta(H) \leq |V(H)| - 2$, then $W = \bigcup_{x \in S} [\{x\} \times T_x]$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, is a resolving set of $G[H]$ if and only if W is a locating set of $G[H]$.*

Proof. Suppose that W is a locating set of $G[H]$. By Theorem 2.4, W is a resolving set of $G[H]$.

Conversely, suppose that W is a resolving set of $G[H]$. Suppose there exists $x \in V(G) \setminus S$. Pick $a, b \in V(H)$, where $a \neq b$, then $(x, a), (x, b) \notin W$ and $(x, a) \neq (x, b)$. Since $\{(x, c) : c \in V(H)\} \cap W = \emptyset$ and $d_{G[H]}((x, a), (y, d)) = d_G(x, y) = d_{G[H]}((x, b), (y, d))$ for any $y \in V(G) \setminus \{x\}$ and for any $d \in V(H)$ by Theorem 5.1(ii), $r_{G[H]}((x, a)/W) = r_{G[H]}((x, b)/W)$. This implies that W is not a resolving set of $G[H]$, contrary to our assumption. Therefore, $S = V(G)$.

Now, let $x \in V(G)$ and suppose that T_x is not a locating set of H , then there exist distinct vertices p and q in $V(H) \setminus T_x$ such that $N_H(p) \cap T_x = N_H(q) \cap T_x$. By Remark 5.1(ii), $r_{G[H]}((x, p)/W) = r_{G[H]}((x, q)/W)$, a contradiction to the assumption. Thus, T_x is a locating set of H .

Let x and y be adjacent vertices of G with $N_G[x] = N_G[y]$. Suppose that T_x and T_y are not strictly locating, then there exist $c \in V(H) \setminus T_x$ and $d \in V(H) \setminus T_y$ such that $N_H(c) \cap T_x = T_x$ and $N_H(d) \cap T_y = T_y$. By Remark 5.1(iii), $r_{G[H]}((x, c)/W) = r_{G[H]}((y, d)/W)$, a contradiction to the assumption that W is a resolving set of $G[H]$. Therefore, T_x or T_y is strictly locating of H .

Let x and y be nonadjacent vertices of G such that $N_G(x) = N_G(y)$. Suppose T_x and T_y are not dominating sets of H , then there exist $a \in V(H) \setminus T_x$, $c \in V(H) \setminus T_y$ such that $ab \notin E(H)$, for all $b \in T_x$ and $cd \notin E(H)$, for all $d \in T_y$. It follows that $(x, a) \notin W$ and $(y, c) \notin W$ and $N_{G[H]}((x, a)) \cap W = \bigcup \{\{z\} \times T_z : z \in N_G(x)\}$. Since $N_G(x) = N_G(y)$, it follows that $\bigcup \{\{z\} \times T_z : z \in N_G(x)\} = N_{G[H]}((y, c)) \cap W$, a contradiction to the assumption. Thus, T_x or T_y is a dominating set of $G[H]$. Therefore, W is a locating set of $G[H]$. $\square$

Theorem 5.3 ([1]). *Let G and H be non-trivial connected graphs with $\Delta(H) \leq |V(H)| - 2$, then $C = \bigcup_{x \in S} [\{x\} \times T_x]$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$ is a locating set of $G[H]$ if and only if*

- (i) $S = V(G)$;
- (ii) T_x is a locating set of H for every $x \in V(G)$;
- (iii) T_x or T_y is strictly locating of H whenever x and y are adjacent vertices of G with $N_G[x] = N_G[y]$.
- (iv) T_x or T_y is (locating) dominating of H whenever x and y are nonadjacent vertices of G with $N_G(x) = N_G(y)$.

The next result follows immediately from Theorem 5.2 and Theorem 5.3.

Theorem 5.4. *Let G and H be non-trivial connected graphs with $\Delta(H) \leq |V(H)| - 2$, then $W = \bigcup_{x \in S} [\{x\} \times T_x]$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, is a resolving set of $G[H]$ if and only if*

- (i) $S = V(G)$;
- (ii) T_x is a locating set of H for every $x \in V(G)$;
- (iii) T_x or T_y is a strictly locating set of H whenever x and y are adjacent vertices of G with $N_G[x] = N_G[y]$.
- (iv) T_x or T_y is a (locating) dominating set of H whenever x and y are non-adjacent vertices of G with $N_G(x) = N_G(y)$.

The following is a direct consequence of Theorem 5.4.

Corollary 5.1. *Let G be a connected totally point determining graph and let H be a non-trivial connected graph, then $W = \bigcup_{x \in S} [\{x\} \times T_x]$ is a minimum resolving set of $G[H]$ if and only if $S = V(G)$ and T_x is a minimum locating set of H for every $x \in V(G)$.*

Corollary 5.2. *Let G be a connected totally point determining graph and let H be a non-trivial connected graph, then $\dim(G[H]) = |V(G)| \cdot \ln(H)$.*

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Biharmonic classification of infinite graphs: Bi-networks and Poisson networks

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Abstract. In an infinite graph, by using potential-theoretic methods based on the discrete Laplace operator, the properties of biharmonic functions are studied; the notion of bipotentials leads to a classification theory of the infinite graphs. Depending on the solvability of the discrete Poisson equation, we introduce the concept of Poisson networks and investigate some special properties of such infinite graphs.

Keywords: infinite graphs, bisuperharmonic functions, Poisson networks.

1. Introduction

On infinite graphs with associated transition functions (as in electrical networks or in random walks), by using the Laplacian operator a general theory of harmonic functions and potentials can be developed [1],[2],[6] and [7]. This leads to a classification of the infinite graphs as hyperbolic and parabolic (as the transient and the recurrent walks in probability). Here we consider two different classifications of infinite graphs: the first, of the graphs X with two associated sets of transition functions, leading to the study of biharmonic functions on X ; and the second, of the graphs X on which the discrete version of the Poisson equation $\Delta u = f$ can be solved.

2. Function theory on infinite graphs: a summary

In an infinite graph X with a countable number of vertices and a countable number of edges, two vertices x and y are said to be neighbours (written $x \sim y$) if there is an edge joining x and y . We say that there is a self loop at x if $x \sim x$. If each vertex has only a finite number of neighbours, X is said to be locally finite. A path $\{x = x_0, x_1, x_2, \dots, x_n = y\}$ connecting the vertices x, y is a collection of distinct vertices x_i such that $x_{i+1} \sim x_i$ for $0 \leq i \leq n - 1$; if there is a path connecting any two vertices in X , then X is said to be connected. The set $\{t(x, y)\}$ of transition functions consists of real valued functions $t(x, y) \geq 0$ for any pair of vertices x, y . If $t(x, y) = t(y, x)$ for any pair of vertices x, y then

we say that the transition function is symmetric. However in keeping with the example of a (not necessarily reversible) random walk, we work here with non-symmetric transition function. Now, for the definition of an infinite network: An infinite graph X is said to be an infinite network $\{X, t\}$ if X is connected, locally finite, without self loops and provided with a transition function $t(x, y)$.

We introduce now the discrete Laplace operator on a subset of the network X . All the functions considered here are real-valued. If a vertex x and all its neighbours are in A , then x is said to be an interior vertex of A . The set $\overset{\circ}{A}$ consists of all interior vertices of A . Denote $\partial A = A \setminus \overset{\circ}{A}$. For a real valued function $u(x)$ on A and for $z \in \overset{\circ}{A}$, the Laplacian of $u(x)$ at z is $\Delta u(z) = \sum_{y \sim z} t(z, y)[u(y) - u(z)]$. Note that $\sum_{y \sim z} t(z, y)[u(y) - u(z)] = \sum_{y \in X} t(z, y)[u(y) - u(z)]$ since $t(z, y) > 0$ if and only if $y \sim z$. The function $u(x)$ on A is said to be harmonic at $z \in \overset{\circ}{A}$ if and only if $\Delta u(z) = 0$. The function $u(x)$ on A is said to be harmonic (respectively superharmonic) on A if $\Delta u(z) = 0$ (respectively $\Delta u(z) \leq 0$) for every $z \in \overset{\circ}{A}$.

Discrete Dirichlet solution: In an infinite network $\{X, t\}$, let F be a subset and $E \subset \overset{\circ}{F}$. Suppose $f(x)$ is a real valued function on $F \setminus E$, such that $v \leq f \leq u$ on $F \setminus E$ where v, u are defined on F , $v \leq u$ on F , $\Delta u \leq 0$ and $\Delta v \geq 0$ at each vertex in E . Then there exists a function $\varphi(x)$ which is uniquely determined by $f(x)$ if E is a finite set such that $\Delta \varphi(z) = 0$ if $z \in E$ and $\varphi(x) = f(x)$ if $x \in F \setminus E$.

We illustrate now with two examples of the presence of the discrete Dirichlet problem in electrical networks and random walks.

1. In a finite connected electrical network X , if a and b are two nodes, then the effective resistance $r(a, b)$ between a and b is the voltage when a unit current enters a and leaves b . What is the value of $r(a, b)$?

The problem is to find the quantity $r(a, b) = [\psi(a) - \psi(b)]$ where $-\Delta \psi(a) = 1$, $-\Delta \psi(b) = -1$ and $\Delta \psi(x) = 0$ if $x \neq a, b$. For this, let us start with the Dirichlet solution $\varphi(x)$ for which $\varphi(a) = 1$, $\varphi(b) = 0$ and $\Delta \varphi(x) = 0$ if $x \neq a, b$ since the transition function is symmetric in an electrical network, $\sum_{x \in X} -\Delta \varphi(x) = 0$. Hence we should have $-\Delta \varphi(a) = \alpha$ and $-\Delta \varphi(b) = -\alpha$ for some α . Now by the Maximum principle, $0 \leq \varphi \leq 1$ on X . Hence $\alpha \geq 0$. But $\alpha = 0$ would mean that $\varphi = 0$ by the maximum principle. Hence $\alpha > 0$. Write $\psi(x) = \frac{1}{\alpha} \varphi(x)$ to obtain the unique function $\psi(x)$ such that $-\Delta \psi(a) = 1$, $-\Delta \psi(b) = -1$ and $\Delta \psi(x) = 0$ for $x \neq a, b$. Hence $r(a, b) = \psi(a) - \psi(b) = \frac{1}{\alpha} \varphi(a) - 0 = \frac{1}{\alpha} = \frac{1}{-\Delta \varphi(a)}$.

2. In a random walk X , let A and B be two disjoint sets. The problem is to find the probability of the walker starting at some state reaches A before arriving in B .

To solve this, suppose $\varphi(x)$ denotes the probability of the walker starting at the state x reaches A before arriving in B . Then clearly $\varphi(x) = 1$ if

$x \in A$ and $\varphi(x) = 0$ if $x \in B$. Suppose now $x \notin A \cup B$. Then $\varphi(x) = \sum_{y \sim x} p(x, y) \varphi(y)$; for the walker goes from x to a neighbour state y with the probability $p(x, y)$ and then from y reaches A first with probability $\varphi(y)$. Since $\sum_{y \sim x} p(x, y) = 1$ we find $\sum_{y \sim x} p(x, y) [\varphi(y) - \varphi(x)]$ that is, $-\Delta\varphi(x) = 0$ for any $x \notin A \cup B$. Thus the probability of the walk starting at x reaches A first is the Dirichlet solution $\varphi(x)$ such that $\varphi(x) = 1$ if $x \in A$, $\varphi(x) = 0$ if $x \in B$ and $\Delta\varphi(x) = 0$ if $x \notin A \cup B$.

The following is the discrete version of an important theorem in $\mathbb{R}^n$.

Poisson kernel of a finite set: Let E be a finite set in a network for any $z \in \partial E$, let $P_E(z, x)$ be the function on E such that $P_E(z, x)$ is harmonic at every vertex x in $\overset{\circ}{E}$, $P_E(z, z) = 1$ and $P_E(z, y) = 0$ for every $y \in \partial E \setminus \{z\}$. We say that $P_E(z, x)$ is the Poisson Kernel associated with the finite set E having the pole at $z \in \partial E$. Suppose now $f(\xi)$ is a function defined on ∂E . Then $h(x) = \sum_{\xi \in \partial E} P_E(\xi, x) f(\xi)$ is defined on E , harmonic at every vertex in $\overset{\circ}{E}$ and $h(\xi) = f(\xi)$ for every ξ in ∂E .

Poisson equation: Given a function $u(x)$ on a network $\{X, t\}$, it is routine to calculate the Laplacian $\Delta u(x)$. The converse is the problem of the discrete Poisson equation: Given a function $f(x)$ on X , is it possible to find a solution $u(x)$ to the Poisson equation $\Delta u(x) = f(x)$? Actually we can find a solution only if certain restrictions are placed on X or on $f(x)$.

In the continuous case, if $f(x)$ is a locally (Lebesgue) integrable function on $\mathbb{R}^n$, $n \geq 2$, then Brelot[4] had proved that a solution $u(x)$ could be found on $\mathbb{R}^n$ satisfying $\Delta u(x) = f(x)$ in the sense of distributions, by using a Runge-type approximation result. Of course, if $f(x)$ is continuously differentiable there always exists locally a solution to the equation $\Delta u(x) = f(x)$ in the classical sense.

In the case of a finite network $\{X, t\}$ with n vertices $x_1, x_2, \dots, x_n$ there exists a vector $(y_1, y_2, y_3, \dots, y_n)$ such that $\Delta u(x_i) = f(x_i)$ has a solution if and only if $\sum_{i=1}^n y_i f(x_i) = 0$, see [3, Theorem 3.5]. This is proved by using Perron Frobenius theorem in matrix theory. If the network is symmetric as in the case of an electrical network, then $y_i = 1$ for all i .

In the case of an infinite network, if X is a tree (that is there is no closed path $[x_0, x_1, \dots, x_n = x_0]$ in X when $n \geq 3$) without terminal vertices, then for any function $f(x)$ there always exists a solution to the equation $\Delta u(x) = f(x)$, see [2, Theorem 5.1.4]. However in the general case, we are able to prove that for an arbitrary function $f(x)$, $\Delta u(x) = f(x)$ has a solution only with some restrictions on X .

3. Binetworks

Let X be an infinite graph and (X, t) an infinite network with the Green function $G(x, y) = G_y(x)$. Let $\{t^*(x, y)\}$ be another set of transition functions on the

graph X such that (X, t^*) is an infinite network with Green function $G^*(x, y)$. Let Δ and Δ^* be the Laplacians on (X, t) and (X, t^*) respectively. We refer to $\{X, t, t^*\}$ as a binetwork.

Definition 3.1. Let $\mathcal{B}$ be the class of pairs of functions (f, g) where $-\Delta f = g$ on X .

1. If $-\Delta^* g = 0$, then (f, g) is said to be a biharmonic pair; and f is referred to as a biharmonic function on X .
2. If $-\Delta^* g \geq 0$, then (f, g) is said to be a bisuperharmonic pair; and f is referred to as a bisuperharmonic function on X .
3. If f is a Δ -potential and g is a Δ^* -potential, then (f, g) is said to be a bipotential pair; and f is referred to as a bipotential on X .
4. If f is a Δ -potential and g is a Δ^* -harmonic function, then (f, g) is said to be a biharmonic potential pair.

Lemma 3.1. Let s be a non-negative bisuperharmonic function that is Δ -superharmonic. Let $-\Delta s = u$. Suppose f is a real-valued function, $0 \leq f \leq u$. Then there exists a Δ -potential q such that $-\Delta q = f$ and $q \leq s$.

Proof. Since s is non-negative Δ -superharmonic and $-\Delta s = u$, then s is the sum of a potential and a harmonic function $h \geq 0$. Hence

$$\begin{aligned} s(x) &= \sum_{y \in X} G_y(x) [-\Delta s(x)] + h(x) \\ &\geq \sum_{y \in X} G_y(x) [-\Delta s(x)] \\ &= \sum_{y \in X} G_y(x) u(x) \\ &\geq \sum_{y \in X} G_y(x) f(x) = q(x). \end{aligned}$$

Consequently $q(x)$ is a Δ -potential, $q(x) \leq s(x)$ and $-\Delta q(x) = f(x)$. $\square$

Proposition 3.1. A non-negative bisuperharmonic function that is Δ -superharmonic is the unique sum of a bipotential and a biharmonic potential up to an additive non-negative Δ -harmonic function.

Proof. Let $s \geq 0$ be bisuperharmonic, that is $-\Delta s = u$ where $u \geq 0$ is Δ^* -superharmonic. Now $u = p + v$, where p is a Δ^* -potential and $v \geq 0$ is a Δ^* -harmonic function. Then by Lemma 3.1, there exist (q, p) and (p_1, v) a bipotential pair and a biharmonic potential pair so that $-\Delta(q + p_1) = p + v = u$. Hence there exists a Δ -harmonic function h such that $s = q + p_1 + h$. Since $s \geq 0$, $-h \leq q + p_1$; since $q + p_1$ is a Δ -potential and h is Δ -harmonic, we have

$-h \leq 0$. Thus the decomposition of s as stated in the proposition.

For the uniqueness, suppose $p' + p'_1 + h'$ is another such decomposition. Since $p + p_1 + h = p' + p'_1 + h'$, we have by the uniqueness of Riesz decomposition $h = h'$. Then $-\Delta p - \Delta p_1 = -\Delta p' - \Delta p'_1$. Here again $-\Delta p$ and $-\Delta p'$ are Δ^* -potentials and $-\Delta p_1$ and $-\Delta p'_1$ are Δ^* -harmonic, so that $-\Delta p = -\Delta p'$ and $-\Delta p_1 = -\Delta p'_1$. Hence $p_1 = p'_1 +$ (a harmonic function). Since p_1, p'_1 are Δ -potentials we conclude that $p_1 = p'_1$. Thus the uniqueness of decomposition of the function s . $\square$

Lemma 3.2. *If $\{(f_n, g_n)\}$ is a sequence from $\mathcal{B}$ and if $f(x) = \lim_{n \rightarrow \infty} f_n(x)$ exists for each $x \in X$, then $g(x) = \lim_{n \rightarrow \infty} g_n(x)$ exists and $(f, g) \in \mathcal{B}$. In this case, we write $(f, g) = \lim_{n \rightarrow \infty} (f_n, g_n)$.*

Proof. For any x ,

$$g_n(x) = -\Delta f_n(x) = \sum t^*(x, y)[f_n(x) - f_n(y)] \rightarrow \sum t^*(x, y)[f(x) - f(y)],$$

when $n \rightarrow \infty$. Hence, $\lim_{n \rightarrow \infty} g_n(x) = -\Delta f(x)$. $\square$

Proposition 3.2. *Let $\{b_n\}$ be a sequence of biharmonic (respectively bisuperharmonic) functions defined on X . If $b(x) = \lim_{n \rightarrow \infty} b_n(x)$ is finite for each $x \in X$ then $b(x)$ is biharmonic on X .*

Proof. Let $\{(b_n, v_n)\}$ be a sequence of biharmonic pairs, that is $-\Delta b_n = v_n$ is Δ^* -harmonic. Suppose $\lim_{n \rightarrow \infty} b_n(x) = b(x)$ for each vertex $x \in X$. Then by the above Lemma, $\lim_{n \rightarrow \infty} v_n(x) = v(x)$ exists. Since each $v_n(x)$ is Δ^* -harmonic, so is $v(x)$; since $-\Delta b_n = v_n$, taking limits we have $-\Delta b = v$. Hence $(b, v) = \lim_{n \rightarrow \infty} (b_n, v_n)$ is a biharmonic pair, that is b is a biharmonic function. Similar proof for bisuperharmonic functions also. $\square$

Definition 3.2. *The infinite graph X with two sets of transition functions $\{t(x, y)\}$ and $\{t^*(x, y)\}$ is said to be a bipotential binetwork if there exists a positive bipotential on $\{X, t, t^*\}$.*

Proposition 3.3. *In a bipotential binetwork X if $p(x)$ is any Δ^* -potential with finite harmonic support, then there exists a unique Δ -potential $q(x)$ on X such that (q, p) is a bipotential.*

Proof. Let X be a bipotential binetwork; that is there exists a positive bipotential (u, v) on X . Now, if p is a Δ^* -potential with finite harmonic support, then for some $\alpha > 0$, $\alpha p(x) \leq v(x)$ on X by the Domination Principle. Now $u(x) = \sum_{y \in X} G(x, y)v(y) \geq \alpha \sum_{y \in X} G(x, y)p(y)$. Hence $q(x) = \sum_{y \in X} G(x, y)p(y)$ is a Δ -potential so that (q, p) is a bipotential. For the uniqueness, suppose (q_1, p) is another bipotential then $-\Delta q = -\Delta q_1 = p$ so that $h = q - q_1$ is a Δ -harmonic function. Since q and q_1 are potentials, $h = 0$. $\square$

Corollary 3.1. *In a bipotential binetwork $\{X, t, t^*\}$, for any given vertex y in X , there exists a unique bipotential $Q_y(x) = Q(x, y)$ such that $-\Delta Q_y(x) = G_y^*(x)$.*

In a bipotential binetwork $\{X, t, t^*\}$, the function $Q(x, y) = Q_y(x)$ is referred to as the biharmonic Green potential with support at the vertex y . We know that a non-negative superharmonic function is the increasing limit of a sequence of potentials. We prove here that a non-negative bisuperharmonic function is the limit of an increasing sequence of bipotentials up to an additive non-negative harmonic function.

Proposition 3.4. *Let $s = (s_1, s_2)$ be a non-negative bisuperharmonic pair. Then up to an additive non-negative harmonic function s is the limit of an increasing sequence of bipotentials. That is, $s = (h, 0) + \lim_{n \rightarrow \infty} (q_n, p_n)$ where h is a non-negative harmonic function and $\{(q_n, p_n)\}$ is an increasing sequence of bipotential pairs.*

Proof. Since $s_1 \geq 0$ and $-\Delta s_1 = s_2 \geq 0$, then s_1 is Δ -superharmonic; hence it is the sum of a Δ -potential p and a non-negative Δ -harmonic function h , so that $-\Delta p = -\Delta s_1 = s_2$. Hence, $p(x) = \sum_{y \in X} G_y(x) [-\Delta p(y)] = \sum_{y \in X} G_y(x) s_2(y)$. Now s_2 is the increasing limit of a sequence of Δ^* -potentials $\{p_n\}$. Since $p_n \leq s_2$, $q_n(x) = \sum_{y \in X} G_y(x) p_n(y)$ is a Δ -potential so that (q_n, p_n) is a bipotential pair; since $\{p_n\}$ is increasing, so is $\{q_n\}$. Also $q_n(x) \leq \sum_{y \in X} G_y(x) s_2(y) = p(x)$. Hence, $\lim_{n \rightarrow \infty} q_n(x) = q(x)$ is Δ -superharmonic; actually $q(x)$ is a Δ -potential since $q_n(x) \leq p(x)$ for each n . Note also that $q(x) = \lim_{n \rightarrow \infty} \sum_{y \in X} G_y(x) p_n(y) = \sum_{y \in X} G_y(x) s_2(y) = p(x)$, a Δ -potential. Thus, $s_1(x) = p(x) + h(x) = q(x) + h(x)$ where $q(x)$ is the limit of the increasing sequence of bipotentials q_n . Hence (s_1, s_2) can be represented as $(s_1, s_2) = (h, 0) + \lim_{n \rightarrow \infty} (q_n, p_n)$. $\square$

Corollary 3.2. *In a binetwork $\{X, t, t^*\}$, if there exists a positive biharmonic potential, then there exist bipotentials on X .*

Proof. Let u be a positive biharmonic potential, hence u is not harmonic. Now by the Proposition 3.4, u is the increasing limit of bipotentials up to an additive harmonic function. Hence there exist bipotentials on X . $\square$

Example 3.1. An example of a binetwork in which bipotentials exist but no positive biharmonic potential. Let $X = \{0, 1, 2, \dots\}$ be the infinite graph in which the transition weights are given by $t(k-1, k) = k(k+1)$, $k \geq 1$ on the edge $k-1, k$ and $t^* = t$. Thus $\{X, t, t^*\}$ is a binetwork with symmetric conductance. The network $\{X, t\}$ is hyperbolic. If $G(x, y) = G_y(x)$ is the symmetric Green function of $\{X, t\}$, we have in particular $G_0(k) = \frac{1}{k+1}$, $k \geq 0$. Further, in the binetwork $\{X, t, t^*\}$

1. Any non-negative harmonic function is constant, For if $h \geq 0$ is harmonic on X , $h(0) = a$ then $0 = \Delta h(0) = t(0, 1)[h(1) - h(0)] \Rightarrow h(1) = a$. Then $0 = \Delta h(1) = t(1, 0)[h(0) - h(1)] + t(1, 2)[h(2) - h(1)] \Rightarrow h(2) = a$. This procedure leads to the conclusion $h(n) = a$ for all n .

2. There is no positive biharmonic potential on X .

For suppose p is a potential and $h > 0$ a harmonic function such that $\Delta p = -h$ on X . Since all positive harmonic functions are constant, we can take $\Delta p = -1$. This means $p(n) = \sum_k [-\Delta p(k)] G_k(n) = \sum_k G_k(n)$ for all $n \geq 0$, hence $p(0) = \sum_k G_k(0) = \sum_k G_0(k)$ because of the symmetry of conductance. Now $G_0(k) = \frac{1}{k+1}$ which shows that $p(0) = \infty$, a contradiction. Hence there cannot be any non-zero biharmonic potential on X .

3. However there exist bipotentials on X . For, let $G_0(k)$ be the Green potential with harmonic support at $\{0\}$. Then $s(n) = \sum_k G_0(k) G_k(n)$ is a potential if $s(n)$ is finite at one vertex. Now $s(0) = \sum_k G_0(k) G_k(0) = \sum_k G_0(k) G_0(k) = \sum \frac{1}{(k+1)^2} < \infty$. Hence $s(n)$ is a potential such that $-\Delta s(n) = G_0(n)$ so that $s(n)$ is a bipotential on X .

Remark 3.1. In the continuous case, the above example is an analogue of the Euclidean spaces $\mathbb{R}^n, n \geq 5$.

Example 3.2. An example of a hyperbolic network which is not a bipotential network: Consider $X = \{1, 2, \dots\}$ in which the symmetric transition functions are given by $t(k, k+1) = t(k+1, k) = \frac{\sqrt{k(k+1)}}{\sqrt{k+1}-\sqrt{k}}$ for $k \geq 1$. Consider the function $u(k) = \frac{1}{\sqrt{k}}$ for $k \geq 1$. Then, $\Delta u(1) = \frac{\sqrt{2}}{\sqrt{2}-1}(\frac{1}{\sqrt{2}} - 1) = -1$ and $\Delta u(k) = \frac{\sqrt{k(k+1)}}{\sqrt{k+1}-\sqrt{k}}(\frac{1}{\sqrt{k+1}} - \frac{1}{\sqrt{k}}) + \frac{\sqrt{k(k-1)}}{\sqrt{k}-\sqrt{k-1}}(\frac{1}{\sqrt{k-1}} - \frac{1}{\sqrt{k}}) = -1 + 1 = 0$. Thus, $u(k) > 0$ is superharmonic on X with harmonic support at $k = 1$. Since $u(k)$ tends to 0 when k tends to ∞ , $u(k)$ is a potential. Consequently if $G_l(k) = G(k, l)$ is the symmetric Green function on X , then $G_1(k) = u(k) = \frac{1}{\sqrt{k}}$. Now suppose X is a bipotential network, then there exists a potential $p > 0$ on X such that $p(k) = \sum_l G(k, l) G_1(l)$ so that $p(1)$ should be finite. But $p(1) = \sum_l G(1, l) G_1(l) = \sum_l [G_1(l)]^2 = \sum_l \frac{1}{l} = \infty$, contradicting the assertion that $p(1)$ should be finite. Hence $\{X, t\}$ is hyperbolic but not a bipotential network.

Remark 3.2. In the continuous case, the above example is an analogue of the Euclidean spaces $\mathbb{R}^3$ and $\mathbb{R}^4$.

Restricting the natural order for the following of real-valued functions on a network $\{X, t\}$, we shall now introduce a specific order for the family of real-valued functions defined on the network $\{X, t\}$. This specific order is useful in the study of functions on a binetwork $\{X, t, t^*\}$.

Definition 3.3. For two real-valued functions f, g defined on a network $\{X, t\}$ write $f \succ g$ if and only if $f \geq g$ and $-\Delta f \geq -\Delta g$. We refer to $\succ$ as the specific order defined on the functions on X .

Proposition 3.5. In a network $\{X, t\}$, $f \succ g$ if and only if there exists a Δ -superharmonic function $s \geq 0$ such that $f = g + s$.

Proof. If there exists a Δ -superharmonic function $s \geq 0$ such that $f = g + s$, then $f \geq g$ and $-\Delta f = -\Delta g - \Delta s$. Since $-\Delta s \geq 0$ we have $-\Delta f \geq -\Delta g$ and $f \geq g$.

Conversely suppose $f \geq g$ and $-\Delta f \geq -\Delta g$. Then $-\Delta(f - g) \geq 0$. That is $f - g = s$ is a superharmonic function. Which is non-negative. $\square$

Proposition 3.6. *A bisuperharmonic function $u \succ 0$ is the unique sum of a bipotential and a biharmonic function $h \succ 0$.*

Proof. $-\Delta u = s \geq 0$ is superharmonic. We have proved (Proposition 3.1) that u is the unique sum of a bipotential q and a biharmonic potential H and a non-negative harmonic function h . Since $H + h \succ 0$ is biharmonic, we have proved the proposition. $\square$

Proposition 3.7. *A biharmonic function $h \succ 0$ is the unique sum of a biharmonic potential and a non-negative harmonic function.*

Proof. Since $h \succ 0$, then h is a non-negative superharmonic function so that $h = (a \text{ potential})p + (a \text{ harmonic function } u \geq 0)$. Now, $-\Delta p = -\Delta h$; since h is biharmonic, Δh is harmonic so that $-\Delta p$ is a non-negative harmonic function. Hence p is a biharmonic potential. $\square$

Proposition 3.8. *A bisuperharmonic function $s \succ 0$ is a bipotential if and only if for any bisubharmonic function $u \prec s$ we have $u \prec 0$.*

Proof. Let s be a bipotential and the bisubharmonic function $u \prec s$. Since $-\Delta u \leq -\Delta s$ where $-\Delta u$ is subharmonic and $-\Delta s$ is a potential, we have $-\Delta u \leq 0$. That is u is subharmonic. Since $u \leq s$ where s is a potential, we have $u \leq 0$. Hence $u \prec 0$.

Conversely, let $s \succ 0$ be bisuperharmonic such that for any bisubharmonic $u \prec s$ we have $u \prec 0$. Then by Proposition 3.6, s is the unique sum of a bipotential and a biharmonic function $h \succ 0$. But $s \succ h$ so that by the present assumption $h \prec 0$. Consequently $h = 0$ which implies that s is a bipotential. $\square$

Consequences:

1. Let $s \succ 0$ be bisuperharmonic. Then s is the unique sum of a bipotential q and a biharmonic function $H \succ 0$. Then H is the greatest biharmonic minorant of s for the order $\succ$. For, let u be biharmonic and $u \prec s = q + H$. Then $u - H \prec q$. Since q is a bipotential, we have $u - H \prec 0$, that is $u \prec H$.
2. The bisuperharmonic function $s \succ 0$ is a bipotential if and only if its greatest biharmonic minorant is 0.

Recall the following known result: Let $\{p_n\}$ be a sequence of potentials such that $\sum p_n(x)$ is finite at one vertex, then $\sum p_n(x)$ is a potential.

Proof. Since $\sum p_n(x)$ is finite at one vertex, the sum represents a non-negative superharmonic function which is finite at every vertex. Let now u be a subharmonic function such that $u(x) \leq \sum_{n=1}^{\infty} p_n(x)$. Then $u(x) - \sum_{n=2}^{\infty} p_n(x) \leq p_1(x)$. Here the left side is subharmonic and the right side is a potential, hence $u(x) \leq \sum_2^{\infty} p_n(x)$. Proceeding in the same way, we conclude that for any given $\epsilon > 0$, $u(x) \leq \sum_m^{\infty} p_n(x) < \epsilon$ for some m . This leads to the conclusion $u \leq 0$ on X , so that $\sum_{n=1}^{\infty} p_n(x)$ is a potential on X . $\square$

Proposition 3.9. *Let $\{s_n\}$ be a sequence of bipotentials. Suppose $\sum_{n=1}^{\infty} s_n(x)$ is finite at one vertex. Then $\sum_{n=1}^{\infty} s_n(x)$ is a bipotential.*

Proof. Since each $s_n(x)$ is a potential, $s(x) = \sum_{n=1}^{\infty} s_n(x)$ is a potential. Write $p_n(x) = \sum_{k=1}^n s_k(x)$. Then

$$\begin{aligned} -\Delta s(x) &= -\Delta \left[\lim_{n \rightarrow \infty} p_n(x) \right] \\ &= \lim_{n \rightarrow \infty} [-\Delta p_n(x)] \\ &= \lim_{n \rightarrow \infty} \left[\sum_{k=1}^n -\Delta s_k(x) \right] \\ &= \sum_{k=1}^{\infty} [-\Delta s_k(x)]. \end{aligned}$$

Since each $-\Delta s_k(x)$ is a potential and $-\Delta s(x)$ is finite. We conclude that $-\Delta s(x)$ is also a potential. Then $s(x)$ and $-\Delta s(x)$ are potentials. That is $s(x) = \sum_{n=1}^{\infty} s_n(x)$ is a bipotential. $\square$

The cone of bipotentials and the cone of biharmonic potentials are lattices for the specific order $\prec$.

Proposition 3.10. *Let $q_1 \succ 0$, $q_2 \succ 0$ be bipotentials. Then $q_1 \vee q_2$ and $q_1 \wedge q_2$ are bipotentials.*

Proof. Write $q_i(x) = \sum_{y \in X} G_y(x) p_i(y)$ where $p_i(x)$ are two potentials. Then, for the natural order

$$p_3(x) = \inf(p_1(x), p_2(x))$$

is a potential and $s(x) = \sum_{y \in X} G_y(x) p_3(y) \succ 0$ is a bipotential. Note that, $s \prec q_1$ and $s \prec q_2$. Now, suppose u is a bipotential such that $u \prec q_1$ and $u \prec q_2$. Then, $-\Delta u \leq \inf(-\Delta q_1, -\Delta q_2) = p_3 = -\Delta s$; also since $-\Delta u \leq -\Delta s$, $(u - s)$ is a subharmonic function v ; since u is a potential and $v \leq u$, then $v \leq 0$. Thus, $u \leq s$. Consequently, $u \prec s$, showing that $s = q_1 \wedge q_2$. On the other hand, let $p_4(x)$ is the infimum for the natural order of the family of superharmonic functions $u(x) \geq \sup(p_1(x), p_2(x))$. Then $p_4(x)$ is a non-negative superharmonic function; in fact, $p_4(x)$ is a potential since it is majorised by the potential $p_1(x) + p_2(x)$. Clearly $q_1 \vee q_2(x) = \sum_{y \in X} G_y(x) p_4(y)$ is a bipotential. $\square$

Remark. The extreme elements for the cone of bipotentials being proportional to $Q_z(x) = \sum_{y \in X} G_y(x)G_z(y)$, $z \in X$, a real-valued function $s(x)$ is a bipotential if and only if it is of the form $s(x) = \sum_z a(z)Q_z(x)$ for suitable non-negative constants $\{a(z)\}$.

Proposition 3.11. *Let $B_1 \succ 0, B_2 \succ 0$ be biharmonic potentials. Then $B_1 \vee B_2$ and $B_1 \wedge B_2$ are biharmonic potentials.*

Proof. Write $B_i(x) = \sum_{y \in X} G_y(x)H_i(y)$ where $H_i(x)$ are two non-negative harmonic functions. Then for the natural order let $H_3(x)$ be the greatest harmonic minorant of the superharmonic function $\inf(H_1, H_2)$; and $H_4(x)$ be the least harmonic majorant of the subharmonic function $\sup(H_1, H_2)$. Clearly, $B_1 \wedge B_2(x) = \sum_{y \in X} G_y(x)H_3(y)$ and $B_1 \vee B_2(x) = \sum_{y \in X} G_y(x)H_4(y)$ are biharmonic potentials. $\square$

4. Poisson networks

Definition 4.1. *An infinite network X is called a Poisson network if for any non-negative subharmonic function u , there exists a solution $v(x)$ to the Poisson equation $\Delta v(x) = u(x)$ for $x \in X$.*

Example 4.1. Let the network X be an infinite tree without any terminal vertices. Then, for any real-valued function $f(x)$ on X , there exists a function $u(x)$ such that $\Delta u(x) = f(x)$ (see, [2][Theorem 5.1.4]). Hence, any infinite tree X , hyperbolic or parabolic, without terminal vertices is a Poisson network.

In an infinite network, let us denote by $\mathfrak{F}$ the class of real-valued functions $f(x)$ such that $-\Delta g(x) = f(x)$ has a solution.

Remark 4.1. Let X be a Poisson network:

1. If h is any harmonic function, there exists $s(x)$ such that $-\Delta s(x) = h(x)$

Proof. Note that, $h = h^+ - h^-$ where h^+ and h^- are non-negative subharmonic functions. This implies there exist v_1, v_2 such that $\Delta v_1 = h^+$ and $\Delta v_2 = h^-$. Then $\Delta(v_1 - v_2) = h$. Take $s = v_2 - v_1$ so that $-\Delta s(x) = h(x)$. $\square$

2. Let s be a superharmonic function majorized by a harmonic function h . Then $s \in \mathfrak{F}$.

Proof. Let $u = h - s$. Then $u \geq 0$ is subharmonic so that for some function f , $\Delta f = u$. Now h being harmonic, $\Delta g = h$ has a solution. Take $v = f - g$ so that $-\Delta v = s$. That is $s \in \mathfrak{F}$. $\square$

Consequently, for any $y \in X$, if $G_y(x)$ is the Green potential with harmonic support $\{y\}$, then $-\Delta v(x) = G_y(x)$ has a solution.

Note. If $v(x)$ is non-negative in the above equation $-\Delta v(x) = G_y(x)$ for some vertex $\{y\}$ then X is a bipotential network.

Recall that a superharmonic function $s(x)$ on an infinite network is said to be admissible if there exists a subharmonic minorant for s outside a finite set.

Definition 4.2. A hyperbolic Poisson network X is said to be singular, if for some vertex y , $-\Delta v(x) = G_y(x)$ and $v(x)$ is not admissible.

Note. Poisson network is a bipotential network if and only if it is non-singular.

Proposition 4.1. In a singular Poisson network, if for a superharmonic function $s > 0$ we have $-\Delta v(x) = s(x)$, then the superharmonic function v is not admissible.

Proof. Since X is singular Poisson, for some vertex y , $-\Delta u(x) = G_y(x)$ and $u(x)$ is not admissible. Now by the Domination Principle (see Theorem 3.3.6 [2]), for some $\alpha > 0$, $\alpha s(x) \geq G_y(x)$. That is $-\Delta[\alpha v(x)] \geq -\Delta u(x)$, hence $u(x) - \alpha v(x) = f(x)$ is subharmonic. Now if v were admissible, then $\alpha v(x)$ will have a subharmonic minorant outside a finite set; Consequently $u(x)$ has a subharmonic minorant outside a finite set. That is $u(x)$ is admissible, contradicting the assumption. $\square$

Homogeneous trees: let T be a homogeneous tree, that is each vertex has only $(q + 1)$ neighbours, $q \geq 2$. Then as mentioned earlier, since there are no terminal vertices, T is a Poisson network. For a vertex $e \in T$, we have as in Cartier [5] the Green potential $G(x, e) = G_e(x)$.

Lemma 4.1. In a hyperbolic network (X, t) , let $h(x)$ be harmonic outside a finite set. Then $h = H + p_1 - p_2$ outside a finite set, where H is harmonic on X and p_1, p_2 are potentials on X with finite harmonic support.

Proof. Let h be defined on $X \setminus \overset{\circ}{E}$ where E is a finite set. Let v be the Dirichlet solution on E such that $\Delta v = 0$ every vertex in $\overset{\circ}{E}$ and $v = h$ on ∂E . Then the function u on X extending v on E by h on $X \setminus E$ is such that $\Delta u = 0$ on X except for a finite number of vertices $\{a_i\}$ on ∂E . Let $\Delta u(a_i) = \alpha_i \neq 0$. Then $H(x) = u(x) + \sum_i \alpha_i G_{a_i}(x)$ is harmonic on X . Take $p_1(x) = \sum_i (-\alpha_i) G_{a_i}(x)$ if $\alpha_i < 0$ and $p_2(x) = \sum_i \alpha_i G_{a_i}(x)$ if $\alpha_i > 0$. Then $p_1(x)$ and $p_2(x)$ are bounded potentials with finite harmonic support. Consequently outside the finite set E , $h(x) = H(x) + p_1(x) - p_2(x)$. $\square$

Proposition 4.2. Let (X, t) be a non-singular Poisson network. Let $b(x)$ be biharmonic outside a finite set. Then $b = B + q_1 - q_2 + p_1 - p_2$ outside a finite set, where B is biharmonic on X ; q_1, q_2 are bipotentials with finite biharmonic support on X ; and p_1, p_2 are potentials with finite harmonic support on X .

Proof. The function $h(x) = -\Delta b(x)$ is harmonic outside a finite set. Hence $h = H' + p'_1 - p'_2$ outside a finite set, as in the Lemma 4.1. Since X is Poisson, if $-\Delta u = H'$, $-\Delta q_1 = p'_1$ and $-\Delta q_2 = p'_2$ then u is biharmonic on X and q_1, q_2 are bipotentials with finite biharmonic support on X . Since $-\Delta b = h = -\Delta(u + q_1 - q_2)$ outside a finite set, $b = u + q_1 - q_2 + v$ outside a finite set where v is harmonic so that $v = H + p_1 - p_2$ as in the Lemma 4.1. Write $B = u + H$ to get the decomposition $b = B + q_1 - q_2 + p_1 - p_2$ outside a finite set. $\square$

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Bach tensor on $N(\kappa)$ -paracontact metric 3-manifolds

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Abstract. In this paper, we characterize the Bach tensor on $N(\kappa)$ -paracontact metric 3-manifold. It is proved that a $N(\kappa)$ -paracontact metric 3-manifold with purely transversal Bach tensor is of constant scalar curvature 6κ .

Keywords: $N(\kappa)$ -paracontact metric 3-manifold, Cotton tensor, Bach tensor, Scalar curvature.

1. Introduction

The notion of Bach tensor was introduced by Rudolf Bach in [1] when studying so-called conformal relativity. That is, instead of using the Hilbert-Einstein functional, one considers the functional

$$(1) \quad \mathcal{W}(g) = \int_{M^4} |W(g)|^2 dv_g,$$

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for a 4-dimensional compact Riemannian manifolds M , where W denotes the Weyl tensor of type $(1, 3)$ defined by

$$\begin{aligned} W(X, Y)Z &= R(X, Y)Z - \frac{1}{2n-1}\{S(Y, Z)X - S(X, Z)Y \\ &+ g(Y, Z)QX - g(X, Z)QY\} \\ (2) \quad &+ \frac{r}{2n(2n-1)}\{g(Y, Z)X - g(X, Z)Y\}, \end{aligned}$$

where R denotes the Riemannian curvature tensor, S is a Ricci tensor and Ricci operator Q is defined by $g(QX, Y) = S(X, Y)$. The Critical points of the functional (1) are characterized by the vanishing of a symmetric trace free $(0, 2)$ type tensor $\mathcal{B}$ is usually referred as Bach tensor and the metric is called Bach flat if $\mathcal{B}$ vanishes. On any Riemannian manifold (M, g) of dimension $(2n + 1)$, the Bach tensor $\mathcal{B}$ is defined by

$$\begin{aligned} \mathcal{B}(X, Y) &= \frac{1}{2n-2} \sum_{i,j=1}^{2n+1} ((\nabla_{e_i} \nabla_{e_j} W)(X, e_i, e_j, Y) \\ (3) \quad &+ \frac{1}{2n-1} \sum_{i,j=1}^{2n+1} S(e_i, e_j)W(X, e_i, e_j, Y), \end{aligned}$$

where $\{e_i\}_{i=1}^{2n+1}$ is a local orthonormal frame on (M, g) . A Riemannian manifold M is said to be Einstein if the Ricci tensor S is a constant multiple of the metric tensor g . Now, from (2) and the contraction of Bianchi second identity it follows that $\text{div}W = \frac{2n-2}{2n-1}\mathcal{C}$, where $\mathcal{C}$ is the $(0, 3)$ -type Cotton tensor tensor defined by [2]

$$\begin{aligned} \mathcal{C}(X, Y)Z &= (\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z) \\ (4) \quad &- \frac{1}{4n}\{dr(X)g(Y, Z) - dr(Y)g(X, Z)\}. \end{aligned}$$

Now, making use of Weyl tensor (2) and Cotton tensor (4), the Bach tensor (3) can be expressed as [3]

$$(5) \quad \mathcal{B}(X, Y) = \frac{1}{2n-1} \left[\sum_{i=1}^{2n+1} (\nabla_{e_i} \mathcal{C})(e_i, X)Y + \sum_{i,j=1}^{2n+1} S(e_i, e_j)W(X, e_i, e_j, Y) \right].$$

In dimension 3, the Weyl tensor W vanishes, and hence the expression of Bach tensor transforms into

$$(6) \quad \mathcal{B}(X, Y) = \sum_{i=1}^3 (\nabla_{e_i} \mathcal{C})(e_i, X)Y.$$

A Riemannian metric g is called Bach flat if the Bach tensor $\mathcal{B}$ of g vanishes. By (3), it is easy to see that Bach flatness is natural generalization of Einstein

and conformal flatness. For more details about Bach tensor, we refer to reader [4, 5, 7, 8] and references therein.

On the other hand, study of nullity distribution on paracontact geometry is one among the most interesting topics in modern paracontact geometry. In 1985, Kaneyuki and Kozai [10] initiated the study of paracontact geometry. The importance of paracontact geometry interplays with the theory of para-Kähler manifolds and its role in pseudo-Riemannian geometry and mathematical physics. A systematic study of paracontact metric manifolds was carried out by Zamkovoy [19]. Further, the study was taken up by many authors, for readers we refer the papers [9, 11, 12, 13, 14, 15, 16, 17, 18, 20] and others.

Recently, Ghosh and Sharma in [6] studied Sasakian manifold with purely transversal Bach tensor. In particular, they proved that if a Sasakian manifold admits a purely transversal Bach tensor, then g has constant scalar curvature $\geq 2n(2n+1)$, with equality holds if and only if g is Einstein, and the Ricci tensor of g has a constant norm. Also, they studied (κ, μ) -contact manifolds with vanishing Bach tensor in [7]. This work turns our attention to study Bach tensor in the framework of certain class of paracontact metric manifolds, particularly, on 3-dimensional $N(\kappa)$ -paracontact metric manifolds (briefly, $N(\kappa)$ -paracontact metric 3-manifolds).

The paper is organized as follows: Section 2 is concerned with the basic formulas and properties of $N(\kappa)$ -paracontact metric manifolds. Section 2 is concerned with the study of $N(\kappa)$ -paracontact metric 3-manifolds with purely transversal Bach tensor. It is proved that a $N(\kappa)$ -paracontact metric 3-manifold with purely transversal Bach tensor is of constant scalar curvature.

2. Preliminaries

An almost paracontact structure on a $(2n+1)$ -dimensional smooth manifold M^{2n+1} is a triplet (ϕ, ξ, η) , where ϕ is a $(1, 1)$ -type tensor field, ξ is a vector field called the Reeb vector field and η is a 1-form such that:

- (i) $\phi^2 X = X - \eta(X)\xi$,
- (ii) $\phi(\xi) = 0, \eta \cdot \phi = 0, \eta(\xi) = 1$,
- (iii) the tensor field ϕ induces an almost paracomplex structure on each fibre of $\mathcal{D} = \ker(\eta)$, that is, the eigendistributions $\mathcal{D}_\phi^+$ and $\mathcal{D}_\phi^-$ of ϕ corresponding to the eigenvalues 1 and -1, respectively, have same dimension n .

An almost paracontact manifold equipped with a pseudo-Riemannian metric g such that

$$(7) \quad g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y),$$

for all $X, Y \in \chi(M)$, is called almost paracontact metric manifold and (ϕ, ξ, η, g) is said to be an almost paracontact metric structure. An almost paracontact structure is normal [30] if and only if the $(1, 2)$ -type torsion tensor $N_\phi = [\phi, \phi] - 2d\eta \otimes \xi = 0$, where $[\phi, \phi](X, Y) = \phi^2[X, Y] - [\phi X, \phi Y]$. An almost paracontact structure is called a paracontact structure if $g(X, \phi Y) = d\eta(X, Y)$

[20]. Any almost paracontact metric manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$ admits a φ basis [20], that is, a pseudo-orthonormal basis of vector fields of the form $\xi, E_1, E_2, \dots, E_n, \varphi E_1, \varphi E_2, \dots, \varphi E_n$, where $\xi, E_1, E_2, \dots, E_n$ are space-like vector fields and then, by (2.1) the vector fields $\varphi E_1, \varphi E_2, \dots, \varphi E_n$ are time-like. For a three-dimensional almost paracontact metric manifold, any (local) pseudo-orthonormal basis of $\ker(\eta)$ determines a φ -basis, up to sign. If e_2, e_3 is a (local) pseudo-orthonormal basis of $\ker(\eta)$, with e_3 , time-like, so by (7) vector field $\varphi e_2 \in \ker(\eta)$ is time-like and orthogonal to e_2 . Therefore, $\varphi e_2 = \pm e_3$ and $\xi, e_2, \pm e_3$ is a φ -basis [13]. In a paracontact metric manifold, one can introduce a symmetric, trace-free $(1, 1)$ -tensor $h = \frac{1}{2}\mathcal{L}_\xi \varphi$ such that $\varphi h + h\varphi = 0$ and $h\xi = 0$ satisfying [12, 16, 19, 20]

$$\begin{aligned} (8) \quad \nabla_X \xi &= -\varphi X + \varphi hX, \\ (9) \quad (\nabla_X \eta)Y &= g(X - hX, \varphi Y), \\ (10) \quad (\nabla_X \varphi)Y &= -g(X - hX, Y)\xi + \eta(Y)(X - hX), \end{aligned}$$

for any $X, Y \in \chi(M)$. Note that, the condition $h = 0$ is equivalent to ξ being Killing vector field and then (φ, ξ, η, g) is said to be K -paracontact structure. A paracontact metric manifold is said to be a (κ, μ) -paracontact metric manifold if the curvature tensor R satisfies

$$(11) \quad R(X, Y)\xi = \kappa(\eta(Y)X - n(X)Y) + \mu(\eta(Y)hX - n(X)hY),$$

for all vector fields $X, Y \in \chi(M)$ and κ, μ are real constants. In particular, if $\mu = 0$, then the (κ, μ) -paracontact metric manifold reduces to an $N(\kappa)$ -paracontact metric manifold. Thus, for an $N(\kappa)$ -paracontact metric manifold the Riemannian curvature tensor R satisfies

$$(12) \quad R(X, Y)\xi = \kappa(\eta(Y)X - n(X)Y).$$

In a 3-dimensional $N(\kappa)$ -paracontact metric manifold $(M^3, \varphi, \xi, \eta, g)$, the following relations hold [14]:

$$\begin{aligned} (13) \quad QX &= \left(\frac{r}{2} - \kappa\right)X + \left(3\kappa - \frac{r}{2}\right)\eta(X)\xi, \\ (14) \quad S(X, Y) &= \left(\frac{r}{2} - \kappa\right)g(X, Y) + \left(3\kappa - \frac{r}{2}\right)\eta(X)\eta(Y), \\ (15) \quad S(X, \xi) &= 2\kappa\eta(X), \\ (16) \quad Q\xi &= 2\kappa\xi, \end{aligned}$$

for all vector fields $X, Y \in \chi(M)$, where Q and S are the Ricci operator and the Ricci tensor, respectively.

The covariant differentiation of (14) along any arbitrary vector field Z gives

$$\begin{aligned} (\nabla_Z S)(X, Y) &= \frac{dr(Z)}{2}[g(X, Y) - \eta(X)\eta(Y)] \\ &+ \left(3\kappa - \frac{r}{2}\right)[(\nabla_Z \eta)(Y)\eta(X) + (\nabla_Z \eta)(X)\eta(Y)]. \end{aligned}$$

Making use of (9) and replacing X by φX in above relation, we obtain

$$(17) \quad \begin{aligned} (\nabla_Z S)(\varphi X, Y) &= \frac{dr(Z)}{2} [g(\varphi X, Y) - \eta(X)\eta(Y)] \\ &+ \left(3\kappa - \frac{r}{2}\right) [g(Z - hZ, X)\eta(Y) - \eta(X)\eta(Y)\eta(Z)]. \end{aligned}$$

In an orthogonal frame field $\{e_i\}_{i=1}^3$ of tangent space M , setting $X = Z = e_i$ in (17) and taking summation over i and then using a fact that $Tr(h) = 0$, we get

$$(18) \quad \begin{aligned} \sum_{i=1}^3 (\nabla_{e_i} S)(\varphi e_i, Y) &= \sum_{i=1}^3 g((\nabla_{e_i} Q)\varphi e_i, Y) \\ &= (6\kappa - r)\eta(Y) - \frac{1}{2}dr(\varphi Y). \end{aligned}$$

Again, setting $X = Y = e_i$ in (17) and taking summation over i and then using the fact that $Tr(\varphi) = 0, h\xi = 0$, we get

$$(19) \quad \sum_{i=1}^3 (\nabla_Z S)(\varphi e_i, e_i) = \sum_{i=1}^3 g((\nabla_Z Q)\varphi e_i, e_i) = 0.$$

Next, replacing X by hX in the relation (17), we obtain

$$(20) \quad (\nabla_Z S)(\varphi hX, Y) = \frac{dr(Z)}{2} g(\varphi hX, Y) + \left(3\kappa - \frac{r}{2}\right) g(Z - hZ, hX)\eta(Y).$$

Taking $X = Z = e_i$ in (20) and taking summation over i and then using a fact that $Tr(h) = Tr(h^2) = 0$, we get

$$(21) \quad \sum_{i=1}^3 (\nabla_{e_i} S)(\varphi h e_i, Y) = \sum_{i=1}^3 g((\nabla_{e_i} Q)\varphi h e_i, Y) = \frac{1}{2}dr(\varphi h Y).$$

Also, by setting $X = Y = e_i$ and taking summation over i and then using the fact that $Tr(\varphi h) = 0, h\xi = 0$, we obtain

$$(22) \quad \sum_{i=1}^3 (\nabla_Z S)(\varphi h e_i, e_i) = \sum_{i=1}^3 g((\nabla_Z Q)\varphi h e_i, e_i) = 0.$$

Now, we recall some results on 3-dimensional $N(\kappa)$ -paracontact metric manifolds which will be useful for the next section:

Lemma 2.1 ([14]). *An $N(\kappa)$ -paracontact metric 3-manifold $(M^3, \varphi, \xi, \eta, g)$ is locally φ -symmetric if and only if the scalar curvature r of g is constant.*

Lemma 2.2 ([18]). *On any $N(\kappa)$ -paracontact metric 3-manifold $\xi r = 0$.*

3. Bach tensor on $N(\kappa)$ -paracontact metric 3-manifolds

In this section, we aim to study $N(\kappa)$ -paracontact metric 3-manifold with purely transversal Bach tensor, i.e., $\mathcal{B}(X, \xi) = 0$. First, we present the following result:

Lemma 3.1. *For a $N(\kappa)$ -paracontact metric 3-manifold M^3 , the following relation hold:*

$$\begin{aligned}
 \sum_{i=1}^3 (\nabla_{e_i} \mathcal{C})(e_i, Y) \xi &= 3(6\kappa - r)\eta(Y) - \frac{3}{2}dr(\varphi Y) - \frac{1}{2}dr(\varphi hY) \\
 &+ \sum_{i=1}^3 g((\nabla_{e_i} \varphi h)Y, Qe_i) - \sum_{i=1}^3 g((\nabla_{e_i} \varphi h)e_i, QY) \\
 (23) \quad &- \frac{1}{4}g(\nabla_\xi Dr, Y).
 \end{aligned}$$

Proof. In a 3-dimensional manifold (M^3, g) , for $X, Y, Z \in \chi(M)$ the Cotton tensor is given by

$$\begin{aligned}
 \mathcal{C}(X, Y)Z &= (\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z) \\
 (24) \quad &- \frac{1}{4}\{dr(X)g(Y, Z) - dr(Y)g(X, Z)\}.
 \end{aligned}$$

Replacing Z by ξ in (24), we get

$$\begin{aligned}
 \mathcal{C}(X, Y)\xi &= (\nabla_X S)(Y, \xi) - (\nabla_Y S)(X, \xi) \\
 (25) \quad &- \frac{1}{4}\{dr(X)\eta(Y) - dr(Y)\eta(X)\}.
 \end{aligned}$$

Differentiating covariantly (16) along an arbitrary vector field X , we obtain

$$(26) \quad (\nabla_X Q)\xi = 2\kappa\nabla_X \xi - Q\nabla_X \xi.$$

Using (8) in the above equation, we get

$$(27) \quad (\nabla_X Q)\xi = -2\kappa(\varphi X - \varphi hX) + Q(\varphi X - \varphi hX).$$

Take an inner product of (27) with respect to vector field Y , we obtain

$$\begin{aligned}
 g((\nabla_X Q)\xi, Y) &= (\nabla_X S)(Y, \xi) = -2\kappa g(\varphi X - \varphi hX, Y) \\
 (28) \quad &+ g(\varphi X - \varphi hX, QY).
 \end{aligned}$$

Making use of (28) in (25) shows that

$$\begin{aligned}
 \mathcal{C}(X, Y)\xi &= -4\kappa g(\varphi X, Y) + 2g(Q\varphi X, Y) - g(Q\varphi hX, Y) + g(Q\varphi hY, X) \\
 (29) \quad &- \frac{1}{4}[dr(X)\eta(Y) - dr(Y)\eta(X)].
 \end{aligned}$$

Taking covariant differentiation of $\mathcal{C}(X, Y)\xi$ along an arbitrary vector field W , we get

$$(30) \quad \begin{aligned} (\nabla_W \mathcal{C})(X, Y)\xi &= \nabla_W \mathcal{C}(X, Y)\xi - \mathcal{C}(\nabla_W X, Y)\xi \\ &- \mathcal{C}(X, \nabla_W Y)\xi - \mathcal{C}(X, Y)\nabla_W \xi. \end{aligned}$$

Using (27) and (29) in (30), we deduce

$$\begin{aligned} (\nabla_W \mathcal{C})(X, Y)\xi &= \mathcal{C}(X, Y)\varphi W - \mathcal{C}(X, Y)\varphi hW - 4\kappa g((\nabla_W \varphi)X, Y) \\ &+ 2[g((\nabla_W Q)\varphi X, Y) + g(Q(\nabla_W \varphi)X, Y)] \\ &- g((\nabla_W Q)\varphi hX, Y) - g(Q(\nabla_W \varphi h)X, Y) \\ &+ g((\nabla_W Q)\varphi hY, X) + g(Q(\nabla_W \varphi h)Y, X) \\ &- \frac{1}{4}[dr(X)(\nabla_W \eta)(Y) - dr(Y)(\nabla_W \eta)(X) \\ &- g(\nabla_W Dr, X)\eta(Y) + g(\nabla_W Dr, Y)\eta(X)]. \end{aligned}$$

From the relation (9), (10) and (24) above equation becomes

$$\begin{aligned} (\nabla_W \mathcal{C})(X, Y)\xi &= (\nabla_X S)(Y, \varphi W) - (\nabla_Y S)(X, \varphi W) - (\nabla_X S)(Y, \varphi hW) \\ &+ (\nabla_Y S)(X, \varphi hW) - 4\kappa g(W - hW, Y)\eta(X) \\ &+ 2g(W - hW, QY)\eta(X) + 2g((\nabla_W Q)\varphi X, Y) \\ &- g((\nabla_W Q)\varphi hX, Y) - g(Q(\nabla_W \varphi h)X, Y) \\ &+ g((\nabla_W Q)\varphi hY, X) + g(Q(\nabla_W \varphi h)Y, X) \\ (31) \quad &+ \frac{1}{4}[g(\nabla_W Dr, X)\eta(Y) - g(\nabla_W Dr, Y)\eta(X)]. \end{aligned}$$

Setting $X = W = e_i$ in (31) and summing over i and then use of the relations (18), (19), (21) and (22) proves our lemma. $\square$

Now, we state and prove the main result of this paper:

Theorem 3.1. *If an $N(\kappa)$ -paracontact metric 3-manifold M^3 with $\kappa > -1$ has purely transversal Bach tensor, then the scalar curvature r is constant.*

Proof. In an $N(\kappa)$ -paracontact metric 3-manifold with $\kappa > -1$, we have

$$(32) \quad \begin{aligned} (\nabla_X \varphi h)Y &= (1 + \kappa)[g(X, Y)\xi - \eta(X)\eta(Y)\xi] - g(hX, Y)\xi \\ &+ (1 + \kappa)\eta(Y)[X - \eta(X)\xi] - \eta(Y)hX. \end{aligned}$$

Taking an inner product of (32) with respect to QZ , we get

$$(33) \quad \begin{aligned} g((\nabla_X \varphi h)Y, QZ) &= 2\kappa(1 + \kappa)[g(X, Y)\eta(Z) - 2\eta(X)\eta(Y)\eta(Z)] \\ &- 2\kappa g(hX, Y)\eta(Z) + (1 + \kappa)g(X, QZ)\eta(Y) \\ &- g(hX, QZ)\eta(Y). \end{aligned}$$

Take an orthogonal frame field $\{e_i\}$, $i = 1, 2, 3$ of tangent space M^3 , setting $X = Y = e_i$ in (33) and then summing over i , we obtain

$$(34) \quad \sum_{i=1}^3 g((\nabla_{e_i} \varphi h)e_i, QZ) = 4\kappa(1 + \kappa)\eta(Z).$$

Again, take an orthogonal frame field $\{e_i\}$, $i = 1, 2, 3$ of tangent space M^3 , setting $X = Z = e_i$ in (33) and then summing over i gives

$$(35) \quad \sum_{i=1}^3 g((\nabla_{e_i} \varphi h)Y, Qe_i) = (1 + \kappa)(r - 2\kappa)\eta(Y).$$

Replacing Z by Y , in (34) and subtracting it in (35), gives

$$(36) \quad \sum_{i=1}^3 g((\nabla_{e_i} \varphi h)Y, Qe_i) - \sum_{i=1}^3 g((\nabla_{e_i} \varphi h)e_i, QY) = (1 + \kappa)(r - 6\kappa)\eta(Y).$$

Making use of (36) in (23) yields

$$(37) \quad \begin{aligned} \sum_{i=1}^3 (\nabla_{e_i} \mathcal{C})(e_i, Y)\xi &= (6\kappa - r)(2 - \kappa)\eta(Y) - \frac{3}{2}dr(\varphi Y) \\ &- \frac{1}{2}dr(\varphi hY) - \frac{1}{4}g(\nabla_\xi Dr, Y). \end{aligned}$$

By employing (37) in (6), we obtain

$$(38) \quad g(\nabla_\xi Dr, Y) = 4(6\kappa - r)(2 - \kappa)\eta(Y) - 6dr(\varphi Y) - 2dr(\varphi hX).$$

Replacing Y by φY in the above equation and simplification leads the following

$$(39) \quad \nabla_\xi Dr = (6 + 2h)\varphi Dr.$$

From Lemma 2.2, we get $dr(\xi) = 0$ implies that $\nabla_\xi Dr = 0$. Therefore from above relation we get

$$(40) \quad (6 + 2h)\varphi Dr = 0,$$

which implies that $\varphi Dr = 0$. If $\varphi Dr = 0$, then $Dr = 0$, which shows that r is constant. This proves the theorem. $\square$

From the Lemma 2.1 and Theorem 3.1, we have the following:

Corollary 3.1. *A 3-dimensional $N(\kappa \neq -1)$ -paracontact metric manifold M^3 with purely transversal Bach tensor is locally φ -symmetric.*

Following the proof of the above theorem 3.1, we are in a position to prove

Corollary 3.2. *If an $N(\kappa)$ -paracontact metric 3-manifold M^3 with $\kappa > -1$ has purely transversal Bach tensor, then the scalar curvature r is equal to 6κ .*

Proof. Since M^3 is a $N(\kappa)$ -paracontact metric 3-manifold with $\kappa > -1$ and has a purely transversal Bach tensor. we have from Theorem 3.1 and (38) that

$$(41) \quad 4(6\kappa - r)(2 - \kappa)\eta(Y) = 0.$$

This gives $r = 6\kappa$. □

Theorem 3.2. *An $N(\kappa)$ -paracontact metric 3-manifold M^3 with $\kappa > -1$ has purely transversal Bach tensor if and only if the manifold M^3 is of constant scalar curvature $r = 6\kappa$.*

Proof. Let M^3 be an $N(\kappa)$ -paracontact metric 3-manifold with $\kappa > -1$ has constant scalar curvature $r = 6\kappa$. Then from (37) we get

$$\sum_{i=1}^3 (\nabla_{e_i} \mathcal{C})(e_i, Y)\xi = 0$$

which intern gives from (6) that $\mathcal{B}(Y, \xi) = 0$. That is, the manifold M^3 has purely transversal Bach tensor. The converse is obvious from Theorem 3.1 and Corollary 3.2. □

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A note on Einstein-like ϵ -LP-Sasakian manifolds

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Abstract. The present paper deals with the study on Einstein-like ϵ -Lorentzian para-Sasakian manifolds. A necessary and sufficient condition for an ϵ -Lorentzian para-Sasakian manifold to be Einstein-like is obtained in terms of its curvature tensor. We also obtain the scalar curvature of an Einstein-like ϵ -Lorentzian para-Sasakian manifold. A necessary and sufficient condition for an ϵ -Lorentzian almost para-contact metric hypersurface of an indefinite locally Riemannian product manifold to be ϵ -Lorentzian para-Sasakian is derived as well as it is shown that the ϵ -Lorentzian para-Sasakian hypersurface of an indefinite locally Riemannian product manifold of almost constant curvature is always Einstein-like.

Keywords: ϵ -LP-Sasakian manifold, Einstein-like manifold, ϵ -LP-Sasakian hypersurfaces.

1. Introduction

Semi-Riemannian geometry has great importance in the field of general relativity [8] and different areas of physics. I. Sato [11] introduced a structure (ϕ, ξ, η) satisfying $\phi^2 = I - \eta \otimes \xi$ and $\eta(\xi) = 1$ on a differentiable manifold in 1976, which is now renowned as an almost para-contact structure. This structure is an analogue of the almost contact structure [5, 10] as well as closely related to the almost product structure. Almost contact manifold is always odd dimen-

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sional but almost para-contact manifold could be even-dimensional as well. T. Adati and K. Motsumoto [1] defined and studied p -Sasakian manifold and special p -Sasakian manifolds in 1977, which are considered as a special kind of an almost contact Riemannian manifold. Further in 1989, Matsumoto [7] used the structure vector field $-\xi$ instead of ξ in an almost para-contact manifold and associated a Lorentzian metric with the resulting structure, called it a Lorentzian almost para-contact manifold. Such structures are also studied by several authors ([2], [4], [13]). The concept of ϵ -Sasakian manifold is introduced by A. Bejancu and K. L. Duggal in 1993 [3]. M. M. Tripathi, Kilic, Perktas and Keles have studied ϵ -almost para-contact manifolds and in particular, ϵ -para-Sasakian manifolds. U. C. De and A. Sarkar [6] introduced ϵ -Kenmotsu manifold in 2009 and studied some curvature properties on the manifold. R. Prasad and V. Srivastava [9] have studied ϵ -Lorentzian para-Sasakian manifold and also shown its existence by an example in 2012. In 2017, Haseeb, Prakash and Siddiqi [14] have studied quarter-symmetric metric connection in an ϵ -Lorentzian para-Sasakian manifolds.

R. Sharma [12] introduced and studied Einstein-like para-Sasakian manifolds in 1982. Motivated by his study, in this paper we introduce and study Einstein-like ϵ - LP -Sasakian manifold. The paper is organized as follows: In section 2, we give some preliminaries about ϵ - LP -Sasakian manifolds. Section 3 contains definition of an Einstein-like para-Sasakian manifold and some basic properties. We also find some results on scalar curvature and a necessary and sufficient condition for ϵ - LP -Sasakian manifolds to be Einstein-like in terms of its curvature tensor is obtain. In section 4, we obtain a necessary and sufficient condition for ϵ -Lorentzian almost para-contact metric hypersurface of an indefinite locally Riemannian product manifold to be ϵ - LP -Sasakian manifold. Finally, we prove that ϵ - LP -Sasakian hypersurface of an indefinite locally Riemannian product manifold of almost constant curvature is always Einstein-like.

2. Preliminaries

A structure on a differential manifold M of dimension n is called an ϵ -Lorentzian almost para-contact metric structure if it admits a $(1, 1)$ -tensor field ϕ , a contravariant vector field ξ , a 1-form η and semi-Riemannian metric g which satisfy

- (1) $\phi^2 X = X + \eta(X)\xi, \quad \eta(\xi) = -1,$
- (2) $g(X, \xi) = \epsilon\eta(X) \Rightarrow g(\xi, \xi) = -\epsilon,$
- (3) $\phi(\xi) = 0, \quad \eta(\phi X) = 0,$
- (4) $g(\phi X, \phi Y) = g(X, Y) + \epsilon\eta(X)\eta(Y),$

for all vector fields $X, Y \in \chi(M)$, where ϵ is 1 or -1 according to the vector field ξ being timelike or spacelike.

If an ϵ -Lorentzian almost para-contact metric structure satisfies

- (5) $(\nabla_X \phi)Y = g(X, Y)\xi + \epsilon\eta(Y)X + 2\epsilon\eta(X)\eta(Y)\xi,$

where ∇ denotes the Levi-Civita connection with respect to g , then M is called an $\epsilon - LP$ -Sasakian manifold.

In an $\epsilon - LP$ -Sasakian manifold, we have [9]

$$(6) \quad \nabla_X \xi = \epsilon \phi X,$$

$$(7) \quad (\nabla_X \eta)Y = g(\phi X, Y) = \Phi(X, Y).$$

Moreover, the curvature tensor R and Ricci tensor S satisfy the following equations:

$$(8) \quad R(X, Y)\xi = \eta(Y)X - \eta(X)Y,$$

$$(9) \quad S(X, \xi) = (n - 1)\eta(X).$$

Example 2.1. Let a 3-dimensional manifold $M_3 = \{(x, y, z) \in R^3\}$, where (x, y, z) are standard coordinates of R^3 . Let e_1, e_2 and e_3 be vector fields on M_3 given by

$$e_1 = \frac{1}{2}e^{2z}\frac{\partial}{\partial y}, \quad e_2 = \frac{1}{2}e^{2z}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) \text{ and } e_3 = \frac{\partial}{\partial z} = \xi,$$

which are linearly independent vector fields at each point of M_3 and form a basis of $T_p M_3$ at each point p of M_3 . Define a semi-Riemannian metric g on M_3 as $g(e_i, e_i) = 1$, for $1 \leq i \leq 2$, $g(e_3, e_3) = -\epsilon$ and $g(e_i, e_j) = 0$, for $i \neq j$ and $1 \leq i, j \leq 3$.

Let η be a 1-form on M_3 defined as $\epsilon\eta(U) = g(U, e_3) = g(U, \xi)$, for all $U \in \chi(M_3)$ and let ϕ be a $(1, 1)$ tensor field on M_3 defined as

$$\phi(e_1) = -\epsilon e_1, \quad \phi(e_2) = -\epsilon e_2, \quad \phi(e_3) = 0.$$

By applying linearity of ϕ and g , we have

$$\eta(\xi) = -1, \quad \phi^2(X) = X + \eta(X)\xi,$$

and

$$g(\phi X, \phi Y) = g(X, Y) + \epsilon\eta(X)\eta(Y) \quad \text{for all } X, Y \in \chi(M_3).$$

Let ∇ be a Levi-Civita connection with respect to the semi-Riemannian metric g . Then we have

$$[e_1, e_2] = 0, \quad [e_2, e_3] = -e_2, \quad [e_1, e_3] = -e_1,$$

The Riemannian connection ∇ of the metric g is given by

$$2g(\nabla_U V, W) = Ug(V, W) + Vg(W, U) - Wg(U, V) \\ - g(U, [V, W]) - g(V, [U, W]) + g(W, [U, V]),$$

which is known as Koszul's formula, we can easily calculate

$$\begin{aligned}\nabla_{e_1}e_1 &= -\epsilon e_3, & \nabla_{e_1}e_2 &= 0, & \nabla_{e_1}e_3 &= -e_1, \\ \nabla_{e_2}e_1 &= 0, & \nabla_{e_2}e_2 &= -\epsilon e_3, & \nabla_{e_2}e_3 &= -e_2, \\ \nabla_{e_3}e_1 &= 0, & \nabla_{e_3}e_2 &= 0, & \nabla_{e_3}e_3 &= 0.\end{aligned}$$

From the above it follows that the manifold satisfies $\nabla_X\xi = \epsilon\phi X$, for $\xi = e_3$ and $(\nabla_X\phi)Y = g(X, Y)\xi + \epsilon\eta(Y)X + 2\epsilon\eta(X)\eta(Y)\xi$. Hence the manifold is an ϵ -LP-Sasakian manifold.

3. Einstein-like ϵ -LP-Sasakian manifolds

We begin with the following definition analogous to Einstein-like para-Sasakian manifolds (Sharma, 1982).

Definition 3.1. *An ϵ -Lorentzian almost para contact metric manifold is said to be Einstein-like if its Ricci tensor S satisfies*

$$(10) \quad S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y) + cg(\phi X, Y),$$

for some real constants a, b and c .

Proposition 3.1. *In an Einstein-like ϵ -Lorentzian almost para contact metric manifold, we have*

$$(11) \quad S(\phi X, Y) = ag(\phi X, Y) + cg(\phi X, \phi Y),$$

$$(12) \quad S(X, \xi) = \epsilon a\eta(X) - b\eta(X).$$

Moreover, if the manifold is ϵ -LP-Sasakian manifold, the following conditions also hold

$$(13) \quad \epsilon a - b = n - 1,$$

$$(14) \quad r = na - \epsilon b + c(\text{trace}(\phi)).$$

Proof. The equations (11) and (12) are obvious from (10).

In an ϵ -LP-Sasakian manifold, it follows that $S(X, \xi) = (n-1)\eta(X)$, which in view of (12) implies (13).

Now, let $\{e_1, e_2, \dots, e_{n-1}, \xi\}$ be a local orthonormal basis. Then from (10), we have

$$r = \sum_{i=1}^n \{a\epsilon_i g(e_i, e_i) + b\epsilon_i \eta(e_i)\eta(e_i) + c\epsilon_i g(\phi e_i, e_i)\},$$

where $\epsilon_i = g(e_i, e_i)$, which gives (14). $\square$

Theorem 3.1. *For an Einstein-like ϵ -LP-Sasakian manifold, the scalar curvature r satisfies the following differential equation*

$$c\xi r + 2br = -2\epsilon(n-1)[c^2 - b(n+b)].$$

Proof. From (10), it follows that the Ricci operator Q satisfies

$$g(QX, Y) = ag(X, Y) + cg(\phi X, Y) + b\epsilon\eta(X)g(\xi, Y),$$

this implies

$$QX = aX + c\phi X + \epsilon b\eta(X)\xi.$$

Differentiating, we find

$$\begin{aligned} (\nabla_Y Q)X + Q(\nabla_Y X) &= a\nabla_Y X + c(\nabla_Y \phi)X + c\phi(\nabla_Y X) \\ &\quad + \epsilon b(\nabla_Y \eta)(X)\xi + \epsilon b\eta(\nabla_Y X)\xi + \epsilon b\eta(X)\nabla_Y \xi. \end{aligned}$$

Using (5), (6) and (7) in the above equation, we get

$$\begin{aligned} (\nabla_Y Q)X &= c[g(X, Y)\xi + \epsilon\eta(X)Y + 2\epsilon\eta(X)\eta(Y)\xi] \\ &\quad + b[\epsilon g(\phi X, Y)\xi + \eta(X)\phi Y]. \end{aligned}$$

Now, let $\{e_1, e_2, \dots, e_{n-1}, \xi\}$ be a local orthonormal basis. Multiplying both side by ϵ_i in the above equation and contracting with respect to Y , we have

$$(15) \quad (\operatorname{div} Q)X = \{b(\operatorname{trace}(\phi)) + \epsilon c(n-1)\}\eta(X).$$

From (13) and (14), we get

$$(16) \quad r = c(\operatorname{trace}(\phi)) + \epsilon(n-1)(n+b).$$

Using $Xr = 2(\operatorname{div} Q)X$ and (16) in (15), we obtain that in an Einstein-like ϵ -LP-Sasakian manifold, the scalar curvature r satisfies the following differential equation

$$(17) \quad c\xi r + 2br = -2\epsilon(n-1)[c^2 - b(n+b)].$$

□

Theorem 3.2. *In an Einstein-like ϵ -LP-Sasakian manifold, if $\operatorname{trace}(\phi)$ is constant then*

$$(18) \quad b(\operatorname{trace}(\phi)) = -\epsilon(n-1)c.$$

Proof. Using $Xr = 2(\operatorname{div} Q)X$ in (15), we get

$$(19) \quad dr = 2\{b(\operatorname{trace}(\phi)) + \epsilon c(n-1)\}\eta.$$

Since $\operatorname{trace}(\phi)$ is constant, then from (14), it follows that r is constant. Hence (19) gives (18). □

From now, on in this section the $\operatorname{trace}(\phi)$ will be assumed to be constant.

Theorem 3.3. *An ϵ -LP-Sasakian manifold with constant $\text{trace}(\phi)$ is Einstein-like if and only if the $(0, 2)$ -tensor field $C_1^1(\phi R)$ is a linear combination of g , Φ and $\eta \otimes \eta$ formed with constant coefficients.*

Proof. Let R be the curvature tensor in an ϵ -LP-Sasakian manifold, we have

$$R(X, Y)\phi Z = \nabla_X \nabla_Y \phi Z - \nabla_Y \nabla_X \phi Z - \nabla_{[X, Y]}\phi Z.$$

Using equations (5), (6) and (7) in the above equation, we have

$$\begin{aligned} (20) \quad R(X, Y)\phi Z &= \phi R(X, Y)Z - \epsilon[\Phi(Y, Z)X - \Phi(X, Z)Y] \\ &\quad + \epsilon[g(Y, Z)\phi X - g(X, Z)\phi Y] \\ &\quad + 2\epsilon[\eta(Y)\Phi(X, Z)\xi - \eta(X)\Phi(Y, Z)\xi] \\ &\quad + 2[\eta(Y)\eta(Z)\phi X - \eta(X)\eta(Z)\phi Y]. \end{aligned}$$

From (20), we have

$$\begin{aligned} (21) \quad {}'R(X, Y, \phi Z, W) - {}'R(X, Y, Z, \phi W) &= \epsilon[\Phi(X, W)g(Y, Z) - \Phi(Y, W)g(X, Z)] \\ &\quad + \epsilon[\Phi(X, Z)g(Y, W) - g(X, W)\Phi(Y, Z)] \\ &\quad + 2\eta(Z)[\eta(Y)\Phi(X, W) - \eta(X)\Phi(Y, W)] \\ &\quad + 2\eta(W)[\eta(Y)\Phi(X, Z) - \eta(X)\Phi(Y, Z)]. \end{aligned}$$

where $'R(X, Y, Z, W) = g(R(X, Y)Z, W)$.

From (21), we have

$$\begin{aligned} (22) \quad {}'R(X, Y, \phi Z, \phi W) - {}'R(X, Y, Z, W) &= \epsilon[g(\phi X, \phi W)g(\phi Y, \phi Z) - g(\phi X, \phi Z)g(\phi Y, \phi W)] \\ &\quad + \epsilon[\Phi(X, Z)\Phi(Y, W) - \Phi(X, W)\Phi(Y, Z)] \\ &\quad + \eta(Z)[\eta(Y)g(X, W) - \eta(X)g(Y, W)] \\ &\quad + \eta(W)[\eta(X)g(Y, Z) - \eta(Y)g(X, Z)]. \end{aligned}$$

From (22), we have

$$(23) \quad {}'R(X, Y, \phi Z, \phi W) = {}'R(\phi X, \phi Y, Z, W).$$

Using (22) and (23), we have

$$\begin{aligned} (24) \quad {}'R(\phi X, \phi Y, \phi Z, \phi W) &= {}'R(X, Y, Z, W) \\ &\quad + \eta(Z)[\eta(Y)g(X, W) - \eta(X)g(Y, W)] \\ &\quad + \eta(W)[\eta(X)g(Y, Z) - \eta(Y)g(X, Z)]. \end{aligned}$$

From (24), we have

$$(25) \quad S(\phi Y, \phi Z) = S(Y, Z) + (n-1)\eta(Y)\eta(Z) \quad \text{and}$$

$$(26) \quad S(Y, \xi) = (n-1)\eta(Y).$$

Taking $Z = \phi Z$, we have

$$(27) \quad S(\phi Y, Z) = S(Y, \phi Z).$$

From (20), we have

$$(28) \quad \begin{aligned} S(Y, \phi Z) &= C_1^1(\phi R)(Y, Z) - \epsilon(n-2)\Phi(Y, Z) \\ &+ [\epsilon g(Y, Z) + 2\eta(Y)\eta(Z)]\text{trace}(\phi). \end{aligned}$$

Now, from (27), we have in $\epsilon - LP$ -Sasakian manifold $S(X, \phi Y) = S(\phi X, Y)$ and also it can be verified that $C_1^1(\phi R)(Y, Z) = C_1^1(\phi R)(Z, Y)$; therefore from this observation relation (28) is consistent.

Now, if the manifold M is Einstein-like then from (11) and (18), the equation (28) reduces to

$$(29) \quad bC_1^1(\phi R) = c(b+n-1)g + \epsilon c[b+2(n-1)]\eta \otimes \eta + (a + \epsilon(n-2))\Phi,$$

which shows that $C_1^1(\phi R)$ is a linear combination of g, Φ and $\eta \otimes \eta$ formed with constant coefficients. The converse is easy to follow. $\square$

Corollary 3.1. *In an Einstein-like $\epsilon - LP$ -Sasakian manifold with constant trace(ϕ), the $(0, 2)$ - tensor field $C_1^1(\phi R)$ is parallel along the vector field ξ .*

Proof. From (5) and (7), we have $\nabla_\xi \eta = 0$, and $\nabla_\xi \phi = 0$.

Now, in an $\epsilon - LP$ -Sasakian manifold

$$\begin{aligned} (\nabla_\xi \Phi)(X, Y) &= \nabla_\xi \Phi(X, Y) - \Phi(\nabla_\xi X, Y) - \Phi(X, \nabla_\xi Y) \\ &= g(\nabla_\xi \phi X - \phi \nabla_\xi X, Y) \\ &= g((\nabla_\xi \phi)X, Y) \end{aligned}$$

this implies $\nabla_\xi \Phi = 0$ for $\epsilon - LP$ -Sasakian manifold.

Since in an $\epsilon - LP$ -Sasakian manifold $\nabla_\xi \Phi = 0$ and $\nabla_\xi \eta = 0$, therefore from (29), we conclude that $C_1^1(\phi R)$ is parallel along the vector field ξ . $\square$

Theorem 3.4. *In an Einstein-like $\epsilon - LP$ -Sasakian manifold, we have*

$$(30) \quad L_\xi S = 2a\epsilon\Phi + 2c(\epsilon g + \eta \otimes \eta).$$

Proof. In a $\epsilon - LP$ -Sasakian manifold, we obtain

$$(31) \quad \begin{aligned} (L_\xi \eta)X &= \xi\eta(X) - \eta[\xi, X] \\ &= \xi\eta(X) - \eta(\nabla_\xi X - \nabla_X \xi) \\ &= (\nabla_\xi \eta)X + \eta(\nabla_X \xi) \\ &= 0 \end{aligned}$$

and

$$(32) \quad \begin{aligned} (L_\xi g)(Y, Z) &= \xi g(Y, Z) - g([\xi, Y], Z) - g(Y, [\xi, Z]) \\ &= 2g(\epsilon \phi Y, Z) = 2\epsilon \Phi(Y, Z). \end{aligned}$$

$$(33) \quad \begin{aligned} (L_\xi \Phi)(Y, Z) &= L_\xi g(\phi Y, Z) - \Phi(L_\xi Y, Z) - \Phi(Y, L_\xi Z) \\ &= (L_\xi g)(\phi Y, Z) + g((L_\xi \phi)Y, Z) \\ &= 2\epsilon \Phi(\phi Y, Z) = 2\epsilon g(Y, Z) + 2\eta(Y)\eta(Z) \end{aligned}$$

Thus, we have $L_\xi \eta = 0$, $L_\xi g = 2\epsilon \Phi$, $L_\xi \phi = 0$ and $L_\xi \Phi = 2(\epsilon g + \eta \otimes \eta)$.

Now, taking Lie derivative of S in the direction of ξ in (10) and using above results, we obtain (30). $\square$

Theorem 3.5. *In an Einstein-like ϵ -LP-Sasakian manifold with constant trace(ϕ), we have*

$$(34) \quad L_\xi(C_1^1(\phi R)) = \frac{2\epsilon c}{b}(b+n-1)\Phi + 2\epsilon(a + \epsilon(n-2))(g + \epsilon\eta \otimes \eta)$$

Proof. Taking Lie derivative of $C_1^1(\phi R)$ in the direction of ξ in (29) and using (31) – (33), we obtain (34). Hence the proof. $\square$

4. ϵ -LP-Sasakian hypersurfaces

Let $\widetilde{M}_{n+1}$ be a real $(n+1)$ dimensional manifold. Suppose $\widetilde{M}_{n+1}$ is endowed with an almost product structure J and a semi-Riemannian metric $\widetilde{g}$ satisfying

$$(35) \quad \widetilde{g}(J\widetilde{X}, J\widetilde{Y}) = \widetilde{g}(\widetilde{X}, \widetilde{Y}),$$

for all vector fields $\widetilde{X}, \widetilde{Y}$ in $\widetilde{M}_{n+1}$. Then we say that $\widetilde{M}_{n+1}$ is an indefinite almost product Riemannian manifold. Moreover, if on $\widetilde{M}_{n+1}$, we have

$$(36) \quad (\widetilde{\nabla}_{\widetilde{X}} \widetilde{J})\widetilde{Y} = 0,$$

for all $\widetilde{X}, \widetilde{Y} \in \Gamma(T\widetilde{M}_{n+1})$, where $\widetilde{\nabla}$ is the Levi-Civita connection with respect to $\widetilde{g}$, we say that $\widetilde{M}_{n+1}$ is an indefinite locally Riemannian product manifold.

Now, let M_n be an orientable non-degenerate hypersurface of $\widetilde{M}_{n+1}$. Suppose that N is the unit normal vector field and ξ is a vector field on M_n such that

$$(37) \quad \widetilde{g}(N, N) = -\epsilon,$$

$$(38) \quad JN = -\xi.$$

Let

$$(39) \quad JX = \phi X + \eta(X)N, \quad \text{where } X \in \chi(M_n).$$

Proposition 4.1. *The set (ϕ, ξ, η, g) is an ϵ -Lorentzian almost para-contact metric structure, where g is induced metric on M_n .*

Proof. We have, from (38) and (39), $X = J^2X = \phi^2X + \eta(\phi X)N - \eta(X)\xi$.

Equating tangential and normal parts, we get $\phi^2X = X + \eta(X)\xi$ and $\eta(\phi X) = 0$, respectively.

Also, from (38) and (39), we have $N = J^2N = -J\xi = -\phi\xi - \eta(\xi)N$,

Equating tangential and normal parts, we get $\phi\xi = 0$ and $\eta(\xi) = -1$, respectively. Finally, we have $g(X, Y) = \tilde{g}(JX, JY)$, where $X, Y \in M_n$, which in view of (39) gives (1). Hence the set (ϕ, ξ, η, g) is an ϵ -Lorentzian almost para-contact metric structure on M_n . $\square$

The Gauss and Weingarten formulas are given respectively by

$$(40) \quad \tilde{\nabla}_X Y = \nabla_X Y - \epsilon g(AX, Y)N,$$

$$(41) \quad \tilde{\nabla}_X N = -AX,$$

where ∇ is the Levi-Civita connection with respect to the Riemannian metric g induced by $\tilde{g}$ on M_n and A is the shape operator of M_n .

Proposition 4.2. *The ϵ -Lorentzian almost para-contact metric structure on M_n satisfies*

$$(42) \quad (\nabla_X \phi)Y = \eta(Y)AX + \epsilon g(AX, Y)\xi,$$

$$(43) \quad (\nabla_X \eta)Y = \epsilon g(AX, \phi Y),$$

$$(44) \quad \nabla_X \xi = \phi AX.$$

Proof. Using the equations (38), (39), (40) and (41) in $(\tilde{\nabla}_X J)Y = 0$, we get $(\nabla_X \phi)Y - \eta(Y)AX - \epsilon g(AX, Y)\xi + ((\nabla_X \eta)Y)N - \epsilon g(AX, \phi Y)N = 0$.

Equating tangential and normal parts, we get (42) and (43), respectively. Equation (42) and (43) implies (44). $\square$

Now, we obtain the following theorems of characterization for ϵ -LP-Sasakian hypersurfaces.

Theorem 4.1. *Let M_n be an orientable hypersurfaces of an indefinite locally Riemannian product manifold. Then M_n is an ϵ -LP-Sasakian manifold if and only if the shape operator is given by*

$$(45) \quad A = \epsilon I + \epsilon \eta \otimes \xi.$$

Proof. Let M_n be a ϵ -LP-Sasakian manifold. Using (6) and (44), we get

$$(46) \quad AX = \epsilon X + \epsilon \eta(X)\xi - \eta(AX)\xi.$$

In particular, we have $A\xi = -\eta(A\xi)\xi$. Thus, we have

$$(47) \quad \eta(AX) = \epsilon g(\xi, AX) = \epsilon g(A\xi, X) = \epsilon g(-\eta(A\xi)\xi, X) = -\eta(A\xi)\eta(X).$$

Using this in (46), we get

$$(48) \quad A = \epsilon I + (\epsilon + \eta(A\xi))\eta \otimes \xi.$$

Using (42) in (48), we have

$$(\nabla_X \phi)Y = \epsilon\eta(Y)\phi X + 2\epsilon\eta(X)\eta(Y)\xi + 2\eta(A\xi)\eta(X)\eta(Y)\xi + g(X, Y)\xi.$$

Using equation (5), we have $\eta(A\xi)\eta(X)\eta(Y)\xi = 0$. From above, we have $\eta(A\xi) = 0$, which when used in (48) yields (45). Conversely, using (45) in (42) we see that M_n is ϵ -LP-Sasakian manifold. $\square$

Now, assume that the indefinite almost product Riemannian manifold $\widetilde{M}_{n+1}$ is of almost constant curvature so that its curvature tensor $\widetilde{R}$ is given (Yano, 1965) by

$$(49) \quad \begin{aligned} {}'\widetilde{R}(\widetilde{X}, \widetilde{Y}, \widetilde{Z}, \widetilde{W}) &= k\{\widetilde{g}(\widetilde{Y}, \widetilde{Z})\widetilde{g}(\widetilde{X}, \widetilde{W}) - \widetilde{g}(\widetilde{X}, \widetilde{W})\widetilde{g}(\widetilde{Y}, \widetilde{Z}) \\ &\quad + \widetilde{g}(J\widetilde{Y}, \widetilde{Z})\widetilde{g}(J\widetilde{X}, \widetilde{W}) - \widetilde{g}(J\widetilde{X}, \widetilde{Z})\widetilde{g}(J\widetilde{Y}, \widetilde{W})\} \end{aligned}$$

for all vector fields $\widetilde{X}, \widetilde{Y}, \widetilde{Z}$ and $\widetilde{W}$ on $\widetilde{M}_{n+1}$. If M_n be a ϵ -LP-Sasakian hypersurface, then in view of (45), (49) and the Gauss equation, we have

$$(50) \quad \begin{aligned} {}'R(X, Y, Z, W) &= (k - \epsilon)\{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\} \\ &\quad + (k)\{g(\phi Y, Z)g(\phi X, W) - g(\phi X, Z)g(\phi Y, W)\} \\ &\quad - g(Y, Z)\eta(X)\eta(W) - g(X, W)\eta(Y)\eta(Z) \\ &\quad + g(X, Z)\eta(Y)\eta(W) + g(Y, W)\eta(X)\eta(Z). \end{aligned}$$

After calculating from (50), we have

$$(51) \quad R(X, Y)\xi = [\epsilon(k - \epsilon) + 1]\{\eta(Y)X - \eta(X)Y\}.$$

Comparing the resulting expression with (8), we find that $k = \epsilon$. With this value of k , from (50), we obtain $S(Y, Z) = c(\text{trace}(\phi))\Phi(Y, Z) - (n-1)\eta(Y)\eta(Z)$. Thus, we have

Theorem 4.2. *An ϵ -LP-Sasakian hypersurface of an indefinite locally Riemannian product manifold of almost constant curvature ($k = \epsilon$) is always Einstein-like.*

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Magic circles cryptosystem

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Abstract. Magic Squares have been the subject of interest among mathematicians for several centuries because of its magical properties. In this paper a type of magic square was constructed and employed it as a cryptosystem. This method was formulated depending on a set of magic circles that are computed using the proposition of the arithmetic modular together with the operations of addition, rotation and reflection. Due to the importance of the magic circles and the existence of many applications in the practical life, we find a link between the magic circles and the encryption processes. For each character in the plaintext, the ASCII codes is given and then write any plaintext (start from any word) as a linear combination of the elements from the arrangement of magic circles to get the cipher text. An algorithm for generating the magic circles and its application in cryptography was written in MATLAB language. The obtained results within an example on magic circles of order 8 are given together with the inverse of modular magic square.

Keywords: Franklin magic, cryptography, algorithms, magic circle.

1. Introduction

Magic squares, being an amusement scientific point in nature, have been the subject of stimulation and enthusiasm to numerous mathematicians and math-lovers alike for many years [3, 5, 9]. The history of the magic squares dates back to about 2200 BC according to the Chinese legend, Emperor Yi was said to have walked near the river, he recognize a turtle have a strange pattern on its cover, the emperor named this style Lu Shu [2] its represent an important pattern in Chinese culture was determined by the principle of yin and yang. The magic square of order 3 was interpreted on the basis of the five elements earth, water, wood, metals and fire [10]. The magic square of order 3, 4, 5, 6, 7, 8 and 9 was constructed by Cornelius Agrippa [1] the magic square of order four was discovered by the German artist Albrecht Dürer's, where he dug in the upper right part of his etching Melencolia I, which included a collection of small details and scientists believe that the explanation is the lack of knowledge possessed by human access to heavenly wisdom and knowledge of the secrets of nature [12]. The famous mathematician Abu Ali Al Hassan bin Al Haytham, who is considered one of the best authors, touched upon the magic squares and methods

of constructing them in his treatise. He mentioned some general methods of construction magic squares. There were many authors who wrote in the magic squares, including the Persian Abu Al-Wafa Buzjani, Abu Hatem Hatem Mazar Asfizar and Abdul Wahab bin Ibrahim Zanjani [16]. Some properties of magic square has been study such if it is possible to generate magic square using integer numbers of cubes or square [14]. Although the magic squares over time have lost the belief that they have magical powers, but it continued to be a favourite subject for mathematicians and also used in entertainment games and puzzles for the possession of the mystery. The magic squares were expanded to three-dimensional by Adams Kochansky probably. And referred to the magic cubes for the first time in a letter of Pierre de Fermat in 1640 at the end of the 19th century, mathematicians applied squares in probabilistic problems. Today, squares are studied in matrices and harmonic mathematics as well as geometry [6, 7, 20]. The concept of water retention on magic squares is a new concept, mentioned for the first time in 2007 by Knecht Most of the magic square that provides the maximum amount of water retaining were discovered in 2010, in the competition that was arranged by Al Zimmerman, [11]. In [18,19] they brought attention to the study of water retention magic squares. Water storage important is due to climate change, which led to the need of finding a way to keep the water within a specific spot. Water stored will be used when needed. It will be used to deliver water to the farthest point in cities as well as to be used in irrigation and agriculture, also can be used in fire extinguishing. In addition it is a very important in generating electricity. Retain of water on the magic squares is one of the ways to reach the greatest possible amount of water that will be obtained. A standard definition of magic circle is not found. Benjamin Franklin, around 1752 developed a variation on the celebrated magic squares which is called a magic (a, r)- circle.

In 2009 Rebecca G. [4] utilizes procedures in combinatorics, enumerative geometry and computational algebraic that develop for counting Franklin magic 8-circles. Mathematical works for Benjamin Franklin are more wonderful than the ordinary magic square; his renowned squares are 8×8 and 16×16 magic squares. Fig. 1 represents one of the Franklin's 8×8 squares; its total sum is 260.

In recreational mathematics, an arrangement of n distinct integer numbers in a square of n by n represents an n order magic square such that, each of its row, column and diagonals is summed to the same number. Many research are discussed the construction of magic squares using different approaches [8, 13, 15, 17]. Song dynasty (960-1279), Chinese mathematician Yang Hui (c. 1238-1298) were invented magic circles. Natural numbers which are arranged on circles have the property of the sum of numbers on the diameter and the sums of numbers on each circle are similar. Since the significance of the magic circles and the presence of numerous applications in the reasonable life, we attempted to discover the connection between the magic circles and the encryption forms. The main reason of considering the magic circle for constructing a system of cryptosystem is due

to the complexity of constructing magic circles in high dimension. The strategy is planned by utilizing a gathering of magic circles that are processed as of late, at that point composing any plaintext as a straight blend of the components in the gathering and encoded plaintext in the wake of finding the ASCII code for each character in the plaintext to get the figure content.

52	61	4	13	20	29	36	45
14	3	62	51	46	35	30	19
53	60	5	12	21	28	37	44
11	6	59	54	43	38	27	22
55	58	7	10	23	26	39	42
9	8	57	56	41	40	25	24
50	63	2	15	18	31	34	47
16	1	64	49	48	33	32	17

Figure 1: Franklin's 8×8 squares

In recreational mathematics, an arrangement of n distinct integer numbers in a square of n by n represents an n order magic square such that, each of its row, column and diagonals is summed to the same number. Many research are discussed the construction of magic squares using different approaches [4,6,7,8]. Song dynasty (960-1279), Chinese mathematician Yang Hui (c. 1238-1298) were invented magic circles. Natural numbers which are arranged on circles have the property of the sum of numbers on the diameter and the sums of numbers on each circle are similar. Since the significance of the magic circles and the presence of numerous applications in the reasonable life, we attempted to discover the connection between the mag-ic circles and the encryption forms. The main reason of considering the magic circle for constructing a system of cryptosystem is due to the complexity of constructing magic circles in high dimension. The strategy is planned by utilizing a gathering of magic circles that are processed as of late, at that point composing any plaintext as a straight blend of the components in the gathering and encoded plaintext in the wake of finding the ASCII code for each character in the plaintext to get the figure content.

2. Basic concepts

Magic circle is a wheel of nonnegative integer numbers in which the spokes and concentric rings add up to the same number. The radii of a concentric annuli is r that defined as a magic (a, r) -circle, with the feature that the aggregate is M on every annular, every radial aggregate is also M . It is called a magic a -circle if $a = r$. Additional sophistication is found for Franklin's magic circle similar to its magic square. For Franklin's magic squares, the bent diagonals has organizations of annuli with the same center included into the major main circle that are eccentric relative to the basic circular grid that is similar for a

Franklin magic circle. Fig. 2 shows a Franklin's original magic circle of order 8, with aggregate Meters =360. A single annulus with the same center is pointed out: four diverse excenters are there labeled N, E, S, and W located as the four directions from the inside. Six concentric circles are around on each excenter, building five annuli. That is, the innermost annulus concentrated at A in Fig. 2 contains, in the positive sense from the bottom, the entries are 42, 59, 19, 66, 21, 68, 28 and 45.

A single annulus eccentric is pointed out: four different excenters are there labeled N, E, S, and W located as north, east, south and west from the center. Six concentric circles are around on each excenter, building five annuli. For example, the innermost annulus concentrated at A in figure 1 contains, in the positive sense from the bottom, the entries are 42, 59, 19, 66, 21, 68, 28 and 45. We call these principles $x_{17}, x_{28}, x_{31}, x_{42}, x_{43}, x_{34}, x_{25}, x_{16}$ in line on the cells that they lie; which is, x_{mn} is the number in the m -th original annulus, counted out, and in the n -th sector, computed in the positive sense in the first quadrant starts with the lower of the two sectors. Now the eccentric circles centered at N: Original Franklin's magic circle complies with the following properties:

- i. The magic constant M is the same as the amount of each radial plus the central quantity.
- ii. every annular aggregate add to it the central number is M Also the sum for every upper- or lower-half annular add to it half the central number is $M = 2$.
- iii. The amount of each 2×2 block add to it half the central number is $M = 2$.
- iv. Half the central quantity plus the sum of 121 or right-half annular of horizontally centered annuli is $M = 2$, and likewise, half the central quantity plus the sum of each left- or right-half annular of horizontally centered eccentric annuli is $M = 2$. Unfortunately, the central number plus the sum of each eccentric annular is M .

3. Formulation of a new proposed method

Constructing the generated magic circles set depends on knowing only one magic circles and constructing all others according on it, from section 2 the first one is given and for this purpose an algorithm is introduced. The purposed method is discussed using the following steps:

For example for $n = 8$, we have the generator magic circles S_1 .

Step 1 : Add to each element in the above magic squares the number $8i + 12$ (means a multiple of 8 plus 12). And then take the modules with respect to 80 which is equal to $10n$, we get $S_2, S_3, S_4, S_5, S_6, S_7, S_8$ and S_9 .

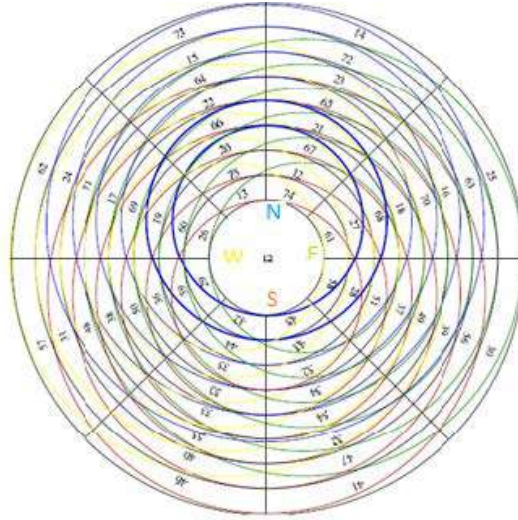


Figure 2: The 8-magic circle

Step 2 :For each of the above magic circles we make use of the transpose for each of them, we get $S_{10}, S_{11}, S_{12}, S_{13}, S_{14}, S_{15}, S_{16}, S_{17}$ and S_{18} , also we take the reflection for each one to obtain $S_{19}, \dots, S_{36}$

4. Implementation and analysis

This method based on writing any plain text after converting it to an ASCII code in the form of linear combination of the elements of the constructed set of magic circles, and then encrypts the cipher text using the inverse for these magic circles. Different sets are obtained depend on the first one. The set is building in light of Algorithm (1).

Example 4.1. I. Encryption: We will take circle magic of order 8.

1. Let the characters of the plaintext be "I Love Mathematics". Its ASCII equivalent is: 73 32 76 111 118 101 32 77 97 116 104 101 109 97 116 105 99 115.
2. We need to add spaces to the end of plain text to get A, since we have 18 ASCII characters.

A. Encryption Algorithm:

Input: plain text, k magic circle (S) of order n.

Output: cipher text.

Begin

step 1. Get the ASCII value of the characters of the plaintext.

step 2. Find the quotient of length of ASCII values over n, if there is a remainder r then add the required number (n-r) of spaces to the end of the plaintext.

step 3. Generate a rectangular matrix A of n columns for ASCII values.

step 4. Set k= number of rows in A.

step 5. Get k magic circle (S) of order n.

step 6. For each row i in A, multiply A_i by S_i . Repeat S if it is necessary (i.e. $k > 36$)

step 7. Reshape the resulting matrix to get a row matrix C. This gives us the cipher text.

End

B. Encryption Algorithm:

Input: cipher text, k magic circle (S) of order n.

Output: plain text.

Begin

step 1. Generate a rectangular matrix C of n columns for cipher text values.

step 2. Set k= number of rows in C.

step 3. Get k magic circle (S) of order n.

step 4. For each row i in C, multiply C_i by (S_i^{-1}) . Repeat S if it is necessary (i.e. $k > 24$)

step 5. Reshape the resulting matrix to get a row matrix A.

step 6. Convert ASCII numbers of A to its equivalent character values. This gives us the plain text.

End

Table 1: An Algorithm For generating semi magic squares set.

3. Put the code of the plain text as a matrix A, where

$$A = \begin{bmatrix} 73 & 32 & 76 & 11 & 118 & 101 & 32 & 77 \\ 97 & 116 & 104 & 101 & 109 & 97 & 116 & 105 \\ 99 & 115 & 32 & 32 & 32 & 32 & 32 & 32 \\ 32 & 32 & 32 & 32 & 32 & 32 & 32 & 32 \\ 32 & 32 & 32 & 32 & 32 & 32 & 32 & 32 \end{bmatrix}$$

4. Then k=5.

5. Use S_1, S_2, S_3, S_4, S_5 .

6. The matrix for the cipher text is

$$C = \begin{bmatrix} 657 & 6887 & 1584 & 1984 & 576 \\ 2240 & 928 & 7245 & 544 & 1952 \\ 4332 & 2392 & 2048 & 448 & 2112 \\ 5994 & 2424 & 1504 & 1056 & 1440 \\ 4838 & 4251 & 1536 & 960 & 1600 \end{bmatrix}$$

7. Convert C in one dimensional matrix.

$$C = [657 \ 6887 \ 1584 \ 1984 \ 576 \ 2240 \ 928 \ \dots]$$

II. Decryption

$$1. \ C = \begin{bmatrix} 657 & 6887 & 1584 & 1984 & 576 \\ 2240 & 928 & 7245 & 544 & 1952 \\ 4332 & 2392 & 2048 & 448 & 2112 \\ 5994 & 2424 & 1504 & 1056 & 1440 \\ 4838 & 4251 & 1536 & 960 & 1600 \end{bmatrix}$$

2. Set k= 5.

3. Take the inverse of S_1, S_2, S_3, S_4, S_5 according to the definition of the inverse for S.

$$4. \ A = \begin{bmatrix} 73 & 32 & 76 & 11 & 118 & 101 & 32 & 77 \\ 97 & 116 & 104 & 101 & 109 & 97 & 116 & 105 \\ 99 & 115 & 32 & 32 & 32 & 32 & 32 & 32 \\ 32 & 32 & 32 & 32 & 32 & 32 & 32 & 32 \\ 32 & 32 & 32 & 32 & 32 & 32 & 32 & 32 \end{bmatrix}$$

- 5 Reshape the resulting matrix to get

$$A = [73 \ 32 \ 76 \ 111 \ 118 \ 101 \ 32 \ \dots]$$

6. After converting numbers of A to its equivalent character values the result is "I Love Mathematics" which is the plain text.

5. Conclusion

In this paper, an important role for the magic circle in cryptosystem is introduced. A method that constructed was given together with an algorithm that constructs the magic circles set and employed for ciphering any plaintext using these set. The resulting text represents a set of numbers, as we shown in example, which is not under-stand able for any one. According to that just the authorized can decrypt the cipher text since they have the inverse of the key which represented by the inverse of semi magic squares. This method wants

many computations which its complexity increases with n . Our computation is implemented by PC using MATLAB language. Our recommendations is to apply permutations on the sequences that used in the bulletined of magic circle cryptosystem. And apply magic circle in another applications other than cryptosystems

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Finite groups with $2pqr^2$ elements of maximal order**Sanbiao Tan**

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Abstract. It is an interesting and difficult topic to determine the structure of a finite group with a given number of elements of maximal order. This topic is related to Thompson's conjecture, that is, if two finite groups have the same order type and one of them is solvable, then the other is solvable. In this article, we continue to this investigation and show that finite groups with $2pqr^2$ elements of maximal order are solvable, where p, q, r are primes and $5 < p < q < r$.

Keywords: finite groups, solvable groups, the order of elements.

1. Introduction

In this paper, all groups considered are finite groups. Let G be a finite group. By $O(x)$ we denote the order of an element $x \in G$, by $\pi(G)$ we denote the set of prime divisors of $|G|$ and by $\pi_e(G)$ we denote the set of orders of elements

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of the group G . k denotes the largest number of $\pi_e(G)$. $M_l(G)$ denotes the set of elements of order l in G , and so $M(G)$ denotes the set of elements with maximal order in G . In the present paper, we also use the following notation: for $p \in \pi(G)$, G_p denotes a Sylow p -subgroup of the group G . In addition, the set of all Sylow p -subgroups of the group G is denoted by $Syl_p(G)$ and $n_p(G) = |Syl_p(G)|$. The Euler function of a positive integer x is denoted by $\varphi(x)$. By $p^n \parallel |G|$ we denote $p^n \parallel |G|$ but $p^{n+1} \nmid |G|$. The other symbols and notations are standard in (cf. [6]).

The topic to study solvability of finite groups is one of the most important topic of finite group theory. The topic in this article is related to the following famous conjecture:

Thompson's conjecture. Let G_1, G_2 be finite groups. Assume that $|M_\ell(G_1)| = |M_\ell(G_2)|$ with $\ell = 1, 2, 3, \dots$. If G_1 is solvable, then G_2 is solvable (see [9, Problem 12.37]).

Up to now, there is no exhaustive answer to the Thompson's Conjecture. It is well known that the properties of elements of maximal or minimal order of a finite group strongly influence the structure of the given group in some scene. Some authors studied if G is solvable while the number of elements of maximal order is specially restricted (see [1], [3], [8]), etc. If G_1 and G_2 satisfies the hypothesis of Thompson's conjecture, then G_1, G_2 have the same number of elements of maximal order. Hence if we proved G is solvable for special $|M(G)|$, then for all groups G_1 and G_2 with $|M_\ell(G_1)| = |M_\ell(G_2)|$ equals this number are solvable. Hence, this study may give useful information to the Thompson's conjecture. The aim of the present paper is to study if a finite group having exactly $2pqr^2$ elements of the maximal order is solvable, we prove the following theorem:

Theorem 1.1. *Let G be a finite group having exactly $2pqr^2$ elements of maximal order, where p, q, r are primes and $5 < p < q < r$, then G is solvable.*

2. Preliminaries

We collect in this section some known results and elementary facts as well as some technical lemmas.

Lemma 2.1. ([12]). *Suppose that G has exactly n cyclic subgroups of order ℓ , then $|M_\ell(G)| = n\varphi(\ell)$. In particular, if n is a number of cyclic subgroups of the group G of order k , then $|M(G)| = n\varphi(k)$.*

Lemma 2.2. ([12]). *Suppose k is the largest order of elements of G . If $|M(G)| = \varphi(k)$, then G is supersolvable.*

Lemma 2.3. ([3]). *There exists a positive number α such that $|G|$ divides $|M(G)|k^\alpha$.*

Lemma 2.4. ([3]). *If there exists a prime divisor p of the number k such that $p(p-1) > |M(G)|$, then the group G contains a unique normal Sylow p -subgroups S and $|S| = p$.*

A finite non-abelian simple group G is called a simple K_n -group if $|\pi(G)| = n$.

Lemma 2.5. ([7]). *Let G be a simple K_3 -group. Then G is isomorphic to one of following simple groups: A_5 , A_6 , $L_2(7)$, $L_2(8)$, $L_2(17)$, $L_3(3)$, $U_3(3)$ and $U_4(2)$. For convenience, we list the information of all K_3 -simple groups in Table 1.*

G	$ G $	$ Out(G) $
A_5	$2^2 \cdot 3 \cdot 5$	2
A_6	$2^3 \cdot 3^2 \cdot 5$	4
$L_2(7)$	$2^3 \cdot 3 \cdot 7$	2
$L_2(8)$	$2^3 \cdot 3^2 \cdot 7$	3
$L_2(17)$	$2^4 \cdot 3^2 \cdot 17$	2
$L_3(3)$	$2^4 \cdot 3^3 \cdot 13$	2
$U_3(3)$	$2^5 \cdot 3^3 \cdot 7$	2
$U_4(2)$	$2^6 \cdot 3^4 \cdot 5$	2

Table 1:

Lemma 2.6. ([10]). *Let G be a simple K_4 -group. Then G is isomorphic to one of following simple groups:*

- (1) $A_7(2^3 \cdot 3^2 \cdot 5 \cdot 7)$, $A_8(2^6 \cdot 3^2 \cdot 5 \cdot 7)$, $A_9(2^6 \cdot 3^4 \cdot 5 \cdot 7)$, $A_{10}(2^7 \cdot 3^4 \cdot 5^2 \cdot 7)$;
- (2) $M_{11}(2^4 \cdot 3^2 \cdot 5 \cdot 11)$, $M_{12}(2^6 \cdot 3^3 \cdot 5 \cdot 11)$, $J_2(2^7 \cdot 3^3 \cdot 5^2 \cdot 7)$;
- (3) $L_2(r)$, where r is a prime number such that $r^2 - 1 = 2^a \cdot 3^b \cdot v^c$ for $a \geq 1, b \geq 1, c \geq 1$, and a prime number $v > 3$;
- (4) $L_2(2^m)$, where the number $m \geq 2$ is such that the number $2^m - 1 = u$ is prime and $2^m + 1 = 3t^b$, moreover, $t > 3$ is a prime number and $b \geq 1$;
- (5) $L_2(3^m)$, where the number $m \geq 2$ is such that $3^m + 1 = 4t, 3^m - 1 = 2u^c$ or $3^m + 1 = 4t^b, 3^m - 1 = 2u$; moreover, the number u and t are odd prime numbers, $b \geq 1$, and $c \geq 1$;
- (6) $L_2(16)(2^4 \cdot 3 \cdot 5 \cdot 17)$, $L_2(25)(2^3 \cdot 3 \cdot 5^2 \cdot 13)$, $L_2(49)(2^4 \cdot 3 \cdot 5^2 \cdot 7^2)$, $L_2(81)(2^4 \cdot 3^4 \cdot 5 \cdot 41)$, $L_3(4)(2^6 \cdot 3^2 \cdot 5 \cdot 7)$, $L_3(5)(2^5 \cdot 3 \cdot 5^3 \cdot 31)$, $L_3(7)(2^5 \cdot 3^2 \cdot 7^3 \cdot 19)$, $L_3(8)(2^9 \cdot 3^2 \cdot 7^2 \cdot 73)$, $L_3(17)(2^9 \cdot 3^2 \cdot 17^3 \cdot 307)$, $L_4(3)(2^7 \cdot 3^6 \cdot 5 \cdot 13)$, $S_4(4)(2^8 \cdot 3^2 \cdot 5^2 \cdot 17)$, $S_4(5)(2^6 \cdot 3^2 \cdot 5^4 \cdot 13)$, $S_4(7)(2^8 \cdot 3^2 \cdot 5^2 \cdot 7^4)$, $S_4(9)(2^8 \cdot 3^8 \cdot 5^2 \cdot 41)$, $S_6(2)(2^9 \cdot 3^4 \cdot 5 \cdot 7)$, $O_8^+(2)(2^{12} \cdot 3^5 \cdot 5^2 \cdot 7)$, $G_2(3)(2^6 \cdot 3^6 \cdot 7 \cdot 13)$, $U_3(4)(2^6 \cdot 3 \cdot 5^2 \cdot 13)$, $U_3(5)(2^4 \cdot 3^2 \cdot 5^3 \cdot 7)$, $U_3(7)(2^7 \cdot 3 \cdot 7^3 \cdot 43)$, $U_3(8)(2^9 \cdot 3^4 \cdot 7 \cdot 19)$, $U_3(9)(2^5 \cdot 3^6 \cdot 5^2 \cdot 73)$, $U_4(3)(2^7 \cdot 3^6 \cdot 5 \cdot 7)$, $U_5(2)(2^{10} \cdot 3^5 \cdot 5 \cdot 11)$, $S_z(8)(2^6 \cdot 5 \cdot 7 \cdot 13)$, $S_z(32)(2^{10} \cdot 5^2 \cdot 31 \cdot 41)$, ${}^3D_4(2)(2^{12} \cdot 3^4 \cdot 7^2 \cdot 13)$ and ${}^2F_4(2)(2^{11} \cdot 3^3 \cdot 5^2 \cdot 13)$.

Lemma 2.7. ([10]). *Let G be a simple K_4 -group with $3 \notin \pi(G)$. Then $G \cong S_z(8)(2^6 \cdot 5 \cdot 7 \cdot 13)$ or $G \cong S_z(32)(2^{10} \cdot 5^2 \cdot 31 \cdot 41)$.*

Lemma 2.8. ([5]). *If $|G| = n$ and $(n, 15) = 1$, then G is solvable.*

Lemma 2.9. ([11]). *If $|M(G)| = 2m$, where $(m, 2) = 1$, then one of the following holds:*

- (1) $k = 4$, s^α or $2s^\alpha$, where s is an odd prime number and $\alpha \in N$;
- (2) If G is non-solvable, then $k = 2s^\alpha$ for some odd prime number s and $2 \parallel \varphi(k) = (s-1)s^{\alpha-1}$; moreover, each Sylow 2-subgroup of group G contains a maximal subgroup which is elementary abelian;
- (3) If $k = 14$, $|G| = 2^\alpha \cdot 3 \cdot 7^\beta$, and G is non-solvable, then $G \cong E \times L_2(7)$, where E is an elementary abelian 2-group.

Lemma 2.10. ([11]). *Suppose that G has n cyclic subgroups A_i of order k , where $1, 2, \dots, n$. Let $\{A_1, A_2, \dots, A_d\}$ be a complete system of representatives of the conjugate classes of n cyclic subgroups of order k , and n_i is the length of the conjugate class containing A_i . Then the following statements hold:*

- (1) $n_i = |G : N_G(A_i)|$, $n = \sum_{i=1}^d n_i$, and $\pi(n_i) \cup \pi(A_i) = \pi(n_1) \cup \pi(A_1)$, where $i = 1, 2, \dots, d$;
- (2) $\pi(C_G(A_i)) = \pi(A_i)$, $|N_G(A_i) : C_G(A_i)| \mid \varphi(k)$ and $|G| = |G : N_G(A_i)| |N_G(A_i) : C_G(A_i)| |C_G(A_i)|$, where $i = 1, 2, \dots, d$;
- (3) Let $A = \langle a \rangle$, $|a| = k$. If $i = 1$, $|M(G)| = 2m$, where $(m, 2) = 1$, then G is solvable.

Lemma 2.11. *Suppose that G is a finite non-solvable group such that $|M(G)| = 2pqr^2 = n \cdot \varphi(k)$. Then $k = 2s^\alpha$, where $\alpha \in N$, $\alpha \leq 3$, and s is an odd prime number. In addition, one of the following statements holds:*

- (1) $\alpha = 3$ and $s = r \in \{2p+1, 2q+1\}$;
- (2) $\alpha = 2$ and either $s = q = 2p+1$, $n = r^2$ or $s = r \in \{2p+1, 2q+1, 2pq+1\}$;
- (3) $\alpha = 1$ and $s \in \{2p+1, 2q+1, 2r+1, 2pq+1, 2pr+1, 2qr+1, 2r^2+1, 2pqr+1, 2pr^2+1, 2qr^2+1\}$.

Proof. Since the G is non-solvable, it follows from Lemma 2.8 that $k = 2s^\alpha$, where s is an odd prime number. Then $2pqr^2 = n \cdot \varphi(2s^\alpha)$ by Lemma 2.1. If $\alpha > 3$, then $s^3 \mid \varphi(s^\alpha)$. Hence, $s^3 \mid pqr^2$, which leads to a contradiction. If $\alpha = 3$, then $s^2(s-1) \mid 2pqr^2$, thus only has $s = r$ and $(s-1)/2 \in \{p, q, pq\}$. Further, if $(s-1)/2 = pq$, then $\varphi(k) = 2pqr^2$. By Lemma 2.2, we know that G is supersolvable, which leads to a contradiction. Hence, $s = r \in \{2p+1, 2q+1\}$. If $\alpha = 2$, then $\frac{s-1}{2} s \mid pqr^2$. Consider the following three possible cases:

If $s = p$, then $(s - 1)/2 < p$. Therefore $(s - 1)/2 = 1$ and $s = p = 3$, which contradicts our assumption.

If $s = q$, then $(s - 1)/2 = p$. Hence $s = q = 2p + 1$ and $n = r^2$.

If $s = r$, then $(s - 1)/2 \in \{p, q, pq\}$.

If $\alpha = 1$, then $(s - 1)/2 \mid pqr^2$. If $s = 2pqr^2 + 1$, then in view of the fact $\varphi(k) = 2pqr^2$. By Lemma 2.2, we know that G is supersolvable, which leads to a contradiction. Hence, $s \in \{2p + 1, 2q + 1, 2r + 1, 2pq + 1, 2pr + 1, 2qr + 1, 2r^2 + 1, 2pqr + 1, 2pr^2 + 1, 2qr^2 + 1\}$.

3. Proof of Theorem 1.1

In the following, we will use the notations in Lemma 2.9- 2.11. By assumption, G has n cyclic subgroups of order k . For any $1 \leq i \leq d$, we choose a complete system of representatives A_i of the conjugate classes of these subgroups and set $n_i = |G : N_G(A_i)|$. Suppose that G is non-solvable, then by Lemma 2.10 we have $d \geq 2$. In addition, by Lemma 2.9, one has that s is an odd prime. We will prove the Theorem 1.1 case by case depending on the values of α and s in Lemma 2.11.

Case 1. Let $\alpha = 3$ and $s = r \in \{2p + 1, 2q + 1\}$.

Subcase 1.1. If $s = r = 2p + 1$, then by Lemma 2.11 we have $k = 2(2p + 1)^3 = 2r^3$. Let $A_i = \langle a_i \rangle$ and $O(a_i) = 2r^3$. By Lemma 2.3, we get $|G| \mid 2^\alpha pqr^\beta$, where $\alpha, \beta > 0$. By Lemma 2.1 and Lemma 2.10, we can get $n = \sum_{i=1}^d n_i = q$, and so $q \nmid n_i$ for any i and $1 \leq i \leq d$. By Lemma 2.10(1), we conclude that $|N_G(A_i) : C_G(A_i)| \mid 2pr^2$. Also, since $\pi(C_G(A_i)) = \{2, r\}$, thus G is a $\{2, p, r\}$ -group. Therefore, one has that $3 \notin \pi(G)$, contradicting to Lemma 2.5.

Subcase 1.2. If $s = r = 2q + 1$, then by Lemma 2.11 we have $k = 2(2q + 1)^3 = 2r^3$. Suppose that $A_i = \langle a_i \rangle$ and $O(a_i) = 2r^3$. By Lemma 2.3, we know that $|G| \mid 2^\alpha pqr^\beta$, where $\alpha, \beta > 0$. By Lemma 2.1 and Lemma 2.10(1), we get that $n = \sum_{i=1}^d n_i = p$, and hence $p \nmid n_i$ for any i and $1 \leq i \leq d$. By Lemma 2.10(2), we conclude that $|N_G(A_i) : C_G(A_i)| \mid 2qr^2$. Since $\pi(C_G(A_i)) = \{2, r\}$, then G is a $\{2, q, r\}$ -group. By Lemma 2.5, we deduce that $3 \nmid |G|$, which leads to a contradiction.

Case 2. Let $\alpha = 2$, $s = q = 2p + 1$ and $n = r^2$. Then by Lemma 2.11 we have $k = 2(2p + 1)^2 = 2q^2$. By Lemma 2.3, we get $|G| \mid 2^\alpha pr^2q^\beta$, where $\alpha, \beta > 0$. Therefore, $\pi(G) \subseteq \{2, p, q, r\}$. Since $5 < p < q < r$, then by Lemma 2.5 and Lemma 2.7, we can conclude that there exists no such normal series of G : $H \trianglelefteq K \trianglelefteq G$ such that K/H is isomorphic to one of some simple K_3 -groups or simple K_4 -groups. Hence G is solvable, a contradiction.

Case 3. Let $\alpha = 2$ and $s = r \in \{2p + 1, 2q + 1\}$. Suppose that $s = r = 2p + 1$. Let $A_i = \langle a_i \rangle$ and $O(a_i) = 2r^2$. Since $k = 2r^2$, then by Lemma 2.3, we have that $\pi(G) \subseteq \{2, p, q, r\}$. If there exists a positive integer i and $1 \leq i \leq d$ such that $q \nmid n_i$, then $|N_G(A_i) : C_G(A_i)| \mid 2pr$. Also, we know $\pi(C_G(A_i)) = \{2, r\}$.

Therefore, G is a $\{2, p, r\}$ -group. By Lemma 2.5, we conclude that G is solvable, a contradiction. If $q|n_i$ for each positive integer i and $1 \leq i \leq d$, then by Lemma 2.1 and Lemma 2.10(1) we have $n = \sum_{i=1}^d n_i = qr$. Consequently, G is a $\{2, p, q, r\}$ -group. Similar to the previous case, a contradiction. Replacing q with p in the previous case, we arrive at a contradiction also in the case where $s = r = 2q + 1$.

Case 4. Let $\alpha = 2$ and $s = r = 2pq + 1$. Then by Lemma 2.11 we have $k = 2(2pq + 1)^2 = 2r^2$. Assume that $A_i = \langle a_i \rangle$ and $O(a_i) = 2r^2$. By Lemma 2.3, we can get $|G| \mid 2^\alpha pqr^\beta$, where $\alpha, \beta > 0$. By hypothesis $n = r$, and by Lemma 2.10 we have $n = \sum_{i=1}^d n_i = r$. Hence, $r \nmid n_i$ for any i and $1 \leq i \leq d$. In addition, it follows from Lemma 2.10(2) that $|N_G(A_i) : C_G(A_i)| \mid 2pqr$. Therefore, $\pi(G) \subseteq \{2, p, q, r\}$. Since $5 < p < q < r$, then by Lemma 2.5 and Lemma 2.7, we can conclude that there exists no such normal series of G : $H \trianglelefteq K \trianglelefteq G$ such that K/H is isomorphic to one of some simple K_3 -groups or simple K_4 -groups. Hence G is solvable, a contradiction.

Case 5. Let $\alpha = 1$ and $s = r \in \{2pqr + 1, 2pr^2 + 1, 2qr^2 + 1\}$. Then by Lemma 2.11 we have $k \in \{2(2pqr + 1), 2(2pr^2 + 1), 2(2qr^2 + 1)\}$.

Subcase 5.1. If $s = 2pqr + 1$, then $k = 2s = 2(2pqr + 1)$. Assume that $A_i = \langle a_i \rangle$ and $O(a_i) = 2s$. Then $|N_G(A_i) : C_G(A_i)| \mid 2pqr$, thus by Lemma 2.10(2) we know that $\pi(G) \subseteq \{2, p, q, r, s\}$. Since $5 < p < q < r < s$, then $3 \nmid |G|$ and $5 \nmid |G|$, and so G is solvable by Lemma 2.8, a contradiction.

Subcase 5.2. Let $s \in \{2pr^2 + 1, 2qr^2 + 1\}$. If $s = 2pr^2 + 1$, since $(2pr^2 + 1)2pr^2 > 2pqr^2$, then by Lemma 2.4 we conclude that $G_{2pr^2+1} \trianglelefteq G$ and $|G_{2pr^2+1}| = 2pr^2 + 1$. Hence, $G_{2pr^2+1}(G)$ is cyclic. Since $(2pr^2 + 1)q > k$, we get $(2pr^2 + 1)q \notin \pi_e(G)$. Therefore, the action of the group G_q on G_{2pr^2+1} is the Frobenius action. This yields the divisibility of $q|2pr^2$, which leads to a contradiction. If we again replacing q with p in the presented above, then we arrive at a contradiction also in the case where $s = 2qr^2 + 1$.

Case 6. Let $\alpha = 1$ and $s \in \{2pr + 1, 2qr + 1\}$. Then by Lemma 2.11 we have $k \in \{2(2pr + 1), 2(2qr + 1)\}$. If $s = 2pr + 1$, then by Lemma 2.3 one has that $\pi(G) \subseteq \{2, p, q, r, s\}$. Assume that $A_i = \langle a_i \rangle$ and $O(a_i) = 2s$. If there exists a positive integer i and $1 \leq i \leq d$ such that $q \nmid n_i$, then $|N_G(A_i) : C_G(A_i)| \mid 2pr$. Therefore, G is a $\{2, p, r, s\}$ -group. Since $5 < p < r < s$, then by Lemma 2.5 and Lemma 2.7, we deduce that G is solvable. Otherwise, by Lemma 2.10(2) we know that $\pi(G) \subseteq \{2, p, q, r, s\}$. Since $5 < p < q < r < s$, then $3 \nmid |G|$ and $5 \nmid |G|$, and hence G is solvable by Lemma 2.8, a contradiction. If we again replacing q with p in the presented above, then we arrive at a contradiction also in the case where $s = 2qr + 1$.

Case 7. Let $\alpha = 1$ and $s = 2pq + 1$. Then by Lemma 2.11 we have $k = 2(2pq + 1)$. Assume that $A_i = \langle a_i \rangle$ and $O(a_i) = 2s$. By Lemma 2.3, we can get $|G| \mid 2^\alpha pqr^2 s^\beta$, where $\alpha, \beta > 0$. Then we know that $\pi(G) \subseteq \{2, p, q, r, s\}$.

Since $5 < p < q < r$ and $5 < s$, then $3 \nmid |G|$ and $5 \nmid |G|$, and so G is solvable by Lemma 2.8, a contradiction.

Case 8. Let $\alpha = 1$ and $s = 2r^2 + 1$. Then by Lemma 2.11 we have $k = 2s$. By Lemma 2.3, we have $\pi(G) \subseteq \{2, p, q, r, s\}$. Let $A_i = \langle a_i \rangle$ and $O(a_i) = 2s$. If there exists a positive integer i and $1 \leq i \leq d$ such that $p \nmid n_i$, then $|N_G(A_i) : C_G(A_i)| \mid 2r^2$. Also, we know $\pi(C_G(A_i)) = \{2, s\}$. Then, $\pi(G) \subseteq \{2, q, r, s\}$. Since $5 < q < r < s$, then by Lemma 2.5 and Lemma 2.7, we infer that G does not admit any simple K_3 -group or simple K_4 -group as its quotient group, which implies G is solvable, a contradiction. If $p \mid n_i$ for any i and $1 \leq i \leq d$, then $n = \sum_{i=1}^d n_i = pq$ by Lemma 2.1 and Lemma 2.10(1). Therefore, there exists a positive integer $1 \leq i \leq d$ such that $q \nmid n_i$. Then G is a $\{2, p, r, s\}$ -group. Reasoning as above, we can get a contradiction.

Case 9. Let $\alpha = 1$ and $s \in \{2p + 1, 2q + 1\}$. Then by Lemma 2.11 we have $k = 2s$. If $s = 2p + 1$, assume that $A_i = \langle a_i \rangle$ and $O(a_i) = 2s$. If there exists a positive integer i and $1 \leq i \leq d$ such that $q \nmid n_i$, then $|N_G(A_i) : C_G(A_i)| \mid 2p$. On the other hand, by Lemma 2.10(2) we have $\pi(C_G(A_i)) = \{2, s\}$. Therefore, G is a $\{2, p, r, s\}$ -group. Since $5 < p < r$ and $5 < s$, thus, by Lemma 2.5 and Lemma 2.7, the group G does not admit any simple K_3 -group or simple K_4 -group as its quotient group. Hence G is solvable, a contradiction. If $q \mid n_i$ for any positive integer i and $1 \leq i \leq d$, then by Lemma 2.1 and Lemma 2.10(1) we have $n = \sum_{i=1}^d n_i = qr^2$. Therefore, we can get that $\pi(G) \subseteq \{2, p, q, r, s\}$. Since $5 < p < q < r$ and $5 < s$, similar to the previous case, we can get a contradiction. If we again replacing q with p in the presented above, then we arrive at a contradiction also in the case where $s = 2q + 1$.

Case 10. Let $\alpha = 1$ and $s = 2r + 1$. Then by Lemma 2.11 we have $k = 2s$. Let $A_i = \langle a_i \rangle$ and $O(a_i) = 2s$. If there exists a positive integer i and $1 \leq i \leq d$ such that $p \nmid n_i$, then $|N_G(A_i) : C_G(A_i)| \mid 2r$. Then, we have $\pi(G) \subseteq \{2, q, r, s\}$. Since $5 < q < r < s$, then by Lemma 2.5 and Lemma 2.7, one has that G does not admit any simple K_3 -group or simple K_4 -group as its quotient group. Hence G is solvable, a contradiction. If $p \mid n_i$ for any i and $1 \leq i \leq d$, then $n = \sum_{i=1}^d n_i = pqr$ by Lemma 2.1 and Lemma 2.10(1). Suppose that there exists a positive integer j and $1 \leq j \leq d$ such that $q \nmid n_j$. Therefore, $\pi(G) \subseteq \{2, p, r, s\}$, again a contradiction. Otherwise, we get that $\pi(G) \subseteq \{2, p, q, r, s\}$. Since $5 < p < q < r < s$, then we have $3 \nmid |G|$ and $5 \nmid |G|$. Thus G is solvable by Lemma 2.8, which contradicts our assumption.

This completes the proof of Theorem 1.1.

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Towards the supercharacter theory of the dicyclic group

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Abstract. The dicyclic group of order $4n$ has a presentation

$$T_{4n} = \langle a, b \mid a^{4n} = 1, a^{2n} = b^2, b^{-1}ab = a^{-1} \rangle$$

and is a non-split extension of a cyclic group of order $2n$ by a cyclic group of order 2. In this paper we investigate a few supercharacter theory for T_{4n} .

Keywords: supercharacter, dicyclic group.

1. Introduction

Supercharacter theory was first defined in [1] and [2] to study the irreducible complex characters of the group $U_n(q)$ of n by n unipotent upper triangular matrices with entries in the Galois field $GF(q)$. But in [4] the authors formally defined the notion of supercharacter theory for an arbitrary finite group and studied supercharacter theory of a family of finite groups known as algebra groups.

Let G be a finite group and $Irr(G)$ be the set of all its complex irreducible characters, and let $Con(G)$ denote the set of all the conjugacy classes of G . The trivial character of G is denoted by 1_G and its identity element is denoted by 1. In [4] a supercharacter theory for G is defined to be a pair $(\mathcal{X}, \mathcal{K})$ where:

1. $\mathcal{X}$ is a partition of $Irr(G)$,
2. $\mathcal{K}$ is a partition of G and $\{1\} \in \mathcal{K}$,
3. $|\mathcal{X}| = |\mathcal{K}|$,
4. For each $X \in \mathcal{X}$ there is a character $\sum_X$ of G such that $\sum_X(x) = \sum_X(y)$ for all $x, y \in K$, $K \in \mathcal{K}$.

In the above situation $\sum_X$ is called a supercharacter and each member of $\mathcal{K}$ is called a superclass. $Sup(G)$ is the set of all the supercharacter theories for G .

Supercharacter theory of a finite group G may be regarded as a general case of the ordinary character theory. In fact, in a supercharacter theory, supercharacters play the role of irreducible ordinary characters and union of conjugacy

classes play the role of the conjugacy classes. In [4] it is shown that $\{1_G\} \in \mathcal{X}$ and if $X \in \mathcal{X}$, then $\sum_X$ is a constant multiple of $\sum_{\chi \in X} \chi(1)\chi$, and that we may assume that $\sum_X = \sum_{\chi \in X} \chi(1)\chi$.

Any finite group G has two extreme supercharacter theories which are denoted by $m(G)$ and $M(G)$ and are called trivial supercharacter theory for G :

$$\begin{aligned} m(G) &= (Con(G), \{\{\chi\} : \chi \in Irr(G)\}) \\ M(G) &= (\{\{1\}, G - \{1\}\}, \{\{1_G\}, Irr(G) - \{1_G\}\}) \end{aligned}$$

These supercharacter theories are distinct for all groups of order greater than 2. In [3] it is shown that the only finite groups with exactly two supercharacter theories are $\mathbb{Z}_3$, $\mathbb{S}_3$ and $SP_6(2)$.

It is proved in [5] that the set of all supercharacter theories of a group forms a lattice in the following natural way. $Sup(G)$ can be made to a poset by defining $(\mathcal{X}, \mathcal{K}) \preceq (\mathcal{Y}, \mathcal{L})$ if $\mathcal{X} \leq \mathcal{Y}$ in the sense that every part of $\mathcal{X}$ is a subset of some part of $\mathcal{Y}$. By [5] this definition is equivalent to $(\mathcal{X}, \mathcal{K}) \preceq (\mathcal{Y}, \mathcal{L})$ if $\mathcal{K} \leq \mathcal{L}$. By this definition $m(G)$ is the least and $M(G)$ is the largest element of $Sup(G)$.

The first author to attempt to classify the supercharacter theories of a given finite group was Hendrickson. In [5] the author classified the supercharacter theories of arbitrary cyclic groups. In [10] the authors study the supercharacter theories for extra special p -groups and also Frobenius groups. In [11] the author attempts to find the supercharacter theories for parabolic subgroups. For other families of finite groups, as extensions of the cyclic groups, the supercharacter theories of the dihedral groups were found in [9]. The dihedral group of order $2n$ has presentation:

$$D_{2n} = \langle a, b \mid a^n = b^2 = 1, b^{-1}ab = a^{-1} \rangle = \langle a \rangle \rtimes \langle b \rangle \cong \mathbb{Z}_n \rtimes \mathbb{Z}_2.$$

The dicyclic group of order $4n$ has presentation

$$T_{4n} = \langle a, b \mid a^{2n} = 1, a^n = b^2, b^{-1}ab = a^{-1} \rangle \cong \mathbb{Z}_{2n} \cdot \mathbb{Z}_2$$

which is a non-split extension of the cyclic group of order $\mathbb{Z}_{2n}$ by the cyclic group of order 2. Our aim in this paper is to provide some supercharacter theories of T_{4n} .

2. Preliminaries

For a non-trivial supercharacter theory of a group the following is described in [4]. Let G be a finite group and $A \leq Aut(G)$. Let

$$Irr(G) = \{\chi_1 = 1_G, \chi_2, \dots, \chi_h\} \text{ and } Con(G) = \{\mathcal{C}_1 = \{1\}, \mathcal{C}_2, \dots, \mathcal{C}_h\}.$$

Suppose that for each $\alpha \in A$, $\mathcal{C}_i^\alpha = \mathcal{C}_j$, $1 \leq i \leq h$, and $\chi_i^\alpha(g) = \chi_i(g^\alpha)$, for all $g \in G$, and $\alpha \in A$. In this case we have an action of A on both $Irr(G)$

and $Con(G)$ and by a Lemma of Brauer mentioned in [7] the number of conjugacy classes of G fixed by α equals the number of irreducible characters fixed by α , moreover the number of orbits of A on $Con(G)$ equals the number of orbits of A on $Irr(G)$. It is easy to see that the orbits of A on $Irr(G)$ and $Con(G)$ yield a supercharacter theory for G . This supercharacter theory for G is called automorphic. There are groups that all of its supercharacter theories are automorphic.

Theorem 2.1. *Every supercharacter theory of $\mathbb{Z}_p$, p prime, is automorphic. Moreover, for each divisor d of $p - 1$, there is a unique supercharacter theory for $\mathbb{Z}_p$ whose non-trivial superclasses all have size d . Therefore the number of supercharacter theories is equal to $\varphi(p - 1)$, where φ denotes the Euler totient function.*

According to [4] there is a method of constructing supercharacter theories for a group G of order n which uses the action of a group A of automorphisms of the cyclotomic field $\mathbb{Q}[\xi]$, where ξ is a primitive n th root of unity. It is known that the values of the irreducible characters of G are all lie in the cyclotomic field $\mathbb{Q}(\xi)$. Let $A \leq \text{Aut}(\mathbb{Q}(\xi))$, then for $\sigma \in \text{Aut}(\mathbb{Q}(\xi))$ we have $\sigma(\xi) = \xi^m$, where $(m, n) = 1$. This defines an automorphism of $\mathbb{Q}(\xi)$ and all automorphisms of $\mathbb{Q}(\xi)$ are obtained in this way. Then σ carries the class of $g \in G$ to the class containing g^m . The Brauer Lemma shows that the number of A -orbits on $Irr(G)$ is equal to the number of A -orbits on $Con(G)$. In this case similarly if $\mathcal{X}$ is the set of A -orbits on $Irr(G)$ and $\mathcal{K}$ is the set of unions of A -orbits on $Con(G)$, then $(\mathcal{X}, \mathcal{K})$ is a supercharacter theory for G . We shall call this supercharacter theory Galois.

At this point we define the supercharacter table. Let $(\mathcal{X}, \mathcal{K})$ be a supercharacter theory for a finite group G . Suppose $\mathcal{X} = \{X_1, X_2, \dots, X_k\}$ is a partition of $Irr(G)$ with corresponding supercharacter $\sum_i = \sum_{\chi \in X_i} \chi(1)\chi$. Let $\mathcal{K} = \{K_1, K_2, \dots, K_k\}$ be the partition of G into superclasses. In fact, $X_1 = \{1_G\}$, $K_1 = \{1\}$ and K_i 's are union of conjugacy classes of G . The supercharacter table of G corresponding to $(\mathcal{X}, \mathcal{K})$ is the following $k \times k$ array.

Table 1

	K_1	K_2	$\dots$	K_j	$\dots$	K_k
$\sum_1$	$\sum_1(K_1)$	$\sum_1(K_2)$	$\dots$	$\sum_1(K_j)$	$\dots$	$\sum_1(K_k)$
$\sum_2$	$\sum_2(K_1)$	$\sum_2(K_2)$	$\dots$	$\sum_2(K_j)$	$\dots$	$\sum_2(K_k)$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
$\sum_i$	$\sum_i(K_1)$	$\sum_i(K_2)$	$\dots$	$\sum_i(K_j)$	$\dots$	$\sum_i(K_k)$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
$\sum_h$	$\sum_h(K_1)$	$\sum_h(K_2)$	$\dots$	$\sum_h(K_j)$	$\dots$	$\sum_h(K_k)$

3. The dicyclic group

In terms of the generators and relations the dicyclic group is defined by

$$T_{4n} = \langle a, b \mid a^{2n} = 1, a^n = b^2, b^{-1}ab = a^{-1} \rangle, n \geq 1.$$

This is a group of order $4n$ and $T_4 \cong \mathbb{Z}_4$, $T_8 \cong Q_8$ the quaternion group of order 8. Since T_{4n} has only one element of order 2, and $\langle a \rangle \trianglelefteq T_{4n}$, therefore $T_{4n} \cong \mathbb{Z}_{2n} \cdot \mathbb{Z}_2$ is a non-split extension of $\mathbb{Z}_{2n}$ by $\mathbb{Z}_2$.

The conjugacy classes and the complex character table of T_{4n} can be found in [8] and are listed below:

$G = T_{4n}$ has $n + 3$ conjugacy classes as follows:

$$\begin{aligned} K_1 &= \{1\} \\ K_2 &= \{a^n\} \\ K_r &= \{a^r, a^{-r}\}, 1 \leq r \leq n-1 \\ K'_3 &= \{a^{2j}b \mid 0 \leq j \leq n-1\} \\ K'_4 &= \{a^{2j+1}b \mid 0 \leq j \leq n-1\}. \end{aligned}$$

If n is odd, then $\frac{G}{G'} \cong \mathbb{Z}_4$ and the character table of G is as follows:

Table 2

$ C_G(x) $	$4n$	$4n$	$2n$	4	4
x	1	a^n	$a^r (1 \leq r \leq n-1)$	b	ab
χ_1	1	1	1	1	1
χ_2	1	-1	$(-1)^r$	i	$-i$
χ_3	1	1	1	-1	-1
χ_4	1	-1	$(-1)^r$	$-i$	i
ψ_j $1 \leq j \leq n-1$	2	$2(-1)^j$	$\omega^{rj} + \omega^{-rj}$	0	0

If n is even, then $\frac{G}{G'} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Table 3

$ C_G(x) $	$4n$	$4n$	$2n$	4	4
x	1	a^n	$a^r (1 \leq r \leq n-1)$	b	ab
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	$(-1)^r$	1	-1
χ_4	1	1	$(-1)^r$	-1	1
ψ_j $1 \leq j \leq n-1$	2	$2(-1)^j$	$\omega^{rj} + \omega^{-rj}$	0	0

Where $\omega = e^{\frac{2\pi i}{n}}$ in both tables.

Lemma 3.1. *The automorphism group of T_{4n} has order $2n\varphi(2n)$ and its elements are*

$$\mathbb{A}ut(T_{4n}) = \{f_{k,l} \mid f_{k,l}(a) = a^k, f_{k,l}(b) = a^l b, (k, 2n) = 1, 0 \leq l < 2n\}.$$

Proof. It is enough to define each automorphism of T_{4n} on a and b . The only elements of order $2n$ in T_{4n} are a^k , where $(k, 2n) = 1$. Therefore if f is an automorphism of T_{4n} we should have $f(a) = a^k$, $(k, 2n) = 1$. But then $f(b)$ cannot be a power of a , and as an element of order 4 we should have $f(b) = a^l b$, $0 \leq l < 2n$. Then it is routine to check that f with above definition is infact an automorphism of T_{4n} . Therefore we set $f = f_{k,l}$ and verify that $f_{k,l}f_{k',l'} = f_{kk',kl'+l}$. If Φ_{2n} denotes the groups of units of $\mathbb{Z}_{2n}$ we have $\mathbb{A}ut(T_{4n}) \cong \mathbb{Z}_{2n} \rtimes \Phi_{2n}$. $\square$

Keeping fixed the previous notation we see that orbits of $\mathbb{A}ut(T_{4n})$ on $Con(T_{4n})$ are:

$$K_1, K_2, K'_3 \cup K'_4, \bigcup_{\substack{k \\ (k, 2n)=1}} K_{rk}.$$

Orbits of $\mathbb{A}ut(T_{4n})$ on $Irr(T_{4n})$ are:

$$\begin{aligned} \chi_1, \chi_2 + \chi_4, \chi_3, \sum_{\substack{j \\ (j, 2n)=1}} \psi_j & \quad \text{if } n \text{ is odd and,} \\ \chi_1, \chi_2, \chi_3 + \chi_4, \sum_{\substack{j \\ (j, 2n)=1}} \psi_j & \quad \text{if } n \text{ is even.} \end{aligned}$$

Therefore, the above is an automorphic supercharacter theory for T_{4n} . To be precise in the following we give a supercharacter table in case of T_{4p} , where p is an odd prime number. In this case the orbits of $\mathbb{A}ut(T_{4p})$ on $Con(T_{4p})$ are:

$$\begin{aligned} K_1 &= \{1\}, \\ K_2 &= \{a^p\}, \\ K'_3 \cup K'_4, \\ K_5 &= \{a^{\pm 1}, a^{\pm 3}, \dots, a^{\pm(2p-1)}\}, \\ K_6 &= \{a^{\pm 2}, a^{\pm 4}, \dots, a^{\pm(2p-2)}\}. \end{aligned}$$

Using tables 2 we find the following partition of $Irr(T_{4n})$:

$$\{\chi_1\}, \{\chi_2, \chi_4\}, \{\chi_3\}, \{\psi_j \mid j = 1, 3, \dots, p-2\}, \{\psi_j \mid j = 2, 4, \dots, p-1\}$$

Therefore $(\mathcal{X}, \mathcal{K})$ is a supercharacter theory for T_{4p} with the following supercharacter table:

Table 4

	K_1	K_2	$K'_3 \cup K'_4$	K_5	K_6
$\sum_1$	1	1	1	1	1
$\sum_2$	2	-2	0	-2	2
$\sum_3$	1	1	-1	1	1
$\sum_4$	$p-1$	$-(p-1)$	0	$2 \cos \frac{\pi}{p}$	$-2 \cos \frac{\pi}{p}$
$\sum_5$	$p-1$	$p-1$	0	$-2 \cos \frac{\pi}{p}$	$2 \cos \frac{\pi}{p}$

If $p = 2$, then $T_8 \cong Q_8$ is the quaternion group of order 8 and this case we have:

$$\mathcal{K} = \{K_1 = \{1\}, K_2 = \{a^2\}, K_3 = \{a\}, K_4 = \{class(b), class(ab)\}\}$$

$$\mathcal{X} = \{\{\chi_1\}, \{\chi_2\}, \{\chi_3, \chi_4\}, \{\psi_1\}\}$$

Table 5

	K_1	K_2	K_3	K_4
$\sum_1$	1	1	1	1
$\sum_2$	1	1	1	-1
$\sum_3$	2	2	-2	0
$\sum_4$	2	-2	0	0

Lemma 3.2. *If n is even, then T_{4n} and D_{4n} have the same number of supercharacter theory.*

Proof. If n is even it is easy to see that T_{4n} and D_{4n} have the same character table. But then groups with the same character tables have the same supercharacter theory. $\square$

Next, we will consider a certain subgroup of $\text{Aut}(T_n)$. Let

$$H = \{f_{1,l} \mid 0 \leq l < 2n\} \cong \mathbb{Z}_{2n},$$

where $f_{1,l}(a) = a$, $f_{1,l}(b) = a^l b$, $0 \leq l < 2n$. Orbits of H on $\text{Con}(T_{4n})$ are:

$$\begin{aligned} K_1 &= \{1\}, \\ K_2 &= \{a^n\}, \\ K_r &= \{a^r\}, 1 \leq r \leq n-1, \\ K'_3 \cup K'_4 &. \end{aligned}$$

Therefore, there are $n+2$ superclasses, the supercharacter are:

$$\begin{cases} \sum_1 = \chi_1, \sum_2 = \chi_2 + \chi_4, \sum_3 = \chi_3, \sum_j = \psi_j, 1 \leq j \leq n-1, & \text{if } n \text{ is odd,} \\ \sum_1 = \chi_1, \sum_2 = \chi_2, \sum_3 = \chi_3 + \chi_4, \sum_j = \psi_j, 1 \leq j \leq n-1, & \text{if } n \text{ is even.} \end{cases}$$

Now, let p be an odd prime and consider the subgroup

$$K = \{f_{k,0} \mid (k, 2p) = 1\} \cong \mathbb{Z}_{p-1},$$

where $f_{k,0}(a) = a^k$, $k \neq p$, $k = 1, 3, \dots, 2p-1$; $f_{k,0} = b$.

In this case, regarding the action of K , the superclasses are $K_1 = \{1\}$, $K_2 = \{a^p\}$, $K_3 = \text{class}(a)$, $K_4 = \text{class}(a^2)$, $K_5 = \text{class}(b)$, $K_6 = \text{class}(b^3)$.

The supercharacters are:

$$\{\chi_1\}, \{\chi_2\}, \{\chi_3\}, \{\chi_4\}, \sum_{\substack{j \text{ odd} \\ 1 \leq j \leq p-1}} 2\psi_j, \sum_{\substack{j \text{ even} \\ 1 \leq j \leq p-1}} 2\psi_j.$$

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Some random iteration processes in modular function space

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Abstract. In this paper, some random iteration processes for some random operators in modular function space are studied. Then the sequences generated by these iterations are strongly convergent to a random fixed point of the ρ -generalized Lipschitzian mapping. Our results extend and improve some recent results in the literature. Finally, some numerical examples are presented to indicate the validity of the results.

Keywords: random iteration, random fixed point, modular function space, strongly convergence.

1. Introduction

The Ishikawa and Mann iteration schemes are two general iterations which have been successfully applied to the operators' fixed point problems and also for obtaining the solutions of operator equations in the linear spaces (see [19, 20]). In recent years, Choudhury [3], has suggested and analyzed random Mann iterative sequence in separable Hilbert spaces for finding random solutions and random fixed points for some kind of random equations and random operators. The study of random fixed point theory is playing an increasing role in mathe-

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matics and applied sciences. At present, it received considerable attention due to enormous applications in many important areas such as Nonlinear Analysis, Probability Theory and for the study of random equations arising in various applied areas. Therefore, random nonlinear analysis is an important mathematical discipline which is mainly concerned with the study of random nonlinear operators and their properties and is much needed for the study of various classes of random equations. The random fixed point results is of great value. Random techniques have played a crucial role in pure mathematics as well as applied sciences. No doubt, famously random methods have revolutionized the financial markets.

In 2001, Jung et al. [6] introduced and studied generalized Lipschitzian mappings in Banach space. The importance of generalized Lipschitzian mapping is because of every Lipschitzian mapping is a generalized Lipschitzian mapping, but vice versa is not necessarily true.

Let us recall that, the theory of modular spaces, as a generalization of a metric space, was initiated by Nakano [14] in 1950 in connection with the theory of order spaces and generalized by Musielak and Orlicz [13] in 1959. These spaces were developed following the successful theory of Orlicz spaces, which Orlicz spaces generalize Lebesgue spaces by considering spaces of functions with some growth properties different from the power type growth control provided by the L_p -norm and obtained by replacing the given integral form of the nonlinear functional which controls the growth of the functions, by an abstract functional, the modular. Modular function spaces are natural generalization of both function and sequence variants of many important, from applications perspective, spaces like Lebesgue, Orlicz, Musielak-Orlicz, Lorentz, Orlicz-Lorentz, Calderon-Lozanovskii spaces and many others. Even though a metric is not defined, many problems in metric fixed point theory can be reformulated and solved in modular spaces.

Our results (in particular) can be applied to $L_1(\Omega, \lambda)$ or Musielak-Orlicz space, for showing that generalized Lipschitzian mappings have a fixed point when they are defined on a convex subset of $L_1(\Omega, \lambda)$ or Musielak-Orlicz space, which sequences generated by the random operators are strongly convergent to a random fixed point of the ρ -generalized Lipschitzian mapping.

The purpose of this paper is to introduce and construct some random iteration processes for ρ -generalized Lipschitzian mapping in modular function spaces. Firstly, a new double random iteration processes for a random operator in modular function space is introduced. Then we show that the sequence generated by the iteration convergence strongly to a random fixed point of the ρ -generalized Lipschitzian mapping. Moreover, we study a random operator iteration for a nonexpansive semigroup and a ρ -generalized Lipschitzian mapping in a modular function space. Finally, we present some examples to show the validity of the results.

At this step, we recall some definitions (see [2, 4, 7, 8, 9, 15, 16, 17, 18, 21, 23, 22, 24]).

Definition 1.1. Let X be an arbitrary vector space over $K = (\mathbb{R} \text{ or } \mathbb{C})$.

a) A functional $\rho : X \rightarrow [0, \infty]$ is called modular if it satisfies the following conditions:

i) $\rho(x) = 0$ if and only if $x = 0$.

ii) $\rho(\alpha x) = \rho(x)$ for $\alpha \in K$ with $|\alpha| = 1$, for all $x \in X$.

iii) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$, for all $x, y \in X$.

If iii) is replaced by:

iii)' $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$ for $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$, for all $x, y \in X$, then the modular ρ is called a convex modular.

b) A modular ρ defines a corresponding modular space, i.e. the space X_ρ given by

$$X_\rho = \{x \in X \mid \rho(\alpha x) \rightarrow 0 \text{ as } \alpha \rightarrow 0\}.$$

c) If ρ is a convex modular, the modular X_ρ can be equipped with a norm called the Luxemburg norm defined by

$$\|x\|_\rho = \inf \left\{ \alpha > 0; \quad \rho\left(\frac{x}{\alpha}\right) \leq 1 \right\}.$$

Remark 1.1. Notice that ρ is an increasing function. Suppose $0 < a < b$, then property (iii) with $y = 0$, shows that $\rho(ax) = \rho(\frac{a}{b}(bx)) \leq \rho(bx)$.

Definition 1.2. Let X_ρ be a modular space.

a) A sequence $(x_n)_{n \in \mathbb{N}}$ in X_ρ is said to be:

i) ρ -convergent to x if $\rho(x_n - x) \rightarrow 0$ as $n \rightarrow \infty$.

ii) ρ -Cauchy if $\rho(x_n - x_m) \rightarrow 0$ as $n, m \rightarrow \infty$.

b) X_ρ is ρ -complete if every ρ -Cauchy sequence is ρ -convergent.

c) A subset $B \subset X_\rho$ is said to be ρ -closed if for any sequence $(x_n)_{n \in \mathbb{N}} \subset B$ and $\rho(x_n - x) \rightarrow 0$ then $x \in B$.

d) A subset $B \subset X_\rho$ is called ρ -bounded if $\delta_\rho(B) = \sup \rho(x - y) < \infty$ for all $x, y \in B$, where $\delta_\rho(B)$ is called the ρ -diameter of B .

e) A function $f : X_\rho \rightarrow X_\rho$ is called ρ -continuous if $\rho(x_n - x) \rightarrow 0$, then $\rho(f(x_n) - f(x)) \rightarrow 0$.

Let Ω be a nonempty set and Σ a nontrivial σ -algebra of subsets of Ω . Suppose P is a δ -ring of subsets of Σ , such that $E \cap A \in P$ for any $E \in P$ and $A \in \Sigma$. Assume that there exists an increasing sequence of sets $K_n \in P$ such that $\Omega = \cup K_n$. By ξ we denote the linear space of all simple functions with supports from P . By M we will denote the space of all measurable functions, i.e. all functions $f : \Omega \rightarrow \mathbb{R}$ such that there exists a sequence $\{g_n\} \in \xi$, $|g_n| \leq |f|$ and $g_n(w) \rightarrow f(w)$ for all $w \in \Omega$. By 1_A we denote the characteristic function of the set A .

Definition 1.3. A function $\rho : \xi \times \Sigma \rightarrow [0, \infty]$ is called a modular function if:

(1) $\rho(0, E) = 0$, for any $E \in \Sigma$,

- (2) $\rho(f, E) \leq \rho(g, E)$ whenever $|f(w)| \leq |g(w)|$, for any $w \in \Omega$, $f, g \in \xi$ and $E \in \Sigma$,
- (3) $\rho(f, \cdot) : \Sigma \longrightarrow [0, \infty]$ is a σ -subadditive measure for every $f \in \xi$,
- (4) $\rho(\alpha, A) \longrightarrow 0$ as α decreases to 0 for every $A \in P$, where $\rho(\alpha, A) = \rho(\alpha 1_A, A)$,
- (5) if there exists $\alpha > 0$ such that $\rho(\alpha, A) = 0$ then $\rho(\beta, A) = 0$ for every $\beta > 0$,
- (6) for any $\alpha > 0$, $\rho(\alpha, \cdot)$ is order continuous on P , that is $\rho(\alpha, A_n) \longrightarrow 0$ if $\{A_n\} \in P$ and decreases to ϕ .

Definition 1.4. Let ρ be a modular function. A modular function space is the vector space $L_\rho(\Omega, \Sigma)$, or briefly L_ρ , defined by

$$L_\rho = \{f \in M; \rho(\alpha f) \rightarrow 0 \text{ as } \alpha \rightarrow 0\}.$$

Let C be a nonempty subset of a modular space X_ρ , $T : C \longrightarrow C$ be a mapping then $F(T) = \{x \in C : Tx = x\}$ denotes the set of fixed points of T .

Lemma 1.1 ([25]). Assume $\{a_n\}$ is a sequence of nonnegative numbers such that

$$a_{n+1} \leq (1 - \alpha_n)a_n + \delta_n, \quad n \geq 0,$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence of real numbers such that

$$(I) \lim_{n \rightarrow \infty} \alpha_n = 0 \text{ and } \sum_{n=1}^{\infty} \alpha_n = \infty.$$

$$(II) \limsup_{n \rightarrow \infty} \frac{\delta_n}{\alpha_n} \leq 0 \text{ or } \sum_{n=1}^{\infty} |\delta_n| < \infty.$$

Then, $\lim_{n \rightarrow \infty} a_n = 0$.

Now, we consider a generalized Lipschitzian mappings in modular space as below (also see [5]).

Definition 1.5. Let X_ρ be a modular space and C be a nonempty subset of X_ρ . A mapping $T : C \rightarrow C$ is said to be ρ -generalized Lipschitzian mapping if there exist $r, l \in \mathbb{R}^+$ with $r > l$ and

$$(1) \quad \begin{aligned} \rho(r(T^n x - T^n y)) &\leq a_n \rho(l(x - y)) \\ &+ b_n [\rho(l(x - T^n x)) + \rho(l(T^n y - y))] \\ &+ c_n [\rho(l(x - T^n y)) + \rho(l(T^n x - y))], \end{aligned}$$

for each $x, y \in C$ and $n \geq 1$, where a_n, b_n and c_n are nonnegative constants such that there exists an integer n_0 such that $b_n + c_n < 1$, for all $n > n_0$.

Definition 1.6. Let X_ρ be a modular space and C be a nonempty subset of X_ρ . A mapping $T : C \rightarrow C$ is said to be uniformly ρ -generalized Lipschitzian mapping, if there exist $r, l \in \mathbb{R}^+$ with $r > l$ and

$$(2) \quad \begin{aligned} \rho(r(Tx - Ty)) &\leq a\rho(l(x - y)) \\ &+ b[\rho(l(x - Tx)) + \rho(l(Ty - y))] \\ &+ c[\rho(l(x - Ty)) + \rho(l(Tx - y))], \end{aligned}$$

for each $x, y \in C$ where a, b and c are nonnegative constants and $b + c < 1$.

2. Random iterations for ρ -generalized Lipschitzian mappings

In 2002, Moore [12], considered the double sequence iteration

$$x_{k,n+1} = (1 - \alpha_n)x_{k,n} + \alpha_n T_k x_{k,n},$$

where $T_k x = (1 - a_k)z + a_k T x$ for all $x \in C$ and C is a bounded, closed, convex and nonempty subset of a real Hilbert space H . Also $\{\alpha_n\}_{n \geq 1}$ and $\{a_k\}_{k \geq 0}$ are sequences in $(0, 1)$. Moore proved the double sequence $\{x_{k,n}\}$ is strongly convergent to a fixed point of T in C .

Definition 2.1 ([12]). Let X_ρ be a modular space where ρ is convex and satisfying the Δ_2 -condition and $\mathbb{N}$ denotes the set of all natural numbers. We consider function $f : \mathbb{N} \times \mathbb{N} \rightarrow X_\rho$ defined by $f(n, m) = x_{n,m} \in X_\rho$. A double sequence $\{x_{n,m}\}$ is said to be strongly ρ -convergent to z if given any $\epsilon > 0$ there exist $N, M > 0$ such that $\rho(x_{n,m} - z) < \epsilon$ for all $n \geq N, m \geq M$. If for all $n, r \geq N$ and $m, t \geq M$, we have that $\rho(x_{n,r} - x_{m,t}) < \epsilon$, then the double sequence is said to be ρ -Cauchy. Furthermore, if for each fixed n , $\rho(x_{n,m} - z_n) \rightarrow 0$ as $m \rightarrow \infty$ and $\rho(z_n - z) \rightarrow 0$ as $n \rightarrow \infty$, then $\rho(x_{n,m} - z) \rightarrow 0$ as $n, m \rightarrow \infty$.

In the sequel, we define random operator in modular space.

Definition 2.2. Let (Ω, Σ) be measurable space and L_ρ a modular function space. Suppose C is a nonempty subset of L_ρ ,

- (1) a mapping $T : \Omega \times C \rightarrow C$ is called a random operator if $T(., x) : \Omega \rightarrow C$ is measurable for each fixed $x \in C$,
- (2) a measurable function $f : \Omega \rightarrow C$ is called a random fixed point of the random operator $T : \Omega \times C \rightarrow C$ if $T(t, f(t)) = f(t)$, for all $t \in \Omega$,
- (3) a random operator $T : \Omega \times C \rightarrow C$ is called continuous if for fixed $t \in \Omega$ $T(t, .) : C \rightarrow C$ is continuous.

We need the following theorem for the main results.

Theorem 2.1. *Let X_ρ be a ρ -complete modular space, where ρ is convex and satisfy the Δ_2 -condition. Suppose C is a nonempty, convex and ρ -closed subset of X_ρ , $T : C \rightarrow C$ is a uniformly ρ -generalized Lipschitzian mapping satisfy inequality (2), such that $\frac{a+b+c}{1-(b+c)} < 1$. Then there exists a unique fixed point $z \in C$ such that $Tz = z$.*

Proof. Let $\alpha \in \mathbb{R}^+$ be the conjugate of $\frac{r}{l}$ and choose $r > 2l$, therefore $\alpha l < r$. Let x be an arbitrary point of C and construct sequence $\{Tx_n\}$ as follows, $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. For each integer $n \geq 1$, we have,

$$\begin{aligned} \rho(r(Tx_{n+1} - Tx_n)) &\leq a\rho(l(x_{n+1} - x_n)) \\ &\quad + b[\rho(l(Tx_{n+1} - x_{n+1})) + \rho(l(Tx_n - x_n))] \\ &\quad + c[\rho(l(Tx_{n+1} - x_n)) + \rho(l(Tx_n - x_{n+1}))], \end{aligned}$$

on the other hand,

$$\begin{aligned} \rho(l(Tx_{n+1} - x_n)) &\leq \rho(\alpha l(Tx_{n+1} - x_{n+1})) + \rho(r(x_{n+1} - x_n)) \\ &\leq \rho(r(Tx_{n+1} - Tx_n)) + \rho(r(x_n - x_{n+1})). \end{aligned}$$

It follows that,

$$\rho(r(Tx_{n+1} - Tx_n)) \leq \frac{a+b+c}{1-(b+c)} \rho(r(x_{n+1} - x_n)) = k\rho(r(x_{n+1} - x_n)),$$

where $k = \frac{a+b+c}{1-(b+c)}$. Continuing this processes, we have $\rho(r(Tx_{n+1} - Tx_n)) \leq k^n \rho(r(Tx - x))$, which implies that $\lim_{n \rightarrow \infty} \rho(r(Tx_{n+1} - Tx_n)) = 0$. If $l < r_0 < 2l$ and since ρ is an increasing function, then $\rho(r_0(Tx_{n+1} - Tx_n)) \leq \rho(r(Tx_{n+1} - Tx_n))$, by Δ_2 -condition $\lim_{n \rightarrow \infty} \rho(r_0(Tx_{n+1} - Tx_n)) = 0$ for $l < r_0 < 2l$. Thus, for any $r > l$ $\lim_{n \rightarrow \infty} \rho(r(Tx_{n+1} - Tx_n)) = 0$.

We show that $\{Tx_n\}$ is ρ -Cauchy sequence. If not, then there exists an $\varepsilon > 0$ and two sequences of integers $\{n(s)\}$, $\{m(s)\}$, with, $n(s) > m(s) \geq s$, such that

$$(3) \quad \rho(r(Tx_{n(s)} - Tx_{m(s)})) \geq \varepsilon \quad \text{for } s = 1, 2, \dots$$

We assume,

$$(4) \quad \rho(r(x_{n(s)} - Tx_{m(s)})) < \varepsilon.$$

In order to show this, suppose $n(s)$ is the smallest number exceeding $m(s)$ for which (3) is hold and

$$\sum_s = \{n \in \mathbb{N} \mid \exists m(s) \in \mathbb{N}; \rho(r(Tx_n - Tx_{m(s)})) \geq \varepsilon \text{ and } n > m(s) \geq s\}.$$

Obviously $\sum_s \neq \emptyset$ and since $\sum_s \subset \mathbb{N}$, then by well ordering principle, the minimum element of $\sum_s$ is denoted by $n(s)$, and clearly (4) is hold. Now

$$\begin{aligned} \rho(r(Tx_{m(s)} - Tx_{n(s)})) &\leq a\rho(l(x_{m(s)} - x_{n(s)})) \\ &\quad + b[\rho(l(Tx_{m(s)} - x_{m(s)})) + \rho(l(Tx_{n(s)} - x_{n(s)}))] \\ &\quad + c[\rho(l(Tx_{m(s)} - x_{n(s)})) + \rho(l(Tx_{n(s)} - x_{m(s)}))], \end{aligned}$$

moreover

$$\rho(l(x_{m(s)} - x_{n(s)})) \leq \rho(\alpha l(Tx_{m(s)} - x_{m(s)})) + \rho(r(Tx_{m(s)} - x_{n(s)})),$$

and

$$\rho(l(Tx_{n(s)} - x_{m(s)})) \leq \rho(r(Tx_{n(s)} - Tx_{m(s)})) + \rho(\alpha l(Tx_{m(s)} - x_{m(s)})),$$

therefore

$$\begin{aligned} \rho(r(Tx_{m(s)} - Tx_{n(s)})) &\leq \frac{a+c}{1-c} \rho(r(Tx_{m(s)} - x_{n(s)})) \\ &\quad + \frac{a+b+c}{1-c} \rho(\alpha l(Tx_{m(s)} - x_{m(s)})) \\ &\quad + \frac{b}{1-c} \rho(l(Tx_{n(s)} - x_{n(s)})). \end{aligned}$$

Since $\frac{a+b+c}{1-(b+c)} < 1$ then $\frac{a+c}{1-c} < 1$, also by using Δ_2 -condition as $s \rightarrow \infty$, we have, $\varepsilon \leq \frac{a+c}{1-c} \varepsilon$, which is a contradiction, therefore $\{Tx_n\}$ is ρ -Cauchy sequence. Since X_ρ is ρ -complete and C , ρ -closed, then there exist a $w \in C$ such that $\rho(Tx_n - w) \rightarrow 0$. Since X_ρ is ρ -complete and C , ρ -closed, then there exists a $w \in C$ such that $\rho(Tx_n - w) \rightarrow 0$. Now, we prove w is a fixed point of T .

$$\begin{aligned} \rho(r(Tx_{n+1} - Tw)) &\leq a\rho(l(x_{n+1} - w)) \\ &\quad + b[\rho(l(Tx_{n+1} - x_{n+1})) + \rho(l(Tw - w))] \\ &\quad + c[\rho(l(Tx_{n+1} - w)) + \rho(l(Tw - x_{n+1}))], \end{aligned}$$

therefore as $n \rightarrow \infty$,

$$\rho(r(Tw - w)) \leq (b+c)\rho(l(Tw - w)) \leq (b+c)\rho(r(Tw - w)),$$

since $b+c < 1$ then $\rho(r(Tw - w)) = 0$ and $Tw = w$. For uniqueness, let z and w be two arbitrary fixed points of T . Then

$$\begin{aligned} \rho(r(Tz - Tw)) &\leq a\rho(l(z - w)) \\ &\quad + b[\rho(l(Tz - z)) + \rho(l(Tw - w))] \\ &\quad + c[\rho(l(Tz - w)) + \rho(l(Tw - z))], \end{aligned}$$

since $\rho(r(z - w)) \leq (a+2c)\rho(l(z - w))$ then $\rho(r(z - w)) = 0$ or $z = w$. $\square$

Let C be a nonempty subset of a modular function L_ρ and $T : \Omega \times C \rightarrow C$ be a uniformly ρ -generalized Lipschitzian random operator, that is,

$$\begin{aligned} \rho(r(T(t, u) - T(t, v))) &\leq a\rho\left(\frac{l}{2}(u - v)\right) \\ (5) \quad &\quad + b[\rho\left(\frac{l}{2}(u - T(t, u))\right) + \rho\left(\frac{l}{2}(T(t, v) - v)\right)] \\ &\quad + c[\rho\left(\frac{l}{2}(u - T(t, v))\right) + \rho\left(\frac{l}{2}(v - T(t, u))\right)], \end{aligned}$$

for each $u, v \in C$ and $t \in \Omega$, also $r > l$. Let $x_{0,0} : \Omega \rightarrow C$ be an arbitrary measurable mapping, consider the following iterative sequence $\{x_{k,n}(t)\}$ defined by,

$$(6) \quad \begin{cases} y_{k,n}(t) = (1 - \beta_n)x_{k,n}(t) + \beta_n T_k(t, x_{k,n}(t)), \\ x_{k,n+1}(t) = (1 - \alpha_n)x_{k,n}(t) + \alpha_n T_k(t, y_{k,n}(t)), \end{cases}$$

for $t \in \Omega$, where $0 < \alpha_n, \beta_n < 1$. For an arbitrary but fixed $z \in C$, define $T_k : \Omega \times C \rightarrow C$ by $T_k(t, x(t)) = (1 - \lambda_k)z(t) + \lambda_k T(t, x(t))$, for all $x \in C$, $t \in \Omega$ and $\lambda_k \in (0, 1)$.

Since C is convex, it follows that $x_{k,n}$ is a mapping from Ω to C , for all $n \geq 0$ and $k \geq 0$.

In the sequel, we can prove that the sequence $\{x_{k,n}(t)\}$ generated by (6), is strongly convergent in a modular function L_ρ .

Theorem 2.2. *Let L_ρ be a ρ -complete modular function space where ρ is convex and satisfies the Δ_2 -condition. Suppose C is a nonempty, ρ -closed and convex subset of L_ρ and*

(I) *T be a continuous random operator and for all $t \in \Omega$, $T(t, \cdot) : C \rightarrow C$ satisfies inequality (5), where $\frac{a+b+c}{1-(b+c)} < 1$, also $r > 2l$.*

(II) $\sum_{n \geq 0} \alpha_n = \infty$, $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$.

(III) $\lim_{k \rightarrow \infty} \lambda_k = 1$ and for all $x \in C$ and $r \in \mathbb{R}^+$, $\rho(r(x(t) - z(t))) \leq m < \infty$.

If $F(T) \neq \emptyset$, then the sequence $\{x_{k,n}(t)\}_{k \geq 0, n \geq 0}$, generated by (6), is strongly convergent to a random fixed point w of T in C .

Proof. For each $k \geq 0$ and for all $t \in \Omega$, we have,

$$\begin{aligned} \rho(r(T_k(t, u(t)) - T_k(t, v(t)))) &= \rho(r\lambda_k(T(t, u(t)) - T(t, v(t)))) \\ &\leq a\lambda_k\rho\left(\frac{l}{2}(u(t) - v(t))\right) \\ &\quad + b\lambda_k[\rho\left(\frac{l}{2}(u(t) - T(t, u(t)))\right) \\ &\quad + \rho\left(\frac{l}{2}(T(t, v(t)) - v(t))\right)] \\ &\quad + c\lambda_k[\rho\left(\frac{l}{2}(u(t) - T(t, v(t)))\right) \\ &\quad + \rho\left(\frac{l}{2}(T(t, u(t)) - v(t))\right)]. \end{aligned}$$

Since $T(t, u(t)) = \frac{1}{\lambda_k}[T_k(t, u(t)) - (1 - \lambda_k)z(t)]$,

$$\rho\left(\frac{l}{2}(T(t, u(t)) - u(t))\right) \leq \rho\left(\frac{l}{\lambda_k}(T_k(t, u(t)) - u(t))\right) + \rho\left(l\frac{1 - \lambda_k}{\lambda_k}(u(t) - z(t))\right).$$

Also, since $\frac{1}{2} < \lambda_k < 1$ then $\frac{l}{\lambda_k} < 2l$ and $\frac{1-\lambda_k}{\lambda_k} < 1$. It follows that,

$$\rho\left(\frac{l}{2}(T(t, u(t)) - u(t))\right) \leq \rho(l_0(T_k(t, u(t)) - u(t))) + \frac{1-\lambda_k}{\lambda_k}m,$$

where $l_0 = 2l$. Therefore, ,

$$\begin{aligned} \rho(r(T_k(t, u(t)) - T_k(t, v(t)))) &\leq a\rho(l_0(u(t) - v(t))) \\ &\quad + b[\rho(l_0(T_k(t, u(t)) - u(t))) \\ &\quad + \rho(l_0(T_k(t, v(t)) - v(t)))] \\ &\quad + c[\rho(l_0(T_k(t, u(t)) - v(t))) \\ &\quad + \rho(l_0(T_k(t, v(t)) - u(t)))] \\ &\quad + 2\frac{1-\lambda_k}{\lambda_k}m(b+c). \end{aligned}$$

By Theorem 2.1, $\lim_{k \rightarrow \infty} \lambda_k = 1$, shows that, for each $k \geq 0$, T_k has a unique fixed point w_k , say, in C . Since C is ρ -closed, it follows that w_k is a mapping from Ω to C . For each $k \geq 0$, we show that $\rho(x_{k,n}(t) - w_k(t)) \rightarrow 0$ as $n \rightarrow \infty$. By equality (6),

$$\begin{aligned} \rho(r(x_{k,n+1}(t) - w_k(t))) &\leq (1 - \alpha_n)\rho(r(x_{k,n}(t) - w_k(t))) \\ &\quad + \alpha_n\rho(r(T_k(t, y_{k,n}(t)) - w_k(t))), \end{aligned}$$

on the other hand,

$$\begin{aligned} \rho(r(T_k(t, y_{k,n}(t)) - w_k(t))) &\leq a\rho(l_0(y_{k,n}(t) - w_k(t))) \\ &\quad + b\rho(l_0(T_k(t, y_{k,n}(t)) - y_{k,n}(t))) \\ &\quad + c[\rho(l_0(y_{k,n}(t) - w_k(t))) \\ &\quad + \rho(l_0(T_k(t, y_{k,n}(t)) - w_k(t)))] \\ &\quad + 2\frac{1-\lambda_k}{\lambda_k}m(b+c), \end{aligned}$$

moreover,

$$\rho(l_0(T_k(t, y_{k,n}(t)) - y_{k,n}(t))) \leq \rho(\alpha l_0(T_k(t, y_{k,n}(t)) - w_k(t))) + \rho(r(y_{k,n}(t) - w_k(t))),$$

where $\alpha \in \mathbb{R}^+$ be the conjugate of $\frac{r}{l_0}$.

Case 1. Let $r > 2l_0$ then $\alpha l_0 < r$, we have,

$$\rho(r(T_k(t, y_{k,n}(t)) - w_k(t))) \leq q\rho(r(y_{k,n}(t) - w_k(t))) + 2\frac{1-\lambda_k}{\lambda_k}m,$$

where $q = \frac{a+b+c}{1-(b+c)}$. Therefore,

$$\begin{aligned} \rho(r(x_{k,n+1}(t) - w_k(t))) &\leq (1 - \alpha_n)\rho(r(x_{k,n}(t) - w_k(t))) \\ (7) \quad &\quad + \alpha_n q \rho(r(y_{k,n}(t) - w_k(t))) \\ &\quad + \alpha_n \Theta_k, \end{aligned}$$

where $\Theta_k = 2\frac{1-\lambda_k}{\lambda_k}m$. By the same argument,

$$(8) \quad \begin{aligned} \rho(r(y_{k,n}(t) - w_k(t))) &\leq (1 - \beta_n)\rho(r(x_{k,n}(t) - w_k(t))) \\ &+ \beta_n q \rho(r(x_{k,n}(t) - w_k(t))) + \beta_n \Theta_k. \end{aligned}$$

By inequalities (7) and (8),

$$(9) \quad \begin{aligned} \rho(r(x_{k,n+1}(t) - w_k(t))) &\leq (1 - \alpha_n)\rho(r(x_{k,n}(t) - w_k(t))) \\ &+ \alpha_n q(1 - \beta_n)\rho(r(x_{k,n}(t) - w_k(t))) \\ &+ \alpha_n q^2 \beta_n \rho(r(x_{k,n}(t) - w_k(t))) \\ &+ \alpha_n (q\beta_n + 1)\Theta_k. \end{aligned}$$

By Lemma 1.1, since $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\lim_{k \rightarrow \infty} \Theta_k = 0$, then $\rho(r(x_{k,n}(t) - w_k(t))) \rightarrow 0$ and by Δ_2 -condition $\rho(x_{k,n}(t) - w_k(t)) \rightarrow 0$ as $n \rightarrow \infty$.

Case 2. Let $l_0 < r_0 < 2l_0$, since ρ is an increasing function, then $\rho(r_0(x_{k,n}(t) - w_k(t))) \leq \rho(r(x_{k,n}(t) - w_k(t)))$, by Δ_2 -condition $\lim_{n \rightarrow \infty} (r_0(x_{k,n}(t) - w_k(t))) = 0$, for $l < r_0 < 2l_0$. Thus, for any $r > l_0$, $\lim_{n \rightarrow \infty} \rho(r(x_{k,n}(t) - w_k(t))) = 0$. Since $w_k(t) = (1 - \lambda_k)z(t) + \lambda_k T(t, w_k(t))$ then

$$\rho(r(w_k(t) - T(t, w_k(t)))) = \rho(r(\frac{1 - \lambda_k}{\lambda_k}(z(t) - w_k(t)))) \leq \frac{1 - \lambda_k}{\lambda_k}m.$$

Since $\lambda_k \rightarrow 1$ then $\rho(w_k(t) - T(t, w_k(t))) \rightarrow 0$ as $k \rightarrow \infty$. Therefore, $\{w_k\}$ is an approximate random fixed point sequence for T .

We show that the sequence $\{w_k\}$ is ρ -Cauchy. For $0 < n \leq m$, we have

$$\begin{aligned} \rho(l(w_m(t) - w_n(t))) &\leq (\lambda_m - \lambda_n)\rho(\beta l(T(t, w_n(t)) - z(t))) \\ &+ \rho(r(T(t, w_m(t)) - T(t, w_n(t)))), \end{aligned}$$

where $\beta \in \mathbb{R}^+$ be the conjugate of $\frac{r}{l}$. On the other hand

$$\begin{aligned} \rho(r(T(t, w_n(t)) - T(t, w_m(t)))) &\leq (a + 2c)\rho(l(w_n(t) - w_m(t))) \\ &+ (b + c)\rho(l(T(t, w_n(t)) - w_n(t))) \\ &+ (b + c)\rho(l(T(t, w_m(t)) - w_m(t))). \end{aligned}$$

Therefore,

$$\begin{aligned} \rho(l(w_n(t) - w_m(t))) &\leq \frac{(\lambda_m - \lambda_n)}{1 - (a + 2c)}m \\ &+ \frac{b + c}{1 - (a + 2c)}\rho(r(T(t, w_m(t)) - w_m(t))) \\ &+ \frac{b + c}{1 - (a + 2c)}\rho(r(T(t, w_n(t)) - w_n(t))). \end{aligned}$$

By $\lambda_k \rightarrow 1$, $\{w_k\}$, is a ρ -Cauchy sequences. Since $\{w_k\} \in C$ and L_ρ is ρ -complete, there exists $w \in C$ such that $\rho(w_k(t) - w(t)) \rightarrow 0$ as $n \rightarrow \infty$.

Notice that T is a ρ -continuous mapping and $\rho(T(t, w_k(t)) - T(t, w(t))) \rightarrow 0$ as $n \rightarrow \infty$. Since

$$\begin{aligned} \rho\left(\frac{1}{3}(T(t, w(t)) - w(t))\right) &\leq \rho(T(t, w(t)) - T(t, w_k(t))) \\ &\quad + \rho(T(t, w_k(t)) - w_k(t)) + \rho(w_k(t) - w(t)), \end{aligned}$$

we have $\rho(T(t, w(t)) - w(t)) \rightarrow 0$ as $n \rightarrow \infty$. Hence $w : \Omega \rightarrow C$ is the limit of a sequence of measurable mapping, is also measurable, and $w : \Omega \rightarrow C$ is a random fixed point of the random operator T . Since

$$\begin{aligned} \rho\left(\frac{1}{3}(T(t, x_{k,n}(t)) - x_{k,n}(t))\right) &\leq \rho(T(t, x_{k,n}(t)) - T(t, w_k(t))) \\ &\quad + \rho(T(t, w_k(t)) - w_k(t)) + \rho(w_k(t) - x_{k,n}(t)), \end{aligned}$$

then $\rho(T(t, x_{k,n}(t)) - x_{k,n}(t)) \rightarrow 0$. $\square$

Now, we can prove Theorem 2.3 (by the same technique of Theorem 2.2).

Theorem 2.3. *Suppose all the conditions of Theorem 2.2 are hold. We consider the following sequence $\{x_{k,n}(t)\}$,*

$$(10) \quad \begin{cases} y_{k,n}(t) = \beta_n x_{k,n}(t) + (1 - \beta_n) T_k(t, x_{k,n}(t)), \\ x_{k,n+1}(t) = \alpha_n x_{k,n}(t) + (1 - \alpha_n) T_k(t, y_{k,n}(t)). \end{cases}$$

If $F(T) \neq \emptyset$, then the sequence $\{x_{k,n}(t)\}_{k \geq 0, n \geq 0}$, generated by (10), is strongly convergent to a random fixed point w of T in C .

Theorem 2.4. *Under the conditions of Theorem 2.2, we consider the following sequence $\{x_{k,n}(t)\}_{k \geq 0, n \geq 0}$,*

$$(11) \quad \begin{cases} x_{k,n}^1(t) = (1 - \beta_{n,0})x_{k,n}(t) + \beta_{n,0}T_k(t, x_{k,n}(t)), \\ x_{k,n}^2(t) = (1 - \beta_{n,1})x_{k,n}(t) + \beta_{n,1}T_k(t, x_{k,n}^1(t)), \\ \vdots \\ x_{k,n}^m(t) = (1 - \beta_{n,m-1})x_{k,n}(t) + \beta_{n,m-1}T_k(t, x_{k,n}^{m-1}(t)), \\ x_{k,n+1}(t) = (1 - \beta_{n,m})x_{k,n}(t) + \beta_{n,m}T_k(t, x_{k,n}^m(t)), \end{cases}$$

where $\{\beta_{n,i}\}$, $i = 0, \dots, m$ be real sequences in $(0, 1)$. Assume the following conditions are satisfied:

- (I) $\sum_{n \geq 0} \beta_{n,m} = \infty$ and $\lim_{n \rightarrow \infty} \beta_{n,m} = 0$.
- (II) $0 < \liminf_{n \rightarrow \infty} \beta_{n,i} \leq \limsup_{n \rightarrow \infty} \beta_{n,i} < 1$, for all $i = 0, \dots, m - 1$.
- (III) $\lim_{k \rightarrow \infty} \lambda_k = 1$.
- (IV) For all $x \in C$ and $r \in \mathbb{R}^+$, $\rho(r(x(t) - z(t))) \leq m < \infty$.

(V) $F(T) \neq \emptyset$.

Then the double sequence $\{x_{k,n}(t)\}_{k \geq 0, n \geq 0}$ generated by (11), is strongly convergent to a random fixed point w of T in C .

Proof. By the same technique of Theorem 2.2,

$$\rho(r(x_{k,n+1}(t) - w_k(t))) \leq \mu_n \rho(r(x_{k,n}(t) - w_k(t))),$$

where

$$\begin{aligned} \mu_n &= (1 - \beta_{n,m}) + \beta_{n,m}q[(1 - \beta_{n,m-1}) \\ &\quad + \beta_{n,m-1}q[(1 - \beta_{n,m-2}) + \beta_{n,m-2}q[\dots[(1 - \beta_{n,1}) \\ &\quad + \beta_{n,1}q[(1 - \beta_{n,0}) + \beta_{n,0}q]] \dots] \\ &\quad + \beta_{n,m}(1 + q\beta_{n,m-1} + \dots + \beta_{n,m-1}\beta_{n,m-2} \dots \beta_{n,0}q^m)\Theta_k. \end{aligned}$$

Since $\lim_{n \rightarrow \infty} \beta_{n,m} = 0$, then $\rho(x_{k,n}(t) - w_k(t)) \rightarrow 0$ as $n \rightarrow \infty$. $\square$

3. Random iteration sequences for ρ -nonexpansive semigroup

In 1992, Khamsi [8] studied the theory of semigroup in modular function space in the context of Musielak-Orlicz space and investigated its applications to differential equations. Since then it has an intensive development, see [1], [10] and [11]. In order to study this space, we recall the definition of ρ -nonexpansive semigroup in modular function space (see [8]).

Let L^φ be the Musielak-Orlicz space and C a nonempty subset of L^φ , and $I : C \rightarrow C$ denotes the identity mapping. A one-parameter family $\zeta = \{T(t) : 0 \leq t < \infty\}$ from self-mappings of a nonempty, closed and convex subset C of L^φ is said to be a ρ -nonexpansive semigroup on C if the following conditions are satisfied:

(I) $T(0)x = x$, for all $x \in C$.

(II) $T(s+t)x = T(s)T(t)x$, for all $x \in C$ and $s, t \geq 0$.

(III) For each $x \in C$, the mapping $t \rightarrow T(t)x$ is continuous on $[0, \infty)$.

(IV) $\rho(T(t)x - T(t)y) \leq \rho(x - y)$, for all $x, y \in C$.

We denote by $Fix(\zeta)$ the set of all common fixed point of ζ ; that is, $Fix(\zeta) = \{x \in C : T(s)x = x, \forall s > 0\}$. Then $Fix(\zeta)$ is nonempty if C is bounded.

In this section, the convergence of the modified new iteration processes is presented. This processes is defined for each arbitrary measurable mapping, $x_1 : \Omega \rightarrow C$, by

$$(12) \quad \begin{cases} u_n(t) = \frac{1}{n+1} \sum_{j=0}^n T^j(t, x_n(t)), \\ y_n(t) = \beta_n x_n(t) + (1 - \beta_n) \frac{1}{w_n} \int_0^{w_n} K(s) u_n(t) ds, \\ x_{n+1}(t) = \alpha_n x_n(t) + (1 - \alpha_n) T^n(t, y_n(t)), \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences in $(0, 1)$.

Theorem 3.1. *Let L^φ be a ρ -complete Musielak-Orlicz space where ρ is convex and satisfies the Δ_2 -condition. Suppose C is a nonempty, ρ -closed and convex subset of L^φ and T is a continuous random operator and for all $t \in \Omega$, $T(t, \cdot) : C \rightarrow C$, satisfies inequality (1), $k_n = \frac{a_n + b_n + c_n}{1 - (b_n + c_n)}$ and $\{k_n\} \subset [1, \infty)$ such that $k_n \rightarrow 1$ as $n \rightarrow \infty$. Let $\zeta = \{K(t) : 0 \leq t < \infty\}$ be a nonexpansive semigroup on C and $\{w_n\}$ a positive real divergent sequence. Also $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$. If $F = F(T) \cap F(\zeta) \neq \emptyset$, then the sequence $\{x_n(t)\}_{n \geq 0}$, generated by (12), is strongly convergent to a random fixed point $v \in F$.*

Proof. For $v \in F$ and $1 \leq j \leq n$,

$$\rho(r(T^j(t, x_n(t)) - v(t))) \leq \frac{a_j + b_j + c_j}{1 - (b_j + c_j)} \rho(r(x_n(t) - v(t))) \leq k_j \rho(r(x_n(t) - v(t))),$$

hence,

$$\begin{aligned} \rho(r(u_n(t) - v(t))) &\leq \frac{1}{n+1} [\rho(r(x_n(t) - v(t))) + \sum_{j=1}^n k_j \rho(r(x_n(t) - v(t)))] \\ &\leq \eta_n \rho(r(x_n(t) - v(t))), \end{aligned}$$

where $\eta_n = \max\{k_j, 1 \leq j \leq n\}$. By process (12);

$$\begin{aligned} \rho(r(x_{n+1}(t) - v(t))) &\leq \alpha_n \rho(r(x_n(t) - v(t))) + (1 - \alpha_n) \rho(r(T^n(t, y_n(t)) - v(t))) \\ &\leq \alpha_n \rho(r(x_n(t) - v(t))) \\ &\quad + (1 - \alpha_n) k_n [\beta_n \rho(r(x_n(t) - v(t))) \\ &\quad + (1 - \beta_n) \eta_n \rho(r(x_n(t) - v(t)))], \end{aligned}$$

therefore

$$(13) \quad \rho(r(x_{n+1}(t) - v(t))) \leq \{\alpha_n + (1 - \alpha_n) k_n [\beta_n + (1 - \beta_n) \eta_n]\} \rho(r(x_n(t) - v(t))).$$

By Lemma 1.1 and inequality (13), $\lim_{n \rightarrow \infty} \rho(x_n(t) - v(t)) = 0$. Since C is ρ -closed, it follows that v is a mapping from Ω into C . Since $v : \Omega \rightarrow C$, is the pointwise limit of the sequence $\{x_n\}$ of measurable functions from Ω into C is also measurable and so, $T(\cdot, v(\cdot)) : \Omega \rightarrow C$ is measurable. By $\rho(\frac{1}{2}(x_n(t) - T(t, x_n(t)))) \leq \rho(x_n(t) - v(t)) + \rho(v(t) - T(t, x_n(t)))$, we have $\lim_{n \rightarrow \infty} \rho(x_n(t) - T(t, x_n(t))) = 0$. $\square$

Now we can prove Theorem 3.2 (by the same argument of Theorem 3.1).

Theorem 3.2. Suppose all the conditions of Theorem 3.1 are hold. Consider the following sequence $\{x_n(t)\}_{n \geq 0}$,

$$(14) \quad \begin{cases} u_n(t) = \frac{1}{n+1} \sum_{j=0}^n T^j(t, x_n(t)), \\ x_n^1(t) = (1 - \beta_{n,0})x_n(t) + \beta_{n,0} \frac{1}{w_n} \int_0^{w_n} K(s)u_n(t)ds, \\ x_n^2(t) = (1 - \beta_{n,1})x_n(t) + \beta_{n,1}T^n(t, x_n^1(t)), \\ \vdots \\ x_n^m(t) = (1 - \beta_{n,m-1})x_n(t) + \beta_{n,m-1}T^n(t, x_n^{m-1}(t)), \\ x_{n+1}(t) = (1 - \beta_{n,m})x_n(t) + \beta_{n,m}T^n(t, x_n^m(t)). \end{cases}$$

Let

(I) $\sum_{n \geq 0} \beta_{n,m} = \infty$ and $\lim_{n \rightarrow \infty} \beta_{n,m} = 0$.

(II) $0 < \liminf_{n \rightarrow \infty} \beta_{n,i} \leq \limsup_{n \rightarrow \infty} \beta_{n,i} < 1$, for all $i = 0, \dots, m-1$.

(III) $F = F(T) \cap F(\zeta) \neq \emptyset$.

Then the sequence $\{x_n(t)\}_{n \geq 0}$, generated by (14), is strongly convergent to a random fixed point $v \in F$.

4. Numerical examples

In this section, some numerical examples (using Matlab software) are presented. First, we show that a generalized Lipschitzian mapping is not necessarily a Lipschitzian mapping.

Example 4.1. Let $X_\rho = \mathbb{R}$ be the set of real numbers and $C = [0, \infty)$. For each $x \in C$, we define

$$(15) \quad Tx = \begin{cases} \frac{rx}{1+x}, & \text{if } x \in [0, \frac{1}{4}], \\ 0, & \text{if } x \in (\frac{1}{4}, \infty), \end{cases}$$

where $0 < r < \frac{1}{4}$. Then $T : C \rightarrow C$ is not continuous at $x = \frac{1}{4}$ and hence T is not a Lipschitzian mapping. Set $C_1 = [0, \frac{1}{4}]$ and $C_2 = (\frac{1}{4}, \infty)$. In order to prove $T(t, \cdot) : C \rightarrow C$ is a generalized Lipschitzian mapping, we need the following steps:

For all $x, y \in C_1$ and $n \geq 1$,

$$|Tx - Ty| = \left| \frac{rx}{1+x} - \frac{ry}{1+y} \right| = \left| \frac{rx(1+y) - ry(1+x)}{(1+x)(1+y)} \right| \leq r|x - y|,$$

and $|T^2x - T^2y| = \left| \frac{rTx}{1+Tx} - \frac{rTy}{1+Ty} \right| \leq r|Tx - Ty| \leq r^2|x - y|$. By induction, for all $n \geq 1$,

$$|T^n x - T^n y| \leq r^n |x - y| + r^n (|x - T^n x| + |y - T^n y|) + r^n (|x - T^n y| + |y - T^n x|).$$

For all $x, y \in C_2$ and $n \geq 1$, $|T^n x - T^n y| = 0 \leq |x - y|$.

For $x \in C_1$ and $y \in C_2$, $|Tx - Ty| = |\frac{rx}{1+x} - 0|$. By induction, for $n \in \mathbb{N}$, $T^{n+1}x = \frac{rT^n x}{1+T^n x} \leq rT^n x \leq r^{n+1}x$ and $|T^n x - T^n y| = |T^n x - 0| \leq r^n|x - y| + r^n(|x - T^n x| + |y - T^n y|) + r^n(|x - T^n y| + |y - T^n x|)$. Therefore, $T : C \rightarrow C$ is a generalized Lipschitzian mapping.

Theorem 4.1. *Let L^φ , be the Musielak-Orlicz space and $I = [0, b'] \subset \mathbb{R}$. Suppose that ρ is convex and satisfies the Δ_2 -condition. Since topologies generated by $\|\cdot\|_\rho$ and ρ are equivalent, then we consider Banach space $(L^\varphi, \|\cdot\|_\rho)$ also $C(I, L^\varphi)$ denotes the space of all $\|\cdot\|_\rho$ -continuous function from I to L^φ with the modular $\|x\|_\sigma = \sup_{t \in I} \|x(t)\|_\rho$ also $C(I, L^\varphi)$ is a real vector space (see [2]). Suppose A is a nonempty, convex and ρ -closed subset of L^φ , $T(t, \cdot) : A \rightarrow A$ is a uniformly ρ -generalized Lipschitzian mapping satisfies inequality (2), such that $k = \frac{a+b+c}{1-(b+c)} < 1$. For each $u, v \in A$ and $t \in I$. Let $U_1 : I \rightarrow A$ be an arbitrary measurable mapping, consider the following iterative sequence $\{U_n(t)\}$ for $n = 2, 4, 6, \dots$, defined by*

$$(16) \quad \begin{cases} U_n(t) = \frac{1}{2n}U_1(t) + \Lambda(t, U_{n-1}(t)) \int_0^t \omega(t, s)\Phi(s, U_{n-1}(s))ds, \\ Y_n(t) = \beta_n U_{n-1}(t) + (1 - \beta_n)\Phi(t, U_n(t)), \\ X_n(t) = \alpha_n U_{n-1}(t) + (1 - \alpha_n)\Lambda(t, Y_n(t)), \\ U_{n+1}(t) = T(t, X_n(t)), \end{cases}$$

for $t \in I$, where $0 < \alpha_n, \beta_n < 1$. Assume the following conditions are satisfied:

- (1) $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$.
- (2) Φ is function from $I \times A$ into A such that $\Phi(t, \cdot) : A \rightarrow A$ is a $\|\cdot\|_\rho$ -continuous and $t \rightarrow \Phi(t, x)$ is measurable for every $x \in A$. Also, for $t \in I$, $x \rightarrow \Phi(t, x)$ is nondecreasing on A . For all $t \in I$ and $x, y \in A$ there exists $q \geq 0$ such that $\|\Phi(t, x) - \Phi(t, y)\|_\rho \leq q\|x - y\|_\rho$. Also, there exists function $\beta \in L^1(I)$ and a nondecreasing continuous function $\gamma : [0, \infty) \rightarrow (0, \infty)$ such that $\|\Phi(t, x)\|_\rho \leq \beta(t)\gamma(\|x\|_\rho)$, for all $t \in I$ and $x \in A$.
- (3) Λ is function from $I \times A$ into A such that $\Lambda(t, \cdot) : A \rightarrow A$ is $\|\cdot\|_\rho$ -continuous and there exists a $h \geq 0$ such that $\|\Lambda(t, x) - \Lambda(t, y)\|_\rho \leq h\|x - y\|_\rho$, for all $t \in I$ and $x, y \in A$. Also for $x \in A$, $t \rightarrow \Lambda(t, x)$ is nondecreasing on I and for $t \in I$, $x \rightarrow \Lambda(t, x)$ is nondecreasing on A .
- (4) ω is function from $I \times I$ into $\mathbb{R}^+$. For each $t \in I$, $\omega(t, s)$ is measurable on $[0, t]$. Also $\overline{\omega(t)} = \text{esssup}|\omega(t, s)|$ is bounded on $[0, b']$ and $r = \sup|\overline{\omega(t)}|$. The map $\omega(\cdot, s) : t \rightarrow \omega(t, s)$ is continuous from I to $L^\infty(I)$. Also for $s \in I$, $t \rightarrow \omega(t, s)$ is nondecreasing on I .

Let $F = F(T) \cap F(\Lambda) \cap F(\Phi) \neq \emptyset$, and there exists a constant $d \geq 0$ such that for all $t \in I$ and $u \in C(I, A)$; $\|u(t)\|_\rho \leq d$ and $\int_0^t \beta(s)ds < \frac{d}{(hd+h')rb'} \int_0^t \frac{1}{\gamma(d)}ds$, where $h' := \sup\{\|\Lambda(t, x)\|_\rho, t \in I, x \in A\}$. For $z \in F$, since

$$\begin{aligned} \|\Lambda(t, U_{n-1}(t)) \int_0^t \omega(t, s)\Phi(s, U_{n-1}(s))ds\|_\rho &= \|(\Lambda(t, U_{n-1}(t)) - \Lambda(t, 0) \\ &\quad + \Lambda(t, 0)) \int_0^t \omega(t, s)\Phi(s, U_{n-1}(s))ds\|_\rho \\ &\leq (h\|U_{n-1}(t)\|_\rho + h')r \int_0^t \beta(s)\gamma(\|U_{n-1}(s)\|_\rho)ds \\ &\leq (dh + h')r \int_0^t \beta(s)\gamma(d)ds \\ &\leq (dh + h')r \int_0^{b'} \frac{d\gamma(d)}{(hd + h')rb'\gamma(d)}ds \\ &\leq d, \end{aligned}$$

and

$$\begin{aligned} \|U_{n+1}(t) - z(t)\|_\rho &\leq k\|X_n(t) - z(t)\|_\rho \\ &\leq \alpha_n\|U_{n-1}(t) - z(t)\|_\rho + kh(1 - \alpha_n)\|Y_n(t) - z(t)\|_\rho \\ &\leq \alpha_n\|U_{n-1}(t) - z(t)\|_\rho \\ &\quad + kh(1 - \alpha_n)\beta_n\|U_{n-1}(t) - z(t)\|_\rho \\ &\quad + khq(1 - \alpha_n)(1 - \beta_n)\frac{1}{2n}\|U_1(t) - z(t)\|_\rho \\ &\quad + dkhq(1 - \alpha_n)(1 - \beta_n), \end{aligned}$$

then the sequence $\{U_n(t)\}_{n \geq 1}$, generated by (16), is strongly convergent to a random fixed point $v \in F$.

Example 4.2. Let $\{U_n\}$ be the sequence defined by (16), T be mapping defined by (15) and $r = \frac{1}{5}$. Also $\alpha_n = \frac{1}{2n}$ and $\beta_n = \frac{n}{2n+1}$. Suppose that $\omega(t, s) = \frac{1}{2}$, $\Lambda(t, x) = \frac{x}{2}$ and $\Phi(t, x) = \frac{x}{2}$. We have

$$(17) \quad \begin{cases} U_n(t) = \frac{1}{2n}U_1(t) + \frac{U_{n-1}(t)}{8} \int_0^t U_{n-1}(s)ds, \\ Y_n(t) = \frac{n}{2n+1}U_{n-1}(t) + \frac{n+1}{2n+1} \frac{U_n(t)}{2}, \\ X_n(t) = \frac{1}{2n}U_{n-1}(t) + \frac{2n-1}{2n} \frac{Y_n(t)}{2} \\ U_{n+1}(t) = \frac{X_n(t)}{5(1 + X_n(t))}. \end{cases}$$

Let $t = \frac{1}{4}$ and $U_1(\frac{1}{4}) = \frac{1}{5}$, the sequence $\{U_n(\frac{1}{4})\}_{n \geq 1}$, is strongly convergent to 0.

n	$U_n(\frac{1}{4})$	n	$U_n(\frac{1}{4})$	n	$U_n(\frac{1}{4})$
1	0.2	14	0.00714285868	27	0.00010176298
2	0.20125	15	0.0001898664	28	0.00357142889
3	0.01579818388	16	0.00625000112	29	0.00009445820
4	0.02500779945	17	0.00016592096	30	0.00333333361
5	0.00160418577	18	0.005555555641	31	0.00008813187
6	0.01666674708	19	0.00014734113	32	0.00312500024
7	0.00050465598	20	0.00500000067	33	0.00008259973
8	0.01250000795	21	0.00013250412	34	0.00294117668
9	0.0051201096	22	0.00454545509	35	0.00007772107
10	0.01000000357	23	0.00012038213	36	0.00277777796
11	0.00026713396	24	0.00416666711	37	0.00007338656
12	0.00833333556	25	0.00011029226	38	0.00263157911
13	0.00022190673	26	0.00384615422	39	0.00006950997

Example 4.3. Let $X_\rho = \mathbb{R}$ be the set of real numbers, $C = [0, \infty)$ and $\{x_{k,n}\}$ be the sequence defined by (6), where

(1) $Tx = \frac{x}{70000}$ and $z = 0$.

(1) Suppose $\alpha_n = \frac{1}{20n}$, $\beta_n = \frac{n}{20n+1}$ and $\lambda_k = \frac{7k}{7k+100}$.

We have $F(T) = \{0\}$ and

$$\begin{cases} y_{k,n} = \frac{19n+1}{20n+1}x_{k,n} + \frac{n}{20n+1} \left(\frac{7k}{7k+100} \right) \frac{x_{k,n}}{70000}, \\ x_{k,n+1} = \frac{20n-1}{20n}x_{k,n} + \frac{1}{20n} \left(\frac{7k}{7k+100} \right) \frac{y_{k,n}}{70000}. \end{cases}$$

If $x_{1,1} = 0.001$, so

n	$x_{1,n}$	n	$x_{1,n}$
1	0.001	11	0.000512
2	0.0009	12	0.000484
3	0.000832	13	0.000457
4	0.000777	14	0.000433
5	0.000728	15	0.00041
6	0.000684	16	0.000388
7	0.000644	17	0.000367
8	0.000607	18	0.000348
9	0.000573	19	0.000329
10	0.000541	20	0.000312

Example 4.4. Let $X_\rho = \mathbb{R}$ be the set of real numbers, $C = [0, \infty)$ and $\{x_{k,n}\}$ be the sequence defined by (10), where $Tx = \frac{x}{70000}$ and $z = 0$. Also $\alpha_n = \frac{1}{20n}$, $\beta_n = \frac{n}{20n+1}$ and $\lambda_k = \frac{7k}{7k+100}$. We have,

$$\begin{cases} y_{k,n} = \frac{n}{20n+1}x_{k,n} + \frac{19n+1}{20n+1}\left(\frac{7k}{7k+100}\right)\frac{z_{k,n}}{70000}, \\ x_{k,n+1} = \frac{1}{20n}x_{k,n} + \frac{20n-1}{20n}\left(\frac{7k}{7k+100}\right)\frac{y_{k,n}}{70000}. \end{cases}$$

If $x_{1,1} = 0.001$, so

n	$x_{1,n}$	n	$x_{1,n}$
1	0.001	11	2.691×10^{-23}
2	5×10^{-5}	12	1.223×10^{-25}
3	1.25×10^{-6}	13	5.097×10^{-28}
4	2.083×10^{-8}	14	1.96×10^{-30}
5	2.604×10^{-10}	15	7.001×10^{-33}
6	2.64×10^{-12}	16	2.333×10^{-35}
7	2.17×10^{-14}	17	7.293×10^{-38}
8	1.55×10^{-16}	18	2.145×10^{-40}
9	9.688×10^{-19}	19	5.959×10^{-43}
10	8.382×10^{-21}	20	1.568×10^{-45}

Corollary 4.1. *Examples 4.3 and 4.4 show that the rate of convergence of sequence $\{x_{k,n}\}$ generated by (10), is faster than the rate of convergence of sequence $\{x_{k,n}\}$ generated by (6).*

Example 4.5. Let $X_\rho = \mathbb{R}$ be the set of real numbers and $C = [0, \infty)$. Consider:

- (1) $Tx = \frac{x}{1000}$.
- (2) $K(s) = e^{-s}$ and $t_n = n$.
- (3) $\alpha_n = \frac{1}{17n}$, $\beta_n = \frac{n}{15n+1}$.

Let $\{x_n\}$ be the sequence defined by (12). So,

$$\begin{cases} u_n = \frac{1}{n+1} \sum_{j=0}^n \frac{x_n}{1000^j}, \\ y_n = \frac{n}{15n+1}x_n + \frac{14n+1}{15n+1}\left(\frac{1-e^{-n}}{n}\right)u_n, \\ x_{n+1} = \frac{1}{17n}x_n + \frac{1}{1000^n}\frac{17n-1}{17n}y_n, \end{cases}$$

and $Fix(\zeta) \cap Fix(T) = \{0\}$. If $x_1 = 1$, then

n	x_n	n	x_n	n	x_n
1	1	11	1.374×10^{-19}	21	1.017×10^{-43}
2	0.05916	12	7.351×10^{-22}	22	2.849×10^{-46}
3	0.00174	13	3.603×10^{-24}	23	7.618×10^{-49}
4	0.00003	14	1.63×10^{-26}	24	1.948×10^{-51}
5	5.017×10^{-7}	15	6.851×10^{-29}	25	4.775×10^{-54}
6	5.902×10^{-9}	16	2.686×10^{-31}	26	1.123×10^{-56}
7	5.787×10^{-11}	17	9.878×10^{-34}	27	2.542×10^{-59}
8	4.863×10^{-13}	18	3.418×10^{-36}	28	5.538×10^{-62}
9	3.575×10^{-15}	19	1.117×10^{-38}	29	1.163×10^{-64}
10	2.337×10^{-17}	20	3.458×10^{-41}	30	2.36×10^{-67}

Example 4.6. Let $X_\rho = \mathbb{R}$ be the set of real numbers and $C = [0, \infty)$, where $Tx = \frac{x}{800}$. Also $\alpha_n = \frac{1}{30n}$ and $\beta_n = \frac{29n+1}{30n+1}$. Let $\{x_n\}$ be the sequence defined by

$$\begin{cases} u_n = \frac{1}{n+1} \sum_{j=0}^n \frac{x_n}{800^j}, \\ y_n = \frac{29n+1}{30n+1} x_n + \frac{n}{30n+1} u_n, \\ x_{n+1} = \frac{1}{30n} x_n + \frac{1}{800^n} \frac{30n-1}{30n} y_n. \end{cases}$$

If $x_1 = 1$, so

n	x_n	n	x_n
1	1	11	4.833×10^{-22}
2	0.0345	12	1.464×10^{-24}
3	5.754×10^{-4}	13	4.068×10^{-27}
4	6.393×10^{-6}	14	1.043×10^{-29}
5	5.327×10^{-8}	15	2.484×10^{-32}
6	3.551×10^{-10}	16	5.52×10^{-35}
7	1.973×10^{-12}	17	1.15×10^{-37}
8	9.396×10^{-15}	18	2.254×10^{-40}
9	3.915×10^{-17}	19	4.175×10^{-43}
10	1.45×10^{-19}	20	7.325×10^{-46}

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Power gamma extending modules

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Abstract. A submodule N of Gamma module M is $PE\Gamma$ -submodule, if for each m in M , ideal I in R if $I\Gamma m \neq 0$, then $I\Gamma(N :_{R_\Gamma} m)\Gamma m \neq 0$. $PE\Gamma$ -submodule is a proper generalization of essential R_Γ -submodule. An R_Γ -module M is called power Gamma extending module ($PE\Gamma$ -module), if every R_Γ -submodule is $PE\Gamma$ -submodule in a direct summand.

Keywords: extending gamma module, power essential gamma submodule, frail closed gamma submodule.

1. Introduction

Let R be a Γ -ring, an additive abelian group M with $\cdot : R \times \Gamma \times M \longrightarrow M$ by $(r, \gamma, m) \longmapsto r\gamma m$ such that $(k_1 + k_2)\rho x = k_1\rho x + k_2\rho x$, $k(\rho + \eta)x = k\rho x + k\eta x$, $k\rho(x_1 + x_2) = k\rho x_1 + k\rho x_2$ and $(k_1\rho k_2)\eta x = k_1\rho(k_2\eta x)$ where $k, k_1, k_2 \in R$, $\rho, \eta \in \Gamma$ and $x, x_1, x_2 \in M$ is called a left R_Γ -module [10]. A Γ -ring R is commutative if $s\beta r = r\beta s$ for any $s, r \in R$ and $\beta \in \Gamma$. M is unitary if there exist $1 \in R$ and $\gamma_o \in \Gamma$ such that $1\gamma_o a = a$ for every $a \in M$ [2]. A nonempty set K is R_Γ -submodule, $K \leq M$ if $R\Gamma K = \{s\gamma k : s \in R, \gamma \in \Gamma, k \in K\} \leq K$ [8]. For $a \in M$, cyclic R_Γ -submodule $Ra = \langle a \rangle = \{\sum_{s=1}^n t_s \gamma_s a : n \in \mathbb{N}, \gamma_s \in \Gamma\}$ [9]. The set of all $r \in R$ such that $r\Gamma x \subseteq N$ denoted by $(N :_{R_\Gamma} x)$ [4]. An R_Γ -submodule N is R_Γ -idempotent, $N \leq_{id} M$ if $N = (N :_{R_\Gamma} M)\Gamma N$ and M is fully idempotent if $N \leq_{id} M$ for each R_Γ -submodule N [3]. An R_Γ -submodule N of R_Γ -module M is essential, $N \leq_e M$ if for each $a \neq 0 \in M$,

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there is $t_1, t_2, \dots, t_n \in R$ and $\nu_1, \nu_2, \dots, \nu_n \in \Gamma$ such that $\sum_{s=1}^n t_s \gamma_s a (\neq 0) \in N$ [2]. For more basics refer to [1,2,5,6,7,8] and [9]. All modules in this work are unitary left gamma module over commutative Γ -ring.

2. Power essential gamma module

In this section, we introduce definition of power essential Gamma submodule and some properties that play an important role in our next definitions.

Definition 2.1. An R_Γ -submodule N of M is called power essential Gamma submodule ($PE\Gamma$ -submodule) in M denoted by $N \leq_{pe} M$, for each $m \in M$ and for each ideal I in R with $I\Gamma m \neq 0$, then $I\Gamma(N :_{R_\Gamma} m)\Gamma m \neq 0$. Power essential Gamma ideal is $PE\Gamma$ -submodule in R as R_Γ -module.

Clearly that $N \leq_{pe} N$ for each $N (\neq 0) \leq M$. Every $PE\Gamma$ -submodule is essential. The converse not true see examples 3.9(1).

Proposition 2.2. The following are equivalent for R_Γ -module:

1. N is $PE\Gamma$ -submodule in M .
2. $\forall m \in M$ and $s \in R$ with $s\Gamma m \neq 0$, we have $s\Gamma(N :_{R_\Gamma} m)\Gamma m \neq 0$.
3. $\forall m \in M$ and $s \in R$ with $s\Gamma m \neq 0$, there exists $r \in R$ such that $r\Gamma m \leq N$ and $s\Gamma r\Gamma m \neq 0$.
4. $\forall m \in M$, $(N :_{R_\Gamma} m)\Gamma m \leq_{pe} Rm$.

Proof. (1) $\rightarrow$ (2) Assume $N \leq_{pe} M$ and $\forall m \in M$ and $s \in R$ with $s\Gamma m \neq 0$, then $\langle s \rangle \Gamma m \neq 0$, hence $\langle s \rangle \Gamma(N :_{R_\Gamma} m)\Gamma m \neq 0$, so $0 \neq \sum_{i=1}^n (r_i \gamma_i s) \gamma r \beta m \in \langle s \rangle \Gamma(N :_{R_\Gamma} m)\Gamma m$ for some $\beta, \gamma, \gamma_i \in \Gamma$, $r \in (N :_{R_\Gamma} m)$, $r_i \in R$. Let $0 \neq \sum_{i=1}^n (r_i \gamma_i s) \gamma r \beta m = \sum_{i=1}^n (s \gamma_i r_i) \gamma r \beta m = s \gamma \circ \sum_{i=1}^n (1 \gamma_i r_i) \gamma r \beta m$, so $s\Gamma(N :_{R_\Gamma} m)\Gamma m \neq 0$ since $\sum_{i=1}^n (1 \gamma_i r_i) \gamma r \in (N :_{R_\Gamma} m)$.

(2) $\rightarrow$ (3) Clear.

(3) $\rightarrow$ (4) Let I be ideal of R with $I\Gamma(\sum_{i=1}^n r_i \gamma_i m) \neq 0$, then there is $s\gamma(\sum_{i=1}^n r_i \gamma_i m) (\neq 0) \in s\Gamma(\sum_{i=1}^n r_i \gamma_i m)$ for some $s \in I$, hence $s\Gamma(\sum_{i=1}^n r_i \gamma_i m) \neq 0$. By (3) there is $r \in R$ such that $r\Gamma \sum_{i=1}^n r_i \gamma_i m \leq N$ and $s\Gamma r\Gamma \sum_{i=1}^n r_i \gamma_i m \neq 0$. Since $r \in (N :_{R_\Gamma} \sum_{i=1}^n r_i \gamma_i m)$, so $r\Gamma \sum_{i=1}^n r_i \gamma_i m \leq (N :_{R_\Gamma} \sum_{i=1}^n r_i \gamma_i m) \Gamma \sum_{i=1}^n r_i \gamma_i m$, hence $s\Gamma r\Gamma \sum_{i=1}^n r_i \gamma_i m (\neq 0) \leq I\Gamma(N :_{R_\Gamma} \sum_{i=1}^n r_i \gamma_i m) \Gamma \sum_{i=1}^n r_i \gamma_i m$.

(4) $\rightarrow$ (1) Let $m \in M$ and I is an ideal of R with $I\Gamma \sum_{i=1}^n r_i \gamma_i m \neq 0$. Since $(N :_{R_\Gamma} \sum_{i=1}^n r_i \gamma_i m) \Gamma \sum_{i=1}^n r_i \gamma_i m \leq N$, then $((N :_{R_\Gamma} \sum_{i=1}^n r_i \gamma_i m) \Gamma \sum_{i=1}^n r_i \gamma_i m :_{R_\Gamma} \sum_{i=1}^n r_i \gamma_i m) \leq (N :_{R_\Gamma} \sum_{i=1}^n r_i \gamma_i m)$. By (4) $(N :_{R_\Gamma} \sum_{i=1}^n r_i \gamma_i m) \Gamma \sum_{i=1}^n r_i \gamma_i m \leq_{pe} Rm$, then $I\Gamma((N :_{R_\Gamma} \sum_{i=1}^n r_i \gamma_i m) \Gamma \sum_{i=1}^n r_i \gamma_i m :_{R_\Gamma} \sum_{i=1}^n r_i \gamma_i m) (\neq 0) \leq I\Gamma(N :_{R_\Gamma} \sum_{i=1}^n r_i \gamma_i m) \Gamma \sum_{i=1}^n r_i \gamma_i m$. $\square$

Proposition 2.3. *Let N and K two R_Γ -submodules of M , then N is $PE\Gamma$ -submodule in K if and only if $(N :_{R_\Gamma} m)\Gamma m$ is $PE\Gamma$ -submodule in $(K :_{R_\Gamma} m)\Gamma m$ for each m in M .*

Proof. Assume that $N \leq_{pe} M$ and $m \in M$, for each $x = r\beta m \in (K :_{R_\Gamma} m)\Gamma m$ for some $r \in (K :_{R_\Gamma} m)$, $\beta \in \Gamma$ and each $s \in R$ with $s\Gamma(r\beta m) \neq 0$, by proposition 2.2(2), $0 \neq s\Gamma(N :_{R_\Gamma} x)\Gamma x = s\Gamma(N :_{R_\Gamma} x)\Gamma r\beta m = s\Gamma(N :_{R_\Gamma} x)\Gamma r\beta 1\gamma_\circ m$ where $x = r\beta m$. To show that $(N :_{R_\Gamma} x)\Gamma r\beta 1 \subseteq (N :_{R_\Gamma} m)$, for each $t\gamma r\beta 1 \in (N :_{R_\Gamma} x)\Gamma r\beta 1$ for some $t \in (N :_{R_\Gamma} x)$ and $\gamma \in \Gamma$, if $\theta \in \Gamma$, then $(t\gamma r\beta 1)\theta m = t\gamma(r\beta 1)\theta 1\gamma_\circ m = t\gamma 1\theta(r\beta 1)\gamma_\circ m = 1\gamma t\theta(r\beta m) = 1\gamma t\theta x \in R\Gamma N \leq N$, so $(N :_{R_\Gamma} x)\Gamma r\beta 1 \subseteq (N :_{R_\Gamma} m)$. Thus $0 \neq s\Gamma(N :_{R_\Gamma} x)\Gamma x = s\Gamma(N :_{R_\Gamma} x)\Gamma r\beta 1\gamma_\circ m \leq s\Gamma(N :_{R_\Gamma} m)\Gamma m$. For other direction, let $k \in K$, then $(N :_{R_\Gamma} k)\Gamma k \leq_{pe} (K :_{R_\Gamma} k)\Gamma k = Rk$, by proposition 2.2(4) $N \leq_{pe} K$. $\square$

Theorem 2.4. *Let M be an R_Γ -module, then:*

1. *Let N be R_Γ -submodule of M , then $N \leq_{pe} M$ if and only if $N \leq_{pe} K$ and $K \leq_{pe} M$.*
2. *If $N_1 \leq_{pe} K_1$ and $N_2 \leq_{pe} K_2$, then $N_1 \cap N_2 \leq_{pe} K_1 \cap K_2$.*
3. *If $f : A \rightarrow M$ is an R_Γ -homomorphism and $N \leq_{pe} M$, then $f^{-1}(N) \leq_{pe} A$.*
4. *If $\{N_\alpha\}$ is an independent family of M and $N_\alpha \leq_{pe} K_\alpha \leq M$ for each α , then $\{K_\alpha\}$ is an independent family of M and $\bigoplus_{\alpha \in \Lambda} N_\alpha \leq_{pe} \bigoplus_{\alpha \in \Lambda} K_\alpha$.*

Proof. 1. Assume that $N \leq_{pe} M$, for each $k \in K, m \in M$ and $s \in R$ with $s\Gamma k \neq 0$, then $s\Gamma(N :_{R_\Gamma} k)\Gamma k \neq 0$, so $N \leq_{pe} K$. Also if $s\Gamma m \neq 0$, then $s\Gamma(N :_{R_\Gamma} m)\Gamma m \neq 0$, hence $s\Gamma(K :_{R_\Gamma} m)\Gamma m \neq 0$, since $(N :_{R_\Gamma} m) \leq (K :_{R_\Gamma} m)$, then $K \leq_{pe} M$. Assume $N \leq_{pe} K \leq_{pe} M$, for each $m \in M$ and $s \in R$ with $s\Gamma m \neq 0$, then $s\Gamma(K :_{R_\Gamma} m)\Gamma m \neq 0$, so there is $s\gamma r\beta m (\neq 0) \in s\Gamma(K :_{R_\Gamma} m)\Gamma m$ for some $r \in (K :_{R_\Gamma} m)$ and $\gamma, \beta \in \Gamma$, so $r\beta m (\neq 0) \in (K :_{R_\Gamma} m)\Gamma m$. By proposition (2.3) $(N :_{R_\Gamma} m)\Gamma m \leq_{pe} (K :_{R_\Gamma} m)\Gamma m$, hence $s\Gamma r\Gamma m \neq 0$.

2. For each $k \in K_1 \cap K_2$ and $s \in R$ with $s\Gamma k \neq 0$, there is $r_1 \in R$ such that $r_1\Gamma k \leq N_1$ and $s\Gamma r_1\Gamma k \neq 0$. Take $0 \neq s\gamma r_1\beta k \in s\Gamma r_1\Gamma k$ for some $\gamma, \beta \in \Gamma$, since $s\gamma r_1 \in R$ and $0 \neq s\gamma r_1\beta k \in (s\gamma r_1)\Gamma k$, then there exists $r_2 \in R$ such that $r_2\Gamma k \leq N_2$ and $(s\gamma r_1)\Gamma r_2\Gamma k \neq 0$, so there is $0 \neq s\gamma r_1\rho r_2\lambda k \in (s\gamma r_1)\Gamma r_2\Gamma k$, since $(r_1\rho r_2)\Gamma k \leq N_1 \cap N_2$ and $0 \neq s\gamma r_1\rho r_2\lambda k \in s\Gamma(r_1\rho r_2)\Gamma k$, then by proposition(2.2)(3), $N_1 \cap N_2 \leq_{pe} K_1 \cap K_2$.

3. For each $a \in A$ and $s \in R$ with $s\Gamma a \neq 0$.

Case (1). If $s\Gamma f(a) \neq 0$, since $N \leq_{pe} M$, there is $r \in R$ such that $r\Gamma f(a) \leq N$ and $s\Gamma r\Gamma f(a) \neq 0$, so $r\Gamma a \leq f^{-1}(N)$ and $s\Gamma r\Gamma a \neq 0$.

Case (2). If $s\Gamma f(a) = 0$, then $s\Gamma a \leq \ker f \leq f^{-1}(N)$, since $s\Gamma a \neq 0$, so there is $0 \neq s\beta a \in s\Gamma a \leq f^{-1}(N)$. Take $r = 1$, then $0 \neq s\Gamma a = 1\gamma_\circ s\Gamma a = s\gamma_\circ 1\Gamma a \leq s\Gamma 1\Gamma a$. Thus, $f^{-1}(N) \leq_{pe} A$.

4. First if $\Lambda = \{1, 2\}$, by (2) $N_1 \cap N_2 = (0) \leq_{pe} K_1 \cap K_2$, then $0 \leq_e K_1 \cap K_2$, so $K_1 \cap K_2 = 0$, that is $\{K_1, K_2\}$ is an independent family of M . Let $\pi_1 : K_1 \oplus K_2 \rightarrow K_1$ and $\pi_2 : K_1 \oplus K_2 \rightarrow K_2$ the projections of $K_1 \oplus K_2$ on K_1 and K_2 , then by (3) $\pi_1^{-1}(N_1) = N_1 \oplus K_2 \leq_{pe} K_1 \oplus K_2$ and $\pi_2^{-1}(N_2) = K_1 \oplus N_2 \leq_{pe} K_1 \oplus K_2$, so by (2) $(K_1 \oplus N_2) \cap (N_1 \oplus K_2) = N_1 \oplus [(K_1 \oplus N_2) \cap K_2] = N_1 \oplus [N_2 \oplus (K_1 \cap K_2)] = N_1 \oplus N_2 \leq_{pe} K_1 \oplus K_2$. Consider $\bigoplus_{\alpha \in \Lambda} N_\alpha \leq_{pe} \bigoplus_{\alpha \in \Lambda} K_\alpha$ for $\Lambda = \{1, 2, \dots, n\}$, using the case above $(K_1 \oplus \dots \oplus K_n) \cap K_{n+1} = 0$, so $\{K_1, K_2, \dots, K_{n+1}\}$ is independent family. By the same way above $\bigoplus_{\alpha \in \Lambda} N_\alpha \leq_{pe} \bigoplus_{\alpha \in \Lambda} K_\alpha$ for all finite index sets. Take distinct indices $\alpha(0), \alpha(1), \dots, \alpha(n)$, then $\{K_{\alpha(0)}, K_{\alpha(1)}, \dots, K_{\alpha(n)}\}$ is independent, for each $k \in \bigoplus K_\alpha$ and each $s \in R$ such that $s\Gamma k \neq 0$, then $k \in K_{\alpha(0)} \oplus \dots \oplus K_{\alpha(n)}$ for some $\alpha(i)$. Hence there is $r \in R$ such that $r\Gamma k \leq N_{\alpha(0)} \oplus \dots \oplus N_{\alpha(n)}$ and $s\Gamma r\Gamma k \neq 0$, thus $r\Gamma k \leq \bigoplus N_\alpha$. This prove for general case. $\square$

Proposition 2.5. *Let $f : M \rightarrow N$ be an R_Γ -monomorphism and A, B be R_Γ -submodules of M with $A \leq B$. If A is $PE\Gamma$ -submodule of B , then $f(A)$ is $PE\Gamma$ -submodule of $f(B)$.*

Proof. Let $b \in B$ and $s \in R$ such that $s\Gamma f(b) \neq 0$, then $f(s\Gamma b) \neq 0$, so $s\Gamma b \neq 0$. By hypothesis there is $r \in R$ such that $r\Gamma b \leq A$ and $s\Gamma r\Gamma b \neq 0$. Hence $f(r\Gamma b) = r\Gamma f(b) \leq f(A)$ and $0 \neq f(s\Gamma r\Gamma b) = s\Gamma r\Gamma f(b)$. $\square$

Proposition 2.6. *Let $N \leq_{pe} L \leq M$, then:*

1. *If $N \cap K = 0$, then $L \cap K = 0$ for each $K \leq M$.*
2. *If $I\Gamma N = 0$, then $I\Gamma L = 0$ for each ideal I in R .*

Proof. 1. If $L \cap K \neq 0$, there is $l = k(\neq 0) \in L \cap K$, so $l = 1\gamma l \in 1\Gamma l \neq 0$, there exists $r \in R$ such that $r\Gamma l \leq N$ and $1\Gamma r\Gamma l \neq 0$, but $r\Gamma l = r\Gamma k \in K \cap N = 0$, then $1\Gamma r\Gamma l = 0$ a contradiction.

2. If $I\Gamma L \neq 0$, then there exists $i\gamma l \neq 0$ for some $i \in I, \gamma \in \Gamma$ and $l \in L$, so $i\Gamma l \neq 0$, then there exists $r \in R$ such that $r\Gamma l \leq N$ and $0 \neq i\Gamma r\Gamma l \leq I\Gamma N = 0$, a contradiction. $\square$

3. Power gamma extending module

In this section, the definitions of frail closed gamma submodule and power Gamma extending module was introduced as a proper generalization of closed and Gamma extending respectively.

Definition 3.1. *An R_Γ -submodule N of M is called frail closed (f -closed), $N \leq_{fc} M$, if N has no proper power essential Gamma extension in M , that is $N \leq_{pe} L \leq M$, then $L = M$.*

Clearly that every closed is f -closed but the converse is not true, see Example (3.9)(1).

Proposition 3.2. *Direct summand of an R_Γ -module is f -closed.*

Proof. Let $M = N \oplus K$ be an R_Γ -module such that N is $PE\Gamma$ -submodule in a submodule $L \leq M$. Hence $L \cap K = 0$ by proposition 2.6, so $L = L \cap (N + K) = N + (L \cap K) = N$. Therefore N has no proper power essential Gamma extensions in M . $\square$

Lemma 3.3. *Let N be an R_Γ -submodule of R_Γ -module M . Then the union of any chain of frail closed extensions of N is also frail closed extensions of N in M .*

Proof. Let $\{L_\alpha\}_{\alpha \in \Lambda}$ be a family of R_Γ -submodules of M such that N is a $PE\Gamma$ -submodule in L_α for each $\alpha \in \Lambda$. For each $x \in \cup_{\alpha \in \Lambda} L_\alpha$ and each $s \in R$, if $s\Gamma x \neq 0$, then there is $\beta \in \Lambda$ such that $x \in L_\beta$, so there is $r \in R$ such that $r\Gamma x \leq L_\beta \leq \cup_{\alpha \in \Lambda} L_\alpha$ and $s\Gamma r\Gamma x \neq 0$. $\square$

Theorem 3.4. *Every R_Γ -submodule N of an R_Γ -module M is contained in an f -closed R_Γ -submodule L of M in which N is a power essential Gamma submodule in L .*

Proof. Let N be an R_Γ -submodule of M and let $\Omega = \{K \leq M : N \leq_{pe} K\}$. Order Ω by inclusion, then by lemma 3.3, Ω is inductive, and hence by Zorn's Lemma Ω has a maximal element L . By Lemma 3.3 L is f -closed in M , that is $N \leq_{pe} L \leq_{fc} M$. $\square$

Proposition 3.5. *Let N be an f -closed submodule of an R_Γ -module and K is $PE\Gamma$ -submodule in M , then $N \cap K$ is f -closed in K .*

Proof. Assume $N \cap K \leq_{pe} L \leq K$. First we show that $N \leq_{pe} N + L$, for each $n + l \in N + L$ where $n \in N, l \in L$ and for each $s \in R$ with $s\Gamma(n + l) \neq 0$. Since $K \leq_{pe} M$, there exists $r \in R$ such that $r\Gamma(n + l) \leq K$ and $s\Gamma r\Gamma(n + l) \neq 0$, so $r\Gamma n \leq K$, hence $r\Gamma n \leq N \cap K \leq L$ with $s\Gamma r\Gamma(n + l) \neq 0$, so $r\Gamma(n + l) \neq 0$. Let $x = r\Gamma(n + l) \neq 0 \in r\Gamma(n + l)$, since $N \cap K \leq_{pe} L$, there exists $r_1 \in R$ such that $r_1\Gamma x \leq N \cap K \leq N$ and $0 \neq s\Gamma r_1\Gamma x = s\Gamma r_1\Gamma r\Gamma(n + l)$. Take $t = r_1\beta r$, then $t\Gamma(n + l) = r_1\beta r\Gamma(n + l) = r_1\gamma_\circ 1\beta r\Gamma 1\gamma_\circ(n + l) = r_1\gamma_\circ 1\Gamma 1\beta r\gamma_\circ(n + l) = r_1\Gamma r\beta 1\gamma_\circ(n + l) = r_1\Gamma r\beta(n + l) = r_1\Gamma x \leq N$ and $s\Gamma t\Gamma(n + l) = s\Gamma r_1\beta r\Gamma(n + l) = s\Gamma r_1\gamma_\circ 1\beta r\Gamma 1\gamma_\circ(n + l) = s\Gamma r_1\gamma_\circ 1\Gamma 1\beta r\gamma_\circ(n + l) = s\Gamma r_1\gamma_\circ 1\Gamma r\beta 1\gamma_\circ(n + l) = s\Gamma r_1\Gamma r\beta(n + l) = s\Gamma r_1\Gamma x \neq 0$, hence $N \leq_{pe} N + L$. Since $N \leq_{fc} M$ it follows that $N = N + L$. Hence $L \leq (N \cap K) + L = (N + L) \cap K = N \cap K \leq L$, so $N \cap K = L$ and hence $N \cap K \leq_{pe} K$. $\square$

Lemma 3.6. *If N is an R_Γ -submodule of M and K is a complement of N in M . Then:*

1. *There is a maximal $PE\Gamma$ -submodule extension L such that $L \cap K = 0$.*
2. *If $N \leq_{fc} M$, then $N = L$.*

Proof. 1. By theorem 3.4 there is a maximal R_Γ -submodule L of M such that $N \leq_{pe} L$. For each $x (\neq 0) \in L \cap K$ and each $s \in R$, if $s\Gamma x \neq 0$, there exists $r \in R$ such that $r\Gamma x \leq N$ and $0 \neq s\Gamma r\Gamma x \leq N \cap K$, a contradiction.

2. Clear. $\square$

Lemma 3.7. *Let N be an R_Γ -submodule of M . If N is f -closed in a direct summand of M , then N is f -closed in M .*

Proof. Let $M = L_1 \oplus L_2$ with $N \leq_{fc} L_1$ and $N \leq_{pe} K \leq M$. Let $\pi : M \rightarrow L_1$ be the projection, then $N = \pi(N) \leq \pi(K) \leq L_1$. To show that $N \leq_{pe} \pi(K)$, for each $\pi(k) \in \pi(K)$ and $s \in R$ such that $s\Gamma\pi(k) \neq 0$, so $s\Gamma k \neq 0$, hence there is $0 \neq s\gamma k \in s\Gamma k \leq K$. Hence there is $r \in R$ such that $r\Gamma k \leq N$ and $s\Gamma r\Gamma k \neq 0$. Now $r\Gamma k = r\Gamma l_1 + r\Gamma l_2 \leq N$ with $k = l_1 + l_2$ for some $l_1 \in L_1$ and $l_2 \in L_2$, so $r\Gamma l_1 = r\Gamma k \leq N$, hence $s\Gamma\pi(k) = s\Gamma l_1 = s\Gamma k \leq N$ and $s\Gamma r\Gamma\pi(k) = s\Gamma r\Gamma l_1 = s\Gamma r\Gamma k \neq 0$. Since $N \leq_{pe} \pi(K) \leq L_1$ and $N \leq_{fc} K$, then $N = \pi(K) \leq K$, so $(1 - \pi)K \leq K$. For each $x = (1 - \pi)k \in N \cap (1 - \pi)K$, $x = (1 - \pi)(l_1 + l_2)$, then $x = l_2 \in L_1 \cap L_2 = 0$. Hence $N \cap (1 - \pi)K = 0$ but $N \leq_e K$, then $(1 - \pi)K = 0$, so $K \leq \pi(K)$ which implice that $K \leq \pi(K) = N$, thus $N \leq_{fc} M$. $\square$

Γ -Extending R_Γ -module was introduce in [4]. An R_Γ -module M is called Γ -Extending if each closed R_Γ -submodule of M is a direct summand of M .

Definition 3.8. *An R_Γ -module M is called power Gamma extending module ($PE\Gamma$ -module), if for each R_Γ -submodule is $PE\Gamma$ -submodule in a direct summand. A Γ -ring R is power Gamma extending if it is power Gamma extending module as R_Γ -module.*

Clearly that every $PE\Gamma$ -module is Γ -Extending but the converse is not true, see Example 3.9(1).

Example 3.9. 1. Let $R = \Gamma = Z$ and $M = Z_4 \oplus Z_2$, then M is Γ -Extending R_Γ -module while M not $PE\Gamma$ -module. Take $N = \langle (0, 1) \rangle$, $K = \langle (2, 1) \rangle$, $L = N \oplus K$ and $I = 2Z$, then $L \not\leq_{pe} M$, since $I\Gamma M \neq 0$ and $I\Gamma L = 0$. Note that $L \leq_e M$ and $L \not\leq_{pe} M$, aslo L is f -closed but not closed in M , in fact every R_Γ -submodule of M is f -closed.

2. Direct sum of $PE\Gamma$ -module may not $PE\Gamma$ -module (see, example (1)).

3. Let $R = \{(n, n), n \in Z\}$ and $\Gamma = \left\{ \begin{pmatrix} \gamma \\ 0 \end{pmatrix}, \gamma \in Z \right\}$. Then R is Γ -ring with $\cdot : R \times \Gamma \times R \rightarrow R$ defined by $(n \ n) \begin{pmatrix} \gamma \\ 0 \end{pmatrix} (m \ m) = (n\gamma m \ n\gamma m)$. For each $(m \ m)$, and $(s \ s) \in R$ with $(s \ s)\Gamma(m \ m) \neq 0$. Then there is $\begin{pmatrix} \gamma \\ 0 \end{pmatrix} \in \Gamma$ such that $0 \neq (s \ s) \begin{pmatrix} \gamma \\ 0 \end{pmatrix} (m \ m) = (n\gamma m \ n\gamma m)$, so $s \neq 0, \gamma \neq 0, m \neq 0$. To show that every ideal $J \neq 0$ in R is $PE\Gamma$ -submodule, let $(j \ j) (\neq 0) \in J$. Choose $(s\gamma j \ s\gamma j) \in R$, for each $(s\gamma j \ s\gamma j) \begin{pmatrix} \beta \\ 0 \end{pmatrix} (m \ m) \in (s\gamma j \ s\gamma j)\Gamma(m \ m)$, then $(s\gamma j \ s\gamma j) \begin{pmatrix} \beta \\ 0 \end{pmatrix} (m \ m) = (s\gamma j\beta m \ s\gamma j\beta m) =$

$(s\gamma m\beta j \ s\gamma m\beta j) = (s\gamma m \ s\gamma m) \begin{pmatrix} \beta \\ 0 \end{pmatrix} (j \ j) \in J$, hence $(s\gamma j \ s\gamma j)\Gamma(m \ m) \leq J$. Since $s \neq 0, \gamma \neq 0, m \neq 0$ and $j \neq 0$, then $s\gamma s\gamma j\gamma m \neq 0$, hence $0 \neq (s \ s) \begin{pmatrix} \gamma \\ 0 \end{pmatrix} (s\gamma j \ s\gamma j) \begin{pmatrix} \gamma \\ 0 \end{pmatrix} (m \ m) \in (s \ s)\Gamma(s\gamma j \ s\gamma j)\Gamma(m \ m)$, therefore $J \leq_{pe} R$. Thus every ideal in R is $PE\Gamma$ -submodule, so R is $PE\Gamma$ -module.

Lemma 3.10. *An R_Γ -module M is $PE\Gamma$ -module if and only if for each R_Γ -submodule N of M , there exists decomposition $M = K \oplus L$ such that $N \leq K$ and $N \oplus L \leq_{pe} M$.*

Proof. For each R_Γ -submodule N of a $PE\Gamma$ -module M , then $N \leq_{pe} K \leq_\oplus M$, by theorem 2.3, $N \oplus L \leq_{pe} K \oplus L = M$. For each $N \leq M$, there exists decomposition $M = K \oplus L$ such that $N \leq K$ and $N \oplus L \leq_{pe} M$. To show that $N \leq_{pe} K$, for each $k \in K$ and $s \in R$ such that $s\Gamma k \neq 0$, then there exists $r \in R$ such that $r\Gamma k \leq N \oplus L$ and $s\Gamma r\Gamma k \neq 0$. For each $r\beta k \in r\Gamma k$, then $r\beta k = n + l$ for some $n \in N$ and $l \in L$, so $r\beta k - n = l \in K \cap L = 0$, thus $r\Gamma k \leq N$. $\square$

Easy to prove the following two results.

Lemma 3.11. *Let M be $PE\Gamma$ -module, then an R_Γ -submodule N of M is $PE\Gamma$ -submodule if and only if N is essential in M .*

Proposition 3.12. *An R_Γ -module M is $PE\Gamma$ -module if and only if every f -closed is a direct summand.*

The following lemma is used in proof of next result.

Lemma 3.13. *Let $M = L_1 \oplus L_2$ be R_Γ -module, if N is R_Γ -idempotent in M , then $N = (N \cap L_1) \oplus (N \cap L_2)$, Furthermore, $\pi_1(N)$ and $\pi_2(N)$ are R_Γ -idempotent in L_1 and L_2 respectively where π_1 and π_2 are the projections of M on to L_1 and L_2 .*

Proof. Let $N = (N :_{R_\Gamma} M)\Gamma N$ and let $\pi_1 : L_1 \oplus L_2 \rightarrow L_1$ and $\pi_2 : L_1 \oplus L_2 \rightarrow L_2$ be the projections of M on to L_1 and L_2 , then $\pi_1(N) = \pi_1((N :_{R_\Gamma} M)\Gamma N) = (N :_{R_\Gamma} M)\Gamma \pi_1(N) \leq N \cap L_1$, similar $\pi_2(N) \leq N \cap L_2$. Now $N \subseteq \pi_1(N) \oplus \pi_2(N) \subseteq (N \cap L_1) \oplus (N \cap L_2) \subseteq N$. Moreover if $r \in (N :_{R_\Gamma} M)$, then $r\Gamma M \leq N$, so $r\Gamma L_1 = \pi_1(r\Gamma M) \subseteq \pi_1(N)$, that is $r \in (\pi_1(N) :_{R_\Gamma} L_1)$. Thus $\pi_1(N) = (N :_{R_\Gamma} M)\Gamma N \pi_1(N) \subseteq (\pi_1(N) :_{R_\Gamma} L_1)\Gamma \pi_1(N) \subseteq \pi_1(N)$. $\square$

Proposition 3.14. *Let M be power Gamma extending module and N be R_Γ -idempotent submodule of M , then N is a power Gamma extending module.*

Proof. For each submodule K of N , by lemma 3.10 there exists a decomposition $M = L_1 \oplus L_2$ such that $K \leq L_1$ and $K \oplus L_2 \leq_{pe} M$. By lemma 3.13, $N = (N \cap L_1) \oplus (N \cap L_2)$ and by theorem 2.4, $N \cap (K \oplus L_2) = K \oplus (N \cap L_2) \leq_{pe} N$. Since $K \leq N \cap L_1$, then $K \leq_{pe} N$ by lemma 3.10. $\square$

Corollary 3.15. *Let M be fully R_Γ -idempotent. If M is a power Gamma extending module, then every R_Γ -submodule is a power Gamma extending module.*

Theorem 3.16. *Let M be a power Gamma extending module. Then every direct summand of M is a power Gamma extending.*

Proof. Let N be a direct summand of M and K an f -closed R_Γ -submodule of N . Then K is also f -closed in M by lemma 3.7. Since M is a power Gamma extending, then $M = K \oplus L$ for some $L \leq M$, so $N = K \oplus (N \cap L)$. Thus N is $PE\Gamma$ -module. $\square$

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Grading and filtrations of gamma rings

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Abstract. The aim of this paper is to introduce and study graded and filtered gamma rings and gamma modules. We prove that the filtered Γ -ring (module) is a generalization of the notion of graded ring (module). Also, we construct a graded Γ -ring from a filtered Γ -ring. We investigate some properties of graded and filtered Gamma rings and Gamma modules. Finally we define and study the strongly graded gamma rings.

Keywords: gamma rings, gamma ring homomorphism, graded Γ -rings, graded Γ -modules, filtered gamma rings, filtered gamma modules, strongly graded gamma rings.

1. Introduction

The concept of Γ -rings was first introduced by Nobusawa in 1964 ([17]). Later, Barnes generalized the notion of Nobusawa's Γ -rings and established a new definition of a Γ -ring ([4]). Nowadays, the notion of Γ -rings means the Γ -rings due to Barnes. Many mathematicians have been involved in extending the concepts and results of ring theory to the broader framework of the Γ -ring setting (see e.g. [2, 7, 13, 12, 14] and references therein).

The graded and filtered rings play a role in various branches of mathematics, and they have become increasingly important. Consequently, the graded analogues of different concept are widely studied (see [3, 5, 6, 9, 16, 11, 19]). For example, S. Shaqaqha, in his dissertation, considered gradings and filtrations on colour Lie superalgebras and obtained more Schreier-type formulas in the case of subalgebras of free colour Lie superalgebras ([19]). So, our results are expected to be useful in various applications.

In [10], a group graded gamma ring is defined to be a Γ -ring R which is the direct sum of additive subgroups R_g , $g \in K$, such that $R_g \Gamma R_h \subseteq R_{gh}$, for all

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$g, h \in K$, where K is a group defined multiplicatively. In this paper we replace the group grading with a semigroup then we introduce the notion of filtered Γ -rings. We extend known results in the case of graded rings to graded Γ -rings.

The article is organized as follows. In Section 2, we recall some definitions and notations that will be used in our work. In Section 3, we introduce the notion of graded gamma rings and we offer some examples. Also, we obtain some properties of semigroup graded gamma rings. We present ways to obtain new graded gamma rings from old ones. In Section 4, we define filtered gamma rings. We give a recipe to form new graded gamma ring from a filtered gamma ring. In Section 5, we study semigroup graded gamma modules. Also, we obtain a way to produce a ring out of old gamma rings. In Section 6, we study filtered gamma modules and obtain some properties. The strongly graded gamma rings are introduced and studied in Section 7.

2. Preliminaries

The notion of gamma rings is a one of the generalizations of rings.

Definition 2.1. *Let Γ be an additive abelian group. A Γ -ring (in the sense of Barnes) is an additive abelian group R together with a map*

$$R \times \Gamma \times R \rightarrow R : (x, \alpha, y) \mapsto x\alpha y$$

and satisfying the following two identities, for any $x, y, z \in R$ and $\alpha, \beta \in \Gamma$:

- (i) $(x + y)\alpha z = x\alpha z + y\alpha z$, $x(\alpha + \beta)z = x\alpha z + x\beta z$, $x\alpha(y + z) = x\alpha y + x\alpha z$.
- (ii) $(x\alpha y)\beta z = x\alpha(y\beta z)$.

Clearly every ring R can be regarded as an R -ring. Let G be an additive group. We denote by $G_{m,n}$ the set of all $m \times n$ matrices over G . Let M be a Γ -ring. Then $M_{m,n}$ forms a $\Gamma_{n,m}$ -ring (see [12]).

Let R be a Γ -ring. An element $1 \in R$ is called a unity of R if $a\gamma 1 = 1\gamma a = a$ for all $a \in R$ and for some $\gamma \in \Gamma$. In this case, we say R is a Γ -ring with unity. One can easily show that if a Γ -ring has a unity with respect to $\gamma \in \Gamma$, then it is unique.

A nonempty subset S of a Γ -ring R is a sub- Γ -ring of R if $a - b \in S$ and $a\gamma b \in S$ for any $a, b \in S$ and $\gamma \in \Gamma$. A subset I of the Γ -ring R is a left (right) ideal of R if I is an additive subgroup of R and $R\Gamma I = \{r\alpha a \mid r \in R, \alpha \in \Gamma, a \in I\}$ ($I\Gamma R$) is contained in I . If I is both a left and a right ideal of I , then we say that I is an ideal (or two-sided ideal) of R . It is clear that the intersection of any number of left (respectively right or two-sided) ideals of R is also a left (respectively right or two-sided) ideal of R .

Let I be a (two-sided) ideal of a Γ -ring R . then the factor group $R/I = x + I : x \in R$ of all cosets of I is a Γ -ring with respect to the following multiplication:

$$(x + I)(\gamma)(y + I) = ((x\gamma y) + I)$$

for $x, y \in R$ and $\gamma \in \Gamma$. Let $R_1, R_2, \dots, R_n$ be Γ -rings. Then the direct product $\prod_{i=1}^n R_i = R_1 \times \dots \times R_n$ has the structure of a Γ -ring under the componentwise addition and the following multiplication

$$(r_1, \dots, r_n)\gamma(s_1, \dots, s_n) = (r_1\gamma s_1, \dots, r_n\gamma s_n),$$

for all $r_i, s_i \in R_i$ ($i = 1, \dots, n$) and $\gamma \in \Gamma$.

Suppose that R_1 and R_2 are Γ -rings and let $\varphi : \Gamma \rightarrow \Gamma$ be a group isomorphism. Then $f : R_1 \rightarrow R_2$ is φ -homomorphism if $f(x + y) = f(x) + f(y)$ and $f(x\gamma y) = f(x)\varphi(\gamma)f(y)$ for all $x, y \in R_1$ and $\gamma \in \Gamma$. In particular, if φ is the identity map, then f is homomorphism.

3. Some properties of graded gamma rings

Definition 3.1. Suppose that G is an abelian semigroup written multiplicatively. Let R be a Γ -ring. Then R is called a graded Γ -ring of type G if there exist additive subgroups R_g of R such that $R = \bigoplus_{g \in G} R_g$ and $R_g \Gamma R_h \subseteq R_{gh}$ for all $g, h \in G$. A nonzero element $r \in R$ is called homogeneous if there is $g \in G$ such that $r \in R_g$, and in this case we say the degree of r is g and we write $d(m) = g$.

So, any nonzero element $r \in R$, where R is a graded Γ -ring of type G , has a unique representation as a sum of a finite set of homogenous elements $r = r_{g_1} + \dots + r_{g_k}$. In this case, we say that $r_{g_1}, r_{g_2}, \dots$, and r_{g_k} are the homogeneous components of r . The set $G_R = \{g \in G \mid R_g \neq \{0\}\}$ is called the support of R .

Throughout the paper we denote by G an abelian semigroup (written multiplicatively) unless otherwise stated.

A two-sided ideal I of a graded Γ -ring R of type G is called a graded Γ -ideal (or homogeneous Γ -ideal) if $I = \bigoplus_{g \in G} I_g$, where $I_g = I \cap R_g$. In other words, I is a graded Γ -ideal if and only if, for each $x \in I$, its homogeneous components belong also to I .

The following two theorems give us recipes for forming new graded gamma ring from old ones.

Theorem 3.1. Let R be a graded Γ -ring of type G where G is a semigroup. If $\varphi : G \rightarrow H$ is an onto homomorphism (epimorphism) of semigroups, then the Γ -ring $S = R$ with gradation

$$S_h = \bigoplus_{g \in \varphi^{-1}(h)} R_g,$$

for all $h \in H$ is a graded Γ -ring of type H (such graded Γ -ring is denoted by $R_{(H)}$).

Proof. Let $a, b \in H$, $x \in S_a$ and $y \in S_b$. Then $x = x_{a_1} + \cdots + x_{a_r}$ and $y = y_{b_1} + \cdots + y_{b_s}$, where $\varphi(a_i) = a$ and $\varphi(b_j) = b$ for all i, j . Now, for $\gamma \in \Gamma$, we obtain

$$x\gamma y = \sum_{1 \leq i \leq r, 1 \leq j \leq s} x_{a_i} \gamma y_{b_j}.$$

Also, for $1 \leq i \leq r$ and $1 \leq j \leq s$, we have $\varphi(a_i b_j) = \varphi(a_i) \varphi(b_j) = ab$. Thus, $x\gamma y \in S_{ab}$. $\square$

Also, we have the following result. We omit the proof since the proof is straightforward.

Theorem 3.2. *Let R be a graded Γ -ring of type G , and let H be a subsemigroup of G . Then*

- (i) $R' = \bigoplus_{h \in H} R_h$ is a graded Γ -ring of type H (such graded is denoted by $R^{(H)}$). In particular, if e is identity of G , then R_e corresponds to the trivial subsemigroup of G .
- (ii) If H is a normal subsemigroup of G , then

$$R = \bigoplus_{gN \in G/N} R_{gN}$$

is a graded Γ -ring of type G/N , where

$$R_{gN} = \bigoplus_{n \in N} R_{gn}.$$

The following theorem extends a well known result to group graded rings.

Theorem 3.3. *Let G be a monoid with the identity element e , and let R be a graded Γ -ring of type G . Then R_e is a sub- Γ -ring of R . Also if 1 is the unity element with respect to $\gamma_0 \in \Gamma$, then $1 \in R_e$.*

Proof. For $a, b \in R_e$, we have $a - b \in R_e$ since M_e is an additive subgroup of R , and also $a\gamma b \in R_{ee} = R_e$ for any $\gamma \in \Gamma$. It follows that R_e is a sub- Γ -ring of R . Next, as $R = \bigoplus_{g \in G} R_g$, we may assume $1 = r_{g_1} + r_{g_2} + \cdots + r_{g_n}$. Pick $\tau \in G$, and a nonzero element $\lambda_\tau \in R_\tau$, then

$$1\gamma_0\lambda_\tau = \lambda_\tau = r_{g_1}\gamma_0\lambda_\tau + r_{g_2}\gamma_0\lambda_\tau + \cdots + r_{g_n}\gamma_0\lambda_\tau.$$

and so, for all $g_i \in G$ with $g_i \neq e$ we have $r_{g_i}\gamma_0\lambda_\tau = 0$. Hence, $1\gamma_0\lambda_\tau = r_e\gamma_0\lambda_\tau$. Therefore, by the uniqueness of the unity, we have $1 = r_e \in R_e$. $\square$

Let R be a Γ -ring with a unity 1 corresponding to $\gamma_0 \in \Gamma$. An element $r \in R$ is called invertible with respect to 1 and γ_0 if there is an element (unique) $r^{-1} \in R$ such that $r^{-1}\gamma_0 r = r\gamma_0 r^{-1} = 1$.

Theorem 3.4. *Let G be a group with the identity element e , and let R be a graded Γ -ring of type G . If $r \in R_g$ is an invertible (corresponding to $\gamma_0 \in \Gamma$) homogeneous element, then r^{-1} is homogeneous too of degree g^{-1} .*

Proof. Let $s = s_{g_1} + \cdots + s_{g_k}$ be the inverse of r . Then

$$1 = r\gamma_0 s = r\gamma_0 s_{g_1} + \cdots + r\gamma_0 s_{g_k},$$

where $r\gamma_0 s_{g_i} \in R_{gg_i}$ for all $i = 1, \dots, k$. By Theorem 3.3, 1 is homogeneous of degree e , and also, the decomposition is unique, so that $r\gamma_0 s_{g_i} = 0$ for each $g_i \neq g^{-1}$. Since r is invertible, $s_{g^{-1}} \neq 0$, and so $s = s_{g^{-1}} \in R_{g^{-1}}$ as desired. $\square$

4. Filtered gamma rings

The following definition produces the notion of filtered Γ -rings. It is a generalization of the notion of filtered rings [15].

Definition 4.1. *Let R be a Γ -ring and let $R^0 \subseteq R^1 \subseteq R^2 \subseteq \cdots$ be a chain of additive subgroups of R . This chain is called an (ascending) filtration of R if*

$$\bigcup_{r \geq 0} R^r = R \text{ and } R^i \Gamma R^j \subseteq R^{i+j}, \quad \forall i, j \geq 0.$$

In this case we say that R is a filtered R_Γ -ring.

Let R and S be filtered Γ -rings. A homomorphism $\varphi : R \rightarrow S$ is called a filtered homomorphism if $\varphi(R_i) \subseteq S_i$ for $i = 0, 1, \dots$.

Given a graded Γ -ring $R = \bigoplus_{i \geq 0} R_i$ of type $\mathbb{Z}$, then there is a corresponding filtration

$$\bigcup_{k=0}^{\infty} R^k,$$

where $R^k = \bigoplus_{j=0}^k R_j$. Conversely, the following result gives us a way to produce a graded Γ -ring out of a filtered Γ -ring.

Theorem 4.1. *Let R be a Γ -ring with filtration $R^0 \subseteq R^1 \subseteq R^2 \subseteq \cdots$. Set $R^{-1} = 0$, and let $\text{gr}R = \bigoplus_{k=0}^{\infty} (R^k / R^{k-1})$. Then, define multiplication*

$$R^m / R^{m-1} \times \Gamma \times R^n / R^{n-1} \rightarrow R^{m+n} / R^{m+n-1}$$

by $(r + R^{m-1})\gamma(s + R^{n-1}) = (r\gamma s) + R^{m+n-1}$. Extending this multiplication by multilinearity makes $\text{gr}R$ is a graded Γ -ring.

Proof. We must show that the map is well defined. Suppose that $x + R^{m-1} = x' + R^{m-1}$, $y + R^{n-1} = y' + R^{n-1}$, and $\gamma \in \Gamma$. Then, $x - x' = r_0$ and $y - y' = r_1$ for some $r_0 \in R^{m-1}$ and $r_1 \in R^{n-1}$. Thus

$$\begin{aligned} x\gamma y - x'\gamma y' &= (x' + r_0)\gamma(y' + r_1) - x'\gamma y' \\ &= x'\gamma r_1 + r_0\gamma y' + r_0\gamma r_1 \end{aligned}$$

is a member of R^{m+n-1} . The remainder of the proof is quite straightforward. $\square$

Remark 4.1. If the filtration R^k , given in the theorem above, comes from a grading (i.e. $R^k = \bigoplus_{i=0}^k R_i$), then $R^k/R^{k-1} \cong R_k$ as additive groups, and so $\text{gr}R \cong R$ as Γ -rings.

5. Graded gamma modules

Let R be a Γ -ring. An abelian group M together with a mapping

$$R \times \Gamma \times M \rightarrow M; (r, \gamma, m) \mapsto r\gamma m,$$

such that the following identities are satisfied for all $r_1, r_2, r \in R$, $\gamma_1, \gamma_2, \gamma \in \Gamma$, and $m_1, m_2, m \in M$:

- (i) $r\gamma(m_1 + m_2) = r\gamma m_1 + r\gamma m_2$,
- (ii) $(r_1 + r_2)\gamma m = r_1\gamma m + r_2\gamma m$,
- (iii) $r(\gamma_1 + \gamma_2)m = r\gamma_1 m + r\gamma_2 m$,
- (iv) $(r_1\gamma_1)(r_2\gamma_2 m) = (r_1\gamma_1 r_2)\gamma_2 m$

is called a (left) R_Γ -module. A right R_Γ -module is defined in analogous manner ([1]). If R and S are Γ -rings, then we say M is an $(R, S)_\Gamma$ -bimodule if it is both left R_Γ -module and right S_Γ -module and simultaneously $(r\alpha m)\beta s = r\alpha(m\beta s)$ for all $r \in R$, $m \in M$, $s \in S$, and $\alpha, \beta \in \Gamma$. A (left) R_Γ -module M is unitary if there exists an element, say 1, in R and $\gamma_0 \in \Gamma$ such that $1\gamma_0 m = m \forall m \in M$. Note that if M is a left R_Γ -module then it is easy to verify that $0\gamma m = r0m = r\gamma 0 = 0$ for any $r \in R$, $\gamma \in \Gamma$ and $m \in M$.

Let R be a Γ -ring. A nonempty subset M_1 of M is called a (left) R_Γ -submodule of M if M_1 is a subgroup of M and $R\Gamma M_1 \subseteq M_1$. In this case we write $M_1 \leq M$. For a Γ -ring R , $\{0\}$ and R are R_Γ -submodules of R_Γ -module R . Also, every ideal of R is a submodule of R .

The following definition is a generalization of the notion of a graded module.

Definition 5.1. Let R be a graded Γ -ring of type G , and M be an R_Γ -module. Then M is said to be graded (left) R_Γ -module if there is a family $\{M_g : g \in G\}$ of additive subgroups of M such that $M = \bigoplus_{g \in G} M_g$ and $R_g\Gamma M_h \subseteq M_{gh}$ for all $g, h \in G$. A nonzero element $m \in M$ is called homogeneous of degree g , and

we write $\deg(m) = g$, if there $g \in G$ such that $m \in M_g$. A submodule N of M is a graded submodule if $N = \bigoplus_{g \in G} (N \cap M_g)$ (equivalently, for any $x \in N$ the homogeneous component(s) of x are again in N).

It is known that if R is a Γ ring, M is a R_Γ -ring, and K is a subgroup of M , then that factor group M/K is an R_Γ -module under the mapping

$$R \times \Gamma \times M/K \rightarrow M/K; (r, \gamma, m + K) \mapsto (r\gamma m) + M$$

(see [1]). Using the same arguments in the graded rings we can prove the following result.

Proposition 5.1. *Let R be a graded Γ -ring of type G and M be a graded R_Γ -module. If K is a submodule of M , then the factor module M/K is a graded R_Γ -module by setting*

$$(M/K)_g = (M_g + K)/K \cong M_g/(K \cap M_g),$$

for any $g \in G$.

Suppose that R is a graded Γ -ring of type G and that M and K are (left) graded R_Γ -module. Let $\varphi : \Gamma \rightarrow \Gamma$ be a group isomorphism. Then $f : M \rightarrow K$ is φ -homomorphism if $f(x + y) = f(x) + f(y)$ and $f(r\gamma x) = r\varphi(\gamma)f(x)$ for all $x, y \in M$, $r \in R$, and $\gamma \in \Gamma$. In particular, if φ is the identity map, then f is homomorphism. It is called isomorphism if it is one to one and onto.

For graded (left) R_Γ -modules M and K . A (φ) -homomorphism $f : M \rightarrow K$ is called homogeneous of degree $h \in G$ if for all $g \in G$, we have $f(M_g) \subseteq K_{hg}$. In particular, for graded Γ -rings R and K . A (φ) -homomorphism of Γ -ring $f : R \rightarrow K$ is called homogeneous of degree $h \in G$ if for all $g \in G$, we have $f(R_g) \subseteq K_{hg}$. A homogeneous (φ) -homomorphism of R_Γ -modules of degree $e \in G$ (the identity element of G) will be called a degree preserving map. Clearly the set $\text{Hom}_R(M, K)$ of all homomorphisms of R_Γ modules from M into K forms a group under the ordinary addition. Also, for graded R_Γ -modules M_1, M_2 , and M_3 the composition of a homogeneous (φ) -homomorphism $f : M_1 \rightarrow M_2$ of degree $h \in G$ with a (φ) -homomorphism $g : M_2 \rightarrow M_3$ of degree $k \in G$ is (φ) -homomorphism of degree hk .

Definition 5.2. *Let R be a Γ -ring and M be an R_Γ -module. Then we say M is finitely generated if there exist $m_1, \dots, m_k$ in M such that if $x \in M$, then there exist $r_1, \dots, r_k \in R$ and $\gamma_1, \dots, \gamma_k \in \Gamma$ with $x = r_1\gamma_1m_1 + \dots + r_k\gamma_km_k$. In this case the set $\{m_1, \dots, m_k\}$ is said a generating set to M .*

The following theorem gives us a procedure to produce a ring out of old Γ -ring.

Theorem 5.1. *Let M and K be graded R_Γ -modules of type G . If M is a finitely generated, then*

$$\text{Hom}_R(M, K) = \bigoplus_{g \in G} (\text{Hom}(M, K))_g$$

is a graded abelian group, where $\text{Hom}_R(M, K)_g$ is the subgroup of $\text{Hom}_R(M, K)$ that consisting of all homomorphisms of from M to K of degree g . Moreover if $M = K$, then $\text{Hom}(M, M)$ is a graded ring of type G .

Proof. Suppose that $f \in \text{Hom}_R(M, K)$. Let $g \in G$. Define a map

$$f_g : M \rightarrow K$$

as follows: for $m = m_{g_1} + m_{g_2} + \cdots + m_{g_k} \in M$ ($g_1, g_2, \dots, g_k \in G$), set

$$f_g(m) = (f(m_{g_1 g^{-1}}))_{g_1} + \cdots + (f(m_{g_k g^{-1}}))_{g_k}.$$

Then $f_g \in \text{Hom}_R(M, K)$. Indeed if $x = \sum_{h \in G} x_h$, $y = \sum_{h \in G} y_h \in M$, then

$$\begin{aligned} f_g(x + y) &= \sum_{h \in G} (f(x + y)_{hg^{-1}})_h \\ &= \sum_{h \in G} ((f(x_{hg^{-1}} + y_{hg^{-1}}))_h \\ &= \sum_{h \in G} ((f(x_{hg^{-1}}) + f(y_{hg^{-1}}))_h \\ &= \sum_{h \in G} ((f(x_{hg^{-1}}))_h + ((f(y_{hg^{-1}}))_h) \\ &= f_g(x) + f_g(y). \end{aligned}$$

Also, for $r \in R$ and $\gamma \in \Gamma$, we have

$$\begin{aligned} f_g(r\gamma x) &= \sum_{h \in G} (f((r\gamma x)_{hg^{-1}}))_h \\ &= \sum_{h \in G} (r\gamma f(x_{hg^{-1}}))_h \\ &= r\gamma \sum_{h \in G} ((f(x_{hg^{-1}}))_h \\ &= r\gamma f_g(x). \end{aligned}$$

In addition, $f_g \in (\text{Hom}_R(M, K))_g$. Indeed for any $h \in G$ and $m \in M_h$, we have $f_g(m) = (f(m))_{hg} \in K_{hg}$.

Let us assume that $\{m_1, \dots, m_s\}$ is a generating set to M . For $x \in M$, there exist $r_1, \dots, r_s \in R$ and $\gamma_1, \dots, \gamma_s \in \Gamma$ with $x = r_1\gamma_1 m_1 + \cdots + r_s\gamma_s m_s$. Thus $f(x) = r_1\gamma_1 f(m_1) + \cdots + r_s\gamma_s f(m_s)$. On the other hand for each $t = 1, 2, \dots, s$, $f_g(m_t) = 0$ for all but a finite number of $g \in G$ and

$$\sum_{g \in G} f_g(m_t) = \sum_{g \in G} (f(m_t))_{hg} = f(m_t).$$

It follows again that $f_g(x) = 0$ for all but finitely number of $g \in G$, and $f = \sum_{g \in G} f_g$. We have so far shown that $\text{Hom}_R(M, K) = \sum_{g \in G} (\text{Hom}_R(M, K))_g$.

On the other hand if $f \in \text{Hom}_R(M, K)_{h_1} \cap \text{Hom}_R(M, K)_{h_2}$ where $h_1, h_2 \in G$ and $h_1 \neq h_2$, then for any $g \in G$ we have $f(M_g) \subseteq K_{h_1g} \cap K_{h_2g} = \{0\}$. Hence $f = 0$. This shows that $\text{Hom}_R(M, K) = \bigoplus_{g \in G} (\text{Hom}_R(M, K))_g$. In particular, if $M = K$, we obtain $\text{Hom}_R(M, M) = \bigoplus_{g \in G} (\text{Hom}_R(M, M))_g$. For $\rho \in \text{Hom}_R(M, M)_{h_1}$ and $\sigma \in \text{Hom}_R(M, M)_{h_2}$ we have $\rho\sigma \in \text{Hom}_R(M, M)_{h_1h_2}$. It is clear now that $\text{Hom}_R(M, M)$ is a graded ring of type G . $\square$

6. Filtered Γ -modules

Let R be a filtered Γ -ring and M be a (left) R_Γ -module. Then M is called a filtered R_Γ module if there is an (ascending) chain

$$M^0 \subseteq M^1 \subseteq M^2 \subseteq \dots$$

of additive subgroups of M such that $\bigcup_{k \geq 0} M^k = M$ and $R^i \Gamma M^j \subseteq M^{i+j}$ for any i, j .

Example 6.1. (i) It is clear that if R is a filtered Γ -ring, then R is a filtered R_Γ -module.

(ii) If the filtration of a graded Γ -ring R is trivial and M is R_Γ -module, then any ascending chain of submodules K^s , where $\bigcup_{s \geq 0} K^s = M$, of M defines a filtration for M .

(iii) Let R be a Γ -ring and $I \subseteq R$ be an ideal of R . Then R has a descending filtration $\bigcup_{i=0}^\infty R^k$, where

$$R^0 = R, R^1 = I, R^2 = I\Gamma I, \text{ and } R^k = R^{k-1}\Gamma I \text{ for } k \geq 3.$$

Such filtration is called I -adic filtration. Let M be R_Γ -module. The corresponding descending filtration of M

$$M \supseteq R^1 \Gamma M \supseteq R^2 \Gamma M \supseteq \dots$$

Let R be a graded Γ -ring and let $M = \bigoplus_{i=0}^\infty M_i$ be a graded R_Γ -module. Consider the corresponding filtration of R as in Example 6.1(iii), then there is a corresponding filtration on M

$$M^0 \subseteq M^1 \subseteq M^2 \subseteq \dots,$$

where $M^k = \bigoplus_{j=0}^k M_j$. Conversely, one can prove the following result.

Theorem 6.1. *If M is a filtered R_Γ module, then*

$$\text{gr}(M) = \bigoplus_{i=0}^\infty M_i / M_{i-1}$$

is a graded $(\text{gr}R)_\Gamma$ module where $\text{gr}R = \bigoplus_{i=0}^\infty R_i / R_{i-1}$.

Let R be a filtered Γ -ring and $M = \bigcup_{k \geq 0} M^k$ be a filtered R_Γ -module. It is trivial that $\bigcup_{k \geq 0} M^k$ is an R_Γ -submodule of M . The following theorem answers the question: Is $\bigcap_{k \geq 0} M^k$ an R_Γ -submodule?

Theorem 6.2. *Let R be a filtered Γ -ring and M be a filtered R_Γ -module by M^k . Then $\bigcap M^k$ is an R_Γ -submodule of M .*

Proof. The intersection of any arbitrary collection of subgroups of a group is subgroup, so $\bigcap M^k$ is a subgroup of M . For $\lambda \in R$, $\gamma \in \Gamma$, and $x \in \bigcap M^k$. There exists $p \in \mathbb{N}_0$ such that $\lambda \in R^p$. As $x \in M^{k-p}$ for all k , we have $\lambda\gamma x \in M^k$ for all k . That is $\lambda\gamma x \in \bigcap M^k$. $\square$

Remark 6.1. One can focus on studying graded Γ -bimodules. Let R and S be graded Γ -rings of type G and let M be an abelian group. Then $M = \bigoplus_{g \in G} M_g$ is a graded $(R, S)_\Gamma$ -bimodule of type G if the following conditions are satisfied:

- (i) M is an $(R, S)_\Gamma$ -bimodule,
- (ii) M is a graded left R_Γ -module,
- (iii) M is a graded right S_Γ -module.

Therefore $N_g \Gamma M_h \Gamma S_k \subseteq M_{ghk}$ for $g, h, k \in G$.

7. Strongly graded gamma rings

Let M be a graded Γ -ring, where G is a monoid, with identity 1. According to Theorem 3.3, we have $1 \in M_e$. Thus $M_g \Gamma M_e = M_g$ and $M_e \Gamma M_g = M_g$ for all $g \in G$. If these equalities hold for any arbitrary two elements g, h, G , we get a strongly graded Γ -ring. In other words we have the following definition.

Definition 7.1. *Let G be a semigroup. A graded Γ -ring M of type G is called a strongly graded Γ -ring if $M_g \Gamma M_h = M_{gh}$ for all $g, h \in G$.*

- Remark 7.1.**
- (i) If M is a strongly graded Γ -ring, then so is M/I for every graded ideal I of M .
 - (ii) If M is a strongly graded Γ -ring of type G and $\phi : G \rightarrow H$ is an epimorphism, then $M_{(H)}$ is strongly graded.
 - (iii) If M is a strongly graded Γ -ring of type G , and H is a sub(semi-)group of G , then $M^{(H)}$ is strongly graded.
 - (iv) Let $(M_i)_{i \in I}$ be a family of graded Γ rings of type G such that I is finite. Then $M = \prod_{i \in I} M_i$ is strongly graded if and only if M_i is strongly graded for each $i \in I$.

Let M be a Γ -ring with unity 1_{γ_0} , and let M^* denotes the set of invertible elements in M with respect to γ_0 . Define multiplication on M^* as follows: $mn = m\gamma_0n$ for $m, n \in M$. Then the set M^* forms a group. For if $m, n \in M^*$, then $mnn^{-1}m^{-1} = (m\gamma_0n)\gamma_0(n^{-1}\gamma_0n^{-1}) = 1_{\gamma_0}$. Also, the associativity follows from one of the properties of Γ -rings. The support of invertible homogeneous elements of M is defined by

$$G_M^* = \{g \in G \mid M_g^* \neq \emptyset\},$$

where $M_g^* = M_g \cap M^*$. Using Theorem 3.4 it is easy to see that G_M^* is a group. Also $G_M^* \subseteq G_M$ where $G_M = \{g \in G \mid M_g \neq \{0\}\}$ is the support of M .

Definition 7.2. A graded Γ -ring $M = \bigoplus_{g \in G} M_g$ with unity 1_{γ_0} , where G is a group, is called crossed product if there is an invertible element in every homogeneous element of M , that is $G_M^* = G$.

Theorem 7.1. Let $M = \bigoplus_{g \in G} M_g$ be a graded Γ -ring of type G , where G is a group, with unity 1_{γ_0} . Then

- (i) M is strongly graded if and only if $1_{\gamma_0} \in M_g \Gamma M_{g^{-1}}$ for all $g \in G$,
- (ii) if M is strongly graded, then the support of M is G ,
- (iii) any crossed product Γ -ring is strongly graded,
- (iv) if N is a graded Γ -ring and $\varphi : M \rightarrow N$ is a degree preserving homomorphism of Γ -rings, and M is strongly graded Γ -ring, then N is strongly graded too.

Proof. (i) Suppose that M is a strongly graded Γ -ring. Then $M_e = M_g \Gamma M_{g^{-1}}$ for any $g \in G$. Now the result follows from Theorem 3.3. Conversely, let $x \in M_e$ and $g \in G$. Then $x = 1_{\gamma_0} \gamma_0 x \in (M_g \Gamma M_{g^{-1}}) \gamma_0 M_e = M_g \Gamma (M_{g^{-1}} \gamma_0 M_e) \subseteq M_g \Gamma M_{g^{-1}}$. Hence $M_e = M_g \Gamma M_{g^{-1}}$ for all $g \in G$. Next for $g, h \in G$ we have

$$\begin{aligned} M_{gh} &\subseteq M_e \Gamma M_{gh} \\ &= (M_g \Gamma M_{g^{-1}}) \Gamma M_{gh} \\ &= M_g \Gamma (M_{g^{-1}} \Gamma M_{gh}) \\ &\subseteq M_g \Gamma M_h. \end{aligned}$$

This proves that $M_{gh} = M_g \Gamma M_h$ for all $g, h \in G$. Therefore M is strongly graded.

- (ii) Let $g \in G$. Then using (i), we have $1_{\gamma_0} \in M_g \Gamma M_{g^{-1}}$. Thus $M_g \neq \{0\}$.
- (iii) Suppose that M is a crossed product Γ -ring. Let $g \in G$. Then, there exists an invertible element (homogeneous) $m \in M_g$. According to Theorem 3.4, we have $m^{-1} \in M_{g^{-1}}$. So $1_{\gamma_0} = m \gamma_0 m^{-1} \in M_g \Gamma M_{g^{-1}}$. Now the result follows from (i).

- (iv) M is strongly graded, so for all $g \in G$ we have $1_{\gamma_0} \in M_g \Gamma M_{g^{-1}}$. It follows $\varphi(1_{\gamma_0}) = 1_{\gamma_0} \in \varphi(M_g) \Gamma \varphi(M_{g^{-1}}) \subseteq N_g \Gamma N_{g^{-1}}$ for all $g \in G$. $\square$

Remark 7.2. Let R be a graded Γ -ring, and M be a graded R_Γ -module. Then M is strongly graded R_Γ -module if $R_g \Gamma M_h = M_{gh}$ for all $g, h \in G$. One can focus on considering such graded Γ -rings. Also, one of the future works will focus on studying tensor product of graded Γ -modules.

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Study of different degrees based on various operations on fuzzy soft graphs

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Abstract. In this paper, two operations, namely conjunction and disjunction on fuzzy soft graphs are defined. The degree and total degree of the vertices of the resultant fuzzy soft graph that are obtained from two given fuzzy soft graphs using these operations are determined. Further, we study properties of degree and total degree of a vertex in fuzzy soft graph based on these operations.

Keywords: degree of a vertex, total degree of a vertex, disjunction and conjunction of two fuzzy soft graphs.

1. Introduction

Graph theory has many interesting applications in various fields of computer science such as image segmentation, data mining, networking etc. Whenever there is uncertainty in the description of objects, fuzziness arises. A. Rosenfeld [1] developed the theory of fuzzy graphs.

In 1975 based on fuzzy sets which were initiated by Zadeh [2] in 1965. The concepts of operations on fuzzy graphs were presented by J. N. Mordeson and C. S. Peng [3]. Roy et al. [4] presented some applications of fuzzy soft sets to decision making problems. Sumit Mohinta and T.K. Samanta [5] introduced the notions of fuzzy soft graph, complete fuzzy soft graph and some of the operations such as union, intersection of two fuzzy soft graphs. M. Akram and Saira Nawaz [6] also introduced fuzzy soft graphs and studied few properties. Akram and Nawaz [7] in 2016, presented the concepts of fuzzy soft graphs and described

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applications of fuzzy soft graphs in social network and road network. Shashikala S and Anil P N [8] discussed connectivity in fuzzy soft graphs in comparison with their complements. Degree, total degree, regularity and total regularity of fuzzy soft graph and its properties are studied by A. Pouhassani and H. Doostie [9]. Anas Al-Masarwah and Majdoleen Abuqamar [10] introduced the concepts of uniform vertex fuzzy soft graphs, uniform edge fuzzy soft graphs, degree of a vertex and total degree of a vertex with some examples. SampathKumar [12] introduced the notion of graph structures. Maximal product of fuzzy graph structures, regular fuzzy graph structures, degree and total degree of a vertex in maximal product of fuzzy graph structures were introduced and some of its properties were presented [13]. Size and degree of fuzzy soft bi-partite graph are discussed and applied in employee selection for an organization by K. Malathy and S. Meenakshi [14].

In this paper, two operations namely, disjunction and conjunction of two fuzzy soft graphs are defined and some results related to degree and total degree of a vertex in fuzzy soft graph based on these operations are obtained.

2. Preliminaries

Definition 2.1 ([7]). A fuzzy soft graph $\tilde{G}$ over a graph $G^* : (V, E)$ is a triple $(\tilde{F}, \tilde{K}, A)$ where

- a) A is a non-empty set of parameters
 - b) $(\tilde{F}, A)$ is a fuzzy soft set over V
 - c) $(\tilde{K}, A)$ is a fuzzy soft set over E
 - d) $((\tilde{F}, (e_i)), (\tilde{K}, (e_i)))$ is a fuzzy graph on $G^* \forall e_i \in A$
- i.e. $\tilde{K}(e_i)(xy) \leq \min\{\tilde{F}(e_i)(x), \tilde{F}(e_i)(y)\}$ for all $e_i \in A$ and $x, y \in V$.

Definition 2.2 ([10]). The underlying crisp graph of a fuzzy soft graph $\tilde{G}$ is denoted by $G^* = (F^*, K^*)$ where $F^* = \{x \in V : \tilde{F}(e_i)(x) > 0\}$, $K^* = \{(x, y) \in V \times V : \tilde{K}(e_i)(x, y) > 0\}$ for some $e_i \in A$.

Definition 2.3 ([10]). Let $\tilde{G}$ be a fuzzy soft graph on G^* . The degree of a vertex x is defined as $\deg_{\tilde{G}}(x) = \sum_{e_i \in A} \sum_{y \in V, y \neq x} \tilde{K}(e_i)(xy)$.

Definition 2.4 ([10]). Let $\tilde{G}$ be a fuzzy soft graph on G^* . The total degree of a vertex x is defined as $tdeg_{\tilde{G}}(x) = \sum_{e_i \in A} \sum_{y \in V, y \neq x} \tilde{K}(e_i)(xy) + \sum_{e_i \in A} \tilde{F}(e_i)(x)$ i.e. $tdeg_{\tilde{G}}(x) = \deg_{\tilde{G}}(x) + \sum_{e_i \in A} \tilde{F}(e_i)(x)$.

Definition 2.5 ([11]). The degree $d_{G^*}(x)$ of a vertex v in G^* is the number of edges incident with x .

In this paper, we assume that $G^* : (V, E)$ of any fuzzy soft graph $\tilde{G}$ is finite and simple.

Notation. Let $\tilde{G}_1 : (\tilde{F}_1, \tilde{K}_1, A_1)$ and $\tilde{G}_2 : (\tilde{F}_2, \tilde{K}_2, A_2)$ be two fuzzy soft graphs. The relation $\tilde{F}_1(e_i) \geq \tilde{K}_2(e_j)$ for all $e_i \in A_1, e_j \in A_2$ means that $\tilde{F}_1(e_i)(x) \geq \tilde{K}_2(e_j)(e) \forall x \in V_1, e \in E_2$ where $\tilde{F}_1$ is a fuzzy soft subset of V_1 and $\tilde{K}_2$ is a fuzzy soft subset of E_2 .

3. Disjunction of two fuzzy soft graphs $\tilde{G}_1 \vee \tilde{G}_2$

Definition 3.1. Let $\tilde{G}_1 : (\tilde{F}_1, \tilde{K}_1, A_1)$ and $\tilde{G}_2 : (\tilde{F}_2, \tilde{K}_2, A_2)$ be two fuzzy soft graphs on G_1^* and G_2^* respectively.

The disjunction $\tilde{G}_1 \vee \tilde{G}_2 : (\tilde{F}_1 \vee \tilde{F}_2, \tilde{K}_1 \vee \tilde{K}_2, A_1 \vee A_2)$ is defined as follows: $(\tilde{F}_1 \vee \tilde{F}_2) : A_1 \times A_2 \rightarrow FS(V_1 \times V_2)$ by $(\tilde{F}_1 \vee \tilde{F}_2)(e_i, e_j)(x_k y_l) = \tilde{F}_1(e_i)(x_k) \vee \tilde{F}_2(e_j)(y_l) \forall e_i \in A_1, e_j \in A_2, x_k y_l \in V_1 \times V_2$ and $(\tilde{K}_1 \vee \tilde{K}_2) : A_1 \times A_2 \rightarrow FS(E_1 \times E_2)$ by

$$\begin{aligned} & (\tilde{K}_1 \vee \tilde{K}_2)(e_i, e_j)(x_k y_l)(x_m y_n) \\ &= \begin{cases} \tilde{F}_1(e_i)(x_k) \vee \tilde{K}_2(e_j)(y_l y_n), & \text{if } x_k = x_m, y_l y_n \in E_2 \\ \tilde{F}_2(e_j)(y_l) \vee \tilde{K}_1(e_i)(x_k x_m), & \text{if } y_l = y_n, x_k x_m \in E_1 \\ \tilde{K}_1(e_i)(x_k x_m) \vee \tilde{K}_2(e_j)(y_l y_n), & \text{if } x_k x_m \in E_1, y_l y_n \in E_2, \\ & x_k \neq x_m, y_l \neq y_n. \end{cases} \end{aligned}$$

Definition 3.2. Let $\tilde{G}_1 : (\tilde{F}_1, \tilde{K}_1, A_1)$ and $\tilde{G}_2 : (\tilde{F}_2, \tilde{K}_2, A_2)$ be two fuzzy soft graphs on $G_1^* : (V_1, E_1)$ and $G_2^* : (V_2, E_2)$ respectively.

Let $A = A_1 \times A_2$ then degree of a vertex $(x_k y_l) \in V_1 \times V_2$ in $\tilde{G}_1 \vee \tilde{G}_2$ is

$$\begin{aligned} \deg_{\tilde{G}_1 \vee \tilde{G}_2}(x_k, y_l) &= \sum_{(e_i, e_j) \in A} \sum_{\substack{x_k x_m \in E_1 \\ y_l y_n \in E_2}} \tilde{K}(e_i, e_j)(x_k y_l)(x_m y_n) \\ &= \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_k = x_m \\ y_l y_n \in E_2}} \tilde{F}_1(e_i)(x_k) \vee \tilde{K}_2(e_j)(y_l y_n) \\ &+ \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{y_l = y_n \\ x_k x_m \in E_1}} \tilde{F}_2(e_j)(y_l) \vee \tilde{K}_1(e_i)(x_k x_m) \\ &+ \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_k x_m \in E_1 \\ y_l y_n \in E_2}} \tilde{K}_1(e_i)(x_k x_m) \vee \tilde{K}_2(e_j)(y_l y_n). \end{aligned}$$

Definition 3.3. Let $\tilde{G}_1 : (\tilde{F}_1, \tilde{K}_1, A_1)$ and $\tilde{G}_2 : (\tilde{F}_2, \tilde{K}_2, A_2)$ be two fuzzy soft graphs on $G_1^* : (V_1, E_1)$ and $G_2^* : (V_2, E_2)$ respectively.

Let $A = A_1 \times A_2$ then total degree of a vertex $(x_k y_l) \in V_1 \times V_2$ in $\widetilde{G}_1 \vee \widetilde{G}_2$ is

$$\begin{aligned}
tdeg_{\widetilde{G}_1 \vee \widetilde{G}_2}(x_k, y_l) &= deg_{\widetilde{G}_1 \vee \widetilde{G}_2}(x_k, y_l) + \sum_{(e_i, e_j) \in A} (\widetilde{F}_1 \vee \widetilde{F}_2)(e_i, e_j)(x_k y_l) \\
&= \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_k = x_m \\ y_l y_n \in E_2}} \widetilde{F}_1(e_i)(x_k) \vee \widetilde{K}_2(e_j)(y_l y_n) \\
&+ \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{y_l = y_n \\ x_k x_m \in E_1}} \widetilde{F}_2(e_j)(y_l) \vee \widetilde{K}_1(e_i)(x_k x_m) \\
&+ \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_k x_m \in E_1 \\ y_l y_n \in E_2}} \widetilde{K}_1(e_i)(x_k x_m) \vee \widetilde{K}_2(e_j)(y_l y_n) + \sum_{(e_i, e_j) \in A} \widetilde{F}_1(e_i)(x_k) \vee \widetilde{F}_2(e_j)(y_l).
\end{aligned}$$

Example 3.1. Consider two fuzzy soft graphs $\widetilde{G}_1 : (\widetilde{F}_1, \widetilde{K}_1, A_1)$ and $\widetilde{G}_2 : (\widetilde{F}_2, \widetilde{K}_2, A_2)$ on $G_1^* : (V_1, E_1)$ and $G_2^* : (V_2, E_2)$ respectively such that $V_1 = \{x_1, x_2, x_3\}$, $E_1 = \{x_1 x_2, x_1 x_3, x_2 x_3\}$, $V_2 = \{y_1, y_2\}$, $E_2 = \{y_1 y_2\}$, $A_1 = \{e_i\}$ where $i = 1, 2$ and $A_2 = \{e_j\}$ where $j = 3, 4$. Let $(\widetilde{F}_1, A_1)$, $(\widetilde{F}_2, A_2)$, $(\widetilde{K}_1, A_1)$ and $(\widetilde{K}_2, A_2)$ be represented by the following Table 1.

$\widetilde{F}_1$	x_1	x_2	x_3	$\widetilde{F}_2$	y_1	y_2
e_1	0.4	0.5	0.6	e_3	0.3	0.3
e_2	0.3	0.4	0.5	e_4	0.2	0.2
$\widetilde{K}_1$	$x_1 x_2$	$x_1 x_3$	$x_2 x_3$	$\widetilde{K}_2$	$y_1 y_2$	
e_1	0.1	0.4	0.3	e_3	0.1	
e_2	0.3	0.3	0.2	e_4	0.3	

Table 1: Tabular representation of two fuzzy soft graphs

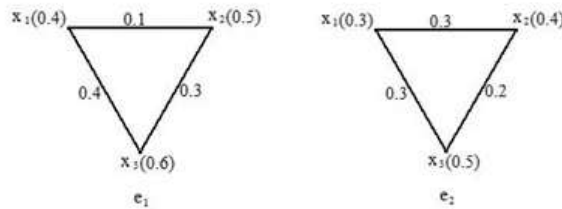


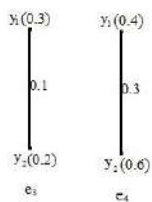
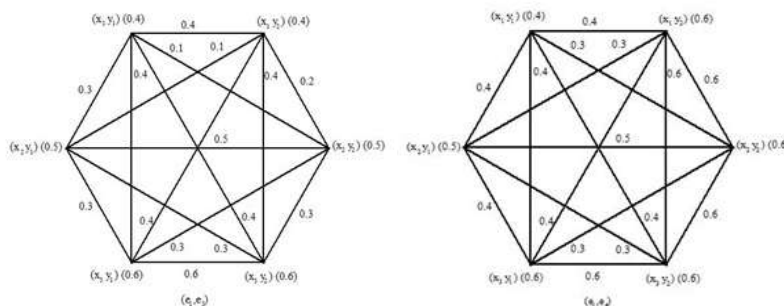
Figure 1: $\widetilde{G}_1 : (\widetilde{F}_1, \widetilde{K}_1, A_1)$

In a similar way, we get for (e_2, e_3) and (e_2, e_4) .

$$deg_{\widetilde{G}_1 \vee \widetilde{G}_2}(x_1, y_1) = 6.7$$

$$tdeg_{\widetilde{G}_1 \vee \widetilde{G}_2}(x_1, y_1) = 6.7 + 0.4 + 0.4 + 0.3 + 0.4 = 8.2$$

Theorem 3.1. Let $\widetilde{G}_1 : (\widetilde{F}_1, \widetilde{K}_1, A_1)$ and $\widetilde{G}_2 : (\widetilde{F}_2, \widetilde{K}_2, A_2)$ be two fuzzy soft graphs on $G_1^* : (V_1, E_1)$ and $G_2^* : (V_2, E_2)$ respectively. If $\widetilde{F}_1(e_i) \geq \widetilde{K}_2(e_j)$,

Figure 2: $\tilde{G}_2 : (\tilde{F}_2, \tilde{K}_2, A_2)$ Figure 3: Disjunction $\tilde{G}_1 \vee \tilde{G}_2$

$\tilde{F}_2(e_j) \geq \tilde{K}_1(e_i)$ and $\tilde{K}_1(e_i) \leq \tilde{K}_2(e_j)$, then

$$\begin{aligned} \deg_{\tilde{G}_1 \vee \tilde{G}_2}(x_k, y_l) &= |e_j| d_{G_2^*}(y_l) \sum_{e_i \in A_1} \tilde{F}_1(e_i)(x_k) \\ &+ |e_i| d_{G_1^*}(x_k) \sum_{e_j \in A_2} \tilde{F}_2(e_j)(y_l) + [d_{\tilde{G}_2}(y_l)] |e_i| d_{G_1^*}(x_k). \end{aligned}$$

Proof. Since $\tilde{F}_1(e_i) \geq \tilde{K}_2(e_j)$, $\tilde{F}_2(e_j) \geq \tilde{K}_1(e_i)$ and $\tilde{K}_1(e_i) \leq \tilde{K}_2(e_j)$, from Definition 3.2,

$$\begin{aligned} \deg_{\tilde{G}_1 \vee \tilde{G}_2}(x_k, y_l) &= \sum_{e_i \in A_1} \sum_{\substack{x_k = x_m \\ y_l y_n \in E_2}} \tilde{F}_1(e_i)(x_k) \\ &+ \sum_{e_j \in A_2} \sum_{\substack{y_l = y_n \\ x_k x_m \in E_1}} \tilde{F}_2(e_j)(y_l) + \sum_{e_j \in A_2} \sum_{\substack{x_k x_m \in E_1 \\ y_l y_n \in E_2}} \tilde{K}_2(e_j)(y_l y_n) \\ &= |e_j| d_{G_2^*}(y_l) \sum_{e_i \in A_1} \tilde{F}_1(e_i)(x_k) + |e_i| d_{G_1^*}(x_k) \sum_{e_j \in A_2} \tilde{F}_2(e_j)(y_l) \\ &+ [d_{\tilde{G}_2}(y_l)] |e_i| d_{G_1^*}(x_k). \end{aligned} \quad \square$$

Corollary 3.1. Let $\tilde{G}_1 : (\tilde{F}_1, \tilde{K}_1, A_1)$ and $\tilde{G}_2 : (\tilde{F}_2, \tilde{K}_2, A_2)$ be two fuzzy soft graphs on $G_1^* : (V_1, E_1)$ and $G_2^* : (V_2, E_2)$ respectively. If $\tilde{F}_1(e_i) \geq \tilde{K}_2(e_j)$,

$\tilde{F}_2(e_j) \geq \tilde{K}_1(e_i)$ and $\tilde{K}_1(e_i) \leq \tilde{K}_2(e_j)$, then

$$\begin{aligned} & \deg_{\tilde{G}_1 \vee \tilde{G}_2}(x_k, y_l) \\ &= |e_j|d_{G_2^*}(y_l) \sum_{e_i \in A_1} \tilde{F}_1(e_i)(x_k) + [td_{\tilde{G}_2}(y_l)]|e_i|d_{G_1^*}(x_k). \end{aligned}$$

Proof. The proof follows from Theorem 3.1. $\square$

Theorem 3.2. Let $\tilde{G}_1 : (\tilde{F}_1, \tilde{K}_1, A_1)$ and $\tilde{G}_2 : (\tilde{F}_2, \tilde{K}_2, A_2)$ be two fuzzy soft graphs on $G_1^* : (V_1, E_1)$ and $G_2^* : (V_2, E_2)$ respectively. If $\tilde{F}_1(e_i) \geq \tilde{K}_2(e_j)$, $\tilde{F}_2(e_j) \geq \tilde{K}_1(e_i)$ and $\tilde{K}_2(e_j) \leq \tilde{K}_1(e_i)$, then

$$\begin{aligned} & \deg_{\tilde{G}_1 \vee \tilde{G}_2}(x_k, y_l) \\ &= |e_j|d_{G_2^*}(y_l) \sum_{e_i \in A_1} \tilde{F}_1(e_i)(x_k) + |e_i|d_{G_1^*}(x_k) \sum_{e_j \in A_2} \tilde{F}_2(e_j)(y_l) + [d_{\tilde{G}_1}(x_k)]|e_j|d_{G_2^*}(y_l). \end{aligned}$$

Proof. Since $\tilde{F}_1(e_i) \geq \tilde{K}_2(e_j)$, $\tilde{F}_2(e_j) \geq \tilde{K}_1(e_i)$ and $\tilde{K}_2(e_j) \leq \tilde{K}_1(e_i)$, from Definition 3.2,

$$\begin{aligned} & \deg_{\tilde{G}_1 \vee \tilde{G}_2}(x_k, y_l) = \sum_{e_i \in A_1} \sum_{\substack{x_k=x_m \\ y_l=y_n \in E_2}} \tilde{F}_1(e_i)(x_k) \\ &+ \sum_{e_j \in A_2} \sum_{\substack{y_l=y_n \\ x_k=x_m \in E_1}} \tilde{F}_2(e_j)(y_l) + \sum_{e_i \in A_1} \sum_{\substack{x_k=x_m \in E_1 \\ y_l=y_n \in E_2}} \tilde{K}_1(e_i)(x_k x_m) \\ &= |e_j|d_{G_2^*}(y_l) \sum_{e_i \in A_1} \tilde{F}_1(e_i)(x_k) + |e_i|d_{G_1^*}(x_k) \sum_{e_j \in A_2} \tilde{F}_2(e_j)(y_l) \\ &+ [d_{\tilde{G}_1}(x_k)]|e_j|d_{G_2^*}(y_l). \end{aligned} \quad \square$$

Corollary 3.2. Let $\tilde{G}_1 : (\tilde{F}_1, \tilde{K}_1, A_1)$ and $\tilde{G}_2 : (\tilde{F}_2, \tilde{K}_2, A_2)$ be two fuzzy soft graphs on $G_1^* : (V_1, E_1)$ and $G_2^* : (V_2, E_2)$ respectively. If $\tilde{F}_1(e_i) \geq \tilde{K}_2(e_j)$, $\tilde{F}_2(e_j) \geq \tilde{K}_1(e_i)$ and $\tilde{K}_2(e_j) \leq \tilde{K}_1(e_i)$, then

$$\deg_{\tilde{G}_1 \vee \tilde{G}_2}(x_k, y_l) = |e_i|d_{G_1^*}(x_k) \sum_{e_j \in A_2} \tilde{F}_2(e_j)(y_l) + [td_{\tilde{G}_1}(x_k)]|e_j|d_{G_2^*}(y_l).$$

Proof. The proof follows from Theorem 3.2. $\square$

Theorem 3.3. Let $\tilde{G}_1 : (\tilde{F}_1, \tilde{K}_1, A_1)$ and $\tilde{G}_2 : (\tilde{F}_2, \tilde{K}_2, A_2)$ be two fuzzy soft graphs on $G_1^* : (V_1, E_1)$ and $G_2^* : (V_2, E_2)$ respectively. If $\tilde{F}_1(e_i) \leq \tilde{K}_2(e_j)$, $\tilde{F}_2(e_j) \geq \tilde{K}_1(e_i)$ and $\tilde{K}_1(e_i) \leq \tilde{K}_2(e_j)$, then $\deg_{\tilde{G}_1 \vee \tilde{G}_2}(x_k, y_l) = |e_i|d_{G_1^*}(x_k) \sum_{e_j \in A_2} \tilde{F}_2(e_j)(y_l) + [td_{\tilde{G}_1}(x_k)]|e_j|d_{G_2^*}(y_l)$.

Proof. Since $\tilde{F}_1(e_i) \leq \tilde{K}_2(e_j)$, $\tilde{F}_2(e_j) \geq \tilde{K}_1(e_i)$ and $\tilde{K}_1(e_i) \leq \tilde{K}_2(e_j)$, from Definition 3.2,

$$\begin{aligned}
 \deg_{\tilde{G}_1 \vee \tilde{G}_2}(x_k, y_l) &= \sum_{e_j \in A_2} \sum_{\substack{x_k = x_m \\ y_l y_n \in E_2}} \tilde{K}_2(e_j)(y_l y_n) \\
 &+ \sum_{e_j \in A_2} \sum_{\substack{y_l = y_n \\ x_k x_m \in E_1}} \tilde{F}_2(e_j)(y_l) + \sum_{e_j \in A_2} \sum_{\substack{x_k x_m \in E_1 \\ y_l y_n \in E_2}} \tilde{K}_2(e_j)(y_l y_n) \\
 &= |e_i| d_{\tilde{G}_2}(y_l) + |e_i| d_{G_1^*}(x_k) \sum_{e_j \in A_2} \tilde{F}_2(e_j)(y_l) + [d_{\tilde{G}_2}(y_l)] |e_i| d_{G_1^*}(x_k) \\
 &= |e_i| d_{\tilde{G}_2}(y_l) [1 + d_{G_1^*}(x_k)] + |e_i| d_{G_1^*}(x_k) \sum_{e_j \in A_2} \tilde{F}_2(e_j)(y_l). \quad \square
 \end{aligned}$$

Corollary 3.3. Let $\tilde{G}_1 : (\tilde{F}_1, \tilde{K}_1, A_1)$ and $\tilde{G}_2 : (\tilde{F}_2, \tilde{K}_2, A_2)$ be two fuzzy soft graphs on $G_1^* : (V_1, E_1)$ and $G_2^* : (V_2, E_2)$ respectively. If $\tilde{F}_1(e_i) \leq \tilde{K}_2(e_j)$, $\tilde{F}_2(e_j) \geq \tilde{K}_1(e_i)$ and $\tilde{K}_1(e_i) \leq \tilde{K}_2(e_j)$, then $\deg_{\tilde{G}_1 \vee \tilde{G}_2}(x_k, y_l) = |e_i| [d_{\tilde{G}_2}(y_l) + td_{\tilde{G}_2}(y_l) d_{G_1^*}(x_k)]$.

Corollary 3.4. Let $\tilde{G}_1 : (\tilde{F}_1, \tilde{K}_1, A_1)$ and $\tilde{G}_2 : (\tilde{F}_2, \tilde{K}_2, A_2)$ be two fuzzy soft graphs on $G_1^* : (V_1, E_1)$ and $G_2^* : (V_2, E_2)$ respectively. If $\tilde{F}_1(e_i) \leq \tilde{K}_2(e_j)$, $\tilde{K}_1(e_i) \leq \tilde{K}_2(e_j)$ and $\tilde{F}_2(e_j)$ is a constant function then $\deg_{\tilde{G}_1 \vee \tilde{G}_2}(x_k, y_l) = |e_i| \{d_{\tilde{G}_2}(y_l) [1 + d_{G_1^*}(x_k)] + c|e_j| d_{G_1^*}(x_k)\}$.

Theorem 3.4. Let $\tilde{G}_1 : (\tilde{F}_1, \tilde{K}_1, A_1)$ and $\tilde{G}_2 : (\tilde{F}_2, \tilde{K}_2, A_2)$ be two fuzzy soft graphs on $G_1^* : (V_1, E_1)$ and $G_2^* : (V_2, E_2)$ respectively. If $\tilde{F}_2(e_j) \leq \tilde{K}_1(e_i)$, $\tilde{F}_1(e_i) \geq \tilde{K}_2(e_j)$ and $\tilde{K}_2(e_j) \leq \tilde{K}_1(e_i)$, then $\deg_{\tilde{G}_1 \vee \tilde{G}_2}(x_k, y_l) = |e_j| d_{\tilde{G}_1}(x_k) [1 + d_{G_2^*}(y_l)] + |e_j| d_{G_2^*}(y_l) \sum_{e_i \in A_1} \tilde{F}_1(e_i)(x_k)$.

Proof. Since $\tilde{F}_2(e_j) \leq \tilde{K}_1(e_i)$, $\tilde{F}_1(e_i) \geq \tilde{K}_2(e_j)$ and $\tilde{K}_2(e_j) \leq \tilde{K}_1(e_i)$, from Definition 3.2,

$$\begin{aligned}
 \deg_{\tilde{G}_1 \vee \tilde{G}_2}(x_k, y_l) &= \sum_{e_i \in A_1} \sum_{\substack{x_k = x_m \\ y_l y_n \in E_2}} \tilde{F}_1(e_i)(x_k) \\
 &+ \sum_{e_i \in A_1} \sum_{\substack{y_l = y_n \\ x_k x_m \in E_1}} \tilde{K}_1(e_i)(x_k x_m) + \sum_{e_i \in A_1} \sum_{\substack{x_k x_m \in E_1 \\ y_l y_n \in E_2}} \tilde{K}_1(e_i)(x_k x_m) \\
 &= |e_j| d_{\tilde{G}_1}(x_k) [1 + d_{G_2^*}(y_l)] + |e_j| d_{G_2^*}(y_l) \sum_{e_i \in A_1} \tilde{F}_1(e_i)(x_k). \quad \square
 \end{aligned}$$

Corollary 3.5. Let $\tilde{G}_1 : (\tilde{F}_1, \tilde{K}_1, A_1)$ and $\tilde{G}_2 : (\tilde{F}_2, \tilde{K}_2, A_2)$ be two fuzzy soft graphs on $G_1^* : (V_1, E_1)$ and $G_2^* : (V_2, E_2)$ respectively. If $\tilde{F}_2(e_j) \leq \tilde{K}_1(e_i)$, $\tilde{F}_1(e_i) \geq \tilde{K}_2(e_j)$ and $\tilde{K}_2(e_j) \leq \tilde{K}_1(e_i)$, then $\deg_{\tilde{G}_1 \vee \tilde{G}_2}(x_k, y_l) = |e_j| [d_{\tilde{G}_1}(x_k) + td_{\tilde{G}_1}(x_k) d_{G_2^*}(y_l)]$.

Corollary 3.6. Let $\tilde{G}_1 : (\tilde{F}_1, \tilde{K}_1, A_1)$ and $\tilde{G}_2 : (\tilde{F}_2, \tilde{K}_2, A_2)$ be two fuzzy soft graphs on $G_1^* : (V_1, E_1)$ and $G_2^* : (V_2, E_2)$ respectively. If $\tilde{F}_2(e_j) \leq \tilde{K}_1(e_i)$, $\tilde{K}_2(e_j) \leq \tilde{K}_1(e_i)$ and $\tilde{F}_1(e_i)$ is a constant function then $\deg_{\tilde{G}_1 \vee \tilde{G}_2}(x_k, y_l) = |e_j| \{d_{\tilde{G}_1}(x_k)[1 + d_{G_2^*}(y_l)] + c|e_i|d_{G_2^*}(y_l)\}$.

Theorem 3.5. Let $\tilde{G}_1 : (\tilde{F}_1, \tilde{K}_1, A_1)$ and $\tilde{G}_2 : (\tilde{F}_2, \tilde{K}_2, A_2)$ be two fuzzy soft graphs on $G_1^* : (V_1, E_1)$ and $G_2^* : (V_2, E_2)$ respectively such that $\tilde{F}_1(e_i) \geq \tilde{K}_2(e_j)$, $\tilde{F}_2(e_j) \geq \tilde{K}_1(e_i)$, $\tilde{K}_1(e_i) \leq \tilde{K}_2(e_j)$ and $\tilde{F}_2(e_j) \geq \tilde{F}_1(e_i)$ then $tdeg_{\tilde{G}_1 \vee \tilde{G}_2}(x_k, y_l) = [\sum_{e_i \in A_1} \tilde{F}_1(e_i)(x_k)]|e_j|d_{G_2^*}(y_l) + [td_{\tilde{G}_2}(y_l)]|e_i|d_{G_1^*}(x_k) + [\sum_{e_j \in A_2} \tilde{F}_2(e_j)(y_l)]|e_i|$.

Proof. From definition,

$$\begin{aligned}
 tdeg_{\tilde{G}_1 \vee \tilde{G}_2}(x_k, y_l) &= \sum_{e_i \in A_1} \sum_{\substack{x_k = x_m \\ y_l y_n \in E_2}} \tilde{F}_1(e_i)(x_k) \\
 &+ \sum_{e_j \in A_2} \sum_{\substack{y_l = y_n \\ x_k x_m \in E_1}} \tilde{F}_2(e_j)(y_l) + \sum_{e_j \in A_2} \sum_{\substack{x_k x_m \in E_1 \\ y_l y_n \in E_2}} \tilde{K}_2(e_j)(y_l y_n) \\
 &+ \sum_{(e_i, e_j) \in A} \tilde{F}_1(e_i)(x_k) \vee \tilde{F}_2(e_j)(y_l) = [\sum_{e_i \in A_1} \tilde{F}_1(e_i)(x_k)]|e_j|d_{G_2^*}(y_l) \\
 &+ [\sum_{e_j \in A_2} \tilde{F}_2(e_j)(y_l)]|e_i|d_{G_1^*}(x_k) + [d_{\tilde{G}_2}(y_l)]|e_i|d_{G_1^*}(x_k) \\
 &+ [\sum_{e_j \in A_2} \tilde{F}_2(e_j)(y_l)]|e_i| \\
 &= tdeg_{\tilde{G}_1 \vee \tilde{G}_2}(x_k, y_l) = [\sum_{e_i \in A_1} \tilde{F}_1(e_i)(x_k)]|e_j|d_{G_2^*}(y_l) \\
 &+ [td_{\tilde{G}_2}(y_l)]|e_i|d_{G_1^*}(x_k) + [\sum_{e_j \in A_2} \tilde{F}_2(e_j)(y_l)]|e_i|. \quad \square
 \end{aligned}$$

Theorem 3.6. Let $\tilde{G}_1 : (\tilde{F}_1, \tilde{K}_1, A_1)$ and $\tilde{G}_2 : (\tilde{F}_2, \tilde{K}_2, A_2)$ be two fuzzy soft graphs on $G_1^* : (V_1, E_1)$ and $G_2^* : (V_2, E_2)$ respectively such that $\tilde{F}_1(e_i) \geq \tilde{K}_2(e_j)$, $\tilde{F}_2(e_j) \geq \tilde{K}_1(e_i)$, $\tilde{K}_2(e_j) \leq \tilde{K}_1(e_i)$ and $\tilde{F}_2(e_j) \geq \tilde{F}_1(e_i)$ then $tdeg_{\tilde{G}_1 \vee \tilde{G}_2}(x_k, y_l) = |e_j|d_{G_2^*}(y_l) + [td_{\tilde{G}_1}(x_k)] + [\sum_{e_j \in A_2} \tilde{F}_2(e_j)(y_l)]|e_i|[1 + d_{G_1^*}(x_k)]$.

Proof. Proof is analogues to proof of Theorem 3.5. $\square$

Theorem 3.7. Let $\tilde{G}_1 : (\tilde{F}_1, \tilde{K}_1, A_1)$ and $\tilde{G}_2 : (\tilde{F}_2, \tilde{K}_2, A_2)$ be two fuzzy soft graphs on $G_1^* : (V_1, E_1)$ and $G_2^* : (V_2, E_2)$ respectively such that $\tilde{F}_1(e_i) \leq \tilde{K}_2(e_j)$, $\tilde{F}_2(e_j) \geq \tilde{K}_1(e_i)$, $\tilde{K}_1(e_i) \leq \tilde{K}_2(e_j)$ and $\tilde{F}_1(e_i) \leq \tilde{F}_2(e_j)$ then $tdeg_{\tilde{G}_1 \vee \tilde{G}_2}(x_k, y_l) = |e_i|td_{\tilde{G}_2}(y_l)[1 + d_{G_1^*}(x_k)]$.

Proof. From the definition,

$$tdeg_{\tilde{G}_1 \vee \tilde{G}_2}(x_k, y_l) = \sum_{e_j \in A_2} \sum_{\substack{x_k = x_m \\ y_l y_n \in E_2}} \tilde{K}_2(e_j)(y_l y_n)$$

$$\begin{aligned}
& + \sum_{\substack{y_l=y_n \\ x_k x_m \in E_1}} \tilde{F}_2(e_j)(y_l) + \sum_{\substack{x_k x_m \in E_1 \\ y_l y_n \in E_2}} \tilde{K}_2(e_j)(y_l y_n) \\
& + \sum_{(e_i, e_j) \in A} \tilde{F}_1(e_i)(x_k) \vee \tilde{F}_2(e_j)(y_l) = |e_i| d_{\tilde{G}_2}(y_l) \\
& + \left[\sum_{e_j \in A_2} \tilde{F}_2(e_j)(y_l) \right] |e_i| d_{G_1^*}(x_k) + [d_{\tilde{G}_2}(y_l)] |e_i| d_{G_1^*}(x_k) \\
& + \left[\sum_{e_j \in A_2} \tilde{F}_2(e_j)(y_l) \right] |e_i| = |e_i| td_{\tilde{G}_2}(y_l) [1 + d_{G_1^*}(x_k)]. \quad \square
\end{aligned}$$

Theorem 3.8. Let $\tilde{G}_1 : (\tilde{F}_1, \tilde{K}_1, A_1)$ and $\tilde{G}_2 : (\tilde{F}_2, \tilde{K}_2, A_2)$ be two fuzzy soft graphs on $G_1^* : (V_1, E_1)$ and $G_2^* : (V_2, E_2)$ respectively such that $\tilde{F}_2(e_j) \leq \tilde{K}_1(e_i)$, $\tilde{F}_1(e_i) \leq \tilde{K}_1(e_i)$, $\tilde{K}_1(e_i) \leq \tilde{K}_2(e_j)$ and $\tilde{F}_1(e_i) \leq \tilde{F}_2(e_j)$ then $td_{\tilde{G}_1 \vee \tilde{G}_2}(x_k, y_l) = |e_j| td_{\tilde{G}_1}(x_k) [1 + d_{G_2^*}(y_l)]$.

Proof. Proof is similar to the proof of Theorem 3.7. $\square$

4. Conjunction of two fuzzy soft graphs $\tilde{G}_1 \wedge \tilde{G}_2$

Definition 4.1. Let $\tilde{G}_1 : (\tilde{F}_1, \tilde{K}_1, A_1)$ and $\tilde{G}_2 : (\tilde{F}_2, \tilde{K}_2, A_2)$ be two fuzzy soft graphs on G_1^* and G_2^* respectively. The conjunction $\tilde{G}_1 \wedge \tilde{G}_2 : (\tilde{F}_1 \wedge \tilde{F}_2, \tilde{K}_1 \wedge \tilde{K}_2, A_1 \times A_2)$ is defined as follows: $(\tilde{F}_1 \wedge \tilde{F}_2) : A_1 \times A_2 \rightarrow FS(V_1 \times V_2)$ by $(\tilde{F}_1 \wedge \tilde{F}_2)(e_i, e_j)(x_k y_l) = \tilde{F}_1(e_i)(x_k) \wedge \tilde{F}_2(e_j)(y_l) \forall e_i \in A_1, e_j \in A_2, x_k y_l \in V_1 \times V_2$ and $(\tilde{K}_1 \wedge \tilde{K}_2) : A_1 \times A_2 \rightarrow FS(E_1 \times E_2)$ by $(\tilde{K}_1 \wedge \tilde{K}_2)(e_i, e_j)(x_k y_l)(x_m y_n) = \tilde{K}_1(e_i)(x_k x_m) \wedge \tilde{K}_2(e_j)(y_l y_n) \forall (x_k y_l)(x_m y_n) \in E_1 \times E_2$.

Definition 4.2. Let $\tilde{G}_1 : (\tilde{F}_1, \tilde{K}_1, A_1)$ and $\tilde{G}_2 : (\tilde{F}_2, \tilde{K}_2, A_2)$ be two fuzzy soft graphs on $G_1^* : (V_1, E_1)$ and $G_2^* : (V_2, E_2)$ respectively. Let $A = A_1 \times A_2$ then degree of a vertex $(x_k y_l) \in V_1 \times V_2$ in $\tilde{G}_1 \wedge \tilde{G}_2$ is $deg_{\tilde{G}_1 \wedge \tilde{G}_2}(x_k, y_l) = \sum_{(e_i, e_j) \in A} \sum_{\substack{x_k x_m \in E_1 \\ y_l y_n \in E_2}} \tilde{K}(e_i, e_j)(x_k y_l)(x_m y_n) = \sum_{e_i \in A_1} \sum_{e_j \in A_2} \sum_{\substack{x_k x_m \in E_1 \\ y_l y_n \in E_2}} \tilde{K}_1(e_i)(x_k x_m) \wedge \tilde{K}_2(e_j)(y_l y_n)$.

Definition 4.3. Let $\tilde{G}_1 : (\tilde{F}_1, \tilde{K}_1, A_1)$ and $\tilde{G}_2 : (\tilde{F}_2, \tilde{K}_2, A_2)$ be two fuzzy soft graphs on $G_1^* : (V_1, E_1)$ and $G_2^* : (V_2, E_2)$ respectively. Let $A = A_1 \times A_2$ then total degree of a vertex $(x_k y_l) \in V_1 \times V_2$ in $\tilde{G}_1 \wedge \tilde{G}_2$ is

$$\begin{aligned}
td_{\tilde{G}_1 \wedge \tilde{G}_2}(x_k, y_l) & = deg_{\tilde{G}_1 \wedge \tilde{G}_2}(x_k, y_l) + \sum_{(e_i, e_j) \in A} (\tilde{F}_1 \wedge \tilde{F}_2)(e_i, e_j)(x_k y_l) \\
& = \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_k x_m \in E_1 \\ y_l y_n \in E_2}} \tilde{K}_1(e_i)(x_k x_m) \wedge \tilde{K}_2(e_j)(y_l y_n) + \sum_{(e_i, e_j) \in A} \tilde{F}_1(e_i)(x_k) \wedge \tilde{F}_2(e_j)(y_l).
\end{aligned}$$

Example 4.1. Consider two fuzzy soft graphs $\tilde{G}_1 : (\tilde{F}_1, \tilde{K}_1, A_1)$ and $\tilde{G}_2 : (\tilde{F}_2, \tilde{K}_2, A_2)$ on $G_1^* : (V_1, E_1)$ and $G_2^* : (V_2, E_2)$ respectively such that $V_1 = x_1, x_2$,

$E_1 = \{x_1x_2\}$, $V_2 = \{y_1, y_2, y_3\}$, $E_2 = \{y_1y_2, y_1y_3, y_2y_3\}$, $A_1 = \{e_i\}$ where $i = 1, 2$ and $A_2 = \{e_j\}$ where $j = 3, 4$. Let $(\tilde{F}_1, A_1)$, $(\tilde{F}_2, A_2)$, $(\tilde{K}_1, A_1)$ and $(\tilde{K}_2, A_2)$ be represented by the following Table 2.

$\tilde{F}_1$	x_1	x_2	$\tilde{F}_2$	y_1	y_2	y_3
e_1	0.3	0.5	e_3	0.3	0.7	0.5
e_2	0.6	0.8	e_4	0.2	0.4	0.5
$\tilde{K}_1$	x_1x_2		$\tilde{K}_2$	y_1y_2	y_1y_3	y_2y_3
e_1	0.3		e_3	0.3	0.2	0.5
e_2	0.5		e_4	0.2	0.1	0.4

Table 2: Tabular representation of two fuzzy soft graphs

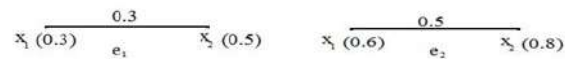


Figure 4: $\tilde{G}_1 : (\tilde{F}_1, \tilde{K}_1, A_1)$

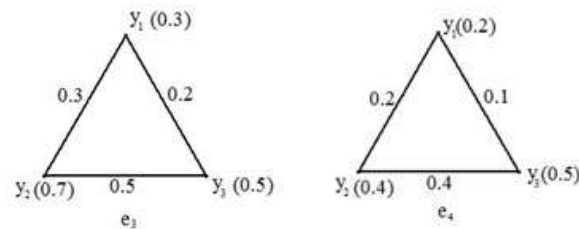


Figure 5: $\tilde{G}_2 : (\tilde{F}_2, \tilde{K}_2, A_2)$

Similarly, we get for (e_2, e_3) and (e_2, e_4) . $\deg_{\tilde{G}_1 \wedge \tilde{G}_2}(x_1, y_1) = 0.5 + 0.3 + 0.3 + 0.5 = 1.6$. $tdeg_{\tilde{G}_1 \wedge \tilde{G}_2}(x_1, y_1) = 1.6 + 0.3 + 0.2 + 0.2 + 0.3 = 2.6$.

Theorem 4.1. Let $\tilde{G}_1 : (\tilde{F}_1, \tilde{K}_1, A_1)$ and $\tilde{G}_2 : (\tilde{F}_2, \tilde{K}_2, A_2)$ be two fuzzy soft graphs on $G_1^* : (V_1, E_1)$ and $G_2^* : (V_2, E_2)$ respectively. If $\tilde{K}_1(e_i) \wedge \tilde{K}_2(e_j)$ is a constant function then $\deg_{\tilde{G}_1 \wedge \tilde{G}_2}(x_k, y_l) = cd_{G_1^*}(x_k)d_{G_2^*}(y_l)|e_i||e_j|$.

Proof. Let $\tilde{K}_1(e_i) \wedge \tilde{K}_2(e_j) = c$.

$$\begin{aligned} \deg_{\tilde{G}_1 \wedge \tilde{G}_2}(x_k, y_l) &= \sum_{(e_i, e_j) \in A} \sum_{\substack{x_k x_m \in E_1 \\ y_l y_n \in E_2}} [\tilde{K}_1(e_i) \wedge \tilde{K}_2(e_j)](x_k y_l)(x_m y_n) = \\ \sum_{(e_i, e_j) \in A} \sum_{\substack{x_k x_m \in E_1 \\ y_l y_n \in E_2}} c &= cd_{G_1^*}(x_k)d_{G_2^*}(y_l)|e_i||e_j|. \quad \square \end{aligned}$$

Theorem 4.2. Let $\tilde{G}_1 : (\tilde{F}_1, \tilde{K}_1, A_1)$ and $\tilde{G}_2 : (\tilde{F}_2, \tilde{K}_2, A_2)$ be two fuzzy soft graphs on $G_1^* : (V_1, E_1)$ and $G_2^* : (V_2, E_2)$ respectively. If $\tilde{K}_1(e_i) \leq \tilde{K}_2(e_j)$ then $\deg_{\tilde{G}_1 \wedge \tilde{G}_2}(x_k, y_l) = [d_{\tilde{G}_1}(x_k)]|e_j|d_{G_2^*}(y_l)$.

$$\begin{aligned}
& + \sum_{(e_i, e_j) \in A} \tilde{F}_1(e_i)(x_k) \wedge \tilde{F}_2(e_j)(y_l), \\
& tdeg_{\tilde{G}_1 \wedge \tilde{G}_2}(x_k, y_l) = \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_k x_m \in E_1 \\ y_l y_n \in E_2}} \tilde{K}_2(e_j)(y_l y_n) \\
& + |e_i| \sum_{e_j \in A_2} \tilde{F}_2(e_j)(y_l) = |e_i| [d_{G_1^*}(x_k) d_{\tilde{G}_2}(y_l) \\
& + \sum_{e_j \in A_2} \tilde{F}_2(e_j)(y_l)]. \quad \square
\end{aligned}$$

Theorem 4.5. Let $\tilde{G}_1 : (\tilde{F}_1, \tilde{K}_1, A_1)$ and $\tilde{G}_2 : (\tilde{F}_2, \tilde{K}_2, A_2)$ be two fuzzy soft graphs on $G_1^* : (V_1, E_1)$ and $G_2^* : (V_2, E_2)$ respectively such that $\tilde{F}_2(e_j) \geq \tilde{F}_1(e_i)$ and $\tilde{K}_2(e_j) \geq \tilde{K}_1(e_i)$ then $tdeg_{\tilde{G}_1 \wedge \tilde{G}_2}(x_k, y_l) = |e_j| [d_{G_2^*}(y_l) d_{\tilde{G}_1}(x_k) + \sum_{e_i \in A_1} \tilde{F}_1(e_i)(x_k)]$.

Proof. The proof is similar to the proof of Theorem 4.4. $\square$

5. Conclusion

In this paper, disjunction and conjunction of two fuzzy soft graphs are defined and illustrated with examples. The degree and total degree of a vertex in the fuzzy soft graph formed by these operations is expressed in terms of the degree and total degree of vertices in the given fuzzy soft graphs in some particular cases. Some properties related to degree and total degree of a vertex in $\tilde{G}_1 \vee \tilde{G}_2$ $\tilde{G}_1 \wedge \tilde{G}_2$ are also analyzed.

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On $K_{\alpha c}$ -spaces

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Abstract. That research is submitted to introduce new type of supra K_c -spaces, it is supra αK (αc)-space, supra L (αc)-space, supra $(\alpha L)c$ -space and supra $\alpha L(\alpha c)$ -space. We showed the relation between these types, also we provided some theorems, propositions and examples about the subjects.

Keywords: supra αK (αc)-space, supra $L(\alpha c)$ -space, supra $(\alpha L)c$ -space, supra $\alpha L(\alpha c)$ -space, supra α -Lindelöf space.

1. Introduction

Wilansky was the first author presented the notion of K_c -spaces [1], after that many authors treated with this space such as in [2]. Mashhour in [3] provided the notion of supra spaces. In [3], [4] the researchers presented the definition of the supra closure and the supra interior to any set in the supra space. In [5] the researcher defined the supra continuous function as (whenever $g^{-1}(V)$ is supra open subset of the supra space (X, μ_X) to any supra open subset V of the supra space (Y, μ_Y) , then $g : (X, \mu_X) \rightarrow (Y, \mu_Y)$ is supra continuous function). The researcher in [6] introduced the definition of supra open function

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as $(g : (X, \mu_X) \rightarrow (Y, \mu_Y))$ is supra open function, whenever for any supra open subset S of X , $g(S)$ is supra open subset of Y). As well as he introduced the definition of the supra homeomorphism functions. In [7] Mukherji and Sarkar presented the concept of Lc-space (when any Lindelöf subset of a space X is closed, thus X is Lc-space). After that [8] the researchers introduced a kind of Lc-space which is stronger than the Kc-space. In this our work we connect between $K(\alpha c)$ -space and the supra space. Also we connect $\alpha K(\alpha c)$ -space and the supra space. By the same way we defined the rest Kind of spaces that we provided in this paper.

2. On $K(\alpha c)$ -spaces

In the next section of the research we will introduce a new types of supra Kc-spaces, we will begin with some definitions that are benefit us in our next topics.

Definition 2.1 ([9]). Consider (X, μ) is a supra space and $S \subseteq X$, when $S \subseteq \text{Int}^\mu(\text{cl}^\mu(\text{int}^\mu(S)))$, so S is supra α -open set. S^c is called supra α -closed.

Definition 2.2 ([9]). A supra α -open cover $\{S_\alpha\}_{\alpha \in \Lambda}$ to (X, μ) is a family of supra α -open subsets of X , where $X \subseteq \bigcup_{\alpha \in \Lambda} S_\alpha$

Definition 2.3 ([9]). Whenever each supra α -open cover for a supra space (X, μ) owns a finite (countable) subcover, then X is supra α -compact (supra α -Lindelöf).

Definition 2.4. When each compact subset of (X, μ) is supra α -closed, so X is supra $K(\alpha c)$ -space.

Example 2.1. The discrete supra space, is supra $K(\alpha c)$ -space.

Remark 2.1. The supra α -compact space is supra α -Lindelöf.

Example 2.2. The supra co-countable space (R, μ_{coc}) is supra α -Lindelöf, while it is not supra α -compact.

Definition 2.5 ([10]). If $X \subseteq \bigcup_{\alpha \in \Lambda} U_\alpha$, where $\{U_\alpha\}_{\alpha \in \Lambda}$ is a family of supra open subsets from (X, μ) , then $\{U_\alpha\}_{\alpha \in \Lambda}$ is a supra open cover for X .

Definition 2.6 ([10]). When each supra open cover to a supra space X owns a finite (or able to be counted) sub cover, then X is supra compact (supra Lindelöf) space.

Remark 2.2 ([10]). Suppose X is Supra α -compact (supra α -Lindelöf) space, so it will be supra compact (supra Lindelöf).

Example 2.3. The discrete supra topology on infinite countable set is a supra Lindelöf space as well as a supra α -Lindelöf space but neither supra compact nor supra α -compact.

Proposition 2.1 ([9]). *Each supra α -closed subset from a supra α -compact space will be supra α -compact set.*

Proposition 2.2 ([9]). *Suppose X is supra α -Lindelöf space, and T is supra α -closed subset of X , hence T is supra α -Lindelöf.*

Definition 2.7. *Consider the supra space (X, μ) and x, y is non-equal points in X , when we can find supra α -open sets V_1 in the supra space X , in which $x \in V_1, y \notin V_1$, so X is supra αT_1 space.*

Example 2.4. The supra co-finite topology on $\mathbb{R}$ is supra αT_1 -space.

Definition 2.8 ([11]). *Suppose (X, μ) is a supra space, x, y are non-equal points in X , we can find are supra α -open sets G_1, G_2 in the supra space X , where $x \in G_1, y \in G_2$ and $G_1 \cap G_2 = \emptyset$ so X is supra αT_2 -space.*

Example 2.5. The discrete supra topology on $\mathbb{R}$ is supra αT_2 -space.

Theorem 2.1. *Consider (X, μ) is a supra αT_2 -space, and T is supra α -compact subset of it, then T is a supra α -closed.*

Proof. Take $d \in T^c$ then for any point $b \in T, d \neq b$. There exist disjoint supra α -open sets G, L in the supra space X containing b, d respectively. We get $H = \{G_b\}_{b \in T}$ is supra α -open cover to T , since T is supra α -compact set, hence H owns a finite sub cover in which $T \subseteq \bigcup_{i=1}^n G_{b_i}$. Suppose $\tilde{L} = \bigcap_{i=1}^n L_{b_i}$ which is supra α -open set having d in it, implies $\tilde{G} = \bigcup_{i=1}^n G_{b_i}$, then $\tilde{G} \cap \tilde{L} = \emptyset$, so that $T \cap \tilde{L} = \emptyset, \tilde{L} \subseteq T^c$, therefore T^c is supra α -open. Hence T will be supra α -closed set. $\square$

Definition 2.9. *Suppose T is a supra α -compact subset of the supra space X , whenever T supra α -closed, hence the space X is called supra αK (αc)-space.*

Example 2.6.

1-The co-countable supra space $(\mathbb{R}, \mu_{coc})$ is supra αK (αc)-space, also it is K_c -space.

2- $(\mathbb{R}, \mu_D)$ is supra αK (αc)-space.

Proposition 2.3. *Consider the supra spade (X, μ) and $T \subseteq X$. T is supra α -closed set iff there is a supra closed set L with $cl^\mu(int^\mu(L)) \subseteq T \subseteq L$.*

Proof. From $cl^\mu(int^\mu(T)) \subseteq T$ ($cl^\mu(T) \subseteq T$ and $\subseteq cl^\mu(T)$), we have $cl^\mu(int^\mu(cl^\mu(T))) \subseteq T \subseteq cl^\mu(T)$, pick $cl^\mu(T) = L$, hence $cl^\mu(int^\mu(L)) \subseteq T \subseteq L$.

Conversely, since $cl^\mu(int^\mu(L)) \subseteq T \subseteq L$, so $cl^\mu(cl^\mu(int^\mu(L))) \subseteq cl^\mu(T) \subseteq cl^\mu(L) = L$, then $cl^\mu(int^\mu(T))(L) \subseteq cl^\mu(T) \subseteq L$, that implies, $int^\mu(cl^\mu(int^\mu(L))) \subseteq int^\mu(cl^\mu(T)) \subseteq int^\mu(L)$, then $cl^\mu(int^\mu(cl^\mu(int^\mu(L)))) \subseteq cl^\mu(int^\mu(cl^\mu(T))) \subseteq cl^\mu(int^\mu(L))$ (because $cl^\mu(int^\mu(L)) \subseteq T \subseteq L$), therefore $cl^\mu(int^\mu(cl^\mu(T))) \subseteq T$, therefore T is supra α -closed. $\square$

Lemma 2.1. *Let T be supra α -closed set in the supra space X which is a domain of a supra homeomorphism function h and the supra space Y its co-domain, then $h(T)$ is a supra α -closed set in Y .*

Proof. Because of T is a supra α -closed subset of X , so $cl^\mu(int^\mu(L)) \subseteq T \subseteq L$, where L is a supra closed subset of X , so $cl^\mu h(int^\mu(L)) = h(cl^\mu(int^\mu(L))) \subseteq h(T) \subseteq h(L)$, but $h(L)$ is supra closed subset of Y , which implies $h(T)$ is supra α -closed. $\square$

Definition 2.10. *If $g(L)$ is supra α -open (or supra α -closed) set in (Y, μ_Y) for each supra open (supra closed) set L in the supra space (X, μ_X) , hence $g : (X, \mu_X) \rightarrow (Y, \mu_Y)$ is supra α -open (supra α -closed) function.*

Definition 2.11. *In Definition 2.10 if L is supra α -open (or supra α -closed) subset of (X, μ_X) and its image is supra open (supra closed) subset of (Y, μ_Y) , then g is supra α^* -open (supra α^* -closed) function.*

Definition 2.12. *In Definition 2.10 whenever L and $g(L)$ are supra α -open (or supra α -closed) subsets of the supra spaces (X, μ_X) and (Y, μ_Y) respectively, hence g is called supra α^{**} -open (supra α^{**} -closed) function.*

Definition 2.13 ([12]). *The function $h : (X, \mu_X) \rightarrow (Y, \mu_Y)$ is supra α -continuous, whenever $h^{-1}(S)$ is supra α -open subset of X for any supra open subset S of Y .*

Definition 2.14. *In Definition 2.13 if S is supra α -open subset and its inverse image is supra open (supra α -open), then g is supra α^* -continuous (supra α^{**} -continuous).*

Definition 2.15. *If any supra compact subset in (X, μ_X) is supra closed set, so X is supra $(\alpha K)c$ -space.*

Example 2.7. (R, μ_D) is supra $(\alpha K)c$ -space.

Theorem 2.2. *Let the supra space (X, μ_X) be supra $(\alpha K)c$ -space and h from (X, μ_X) onto the supra space (Y, μ_Y) is injective, supra α^* -open and supra closed function, hence the space Y is supra $(\alpha K)c$ -space.*

Proof. Consider B is a supra α -compact set in Y . Also $\{H_i, i \in I\}$ be a supra α -open cover for $h^{-1}(B) \implies h^{-1}(B) \subseteq \cup_{i \in I} H_i \implies h(h^{-1}(B)) \subseteq h(\cup_{i \in I} H_i) \implies B \subseteq \cup_{i \in I} h(H_i)$ where all $h(H_i), i \in I$ are supra α -open subsets of Y , so there is a finite collection from the sets $h(H_i)$ in which $B \subseteq \cup_{i=1}^n h(H_i) \implies h^{-1}(B) \subseteq h^{-1}(\cup_{i=1}^n h(H_i)) \implies h^{-1}(B) \subseteq \cup_{i=1}^n h^{-1}(h(H_i)) \implies h^{-1}(B) \subseteq \cup_{i=1}^n H_i \implies h^{-1}(B)$ is supra α -compact subset of X , therefore $h^{-1}(B)$ is a supra α -closed (because X is supra $(\alpha K)c$ -space). So $h(h^{-1}(B)) = B$ is supra closed subset of Y , there upon B is a supra α -closed, so (Y, μ) will be a supra $(\alpha K)c$ -space. $\square$

Corollary 2.1. *Let the supra space (X, μ_X) be supra $(\alpha K)c$ -space and g from (X, μ_X) onto the supra space (Y, μ_Y) is injective, supra open (supra open) and supra α^* -closed (supra α^{**} -closed) function, hence the space Y is supra $(\alpha K)c$ -space.*

Proposition 2.4. *Let the supra space (X, μ_X) be supra $\alpha K(\alpha c)$ -space and h from (X, μ_X) onto the supra space (Y, μ_Y) is injective, supra α -open and supra α^{**} -closed function, hence the space Y is supra $\alpha K(\alpha c)$ -space.*

Proof. Consider $\{H_i \mid i \in I\}$ is a supra α -open covering for the inverse image of $B \implies h^{-1}(B) \subseteq \cup_{i \in I} H_i \implies h(h^{-1}(B)) \subseteq h(\cup_{i \in I} H_i) \implies B \subseteq \cup_{i \in I} h(H_i)$, in which each $h(H_i)$ is supra α -open set in Y , but B is supra α -compact, so there is a finite subfamily of the sets $h(H_i)$ in which $B \subseteq \cup_{i=1}^n h(H_i) \implies h^{-1}(B) \subseteq \cup_{i=1}^n H_i \implies h^{-1}(B)$ is supra α -compact subset from the supra space X , therefore it is a supra α -closed (since X is a supra $\alpha K(\alpha c)$ -space). Hence $h(h^{-1}(B)) = B$ is supra α -closed subset from the supra space Y , so Y is a supra $\alpha K(\alpha c)$ -space. $\square$

Proposition 2.5. *When $h : (X, \mu_X) \rightarrow (Y, \mu_Y)$ is onto supra α^* -continuous function, where X is supra α -compact (or supra α -Lindelöf) space, hence Y is also supra α -compact (supra α -Lindelöf).*

Proof. Consider the supra α -open cover $\{C_i \mid i \in I\}$ to the supra space Y , so $Y \subseteq \cup_{i \in I} C_i \implies h^{-1}(Y) \subseteq h^{-1}(\cup_{i \in I} C_i) \implies X \subseteq \cup_{i \in I} h^{-1}(C_i)$ (since h is onto), where each of $h^{-1}(C_i), i \in I$ is supra α -open subset from the supra space X , hence the collection of $h^{-1}(C_i), i \in I$ is supra α -open cover to the supra space X which is supra α -compact, so there is a finite sub collection from $h^{-1}(C_i) \mid i \in I$ in which $X \subseteq \cup_{i=1}^n h^{-1}(C_i) \implies h(X) \subseteq h(\cup_{i=1}^n h^{-1}(C_i)) \rightarrow Y \subseteq \cup_{i=1}^n h(h^{-1}(C_i)) \implies Y \subseteq \cup_{i=1}^n C_i$, which means Y is supra α -compact space. The other possibility can be proved by the same way. $\square$

Proposition 2.6 ([9]). *A finite supra space, is supra α -compact space.*

Proposition 2.7. *Suppose (X, μ_X) is supra $K(\alpha c)$ -space and $h : (X, \mu_X) \rightarrow (Y, \mu_Y)$ is supra homeomorphism function, hence (Y, μ_Y) is also supra $K(\alpha c)$ -space.*

Proof. Suppose S is a supra compact subset from the supra space (Y, μ_Y) , $h^{-1}(S)$ is supra compact sub of (X, μ_X) , but X is supra $K(\alpha c)$ -space, thus $h^{-1}(S)$ is supra α -closed. Now $h(h^{-1}(S)) = S$ is supra α -closed subset from Y (h is surjective function), and this lead us to the required. $\square$

Definition 2.16. *Whenever any supra Lindelöf subset from the supra space (X, μ_X) is supra α -closed, thus X is supra $L(\alpha c)$ -space.*

Example 2.8. (Z, μ_{ind}) is supra $L(\alpha c)$ -space.

Remark 2.3. Any supra Lc-space is supra $L(\alpha c)$ -space. (Z, μ_{ind}) is an example about the converse,

Remark 2.4. Suppose the supra space X is supra $L(\alpha c)$ -space, hence it will be supra $K(\alpha c)$ -space, because when B is supra compact subset from the space X , it will be supra Lindelöf, Since X is supra $L(\alpha c)$ -space, B will be supra α -closed set, then X is supra $K(\alpha c)$ -space.

Remark 2.5. Suppose X is supra Lindelöf space and Y is a subspace of X , so Y is not necessary supra Lindelöf. But if any sub space of supra Lindelöf (supra α -Lindelöf) is supra Lindelöf (supra α -Lindelöf) sub space we called it a supra hereditary Lindelöf (α -Lindelöf).

Example 2.9. The excluded point supra space (R, μ_{EX}) is supra Lindelöf space but $(R - \{x_0\}, \mu_{EX})$ is not supra Lindelöf.

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A generalized BL-ring

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Abstract. The purpose of this work is to extend the commutative rings whose lattice of ideals can be equipped with a structure of BL-algebra as carried out by Heubo-Kwegna et al in 2018, to non-commutative ones which are called pseudo BL-rings in

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this paper. We study and characterize rings whose ideals form a pseudo BL-algebra, we describe them in terms of their subdirectly irreducible factors. We obtain that every unitary pseudo BL-ring with left and right unit is isomorphic to a subring of a direct product of special primary ring and is also isomorphic to a subring of a direct product of discrete valuation rings.

Keywords: BL-rings, pseudo-BL algebras, pseudo BL-ring, BL-algebras, multiplication ring.

1. Introduction

It is well known that the algebraic study of classical logic proceeds via Boolean algebras [1]. For the infinite valued logic of Łukasiewicz, the algebraic analogues of Boolean algebras are the MV-algebras. It is also well known that Boolean algebras can be subsumed within the theory of rings. From their inception there was the question of whether or not the same is true of MV-algebras. In 2009 Belluce and Di Nola wrote an article on commutative rings whose ideals form an MV-algebra [1], in that work they introduced and gave some properties of a class of commutative rings, whose lattice of ideals forms an MV-algebra, which were called Łukasiewicz rings. An important non-commutative generalization of MV-algebra, known as pseudo MV-algebra was introduced by Georgescu and Iorgulescu [5]. The natural question that arises is what happens if one drops the commutativity assumption on Łukasiewicz rings. The answer of that question has been the goal of Kadji, Lele and Nganou [7], where they studied and characterized all rings whose ideals form a pseudo MV-algebra, which were called generalized Łukasiewicz rings, this was in 2016. Since the class of BL-algebras contains the MV-algebras, from then in 2018, the authors in [8] initiated the study of commutative rings, whose lattice of ideals can be equipped with a structure of BL-algebra. Recall that BL-algebras introduced by Peter Hájek in 1998 as algebraic structures for the basic logic, he used it to formalize many-valued logics (in short MV logics) induced by continuous t -norms on the real unit interval $[0,1]$ and they were generalized to pseudo BL-algebras by Di Nola, Georgescu and Iorgulescu [3].

In the idea to continue the investigation of the classes of rings for which the residuated lattice $A(R)$ of their ideals is an algebra of a well-known subvariety of residuated lattices, in this work we generalize the notion of commutative BL-ring to non commutative case, and we call it *pseudo BL-ring*.

The paper is organized as follows. In Section 2, we introduce and recall some definitions and preliminary results that we will use in the paper. In Section 3, we introduce and study the main properties of pseudo BL-ring, also its connections with some class of non commutative rings. In Section 4, we proved that this class is closed under finite direct products, arbitrary direct sums and homomorphic images. Furthermore, a description of subdirectly irreducible pseudo BL-rings is obtained.

2. General preliminaries

We recall here some definitions and properties that will be used throughout the article.

Definition 2.1 ([6]). A *BL-algebra* is a structure $(A, \vee, \wedge, \odot, 0, 1)$ such that

- (1) $(A, \vee, \wedge, 0, 1)$ is a bounded lattice,
- (2) $(A, \odot, 1)$ is an abelian monoid, that is, $\odot$ is commutative, associative and $x \odot 1 = 1 \odot x = x$,
- (3) the following conditions hold for all $x, y, z \in A$,
 (BL-1) $x \odot y \leq z$ iff $x \leq y \rightarrow z$ (residuation),
 (BL-2) $x \wedge y = x \odot (x \rightarrow y)$ (divisibility),
 (BL-3) $(x \rightarrow y) \vee (y \rightarrow x) = 1$ (prelinearity).

An *MV-algebra* is a BL-algebra A satisfying the double negation law: $x^{**} = x$, for all $x \in A$, where $x^* = x \rightarrow 0$.

Given any BL-algebra A , $MV(A) := \{x^* \mid x \in A\}$ is an MV-algebra, and is indeed the largest BL-subalgebra of A satisfying the double negation. This MV-algebra is called the *MV-center* of A .

We recall that a BL-algebra A is an MV-algebra iff $(x^*)^* = x$, for any $x \in A$.

Definition 2.2 ([5]). A pseudo MV-algebra is a structure $(A, \odot, \oplus, (*), (-), 0, 1)$ of type $(2, 2, 1, 1, 0, 0)$, such that the following axioms are satisfied for all $x, y, z \in A$,

- (1) $x \odot (y \odot z) = (x \odot y) \odot z$,
- (2) $x \odot 1 = 1 \odot x = x$,
- (3) $x \odot 0 = 0 \odot x = 0$,
- (4) $0^* = 1, 0^- = 1$,
- (5) $(x^- \odot y^-)^* = (x^* \odot y^*)^-$,
- (6) $x \odot (x^- \oplus y) = y \odot (y^- \oplus x) = (x \oplus y^*) \odot y = (y \oplus x^*) \odot x$,
- (7) $x \oplus (x^* \odot y) = (x \odot y^*) \oplus y$,
- (8) $(x^*)^- = x$, where $y \oplus x := (x^* \odot y^*)^- = (x^* \odot y^*)^-$.

Definition 2.3 ([3]). A pseudo BL-algebra is a structure $\mathcal{A} = (A, \vee, \wedge, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ of type $(2, 2, 2, 2, 2, 0, 0)$ which satisfies the following axioms, for all $x, y, z \in A$,

- (1) $(A, \vee, \wedge, 0, 1)$ is a bounded lattice,
- (2) $(A, \odot, 1)$ is a monoid, that is, $\odot$ is associative and $x \odot 1 = 1 \odot x = x$,
- (3) $x \odot y \leq z$ iff $x \leq y \rightarrow z$ iff $y \leq x \rightsquigarrow z$,
- (4) $x \wedge y = (x \rightarrow y) \odot x = x \odot (x \rightsquigarrow y)$,
- (5) $(x \rightarrow y) \vee (y \rightarrow x) = (x \rightsquigarrow y) \vee (y \rightsquigarrow x) = 1$.

A ring R is said to be *generated by idempotents*, if for every $x \in R$, there exists an idempotent element $e \in R$ (that is $e^2 = e$) such that $e \cdot x = x \cdot e = x$. There is a residuated lattice formed by the two-sided ideals of the ring R , this lattice is defined by

$$A(R) := \langle Id(R), \wedge, \vee, \otimes, \rightarrow, \rightsquigarrow, \{0\}, R \rangle,$$

where

$$\begin{aligned} I \wedge J &= I \cap J, \quad I \vee J = I + J, \quad I \otimes J = I \cdot J, \\ I \rightarrow J &:= \{x \in R \mid xI \subseteq J\}, \quad I \rightsquigarrow J := \{x \in R \mid Ix \subseteq J\}. \end{aligned}$$

Note that $I^* := \{x \in R \mid xI = \{0\}\}$ denotes the right annihilator of I in R and $I^- := \{x \in R \mid Ix = \{0\}\}$ denotes the left annihilator of I in R , where I is an ideal of R . Given a ring R , recall that an ideal I of R is called an annihilator ideal (resp. a dense ideal) if $I = J^*$ and $I = K^-$ for some ideals J, K of R (resp. $I^* = I^- = \{0\}$).

$AN^*(R) := \{I^* \mid I \in A(R)\}$ and $AN^-(R) := \{I^- \mid I \in A(R)\}$ denote the sets of left and right annihilator ideals of R , respectively.

$D^*(R) := \{I \in A(R) \mid I^* = \{0\}\}$ and $D^-(R) := \{I \in A(R) \mid I^- = \{0\}\}$ denote the sets of left and right dense ideals of R , respectively.

Definition 2.4 ([11]). A ring R is called a multiplication ring, if for every ideals I, J of R such that $I \subseteq J$, there exist ideals K, K' such that $I = J \cdot K = K' \cdot J$.

3. Pseudo BL-rings

Definition 3.1. A ring R generated by idempotents is called a *pseudo BL-ring* if it satisfies,

$$\text{PBLR-1} \quad I \cap J = I \cdot (I \rightsquigarrow J) = (I \rightarrow J) \cdot I,$$

$$\text{PBLR-2} \quad (I \rightarrow J) + (J \rightarrow I) = (I \rightsquigarrow J) + (J \rightsquigarrow I) = R,$$

for all ideals I, J of R .

Lemma 3.1. Let I and J be two ideals of a ring R . Then

- (1) $I \rightarrow J = I \rightarrow (I \cap J)$ and $I \rightsquigarrow J = I \rightsquigarrow (I \cap J)$.
- (2) $(I + J) \rightarrow J = I \rightarrow J$ and $(I + J) \rightsquigarrow I = J \rightsquigarrow I$.

Proof. These are easily derived from the definitions and the operations involved. $\square$

Lemma 3.2. Let I and J be two ideals of a ring R . Then $(I \rightarrow J) \cdot I \subseteq I \cap J$ and $I \cdot (I \rightsquigarrow J) \subseteq I \cap J$.

Proof. Let $x \in (I \rightarrow J) \cdot I$, then $x = \sum_{i=1}^n a_i b_i$ with $a_i \in I \rightarrow J$, $b_i \in I$ and $n \in \mathbb{N}$. By the definition of ideal and the definition of $I \rightarrow J$, we have that $x \in I$ and $x \in J$. Therefore, $x \in I \cap J$ and $(I \rightarrow J) \cdot I \subseteq I \cap J$.

Let $y \in I \cdot (I \rightsquigarrow J)$, then $y = \sum_{i=1}^n a_i b_i$ with $a_i \in I$, $b_i \in I \rightsquigarrow J$ and $n \in \mathbb{N}$. By the definition of ideal and the definition of $I \rightsquigarrow J$, we have that $y \in I$ and $y \in J$. Therefore, $y \in I \cap J$ and $I \cdot (I \rightsquigarrow J) \subseteq I \cap J$. $\square$

The following proposition gives some axioms which are equivalent to PBLR-1 or PBLR-2.

Proposition 3.1. *Let R be a pseudo BL-ring. For all ideals I, J, K of R , we have that,*

- (1) *PBLR-1 iff $I \cap J \subseteq (I \rightarrow J) \cdot I$ and $I \cap J \subseteq I \cdot (I \rightsquigarrow J)$.*
- (2) *PBLR-2 iff $(I \cap J) \rightarrow K = (I \rightarrow K) + (J \rightarrow K)$ iff $(I \cap J) \rightsquigarrow K = (I \rightsquigarrow K) + (J \rightsquigarrow K)$.*
- (3) *PBLR-2 iff $I \rightarrow (J + K) = (I \rightarrow J) + (I \rightarrow K)$ iff $I \rightsquigarrow (J + K) = (I \rightsquigarrow J) + (I \rightsquigarrow K)$.*

Proof. (1) By using Lemma 3.2.

(2) Assume that $(I \cap J) \rightarrow K = (I \rightarrow K) + (J \rightarrow K)$. Let's take $K = I \cap J$, then $(I \cap J) \rightarrow (I \cap J) = (I \rightarrow (I \cap J)) + (J \rightarrow (I \cap J))$. Since $(I \cap J) \rightarrow (I \cap J) = R$, $I \rightarrow (I \cap J) = I \rightarrow J$ and $J \rightarrow (I \cap J) = J \rightarrow I$, by Lemma 3.1, we have $R = (I \rightarrow J) + (J \rightarrow I)$.

Assume now that $R = (I \rightarrow J) + (J \rightarrow I)$.

Let $x \in (I \cap J) \rightarrow K$, then x can be written as $x = x \cdot i + x \cdot j$, with $i \in I$, $j \in J$, and $i + j = 1$ (1 is an idempotent). Thus, $x \cdot j \in I \rightarrow K$ and $x \cdot i \in J \rightarrow K$. Therefore, $(I \cap J) \rightarrow K \subseteq (I \rightarrow K) + (J \rightarrow K)$.

Let $z \in (I \rightarrow K) + (J \rightarrow K)$, then $z = x + y$ with $x \in I \rightarrow K$ and $y \in J \rightarrow K$, that is $xI \subseteq K$ and $yJ \subseteq K$. Thus, $x(I \cap J) \subseteq K$ and $y(I \cap J) \subseteq K$, and we have that $z(I \cap J) = (x + y)(I \cap J) \subseteq K$. Therefore, $(I \rightarrow K) + (J \rightarrow K) \subseteq (I \cap J) \rightarrow K$.

Using the same method with $\rightsquigarrow$ it is straightforward that PBLR-2 holds iff $(I \cap J) \rightsquigarrow K = (I \rightsquigarrow K) + (J \rightsquigarrow K)$.

(3) Assume that PBLR-2 is true, then for ideals J and K of R , we have $J \rightarrow K + K \rightarrow J = R$. Let $y \in I \rightarrow (J + K)$, then $y \in R$. This means that y can be written as $y = y \cdot j + y \cdot k$, with $j \in J$ and $k \in K$, such that $j + k = 1$ (1 is an idempotent). Since $y \cdot j \in I \rightarrow J$ and $y \cdot k \in I \rightarrow K$, then $I \rightarrow (J + K) \subseteq (I \rightarrow J) + (I \rightarrow K)$.

Let $x \in (I \rightarrow J) + (I \rightarrow K)$, then $x = a + b$ with $a \in I \rightarrow J$ and $b \in I \rightarrow K$. Thus, $aI \subseteq J$ and $bI \subseteq K$, so $aI + bI \subseteq J + K$. Therefore, $xI \subseteq J + K$, and $I \rightarrow (J + K) \supseteq (I \rightarrow J) + (I \rightarrow K)$.

Conversely, assume now that $I \rightarrow (J + K) = (I \rightarrow J) + (I \rightarrow K)$. Set $I = J + K$, then by Lemma 3.1, we have $J \rightarrow K + K \rightarrow J = R$ for all ideals J, K of R . Thus, PBLR-2 holds.

Using the same method with " $\rightsquigarrow$ " it is straightforward that PBLR-2 holds iff $I \rightsquigarrow (J + K) = (I \rightsquigarrow J) + (I \rightsquigarrow K)$. $\square$

Remark 3.1. Let R be a commutative ring. If $I \rightarrow J = I \rightsquigarrow J$, then R is a BL-ring.

By Definitions 3.1 and 2.4, we have the following results that give a characterization of the pseudo BL-rings.

Proposition 3.2. *Let R be a ring. R satisfies PBLR-1 if and only if it is a multiplication ring.*

Proof. Let R be a ring. Assume that R satisfies PBLR-1, then $I \cap J = I \cdot (I \rightsquigarrow J) = (I \rightarrow J) \cdot I$. Let I and J be two ideals of R such that $I \subseteq J$. We can find two ideals K and K' of R , such that $I = J \cdot K = K' \cdot J$.

By PBLR-1, $I = I \cap J = J \cdot (J \rightsquigarrow I) = (J \rightarrow I) \cdot J$. Then $K = J \rightsquigarrow I$ and $K' = J \rightarrow I$.

Conversely, assume that R is a multiplication ring and let I, J be ideals of R . Since $I \cap J \subseteq J$, there exist two ideals K and K' of R such that $I \cap J = J \cdot K = K' \cdot J$. Hence $J \cdot K \subseteq I$, $K' \cdot J \subseteq I$ and it follows that $K \subseteq J \rightsquigarrow I$ and $K' \subseteq J \rightarrow I$.

Thus, $I \cap J \subseteq J \cdot (J \rightsquigarrow I)$ and $I \cap J \subseteq (J \rightarrow I) \cdot J$. Therefore, PBLR-1 holds. $\square$

Proposition 3.3. *Let R be a ring. Then the followings are equivalent,*

- (1) R is a pseudo BL-ring.
- (2) $A(R)$ is a pseudo BL-algebra.

Proof. Assume that R is a pseudo BL-ring. Then PBLR-1 and PBLR-2 hold.

By the definition of $A(R)$, $(A(R); \wedge, \vee, \{0\}, R)$ is a bounded lattice and $(A(R); \otimes, R)$ is an associative monoid.

Let I, J and K be ideals of R , then $I \otimes J \subseteq K \Leftrightarrow I \subseteq J \rightarrow K \Leftrightarrow J \subseteq I \rightsquigarrow K$. Since PBLR-1 and PBLR-2 hold, then prelinearity and divisibility hold. Thus, $A(R)$ is a pseudo BL-algebra.

Conversely assume that $A(R)$ is a pseudo BL-algebra.

Since for all x, y in R , $x \wedge y = (x \rightarrow y) \cdot x = x \cdot (x \rightsquigarrow y)$ and $(x \rightarrow y) \vee (y \rightarrow x) = (x \rightsquigarrow y) \vee (y \rightsquigarrow x) = 1$, then PBLR-1 and PBLR-2 hold. Therefore, by Proposition 3.2, R is a multiplication ring, so generated by idempotents. Thus, R is a pseudo BL-ring. $\square$

We know that if R is a ring and $M_n(R)$ the ring of $n \times n$ matrices over R , then any ideal $\mathcal{I}$ of $M_n(R)$ has the form $M_n(I)$ for a uniquely determined ideal I of R [9].

Lemma 3.3. *If R is a multiplication ring, then $M_n(R)$ is also a multiplication ring.*

Proof. Let $M_n(I)$ and $M_n(J)$ be two ideals of $M_n(R)$ such that $M_n(I) \subseteq M_n(J)$, with I, J two ideals of R .

It is easy to verify that $M_n(I) \subseteq M_n(J)$ if and only if $I \subseteq J$. Since R is a multiplicative ring, then there exist two ideals K and K' of R , such that $I = J \cdot K = K' \cdot J$ which allows that $M_n(I) = M_n(J) \cdot M_n(K) = M_n(K') \cdot M_n(J)$. Thus, $M_n(R)$ is a multiplicative ring. $\square$

Lemma 3.4. *If R is a multiplication ring, then $M_n(R)$ satisfies PBLR-1.*

Proof. Let R be a multiplication ring, then by using Lemma 3.3 and Proposition 3.2, we have that PBLR-1 holds for $M_n(R)$. $\square$

Lemma 3.5. *Let I and J be two ideals of a ring R , then*

- (1) $M_n(I + J) = M_n(I) + M_n(J)$.
- (2) $M_n(I \rightarrow J) = M_n(I) \rightarrow M_n(J)$ and $M_n(I \rightsquigarrow J) = M_n(I) \rightsquigarrow M_n(J)$.

Proof. Straightforward, just use the definition of the operations involved. $\square$

Example 3.1. (1) Let R be a Noetherian multiplication ring, then $M_n(R)$ is a pseudo BL-ring.

(2) Let R be a discrete valuation ring or a Łukasiewicz ring, then $M_n(R)$ is a pseudo BL-ring.

Now given a ring R and a proper ideal I of R , let's consider the ring R/I . An ideal of R/I is a set $J/I = \{x/I \mid x \in J\}$, where J is an ideal of R and $I \subseteq J$. We also define the left annihilator (respectively the right annihilator) of an ideal I by $I^* = \{x \in R \mid xI = \{0\}\} = \{x \rightarrow 0 \mid x \in I\}$ (respectively $I^- = \{x \in R \mid Ix = \{0\}\} = \{x \rightsquigarrow 0 \mid x \in I\}$).

Lemma 3.6. *Let I , J and K be ideals of a ring R , such that $I \subseteq J$ and $I \subseteq K$. Then,*

- (1) $(J/I)^* = (J \rightarrow I)/I$ and $(J/I)^- = (J \rightsquigarrow I)/I$.
- (2) $(J/I) \rightarrow (K/I) = (J \rightarrow K)/I$ and $(J/I) \rightsquigarrow (K/I) = (J \rightsquigarrow K)/I$.

Proof. (1) Let $x/I \in (J/I)^*$, then for all a/I with $a \in J$, $xa \in I$. This means that $xJ \subseteq I$. Thus, $x/I \in (J \rightarrow I)/I$. Therefore, $(J/I)^* \subseteq (J \rightarrow I)/I$.

Conversely, let $y/I \in (J \rightarrow I)/I$, then for any $a/I \in J/I$, $ya \in I$ because, $yJ \subseteq I$ by the definition of $y/I \in (J \rightarrow I)/I$. This means that $y/I(J/I) = I$. Therefore, $(J/I)^* \supseteq (J \rightarrow I)/I$.

To show that $(J/I)^- = (J \rightsquigarrow I)/I$ holds, we just have to use the same idea as in $(J/I)^* = (J \rightarrow I)/I$.

(2) Let $y/I \in (J \rightarrow K)/I$, then $yJ \subseteq K$. This implies that $(xJ)/I \subseteq K/I$ which is $(y/I) \cdot (J/I) \subseteq K/I$. Thus, $(J \rightarrow K)/I \subseteq (J/I) \rightarrow (K/I)$.

Conversely, let $x/I \in (J/I) \rightarrow (K/I)$, then for all $a \in J$ $xa \in K$, which implies that $x/I \in (J \rightarrow K)/I$. Therefore, $(J/I) \rightarrow (K/I) \subseteq (J \rightarrow K)/I$.

With the same idea as for $(J/I) \rightarrow (K/I) = (J \rightarrow K)/I$, we can show that $(J/I) \rightsquigarrow (K/I) = (J \rightsquigarrow K)/I$. $\square$

It is possible to observe if the property PBLR-2 holds in a ring R by observing its quotient by a proper ideal. This is the main purpose of the following propositions.

Proposition 3.4. *Let I and J be two ideals of a ring R . If the ring R satisfies PBLR-2, then $I \cap J = \{0\}$ implies that $I^* + J^* = R$ and $I^- + J^- = R$.*

Proof. Assume that $I \cap J = \{0\}$.

From PBLR-2 we have that $I \rightarrow J + J \rightarrow I = R$ and $I \rightsquigarrow J + J \rightsquigarrow I = R$. Then $I^* + J^* = R$ and $I^- + J^- = R$ by using Lemma 3.1, definitions and operations involved. $\square$

We denote PBLR-3: $I \cap J = \{0\}$ implies that $I^* + J^* = R$ and $I^- + J^- = R$.

Proposition 3.5. *A ring R satisfies PBLR-2 if and only if any quotient of it by an ideal satisfies PBLR-3.*

Proof. Let I be an ideal of the ring R which satisfies PBLR-2. Let J, K be two ideals of R such that $I \subseteq J$, $J \subseteq K$ and $(J/I) \cap (K/I) = I$. It is clear that $J \cap K = I$. Since $(J/I)^- + (K/I)^- = (J \rightarrow I)/I + (K \rightarrow I)/I = (J \rightarrow (J \cap K))/I + (K \rightarrow (J \cap K))/I = ((J \rightarrow K) + (K \rightarrow J))/I = R/I$ (by using Lemmas 3.1, 3.6). In the same way, $(J/I)^* + (K/I)^* = R/I$. Thus, PBLR-3 holds for R/I .

Conversely, suppose that in every factor of R , PBLR-3 holds. Let I, J be ideals of R , then $R/(I \cap J)$ satisfies PBLR-3. Clearly $(I/(I \cap J)) \cap (J/(I \cap J)) = (I \cap J)$, then $(I/(I \cap J))^- + (J/(I \cap J))^- = R/(I \cap J)$ and $(I/(I \cap J))^* + (J/(I \cap J))^* = R/(I \cap J)$ (by using Lemma 3.6). That is $(I \rightsquigarrow J)/(I \cap J) + (J \rightsquigarrow I)/(I \cap J) = (I \rightarrow J)/(I \cap J) + (J \rightarrow I)/(I \cap J) = R/(I \cap J)$. Thus, $((I \rightarrow J) + (J \rightarrow I))/(I \cap J) = ((I \rightsquigarrow J) + (J \rightsquigarrow I))/(I \cap J) = R/(I \cap J)$ that implies $(I \rightarrow J) + (J \rightarrow I) = R$ and $(I \rightsquigarrow J) + (J \rightsquigarrow I) = R$. Finally R satisfies PBLR-2, which concludes the proof. $\square$

Proposition 3.6. *Every quotient of a multiplication ring is a multiplication ring.*

Proof. Let R be a multiplication ring and K an ideal of R . Let I/K and J/K be two ideals of R/K such that $I/K \subseteq J/K$, then $I \subseteq J$. Hence, there exist ideals T and T' of R (because R is a multiplication ring) such that $I = J \cdot T = T' \cdot J$. Thus, $I/K = (J \cdot T)/K = (T' \cdot J)/K$ and $I/K = J/K \cdot T/K = T'/K \cdot J/K$. Therefore, R/K is a multiplication ring. $\square$

Proposition 3.7. *Let R be a multiplication ring such that its quotient ring satisfies PBLR-3, then R satisfies PBLR-3.*

Proof. Let R be a multiplication ring, then its quotient ring is a multiplication ring (by Proposition 3.6). Since the quotient of R satisfies PBLR-3, then the ring R satisfies PBLR-2 (by Proposition 3.5). We use Proposition 3.4 to conclude that PBLR-3 holds for the ring R . $\square$

Corollary 3.1. *A multiplication ring satisfies PBLR-2 if and only if it satisfies PBLR-3.*

Corollary 3.2. *Any quotient of a pseudo BL-ring by a proper ideal is also a pseudo BL-ring.*

Definition 3.2 ([9]). A ring R is called a prime ring if its null ideal is a prime ring.

In the following, we sometimes use 0 to denote the null ideal of the ring R , which can be easily recognized from the context.

Proposition 3.8. *Prime ideals of pseudo BL-rings are maximal.*

Proof. Let I, J be two ideals of a pseudo BL-ring R , such that I is a prime ideal and $I \subseteq J \subseteq R$ (with $I \neq J$). Since I is prime, it is known that R/I is a prime ring, so $(J/I)^* = (J/I)^- = 0/I$.

Since R/I is a pseudo BL-ring (by Corollary 3.2), $(J/I) = ((J/I)^*)^* = ((J/I)^-)^- = (0/I)^* = (0/I)^- = R/P$. Thus, $J = R$ and I is a maximal ideal of R . $\square$

Proposition 3.9. *Let R be a pseudo BL-ring and I an ideal of R such that $I \cap I^* = \{0\}$ and $I \cap I^- = \{0\}$. Then I is a pseudo BL-ring.*

Proof. It is well known that an ideal is a subring, thus $Id(I) \subseteq Id(R)$ because

$$\begin{cases} I \cap I^* = \{0\}, \\ I \cap I^- = \{0\}, \end{cases}$$

imply that,

$$\begin{cases} I + I^* = I \vee I^* = R, \\ I + I^- = I \vee I^- = R, \end{cases}$$

Since R is generated by idempotents, for any $x \in I$, there are right and left idempotents $e, e' \in R$, $m_1, m_3 \in I$ and $m_2 \in I^*$, $m_4 \in I^-$ such that $ex = x, e'x = x$ and

$$\begin{cases} m_1 + m_2 = e, \\ m_3 + m_4 = e', \end{cases}$$

Then $m_1e + m_2e = e^2 = e = m_1 + m_2$ implies $m_1 - m_1e = m_2e - m_2 \in I \cap I^* = \{0\}$ and $e'm_3 + e'm_4 = (e')^2 = e' = m_3 + m_4$ implies $m_3 - e'm_3 = e'm_4 - m_4 \in I \cap I^- = \{0\}$.

It follows that

$$\begin{cases} m_1 = m_1e, & m_2e = m_2 \\ e'm_3 = m_3, & e'm_4 = m_4 \end{cases}$$

$x = m_1x, x = xm_3, (m_1)^2 = m_1, (m_3)^2 = m_3$ since $m_2 \in I^*, m_4 \in I^-$. Thus, m_1 and m_3 are idempotents in I , and I is generated by idempotents.

Since $Id(M) \subseteq Id(R)$, then by Proposition 3.7, PBLR-2 holds for I . Hence I is a pseudo BL-ring. $\square$

Proposition 3.10. *Let R be a pseudo BL-rings. R is closed under each of the following operations,*

- (1) *finite direct products,*
- (2) *arbitrary direct sums,*
- (3) *homomorphic images.*

Proof. (1) Let $R = \prod_{k=1}^n R_k$ where each R_k is a pseudo BL-ring. Since each R_k is generated by idempotents, one gets that any ideal I of R is of the form

$$I = \prod_{k=1}^n I_k,$$

where I_k is an ideal of R_k .

Let $I = \prod_{k=1}^n I_k$ and $J = \prod_{k=1}^n J_k$ to be two ideals of R , then we get,

$$I \cdot J = \prod_{k=1}^n I_k \cdot \prod_{k=1}^n J_k = \prod_{k=1}^n I_k \cdot J_k.$$

Analogously, we also have $I \rightarrow J = \prod_{k=1}^n I_k \rightarrow J_k$ and $I \rightsquigarrow J = \prod_{k=1}^n I_k \rightsquigarrow J_k$. Then

$$\begin{aligned} I \cap J &= \prod_{k=1}^n I_k \cap \prod_{k=1}^n J_k = \prod_{k=1}^n (I_k \cap J_k) = \prod_{k=1}^n (I_k \cdot (I_k \rightsquigarrow J_k)) \\ &= \prod_{k=1}^n I_k \cdot \prod_{k=1}^n (I_k \rightsquigarrow J_k) = I \cdot (I \rightsquigarrow J) \end{aligned}$$

and

$$\prod_{k=1}^n (I_k \cap J_k) = \prod_{k=1}^n (I_k \rightarrow J_k) \cdot I_k = \prod_{k=1}^n (I_k \rightarrow J_k) \cdot \prod_{k=1}^n I_k = (I \rightarrow J) \cdot I.$$

Thus, R satisfies PBLR-1, and because

$$I + J = \prod_{k=1}^n I_k + \prod_{k=1}^n J_k = \prod_{k=1}^n (I_k + J_k),$$

we have,

$$\begin{aligned} (I \rightarrow J) + (J \rightarrow I) &= \prod_{k=1}^n (I_k \rightarrow J_k) + \prod_{k=1}^n (J_k \rightarrow I_k) \\ &= \prod_{k=1}^n ((I_k \rightarrow J_k) + (J_k \rightarrow I_k)) = \prod_{k=1}^n R_k = R, \end{aligned}$$

and

$$\begin{aligned} (I \rightsquigarrow J) + (J \rightsquigarrow I) &= \prod_{k=1}^n (I_k \rightsquigarrow J_k) + \prod_{k=1}^n (J_k \rightsquigarrow I_k) \\ &= \prod_{k=1}^n ((I_k \rightsquigarrow J_k) + (J_k \rightsquigarrow I_k)) = \prod_{k=1}^n R_k = R. \end{aligned}$$

Thus, R satisfies PBLR-2. Therefore, R is a pseudo BL-ring.

(2) Let $R = \bigoplus_{k=1}^n R_k$, where each R_k is a pseudo BL-ring. Since each R_k is generated by idempotents, one gets that any ideal I of R is of the form,

$$I = \bigoplus_{k=1}^n I_k,$$

where I_k is an ideal of R_k .

Let $I = \bigoplus_{k=1}^n I_k$ and $J = \bigoplus_{k=1}^n J_k$ to be two ideals of R , we get,

$$I \cdot J = \bigoplus_{k=1}^n I_k \cdot \bigoplus_{k=1}^n J_k = \bigoplus_{k=1}^n I_k \cdot J_k;$$

analogously, we also have $I \rightarrow J = \bigoplus_{k=1}^n I_k \rightarrow J_k$ and $I \rightsquigarrow J = \bigoplus_{k=1}^n I_k \rightsquigarrow J_k$. Then

$$\begin{aligned} I \cap J &= \bigoplus_{k=1}^n I_k \cap \bigoplus_{k=1}^n J_k = \bigoplus_{k=1}^n (I_k \cap J_k) = \bigoplus_{k=1}^n (I_k \cdot (I_k \rightsquigarrow J_k)) \\ &= \bigoplus_{k=1}^n I_k \cdot \bigoplus_{k=1}^n (I_k \rightsquigarrow J_k) = I \cdot (I \rightsquigarrow J) \end{aligned}$$

and

$$\bigoplus_{k=1}^n (I_k \cap J_k) = \bigoplus_{k=1}^n (I_k \rightarrow J_k) \cdot I_k = \bigoplus_{k=1}^n (I_k \rightarrow J_k) \cdot \bigoplus_{k=1}^n I_k = (I \rightarrow J) \cdot I.$$

Thus, R satisfies PBLR-1, and because $I + J = \bigoplus_{k=1}^n I_k + \bigoplus_{k=1}^n J_k = \bigoplus_{k=1}^n (I_k + J_k)$, we have,

$$\begin{aligned} (I \rightarrow J) + (J \rightarrow I) &= \bigoplus_{k=1}^n (I_k \rightarrow J_k) + \bigoplus_{k=1}^n (J_k \rightarrow I_k) \\ &= \bigoplus_{k=1}^n ((I_k \rightarrow J_k) + (J_k \rightarrow I_k)) = \bigoplus_{k=1}^n R_k = R \end{aligned}$$

and

$$\begin{aligned} (I \rightsquigarrow J) + (J \rightsquigarrow I) &= \bigoplus_{k=1}^n (I_k \rightsquigarrow J_k) + \bigoplus_{k=1}^n (J_k \rightsquigarrow I_k) \\ &= \bigoplus_{k=1}^n ((I_k \rightsquigarrow J_k) + (J_k \rightsquigarrow I_k)) = \bigoplus_{k=1}^n R_k = R. \end{aligned}$$

Thus, R satisfies PBLR-2. Therefore, R is a pseudo BL-ring.

(3) Let R be a PBL-ring and I be an ideal of R . We recall that the ideals of R/I are of the form J/I , where J is an ideal of R with $I \subseteq J$.

Let J and K be two ideals of R such that $I \subseteq J, K$. Then $(J/I) \cap (K/I) = (J \cap K)/I = ((J \rightarrow K) \cdot J)/I = (J \rightarrow K)/I \cdot (J/I) = ((J/I) \rightarrow (K/I)) \cdot (J/I)$; furthermore, $(J \cap K)/I = J \cdot (J \rightsquigarrow K)/I = (J/I) \cdot (J \rightsquigarrow K)/I = (J/I) \cdot (J/I \rightsquigarrow K/I)$, thus R/I satisfies PBLR-1.

We also have $(J/I \rightarrow K/I) + (K/I \rightarrow J/I) = (J + K)/I \rightarrow (K + J)/I = ((J \rightarrow K) + (K \rightarrow J))/I = R/I$ and $(J/I \rightsquigarrow K/I) + (K/I \rightsquigarrow J/I) = (J + K)/I \rightsquigarrow (K + J)/I = ((J \rightsquigarrow K) + (K \rightsquigarrow J))/I = R/I$, thus R/I satisfies PBLR-2. Therefore, R/I is a pseudo BL-ring. $\square$

We are going to establish the connection between pseudo BL-ring and some well-known rings.

Definition 3.3. A unitary ring is called a Von Neumann ring if R/P is a division ring for all prime ideals of R .

Definition 3.4. A non commutative Bear ring is a ring in which for every ideal I of R , there exist idempotents $e, e' \in R$ such that $I^* = eR$ and $I^- = Re'$.

An ideal in a Bear ring (commutative or not) is called a *Bear-ideal* if for every $a, b \in R$ such that $a - b \in I$, then $a^* - b^* \in I$ and $a^- - b^- \in I$.

Definition 3.5 ([8]). A reducing ring is a ring in which 0 is the only nilpotent element, that is, the only element $x \in R$ for which there exists an integer $n \geq 1$ such that $x^n = 0$.

Lemma 3.7. Let R be a ring and I, J be ideals of R . If $I \cap J = \{0\}$, then $I \subseteq J^*$, $I \subseteq J^-$.

Proof. This is easily derived from the definition of ideal and the operations $(*)$ and $(-)$ involved. $\square$

Proposition 3.11. In a reduced ring with identities, PBLR-3 holds if and only if this ring is a Bear ring.

Proof. Let R be a reduced ring with identities in which PBLR-3 holds. Let K be an ideal of R , then $K \cap K^* = \{0\}$ and $K \cap K^- = \{0\}$ implies $K^* + (K^*)^* = R$ and $K^- + (K^-)^- = R$. Thus, there are $k_1 \in K^*$, $k'_1 \in (K^*)^*$, $k_2 \in K^-$, and $k'_2 \in (K^-)^-$ such that $1_L = k_1 + k'_1$ and $1_R = k_2 + k'_2$. So,

$$\begin{cases} k_1 = k_1 \cdot 1_L = k_1(k_1 + k'_1) = k_1^2, \\ k_2 = 1 \cdot k_2 = (k_2 + k'_2)k_2 = k_2^2, \end{cases}$$

from which we get that k_1 and k_2 are idempotents.

It remains to show that $K^* = k_1R$ and $K^- = Rk_2$. Let $x \in K^*$, then $x = 1_L \cdot x = (k_1 + k'_1)x = k_1x \in k_1R$.

Let $y \in k_1R$, then there is an $r \in R$ such that $y = k_1r \in K^*$ because $k_1 \in K^*$. Therefore, $K^* = k_1R$.

By the similar way, $K^- = Rk_2$. Then, R is a non commutative Bear ring.

Conversely, assume that I and J are two ideals of a Bear ring such that $I \cap J = \{0\}$. We have to show that $I^* + J^* = R = I^- + J^-$.

Since R is a Bear ring, there exist two idempotents e and e' in R , such that $J^* = eR$ and $J^- = Re'$. Thus, $I \subseteq eR$ and $I \subseteq Re'$ (by Lemma 3.7). Hence for all $i \in I$, $i = er = r'e'$ for some $r, r' \in R$ and

$$\begin{cases} 0 = er - er = er - e^2r = (1_L - e)er, \\ 0 = r'e' - e'r' = r'e' - (e')^2r' = r'e'(1_R - e'), \end{cases}$$

which implies $1_L - e \in I^*$ and $1_R - e' \in I^-$. Therefore,

$$\begin{cases} 1_L = (1_L - e) + e \in I^* + (I^*)^*, \\ 1_R = (1_R - e') + e' \in I^- + (I^-)^-, \end{cases}$$

Since $(I^*)^*$ and $I^- + (I^-)^-$, then $I^* + J^* = R$ and $I^- + J^- = R$. $\square$

Proposition 3.12. *For any Bear-ideals I, J of a Bear ring R , we have that, $(I \rightarrow J) + (J \rightarrow I) = R$ and $(I \rightsquigarrow J) + (J \rightsquigarrow I) = R$.*

Proof. Straightforward by using Propositions 3.11 and 3.5. $\square$

The same result as in Proposition 3.6 holds for Bear rings and Bear-ideals.

Proposition 3.13. *Every quotient of a Bear ring is a multiplication Bear ring.*

Lemma 3.8. *Every Von Neumann is a multiplication ring.*

Proof. Let I and J be two ideals of a non commutative Von Neumann ring R and P a prime ideal of R . Since all multiplication ring have the PBLR-1 property and vice-versa (by Proposition 3.2), then by Definition 3.3 of a non commutative Von Neumann ring, we only have to show that $I/P \cap J/P = (I/P) \cdot ((I \rightsquigarrow J)/P)$ and $I/P \cap J/P = ((I \rightarrow J)/P) \cdot (I/P)$. Since in the division ring R/P the ideals I/P and J/P are trivials, we just have to check four cases.

Case 1. If $I/P = J/P = \{0/P\}$, then $I/P \cap J/P = \{0/P\}$ and $(I/P) \cdot ((I \rightsquigarrow J)/P) = ((I \rightarrow J)/P) \cdot (I/P) = \{0/P\}$. Hence the property PBLR-1 holds.

Case 2. If $I/P = \{0/P\}$ and $J/P = R/P$, then $I = \{0\}$, which implies $I/P \cap J/P = \{0/P\}$ and $(I/P) \cdot ((I \rightsquigarrow J)/P) = ((I \rightarrow J)/P) \cdot (I/P) = \{0/P\}$. Hence the property PBLR-1 holds.

Case 3. If $I/P = R/P$ and $J/P = \{0/P\}$, using the definitions of $(I \rightsquigarrow J)/P$ and $(I \rightarrow J)/P$, we easily prove that $(I/P) \cdot ((I \rightsquigarrow J)/P) = ((I \rightarrow J)/P) \cdot (I/P) = \{0/P\}$. Hence the property PBLR-1 holds.

Case 4. If $I/P = R/P$ and $J/P = R/P$, using the definitions of $(I \rightsquigarrow J)/P$ and $(I \rightarrow J)/P$, we easily prove that $(I/P) \cdot ((I \rightsquigarrow J)/P) = ((I \rightarrow J)/P) \cdot (I/P) = R/P$. Hence the property PBLR-1 holds.

We complete the proof. $\square$

Proposition 3.14. *Let R be a unitary Von Neumann ring. Then R is a pseudo BL-ring if and only if it satisfies $I/P = J/P = \{0\}$ implies $(I \rightarrow J)/P = R/P$, for all ideals I, J of R and all prime ideals P of R .*

Proof. This result follows directly from Lemma 3.8. $\square$

4. Representation of pseudo-BL algebras

The purpose of this section is to find a representation of pseudo BL-ring in the sense of subdirectly irreducible algebras. We start this section with the definitions of subdirectly irreducible ring and special primary ring.

Definition 4.1. A ring R is said to be subdirectly irreducible if every subdirect product of R is trivial.

Equivalently, a ring R is subdirectly irreducible if and only if the intersection of all non-zero ideals of R is non-zero.

Definition 4.2. A ring R is a special primary ring if R has a unique maximal ideal M and if each proper ideal of R is a power of M .

If P is a prime ideal of ring R and if $S = R \setminus P$, we shall denote 0_S^* (respectively 0_S^-), the S -component of the right zero ideal, by $N^*(P)$ (respectively $N^-(P)$), that is, $N^*(P) = \{x \in R \mid xs = 0 \text{ for some } s \in R \setminus P\}$ (respectively $N^-(P) = \{x \in R \mid sx = 0 \text{ for some } s \in R \setminus P\}$).

Lemma 4.1. *Let R be a ring that is generated by idempotents and P be a prime ideal of R . Then,*

$$\bigcap_P N^*(P) = \{0\} \text{ and } \bigcap_P N^-(P) = \{0\}.$$

Proof. To show that $\bigcap_P N^*(P) = \{0\}$ and $\bigcap_P N^-(P) = \{0\}$, it is sufficient to prove that if $x \neq 0$, then there exists a prime ideal P such that $x \notin N^*(P)$ and $x \notin N^-(P) = 0$.

Let $x \neq 0$, then $(xR)^* \neq R$ and $(Rx)^- \neq R$ (because if we assume that they are all equal, then for all $z \in R$, $z \cdot x = 0$ and $x \cdot z = 0$ with $x \neq 0$, which means that for all $z \in R$, $z = 0$. So $R = \{0\}$, a contradiction). Thus, by Proposition 2.10, [8] there exists a prime ideal Q of R such that $(xR)^* \subseteq Q$ and $(Rx)^- \subseteq Q$.

Assume that $x \in N^*(Q)$ and $x \in N^-(Q)$, then $sx = 0$ and $xs = 0$ for some $s \notin Q$.

If $s \cdot x = 0$, then $(s \cdot x)R = \{0\}$, thus $s \in (xR)^* \subseteq Q$. Therefore, $x \notin N^*(Q)$.

If $x \cdot s = 0$, then $R(x \cdot s) = \{0\}$, thus $s \in (xR)^- \subseteq Q$. Therefore, $x \notin N^-(Q)$.

Hence, $\bigcap_P N^*(P) = \bigcap_P N^-(P) = \{0\}$. $\square$

Proposition 4.1. *Every unitary pseudo BL-ring with left and right unit is isomorphic to a subring of a direct product of special primary rings.*

Proof. Let R be a pseudo BL-ring with left and right unit, and P a prime ideal of R . Then by Proposition 3.2 we know that R is a multiplicative ring.

Let's consider the mappings $f : R \rightarrow \prod_P R/P$ such that $f(x) = (\frac{x}{1_R})/P$ a sequence on the prime ideals of R and $g : R \rightarrow \prod_P R/P$ such that $g(x) = (\frac{x}{1_L})/P$ a sequence on the prime ideals of R . We easily check that f and g are ring homomorphisms. Using the previous Lemma 4.1, the kernels of f and g are defined to be $\ker(f) = \bigcap_P N^*(P) = \{0\}$ and $\ker(g) = \bigcap_P N^-(P) = \{0\}$. Hence R is isomorphic to $f(R)$ and $g(R)$ two subrings of $\prod_P R/P$. Since R is a multiplicative ring, then by Theorem 9.23 and Propositions 9.25 and 9.26 in [10], each R/P is a special primary ring. $\square$

Proposition 4.2. *Every unitary pseudo BL-ring with left and right unit is isomorphic to a subring of a direct product of discrete valuation rings.*

Proof. We use the same idea as in Proposition 4.1 and [10]. $\square$

Proposition 4.3. *Let R be a subdirectly irreducible pseudo BL-ring with minimal ideal M , then*

- (1) *M is an annihilator ideal.*
- (2) *the annihilator ideals of R are linearly ordered and finite in number.*

Proof. (1) We know by definition of M^* that it is an ideal of R . Since M is the minimal ideal of R , then $M \subseteq M^*$, and because $M \neq \{0\}$, then $M^* \neq R$. Therefore, by the maximality of M , we obtain that $M = M^*$; similarly, we show that in the case of right ideal M^- of R , we also have $M = M^-$; therefore M is an annihilator ideal.

(2) Let R be a pseudo BL-ring, then $A(R)$ is a pseudo BL-algebra and also a pseudo MV-algebra. Consequently, $AN^*(R)$ and $AN^-(R)$ are respectively the left pseudo MV-center and right pseudo MV-center of $A(R)$. Moreover, for all ideal I of R , we have $(\bigvee I)^* = \bigcap I^*$ and $(\bigvee I)^- = \bigcap I^-$ which imply that every subset of $AN^*(R)$ and $AN^-(R)$ have an infimum. Thus, every subset of $AN^*(R)$ and $AN^-(R)$ also have a supremum because if S is a subset of $AN^*(R)$ (or $AN^-(R)$), then $\bigwedge S = (\bigvee S^*)^*$ (or $\bigwedge S = (\bigvee S^-)^-$). Hence $AN^*(R)$ and $AN^-(R)$ are complete pseudo MV-algebras. Also, for every non zero ideal I of R , $M \subseteq I$ implies that $I^* \subseteq M^*$ and $I^- \subseteq M^-$.

Therefore, for every proper ideals J in $AN^*(R)$ and K in $AN^-(R)$, we have $J^* \subseteq M^*$ and $K^- \subseteq M^-$ which induce that $AN^*(R) \setminus \{R\}$ and $AN^-(R) \setminus \{R\}$ have both of them a maximum element, namely $M^* \neq R$ and $M^- \neq R$ because $M \neq \{0\}$.

Let $X, Y \in AN^*(R)$ such that X is not included in Y and Y is not included in X , then $X \rightarrow Y, Y \rightarrow X \neq R$ and $X \rightarrow Y, Y \rightarrow X \subseteq M^*$; hence, by pre-linearity axiom, $R = X \rightarrow Y \vee Y \rightarrow X \subseteq M^*$; thus $M^* = R$, which is a contradiction, so $(AN^*(R), \subseteq)$ is a pseudo MV-chain.

By the same way we prove that $(AN^-(R), \subseteq)$ is a pseudo MV-chain. $\square$

The next result is the Representation Theorem for pseudo BL-ring.

Theorem 4.1. *Every pseudo BL-ring R is a subdirect product of a family $\{R_x | x \in R \setminus \{0\}\}$ of subdirectly irreducible pseudo BL-rings satisfying, then*

- (1) *$A(R_x)$ is isomorphic to $AN^*(R_x) \oplus D^*(R_x)$ and to $AN^-(R_x) \oplus D^-(R_x)$ for all $x \neq 0$.*
- (2) *every ideal of each R_x is either an annihilator ideal or dense.*
- (3) *$A(R)$ is a subdirect product of $\{A(R_x) | x \in R \setminus \{0\}\}$.*
- (4) *$A(R_x)$ is a pseudo BL-algebra with a unique atom.*

Proof. Let R be a pseudo BL-ring, it is well known that R is a subdirect product of subdirectly irreducible homomorphic images of R . Using the Zorn's lemma, for all $x \in R$ and $x \neq 0$, there is an maximal ideal K_x that does not contain x , which implies that $\bigcap_x K_x = \{0\}$. Hence each factor R/K_x is subdirectly irreducible and R is the product of the family $\{R/K_x | x \in R \setminus \{0\}\}$. Since R/K_x is a pseudo BL-ring by Corollary 3.2 and Proposition 3.10. Let pick up the set $\{R_x | x \in R \setminus \{0\}\}$ to be $\{R/K_x | x \in R \setminus \{0\}\}$. Then the rest of the proof is similar to Theorem 4.2 in [8]. $\square$

The following corollary is similar to Corollary 4.3 in [8].

Corollary 4.1. *Let R be a subdirectly irreducible BL-ring. Then*

- (1) *for every annihilator ideal $I \neq R$ and every dense ideal J with $I \subseteq J$, we have $J \rightarrow I = I$ and $J \rightsquigarrow I = I$.*
- (2) *for ideals I, J of R , either $(I \rightarrow J \text{ and } I \rightsquigarrow J)$ or $(J \rightarrow I \text{ and } J \rightsquigarrow I)$ are denses.*

Proof. Let R be a subdirectly irreducible BL-ring. As established in the proof of Theorem 4.1, $A(R)$ is isomorphic to $AN^*(R) \oplus D^*(R)$ and to $AN^-(R) \oplus D^-(R)$.

(1) By the definition of the implication in the ordinal sum of hoops, the condition stated holds.

(2) Since $A(R)$ is a pseudo BL-algebra, using the same way as in [2] and [3], we prove that the sets $D^*(M)$ and $D^-(M)$ of dense elements of any pseudo BL-algebra M is an implicative filter, $M/D^*(R)$ is isomorphic to $AN^*(M)$ and $M/D^-(R)$ is isomorphic to $AN^-(M)$.

Therefore, $A(R)/D^*(R)$ is isomorphic to $AN^*(R)$, and $A(R)/D^-(R)$ is isomorphic to $AN^-(R)$. Furthermore $A(R)/D^*(R)$ and $A(R)/D^-(R)$ are linearly ordered because $AN^*(R)$ and $AN^-(R)$ are linearly ordered. Now using the definitions of order, $\rightsquigarrow$ and $\rightarrow$ on $A(R)/D^*(R)$ and $A(R)/D^-(R)$, we finish the proof. $\square$

5. Conclusion

It is a well-known result that a t -norm has a residuum if and only if it is left-continuous. Therefore, this shows that Basic Logic is not the most general t -norm-based logic. In fact, a logic weaker than Basic Logic, called *Monoidal t -norm-based logic* (MTL for short), was defined by Esteva and Godo in [4] and

proved to be the logic of left-continuous t -norms and their residua. So MTL-algebra generalized BL-algebra. In this paper, we study a ring R , no matter it is commutative or non-commutative, with which $A(R)$ is a BL-algebra or pseudo BL-algebra.

Our future research in that way will consist to study commutative rings and non commutative rings R for which $A(R)$ is a MTL-algebra and pseudo MTL-algebra.

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Fredholm criteria for a class of regular hypoelliptic operators in multianisotropic spaces in $\mathbb{R}^n$

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Abstract. We study the Fredholm property of regular hypoelliptic operators, which are the special subclass of Hyormander’s hypoelliptic operators. In this paper necessary and sufficient conditions are obtained for the Fredholm property of regular hypoelliptic operators with special variable coefficients in multianisotropic weighted spaces in $\mathbb{R}^n$.

Keywords: Fredholm property, regular hypoelliptic operator, a priori estimate, multianisotropic weighted space.

1. Introduction

In this paper, we study the Fredholm property of regular hypoelliptic operators in multianisotropic weighted Sobolev spaces. The class of regular hypoelliptic operators is an important subclass of Hyormander’s hypoelliptic operators (see [1]). They were introduced in late 60s-70s and studied by many authors: V. P. Mikhailov [2], J. Friberg [3], L. R. Volevich, S. G. Gindikin. [4]. Corresponding characteristic polynomials of regular hypoelliptic operators are ”multi-quasi-elliptic”, so they are natural generalization of elliptic and quasielliptic polynomials.

The analysis of the Fredholm property of regular hypoelliptic operators in Sobolev spaces in $\mathbb{R}^n$ has certain difficulties - characteristic polynomials of such operators are not homogeneous as in elliptic case and Fredholm theorems for compact manifolds cannot always be used in this case.

The Fredholm property of elliptic operators in special weighted spaces is studied in the works of L. A. Bagirov [5], R. B. Lockhart, R. C. McOwen [6, 7], E. Schrohe [8] and others.

L. A. Bagirov [9], G. A. Karapetyan, A. A. Darbinyan [10], A. G. Tumanyan [11, 12] studied the Fredholm property of quasielliptic operators in weighted anisotropic spaces. For quasielliptic operators with constant coefficients isomorphism properties in some special scales of weighted Sobolev spaces are studied in G. V. Demidenko’s works (see [13, 14]). In the works of L. Rodino, P. Boggiatto, E. Buzano (see [15]) the Fredholm property is studied for the special

classes of pseudodifferential operators acting in multianisotropic spaces with special polynomial weights.

In this work, necessary conditions are obtained for fulfillment of special a priori estimates for differential operators acting in multianisotropic Sobolev spaces in $\mathbb{R}^n$ (Theorem 3.1 and Theorem 3.4). Necessary and sufficient conditions are obtained for the Fredholm property of regular hypoelliptic operators with variable coefficients acting in multianisotropic Sobolev spaces in $\mathbb{R}^n$ with certain weight functions (Theorem 3.6).

2. Basic notions and definitions

Definition 2.1. *A bounded linear operator A , acting from a Banach space X to a Banach space Y , is called an n -normal operator, if the following conditions hold:*

1. *the image of operator A is closed ($\text{Im}(A) = \overline{\text{Im}(A)}$);*
2. *the kernel of operator A is finite dimensional ($\dim \text{Ker}(A) < \infty$).*

An operator A is called a Fredholm operator if conditions 1-2 hold and

3. *the cokernel of operator A is finite dimensional ($\dim \text{coker}(A) = \dim Y / \text{Im}(A) < \infty$).*

The difference between the dimension of the kernel and the cokernel of operator A is called index of the operator:

$$\text{ind}(A) = \dim \text{Ker}(A) - \dim \text{coker}(A).$$

Definition 2.2. *For a bounded linear operator A , acting from a Banach space X to a Banach space Y , bounded linear operator $R_1 : Y \rightarrow X$ and $R_2 : Y \rightarrow X$ are called respectively left and right regularizers if the following holds: $R_1 A = I_X + T_1$, $A R_2 = I_Y + T_2$, where I_X, I_Y – identity operators, $T_1 : X \rightarrow X$ and $T_2 : Y \rightarrow Y$ are compact operators.*

Definition 2.3. *For a bounded linear operator A , acting from a whole Banach space X to a Banach space Y , bounded linear operator $R : Y \rightarrow X$ is called a regularizer for operator A , if it is left and right regularizer.*

Let $n \in \mathbb{N}$ and $\mathbb{R}^n$ be Euclidean n -dimensional space, $\mathbb{Z}_+^n, \mathbb{N}^n$ be the sets of n -dimensional multi-indices and multi-indices with natural components respectively.

Let $\mathcal{N} \subset \mathbb{Z}_+^n$ be a finite set of multi-indices, $\mathcal{R} = \mathcal{R}(\mathcal{N})$ be a minimum convex polyhedron containing all the points $\mathcal{N}$.

Definition 2.4. *A polyhedron $\mathcal{R}$ is called a completely regular if the following holds: a) $\mathcal{R}$ is a complete polyhedron: $\mathcal{R}$ has a vertex at the origin and further vertices on each coordinate axes in $\mathbb{R}^n$; b) all components of the outer normals of $(n-1)$ -dimensional non-coordinate faces of $\mathcal{R}$ are positive.*

Let $\mathcal{R}$ be a completely regular polyhedron. Denote by $\mathcal{R}_j^{n-1} (j = 1, \dots, I_{n-1})$ $(n-1)$ -dimensional non-coordinate faces of $\mathcal{R}$ with corresponding outer normal μ^j such that all multi-indices $\alpha \in \mathcal{R}_j^{n-1}$ satisfy $(\alpha : \mu^j) = \frac{\alpha_1}{\mu_1^j} + \dots + \frac{\alpha_n}{\mu_n^j} = 1$, $\partial' \mathcal{R} = \bigcup_{j=1}^{I_{n-1}} \mathcal{R}_j^{n-1}$. For $k > 0$ denote by $k\mathcal{R} := \{k\alpha = (k\alpha_1, k\alpha_2, \dots, k\alpha_n) : \alpha \in \mathcal{R}\}$.

Consider the differential form

$$(1) \quad P(x, \mathbb{D}) = \sum_{\alpha \in \mathcal{R}} a_\alpha(x) D^\alpha,$$

where $D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}$, $D_j = i^{-1} \frac{\partial}{\partial x_j}$, $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $a_\alpha(x) \in C(\mathbb{R}^n)$.

Denote by

$$(2) \quad P(x, \xi) = \sum_{\alpha \in \mathcal{R}} a_\alpha(x) \xi^\alpha.$$

For each $(n-1)$ -dimensional non-coordinate face $\mathcal{R}_j^{n-1} (j = 1, \dots, I_{n-1})$ denote by

$$P_j(x, \mathbb{D}) = \sum_{\alpha \in \mathcal{R}_j^{n-1}} a_\alpha(x) D^\alpha, \quad P_j(x, \xi) = \sum_{\alpha \in \mathcal{R}_j^{n-1}} a_\alpha(x) \xi^\alpha.$$

For $\xi \in \mathbb{R}^n$ denote by

$$|\xi|_{\mathcal{R}} = \sum_{\alpha \in \mathcal{R}} |\xi^\alpha|, \quad |\xi|_{\partial' \mathcal{R}} = \sum_{\alpha \in \partial' \mathcal{R}} |\xi^\alpha|.$$

Definition 2.5. A differential form $P(x, \mathbb{D})$ is called regular at a point $x_0 \in \mathbb{R}^n$, if there exists a constant $\delta > 0$ such that:

$$1 + |P(x_0, \xi)| \geq \delta |\xi|_{\mathcal{R}}, \quad \forall \xi \in \mathbb{R}^n.$$

Definition 2.6. A differential form $P(x, \mathbb{D})$ is called regular in $\mathbb{R}^n$, if $P(x, \mathbb{D})$ is regular at each point $x \in \mathbb{R}^n$.

Definition 2.7. A differential form $P(x, \mathbb{D})$ is called uniformly regular in $\mathbb{R}^n$, if there exists a constant $\delta > 0$ such that:

$$1 + |P(x, \xi)| \geq \delta |\xi|_{\mathcal{R}}, \quad \forall \xi \in \mathbb{R}^n, \quad \forall x \in \mathbb{R}^n.$$

Let's consider some examples of regular differential forms.

1. Let $m \in \mathbb{N}$ and $\mathcal{R}$ be a Newton polyhedron for the set of points $(0, 0, \dots, 0)$, $(m, 0, \dots, 0), \dots, (0, 0, \dots, m)$. In this case conditions from definitions 2.5–2.7 coincide with ellipticity conditions and differential form $P(x, \mathbb{D})$ is elliptical.

2. Let $\nu \in \mathbb{N}^n$ and $\mathcal{R}$ be a Newton polyhedron for the set of points $(0, 0, \dots, 0)$, $(\nu_1, 0, \dots, 0), \dots, (0, 0, \dots, \nu_n)$. In this case conditions from definitions 2.5–2.7 coincide with quasiellipticity of differential form $P(x, \mathbb{D})$.
3. Let $n = 2$ and $\mathcal{R}$ be a Newton polyhedron for the points $(0, 0)$, $(8, 0)$, $(0, 8)$ and $(6, 4)$. Then,

$$P(x, \mathbb{D}) = a_1 D_1^8 + a_2 D_1^6 D_2^4 + a_3 D_2^8 + q(x)$$

is a regular differential form in $\mathbb{R}^2$ with some $a_1, a_2, a_3 > 0$ and $q \in C(\mathbb{R}^2)$.

4. Let $n = 3$ and $\mathcal{R}$ be a Newton polyhedron for the points $(0, 0, 0)$, $(8, 0, 0)$, $(0, 8, 0)$, $(6, 4, 0)$, $(6, 0, 6)$, $(0, 6, 6)$ and $(0, 0, 12)$. Then,

$$P(x, \mathbb{D}) = D_1^8 + D_1^6 D_2^4 + D_2^8 + D_1^6 D_3^6 + D_2^6 D_3^6 + D_3^{12} + q(x)$$

is a regular differential form in $\mathbb{R}^3$ with $q \in C(\mathbb{R}^3)$.

For $k \in \mathbb{R}$ and completely regular polyhedron $\mathcal{R}$ denote

$$H^{k, \mathcal{R}}(\mathbb{R}^n) := \{u \in S' : \|u\|_{k, \mathcal{R}} := (\int |\widehat{u}(\xi)|^2 (1 + |\xi|_{\partial' \mathcal{R}})^{2k} d\xi)^{\frac{1}{2}} < \infty\},$$

where S' is the set of tempered distributions, $\widehat{u}$ is a Fourier transform of function u .

Denote

$$Q := \{g \in C(\mathbb{R}^n) : \exists c > 0 \text{ such that } g(x) \geq c > 0, \forall x \in \mathbb{R}^n\}.$$

For $m \in \mathbb{Z}_+$ and completely regular polyhedron $\mathcal{R}$ denote

$$Q^{m, \mathcal{R}} := \{g(x) \in Q : D^\beta g(x) \in C(\mathbb{R}^n), \frac{1}{g(x)} \Rightarrow 0, \max_{|x-y| \leq 1} \frac{|g(x) - g(y)|}{g(y)} \Rightarrow 0, \\ \frac{|D^\beta g(x)|}{g(x)^{1+(\beta; \mu^j)}} \Rightarrow 0 \text{ as } |x| \rightarrow \infty, \forall \beta \in m\mathcal{R}, \beta \neq 0, j = 1, \dots, I_{n-1}\}.$$

The examples of weight functions from $Q^{m, \mathcal{R}}$ include polynomial functions as well as special exponential functions, for example:

$$(1 + |x|_{\mathcal{R}})^l, l > 0, \exp(1 + |x|_{\mathcal{R}})^r, 0 < r < \frac{1}{\mu_{\max}},$$

where $\mu_{\max} = \max_{1 \leq i \leq I_{n-1}} \max_{1 \leq s \leq n} \{\mu_s^i\}$. For $k \in \mathbb{Z}_+$, $q \in Q$, completely regular polyhedron $\mathcal{R}$ and $\Omega \subset \mathbb{R}^n$ denote

$$H_q^{k, \mathcal{R}}(\mathbb{R}^n) := \\ \{u : \|u\|_{H_q^{k, \mathcal{R}}(\mathbb{R}^n)} := \|u\|_{k, \mathcal{R}, q} := \sum_{\alpha \in k\mathcal{R}} \|D^\alpha u \cdot q^{k - \max_i(\alpha; \mu^i)}\|_{L_2(\mathbb{R}^n)} < \infty\},$$

$$H_q^{k,\mathcal{R}}(\Omega) := \{u : \|u\|_{H_q^{k,\mathcal{R}}(\Omega)} := \sum_{\alpha \in k\mathcal{R}} \|D^\alpha u \cdot q^{k-\max_i(\alpha;\mu^i)}\|_{L_2(\Omega)} < \infty\}.$$

Let $k \in \mathbb{Z}_+$, $q \in Q$ and the coefficients of differential expression $P(x, \mathbb{D})$ of the form (1) satisfy the following conditions:

$$(3) \quad |D^\beta a_\alpha(x)| \leq C_{\alpha,\beta} q(x)^{1-\max_i(\alpha-\beta;\mu^i)} \quad \forall \alpha \in \mathcal{R}, \beta \in k\mathcal{R}.$$

Then, it is easy to check that $P(x, \mathbb{D})$ generates a bounded linear operator, acting from $H_q^{k+1,\mathcal{R}}(\mathbb{R}^n)$ to $H_q^{k,\mathcal{R}}(\mathbb{R}^n)$.

3. Main results

For $N > 0$ and $x_0 \in \mathbb{R}^n$ denote

$$K_N(x_0) := \{x \in \mathbb{R}^n : |x - x_0| \leq N\}, K_N := K_N(0).$$

Theorem 3.1. *Let $k \in \mathbb{Z}_+$, $q \in Q$ and the differential form $P(x, \mathbb{D})$ with some constant $\kappa > 0$ satisfies the following estimate:*

$$(4) \quad \|u\|_{k+1,\mathcal{R},q} \leq \kappa(\|Pu\|_{k,\mathcal{R},q} + \|u\|_{L_2(\mathbb{R}^n)}), \forall u \in H_q^{k+1,\mathcal{R}}(\mathbb{R}^n).$$

Then, $P(x, \mathbb{D})$ is uniformly regular in $\mathbb{R}^n$.

Proof. Let $x_0, \xi \in \mathbb{R}^n$, $|\xi| \neq 0$, $N > 0$ and $\varphi \in C_0^\infty(\mathbb{R}^n)$ such that $\text{supp } \varphi \subset K_N(x_0)$ and $\|\varphi\|_{L_2(\mathbb{R}^n)} = 1$. For $\lambda > 0$ and $j \in \{1, \dots, I_{n-1}\}$ denote by $\lambda^{\frac{1}{\mu^j}} \xi = (\lambda^{\frac{1}{\mu^1}} \xi_1, \dots, \lambda^{\frac{1}{\mu^n}} \xi_n)$, where μ^j is outer normal of $\mathcal{R}_j^{n-1}$. Denote by $u_{\lambda,j}(x) = \exp(i(\lambda^{\frac{1}{\mu^j}} \xi, x))\varphi(x)$. For $\beta \in (k+1)\mathcal{R}$, we have

$$\begin{aligned} D^\beta u_{\lambda,j}(x) &= \lambda^{(\beta;\mu^j)} \xi^\beta \exp(i(\lambda^{\frac{1}{\mu^j}} \xi, x))\varphi(x) \\ &\quad + \sum_{0 \leq \gamma < \beta} C_\beta^\gamma \lambda^{(\gamma;\mu^j)} \xi^\gamma \exp(i(\lambda^{\frac{1}{\mu^j}} \xi, x)) D^{\beta-\gamma} \varphi(x). \end{aligned}$$

From last equality, for $\beta \in (k+1)\mathcal{R}$, we get

$$\begin{aligned} \|D^\beta u_{\lambda,j} \cdot q^{k+1-\max_i(\beta;\mu^i)}\|_{L_2(\mathbb{R}^n)} &= \lambda^{(\beta;\mu^j)} |\xi^\beta| \|\varphi \cdot q^{k+1-\max_i(\beta;\mu^i)}\|_{L_2(\mathbb{R}^n)} \\ &\quad + o(\lambda^{(\beta;\mu^j)}), \end{aligned}$$

when $\lambda \rightarrow \infty$. Then, we get

$$(5) \quad \|u_{\lambda,j}\|_{k+1,\mathcal{R},q} = \sum_{\beta \in (k+1)\mathcal{R}} \lambda^{(\beta;\mu^j)} |\xi^\beta| \|\varphi \cdot q^{k+1-\max_i(\beta;\mu^i)}\|_{L_2(\mathbb{R}^n)} + o(\lambda^{k+1}),$$

when $\lambda \rightarrow \infty$.

Notice that, for $\alpha \in \mathcal{R}$ the following holds: $(\alpha : \mu^j) \leq 1$ and $(\alpha : \mu^j) = 1$ if only if $\alpha \in \mathcal{R}_j^{n-1}$. Taking it into account along with the fact that $\varphi \in C_0^\infty(\mathbb{R}^n)$, $\text{supp } \varphi \subset K_N(x_0)$ for $\beta \in k\mathcal{R}$ we get

$$\begin{aligned} & \|D^\beta(P(x, \mathbb{D})u_{\lambda,j})q^{k-\max_i(\beta:\mu^i)}\|_{L_2(\mathbb{R}^n)} \\ & \leq \lambda^{(\beta:\mu^j)}\lambda|\xi^\beta| \max_{x \in K_N(x_0)} |P_j(x, \xi)| \|\varphi \cdot q^{k-\max_i(\beta:\mu^i)}\|_{L_2(\mathbb{R}^n)} + o(\lambda^{k+1}). \end{aligned}$$

From the last estimate, we get

$$\begin{aligned} \|Pu_{\lambda,j}\|_{k,\mathcal{R},q} & \leq \sum_{\beta \in k\mathcal{R}} \lambda^{(\beta:\mu^j)+1} |\xi^\beta| \max_{x \in K_N(x_0)} |P_j(x, \xi)| \\ (6) \quad & \times \|\varphi \cdot q^{k-\max_i(\beta:\mu^i)}\|_{L_2(\mathbb{R}^n)} + o(\lambda^{k+1}), \end{aligned}$$

when $\lambda \rightarrow \infty$. Then, from (4) and (5)–(6) we get

$$\begin{aligned} & \sum_{\beta \in (k+1)\mathcal{R}_j^{n-1}} \lambda^{k+1} |\xi^\beta| \|\varphi\|_{L_2(\mathbb{R}^n)} + o(\lambda^{k+1}) \\ & \leq \kappa \sum_{\beta \in k\mathcal{R}_j^{n-1}} \lambda^{k+1} |\xi^\beta| \max_{x \in K_N(x_0)} |P_j(x, \xi)| \|\varphi\|_{L_2(\mathbb{R}^n)} + o(\lambda^{k+1}), \end{aligned}$$

when $\lambda \rightarrow \infty$.

In last inequality we take into account that $\|\varphi\|_{L_2(\mathbb{R}^n)} = 1$, divide by λ^{k+1} and tend $\lambda \rightarrow \infty$. Then the following is obtained:

$$\sum_{\beta \in (k+1)\mathcal{R}_j^{n-1}} |\xi^\beta| \leq \kappa \sum_{\beta \in k\mathcal{R}_j^{n-1}} |\xi^\beta| \max_{x \in K_N(x_0)} |P_j(x, \xi)|.$$

Since $\mathcal{R}$ is completely regular polyhedron for $k \in \mathbb{Z}_+$ and $j \in \{1, \dots, I_{n-1}\}$, there exist such constants $\delta_1 > 0, \delta_2 > 0$ such that

$$\begin{aligned} & \sum_{\beta \in (k+1)\mathcal{R}_j^{n-1}} |\xi^\beta| \geq \delta_1 \left(\sum_{\beta \in \mathcal{R}_j^{n-1}} |\xi^\beta| \right)^{k+1}, \\ (7) \quad & \sum_{\beta \in k\mathcal{R}_j^{n-1}} |\xi^\beta| \leq \delta_2 \left(\sum_{\beta \in \mathcal{R}_j^{n-1}} |\xi^\beta| \right)^k. \end{aligned}$$

From (7), we get

$$\delta_1 \sum_{\beta \in \mathcal{R}_j^{n-1}} |\xi^\beta| \leq \kappa \delta_2 \max_{x \in K_N(x_0)} |P_j(x, \xi)|.$$

Since the coefficients of $P(x, \mathbb{D})$ are continuous, tending $N \rightarrow 0$, we obtain

$$(8) \quad |P_j(x_0, \xi)| \geq \delta_3 \sum_{\beta \in \mathcal{R}_j^{n-1}} |\xi^\beta|, \forall \xi \in \mathbb{R}^n,$$

where $\delta_3 = \frac{\delta_1}{\kappa\delta_2} > 0$.

Since constant δ_3 does not depend on the choice of $x_0 \in \mathbb{R}^n$, we get

$$(9) \quad |P_j(x, \xi)| \geq \delta_3 \sum_{\beta \in \mathcal{R}_j^{n-1}} |\xi^\beta|, \forall x \in \mathbb{R}^n, \forall \xi \in \mathbb{R}^n.$$

Similarly analogous inequalities can be obtained for all $j \in \{1, \dots, I_{n-1}\}$. Then applying Theorem 6.1 from [2] we obtain that $P(x, \mathbb{D})$ is uniformly regular in $\mathbb{R}^n$. $\square$

Theorem 3.2. (Theorem 7.1 [19]) *Let E , F and E_0 be Banach spaces such that E is compactly embedded in E_0 . Let A be a bounded linear operator acting from E to F . An operator $A : E \rightarrow F$ is an n -normal operator if and only if there exists a constant $C > 0$ such that*

$$\|x\|_E \leq C(\|Ax\|_F + \|x\|_{E_0}), \forall x \in E.$$

Applying the last theorem for operator $P(x, \mathbb{D})$, acting from $H_q^{k+1, \mathcal{R}}(\mathbb{R}^n)$ to $H_q^{k, \mathcal{R}}(\mathbb{R}^n)$, we get

Theorem 3.3. *Let $k \in \mathbb{Z}_+$, $q \in Q$ and $P(x, \mathbb{D})$ be differential form (1). Then operator $P(x, \mathbb{D})$, acting from $H_q^{k+1, \mathcal{R}}(\mathbb{R}^n)$ to $H_q^{k, \mathcal{R}}(\mathbb{R}^n)$, is an n -normal operator if and only if there exist constants $\kappa > 0$ and $N > 0$ such that the following holds*

$$\|u\|_{k+1, \mathcal{R}, q} \leq \kappa(\|Pu\|_{k, \mathcal{R}, q} + \|u\|_{L_2(K_N)}), \forall u \in H_q^{k+1, \mathcal{R}}(\mathbb{R}^n).$$

Corollary 3.1. *Let $k \in \mathbb{Z}_+$, $q \in Q$ and operator $P(x, \mathbb{D})$, acting from $H_q^{k+1, \mathcal{R}}(\mathbb{R}^n)$ to $H_q^{k, \mathcal{R}}(\mathbb{R}^n)$, be a Fredholm operator. Then $P(x, \mathbb{D})$ is uniformly regular in $\mathbb{R}^n$.*

Proof. Since operator $P(x, \mathbb{D})$, acting from $H_q^{k+1, \mathcal{R}}(\mathbb{R}^n)$ to $H_q^{k, \mathcal{R}}(\mathbb{R}^n)$ is a Fredholm operator, then it is an n -normal operator. From Theorem 3.3, we get that there exist such constants $\kappa > 0$ and $N > 0$ that the following estimate holds

$$\|u\|_{k+1, \mathcal{R}, q} \leq \kappa(\|Pu\|_{k, \mathcal{R}, q} + \|u\|_{L_2(K_N)}) \leq \kappa(\|Pu\|_{k, \mathcal{R}, q} + \|u\|_{L_2(\mathbb{R}^n)}),$$

for all $u \in H_q^{k+1, \mathcal{R}}(\mathbb{R}^n)$. From last estimate using Theorem 3.1 we obtain that $P(x, \mathbb{D})$ is uniformly regular in $\mathbb{R}^n$. $\square$

Remark 3.1. It is easy to check that in the case $q \equiv 1$ inverse statement is true with some smoothness conditions on the coefficients of $P(x, \mathbb{D})$. In next theorem 3.4 it is proved that under the special conditions on the weight function q and coefficients of the differential form $P(x, \mathbb{D})$ uniform regularity in $\mathbb{R}^n$ (in the sense of definition (2.7)) does not imply the fulfillment of a priori estimate of the form (4) and stronger conditions are necessary for it.

Let $k \in \mathbb{Z}_+$ and $q \in Q^{k, \mathcal{R}}$. Consider the differential form $P(x, \mathbb{D})$ (see (1)), which is expressed in the following way:

$$(10) \quad P(x, \mathbb{D}) = \sum_{\alpha \in \mathcal{R}} a_\alpha(x) D^\alpha = \sum_{\alpha \in \mathcal{R}} (a_\alpha^0(x) q(x)^{1-\max_i(\alpha: \mu^i)} + a_\alpha^1(x)) D^\alpha,$$

where $a_\alpha(x) = a_\alpha^0(x) q(x)^{1-\max_i(\alpha: \mu^i)} + a_\alpha^1(x)$, $D^\beta(a_\alpha^0(x)) = O(q(x)^{\min_i(\beta: \mu^i)})$ and $D^\beta(a_\alpha^1(x)) = o(q(x)^{1-\max_i(\alpha: \mu^i)})$, when $|x| \rightarrow \infty$ for all $\alpha \in \mathcal{R}, \beta \in k\mathcal{R}$.

Theorem 3.4. *Let $k \in \mathbb{Z}_+, q \in Q^{k, \mathcal{R}}$ and $P(x, \mathbb{D})$ be the differential form (10) with the coefficients that satisfy $\lim_{|x| \rightarrow \infty} \max_{|x-y| \leq 1} |a_\alpha^0(x) - a_\alpha^0(y)| = 0$ for all $\alpha \in \mathcal{R}$.*

Let there exists a constant $\kappa > 0$ such that:

$$(11) \quad \|u\|_{k+1, \mathcal{R}, q} \leq \kappa (\|Pu\|_{k, \mathcal{R}, q} + \|u\|_{L_2(\mathbb{R}^n)}), \forall u \in H_q^{k+1, \mathcal{R}}(\mathbb{R}^n).$$

Then, $P(x, \mathbb{D})$ is regular in $\mathbb{R}^n$ and there exist constants $\delta > 0$ and $M > 0$ such that

$$(12) \quad \left| \sum_{\alpha \in \mathcal{R}} a_\alpha^0(x) \lambda^{1-\max_i(\alpha: \mu^i)} \xi^\alpha \right| \geq \delta (\lambda + |\xi|_{\partial' \mathcal{R}}), \forall \xi \in \mathbb{R}^n, \lambda > 0, |x| > M.$$

Proof. Regularity of $P(x, \mathbb{D})$ in $\mathbb{R}^n$ follows from Theorem 3.1. It remains to prove that inequality (12) holds.

Let $M \in \mathbb{R}_+, x_M \in \mathbb{R}^n \setminus K_M, \varphi \in C_0^\infty(\mathbb{R}^n), \text{supp } \varphi \subset K_1(x_M), \|\varphi\|_{L_2(\mathbb{R}^n)} = 1$ and $\xi \in \mathbb{R}^n$. Let $j \in \{1, \dots, I_{n-1}\}$.

Consider the function $\tilde{u}_j(x) = \exp(i(q(x_M)^{\frac{1}{\mu^j}} \xi, x)) \varphi(x)$ where μ^j is an outer normal of non-coordinate face $\mathcal{R}_j^{n-1}$ such that all multi-indices $\alpha \in \mathcal{R}_j^{n-1}$ satisfy $(\alpha: \mu^j) = 1$.

Denote by $\mathcal{R}_j = \{\alpha \in \mathcal{R} : (\alpha: \mu^j) = \max_{1 \leq i \leq I_{n-1}} (\alpha: \mu^i)\}$.

Since $\lim_{|x| \rightarrow \infty} \max_{|x-y| \leq 1} \frac{|q(x) - q(y)|}{q(y)} = 0$, then for any $r \in \mathbb{R}_+$ the following inequality is fulfilled

$$(13) \quad |q(x)^r - q(x_M)^r| \leq \varepsilon_r(M) q(x_M)^r, \forall x \in K_1(x_M),$$

where $\varepsilon_r(M) \rightarrow 0$ when $M \rightarrow \infty$.

Using inequality (13) and the fact that $\text{supp } \tilde{u}_j \subset K_1(x_M)$ it is easy to see that there exists a function $\varepsilon(M)$ such that $\varepsilon(M) \rightarrow 0$ when $M \rightarrow \infty$ and the following inequalities hold:

$$(14) \quad \|\tilde{u}_j\|_{k+1, \mathcal{R}, q} \geq (1 - \varepsilon(M)) \|\tilde{u}_j\|_{k+1, \mathcal{R}, q(x_M)},$$

$$(15) \quad \|P\tilde{u}_j\|_{k, \mathcal{R}, q} \leq (1 + \varepsilon(M)) \|P\tilde{u}_j\|_{k, \mathcal{R}, q(x_M)}.$$

Taking into consideration the definition of function $\tilde{u}_j$ one can check that for any $\beta \in (k+1)\mathcal{R}_j$ with some constants $C_1 > 0$ and $\sigma = \sigma(\mathcal{R}) > 0$ the following holds

$$\begin{aligned} \|D^\beta \tilde{u}_j\|_{L_2(\mathbb{R}^n)} q(x_M)^{k+1-(\beta:\mu^j)} &\geq |\xi^\beta| q(x_M)^{k+1} \|\varphi\|_{L_2(\mathbb{R}^n)} \\ &\quad - C_1 \sum_{0 \leq \gamma < \beta} |\xi^\gamma| q(x_M)^{k+1-\sigma}. \end{aligned}$$

For $\beta \in (k+1)(\mathcal{R} \setminus \mathcal{R}_j)$ with some constants $C_2 > 0$ and $\sigma = \sigma(\mathcal{R}) > 0$ the following holds

$$\begin{aligned} \|D^\beta \tilde{u}_j\|_{L_2(\mathbb{R}^n)} q(x_M)^{k+1-\max_i(\beta:\mu^i)} \\ \geq |\xi^\beta| q(x_M)^{k+1-\sigma} \|\varphi\|_{L_2(\mathbb{R}^n)} - C_2 \sum_{0 \leq \gamma < \beta} |\xi^\gamma| q(x_M)^{k+1-\sigma}. \end{aligned}$$

From last two inequalities, taking into account that $\|\varphi\|_{L_2(\mathbb{R}^n)} = 1$, with some constants $C_3 > 0$ and $\sigma > 0$ we get

$$\begin{aligned} \|\tilde{u}_j\|_{k+1, \mathcal{R}, q(x_M)} &\geq \sum_{\beta \in (k+1)\mathcal{R}_j} |\xi^\beta| q(x_M)^{k+1} \\ (16) \quad &- C_3 \sum_{\gamma \in (k+1)(\mathcal{R} \setminus \mathcal{R}_j^{n-1})} |\xi^\gamma| q(x_M)^{k+1-\sigma}. \end{aligned}$$

Taking into account, conditions (3) on the coefficients of $P(x, \mathbb{D})$ and inequality (13) we get that for $\beta \in k\mathcal{R}$ and $\alpha \in \mathcal{R}$ with some $C_{\alpha, \beta}, C'_{\alpha, \beta} > 0$ holds

$$\begin{aligned} &|D^\beta(a_\alpha(x)D^\alpha \tilde{u}_j(x))| q(x_M)^{k-\max_i(\beta:\mu^i)} \\ &\leq C_{\alpha, \beta} \sum_{\beta_1 + \beta_2 = \beta} |D^{\beta_1}(a_\alpha(x))D^{\alpha+\beta_2} \tilde{u}_j(x)| q(x_M)^{k-\max_i(\beta:\mu^i)} \\ &\leq C'_{\alpha, \beta} \sum_{\beta_1 + \beta_2 = \beta} |D^{\alpha+\beta_2} \tilde{u}_j(x)| q(x_M)^{1-\max_i(\alpha-\beta_1:\mu^i)} q(x_M)^{k-\max_i(\beta:\mu^i)}. \end{aligned}$$

Let $\beta \in k(\mathcal{R} \setminus \mathcal{R}_j)$ and $\alpha \in \mathcal{R}$. Then, for $\beta_1, \beta_2 \in k\mathcal{R}, \beta_1 + \beta_2 = \beta$ with some $\sigma > 0$ the following holds

$$\begin{aligned} &\max_i(\alpha - \beta_1 : \mu^i) + \max_i(\beta : \mu^i) - (\alpha + \beta_2 : \mu^j) \\ &\geq (\max_i(\alpha - \beta_1 : \mu^i) - (\alpha - \beta_1 : \mu^j)) + (\max_i(\beta : \mu^i) - (\beta : \mu^j)) \geq \sigma > 0. \end{aligned}$$

So, for $\beta \in k(\mathcal{R} \setminus \mathcal{R}_j)$ and $\alpha \in \mathcal{R}$ with some constants $C''_{\alpha, \beta}, C'''_{\alpha, \beta} > 0$ and $\sigma > 0$ we get

$$|D^\beta(a_\alpha(x)D^\alpha \tilde{u}_j(x))| q(x_M)^{k-\max_i(\beta:\mu^i)}$$

$$\begin{aligned}
&\leq C''_{\alpha,\beta} \sum_{\beta_1+\beta_2=\beta} |D^{\alpha+\beta_2}\tilde{u}_j(x)|q(x_M)^{k+1-\sigma-\max_i(\alpha+\beta_2;\mu^i)}, \\
&\|D^\beta(a_\alpha(x)D^\alpha\tilde{u}_j)\|_{L_2(\mathbb{R}^n)}q(x_M)^{k-\max_i(\beta;\mu^i)} \\
&\leq C'''_{\alpha,\beta} \sum_{\gamma\in(k+1)(\mathcal{R}\setminus\mathcal{R}_j^{n-1})} |\xi^\gamma|q(x_M)^{k+1-\sigma}.
\end{aligned}$$

From the last estimate we obtain that for $\beta \in k(\mathcal{R}\setminus\mathcal{R}_j)$ with some constants $C_4 > 0$ and $\sigma > 0$

$$\begin{aligned}
(17) \quad &\|D^\beta(P(x, \mathbb{D})\tilde{u}_j)\|_{L_2(\mathbb{R}^n)}q(x_M)^{k-\max_i(\beta;\mu^i)} \\
&\leq C_4q(x_M)^{k+1-\sigma} \sum_{\gamma\in(k+1)(\mathcal{R}\setminus\mathcal{R}_j^{n-1})} |\xi^\gamma|
\end{aligned}$$

For $\beta \in k\mathcal{R}_j$, we have

$$\begin{aligned}
(18) \quad &\|D^\beta(P(x, \mathbb{D})\tilde{u}_j)\|_{L_2(\mathbb{R}^n)}q(x_M)^{k-(\beta;\mu^j)} \\
&\leq \|D^\beta(\sum_{\alpha\in\mathcal{R}_j} a_\alpha^0(x)q(x)^{1-(\alpha;\mu^j)}D^\alpha\tilde{u}_j)\|_{L_2(\mathbb{R}^n)}q(x_M)^{k-(\beta;\mu^j)} \\
&\quad + \|D^\beta(\sum_{\alpha\in\mathcal{R}\setminus\mathcal{R}_j} a_\alpha^0(x)q(x)^{1-\max_i(\alpha;\mu^i)}D^\alpha\tilde{u}_j)\|_{L_2(\mathbb{R}^n)}q(x_M)^{k-(\beta;\mu^j)} \\
&\quad + \|D^\beta(\sum_{\alpha\in\mathcal{R}_j} a_\alpha^1(x)D^\alpha\tilde{u}_j)\|_{L_2(\mathbb{R}^n)}q(x_M)^{k-(\beta;\mu^j)}.
\end{aligned}$$

Similarly, to (17) with some constants $C_5 > 0$ and $\sigma > 0$ holds

$$\begin{aligned}
(19) \quad &\|D^\beta(\sum_{\alpha\in\mathcal{R}\setminus\mathcal{R}_j} a_\alpha^0(x)q(x)^{1-\max_i(\alpha;\mu^i)}D^\alpha\tilde{u}_j)\|_{L_2(\mathbb{R}^n)}q(x_M)^{k-(\beta;\mu^j)} \\
&\leq C_5 \sum_{\gamma\in(k+1)(\mathcal{R}\setminus\mathcal{R}_j^{n-1})} |\xi^\gamma|q(x_M)^{k+1-\sigma}.
\end{aligned}$$

Using conditions $D^\beta(a_\alpha^1(x)) = o(q(x)^{1-\max_i(\alpha-\beta;\mu^i)})$ when $|x| \rightarrow \infty$ for all $\alpha \in \mathcal{R}, \beta \in k\mathcal{R}$ it is easy to check that for $\beta \in k\mathcal{R}_j$ there exists such a function $\tau_1(M)$ that $\tau_1(M) \rightarrow 0$ when $M \rightarrow \infty$ and the following holds

$$\begin{aligned}
(20) \quad &\|D^\beta(\sum_{\alpha\in\mathcal{R}} a_\alpha^1(x)D^\alpha\tilde{u}_j)\|_{L_2(\mathbb{R}^n)}q(x_M)^{k-(\beta;\mu^j)} \\
&\leq \tau_1(M) \sum_{\gamma\in(k+1)\mathcal{R}} |\xi^\gamma|q(x_M)^{k+1}.
\end{aligned}$$

For the first term in (18) the following estimate holds

$$\begin{aligned}
& \|D^\beta \left(\sum_{\alpha \in \mathcal{R}_j} a_\alpha^0(x) q(x)^{1-(\alpha;\mu^j)} D^\alpha \tilde{u}_j \right)\|_{L_2(\mathbb{R}^n)} q(x_M)^{k-(\beta;\mu^j)} \\
& \leq \| \sum_{\alpha \in \mathcal{R}_j} a_\alpha^0(x_M) q(x_M)^{1-(\alpha;\mu^j)} D^{\alpha+\beta} \tilde{u}_j \|_{L_2(\mathbb{R}^n)} q(x_M)^{k-(\beta;\mu^j)} \\
& + \| \sum_{\alpha \in \mathcal{R}_j} D^\beta ((a_\alpha^0(x) q(x)^{1-(\alpha;\mu^j)} - a_\alpha^0(x_M) q(x_M)^{1-(\alpha;\mu^j)}) D^\alpha \tilde{u}_j) \|_{L_2(\mathbb{R}^n)} \\
& \times q(x_M)^{k-(\beta;\mu^j)}.
\end{aligned}$$

It is easy to check that with some constants $C_6 > 0$ and $\sigma > 0$ the following holds

$$\begin{aligned}
(21) \quad & \| \sum_{\alpha \in \mathcal{R}_j} a_\alpha^0(x_M) q(x_M)^{1-(\alpha;\mu^j)} D^{\alpha+\beta} \tilde{u}_j \|_{L_2(\mathbb{R}^n)} q(x_M)^{k-(\beta;\mu^j)} \\
& \leq | \sum_{\alpha \in \mathcal{R}_j} a_\alpha^0(x_M) \xi^\alpha | |\xi^\beta| q(x_M)^{k+1} \\
& + C_6 \sum_{\gamma \in (k+1)(\mathcal{R} \setminus \mathcal{R}_j^{n-1})} |\xi^\gamma| q(x_M)^{k+1-\sigma}.
\end{aligned}$$

Taking into account, conditions in (10), $\lim_{|x| \rightarrow \infty} \max_{|x-y| \leq 1} |a_\alpha^0(x) - a_\alpha^0(y)| = 0$ for $\alpha \in \mathcal{R}$, $q \in Q^{k,\mathcal{R}}$ and inequality (13) we get that for $\alpha \in \mathcal{R}$ and $\beta \in k\mathcal{R}$ there exists $\tau_{\alpha,\beta}(M)$ such that $\tau_{\alpha,\beta}(M) \rightarrow 0$ when $M \rightarrow \infty$ and the following holds

$$|D^\beta(a_\alpha^0(x) q(x)^{1-(\alpha;\mu^j)} - a_\alpha^0(x_M) q(x_M)^{1-(\alpha;\mu^j)})| \leq \tau_{\alpha,\beta}(M) q(x_M)^{1-(\alpha;\mu^j) + (\beta;\mu^j)}.$$

From last inequality we get that there exist a function $\tau_2(M)$ such that $\tau_2(M) \rightarrow 0$ when $M \rightarrow \infty$, constants $C_7 > 0$ and $\sigma > 0$ that the following estimate is satisfied

$$\begin{aligned}
(22) \quad & \| \sum_{\alpha \in \mathcal{R}_j} D^\beta (a_\alpha^0(x) q(x)^{1-(\alpha;\mu^j)} - a_\alpha^0(x_M) q(x_M)^{1-(\alpha;\mu^j)}) D^\alpha \tilde{u}_j \|_{L_2(\mathbb{R}^n)} \\
& \times q(x_M)^{k-(\beta;\mu^j)} \leq \tau_2(M) \sum_{\alpha \in \mathcal{R}_j} |\xi^{\alpha+\beta}| q(x_M)^{k+1} \\
& + C_7 \sum_{\gamma \in (k+1)(\mathcal{R} \setminus \mathcal{R}_j^{n-1})} |\xi^\gamma| q(x_M)^{k+1-\sigma}.
\end{aligned}$$

So, using last estimate and estimates (17)–(22) we obtain that there exist such a function $\tau_3(M)$, $\tau_3(M) \rightarrow 0$ when $M \rightarrow \infty$ and constants $C_8 > 0$ and

$\sigma > 0$ that the following holds

$$(23) \quad \begin{aligned} & \|P\tilde{u}_j\|_{k,\mathcal{R},q(x_M)} \leq \left| \sum_{\alpha \in \mathcal{R}_j} a_\alpha^0(x_M) \xi^\alpha \right| \sum_{\beta \in k\mathcal{R}_j} |\xi^\beta| q(x_M)^{k+1} \\ & + C_8 \sum_{\gamma \in (k+1)(\mathcal{R} \setminus \mathcal{R}_j^{n-1})} |\xi^\gamma| q(x_M)^{k+1-\sigma} + \tau_3(M) \sum_{\gamma \in (k+1)\mathcal{R}} |\xi^\gamma| q(x_M)^{k+1}. \end{aligned}$$

Then, from (11) using (16) and (23) we get

$$\begin{aligned} & \sum_{\beta \in (k+1)\mathcal{R}_j} |\xi^\beta| q(x_M)^{k+1} - C_3 \sum_{\gamma \in (k+1)(\mathcal{R} \setminus \mathcal{R}_j^{n-1})} |\xi^\gamma| q(x_M)^{k+1-\sigma} \\ & \leq \kappa \left(\left| \sum_{\alpha \in \mathcal{R}_j} a_\alpha^0(x_M) \xi^\alpha \right| \sum_{\beta \in k\mathcal{R}_j} |\xi^\beta| q(x_M)^{k+1} \right. \\ & \left. + C_8 \sum_{\gamma \in (k+1)(\mathcal{R} \setminus \mathcal{R}_j^{n-1})} |\xi^\gamma| q(x_M)^{k+1-\sigma} + \tau_3(M) \sum_{\gamma \in (k+1)\mathcal{R}} |\xi^\gamma| q(x_M)^{k+1} \right). \end{aligned}$$

Since $\{a_\alpha^0(x) : \alpha \in \mathcal{R}_j\}$ are bounded functions and $x_M \rightarrow \infty$ when $M \rightarrow \infty$, there exist convergent subsequences of sequences $\{a_\alpha^0(x_M) : \alpha \in \mathcal{R}_j\}$. Without loss of generality assume that sequences $\{a_\alpha^0(x_M) : \alpha \in \mathcal{R}_j\}$ are convergent, so for each $\alpha \in \mathcal{R}_j$ there exists a constant $\tilde{a}_\alpha^0$ such that $a_\alpha^0(x_M) \rightrightarrows \tilde{a}_\alpha^0$ when $M \rightarrow \infty$.

Dividing by $q(x_M)^{k+1}$ and tending M to infinity we get

$$\sum_{\alpha \in (k+1)\mathcal{R}_j} |\xi^\alpha| \leq \kappa \left| \sum_{\alpha \in \mathcal{R}_j} \tilde{a}_\alpha^0 \xi^\alpha \right| \sum_{\beta \in k\mathcal{R}_j} |\xi^\beta|.$$

From last inequality similarly to (7) from Theorem 3.1 we obtain that for each $j \in 1, \dots, I_{n-1}$ there exists a constant $\delta_j > 0$ such that

$$\left| \sum_{\alpha \in \mathcal{R}_j} \tilde{a}_\alpha^0 \xi^\alpha \right| \geq \delta_j (1 + |\xi|_{\mathcal{R}_j^{n-1}}),$$

where $|\xi|_{\mathcal{R}_j^{n-1}} = \sum_{\beta \in \mathcal{R}_j^{n-1}} |\xi^\beta|$. For $\lambda > 0$ substituting $\xi = (\xi_1, \dots, \xi_n)$ with $\lambda^{-\frac{1}{\mu^j}} \xi = (\lambda^{-\frac{1}{\mu_1^j}} \xi_1, \dots, \lambda^{-\frac{1}{\mu_n^j}} \xi_n)$ we get

$$\left| \sum_{\alpha \in \mathcal{R}_j} \tilde{a}_\alpha^0 \xi^\alpha \lambda^{1-(\alpha:\mu^j)} \right| \geq \delta_j (\lambda + |\xi|_{\mathcal{R}_j^{n-1}}).$$

Similarly, analogous inequalities can be obtained for all $j \in \{1, \dots, I_{n-1}\}$.

Using Theorem 6.1, from [2], we get

$$\left| \sum_{\alpha \in \mathcal{R}} \tilde{a}_\alpha^0 \xi^\alpha \lambda^{1-\max_i(\alpha:\mu^i)} \right| \geq \delta (\lambda + |\xi|_{\partial'\mathcal{R}}), \forall \lambda > 0, \forall \xi \in \mathbb{R}^n.$$

Since the last inequality holds for all subsequential limit values of sequences $\{a_\alpha^0(x_M) : \alpha \in \mathcal{R}\}$ where $x_M \rightarrow \infty$, we obtain that there exist such constants $\delta > 0$ and $M > 0$ that the following holds

$$|\sum_{\alpha \in \mathcal{R}} a_\alpha^0(x) \xi^\alpha \lambda^{1-\max_i(\alpha; \mu^i)}| \geq \delta(\lambda + |\xi|_{\partial' \mathcal{R}}), \forall \lambda > 0, \forall \xi \in \mathbb{R}^n, |x| > M.$$

□

Further, we use the following criteria for the Fredholm property (equivalent formulation and proof can be found for example in work [16], Theorem 3.14):

Theorem 3.5. *Let A be a bounded linear operator acting from a Banach space X to a Banach space Y . Then the following holds:*

1. *if operator A has left regularizer, then kernel of operator A in X is finite dimensional;*
2. *if operator A has right regularizer, then the image of operator A is closed in Y and cokernel is finite dimensional;*
3. *operator A has left and right regularizers if and only if A is a Fredholm operator.*

It is easy to check that the following proposition holds:

Proposition 3.1. *Let $k \in \mathbb{Z}_+, q \in Q^{k, \mathcal{R}}, \varphi \in C_0^\infty(\mathbb{R}^n)$ and $P(x, \mathbb{D})$ be the differential form (1) with the coefficients that satisfy conditions (3). Then operator*

$$Tu := P(u\varphi) - \varphi Pu, u \in H_q^{k+1, \mathcal{R}}(\mathbb{R}^n)$$

is a compact operator acting from $H_q^{k+1, \mathcal{R}}(\mathbb{R}^n)$ to $H_q^{k, \mathcal{R}}(\mathbb{R}^n)$.

Lemma 3.1. *Let $k \in \mathbb{Z}_+, q \equiv 1, x_0 \in \mathbb{R}^n$ and $P(x, \mathbb{D})$ be regular in $\mathbb{R}^n$ differential form (1) with the coefficients that satisfy conditions (3). Then, there exists $\eta_0 > 0$, such that when $\max_{\alpha \in \partial' \mathcal{R}} \sup_{x \in \mathbb{R}^n} |a_\alpha(x) - a_\alpha(x_0)| < \eta_0$, for operator $P(x, \mathbb{D}) : H^{k+1, \mathcal{R}}(\mathbb{R}^n) \rightarrow H^{k, \mathcal{R}}(\mathbb{R}^n)$ exists an operator $R : H^{k, \mathcal{R}}(\mathbb{R}^n) \rightarrow H^{k+1, \mathcal{R}}(\mathbb{R}^n)$ such that*

$$RP(x, \mathbb{D}) = I + T_1, P(x, \mathbb{D})R = I + T_2,$$

where $T_1 : H^{k+1, \mathcal{R}}(\mathbb{R}^n) \rightarrow H^{k+1+\sigma, \mathcal{R}}(\mathbb{R}^n)$, $T_2 : H^{k, \mathcal{R}}(\mathbb{R}^n) \rightarrow H^{k+\sigma, \mathcal{R}}(\mathbb{R}^n)$ are bounded linear operators with some $\sigma = \sigma(\mathcal{R}) > 0$.

Proof. For differential form $P(x, \mathbb{D})$ (see (1)) denote

$$P^0(x, \mathbb{D}) = \sum_{\alpha \in \partial' \mathcal{R}} a_\alpha(x) D^\alpha, Q(x, \mathbb{D}) = \sum_{\alpha \in \mathcal{R} \setminus \partial' \mathcal{R}} a_\alpha(x) D^\alpha.$$

Consider

$$R^0 := F^{-1} \frac{|\xi|_{\partial' \mathcal{R}}}{(1 + |\xi|_{\partial' \mathcal{R}}) P^0(x_0, \xi)} F.$$

Since $P(x, \mathbb{D})$ is regular in $\mathbb{R}^n$, it is easy to check that $|\xi|_{\partial' \mathcal{R}}/P^0(x_0, \xi)$ is a Fourier multiplier (see [17]) and for operator $R^0 : H^{k, \mathcal{R}}(\mathbb{R}^n) \rightarrow H^{k+1, \mathcal{R}}(\mathbb{R}^n)$ holds

$$(24) \quad R^0 P^0(x_0, \mathbb{D}) = I + T',$$

where $T' : H^{k+1, \mathcal{R}}(\mathbb{R}^n) \rightarrow H^{k+2, \mathcal{R}}(\mathbb{R}^n)$ is a bounded linear operator.

Using (24) and estimates similar to the ones used in the proof of Lemma 4.2 from [18], it can be checked that there exists $\eta_0 > 0$ such that when $\max_{\alpha \in \partial' \mathcal{R}} \sup_{x \in \mathbb{R}^n} |a_\alpha(x) - a_\alpha(x_0)| < \eta_0$ the following holds

$$(25) \quad \begin{aligned} R^0 P(x, \mathbb{D}) &= R^0 P^0(x_0, \mathbb{D}) + R^0 (P^0(x, \mathbb{D}) - P^0(x_0, \mathbb{D})) + R^0 Q(x, \mathbb{D}) \\ &= I + T'_1 + T'_2, \end{aligned}$$

where $T'_1 : H^{k+1, \mathcal{R}}(\mathbb{R}^n) \rightarrow H^{k+1+\sigma, \mathcal{R}}(\mathbb{R}^n)$ with some $\sigma = \sigma(\mathcal{R}) > 0$ and operator $T'_2 : H^{k+1, \mathcal{R}}(\mathbb{R}^n) \rightarrow H^{k+1, \mathcal{R}}(\mathbb{R}^n)$ satisfies $\|T'_2\| < 1$.

Consider

$$R := (I + T'_2)^{-1} R^0.$$

From (25), we get

$$RP(x, \mathbb{D}) = I + T_1,$$

where $T_1 : H^{k+1, \mathcal{R}}(\mathbb{R}^n) \rightarrow H^{k+1+\sigma, \mathcal{R}}(\mathbb{R}^n)$ is a bounded linear operator with some $\sigma = \sigma(\mathcal{R}) > 0$.

Similarly, we get

$$P(x, \mathbb{D})R = I + T_2,$$

where $T_2 : H^{k, \mathcal{R}}(\mathbb{R}^n) \rightarrow H^{k+\sigma, \mathcal{R}}(\mathbb{R}^n)$ is a bounded linear operator with some $\sigma = \sigma(\mathcal{R}) > 0$. $\square$

Theorem 3.6. *Let $k \in \mathbb{Z}_+$, $q \in Q^{k, \mathcal{R}}$ and $P(x, \mathbb{D})$ be the differential form (10) with the coefficients that satisfy $\lim_{|x| \rightarrow \infty} \max_{|x-y| \leq 1} |a_\alpha^0(x) - a_\alpha^0(y)| = 0$ for all $\alpha \in \mathcal{R}$. Then, operator $P(x, \mathbb{D}) : H_q^{k+1, \mathcal{R}}(\mathbb{R}^n) \rightarrow H_q^{k, \mathcal{R}}(\mathbb{R}^n)$ is a Fredholm operator if and only if $P(x, \mathbb{D})$ is regular in $\mathbb{R}^n$ and there exist constants $\delta > 0$ and $M > 0$ such that*

$$(26) \quad \left| \sum_{\alpha \in \mathcal{R}} a_\alpha^0(x) \lambda^{1 - \max_i(\alpha; \mu^i)} \xi^\alpha \right| \geq \delta(\lambda + |\xi|_{\partial' \mathcal{R}}), \forall \xi \in \mathbb{R}^n, \lambda > 0, |x| > M.$$

Proof. Let's first prove sufficient part.

Let $\delta_0 > 0, \varphi(x) \in C_0^\infty(\mathbb{R}^n)$ be such that $0 \leq \varphi(x) \leq 1$ for all $x \in \mathbb{R}^n$ and $\varphi(x) = 1$ for $x \in K_{\frac{\delta_0}{2}}$, $\varphi(x) = 0$ for $|x| \geq \delta_0$ and $\psi \in C_0^\infty(\mathbb{R}^n)$ such that $\text{supp } \psi \subset K_{2\delta_0}$ and $\psi(x) = 1$ for $x \in K_{\delta_0}$. Let $\omega > 0$ be such that $\omega\sqrt{n} < \delta_0$. Let's denote $\{z_m\}_{m=0}^\infty$ points on the lattice in $\mathbb{R}^n$ with a side equals to ω .

Denote

$$\varphi_m(x) := \varphi(x - z_m) \left(\sum_{l=0}^{\infty} \varphi(x - z_l) \right)^{-1}, \psi_m(x) := \psi(x - z_m), \quad m \in \mathbb{Z}_+.$$

Then, $\{\varphi_m\}_{m=0}^{\infty}$ and $\{\psi_m\}_{m=0}^{\infty}$ satisfy the following conditions:

- (i) $\max_{x,y \in \text{supp } \varphi_m} |x - y| < \delta_0$,
- (ii) there exists $r \in \mathbb{N}$ such that for any number i there are no more than r functions $\varphi_j(x)$ such that $\text{supp } \varphi_i \cap \text{supp } \varphi_j \neq \emptyset$;
- (iii) $\varphi_m(x)\psi_m(x) \equiv \varphi_m(x)$ for all $m \in \mathbb{Z}_+$;
- (iv) for any $\alpha \in \mathbb{Z}_+^n$ there exists some constant $C_\alpha > 0$ such that $|D^\alpha \varphi_m(x)| \leq C_\alpha$, $|D^\alpha \psi_m(x)| \leq C_\alpha$, $\forall x \in \mathbb{R}^n, \forall m \in \mathbb{Z}_+$;
- (v) $\sum_{m=0}^{\infty} \varphi_m(x) \equiv 1$.

Denote $W_m = \text{supp } \varphi_m, m \in \mathbb{Z}_+$. Let $x_m \in W_m$ and $m_0 \in \mathbb{N}$. For $m \leq m_0$ denote

$$P^m(x, \mathbb{D}) := \sum_{\alpha \in \mathcal{R}} (\psi_m(x)(a_\alpha(x) - a_\alpha(x_m)) + a_\alpha(x_m)) D^\alpha.$$

For $m > m_0$, denote

$$P^m(x, \mathbb{D}) := \sum_{\alpha \in \mathcal{R}} [\psi_m(x)(a_\alpha^0(x)q(x)^{1-\max_i(\alpha:\mu^i)} - a_\alpha^0(x_m)q(x_m)^{1-\max_i(\alpha:\mu^i)}) + a_\alpha^0(x_m)q(x_m)^{1-\max_i(\alpha:\mu^i)}] D^\alpha.$$

Since $q \in Q^{k,\mathcal{R}}$ and $\lim_{m \rightarrow \infty} \max_{|x-x_m| \leq 1} |a_\alpha^0(x) - a_\alpha^0(x_m)| = 0$, according to Theorem 2.2, from [10], we can choose m_0 big enough such that for $m > m_0$ operator $P^m : H_q^{k+1,\mathcal{R}}(\mathbb{R}^n) \rightarrow H_q^{k,\mathcal{R}}(\mathbb{R}^n)$ has the inverse operator $R^m : H_q^{k,\mathcal{R}}(\mathbb{R}^n) \rightarrow H_q^{k+1,\mathcal{R}}(\mathbb{R}^n)$.

Since $P(x, \mathbb{D})$ is regular in $\mathbb{R}^n$, according to Lemma 3.1, we get that for a small enough δ_0 from the condition (i) and any $m \leq m_0$ there exists operator $R^m : H^{k,\mathcal{R}}(\mathbb{R}^n) \rightarrow H^{k+1,\mathcal{R}}(\mathbb{R}^n)$ such that

$$(27) \quad R^m P^m = I + T^m,$$

where $T^m : H^{k,\mathcal{R}}(\mathbb{R}^n) \rightarrow H^{k+\sigma,\mathcal{R}}(\mathbb{R}^n)$ with some number $\sigma = \sigma(\mathcal{R}) > 0$.

Denote

$$Rf := \sum_{l=0}^{\infty} \psi_l R^l(\varphi_l f), f \in H_q^{k,\mathcal{R}}(\mathbb{R}^n).$$

Since (26) holds one can check that the norms of operators R^l , acting from $H_q^{k,\mathcal{R}}(\mathbb{R}^n)$ to $H_q^{k+1,\mathcal{R}}(\mathbb{R}^n)$, are uniformly bounded. From this fact, taking into account that $\frac{1}{q(x)} \rightrightarrows 0$ when $|x| \rightarrow \infty$ and properties (i)–(v) of the functions

$\{\varphi_m\}_{m=0}^\infty, \{\psi_m\}_{m=0}^\infty$, it is easy to check that R is a bounded linear operator, acting from $H_q^{k,\mathcal{R}}(\mathbb{R}^n)$ to $H_q^{k+1,\mathcal{R}}(\mathbb{R}^n)$.

In (10) denote

$$L(x, \mathbb{D}) = \sum_{\alpha \in \mathcal{R}} a_\alpha^1(x) D^\alpha.$$

For $P(x, \mathbb{D})$ and $RP(x, \mathbb{D})$, using properties (i)–(v) of the functions $\{\varphi_m\}_{m=0}^\infty$ and $\{\psi_m\}_{m=0}^\infty$, we get

$$\begin{aligned} P(x, \mathbb{D})u &= \sum_{m=0}^\infty \varphi_m P(x, \mathbb{D})(\psi_m u) = \sum_{m=0}^{m_0} \varphi_m P^m(x, \mathbb{D})(\psi_m u) \\ &\quad + \sum_{m=m_0+1}^\infty \varphi_m P^m(x, \mathbb{D})(\psi_m u) + \sum_{m=m_0+1}^\infty \varphi_m L(x, \mathbb{D})(\psi_m u), \\ RP(x, \mathbb{D})u &= \sum_{l=0}^{m_0} \sum_{m=0}^{m_0} \psi_l R^l(\varphi_l \varphi_m P^m(\psi_m u)) \\ &\quad + \sum_{l=0}^{m_0} \sum_{m=m_0+1}^\infty \psi_l R^l(\varphi_l \varphi_m P^m(\psi_m u)) + \sum_{l=m_0+1}^\infty \sum_{m=0}^{m_0} \psi_l R^l(\varphi_l \varphi_m P^m(\psi_m u)) \\ (28) \quad &+ \sum_{l=m_0+1}^\infty \sum_{m=m_0+1}^\infty \psi_l R^l(\varphi_l \varphi_m P^m(\psi_m u)) \\ &\quad + \sum_{l=0}^\infty \sum_{m=m_0+1}^\infty \psi_l R^l(\varphi_l \varphi_m L(\psi_m u)), \end{aligned}$$

where $u \in H_q^{k+1,\mathcal{R}}(\mathbb{R}^n)$.

Based on definitions of $P^m(x, \mathbb{D})$ and properties of functions $\{\varphi_m\}_{m=0}^\infty, \{\psi_m\}_{m=0}^\infty$, the following equalities hold

- a) for $0 \leq l \leq m_0$ and $0 \leq m \leq m_0$,
 $\varphi_l \varphi_m P^m(x, \mathbb{D})(\psi_m u) = \varphi_l \varphi_m P(x, \mathbb{D})(\psi_m u) = \varphi_l \varphi_m P^l(x, \mathbb{D})(\psi_m u);$
- b) for $0 \leq l \leq m_0$ and $m > m_0$,
 $\varphi_l \varphi_m P^m(x, \mathbb{D})(\psi_m u) = \varphi_l \varphi_m (P(x, \mathbb{D}) - L(x, \mathbb{D}))(\psi_m u)$
 $= \varphi_l \varphi_m P^l(x, \mathbb{D})(\psi_m u) - \varphi_l \varphi_m L(x, \mathbb{D})(\psi_m u);$
- c) for $l > m_0$ and $0 \leq m \leq m_0$,
 $\varphi_l \varphi_m P^m(x, \mathbb{D})(\psi_m u) = \varphi_l \varphi_m P^l(x, \mathbb{D})(\psi_m u) + \varphi_l \varphi_m L(x, \mathbb{D})(\psi_m u);$
- d) for $l > m_0$ and $m > m_0$,
 $\varphi_l \varphi_m P^m(x, \mathbb{D})(\psi_m u) = \varphi_l \varphi_m P^l(x, \mathbb{D})(\psi_m u),$

where $u \in H_q^{k+1,\mathcal{R}}(\mathbb{R}^n)$.

Consider the first three sums in (28). They contain finite number of terms such that $\varphi_l \varphi_m \neq 0$. Using equalities a)-c), (27) and Proposition 3.1 similarly to the proof of Theorem 2.6 from work [12] it can be checked that the following holds:

$$\begin{aligned}
 (29) \quad & \sum_{l=0}^{m_0} \sum_{m=0}^{m_0} \psi_l R^l(\varphi_l \varphi_m P^m(\psi_m u)) = \sum_{l=0}^{m_0} \sum_{m=0}^{m_0} \varphi_l \varphi_m u + T_1 u, \\
 & \sum_{l=0}^{m_0} \sum_{m=m_0+1}^{\infty} \psi_l R^l(\varphi_l \varphi_m P^m(\psi_m u)) = \sum_{l=0}^{m_0} \sum_{m=m_0+1}^{m_1} \psi_l R^l(\varphi_l \varphi_m P^l(\psi_m u)) \\
 (30) \quad & - \sum_{l=0}^{m_0} \sum_{m=m_0+1}^{m_1} \psi_l R^l(\varphi_l \varphi_m L(x, \mathbb{D})(\psi_m u)) = \sum_{l=0}^{m_0} \sum_{m=m_0+1}^{\infty} \varphi_l \varphi_m u + T_2 u \\
 & - \sum_{l=0}^{m_0} \sum_{m=m_0+1}^{\infty} \psi_l R^l(\varphi_l \varphi_m L(x, \mathbb{D})(\psi_m u)), \\
 & \sum_{l=m_0+1}^{\infty} \sum_{m=0}^{m_0} \psi_l R^l(\varphi_l \varphi_m P^m(\psi_m u)) = \sum_{l=m_0+1}^{m_1} \sum_{m=0}^{m_0} \psi_l R^l(\varphi_l \varphi_m P^l(\psi_m u)) \\
 (31) \quad & + \sum_{l=m_0+1}^{m_1} \sum_{m=0}^{m_0} \psi_l R^l(\varphi_l \varphi_m L(x, \mathbb{D})(\psi_m u)) = \sum_{l=m_0+1}^{\infty} \sum_{m=0}^{m_0} \varphi_l \varphi_m u + T_3 u \\
 & + \sum_{l=m_0+1}^{m_1} \sum_{m=0}^{m_0} \psi_l R^l(\varphi_l \varphi_m L(x, \mathbb{D})(\psi_m u)),
 \end{aligned}$$

where $u \in H_q^{k+1, \mathcal{R}}(\mathbb{R}^n)$, $m_1 := \max_{m > m_0} \{m : \text{supp } \varphi_m \cap (\bigcup_{s=0}^{m_0} \text{supp } \varphi_s) \neq \emptyset\}$ and operators T_1, T_2 and T_3 are compact operators acting from $H_q^{k+1, \mathcal{R}}(\mathbb{R}^n)$ to $H_q^{k+1, \mathcal{R}}(\mathbb{R}^n)$.

From equality d), and the fact that, for $m > m_0$ operators $R^m : H_q^{k, \mathcal{R}}(\mathbb{R}^n) \rightarrow H_q^{k+1, \mathcal{R}}(\mathbb{R}^n)$ are the inverse operators of $P^m : H_q^{k+1, \mathcal{R}}(\mathbb{R}^n) \rightarrow H_q^{k, \mathcal{R}}(\mathbb{R}^n)$ we get

$$\begin{aligned}
 & \sum_{l=m_0+1}^{\infty} \sum_{m=m_0+1}^{\infty} \psi_l R^l(\varphi_l \varphi_m P^m(\psi_m u)) \\
 & = \sum_{l=m_0+1}^{\infty} \sum_{m=m_0+1}^{\infty} \psi_l R^l(\varphi_l \varphi_m P^l(\psi_m u)) = \sum_{l=m_0+1}^{\infty} \sum_{m=m_0+1}^{\infty} \varphi_l \varphi_m u \\
 & + \sum_{l=m_0+1}^{\infty} \sum_{m=m_0+1}^{\infty} \psi_l R^l(\varphi_l \varphi_m P^l(\psi_m u) - P^l(\varphi_l \varphi_m \psi_m u)),
 \end{aligned}$$

where $u \in H_q^{k+1, \mathcal{R}}(\mathbb{R}^n)$.

Taking into account (26), definitions of $P^l(x, \mathbb{D})$ and properties of functions $\{\varphi_m\}_{m=0}^{\infty}$ and $\{\psi_m\}_{m=0}^{\infty}$, for $l > m_0$ and $m > m_0$ with some constant $C_1 > 0$

we get

$$\begin{aligned}
& \|\varphi_l \varphi_m P^l(\psi_m u) - P^l(\varphi_l \varphi_m \psi_m u)\|_{k, \mathcal{R}, q} \\
& \leq C_1 \left\| \sum_{\alpha \in \mathcal{R}} \sum_{\substack{\beta + \gamma = \alpha, \\ |\gamma| > 0}} a_\alpha^0(x) D^\beta(\psi_m u) D^\gamma(\varphi_l \varphi_m) q(x)^{1 - \max_i(\alpha; \mu^i)} \right\|_{k, \mathcal{R}, q} \\
& \leq C_1 \left\| \sum_{\alpha \in \mathcal{R}} \sum_{\substack{\beta + \gamma = \alpha, \\ |\gamma| > 0}} a_\alpha^0(x) D^\gamma(\varphi_l \varphi_m) \frac{1}{q(x)^{\min_i(\gamma; \mu^i)}} D^\beta(\psi_m u) q(x)^{1 - \max_i(\beta; \mu^i)} \right\|_{k, \mathcal{R}, q}.
\end{aligned}$$

From the last inequiliaty, taking into account that $\frac{1}{q(x)} \rightrightarrows 0$ when $|x| \rightarrow \infty$, properties (i)–(v) of the functions $\{\varphi_m\}_{m=0}^\infty, \{\psi_m\}_{m=0}^\infty$ and the conditions on the coefficients $\{a_\alpha^0(x)\}$ (see (10)) we get

$$(32) \quad \|\varphi_l \varphi_m P^l(\psi_m u) - P^l(\varphi_l \varphi_m \psi_m u)\|_{k, \mathcal{R}, q} \leq \omega(m_0) \|u\|_{H_q^{k+1, \mathcal{R}}(W_l \cap W_m)},$$

where $\omega(m_0)$ is such a function that $\omega(m_0) \rightarrow 0$ when $m_0 \rightarrow \infty$. Since (26) holds, the norms of operators R^l , acting from $H_q^{k, \mathcal{R}}(\mathbb{R}^n)$ to $H_q^{k+1, \mathcal{R}}(\mathbb{R}^n)$, are uniformly bounded. Using this fact, inequality (32) and properties (i)–(v) of the functions $\{\varphi_m\}_{m=0}^\infty, \{\psi_m\}_{m=0}^\infty$, it is easy to check that for a big enough m_0 the operator

$$T_4 := \sum_{l=m_0+1}^\infty \sum_{m=m_0+1}^\infty \psi_l R^l[\varphi_l \varphi_m P^l(\psi_m \cdot) - P^l(\varphi_l \varphi_m \psi_m \cdot)],$$

acting from $H_q^{k+1, \mathcal{R}}(\mathbb{R}^n)$ to $H_q^{k+1, \mathcal{R}}(\mathbb{R}^n)$, satisfies $\|T_4\| < \frac{1}{2}$.

Similarly, for remained terms from (28), (30) and (31), taking into account that $D^\beta(a_\alpha^1(x)) = o(q(x)^{1 - \max_i(\alpha - \beta; \mu^i)})$ when $|x| \rightarrow \infty$, $\alpha \in \mathcal{R}, \beta \in k\mathcal{R}$, for a big enough m_0 we get that the operator

$$\begin{aligned}
T_5 &:= \sum_{l=0}^\infty \sum_{m=m_0+1}^\infty \psi_l R^l(\varphi_l \varphi_m L(\psi_m \cdot)) - \sum_{l=0}^{m_0} \sum_{m=m_0+1}^\infty \psi_l R^l(\varphi_l \varphi_m L(\psi_m \cdot)) \\
&+ \sum_{l=m_0+1}^\infty \sum_{m=0}^{m_0} \psi_l R^l(\varphi_l \varphi_m L(\psi_m \cdot)),
\end{aligned}$$

acting from $H_q^{k+1, \mathcal{R}}(\mathbb{R}^n)$ to $H_q^{k+1, \mathcal{R}}(\mathbb{R}^n)$, has a norm that satisfies $\|T_5\| < \frac{1}{2}$.

Denote

$$T' := T_1 + T_2 + T_3, T'' := T_4 + T_5.$$

From the representation (28), we get

$$R P u = \sum_{l=0}^\infty \sum_{m=0}^\infty \varphi_l \varphi_m u + T_1 u + T_2 u + T_3 u + T_4 u + T_5 u = u + T' u + T'' u,$$

where $u \in H_q^{k+1, \mathcal{R}}(\mathbb{R}^n)$, $T' : H_q^{k+1, \mathcal{R}}(\mathbb{R}^n) \rightarrow H_q^{k+1, \mathcal{R}}(\mathbb{R}^n)$ is a compact operator and for operator $T'' : H_q^{k+1, \mathcal{R}}(\mathbb{R}^n) \rightarrow H_q^{k+1, \mathcal{R}}(\mathbb{R}^n)$ we have $\|T''\| < 1$. Therefore,

$$(I + T'')^{-1}RP = I + (I + T'')^{-1}T',$$

where $T := (I + T'')^{-1}T' : H_q^{k+1, \mathcal{R}}(\mathbb{R}^n) \rightarrow H_q^{k+1, \mathcal{R}}(\mathbb{R}^n)$ is a compact operator. So we get that operator $(I + T'')^{-1}R : H_q^{k, \mathcal{R}}(\mathbb{R}^n) \rightarrow H_q^{k+1, \mathcal{R}}(\mathbb{R}^n)$ is a left regularizer.

Analogously we can construct a right regularizer.

Since right and left regularizers exist, applying Theorem 3.5, we obtain the Fredholm property for operator $P(x, \mathbb{D})$ acting from $H_q^{k+1, \mathcal{R}}(\mathbb{R}^n)$ to $H_q^{k, \mathcal{R}}(\mathbb{R}^n)$. For necessary part regularity of $P(x, \mathbb{D})$ follows from Corollary 3.1. Necessity of condition (26) for the Fredholm property of operator $P(x, \mathbb{D})$, acting from $H_q^{k+1, \mathcal{R}}(\mathbb{R}^n)$ to $H_q^{k, \mathcal{R}}(\mathbb{R}^n)$, follows from Theorem 3.3 and Theorem 3.4. $\square$

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Strong block-block domination of a graph

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Abstract. Let $B(G)$ denotes the set of all blocks of a graph G . Two blocks in G are adjacent if there is a common cutvertex incident on them. Two blocks $b_1, b_2 \in B(G)$ are said to *bb-dominate* each other if there is a common vertex incident with b_1 and b_2 . A set $L \subseteq B(G)$ is said to be a *bb-dominating set (BBD set)* if every block in G is bb-dominated by some block in L . The *bb-domination number* $\gamma_{bb} = \gamma_{bb}(G)$ is the cardinality of a minimum bb-dominating set of G .

In this paper we define strong (weak) bb-dominating set and strong (weak) bb-full set and obtained relationship between them. We also obtain the relation with existing graph parameters.

Keywords: strong bb-domination, weak bb-domination, strong bb-full domination, weak bb-full domination.

1. Introduction

The terminologies and notations used here are as in [10]. By a graph $G(V, E)$ we mean a connected finite simple graph of order p and size q . The independence number $\beta_0 = \beta_0(G)$ is the maximum order of a set $S \subseteq V$ in which no two vertices are adjacent. On the other hand, minimum number of vertices that cover all the edges of G is called vertex covering number $\alpha_0 = \alpha_0(G)$. These

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two parameters are related by $\alpha_0(G) + \beta_0(G) = p$ which is now referred as the classical Gallai's Theorem [10]. A set $S \subseteq V$ is a dominating set of G if every vertex not in S is adjacent to some vertex in S . The domination number $\gamma = \gamma(G)$ is the order of a minimum dominating set of G . The domination number is a well studied parameter in literature and for a survey refer [2], [7], [8]. Sampathkumar and Pushpa Latha [6] introduced strong (weak) dominating sets. For any two adjacent vertices u and v , we say that u strongly dominates v if $\deg(u) \geq \deg(v)$. A set $D \subseteq V$ is a strong dominating set (sd-set) if every vertex $v \in V - D$ is strongly dominated by some $u \in D$. The strong domination number $\gamma_s = \gamma_s(G)$ is the order of a minimum sd-set of G . Similarly, weak domination number $\gamma_w = \gamma_w(G)$ is defined. The strong domination is later studied in [3], [4]. Similar to strong (weak) domination S. S. Kamath and R. S. bhat [5] studied strong (weak) independent sets. A vertex $v \in V$ is a cutvertex if $G - \{v\}$ is disconnected. A graph which has no cutvertex is called non separable. A maximal non-separable subgraph is a block of G . Let $B(G)$ and $C(G)$ respectively denote the set of all blocks and cutvertices of G with $|B(G)| = m$ and $|C(G)| = n$. Two blocks in G are adjacent if there is a common cutvertex incident on them. A block-graph $B_G(G)$ is a graph with vertex set $B(G)$ and any two vertices in $B_G(G)$ are adjacent if and only if corresponding blocks are adjacent in G . P. G. Bhat and R. S. Bhat [1] defined bb-dominating sets. Two blocks $b_1, b_2 \in B(G)$ are said to *bb-dominate* each other if there is a common vertex incident with b_1 and b_2 . A set $L \subseteq B(G)$ is said to be a *bb-dominating set* (*BBD set*) if every block in G is bb-dominated by some block in L . The *bb-domination number* $\gamma_{bb} = \gamma_{bb}(G)$ is the cardinality of a minimum bb-dominating set of G . For any tree T , $\gamma_{bb}(T) = \gamma'(T)$ where $\gamma'(T)$ is the line domination number of T . A set $L \subseteq B(G)$ is *bb-full* if every block in L is adjacent to some block in $B(G) - L$. Then *bb-full number* $f_{bb} = f_{bb}(G)$ is the cardinality of a maximum bb-full set of G .

2. Strong bb-degree, weak bb-degree and regular bb-degree of a vertex

A block $g \in B(G)$ strongly (weakly) b-dominates a block $h \in B(G)$ if g is adjacent to h and $d_{bb}(g) \geq d_{bb}(h)$ ($d_{bb}(g) \leq d_{bb}(h)$). Then strong (weak) bb-degree of a block g , denoted as $d_{sbb}(g)$ ($d_{wbb}(g)$) is the number of blocks strongly (weakly) b-dominated by a block g . A block $g \in B(G)$ regularly b-dominates a block $h \in B(G)$ if g is adjacent to h and $d_{bb}(g) = d_{bb}(h)$. Then regular bb-degree of a block g , denoted as $d_{rbb}(g)$ is the number of blocks regularly b-dominated by a block g . Also $\Delta_{sbb}(G) = \max_{h \in B(G)} \{d_{sbb}(h)\}$ and $\delta_{sbb}(G) = \min_{h \in B(G)} \{d_{sbb}(h)\}$. Analogously other parameters are defined.

Example 2.1. In the Figure 1, The first, second and third elements of a vertex label represent strong bb-degree, weak bb-degree and regular bb-degree of the corresponding vertices respectively. For the graph G in the above Figure 1, $\Delta_{sbb}(G) = 4$, $\Delta_{wbb}(G) = 1$, $\Delta_{rbb}(G) = 0$ and $\delta_{sbb}(G) = \delta_{wbb}(G) = \delta_{rbb}(G) = 0$.

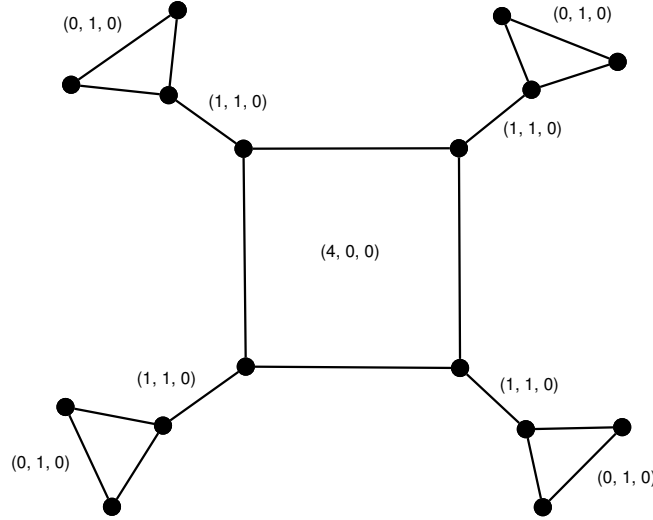


Figure 1: Strong bb-degree, Weak bb-degree and Regular bb-degree of a vertex

These various degrees are related by the following proposition.

Proposition 2.1. *Let $G = (V, E)$ be a graph. Then for any block $h \in B(G)$*

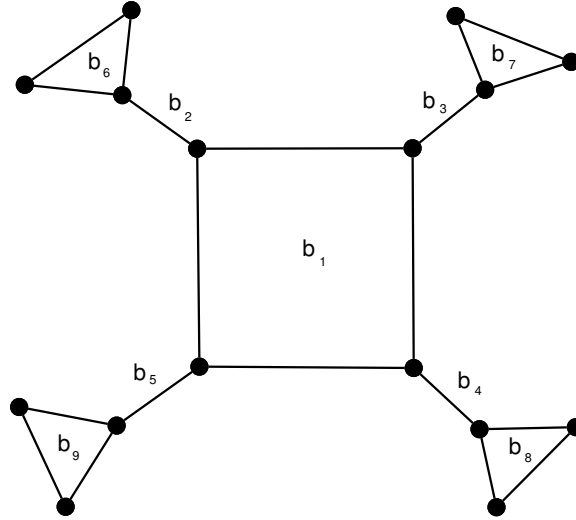
$$(1) \quad d_{bb}(h) = d_{sbb}(h) + d_{wbb}(h) - d_{rbb}(h).$$

Proof. For any block $h \in B(G)$, Let A =set of blocks bb-dominated by h , S =set of blocks strongly bb-dominated by h , W =set of blocks weakly bb-dominated by h and R =set of blocks regularly bb-dominated by h . We observe that $A = S \cup W$, $S \cap W = R$. Therefore $d_{bb}(h) = |A| = |S \cup W| = |S| + |W| - |S \cap W| = d_{sbb}(h) + d_{wbb}(h) - d_{rbb}(h)$. $\square$

2.1 bb-strong number, bb-weak number, bb-balanced number and bb-regular number

A block $g \in B(G)$ is said to be bb-strong block (bb-weak block) if $d_{bb}(g) \geq d_{bb}(h)$ ($d_{bb}(g) \leq d_{bb}(h)$) for every h adjacent g . A block $g \in B(G)$ is said to be bb-balanced block if it is neither bb-strong block nor b-weak block. A block $g \in B(G)$ is said to be bb-regular block if it is both bb-strong block and bb-weak block.

A set $L \subseteq B(G)$ is said to be bb-strong, bb-weak, bb-balanced and bb-regular set if every block in L is respectively bb-strong, bb-weak, bb-balanced and bb-regular block in G . The bb-strong number $s_{bb} = s_{bb}(G)$, The bb-weak number $w_{bb} = w_{bb}(G)$, The bb-balanced number $b_{bb} = b_{bb}(G)$ and The bb-regular number $r_{bb} = r_{bb}(G)$ are respectively the cardinalities of the bb-strong, bb-weak, bb-balanced and bb-regular set of G . A block $g \in B(G)$ is strictly bb-strong (strictly bb-weak) if $d_{bb}(g) > d_{bb}(h)$ ($d_{bb}(g) < d_{bb}(h)$) for every h adjacent to g .

Figure 2: Graph G

Example 2.2. In the Figure 2, the bb-strong set is $S = \{b_1\}$ and $s_{bb}(G) = 1$. The bb-weak set is $W = \{b_6, b_7, b_8, b_9\}$ and $w_{bb}(G) = 4$. The bb-balanced set is $B = \{b_2, b_3, b_4, b_5\}$ and $b_{bb}(G) = 4$. The bb-regular set is $R = \{\}$ and $r_{bb}(G) = 0$.

Proposition 2.2. For any graph $G = (V, E)$ with m blocks,

$$(2) \quad s_{bb}(G) + w_{bb}(G) + b_{bb}(G) - r_{bb}(G) = m.$$

Proof. Let G be a graph with m blocks. Let S_{bb} be the bb-strong set, W_{bb} be the bb-weak set, B_{bb} be the bb-balanced set and R_{bb} be the bb-regular set. We have $S_{bb} \cap W_{bb} = R_{bb}$, by definition. Since any bb-balanced block is neither bb-strong nor bb-weak, therefore it cannot be either in S_{bb} set or in W_{bb} set. Therefore $(S_{bb} \cup W_{bb}) \cap B_{bb} = \phi$. Also we have $B(G) = (S_{bb} \cup W_{bb}) \cup B_{bb}$. Then $|B(G)| = m = |S_{bb} \cup W_{bb}| + |B_{bb}| = |S_{bb}| + |W_{bb}| - |S_{bb} \cap W_{bb}| + |B_{bb}| = s_{bb}(G) + w_{bb}(G) - r_{bb}(G) + b_{bb}(G)$. $\square$

3. Strong (weak) bb-dominating sets of a graph

Two blocks $b_1, b_2 \in B(G)$ strongly (weakly) bb-dominates each other if there is a common cutvertex incident on b_1 and b_2 , and $d_{bb}(b_1) \geq d_{bb}(b_2)$ ($d_{bb}(b_1) \leq d_{bb}(b_2)$). A set $L \subseteq B(G)$ is said to be a strong block-block dominating set (SBBD-set) (weak block-block dominating set (WBBD-set)) if every block in $B(G) - L$ is strongly (weakly) bb-dominated by some block in L . The strong block-block domination number $\gamma_{sbb} = \gamma_{sbb}(G)$ (weak block-block domination number $\gamma_{wbb} = \gamma_{wbb}(G)$) is the cardinality of a minimum SBBD-set (WBBD-set) of G .

A set $L \subseteq B(G)$ is said to be strong bb-full set (SBBF-set) (weak bb-full set (WBBF-set)) if every block in L strongly (weakly) bb-dominates at least

one block in $B(G) - L$. The strong bb-full number $f_{sbb} = f_{sbb}(G)$ (weak bb-full number $f_{wbb} = f_{wbb}(G)$) is the cardinality of a maximum SBBF-set (WBBF-set) of G .

Example 3.1. In the Figure 2, BBD-set is $S_1 = \{b_2, b_3, b_4, b_5\}$, and BBF-set is $B(G) - S_1$. Therefore $\gamma_{bb}(G) = 4$ and $f_{bb}(G) = 5$. Also SBBD-set is $S_2 = \{b_1, b_2, b_3, b_4, b_5\}$ and WBBF-set is $B(G) - S_2$. Therefore $\gamma_{sbb}(G) = 5$ and $f_{wbb}(G) = 4$. Also WBBD-set is $S_3 = \{b_6, b_7, b_8, b_9, b_2\}$ and SBBF-set is $B(G) - S_3$. Therefore $\gamma_{wbb}(G) = 5$ and $f_{sbb}(G) = 4$.

3.1 Gallai type results

It is known that for any graph G with m blocks, $\gamma_{bb}(G) + f_{bb}(G) = m$. We now obtain similar results for the new parameters defined.

Observation 3.1. For any set $L \subseteq B(G)$,

- (i) L is SBBF-set if, and only if, $B(G) - L$ is a WBBD-set,
- (ii) L is WBBF-set if, and only if, $B(G) - L$ is a SBBD-set.

Analogous to Gallai's theorem, we now prove the following.

Proposition 3.1. For any graph $G = (V, E)$ with m blocks,

$$(3) \quad \gamma_{sbb}(G) + f_{wbb}(G) = m,$$

$$(4) \quad \gamma_{wbb}(G) + f_{sbb}(G) = m.$$

Proof. Let S be a minimum SBBD set of G . Then $B(G) - S$ is an WBBF set by Observation 3.1. Hence $f_{wbb}(G) \geq |B(G) - S|$. Therefore $\gamma_{sbb}(G) + f_{wbb}(G) \geq m$. Again, if D is a maximum WBBF set of G , then $B(G) - D$ is an SBBD set by Observation 3.1. Hence $\gamma_{sbb}(G) + f_{wbb}(G) \leq m$. Then (3) follows from the above inequalities. The proof of (4) is similar. $\square$

Proposition 3.2. For any graph $G = (V, E)$,

$$(5) \quad \gamma_{sbb}(G) \leq \gamma_{wbb}(G),$$

$$(6) \quad f_{sbb}(G) \leq f_{wbb}(G).$$

Proof. Proof is by induction on the number of cutvertices n of G . If G contains only one cutvertex, then clearly $\gamma_{sbb}(G) = \gamma_{wbb}(G) = 1$. If G contains two cutvertices say u_1 and u_2 . Then observe that there is only one support block say b which is incident on u_1 and u_2 , and is adjacent to all the remaining blocks of G . Hence b strongly bb-dominates all the blocks of G . Therefore $\gamma_{sbb}(G) = 1$. On the other hand, in order to weakly bb-dominates all the blocks of G , we have to consider two pendant blocks say b_1 and b_2 which are incident on u_1 and u_2 respectively. Hence $\gamma_{wbb}(G) = 2$. Therefore $\gamma_{sbb}(G) < \gamma_{wbb}(G)$. Assume that

the result is true for $n = k$. Now we prove the result for $n = k + 1$ with $n \geq 3$. Now, consider a graph G with $n = k + 1$ cutvertices. Observe that there is at least one cutvertex u incident on only one support block and any number of pendant blocks. Let b_1 is any one pendant block incident on the cutvertex u . Now, removal of all pendant blocks incident on a cutvertex u from G results in a graph G' with $n = k$ cutvertices. Let W' be γ_{wbb} -set of G' , then $W = W' \cup \{b_1\}$ is the γ_{wbb} -set of G . Observe that, number of blocks in γ_{wbb} -set of G is one greater than the number of blocks in γ_{wbb} -set of G' . Now, Let S' be the γ_{sbb} -set of G' . Then $S = S' \cup \{b\}$ is the γ_{sbb} -set of G with cardinality one greater than that of S' of G' . Therefore $|S| \leq |W|$, i.e. $\gamma_{svb}(G) \leq \gamma_{wvb}(G)$. Hence the result is true for $n = k + 1$. Therefore by principle of mathematical induction, the result is true for all n . (6) follows from the Proposition (3.1). $\square$

Corollary 3.1. *For any graph $G = (V, E)$ with m blocks,*

$$(7) \quad \gamma_{wbb}(G) + f_{wbb}(G) \geq m,$$

$$(8) \quad \gamma_{sbb}(G) + f_{sbb}(G) \leq m.$$

Proof. Since $\gamma_{sbb}(G) + f_{wbb}(G) = m$ from the Proposition 3.1. Also $\gamma_{sbb}(G) \leq \gamma_{wbb}(G)$ from the Proposition 3.2. Then (7) follows from the above inequalities. Proof of (8) is similar. $\square$

3.2 Bounds on strong (weak) bb-dominating sets

We now obtain some elementary bounds for $\gamma_{sbb}(G)$ and $\gamma_{wbb}(G)$.

For any $h \in B(G)$, $N_{bb}(h) = \{g \in B(G) \mid g \text{ is adjacent to } h\}$. Let $\Delta_{bb}(G)$ and $\delta_{bb}(G)$ respectively be the maximum and minimum bb-degrees of a graph G .

Proposition 3.3. *For any graph G with m vertices,*

$$(9) \quad \gamma_{bb}(G) \leq \gamma_{sbb}(G) \leq m - \Delta_{bb}(G),$$

$$(10) \quad \gamma_{bb}(G) \leq \gamma_{wbb}(G) \leq m - \delta_{bb}(G),$$

$$(11) \quad \Delta_{bb}(G) \leq f_{wbb}(G) \leq f_{bb}(G),$$

$$(12) \quad \delta_{bb}(G) \leq f_{sbb}(G) \leq f_{bb}(G).$$

Proof. Since every SBBD-set or WBBD-set is a BBD-set, we have $\gamma_{bb}(G) \leq \gamma_{sbb}(G)$ and $\gamma_{bb}(G) \leq \gamma_{wbb}(G)$. For any $g, h \in B(G)$, let $d_{bb}(g) = \Delta_{bb}(G)$ and $d_{bb}(h) = \delta_{bb}(G)$. It is clear that $B(G) - N_{bb}(g)$ is a SBBD-set and $B(G) - N_{bb}(h)$ is a WBBD-set. Hence bound in (9) and (10) follows. (11) and (12) follows from the Proposition 3.1. $\square$

For any block $h \in B(G)$, be-degree $d_{be}(h)$ is the number of edges in the block h and $\delta_{be}(G) = \min_{h \in B(G)} d_{be}(h)$.

Proposition 3.4. *For any graph $G = (V, E)$ with q edges,*

$$(13) \quad \gamma_{bb}(G) \leq \gamma_{sbb}(G) \leq \frac{q}{\delta_{be}(G)}.$$

Proof. Since every SBBD-set is a BBD-set, we have $\gamma_{bb}(G) \leq \gamma_{sbb}(G)$. Since edges of G can be partitioned into blocks of G , we have $\sum_{h \in B(G)} d_{be}(h) = q \geq m\delta_{be}(G)$. As set of all blocks $B(G)$ is a SBBD-set of G , we have $\gamma_{sbb}(G) \leq |B(G)| = m \leq \frac{q}{\delta_{be}(G)}$ $\square$

4. Conclusion

Block domination is a well-studied parameter in literature. We modified and studied strong block-block domination in graphs. Few Gallai's theorem type results are obtained. Several bounds for strong block-block domination parameters are obtained. We also obtained the relationship between these newly defined parameters. Characterization of the graphs attaining these bounds are not studied in full and one may take this as an open problem for further research.

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Composition of Abel-Grassmann's strong root of band

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Abstract. As the role of band in the study of semigroups, AG-band plays an important role in the study of AG-groupoids. In this paper, the concept of strong root of band which is an extension of AG-band, is proposed and its structure is studied in depth. We investigate decomposition of strong root of band induced by an equivalence relation and prove that every strong root of band is the disjoint union of its sub-AG-groups. Furthermore, two different ways how to make a strong root of band are obtained which illuminate the structure of strong root of band.

Keywords: strong root of band, root of band, AG-3-band, AG-4-band, AG-band.

1. Introduction

The concept of an Abel-Grassmann's groupoid (AG-groupoid) was first given by Kazim and Naseeruddin [1], in 1972. Moreover, further AG-groupoid theoretical studies can be found in [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14] and some important results have been gotten.

As in Semigroup Theory, bands and band decompositions [15, 16, 17, 18] are one of the most useful method for research on AG-groupoids. In [16], the concept of AG-3-band was introduced and its basic properties are investigated. AG-3-band is the disjoint union of its sub Abelian groups. In [18], the concept of root of band and unipotent radical were introduced, and the decomposition theorem of root of band was proved. As a continuation of [18], we propose strong root of band, study its properties, and analyze its relationship with other AG-groupoid bands.

The rest of this paper is arranged as follows. In Section 2, some definitions and properties on AG-groupoid bands are given. The relationships among strong

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root of band and other AG-groupoid bands are discussed in Section 3. We establish the decomposition theorem and construction methods of strong root of band in Section 4. Lastly, Section 5 presents the summary and the direction of future efforts.

2. Preliminaries

This section contains the related research and results of AG-groupoid bands and some related notions are introduced first.

An AG-groupoid is a generalization of commutative semigroup. A groupoid $(G, *)$ is called an AG-groupoid if it holds the left invertive law, that is, for all $a, b, c \in G$, $(a * b) * c = (c * b) * a$. In an AG-groupoid $(G, *)$ the medial law holds, for all $a, b, c, d \in G$, $(a * b) * (c * d) = (a * c) * (b * d)$. In an AG-groupoid $(G, *)$, $\forall x \in G$, $k \in \mathbb{Z}^+$, x^k is defined as follows: $x^1 = x$, $x^2 = x * x$, $x^3 = x^2 * x$, $x^4 = x^3 * x$, ..., $x^k = x^{k-1} * x$.

Let G be an AG-groupoid, then a is an idempotent in G if $a \in G$, $a^2 = a$. The set of all idempotents in G is denoted by $E(G)$. An AG-groupoid G is called an AG-band if $G = E(G)$. For an AG-groupoid G we define the set $\sqrt{E(G)} = \{a \in G : a^2 * a^2 = a^2\}$.

Definition 2.1 ([18]). *An AG-groupoid G with a unique idempotent $e \in G$ is called a unipotent radical if for all $a \in G$, $a^2 = e$.*

Definition 2.2 ([18]). *The AG-groupoid G is called a root of band if $\sqrt{E(G)} = G$.*

Definition 2.3 ([16]). *An AG-groupoid $(G, *)$ is called an AG-3-band, if for all $a \in G$, $a^2 * a = a * a^2 = a$.*

Definition 2.4 ([19]). *An AG-groupoid G with a left identity e is called an AG-group if every $a \in G$ has an inverse element a' .*

Definition 2.5 ([18]). *An AG-groupoid $(G, *)$ is called an AG-4-band, if for all $a \in G$, $a^3 * a = a * a^3 = a^2 * a^2 = a * (a * a^2) = (a * a^2) * a = a^2$, that is, all products of a of length 4 are equal to a^2 .*

3. The relationships among strong root of band and other bands

Definition 3.1. *An AG-groupoid $(G, *)$ is called a strong root of band if it satisfies: $\forall a \in G$, $a^2 * a^2 = a^2$, $a^2 * a = a$.*

Theorem 3.1. *Let $(G, *)$ be a groupoid. Then, G is an AG-3-band if it is both a strong root of band and an AG-4-band.*

Proof. Suppose G is both a strong root of band and an AG-4-band, from Definition 3.1 and Definition 2.5, for any $a \in G$, $a^3 * a = a * a^3 = a^2 * a^2 = a * (a * a^2) = (a * a^2) * a = a^2$, $a^2 * a = a$.

we have

$$\begin{aligned}
 a * a^2 &= (a^2 * a) * (a^2 * a^2) \\
 &= (a^2 * a^2) * (a * a^2) \quad (\text{by the medial law}) \\
 &= a^2 * (a * a^2) \\
 &= (a * a) * (a * a^2) \\
 &= ((a * a^2) * a) * a \quad (\text{by the left invertive law}) \\
 &= a^2 * a = a.
 \end{aligned}$$

By Definition 2.3, G is an AG-3-band. $\square$

Theorem 3.2. *Let $(G, *)$ be a root of band. Then, G is an AG-3-band if it satisfies: $\forall a \in G, a * a^2 = a$.*

Proof. Suppose G is a root of band, from Definition 2.2, for any $a \in G$, $a^2 * a^2 = a^2$. Being $a * a^2 = a$, we have $a^2 * a = (a^2 * a^2) * a = (a * a^2) * a^2 = a * a^2 = a$. By Definition 2.3, G is an AG-3-band. $\square$

From Definition 2.2, Definition 2.5 and Definition 3.1, it is easy to see that both an AG-4-band and a strong root of band are a root of band. The following example illustrates that a root of band may be neither an AG-4-band nor a strong root of band.

Example 3.1. Let $G = \{1, 2, 3, 4, 5, 6, 7\}$, the definition of operation $*$ on G is shown in Table 1. From Definition 2.2 G is a root of band. However, $(2 * 2^2) * 2 = 3 \neq 2^2$, $4^2 * 4 = 3 \neq 4$, from Definition 2.5 and Definition 3.1 it is neither an AG-4-band nor a strong root of band.

Table 1: The operation table of Example 3.1.

$*$	1	2	3	4	5	6	7
1	1	2	3	3	1	1	1
2	3	1	2	2	3	3	3
3	2	3	1	1	2	2	2
4	2	3	1	1	2	2	2
5	1	2	3	3	5	1	1
6	1	2	3	3	1	6	1
7	1	2	3	3	1	1	7

From Definition 2.5 and Definition 3.1, the notions of AG-4-band and strong root of band are different. The following examples illustrate that an AG-4-band may not be a strong root of band and a strong root of band may not be an AG-4-band.

Example 3.2. Let $G = \{1, 2, 3, 4, 5, 6, 7\}$, the definition of operation $*$ on G is shown in Table 2. From Definition 2.5 G is an AG-4-band. However, $2^2 * 2 = 1 \neq 2$, thus, From Definition 3.1 it is not a strong root of band.

Table 2: The operation table of Example 3.2.

$*$	1	2	3	4	5	6	7
1	1	1	1	4	1	1	1
2	1	1	3	4	1	1	3
3	1	2	1	4	1	1	2
4	4	4	4	1	4	4	4
5	1	1	1	4	5	1	1
6	1	1	1	4	1	6	1
7	1	2	3	4	1	1	7

Example 3.3. Let $G = \{1, 2, 3, 4, 5, 6, 7\}$, the definition of operation $*$ on G is shown in Table 3. From Definition 3.1 G is a strong root of band. However, $(3 * 3^2) * 3 = 4 \neq 3^2$, thus, From Definition 2.5 it is not an AG-4-band.

Table 3: The operation table of Example 3.3.

$*$	1	2	3	4	5	6	7
1	1	1	3	4	1	1	1
2	1	2	3	4	1	1	2
3	4	4	1	3	4	4	4
4	3	3	4	1	3	3	3
5	1	1	3	4	5	1	1
6	1	1	3	4	1	6	6
7	1	2	3	4	1	6	7

Figure 1 can be used to express the relationships among strong root of band and other bands. Here, A represents an AG-band; B represents an AG-3-band, but it is not an AG-band; as shown in Example 3.2, C represents an AG-4-band, but it is not a strong root of band; as shown in Example 3.3, D represents a strong root of band, but it is not an AG-4-band; and as shown in Example 3.1, E represents a root of band, but it is neither a strong root of band nor an AG-4-band. $A + B$ represents an AG-3-band, $A + B + C$ represents an AG-4-band, $A + B + D$ represents a strong root of band, and $A + B + C + D + E$ represents a root of band.

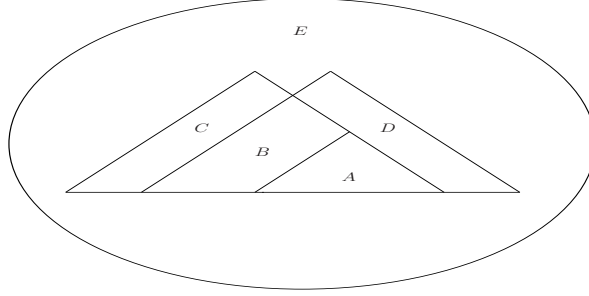


Figure 1: The relationships among strong root of band and other bands.

4. Band decomposition of strong root of band

In [18], a relation $\approx$ on arbitrary AG-groupoids was introduced as follows,

$$(1) \quad a \approx b \Leftrightarrow a^2 = b^2.$$

From the binary $\approx$ definition, it is easy to verify that $\approx$ has the properties of reflexive, symmetric and transitive. Thus, it is an equivalence relation.

From now on we assume that G is an AG-groupoid. Let Y be an AG-band, $Y \subset G$, for any $\alpha \in Y$, the equivalent class contained α will be denoted by S_α , and the elements of S_α will be denoted by $a_\alpha, b_\alpha, \dots$.

Theorem 4.1. *Let G be an AG-groupoid, Y be an AG-band, $Y \subset G$. G is a strong root of band if and only if $G = \bigcup_{\alpha \in Y} S_\alpha$ where S_α is an AG-group satisfying $a_\alpha^2 = e_\alpha$ (e_α is the left identity of S_α) for all $a_\alpha \in S_\alpha$.*

Proof. Let G be an AG-groupoid, Y be an AG-band, $Y \subset G$. $\forall a_\alpha, b_\alpha, c_\alpha \in S_\alpha$, the left invertive law holds directly.

$\forall \alpha \in Y$, let $e_\alpha = \alpha$, we have $e_\alpha \in S_\alpha$ and e_α is unique idempotent of S_α .

$\forall a_\alpha, b_\alpha \in S_\alpha$, we have $a_\alpha^2 = b_\alpha^2 = e_\alpha$, $(a_\alpha * b_\alpha) * (a_\alpha * b_\alpha) = (a_\alpha * a_\alpha) * (b_\alpha * b_\alpha) = e_\alpha^2 = e_\alpha$. Thus, $a_\alpha * b_\alpha \in S_\alpha$.

$\forall a_\alpha \in S_\alpha$, $a_\alpha^2 = e_\alpha^2 = e_\alpha$, since G is a strong root of band, by Definition 3.1, we have $a_\alpha^2 * a_\alpha = a_\alpha$, $e_\alpha * a_\alpha = a_\alpha^2 * a_\alpha = a_\alpha$. Thus, e_α is a left identity of S_α and a_α is an inverse element of a_α . From Definition 2.4, S_α is a sub-AG-group of G . Since e_α is unique idempotent of S_α , $G = \bigcup_{\alpha \in Y} S_\alpha$.

Conversely, suppose that G is an AG-groupoid, Y is an AG-band, $Y \subset G$. If $G = \bigcup_{\alpha \in Y} S_\alpha$, S_α is an AG-group satisfying $a_\alpha^2 = e_\alpha$ for all $a_\alpha \in S_\alpha$. We are going to prove that G is a strong root of band.

Let $x \in G$ be an arbitrary element, then there exist $\alpha \in Y$ such that $x \in S_\alpha$. Since e_α is the left identity element of S_α , we have $x^2 = e_\alpha$ and $(x * x) * x = e_\alpha * x = x$. Moreover, $x^2 * x^2 = e_\alpha * e_\alpha = e_\alpha = x^2$, thus G is a strong root of band. $\square$

Example 4.1. Let $G = \{1, 2, 3, 4, 5, 6, 7, 8\}$, the definition of operation $*$ on G is shown in Table 4. From Definition 3.1, G is a strong root of band. An

AG-band $Y = \{1, 6\}$ and $Y \subset G$. $S_1 = \{1, 2, 3, 4, 5\}$ and $S_6 = \{6, 7, 8\}$, both S_1 and S_6 are sub-AG-groups of G . It is easy to verify that $G = S_1 \cup S_6$.

Table 4: The operation table of Example 4.1.

*	1	2	3	4	5	6	7	8
1	1	2	3	4	5	1	1	1
2	3	1	5	2	4	3	3	3
3	2	4	1	5	3	2	2	2
4	5	3	4	1	2	5	5	5
5	4	5	2	3	1	4	4	4
6	1	2	3	4	5	6	7	8
7	1	2	3	4	5	8	6	7
8	1	2	3	4	5	7	8	6

Theorem 4.2. Let $(G, *)$ be a groupoid, Y be an AG-band, $Y \subset G$. $G = \bigcup_{\alpha \in Y} S_\alpha$, $(S_\alpha, *)$ is an AG-group, for all $a_\alpha \in S_\alpha$, $a_\alpha^2 = e_\alpha$ (e_α is the left identity of S_α) and $\forall \alpha, \beta \in Y, \alpha \neq \beta$, $S_\alpha \cap S_\beta = \emptyset$. If $\forall a_\alpha \in S_\alpha, \forall b_\beta \in S_\beta$, $a_\alpha * b_\beta = a_\alpha * e_\alpha, b_\beta * a_\alpha = a_\alpha$, then G is a strong root of band.

Proof. Suppose $G = \bigcup_{\alpha \in Y} S_\alpha$ is the groupoid, Y is an AG-band, for each $\alpha \in Y$, S_α is an AG-group and $S_\alpha \cap S_\beta = \emptyset$ if $\alpha \neq \beta$ in Y .

We first prove that $(G, *)$ is an AG-groupoid. Let $a_\alpha \in S_\alpha$, $b_\beta \in S_\beta$ and $c_\gamma \in S_\gamma$ be arbitrary elements. Suppose e_α is the left identity of S_α , being S_α is an AG-group, we have $a_\alpha * e_\alpha \in S_\alpha$. Moreover,

$$\begin{aligned}
 (a_\alpha * b_\beta) * c_\gamma &= (a_\alpha * e_\alpha) * c_\gamma \\
 &= (a_\alpha * e_\alpha) * e_\alpha \\
 &= (e_\alpha * e_\alpha) * a_\alpha \quad (\text{by the left invertive law}) \\
 &= e_\alpha * a_\alpha \\
 &= a_\alpha,
 \end{aligned}$$

$(c_\gamma * b_\beta) * a_\alpha = b_\beta * a_\alpha = a_\alpha = (a_\alpha * b_\beta) * c_\gamma$. Since S_α is an AG-group, the left invertive law holds directly for elements $a_\alpha, b_\alpha, c_\alpha \in S_\alpha$. Thus, G is an AG-groupoid.

Let $a_\alpha \in S_\alpha$ be an arbitrary element. Since e_α is the left identity of S_α and $a_\alpha^2 = e_\alpha$, it follows that $(a_\alpha * a_\alpha) * a_\alpha = e_\alpha * a_\alpha = a_\alpha$, and $a_\alpha^2 * a_\alpha^2 = e_\alpha * e_\alpha = e_\alpha = a_\alpha^2$. Consequently, G is a strong root of band. $\square$

Example 4.2. Let $G = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, the definition of operation $*$ on G is shown in Table 5. An AG-band $Y = \{1, 4, 7\}$ and $Y \subset G$. $S_1 = \{1, 2, 3\}$, $S_4 = \{4, 5, 6\}$, $S_7 = \{7, 8, 9, 10\}$, S_1, S_4 and S_7 are sub-AG-groups of G . It is easy to verify that $G = S_1 \cup S_4 \cup S_7$. For any $a_1 \in S_1, b_4 \in S_4, c_7 \in S_7$, without

losing generality, let $a_1 = 2, b_4 = 5, c_7 = 9$, we can get $2 * 5 = 2 * 1, 5 * 2 = 1 * 2; 2 * 9 = 2 * 1, 9 * 2 = 1 * 2; 5 * 9 = 5 * 4, 9 * 5 = 4 * 5$; and $(2 * 5) * 9 = (9 * 5) * 2$. The other cases can be verified, thus G is an AG-groupoid. $\forall x \in G$, we have $x^2 * x^2 = x^2, x^2 * x = x$. From Definition 3.1, G is a strong root of band.

Table 5: The operation table of Example 4.2.

*	1	2	3	4	5	6	7	8	9	10
1	1	2	3	1	1	1	1	1	1	1
2	3	1	2	3	3	3	3	3	3	3
3	2	3	1	2	2	2	2	2	2	2
4	1	2	3	4	5	6	4	4	4	4
5	1	2	3	6	4	5	6	6	6	6
6	1	2	3	5	6	4	5	5	5	5
7	1	2	3	4	5	6	7	8	9	10
8	1	2	3	4	5	6	8	7	10	9
9	1	2	3	4	5	6	10	9	7	8
10	1	2	3	4	5	6	9	10	8	7

Theorem 4.3. Let $(G, *)$ be a groupoid, Y be an AG-band, $Y \subset G$. $G = \bigcup_{\alpha \in Y} S_\alpha$, $(S_\alpha, *)$ is an AG-group, for all $a_\alpha \in S_\alpha$, $a_\alpha^2 = e_\alpha$ (e_α is the left identity of S_α) and $\forall \alpha, \beta \in Y, \alpha \neq \beta, S_\alpha \cap S_\beta = \emptyset$. If $\forall a_\alpha \in S_\alpha, \forall b_\beta \in S_\beta$, $a_\alpha * b_\beta = b_\beta, b_\beta * a_\alpha = b_\beta * e_\beta$, then G is a strong root of band.

Proof. It is similar to Theorem 4.2. □

The strong root of band constructed by Theorem 4.2 is not isomorphic to the strong root of band constructed by Theorem 4.3.

Corollary 4.1. Let $(G, *)$ be an AG-groupoid, Y be an AG-band, $Y \subset G$. $G = \bigcup_{\alpha \in Y} S_\alpha$, $(S_\alpha, *)$ is an AG-group, for all $a_\alpha \in S_\alpha$, $a_\alpha^2 = e_\alpha$ (e_α is the left identity of S_α) and $\forall \alpha, \beta \in Y, \alpha \neq \beta, S_\alpha \cap S_\beta = \emptyset$. If $|Y| = 2$, then G is a strong root of band with the left identity.

Proof. Suppose that $G = \bigcup_{\alpha \in Y} S_\alpha$ is an AG-groupoid satisfying $a_\alpha^2 = e_\alpha$ for all $a_\alpha \in S_\alpha$, Y is an AG-band and $|Y| = 2$. By Theorem 4.1, we have G is a strong root of band. If $Y = \{\alpha, \beta\}$, and $\alpha = e_\alpha \in S_\alpha, \beta = e_\beta \in S_\beta$, we have $(e_\alpha * e_\beta) * (e_\alpha * e_\beta) = (e_\alpha * e_\alpha) * (e_\beta * e_\beta) = e_\alpha * e_\beta$, thus, $e_\alpha * e_\beta \in Y$.

Case 1: $e_\alpha * e_\beta = e_\beta$. Let $a_\alpha \in S_\alpha$ be an arbitrary element, $a_\alpha * a_\alpha = e_\alpha, e_\alpha * a_\alpha = a_\alpha$. Let $b_\beta \in S_\beta$ be an arbitrary element, $b_\beta * b_\beta = e_\beta, e_\beta * b_\beta = b_\beta$,

$$e_\beta * e_\alpha = (e_\beta * e_\beta) * e_\alpha = (e_\alpha * e_\beta) * e_\beta = e_\beta * e_\beta = e_\beta.$$

$$\begin{aligned} e_\alpha * b_\beta &= (e_\alpha * e_\alpha) * (e_\beta * b_\beta) \\ &= (e_\alpha * e_\beta) * (e_\alpha * b_\beta) \quad (\text{by the medial law}) \\ &= e_\beta * (e_\alpha * b_\beta) \\ &= (e_\beta * e_\beta) * (e_\alpha * b_\beta) \\ &= (e_\beta * e_\alpha) * (e_\beta * b_\beta) \quad (\text{by the medial law}) \\ &= e_\beta * b_\beta = b_\beta. \end{aligned}$$

Thus, e_α is the left identity of G .

Case 2: $e_\alpha * e_\beta = e_\alpha$. Let $a_\alpha \in S_\alpha$ be an arbitrary element, $a_\alpha * a_\alpha = e_\alpha$, $e_\alpha * a_\alpha = a_\alpha$. Let $b_\beta \in S_\beta$ be an arbitrary element, $b_\beta * b_\beta = e_\beta$, $e_\beta * b_\beta = b_\beta$, $e_\beta * e_\alpha = (e_\beta * e_\beta) * e_\alpha = (e_\alpha * e_\beta) * e_\beta = e_\alpha * e_\beta = e_\alpha$.

$$\begin{aligned} e_\beta * a_\alpha &= (e_\beta * e_\beta) * (e_\alpha * a_\alpha) \\ &= (e_\beta * e_\alpha) * (e_\beta * a_\alpha) \quad (\text{by the medial law}) \\ &= e_\alpha * (e_\beta * a_\alpha) \\ &= (e_\alpha * e_\alpha) * (e_\beta * a_\alpha) \\ &= (e_\alpha * e_\beta) * (e_\alpha * a_\alpha) \quad (\text{by the medial law}) \\ &= e_\alpha * a_\alpha = a_\alpha. \end{aligned}$$

Thus, e_β is the left identity of G .

From all the above cases, G is a strong root of band with the left identity. $\square$

Example 4.3. Let $G = \{1, 2, 3, 4, 5, 6, 7, 8\}$, the definition of operation $*$ on G is shown in Table 6. An AG-band $Y = \{1, 5\}$ and $Y \subset G$. $S_1 = \{1, 2, 3, 4\}$, $e_1 = 1$ and $S_5 = \{5, 6, 7, 8\}$, $e_5 = 5$. Both S_1 and S_5 are sub-AG-groups of G . It is easy to verify that $G = S_1 \cup S_5$. From Definition 3.1, G is a strong root of band. Being $e_1 * e_5 = e_1$, we can get e_5 is the left identity of G .

Table 6: The operation table of Example 4.3.

$*$	1	2	3	4	5	6	7	8
1	1	2	3	4	1	1	1	1
2	2	1	4	3	2	2	2	2
3	4	3	1	2	4	4	4	4
4	3	4	2	1	3	3	3	3
5	1	2	3	4	5	6	7	8
6	1	2	3	4	6	5	8	7
7	1	2	3	4	8	7	5	6
8	1	2	3	4	7	8	6	5

5. Conclusion

The construction problem is the basic problem in the study of algebraic systems. In this paper, the construction problem of a kind of AG-groupoid is completely solved. With the introduction to strong root of band and the thorough study of its structure, some important results can be obtained. Figure 1 gives the relationships among strong root of band and other bands. In Figure 1, as strong root of band is a special root of band, strong root of band is a generalization of both AG-band and AG-3-band. AG-3-band is the intersection of AG-4-band and strong root of band. The decomposition theorem of strong root of band has been investigated, where every strong root of band is the disjoint union of its sub-AG-groups (see Theorem 4.1). Furthermore, two different ways how to make a strong root of band are obtained (see Theorem 4.2 and Theorem 4.3) and the structure of strong root of band has been illuminated. As the next research direction, we will discuss the relationship between the strong root of band and other related AG-groupoid, such as Abel-Grassmann's neutrosophic extended triplet loop (AG-NET-Loop), generalized Abel-Grassmann's neutrosophic extended triplet loop (GAG-NET-Loop) (see [20], [21]).

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On periodic solutions of Mackey-Glass hematopoiesis model via concave and increasing operator

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Abstract. The paper is concerned with a Mackey-Glass hematopoiesis model. By employing the fixed point theorem of u_0 concave and increasing operator, we provide easily verifiable sufficient conditions for the existence of a unique positive periodic solution for the considered model. We provide numerical examples along with illustrative figures to demonstrate the theory. Our approach is new and it is different from previously considered methods.

Keywords: Mackey-Glass hematopoiesis model; Periodic solution; Concave and increasing operator.

1. Introduction

One of the most important models in field of biomathematics is the celebrated hematopoiesis model which has been introduced and thus referred to by Mackey and Glass in their pioneering paper [1]. The model deals with the physiological processes that study the regulation of hematopoiesis in which time delays play a significant role. The proposed model not only exhibits a wide range of periodic behaviors but also exposes chaotic attitude. Indeed its importance lies in the

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fact that the early detection of some diseases are associated with alterations in the periodicity of certain physiological variables [2, 3, 4, 5, 6].

Due to its widespread applications, the Mackey-Glass hematopoiesis model has been the object of many researchers who significantly contributed abundant number of papers which study its dynamic behavior. Researchers have provided their results by the help of several methods and different techniques such as the Banach contraction fixed point theorem, decreasing operator fixed point theorem, the method of Hilbert projective metric in a cone, the fixed point theory and Lyapunov functional and the Mawhin's coincidence degree theory [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19].

In this paper, we investigate Mackey-Glass hematopoiesis model described by the the following equation

$$(1) \quad x'(t) = -\alpha(t)x(t) + \frac{\beta(t)x^n(t-\tau)}{1+x^n(t-\tau)}, \quad n \in \mathbb{R}^+,$$

where $\alpha \in C(\mathbb{R}, \mathbb{R})$, $\beta \in C(\mathbb{R}, \mathbb{R}^+)$, α and β are ω -periodic functions with $\int_0^\omega \alpha(t)dt > 0$ and τ and ω are positive constants. Here $\mathbb{R}$ indicates the set of all real numbers while $\mathbb{R}^+$ denotes the set of all positive real numbers. Due to biological reasonings, we restrict our attention to positive solutions of equation (1). The associated initial condition is $x(t) = \phi(t) > 0$, $t \in [-\tau, 0]$.

Unlike previous methods used in the literature, we employ a novel approach based on the u_0 concave and increasing operator fixed point theorem to prove the existence of a unique ω -periodic positive solution for (1). The main results are proved for the cases when $0 < n < 1$, $n = 1$ and $n > 1$. Throughout the proofs, we accommodate some easily verifiable algebraic computations to complete the steps. Numerical examples along with illustrative figures are provided to demonstrate consistency with the theory.

2. Preliminaries and essential results

In this part, we assemble some notations, fundamental definitions and lemmas which will be served as underpinning framework prior to proving the main results. Meanwhile, the u_0 concave and increasing fixed point theorem is stated.

Definition 2.1 ([20]). *Let X be a Banach space and P be a closed and nonempty subset of X . Then P is called a cone if*

- (i) $x \in P$, $\lambda \geq 0$, it implies that $\lambda x \in P$;
- (ii) $x \in P$ and $-x \in P$, it implies that $x = 0$.

Every cone $P \subset X$ induces an ordering in X . We define " $\leq$ " with respect to P by $x \leq y$ if and only if $y - x \in P$.

Definition 2.2 ([20]). *A cone P of X is called normal cone if there exists a positive constant σ such that $\|x + y\| \geq \sigma$ for any $x, y \in P$ and $\|x\| = \|y\| = 1$.*

Definition 2.3. Let $A : P \rightarrow P$ be an operator where P is a cone of X . Then, A is said to be an increasing operator if for $y \geq x \geq 0$, it implies that $Ay \geq Ax$.

Let $P = \{x \in X : x(t) \geq 0\}$ and assume that $u_0 \in P$, $u_0 > 0$, that is, u_0 is not identically vanishing. Define

$$P_{u_0} = \{x : x \in X, \text{ there exist } \mu_1 > 0 \text{ and } \mu_2 > 0, \text{ such that } \mu_1 u_0 \leq x(t) \leq \mu_2 u_0\}.$$

Definition 2.4 ([20]). Let $A : P \rightarrow P$ and $u_0 > 0$. Then A is said to be u_0 concave operator if A satisfies:

- (i) For all $x > 0$, it implies that $Ax \in P_{u_0}$;
- (ii) There exists $\eta = \eta(\lambda, x) > 0$ such that $A(\lambda x) \geq \lambda[1 + \eta(\lambda, x)]Ax$, for all $x \in P_{u_0}$ and $0 < \lambda < 1$.

Our approach is based on the following fixed point theorem of u_0 concave and increasing operator [21].

Theorem 2.1 ([21]). Suppose that

- (i) P is a normal cone of Banach space X , $u_0 > 0$ and the operator $A : P_{u_0} \rightarrow P_{u_0}$ is increasing;
- (ii) there exists $\eta = \eta(\lambda) > 0$ such that $A(\lambda x) \geq \lambda[1 + \eta(\lambda)]Ax$, for all $x \in P_{u_0}$ and $0 < \lambda < 1$.

Then, A has a unique fixed point $x^* \in P_{u_0}$ if and only if there exist w_0 and v_0 in P_{u_0} such that $w_0 \leq Aw_0 \leq Av_0 \leq v_0$.

Remark. The fixed point x^* can be approximated by the help of an iterative scheme. That is, $x_k \rightarrow x^*$ as $k \rightarrow \infty$ where $x_k = Ax_{k-1}$, $k = 1, 2, \dots$ for any initial point $x_0 \in [w_0, v_0]$.

Let $X = \{x(t) : x \in C(\mathbb{R}, \mathbb{R}), x(t + \omega) = x(t)\}$ with the norm $\|x\| = \sup_{t \in [0, \omega]} |x(t)|$. Then, X is a Banach space.

Notations: We make use of the following notations

$$G(t, s) = \frac{\exp\{\int_t^s \alpha(t)\xi d\xi\}}{\exp\{\int_t^s \alpha(t)\xi d\xi\} - 1},$$

$$H = [t, t + \omega], H_1 = \{t \in H : \alpha(t) \geq 0\}, H_2 = \{t \in H : \alpha(t) < 0\},$$

$$\bar{\alpha} = \max\{0, \alpha(t)\}, \underline{\alpha} = \min\{0, \alpha(t)\}$$

and

$$B = \frac{\exp\{\int_0^\omega \bar{\alpha}(\xi) d\xi\}}{\exp\{\int_0^\omega \alpha(t)\xi d\xi\} - 1}, m = \frac{\exp\{\int_0^\omega \underline{\alpha}(\xi) d\xi\}}{\exp\{\int_0^\omega \alpha(t)\xi d\xi\} - 1}.$$

Moreover, we observe that

$$\begin{aligned} G(t, s) &= \frac{\exp\left\{\left(\int_{[t,s]\cap H_1} \alpha(t)\xi d\xi + \int_{[t,s]\cap H_2} \alpha(t)\xi d\xi\right)\right\}}{\exp\left\{\int_0^\omega \alpha(t)\xi d\xi\right\} - 1} \\ &\leq \frac{\exp\left\{\int_{[t,s]\cap H_1} \alpha(t)\xi d\xi\right\}}{\exp\left\{\int_0^\omega \alpha(t)\xi d\xi\right\} - 1} \leq \frac{\exp\left\{\int_0^\omega \bar{\alpha}(\xi) d\xi\right\}}{\exp\left\{\int_0^\omega \alpha(t)\xi d\xi\right\} - 1} = B, \end{aligned}$$

and

$$G(t, s) \geq \frac{\exp\left\{\int_{[t,s]\cap H_2} \alpha(t)\xi d\xi\right\}}{\exp\left\{\int_0^\omega \alpha(t)\xi d\xi\right\} - 1} \geq \frac{\exp\left\{\int_0^\omega \underline{\alpha}(\xi) d\xi\right\}}{\exp\left\{\int_0^\omega \alpha(t)\xi d\xi\right\} - 1} = m.$$

Therefore, we conclude that $0 < m \leq G(t, s) \leq B$, $s \in [t, t + \omega]$.

It is easy to verify that x is the ω -periodic solution of equation (1) if and only if x is the ω -periodic solution of the integral equation

$$x(t) = \int_t^{t+\omega} G(t, s) \frac{\beta(s)x^n(s-\tau)}{1+x^n(s-\tau)} ds.$$

We define the operator $A : X \rightarrow X$ by

$$(Ax)(t) = \int_t^{t+\omega} G(t, s) \frac{\beta(s)x^n(s-\tau)}{1+x^n(s-\tau)} ds.$$

Clearly, $x \in C(\mathbb{R}, \mathbb{R})$ is the ω -periodic solution of equation (1) if and only if x is the fixed point of the operator A .

We refer to β^- and β^+ as follows:

$$\beta^- = \min_{t \in [0, \omega]} \{\beta(t)\} \quad \text{and} \quad \beta^+ = \max_{t \in [0, \omega]} \{\beta(t)\}.$$

Let $f(x) = \frac{x^n}{1+x^n}$ and $g(x) = \frac{1}{\beta^- m \omega} x$. In order to prove the main results, we first consider the existence of roots for the algebraic equation $\frac{c^n}{1+c^n} = \frac{1}{\beta^- m \omega} c$. We discuss this in the following three lemmas whose proofs are entirely based on calculus background.

Lemma 2.1. *Let $0 < n < 1$. Then there exists a unique positive constant $c > 0$ such that $\frac{c^n}{1+c^n} = \frac{1}{\beta^- m \omega} c$.*

Proof. Since $0 < n < 1$, we have $f'(x) = \frac{n}{x^{1-n}(1+x^n)^2}$, $\lim_{x \rightarrow 0^+} f'(x) = +\infty$, $\lim_{x \rightarrow \infty} f(x) = 1$ and $f''(x) = \frac{nx^{n-2}[n-1-(n+1)x^n]}{(1+x^n)^3} < 0$ for $x \in (0, +\infty)$. Then, the curve of $f(x)$ is convex on $(0, +\infty)$. We conclude that the curve of f and the straight line g has a unique intersection point on $(0, +\infty)$. Thus, there exists a unique positive constant $c > 0$ such that $\frac{c^n}{1+c^n} = \frac{1}{\beta^- m \omega} c$. $\square$

Lemma 2.2. *Let $n = 1$, and $\frac{1}{\beta^- m \omega} < 1$. Then there exists a unique positive constant $c > 0$ such that $\frac{c^n}{1+c^n} = \frac{1}{\beta^- m \omega} c$.*

Proof. Since $n = 1$, we have $f'(x) = \frac{1}{(1+x^n)^2}$, $f'(0) = 1$, $\lim_{x \rightarrow \infty} f(x) = 1$ and $f''(x) = \frac{-2}{(1+x)^2} < 0$, for $x \in (0, +\infty)$. Then, the curve of $f(x)$ is convex on $(0, +\infty)$. From the assumption $\frac{1}{\beta^{-mw}} < 1$, we know the curve of f and the straight line g has a unique intersection point on $(0, +\infty)$. Thus, there exists a unique positive constant $c > 0$ such that $\frac{c^n}{1+c^n} = \frac{1}{\beta^{-mw}} c$. $\square$

Lemma 2.3. Let $n > 1$, and $\frac{1}{\beta^{-mw}} \leq \frac{\frac{n-1}{2n}}{\sqrt[n]{\frac{n-1}{n+1}}}$. Then there exists a unique positive constant $c > 0$ such that $\frac{c^n}{1+c^n} = \frac{1}{\beta^{-mw}} c$.

Proof. Since $n > 1$, we have $f''(x) = \frac{nx^{n-2}[n-1-(n+1)x^n]}{(1+x^n)^3}$. Thus, $f''(x) > 0$ for $0 < x < \sqrt[n]{\frac{n-1}{n+1}}$ and $f''(x) < 0$ for $x > \sqrt[n]{\frac{n-1}{n+1}}$. Then, the curve of $f(x)$ is convex on $(\sqrt[n]{\frac{n-1}{n+1}}, +\infty)$, and concave on $(0, \sqrt[n]{\frac{n-1}{n+1}})$. Moreover, $f'(0) = 0$, $\lim_{x \rightarrow \infty} f(x) = 1$ and $f(\sqrt[n]{\frac{n-1}{n+1}}) = \frac{n-1}{2n}$. By the assumption $\frac{1}{\beta^{-mw}} \leq \frac{\frac{n-1}{2n}}{\sqrt[n]{\frac{n-1}{n+1}}}$, it is clear that the curve of f and the straight line g has at least one intersection point on $(0, +\infty)$. Thus, the desired conclusion holds. $\square$

3. Existence results

The existence of a unique positive periodic solution for equation (1) is discussed in this section. We prove the main results by the help of Theorem 2.1.

Theorem 3.1. Let $0 < n < 1$. Then equation (1) has a unique ω -periodic positive solution.

Proof. Let $X = \{x(t) : x \in C(\mathbb{R}, \mathbb{R}), x(t+\omega) = x(t)\}$ with the norm $\|x\| = \sup_{t \in [0, \omega]} |x(t)|$, then X is a Banach space. Define $P = \{x \in X : x(t) \geq 0\}$, then P is a normal cone of the Banach space X . Let $u_0 = 1$ then $P_{u_0} = \{x : x \in X, \text{ there exist } \mu_1 > 0 \text{ and } \mu_2 > 0, \text{ such that } \mu_1 \leq x(t) \leq \mu_2\}$. Thus, we have

$$(Ax)(t) = \int_t^{t+\omega} G(t, s) \frac{\beta(s)x^n(s-\tau)}{1+x^n(s-\tau)} ds \leq \int_t^{t+\omega} G(t, s)\beta(s)ds \leq \beta^+ B\omega$$

and

$$\begin{aligned} (Ax)(t) &= \int_t^{t+\omega} G(t, s) \frac{\beta(s)x^n(s-\tau)}{1+x^n(s-\tau)} ds \geq \int_t^{t+\omega} G(t, s)\beta(s) \frac{\mu_1^n}{1+\mu_1^n} ds \\ &\geq m\beta^- B\omega \frac{\mu_1^n}{1+\mu_1^n}. \end{aligned}$$

The above two inequalities yield $m\beta^- B\omega \frac{\mu_1^n}{1+\mu_1^n} \leq (Ax)(t) \leq \beta^+ B\omega$, which implies that $Ax \in P_{u_0}$. Therefore, we get $A : P_{u_0} \rightarrow P_{u_0}$. In addition, it is clear that A is an increasing operator.

For all $x \in P_{u_0}$, there exists $\mu_1 > 0$ and $\mu_2 > 0$ such that $\mu_1 \leq x(t) \leq \mu_2$. Let $0 < \lambda < 1$, then

$$\begin{aligned} A(\lambda x)(t) &= \int_t^{t+\omega} G(t, s) \frac{\beta(s) \lambda^n x^n(s-\tau)}{1 + \lambda^n x^n(s-\tau)} ds \\ (2) \quad &= \int_t^{t+\omega} G(t, s) \frac{\beta(s) x^n(s-\tau)}{1 + x^n(s-\tau)} \cdot \frac{\lambda^n [1 + x^n(s-\tau)]}{1 + \lambda^n x^n(s-\tau)} ds. \end{aligned}$$

However, we observe that $\frac{\lambda^n [1 + x^n(s-\tau)]}{1 + \lambda^n x^n(s-\tau)} = 1 + \frac{\lambda^n - 1}{1 + \lambda^n x^n(s-\tau)} \geq 1 + \frac{\lambda^n - 1}{1 + \lambda^n \mu_1^n} = \frac{\lambda^n (1 + \mu_1^n)}{1 + \lambda^n \mu_1^n}$. It follows from (2) that

$$\begin{aligned} A(\lambda x)(t) &\geq \int_t^{t+\omega} G(t, s) \frac{\beta(s) \lambda^n x^n(s-\tau)}{1 + x^n(s-\tau)} \frac{\lambda^n (1 + \mu_1^n)}{1 + \lambda^n \mu_1^n} ds \\ &= \lambda \frac{\lambda^{n-1} (1 + \mu_1^n)}{1 + \lambda^n \mu_1^n} \int_t^{t+\omega} G(t, s) \frac{\beta(s) x^n(s-\tau)}{1 + x^n(s-\tau)} ds \\ &= \lambda \left(1 + \frac{\lambda^{n-1} + \lambda^{n-1} \mu_1^n - 1 - \lambda^n \mu_1^n}{1 + \lambda^n \mu_1^n} \right) Ax(t) = \lambda(1 + \eta) Ax(t), \end{aligned}$$

in which $\eta := \eta(\lambda) = \frac{\lambda^{n-1} + \lambda^{n-1} \mu_1^n - 1 - \lambda^n \mu_1^n}{1 + \lambda^n \mu_1^n}$. Consider the function $h(x) = x^{n-1} + x^{n-1} \mu_1^n - 1 - \lambda^n \mu_1^n$. It follows that $h'(x) = x^{n-2} [n - 1 + (n - 1) \mu_1^n - nx \mu_1^n]$, and thus $h'(x) < 0$ for $x \in (0, +\infty)$, which means that $h(x)$ is decreasing on $(0, +\infty)$. So we have $h(\lambda) > h(1) = 0$. From this, we conclude that $\eta(\lambda) > 0$.

By Lemma 2.1, there exists a unique positive constant $c > 0$ such that $\frac{c^n}{1+c^n} = \frac{1}{\beta^- m \omega} c$. Let

$$w_0 = \int_t^{t+\omega} G(t, s) \frac{\beta(s) c^n}{1 + c^n} ds, v_0 = \int_t^{t+\omega} G(t, s) \beta(s) ds.$$

Then, $w_0 < v_0$, thus we have $Aw_0 < Av_0$. From this, we deduce that

$$\begin{aligned} w_0 &\leq \int_t^{t+\omega} G(t, s) \beta(s) ds \leq \beta^+ B \omega, \\ w_0 &\geq \int_t^{t+\omega} m \beta^- \frac{c_n}{1 + c^n} ds = \omega m \beta^- \frac{c_n}{1 + c^n} = c, \end{aligned}$$

and

$$\begin{aligned} v_0 &\leq \int_t^{t+\omega} B \beta^+ ds = B \beta^+ \omega, \\ v_0 &\geq \int_t^{t+\omega} m \beta^- ds = \omega m \beta^- > \frac{c_n}{1 + c^n} \omega m \beta^- = c. \end{aligned}$$

Hence, we obtain that

$$w_0(t) \leq Aw_0(t) < Av_0(t) < v_0(t).$$

By Theorem 2.1, the operator A has a unique fixed point $x^* \in P_{u_0}$, which means that equation (1) has a unique ω -periodic positive solution x^* . The proof is complete. $\square$

By the help of Lemma 2.2 and Lemma 2.3, we can prove the following theorems by applying the same arguments done in the proof of Theorem 3.1.

Theorem 3.2. *Let $n = 1$, and $\frac{1}{\beta - m\omega} < 1$. Then equation (1) has a unique ω -periodic positive solution.*

Theorem 3.3. *Let $n > 1$, and $\frac{1}{\beta - m\omega} \leq \frac{\frac{n-1}{2n}}{\sqrt[n]{\frac{n-1}{n+1}}}$. Then equation (1) has a unique ω -periodic positive solution.*

4. Illustrative examples

Example 1. For $n = \frac{1}{2}$, consider the equation

$$(3) \quad x'(t) = -(0.5 + \sin(2t))x(t) + \frac{650(1.2 + \cos(2t))x^{\frac{1}{2}}(t-2)}{1 + x^{\frac{1}{2}}(t-2)},$$

where the initial condition can be defined as $x(t) = 5t$ on $[-2, 0]$. The parameters $0.5 + \sin(2t)$ and $650(1.2 + \cos(2t))$ are π -periodic continuous functions on $\mathbb{R}$. It is clear that $\int_0^\pi (0.5 + \sin(2t))dt = \frac{\pi}{2} > 0$. By the conclusion of Theorem 3.1, equation (3) has a unique π -periodic solution. Figure 1 illustrates the the solution behavior of equation (3).

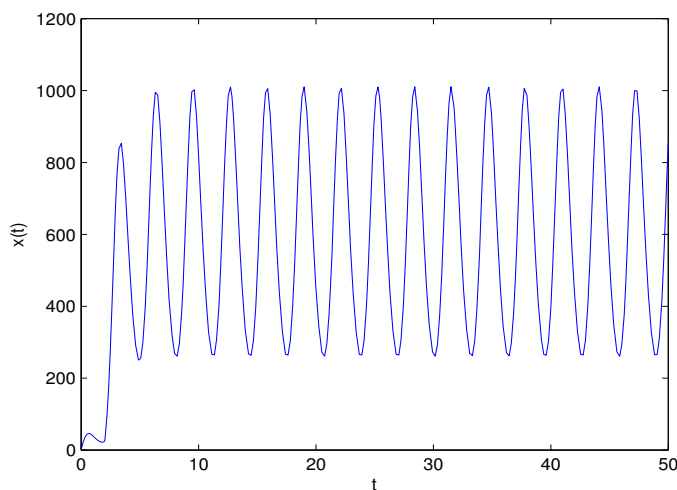


Figure 1: Behavior of the solution of Example 1 for the case $n = 0.5$.

Example 2. For $n = 1$, consider the equation

$$(4) \quad x'(t) = -(1 + \sin(2t))x(t) + \frac{700(1.5 + \cos(2t))x(t - 3\pi)}{1 + x(t - 3\pi)},$$

where the initial condition can be defined as $x(t) = \frac{t}{4}$ on $[-3\pi, 0]$. One can easily see that the parameters $1 + \sin(2t)$ and $700(1.5 + \cos(2t))$ are π -periodic continuous functions on $\mathbb{R}$ and the integral $\int_0^\pi (1 + \sin(2t))dt = \pi > 0$. Besides, the assumption $\frac{1}{\beta^- m \omega} < 1$ is preserved since $m = \frac{1}{e^\pi - 1}$, $\omega = \pi$ and $\beta^- = 350$. By the conclusion of Theorem 3.2, equation (4) has a unique π -periodic solution which is described in the following figure.

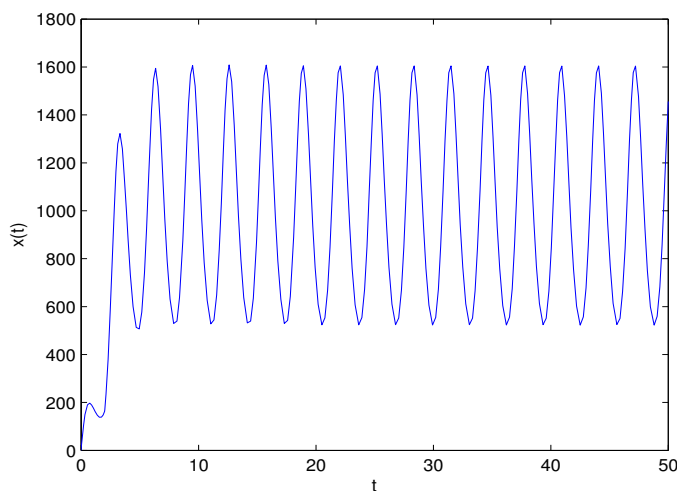


Figure 2: Behavior of the solution of Example 2 for the case $n = 1$.

Example 3. For $n = 3$, consider the equation

$$(5) \quad x'(t) = -(1 + \sin(2t))x(t) + \frac{600(1.1 + \cos(2t))x^3(t - 5.82)}{1 + x^3(t - 5.82)},$$

where the initial condition can be chosen as $x(t) = |t|$ on $[-5.82, 0]$. The assumptions of Theorem 3.3 are clearly satisfied. Moreover, that $\frac{1}{\beta^- m \omega} \leq \frac{\frac{n-1}{2n}}{\sqrt[n]{\frac{n-1}{n+1}}}$ holds since $n = 3$, $m = \frac{1}{e^\pi - 1}$, $\omega = \pi$ and $\beta^- = 60$. Therefore, Theorem 3.3 guarantees the existence of a unique π -periodic solution for equation (5). Figure 3 depicts the solution behavior.

Conclusion

In this paper, we prove the existence and uniqueness of ω -periodic positive solution of a Mackey-Glass hematopoiesis model. Different from the methods

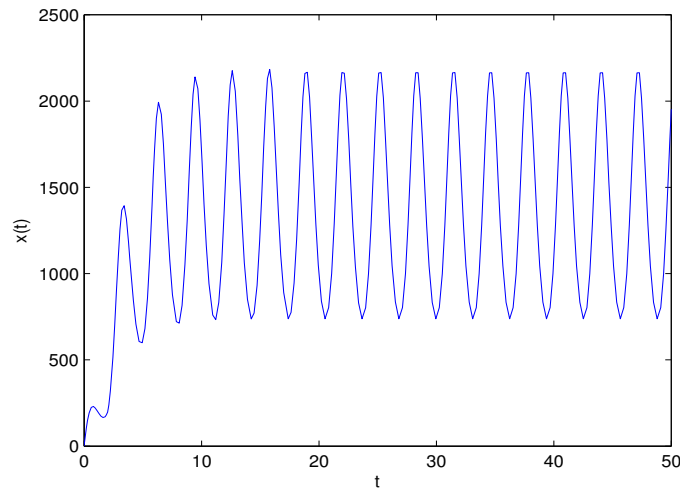


Figure 3: Behavior of the solution of Example 3 for the case $n = 3$.

used in the previous works, easily variable technique is utilized to prove the main results. We conclude here some features that make this research distinctive:

1. The proofs of the main results are given by the help of a novel approach depending on u_0 concave and increasing operator fixed point theorem that has not been employed before.
2. The main theorems exhibit that the periodic coefficients of the addressed model admit a unique positive periodic solution without additional restrictions.
3. The provided examples are illustrative and demonstrate consistency to the theoretical findings.

The approach employed in this paper can be used to prove existence results for more general population models.

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Two equations in unequal powers of primes and powers of 2

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Abstract. In this paper, we obtained when $k = 176$, every pair of large even integers satisfying some necessary conditions can be represented in the form of a pair of two prime squares, two prime cubes, two prime fourth powers and k powers of 2.

Keywords: Hardy-Littlewood method, Goldbach-Linnik problem, powers of 2.

1. Introduction

As an approximation to Goldbach's problem Linnik proved in 1951 [8] under the assumption of the Generalized Riemann Hypothesis (GRH), and later in 1953 [9] unconditionally, that each large even integer N is a sum of two primes p_1, p_2 and a bounded number of powers of 2, namely

$$N = p_1 + p_2 + 2^{\nu_1} + \cdots + 2^{\nu_k}.$$

In 2002, Heath-Brown and Puchta [1] applied a rather different approach to this problem and showed that $k = 13$ and, on the GRH, $k = 7$. In 2003, Pintz and Ruzsa [16] established this latter result and announced that $k = 8$ is acceptable unconditionally. This paper is yet to appear in print. Elsholtz, in an unpublished manuscript, showed that $k = 12$; this was proved independently by Liu and Lü [15].

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In 1999, Liu, Liu and Zhan [11] proved that every large even integer N can be written as a sum of four squares of primes and a bounded number of powers of 2, namely

$$N = p_1^2 + p_2^2 + p_3^2 + p_4^2 + 2^{v_1} + \cdots + 2^{v_k}.$$

Subsequently Liu and Liu [10] got that $k = 8330$ suffices. Later Liu and Lü [12] improved the value of k of (1.2) to 165, Li [7] improved it to 151 and Zhao [18] improved it to 46. Finally Platt and Trudgian [17] revised it to 45. In 2011, Liu and Lü [14] considered the problem on the representation of large even integers N in the form

$$N = p_1 + p_2^2 + p_3^3 + p_4^3 + 2^{v_1} + \cdots + 2^{v_k}.$$

They showed that $k = 161$ is acceptable and Zhao [19] improved it to 18. Also in 2017, Liu [13] construct that every large even integers N can be represented in the form

$$(1.1) \quad N = p_1^2 + p_2^2 + p_3^3 + p_4^3 + p_5^4 + p_6^4 + 2^{v_1} + \cdots + 2^{v_k}.$$

He showed that $k = 41$ is acceptable.

As a comparison, Kong [5] first considered the result on pairs of linear equations in four prime variables and powers of 2, in the form

$$\begin{cases} N_1 = p_1 + p_2 + 2^{v_1} + \cdots + 2^{v_k}, \\ N_2 = p_3 + p_4 + 2^{v_1} + \cdots + 2^{v_k}, \end{cases}$$

where k is a positive integer. She proved that the simultaneous equations are solvable for $k = 63$. Then Platt and Trudgian [17] and Kong and Liu [6] revised it to 62 and 34. Later, Hu and Liu [2], Hu and Yang [3][4] considered other equations

$$\begin{cases} N_1 = p_1^2 + p_2^2 + p_3^2 + p_4^2 + 2^{v_1} + \cdots + 2^{v_k}, \\ N_2 = p_5^2 + p_6^2 + p_7^2 + p_8^2 + 2^{v_1} + \cdots + 2^{v_k}, \\ N_1 = p_1 + p_2^2 + p_3^2 + 2^{v_1} + \cdots + 2^{v_k}, \\ N_2 = p_4 + p_5^2 + p_6^2 + 2^{v_1} + \cdots + 2^{v_k}, \\ N_1 = p_1 + p_2^2 + p_3^3 + p_4^3 + 2^{v_1} + \cdots + 2^{v_k}, \\ N_2 = p_5 + p_6^2 + p_7^3 + p_8^3 + 2^{v_1} + \cdots + 2^{v_k}, \end{cases}$$

and proved that these equations are solvable for $k = 142, 332, 455$.

In this paper, comparing to (1.1), we shall consider the simultaneous representation of pairs of positive even integers $N_2 \gg N_1 > N_2$, in the form

$$(1.2) \quad \begin{cases} N_1 = p_1^2 + p_2^2 + p_3^3 + p_4^3 + p_5^4 + p_6^4 + 2^{v_1} + \cdots + 2^{v_k}, \\ N_2 = p_7^2 + p_8^2 + p_9^3 + p_{10}^3 + p_{11}^4 + p_{12}^4 + 2^{v_1} + \cdots + 2^{v_k}, \end{cases}$$

where k is a positive integer. Our result is stated as follows.

Theorem 1.1. *For $k = 176$, the equations (1.2) are solvable for every pair of sufficiently large positive even integers N_1 and N_2 satisfying $N_2 \gg N_1 > N_2$.*

Our proof of Theorem 1.1 uses the Hardy-Littlewood circle method and draws on the strategies adopted in the works of Hu and Yang [4] and Liu [13].

Throughout this paper, the letter ϵ denotes a positive constant which is arbitrarily small but may not be the same at different occurrences. And p and v denote a prime number and a positive integer, respectively.

2. Outline of the method

Here we give an outline for the proof of Theorem 1.1.

Throughout, we assume that $N_i, i = 1, 2$ are sufficient large even integers. Then we set

$$U_i = \sqrt{(1-\eta)N_i}, \quad V_i = (\eta N/2)^{1/3}, \quad W_i = (\eta N/2)^{1/4}$$

for $i = 1, 2$, where η is a small positive constant. We set

$$\begin{aligned} f(\alpha_i, U_i) &= \sum_{U_i/2 < p \leq U_i} (\log p) e(p^2 \alpha_i), \\ g(\alpha_i, V_i) &= \sum_{V_i/2 < p \leq V_i} (\log p) e(p^3 \alpha_i), \\ (2.1) \quad h(\alpha_i, W_i) &= \sum_{W_i/2 < p \leq W_i} (\log p) e(p^4 \alpha_i), \end{aligned}$$

$$(2.2) \quad G(\alpha_i) = \sum_{v \leq L} e(2^v \alpha_i), \quad \mathcal{E}_\lambda := \{\alpha_i \in [0, 1] : |G(\alpha_i)| \geq \lambda L\}.$$

where $i = 1, 2$, $e(x) := \exp(2\pi i x)$ and $L = \log_2 N_1$.

In order to apply the circle method, we set

$$(2.3) \quad P_i = N_i^{3/20-2\epsilon}, \quad Q_i = N_i^{17/20+\epsilon},$$

for $i = 1, 2$. For any integers a_1, a_2, q_1, q_2 satisfying

$$1 \leq a_1 \leq q_1 \leq P_1, (a_1, q_1) = 1, 1 \leq a_2 \leq q_2 \leq P_2, (a_2, q_2) = 1,$$

we define the major arcs $\mathcal{M}_1, \mathcal{M}_2$ and minor arcs $C(\mathcal{M}_1), C(\mathcal{M}_2)$ as usual, namely

$$(2.4) \quad \mathcal{M}_i = \bigcup_{q_i \leq P_i} \bigcup_{\substack{1 \leq a_i \leq q_i \\ (a_i, q_i) = 1}} \mathcal{M}_i(a_i, q_i), \quad C(\mathcal{M}_i) = \left[\frac{1}{Q_i}, 1 + \frac{1}{Q_i} \right] \setminus \mathcal{M}_i,$$

where $i = 1, 2$ and

$$\mathcal{M}_i(a_i, q_i) = \left\{ \alpha_i \in [0, 1] : \left| \alpha_i - \frac{a_i}{q_i} \right| \leq \frac{1}{q_i Q_i} \right\}.$$

It follows from $2P_i \leq Q_i$ that the arcs $\mathcal{M}_1(a_1, q_1)$ and $\mathcal{M}_2(a_2, q_2)$ are mutually disjoint respectively.

Let

$$R(N_1, N_2) = \sum \log p_1 \log p_2 \cdots \log p_{12}$$

be the weighted number of solutions of (1.2) in $(p_1, \dots, p_{12}, v_1, \dots, v_k)$ with

$$\begin{aligned} p_1, p_2 &\sim U_1, & p_3, p_4 &\sim V_1, & p_5, p_6 &\sim W_1, \\ p_7, p_8 &\sim U_2, & p_9, p_{10} &\sim V_2, & p_{11}, p_{12} &\sim W_2, \\ v_j &\leq L, \end{aligned}$$

for $j = 1, 2, \dots, k$. Then $R(N_1, N_2)$ can be written as

$$\begin{aligned} R(N_1, N_2) &= \int_0^1 \int_0^1 f^2(\alpha_1, U_1) g^2(\alpha_1, V_1) h^2(\alpha_1, W_1) f^2(\alpha_2, U_2) g^2(\alpha_2, V_2) \\ &\quad \times h^2(\alpha_2, W_2) G^k(\alpha_1 + \alpha_2) e(-\alpha_1 N_1 - \alpha_2 N_2) d\alpha_1 d\alpha_2 \\ &= \left\{ \int_{\mathcal{M}_1} + \int_{C(\mathcal{M}_1) \cap \mathcal{E}_\lambda} + \int_{C(\mathcal{M}_1) \setminus \mathcal{E}_\lambda} \right\} \left\{ \int_{\mathcal{M}_2} + \int_{C(\mathcal{M}_2) \cap \mathcal{E}_\lambda} + \int_{C(\mathcal{M}_2) \setminus \mathcal{E}_\lambda} \right\} \\ &\quad \times f^2(\alpha_1, U_1) g^2(\alpha_1, V_1) h^2(\alpha_1, W_1) f^2(\alpha_2, U_2) g^2(\alpha_2, V_2) \\ &\quad \times h^2(\alpha_2, W_2) G^k(\alpha_1 + \alpha_2) e(-\alpha_1 N_1 - \alpha_2 N_2) d\alpha_1 d\alpha_2 \\ &:= \sum_{s=1}^3 \sum_{t=1}^3 R_{st}(N_1, N_2), \end{aligned}$$

where $R_{st}(N_1, N_2)$ denotes the combination of s -th term in the first bracket and the t -th term in the second bracket.

We will establish Theorem 1.1 by estimating the term $R_{st}(N_1, N_2)$ for all $1 \leq s, t \leq 3$. We need to show that $R(N_1, N_2) > 0$ for every pair of sufficiently large even positive integers $N_2 \gg N_1 > N_2$.

We need the following lemmas to prove Theorem 1.1.

Lemma 2.1. *Let $\mathcal{E}_\lambda = \{\alpha \in (0, 1] : |G(\alpha)| \geq \lambda L\}$. We have $\text{meas}(\mathcal{E}_\lambda) \ll N_i^{-E(\lambda)}$, with $E(0.90365) > 131/168 + 10^{-10}$.*

Proof. This is Lemma 2.3 in Liu [13]. □

Let

$$(2.5) \quad \mathfrak{S}(n) = \sum_{q=1}^{\infty} A(n, q),$$

where

$$A(n, q) = \frac{1}{\varphi^6(q)} \sum_{\substack{a=1 \\ (a, q)=1}}^q C_2^2(q, a) C_3^2(q, a) C_4^2(q, a) e\left(-\frac{an}{q}\right),$$

$$C_k(q, a) = \sum_{\substack{m=1 \\ (m, q)=1}}^q e\left(\frac{am^k}{q}\right).$$

Lemma 2.2. *Let $\mathcal{M}_i$ be as in (2.4), with P_i and Q_i determined by (2.3). Then for $2 \leq n \leq N_i$, we have*

$$\int_{\mathcal{M}_i} f^2(\alpha_i, U_i) g^2(\alpha_i, V_i) h^2(\alpha_i, W_i) e(-\alpha n) d\alpha = \frac{1}{2^2 \cdot 3^2 \cdot 4^2} \mathfrak{S}(n) \mathfrak{J}(n) + O(N_i^{7/6} L^{-1}).$$

Here, the singular series $\mathfrak{S}(n)$ is defined as in (2.5) and satisfies $\mathfrak{S}(n) \gg 1$ for $n \equiv 0 \pmod{2}$. $\mathfrak{J}(n)$ is defined as

$$\mathfrak{J}(n) := \sum_{\substack{m_1 + \dots + m_6 = n \\ (U_i/2)^2 < m_1, m_2 \leq U_i^2 \\ (V_i/2)^2 < m_3, m_4 \leq V_i^2 \\ (W_i/2)^2 < m_5, m_6 \leq W_i^2}} (m_1 m_2)^{-1/2} (m_3 m_4)^{-2/3} (m_5 m_6)^{-3/4},$$

and satisfies $N_i^{7/6} \ll \mathfrak{J}(n) \ll N_i^{7/6}$.

Proof. This is Lemma 2.1 in Liu [13]. $\square$

Lemma 2.3. *For all integers $n \equiv 0 \pmod{2}$, we have $\mathfrak{S}(n) > 1.072808$.*

Proof. This result can be found in Section 3 in Liu [13]. $\square$

Lemma 2.4. *Let $\mathcal{B}(N_i, k) = \{(1 - \eta)N_i \leq n_i \leq N_i : n_i = N_i - 2^{v_1} - \dots - 2^{v_k}\}$ with $k \geq 2$. Then, for $N_1 \equiv N_2 \equiv 0 \pmod{2}$, we have*

$$\sum_{\substack{n_1 \in \mathcal{B}(N_1, k) \\ n_2 \in \mathcal{B}(N_2, k) \\ n_1 \equiv n_2 \equiv 0 \pmod{2}}} \mathfrak{J}(n_1) J(n_2) \geq 57.877 N_1^{7/6} N_2^{7/6} L^k.$$

Proof. Using [13, Lemma 4.2], we have

$$\sum_{\substack{n_1 \in \mathcal{B}(N_1, k) \\ n_2 \in \mathcal{B}(N_2, k) \\ n_1 \equiv n_2 \equiv 0 \pmod{2}}} \mathfrak{J}(n_1) \mathfrak{J}(n_2) \geq (7.607695)^2 N_1^{7/6} N_2^{7/6} \sum_{((v))} 1,$$

where $((v))$ means that $v_1, \dots, v_k$ satisfies

$$1 \leq v_1, \dots, v_k \leq L, \quad 2^{v_1} + \dots + 2^{v_k} \equiv N_1 \pmod{2}.$$

Then following the argument of [13, Lemma 4.1], we have

$$\sum_{(v)} 1 \geq (1 - \epsilon)L^k.$$

Then we get the proof of this lemma. $\square$

Lemma 2.5. *Let $f(\alpha_i, U_i)$, $g(\alpha_i, V_i)$, be defined by (2.1), $C(\mathcal{M}_i)$ by (2.4). Then*

$$\sup_{\alpha \in C(\mathcal{M}_i)} |f(\alpha_i, U_i)| \ll N_i^{7/16+\epsilon}, \quad \sup_{\alpha \in C(\mathcal{M}_i)} |g(\alpha_i, V_i)| \ll N_i^{13/42+\epsilon}.$$

Proof. The proof of this lemma can be found in [13, Lemma 2.4], which is based on the estimate of exponential sums over primes. $\square$

Lemma 2.6. *Let $f(\alpha_i, U_i)$, $g(\alpha_i, V_i)$ and $h(\alpha_i, W_i)$ be defined by (2.1). Then we have*

$$\int_0^1 |f(\alpha, U_i)g(\alpha, V_i)h(\alpha, W_i)|^2 d\alpha \leq 0.8485 N_i^{7/6}.$$

Proof. This is Lemma 2.2 in [13]. $\square$

3. Proof of Theorem 1.1

In this section, we will give the proof of Theorem 1.1.

We begin with the estimate for $R_{11}(N_1, N_2)$. Applying Lemmas 2.2, 2.3 and 2.4 and introducing the notation $\mathcal{B}(N_i, k)$, we can get

$$\begin{aligned} & R_{11}(N_1, N_2) \\ &= \int_{\mathcal{M}_1} f^2(\alpha_1, U_1) g^2(\alpha_1, V_1) h^2(\alpha_1, W_1) G^k(\alpha_1) e(-\alpha_1 N_1) d\alpha_1 \\ &\quad \times \int_{\mathcal{M}_2} f^2(\alpha_2, U_2) g^2(\alpha_2, V_2) h^2(\alpha_2, W_2) G^k(\alpha_2) e(-\alpha_2 N_2) d\alpha_2 \\ (3.1) \quad &= \sum_{\substack{n_1 \in \mathcal{B}(N_1, k) \\ n_2 \in \mathcal{B}(N_2, k)}} \int_{\mathcal{M}_1} f^2(\alpha_1, U_1) g^2(\alpha_1, V_1) h^2(\alpha_1, W_1) e(-\alpha_1 n_1) d\alpha_1 \\ &\quad \times \int_{\mathcal{M}_2} f^2(\alpha_2, U_2) g^2(\alpha_2, V_2) h^2(\alpha_2, W_2) e(-\alpha_2 n_2) d\alpha_2 \\ &\geq \left(\frac{1}{2^2 \cdot 3^2 \cdot 4^2} \right)^2 \sum_{\substack{n_1 \in \mathcal{B}(N_1, k) \\ n_2 \in \mathcal{B}(N_2, k)}} \mathfrak{S}(n_1) \mathfrak{S}(n_2) J(n_1) J(n_2) + O(N_1^{7/6} N_2^{7/6} L^{k-1}) \\ &\geq \frac{1}{331776} \cdot (1.072808)^2 \cdot 57.8771 N_1^{7/6} N_2^{7/6} L^k, \end{aligned}$$

where we used $\frac{n_i}{N_i} = 1 + O(L^{-1})$ for $n_i \in \mathcal{B}(N_i, k)$.

Now, we turn to give an upper bound for $R_{12}(N_1, N_2)$. The estimate for $R_{21}(N_1, N_2)$ is similar. By Cauchy's inequality, we can get $|G(\alpha_1 + \alpha_2)| \leq \sqrt{|G(2\alpha_1)G(2\alpha_2)|}$. For $\alpha_2 \in C(\mathcal{M}_2) \setminus \mathcal{E}_\lambda$ and sufficiently large N_1 , we have

$$|G(2\alpha_i)| \leq |G(\alpha_i)| + 2 \leq \lambda L + 2 \leq (1 + o(1))\lambda L.$$

Then, using the definition of $\mathcal{E}_\lambda$, the trivial bound of $G(\alpha_i)$, Lemmas 2.1, 2.5 and 2.6, we have

$$\begin{aligned} & R_{12}(N_1, N_2) \\ &= \int_{\mathcal{M}_1} \int_{C(\mathcal{M}_2) \cap \mathcal{E}_\lambda} f^2(\alpha_1, U_1) g^2(\alpha_1, V_1) h^2(\alpha_1, W_1) f^2(\alpha_2, U_2) g^2(\alpha_2, V_2) \\ &\quad \times h^2(\alpha_2, W_2) G^k(\alpha_1 + \alpha_2) e(-\alpha_1 N_1 - \alpha_2 N_2) d\alpha_1 d\alpha_2 \\ &\ll \int_{\mathcal{M}_1} |f^2(\alpha_1, U_1) g^2(\alpha_1, V_1) h^2(\alpha_1, W_1) G^{k/2}(2\alpha_1)| d\alpha_1 \\ (3.2) \quad &\times \int_{C(\mathcal{M}_2) \cap \mathcal{E}_\lambda} |f^2(\alpha_2, U_2) g^2(\alpha_2, V_2) h^2(\alpha_2, W_2) G^{k/2}(2\alpha_2)| d\alpha_2 \\ &\ll L^{k/2-2} \int_{\mathcal{M}_1} |f^2(\alpha_1, U_1) g^2(\alpha_1, V_1) h^2(\alpha_1, W_1) G^2(2\alpha_1)| d\alpha_1 \\ &\quad \times \int_{C(\mathcal{M}_2) \cap \mathcal{E}_\lambda} |f^2(\alpha_2, U_2) g^2(\alpha_2, V_2) h^2(\alpha_2, W_2) G^{k/2}(2\alpha_2)| d\alpha_2 \\ &\ll N_1^{7/6} L^{k/2} L^{k/2} \left(\max_{\alpha_2 \in C(\mathcal{M}_2)} |f(\alpha_2, U_2) g^2(\alpha_2, V_2)| \right. \\ &\quad \times \left. \left(\int_0^1 |f^2(\alpha_2, U_2) g^2(\alpha_2, V_2) \times h^2(\alpha_2, W_2)| d\alpha_2 \right)^{1/2} \right. \\ &\quad \times \left. \left(\int_0^1 |f^2(\alpha_2, U_2) h^4(\alpha_2, W_2)| d\alpha_2 \right)^{1/2} \right)^{2/3} \\ &\quad \times \left(\int_{\mathcal{E}_\lambda} 1 d\alpha_2 \right)^{1/3} \ll N_1^{7/6} L^k N_2^{719/504+\epsilon} N_2^{-E(\lambda)/3} \ll N_1^{7/6} N_2^{7/6} L^{k-1}. \end{aligned}$$

Similarly, we can get

$$(3.3) \quad R_{21}(N_1, N_2) \ll N_1^{7/6} N_2^{7/6} L^{k-1}.$$

Next, we give an upper bound for $R_{13}(N_1, N_2)$. By Lemma 2.6, using the trivial bound $|G(2\alpha)| \leq L$ when $\alpha \in \mathcal{M}_1$ and the bound $|G(2\alpha)| \leq (1 + o(1))\lambda L$ when $\alpha \in C(\mathcal{M}_2) \setminus \mathcal{E}_\lambda$, we have

$$\begin{aligned} & |R_{13}(N_1, N_2)| \\ &= \left| \int_{\mathcal{M}_1} \int_{C(\mathcal{M}_2) \setminus \mathcal{E}_\lambda} f^2(\alpha_1, U_1) g^2(\alpha_1, V_1) h^2(\alpha_1, W_1) f^2(\alpha_2, U_2) g^2(\alpha_2, V_2) \right. \\ &\quad \left. \times h^2(\alpha_2, W_2) G^k(\alpha_1 + \alpha_2) e(-\alpha_1 N_1 - \alpha_2 N_2) d\alpha_1 d\alpha_2 \right| \end{aligned}$$

$$\begin{aligned}
(3.4) \quad & \leq \int_{\mathcal{M}_1} |f^2(\alpha_1, U_1)g^2(\alpha_1, V_1)h^2(\alpha_1, W_1)G^{k/2}(2\alpha_1)|d\alpha_1 \\
& \times \int_{C(\mathcal{M}_2) \setminus \mathcal{E}_\lambda} |f^2(\alpha_2, U_2)g^2(\alpha_2, V_2)h^2(\alpha_2, W_2)G^{k/2}(2\alpha_2)|d\alpha_2 \\
& \leq L^{k/2} \int_{\mathcal{M}_1} |f^2(\alpha_1, U_1)g^2(\alpha_1, V_1)h^2(\alpha_1, W_1)|d\alpha_1 \\
& \times (\lambda L)^{k/2} \int_{C(\mathcal{M}_2) \setminus \mathcal{E}_\lambda} |f^2(\alpha_2, U_2)g^2(\alpha_2, V_2)h^2(\alpha_2, W_2)|d\alpha_2 \\
& \leq (0.8485)^2 \lambda^{k/2} N_1^{7/6} N_2^{7/6} L^k.
\end{aligned}$$

We can obtain the estimate for $R_{31}(N_1, N_2)$ analogously,

$$(3.5) \quad |R_{31}(N_1, N_2)| \leq (0.8485)^2 \lambda^{k/2} N_1^{7/6} N_2^{7/6} L^k.$$

Similar with the estimate of $R_{12}(N_1, N_2)$, We give the estimate for $R_{22}(N_1, N_2)$ by the trivial bound for $G(\alpha)$, Lemma 2.1, 2.5, 2.6 and the definition of $\mathcal{E}_\lambda$,

$$\begin{aligned}
(3.6) \quad & R_{22}(N_1, N_2) \\
& = \int_{C(\mathcal{M}_1) \cap \mathcal{E}_\lambda} \int_{C(\mathcal{M}_2) \cap \mathcal{E}_\lambda} f^2(\alpha_1, U_1)g^2(\alpha_1, V_1)h^2(\alpha_1, W_1)f^2(\alpha_2, U_2) \\
& \times g^2(\alpha_2, V_2)h^2(\alpha_2, W_2)G^k(\alpha_1 + \alpha_2)e(-\alpha_1 N_1 - \alpha_2 N_2)d\alpha_1 d\alpha_2 \\
& \ll \int_{C(\mathcal{M}_1) \cap \mathcal{E}_\lambda} |f^2(\alpha_1, U_1)g^2(\alpha_1, V_1)h^2(\alpha_1, W_1)G^{k/2}(2\alpha_1)|d\alpha_1 \\
& \times \int_{C(\mathcal{M}_2) \cap \mathcal{E}_\lambda} |f^2(\alpha_2, U_2)g^2(\alpha_2, V_2)h^2(\alpha_2, W_2)G^{k/2}(2\alpha_2)|d\alpha_2 \\
& \ll N_1^{7/6} L^{k/2-1} N_2^{7/6} L^{k/2-1} \ll N_1^{7/6} N_2^{7/6} L^{k-1}.
\end{aligned}$$

For $R_{23}(N_1, N_2)$, we can easily get

$$\begin{aligned}
(3.7) \quad & R_{23}(N_1, N_2) \\
& = \int_{C(\mathcal{M}_1) \cap \mathcal{E}_\lambda} \int_{C(\mathcal{M}_2) \cap \mathcal{E}_\lambda} f^2(\alpha_1, U_1)g^2(\alpha_1, V_1)h^2(\alpha_1, W_1)f^2(\alpha_2, U_2) \\
& \times g^2(\alpha_2, V_2)h^2(\alpha_2, W_2)G^k(\alpha_1 + \alpha_2)e(-\alpha_1 N_1 - \alpha_2 N_2)d\alpha_1 d\alpha_2 \\
& \ll \int_{C(\mathcal{M}_1) \cap \mathcal{E}_\lambda} |f^2(\alpha_1, U_1)g^2(\alpha_1, V_1)h^2(\alpha_1, W_1)G^{k/2}(2\alpha_1)|d\alpha_1 \\
& \times \int_{C(\mathcal{M}_2) \setminus \mathcal{E}_\lambda} |f^2(\alpha_2, U_2)g^2(\alpha_2, V_2)h^2(\alpha_2, W_2)G^{k/2}(2\alpha_2)|d\alpha_2 \\
& \ll N_1^{7/6} L^{k/2-1} N_2^{7/6} L^{k/2} \ll N_1^{7/6} N_2^{7/6} L^{k-1}.
\end{aligned}$$

Similarly, we have

$$(3.8) \quad R_{32}(N_1, N_2) \ll N_1^{7/6} N_2^{7/6} L^{k-1}.$$

In the end, we provide the upper bound for $R_{33}(N_1, N_2)$.

$$\begin{aligned}
 & |R_{33}(N_1, N_2)| \\
 &= \left| \int_{C(\mathcal{M}_1) \setminus \mathcal{E}_\lambda} \int_{C(\mathcal{M}_2) \setminus \mathcal{E}_\lambda} f^2(\alpha_1, U_1) g^2(\alpha_1, V_1) h^2(\alpha_1, W_1) f^2(\alpha_2, U_2) \right. \\
 (3.9) \quad & \times g^2(\alpha_2, V_2) h^2(\alpha_2, W_2) G^k(\alpha_1 + \alpha_2) e(-\alpha_1 N_1 - \alpha_2 N_2) d\alpha_1 d\alpha_2 \Big| \\
 &\leq \int_{C(\mathcal{M}_1) \setminus \mathcal{E}_\lambda} |f^2(\alpha_1, U_1) g^2(\alpha_1, V_1) h^2(\alpha_1, W_1) G^{k/2}(2\alpha_1)| d\alpha_1 \\
 &\times \int_{C(\mathcal{M}_2) \setminus \mathcal{E}_\lambda} |f^2(\alpha_2, U_2) g^2(\alpha_2, V_2) h^2(\alpha_2, W_2) G^{k/2}(2\alpha_2)| d\alpha_2 \\
 &\leq \left((\lambda L)^{k/2} \times 0.8485 N_1^{7/6} \right) \left((\lambda L)^{k/2} \times 0.8485 N_2^{7/6} \right) \\
 &\leq \lambda^k (0.8485)^2 N_1^{7/6} N_2^{7/6} L^k.
 \end{aligned}$$

Combining (3.1)-(3.9), we can obtain

$$\begin{aligned}
 R(N_1, N_2) &> R_{11}(N_1, N_2) - R_{13}(N_1, N_2) - R_{31}(N_1, N_2) - R_{33}(N_1, N_2) \\
 &\quad + O(N_1^{7/6} N_2^{7/6} L^{k-1}) \\
 &> \frac{1}{331776} \cdot (1.072808)^2 \cdot 57.8771 N_1^{7/6} N_2^{7/6} L^k \\
 &\quad - 2 \times (0.8485)^2 \lambda^{k/2} N_1^{7/6} N_2^{7/6} L^k \\
 &\quad - \lambda^k (0.8485)^2 N_1^{7/6} N_2^{7/6} L^k + O(N_1^{7/6} N_2^{7/6} L^{k-1}).
 \end{aligned}$$

We therefore solve the inequality $R(N_1, N_2) > 0$ and get $k \geq 176$. Consequently we deduce that every pair of large even integers N_1, N_2 satisfying $N_2 \gg N_1 > N_2$ and $N_1 \equiv N_2 \equiv 0 \pmod{2}$ can be written in the form of (1.3) for $k \geq 176$. Thus, Theorem 1.1 follows.

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A new characterization of sporadic groups

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Abstract. Let G be a finite group, n a positive integer. $\pi(n)$ denotes the set of all prime divisors of n and $\pi(G) = \pi(|G|)$. The prime graph $\Gamma(G)$ of G , defined by Grenberg and Kegel, is a graph whose vertex set is $\pi(G)$, two vertices p, q in $\pi(G)$ joined by an edge if and only if G contains an element of order pq . In this article, a new characterization of sporadic simple groups is obtained, that is, if G is a finite group and S a sporadic simple group. Then $G \cong S$ if and only if $|G| = |S|$ and $\Gamma(G)$ is disconnected. This characterization unifies the several characterizations that can conclude the group has disconnected prime graphs, hence several known characterizations of sporadic simple groups become the corollaries of this new characterization.

Keywords: sporadic groups, order, prime graph, characterization.

1. Introduction

In the past three decades, as a very interesting topic, quantitative characterization of a group, especially a simple group, has been being an active topic

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in the theory of finite simple group since classification of finite simple groups completed in the early of 1980s. When Wujie Shi began to investigate the topic whether a finite simple group can be uniquely determined by its order and the set of its element orders, he proposed a famous conjecture in 1987, which was recorded as Problem 12.39 in [1].

Shi's Conjecture. Let G be a finite group, S a finite simple group, then $G \cong S$ if and only if $|G| = |S|$ and $\pi_e(G) = \pi_e(S)$, where $\pi_e(G)$ denotes the set of element orders in G .

Research on Shi's conjecture began an era of quantitative characterization of finite simple groups. At last, this conjecture was completely proved in 2009. In the series of papers to prove Shi's conjecture, an important concept the prime graph of a finite group was frequently used for dealing with those simple groups with disconnected prime graph, which was defined by Gruenberg and Kegel in [2] as following:

Let G be a finite group, n a positive integer. $\pi(n)$ denotes the set of all prime divisors of n and $\pi(G) = \pi(|G|)$. The prime graph $\Gamma(G)$ of G , defined by Gruenberg and Kegel, is a graph whose vertex set is $\pi(G)$, two vertices p, q in $\pi(G)$ joined by an edge if and only if G contains an element of order pq . We denote the number of connected components of $\Gamma(G)$ as $t(G)$, the connected components of $\Gamma(G)$ as $\{\pi_i(G), i = 1, \dots, t(G)\}$, and we always assume 2 is in $\pi_1(G)$ if $2 \mid |G|$. The components of prime graphs of all finite simple groups were given by J. S. William, A. S. Kondratév, M. Suzuki, N. Iiyora and H. Yamaki etc. (see [2-5]).

The prime graph once be used by the second author to study the famous Thompson's conjecture:

Thompson's Conjecture. Let G be a finite group with $Z(G) = 1$, $N(G) = \{n \in \mathbf{N} \mid G \text{ has a conjugacy class of length } n\}$. If M is a finite simple group such that $N(G) = N(M)$, then $G \cong M$.

During the second author studying Thompson's conjecture, he proved that $|G| = |M|$ if G and M satisfy conditions of Thompson's Conjecture and the prime graph $\Gamma(M)$ is disconnected. For a finite group with disconnected prime graph, he divided its order into co-prime divisors, each of them exactly corresponding to each of components of the prime graph, and called these divisors the order components of the finite group and found that many finite simple groups can be uniquely determined by their order components. Actually, many simple groups with disconnected prime graphs have been proved to be uniquely determined by their order components, for example, in a series of articles, for example [6-11], etc, it is shown many simple groups with disconnected prime graphs are characterized by order components of their prime graphs. There are some other topics on characterization of a finite simple group by its order and some other quantitative properties related to disconnected prime graph, for example, in [12-16], characterization of a finite simple group by its order and

maximal element order (the largest element order or the second largest element order, or both of them), characterization of a finite simple group by its order and the set of orders of maximal abelian subgroups, etc. In these topics, the discussed finite group are usually ascribed to a finite group having disconnected prime graph, whose order is the same as some finite simple group. Therefore, it is a meaningful topic to study the finite group having its prime graph disconnected and its order being equal to a finite simple group. In this article, we shall discuss this topic and specially focus on a finite group having its prime graph disconnected and its order being equal to a sporadic group. we shall prove the following theorem:

Main theorem. *Let G be a finite group and S a sporadic simple group. Then $G \cong S$ if and only if $|G| = |S|$ and the prime graph of G is disconnected.*

By above theorem, the following known characterizations of sporadic simple groups including Shi' Conjecture and Thompson Conjecture, become its corollaries since under the hypothesis the prime graphs of the groups are disconnected, for example:

Corollary 1.1. *Let G be a finite group and S a sporadic simple group. Then $G \cong S$ if and only if $|G| = |S|$ and $\pi_e(G) = \pi_e(S)$.*

Corollary 1.2. *Let G be a finite group and S a sporadic simple group. Then $G \cong S$ if and only if $Z(G) = 1$ and $N(G) = N(S)$.*

Corollary 1.3. *Let G be a finite group and S a sporadic simple group. Then $G \cong S$ if and only if G and S have the same order components.*

Corollary 1.4. *Let G be a finite group and S a sporadic simple group. Then $G \cong S$ if and only if $|G| = |S|$ and the sets of orders of maximal abelian subgroups of G and S are equal.*

Throughout the paper, the actions unspecified always means conjugate action. Let G be a finite group, for a prime $p \in \pi(G)$, G_p denotes the Sylow p -subgroup of G . In addition, $G = U \rtimes V$ denotes G is the semidirect product of U and V , especially, $V \trianglelefteq G$.

2. The Proof of the Main Theorem

In order to prove the Main Theorem, we first present some lemmas which are useful in our proof.

Lemma 2.1 ([2], Theorem A). *If G is a finite group whose prime graph has more than one components, then one of the following holds:*

- (1) G is a Frobenius group;
- (2) G is a 2-Frobenius group;
- (3) G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are $\pi_1(G)$ -groups, K/H is a non-abelian simple group, H is a nilpotent group, where $2 \in \pi_1(G)$. And $|G/K| \mid |\text{Out}(K/H)|$.

Lemma 2.2 ([18], Lemma 2.6(1)). *Let G be a Frobenius group with Frobenius kernel K and Frobenius complement H . Then K is nilpotent and $|H||K| - 1$. Moreover, $t(G) = 2$, and $\{\pi_1(G), \pi_2(G)\} = \{\pi(H), \pi(K)\}$.*

Lemma 2.3 ([18], Lemma 2.6(2)). *Let G be a 2-Frobenius group, then $G = DEF$, where D and DE are normal subgroups in G , DE and EF are Frobenius groups with kernels D and E , respectively. Moreover $t(G) = 2$, $\pi_1(G) = \pi(D) \cup \pi(F)$ and $\pi_2(G) = \pi(E)$.*

Lemma 2.4 ([4], Lemma 14). *Any odd number from $\pi(\text{Out}(L))$, where L is a non-abelian simple group, either lies in $\pi(L)$ or does not exceed $\frac{p}{2}$, where $p = \max(\pi(L))$.*

Lemma 2.5. *Let G be a finite group, L a non-abelian simple group such that $|L||G|$, $|G||\text{Aut}(L)|$. Then any prime divisor r of $|G|$ with $\frac{p}{2} \leq r \leq p$, where $p = \max_{t \in \pi(G)} t$, lies in $\pi(L)$.*

Proof. It follows straight forward from $t \in \pi(L)$ by Lemma 2.4. $\square$

Lemma 2.6 ([19], 8.2.3). *Let G and A be two groups. Let p be a prime divisor of $|G|$. Suppose that the action of A on G is coprime. Then there exists an A -invariant Sylow p -subgroup of G .*

Lemma 2.7 ([18], Lemma 2.4). *Let G be a p -group of order p^n , and G act on a q -group H of order q^m , where p, q are two distinct primes. If $|G| \nmid |\text{GL}(m, q)|$, then $pq \in \pi_e(G \ltimes H)$.*

By the above Lemma, we have the following corollary.

Corollary 2.1. *Let G be a p -group of order p^n , and G act on a q -group H of order q^m . If $p^r \nmid \prod_{i=1}^m (q^i - 1)$, where $1 \leq r \leq n$, then $pq \in \pi_e(G \ltimes H)$.*

Remark. Let G be a finite group satisfying the hypothesis of the Main Theorem, then by Lemma 2.1, G may be a Frobenius group or a 2-Frobenius group. In order to prove the Main Theorem, we first show that G cannot be a Frobenius group or a 2-Frobenius group, the proof will be separated into six lemmas. In the proofs of next six lemmas, when we mention a Frobenius group or a 2-Frobenius group, we are referring groups and notations in Lemma 2.2 and Lemma 2.3 without explanations. Moreover, we get that K is a Hall subgroup of $G = HK$, and E is a Hall subgroup of $G = DEF$, therefore, the following property holds:

(*) G has no normal Sylow subgroup of the smallest order.

2.1 To prove that G is neither a Frobenius group nor a 2-Frobenius group

Now we start to show that G is neither a Frobenius group nor a 2-Frobenius group.

Lemma 2.8. *Let G be a group and S a Mathieu group. If $|G| = |S|$, and the prime graph of G is disconnected, then G is neither a Frobenius group nor a 2-Frobenius group.*

Proof. From [20], we get the orders of Mathieu groups are the following:

$$|M_{11}| = 2^4 \cdot 3^2 \cdot 5 \cdot 11, |M_{12}| = 2^6 \cdot 3^3 \cdot 5 \cdot 11, |M_{22}| = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11, \\ |M_{23}| = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23, |M_{24}| = 2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23.$$

(1) Since $|G| = |S|$, it follows that $11 \in \pi(G)$. Now, we prove that G has no nilpotent normal subgroup with order divided by 11. Otherwise, let T be such a group. Then $T_{11} \trianglelefteq G$, so that $\pi(G) \setminus 5 \subseteq \pi(C_G(T_{11}))$ by $|G/C_G(T_{11})| \nmid 10$. In view of the fact that $t(G) > 1$, we conclude that $5 \notin \pi(C_G(T_{11}))$. Note that $3 \in \pi(C_G(T_{11}))$, considering the action on $C_G(T_{11})$ by an element of order 5 and using Lemmas 2.6 and 2.7, we come to that $15 \in \pi_e(G)$. Hence, $t(G) = 1$, a contradiction. Similarly, we get that if $23 \in \pi(G)$, then G has no nilpotent normal subgroup with order divided by 23.

(2) Assume first that G is a Frobenius group, then $G = KH$ and $11 \in \pi(H)$ by (1). Applying Lemma 2.2, we obtain that, for any $r \in \pi(K)$, $t \in \pi(H)$, $|H_t| \nmid (|K_r| - 1)$. Therefore, we get that $|K_2| = 2^{10}$ or $|K_2| = 1$ after checking calculation. Moreover, $|K_3| = 1$. Since $|G_2| > |G_r|$ for any $r \neq 2$, we conclude that $|K_2| = 2^{10}$. But $3^2 \nmid |H|$, which contradicts Lemma 2.2 as $2^{10} - 1 = 3 \times 11 \times 31$. Therefore, G is not a Frobenius group.

(3) Now suppose that $G = DEF$ is a 2-Frobenius group. Using (1), we get that $11 \in \pi(E)$ or $11 \in \pi(F)$. If the former holds, then $|F| \nmid 10$. Further, by calculating and using the fact $23 \notin \pi(D)$ by (1), we come to that $D_r = 1$ for $r \neq 2$. Hence $|D| = 2^{10}$ and $|G| = |M_{24}|$. Therefore, $7 \in \pi(E)$ for $|F| \nmid 10$, contradicting $|E| \nmid (|D| - 1)$ as $2^{10} - 1 = 3 \times 11 \times 31$. Thus, $11 \notin \pi(E)$. So that $11 \in \pi(F)$, forcing that $|E| = 23$ and $|G| = |M_{24}|$, which indicates that $|F| \nmid 22$, yielding that $|D_5| = 5 < |E|$, contradicting Lemma 2.2. The lemma holds. $\square$

Lemma 2.9. *Let G be a group and S a Janko group. If $|G| = |S|$ and the prime graph of G is disconnected, then G is neither a Frobenius group nor a 2-Frobenius group.*

Proof. We write the proof in several steps.

Step 1. To show the lemma follows while $|G| = |J_1|$ or $|J_3|$.

Noticing $|J_1| = 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$, $|J_3| = 2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19$, similarly to Lemma 2.8(1), we can prove that G has no nilpotent normal subgroup with order divided by 19.

Suppose that $G = KH$ is a Frobenius group, then $19 \in \pi(H)$. By calculating, we have, $|G_r| \not\equiv 1 \pmod{19}$ for each $r \in \pi(G)$, a contradiction to Lemma 2.2. Hence, G can not be a Frobenius group. Now Assume that G is a 2-Frobenius group, referring to above and using the fact that E is a *Hall* subgroup of G , we conclude that $19 \notin \pi(F)$. Clearly, $19 \notin \pi(D)$, therefore, $19 \in \pi(E)$, which is impossible as $19 \nmid |\mathrm{GL}(7, 2)|$, $19 \nmid |\mathrm{GL}(5, 3)|$ and $5, 7, 11 < 19$. Consequently, G is not a 2-Frobenius group.

Step 2. To show the lemma follows while $|G| = |J_2|$ or $|J_4|$.

(1) Assume that $|G| = |J_2| = 2^7 \cdot 3^3 \cdot 5^2 \cdot 7$. If $G = KH$ is Frobenius group, then $7 \in \pi(H)$ by (*). But $|G_r| \not\equiv 1 \pmod{7}$ for $r = 2, 3, 5$, a contradiction to Lemma 2.2. Now, assume that $G = DEF$ is a 2-Frobenius group, similarly, $7 \notin \pi(F)$, so $7 \in \pi(E)$ by (*), which implies $|D| = 2^3 < 3^3$, also a contradiction. As a result, G is neither a Frobenius group nor a 2-Frobenius group.

(2) Assume that $|G| = |J_4| = 2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43$. If $G = KH$ is a Frobenius group, using (*), we see that $5 \in \pi(H)$, so that $43 \in \pi(H)$ for $5 \nmid 42$. Which is impossible, since $|G_r| \not\equiv 1 \pmod{43}$ for any $r \in \pi(G)$. If $G = DEF$ is a 2-Frobenius group, similarly, we get that $43 \notin \pi(F)$ for E is a *Hall* subgroup of G . Consequently, $43 \in \pi(D)$ or $43 \in \pi(E)$. If the former holds, then $|E| = 7$ for $|E|$ is odd and E is a *Hall* subgroup of G . Using (*), we have, $5 \notin \pi(D)$, so that $5 \in \pi(F)$, this is impossible since EF cannot be a Frobenius group. Thus, $43 \in \pi(E)$, which forces $|D| = 2^{14}$ by calculating, so that $|F_2| = 2^7 > |E_{43}|$, a contradiction to Lemma 2.2. Hence, G is neither a Frobenius group nor a 2-Frobenius group. The lemma follows from Steps 1-2. $\square$

Lemma 2.10. *Let G be a group and S a Conway group. If $|G| = |S|$ and the prime graph of G is disconnected, then G is neither a Frobenius group nor a 2-Frobenius group.*

Proof. From [20], we get the orders of Conway groups are the following:

$$|Co_1| = 2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23, |Co_2| = 2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23, |Co_3| = 2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23.$$

In the same way as Lemma 2.8(1), we get that G has no nilpotent normal subgroup with order divided by 23. Therefore, if $G = KH$ is a Frobenius group, then $23 \in \pi(H)$, this is impossible since $|G_r| \not\equiv 1 \pmod{23}$ for any $r \in \pi(G)$. Now assume $G = DEF$ is a 2-Frobenius group, then $23 \in \pi(E)$ or $23 \in \pi(F)$. If $23 \in \pi(F)$, then we get a contradiction by $|E| = 2^{11}$. If $23 \in \pi(E)$, then $|D| = 2^{11}$, yielding that $|G| \neq |Co_3|$ and $2^7 \nmid |F|$, contradicting $|F| \mid 22$. To summarize, G is neither a Frobenius group nor a 2-Frobenius group. $\square$

Lemma 2.11. *Let G be a group and S a Fischer group. If $|G| = |S|$ and the prime graph of G is disconnected, then G is neither a Frobenius group nor a 2-Frobenius group.*

Proof. From [20], we get the orders of Fischer groups are the following:

$$|Fi_{22}| = 2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13, |Fi_{23}| = 2^{18} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23, \\ |Fi'_{24}| = 2^{21} \cdot 3^{16} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29.$$

Since $t(G) > 1$, we deduce that G has no nilpotent normal subgroup with order divided by 11 or 13. Suppose that $G = KH$ is a Frobenius group, then $11, 13 \in \pi(H)$, which is impossible since $|G_r| \not\equiv 1 \pmod{11 \times 13}$ for any $r \in \pi(G)$. Now assume that $G = DEF$ is a 2-Frobenius group. In this case, $11, 13 \notin \pi(D)$. If $13 \in \pi(E)$, then $|D_3| = 3^9$. Since $3^9 - 1 = 2 \times 13 \times 757$, we conclude that $11 \notin \pi(E)$, yielding that $11 \in \pi(F)$, contradicting the fact that $|F||E_{13}| - 1$. Therefore, $13 \in \pi(F)$, implying that $|E| = 3^9$ by calculating. Hence, $|G| = |Fi_{22}|$ and $|D_5| = 5^2 < |E|$, a contradiction. $\square$

Lemma 2.12. *Let G be a group and S the Monster group or the Baby group. If $|G| = |S|$ and the prime graph of G is disconnected, then G is neither a Frobenius group nor a 2-Frobenius group.*

Proof. From [20], the orders of the Monster group and the Baby group are the following:

$$|M| = 2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71, |B| = 2^{41} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47.$$

(1) It can be easily shown that $t(G) = 1$ if G_p is normal in G , where $p = 17, 19, 31$, hence, G has no normal subgroup with order p .

(2) Assume that $G = KH$ is a Frobenius group, then $17, 19, 31 \in \pi(H)$ by (1). By calculating, we get that $|G_r| \not\equiv 1 \pmod{17 \times 19 \times 31}$ for any $r \in \pi(G)$, which contradicts Lemma 2.2. Now assume that G is a 2-Frobenius group. Then $17, 19, 31 \notin \pi(D)$ by (1). If $47 \in \pi(D)$, then $|E| = 23$, hence, $|F||2 \cdot 11$, from which it follows that $17 \in \pi(D)$, a contradiction to (1). If $47 \in \pi(E)$, then $|F|$ divides $2 \cdot 23$, and D can only be a 2-group by calculating. Consequently, for any $p \in \pi(G) \setminus \{2, 23\}$, all p -elements are contained in E , so $|E| \geq 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 31 \cdot 47 > 2^{46}$, contradicting $|E| \mid (|D| - 1)$. Hence, $47 \in \pi(F)$, then, similarly, E can only be a 2-group since EF_{47} is a Frobenius group, a contradiction to $|E|$ is odd. $\square$

Lemma 2.13. *Let G be a group and S one of Suz , HS , M^cL , He , HN , Th , $O'N$, Ly and Ru . If $|G| = |S|$ and the prime graph of G is disconnected, then G is neither a Frobenius group nor a 2-Frobenius group.*

Proof. We write the proof in three steps.

Step 1. To show the lemma follows while $|G|$ equals one of $|Suz|$, $|HS|$, $|M^cL|$, $|HN|$, $|O'N|$, $|Ly|$.

From [20], we get that: $|Suz| = 2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$; $|HS| = 2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$; $|M^cL| = 2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$; $|HN| = 2^{14} \cdot 3^6 \cdot 5^6 \cdot 7 \cdot 11 \cdot 19$; $|O'N| = 2^9 \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31$; $|Ly| = 2^8 \cdot 3^7 \cdot 5^6 \cdot 7 \cdot 11 \cdot 31 \cdot 37 \cdot 67$.

(1) Since $t(G) > 1$, we deduce that G has no nilpotent subgroup with order divided by 7 or 11.

(2) Suppose that $G = KH$ is a Frobenius group, then (1) implies that $11 \in \pi(H)$. By calculating, we conclude that $11 \nmid |G_r| - 1$ for any $r \in \pi(G)$, which contradicts the facts that K is a *Hall* subgroup and $|H||K| - 1$. Now assume that $G = DEF$ is a 2-Frobenius group. Then $7, 11 \notin \pi(D)$ by (1). Referring to above and using the fact that $|E|$ is a *Hall* subgroup of G , we see that $11 \notin \pi(F)$. Therefore, $11 \in \pi(E)$, then $|F| \mid 2 \cdot 5$. Moreover, by calculating, we come to that $|D_2| = 2^{10}$ or $D_3 = 3^5$, $D_r = 1$ while $r \neq 2, 3$. This indicates that $7 \in \pi(E)$. But $7 \nmid 2^{10} - 1$ and $7 \nmid 3^5 - 1$, a contradiction.

Step 2. To show the lemma follows while $|G| = |He|$

By $|He| = 2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$, one has that $|G_{17}| = 17$, hence, G has no nilpotent normal subgroup with order divided by 17 for $t(G) > 1$. Assume that G is a Frobenius group. Then $17 \in \pi(H)$, but $17 \nmid |G_r| - 1$ for every $r \in \pi(G)$, a contradiction to Lemma 2.2. Now Assume that G is a 2-Frobenius group. Note that E is a *Hall* subgroup of G and D is a nilpotent normal subgroup of G , for the same reason as above, we conclude that $17 \in \pi(E)$. Which forces that $|D| = 2^8$, so that $|F| = 2^2$, thereby, $|E_5| = 5^2$, it follows by Lemma 2.2 that $5^2 \mid 2^8 - 1$, contradicting $2^8 - 1 = 3 \times 5 \times 17$.

Step 3. To show the lemma follows while $|G|$ equals $|Th|$ or $|Ru|$.

(1) By [20], $|Th| = 2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31$ and $|Ru| = 2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29$, we see that, $|G_{13}| = 13$, and G_{13} is not normal in G , otherwise, $t(G) = 1$.

(2) Assume that G is a Frobenius group, then (1) follows that $13 \mid |H|$, so that $|K| = 3^3$, which yields $|K| < |H|$, contradicting $|H| \mid (|K| - 1)$. Therefore, G is not a Frobenius group.

(3) Assume that G is a 2-Frobenius group. Then $13 \in \pi(E)$ or $13 \in \pi(F)$ by (1). Suppose that $13 \in \pi(E)$, and if $|G| = |Th|$, then, we get that $19 \in \pi(E)$. This is impossible since 19 does not divide $|\text{GL}(14, 2)|$, $|\text{GL}(10, 3)|$, $|\text{GL}(3, 5)|$ and $|\text{GL}(2, 7)|$. If $|G| = |Ru|$, then $29 \in \pi(E)$. However, $29 \nmid |\text{GL}(14, 2)|$, we deduce that $D_2 = 1$, so that $|F_2| = |G_2| > |E_{13}|$, a contradiction. Hence, $13 \in \pi(F)$. Then, $|E| = 3^3$ by calculating, so $|G| = |Ru|$. Noticing that $3^3 - 1 = 2 \cdot 13$, we deduce that $|D_{29}| = 29$, which is impossible for ED_{29} cannot be a Frobenius group. which implies G is not a 2-Frobenius group.

This Lemma follows from Steps 1-3. $\square$

Lemmas 2.1 and Lemmas 2.8–2.13 imply the following.

Corollary 2.2. *Let G be a finite group and S one of 26 sporadic groups. If $|G| = |S|$ and the prime graph of G is disconnected, then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, where H and G/K are $\pi_1(G)$ -groups, K/H is a non-abelian simple group with $t(K/H) \geq t(G)$, H is a nilpotent group, and $G/H \leq \text{Aut}(K/H)$.*

2.2 To prove $G \cong S$

The necessity of the Main Theorem is obvious. Now we start to prove the sufficiency of The Main Theorem. In the following discussion, K and H always means subgroups of G in Corollary 2.2. We shall check the possibili-

ties of K/H based on $|K/H||S|$ by using the list of simple groups in [20], which means we need to find out all simple groups T such that $|T||S|$. In [20] there is a list of non-abelian simple groups of orders $< 10^{25}$, except that $L_2(q), L_3(q), U_3(q), L_4(q), U_4(q), S_4(q)$ and $G_2(q)$, which are stopped at orders $10^6(q \leq 125), 10^{12}(q \leq 31), 10^{12}(q \leq 32), 10^{16}(q \leq 11), 10^{16}(q \leq 11), 10^{16}(q \leq 41), 10^{20}(q \leq 25)$, respectively. For a sporadic simple group $S \not\cong B$ and M , it follows that $|S| < 10^{25}$, hence we need to find out a simple group T such that $|T||S|$ from the mentioned series of simple groups of Lie type. For $S = B$ or M , since $|B||M|$ and $10^{53} < |M| < 10^{54}$, so it is enough for us to find out simple group T such that $|T||M|$. Therefore, we set up the following lemma:

Lemma 2.14. *Let S be a sporadic group, and T a non-abelian simple group such that $|T||S|$, then*

(I) *If $S = M$, then T is either contained in the list of [20] or one of $A_n(26 \leq n \leq 32), L_2(2^{10})$ and $L_2(13^2)$.*

(II) *If $S \neq M$, then T is contained in the list of [20].*

Proof. (I) For a simple group T satisfying the hypothesis of (I), since $10^{53} < |M| < 10^{54}$, we have that $|T| \leq 10^{54}$, hence we have to check if there exists T of order $\leq 10^{25}$ among the series of $L_2(q)(q > 125), L_3(q)(q > 31), U_3(q)(q > 32), L_4(q)(q > 11), U_4(q)(q > 11), S_4(q)(q > 41)$ and $G_2(q)(q > 25)$, and to check if there exists T of order greater than 10^{25} and smaller than 10^{54} among all non-abelian simple groups except sporadic simple groups. In order to make the proof easy to read, we check all series of non-abelian simple groups except sporadic simple groups. By Classification Theorem of Simple Groups, we have the following steps.

(1) Assume that $T \cong A_n$, an alternating group. Note that $|A_{25}| < 10^{25} < |A_{26}|$, we have $n \geq 26$. On the other hand, in view of the fact that $11^2||M|$, we come to that $26 \leq n \leq 32$. By calculating the order of $A_n(26 \leq n \leq 32)$, we get that $T \cong A_n(26 \leq n \leq 32)$.

(2) Assume that $T \cong A_n(q)$. Then $|T| = q^{\frac{n(n+1)}{2}} \prod_{i=1}^n (q^{i+1} - 1)/(n+1, q-1)$. It follows from $|T||M|$ and the largest exponent of prime power of $|M|$ is 46 that $n \leq 9$. Therefore, T can only be one of the Lie type simple groups: $A_1(q)(q > 125), A_2(q)(q > 31), A_3(q)(q > 11), A_4(q)(q > 11), A_5(q)(q > 5), A_6(q)(q > 3), A_7(q)(q > 2), A_8(q)(q > 2), A_9(q)(q < 4)$.

If $T = A_1(q)(q > 125)$, since $|T|$ divides $|M|$, we deduce that one of the cases holds: $q = 2^t(7 \leq t \leq 46), q = 3^t(5 \leq t \leq 20), q = 5^t(4 \leq t \leq 9), q = 7^t(3 \leq t \leq 6), q = 13^t(2 \leq t \leq 3)$. Now calculating the order of these groups by using Maple, we come to that q is either 2^{10} or 13^2 .

If $T = A_2(q)(q > 31)$, then q^3 divides one of $2^{45}, 3^{18}, 5^9$ and 7^6 . Hence, q is one of: $2^t(5 \leq t \leq 15), q = 3^t(4 \leq t \leq 6), q = 5^3$ and $q = 7^2$. Now calculating the order of these groups, for example, by using Maple, we have $|T| \nmid |M|$, a contradiction.

By the same approach above, we can get contradictions if $T = A_n(q)$, where $3 \leq n \leq 9$.

Suppose that T is one of the remaining cases, we can get contradictions by $|T| \nmid |M|$ in the same way as above. The checking procedures is trivial, we omit them here. We come to that (I) follows.

(II) For a sporadic group S , we have that if $S \neq B$ and M , then $|S| < 10^{25}$; and if $S \not\cong Ly$ or J_4 , then $|S| \nmid |M|$. Hence if $S \not\cong Ly$ and J_4 , then by (I) T is contained in the list of [20], or T is one of $A_n (26 \leq n \leq 32)$, $L_2(2^{10})$ and $L_2(13^2)$. But $|A_n| > 10^{25}$ ($26 \leq n \leq 32$), $|L_2(2^{10})|$ and $|L_2(13^2)|$ do not divide the order of any sporadic group except M . Therefore (II) follows for $S \not\cong Ly$ and J_4 . If $S \cong Ly$ or J_4 , since both of $|Ly|$ and $|J_4| < 10^{25}$, we have that if T is not in the list of [20], then T is contained in the series of $L_2(q) (q > 125)$, $L_3(q) (q > 31)$, $U_3(q) (q > 32)$, $L_4(q) (q > 11)$, $U_4(q) (q > 11)$, $S_4(q) (q > 41)$ and $G_2(q) (q > 25)$. By using Maple to calculate the orders of groups with orders $< 10^{25}$ among these series, we can show that $|T|$ does not divide $|S|$. $\square$

Now, we begin to prove $G \cong S$, in the following proofs, we use (2) in Corollary 2.2 and Lemma 2.5 frequently.

Lemma 2.15. *Let G be a group and S one of groups: M_{11} , M_{12} , M_{22} , M_{23} and M_{24} . If $|G| = |S|$ and the prime graph of G is disconnected, then $G \cong S$.*

Proof. By assumptions, $|G|$ is one of $|M_{11}|$, $|M_{12}|$, $|M_{22}|$, $|M_{23}|$ and $|M_{24}|$, also, $\Gamma(G)$ is disconnected. Applying Corollary 2.2, G has a unique simple section K/H such that K and H satisfying (1) in Corollary 2.2. By Lemma 2.15, K/H can only be simple groups in the list of [20]. By comparing the order of G with the orders of the simple groups in [20], we can obtain all possibilities of K/H . We discuss step by step following what S is.

Step 1. If $S = M_{11}$ or M_{12} , then $G \cong S$.

Observe that $|M_{11}| = 2^4 \cdot 3^2 \cdot 5 \cdot 11$, $|M_{12}| = 2^6 \cdot 3^3 \cdot 5 \cdot 11$, and $|G| = |S|$. By $t(G) > 1$, we assert that $11 \notin \pi(H)$. Otherwise, $|H_{11}| = 11$ and $H_{11} \leq G$. Since $G/C_G(H_{11}) \leq \text{Aut}(H_{11})$, it follows that $\pi(G) \setminus \{5\} \in \pi(C_G(H_{11}))$. As $t(G) > 1$, we have $5 \notin \pi(C_G(H_{11}))$. Note that $3 \in \pi(C_G(H_{11}))$, considering the action of an element of order 5 on $C_G(H_{11})$. Apply Lemmas 2.6 and 2.7, we get that $15 \in \pi_e(G)$, thus $t(G) = 1$, a contradiction. Hence $11 \in \pi(G/H)$ as desired. By Corollary 2.2 and Lemma 2.5, we get that $11 \in \pi(K/H)$. Checking simple groups in the list of [20], we get that K/H has the following cases:

(1) Suppose that $S \cong M_{11}$, then $K/H \cong L_2(11)$ or M_{11} . As $|L_2(11)| = 2^2 \cdot 3 \cdot 5 \cdot 11$, $|\text{Out}(L_2(11))| = 2$, we see that $|H_3| = 3$, thus $t(G) = 1$, a contradiction. So $K/H \cong M_{11}$, hence, $G \cong S \cong M_{11}$ by $|G| = |M_{11}|$.

(2) Suppose that $S \cong M_{12}$, then $K/H \cong L_2(11)$, M_{11} or M_{12} . If $K/H \cong M_{12}$, then $G \cong S \cong M_{12}$, we are done. Note that $|\text{Out}(M_{11})| = 1$ and refer to (1), we get $|H_3| = 3$ or 3^2 if $K/H \cong L_2(11)$ or M_{11} . This implies that $t(G) = 1$, a contradiction. Step (1) follows.

Step 2. If $S = M_{22}$, then $G \cong M_{22}$.

Since $|G| = |M_{22}| = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$, it follows by $t(G) > 1$ that $7, 11 \in \pi(G/H)$. Using Corollary 2.2 and Lemma 2.5, $7, 11 \in \pi(K/H)$, so K/H is uniquely isomorphic to M_{22} by [20]. Therefore, $G \cong S$, and Step (2) follows.

Step 3. If $S = M_{23}$ or M_{24} , then $G \cong S$.

In these cases, either $|G| = |M_{23}| = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$ or $|G| = |M_{24}| = 2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$. For the same reason as above, $23 \in \pi(K/H)$. By [20], the following hold:

(1) Suppose that $S \cong M_{23}$, then $K/H \cong M_{23}$ or $L_2(23)$. If $K/H \cong M_{23}$, then $G \cong S$, we are done. If $K/H \cong L_2(23)$, then by $G/H \leq \text{Aut}(K/H)$, we get that $|H_7| = 7$, one has that $t(G) = 1$, a contradiction.

(2) Suppose that $S \cong M_{24}$, then K/H is one of M_{24} , M_{23} and $L_2(23)$. Note that if $K/H \cong M_{23}$ or $L_2(23)$, then $|H_3| = 3$ or $|H_7| = 7$. But in either case, one has that $t(G) = 1$, this is impossible. Therefore $K/H \cong M_{24}$, thus $G \cong M_{24}$. $\square$

Lemma 2.16. *Let G be a group and S one of groups: J_1 , J_2 , J_3 and J_4 . If $|G| = |S|$ and the prime graph of G is disconnected, then $G \cong S$.*

Proof. Similar to Lemma 2.15, we need consider the possibilities of the unique simple section K/H for each S , and have the following steps.

Step 1. If S is one of J_1 , J_3 and J_4 , then $G \cong S$.

Since $|J_1| = 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$, $|J_3| = 2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19$, and $|J_4| = 2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43$, it follows by $t(G) > 1$ and Corollary 2.2 and Lemma 2.5 that: if $S = J_1$, then $11, 19 \in \pi(K/H)$; if $S = J_3$, then $17, 19 \in \pi(K/H)$; if $S = J_4$, then $23, 29, 31, 37, 43 \in \pi(K/H)$. Hence, by [20], K/H can only be isomorphic to S . Therefore, $G \cong S$ by $|G| = |K/H|$.

Step 2. If $S = J_2$, then $G \cong S$.

In such case, $|G| = |J_2| = 2^7 \cdot 3^3 \cdot 5^2 \cdot 7$. similarly, $5, 7 \in \pi(K/H)$. We check the list of simple groups in [20] and come to that K/H is one of the following simple groups: $A_7(|A_7| = 2^3 \cdot 3^2 \cdot 5 \cdot 7, |\text{Out}(A_7)| = 2)$, $A_8(|A_8| = 2^6 \cdot 3^2 \cdot 5 \cdot 7, |\text{Out}(A_8)| = 2)$, $L_3(4)(|L_3(4)| = 2^6 \cdot 3^2 \cdot 5 \cdot 7, |\text{Out}(L_3(4))| = 2^2 \cdot 3)$, J_2 . If $K/H \not\cong J_2$, then $|H_5| = 5$, which yields $t(G) = 1$, a contradiction. Hence, $K/H \cong J_2$. Therefore, $G \cong S$, as desired. $\square$

Lemma 2.17. *Let G be a group and S one of Co_1 , Co_2 and Co_3 . If $|G| = |S|$ and the prime graph of G is disconnected, then $G \cong S$.*

Proof. We write the proof step by step upon what S is .

Step 1. If $S = Co_1$, then $G \cong Co_1$.

By assumption, $|G| = |Co_1| = 2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23$. So $13, 23 \in \pi(K/H)$ by $t(G) > 1$ and Corollary 2.2 and Lemma 2.5. Applying Lemma 2.14 and checking simple groups of orders dividing $|Co_1|$ in [20], we get that $K/H \cong S = Co_1$.

Step 2. If S is one of Co_2 and Co_3 then $G \cong S$.

Since $|G|$ equals $|Co_2| = 2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$ or $|Co_3| = 2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$, we get that $23 \in \pi(K/H)$. By the same reason, we get that K/H is one of the simple groups: S , $L_2(23)(|L_2(23)| = 2^3 \cdot 3 \cdot 11 \cdot 23, |\text{Out}(L_2(23))| = 2)$, $M_{23}(|M_{23}| =$

$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$, $\text{Out}(M_{23}) = 1$), $M_{24}(|M_{24}| = 2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23, \text{Out}(M_{24}) = 1)$. Comparing $|G|$ with $|\text{Aut}(K/H)|$, we conclude that if $K/H \not\cong S$, then $|H_5| = 5^r$, where $1 \leq r \leq 3$. Note that $|\text{GL}(3, 5)| = 2^7 \times 3 \times 31$, by $G/C_G(H_5) \leq \text{Aut}(H_5)$ and using Lemma 2.7, we get that $t(G) = 1$, a contradiction. Therefore, $K/H \cong S$, hence $G \cong S$, we are done. $\square$

Lemma 2.18. *Let G be a group and $S = Fi_{22}$, Fi_{23} or Fi'_{24} . If $|G| = |S|$ and the prime graph of G is disconnected, then $G \cong S$.*

Proof. The proof is divided into two steps.

Step 1. If $S = Fi_{22}$, then $G \cong Fi_{22}$.

In this case, $|G| = |Fi_{22}| = 2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$, so $7, 11, 13 \in \pi(K/H)$. By the same reason, checking the list in [20], we get that K/H is isomorphic to one of the groups: Fi_{22} , $A_{13}(|A_{13}| = 2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13, |\text{Out}(A_{13})| = 2)$, $Suz(|Suz| = 2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13, |\text{Out}(Suz)| = 2)$. Checking the orders of $\text{Aut}(K/H)$ and using the fact that $|G/H| \mid |\text{Aut}(K/H)|$, we get that if $K/H \not\cong S$, then $|H_3| = 3^r$ and $2 \leq r \leq 4$. Observe that $|\text{GL}(4, 3)| = 2^9 \times 3^6 \times 5 \times 13$, by $G/C_G(H_3) \leq \text{Aut}(H_3)$ and $t(G) > 1$, we have $\pi(C_G(H_3)) = \pi(G) \setminus 13$. In particular, $7 \in \pi(C_G(H_3))$. Consider the action of an element of order 13 on $C_G(H_3)$. Using Lemmas 2.6 and 2.7, $91 \in \pi_e(G)$, which implies that $t(G) = 1$, a contradiction. Therefore, $K/H \cong S$, hence $G \cong S$.

Step 2. If $S = Fi_{23}$ or Fi'_{24} , then $G \cong S$.

Note that $|Fi_{23}| = 2^{18} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23$, $|Fi'_{24}| = 2^{21} \cdot 3^{16} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29$, similarly we conclude $13, 17, 23 \in \pi(K/H)$ if $S = Fi_{23}$, and $17, 23, 29 \in \pi(K/H)$ if $S = Fi'_{24}$. By the same reason, checking the list in [20], we get that K/H can only be isomorphic to S , so that $G \cong S$. $\square$

Lemma 2.19. *Let G be a group and S a Monster group or Baby Monster group. If $|G| = |S|$ and the prime graph of G is disconnected, then $G \cong S$.*

Proof. Because $|M| = 2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$ and $|B| = 2^{41} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47$, by assumptions, $t(G) > 1$, apply Corollary 2.2 and Lemma 2.5, we come to that, if $S = M$, then $41, 47, 59, 71 \in \pi(K/H)$; if $S = B$, then $31, 47 \in \pi(K/H)$. By Lemma 2.14 and [20], we get that K/H can only be isomorphic to S . Consequently, $G \cong S$. $\square$

Lemma 2.20. *Let G be a group and S one of the groups: Ly , $O'N$, M^cL , Th , HN , He , Ru , HS , and Suz . If $|G| = |S|$ and the prime graph of G is disconnected, then $G \cong S$.*

Proof. We have the following steps.

Step 1. If $S = Ly$, $O'N$, or Th , then $G \cong S$.

In this case, $|G|$ is one of $|Ly| = 2^8 \cdot 3^7 \cdot 5^6 \cdot 7 \cdot 11 \cdot 31 \cdot 37 \cdot 67$, $|O'N| = 2^9 \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31$ and $|Th| = 2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31$. Similarly, we see that: If $S = Ly$, then $37, 67 \in \pi(K/H)$; if $S = O'N$ or Th , then

$19, 31 \in \pi(K/H)$. In either case, we get that $K/H \cong S$ by [20] and Lemma 2.14. Hence, $G \cong S$.

Step 2. If $S = M^cL$, then $G \cong M^cL$.

$|G| = |M^cL| = 2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$ indicates $7, 11 \in \pi(K/H)$. Applying Lemma 2.14 and by [20], we see that K/H is one of: M^cL , $M_{22}(|M_{22}| = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11, |\text{Out}(M_{22})| = 2)$; $A_{11}(|A_{11}| = 2^7 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11, |\text{Out}(A_{11})| = 2)$. If $K/H \cong M^cL$, then $G \cong S$, we are done. If $K/H \cong A_{11}$ or M_{22} , by calculating, we get that $|H_5| = 5$ or 5^2 . But in either case, we can get that $t(G) = 1$ by $G/C_G(H_5) \leq \text{Aut}(H_5)$, a contradiction to our hypothesis.

Step 3. If $S = HN$, then $G \cong HN$.

Since $|G| = |HN| = 2^{14} \cdot 3^6 \cdot 5^6 \cdot 7 \cdot 11 \cdot 19$, we have $11, 19 \in \pi(K/H)$. Then apply Lemma 2.14 and [20], we get that, $K/H \cong HN$ or J_1 . If $K/H \cong HN$, then $G \cong S$ as desired. If $K/H \cong J_1$, since $|J_1| = 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$ and $|\text{Out}(J_1)| = 2$, it follows by Corollary 2.2 that $|H|_3 = 3^5$. Note that $|\text{GL}(5, 3)| = 2^9 \cdot 3^{10} \cdot 5 \cdot 13$, use the same method, we get that $t(G) = 1$, a contradiction. Hence $K/H \cong S$, so $G \cong S$.

Step 4. If $S = He$, then $G \cong He$.

In this case, $|G| = |He| = 2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$, so that $17 \in \pi(K/H)$. By [20] and Lemma 2.14, we get that K/H is one of the groups: He ; $L_2(16)(|L_2(16)| = 2^4 \cdot 3 \cdot 5 \cdot 17, |\text{Out}(L_2(16))| = 4)$; $L_2(17)(|L_2(17)| = 2^4 \cdot 3^2 \cdot 17, |\text{Out}(L_2(17))| = 2)$; $S_4(4)(|S_4(4)| = 2^8 \cdot 3^2 \cdot 5 \cdot 17, |\text{Out}(S_4(4))| = 2)$.

Suppose that K/H is isomorphic to one of the groups except He , then $|H_3| = 3$ or 3^2 , this yields $t(G) = 1$, a contradiction. Now we have $K/H \cong He$, hence $G \cong S$.

Step 5. If $S = Ru$, then $S \cong Ru$.

By $|G| = |Ru| = 2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29$, we have $29 \in \pi(K/H)$. Lemma 2.14 and [20] show that $K/H \cong Ru$ or $L_2(29)(|L_2(29)| = 2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 29, |\text{Out}(L_2(29))| = 2)$. If $K/H \cong L_2(29)$, then $|H_{13}| = 13$, hence $t(G) = 1$, a contradiction. Thus $K/H \cong Ru$, so $G \cong S$, as desired.

Step 6. If $S = HS$, then $G \cong HS$.

In this case, $|G| = |HS| = 2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$, thus, $7, 11 \in \pi(K/H)$. Applying Lemma 2.14 and [20], we get that $K/H \cong HS$ or $M_{22}(|M_{22}| = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11, |\text{Out}(M_{22})| = 2)$. If $K/H \cong M_{22}$, then $G/H \leq \text{Aut}(K/H)$ imply $|H_5| = 5^2$. By $G/C_G(H_5) \leq \text{Aut}(H_5)$, we get that $t(G) = 1$, a contradiction. Thus $K/H \cong HS$, so $G \cong S$.

Step 7. If $S = Suz$, then $G \cong Suz$.

In this case, $|G| = |Suz| = 2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$, it follows by $t(G) = 1$ and Corollary 2.2 and Lemma 2.5 that $7, 11, 13 \in \pi(K/H)$. By the same reason, we get that $K/H \cong Suz$ or $A_{13}(|A_{13}| = 2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13, |\text{Out}(A_{13})| = 2)$. Again by $t(G) = 1$, we get that $K/H \cong Suz$, so that $G \cong S$, we are done.

The lemma holds by Step 1-7. □

2.3 Conclusion

Proof of the Main Theorem. The Main Theorem follows from Lemma 2.8-2.13 and 2.15–2.20.

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A new secant type method for solving one variable functions

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Abstract. For solving one variable functions, Newton method is an important and basic method which convergences quadratically. In this paper, we deriving a new secant type methods based on the approximating the second derivative information for solving one minimization problem. The new secant type iterative method has convergence of order two. Application examples are given for demonstrated that the proposed method has better numerical characteristics compared to Newton method.

Keywords: Newton's methods, secant method, test functions.

1. Introduction

The majority useful optimization problems have many variables, know the study of single variable minimization may seem academic. For function of one variable, the classical approach finds the values of σ at the turning points of $f(\sigma)$ as the solutions of:

$$(1) \quad f'(\sigma) = 0.$$

For more details can be found in [3].

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Newton's classic method is the best broadly use of these types of methods, which is given as follows:

$$(2) \quad \sigma_{\ell+1} = \sigma_{\ell} - \frac{f'(\sigma_{\ell})}{f''(\sigma_{\ell})}.$$

For details see [14]. Secant method [4,5] is derived from the Newton-Raphson Method. Sometimes evaluating the second derivative of the function could be very tedious and cumbersome. To overcome this, the derivative is approximated and the next estimate is given as:

$$(3) \quad \sigma_{\ell+1} = \sigma_{\ell} - \frac{f'(\sigma_{\ell})}{f'(\sigma_{\ell}) - f'(\sigma_{\ell-1})}(\sigma_{\ell} - \sigma_{\ell-1}).$$

Lately, some authors have derived new iterative methods which are more efficient than of Newton's [4,6-10,13 and 15].

The new update deriving for trying to approximate the secant update by approximating the second derivative $f''(\sigma_{\ell})$ information.

2. A new modification into Newton's method

Theoretically, the Taylor's theorem has an important role in iterative methods. Taking Three-order Taylor expansion around the point σ_{ℓ} , as follows [12]:

$$(4) \quad f(\sigma) \cong f(\sigma_{\ell}) + (\sigma - \sigma_{\ell})f'(\sigma_{\ell}) + \frac{1}{2}(\sigma - \sigma_{\ell})^2 f''(\sigma_{\ell}) + \frac{1}{6}(\sigma - \sigma_{\ell})^3 f'''(\sigma_{\ell})$$

finding the minimum point where:

$$(5) \quad f(\sigma) \cong f'(\sigma_{\ell}) + (\sigma - \sigma_{\ell})f''(\sigma_{\ell}) + \frac{1}{2}(\sigma - \sigma_{\ell})^2 f'''(\sigma_{\ell}) = 0.$$

It reduces to:

$$(6) \quad (\sigma - \sigma_{\ell})^2 f'''(\sigma_{\ell}) = -2(f'(\sigma_{\ell}) + (\sigma - \sigma_{\ell})f''(\sigma_{\ell})).$$

Putting Eq. 6 in Eq. 4 we get:

$$(\sigma - \sigma_{\ell})^2 f'''(\sigma_{\ell}) = 6((f(\sigma) - f(\sigma_{\ell})) - 4(\sigma - \sigma_{\ell})f'(\sigma_{\ell})).$$

This yields:

$$(7) \quad f'''(\sigma_{\ell}) = \frac{6((f(\sigma) - f(\sigma_{\ell})) - 4(\sigma - \sigma_{\ell})f'(\sigma_{\ell}))}{(\sigma - \sigma_{\ell})^2}.$$

Substituting σ with $\sigma_{\ell-1}$ in the above equation, we get:

$$(8) \quad f'''(\sigma_{\ell}) = \frac{6((f(\sigma_{\ell-1}) - f(\sigma_{\ell})) - 4(\sigma_{\ell-1} - \sigma_{\ell})f'(\sigma_{\ell}))}{(\sigma_{\ell-1} - \sigma_{\ell})^2}.$$

Putting Eq. 8 into Eq. 1, we get:

$$(9) \quad \sigma_{\ell+1} = \sigma_{\ell} - \frac{(\sigma_{\ell-1} - \sigma_{\ell})^2 f'(\sigma_{\ell})}{6(f(\sigma_{\ell-1}) - f(\sigma_{\ell})) - 4((\sigma_{\ell-1} - \sigma_{\ell})f'(\sigma_{\ell}))},$$

which is new modification into Newton's method.

Now, we are prepared to state the new algorithm based on the above preparation.

Stage 1. Let σ_0 be the estimate of a minimum of $f(\sigma)$.

Stage 2. Set $\ell = \ell + 1$.

Stage 3. Set $\sigma_{\ell} = \sigma_{\ell-1} - \frac{f'(\sigma_{\ell-1})}{f''(\sigma_{\ell-1})}$.

Stage 4. Compute $\sigma_{\ell+1} = \sigma_{\ell} - \frac{(\sigma_{\ell-1} - \sigma_{\ell})^2 f'(\sigma_{\ell})}{6(f(\sigma_{\ell-1}) - f(\sigma_{\ell})) - 4((\sigma_{\ell-1} - \sigma_{\ell})f'(\sigma_{\ell}))}$.

Stage 5. If stopping criteria satisfies, then stop, otherwise go to 2.

3. Convergence analysis

In the next theorem, we will study the new method's convergence.

Theorem 3.1. *Let $f : I \rightarrow R$ be a sufficiently differentiable function where I is an open interval, assume $\sigma^* \in R$ be a zero of $f(\sigma)$. If σ_0 is sufficiently close to this zero, then the convergence of the new method is quadratic.*

Proof. The modification into Newton method is as:

$$(10) \quad \sigma_{\ell+1} = \sigma_{\ell} - \frac{(\sigma_{\ell-1} - \sigma_{\ell})^2 f'(\sigma_{\ell})}{6(f(\sigma_{\ell-1}) - f(\sigma_{\ell})) - 4((\sigma_{\ell-1} - \sigma_{\ell})f'(\sigma_{\ell}))}.$$

By subtracting σ^* from both sides of equation 10, we obtain:

$$(11) \quad e_{\ell+1} = e_{\ell} - \frac{f'(\sigma_{\ell})(e_{\ell-1} - e_{\ell})^2}{6(f(\sigma_{\ell-1}) - f(\sigma_{\ell})) - 4((e_{\ell-1} - e_{\ell})f'(\sigma_{\ell}))},$$

where $e_{\ell} = \sigma_{\ell+1} - \sigma^*$. Applying expanding Taylor's series, we get:

$$(12) \quad \begin{aligned} f(\sigma^*) &= f(\sigma_{\ell}) + (\sigma^* - \sigma_{\ell})f'(\sigma_{\ell}) + \frac{1}{2!}(\sigma^* - \sigma_{\ell})^2 f^{(2)}(\sigma_{\ell}) + \frac{1}{3!}(\sigma^* - \sigma_{\ell})^3 f^{(3)}(\sigma_{\ell}) \\ &+ \frac{1}{4!}(\sigma^* - \sigma_{\ell})^4 f^{(4)}(z), \end{aligned}$$

where z lies between σ^* and σ_{ℓ} , the derivative of equation (12), equal to zero, we obtained:

$$(13) \quad f'(\sigma^*) = f'(\sigma_{\ell}) + (\sigma^* - \sigma_{\ell})f^{(2)}(\sigma_{\ell}) + \frac{1}{2}(\sigma^* - \sigma_{\ell})^2 f^{(3)} + \frac{1}{6}(\sigma^* - \sigma_{\ell})^3 f^{(4)}(z)$$

and using:

$$(14) \quad -f^{(1)}(\sigma^*) = -e_{\ell}f^{(2)}(\sigma_{\ell}) + \frac{e_{\ell}^2}{2}f^{(3)}(\sigma_{\ell}) - \frac{e_{\ell}^3}{6}f^{(4)}(z)$$

from equation 11 we get:

$$(15) \quad e_{\ell+1} = \frac{e_{\ell}^2 f^{(3)}(\sigma_{\ell})}{2f^{(2)}(\sigma_{\ell})} - \frac{e_{\ell}^3 f^{(4)}(z)}{6f^{(2)}(\sigma_{\ell})}.$$

This means the order of convergence is quadratically. $\square$

4. Computational results

Applying the iterative methods to solve seven unimodal function, in Matlab, it is implemented. In addition, there are other test functions for multidimensional case as in [1,2]. Using the execution time and number of iteration for the various functions for comparison between the iterative methods. We use accuracy is $\epsilon = 10^{-10}$, for computer programs.

Problem 1. Function $f(\sigma) = \cos(\sigma) + (\sigma - 2)^2$ **starting point** $\sigma_0 = 2$.

Methods	Number of iterations	σ_{ℓ}	Execution time
Newton method	4	2.3542	0.5790
New method	3	2.3684	0.1410

Problem 2. Function $f(\sigma) = e^{\sigma} - 3\sigma^2$ **starting point** $\sigma_0 = 0.25$.

Methods	Number of iterations	σ_{ℓ}	Execution time
Newton method	4	0.2045	0.2660
New method	3	0.2043	0.1880

Problem 3. Function $f(\sigma) = e^{\sigma} + \sigma^2$ **starting point** $\sigma_0 = 1$.

Methods	Number of iterations	σ_{ℓ}	Execution time
Newton method	5	0.3517	0.3130
New method	3	0.3257	0.1570

Problem 4. Function $f(\sigma) = -\sigma e^{-\sigma}$ **starting point** $\sigma_0 = 0$.

Methods	Number of iterations	σ_{ℓ}	Execution time
Newton method	7	1	0.3900
New method	4	0.5630	0.2030

Problem 5. Function $f(\sigma) = 0.65 - 0.75/(1 + \sigma^2) - 0.65\sigma \tan^{-1}(1/\sigma)$
starting point $\sigma_0 = 0.1$.

Methods	Number of iterations	σ_{ℓ}	Execution time
Newton method	4	0.4809	0.7970
New method	4	0.3971	0.2810

Problem 6. Function $f(\sigma) = 0.5\sigma^2 - \sin(\sigma)$ **starting point** $\sigma_0 = 2$.

Methods	Number of iterations	σ_{ℓ}	Execution time
Newton method	5	0.7391	0.4210
New method	3	0.7359	0.1560

Problem 7. Function $f(\sigma) = \sigma^4 + 2\sigma^2 - \sigma - 3$ **starting point** $\sigma_0 = 1$.

Methods	Number of iterations	σ_ℓ	Execution time
Newton method	7	0.2367	0.6870
New method	4	0.5124	0.1880

5. Conclusion

A new secant type method for solving one minimization problem has been derived and analyzed. In certain examples, both strategies which fail to converge to the minimum and may not actually converge to the global minimum, either.

Based on our results, we now believe that the most powerful of the Newton methods we studied in this analysis is the new method. At each step, we need to determine the first derivative and it is likely that we will have trouble gaining the global minimum. It's not much different than its affinity, in the worst situation. The new iterative method is performing very well in comparison to Newton method as noted in the tables.

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An open problem on Tarski associative NET-groupoids and GTA-NET-groupoids

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Abstract. Tarski associative groupoid (TA-groupoid) is a kind of non-associative groupoid satisfying Tarski associative law. Tarski associative NET-groupoid (TA-NET-groupoid) is a neutrosophic extended triplet set with Tarski associative law. In this paper, we prove that every TA-NET-groupoid is a semigroup, thus successfully solving an open problem. Moreover, we propose the new notions of generalized TA-NET-groupoid (GTA-NET-groupoid) and weak commutative GTA-NET-groupoid, and investigate their properties and structural characteristics. We prove that GTA-NET-groupoid is equivalent to quasi strong regular TA-groupoid, and give the necessary and sufficient conditions for TA-groupoid to be a weak commutative GTA-NET-groupoid.

Keywords: Tarski associative groupoid (TA-groupoid), semigroup, TA-NET-groupoid, generalized TA-NET-groupoid (GTA-NET-groupoid), strong regular TA-groupoid.

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1. Introduction

Neutrosophic set (NS) was first proposed by Smarandache in [1]. In the past 15 years, NS theory has been developing rapidly. Until now, NS theory has been successfully applied to many fields such as professional selection [2], data filling [3], analysis of age group and time [4], automated skin lesion segmentation [5], clustering methods [6], etc. Moreover, further new theoretical studies on NS in [7, 8, 9, 10] have been performed, and some important results have been gotten.

As early as 1929, Suschkewitsch [11] studied the following generalized associative law which was originally called “Postulate A”. In a finite group $(G, *)$, $\forall x, a, b \in G, \exists c \in G, (x * a) * b = x * c$, where the element c depends upon the element a and b only, and not upon x . When $c = b * a$, we can get the identity:

$$(x * a) * b = x * (b * a) \quad (\text{Tarski associative law}).$$

Apparently, Tarski associative law is a special case of this Postulate A.

Function equations satisfying Tarski associative law were first discussed by Hosszú [12] in 1954. Continuous and strictly monotonic solutions of function equations satisfying Tarski associative law were investigated in [13]. Schölzel and Tomaschek [14] characterized the power series solutions of function equations satisfying Tarski associative law in the complex domain. A class of rings symmetric to Tarski associative law is studied in [15]. Pushkashu [16] studied properties of left (right) division (cancellative) groupoids satisfying Tarski associative law.

The concept of Tarski associative groupoid (TA-groupoid) and Tarski associative neutrosophic extended triplet groupoid (TA-NET-groupoid) were first given by Xiaohong Zhang et al. [17] in 2020. A groupoid $(G, *)$ is called a TA-groupoid, if it holds the Tarski associative law, that is, $\forall x, y, z \in G, (x * y) * z = x * (z * y)$.

Yingcang Ma et al. [18] put forward the concept of generalized neutrosophic extended triplet (GNET) set, which is an extension of the NS. In this paper, we study the algebraic structure with the Tarski associative law on GNET set, thus extending TA-NET-groupoid and obtaining generalized Tarski associative neutrosophic extended triplet groupoid (GTA-NET-groupoid). That is, GTA-NET-groupoid is both TA-groupoid and GNET set. GTA-NET-groupoid relaxes the restrictions on the elements in TA-groupoid as compared to TA-NET-groupoid. This is also a concrete embodiment of the research method of regular semigroups to quasi-regular semigroups in TA-groupoid.

The rest of this paper is arranged as follows. In Section 2, some definitions and properties on TA-NET-groupoid are given. The relations between TA-NET-groupoid and regular TA-groupoid are discussed in Section 3. We propose the GTA-NET-groupoid and discuss its properties and structure in Section 4. Lastly, Section 5 presents the summary and the direction of future efforts.

2. Preliminaries

In this section, the related research and results of the TA-groupoid and TA-NET-groupoid are presented. Some related notions are introduced first.

A TA-groupoid $(G, *)$ is called locally associative if $\forall x \in G, (x * x) * x = x * (x * x)$. It is readily verified that every TA-groupoid is locally associative. In a TA-groupoid $(G, *)$, $\forall x \in G, k \in \mathbb{Z}^+$, x^k is defined as follows: $x^1 = x, x^2 = x * x, x^3 = x^2 * x, x^4 = x^3 * x, \dots, x^k = x^{k-1} * x$.

Definition 2.1 ([19]). Let G be a non-empty set with a binary operation $*$. Then, G is called a neutrosophic extended triplet set if $\forall x \in G$, there exist a neutral of “ x ” and an opposite of “ x ” (denoted by $neut(x)$ and $anti(x)$ respectively), such that $neut(x), anti(x) \in G$ and: $neut(x) * x = x * neut(x) = x, anti(x) * x = x * anti(x) = neut(x)$. The triplet $(x, neut(x), anti(x))$ is called a neutrosophic extended triplet (NET).

Definition 2.2 ([20]). A NET set $(G, *)$ is called a NET-Loop, if $\forall x, y \in G$, one has $x * y \in G$.

Definition 2.3 ([17]). A TA-groupoid $(G, *)$ is called a TA-NET-groupoid if it is a NET-Loop.

A TA-NET-groupoid $(G, *)$ is called a commutative TA-NET-groupoid if $\forall x, y \in G, x * y = y * x$. Example 2.1 illustrates that a TA-NET-groupoid is not always a commutative TA-NET-groupoid.

Example 2.1. Let $G = \{a, b, c, d, e, f\}$, the definition of operation $*$ on G is shown in Table 1. From Definition 2.3, G is a TA-NET-groupoid. However, it is not a commutative TA-NET-groupoid since $a * c \neq c * a$.

Table 1: A TA-NET-groupoid of Example 2.1.

$*$	a	b	c	d	e	f
a	a	a	a	a	a	a
b	b	b	b	b	b	b
c	c	c	c	c	c	c
d	a	a	a	d	e	e
e	a	a	a	e	e	e
f	c	c	c	f	f	f

Proposition 2.1 ([17]). Let $(G, *)$ be a TA-groupoid. Then, $\forall a, b, c, d \in G$, $(a * b) * (c * d) = (a * d) * (c * b)$.

Theorem 2.1 ([17]). Let $(G, *)$ be a TA-NET-groupoid. Then, $\forall x \in G$, for any $a, b \in \{anti(x)\}$,

- (1) $\text{neut}(x) * \text{neut}(x) = \text{neut}(x)$, and $\text{neut}(x)$ is unique.
- (2) $\text{neut}(x) * a = a * \text{neut}(x) = \text{neut}(x) * b$, that is, $\text{neut}(x) * \text{anti}(x)$ is unique.
- (3) $(\text{neut}(x) * a) * x = x * (\text{neut}(x) * a) = \text{neut}(x)$.

Definition 2.4. A TA-groupoid $(G, *)$ is called a TA-(l,l)-groupoid, if $\forall a \in G$, $\exists b, c \in G$, $b * a = a$, and $c * a = b$.

Definition 2.5. A TA-groupoid $(G, *)$ is called a TA-(r,r)-groupoid, if $\forall a \in G$, $\exists b, c \in G$, $a * b = a$, and $a * c = b$.

Definition 2.6. A TA-groupoid $(G, *)$ is called a TA-(l,r)-groupoid, if $\forall a \in G$, $\exists b, c \in G$, $b * a = a$, and $a * c = b$.

Definition 2.7. A TA-groupoid $(G, *)$ is called a TA-(r,l)-groupoid, if $\forall a \in G$, $\exists b, c \in G$, $a * b = a$, and $c * a = b$.

From Definition 2.5 and Definition 2.6, it is readily verified that a TA-(r,r)-groupoid is equal to a TA-(l,r)-groupoid.

3. Regular TA-groupoid and TA-NET-groupoid

In this section, the relations between regular TA-groupoid and TA-NET-groupoid are established. Moreover, we prove that every TA-NET-groupoid is a semi-group, thus successfully solving an open problem.

Theorem 3.1. Let $(G, *)$ be a finite TA-groupoid. Then, $\exists a \in G$, $a^2 = a$. That is, there exists an idempotent element in G .

Proof. Assume that $(G, *)$ is a finite TA-groupoid. Then, $\forall a \in G$, $n \in \mathbb{Z}^+$, $a^n \in G$. Since G is finite, there exist $i, j \in \mathbb{Z}^+$ such that $a^i = a^{i+j}$. According to the value of i and j , we have three cases to discuss.

Case 1: if $j = i$, then $a^i = a^{2i}$, that is, $a^i = a^i * a^i$, a^i is an idempotent element in G .

Case 2: if $j > i$, then from $a^i = a^{i+j}$ we have

$$a^j = a^i * a^{j-i} = a^{i+j} * a^{j-i} = a^{2j} = a^j * a^j.$$

This means that a^j is an idempotent element in G .

Case 3: if $j < i$, then from $a^i = a^{i+j}$ we have

$$a^i = a^{i+j} = a^i * a^j = a^{i+j} * a^j = a^{i+2j};$$

$$a^i = a^{i+2j} = a^i * a^{2j} = a^{i+j} * a^{2j} = a^{i+3j};$$

.....

$$a^i = a^{i+ij}.$$

Since $i, j \in \mathbb{Z}^+$, then $ij \geq i$. For $a^i = a^{i+ij}$, Case 3 becomes Case 1 when $ij = i$, and Case 3 becomes Case 2 when $ij > i$. Thus, we know that there exists an idempotent element in G . $\square$

Theorem 3.2. *Let $(G, *)$ be a TA-groupoid. Then, the following three statements are equivalent:*

- (1) G is a TA-(l,l)-groupoid;
- (2) G is a TA-(l,r)-groupoid;
- (3) G is a TA-NET-groupoid.

Proof. (1) $\Rightarrow$ (2). Suppose G is a TA-(l,l)-groupoid, from Definition 2.4, $\forall a \in G$, $\exists b, c \in G$, $b * a = a$, and $c * a = b$. We have $a * c = (b * a) * c = b * (c * a) = b * b = (c * a) * b = c * (b * a) = c * a = b$. By Definition 2.6, G is a TA-(l,r)-groupoid.

(2) $\Rightarrow$ (3). Assume that G is a TA-(l,r)-groupoid, from Definition 2.6, $\forall a \in G$, $\exists b, c \in G$, $b * a = a$, and $a * c = b$. We have $a * b = a * (a * c) = (a * c) * a = b * a = a$. Let $d = b^2$, $f = b * c$, there are $d * a = b^2 * a = (b * b) * a = b * (a * b) = b * a = a$, $a * d = a * (b * b) = (a * b) * b = a * b = a$, $a * f = a * (b * c) = (a * c) * b = b^2 = d$, $f * a = (b * c) * a = b * (a * c) = b^2 = d$. By Definition 2.3, G is a TA-NET-groupoid.

(3) $\Rightarrow$ (1). It is obvious that a TA-NET-groupoid is a TA-(l,l)-groupoid. $\square$

Definition 3.1. *A TA-groupoid $(G, *)$ is called a regular TA-groupoid if $\forall a \in G$, $\exists x \in G$, $a = a * (x * a)$.*

From Definition 3.1 and Definition 2.7, it is readily verified that a TA-(r,l)-groupoid is equal to a regular TA-groupoid. Example 3.1 illustrates that a TA-groupoid is not always a regular TA-groupoid.

Example 3.1. Let $G = \{1, 2, 3, 4, 5\}$, the definition of operation $*$ on G is shown in Table 2. G is a TA-groupoid. However, there is no element $x \in G$ such that $4 = 4 * (x * 4)$. From Definition 3.1, G is not a regular TA-groupoid.

Table 2: A TA-groupoid of Example 3.1.

$*$	1	2	3	4	5
1	1	3	1	1	3
2	2	2	2	2	2
3	1	3	1	1	3
4	1	3	1	3	4
5	2	2	2	2	5

Definition 3.2. *A TA-groupoid $(G, *)$ is called a strong regular TA-groupoid if for any $x \in G$, there is a unary operation $x \rightarrow x^{-1}$ in G such that*

$$x = (x^{-1})^{-1}, \quad x * x^{-1} = x^{-1} * x, \quad x = x * (x^{-1} * x).$$

Example 3.2 illustrates that a regular TA-groupoid is not always a strong regular TA-groupoid.

Example 3.2. Let $G = \{1, 2, 3, 4, 5\}$, the definition of operation $*$ on G is shown in Table 3. From Definition 3.1, G is a regular TA-groupoid. However, for element 3, $3 = 3 * (5 * 3)$, $5 * 3 \neq 3 * 5$. Thus, G is not a strong regular TA-groupoid.

Table 3: A regular TA-groupoid of Example 3.2.

$*$	1	2	3	4	5
1	1	2	2	1	3
2	2	1	1	3	1
3	2	1	1	3	1
4	4	5	5	4	5
5	5	4	4	5	4

Proposition 3.1. Let $(G, *)$ be a TA-NET-groupoid. Then, $\forall x \in G$, for any $a \in \{anti(x)\}$,

$$neut(neut(x) * a) * anti(neut(x) * a) = x.$$

Proof. For any $a \in \{anti(x)\}$, we can get

$$\begin{aligned}
 & (neut(x) * a) * neut(x) \\
 &= neut(x) * (neut(x) * a) \quad (\text{applying the Tarski associative law}) \\
 &= neut(x) * (a * neut(x)) \quad (\text{by Theorem 2.1 (2)}) \\
 &= (neut(x) * neut(x)) * a \quad (\text{applying the Tarski associative law}) \\
 &= neut(x) * a. \quad (\text{by Theorem 2.1 (1)}).
 \end{aligned}$$

From Theorem 2.1 (3) $(neut(x) * a) * x = x * (neut(x) * a) = neut(x)$, we can get

$$neut(neut(x) * a) = neut(x), x \in anti\{neut(x) * a\}.$$

From Theorem 2.1 (2) $neut(x) * anti(x)$ is unique, we have

$$neut(neut(x) * a) * anti(neut(x) * a) = neut(x) * x = x. \quad \square$$

Example 3.3. Let $G = \{1, 2, 3, 4, 5\}$, the definition of operation $*$ on G is shown in Table 4. By Definition 2.3, G is a TA-NET-groupoid. For element 1, $neut(1) = 1$, $\{anti(1)\} = \{1, 4, 5\}$, $1 * 1 = 1 * 4 = 1 * 5 = 1$,

$$\begin{aligned}
 neut(neut(1) * 1) * anti(neut(1) * 1) &= neut(neut(1) * 4) * anti(neut(1) * 4) \\
 &= neut(neut(1) * 5) * anti(neut(1) * 5) = 1.
 \end{aligned}$$

The other cases can be checked and verified, that is $\forall x \in G$, for any $a \in \{anti(x)\}$, $neut(neut(x) * a) * anti(neut(x) * a) = x$.

Table 4: A TA-NET-groupoid of Example 3.3.

*	1	2	3	4	5
1	1	2	3	1	1
2	2	3	1	2	2
3	3	1	2	3	3
4	1	2	3	4	4
5	1	2	3	5	5

Theorem 3.3. *Let $(G, *)$ be a TA-groupoid. Then, G is a TA-NET-groupoid iff it is a strong regular TA-groupoid.*

Proof. Suppose G is a TA-NET-groupoid, from Definition 2.3, $\forall x \in G$, there is a NET $(x, neut(x), anti(x))$ in G .

Set $x^{-1} = neut(x) * anti(x)$, by Theorem 2.1 (2), x^{-1} is unique.

By Proposition 3.1, there are $(x^{-1})^{-1} = neut(neut(x) * anti(x)) * anti(neut(x) * anti(x)) = x$,

$$\begin{aligned}
 x^{-1} * x &= (neut(x) * anti(x)) * x \\
 &= neut(x) * (x * anti(x)) \quad (\text{applying the Tarski associative law}) \\
 &= neut(x) * neut(x) \\
 &= (x * anti(x)) * neut(x) \\
 &= x * (neut(x) * anti(x)) \quad (\text{applying the Tarski associative law}) \\
 &= x * x^{-1},
 \end{aligned}$$

$$x * (x^{-1} * x) = x * (neut(x) * neut(x)) = x * neut(x) = x.$$

From Definition 3.2, G is a strong regular TA-groupoid.

In contrast, suppose G is a strong regular TA-groupoid and $x^{-1} \in G$, such that $x^{-1} * x = x * x^{-1}$ and $x = x * (x^{-1} * x)$. Set $neut(x) = x * x^{-1}$, then $x * x^{-1} = x^{-1} * x = neut(x)$,

$$\begin{aligned}
 neut(x) * x &= (x * x^{-1}) * x = x * (x * x^{-1}) = x * neut(x) \\
 &= x * (x^{-1} * x) = x.
 \end{aligned}$$

By Definition 2.3, G is a TA-NET-groupoid and $x^{-1} \in \{anti(x)\}$. □

Example 3.4. Let $G = \{1, 2, 3, 4, 5\}$, an operation $*$ on G is defined as in Table 5. By Definition 2.3, G is a TA-NET-groupoid. For element 2 and element 3, $neut(2) = 4, anti(2) = 3, neut(3) = 4, anti(3) = 2$, we can get

$$\begin{aligned}
 2^{-1} &= neut(2) * anti(2) = 3, \quad 3^{-1} = neut(3) * anti(3) = 2, \\
 (2^{-1})^{-1} &= 3^{-1} = 2, \quad 2 * (2^{-1} * 2) = 2, \quad 2 * 2^{-1} = 4 = 2^{-1} * 2, \\
 (3^{-1})^{-1} &= 2^{-1} = 3, \quad 3 * (3^{-1} * 3) = 3, \quad 3 * 3^{-1} = 4 = 3^{-1} * 3.
 \end{aligned}$$

The other cases can be checked and verified. Thus, G is a strong regular TA-groupoid by Definition 3.2.

Table 5: A strong regular TA-groupoid of Example 3.4.

$*$	1	2	3	4	5
1	1	1	1	1	1
2	1	3	4	2	1
3	1	4	2	3	1
4	1	2	3	4	1
5	5	5	5	5	5

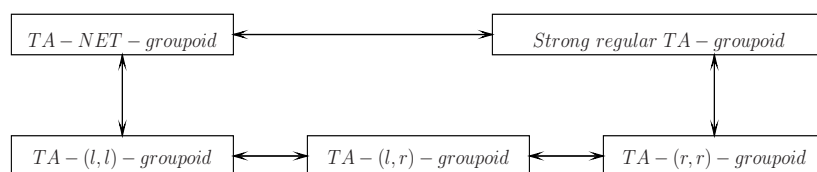


Figure 1: The relationship between TA-NET-groupoid and strong regular TA-groupoid.

Figure 1 shows that the relationship between TA-NET-groupoid and strong regular TA-groupoid. In Figure 1, a TA-NET-groupoid, a strong regular TA-groupoid, a TA-(l,l)-groupoid, a TA-(l,r)-groupoid, and a TA-(r,r)-groupoid are equivalent to each other.

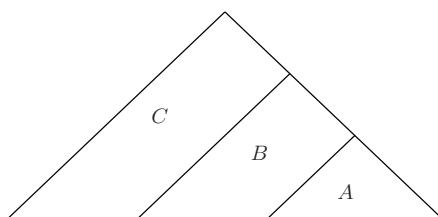


Figure 2: The relationship between regular TA-groupoid and TA-NET-groupoid.

Figure 2 can be used to express the relationship between regular TA-groupoid and TA-NET-groupoid. Here, A stands for TA-NET-groupoid; B stands for regular TA-groupoid shown in Example 3.2 rather than TA-NET-groupoid; and C stands for TA-groupoid shown in Example 3.1 rather than regular TA-groupoid. $A+B$ stands for regular TA-groupoid, and $A+B+C$ stands for TA-groupoid.

Xiaohong Zhang et al. [17] have proved the decomposition theorem of TA-NET-groupoid (see, Theorem 3.4) and proposed an open problem, “are there some TA-NET-groupoids which are not semigroups?”. We try to look for such an example by computer programming in finite TA-NET-groupoids, which is not a semigroup. But it is not found. So we consider another possibility, a TA-NET-groupoid is a semigroup. Fortunately, this conjecture can prove to be correct.

Theorem 3.4 ([17]). *Let $(G, *)$ be a TA-NET-groupoid. The set of all different neutral element in G is denoted as $E(G)$, $\forall e \in E(G), G(e) = \{a \in G \mid \text{neut}(a) = e\}$. Then:*

- (1) $G(e)$ is a sub Abelian group of G ;
- (2) for any $e_1, e_2 \in E(G)$, if $e_1 \neq e_2$, then $G(e_1) \cap G(e_2) = \emptyset$;
- (3) $G = \bigcup_{e \in E(G)} G_e$.

Theorem 3.5. *A TA-NET-groupoid is a semigroup.*

Proof. Let $(G, *)$ be a TA-NET-groupoid. Denote $E(G)$ is the set of all different neutral elements in G , $\forall e \in E(G), G(e) = \{a \in G \mid \text{neut}(a) = e\}$. Suppose $E(G) = \{e_1, e_2, e_3, \dots\}$, from Theorem 3.4 (1), $G(e_1), G(e_2)$ and $G(e_3)$ are all sub Abelian groups of G . Let $a \in G(e_1), b \in G(e_2)$ and $c \in G(e_3)$ be arbitrary elements. Set $d = (a * b) * c$ and $f = a * (b * c)$, we can get

$$\begin{aligned}
 d * d &= ((a * b) * c) * ((a * b) * c) \\
 &= (a * b) * ((a * b) * c) * c \quad (\text{applying the Tarski associative law}) \\
 &= (a * b) * ((a * b) * (c * c)) \quad (\text{applying the Tarski associative law}) \\
 &= (a * b) * ((a * b) * c^2) \\
 &= a * (((a * b) * c^2) * b) \quad (\text{applying the Tarski associative law}) \\
 &= a * ((a * b) * (b * c^2)) \quad (\text{applying the Tarski associative law}) \\
 &= a * ((a * c^2) * (b * b)) \quad (\text{by Proposition 2.1}) \\
 &= a * ((a * c^2) * b^2) \\
 &= (a * b^2) * (a * c^2),
 \end{aligned}$$

$$\begin{aligned}
 f * f &= (a * (b * c)) * (a * (b * c)) \\
 &= a * ((a * (b * c)) * (b * c)) \quad (\text{applying the Tarski associative law}) \\
 &= a * (a * ((b * c) * (b * c))) \quad (\text{applying the Tarski associative law}) \\
 &= a * (a * (b * ((b * c) * c))) \quad (\text{applying the Tarski associative law}) \\
 &= a * (a * (b * (b * (c * c)))) \quad (\text{applying the Tarski associative law}) \\
 &= a * (a * (b * (b * c^2)))
 \end{aligned}$$

$$\begin{aligned}
&= a * (a * ((b * c^2) * b)) \quad (\text{applying the Tarski associative law}) \\
&= a * ((a * b) * (b * c^2)) \quad (\text{applying the Tarski associative law}) \\
&= a * ((a * c^2) * (b * b)) \\
&= a * ((a * c^2) * b^2) \\
&= (a * b^2) * (a * c^2),
\end{aligned}$$

$$\begin{aligned}
d * f &= ((a * b) * c) * (a * (b * c)) \\
&= (a * b) * ((a * (b * c)) * c) \quad (\text{applying the Tarski associative law}) \\
&= (a * b) * (a * (c * (b * c))) \quad (\text{applying the Tarski associative law}) \\
&= (a * b) * (a * ((c * c) * b)) \quad (\text{applying the Tarski associative law}) \\
&= (a * b) * (a * (c^2 * b)) \\
&= a * ((a * (c^2 * b)) * b) \quad (\text{applying the Tarski associative law}) \\
&= a * (a * (b * (c^2 * b))) \quad (\text{applying the Tarski associative law}) \\
&= a * (a * ((b * b) * c^2)) \quad (\text{applying the Tarski associative law}) \\
&= a * (a * (b^2 * c^2)) \\
&= a * ((a * c^2) * b^2) \quad (\text{applying the Tarski associative law}) \\
&= (a * b^2) * (a * c^2).
\end{aligned}$$

Since $d * d = f * f$, from Theorem 3.4, we have d and f belong to the same subgroup $G(e)$. Since $d * d = d * f$ and the subgroup $G(e)$ satisfies the cancellation law, we have $d = f$. Thus, $(a * b) * c = a * (b * c)$, that is, G is a semigroup. $\square$

Example 3.5. Let $G = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, the definition of operation $*$ on G is shown in Table 6. From Definition 2.3, G is a TA-NET-groupoid. $E(G) = \{1, 4, 7\}$, $G(e_1) = \{1, 2, 3\}$, $G(e_2) = \{4, 5, 6\}$, $G(e_3) = \{7, 8, 9\}$. $G(e_1)$, $G(e_2)$ and $G(e_3)$ are all sub Abelian groups of G . It is easy to verify that $G(e_1) \cap G(e_2) = \emptyset$, $G(e_2) \cap G(e_3) = \emptyset$, $G(e_1) \cap G(e_3) = \emptyset$, and $G = G(e_1) \cup G(e_2) \cup G(e_3)$. For any $a \in G(e_1)$, $b \in G(e_2)$ and $c \in G(e_3)$, without losing generality, let $a = 2$, $b = 5$ and $c = 7$. So, we can get

$$d = (2 * 5) * 7 = 3, \quad f = 2 * (5 * 7) = 3, \quad d * d = f * f = d * f = 2, \quad d, f \in G(e_1).$$

The other cases can be verified, thus G is a semigroup.

Example 3.6 illustrates that a semigroup is not always a TA-groupoid.

Example 3.6. Let $G = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, the definition of operation $*$ on G is shown in Table 7. G is a semigroup. However, it is not a TA-groupoid since $(x_3 * x_2) * x_3 \neq x_3 * (x_3 * x_2)$.

Table 6: A TA-NET-groupoid of Example 3.5.

$*$	1	2	3	4	5	6	7	8	9
1	1	2	3	1	2	3	1	2	3
2	2	3	1	2	3	1	2	3	1
3	3	1	2	3	1	2	3	1	2
4	4	5	6	4	5	6	4	5	6
5	5	6	4	5	6	4	5	6	4
6	6	4	5	6	4	5	6	4	5
7	7	8	9	7	8	9	7	8	9
8	8	9	7	8	9	7	8	9	7
9	9	7	8	9	7	8	9	7	8

Table 7: A semigroup of Example 3.6.

$*$	x_1	x_2	x_3	x_4	x_5	x_6
x_1	x_1	x_1	x_1	x_1	x_1	x_1
x_2	x_1	x_1	x_1	x_1	x_2	x_1
x_3	x_1	x_2	x_3	x_3	x_2	x_3
x_4	x_1	x_2	x_4	x_4	x_2	x_4
x_5	x_1	x_1	x_1	x_1	x_5	x_1
x_6	x_1	x_2	x_3	x_3	x_2	x_6

Example 3.7 illustrates that a groupoid which is both a semigroup and a TA-groupoid is not always a TA-NET-groupoid.

Example 3.7. Let $G = \{a, b, c, d, e, f\}$, the definition of operation $*$ on G is shown in Table 8. G is both a semigroup and a TA-groupoid. However, there is no element $x \in G$ such that $x * b = b$. From Definition 2.3 G is not a TA-NET-groupoid.

Table 8: A semigroup of Example 3.7.

$*$	a	b	c	d	e	f
a	a	a	a	a	a	a
b	a	a	b	b	b	a
c	a	a	c	e	e	a
d	a	a	e	d	e	f
e	a	a	e	e	e	a
f	a	a	a	f	a	f

Figure 3 can be used to express the relationship between TA-groupoid and semigroup. Here, A stands for TA-NET-groupoid; B stands for both TA-

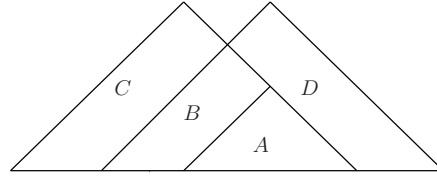


Figure 3: The relationship between TA-groupoid and semigroup.

groupoid and semigroup shown in Example 3.7 rather than TA-NET-groupoid; C stands for semigroup shown in Example 3.6 rather than TA-groupoid; and D stands for TA-groupoid rather than semigroup. $A+B+C$ stands for semigroup, and $A+B+D$ stands for TA-groupoid.

4. Generalized Tarski associative neutrosophic extended triplet groupoid

In this section, the new notion of generalized Tarski associative neutrosophic extended triplet groupoid (GTA-NET-groupoid) is proposed, and some properties and structural characteristics are discussed.

Definition 4.1 ([18, 21]). *Let G be a non-empty set with a binary operation $*$. Then, G is called a generalized neutrosophic extended triplet set if for any $a \in G$, there is at least $n \in \mathbb{Z}^+$, such that a^n has neutral element and opposite element (denoted by $\text{neut}(a^n)$ and $\text{anti}(a^n)$ respectively). The triplet $(a, \text{neut}(a^n), \text{anti}(a^n))$ is called a generalized neutrosophic extended triplet (GNET) with degree n .*

Definition 4.2. *Let $(G, *)$ be a GNET set. Then, G is called a generalized Tarski associative neutrosophic extended triplet groupoid (GTA-NET-groupoid), if $\forall a, b, c \in G$, $a * (c * b) = (a * b) * c$.*

A GTA-NET-groupoid $(G, *)$ is called a commutative GTA-NET-groupoid if $\forall x, y \in G$, $y * x = x * y$.

Example 4.1. Let $G = \{1, 2, 3, 4, 5\}$, the definition of operation $*$ on G is shown in Table 9. For element 4, there is no element $x \in G$ such that $x * 4 = 4$. Thus, G is not a TA-NET-groupoid. Moreover, there is a GNET $(4, \text{neut}(4^2), \text{anti}(4^2))$ with degree 2 in G , thus G is a GTA-NET-groupoid. Since $5 * 4 \neq 4 * 5$ it is not a commutative GTA-NET-groupoid.

Proposition 4.1. *Let $(G, *)$ be a GTA-NET-groupoid, $x \in G, n \in \mathbb{Z}^+$, and $(x, \text{neut}(x^n), \text{anti}(x^n))$ is a GNET with degree n . Then,*

- (1) $\text{neut}(x^n)$ is unique.
- (2) $\text{neut}(x^n) * \text{neut}(x^n) = \text{neut}(x^n)$, $\text{neut}(\text{neut}(x^n)) = \text{neut}(x^n)$.

Table 9: A GTA-NET-groupoid of Example 4.1.

*	1	2	3	4	5
1	1	1	3	1	1
2	1	2	3	2	2
3	3	3	1	3	3
4	1	2	3	2	4
5	1	2	3	2	5

- (3) $neut(x^n) * a = a * neut(x^n) = neut(x^n) * b$, for any $a, b \in \{anti(x^n)\}$.
- (4) $x^n * (neut(x^n) * a) = (neut(x^n) * a) * x^n = neut(x^n)$, for any $a \in \{anti(x^n)\}$.
- (5) $(neut(x^n) * a) * neut(x^n) = neut(x^n) * (neut(x^n) * a) = neut(x^n) * a$, for any $a \in \{anti(x^n)\}$.
- (6) $neut(neut(x^n) * a) * anti(neut(x^n) * a) = x^n$, for any $a \in \{anti(x^n)\}$.

Proof.

- (1) Assume $c, d \in \{neut(x^n)\}$, so $x^n * c = c * x^n = x^n, x^n * d = d * x^n = x^n$, and there exist $a, b \in G$ such that $x^n * a = a * x^n = c, x^n * b = b * x^n = d$. We have $c * d = (a * x^n) * d = a * (d * x^n) = a * x^n = c, c * d = c * (b * x^n) = (c * x^n) * b = x^n * b = d$. Thus, $c = d$, $neut(x^n)$ is unique.

(2)

$$\begin{aligned}
neut(x^n) * neut(x^n) &= (anti(x^n) * x^n) * neut(x^n) \\
&= anti(x^n) * (neut(x^n) * x^n) \\
&\quad (\text{applying the Tarski associative law}) \\
&= anti(x^n) * x^n \\
&= neut(x^n).
\end{aligned}$$

Thus, $neut(neut(x^n)) = neut(x^n)$.

- (3) For any $a, b \in \{anti(x^n)\}$, $neut(x^n) * a = (a * x^n) * a = a * (a * x^n) = a * neut(x^n) = a * (b * x^n) = (a * x^n) * b = neut(x^n) * b$.
- (4) For any $a \in \{anti(x^n)\}$, we have $x^n * (neut(x^n) * a) = (x^n * a) * neut(x^n) = neut(x^n) * neut(x^n) = neut(x^n)$, $(neut(x^n) * a) * x^n = neut(x^n) * (x^n * a) = neut(x^n) * neut(x^n) = neut(x^n)$. Thus, $x^n * (neut(x^n) * a) = (neut(x^n) * a) * x^n = neut(x^n)$.

(5) For any $a \in \{\text{anti}(x^n)\}$, we have

$$\begin{aligned}
 & (\text{neut}(x^n) * a) * \text{neut}(x^n) \\
 &= \text{neut}(x^n) * (\text{neut}(x^n) * a) \quad (\text{applying the Tarski associative law}) \\
 &= \text{neut}(x^n) * (a * \text{neut}(x^n)) \quad (\text{applying (3)}) \\
 &= (\text{neut}(x^n) * \text{neut}(x^n)) * a \quad (\text{applying the Tarski associative law}) \\
 &= \text{neut}(x^n) * a. \quad (\text{applying (2)})
 \end{aligned}$$

Thus, $(\text{neut}(x^n) * a) * \text{neut}(x^n) = \text{neut}(x^n) * (\text{neut}(x^n) * a) = \text{neut}(x^n) * a$.

(6) For any $a \in \{\text{anti}(x^n)\}$, from (4) and (5), we have $\text{neut}(\text{neut}(x^n) * a) = \text{neut}(x^n)$, $x^n \in \text{anti}\{\text{neut}(x^n) * a\}$. From (3), $\text{neut}(x^n) * \text{anti}(x^n)$ is unique, we get $\text{neut}(\text{neut}(x^n) * a) * \text{anti}(\text{neut}(x^n) * a) = \text{neut}(\text{neut}(x^n) * a) * x^n = \text{neut}(x^n) * x^n = x^n$. $\square$

Example 4.2. Let $G = \{1, 2, 3, 4, 5\}$, the definition of operation $*$ on G is shown in Table 10. For element 1, $\text{neut}(1) = 1, \{\text{anti}(1)\} = \{1, 3\}$; for element 2, $\text{neut}(2) = 2, \{\text{anti}(2)\} = \{2, 4\}$; and for element 5, $\text{neut}(5) = \text{anti}(5) = 5$. There is no element $x \in G$ such that $x * 3 = 3, x * 4 = 4$. However, $3^2 = 1, 4^2 = 2$. Thus, G is a GTA-NET-groupoid. We can get the results that correspond to Proposition 4.1:

Table 10: A GTA-NET-groupoid of Example 4.2.

$*$	1	2	3	4	5
1	1	3	1	3	3
2	4	2	4	2	4
3	1	3	1	3	3
4	4	2	4	2	4
5	5	5	5	5	5

(1) It is easily to verify that $\text{neut}(x^n)$ is unique.

(2) Being $\text{neut}(1^1) * \text{neut}(1^1) = \text{neut}(1^1), \text{neut}(2^1) * \text{neut}(2^1) = \text{neut}(2^1)$,
 $\text{neut}(3^2) * \text{neut}(3^2) = \text{neut}(3^2), \text{neut}(4^2) * \text{neut}(4^2) = \text{neut}(4^2), \text{neut}(5^1) * \text{neut}(5^1) = \text{neut}(5^1)$,
that is, $\text{neut}(x^n) * \text{neut}(x^n) = \text{neut}(x^n)$ and $\text{neut}(\text{neut}(x^n)) = \text{neut}(x^n)$.

(3) For element 3, $\text{neut}(3^2) = 1, \{\text{anti}(3^2)\} = \{1, 3\}$, we have

$$\text{neut}(3^2) * 1 = 1 * \text{neut}(3^2) = \text{neut}(3^2) * 3 = 3 * \text{neut}(3^2) = 1.$$

The other cases can be checked and verified. That is $\text{neut}(x^n) * a = a * \text{neut}(x^n) = \text{neut}(x^n) * b$, for any $a, b \in \{\text{anti}(x^n)\}$.

(4) For element 4, $neut(4^2) = 2, \{anti(4^2)\} = \{2, 4\}$, we have

$$4^2 * (neut(4^2) * 2) = (neut(4^2) * 2) * 4^2 = 4^2 * (neut(4^2) * 4) = (neut(4^2) * 4) * 4^2 = neut(4^2).$$

The other cases can be checked and verified. That is, $x^n * (neut(x^n) * a) = (neut(x^n) * a) * x^n = neut(x^n)$, for any $a \in \{anti(x^n)\}$.

(5) For element 3, $neut(3^2) = 1, \{anti(3^2)\} = \{1, 3\}$, we have

$$(neut(3^2) * 1) * neut(3^2) = neut(3^2) * (neut(3^2) * 1) = neut(3^2) * 1,$$

$$(neut(3^2) * 3) * neut(3^2) = neut(3^2) * (neut(3^2) * 3) = neut(3^2) * 3.$$

The other cases can be checked and verified. That is, $(neut(x^n) * a) * neut(x^n) = neut(x^n) * (neut(x^n) * a) = neut(x^n) * a$, for any $a \in \{anti(x^n)\}$.

(6) For element 3, $neut(3^2) = 1, \{anti(3^2)\} = \{1, 3\}$,

$$neut(neut(3^2) * 1) * anti(neut(3^2) * 1) = 3^2,$$

$$neut(neut(3^2) * 3) * anti(neut(3^2) * 3) = 3^2.$$

For element 4, $neut(4^2) = 2, \{anti(4^2)\} = \{2, 4\}$,

$$neut(neut(4^2) * 2) * anti(neut(4^2) * 2) = 4^2,$$

$$neut(neut(4^2) * 4) * anti(neut(4^2) * 4) = 4^2.$$

The other cases can be checked and verified. That is, $neut(neut(x^n) * a) * anti(neut(x^n) * a) = x^n$, for any $a \in \{anti(x^n)\}$.

Definition 4.3. A TA-groupoid $(G, *)$ is called a quasi strong regular TA-groupoid, if $\forall x \in G$, there is a positive integer n , $x^n \in G$, and a unary operation $x^n \rightarrow (x^n)^{-1}$ on G such that $((x^n)^{-1})^{-1} = x^n$, $(x^n)^{-1} * x^n = x^n * (x^n)^{-1}$, $x^n = x^n * ((x^n)^{-1} * x^n)$.

Theorem 4.1. Let $(G, *)$ be a groupoid. Then, G is a GTA-NET-groupoid iff it is a quasi strong regular TA-groupoid.

Proof. Suppose G is a GTA-NET-groupoid, from Definition 4.2, for each $x \in G$, there exists a GNET $(x, neut(x^n), anti(x^n))$ with degree n . Set $(x^n)^{-1} = neut(x^n) * anti(x^n)$. From Proposition 4.1 (6), we get $((x^n)^{-1})^{-1} = neut(neut(x^n) * anti(x^n)) * anti(neut(x^n) * anti(x^n)) = x^n$. Being

$$\begin{aligned} (x^n)^{-1} * x^n &= (neut(x^n) * anti(x^n)) * x^n \\ &= neut(x^n) * (x^n * anti(x^n)) \quad (\text{applying the Tarski associative law}) \\ &= neut(x^n) * neut(x^n) \\ &= (x^n * anti(x^n)) * neut(x^n) \\ &= x^n * (neut(x^n) * anti(x^n)) \quad (\text{applying the Tarski associative law}) \\ &= x^n * (x^n)^{-1}, \end{aligned}$$

$x^n * ((x^n)^{-1} * x^n) = x^n * (neut(x^n) * neut(x^n)) = x^n * neut(x^n) = x^n$, from Definition 4.3, G is a quasi strong regular TA-groupoid.

In contrast, if G is a quasi strong regular TA-groupoid and $(x^n)^{-1} \in G$, such that $x^n * (x^n)^{-1} = (x^n)^{-1} * x^n$ and $x^n * ((x^n)^{-1} * x^n) = x^n$. Set $neut(x^n) = x^n * (x^n)^{-1}$, then $x^n * (x^n)^{-1} = (x^n)^{-1} * x^n = neut(x^n)$. Being $neut(x^n) * x^n = (x^n * (x^n)^{-1}) * x^n = x^n * (x^n * (x^n)^{-1}) = x^n * neut(x^n) = x^n * ((x^n)^{-1} * x^n) = x^n$, from Definition 4.2, G is a GTA-NET-groupoid and $(x^n)^{-1} \in \{anti(x^n)\}$. $\square$

Example 4.3. Let $G = \{1, 2, 3, 4, 5\}$, the definition of operation $*$ on G is shown in Table 11. G is a GTA-NET-groupoid. For element 3, $neut(3^2) = 1$, $\{anti(3^2)\} = \{1, 3\}$, $((3^2)^{-1})^{-1} = 3^2$, $3^2 * ((3^2)^{-1} * 3^2) = 3^2$, $3^2 * (3^2)^{-1} = (3^2)^{-1} * 3^2$. For element 4, $neut(4^2) = 2$, $\{anti(4^2)\} = \{2, 4\}$, $((4^2)^{-1})^{-1} = 4^2$, $4^2 * ((4^2)^{-1} * 4^2) = 4^2$, $4^2 * (4^2)^{-1} = (4^2)^{-1} * 4^2$. The other cases can be checked and verified. Thus G is a quasi strong regular TA-groupoid by Definition 4.3.

Table 11: A GTA-NET-groupoid of Example 4.3.

$*$	1	2	3	4	5
1	1	3	1	3	1
2	4	2	4	2	4
3	1	3	1	3	1
4	4	2	4	2	4
5	5	5	5	5	5

Definition 4.4. Let $(G, *)$ be a GTA-NET-groupoid. G is called a weak commutative GTA-NET-groupoid (WC-GTA-NET-groupoid) if $\forall x, y \in G$, there exist a $GNET(x, neut(x^n), anti(x^n))$ with degree n and a $GNET(y, neut(y^m), anti(y^m))$ with degree m ($n, m \in \mathbb{Z}^+$), $x^n * neut(y^m) = neut(y^m) * x^n$.

Example 4.4. Let $G = \{1, 2, 3, 4, 5\}$, the definition of operation $*$ on G is shown in Table 12. From Definition 4.2 G is a GTA-NET-groupoid. It is not a commutative GTA-NET-groupoid since $5 * 4 \neq 4 * 5$. We can show that it is a WC-GTA-NET-groupoid. For element 1 and element 2, $neut(1) = neut(2) = 1$, we have $1 * neut(2) = neut(2) * 1$, $2 * neut(1) = neut(1) * 2$. For element 3 and element 5, $neut(3^2) = 1$, $neut(5) = 5$, we have $3^2 * neut(5) = neut(5) * 3^2$, $5 * neut(3^2) = neut(3^2) * 5$. The other cases can be checked and verified. Thus, G is a WC-GTA-NET-groupoid.

Example 4.5 illustrates that a commutative GTA-NET-groupoid may not be a commutative TA-NET-groupoid.

Example 4.5. Let $G = \{a, b, c, d, e, f\}$, the definition of operation $*$ on G is shown in Table 13. G is a commutative GTA-NET-groupoid. Since there is no element x in G such that $(x * b) * b = b$, it is not a commutative TA-NET-groupoid.

Table 12: A WC-GTA-NET-groupoid of Example 4.4.

$*$	1	2	3	4	5
1	1	2	2	2	1
2	2	1	1	1	2
3	2	1	1	1	3
4	2	1	1	1	4
5	1	2	3	3	5

Table 13: A commutative GTA-NET-groupoid of Example 4.5.

$*$	a	b	c	d	e	f
a	a	a	a	a	a	a
b	a	a	b	a	b	b
c	a	b	c	a	c	c
d	a	a	a	d	d	a
e	a	b	c	d	e	c
f	a	b	c	a	c	f

Example 4.6 illustrates that a GTA-NET-groupoid may not be either a WC-GTA-NET-groupoid or a TA-NET-groupoid.

Example 4.6. Let $G = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, the definition of operation $*$ on G is shown in Table 14. From Definition 4.2, G is a GTA-NET-groupoid. There is no element a in G such that $a * x_2 = x_2$, and $x_1 * neut(x_3) \neq neut(x_3) * x_1$. Thus, it is neither a TA-NET-groupoid nor a WC-GTA-NET-groupoid.

Table 14: A GTA-NET-groupoid of Example 4.6.

$*$	x_1	x_2	x_3	x_4	x_5	x_6
x_1	x_1	x_1	x_2	x_2	x_1	x_1
x_2	x_1	x_1	x_2	x_2	x_1	x_1
x_3	x_3	x_3	x_3	x_3	x_3	x_3
x_4	x_3	x_3	x_3	x_4	x_3	x_3
x_5	x_1	x_1	x_2	x_2	x_5	x_6
x_6	x_1	x_1	x_2	x_2	x_6	x_6

Figure 4 shows the relationships among WC-GTA-NET-groupoid and TA-NET-groupoid. Here, A stands for commutative TA-NET-groupoid; B stands for commutative GTA-NET-groupoid shown in Example 4.5 rather than commutative TA-NET-groupoid; C stands for WC-GTA-NET-groupoid shown in Example 4.4 rather than commutative GTA-NET-groupoid; D stands for non-

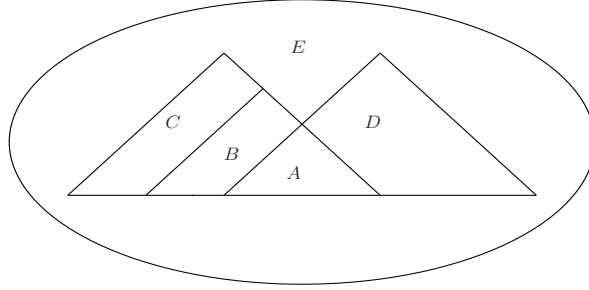


Figure 4: The relationships among WC-GTA-NET-groupoid and TA-NET-groupoid.

commutative TA-NET-groupoid shown in Example 2.1; and E stands for GTA-NET-groupoid shown in Example 4.6, it is, however, not either TA-NET-groupoid or WC-GTA-NET-groupoid. A+B stands for commutative GTA-NET-groupoid, A+B+C stands for WC-GTA-NET-groupoid, A+D stands for TA-NET-groupoid, and A+B+C+D+E stands for GTA-NET-groupoid.

Proposition 4.2. *Let $(G, *)$ be a GTA-NET-groupoid. Then, G is a WC-GTA-NET-groupoid iff $\forall x, y \in G$, there exist a GNET $(x, \text{neut}(x^n), \text{anti}(x^n))$ with degree n and a GNET $(y, \text{neut}(y^m), \text{anti}(y^m))$ with degree m ($n, m \in \mathbb{Z}^+$), $x^n * y^m = y^m * x^n$.*

Proof. If $(G, *)$ is a GTA-NET-groupoid, then $\forall x, y \in G$, there exist a GNET $(x, \text{neut}(x^n), \text{anti}(x^n))$ with degree n and a GNET $(y, \text{neut}(y^m), \text{anti}(y^m))$ with degree m ($n, m \in \mathbb{Z}^+$). Suppose G is a WC-GTA-NET-groupoid, $x^n * \text{neut}(y^m) = \text{neut}(y^m) * x^n$, we have

$$\begin{aligned}
 x^n * y^m &= (\text{neut}(x^n) * x^n) * (\text{neut}(y^m) * y^m) \\
 &= (\text{neut}(x^n) * y^m) * (\text{neut}(y^m) * x^n) \quad (\text{by Proposition 2.1}) \\
 &= (y^m * \text{neut}(x^n)) * (x^n * \text{neut}(y^m)) \\
 &= (y^m * \text{neut}(y^m)) * (x^n * \text{neut}(x^n)) \quad (\text{by Proposition 2.1}) \\
 &= y^m * x^n.
 \end{aligned}$$

In contrast, $(x, \text{neut}(x^n), \text{anti}(x^n))$ and $(y, \text{neut}(y^m), \text{anti}(y^m))$ are two GNETs in G . From Proposition 4.1 (2), we can get $(\text{neut}(y^m), \text{neut}(y^m), \text{neut}(y^m))$ is a GNET with degree 1. Suppose that G satisfies the condition $x^n * y^m = y^m * x^n$, we get $x^n * \text{neut}(y^m) = \text{neut}(y^m) * x^n$. From Definition 4.4, G is a WC-GTA-NET-groupoid. $\square$

Definition 4.5. *A quasi strong regular TA-groupoid $(G, *)$ is called a quasi Clifford TA-groupoid, if $\forall x, y \in G$, there exist $n, m \in \mathbb{Z}^+$, $(x^n)^{-1}, (y^m)^{-1} \in G$, and $x^n * (y^m * (y^m)^{-1}) = (y^m * (y^m)^{-1}) * x^n$.*

Theorem 4.2. *Let $(G, *)$ be a groupoid. Then G is a WC-GTA-NET-groupoid iff it is a quasi Clifford TA-groupoid.*

Proof. Suppose that G is a WC-GTA-NET-groupoid, by Theorem 4.1, G is a quasi strong regular TA-groupoid. $\forall x, y \in G$, there exist $n, m \in \mathbb{Z}^+$, $(x^n)^{-1}, (y^m)^{-1} \in G$. Since $(y, \text{neut}(y^m), \text{anti}(y^m))$ is a GNET with degree m , we have $(y^m, \text{neut}(y^m), \text{anti}(y^m))$ is a GNET with degree 1.

Set $(y^m)^{-1} = \text{neut}(y^m) * \text{anti}(y^m)$, so, we have

$$x^n * (y^m * (y^m)^{-1}) = x^n * \text{neut}(y^m) = \text{neut}(y^m) * x^n = (y^m * (y^m)^{-1}) * x^n.$$

Thus, G is a quasi Clifford TA-groupoid.

In contrast, assume that G is a quasi Clifford TA-groupoid, then $\forall x, y \in G$, there exist $n, m \in \mathbb{Z}^+$, $(x^n)^{-1}, (y^m)^{-1} \in G$. By Theorem 4.1, G is a GTA-NET-groupoid, so $(x^n, \text{neut}(x^n), \text{anti}(x^n))$ and $(y^m, \text{neut}(y^m), \text{anti}(y^m))$ are two GNETs with degree 1.

Set $\text{neut}(x^n) = x^n * (x^n)^{-1}$, $\text{neut}(y^m) = y^m * (y^m)^{-1}$. From Definition 4.5, being $x^n * (y^m * (y^m)^{-1}) = (y^m * (y^m)^{-1}) * x^n$, we have $x^n * \text{neut}(y^m) = \text{neut}(y^m) * x^n$. Thus, G is a WC-GTA-NET-groupoid by Definition 4.4. $\square$

Example 4.7. Let $G = \{1, 2, 3, 4, 5\}$, the definition of operation $*$ on G is shown in Table 15. It is a WC-GTA-NET-groupoid. From Theorem 4.1, G is a quasi strong regular TA-groupoid. For element 1 and element 2, $1^{-1} = 1, 2^{-1} = 2$, $1 * (2 * 2^{-1}) = (2 * 2^{-1}) * 1$, $2 * (1 * 1^{-1}) = (1 * 1^{-1}) * 2$. For element 3 and element 5, $(3^2)^{-1} = 1, 5^{-1} = 5$, $3^2 * (5 * 5^{-1}) = (5 * 5^{-1}) * 3^2$, $5 * (3^2 * (3^2)^{-1}) = (3^2 * (3^2)^{-1}) * 5$.

The other cases can be checked and verified, thus G is a quasi Clifford TA-groupoid.

Table 15: A WC-GTA-NET-groupoid of Example 4.7.

$*$	1	2	3	4	5
1	1	2	2	2	1
2	2	1	1	1	2
3	2	1	1	1	3
4	2	1	1	1	4
5	1	2	2	4	5

5. Conclusion

In this paper, the GTA-NET-groupoid has been introduced, the structures of TA-groupoid and GTA-NET-groupoid have been studied in detail, and some important results have been obtained. We prove that TA-NET-groupoid, TA-(l,l)-groupoid, TA-(r,r)-groupoid, TA-(l,r)-groupoid, and strong regular TA-groupoid

are equivalent (see, Theorem 3.2 and Theorem 3.3); quasi strong regular TA-groupoid and GTA-NET-groupoid are equivalent (see, Theorem 4.1); quasi Clifford TA-groupoid and WC-GTA-NET-groupoid are equivalent (see, Theorem 4.2).

Furthermore, using the research results of Theorem 3.4, we successfully solve an open problem in [17]. We prove that a TA-NET-groupoid is a semigroup (see, Theorem 3.5), thus illuminating the structure of TA-NET-groupoid. Figure 5 shows the main results of this paper. Future efforts will be directed towards discussing the relationships between TA-groupoid and related logic algebras (see, [22, 23]).

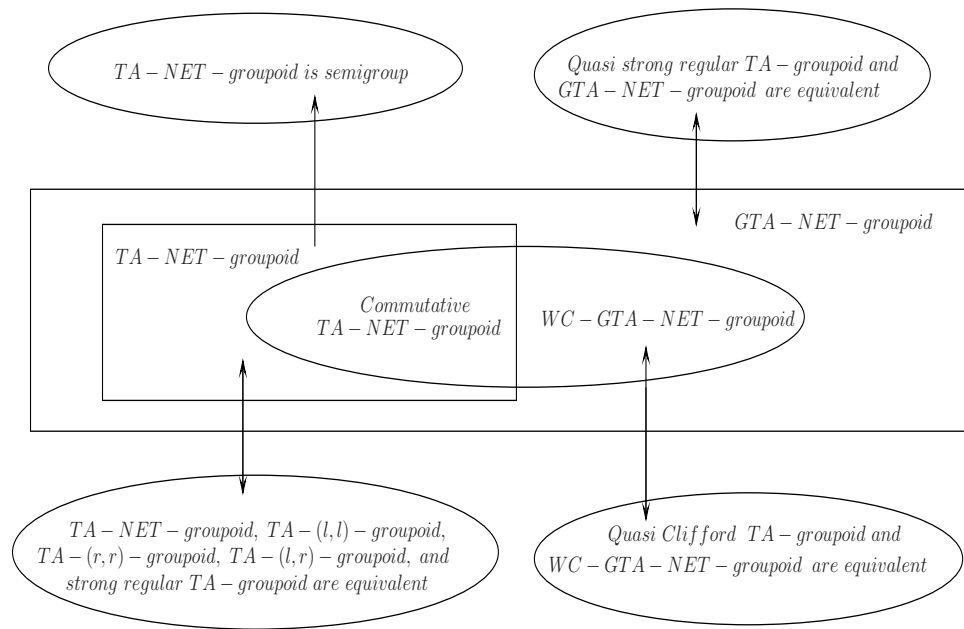


Figure 5: The main results of this paper.

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On the solutions of the Diophantine equation $M^x + (M - 1)^y = z^2$ **William S. Gayo, Jr.***

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Abstract. A Mersenne prime M is a prime number of the form $2^p - 1$, where p is also a prime number. In this study, we consider the Mersenne prime-involved exponential Diophantine equation $M^x + (M - 1)^y = z^2$. The main goal is to find the complete set of solutions of the given Diophantine equation in non-negative integers x , y and z . Though it is not yet known whether there are infinitely many Mersenne primes, we have proven that the equation under consideration has only finitely many solutions. The proof uses techniques on quadratic congruence, factorization and modular arithmetic.

Keywords: Mersenne primes, exponential diophantine equation, quadratic residue, non-negative integer solutions.

1. Introduction

Number theory is considered as the Queen of Mathematics and one of the precious jewels on its crown is the family of Diophantine equations. Diophantine equations are simply equations with the condition that only integer solutions are required. They come in different forms. The simplest type is the two-variable linear Diophantine equation $ax + by = c$, where x and y are the unknowns and the rest of the parameters are integer constants. There is also the class of exponential Diophantine equations, wherein the exponents are the unknowns. Recently, exponential Diophantine equations of the form $a^x + b^y = z^2$ are widely studied. Researchers study these equations in connection with several prime pairs, such as the twin primes, sexy primes and cousin primes. The literature

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contains a large number of published articles (cf. [1], [2], [3], [4], [5], [6] and [7].) In this work, we consider the exponential Diophantine equation

$$(1) \quad M^x + (M - 1)^y = z^2,$$

where M is a Mersenne prime. It is conjectured that there are infinitely many Mersenne primes but as of now, there are only 51 Mersenne primes being discovered. The search for other Mersenne primes is still an active research area. Fortunately, we have shown that no large Mersenne primes that satisfy equation (1). More precisely, we found out that there are only a finite number of solutions to (1).

2. Preliminaries

The following definitions, lemmas and a theorem are needed for the main result of this study.

Definition 2.1. *A Mersenne prime is a prime number of the form $2^p - 1$, where p is also a prime number.*

Lemma 2.1. *For all Mersenne primes $M > 3$, the congruence $M \equiv 7 \pmod{8}$ always hold.*

Proof. From $M > 3$, we get $2^p - 1 > 3$. It follows that $p > 2$ and thus, $2^p \equiv 0 \pmod{8}$. This yields to $M = 2^p - 1 \equiv -1 \pmod{8}$ which is equivalent to $M \equiv 7 \pmod{8}$. $\square$

Definition 2.2. *Let p be an odd prime and $\gcd(a, p) = 1$. If the quadratic congruence $z^2 \equiv a \pmod{p}$ has a solution, then a is said to be a quadratic residue of p . Otherwise, a is called a quadratic nonresidue of p .*

Lemma 2.2 (Euler's criterion). *Let p be an odd prime and $\gcd(a, p) = 1$. Then, a is a quadratic residue of p if and only if $a^{(p-1)/2} \equiv 1 \pmod{p}$.*

Lemma 2.3 ([8]). *The Diophantine equation $3^x + 2^y = z^2$ has exactly three nonnegative integer solutions (x, y, z) , namely, $(1, 0, 2)$, $(0, 3, 3)$, $(2, 4, 5)$.*

Theorem 2.1 (Mihailescu's Theorem [9]). *The quadruple $(3, 2, 2, 3)$ is the unique solution for the Diophantine equation $a^x - b^y = 1$, where a, b, x and y are integers with $\min\{a, b, x, y\} > 1$.*

3. Main results

We now state the main result which is about the solutions of the Diophantine equation (1).

Theorem 3.1. *The quadruples $(3, 1, 0, 2)$, $(3, 0, 3, 3)$ and $(3, 2, 4, 5)$ are the only nonnegative integer solutions (M, x, y, z) of the exponential Diophantine equation $M^x + (M - 1)^y = z^2$, where M is a Mersenne prime.*

Proof. The proof is divided into two cases for the Mersenne prime M .

Case 1. If $M = 3$, then equation (1) becomes

$$(2) \quad 3^x + 2^y = z^2.$$

By Lemma 2.3, equation (2) has exactly three solutions (x, y, z) , which are $(1, 0, 2)$, $(0, 3, 3)$ and $(2, 4, 5)$. Thus, the quadruples in the set

$$\{M, x, y, z\} = \{(3, 1, 0, 2), (3, 0, 3, 3), (3, 2, 4, 5)\}$$

are solutions of equation (1).

Case 2. For $M > 3$, we consider two cases for the exponent y .

Subcase 2.1. If y is even, let $y = 2t$ for some integer $t \geq 0$. Then, equation (1) becomes

$$M^x + (M - 1)^{2t} = z^2.$$

This can be expressed as $M^x = z^2 - (M - 1)^{2t} = (z + (M - 1)^t)(z - (M - 1)^t)$. There exist nonnegative integers α and β with $\alpha + \beta = x$ and $\alpha > \beta$ such that $z + (M - 1)^t = M^\alpha$ and $z - (M - 1)^t = M^\beta$. Combining these two equations leads to $2(M - 1)^t = M^\alpha - M^\beta$. By factoring out M^β , we get $2(M - 1)^t = M^\beta(M^{\alpha-\beta} - 1)$. Since M is an odd prime and $\gcd(M, M - 1) = 1$, it follows that

$$(3) \quad \begin{cases} M^\beta = 1, \\ M^{\alpha-\beta} - 1 = 2(M - 1)^t. \end{cases}$$

From the first equation of the system (3), it can be implied that $\beta = 0$. Hence, $\alpha = x > 0$ and $z = (M - 1)^t + 1$. The second equation of the system (3) becomes

$$M^x - 1 = 2(M - 1)^t.$$

Note that $M^x - 1 \equiv -1 \pmod{M}$, and $2(M - 1)^t \equiv 2(-1)^t \pmod{M}$. These two congruences imply that $M^x - 1 = -1 + nM$ and $2(M - 1)^t = 2(-1)^t + mM$ for some integers m and n . Thus,

$$(4) \quad -1 + nM = 2(-1)^t + mM.$$

If t is odd, then equation (4) becomes $-1 + nM = -2 + mM$ which can be written as $M(n - m) = -1$. But this impossible because $M > 3$. On the other hand, if t is even, then equation (4) becomes $-1 + nM = 2 + mM$ which is equivalent to $M(m - n) = 3$. This means that $M = 3$, which contradicts the assumption that $M > 3$. Thus, equation (1) has no solution for even y and $M > 3$.

Subcase 2.2. If y is odd, then $M^x + (M - 1)^y \equiv -1 \pmod{M}$ for positive integer x ; hence $z^2 \equiv -1 \pmod{M}$. Since $M > 3$, it follows that $M \equiv 7 \pmod{8}$ by Lemma 2.1. So, there exists an integer m such that $M = 8m + 7$. Using Euler's criterion, $(-1)^{(M-1)/2} = (-1)^{(8m+6)/2} = (-1)^{4m+3} = -1$. Hence, -1 is a quadratic nonresidue of M . This means that quadratic congruence

$z^2 \equiv -1 \pmod{M}$ has no solution and so does the equation (1) for positive integer x , odd integer y and $M > 3$. For the case when $x = 0$, the equation (1) becomes $1 + (M - 1)^y = z^2$. By Mihailescu's Theorem, $M = 3, y = 3$ and $z = 3$. This is a contradiction to $M > 3$. Hence, equation (1) has no solution for odd integer y and $M > 3$.

To sum up all the cases, the Diophantine equation (1) has exactly three non-negative solutions (M, x, y, z) which are $(3, 1, 0, 2)$, $(3, 0, 3, 3)$ and $(3, 2, 4, 5)$. $\square$

Corollary 3.1. *The quintuples $(3, 1, 0, 2, 1)$, $(3, 0, 3, 3, 1)$ and $(3, 2, 4, 5, 1)$ are the only nonnegative integer solutions (M, x, y, z, n) of the exponential Diophantine equation $M^x + (M - 1)^y = z^{2n}$, where M is a Mersenne prime and n is a natural number.*

Proof. Let x, y and z be nonnegative integers, n be natural number, and M be a Mersenne prime such that $M^x + (M - 1)^y = (z^n)^2$. By Theorem 3.1, we can see that $(M, x, y, z^n) = (3, 1, 0, 2)$, $(3, 0, 3, 3)$ and $(3, 2, 4, 5)$. By the primality of 2, 3 and 5, we get $n = 1$. Thus, we have the solutions $(3, 1, 0, 2, 1)$, $(3, 0, 3, 3, 1)$ and $(3, 2, 4, 5, 1)$. $\square$

4. Concluding remarks

In this study, we have proven that there are exactly three nonnegative integer solutions of the Diophantine equation (1), and these solutions exist only for the Mersenne prime $M = 3$. For other Mersenne primes larger than 3, the equation (1) will have no solution. A list of some exponential Diophantine equations with no solutions in the set of nonnegative integers is given in Table 1.

Table 1: List of Some Exponential Diophantine Equations of the Form $M^x + (M - 1)^y = z^2$ with No Solution

1.) $7^x + 6^y = z^2$	6.) $(2^{19} - 1)^x + (2^{19} - 2)^y = z^2$
2.) $31^x + 30^y = z^2$	7.) $(2^{31} - 1)^x + (2^{31} - 2)^y = z^2$
3.) $127^x + 126^y = z^2$	8.) $(2^{61} - 1)^x + (2^{61} - 2)^y = z^2$
4.) $8191^x + 8190^y = z^2$	9.) $(2^{89} - 1)^x + (2^{89} - 2)^y = z^2$
5.) $131071^x + 131070^y = z^2$	10.) $(2^{107} - 1)^x + (2^{107} - 2)^y = z^2$

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Numerical solution of electromagnetic problem in horizontal porous medium

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Abstract. A model of heat transfer in a channel of porous walls under the influence of a vertical electromotive force (EMF) with natural convection and thermal radiation of a dissipative fluid is discussed in the channel plane. Using the finite difference method, the partial differential equations that govern the issue have been formulated and solved. Where, using successive iterations of the above method on the equations that are in the time-dependent form, we were able to reach the required solution to the problem. Besides, the temperature distribution behaviour inside the channel was noticed, and the effect of the thermal behaviour of Rayleigh number, Brinkman number, Prandtl number, Eckert number and Hartmann number was also studied. By creating a computer program using MATLAB, we solved the system.

Keywords: transfer of heat, porous medium, Brinkman number, Eckert number, Hartmann number, Prandtl number.

1. Introduction

The study of fluid flow in a porous wall channel is very interesting since it has a wide range of implementation, and this is because these flows have many engineering and geophysical applications that include geothermal resources, blood flow inside human bodies, building insulation, oil extraction, heat salt leaching in soils, lymph transport flow system, urinary circulatory system [1]. A analysis of some previous work in this field is provided below.

In enclosures with localized heating natural convection was studied from below the creeping flow to the onset of laminar instability (see [2]).

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The convection in a rotating cylindrical annulus under the influence of its symbol MF only mention in [3]. They examined the effect MFs radial and azimuthal components on the convection columns in the fluid-filled gap between two cylinders rotating stiffly numerical around their shared vertical axis, cooling the inner cylinder, heating the outer one so that the centrifugal force provides the buoyancy force driving the convection.

The effects of radiative in magneto fluid-dynamic channel flow demonstrate ref. [4], he has expanded plane Hartmann flow to account for thermally radiative effects with variable absorption coefficient and non-uniform temperature of channel walls. In addition, several dimensions of stability have been reviewed.

While uses a 2-dimensional Galerkin formulation of the 3-dimensional Oberbeck-Boussinesq equations to evaluate the convection beginning in an infinite rigid horizontal channel that has a uniformly lower heat source [5]. In order to include convection patterns, they expanded the previous findings to higher truncation levels.

In a fluid layer with a lower heat source, the thermal convection issue is introduced [6] and numerically resolved when the layer has high vertical MF parameters. The stability of 2-dimensional convection rolls is studied by using various values of the Hartmann number within the area (200-400).

The convection inside a rotating cylindrical annulus, by a system that contains three coupled amplitude equations, is investigated [7]. They described many features to a good approximation, and show that the time integrations based on the Galerkin expansion display transitions to chaotic convection at a high number of Rayleigh.

The fully defined free convection problem of two fluid magneto hydrodynamic (MHD) flow in a slanted channel is addressed, its observed that the flow can be effectively dominated by adequate adjustment of the values for the heights, electrical conductivity, and viscosity ratios of two fluids [8].

In [9] studied linear stability to obtain the numerical solution of a multi-layer system consisting of the onset of finger convection in a fluid layer overlying a porous layer using the spectral Chebyshev polynomial process.

In [10] the investigation of combined effects of a transverse magnetic field and irradiative heat transfer on the unsteady flow of a conducting optically thin fluid through a channel filled with the saturated porous medium the temperature of a non-uniform wall, his results showed that increasing magnetic field intensity reduce wall shear stress while increasing radiation parameter through heat absorption causes an increase in the magnetic of wall shear stress.

In [11] proposed a solution to the magneto hydrodynamic(MHD) problem analytically free convection flow of an electrically conducting fluid between two heated parallel plates in the existence of an induced MF. It has been noted that the skin-friction increases first then gradually decreases with the increase of Hartmann number at $y=1$.

The impacts of radiation on (MHD) flow of Maxwell fluid in a channel with a porous medium by employing the homotopy analysis method (HAM) discussed [12].

In [13] investigated the 2-dimensional steady flow of electrically conducting incompressible power-law fluid past an infinite porous flat plate subjected to suction or blowing, they are also analyzed the heat transfer flow in the case when the plate is held at a fixed temperature. In the same year, the homotopy analysis method (HAM) is applied to obtain the analytic solution of partial differential equations. Numerical results and graphical representation strongly reconfirm the efficiency of the proposed scheme [14].

In [15] addressed the stability analysis in the glass cavity, and this analysis was achieved by finding the self-values for the system that could or could not be determined by the growth of the disturbance after making linear equations.

In [16] Studied the approximated solutions for heat transfer over a porous plate and steady MHD mixed convection boundary layer flow in the existence of the thermal and velocity stumble, they found the MF impact on a viscous incompressible fluid increase the fluid velocity by reducing the drag on the flow that causes a decrease in the temperature fluid.

In [17] Presented the mass transfer impact, viscous suction parameter, and dissipative on the 2-dimensional steady hydro magnetic viscous fluid flow between two parallel plates in the presence of thermal radiation and we find that velocity, temperature, and concentration decrease with the increase of variable suction parameter and Reynolds number as well as the relation between the different quality physicals.

In [18] Presented a Crank-Nicolson finite difference method to solve the time-fractional 2-dimensional sub-diffusion equation in the case where the Grunwald-Letnikov definition is used for the time-fractional derivative. The stability and convergence of the proposed Crank-Nicolson scheme are also analyzed as well as the numerical examples are presented to test that the numerical scheme is accurate and feasible. In the same year, The modified implicit finite difference approximation of the stability and convergence of the proposed scheme is analyzed [19]. Its founded that show that the scheme is unconditionally stable and the approximate solution converges to the exact solution.

Numerical Solution the motion, heat transfer, and diffusion of the equation in the Porous medium with the presence of Radiation and magnetic field (MF) was studied [20]. It has been found that the parameters have a major effect on the Solution of these Equations. The goal of this research is to study the flow of fluids in a cross-section under the influence of the electromagnetic field(EMF) and to solve the partial differential equations that regulate the problem using the method of (ADI). In addition to showing the behaviour of temperatures inside the cross-section and the effect of physical quantities.

2. Statement of issue

After applying a continuous (EMF) to the fluid, we can research the flow of an electrically conductive fluid inside a segment of a horizontal channel so that the fluid transforms into a magnetized fluid. Assume that the temperature of the horizontal and vertical walls is expressed by τ_c, τ_h and $\tau_h > \tau_c$. The distance between the lower and upper walls is expressed by H and the cross-section length is represented by L . The y -axis is defined by reprinted vertical projected electromagnetic field (EMF) and a magnetic fluid (MF) is created after the electromagnetic field (EMF) is shed that flows to the x -axis and an (EMF) is produced in the direction of the z -axis that is small enough to be ignored as shown in Figure 1:

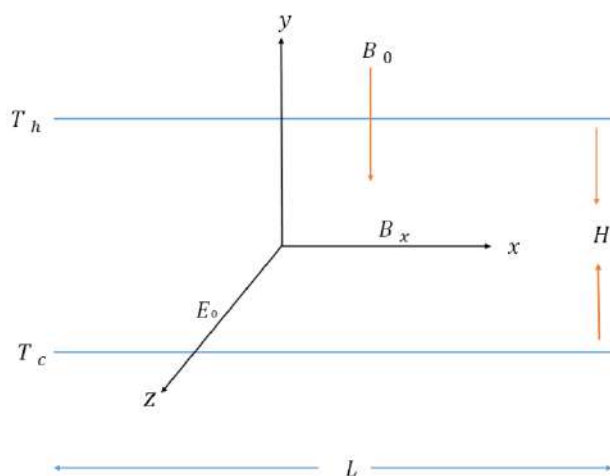


Figure 1: Physical Model and System of Coordinates.

$$(1) \quad \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

$$(2) \quad \begin{aligned} & \frac{\partial u}{\partial t} + v \frac{\partial v}{\partial x} + \nu \frac{\partial v}{\partial y} \\ &= -\frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{\mu}{\rho} \nabla^2 v + \frac{\sigma E_0 \beta_0}{\rho} + \frac{\sigma E_0 \beta_0 v}{\rho} - \frac{\sigma \beta_x \beta_0 \nu}{\rho}, \end{aligned}$$

$$(3) \quad \begin{aligned} & \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + \nu \frac{\partial v}{\partial y} \\ &= -\frac{1}{\rho} \frac{\partial p}{\partial y} + \frac{\mu}{\rho} \nabla^2 v - \frac{\sigma E_0 \beta_x}{\rho} - \frac{\sigma \beta_x \beta_0 v}{\rho} + \frac{\sigma \beta_x^2 \nu}{\rho} - g\beta [\tau - \tau_c], \end{aligned}$$

$$(4) \quad \frac{\partial \tau}{\partial t} + v \frac{\partial \tau}{\partial x} + \nu \frac{\partial \tau}{\partial y}$$

$$= \alpha \nabla^2 \tau + \frac{1}{\rho C_p} \left[\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} \right] + \frac{\mu}{\rho C_p} \left[\left[\frac{\partial v}{\partial x} \right]^2 + \left[\frac{\partial v}{\partial y} \right]^2 \right].$$

We differentiate the equation (2) for y and the equation (3) for x in order to get the general motion equation, and then subtract the two equations, we get:

$$(5) \quad \frac{\partial}{\partial t} \left[\frac{\partial v}{\partial x} - \frac{\partial v}{\partial y} \right] + \left[\frac{\partial}{\partial x} \left[v \frac{\partial v}{\partial x} + \nu \frac{\partial v}{\partial y} \right] - \frac{\partial}{\partial y} \left[v \frac{\partial v}{\partial x} + \nu \frac{\partial v}{\partial y} \right] \right] \\ = \frac{\mu}{\rho} \nabla^2 \left[\frac{\partial v}{\partial x} - \frac{\partial v}{\partial y} \right] - g \beta \frac{\partial \tau}{\partial x} + \frac{\sigma \beta_x \beta_0}{\rho} \left[\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x} \right] + \frac{\sigma \beta_x^2}{\rho} \frac{\partial v}{\partial x} - \frac{\sigma E_0 \beta_0}{\rho} \frac{\partial v}{\partial y},$$

where v, ν are the velocity components in x, y respectively, t is the time and $\tau, P, \rho, \sigma, E_0, \beta_0, \beta_x, \mu, g, \beta, C_p, \alpha, q_x, q_y$ temperature, pressure, density, kinematic viscosity, Electric field vector, The magnetic field vector is in the direction of x and y , viscosity, gravitational acceleration, thermal expansion coefficient, specific heat at constant pressure, thermal diffusivity, radiation components in x, y respectively. With the boundary conditions below:

$$(6) \quad \begin{aligned} \tau [x, (y = 0, H)] &= \tau_c, \tau_h, \\ \frac{\partial \tau}{\partial x} &= 0, x = 0, L \\ v [(x = 0, L), y] &= \nu [(x = 0, L), y] = 0, \\ v [x, (y = 0, H)] &= \nu [x, (y = 0, H)] = 0. \end{aligned}$$

3. Non-dimensional form

We need to add the following non-dimensional quantities to solve the governing equations (1), (4) and (5) with the boundary conditions (6) [24].

$$(7) \quad \begin{aligned} u &= v_0 v^*, & \nu &= v_0 \nu^*, & v_0 &= \sqrt{g \beta \Delta \tau L}, & t &= \frac{L}{v_0} t^*, \\ x &= L x^*, & y &= L y^*, & \vartheta &= \frac{\tau - \tau_c}{\tau_h - \tau_c}, & \vec{q} &= \tau_1^4 \sigma \vec{Q}. \end{aligned}$$

When these quantities are substituted into equations (1), (4) and (5), the governing equations become:

$$(8) \quad \frac{\partial v^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0, \\ \frac{\partial}{\partial t^*} \left(\frac{\partial v^*}{\partial x^*} - \frac{\partial v^*}{\partial y^*} \right) + \frac{\partial}{\partial x^*} \left(v^* \frac{\partial v^*}{\partial x^*} + \nu^* \frac{\partial v^*}{\partial y^*} \right) - \frac{\partial}{\partial y^*} \left(v^* \frac{\partial v^*}{\partial x^*} + \nu^* \frac{\partial v^*}{\partial y^*} \right) \\ (9) \quad = \sqrt{\frac{P_r}{R_a}} \nabla^2 \left(\frac{\partial v^*}{\partial x^*} - \frac{\partial v^*}{\partial y^*} \right) - \frac{\partial \vartheta}{\partial x^*} - (H_a)^2 \sqrt{\frac{P_r}{R_a}} \left(\frac{\partial v^*}{\partial y^*} - \frac{\partial v^*}{\partial x^*} \right) \\ + (H_a)^2 \sqrt{\frac{P_r}{R_a}} \frac{\partial v^*}{\partial x^*} - I(H_a)^2 \sqrt{\frac{P_r}{R_a}} \frac{\partial v^*}{\partial y^*}.$$

But $v^* = \frac{\partial \Psi^*}{\partial y^*}$ and $\nu^* = -\frac{\partial \Psi^*}{\partial x^*}$ stream function [23], then the equation of general motion becomes: $-\frac{\partial}{\partial t^*} \nabla^2 \Psi^* = -\sqrt{\frac{P_r}{R_a}} \nabla^4 \Psi^* - (H_a)^2 \sqrt{\frac{P_r}{R_a}} \nabla^2 \Psi^*$. Put $\varepsilon^* = -\nabla^2 \Psi^*$, and then:

$$(10) \quad \frac{\partial \varepsilon^*}{\partial t^*} = -\sqrt{\frac{P_r}{R_a}} \nabla^2 \varepsilon^* - (H_a)^2 \sqrt{\frac{P_r}{R_a}} \varepsilon^*,$$

$$(11) \quad \frac{\partial \vartheta}{\partial t^*} + v^* \frac{\partial \vartheta}{\partial x^*} + \nu^* \frac{\partial \vartheta}{\partial y^*} = \frac{1}{\sqrt{P_r R_a}} \nabla^2 \vartheta - \frac{F_s \beta_0 R_e}{\sqrt{P_r R_a}} \nabla \vec{Q} + 2 \frac{F_s}{\sqrt{P_r R_a}} \left(\left(\frac{\partial v^*}{\partial x^*} \right)^2 + \left(\frac{\partial \nu^*}{\partial y^*} \right)^2 \right)$$

and that is about it $\nabla \cdot \vec{Q} = 16\omega^* \vartheta - 12\omega^*$ [4], equation (11) then becomes:

$$(12) \quad \frac{\partial \vartheta}{\partial t^*} + v^* \frac{\partial \vartheta}{\partial x^*} + \nu^* \frac{\partial \vartheta}{\partial y^*} = \frac{1}{\sqrt{P_r R_a}} \nabla^2 \vartheta - \frac{F_s \beta_0 R_e}{\sqrt{P_r R_a}} (16\omega^* \vartheta - 12\omega^*) + \phi.$$

Let $\phi = \frac{2F_s}{\sqrt{P_r R_a}} \left(\left(\frac{\partial v^*}{\partial x^*} \right)^2 + \left(\frac{\partial \nu^*}{\partial y^*} \right)^2 \right)$ physical quantities, and $I = \frac{E_0}{\beta_0}$, $E_c = \frac{v^{*2}}{C_p \tau_1}$, $R_a = \frac{\rho g \beta (\tau_h - \tau_c) L^3}{\mu \alpha}$, $R_e = \frac{h \rho v^*}{\mu}$, $\omega^* = \alpha_0 h$, $\beta_0 = \frac{\rho v^{*3}}{\tau_1^4 \sigma}$, $P_r = \frac{\mu C_p}{\kappa}$, $F_s = P_r E_c$, $H_a = \beta_0 L \sqrt{\frac{\sigma}{\mu}}$, are Inductance, Eckert number, Rayleigh number, Reynolds number, Bouger number, Boltzmann number, Prandtl number, Brinkman number, and Hartmann number respectively. The non-dimensional boundary condition becomes:

$$(13) \quad \begin{aligned} \vartheta &= 0, 1.0 & \text{at} & y^* = 0, 1, \\ \frac{\partial \vartheta}{\partial x^*} &= 0 & \text{at} & x^* = 0, 1, \\ v^* &= \nu^* = 0.0 & \text{at} & y^* = 0, 1. \end{aligned}$$

4. Solution process

To solve the system of equations (8), (10) and (12) with the boundary conditions (13), we use the ADI finite difference method [22], and achieve this we have to start with the last equation(12), heat equation and finally equation(10), motion equation as follows:

4.1 Equation of heat

$$(14) \quad \begin{aligned} & \frac{\vartheta_{i,j}^* - \vartheta_{i,j,n}}{\frac{\Delta t}{2}} + v_{i,j,n}^* \frac{\vartheta_{i+1,j}^* - \vartheta_{i-1,j}^*}{2\Delta x^*} + \nu_{i,j,n}^* \frac{\vartheta_{i,j+1,n} - \vartheta_{i,j-1,n}}{2\Delta y^*} \\ &= -\frac{F_s \beta_0 R_e}{\sqrt{P_r R_a}} (16\omega^* \vartheta_{i,j,n} - 12\omega^*) - \frac{1}{\sqrt{P_r R_a}} \left(\frac{\vartheta_{i+1,j}^* - 2\vartheta_{i,j}^* + \vartheta_{i-1,j}^*}{(\Delta x^*)^2} \right. \\ & \quad \left. + \frac{\vartheta_{i,j+1,n} - 2\vartheta_{i,j,n} + \vartheta_{i,j-1,n}}{(\Delta y^*)^2} \right) + \phi, \end{aligned}$$

$$\begin{aligned}
& \frac{\vartheta_{i,j,n+1} - \vartheta_{i,j}^*}{\frac{\Delta t}{2}} + \nu_{i,j,n}^* \frac{\vartheta_{i+1,j}^* - \vartheta_{i-1,j}^*}{2\Delta x^*} + \nu_{i,j,n}^* \frac{\vartheta_{i,j+1,n+1} - \vartheta_{i,j-1,n+1}}{2\Delta y^*} \\
(15) \quad &= \frac{1}{\sqrt{P_r R_a}} \left(\frac{\vartheta_{i+1,j}^* - 2\vartheta_{i,j}^* + \vartheta_{i-1,j}^*}{(\Delta x^*)^2} + \frac{\vartheta_{i,j+1,n+1} - 2\vartheta_{i,j,n+1} + \vartheta_{i,j-1,n+1}}{(\Delta y^*)^2} \right) \\
& - \frac{F_s \beta_0 R_e}{\sqrt{P_r R_a}} (16\omega^* \vartheta_{i,j,n+1} - 12\omega^*) + \phi.
\end{aligned}$$

Under boundary conditions,

$$(16) \quad \vartheta_{i,0,n} = 0, \vartheta_{i,N,n} = 1.0.$$

Equations (14) and (15) can be reduced in order to give,

$$(17) \quad A_1(i) \vartheta_{i-1,j}^* + B_1(i) \vartheta_{i,j}^* + C_1(i) \vartheta_{i+1,j}^* = D_1(i)$$

where

$$\begin{aligned}
A_1(i) &= - \left(\frac{1}{\sqrt{P_r R_a}} + \frac{\Delta x^*}{2} \nu_{i,j,n}^* \right), \\
B_1(i) &= 2 \left(\frac{(\Delta x^*)^2}{\Delta t} + \frac{1}{\sqrt{P_r R_a}} \right) \\
C_1(i) &= \left(\frac{1}{\sqrt{P_r R_a}} - \frac{\Delta x^*}{2} \nu_{i,j,n}^* \right), \\
D_1(i) &= \left(\frac{1}{\sqrt{P_r R_a}} + \frac{\Delta y^*}{2} \nu_{i,j,n}^* \right) \vartheta_{i,j-1,n} + \frac{12F_s \beta_0 R_e \omega^* (\Delta x^*)^2}{\sqrt{P_r R_a}} \\
&+ 2 \left(\frac{(\Delta y^*)^2}{\Delta t} - \frac{1}{\sqrt{P_r R_a}} - \frac{8F_s \beta_0 R_e \omega^* (\Delta y^*)^2}{\sqrt{P_r R_a}} \right) \vartheta_{i,j,n} \\
&+ \left(\frac{1}{\sqrt{P_r R_a}} - \frac{\Delta y^*}{2} \nu_{i,j,n}^* \right) \vartheta_{i,j+1,n} + (\Delta x^*)^2 \phi.
\end{aligned}$$

Followed,

$$(18) \quad A_2(i) \vartheta_{i,j-1,n+1} + B_2(i) \vartheta_{i,j,n+1} + C_2(i) \vartheta_{i,j+1,n+1} = D_2(i)$$

where

$$\begin{aligned}
A_2(i) &= - \left[\frac{\Delta t}{2(\Delta y^*)^2 \sqrt{P_r R_a}} + \frac{\Delta t}{4\Delta y^*} \nu_{i,j,n}^* \right], \\
B_2(i) &= \left(1 + \frac{\Delta t}{(\Delta y^*)^2 \sqrt{P_r R_a}} + \frac{8F_s \beta_0 R_e \omega^* \Delta t}{\sqrt{P_r R_a}} \right), \\
C_2(i) &= - \left(\frac{\Delta t}{2(\Delta y^*)^2 \sqrt{P_r R_a}} - \frac{\Delta t}{4\Delta y^*} \nu_{i,j,n}^* \right),
\end{aligned}$$

$$\begin{aligned}
 D_2(i) = & \left(\frac{\Delta t}{2(\Delta x^*)^2 \sqrt{P_r R_a}} + \frac{\Delta t}{4\Delta x^*} v_{i,j,n}^* \right) \vartheta_{i-1,j}^* \\
 & + \left(1 - \frac{\Delta t}{(\Delta x^*)^2 \sqrt{P_r R_a}} \right) \vartheta_{i,j}^* + \frac{16F_s \beta_0 R_e \omega^* \Delta t}{\sqrt{P_r R_a}} \\
 & + \left(\frac{\Delta t}{2(\Delta x^*)^2 \sqrt{P_r R_a}} - \frac{\Delta t}{4\Delta x^*} v_{i,j,n}^* \right) \vartheta_{i+1,j}^* + \frac{\Delta t}{2} \phi.
 \end{aligned}$$

4.2 Equation of motion

$$\begin{aligned}
 \frac{\varepsilon_{i,j}^* - \varepsilon_{i,j,n}}{\frac{\Delta t}{2}} = & \sqrt{\frac{P_r}{R_a}} \left(\frac{\varepsilon_{i+1,j}^* - 2\varepsilon_{i,j}^* + \varepsilon_{i-1,j}^*}{(\Delta x^*)^2} + \frac{\varepsilon_{i,j+1,n} - 2\varepsilon_{i,j,n} + \varepsilon_{i,j-1,n}}{(\Delta y^*)^2} \right) \\
 (19) \quad & + (H_a)^2 \sqrt{\frac{P_r}{R_a}} \varepsilon_{i,j,n}
 \end{aligned}$$

$$\begin{aligned}
 \frac{\varepsilon_{i,j,n+1} - \varepsilon_{i,j}^*}{\frac{\Delta t}{2}} = & \sqrt{\frac{P_r}{R_a}} \left(\frac{\varepsilon_{i+1,j}^* - 2\varepsilon_{i,j}^* + \varepsilon_{i-1,j}^*}{(\Delta x^*)^2} + \frac{\varepsilon_{i,j+1,n+1} - 2\varepsilon_{i,j,n+1} + \varepsilon_{i,j-1,n+1}}{(\Delta y^*)^2} \right) \\
 (20) \quad & + (H_a)^2 \sqrt{\frac{P_r}{R_a}} \varepsilon_{i,j,n+1}
 \end{aligned}$$

Under boundary conditions,

$$\begin{aligned}
 v^* &= cons. & v_{0,j,n}^* &= 0, & v_{i,0,n}^* &= 0 \\
 (21) \quad \nu^* &= cons. & \nu_{0,j,n}^* &= 0, & \nu_{i,0,n}^* &= 0
 \end{aligned}$$

Equations (19) and (20) can be reduced in order to give,

$$(22) \quad A_3(i) \varepsilon_{i-1,j}^* + B_3(i) \varepsilon_{i,j}^* - C_3(i) \varepsilon_{i+1,j}^* = D_3(i),,$$

where

$$\begin{aligned}
 A_3(i) &= - \left(\frac{\Delta t}{2(\Delta x^*)^2} \sqrt{\frac{P_r}{R_a}} \right), \\
 B_3(i) &= \left(1 + \frac{\Delta t}{(\Delta x^*)^2} \sqrt{\frac{P_r}{R_a}} \right), \\
 C_3(i) &= - \left(\frac{\Delta t}{2(\Delta x^*)^2} \sqrt{\frac{P_r}{R_a}} \right), \\
 D_3(i) &= \left(\frac{\Delta t}{2(\Delta x^*)^2} \sqrt{\frac{P_r}{R_a}} \right) \varepsilon_{i,j-1,n} + \frac{\Delta t}{2(\Delta x^*)^2} \sqrt{\frac{P_r}{R_a}} \varepsilon_{i,j+1,n} \\
 &+ \left(1 - \frac{\Delta t}{(\Delta x^*)^2} \sqrt{\frac{P_r}{R_a}} + \frac{\Delta t (H_a)^2}{2} \sqrt{\frac{P_r}{R_a}} \right) \varepsilon_{i,j,n}.
 \end{aligned}$$

Followed,

$$(23) \quad A_4(i)\varepsilon_{i,j-1,n+1} + B_4(i)\varepsilon_{i,j,n+1} + C_4(i)\varepsilon_{i,j+1,n+1} = D_4(i)$$

where

$$\begin{aligned} A_4(i) &= - \left(\frac{\Delta t}{2(\Delta x^*)^2} \sqrt{\frac{P_r}{R_a}} \right), \\ B_4(i) &= \left(1 + \frac{\Delta t}{(\Delta x^*)^2} \sqrt{\frac{P_r}{R_a}} - \frac{\Delta t(H_a)^2}{2} \sqrt{\frac{P_r}{R_a}} \right), \\ C_4(i) &= - \left(\frac{\Delta t}{2(\Delta x^*)^2} \sqrt{\frac{P_r}{R_a}} \right), D_4(i) = \left(\frac{\Delta t}{2(\Delta x^*)^2} \sqrt{\frac{P_r}{R_a}} \right) \varepsilon_{i-1,j}^* \\ &\quad + \left(1 - \frac{\Delta t}{(\Delta x^*)^2} \sqrt{\frac{P_r}{R_a}} \right) \varepsilon_{i,j}^* + \frac{\Delta t}{2(\Delta x^*)^2} \sqrt{\frac{P_r}{R_a}} \varepsilon_{i+1,j}^*. \end{aligned}$$

The coefficients v^*, ν^* are treated as constants during any one time-step of the calculation [4], each of the equations (heat, motion) producing a tridiagonal system that is solved by the Gauss elimination process, all details are given in reference no. [18].

5. Results and data

In this section, we present some of the results obtained from the calculation carried out, and these results have been expressed by figures to explain how the solution for various cases becomes, as well as the effects of different parameters as follows figures:

6. Conclusions

In this work, we have used the ADI method in the solution of the governing equations completely without reducing or modifying, and from the results obtained we conclude that the steady-state can be reached after some iterations for all equations and this is clear from the figures given in the previous section, and the most important thing that has been achieved is the ADI method:

1. The higher the time stage, the further away from stability, as seen in the two Figures 2, 10.
2. The temperature within the boundary layer increases as the Brinkman number F_s increases. This is shown in Figure 4.
3. The temperature within the boundary layer decreases as the Boltzmann number β_0 increases. This is shown in Figure 5.

4. The temperature within the boundary layer increases as the amount of Reynolds number R_e increases. This is shown in Figure 6.
5. Figures 7, 12 shows that the effect of Rayleigh number R_a is constant in energy equation and motion, the higher the number of Rayleigh, the closer to equilibrium.
6. The effect of Prandtl number P_r on the equation energy was different from its effect on the equation of motion, as the more in the energy equation the further we moved away from stability, while we approached stability in the equation of motion by increasing the number as shown in Figures 8, 11.
7. The increase in Eckert number E_c is balanced by an increase in stability as shown in Fig. 9.
8. By reducing the Hartman number H_a , he offsets his stability convergence as shown in Fig.(13).

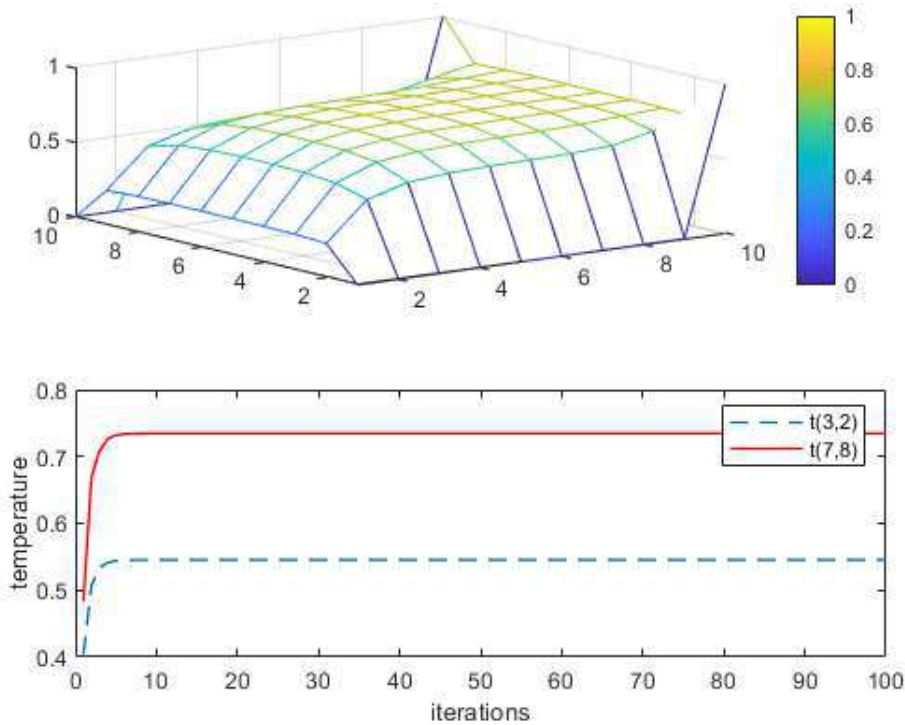


Figure 2: Temperature conduct

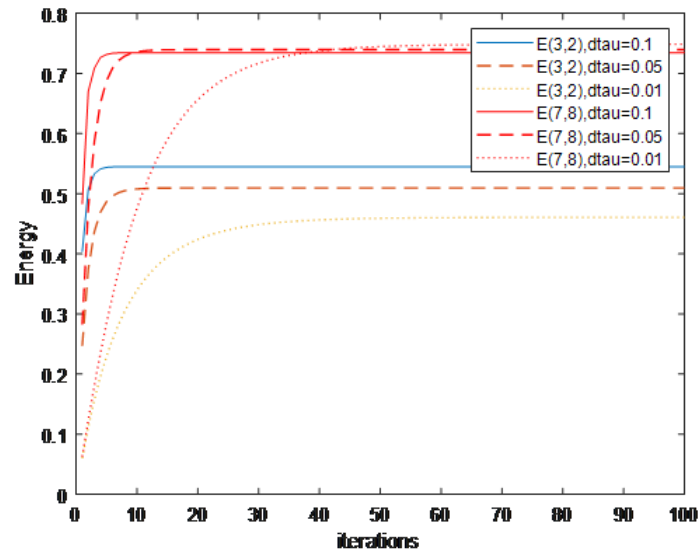
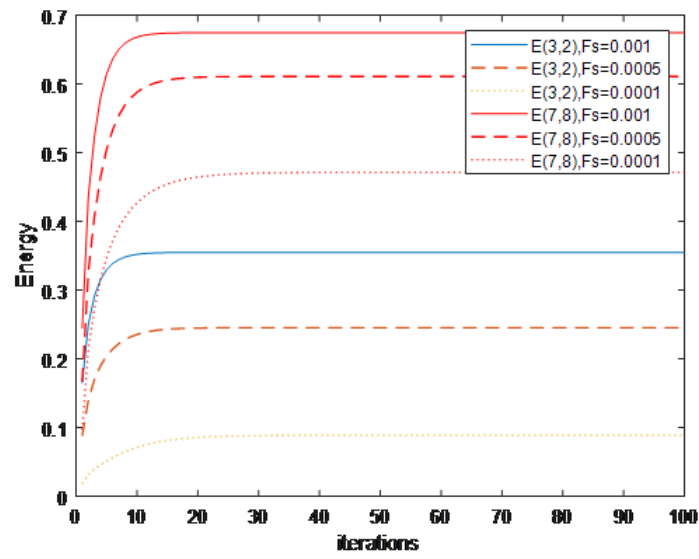


Figure 3: Impact of time-step on energy equation

Figure 4: Effect on the energy equation of Brinkman number F_s

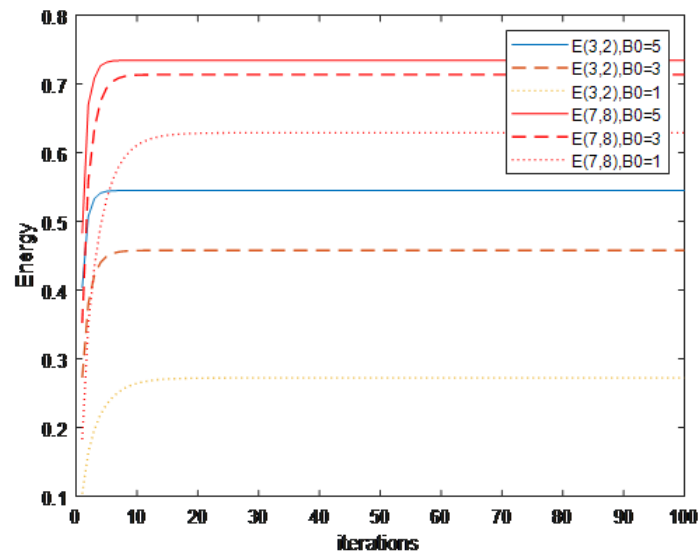


Figure 5: Effect on the energy equation of Boltzmann number B_0

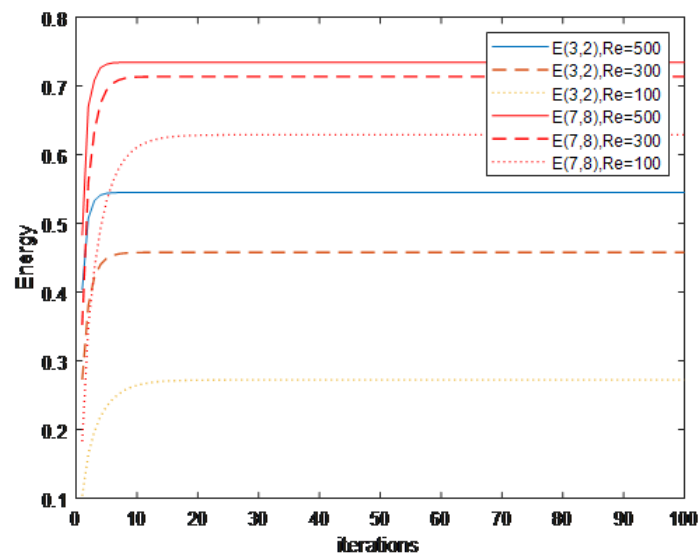


Figure 6: Effect of Reynolds number Re in the energy equation

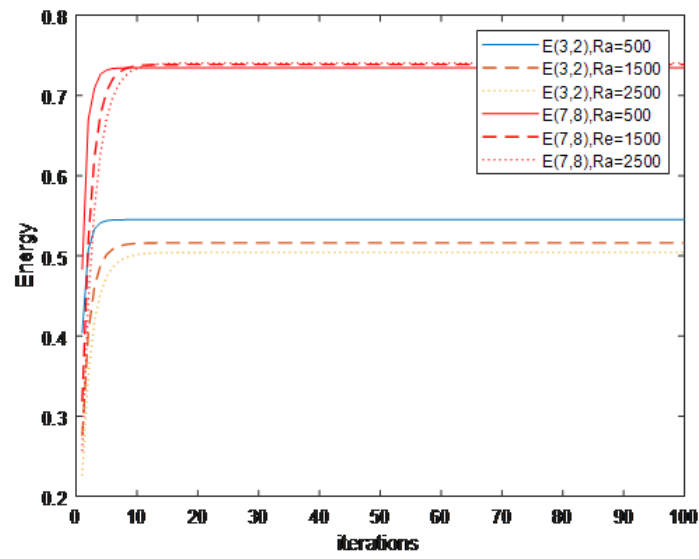


Figure 7: Effect of Rayleigh number Ra in energy equation number

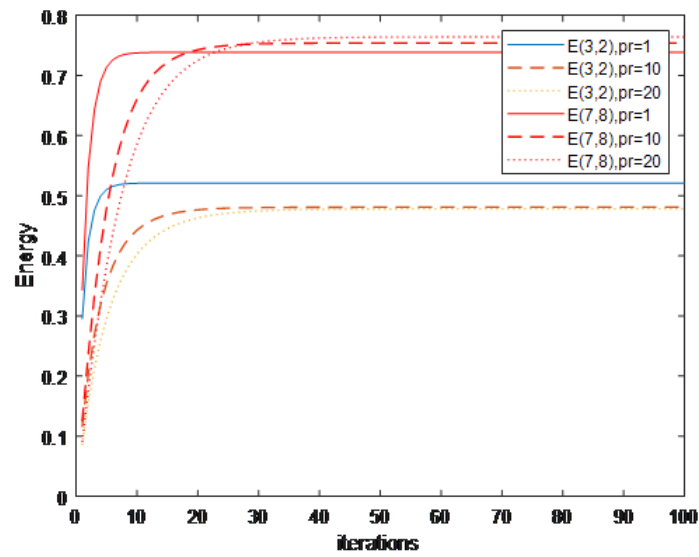


Figure 8: The impact of Prandtl number Pr on energy equations

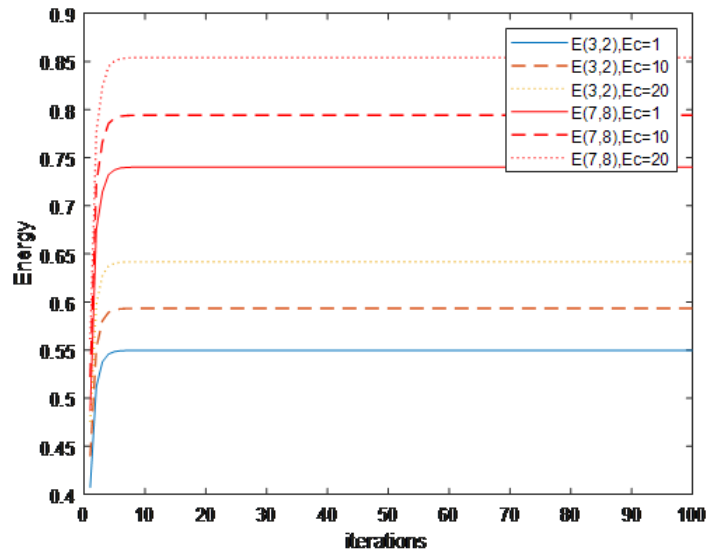


Figure 9: Impact of Eckert number E_c on energy equation

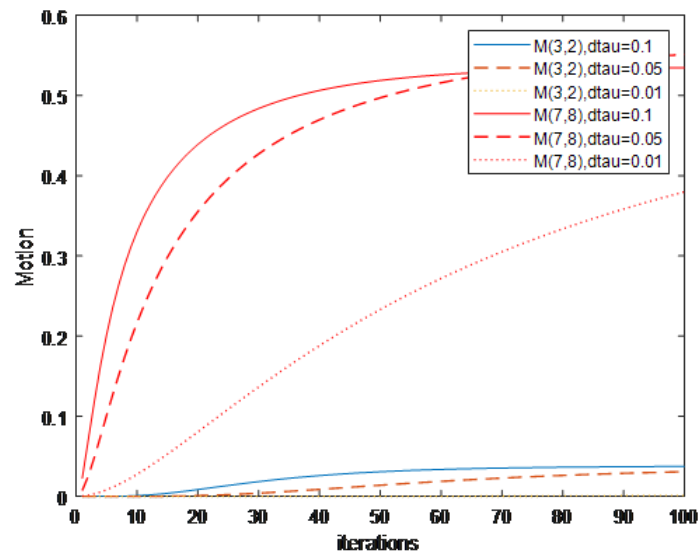


Figure 10: Time effect-step ($dtua$) on the motion equation

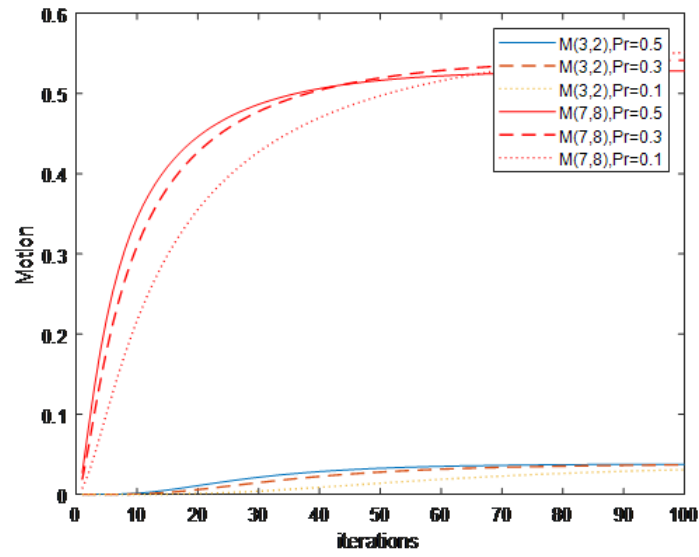


Figure 11: Influence of the Prandtl number P_r on the motion equation

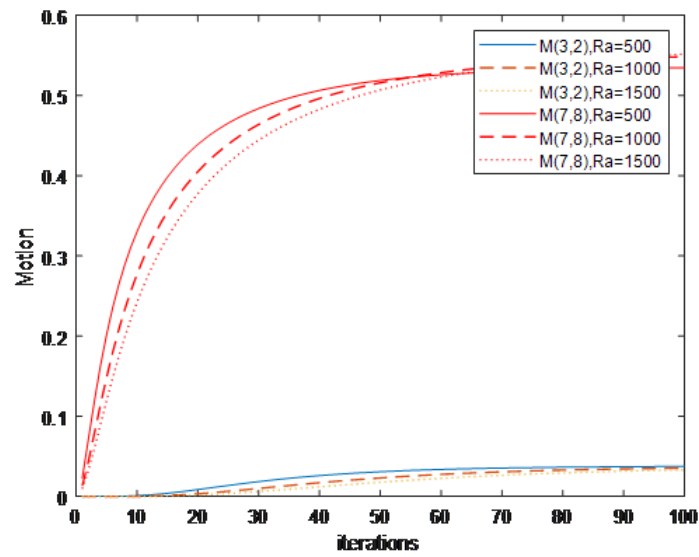


Figure 12: Effect of Rayleigh R_a number on motion equation

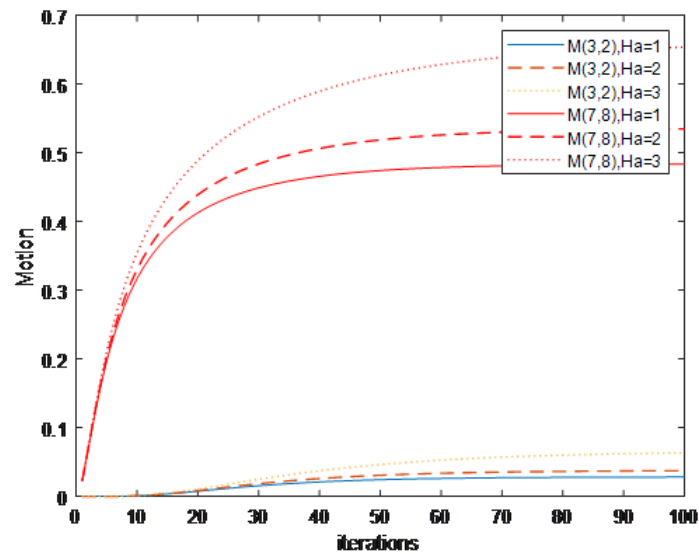


Figure 13: Effect of Hartman number H_a on motion equation

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Finslerian hypersurfaces of a Finsler space with special (α, β) -metric

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Abstract. In the present paper we have studied the Finslerian hypersurfaces of a Finsler space with the special (α, β) metric. We have examined the hypersurfaces of this special metric as a hyperplane of first, second and third kinds. Further, we the condition under which this hypersurface will be C2-like Finsler space also obtained the value of Main Scalar I in two-Dimensional frame of Finsler Space.

Keywords: Finslerian hypersurface, (α, β) metric, Douglas metric, infinite series metric.

1. Introduction

In 1992 Matsumoto [5] summarised all the result of a Finsler space based on a Finsler metric $L(\alpha, \beta)$ in a differentiable manifold M^n by considering L is a positively homogeneous function of degree one in two variable α which was a Riemannian metric and β one-form metric on M^n . Further in 1998 he [7] introduced a special (α, β) - metric which was known as Douglas metric and studied the conditions for some special Finsler spaces. Since the Douglas space is the generalization of Berwald space form the viewpoint of geodesic equation, so this metric is very important in the development of the Finsler geometry. Further in 2004 Lee and Park [1] introduced a r -th series (α, β) -metric and by considering $r = \infty$ he defined the above metric as infinite series metric.

The concept of Finslerian hypersurface was first introduced by Matsumoto in 1985 and further he defined three types of hypersurfaces that were called a hyperplane of the first, second and third kinds. Further many authors studied

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these hyperplanes in different changes of the Finsler metric [2, 3, 8, 9, 10, 11, 12] and obtained different important results.

In the present paper we combine the metric Douglas metric and Infinite series metric in a special manner and introduce a generalized metric and examined the hypersurfaces of this metric as a hyperplane of first, second and third kinds. Further we obtained the condition under which this hypersurface will be C2-like Finsler space and the value of Main Scalar I in two-Dimensional frame of Finsler Space.

2. Preliminaries

In 1998 Matsumoto introduced the concept of Douglas type metric which was defined as

$$(1) \quad L = \alpha + \frac{\beta^2}{\alpha},$$

where $\alpha^2 = a_{ij}(x)y^i y^j$ is a Riemannian metric and $\beta = b_i(x)y^i$ one-form metric.

In 2004 Lee and Park [1] introduced a r -th series (α, β) -metric

$$(2) \quad L(\alpha, \beta) = \beta \sum_{k=0}^r \left(\frac{\alpha}{\beta} \right)^k,$$

where they assume $\alpha < \beta$. If we consider the above infinite series metric in the case of $\beta < \alpha$ and $r = \infty$ then the above metric becomes

$$(3) \quad L(\alpha, \beta) = \frac{\beta^2}{\alpha - \beta}$$

and the metric given by (3) named as special infinite series (α, β) -metric.

In the present paper we combine the metric (1) and (3) in a special manner and introduce a new combination of metric in following way

$$(4) \quad L(\alpha, \beta) = A_1 \left(\alpha + \frac{\beta^2}{\alpha} \right) + A_2 \left(\frac{\beta^2}{\alpha - \beta} \right),$$

where A_1 and A_2 are any constant. If $A_1 = 0$ then the above metric becomes special infinite series metric also if $A_2 = 0$ the above metric becomes Douglas type.

Now, differentiating equation (4) partially w.r.t α and β are given by

$$\begin{aligned} L_\alpha &= A_1 \left(1 - \frac{\beta^2}{\alpha^2} \right) + A_2 \left(-\frac{\beta^2}{(\alpha - \beta)^2} \right), & L_\beta &= A_1 \left(\frac{2\beta}{\alpha} \right) + A_2 \left(\frac{2\alpha\beta - \beta^2}{(\alpha - \beta)^2} \right), \\ L_{\alpha\alpha} &= A_1 \left(\frac{2\beta^2}{\alpha^3} \right) + A_2 \left(\frac{2\beta^2}{(\alpha - \beta)^3} \right), & L_{\beta\beta} &= A_1 \left(\frac{2}{\alpha} \right) + A_2 \left(\frac{2\alpha^2}{(\alpha - \beta)^3} \right) \\ L_{\alpha\beta} &= A_1 \left(\frac{-2\beta}{\alpha^2} \right) + A_2 \left(\frac{-2\alpha\beta}{(\alpha - \beta)^3} \right), \end{aligned}$$

where $L_\alpha = \frac{\partial L}{\partial \alpha}$, $L_\beta = \frac{\partial L}{\partial \beta}$, $L_{\alpha\alpha} = \frac{\partial L_\alpha}{\partial \alpha}$, $L_{\beta\beta} = \frac{\partial L_\beta}{\partial \beta}$, $L_{\alpha\beta} = \frac{\partial L_\alpha}{\partial \beta}$.

In the Finsler space $F^n = \{M^n, L(\alpha, \beta)\}$ the normalized element of support $l_i = \dot{\partial}_i L$ and angular metric tensor h_{ij} are given by [5]:

$$l_i = \alpha^{-1} L_\alpha Y_i + L_\beta b_i,$$

$$h_{ij} = p a_{ij} + q_0 b_i b_j + q_{-1} (b_i Y_j + b_j Y_i) + q_{-2} Y_i Y_j,$$

where $Y_i = a_{ij} y^j$.

For the fundamental function (4) above constants are

$$\begin{aligned} p &= L L_\alpha \alpha^{-1} = A_1^2 \frac{(\alpha^4 - \beta^4)}{\alpha^4} - A_1 A_2 \frac{(\alpha + \beta) \beta^3}{\alpha^2 (\alpha - \beta)^2} - A_2^2 \frac{\beta^4}{\alpha (\alpha - \beta)^3}, \\ q_0 &= L L_{\beta\beta} = A_1^2 \frac{(2\alpha^2 + 2\beta^2)}{\alpha^2} + A_1 A_2 \frac{(\alpha^2 + \beta^2)^2 - 4\alpha\beta^3}{\alpha (\alpha - \beta)^3} + A_2^2 \frac{2\alpha^2 \beta^2}{(\alpha - \beta)^4}, \\ q_{-1} &= L L_{\alpha\beta} \alpha^{-1} = -A_1^2 \frac{(2\alpha^2 + 2\beta^2) \beta}{\alpha^4} - A_2^2 \frac{2\beta^3}{(\alpha - \beta)^4} \\ (5) \quad &- A_1 A_2 \frac{(2\alpha^4 \beta + 4\alpha^2 \beta^3 + 2\beta^5 - 4\alpha\beta^4)}{\alpha^3 (\alpha - \beta)^3}, \\ q_{-2} &= L \alpha^{-2} (L_{\alpha\alpha} - L_\alpha \alpha^{-1}) = A_1^2 \frac{(2\alpha^2 \beta^2 + 3\beta^4 - \alpha^4)}{\alpha^6} + A_2^2 \frac{(3\alpha - \beta) \beta^4}{\alpha^3 (\alpha - \beta)^4} \\ &- A_1 A_2 \frac{(2\alpha^4 - 7\alpha\beta^3 + 2\alpha^3 \beta + 3\beta^4) \beta^2}{\alpha^5 (\alpha - \beta)^3} \end{aligned}$$

Fundamental metric tensor $g_{ij} = \frac{1}{2} \dot{\partial}_i \dot{\partial}_j L^2$ and its reciprocal tensor g^{ij} for $L = L(\alpha, \beta)$ are given by [4, 5]

$$(6) \quad g_{ij} = p a_{ij} + p_0 b_i b_j + p_{-1} (b_i Y_j + b_j Y_i) + p_{-2} Y_i Y_j,$$

where

$$(7) \quad p_0 = q_0 + L_\beta^2, \quad p_{-1} = q_{-1} + L^{-1} p L_\beta, \quad p_{-2} = q_{-2} + p^2 L^{-2}.$$

The reciprocal tensor g^{ij} of g_{ij} is given by

$$(8) \quad g^{ij} = p^{-1} a^{ij} - s_0 b^i b^j - s_{-1} (b^i y^j + b^j y^i) - s_{-2} y^i y^j$$

where $b^i = a^{ij} b_j$ and $b^2 = a_{ij} b^i b^j$

$$\begin{aligned} s_0 &= \frac{1}{\tau p} \{p p_0 + (p_0 p_{-2} - p_{-1}^2) \alpha^2\}, \\ s_{-1} &= \frac{1}{\tau p} \{p p_{-1} + (p_0 p_{-2} - p_{-1}^2) \beta\}, \\ (9) \quad s_{-2} &= \frac{1}{\tau p} \{p p_{-2} + (p_0 p_{-2} - p_{-1}^2) b^2\}, \\ \tau &= p(p + p_0 b^2 + p_{-1} \beta) + (p_0 p_{-2} - p_{-1}^2) (\alpha^2 b^2 - \beta^2). \end{aligned}$$

The hv-torsion tensor $C_{ijk} = \frac{1}{2}\dot{\partial}_k g_{ij}$ is given by [9]

$$(10) \quad 2pC_{ijk} = p_{-1}(h_{ij}m_k + h_{jk}m_i + h_{ki}m_j) + \gamma_1 m_i m_j m_k,$$

where

$$(11) \quad \gamma_1 = p \frac{\partial p_0}{\partial \beta} - 3p_{-1}q_0, \quad m_i = b_i - \alpha^{-2}\beta Y_i.$$

Here m_i is a non-vanishing covariant vector orthogonal to the element of support y^i .

Let $\{\overset{i}{j}{}_k\}$ be the component of christoffel symbols of the associated Riemannian space R^n and ∇_k be the covariant derivative with respect to x^k relative to this christoffel symbol. Now we define,

$$(12) \quad 2E_{ij} = b_{ij} + b_{ji}, \quad 2F_{ij} = b_{ij} - b_{ji},$$

where $b_{ij} = \nabla_j b_i$.

Let $C\Gamma = (\Gamma_{jk}^{*i}, \Gamma_{0k}^{*i}, \Gamma_{jk}^i)$ be the cartan connection of F^n . The difference tensor $D_{jk}^i = \Gamma_{jk}^{*i} - \{\overset{i}{j}{}_k\}$ of the special Finsler space F^n is given by

$$(13) \quad \begin{aligned} D_{jk}^i &= B^i E_{jk} + F_k^i B_j + F_j^i B_k + B_j^i b_{0k} + B_k^i b_{0j} - b_{0m} g^{im} B_{jk} \\ &\quad - C_{jm}^i A_k^m - C_{km}^i A_j^m + C_{jkm} A_s^m g^{is} + \lambda^s (C_{jm}^i C_{sk}^m \\ &\quad + C_{km}^i C_{sj}^m - C_{jk}^m C_{ms}^i), \end{aligned}$$

where

$$(14) \quad \begin{aligned} B_k &= p_0 b_k + p_{-1} Y_k, \quad B^i = g^{ij} B_j, \quad F_i^k = g^{kj} F_{ji}, \\ B_{ij} &= \frac{\{p_{-1}(a_{ij} - \alpha^{-2} Y_i Y_j) + \frac{\partial p_0}{\partial \beta} m_i m_j\}}{2}, \quad B_i^k = g^{kj} B_{ji}, \\ A_k^m &= B_k^m E_{00} + B^m E_{k0} + B_k F_0^m + B_0 F_k^m, \\ \lambda^m &= B^m E_{00} + 2B_0 F_0^m, \quad B_0 = B_i y^i, \end{aligned}$$

where 0 denote contraction with y^i except for the quantities p_0, q_0 and s_0 .

3. Induced Cartan connection

Let F^{n-1} be a hypersurface of F^n given by the equation $x^i = x^i(u^\alpha)$ (where $\alpha = 1, 2, 3, \dots, (n-1)$). The element of support y^i of F^n is to be taken tangential to F^{n-1} , that is [6],

$$(15) \quad y^i = B_\alpha^i(u) v^\alpha$$

the metric tensor $g_{\alpha\beta}$ and hv-tensor $C_{\alpha\beta\gamma}$ of F^{n-1} are given by

$$g_{\alpha\beta} = g_{ij} B_\alpha^i B_\beta^j, \quad C_{\alpha\beta\gamma} = C_{ijk} B_\alpha^i B_\beta^j B_\gamma^k$$

and at each point (u^α) of F^{n-1} , a unit normal vector $N^i(u, v)$ is defined by

$$g_{ij}\{x(u, v), y(u, v)\}B_\alpha^i N^j = 0, \quad g_{ij}\{x(u, v), y(u, v)\}N^i N^j = 1.$$

Angular metric tensor $h_{\alpha\beta}$ of the hypersurface are given by

$$(16) \quad h_{\alpha\beta} = h_{ij}B_\alpha^i B_\beta^j, \quad h_{ij}B_\alpha^i N^j = 0, \quad h_{ij}N^i N^j = 1,$$

(B_i^α, N_i) inverse of (B_α^i, N^i) is given by

$$\begin{aligned} B_i^\alpha &= g^{\alpha\beta} g_{ij} B_\beta^j, & B_\alpha^i B_i^\beta &= \delta_\alpha^\beta, & B_i^\alpha N^i &= 0, & B_\alpha^i N_i &= 0, \\ N_i &= g_{ij} N^j, & B_i^k &= g^{kj} B_{ji}, & B_\alpha^i B_j^\alpha + N^i N_j &= \delta_j^i. \end{aligned}$$

The induced connection $ICT = (\Gamma_{\beta\gamma}^{\alpha}, G_\beta^\alpha, C_{\beta\gamma}^\alpha)$ of F^{n-1} from the Cartan's connection $CT = (\Gamma_{jk}^{*i}, \Gamma_{0k}^{*i}, C_{jk}^{*i})$ is given by [6].

$$\begin{aligned} \Gamma_{\beta\gamma}^{\alpha} &= B_i^\alpha (B_{\beta\gamma}^i + \Gamma_{jk}^{*i} B_\beta^j B_\gamma^k) + M_\beta^\alpha H_\gamma, \\ G_\beta^\alpha &= B_i^\alpha (B_{0\beta}^i + \Gamma_{0j}^{*i} B_\beta^j), & C_{\beta\gamma}^\alpha &= B_i^\alpha C_{jk}^i B_\beta^j B_\gamma^k, \end{aligned}$$

where

$$M_{\beta\gamma} = N_i C_{jk}^i B_\beta^j B_\gamma^k, \quad M_\beta^\alpha = g^{\alpha\gamma} M_{\beta\gamma}, \quad H_\beta = N_i (B_{0\beta}^i + \Gamma_{0j}^{*i} B_\beta^j)$$

and

$$B_{\beta\gamma}^i = \frac{\partial B_\beta^i}{\partial u^\gamma}, \quad B_{0\beta}^i = B_{\alpha\beta}^i v^\alpha.$$

The quantities $M_{\beta\gamma}$ and H_β are called the second fundamental v-tensor and normal curvature vector respectively [6]. The second fundamental h-tensor $H_{\beta\gamma}$ is defined as [6]

$$(17) \quad H_{\beta\gamma} = N_i (B_{\beta\gamma}^i + \Gamma_{jk}^{*i} B_\beta^j B_\gamma^k) + M_\beta H_\gamma,$$

where

$$(18) \quad M_\beta = N_i C_{jk}^i B_\beta^j N^k.$$

The relative h and v-covariant derivatives of projection factor B_α^i with respect to ICT are given by

$$B_{\alpha|\beta}^i = H_{\alpha\beta} N^i, \quad B_\alpha^i|_\beta = M_{\alpha\beta} N^i.$$

It is obvious from the equation (17) that $H_{\beta\gamma}$ is generally not symmetric and

$$(19) \quad H_{\beta\gamma} - H_{\gamma\beta} = M_\beta H_\gamma - M_\gamma H_\beta.$$

The above equation yield

$$(20) \quad H_{0\gamma} = H_\gamma, \quad H_{\gamma 0} = H_\gamma + M_\gamma H_0.$$

We shall use following lemmas which are due to Matsumoto [6] in the coming section

Lemma 3.1. *The normal curvature $H_0 = H_\beta v^\beta$ vanishes if and only if the normal curvature vector H_β vanishes.*

Lemma 3.2. *A hypersurface $F^{(n-1)}$ is a hyperplane of the first kind with respect to connection CT if and only if $H_\alpha = 0$.*

Lemma 3.3. *A hypersurface $F^{(n-1)}$ is a hyperplane of the second kind with respect to connection CT if and only if $H_\alpha = 0$ and $H_{\alpha\beta} = 0$.*

Lemma 3.4. *A hypersurface $F^{(n-1)}$ is a hyperplane of the third kind with respect to connection CT if and only if $H_\alpha = 0$ and $H_{\alpha\beta} = M_{\alpha\beta} = 0$.*

4. Hypersurface $F^{(n-1)}(c)$ of a special Finsler space

Let us consider a Finsler space with the metric $L(\alpha, \beta) = A_1(\alpha + \frac{\beta^2}{\alpha}) + A_2(\frac{\beta^2}{\alpha - \beta})$, where, vector field $b_i(x) = \frac{\partial b}{\partial x^i}$ is a gradient of some scalar function $b(x)$. Now, we consider a hypersurface $F^{(n-1)}(c)$ given by equation $b(x) = c$, a constant [9].

From the parametric equation $x^i = x^i(u^\alpha)$ of $F^{n-1}(c)$, we get

$$\frac{\partial b(x)}{\partial u^\alpha} = 0, \quad \frac{\partial b(x)}{\partial x^i} \frac{\partial x^i}{\partial u^\alpha} = 0, \quad b_i B_\alpha^i = 0.$$

Above shows that $b_i(x)$ are covariant component of a normal vector field of hypersurface $F^{n-1}(c)$. Further, we have

$$(21) \quad b_i B_\alpha^i = 0 \quad \text{and} \quad b_i y^i = 0 \quad \text{i.e.} \quad \beta = 0$$

and induced metric $L(u, v)$ of $F^{n-1}(c)$ is given by

$$(22) \quad L(u, v) = a_{\alpha\beta} v^\alpha v^\beta, \quad a_{\alpha\beta} = a_{ij} B_\alpha^i B_\beta^j$$

which is a Riemannian metric.

Writing $\beta = 0$ in the equations (6), (7) and (9) we get

$$(23) \quad \begin{aligned} p &= A_1^2, \quad q_0 = 2(A_1^2 + A_1 A_2), \quad q_{-1} = 0 \quad q_{-2} = -A_1^2 \alpha^{-2}, \\ p_0 &= 2(A_1^2 + A_1 A_2) \quad p_{-1} = 0 \quad p_{-2} = 0 \quad \tau = A_1^3 (3A_1 + 2A_2) b^2, \\ s_0 &= \frac{2(A_1 + A_2)}{A_1^2 (3A_1 + 2A_2) b^2} \quad s_{-1} = 0 \quad s_{-2} = 0, \end{aligned}$$

from (8) we get,

$$(24) \quad g^{ij} = \frac{1}{A_1^2} a^{ij} - \frac{2(A_1 + A_2)}{A_1^2 (3A_1 + 2A_2) b^2} b^i b^j$$

thus, along $F^{n-1}(c)$, (24) and (21) leads to

$$g^{ij} b_i b_j = \frac{b^2}{A_1 (3A_1 + 2A_2)}.$$

So, we get

$$(25) \quad b_i(x(u)) = \sqrt{\frac{b^2}{A_1(3A_1 + 2A_2)}} N_i, \quad b^2 = a^{ij} b_i b_j,$$

where b is the length of the vector b^i .

Again from (24) and (25), we get

$$(26) \quad b^i = a^{ij} b_j = A_1(3A_1 + 2A_2) N^i$$

thus, we have

Theorem 4.1. *In a special Finsler hypersurface $F^{(n-1)}(c)$, the Induced Riemannian metric is given by (22) and the scalar function $b(x)$ is given by (25) and (26).*

Now, the angular metric tensor h_{ij} and metric tensor g_{ij} of F^n are given by

$$(27) \quad h_{ij} = A_1^2 a_{ij} + 2A_1(A_1 + A_2) b_i b_j - \frac{A_1^2}{\alpha^2} Y_i Y_j$$

and

$$g_{ij} = A_1^2 a_{ij} + 2A_1(A_1 + A_2) b_i b_j.$$

From equation (21), (27) and (16) it follows that if $h_{\alpha\beta}^{(a)}$ denote the angular metric tensor of the Riemannian $a_{ij}(x)$ then we have along $F_{(c)}^{n-1}$, $h_{\alpha\beta} = h_{\alpha\beta}^{(a)}$.

Thus, along $F_{(c)}^{n-1}$, $\frac{\partial p_0}{\partial \beta} = \frac{6A_1 A_2}{\alpha}$ from equation (12) we get $r_1 = \frac{6A_1^3 A_2}{\alpha}$, $m_i = b_i$ then hv-torsion tensor becomes

$$(28) \quad C_{ijk} = \frac{3A_1 A_2}{\alpha} b_i b_j b_k$$

in the special Finsler hypersurface $F_{(c)}^{(n-1)}$. Due to fact from (16), (17), (19), (21) and (28) we have

$$(29) \quad M_{\alpha\beta} = 0, \quad M_\alpha = 0.$$

Therefore, from equation (20) it follows that $H_{\alpha\beta}$ is symmetric. Thus we have

Theorem 4.2. *The second fundamental v-tensor of the special Finsler hypersurface $F_{(c)}^{(n-1)}$ is given by (29) and the second fundamental h-tensor $H_{\alpha\beta}$ is symmetric.*

Now, from (21) we have $b_i B_\alpha^i = 0$. Then we have $b_{i|\beta} B_\alpha^i + b_i B_{\alpha|\beta}^i = 0$. Therefore, from (19) and using $b_{i|\beta} = b_{i|j} B_\beta^j + b_i|_j N^j H_\beta$, we have

$$(30) \quad b_{i|j} B_\alpha^i B_\beta^j + b_{i|j} B_\alpha^i N^j H_\beta + b_i H_{\alpha\beta} N^i = 0$$

since $b_i|_j = -b_h C_{ij}^h$, we get $b_i|_j B_\alpha^i N^j = 0$. Therefore, from equation (30), we have

$$(31) \quad \sqrt{\frac{b^2}{A_1(3A_1 + 2A_2)}} H_{\alpha\beta} + b_i|_j B_\alpha^i B_\beta^j = 0,$$

because $b_i|_j$ is symmetric. Now, contracting (31) with v^β and using (15) we get

$$(32) \quad \sqrt{\frac{b^2}{A_1(3A_1 + 2A_2)}} H_\alpha + b_i|_j B_\alpha^i y^j = 0.$$

Again, contracting by v^α equation (32) and using (15), we have

$$(33) \quad \sqrt{\frac{b^2}{A_1(3A_1 + 2A_2)}} H_0 + b_i|_j y^i y^j = 0.$$

From Lemma 3.1 and 3.2, it is clear that the hypersurface $F_{(c)}^{(n-1)}$ is a hyperplane of first kind if and only if $H_0 = 0$. Thus from (33) it is obvious that $F_{(c)}^{n-1}$ is a hyperplane of first kind if and only if $b_i|_j y^i y^j = 0$. This $b_i|_j$ being the covariant derivative with respect to CT of F^n defined on y^i , but $b_{ij} = \nabla_j b_i$ is the covariant derivative with respect to Riemannian connection $\{_{jk}^i\}$ constructed from $a_{ij}(x)$. Hence b_{ij} does not depend on y^i . We shall consider the difference $b_i|_j - b_{ij}$ where $b_{ij} = \nabla_j b_i$ in the following. The difference tensor $D_{jk}^i = \Gamma_{jk}^{*i} - \{_{jk}^i\}$ is given by (14). Since b_i is a gradient vector, then from (13) we have $E_{ij} = b_{ij}$, $F_{ij} = 0$ and $F_j^i = 0$. Thus, (14) reduces to

$$(34) \quad \begin{aligned} D_{jk}^i &= B^i b_{jk} + B_j^i b_{0k} + B_k^i b_{0j} - b_{0m} g^{im} B_{jk} - C_{jm}^i A_k^m \\ &\quad - C_{km}^i A_j^m + C_{jkm} A_s^m g^{is} + \lambda^s (C_{jm}^i C_{sk}^m + C_{km}^i C_{sj}^m - C_{jk}^m C_{ms}^i), \end{aligned}$$

where

$$(35) \quad \begin{aligned} B_i &= 2A_1(A_1 + A_2)b_i, \quad B^i = \frac{2(A_1 + A_2)}{(3A_1 + 2A_2)}b^i, \\ \lambda^m &= B^m b_{00}, \quad B_{ij} = \frac{3A_1 A_2}{\alpha} b_i b_j, \\ B_j^i &= \frac{3A_2}{\alpha(3A_1 + A_2)} b^i b_j, \\ A_k^m &= B_k^m b_{00} + B^m b_{k0}. \end{aligned}$$

In view of (23) and (24), the relation in (15) becomes to by virtue of (35) we have $B_0^i = 0$, $B_{i0} = 0$ which leads $A_0^m = B^m b_{00}$.

Now, contracting (34) by y^k we get

$$D_{j0}^i = B^i b_{j0} + B_j^i b_{00} - B^m C_{jm}^i b_{00}.$$

Again, contracting the above equation with respect to y^j we have

$$D_{00}^i = B^i b_{00} = \frac{2(A_1 + A_2)}{(3A_1 + 2A_2)} b^2 b_{00}.$$

Paying attention to (21), along $F_{(c)}^{(n-1)}$, we get

$$(36) \quad \begin{aligned} b_i D_{j0}^i &= \frac{2(A_1 + A_2)}{(3A_1 + 2A_2)} b^2 b_{j0} + \frac{3A_2 b^2}{\alpha(3A_1 + 2A_2)} b_j b_{00} \\ &- \frac{2(A_1 + A_2)}{(3A_1 + 2A_2)} b_i b^m C_{jm}^i b_{00}. \end{aligned}$$

Now, we contract (36) by y^j we have

$$(37) \quad b_i D_{00}^i = \frac{2(A_1 + A_2)}{(3A_1 + 2A_2)} b^2 b_{00}.$$

From (17), (25), (26), (29) and $M_\alpha = 0$, we have

$$b_i b^m C_{jm}^i B_\alpha^j = b^2 M_\alpha = 0.$$

Thus, the relation $b_{i|j} = b_{ij} - b_r D_{ij}^r$ the equation (36) and (37) gives

$$b_{i|j} y^i y^j = b_{00} - b_r D_{00}^r = \left\{ \frac{A_1(3 - 2b^2) + 2A_2(1 - b^2)}{(3A_1 + 2A_2)} \right\} b_{00}.$$

Consequently (32) and (33) may be written as

$$(38) \quad \begin{aligned} \sqrt{\frac{b^2}{A_1(3A_1 + 2A_2)}} H_\alpha + \left\{ \frac{A_1(3 - 2b^2) + 2A_2(1 - b^2)}{(3A_1 + 2A_2)} \right\} b_{i0} B_\alpha^i &= 0, \\ \sqrt{\frac{b^2}{A_1(3A_1 + 2A_2)}} H_0 + \left\{ \frac{A_1(3 - 2b^2) + 2A_2(1 - b^2)}{(3A_1 + 2A_2)} \right\} b_{00} &= 0. \end{aligned}$$

Thus, the condition $H_0 = 0$ is equivalent to $b_{00} = 0$. Using the fact $\beta = b_i y^i = 0$ the condition $b_{00} = 0$ can be written as $b_{ij} y^i y^j = b_i y^i b_j y^j$ for some $c_j(x)$. Thus, we can write

$$(39) \quad 2b_{ij} = b_i c_j + b_j c_i.$$

Now, from (21) and (39) we get $b_{00} = 0$, $b_{ij} B_\alpha^i B_\beta^j = 0$, $b_{ij} B_\alpha^i y^j = 0$. Hence, from (38) we get $H_\alpha = 0$, again from (39) and (35) we get $b_{i0} b^i = \frac{c_0 b^2}{2}$, $\lambda^m = 0$, $A_j^i B_\beta^j = 0$ and $B_{ij} B_\alpha^i B_\beta^j = 0$.

Now, we use equation (17), (24), (25), (26), (29) and (34) then, we have

$$(40) \quad b_r D_{ij}^r B_\alpha^i B_\beta^j = 0.$$

Thus, the equation (31) reduces to

$$(41) \quad \sqrt{\frac{b^2}{A_1(3A_1 + 2A_2)}} H_{\alpha\beta} = 0.$$

Hence, the hypersurface $F_{(c)}^{n-1}$ is umbilic.

Theorem 4.3. *The necessary and sufficient condition for in a Finslerian hypersurface $F^{(n-1)}(c)$ of the Finsler space F^n equipped with a special (α, β) -metric defined in equation (4) to be a hyperplane of first kind is (39).*

Now, from Lemma 3.3, $F_{(c)}^{(n-1)}$ is a hyperplane of second kind if and only if $H_\alpha = 0$ and $H_{\alpha\beta} = 0$. Thus from (40), we get $c_0 = c_i(x)y^i = 0$. Therefore, there exist a function $\psi(x)$ such that $c_i(x) = \psi(x)b_i(x)$. Therefore (39) we get $2b_{ij} = b_i(x)\psi(x)b_j(x) + b_j(x)\psi(x)b_i(x)$. This can also be written as $b_{ij} = \psi(x)b_ib_j$.

Theorem 4.4. *The necessary and sufficient condition for in a Finslerian hypersurface $F^{(n-1)}(c)$ of the Finsler space F^n equipped with a special (α, β) -metric defined in equation (4) to be a hyperplane of second kind is (41).*

Again Lemma 3.4, together with (29) and $M_\alpha = 0$ shows that $F_{(c)}^{n-1}$ become a hyperplane of third kind.

Theorem 4.5. *The necessary and sufficient condition for in a Finslerian hypersurface $F^{(n-1)}(c)$ of the Finsler space F^n equipped with a special (α, β) -metric defined in equation (4) is a hyperplane of the third kind.*

5. Some important result of hypersurface $F^{(n-1)}(c)$ of a Finsler space $F^n(c)$ with metric $A_1(\alpha + \frac{\beta^2}{\alpha}) + A_2(\frac{\beta^2}{\alpha - \beta})$

The hv-torsion tensor C_{ijk} of $F^{(n-1)}(c)$ with deformed infinite series metric written in equation (28) as

$$C_{ijk} = \frac{3A_1A_2}{\alpha} b_ib_jb_k.$$

Contracting by g^{jk} , we have

$$C_i = C_{ijk}g^{jk} = \frac{3A_1A_2}{\alpha} b^2b_i.$$

This implies that $b_i = \frac{\alpha}{3A_1A_2b^2} C_i$. Therefore, equation (28) becomes

$$(42) \quad C_{ijk} = \frac{\alpha^2}{9A_1^2A_2^2b^6} C_iC_jC_k.$$

Definition 5.1. *A Finsler space F^n is called C2-like, if the (h) hv-torsion tensor C_{ijk} is written in the form $C_{ijk} = \frac{1}{C^2} C_iC_jC_k$.*

Thus, using the above definition and equation (42) we have

$$(43) \quad C = \frac{3A_1 A_2 b^3}{\alpha}.$$

Thus

Proposition 5.1. *The Finslerian hypersurface $F^{(n-1)}(c)$ of the Finsler space F^n equipped with a special (α, β) -metric defined in equation (4) is always be a C2-like Finsler space if equation (43) is satisfied.*

Since the main scalar of two dimensional Finsler space is defined as $LC_{ijk} = Im_i m_j m_k$. Since $m_i = b_i$ then we have $LC_{ijk} = Ib_i b_j b_k$. Contracting g^{jk} we have $LC_i = Ib^2 b_i$ which gives $b_i = \frac{L}{b^2} C_i$.

Now, the main scalar of two dimensional Finsler

$$(44) \quad LC_{ijk} = \frac{IL^3}{b^6} C_i C_j C_k.$$

From equation (5.1) and (5.3), we have

$$(45) \quad I = \frac{\alpha^2}{9A_1 A_2 L^2}.$$

Proposition 5.2. *The main scalar I of a Finslerian hypersurface $F^{(n-1)}(c)$ for the Finsler space F^n equipped with a special (α, β) -metric defined in equation (4) in a two dimensional case is given by (45).*

6. Conclusion

In the present paper we have combined Matsumoto douglas type Finsler metric and special Infinite series metric and introduced a special Finsler metric (4) with certain scalar A_1 and A_2 . Further we obtained the necessary and sufficient condition for a Finslerian Hypersurface $F^{(n-1)}(c)$ of a Finsler space F^n equipped with a special (α, β) metric will be hyperplane of first, second and third kind in the Theorem 4.3, 4.4 and 4.5 respectively. Further as application point of view for the above deformed metric we obtain a Proposition 5.1 in which it stated that this hypersurface will be C2-like Finsler space and in propostion (5.2) the value of main scalar I in two-dimensional frame of Finsler Space is obtained.

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A note on conjugacy degrees of a finite group

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Abstract. Let G be a group, $g_1, g_2, \dots, g_k$ are a complete set of representatives for the conjugacy classes of G . Then

$$k(G) = \frac{1}{|G|^2} \sum_{i=1}^k |(g_i)^G|^2 = \sum_{i=1}^k \frac{1}{|C_G(g_i)|^2}$$

is called the conjugate degree of G (see [6]). In this short paper, we investigate the influence of the conjugacy degree of G on the structure of finite groups. The authors get the formulas of the conjugacy degree of the dihedral group and the generalized quaternion group and classify those groups G such that $k(G) \geq \frac{1}{3}$.

Keywords: conjugacy degree, dihedral group, generalized quaternion group.

1. Introduction

All groups considered in this paper are finite and G always denotes a group. Let $\pi(|G|)$ stand for the set of all prime divisors of the order of G . The symbol $[A]B$ denotes the semidirect product of the groups A and B , where B is an operator group of A . Let C_n be the cyclic group of order n , where n is a positive integer. The other notions and notations are standard, as in [1].

In the past few years, there has been a growing interest in the application of probability in finite group theory. One of the most important aspects that have

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been studied is the probability that two elements of a finite group G commute. In 1979, D.J. Rusin [2] introduced the commutativity degree of G denoted by $d(G)$.

$$(*) \quad d(G) = \frac{1}{|G|^2} |\{(x, y) \in (G, G) | xy = yx\}|.$$

Evidently, G is an abelian group if and only if $d(G) = 1$, and the author also get, if $\frac{1}{2} < d(G) < 1$, then $d(G) \in \{\frac{1}{2}(1 + \frac{1}{4^n}) | n \in \mathbf{N}, n \geq 1\}$. Later, P. Lescot continues to research on this topic, and get the following results. If G is non abelian, then $d(G) \leq \frac{5}{8}$; if $d(G) > \frac{1}{2}$, then G is nilpotent (see [5]). If $d(G) = \frac{1}{2}$, and G is not nilpotent, then $G/Z(G) \cong \sum_3$ and $G' \cong Z_3$ (see [4]).

As we know, the elements of any group may be partitioned into conjugacy classes, so the conjugacy relation plays an important role in group theory. In [6], S. Blackburn, J. Britnell and M. Wildon replace ‘community of elements’ with ‘conjugation of elements’, introduce the concept of the probability that a pair of elements of a finite group are conjugate in 2012, we call it conjugate degree of G , denoted by

$$k(G) = \frac{1}{|G|^2} |\{(x, y) \in (G, G) | x \sim y\}| = \frac{1}{|G|^2} \sum_{i=1}^k |(g_i)^G|^2 = \sum_{i=1}^k \frac{1}{|C_G(g_i)|^2},$$

where $\sim$ is the conjugacy relation, $g_1, g_2, \dots, g_k$ are a complete set of representatives for the conjugacy classes of G . Using this concept, the authors investigate the structure of finite groups, and show that G is abelian whenever $k(G)|G| < \frac{7}{4}$. Specializing to the symmetric group S_n , they show that $k(S_n) \leq C/n^2$ for an explicitly determined constant C . This bound leads to an elementary proof of a result of Flajolet et al. (see [3]), that $k(S_n) \sim A/n^2$ as $n \rightarrow \infty$ for some constant A . In this paper, we continue to research on this topic. We investigate the influence of the conjugacy degree of G on the structure of finite groups, and get the formulas of the conjugacy degree of the dihedral group and the generalized quaternion group and classify those groups G such that $k(G) \geq \frac{1}{3}$.

2. Preliminaries

For the same of convenience, we list here some known results which will be useful in the sequel.

Lemma 2.1 ([6, Proposition 4.2]). *Suppose that G is a finite group and that $t \in G$ is a self-centralizing involution. Then $k(G) = 1/4 + 1/|G| - 1/|G|^2$ and G has a normal abelian subgroup A of odd order such that $|G : A| = 2$.*

To state our following lemma, we shall need the majorization (or dominance) order, denoted by $\succsim$, which is defined on R^k by setting

$$(x_1, x_2, \dots, x_k) \succsim (y_1, y_2, \dots, y_k),$$

if and only if $\sum_{i=1}^j x_i \geq \sum_{i=1}^j y_i$ for all j such that $1 \leq j \leq k$.

Lemma 2.2 ([6, Lemma 3.3]). *Let $x, y \in R^k$ be decreasing k -tuples of real numbers such that $\sum_{i=1}^k x_i = \sum_{i=1}^k y_i = 1$. Suppose that $x \succ y$. Then $\sum_{i=1}^k x_i^2 \geq \sum_{i=1}^k y_i^2$, and equality holds if and only if $x = y$.*

3. Main results

The dihedral group D_{2n} ($n \geq 2$) is the symmetry group of a regular polygon with n sides and it has the order $2n$. The most convenient abstract description of D_{2n} is obtained by using its generators: a rotation a of order n and a reflection b of order 2. Under these notations, we have

$$D_{2n} = \langle a, b | a^n = b^2 = 1, bab = a^{-1} \rangle = [\langle a \rangle] \langle b \rangle.$$

Proposition 3.1. *The conjugacy degree $k(D_{2n})$ of the dihedral group D_{2n} is given by the following equality:*

$$k(D_{2n}) = \begin{cases} \frac{n^2 + 4n - 4}{8n^2}, & n \text{ is even}; \\ \frac{n^2 + 2n - 1}{4n^2}, & n \text{ is odd}. \end{cases}$$

Proof. By the definition of the dihedral group D_{2n} , we have that $D_{2n} = \{e, a, \dots, a^{n-1}, b, ba, \dots, ba^{n-1}\}$.

Case 1. n is even. Then $\frac{n}{2}$ is integer, $Z(D_{2n}) = \{e, a^{\frac{n}{2}}\}$. Observing that each element of the center $Z(D_{2n})$ forms a conjugacy class containing just itself. And $\{a^i, (a^i)^{-1}\}$ forms a conjugate class, where $i < n$ and $i \neq 0, \frac{n}{2}$, so there are $\frac{n-2}{2}$ conjugate classes of this type. Since $a^n = b^2 = 1, bab = a^{-1}$, we have that $ba = a^{-1}ba = baa = ba^2$, $(ba^2)^a = a^{-1}(ba^2)a = ba^4$, $\dots$, $(ba^{n-2})^a = a^{-1}(ba^{n-2})a = ba^n = b$; And $(ba)^a = a^{-1}(ba)a = baaa = ba^3$, $(ba^3)^a = a^{-1}(ba^3)a = ba^5$, $\dots$, $(ba^{n-1})^a = a^{-1}(ba^{n-1})a = ba$. So $\{b, ba^2, \dots, ba^{n-2}\}$ forms a conjugate class and $\{ba, ba^3, \dots, ba^{n-1}\}$ forms a conjugate class. By the definition of conjugacy degree of G , we get that $k(D_{2n}) = \frac{1 + 1 + \frac{n-2}{2} \times 2^2 + 2 \times (\frac{n}{2})^2}{(2n)^2} = \frac{n^2 + 4n - 4}{8n^2}$.

Case 2. n is odd. By the similar argument as in Case 1, $\{e\}$ forms a conjugate class; $\{a^i, (a^i)^{-1}\}$ forms a conjugate class, where $i < n$ and $i \neq 0$, thus there are $\frac{n-1}{2}$ conjugate classes of this type; And $\{b, ba, \dots, ba^{n-1}\}$ forms a conjugate class by n is odd; Thus by the definition of conjugacy degree of G , we get that $k(D_{2n}) = \frac{1 + \frac{n-1}{2} \times 2^2 + n^2}{(2n)^2} = \frac{n^2 + 2n - 1}{4n^2}$.

Therefore,

$$k(D_{2n}) = \begin{cases} \frac{n^2 + 4n - 4}{8n^2}, & n \text{ is even}; \\ \frac{n^2 + 2n - 1}{4n^2}, & n \text{ is odd}. \end{cases}$$

□

It is easy to check that $k(D_{2n}) = \frac{n^2 + 4n - 4}{8n^2}$ or $\frac{n^2 + 2n - 1}{4n^2}$, is a monotonic decreasing function. If n is even, then $\lim_{n \rightarrow \infty} k(D_{2n}) = \frac{1}{8}$, so $\frac{1}{8} < k(D_{2n}) \leq \frac{1}{4}$.

If n is odd and $n \geq 2$, then $\lim_{n \rightarrow \infty} k(D_{2n}) = \frac{1}{4}$, so $\frac{1}{4} < k(D_{2n}) \leq \frac{7}{18}$. Thus for any integer n , we have $\frac{1}{8} < k(D_{2n}) \leq \frac{7}{18}$.

We can also calculate the conjugate degree of D_{2n} by using GAP (see [7]), the following is calculation program.

```
gap > G := DihedralGroup(2n);
gap > CC := ConjugacyClasses(G);
gap > a := 0;;
gap > i := 1;;
gap > A := 0;;
gap > L := Length(CC);
gap > while i <= L do
> A := Size(Centralizer(CC[i]));
> a := a + 1/A^2;
> Print(a, " n");
> i := i + 1;
od;
```

Some authors define generalized quaternion group to be the same as dicyclic group.

$$\langle a, b | a^{2n} = 1, b^2 = a^n, b^{-1}ab = a^{-1} \rangle,$$

for some integer $n \geq 2$. This group is denoted Q_{4n} and has order $4n$.

Proposition 3.2. *The conjugacy degree $k(Q_{4n})$ of the generalized quaternion group Q_{4n} is given by the following equality:*

$$k(Q_{4n}) = \frac{n^2 + 2n - 1}{8n^2}.$$

Proof. By the definition of the generalized quaternion group Q_{4n} , we have that $Q_{4n} = \{e, a, \dots, a^{2n-1}, b, ab, \dots, a^{2n-1}b\}$. By the similar argument as in Proposition 3.1, $Z(Q_{4n}) = \{e, a^n\}$, then each element of the center $Z(Q_{4n})$ forms a conjugacy class containing just itself; $\{a^i, a^{2n-i}\}$ forms a conjugate class, where $0 < i < 2n$ and $i \neq n$, so there are $\frac{2n-2}{2}$ conjugate classes of this type; Since $ab = ba^{2n-1}$, $ba = a^{2n-1}b$, we get that $b^a = a^{-1}ba = a^{2n-2}b$, $(a^{2n-2}b)^a = a^{-1}(a^{2n-2}b)a = a^{2n-4}b$, $\dots$, $(a^2b)^a = a^{-1}(a^2b)a = b$. Thus $\{b, a^2b, \dots, a^{2n-2}b\}$ forms a conjugate class. And $(ab)^a = a^{-1}(ab)a = a^{2n-1}b$, $(a^{2n-1}b)^a = a^{-1}(a^{2n-1}b)a = a^{2n-3}b$, $\dots$, $(a^3b)^a = a^{-1}(a^3b)a = ab$. So $\{ab, a^3b, \dots, a^{2n-1}b\}$ forms a conjugate class. By the definition of conjugacy degree of G , we get that $k(Q_{4n}) = \frac{1+1+\frac{2n-2}{2} \times 2^2 + 2 \times n^2}{(4n)^2} = \frac{n^2+2n-1}{8n^2}$. $\square$

It is easy to check that $k(Q_{4n}) = \frac{n^2+2n-1}{8n^2}$ is a monotonic decreasing function. And $\lim_{n \rightarrow \infty} k(Q_{4n}) = \frac{1}{8}$, so $\frac{1}{8} < k(Q_{4n}) \leq \frac{7}{32}$ by $n \geq 2$.

Of course, we can also calculate the conjugate degree of Q_{4n} by using GAP (see [7]), the calculation program is similar to the program for calculating D_{2n} . In the calculation program of D_{2n} , we just only replace ' $G :=$

DihedralGroup($2n$)' with ' $G := \text{QuaternionGroup}(4n)$ ', then we can get what we want.

Now, we can get the following result by using the Propositions.

Theorem 3.1. *Let G be a non-trivial finite group. Then $k(G) \geq 1/3$ if and only if G is isomorphic to one of the following groups: C_2 , C_3 , D_6 and D_{10} .*

Proof. Let $c_i(G)$ be the size of the i th smallest centralizer in a finite group G , m be the number of conjugacy classes of G and let

$$r(G) = (1/c_1(G), 1/c_2(G), \dots, 1/c_m(G)).$$

Since the size of the i th largest conjugacy class of G is $|G|/c_i(G)$, we have that $\sum_{i=1}^m 1/c_i(G) = 1$.

If $c_1(G) > 2$, then we have that $(1/3, 1/3, 1/3) \succ r(G)$. Hence, either $(1/3, 1/3, 1/3) = r(G)$, that is, $|G| = 3$, $G \cong C_3$ or by Lemma 2.2, we have that

$$k(G) = \sum_{i=1}^m 1/c_i(G)^2 < 1/3.$$

So, $c_1(G) = 2$, that is, G contains an element t such that $|C_G(t)| = 2$. Since $\langle t \rangle \leq C_G(t)$, we get that $\langle t \rangle = C_G(t)$. Thus we have that t is a self-centralizing involution. By Lemma 2.1 and the hypothesis, we get that $k(G) = 1/4 + 1/|G| - 1/|G|^2 \geq 1/3$, that is, $|G|^2 - 12|G| + 12 \leq 0$, so $6 - 2\sqrt{6} \leq |G| \leq 6 + 2\sqrt{6}$. By Lemma 2.1, we get that $2 \in \pi(|G|)$ and $2 \notin \pi(|G|/2)$, so $|G| = 2, 6, 10$. It is clear that, if $|G| = 6$, then $G \cong C_6$ or D_6 . If $|G| = 10$, then $G \cong C_{10}$ or D_{10} . By Proposition 3.1, we get that $k(D_6) = 7/18 > 1/3$, $k(D_{10}) = 17/50 > 1/3$, while, $k(C_2) = 1/2$, $k(C_6) = 1/6$, $k(C_{10}) = 1/10$. By $k(G) \geq 1/3$, we get that G is isomorphic to one of C_2 , D_6 and D_{10} .

The proof of Theorem is completed. $\square$

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Stability aspects for linear Hamiltonian systems and its applications in the three body problem

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Abstract. This paper concerns stability and strong stability of the Lagrange equilateral solutions in the three body problem. For given three masses, there exists a family of periodic solutions for which each mass is at the vertex of an equilateral triangle and traveling along an elliptic Kepler orbit. By means of symmetry, we reduce the dimensions of our problem from 12 to 6 dimensions. After making a clever change of coordinates a 4 dimensional system is obtained. We show that this system is G-Hamiltonian where G can be found by the restriction of an invariant subspace to the linear system. We study the bifurcation of this system as parameter within the system are changed. In particular, we prove that the strong stability of this system at $e > 0$ is lost in two ways, through period doubling bifurcation, and through two $+1$ eigenvalues. We also prove that stability and instability regions occur in the circular case ($e = 0$) as the determinant of the characteristic polynomial of the system changes.

Keywords: Hamiltonian, N -body problem, periodic orbits.

1. Introduction

In this paper, some basic definitions and propositions from the Theory of Linear transformation finite dimensional spaces with indefinite metric are presented. An interesting example comes from Celestial Mechanics and the lagrangian equilateral solutions of the three body problem is discussed and stability type is investigated. The system we are considering is nonlinear, and the linearization of the vector field along the periodic solution gives rise to a linear system of Hamiltonian equations. In the circular case $e = 0$, this leads to an investigation of an equilibrium system of linear equations where as in the elliptic case $e > 0$, the linearized system is of time periodic type.

In this example, we focus on an analytic argument for stability and strong stability of the Lagrange equilateral triangle solutions in the three body problem. This is challenging since strong stability is not always preserved under

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parameter changes in this example. Moreover, the analytic calculation of the characteristic multipliers for the elliptic orbits requires computing the fundamental matrix solutions to the associated time-dependant linear system. This is difficult in general and typically requires the use of numerical methods. However, our techniques allow us to pinpoint the likely bifurcations which may occur in numerical simulation. Stability of the restricted three body problem was studied by Danby in [6] where numerical integration has been used throughout. He used the mass value μ and the eccentricity e as parameters to achieve the stability type. In fact, he proved that the elliptic orbits are linearly stable while the circular solutions are not. Roberts in [3, 4] has used similar techniques to prove linear stability in the three body problem. In [9, 11], the authors studied minimization of different types of Linear Hamiltonian systems which can be used to study the question of stability or instability of some periodic solutions. The authors in [7, 8, 10, 12] have discussed the Instability of symmetric periodic family of orbits together with techniques from symplectic geometry and in particular the Maslov index along with the minimization properties applied to preclude the existence of eigenvalues of the monodromy matrix on the unit circle. In this paper, we show that the four dimensional system is G-Hamiltonian where the matrix G will be found by the restriction of the symplectic form to an invariant subspace of the linear system. We then analyze the behavior of the roots of the characteristic polynomial of the monodromy matrix. We show that the strong stability of this system is lost in three ways, through period doubling bifurcation, through two $+1$ eigenvalues and through Krein collision. Our arguments give an analytical proof of the absence of strong stability in the first two cases when $e > 0$.

We also analyze the behavior of the roots of the characteristic polynomial of the G-Hamiltonian system and how they vary with the eccentricity e and its determinant. We show that as the determinant of the characteristic polynomial increases from 0 to $\frac{1}{4}$ at $e = 0$, the eigenvalues remain on the imaginary axis leading to stability. We prove this analytically by showing these roots are purely imaginary. Instability also occurs at $e = 0$ as the determinant of the characteristic polynomial passes through krein collision. We prove this analytically by showing the real part of these eigenvalues are non-zero.

The rest of the paper is organized as follows: In section 2, we present some propositions from the Theory of linear transformation of finite dimensional spaces with Indefinite metric. In section 3, we first reduce the dimensions of our problem to four degrees of freedom system and then change of coordinates will decouple our system into a 2 by 2 system and a 4 by 4 system. In section 4, a result about stability and strong stability of our decoupled system is discussed.

2. Some propositions from the theory of linear transformation of finite dimensional spaces with indefinite metric

In this section we present some basic definitions that are used frequently throughout this paper. Let G be a nonsingular Hermitian matrix (i.e $\det G \neq 0, G^* = G$) which is neither positive nor negative definite. the matrix G in many applications is equal to iJ , where i is the imaginary unit and J has the form

$$(1) \quad \mathbf{J} = \begin{pmatrix} 0 & -I_k \\ I_k & 0 \end{pmatrix}.$$

Throughout this paper, we assume that the matrix G is Hermitian. An important special case occurs when $G = iJ$, which is connected with the linear Hamiltonian equation

$$(2) \quad \frac{dx}{dt} = JH(t)x,$$

where x is a $2n$ column vector, $H(t)$ is the $n \times n$ matrix, and the matrix J is given in (1). Below we will show that the number $\langle e, e \rangle$ is real but unlike (e, e) it need not be positive, hence $\langle e, d \rangle$ is called an *indefinite scalar product*. Given a matrix $X \in \mathbb{C}^{n \times n}$, the matrix X^G defined by $\langle Xe, d \rangle = \langle e, X^G d \rangle$, (e, d) are arbitrary vectors in $\mathbb{C}^{n \times n}$ is said to be G -adjoint to X . The explicit formula for X^G is given by $X^G = G^{-1}X^*G$.

From here we consider the following definition:

Definition 2.1. A matrix $X \in \mathbb{C}^{n \times n}$ is called G -unitary if it satisfies the relation $X^G X = I$.

Definition 2.2. A matrix $Y \in \mathbb{C}^{n \times n}$ is called G -Hamiltonian (or G -anti-hermitian) if it satisfies the relation $Y^G = -Y$.

Each G -Hamiltonian Y has the representation $Y = i^{-1}GH$ with some $H = H^*$. For a G -Unitary matrix, we have $X = G^{-1}(X^{-1})^*G$, the set of all multipliers of X is symmetric with respect to the unit circle.

The following proposition, is very important for what follows and its proof is straightforward and left to the reader:

Proposition 2.1. The matrizant $X(t)$ of a Hamiltonian equation (2) is a G -unitary matrix for any t with $G = iJ$.

3. The Lagrangian equilateral triangle solutions and elimination of dimensions

We let the mass and position of the N bodies be given by m_i and $q_i \in \mathbb{R}^2$, respectively. Interaction between the masses is determined by the Newtonian potential function

$$(3) \quad U(q) = \sum_{i < j} \frac{m_i m_j}{\|q_j - q_i\|}.$$

on the set of non-collision configurations (where $q_i \neq q_j$, $i \neq j$). The Hamiltonian for the N-body problem is given as

$$(4) \quad H = \sum_i^N \frac{1}{2m_i} \|p_i\|^2 - U(q).$$

The Hamiltonian equations of the N-body problem are

$$(5) \quad \dot{q}_i = \frac{1}{m_i} p_i, \quad \dot{p}_i = -\frac{\partial U(q)}{\partial q_i}, \quad i = 1, 2, \dots, N.$$

These equations are equivalent to the second-order differential equation system

$$(6) \quad m_i \ddot{q}_i = \sum_{j \neq i} \frac{m_i m_j (q_j - q_i)}{\|q_j - q_i\|^3} = -\frac{\partial U}{\partial q_i}.$$

We are looking for solutions of the form

$$(7) \quad q_i(t) = \phi(t) a_i,$$

where a_i is a constant vector and $\phi(t)$ is a scalar function. Identify $\mathbb{R}^2$ with the complex plane by considering the $q_i, a_i, \phi(t)$ as complex numbers. Substituting this into equation (6), we get

$$m_i \ddot{\phi} a_i = \sum_{j \neq i} \frac{m_i m_j (\phi a_j - \phi a_i)}{\|\phi a_j - \phi a_i\|^3} = \sum_{j \neq i} \frac{m_i m_j (a_j - a_i) \phi}{\|a_j - a_i\|^3 |\phi|^3},$$

where this equation is solvable up to quadrature ([2], page 100). In polar coordinates (r, θ) , the solution with $\lambda = 1$ is given by

$$(8) \quad r(t) = \frac{\omega^2}{1 + e \cos \theta(t)}, \quad \dot{\theta} = \frac{\omega}{r^2}, \quad \theta(0) = 0,$$

where ω , the angular momentum, and e the eccentricity of the ellipse, are two parameters.

Remark. A periodic solution of the planar N-body problem has 8 trivial characteristic multipliers of value +1. The solution is linearly stable if the remaining multipliers lie on the unit circle, the monodromy matrix $X(T)$, the matrix satisfying $X(t+T) = X(t)M$ (for example see [5]) restricted to the reduced space is diagonalizable, and strongly stable, if in addition, the multipliers of $X(T)$ are of definite kind.

3.1 Eliminating the standard integrals

We follow the discussion in Roberts [3] to reduce the system. We let the momenta of each body be $p_i = m_i \dot{q}_i$ and let $p = (p_1, p_2, p_3) \in \mathbb{R}^6$. We now start reducing the dimensions of our problem by using the following general fact.

For canonical (symplectic) transformations, the transformed equations of motion are Hamiltonian, with the new Hamiltonian obtained by substitution with the new coordinates.

Using Jacobi coordinates (see [2]), we can eliminate the center of mass and total linear momentum by setting

$$\begin{aligned} u_1 &= q_2 - q_1, & v_1 &= -\frac{m_2}{m_1 + m_2} p_1, \\ u_2 &= q_3 - \frac{1}{m_1 + m_2} (m_1 q_1 + m_2 q_2), & v_2 &= -\frac{m_3}{M} (p_1 + p_2 + p_3) + p_3, \\ u_3 &= \frac{1}{M} (m_1 q_1 + m_2 q_2 + m_3 q_3), & v_3 &= p_1 + p_2 + p_3. \end{aligned}$$

Using $u_3 = v_3 = 0$, the new Hamiltonian will has the following form

$$H_1(u_1, u_2, v_1, v_2) = \frac{\|v_1\|^2}{2M_1} + \frac{\|v_2\|^2}{2M_2} - \frac{m_1 m_2}{\|u_1\|} - \frac{m_1 m_3}{\|u_2 + M_3 u_1\|} - \frac{m_2 m_3}{\|u_2 + M_4 u_1\|}.$$

where

$$\begin{aligned} M_1 &= \frac{m_1 m_2}{m_1 + m_2}, & M_2 &= \frac{m_3(m_1 + m_2)}{M}, \\ M_3 &= \frac{m_2}{m_1 + m_2}, & M_4 &= -\frac{m_1}{m_1 + m_2}. \end{aligned}$$

This Hamiltonian is independent of u_3, v_3 that is $u_3 = v_3 = 0$. This reduces the dimension by 4 from 12 to 8.

Next, using polar coordinates (for example see [2]), we can eliminate the integrals due to the angular momentum and rotational symmetry. The new Hamiltonian will has the form

$$H_2 = \frac{1}{2M_1} (R_1^2 + \frac{\omega_1^2}{r_1^2}) + \frac{1}{2M_2} (R_2^2 + \frac{\omega_2^2}{r_2^2}) - \frac{m_1 m_2}{r_1} - \frac{m_1 m_3}{\delta_1} - \frac{m_2 m_3}{\delta_2}$$

where

$$\begin{aligned} \delta_1 &= \sqrt{r_2^2 + M_3^2 r_1^2 + 2M_3 r_1 r_2 \cos(\theta_2 - \theta_1)}, \\ \delta_2 &= \sqrt{r_2^2 + M_4^2 r_1^2 + 2M_4 r_1 r_2 \cos(\theta_2 - \theta_1)}. \end{aligned}$$

Now, making a final change of coordinates by setting

$$\omega_1 + \omega_2 = c, \quad \theta_2 - \theta_1 = \phi, \quad \omega_2 = \Phi.$$

The new Hamiltonian will be independent of θ_1 . Substituting this into the Hamiltonian H_2 yields

$$H = \frac{1}{2M_1}(R_1^2 + \frac{(c - \Phi)^2}{r_1^2}) + \frac{1}{2M_2}(R_2^2 + \frac{\Phi^2}{r_2^2}) - \frac{m_1 m_2}{r_1} - \frac{m_1 m_3}{\delta_1} - \frac{m_2 m_3}{\delta_2}$$

This reduces the system to six dimensions, in the variables $(r_1, r_2, \phi, R_1, R_2, \Phi)$. The equations of motion in these new variables are

$$\begin{aligned} r_1 &= \frac{R_1}{M_1}, r_2 = \frac{R_2}{M_2}, \dot{\phi} = \frac{\Phi - c}{M_1 r_1^2} + \frac{\Phi}{M_2 r_2^2}, \dot{R}_1 = \frac{(\Phi - c)^2}{M_1 r_1^3} - s_1, \\ \dot{R}_2 &= \frac{\Phi^2}{M_2 r_2^3} - s_2, \dot{\Phi} = m_3 r_1 r_2 \sin \phi s_3 \end{aligned}$$

where

$$\begin{aligned} s_1 &= \frac{m_1 m_2}{r_1^2} + \frac{m_1 m_3 M_3 (r_1 M_3 + r_2 \cos \phi)}{\delta_1^3} + \frac{m_2 m_3 M_4 (r_1 M_4 + r_2 \cos \phi)}{\delta_2^3}, \\ s_2 &= \frac{m_1 m_3 (r_2 + r_1 M_3 \cos \phi)}{\delta_1^3} + \frac{m_2 m_3 (r_2 + r_1 M_4 \cos \phi)}{\delta_2^3}, \quad s_3 = \frac{m_1 M_3}{\delta_1^3} + \frac{m_2 M_4}{\delta_2^3}. \end{aligned}$$

The Kepler periodic solution, denoted in general by $\gamma(t)$, can be obtained using a simple calculation (see [3]) as

$$(9) \quad \begin{aligned} r_1(t) &= A_1 r(t), \quad R_1(t) = M_1 A_1 R(t), \quad r_2(t) = A_2 r(t), \\ R_2(t) &= M_2 A_2 R(t), \quad \phi = \bar{\theta}_3 + \eta, \quad \Phi = \omega M_2 A_2^2, \end{aligned}$$

where $A_1 = M^{1/3}$, $A_2 = \bar{r}_3 M / (m_1 + m_2)$, and η is the angle between the vectors $a_2 - a_1$ and the positive horizontal axis. The total angular momentum for the full problem has the value $c = \omega(M_1 A_1^2 + M_2 A_2^2)$.

In order to discuss linear stability of the Kepler solutions $\gamma(t)$ given by (9), we will linearize the reduced Hamiltonian equation along the Kepler solution, and then decouple the system before we discuss the fundamental solution, and its monodromy matrix.

Linearization of the six-dimensional system about the periodic solution $\gamma(t)$ leads to the time-dependant periodic linear Hamiltonian system which has the following form

$$\dot{X}(t) = J_3 D^2 H(\gamma(t)) X = J_3 A(t) X.$$

Using the solution described in (9), and after a simple calculation and simplification, we get

$$J_3 \mathbf{A}(\mathbf{t}) = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{M_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{M_2} & 0 \\ \frac{2\omega}{A_1 r^3} & \frac{-2\omega}{A_2 r^3} & 0 & 0 & 0 & \frac{c}{\omega M_1 M_2 A_1^2 A_2^2 r^2} \\ a_{11} & a_{12} & a_{13} & 0 & 0 & \frac{-2\omega}{A_1 r^3} \\ a_{12} & a_{22} & \frac{-A_1}{A_2} a_{13} & 0 & 0 & \frac{2\omega}{A_2 r^3} \\ a_{13} & \frac{-A_1}{A_2} a_{13} & a_{33} & 0 & 0 & 0 \end{pmatrix}$$

where

$$a_{12} = \frac{9m_1m_2m_3}{4M\sqrt{m_1^2 + m_1m_2 + m_2^2}} \cdot \frac{1}{r^3}, \quad a_{13} = \frac{3\sqrt{3}m_1m_2m_3(m_1 - m_2)}{4M^{2/3}(m_1 + m_2)^2} \cdot \frac{1}{r^2}$$

and

$$a_{33} = \frac{9m_1m_2m_3}{4M^{1/3}(m_1 + m_2)} \cdot \frac{1}{r}.$$

3.2 Decoupling the linear system

A linear time-dependant periodic Hamiltonian system is one of the form

$$(10) \quad \dot{X}(t) = JD^2H(\gamma(t))X,$$

where J is the standard symplectic matrix, and $D^2H(t+T) = D^2H(t)$.

Define the skew-inner product of two vectors $v, w \in \mathbb{C}^{4n}$ as $\Omega(v, w) = v^T J w$. A key trait of linear Hamiltonian systems is that the skew-orthogonal complement of an invariant subspace is also invariant, where the skew-orthogonal complement is defined as $W^\perp = \{v \in \mathbb{C}^{4n} : \Omega(v, w) = 0, \forall w \in W\}$.

Lemma 3.1. *Suppose W is an invariant subspace of the matrix $B = JD^2H(\gamma(t))$ in system (10), then the skew-orthogonal complement $W^\perp$ of W is also an invariant subspace of the matrix B in (10).*

The above lemma shows that a simple linear change of variables will decouple the system. The characteristic multipliers remain the same since the transformation is linear. We need to find an invariant subspace for $J_3A(t)$, to apply the same idea. We make use of the fact that the Kepler periodic solution $r(t)$ satisfies

$$(11) \quad \ddot{r}(t) = \frac{\omega^2}{r^3} - \frac{1}{r^2}, \quad \ddot{\dot{r}}(t) = \left(-\frac{3\omega^2}{r^4} + \frac{2}{r^3}\right)\dot{r}.$$

From these we have

$$\gamma = \begin{pmatrix} A_1 r(t) \\ A_2 r(t) \\ \bar{\theta}_3 + \eta \\ M_1 A_1 \dot{r}(t) \\ M_2 A_2 \dot{r}(t) \\ \omega M_2 A_2^2 \end{pmatrix}, \quad \dot{\gamma} = \begin{pmatrix} A_1 \dot{r} \\ A_2 \dot{r} \\ 0 \\ M_1 A_1 \ddot{r} \\ M_2 A_2 \ddot{r} \\ 0 \end{pmatrix}, \quad \ddot{\gamma} = \begin{pmatrix} A_1 \ddot{r} \\ A_2 \ddot{r} \\ 0 \\ M_1 A_1 \left(-\frac{3\omega^2}{r^4} + \frac{2}{r^3}\right) \dot{r} \\ M_2 A_2 \left(-\frac{3\omega^2}{r^4} + \frac{2}{r^3}\right) \dot{r} \\ 0 \end{pmatrix}$$

as expressions for the first and second derivatives of the periodic orbit. A short calculation gives

$$J_3 A(t) \dot{\gamma}(t) = \ddot{\gamma}(t), \quad J_3 A(t) \ddot{\gamma}(t) = \left(-\frac{3\omega^2}{r^4} + \frac{2}{r^3}\right) \dot{\gamma}(t).$$

Then the vectors $w_1 := [A_1, A_2, 0, 0, 0, 0]$, $w_2 := [0, 0, 0, M_1 A_1, M_2 A_2, 0]$ will span an invariant subspace W for $J_3 A(t)$. This is the basis for an application of lemma (3.1).

Consider the change of variables determined by

$$(12) \quad \begin{pmatrix} r_1 \\ r_2 \\ \phi \\ R_1 \\ R_2 \\ \Phi \end{pmatrix} = \begin{pmatrix} A_1 & 0 & \frac{\omega A_2 M_2}{c} & 0 & 0 & 0 \\ A_2 & 0 & -\frac{\omega A_1 M_1}{c} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & M_1 A_1 & 0 & \frac{\omega A_2}{c} & 0 & 0 \\ 0 & M_2 A_2 & 0 & -\frac{\omega A_1}{c} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ \chi \\ y \\ Y \\ z \\ Z \end{pmatrix}.$$

(Recall that $\frac{\omega}{c}(M_1 A_1^2 + M_2 A_2^2) = 1$). The linear transformation matrix is non-singular, because the determinant is non-zero. The columns

$$w_3 = [\frac{\omega A_2 M_2}{c}, \frac{-\omega A_1 M_1}{c}, 0, 0, 0, 0],$$

$w_4 = [0, 0, 0, \frac{\omega A_2}{c}, -\frac{\omega A_1}{c}, 0]$, $w_5 = [0, 0, 1, 0, 0, 0]$ and $w_6 = [0, 0, 0, 0, 0, 1]$ of the above matrix are chosen to form a basis for $W^\perp$. Consequently, this change of variables will decouple our linear system into a 2×2 and a 4×4 system. After computing the restriction of $J_3 A(t)$ to the space spanned by four columns of the matrix in system (12), we get the new differential equation system

$$(13) \quad \begin{pmatrix} \dot{y} \\ \dot{Y} \\ \dot{z} \\ \dot{Z} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{M_1 M_2} & 0 & 0 \\ M_1 M_2 (\frac{-3w^2}{r^4} + \frac{2}{r^3}) - \frac{a_{12}c}{w A_1 A_2} & 0 & \frac{a_{13}c}{w A_2} & -\frac{2c}{A_1 A_2 r^3} \\ \frac{2w}{A_1 A_2 r^3} & 0 & 0 & d \\ \frac{a_{13}}{A_2} & 0 & a_{33} & 0 \end{pmatrix} \begin{pmatrix} y \\ Y \\ z \\ Z \end{pmatrix}.$$

An important observation here is that this system (13) is no longer Hamiltonian (due to the fact that the transformation (12) is not canonical). We denote the 4×4 system in equation (13) by the matrix B where all the parameters are described in section (3.1). To study the stability properties of this system, we first show that the system is G-Hamiltonian where the matrix G will be found by the restriction of W to the invariant subspace $W^\perp$. We proceed by considering this calculation, the skew symmetric form can be determined from the following:

$$\Omega(v, w) = v^T G w$$

where $v = (y, Y, z, Z)^T$, $w = (y, Y, z, Z)$. After a good calculations, it is clear that G has the form

$$(14) \quad \mathbf{G} = \begin{pmatrix} 0 & \frac{w}{c} & 0 & 0 \\ -\frac{w}{c} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

where iG is Hermitian, non-singular and symplectic with multiplier $\frac{w}{c}$, $\frac{w}{c} > 0$.

The following lemma states the G-Hamiltonian structure of the matrix B in equation (13). The explicit formula for B^G is given by $B^G = G^{-1}B^*G$ so that

$$(15) B^G = \begin{pmatrix} 0 & \frac{-1}{M_1 M_2} & 0 & 0 \\ -M_1 M_2 \left(\frac{-3w^2}{r^4} + \frac{2}{r^3} \right) + \frac{a_{12}c}{wA_1 A_2} & 0 & \frac{-a_{13}c}{wA_2} & \frac{2c}{A_1 A_2 r^3} \\ \frac{-2w}{A_1 A_2 r^3} & 0 & 0 & -d \\ \frac{-a_{13}}{A_2} & 0 & -a_{33} & 0 \end{pmatrix}.$$

Theorem 3.2. *The 4×4 system B given in equation (13) is G-Hamiltonian.*

Proof. In order to prove this theorem, we need to check only that $B^G = -B$. This is clear by comparing (15) with (13). $\square$

4. Stability and strong stability analysis

The characterization of linear stability of the elliptic kepler orbits through system (13) was considered by Roberts in 2000 without consideration of strong stability [3].

Now, we wish to study the strong stability of the elliptic orbits through the same system. This requires that all the roots of the characteristic polynomial of the monodromy matrix must lie on the unit circle, and in addition, all roots of the characteristic polynomial must be of one kind. The monodromy matrix is denoted here by M , the eigenvalues of M are the characteristic multipliers.

The characteristic polynomial of M is reciprocal and has the form

$$(16) \quad P(\lambda) = \det(\lambda I - M) = \lambda^4 + a\lambda^3 + b\lambda^2 + a\lambda + 1$$

where

$$a = -\text{tr}(M), \quad b = \frac{1}{2}((\text{tr}(M))^2 - \text{tr}(M^2)), \quad \det M = 1.$$

In order to have strong stability, we require that there are no G-isotropic eigenvectors. This simple remark leads us to a characterization of strong stability of the monodromy matrix M . There are three ways, in which strong stability can be lost:

- Periodic doubling bifurcation (two -1 eigenvalues), occurring when $b = 2a - 2$.
- Two +1 eigenvalues, occurring when $b = -(2a + 2)$.
- Krein collision, occurring when $b = \frac{a^2}{4} + 2$.

We now consider the first two cases in the following theorems:

Theorem 4.1. *Consider the characteristic polynomial (16) in the case where $b = 2a - 2$. There are two -1 eigenvalues occurring in this case, which are indefinite and therefore not strongly stable.*

Proof. Since $\lambda = -1$ is real, then the root subspace $\mathcal{L}_\lambda$ is also real and even dimensional. We define the Hermitian form to be $f(e) = (iGe, e)$. Note that, since the imaginary part of $f(e)$ is zero, then $(iG\bar{e}, \bar{e}) = -(iGe, e)$.

Now, we can consider in the root subspace $\mathcal{L}_\lambda$ iG -orthogonal basis of $2m$ vectors $e_1, e_2, \dots, e_{2m}$, chosen and normalized by the condition

$$(iGe_i, e_j) = 0 \quad \forall i \neq j, \quad (iGe_i, e_i) = \pm 1 \quad i = 1, 2, \dots, 2m.$$

Then the vectors $\bar{e}_1, \bar{e}_2, \dots, \bar{e}_{2m}$ form iG -orthogonal basis in the corresponding root subspace $\mathcal{L}_\lambda$. Moreover, the sign of the Hermitian form has been changed, that is $(iG\bar{e}_i, \bar{e}_i) = -(iGe_i, e_i)$ which implies that the signature of $f(e)$ is zero, where the signature of $f(e)$ is defined to be the difference between the positive and negative eigenvalues of the Hermitian form. Therefore the number of positive and negative entries of the Hermitian form are the same. Hence $\lambda = -1$ is indefinite eigenvalue and thus the system is not strongly stable. $\square$

Theorem 4.2. *Consider the characteristic polynomial (16) in the case where $b = -(2a + 2)$. There are two $+1$ eigenvalues occurring in this case, which are indefinite and therefore not strongly stable.*

Proof of this theorem is similar to that of theorem (4.1)

We are also interested to study the Krein collision case, which occurs when $b = a^2/4 + 2$. It is not easy in general to show these Krein collision eigenvalues are of different kind. Henceforth, we will restrict this discussion to the special case when $e = 0$. We study the stability properties of the uniformly rotating Lagrangian solutions when the eccentricity of the periodic orbit is zero. The important observation here is that the matrix B is time independent when $e = 0$. Stability then requires that all the roots of the characteristic polynomial of the G -Hamiltonian matrix B in system (13) must lie on the imaginary axis.

The characteristic polynomial of B is symmetric and has the form

$$(17) \quad P(\lambda) = \lambda^4 + b\lambda^2 + \det B$$

where

$$(18) \quad b = 1/2((\text{tr}(B))^2 - \text{tr}(B^2)).$$

It is furthermore easy to see that the trace of B is zero and the trace of B^2 is -2 . Substituting this in equation (18), we get $b = 1$. The characteristic polynomial (17) can then be written as

$$(19) \quad \lambda^4 + \lambda^2 + \det B.$$

It has the roots

$$(20) \quad \lambda^2 = \frac{1}{2}[-1 \pm \sqrt{1 - 4 \det B}].$$

We follow the behavior of the eigenvalues of (19) as $\det B$ varies. In order to have stability, the roots (20) must be purely imaginary. It is easy to see that this occurs only when $0 \leq \det B \leq \frac{1}{4}$.

For simplicity, Let $q(\rho) = \rho^2 + \rho + \det B$ and denote the two roots of $q(\rho)$ as ρ_1 and ρ_2 where $-\pi^2 < \rho_1 \leq \rho_2 < 0$. As $\det B$ increases from 0 to $1/4$, ρ_1 increases from -1 to $-\frac{1}{2}$ and ρ_2 decreases from 0 to $-\frac{1}{2}$. In the interval $(0, \frac{1}{4})$, the multipliers come in the pairs $\lambda_1 = \pm\sqrt{\rho_1}$, $\lambda_2 = \pm\sqrt{\rho_2}$ where ρ_1 and ρ_2 are given in equation (20).

When the value of $\det B = \frac{1}{4}$ in equation (20), we have the double eigenvalues $\pm i\frac{\sqrt{2}}{2}$. In other words, as $\det B$ increases from 0 to $\frac{1}{4}$, two pairs of eigenvalues meet on the imaginary axis at $\pm i\frac{\sqrt{2}}{2}$. For values of $\det B > \frac{1}{4}$, the characteristic polynomial (19) has four complex roots and thus there will be a complex quartet of eigenvalues $(\lambda, -\lambda, \bar{\lambda}, -\bar{\lambda})$ leading to instability. Since the quartet of eigenvalues move off the imaginary axis and contravene the stability condition, then the system becomes unstable.

We visualize the result of the above argument in Figure 1.

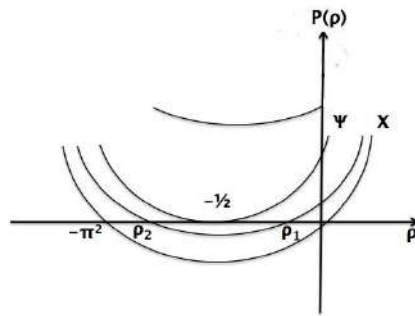


Figure 1: Stability Region for the Case $e = 0$

Beginning on the parabola curve X (the first curve from the bottom) corresponding to $e = 0$, when $\det B$ passes the point zero, the calculations in the above argument show that the curve moves up to intersect the real line at ρ_1 and ρ_2 where $-\pi^2 < \rho_1 \leq \rho_2 < 0$ (the second curve from the bottom), hence this pushes us into the stability region. For $\det B = \frac{1}{4}$, the curve moves up to intersect the real line at $\rho_1 = \rho_2 = -\frac{1}{2}$, which is the point of Krein collision. In contrast as $\det B$ increases from $\frac{1}{4}$, the square root in (20) produces imaginary values, and so λ will be complex with a non-zero real part. We prove that easily by assuming $\lambda = A + iB$ in the above argument, and

$$(21) \quad \lambda^2 = \rho,$$

where $\rho = \frac{-1 \pm i\sqrt{4\det B - 1}}{2}$ the root of $q(\rho)$. Since $\det B > \frac{1}{4}$, the imaginary part of ρ never can be zero, it follows that the real part A of λ is also non-zero. This

is clear by comparing both sides of equation (21). The eigenvalues of (20) lie off the imaginary axis and the system can't be stable. Thus, we have proved that for $\det B > \frac{1}{4}$, stability is immediately lost for $e = 0$ while for $0 < \det B < \frac{1}{4}$, the system is strongly stable.

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The Gel'fand spaces of discrete Beurling algebras on $\mathbb{Z}_+^2$ and $\mathbb{Z}^2$

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Abstract. It is well known that the Gel'fand spaces of the Beurling algebras $l^1(\mathbb{Z}_+, \omega)$ and $l^1(\mathbb{Z}, \omega)$ can be identified, respectively, with some closed disc and closed annulus in the complex plane $\mathbb{C}$ (see [6, P.104]). Therefore, it is natural to investigate the Gel'fand spaces of the Beurling algebras $l^1(\mathbb{Z}_+^2, \omega)$ and $l^1(\mathbb{Z}^2, \omega)$. Surprisingly, their Gel'fand spaces are isomorphic to the union (possibly, uncountable) of product of closed discs and closed annuli in $\mathbb{C}^2$, respectively.

Keywords: Beurling algebras, complex homomorphism, Gel'fand space, Gel'fand transform, holomorphic functions.

1. Introduction

Let $\mathbb{Z}$, $\mathbb{Z}_+$, and $\mathbb{N}$ denote the sets of integers, non-negative integers, and positive integers, respectively. Then, the set $\mathbb{Z}_+^2$ (respectively, $\mathbb{Z}^2$) is a unital abelian semigroup (respectively, group) with respect to co-ordinatewise addition. Let $S = \mathbb{Z}_+^2$ or $\mathbb{Z}^2$. A *weight* on S is a strictly positive function $\omega : S \rightarrow (0, \infty)$ satisfying the submultiplicativity $\omega(s+t) \leq \omega(s)\omega(t)$, for all $s, t \in S$. Let $l^1(S, \omega)$ be the set of all functions $f : S \rightarrow \mathbb{C}$ such that $\sum\{|f(s)|\omega(s) : s \in S\} < \infty$. For $f, g \in l^1(S, \omega)$, the *convolution product* $f * g$ of f and g is defined as

$$f * g(s) = \sum\{f(u)g(v) : u, v \in S \text{ and } u + v = s\}, \quad (s \in S).$$

Then $l^1(S, \omega)$ is a unital, commutative Banach algebra with respect to pointwise linear operations, the convolution product, and the norm $\|f\|_\omega = \sum\{|f(s)|\omega(s) :$

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$s \in S\}$. The Banach algebras $l^1(\mathbb{Z}_+, \omega)$ and $l^1(\mathbb{Z}, \omega)$ have been studied extensively in Banach algebra theory as well as in Fourier analysis [1, 2, 3, 4]. For $r_+ \geq r_- > 0$ and $r \geq 0$, let $\mathbb{D}_r = \{z \in \mathbb{C} : |z| \leq r\}$ and $\Gamma(r_-, r_+) = \{z \in \mathbb{C} : r_- \leq |z| \leq r_+\}$. In particular, the Gel'fand space of $\Delta(l^1(\mathbb{Z}_+, \omega))$ and $\Delta(l^1(\mathbb{Z}, \omega))$ can be identified with the closed disc $\mathbb{D}_r$ and the closed annulus $\Gamma(r_-, r_+)$, respectively, where $0 \leq r < \infty$ and $0 < r_- \leq r_+ < \infty$. It is quite natural to believe that, the Gel'fand spaces of $l^1(\mathbb{Z}_+^2, \omega)$ and $l^1(\mathbb{Z}^2, \omega)$ might be isomorphic to either finite product of closed discs or annuli, respectively. However, it turns out that this is not true in general. In this article, we prove that their Gel'fand space could be an infinite (possibly, uncountable) union of products of closed discs and closed annuli, respectively.

Notations: Throughout we shall consider the following notations:

- (1) Let $\mathbb{C}^\bullet = \mathbb{C} \setminus \{0\}$; $\mathbb{U}_r = \{z \in \mathbb{C} : |z| < r\}$; $\mathbb{D}_r = \{z \in \mathbb{C} : |z| \leq r\}$;
- (2) $e_1 = (1, 0)$, $e_2 = (0, 1)$, and $e = (1, 1)$.

2. Some results on weights

If ω is a weight on $\mathbb{Z}_+$, then there is only one non-negative real number ρ_ω (which is so-called ω -spectral radius) associated with the weight ω and it is equal to the limit $\lim_{n \rightarrow \infty} \omega(n)^{1/n} = \inf_{n \in \mathbb{N}} \omega(n)^{1/n}$ [6, P.103]. On the other hand, if ω is a weight on $\mathbb{Z}_+^2$, then $\omega_1(m) = \omega(m, 0)$ and $\omega_2(n) = \omega(0, n)$ both are weights on $\mathbb{Z}_+$ and $\omega(m, n) \leq \omega_1(m)\omega_2(n)$, for all $m, n \in \mathbb{Z}_+$. In the case of weight ω on $\mathbb{Z}_+^2$, we can define several numbers related to ω as follow. However, none of these numbers is related to the numbers ρ_{ω_1} and ρ_{ω_2} related to the weights ω_1 and ω_2 defined by ω as above.

Definition 2.1. Let ω be a weight on $\mathbb{Z}_+^2$. Then define

$$\begin{aligned}
 \beta_{1,k} &= \inf\{\omega(m, k)^{1/m} : m \in \mathbb{N}\} \quad (k \in \mathbb{Z}_+) \\
 \beta_{2,k} &= \inf\{\omega(k, n)^{1/n} : n \in \mathbb{N}\} \quad (k \in \mathbb{Z}_+) \\
 \rho_2 &= \inf\{\omega(n, n)^{1/n} : n \in \mathbb{N}\} \\
 \beta_1 &= \inf\{\omega(m, n)^{1/m} : m \in \mathbb{N}, n \in \mathbb{Z}_+\} \\
 \beta_2 &= \inf\{\omega(m, n)^{1/n} : m \in \mathbb{Z}_+, n \in \mathbb{N}\} \\
 \mu_2 &= \inf\{\omega(m, n)^{\frac{1}{m+n}} : m, n \in \mathbb{Z}_+ \text{ and } (m, n) \neq (0, 0)\} \\
 \nu_2 &= \inf\{\omega(m, n)^{1/mn} : m, n \in \mathbb{Z}_+ \text{ and } mn \neq 0\} \\
 \sigma_1 &= \inf_{n \in \mathbb{Z}_+} \left[\liminf_{m \rightarrow \infty} \omega(m, n)^{1/m} \right] \\
 \sigma_2 &= \inf_{m \in \mathbb{Z}_+} \left[\liminf_{n \rightarrow \infty} \omega(m, n)^{1/n} \right]
 \end{aligned}$$

If ω is a weight on $\mathbb{Z}^2$, then, further, define

$$\begin{aligned}
 \alpha_{1,k} &= \sup\{\omega(-m, k)^{-1/m} : m \in \mathbb{N}\} \quad (k \in \mathbb{Z}_-) \\
 \alpha_{2,k} &= \sup\{\omega(k, -n)^{-1/n} : n \in \mathbb{N}\} \quad (k \in \mathbb{Z}_-)
 \end{aligned}$$

$$\begin{aligned}
\rho_1 &= \sup\{\omega(-n, -n)^{-1/n} : n \in \mathbb{N}\} \\
\alpha_1 &= \sup\{\omega(-m, n)^{-1/m} : m \in \mathbb{N}, n \in \mathbb{Z}_-\} \\
\alpha_2 &= \sup\{\omega(m, -n)^{-1/n} : m \in \mathbb{Z}_-, n \in \mathbb{N}\} \\
\mu_1 &= \sup\{\omega(m, n)^{\frac{1}{m+n}} : m, n \in \mathbb{Z}_- \text{ and } (m, n) \neq (0, 0)\} \\
\nu_1 &= \sup\{\omega(m, n)^{1/mn} : m, n \in \mathbb{Z}_- \text{ and } mn \neq 0\} \\
\gamma_1 &= \sup_{n \in \mathbb{Z}_-} \left[\limsup_{m \rightarrow -\infty} \omega(m, n)^{1/m} \right] \\
\gamma_2 &= \sup_{m \in \mathbb{Z}_-} \left[\limsup_{n \rightarrow -\infty} \omega(m, n)^{1/n} \right]
\end{aligned}$$

Note that it may happen that $\alpha_1 = \alpha_2 = \infty$. For example, consider $\omega(m, n) = e^{m+n}$ ($m, n \in \mathbb{Z}$).

Lemma 2.1. *Let ω be a weight on $\mathbb{Z}_+^2$ and $k \in \mathbb{Z}_+$. Then*

- (i) $\beta_i \leq \beta_{i,k} \leq \beta_{i,0}$ ($i = 1, 2$);
- (ii) $\beta_{1,0} = \lim_{m \rightarrow \infty} \omega(m, 0)^{1/m}$ and $\beta_{2,0} = \lim_{n \rightarrow \infty} \omega(0, n)^{1/n}$;
- (iii) $\rho_2 = \lim_{n \rightarrow \infty} \omega(n, n)^{1/n}$;
- (iv) $\beta_i \leq \sigma_i \leq \beta_{i,0}$ ($i = 1, 2$).

Proof. (i) It is clear from the definition that $\beta_i \leq \beta_{i,k}$ ($i = 1, 2$ and $k \in \mathbb{Z}_+$). Now, let $\varepsilon > 0$ and choose $k_0 \in \mathbb{N}$ such that $\omega(k_0, 0)^{1/k_0} < \beta_{1,0} + \varepsilon$. Write $m \in \mathbb{N}$ in the form $m = p(m)k_0 + q(m)$, where $p(m), q(m) \in \mathbb{Z}_+$ and $q(m) < k_0$. Then

$$\frac{1}{k_0} = \lim_{m \rightarrow \infty} \left(\frac{p(m)}{m} + \frac{q(m)}{k_0 m} \right) = \lim_{m \rightarrow \infty} \frac{p(m)}{m} = \lim_{m \rightarrow \infty} \frac{p(m) - 1}{m}.$$

Now,

$$\begin{aligned}
\omega(m, k)^{1/m} &= \omega(p(m)k_0 + q(m), k)^{1/m} \\
&\leq \omega(p(m)k_0, k)^{1/m} \omega(q(m), 0)^{1/m} \\
(1) \quad &\leq \omega(k_0, k)^{1/m} \omega(k_0, 0)^{\frac{p(m)-1}{m}} \omega(q(m), 0)^{1/m}
\end{aligned}$$

This implies $\liminf_{m \rightarrow \infty} \omega(m, k)^{1/m} \leq \omega(k_0, 0)^{\frac{1}{k_0}}$, and hence $\beta_{1,k} \leq \beta_{1,0} + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, $\beta_{1,k} \leq \beta_{1,0}$. Similarly, $\beta_{2,k} \leq \beta_{2,0}$.

(ii) Substituting $k = 0$ in the inequality (1), we get

$$\begin{aligned}
\omega(m, 0)^{1/m} &\leq \omega(k_0, 0)^{\frac{p(m)}{m}} \omega(q(m), 0)^{1/m} \\
\Rightarrow \limsup_{m \rightarrow \infty} \omega(m, 0)^{1/m} &\leq \omega(k_0, 0)^{\frac{1}{k_0}} < \beta_{1,0} + \varepsilon.
\end{aligned}$$

But $\beta_{1,0} \leq \liminf_{m \rightarrow \infty} \omega(m, 0)^{1/m}$. Hence $\beta_{1,0} = \lim_{m \rightarrow \infty} \omega(m, 0)^{1/m}$. Similarly, we can show that $\beta_{2,0} = \lim_{n \rightarrow \infty} \omega(0, n)^{1/n}$.

(iii) Define $\gamma(n) = \omega(n, n)$ ($n \in \mathbb{Z}_+$). Then γ is a weight on $\mathbb{Z}_+$ and

$$\rho_2 = \inf\{\gamma(n)^{1/n} : n \in \mathbb{N}\} = \lim_{n \rightarrow \infty} \gamma(n)^{1/n} = \lim_{n \rightarrow \infty} \omega(n, n)^{1/n}.$$

(iv) Let $k \in \mathbb{N}$. Then

$$\begin{aligned} \sigma_1 &= \inf_{n \in \mathbb{N}} \left(\liminf_{m \rightarrow \infty} \omega(m, n)^{1/m} \right) \leq \liminf_{m \rightarrow \infty} \omega(m, k)^{1/m} \\ &\leq \liminf_{m \rightarrow \infty} \omega(m, 0)^{1/m} \omega(0, k)^{1/m} = \beta_{1,0}. \end{aligned}$$

Clearly $\beta_1 \leq \sigma_1 \leq \beta_{1,0}$. Similarly, $\beta_2 \leq \sigma_2 \leq \beta_{2,0}$. $\square$

Lemma 2.2. *Let ω be a weight on $\mathbb{Z}^2$ and let $k \in \mathbb{Z}_-$. Then*

- (i) $\alpha_{i,0} \leq \alpha_{i,k} \leq \alpha_i$ ($i = 1, 2$);
- (ii) $\alpha_{1,0} = \lim_{m \rightarrow \infty} \omega(-m, 0)^{-1/m}$ and $\alpha_{2,0} = \lim_{n \rightarrow \infty} \omega(0, -n)^{-1/n}$;
- (iii) $\rho_1 = \lim_{n \rightarrow \infty} \omega(-n, -n)^{-1/n}$;
- (iv) $\alpha_{i,0} \leq \gamma_i \leq \alpha_i$ ($i = 1, 2$).

Proof. (i) Note that $\beta_i \leq \beta_{i,k} \leq \beta_{i,0}$ is proved in Lemma 2.1(i). Similarly, we can prove that $\alpha_{i,0} \leq \alpha_{i,k} \leq \alpha_i$ ($i = 1, 2$). Statements (ii), (iii), and (iv) can be proved as per the proofs of Lemma 2.1(ii), Lemma 2.1(iii), and Lemma 2.1(iv), respectively. $\square$

Remark. We make following remarks for a weight ω on $\mathbb{Z}_+^2$.

(1) In general, $\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \omega(m, n)^{\frac{1}{m+n}}$ exists, but $\lim_{m, n \rightarrow \infty} \omega(m, n)^{\frac{1}{m+n}}$ may not exist.

Similarly, $\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \omega(m, n)^{\frac{1}{mn}}$ exists but not $\lim_{m, n \rightarrow \infty} \omega(m, n)^{\frac{1}{mn}}$.

(2) In general, $\mu_2 \neq \lim_{m, n \rightarrow \infty} \omega(m, n)^{\frac{1}{m+n}}$ and $\nu_2 \neq \lim_{m, n \rightarrow \infty} \omega(m, n)^{\frac{1}{mn}}$.

(3) Suppose that the double limits of $\omega(m, n)^{\frac{1}{m+n}}$ and $\omega(m, n)^{\frac{1}{mn}}$ exist. In this case, we do not know whether $\mu_2 = \lim_{m, n \rightarrow \infty} \omega(m, n)^{\frac{1}{m+n}}$ and $\nu_2 = \lim_{m, n \rightarrow \infty} \omega(m, n)^{\frac{1}{mn}}$.

Lemma 2.3. *Let ω_1 and ω_2 be weights on $\mathbb{Z}_+$. Define $\omega(m, n) = \omega_1(m)\omega_2(n)$ for $m, n \in \mathbb{Z}_+$. Then*

- (i) $\lim_{m \rightarrow \infty} \omega(m, k)^{1/m} = \beta_{1,0}$ and $\lim_{n \rightarrow \infty} \omega(k, n)^{1/n} = \beta_{2,0}$, for all $k \in \mathbb{Z}_+$; but they may not be equal to $\beta_{1,k}$ and $\beta_{2,k}$, respectively;
- (ii) $\lim_{m, n \rightarrow \infty} \omega(m, n)^{\frac{1}{m+n}}$ may not exist;
- (iii) $\lim_{m, n \rightarrow \infty} \omega(m, n)^{\frac{1}{mn}}$ may not exist.

Proof. (i) Let $k \in \mathbb{N}$. Then, by Lemma 2.1(ii), we have

$$\begin{aligned}\beta_{1,0} &= \lim_{m \rightarrow \infty} \omega(m, 0)^{\frac{1}{m}} \\ &= \lim_{m \rightarrow \infty} \omega_1(m)^{\frac{1}{m}} \omega_2(0)^{\frac{1}{m}} \\ &= \lim_{m \rightarrow \infty} \omega_1(m)^{\frac{1}{m}} \omega_2(k)^{\frac{1}{m}} \\ &= \lim_{m \rightarrow \infty} \omega(m, k)^{\frac{1}{m}}.\end{aligned}$$

Now, $\beta_{1,0}$ may not be equal to $\beta_{1,k}$. For example, $\omega(m, n) = \omega_1(m)\omega_2(n)$ ($m, n \in \mathbb{Z}_+$), where $\omega_1(m) = e^{-m}$ and $\omega_2(n) = e^{-n}$. Then $\beta_{1,0} = \lim_{m \rightarrow \infty} \omega(m, 0)^{1/m} = e^{-1}$, while for $k \in \mathbb{N}$,

$$\beta_{1,k} = \inf\{\omega(m, k)^{1/m} : m \in \mathbb{N}\} = \inf\{e^{-1}e^{-k/m} : m \in \mathbb{N}\} = e^{-k-1}.$$

Similarly, we can show that $\beta_{2,0} = \lim_{n \rightarrow \infty} \omega(k, n)^{\frac{1}{n}} \neq \beta_{2,k}$.

An example for statements (ii) and (iii) is $\omega(m, n) = \omega_1(m)\omega_2(n)$ ($m, n \in \mathbb{Z}_+$), where $\omega_1(m) = 1$ and $\omega_2(n) = e^n + e^{-n}$. $\square$

Lemma 2.4. *Let ω_1 and ω_2 be weights on $\mathbb{Z}$. Define $\omega(m, n) = \omega_1(m)\omega_2(n)$ for $m, n \in \mathbb{Z}$. Then*

(i) $\lim_{m \rightarrow \infty} \omega(-m, k)^{-1/m} = \alpha_{1,0}$ and $\lim_{n \rightarrow \infty} \omega(k, -n)^{-1/n} = \alpha_{2,0}$, for all $k \in \mathbb{Z}_-$; but they may not be equal to $\alpha_{1,k}$ and $\alpha_{2,k}$, respectively;

(ii) $\lim_{m, n \rightarrow \infty} \omega(m, n)^{\frac{1}{m+n}}$ may not exist;

(iii) $\lim_{m, n \rightarrow \infty} \omega(m, n)^{\frac{1}{mn}}$ may not exist.

Proof. Statement (i) can be proved as per the proof of Lemma 2.3(i) and by taking the weight $\omega(m, n) = \omega_1(m)\omega_2(n)$ ($m, n \in \mathbb{Z}$), where $\omega_1(m) = e^m$ and $\omega_2(n) = e^n$. A counter example for (ii) and (iii) is $\omega(m, n) = \omega_1(m)\omega_2(n)$ ($m, n \in \mathbb{Z}$), where $\omega_1(m) = 1$ and $\omega_2(n) = e^n + e^{-n}$. $\square$

Lemma 2.5. *Let ω_1 and ω_2 be weights on $\mathbb{Z}_+$.*

Define $\omega(m, n) = \max\{\omega_1(m), \omega_2(n)\}$ for $m, n \in \mathbb{Z}_+$. Then

(i) $\lim_{m \rightarrow \infty} \omega(m, k)^{1/m} = \beta_{1,0}$ and $\lim_{n \rightarrow \infty} \omega(k, n)^{1/n} = \beta_{2,0}$, for all $k \in \mathbb{Z}_+$; but they may not be equal to $\beta_{1,k}$ and $\beta_{2,k}$, respectively;

(ii) $\lim_{m, n \rightarrow \infty} \omega(m, n)^{\frac{1}{m+n}}$ may not exist;

(iii) $\lim_{m, n \rightarrow \infty} \omega(m, n)^{\frac{1}{mn}}$ may not exist.

Proof. (i) Let $k \in \mathbb{Z}_+$. Then $\omega(m, k) = \max\{\omega_1(m), \omega_2(k)\}$. This implies that $\omega(m, k)^{1/m} = \max\{\omega_1(m)^{1/m}, \omega_2(k)^{1/m}\}$. Therefore

$$\begin{aligned} \lim_{m \rightarrow \infty} \omega(m, k)^{1/m} &= \max\left\{\lim_{m \rightarrow \infty} \omega_1(m)^{1/m}, \lim_{m \rightarrow \infty} \omega_2(k)^{1/m}\right\} \\ &= \max\left\{\lim_{m \rightarrow \infty} \omega_1(m)^{1/m}, \lim_{m \rightarrow \infty} \omega_2(0)^{1/m}\right\} \\ &= \lim_{m \rightarrow \infty} \omega(m, 0)^{1/m} = \beta_{1,0} \end{aligned}$$

Similarly, we have $\lim_{n \rightarrow \infty} \omega(k, n)^{1/n} = \beta_{2,0}$. Next we give an example exhibiting that these limits may not be equal to $\beta_{1,k}$ and $\beta_{2,k}$, respectively. For example, let $\omega(m, n) = \max\{e^{-m^2}, e^{-n^2}\}$ ($m, n \in \mathbb{Z}_+$). Then it is clear that $\lim_{m \rightarrow \infty} \omega(m, k)^{1/m} = \max\{\lim_{m \rightarrow \infty} (e^{-m^2})^{1/m}, 1\} = 1$, while

$$\begin{aligned} \beta_{1,k} &= \inf\{\omega(m, k)^{1/m} : m \in \mathbb{N}\} \\ &= \inf\{\max\{e^{-m^2}, e^{-k^2}\}^{1/m} : m \in \mathbb{N}\} \\ &= \inf\{e^{-1}, e^{-\frac{2^2}{2}}, \dots, e^{-\frac{k^2}{k}}, e^{-\frac{k^2}{k+1}}, e^{-\frac{k^2}{k+2}}, \dots\} = e^{-k}. \end{aligned}$$

Similarly, we can prove that $\lim_{n \rightarrow \infty} \omega(k, n)^{1/n} = 1$ and $\beta_{2,k} = e^{-k}$.

An example for (ii) and (iii) is $\omega(m, n) = \max\{\omega_1(m), \omega_2(n)\}$ ($m, n \in \mathbb{Z}_+$), where $\omega_1(m) = 1$ and $\omega_2(n) = e^n + e^{-n}$. $\square$

Lemma 2.6. *Let ω_1 and ω_2 be weights on $\mathbb{Z}$.*

Define $\omega(m, n) = \max\{\omega_1(m), \omega_2(n)\}$ for $m, n \in \mathbb{Z}$. Then

(i) $\lim_{m \rightarrow \infty} \omega(-m, k)^{-1/m} = \alpha_{1,0}$ and $\lim_{n \rightarrow \infty} \omega(k, -n)^{-1/n} = \alpha_{2,0}$, for all $k \in \mathbb{Z}_-$; but they may not be equal to $\alpha_{1,k}$ and $\alpha_{2,k}$ respectively;

(ii) $\lim_{m, n \rightarrow \infty} \omega(m, n)^{\frac{1}{m+n}}$ may not exist;

(iii) $\lim_{m, n \rightarrow \infty} \omega(m, n)^{\frac{1}{mn}}$ may not exist.

Proof. (i) As in the proof of Lemma 2.5(i), we have $\lim_{m \rightarrow \infty} \omega(-m, k)^{-1/m} = \alpha_{1,0}$. Now we show that it may not be equal to $\alpha_{1,k}$. For example, let $\omega(m, n) = \max\{e^m, e^n\}$ ($m, n \in \mathbb{Z}$). Then

$$\lim_{m \rightarrow \infty} \omega(-m, k)^{-1/m} = \max\left\{\lim_{m \rightarrow \infty} (e^{-m})^{1/m}, 1\right\}^{-1} = 1,$$

$$\begin{aligned} \text{while } \alpha_{1,k} &= \sup\{\omega(-m, k)^{-1/m} : m \in \mathbb{N}\} \\ &= \sup\{\max\{e^{-m}, e^k\}^{-1/m} : m \in \mathbb{N}\} \\ &= \sup\{e, e^{\frac{-k}{(-k+1)}}, e^{\frac{-k}{(-k+2)}}, \dots\} = e. \end{aligned}$$

Similarly, we can show that $\lim_{n \rightarrow \infty} \omega(k, -n)^{-1/n} = 1 = \alpha_{2,0}$ but $\alpha_{2,k} = e$ ($k \leq -1$).

The statements (ii) and (iii) follow from example given in Lemma 2.5. $\square$

Remark. We note the following:

- (1) If ω_1 and ω_2 are weights on $\mathbb{Z}_+$ or $\mathbb{Z}$, then $\omega(m, n) = \min\{\omega_1(m), \omega_2(n)\}$ may not be a weight on $\mathbb{Z}_+^2$ or $\mathbb{Z}^2$.
- (2) Let ω_1 and ω_2 be two weights on $\mathbb{Z}_+$ or $\mathbb{Z}$. Then Lemma 2.5 and Lemma 2.6 are also true for the weight $\omega = \omega_1 + \omega_2$ because $\max\{\omega_1(m), \omega_2(n)\} \leq \omega_1(m) + \omega_2(n) \leq 2 \max\{\omega_1(m), \omega_2(n)\}$, for all $m, n \in \mathbb{Z}_+$ or $\mathbb{Z}$.

3. The Gel'fand spaces of $l^1(\mathbb{Z}_+^2, \omega)$ and $l^1(\mathbb{Z}^2, \omega)$

The Russian mathematician Gel'fand developed the Gel'fand theory for commutative Banach algebras. He showed the importance of the Gel'fand space in the theory of commutative Banach algebras [5]. So it is always interesting to identify the Gel'fand space with some subset of $\mathbb{C}^n$. From this point of view, in this section, we have made an attempt to identify the Gel'fand spaces of the Beurling algebras $l^1(\mathbb{Z}_+^2, \omega)$ and $l^1(\mathbb{Z}^2, \omega)$. Such an attempt was made by Dabhi for $\omega \geq 1$ on $\mathbb{Z}^2$ [2]. Though we could identify them with some subsets of $\mathbb{C}^2$, they are found to be quite complicated for some weights. We start with the following two simple lemmas:

Lemma 3.1. *Let $z, w \in \mathbb{C}^\bullet$ and ω be a weight on $\mathbb{Z}_+^2$. Suppose that there exists a constant $M > 0$ such that $|z|^m |w|^n \leq M\omega(m, n)$ ($m, n \in \mathbb{Z}_+$). Then*

$$(2) \quad |z|^m |w|^n \leq \omega(m, n) \quad (m, n \in \mathbb{Z}_+).$$

Similar statement holds true for a weight on $\mathbb{Z}^2$.

Proof. Let $k \in \mathbb{N}$. Then $(|z|^m |w|^n)^k = |z|^{km} |w|^{kn} \leq M\omega(km, kn) \leq M\omega(m, n)^k$. Hence $|z|^m |w|^n \leq M^{\frac{1}{k}} \omega(m, n)$. Since k is arbitrary, $|z|^m |w|^n \leq \omega(m, n)$, for all m, n . Therefore, $|z|^m |w|^n \leq M\omega(m, n) \Rightarrow |z|^m |w|^n \leq \omega(m, n)$. $\square$

Lemma 3.2. *Let ω be a weight on $\mathbb{Z}_+^2$. Then μ_2 is the largest number satisfying the following*

$$|z|^m |w|^n \leq \omega(m, n) \quad (z, w \in \mathbb{D}_{\mu_2}; m, n \in \mathbb{Z}_+).$$

Proof. Let $r \geq 0$ such that $|z|^m |w|^n \leq \omega(m, n)$ ($z, w \in \mathbb{D}_r; m, n \in \mathbb{Z}_+$). Take $z = w = r \in \mathbb{D}_r$. Then $r^{m+n} \leq \omega(m, n)$ ($m, n \in \mathbb{Z}_+$). So that $r \leq \omega(m, n)^{\frac{1}{m+n}}$, for all $(m, n) \in \mathbb{Z}_+^2 \setminus \{(0, 0)\}$. Hence $r \leq \mu_2$. $\square$

Notations: We set the following notations:

1. Let ω be a weight on $\mathbb{Z}_+^2$. Then define

- (a) $\mathbb{S}_0(\omega) = \{(z, w) \in \mathbb{C}^{\bullet 2} : |z|^m |w|^n \leq \omega(m, n) \quad (m, n \in \mathbb{Z}_+)\}$.
- (b) $\mathbb{S}_1(\omega) = \{(z, 0) \in \mathbb{C}^2 \setminus \{(0, 0)\} : |z|^m \leq \omega(m, 0) \quad (m \in \mathbb{Z}_+)\}$.
- (c) $\mathbb{S}_2(\omega) = \{(0, w) \in \mathbb{C}^2 \setminus \{(0, 0)\} : |w|^n \leq \omega(0, n) \quad (n \in \mathbb{Z}_+)\}$.
- (d) $\mathbb{G}(\omega) = \mathbb{S}_0(\omega) \cup \mathbb{S}_1(\omega) \cup \mathbb{S}_2(\omega) \cup \{(0, 0)\}$.

2. Let $z, w \in \mathbb{C}^\bullet$. Then, for $f \in l^1(\mathbb{Z}_+^2, \omega)$, define

$$\begin{aligned}\phi_{z,w}(f) &= \sum \{f(m, n)z^m w^n : m, n \in \mathbb{Z}_+\} \\ \phi_{z,0}(f) &= \sum \{f(m, 0)z^m : m \in \mathbb{Z}_+\} \\ \phi_{0,w}(f) &= \sum \{f(0, n)w^n : n \in \mathbb{Z}_+\} \\ \phi_{0,0}(f) &= f(0, 0).\end{aligned}$$

The set $\mathbb{G}(\omega)$ will be called the *Gel'fand set* of ω . It is routine to check that if (z, w) belongs to the Gel'fand set $\mathbb{G}(\omega)$, then $\phi_{z,w}$ is a complex homomorphism on $l^1(\mathbb{Z}_+^2, \omega)$. The following result asserts that if $\varphi \in \Delta(l^1(\mathbb{Z}_+^2, \omega))$, then there exists a unique pair $(z, w) \in \mathbb{G}(\omega)$ such that $\varphi = \phi_{z,w}$.

Theorem 3.1. *The Gel'fand space $\Delta(l^1(\mathbb{Z}_+^2, \omega))$ is homeomorphic to $\mathbb{G}(\omega)$.*

Proof. Define $\Lambda : \Delta(l^1(\mathbb{Z}_+^2, \omega)) \rightarrow \mathbb{C}^2$ as $\Lambda(\varphi) = (\varphi(\delta_{e_1}), \varphi(\delta_{e_2}))$.

First, we show that the range of Λ is contained in $\mathbb{G}(\omega)$. Fix $\varphi \in \Delta(l^1(\mathbb{Z}_+^2, \omega))$. Let $\Lambda(\varphi) = (\varphi(\delta_{e_1}), \varphi(\delta_{e_2})) = (z, w)$. If $z = w = 0$, then clearly $\Lambda(\varphi) \in \mathbb{G}(\omega)$. Assume that $z \neq 0$ and $w \neq 0$. Then, for $m, n \in \mathbb{Z}_+$,

$$|z|^m |w|^n = |\varphi(\delta_{e_1})|^m |\varphi(\delta_{e_2})|^n = |\varphi(\delta_{(m,n)})| \leq \|\delta_{(m,n)}\|_\omega = \omega(m, n).$$

Hence $\Lambda(\varphi) = (z, w) \in \mathbb{S}_0(\omega) \subset \mathbb{G}(\omega)$. Now assume that $z \neq 0$ and $w = 0$. Then, for $m \in \mathbb{Z}_+$,

$$|z|^m = |\varphi(\delta_{e_1})|^m = |\varphi(\delta_{(m,0)})| \leq \omega(m, 0).$$

So, $\Lambda(\varphi) = (z, 0) \in \mathbb{S}_1(\omega) \subset \mathbb{G}(\omega)$. Similarly, if $z = 0$ and $w \neq 0$, then it can be shown that $\Lambda(\varphi) = (0, w) \in \mathbb{S}_2(\omega) \subset \mathbb{G}(\omega)$. Thus $\Lambda : \Delta(l^1(\mathbb{Z}_+^2, \omega)) \rightarrow \mathbb{G}(\omega)$.

Next, we show that Λ is one-one. Let $\varphi_1, \varphi_2 \in \Delta(l^1(\mathbb{Z}_+^2, \omega))$ such that $\Lambda(\varphi_1) = \Lambda(\varphi_2)$. Then $\varphi_1(\delta_{e_1}) = \varphi_2(\delta_{e_1})$ and $\varphi_1(\delta_{e_2}) = \varphi_2(\delta_{e_2})$. Let $f \in l^1(\mathbb{Z}_+^2, \omega)$. Then

$$\begin{aligned}\varphi_1(f) &= \sum_{m,n \in \mathbb{Z}_+} f(m, n) \varphi_1(\delta_{(m,n)}) = \sum_{m,n \in \mathbb{Z}_+} f(m, n) \varphi_1(\delta_{e_1})^m \varphi_1(\delta_{e_2})^n \\ &= \sum_{m,n \in \mathbb{Z}_+} f(m, n) \varphi_2(\delta_{e_1})^m \varphi_2(\delta_{e_2})^n = \sum_{m,n \in \mathbb{Z}_+} f(m, n) \varphi_2(\delta_{(m,n)}) = \varphi_2(f).\end{aligned}$$

Thus, $\varphi_1 = \varphi_2$, i.e. Λ is one-one.

Next, we show that Λ is onto. Let $(z, w) \in \mathbb{G}(\omega)$. Then it is easy to see that $\phi_{z,w} \in \Delta(l^1(\mathbb{Z}_+^2, \omega))$ and $\Lambda(\phi_{z,w}) = (z, w)$. Thus, Λ is onto.

Finally, we show that Λ is a homeomorphism. Consider a convergent net $\varphi_\alpha \rightarrow \varphi$ in $\Delta(l^1(\mathbb{Z}_+^2, \omega))$. By the definition of Gel'fand topology [6, P.52], $\varphi_\alpha(f) \rightarrow \varphi(f)$, for all $f \in l^1(\mathbb{Z}_+^2, \omega)$. In particular, $\varphi_\alpha(\delta_{e_1}) \rightarrow \varphi(\delta_{e_1})$ and $\varphi_\alpha(\delta_{e_2}) \rightarrow \varphi(\delta_{e_2})$. Hence $\Lambda(\varphi_\alpha) = (\varphi_\alpha(\delta_{e_1}), \varphi_\alpha(\delta_{e_2})) \rightarrow (\varphi(\delta_{e_1}), \varphi(\delta_{e_2})) = \Lambda(\varphi)$. So Λ is continuous. Since $l^1(\mathbb{Z}_+^2, \omega)$ has identity, the Gel'fand space is compact [6, P.52]. Since $\mathbb{G}(\omega)$ is Hausdorff, Λ is a homeomorphism. $\square$

Theorem 3.2. *Let ω be a weight on $\mathbb{Z}_+^2$. Then*

- (i) $\mathbb{D}_{\mu_2} \times \mathbb{D}_{\mu_2} \subseteq \mathbb{G}(\omega)$;
- (ii) $\mathbb{G}(\omega) \subseteq \mathbb{D}_{\beta_{1,0}} \times \mathbb{D}_{\beta_{2,0}}$;
- (iii) $\mathbb{S}_0(\omega) \subseteq \mathbb{D}_{\sigma_1} \times \mathbb{D}_{\sigma_2}$;
- (iv) $(\mathbb{D}_{\beta_{1,0}} \times \mathbb{D}_{s_2}) \cup (\mathbb{D}_{s_1} \times \mathbb{D}_{\beta_{2,0}}) \subseteq \mathbb{G}(\omega)$ for some $s_1, s_2 \geq 0$.

Proof. (i) Let $(z, w) \in \mathbb{D}_{\mu_2} \times \mathbb{D}_{\mu_2}$. Then $|z| \leq \mu_2$ and $|w| \leq \mu_2$. This implies that $|z| \leq \omega(m, n)^{\frac{1}{m+n}}$ and $|w| \leq \omega(m, n)^{\frac{1}{m+n}}$, for all $(m, n) \neq (0, 0)$. Thus $|z|^m |w|^n \leq \omega(m, n)$. Hence $(z, w) \in \mathbb{G}(\omega)$.

(ii) Let $(z, w) \in \mathbb{G}(\omega)$. It is easy to check, if $(z, w) \in \{(0, 0)\} \cup \mathbb{S}_1(\omega) \cup \mathbb{S}_2(\omega)$. Now, if $(z, w) \in \mathbb{S}_0(\omega)$, then $|z|^m |w|^n \leq \omega(m, n)$, for all $m, n \in \mathbb{Z}_+$. This implies $|z|^m \leq \omega(m, 0)$, for all $m \in \mathbb{Z}_+$. Therefore $|z| \leq \beta_{1,0}$. Similarly, $|w| \leq \beta_{2,0}$.

(iii) Let $(z, w) \in \mathbb{S}_0(\omega)$. Then, as usual, $|z|^m |w|^n = |\varphi(\delta_{(m,n)})| \leq \omega(m, n)$, for all $m, n \in \mathbb{Z}_+$. Fix any $k \in \mathbb{Z}_+$. Then $|z|^m |w|^k \leq \omega(m, k)$ ($m \in \mathbb{N}$). This implies that

$$|z| = \liminf_{m \rightarrow \infty} |z| |w|^{\frac{k}{m}} \leq \liminf_{m \rightarrow \infty} \omega(m, k)^{\frac{1}{m}}.$$

Since k is arbitrary, we get $|z| \leq \sigma_1$. Similarly, we have $|w| \leq \sigma_2$. Hence $(z, w) \in \mathbb{D}_{\sigma_1} \times \mathbb{D}_{\sigma_2}$.

(iv) Define $s_2 = \sup\{|w| : (\beta_{1,0}, w) \in \mathbb{G}(\omega)\} \geq 0$. Let $|z| \leq \beta_{1,0}$ and $|w| \leq s_2$. By definition of s_2 , we have $(\beta_{1,0}, w) \in \mathbb{G}(\omega)$. Hence $\mathbb{D}_{\beta_{1,0}} \times \mathbb{D}_{s_2} \subset \mathbb{G}(\omega)$. Similarly, we can define s_1 and prove that $\mathbb{D}_{s_1} \times \mathbb{D}_{\beta_{2,0}} \subset \mathbb{G}(\omega)$. $\square$

Theorem 3.3. *Let ω be a weight on $\mathbb{Z}_+^2$. Then there is a subset $\{(r_i, s_i) : i \in \Lambda\}$ of $\mathbb{R}_+^2$ such that*

$$\mathbb{G}(\omega) = \cup_{i \in \Lambda} (\mathbb{D}_{r_i} \times \mathbb{D}_{s_i}).$$

Proof. Let us define $\mathcal{D} = \{(r, s) \in \mathbb{R}_+^2 : \phi_{r,s} \in \Delta(l^1(\mathbb{Z}_+^2, \omega))\}$. For $(r, s), (u, v) \in \mathcal{D}$, we have $(r, s) \leq (u, v)$ if and only if $r \leq u$ and $s \leq v$. Then $(\mathcal{D}, \leq)$ is a partially ordered set. Let $M = \{(r_i, s_i) : i \in \Lambda\}$ be the set of all maximal elements of $\mathcal{D}$. Now, we shall prove that the set M is non-empty. Let $\mathcal{C} \subset \mathcal{D}$ be any chain in $\mathcal{D}$. Set $(\bar{r}, \bar{s}) = \sup\{(r, s) : (r, s) \in \mathcal{C}\}$. We claim that $(\bar{r}, \bar{s}) \in \mathcal{D}$.

Case (i). $\bar{r} = 0$ and $\bar{s} = 0$. This implies that $\mathcal{C} = \{(0, 0)\}$ and $(0, 0) \in \mathcal{D}$.

Case (ii). $\bar{r} > 0$ and $\bar{s} = 0$. Then there exists a sequence $(u_n, 0) \in \mathcal{C}$ such that $u_n \rightarrow \bar{r}$ and $\phi_{u_n,0} \in \Delta(l^1(\mathbb{Z}_+^2, \omega))$. Now,

$$\phi_{\bar{r},0}(\delta_{(m,n)}) = \bar{r}^m = \lim_{n \rightarrow \infty} u_n^m = \lim_{n \rightarrow \infty} \phi_{u_n,0}(\delta_{(m,n)}).$$

Thus $\phi_{u_n,0}(\delta_{(m,n)}) \rightarrow \phi_{\bar{r},0}(\delta_{(m,n)})$. This implies that $\phi_{u_n,0}(f) \rightarrow \phi_{\bar{r},0}(f)$, for all $f \in l^1(\mathbb{Z}_+^2, \omega)$ with $\text{supp} f$ being finite. Hence $\phi_{u_n,0}(f) \rightarrow \phi_{\bar{r},0}(f)$ ($f \in l^1(\mathbb{Z}_+^2, \omega)$) and $\phi_{\bar{r},0} \in \Delta(l^1(\mathbb{Z}_+^2, \omega))$.

Case (iii). $\bar{r} = 0$ and $\bar{s} > 0$. By similar arguments as given in Case (ii), we have $\phi_{0,\bar{s}} \in \Delta(l^1(\mathbb{Z}_+^2, \omega))$.

Case (iv). $\bar{r} > 0$ and $\bar{s} > 0$. Choose $k_0 \in \mathbb{N}$ sufficiently large enough so that $0 < \frac{1}{k_0} < \bar{r}$ and $0 < \frac{1}{k_0} < \bar{s}$. For any $k \geq k_0$, there exists $(u_k, v_k) \in \mathcal{C}$ such that $(\bar{r} - \frac{1}{k}, \bar{s} - \frac{1}{k}) < (u_k, v_k) \leq (\bar{r}, \bar{s})$. This implies that $\phi_{\bar{r}, \bar{s}}(\delta_{(m,n)}) = \bar{r}^m \bar{s}^n = \lim_{k \rightarrow \infty} u_k^m v_k^n = \lim_{k \rightarrow \infty} \phi_{u_k, v_k}(\delta_{(m,n)})$. Thus $\phi_{u_k, v_k}(\delta_{(m,n)}) \rightarrow \phi_{\bar{r}, \bar{s}}(\delta_{(m,n)})$. This implies that $\phi_{u_k, v_k}(f) \rightarrow \phi_{\bar{r}, \bar{s}}(f)$ ($f \in l^1(\mathbb{Z}_+^2, \omega)$). Hence $(\bar{r}, \bar{s}) \in \mathcal{D}$.

Therefore, by Zorn's Lemma, $\mathcal{D}$ has a maximal element. Hence $M \neq \emptyset$.

Finally, we claim that $\mathbb{G}(\omega) = \cup_{i \in \Lambda} (\mathbb{D}_{r_i} \times \mathbb{D}_{s_i})$. Let $(z, w) \in \mathbb{G}(\omega)$. Then $(z, w) \in \mathbb{D}_{r_i} \times \mathbb{D}_{s_i}$ for some $i \in \Lambda$. This implies that $|z| \leq r_i$ and $|w| \leq s_i$. Therefore $\phi_{|z|, |w|} \in \Delta(l^1(\mathbb{Z}_+^2, \omega))$ because $(r_i, s_i) \in \mathcal{D}$. Hence $(z, w) \in \mathbb{G}(\omega)$.

Conversely, let $(z, w) \in \mathbb{G}(\omega)$. Then $(|z|, |w|) \in \mathcal{D}$. Since $\mathcal{D}$ has maximal elements, there exists $i \in \Lambda$ such that $(|z|, |w|) \leq (r_i, s_i)$. So that $(z, w) \in \mathbb{D}_{r_i} \times \mathbb{D}_{s_i}$. Hence $\mathbb{G}(\omega) \subseteq \cup_{i \in \Lambda} (\mathbb{D}_{r_i} \times \mathbb{D}_{s_i})$. This completes the proof. $\square$

Now we identify the Gel'fand space of $l^1(\mathbb{Z}^2, \omega)$ with a closed subset of $\mathbb{C}^{\bullet 2}$. Note that Dabhi identified the Gel'fand space $\Delta(l^1(\mathbb{Z}_+^2, \omega))$ with $\mathbb{T}(\omega)$ for those weights satisfying $\omega(m, n) \geq 1$ ($m, n \in \mathbb{Z}$) [2, Corollary 2]. Here we do not put any condition on ω . In order to this, we need to introduce some notations.

Notations. Let $r, s > 0$ and ω be a weight on $\mathbb{Z}^2$. Then

$$(i) \quad \Gamma(r) = \{z \in \mathbb{C} : |z| = r\};$$

$$(ii) \quad \Gamma(r, s) = \{z \in \mathbb{C} : r \leq |z| \leq s\};$$

$$(iii) \quad \mathbb{T}(\omega) = \{(z, w) \in \mathbb{C}^2 : |z|^m |w|^n \leq \omega(m, n) \quad (m, n \in \mathbb{Z})\}.$$

It is clear that $\Gamma(r, s) \neq \emptyset$ iff $r \leq s$. Moreover, $\mathbb{T}(\omega) \subset \mathbb{C}^{\bullet 2}$.

Theorem 3.4. *The Gel'fand space $\Delta(l^1(\mathbb{Z}^2, \omega))$ is homeomorphic to $\mathbb{T}(\omega)$.*

Proof. Define $\Lambda : \Delta(l^1(\mathbb{Z}^2, \omega)) \rightarrow \mathbb{C}^2$ as $\Lambda(\varphi) = (\varphi(\delta_{e_1}), \varphi(\delta_{e_2}))$. It is enough to prove that $\varphi(\delta_{e_1}) \neq 0$ and $\varphi(\delta_{e_2}) \neq 0$. The rest of the proof is similar to the proof of Theorem 3.1. Since $l^1(\mathbb{Z}^2, \omega)$ is a unital, commutative, Banach algebra with identity $\delta_{(0,0)}$, we have $\varphi(\delta_{(0,0)}) = 1$, for all $\varphi \in \Delta(l^1(\mathbb{Z}^2, \omega))$. Now, $\delta_{e_1} * \delta_{-e_1} = \delta_{(0,0)}$. This implies $\varphi(\delta_{e_1})\varphi(\delta_{-e_1}) = \varphi(\delta_{(0,0)}) = 1$. Hence $\varphi(\delta_{e_1}) \neq 0$. Similarly, we can show that $\varphi(\delta_{e_2}) \neq 0$. $\square$

Theorem 3.5. *Let $\omega \geq 1$ be a weight on $\mathbb{Z}^2$. Then*

$$(i) \quad \Gamma(\sqrt{\alpha_1}, \sqrt{\beta_1}) \times \Gamma(\sqrt{\alpha_2}, \sqrt{\beta_2}) \subseteq \mathbb{T}(\omega);$$

$$(ii) \quad \mathbb{T}(\omega) \subseteq \Gamma(\alpha_{1,0}, \beta_{1,0}) \times \Gamma(\alpha_{2,0}, \beta_{2,0});$$

$$(iii) \quad (\Gamma(\alpha_1) \times \Gamma(1)) \cup (\Gamma(1) \times \Gamma(\alpha_2)) \cup (\Gamma(\beta_1) \times \Gamma(1)) \cup (\Gamma(1) \times \Gamma(\beta_2)) \subseteq \mathbb{T}(\omega).$$

Proof. (i) Let $(z, w) \in \Gamma(\sqrt{\alpha_1}, \sqrt{\beta_1}) \times \Gamma(\sqrt{\alpha_2}, \sqrt{\beta_2})$. Then

$$\sqrt{\alpha_1} \leq |z| \leq \sqrt{\beta_1}$$

$$\Rightarrow |z|^2 \leq \omega(m, n)^{1/m} (m \in \mathbb{N}; n \in \mathbb{Z}) \text{ and } \omega(-m, n)^{-1/m} \leq |z|^2 (m \in \mathbb{N}; n \in \mathbb{Z})$$

$$\Rightarrow |z|^{2m} \leq \omega(m, n) (m \in \mathbb{N}; n \in \mathbb{Z}) \text{ and } |z|^{-2m} \leq \omega(-m, n) (m \in \mathbb{N}; n \in \mathbb{Z})$$

$$\Rightarrow |z|^{2m} \leq \omega(m, n), (m, n \in \mathbb{Z}).$$

Similarly, we get, $|w|^{2n} \leq \omega(m, n)$, $m \in \mathbb{Z}; n \in \mathbb{Z}$. It follows that $|z|^m |w|^n \leq \omega(m, n)$ $m, n \in \mathbb{Z}$. Therefore, $(z, w) \in \Gamma(\sqrt{\alpha_1}, \sqrt{\beta_1}) \times \Gamma(\sqrt{\alpha_2}, \sqrt{\beta_2})$.

(ii) Let $(z, w) \in \mathbb{T}(\omega)$. Define $z = \varphi(\delta_{e_1})$ and $w = \varphi(\delta_{e_2})$. Then $\varphi = \phi_{z,w}$. Since $\phi_{z,w}$ is continuous, $|z|^m = |\varphi(\delta_{(m,0)})| \leq \|\delta_{(m,0)}\| = \omega(m, 0)$, for all $m \in \mathbb{Z}$. Therefore $\sup_{m \in \mathbb{N}} \omega(-m, 0)^{\frac{-1}{m}} \leq |z| \leq \inf_{m \in \mathbb{N}} \omega(m, 0)^{\frac{1}{m}}$ and similarly, $\sup_{n \in \mathbb{N}} \omega(0, -n)^{\frac{-1}{n}} \leq |w| \leq \inf_{n \in \mathbb{N}} \omega(0, n)^{\frac{1}{n}}$. Hence $(z, w) \in \Gamma(\alpha_{1,0}, \beta_{1,0}) \times \Gamma(\alpha_{2,0}, \beta_{2,0})$.

(iii) Let $|z| = \alpha_1$ and $|w| = 1$. Then, for $m \in \mathbb{Z}_+, n \in \mathbb{Z}$, we have, $|z|^m \leq \omega(m, n)$ as $\omega \geq 1$. Now, for $m \in \mathbb{N}, n \in \mathbb{Z}$, we have $|z|^{-m} \leq \omega(-m, n)$. Hence $\Gamma(\alpha_1) \times \Gamma(1) \subseteq \mathbb{T}(\omega)$. Similarly, $\Gamma(1) \times \Gamma(\alpha_2) \subseteq \mathbb{T}(\omega)$. Now let $|z| = \beta_1$ and $|w| = 1$. Then for $m \in \mathbb{N}, n \in \mathbb{Z}$, we have $|z|^m \leq \omega(m, n)$. For $m < 0, n \in \mathbb{Z}$, $|z|^m \leq \omega(m, n)$ as $\omega \geq 1$. Therefore, $\Gamma(\beta_1) \times \Gamma(1) \subseteq \mathbb{T}(\omega)$. Similarly, $\Gamma(1) \times \Gamma(\beta_2) \subseteq \mathbb{T}(\omega)$. $\square$

Theorem 3.6. Let ω be any weight on $\mathbb{Z}^2$. Then there exist a subset Λ of $\mathbb{R}_+^\bullet$ and a subset $\{(s_r, t_r) : r \in \Lambda\}$ of $\mathbb{R}_+^{\bullet 2}$ such that

$$\mathbb{T}(\omega) = \cup_{r \in \Lambda} (\Gamma(r) \times \Gamma(s_r, t_r)).$$

Proof. Let $\mathcal{D} = \{(r, s) \in \mathbb{R}_+^2 : r^m s^n \leq \omega(m, n) \ (m, n \in \mathbb{Z})\}$. Then $\mathcal{D}$ is a partially ordered set with respect to the product order. Next we define $\Lambda = \{r \in \mathbb{R}_+^\bullet : (r, s) \in \mathcal{D} \text{ for some } s > 0\}$. Then, for each $r \in \Lambda$, the set $\mathcal{C}_r = \{(r, s) : (r, s) \in \mathcal{D}\}$ is a chain in $\mathcal{D}$, and so it has minimum and maximum elements, say (r, s_r) and (r, t_r) , respectively.

Now, we claim that $\mathbb{T}(\omega) = \cup_{r \in \Lambda} (\Gamma(r) \times \Gamma(s_r, t_r))$. Let $(z, w) \in \mathbb{T}(\omega)$ and set $|z| = r$. Then $(r, |w|) \in \mathcal{D}$ and so $r \in \Lambda$. This implies that $(z, w) \in \Gamma(r) \times \Gamma(s_r, t_r)$. Therefore $\mathbb{T}(\omega) \subset \cup_{r \in \Lambda} (\Gamma(r) \times \Gamma(s_r, t_r))$. Conversely, let $r \in \Lambda$ be arbitrary. Let $(z, w) \in \Gamma(r) \times \Gamma(s_r, t_r)$. Then $|z| = r$ and $s_r \leq |w| \leq t_r$. Let $m, n \in \mathbb{Z}$.

(1) If $m \geq 0$ and $n \geq 0$, then $|z|^m |w|^n \leq r^m t_r^n \leq \omega(m, n)$;

(2) If $m \geq 0$ and $n \leq 0$, then $|z|^m |w|^n \leq r^m s_r^n \leq \omega(m, n)$;

(3) If $m \leq 0$ and $n \geq 0$, then $|z|^m |w|^n \leq r^m t_r^n \leq \omega(m, n)$;

(4) If $m \leq 0$ and $n \leq 0$, then $|z|^m |w|^n \leq r^m s_r^n \leq \omega(m, n)$.

Therefore, $(z, w) \in \mathbb{T}(\omega)$. Hence $\cup_{r \in \Lambda} (\Gamma(r) \times \Gamma(s_r, t_r)) \subset \mathbb{T}(\omega)$. $\square$

Theorem 3.7. Let ω_1 and ω_2 be any two weights either on $\mathbb{Z}_+$ or on $\mathbb{Z}$. Let $\omega(m, n) = \omega_1(m) \omega_2(n)$.

(i) If ω is defined on $\mathbb{Z}_+^2$, then $\mathbb{G}(\omega) = \mathbb{D}_{\beta_{1,0}} \times \mathbb{D}_{\beta_{2,0}}$;

(ii) If ω is defined on $\mathbb{Z}^2$, then $\mathbb{T}(\omega) = \Gamma(\alpha_{1,0}, \beta_{1,0}) \times \Gamma(\alpha_{2,0}, \beta_{2,0})$.

Proof. (i) By Theorem 3.2, it is clear that $\mathbb{G}(\omega) \subseteq \mathbb{D}_{\beta_{1,0}} \times \mathbb{D}_{\beta_{2,0}}$. Conversely, let $(z, w) \in \mathbb{D}_{\beta_{1,0}} \times \mathbb{D}_{\beta_{2,0}}$. For $m, n \in \mathbb{Z}_+$, we have $|z|^m \leq \omega(m, 0) = \omega_1(m)\omega_2(0)$ and $|w|^n \leq \omega(0, n) = \omega_1(0)\omega_2(n)$. This implies that

$$|z|^m |w|^n \leq \omega_1(m)\omega_2(n)\omega_1(0)\omega_2(0) = c\omega(m, n),$$

where $c = \omega_1(0)\omega_2(0)$. Hence $(z, w) \in \mathbb{G}(\omega)$.

(ii) By Theorem 3.5, it is clear that $\mathbb{T}(\omega) \subseteq \Gamma(\alpha_{1,0}, \beta_{1,0}) \times \Gamma(\alpha_{2,0}, \beta_{2,0})$. Conversely, let $(z, w) \in \Gamma(\alpha_{1,0}, \beta_{1,0}) \times \Gamma(\alpha_{2,0}, \beta_{2,0})$. Then, for $m, n \in \mathbb{Z}$, we have, $\omega_1(-m)^{-1/m}\omega_2(0)^{-1/m} \leq |z| \leq \omega_1(m)^{1/m}\omega_2(0)^{1/m}$ and $\omega_1(0)^{-1/n}\omega_2(-n)^{-1/n} \leq |w| \leq \omega_1(0)^{1/n}\omega_2(n)^{1/n}$. Therefore, $|z|^m |w|^n \leq \omega_1(m)\omega_2(n)\omega_1(0)\omega_2(0) = c\omega(m, n)$, for all $m, n \in \mathbb{Z}$, where $c = \omega_1(0)\omega_2(0)$. Hence $(z, w) \in \mathbb{T}(\omega)$ by Lemma 3.1. $\square$

4. Examples of weights

Here, we discuss some examples of weights on $\mathbb{Z}_+^2$ and $\mathbb{Z}^2$ and then find the Gel'fand spaces of the corresponding Beurling algebras $l^1(\mathbb{Z}_+^2, \omega)$ and $l^1(\mathbb{Z}^2, \omega)$, i.e., $\mathbb{G}(\omega)$ and $\mathbb{T}(\omega)$. Note that a weight on $\mathbb{Z}^2$ is also a weight on $\mathbb{Z}_+^2$. So we shall take examples of weights on $\mathbb{Z}^2$.

Example 1. Let $\omega(m, n) = 1$ be a weight on $\mathbb{Z}^2$. Then $\mathbb{G}(\omega) = \mathbb{D}_1 \times \mathbb{D}_1$ and $\mathbb{T}(\omega) = \Gamma(1) \times \Gamma(1)$. Because $\mu_2 = \beta_{1,0} = \beta_{2,0} = 1$. Also, $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = \alpha_{1,0} = \alpha_{2,0} = \beta_{1,0} = \beta_{2,0} = 1$.

Example 2. Let $\omega(m, n) = e^{|m|} + e^{|n|}$ ($m, n \in \mathbb{Z}$) be a weight on $\mathbb{Z}^2$. Then

- (i) $\alpha_1 = \alpha_2 = \alpha_{1,0} = \alpha_{2,0} = \rho_1 = e^{-1}$;
- (ii) $\beta_1 = \beta_2 = \beta_{1,0} = \beta_{2,0} = \rho_2 = e$;
- (iii) $\mu_2 = \sqrt{e}$;
- (iv) $\mathbb{G}(\omega) = (\mathbb{U}_{\sqrt{e}} \times \mathbb{U}_{\sqrt{e}}) \cup (\mathbb{U}_1 \times \Gamma(\sqrt{e}, e)) \cup (\Gamma(\sqrt{e}, e) \times \mathbb{U}_1) \cup (\mathbb{S}_4 \cup \mathbb{S}_4^{op})$,
 where $\mathbb{S}_4 = \{(z, w) : 1 \leq |z| \leq \sqrt{e} \leq |w| \leq e \text{ and } |z||w| \leq e\}$
 and $\mathbb{S}_4^{op} = \{(w, z) : (z, w) \in \mathbb{S}_4\}$
 i.e. $\mathbb{D}_{\mu_2} \times \mathbb{D}_{\mu_2} \subsetneq \mathbb{G}(\omega) \subsetneq \mathbb{D}_{\beta_{1,0}} \times \mathbb{D}_{\beta_{2,0}}$;
- (v) $\mathbb{T}(\omega) = \Gamma(\frac{1}{\sqrt{e}}, \sqrt{e}) \times \Gamma(\frac{1}{\sqrt{e}}, \sqrt{e}) \cup [\cup_{i=1}^4 (\mathbb{T}_i \cup \mathbb{T}_i^{op})]$
 where $\mathbb{T}_1 = \{(z, w) : 1 \leq |z| \leq \sqrt{e} \leq |w| \leq e \text{ and } |z||w| \leq e\}$,
 $\mathbb{T}_2 = \{(z, w) : \frac{1}{\sqrt{e}} \leq |z| \leq 1, \sqrt{e} \leq |w| \leq e \text{ and } |z|^{-1}|w| \leq e\}$,
 $\mathbb{T}_3 = \{(z, w) : \frac{1}{e} \leq |z| \leq \frac{1}{\sqrt{e}}, 1 \leq |w| \leq \sqrt{e} \text{ and } |z|^{-1}|w| \leq e\}$,
 $\mathbb{T}_4 = \{(z, w) : \frac{1}{e} \leq |z| \leq \frac{1}{\sqrt{e}} \leq |w| \leq 1 \text{ and } |z||w| \geq \frac{1}{e}\}$, and
 $\mathbb{T}_i^{op} = \{(w, z) : (z, w) \in \mathbb{T}_i\}$ ($i = 1, 2, 3, 4$)
 i.e. $\Gamma(\sqrt{\alpha_1}, \sqrt{\beta_1}) \times \Gamma(\sqrt{\alpha_2}, \sqrt{\beta_2}) \subsetneq \mathbb{T}(\omega) \subsetneq \Gamma(\alpha_{1,0}, \beta_{1,0}) \times \Gamma(\alpha_{2,0}, \beta_{2,0})$.

Solution. Let $\eta(m, n) = \log \omega(m, n)$ ($m, n \in \mathbb{Z}$).

(i) $\alpha_1 = \alpha_2 = \alpha_{1,0} = \alpha_{2,0} = \rho_1 = e^{-1}$.

It is enough to prove that $\sup\{\frac{\eta(-m,n)}{-m} : m \in \mathbb{N}, n \in \mathbb{Z}\} = -1$. Note that, for $m \in \mathbb{N}$ and $n \in \mathbb{Z}$,

$$\begin{aligned} \frac{\eta(-m,n)}{-m} &= \frac{\log \omega(-m,n)}{-m} = \frac{\log(e^m + e^{|n|})}{-m} > \frac{\log(e^m + e^{|n|+1})}{-m} \\ &> \frac{\log(e^m e^{|n|+1})}{-m} = \frac{\log e^{m+|n|+1}}{-m} = \frac{m+|n|+1}{-m}. \end{aligned}$$

Hence, $\sup\{\frac{\eta(-m,n)}{-m} : m \in \mathbb{N}, n \in \mathbb{Z}\} \geq \limsup_{m \rightarrow \infty} \frac{\eta(-m,n)}{-m} \geq \limsup_{m \rightarrow \infty} \frac{m+|n|+1}{-m} = -1$. On the other hand, for $m \in \mathbb{N}$ and $n \in \mathbb{Z}$, we have

$$\frac{\eta(-m,n)}{-m} = \frac{\log \omega(-m,n)}{-m} = \frac{\log(e^m + e^{|n|})}{-m} < \frac{\log e^m}{-m} = -1.$$

Therefore, $\alpha_1 = e^{-1}$.

Similarly, $\alpha_2 = e^{-1}$, because $\omega(m,n) = \omega(n,m)$.

Since $\alpha_{1,0} = \lim_{m \rightarrow \infty} \omega(-m,0)^{-1/m} = \lim_{m \rightarrow \infty} (e^m + 1)^{-1/m} = e^{-1}$. Similarly, we have, $\alpha_{2,0} = e^{-1}$.

We know that $\rho_1 = \lim_{n \rightarrow \infty} \omega(-n,-n)^{-1/n} = \lim_{n \rightarrow \infty} (e^n + e^n)^{-1/n} = e^{-1}$.

(ii) This can be proved by similar arguments given in (i) above, we get, $\beta_1 = \beta_2 = \beta_{1,0} = \beta_{2,0} = \rho_2 = e$.

(iii) $\mu_2 = \sqrt{e}$.

It is enough to show that $\inf\{\frac{\eta(m,n)}{m+n} : (m,n) \neq (0,0)\} = \frac{1}{2}$. For $m = n$, note that,

$$\frac{\eta(n,n)}{n+n} = \frac{\log(e^n + e^n)}{2n} = \frac{\log 2 + n}{2n}.$$

Hence, $\inf\{\frac{\eta(m,n)}{m+n} : (m,n) \neq (0,0)\} \leq \lim_{n \rightarrow \infty} \frac{\eta(n,n)}{2n} \leq \lim_{n \rightarrow \infty} \frac{\log 2 + n}{2n} = \frac{1}{2}$. Thus $\mu_2 \leq \frac{1}{2}$. On the other hand, we have

$$\frac{\eta(m,n)}{m+n} = \frac{\log \omega(m,n)}{m+n} = \frac{\log(e^m + e^n)}{m+n} > \frac{1}{2}.$$

Hence $\mu_2 = e^{\frac{1}{2}}$.

(iv) $\mathbb{G}(\omega) = (\mathbb{U}_{\sqrt{e}} \times \mathbb{U}_{\sqrt{e}}) \cup (\mathbb{U}_1 \times \Gamma(\sqrt{e}, e)) \cup (\Gamma(\sqrt{e}, e) \times \mathbb{U}_1) \cup (\mathbb{S}_4 \cup \mathbb{S}_4^{op})$.

Note that $\mu_2 = \sqrt{e}$. So, by Theorem 3.2(i), we have $\mathbb{U}_{\sqrt{e}} \times \mathbb{U}_{\sqrt{e}} \subset \mathbb{G}(\omega)$.

Let $(z, w) \in \mathbb{U}_1 \times \Gamma(\sqrt{e}, e)$. Then $|z| \leq 1$ and $|w| \leq e$. This implies that

$$|z|^m |w|^n \leq |w|^n \leq e^n < e^m + e^n = \omega(m, n).$$

Thus $\mathbb{U}_1 \times \Gamma(\sqrt{e}, e) \subset \mathbb{G}(\omega)$. Similarly $\Gamma(\sqrt{e}, e) \times \mathbb{U}_1 \subset \mathbb{G}(\omega)$.

Let $(z, w) \in \mathbb{S}_4$ and let $m, k \in \mathbb{Z}_+$ and $n = m + k$. Then

$$\begin{aligned} |z|^m |w|^n &= |z|^m |w|^m |w|^k \leq e^m |w|^k \\ &\leq e^m e^k < e^m (1 + e^k) = e^m + e^{m+k} \\ &= e^m + e^n = \omega(m, n). \end{aligned}$$

Similarly if $(z, w) \in \mathbb{S}_4^{op}$. Then $|z|^m |w|^n \leq \omega(m, n)$.

Conversely, let $(z, w) \in \mathbb{G}(\omega)$. Then $|z|^m |w|^m \leq \omega(m, m) = 2e^m$ ($m \in \mathbb{Z}_+$).

This implies that $|z||w| \leq e$. Now, we have following two cases:

Case (i). $0 < |z| < \sqrt{e}$. Then we have either $0 < |w| < \sqrt{e}$ or $\sqrt{e} \leq |w| \leq e$. Thus (z, w) belongs to $\mathbb{U}_{\sqrt{e}} \times \mathbb{U}_{\sqrt{e}}$ or $\mathbb{S}_4(\omega)$ or $\mathbb{U}_1 \times \Gamma(\sqrt{e}, e)$.

Case (ii). $\sqrt{e} \leq |z| \leq e$. Then $\sqrt{e}|w| \leq |z||w| \leq e$. This implies $|w| \leq \sqrt{e}$. Thus (z, w) belongs to $\Gamma(\sqrt{e}, e) \times \mathbb{U}_1$ or $(z, w) \in \mathbb{S}_4^{op}$.

$$(v) \mathbb{T}(\omega) = \Gamma(\frac{1}{\sqrt{e}}, \sqrt{e}) \times \Gamma(\frac{1}{\sqrt{e}}, \sqrt{e}) \cup [\cup_{i=1}^4 (\mathbb{T}_i \cup \mathbb{T}_i^{op})];$$

Note that, $\alpha_1 = \alpha_2 = e^{-1}$ and $\beta_1 = \beta_2 = e$. By Theorem 3.5(i), we have $\Gamma(\frac{1}{\sqrt{e}}, \sqrt{e}) \times \Gamma(\frac{1}{\sqrt{e}}, \sqrt{e}) \subset \mathbb{T}(\omega)$. Now let $(z, w) \in \mathbb{T}_1$.

Let $m, k \in \mathbb{Z}_+$ or $\mathbb{Z}_-$ and $n = m + k$. Then

$$\begin{aligned} |z|^m |w|^n &= |z|^m |w|^m |w|^k \leq |z|^{|m|} |w|^{|m|} |w|^{|k|} \\ &\leq e^{|m|} |w|^{|k|} < e^{|m|} (1 + e^{|k|}) = e^{|m|} + e^{|m|+|k|} \\ &= e^{|m|} + e^{|n|} = \omega(m, n). \end{aligned}$$

Let $m \in \mathbb{Z}_+$, $k \in \mathbb{Z}_-$, and $n = m + k$. Then, it is clear that $|w|^k \leq 1 + e^k$, for all $k \in \mathbb{Z}$. By repeating above arguments, we get

$$|z|^m |w|^n \leq e^m |w|^k \leq e^m (1 + e^k) = e^m + e^{|n|} = \omega(m, n).$$

If $m \in \mathbb{Z}_-$ and $k \in \mathbb{Z}_+$, then by reversing the role of m and k in above argument, we can prove that $|z|^m |w|^n \leq \omega(m, n)$. Hence $\mathbb{T}_1 \subset \mathbb{T}(\omega)$.

By similar argument we can show that $\mathbb{T}_1^{op} \cup \mathbb{T}_2 \cup \mathbb{T}_2^{op} \cup \mathbb{T}_3 \cup \mathbb{T}_3^{op} \cup \mathbb{T}_4 \cup \mathbb{T}_4^{op} \subset \mathbb{T}(\omega)$.

Conversely, let $(z, w) \in \mathbb{T}(\omega)$. Then $|z|^m |w|^m \leq \omega(m, m) = 2e^m$, $|z|^{-m} |w|^{-m} \leq 2e^m$, and $|z|^{-m} |w|^m \leq 2e^m$, for all $m \in \mathbb{Z}_+$. This implies that $|z||w| \leq e$, $|z|^{-1}|w| \leq e$, $|z||w|^{-1} \leq e$, and $|z|^{-1}|w|^{-1} \leq e$. Since $\alpha_1 = \alpha_{1,0} = \alpha_2 = \alpha_{2,0} = e^{-1}$ and $\beta_1 = \beta_{1,0} = \beta_2 = \beta_{2,0} = e$, we have the following cases for z and w :

Case (i). Let $\frac{1}{e} \leq |z| \leq \frac{1}{\sqrt{e}}$. Then $|z|^{-1}|w|^{-1} \leq e$ implies that $\frac{1}{\sqrt{e}} \leq |w| \leq 1$ and $|z|^{-1}|w| \leq e$ implies that $1 \leq |w| \leq \sqrt{e}$. Therefore $(z, w) \in \mathbb{T}_3 \cup \mathbb{T}_4$.

Case (ii). Let $\frac{1}{\sqrt{e}} \leq |z| \leq \sqrt{e}$. Then $\frac{1}{e} \leq |w| \leq e$.

If $\frac{1}{\sqrt{e}} \leq |z| \leq 1$ and $\frac{1}{e} \leq |w| \leq \frac{1}{\sqrt{e}}$. Then $(z, w) \in \mathbb{T}_4^{op}$.

If $\frac{1}{\sqrt{e}} \leq |z| \leq 1$ and $\frac{1}{\sqrt{e}} \leq |w| \leq \sqrt{e}$. Then $(z, w) \in \Gamma(\frac{1}{\sqrt{e}}, \sqrt{e}) \times \Gamma(\frac{1}{\sqrt{e}}, \sqrt{e})$.

If $\frac{1}{\sqrt{e}} \leq |z| \leq 1$ and $\sqrt{e} \leq |w| \leq e$. Then $(z, w) \in \mathbb{T}_2$.

If $1 \leq |z| \leq \sqrt{e}$ and $\frac{1}{e} \leq |w| \leq \frac{1}{\sqrt{e}}$. Then $(z, w) \in \mathbb{T}_3^{op}$.

If $1 \leq |z| \leq \sqrt{e}$ and $\frac{1}{\sqrt{e}} \leq |w| \leq \sqrt{e}$. Then $(z, w) \in \Gamma(\frac{1}{\sqrt{e}}, \sqrt{e}) \times \Gamma(\frac{1}{\sqrt{e}}, \sqrt{e})$.

If $1 \leq |z| \leq \sqrt{e}$ and $\sqrt{e} \leq |w| \leq e$. Then $(z, w) \in \mathbb{T}_1$.

Case (iii). Let $\sqrt{e} \leq |z| \leq e$. Then $|z||w| \leq e$ implies that $1 \leq |w| \leq \sqrt{e}$ and $|z||w|^{-1} \leq e$ implies that $\frac{1}{\sqrt{e}} \leq |w| \leq 1$. Therefore, $(z, w) \in \mathbb{T}_1^{op} \cup \mathbb{T}_2^{op}$.

Example 3. Let $\omega(m, n) = e^{|m|} + e^{-n}$ ($m, n \in \mathbb{Z}$). Then

$$(i) \quad \alpha_1 = \alpha_{1,0} = \rho_1 = e^{-1}; \alpha_2 = \alpha_{2,0} = 1;$$

$$(ii) \quad \beta_1 = \beta_{1,0} = \rho_2 = e; \beta_2 = \beta_{2,0} = 1;$$

$$(iii) \quad \mu_2 = 1;$$

$$(iv) \quad \mathbb{G}(\omega) = \mathbb{D}_e \times \mathbb{D}_1. \text{ i.e } \mathbb{D}_{\mu_2} \times \mathbb{D}_{\mu_2} \subsetneq \mathbb{G}(\omega) = \mathbb{D}_{\beta_{1,0}} \times \mathbb{D}_{\beta_{2,0}}.$$

$$(v) \quad \mathbb{T}(\omega) = \Gamma(\frac{1}{e}, e) \times \Gamma(1).$$

$$\text{i.e. } \Gamma(\sqrt{\alpha_1}, \sqrt{\beta_1}) \times \Gamma(\sqrt{\alpha_2}, \sqrt{\beta_2}) \subsetneq \mathbb{T}(\omega) = \Gamma(\alpha_{1,0}, \beta_{1,0}) \times \Gamma(\alpha_{2,0}, \beta_{2,0}).$$

Solution. Define $\eta(m, n) = \log \omega(m, n)$ ($m, n \in \mathbb{Z}$).

(i) $\alpha_1 = \alpha_{1,0} = \rho_1 = e^{-1}; \alpha_2 = \alpha_{2,0} = 1$ can be proved as per the argument given in (ii) below.

$$(ii) \quad \beta_1 = \beta_{1,0} = \rho_2 = e; \beta_2 = \beta_{2,0} = 1;$$

It is enough to prove that $\inf\{\frac{\eta(m,n)}{m} : m \in \mathbb{N}, n \in \mathbb{Z}\} = 1$. Note that, for $m \in \mathbb{N}$ and $n \in \mathbb{Z}$,

$$\begin{aligned} \frac{\eta(m, n)}{m} &= \frac{\log \omega(m, n)}{m} = \frac{\log(e^m + e^{-n})}{m} \\ &\leq \frac{\log(e^m + e^m)}{m} \leq \frac{\log 2 + m}{m}. \end{aligned}$$

Hence, $\inf\{\frac{\eta(m,n)}{m} : m \in \mathbb{N}, n \in \mathbb{Z}\} \leq \liminf_{m \rightarrow \infty} \frac{\eta(m,n)}{m} \leq \liminf_{m \rightarrow \infty} \frac{\log 2 + m}{m} = 1$. On the other hand, for $m \in \mathbb{N}$ and $n \in \mathbb{Z}$, we have

$$\frac{\eta(m, n)}{m} = \frac{\log(e^m + e^{-n})}{m} \geq \frac{\log e^m}{m} = 1.$$

Thus $\beta_1 = e$.

It is enough to prove that $\inf\{\frac{\eta(m,n)}{n} : m \in \mathbb{Z}, n \in \mathbb{N}\} = 0$. Note that, for $m \in \mathbb{Z}$ and $n \in \mathbb{N}$,

$$\begin{aligned} \frac{\eta(m, n)}{n} &= \frac{\log \omega(m, n)}{n} = \frac{\log(e^{|m|} + e^{-n})}{n} \\ &\leq \frac{\log(e^{|m|} + e^{|m|})}{n} \leq \frac{\log 2 + |m|}{n} \end{aligned}$$

Hence, $0 \leq \inf\{\frac{\eta(m,n)}{n} : m \in \mathbb{Z}, n \in \mathbb{N}\} \leq \liminf_{n \rightarrow \infty} \frac{\eta(m,n)}{n} \leq \liminf_{n \rightarrow \infty} \frac{\log 2 + |m|}{n} = 0$. Therefore, $\beta_2 = 1$.

We know that $\beta_{1,0} = \lim_{m \rightarrow \infty} \omega(m, 0)^{1/m} = \lim_{m \rightarrow \infty} (e^{|m|} + 1)^{1/m} = e$.

Also, $\beta_{2,0} = \lim_{n \rightarrow \infty} \omega(0, n)^{1/n} = \lim_{n \rightarrow \infty} (1 + e^{-n})^{1/n} = 1$ and $\rho_2 = \lim_{n \rightarrow \infty} \omega(n, n)^{1/n} = \lim_{n \rightarrow \infty} (e^{|n|} + e^{-n})^{1/n} = e$.

(iii) $\mu_2 = 1$.

It is enough to prove that $\inf\{\frac{\eta(m,n)}{m+n} : (m,n) \neq (0,0)\} = 0$. Note that, for $(m,n) \neq (0,0)$, we have

$$\frac{\eta(m,n)}{m+n} = \frac{\log(e^m + e^{-n})}{m+n} \leq \frac{\log(e^m + e^m)}{m+n} \leq \frac{\log 2 + m}{m+n}$$

Hence, $0 \leq \inf\{\frac{\eta(m,n)}{m+n} : (m,n) \neq (0,0)\} \leq \liminf_{n \rightarrow \infty} \frac{\log 2 + m}{m+n} = 0$. Thus $\mu_2 = 1$.

(iv) $\mathbb{G}(\omega) = \mathbb{D}_e \times \mathbb{D}_1$.

Here, $\beta_{1,0} = e$ and $\beta_{2,0} = 1$. Now let $(z, w) \in \mathbb{D}_e \times \mathbb{D}_1$ and $f \in l^1(\mathbb{Z}_+^2, \omega)$, $m, n \in \mathbb{Z}_+$. Then

$$|z|^m |w|^n \leq e^m \leq (e^m + e^{-n}) = \omega(m, n).$$

This implies that $\mathbb{G}(\omega) = \mathbb{D}_e \times \mathbb{D}_1$.

(v) $\mathbb{T}(\omega) = \Gamma(\frac{1}{e}, e) \times \Gamma(1)$;

Since $\alpha_1 = \alpha_{1,0} = e^{-1}$, $\beta_1 = \beta_{1,0} = e$ and $\alpha_2 = \alpha_{2,0} = \beta_2 = \beta_{2,0} = 1$. By Theorem 3.5(i), we have $\Gamma(\frac{1}{\sqrt{e}}, \sqrt{e}) \times \Gamma(1) \subset \mathbb{T}(\omega)$.

Let $\frac{1}{e} \leq |z| \leq e$ and $|w| = 1$. Then for $m, n \in \mathbb{Z}$, note that

$$|z|^m |w|^n \leq |z|^m \leq e^{|m|} \leq e^{|m|} + e^{-n} = \omega(m, n).$$

Hence $\mathbb{T}(\omega) = \Gamma(\frac{1}{e}, e) \times \Gamma(1)$.

Example 4. Let $\omega(m, n) = e^{|m|} + |n|$ be a weight on $\mathbb{Z}^2$. Then

(i) $\alpha_1 = \alpha_{1,0} = \rho_1 = e^{-1}$, $\alpha_2 = \alpha_{2,0} = 1$;

(ii) $\beta_1 = \beta_{1,0} = \rho_2 = e$, $\beta_2 = \beta_{2,0} = 1$;

(iii) $\mu_2 = 1$;

(iv) The Gel'fand set $\mathbb{G}(\omega) = \mathbb{D}_e \times \mathbb{D}_1$. i.e. $\mathbb{D}_{\mu_2} \times \mathbb{D}_{\mu_2} \subsetneq \mathbb{G}(\omega) = \mathbb{D}_{\beta_{1,0}} \times \mathbb{D}_{\beta_{2,0}}$.

(v) The Gel'fand set $\mathbb{T}(\omega) = \Gamma(\frac{1}{e}, e) \times \Gamma(1)$. i.e. $\Gamma(\sqrt{\alpha_1}, \sqrt{\beta_1}) \times \Gamma(\sqrt{\alpha_2}, \sqrt{\beta_2}) \subsetneq \mathbb{T}(\omega) = \Gamma(\alpha_{1,0}, \beta_{1,0}) \times \Gamma(\alpha_{2,0}, \beta_{2,0})$.

Solution. By the arguments given in Example 3 (i), (ii), and (iii), the first three statements can be proved easily.

(iv) Let $(z, w) \in \mathbb{D}_e \times \mathbb{D}_1$ and $m, n \in \mathbb{Z}_+$. Then

$$|z|^m |w|^n \leq e^m \leq (e^m + n) = \omega(m, n).$$

This implies that $\mathbb{G}(\omega) = \mathbb{D}_e \times \mathbb{D}_1$.

(v) Let $(z, w) \in \Gamma(\frac{1}{e}, e) \times \Gamma(1)$ and $m, n \in \mathbb{Z}$. Then

$$|z|^m |w|^n \leq e^{|m|} \leq (e^{|m|} + |n|) = \omega(m, n).$$

Hence $\mathbb{T}(\omega) = \Gamma(\frac{1}{e}, e) \times \Gamma(1)$.

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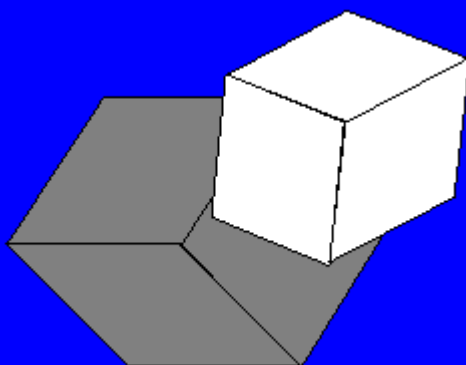
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