N° 46 – September 2021

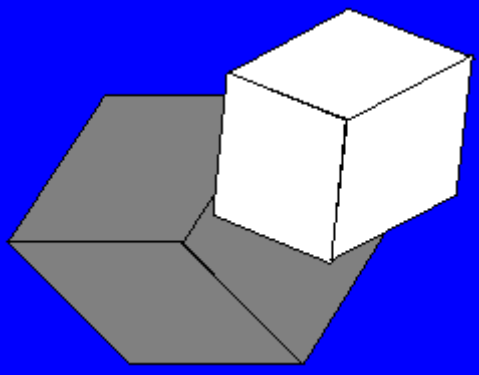
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ISSN 2239-0227

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In memoriam of Professors **Vasile Oproiu** and **James Jantosciak**

The "Italian Journal of Pure and Applied Mathematics" cannot take anymore advantage of the precious collaboration of professors **Vasile Oproiu** and **James Jantosciak** who passed away in the past years.

The Editorial Board and Managing Board members express their deep sorrow for the loss of these two brilliant mathematicians.

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On primary subgroups and the structure of finite groups

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Abstract. If P is a p-group for some prime p we shall write $\mathcal{U}(P)$ to denote the set of all maximal subgroups of P and $\mathcal{U}_d(P) = \{P_1, \ldots, P_d\}$ to denote any set of maximal subgroups of P such that $\bigcap_{i=1}^{d} P_i = \Phi(P)$ and d is as small as possible. In this paper, the structure of a finite group G under some assumptions on the c-normal or ss-quasinormal subgroups in $\mathcal{U}_d(P)$, for each prime p, and Sylow p-subgroups P of G is researched. Some known results are generalized.

Keywords: c-normal subgroup, ss-quasinormal subgroup, supersolvable groups.

1. Introduction

All groups considered in this paper are finite. Let G be a group and let $\mathcal{U}(G)$ be the set of all maximal subgroups of all Sylow subgroups of G. A interesting topic in group theory is to study the influence of the elements of $\mathcal{U}(G)$ on the structure of G. A typical result in this direction is due to Srinivasan [1]. He proved that G is supersolvable provided that every member of $\mathcal{U}(G)$ is normal in G. This result has been widely generalized.

A subgroup H of G is called s-quasinormal in G provided H permutes with all Sylow subgroups of G, i.e, $HP = PH$ for any Sylow subgroup P of G. This concept was introduced by Kegel in [2] and has been studied extensively by Deskins [3] and Schmidt [4].

More recently, Li et al. [5] generalized s-quasinormal subgroups to ss-quasinormal subgroups. A subgroup H of G is said to be ss-quasinormal subgroup of G, if there exists a subgroup B of G such that $G = HB$ and H permutes with every Sylow subgroup of B. In [5], They showed that, let p be the smallest prime dividing the order of a group G and P a Sylow p-subgroup of G. If every member of some fixed $\mathcal{U}_d(P)$ is ss-quasinormal in G, then G is p-nilpotent. Furthermore, they showed that, for every prime p dividing the order of G and $P \in \mathrm{Syl}_p(G)$, if every member of some fixed $\mathcal{U}_d(P)$ is ss-quasinormal in G, then G is supersolvable.

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As another generalization of the normality, Wang [6] introduced the following concept: A subgroup H of G is called c-normal in G if there is a normal subgroup K such that $G = HK$ and $H \cap K \leq H_G$, where H_G is the normal core of H in G. In [6], Wang showed that G is supersolvable if every member of $\mathcal{U}(G)$ is c-normal. Wang′s result has been generalized by some authors(see [7-11], etc). For example, Guo and Shum showed in [7] the following result. Let p be the smallest prime dividing the order of G and let P be a Sylow p-subgroup of G. If every member of $\mathcal{U}(P)$ is c-normal, then G is p-nilpotent. The research on c-normal subgroups has formed a series, which is similar to the series of s-quasinormal subgroups. However, the two series are independent of each other. The aim of this article is to unify and improve the results of [1], [5], [6] and some of [7].

If P is a p-group for some prime p we shall write $\mathcal{U}(P)$ to denote the set of all maximal subgroups of P and $\mathcal{U}_d(P) = \{P_1, \ldots, P_d\}$ to denote any set of maximal subgroups of P such that $\bigcap_{i=1}^{d} P_i = \Phi(P)$ and d is as small as possible.

Such subset $\mathcal{U}_d(P)$ is not unique for a fixed P in general. We know that

$$|\mathcal{U}(P)| = \frac{p^d - 1}{p - 1}, |\mathcal{U}_d(P)| = d, \lim_{d \to \infty} \frac{p^d - 1}{(p - 1)d} = \infty,$$

so $|\mathcal{U}(P)| \gg |\mathcal{U}_d(P)|$.

In this paper, we study the influence of the members of some fixed $\mathcal{U}_d(P)$ on the structure of group G. Our results are more general.

2. Basic definitions and preliminary results

In this section, we collect some known results that are useful later.

Lemma 2.1 ([5]). *Suppose that H is an ss-quasinormal subgroup of G, $K \leq G$ and N is a normal subgroup of G. Then, we have the following:*

(1) *If $H \leq K$, then H is an ss-quasinormal subgroup of K;*

(2) *HN/N is an ss-quasinormal subgroup of G/N.*

Lemma 2.2 ([6]). *Let $X \leq H \leq G$ and $N \trianglelefteq G$. Then:*

(a) *If X is c-normal in G, then X is also c-normal in H;*

(b) *If X is c-normal in G, then XN/N is c-normal in G/N.*

Lemma 2.3 ([12]). *If P is a Sylow p-subgroup of G and $N \trianglelefteq G$ such that $P \bigcap N \leq \Phi(P)$, then N is p-nilpotent.*

Lemma 2.4 ([13]). *A group G is superslovable if and only if there exists a subgroup of order dividing $|H|$ for every subgroup H of G.*

Lemma 2.5 ([14]). *Let p_1 be the minimal prime dividing $|G|$ and p_s the maximal prime dividing $|G|$. If G possesses two supersolvable subgroups H and K with $|G : H| = p_1$ and $|G : K| = p_s$, then G is superslovable.*

Lemma 2.6 ([5]). *Let H be a p-subgroup of G. Then, the following statements are equivalent:*

(a) *H is s-quasinormal in G;*

(b) *$H \leq O_p(G)$, and H is ss-quasinormal in G.*

Lemma 2.7 ([15]). *Let G be a group and let P_0 be a maximal subgroup of P. Then the following two statements are equivalent:*

(a) *P_0 is normal in G;*

(b) *P_0 is s-quasinormal in G.*

3. Main results

Theorem 3.1. *Let p be the smallest prime dividing the order of a group G and P be a Sylow p-subgroup of G. Assume that every member of some fixed $\mathcal{U}_d(P)$ is either c-normal or ss-quasinormal in G. Then G is p-nilpotent.*

Proof. Assume that the result is not true and let G be a counterexample of minimal order. Let $\mathcal{U}_d(P) = \{P_1, \ldots, P_d\}$. By hypothesis, each P_i is either c-normal or ss-quasinormal in G. Without loss of generality, let I_1 be the subset of $\{1, ..., d\}$ such that every $P_i (i \in I_1)$ is c-normal in G and I_2 is the subset such that every $P_i (i \in I_2)$ is ss-quasinormal in G. We prove the theorem by the following claims:

(1) $O_{p'}(G) = 1$.

Set $N = O_{p'}(G)$. Consider the quotient group G/N. We know that PN/N is a Sylow p-subgroup of G/N, $N_{G/N}(PN/N) = N_G(P)N/N$ and $\mathcal{U}(PN/N) = \{P_1N/N, \ldots, P_mN/N\}$. Now, by Lemma 2.1 and Lemma 2.2, we see easily that G/N satisfies the condition. If $O_{p'}(G) > 1$, then $G/O_{p'}(G)$ is p-nilpotent and hence G itself is p-nilpotent, a contradiction. Thus claim (1) holds.

(2) G/P_{iG} is p-nilpotent for all $i \in I_1$, where P_{iG} is the core of P_i in G.

In this case, P_i is a c-normal subgroup of G. We know that there exists a normal subgroup K_i of G such that $G = P_iK_i$ and $P_i \cap K_i = P_{iG}$. Hence,

$$G/P_{iG} = P_i/P_{iG} \cdot K_i/P_{iG}, P_i \cap K_i = P_{iG}.$$

Therefore,

$$|K_i/P_{iG}|_p = |G : P_i|_p = |P : P_i| = p.$$

As p is the smallest prime dividing $|G|$, we know that K_i/P_{iG} is p-nilpotent by Burnside′s theorem. Therefore, K_i/P_{iG} has a normal Hall p'-subgroup H/P_{iG}. We see that H/P_{iG} is also a normal Hall p'-subgroup of G/P_{iG} because K_i/P_{iG} is normal in G/P_{iG}. It follows that G/P_{iG} is p-nilpotent for all $i \in I_1$.

(3) For every $P_i (i \in I_2)$, there exists a normal subgroup H_i of G such that G/H_i is p-nilpotent.

By the condition, there is a subgroup $B \leq G$ such that $G = P_iB$ and and P_i permutes with every Sylow subgroup of B. From $G = P_iB$, we obtain

$$|B : P_i \cap B|_p = |G : P_i|_p = p,$$

and hence, $P_i \cap B$ is of index p in B_p, a Sylow p-subgroup of B containing $P_i \cap B$. Thus, $S \nleq P_i$ for all $S \in \mathrm{Syl}_p(B)$, and $P_iS = SP_i$ is a Sylow p-subgroup of G. In view of $|P : P_i| = p$ and by comparison of orders, $S \cap P_i = B \cap P_i$ for all $S \in \mathrm{Syl}_p(B)$. Therefore,

$$B \cap P_i = \bigcap_{b \in B} (S^b \cap P_i) \leq \bigcap_{b \in B} S^b = O_p(B).$$

We claim that B has a Hall p'-subgroup. In fact, because $|O_p(B) : B \cap P_i| = p$ or 1, it follows that $|B/O_p(B)|_p = p$ or 1. As p is the smallest prime dividing $|G|$, by a well-known theorem of Burnside, $B/O_p(B)$ is p-nilpotent, and hence, B is p-solvable. Therefore, B has a Hall p'-subgroup K. Now, set $\pi(K) = \{p_2, ..., p_s\}$ and $S_j \in \mathrm{Syl}_{p_j}(K)$. By the condition, P_i permutes with every S_j, and so, P_i permutes with the subgroup K. Thus, $P_iK \leq G$. It is easy to see that K is a Hall p'-subgroup of G, and P_iK is a subgroup of index p in G. Let $H_i = P_iK$, where H_i is a subgroup of index p in G. As p is the smallest prime dividing $|G|$, it follows that $H_i \trianglelefteq G$, and hence, G/H_i is a p-nilpotent.

Let

$$N = (\bigcap_{i \in I_1} P_{iG}) \bigcap (\bigcap_{i \in I_2} H_i).$$

(4) N is p-nilpotent.

First, as all P_{iG} and H_i are normal in G, we get $N \trianglelefteq G$. Second, we consider the subgroup $P \cap N$. Recall that P_i is a Sylow p-subgroup of H_i and $P_i \leq P$, so $P \cap H_i \leq P_i$. Moreover, $P_i \leq P \cap H_i$. We have $P \cap H_i = P_i$. Therefore,

$$P \cap N = (\bigcap_{i \in I_1} P_{iG}) \bigcap (\bigcap_{i \in I_2} H_i \cap P) = (\bigcap_{i \in I_1} P_{iG}) \bigcap (\bigcap_{i \in I_2} P_i) = \Phi(P).$$

Applying Lemma 2.3, we know that N is p-nilpotent.

(5) Final contradiction. □

Now, N possesses a Hall p'-normal subgroup $N_{p'}$ such that $N = N_pN_{p'}$, where N_p is a Sylow p-subgroup of N. Then, $N_{p'}$ char $N \trianglelefteq G$, so $N_{p'}$ is normal in G, and hence, $N_{p'} \leq O_{p'}(G)$. It follows by $O_{p'}(G) = 1$ that $N_{p'} = 1$. Consequently, N is a normal p-subgroup of G, and so, $N = P \cap N = \Phi(P)$. Also, note that the class of p-nilpotent groups is a formation, by steps (2) and (3), we have G/N must be p-nilpotent. It follows that $G/\Phi(P)$ is p-nilpotent. Moreover, by III, 3.3 Hilfs-Satz in [16], $\Phi(P) \leq \Phi(G)$, so $G/\Phi(G)$ is p-nilpotent. It follows that G would be p-nilpotent, contrary to the choice of G.

The following corollaries are immediate from Theorem 3.1.

Corollary 3.2 ([6]). *Let p be the smallest prime dividing the order of a group G and P a Sylow p-subgroup of G. If every member of $\mathcal{U}(P)$ is c-normal, then G is p-nilpotent.*

Corollary 3.3 ([5]). *Let p be the smallest prime dividing the order of a group G and P a Sylow p-subgroup of G. If every member of some fixed $\mathcal{U}_d(P)$ is ss-quasinormal in G, then G is p-nilpotent.*

Theorem 3.4. *Let G be a group. If there exists a normal subgroup H of G such that G/H is supersolvable, and for each Sylow subgroup P of H, every member in some fixed $\mathcal{U}_d(P)$ is either c-normal or ss-quasinormal in G, then G is supersolvable.*

Proof. Suppose that the theorem is false so that there exists a counterexample G of minimal order. We shall finish the proof by the following claims.

(1) H is a q-group for some prime q.

By hypothesis and Theorem 3.1, we have that H has a Sylow tower of supersolvable type. Let q be the largest prime dividing $|H|$, and let Q be a Sylow q-subgroup of H. The property that H possesses an order Sylow tower property implies that Q is normal in H. Now, Q char H and $H \trianglelefteq G$, so $Q \trianglelefteq G$. Furthermore, $(G/Q)/(H/Q) \cong G/H$, and Lemmas 2.1 and 2.2 show that G/Q satisfies the condition of the theorem, by the choice of G, G/Q is supersolvable. Hence, $H = Q$ by the choice of H.

(2) Q is a Sylow q-subgroup of G.

Suppose that Q is not a Sylow q-subgroup of G. Let p be the smallest prime dividing $|G/Q|$ and r the largest prime dividing $|G/Q|$. By (1), G/Q is supersolvable. By Lemma 2.4, G/Q contains two subgroups M_1/Q and M_2/Q with $|G : M_1| = p$ and $|G : M_2| = r$. By Lemmas 2.1 and 2.2, $(M_i, Q)(i = 1, 2)$ satisfy the condition of the theorem. By the choice of G, M_1 and M_2 are supersolvable. Now, by Lemma 2.5, G would be supersolvable, which is a contradiction. Thus, (2) holds.

(3) $\Phi(Q) = 1$.

Otherwise, by Lemmas 2.1 and 2.2, $G/\Phi(Q)$ satisfies the hypothesis, applying induction, we have $G/\Phi(Q)$ is supersolvable. Furthermore, $\Phi(Q) \leq \Phi(G)$ by III, 3.3 Hilfs-satz in [16], so $G/\Phi(G)$ is supersolvable. It follows that G is supersolvable, which is a contradiction.

(4) Q is a minimal normal subgroup of G.

Let N be a minimal normal subgroup of G contained in Q. Clearly the quotient group $(G/N, Q/N)$ satisfies the condition, so G/N is supersolvable. As the class of supersolvable groups is a formation, N must be the unique minimal normal subgroup of G which is contained in Q and $N \nsubseteq (G)$. So there is a maximal subgroup M of G such that $G = NM$ and $N \cap M = 1$. Thus $Q = N(Q \cap M)$. As $G = QM$ and Q is normal abelian in G, we know that $Q \cap M$ is normal in G. If $Q \cap M > 1$, let N_1 be a minimal normal subgroup of

G such that $N_1 \leq Q \cap M$, then $N_1 \leq Q$ and $N \neq N_1$, this is a contradiction. Hence $Q \cap M = 1$, which implies $Q = N$.

(5) Every $Q_i \in \mathcal{U}_d(Q) = \{Q_1, \ldots, Q_d\}$ is ss-quasinormal in G.

Assume that there is a Q_i in $\mathcal{U}_d(Q)$ such that Q_i is c-normal in G. By definition, there is a normal subgroup K_i of G such that $G = Q_iK_i$ and $Q_i \cap K_i = Q_{iG}$ is a normal subgroup of G. By (4), $Q_i \cap K_i = 1$ or Q. If $Q_i \cap K_i = Q$, then $Q_i = Q$, a contradiction. If $Q_i \cap K_i = 1$, then $Q = Q_i(Q \cap Ki)$. But then $Q \cap K_i$ is a normal subgroup of order q of G. So $Q = Q \cap K_i$ by (4). As the class of supersolvable groups is a formation, thus G is supersolvable, contrary to the choice of G.

(6) $Q_i (i = 1, 2, ..., d)$ are normal subgroups of G.

Lemmas 2.6 and 2.7 imply that $Q_i (i = 1, 2, ..., d)$ are normal subgroups of G.

(7) The final contradiction. □

Now,

$$(G/Q_i)/(Q/Q_i) \cong G/Q,$$

by (1), G/Q is supersolvable. As Q/Q_i is cyclic of order q, it follows that G/Q_i is supersolvable. Set

$$N = \bigcap_{i=1}^{d} Q_i.$$

By the definition of $\mathcal{U}_d(Q)$,

$$\bigcap_{i=1}^{d} Q_i = \Phi(Q),$$

so $N = \Phi(Q)$. Now, by the class of supersolvable groups is a formation, $G/\Phi(Q)$ is supersolvable. It follows that $G/\Phi(G)$ is supersolvable, and hence, G is supersolvable. which is a final contradiction. The proof is now completed.

The following corollaries are immediate from Theorem 3.4.

Corollary 3.5 ([7]). *Let G be a group. If every member of $\mathcal{U}(G)$ is c-normal, then G is supersolvable.*

Corollary 3.6 ([5]). *Let G be a group. For every prime p dividing the order of G and $P \in Syl_p(G)$, if every member of some fixed $\mathcal{U}_d(P)$ is ss-quasinormal in G, then G is supersolvable.*

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Accepted: January 21, 2016

Trapezoidal cubic fuzzy weighted geometric operator and their application to multiple attribute group decision making

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Abstract. In this paper, the score function and the accuracy function of trapezoidal cubic fuzzy number are developed. We extend cubic fuzzy aggregation operators as the trapezoidal cubic fuzzy weighted geometric operator. Finally, we give an illustrate example to show the multiple attribute group decision-making method with trapezoidal cubic fuzzy weighted geometric operator.

Keywords: trapezoidal cubic fuzzy number, trapezoidal cubic fuzzy weighted geometric operator, algorithm, numerical applications.

1. Introduction

The concept of fuzzy set was first proposed by Zadeh [8]. Atanassov [1, 2] initiated the notion of intuitionistic fuzzy set. Later on Bustince and Burillo [3] showed that vague sets are intuitionistic fuzzy sets. Szmidt and Kacprzyk [5] proposed some concepts such as the intuitionistic fuzzy core and developed an approach to aggregate the specific intuitionistic fuzzy preference relations into a

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common fuzzy fondness relation based on fuzzy most corresponded with a fuzzy linguistic calculation.

The multiple attribute group decision-making (MAGDM) problems are the most interesting problems for many decision-making experts. Multiple attribute group decision-making problems have some interesting scientific application in field of management science, decision-making theory, system theory, operational research and economics.

Altered command of aggregation operators is enter in the agree work for aggregating the escape. Namely, their valour occur confident uncertainty degree, which is a very significant negotiator to be taken into description when irritating to suggestion indeed a required role model and winner of judgement problems. Wang and Zhang [7] defined the trapezoidal Intuitionistic fuzzy number (TRIFN) and gave ranking method which transformed the ranking of TRIFNs into ranking of interval numbers. Varanelli et al. [9] introduced the constant starting temperatures for the low-temperature annealing phase.

The idea of cubic set was first proposed by Jun et al. in [4]. The concept of cubic set is a generalization of intuitionistic fuzzy sets, in which there are two representations, one is used for the degree of membership and other is used for the degree of non-membership. The membership function is hold in the technique of retreat, while non-membership is concern the normal fuzzy set .

In this paper, we introduced the concept of trapezoidal cubic fuzzy number and develop their operational laws. Further we give the concept of trapezoidal cubic weighted geometric operator, construct its algorithm and give numerical applications to verify the data.

2. Preliminaries

Definition 2.1 ([1]). *Let H be a universe. Then a fuzzy set J over H is a function defined as follows:$J = \{h, \tilde{\Omega}_J(h) | h \in H\}$,where $\tilde{\Omega}_J : H \to I$. Here $\tilde{\Omega}_J$ called membership function of J and the value $\tilde{\Omega}_J(h)$ is called the grade of membership of $h \in H$.*

Definition 2.2 ([8]). *Let H be a nonempty set. A function $A : H \to [I]$ is called an interval-valued fuzzy set (briefly, an IVF set) in H. Let $[I]^H$ stand for the set of all IVF sets in H. For every $A \in [I]^H$ and $\tilde{h} \in H$, $A(\tilde{h}) = [A^-(\tilde{h}), A^+(\tilde{h})]$ is called the degree of membership of an element $\tilde{h}$ to A, where $A^- : H \to I$ and $A^+ : H \to I$ are fuzzy sets in H which are called a lower fuzzy set and an upper fuzzy set in H, respectively. For simpleness, we denote $A = [A^-, A^+]$. For every $A, B \in [I]^H$, we define $A \subseteq B \Leftrightarrow A(\tilde{h}) \leq B(\tilde{h}) \quad \forall h \in H$ and $A = B \Leftrightarrow A(\tilde{h}) = B(\tilde{h}) \; \forall \tilde{h} \in H$.*

Definition 2.3 ([2]). *An Atanassov intuitionistic fuzzy set on H is a set*

$$J = \{h, \Gamma_J(h), \eta_J(h) : h \in H\},$$

where Γ_J and η_j are membership and non-membership function respectively $\Gamma_J(h) : h \to [0,1],\ h \in H \to \ \Gamma_J(h) \in [0,1],\ \eta_J(h) : h \to [0,1],\ h \in H \to \eta_J(h) \in [0,1]$ and $0 \leq \Gamma_J(h) + \eta_J(h) \leq 1 \quad \forall h \in H, \pi_J(h) = 1 - \Gamma_J(h) - \eta_J(h)$.

Definition 2.4 ([4]). *Let H be a nonempty set. By a cubic set in H we mean a structure $F = \{\langle h, \tilde{\Omega}(\tilde{h}), \tilde{\varkappa}(\tilde{h}) : h \in H\rangle\}$ in which $\tilde{\Omega}$ is an IVF set in H and $\varkappa$ is a fuzzy set in H. A cubic set $\tilde{F} = \{h, \tilde{\Omega}(\tilde{h}), \tilde{\varkappa}(\tilde{h}) : \tilde{h} \in H\}$ is simply denoted by $\tilde{F} = \langle\tilde{\Omega}, \tilde{\varkappa}\rangle$. Denote by C^H the collection of all cubic sets in H. A cubic set $\tilde{F} = \langle\tilde{\Omega}, \tilde{\varkappa}\rangle$ in which $\tilde{\Omega}(\tilde{h}) = 0$ and $\tilde{\varkappa}(\tilde{h}) = 1$(resp.$\tilde{\Omega}(\tilde{h}) = 1$ and $\tilde{\varkappa}(\tilde{h}) = 0$ for all $\tilde{h} \in H$ is denoted by 0 (resp. 1). A cubic set $\tilde{D} = \langle\tilde{\lambda}, \tilde{\xi}\rangle$ in which $\tilde{\lambda}(\tilde{h}) = 0$ and $\tilde{\xi}(\tilde{h}) = 0$ (resp.$\tilde{\lambda}(\tilde{h}) = 1$ and $\tilde{\xi}(\tilde{h}) = 1$) for all $\tilde{h} \in H$ is denoted by 0 (resp. 1).*

Definition 2.5 ([4]). *Let H is a non empty set. A cubic set $F = (\tilde{\vartheta}, \tilde{\lambda})$ in H is said to be an internal cubic set if $\tilde{\vartheta}^-(\tilde{h}) \leq \tilde{\lambda}(\tilde{h}) \leq \tilde{\vartheta}^+(\tilde{h}) \ \ \forall \tilde{h} \in H$.*

Definition 2.6 ([4]). *Let H be a non empty set. A cubic set $F = (\tilde{\vartheta}, \tilde{\lambda})$ on H is said to be an external cubic set if $\tilde{\lambda}(\tilde{h}) \notin (\tilde{\vartheta}^-(\tilde{h}), \tilde{\vartheta}^+(\tilde{h})) \ \ \forall \tilde{h} \in H$.*

Definition 2.7 ([6]). *Let $\tilde{b}_i = \langle[(\tilde{p}_i, \tilde{q}_i, \tilde{r}_i, \tilde{s}_i); (w_{\tilde{b}_i}, \eta_{\tilde{b}_i})]\rangle \ (i = 1, 2, ..., n)$ be a collection of trapezoidal intuitionistic fuzzy number and let TrIFWGO:$\Lambda^n \to \Lambda$,*

$$\text{if } TrIFWGO(\tilde{b}_1, \tilde{b}_2, ...\tilde{b}_n) = \prod_{i=1}^{n} \tilde{b}_i^{w_j} = \left\{ \begin{array}{c} [\prod_{i=1}^{n}(\tilde{p}_i)^{w_i}, \prod_{i=1}^{n}(\tilde{q}_i)^{w_i}, \prod_{i=1}^{n}(\tilde{r}_i)^{w_i}, \\ \prod_{i=1}^{n}(\tilde{s}_i)^{w_i}]; \prod_{i=1}^{n}(w_{b_i^-})^{w_j}, \\ [1 - \prod_{i=1}^{n}(1 - \eta_{b_i})^{w_j}] \end{array} \right\}$$

then TrIFWGO is called trapezoidal intuitionstic fuzzy weighted geometric operator of dimension n, where $w = (w, w_2, ..., w_n)^T$ is the weight vector of $\tilde{b}_i (i = 1, 2, ..., n)$, $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$. Specifically $w = (\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n})^T$, the TrIFWGO operator is reduced to trapezoidal intuitionstic fuzzy geometric averaging (TrIFGA) operator, which is defined as follows:$TrIFGAO(\tilde{b}_1, \tilde{b}_2, ..., \tilde{b}_n) = (\tilde{b}_1 \otimes \tilde{b}_2 \otimes .... \otimes \tilde{b}_n)^{1/n}$.

3. Trapezoidal cubic fuzzy number

Definition 3.1. *The interval valued trapezoidal fuzzy number is defined as:*

$$\lambda_{\tilde{b}}(h) = \begin{cases} \left\{ \frac{(h-r^-)}{(s^- - r^-)}\omega_{b^-}, \frac{(h-r^+)}{(s^+ - r^+)}\omega_{b}^{+} \right\}, & (r^-, s^-) \leq h < (r^+, s^+) \\ \{\omega_{b^-}, \omega_{b^+}\}, & (s^-, t^-) \leq h < (s^+, t^+) \\ \left\{ \frac{(t^- - h)}{(d^- - s^-)}\omega_{b^-}, \frac{(t^- - h)}{(d^- - s^-)}\omega_{b^+} \right\}, & (t^-, u^-) \leq h < (t^+, u^+) \\ 0, & h < (r^-, u^-) \text{ or } h > (r^+, u^+) \end{cases}$$

and its trapezoidal fuzzy set

$$\Gamma_{\tilde{b}}(h)=\begin{cases}\frac{s-h+(h-r)\eta_{\tilde{b}}}{(r-s)}, & r\leq h<s\\ \eta_{\tilde{b}}, & s\leq h\leq t\\ \frac{h-t+(u-h)\eta_{\tilde{b}}}{(u-t)}, & t<h\leq u\\ 0, & h<r \text{ or } h>u)\end{cases}$$

where $0\leq\lambda_{\tilde{b}}(h)\leq 1, 0\leq\Gamma_{\tilde{b}}(h)\leq 1$ *and* $r^-,s^-,t^-,u^-,r^+,s^+,t^+,u^+$ r,s,t,u *are real numbers,* $(r^-,s^-)\leq(t^-,u^-)\leq(r^+,s^+)\leq(t^+,u^+)\leq r\leq s\leq t\leq u$. *The values of* $[\omega_{b^-},\omega_{b^+}]$ *consequently the maximum values of interval valued trapezoidal fuzzy set and* $\eta_{\tilde{b}}$ *minimum value of trapezoidal fuzzy set. Then the TrCFN* $\widetilde{b}$ *basically denoted by* $\tilde{b}=\langle[(r^-,s^-,t^-,u^-),(\omega_{b^-})],[(r^+,s^+,t^+,u^+),(\omega_{b^+})],[(r,s,t,u),(\eta_{\tilde{b}})]\rangle$. *Then* $\tilde{b}$ *is called trapezoidal cubic fuzzy number.*

Definition 3.2. *Let* $\widetilde{b}_1=\left\langle\begin{array}{c}[(p_1^-,q_1^-,\\ r_1^-,s_1^-),(\omega_{b_1^-})],\\ [(p_1^+,q_1^+,\\ r_1^+,s_1^+),(\omega_{b_1^+})],\\ [(p_1,q_1,\\ r_1,s_1);(\eta_{\tilde{b}_1})]\end{array}\right\rangle$ *and* $\tilde{b}_2=\left\langle\begin{array}{c}[(p_2^-,q_2^-,\\ r_2^-,s_2^-),(\omega_{b_2^-})],\\ [(p_2^+,q_2^+,\\ r_2^+,s_2^+),(\omega_{b_2^+})],\\ [(p_2,q_2,\\ q_2,s_2);(\eta_{\tilde{b}_2})]\end{array}\right\rangle$
be two TrCFNs and ξ *be any real number. The operational rules over TrCFNs are solid as under:*

$$(1)\ \widetilde{b}_1+\widetilde{b}_2=\left\langle\begin{array}{c}\{[(\bar{p}_1+\bar{p}_2),(\bar{q}_1+\bar{q}_2),(\bar{r}_1+\bar{r}_2),\\ (\bar{s}_1+\bar{s}_2)],\min(\omega_{b_1^-},\omega_{b_2^-})\},\{[(\overset{+}{p}_1+\overset{+}{p}_2),\\ (\overset{+}{q}_1+\overset{+}{q}_2),(\overset{+}{r}_1+\overset{+}{r}_2),(\overset{+}{s}_1+\overset{+}{s}_2)],\\ \min(\omega_{b_1^+},\omega_{b_2^+})\},\{[(p_1+p_2),(q_1+q_2),\\ (r_1+r_2),(s_1+s_2)];\max(\eta_{\tilde{b}_1},\eta_{\tilde{b}_2})\}\end{array}\right\rangle,$$

$$(2)\ \xi\tilde{b}_1=\left\langle\begin{array}{c}\{[\xi\bar{p}_1,\xi\bar{q}_1,\xi\bar{r}_1,\xi\bar{s}_1],(\omega_{b_1^-})\},\\ \{[\xi\overset{+}{p}_1,\xi\overset{+}{q}_1,\xi\overset{+}{r}_1,\xi\overset{+}{s}_1],(\omega_{b_1^+})\}\\ ,\{[\xi p_1,\xi q_1,\xi r_1,\xi s_1],\eta_{\tilde{b}_1}\}\end{array}\right\rangle,$$

$$(3)\ \widetilde{b}_1.\widetilde{b}_2=\left\langle\begin{array}{c}\{[(\bar{p}_1,\bar{p}_2),(\bar{q}_1,\bar{q}_2),(\bar{r}_1,\bar{r}_2),(\bar{s}_1,\bar{s}_2)],\\ \min(\omega_{b_1^-},\omega_{b_2^-})\},\{[(\overset{+}{p}_1,\overset{+}{p}_2),(\overset{+}{q}_1,\overset{+}{q}_2),(\overset{+}{r}_1,\overset{+}{r}_2),\\ (\overset{+}{s}_1,\overset{+}{s}_2)],\min(\omega_{b_1^+},\omega_{b_2^+})\},\{[(p_1,p_2),(q_1,q_2),\\ (r_1,r_2),(s_1,s_2)];\max(\eta_{\tilde{b}_1},\eta_{\tilde{b}_2})\}\end{array}\right\rangle.$$

Example 1. Let $\widetilde{b}_1=\left\langle\begin{array}{c}[(0.14,0.18,\\ 0.21,0.26),(0.10)],\\ [(0.27,0.31,\\ 0.33,0.38),(0.5)],\\ [(0.35,0.41,\\ 0.45,0.47),(0.2)]\end{array}\right\rangle$ and $\tilde{b}_2=\left\langle\begin{array}{c}[(0.10,0.16,0.20,\\ 0.28),(0.4)],\\ [(0.15,0.19,\\ 0.24,0.32),(0.2)],\\ [(0.12,0.17,0.22,\\ 0.30),(0.6)]\end{array}\right\rangle$
be two TrCFNs and ξ be any real number, then

$$(1)\widetilde{b_1}+\widetilde{b_2}=\left\langle \begin{array}{c}[(0.14+0.10,0.18+0.16,0.21+0.20,\\ 0.26+0.28),mim(0.10,0.4)],[(0.27+0.15,\\ 0.31+0.19,0.33+0.24,0.38+0.32),\\ min(0.5,0.2)],[(0.35+0.12,0.41+0.17,\\ 0.45+0.22,0.47+0.30),max(0.2,0.6)]\end{array}\right\rangle$$

$$=\left\langle \begin{array}{c}[(0.24,0.34,0.41,0.54),0.4],\\ [(0.42,0.50,0.57,0.70),0.2],\\ [(0.47,0.58,0.67,0.77),0.6]\end{array}\right\rangle$$

$(2)\xi=0.2,0.3,0.5,$ then

$$\xi\tilde{b_1}=\left\langle \begin{array}{c}[(0.028,0.036,0.042,0.052),0.02],\\ [(0.081,0.093,0.099,0.114),0.15],\\ [0.175,0.205,0.225,0.235),0.1]\end{array}\right\rangle$$

$$(3)\ \widetilde{b}_1.\widetilde{b}_2=\left\langle \begin{array}{c}[(0.014,0.0288,0.042,0.0728),0.4],\\ [(0.0405,0.0589,0.0792,0.1216),0.2],\\ [(0.042,0.0697,0.99,0.141),0.6]\end{array}\right\rangle.$$

4. Trapezoidal cubic fuzzy weighted geometric operator

Definition 4.1. *Let* $\tilde{b}_i=\left\langle \begin{array}{c}[(\tilde{p}_i^-,\tilde{q}_i^-,\tilde{r}_i^-,\tilde{s}_i^-);(w_{b_i^-}),\\ (\tilde{p}_i^+,\tilde{q}_i^+,\tilde{r}_i^+,\tilde{s}_i^+);(w_{b_i^+})],\\ [(\tilde{p}_i,\tilde{q}_i,\tilde{r}_i,\tilde{s}_i);(\eta_{\tilde{b}_i})]\end{array}\right\rangle (i=1,2,...,n)$ *be a collection of trapezoidal cubic fuzzy number and let TrCFWGO*: $\Lambda^n\to\Lambda$ *if* $TrCFWGO(\tilde{b}_1,\tilde{b}_2,...,\tilde{b}_n)=\prod_{i=1}^n\tilde{b}_i^{w_j}=$

$$\left\langle \begin{array}{l}[\prod_{i=1}^n(\tilde{p}_i^-)^{w_i},\prod_{i=1}^n(\tilde{q}_i^-)^{w_i},\prod_{i=1}^n(\tilde{r}_i^-)^{w_i},\prod_{i=1}^n(\tilde{s}_i^-)^{w_i}];\\ \prod_{i=1}^n(w_{b_i^-})^{w_j};[\prod_{i=1}^n(\tilde{p}_i^+)^{w_i},\prod_{i=1}^n(\tilde{q}_i^+)^{w_i},\\ \prod_{i=1}^n(\tilde{r}_i^+)^{w_i},\prod_{i=1}^n(\tilde{s}_i^+)^{w_i}];\ \prod_{i=1}^n(w_{b_i^+})^{w_j}],\\ [1-\prod_{i=1}^n(1-\tilde{p}_i)^{w_i},1-\prod_{i=1}^n(1-\tilde{q}_i)^{w_i},\\ 1-\prod_{i=1}^n(1-\tilde{r}_i)^{w_i},1-\prod_{i=1}^n(1-\tilde{s}_i)^{w_i}];\\ [1-\prod_{i=1}^n(1-\eta_{b_i})^{w_j}]\end{array}\right\rangle.$$

Then TrCFWGO is called trapezoidal cubic fuzzy weighted geometric operator of dimension n, where $w=(w_1,w_2,...,w_n)^T$ *is the weight vector of* $\tilde{b}_i(i=1,2,...,n)$, $w_i\in[0,1]$ *and* $\sum_{i=1}^n w_i=1$. *Especially* $w=(\frac{1}{n},\frac{1}{n},...,\frac{1}{n})^T$, *the TrCFWG operator is reduced to trapezoidal cubic fuzzy geometric averaging operator, which is defined as follows:*

$$TrCFGAO(\tilde{b}_1,\tilde{b}_2,...,\tilde{b}_n)=(\tilde{b}_1\otimes\tilde{b}_2\otimes,...,\otimes\tilde{b}_n)^{1/n}$$

Example 2.

Attributes	Table
	$\langle[0.1, 0.4, 0.8, 0.10], 0.3$
b_1	$[0.3, 0.6, 0.10, 0.15], 0.12,$
	$[0.2, 0.5, 0.9, 0.12], 0.97\rangle$
	$\langle[0.33, 0.39, 0.46, 0.49], 0.67$
b_2	$[0.36, 0.44, 0.48, 0.57], 0.56,$
	$[0.34, 0.41, 0.47, 0.52], 0.18\rangle$
	$\langle[0.23, 0.25, 0.27, 0.29], 0.99$
b_3	$[0.25, 0.27, 0.29, 0.32], 0.59,$
	$[0.24, 0.26, 0.28, 0.31], 0.73\rangle$
	$\langle[0.42, 0.48, 0.44, 0.50], 0.16$
b_4	$[0.45, 0.50, 0.58, 0.64], 0.92,$
	$[0.43, 0.49, 0.52, 0.62], 0.2\rangle$
	$\langle[0.5, 0.7, 0.9, 0.11], 0.10$
b_5	$[0.7, 0.9, 0.12, 0.14], 0.77,$
	$[0.6, 0.8, 0.10, 0.12], 0.83\rangle$
	$\langle[0.20, 0.28, 0.33, 0.39], 0.52$
b_6	$[0.25, 0.32, 0.38, 0.45], 0.31,$
	$[0.22, 0.30, 0.35, 0.42], 0.80\rangle$

$$w = [0.27, 0.49, 0.24], \text{ then } \left\langle \begin{array}{c} [(0.1137, 0.2232, 0.3094, 0.1123), 0.1774], \\ [(0.0490, 0.1060, 0.0207, 0.0355), 0.0978], \\ [(0.4693, 0.6316, 0.6633, 0.4981), 0.8736] \end{array} \right\rangle.$$

Theorem 1. *Let*

$$\tilde{b}_1 = \left\langle \begin{array}{c} [(\tilde{p}_1^-, \tilde{q}_1^-, \tilde{r}_1^-, \tilde{s}_1^-); (w_{b_1^-}), \\ (\tilde{p}_1^+, \tilde{q}_1^+, \tilde{r}_1^+, \tilde{s}_1^+); (w_{b_1^+})], \\ [(\tilde{p}_1, \tilde{q}_1, \tilde{r}_1, \tilde{s}_1); (\eta_{\tilde{b}_1})] \end{array} \right\rangle \text{ and } \tilde{b}_2 = \left\langle \begin{array}{c} [(\tilde{p}_2^-, \tilde{q}_2^-, \tilde{r}_2^-, \tilde{s}_2^-); (w_{b_2^-}), \\ (\tilde{p}_2^+, \tilde{q}_2^+, \tilde{r}_2^+, \tilde{s}_2^+); (w_{b_2^+})], \\ [(\tilde{p}_2, \tilde{q}_2, \tilde{r}_2, \tilde{s}_2); (\eta_{\tilde{b}_2})] \end{array} \right\rangle$$

be TrCLFNs. Then the score function S and the accuracy function G of H are defined by follows:

$$S(H) = \frac{\begin{array}{c}[(w_{\tilde{b}_i^-})(p_i^- + q_i^- + r_i^- + s_i^-) + (w_{\tilde{b}_i^+})(p_i^+ + q_i^+ + r_i^+ + s_i^+)] \\ -(p_i + q_i + r_i)(1 - \eta_{\tilde{b}_i})\end{array}}{12}$$

and

$$G(H) = \frac{\begin{array}{c}[(w_{\tilde{b}_i^-})(p_i^- + q_i^- + r_i^- + s_i^-) + (w_{\tilde{b}_i^+})(p_i^+ + q_i^+ + r_i^+ + s_i^+)] \\ +(p_i + q_i + r_i + s_i)(1 - \eta_{\tilde{b}_i})\end{array}}{12}$$

respectively.

It is obvious that $S(H) \in [-1,1]$ and $G(H) \in [0,1]$ for any TrCFN H.
(1) If $S(h_1) > S(h_2)$, then $s(h_1) > s(h_2)$.
(2) If $S(h_1) = S(h_2)$, then the following hold.
(a) If $g(h_1) > g(h_2)$, then $h_1 > h_2$.
(b) If $g(h_1) = g(h_2)$, then $h_1 = h_2$.
(c) If $g(h_1) > g(h_1)$, then $h_1 > h_2$.

Theorem 2. *Let $b_i(i = 1,2,...,n)$ be a collection of trapezoidal cubic fuzzy number. Then their aggregated value by using the TrCFWG operator is also an trapezoidal cubic fuzzy number and TrCFWG*

$$(\tilde{b}_1, \tilde{b}_2, ..., \tilde{b}_n) = \left\langle \begin{array}{c} [\prod_{i=1}^{n}(\tilde{p}_i^-)^{w_i}, \prod_{i=1}^{n}(\tilde{q}_i^-)^{w_i}, \prod_{i=1}^{n}(\tilde{r}_i^-)^{w_i}, \\ \prod_{i=1}^{n}(\tilde{s}_i^-)^{w_i}], \prod_{i=1}^{n}(w_{b_i^-})^{w_j} \\ [\prod_{i=1}^{n}(\tilde{p}_i^+)^{w_i}, \prod_{i=1}^{n}(\tilde{q}_i^+)^{w_i}, \prod_{i=1}^{n}(\tilde{r}_i^+)^{w_i}, \\ \prod_{i=1}^{n}(\tilde{s}_i^+)^{w_i}]; \prod_{i=1}^{n}(w_{b_i^+})^{w_j}], \\ 1 - \prod_{i=1}^{n}(1-\tilde{p}_i)^{w_i}, 1 - \prod_{i=1}^{n}(1-\tilde{q}_i)^{w_i}, \\ 1 - \prod_{i=1}^{n}(1-\tilde{r}_i)^{w_i}, 1 - \prod_{i=1}^{n}(1-\tilde{s}_i)^{w_i}]; \\ [1 - \prod_{i=1}^{n}(1-\eta_{b_i})^{w_j}] \end{array} \right\rangle \text{ where } w=(w_1, w_2, ...w_n)^T$$

is the weight vector of $\tilde{b}_i(i = 1,2,,...,n)$, with $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$.

Proof. We prove that by using matematical induction on n.

For $n = 2$.

Since $\tilde{b}_1 = \langle[(p_1^-, q_1^-, r_1^-, s_1^-)(w_{b_1^-}), (p_1^+, q_1^+, r_1^+, s_1^+)(w_{b_1^+})], [(p_1, q_1, r_1, s_1); (\eta_{b_1})]\rangle$ and $\tilde{b}_2 = \langle[(p_2^-, q_2^-, r_2^-, r_2^-)(w_{b_2^-}), (p_2^+, q_2^+, r_2^+, s_2^+)(w_{b_2^+})], [(p_2, q_2, r_2, s_2); (\eta_{b_2})]\rangle$ then, $\text{TCFWG}(\tilde{b}_1, \tilde{b}_2) = \tilde{b}_1 \otimes \tilde{b}_2 = \langle[(p_1^-, q_1^-, r_1^-, s_1^-)(w_{b_1^-}), (p_1^+, q_1^+, r_1^+, s_1^+)(w_{b_1^+})], [(p_1, q_1, r_1, s_1); (\eta_{b_1})]\rangle \otimes \langle[(p_2^-, q_2^-, r_2^-, r_2^-)(w_{b_2^-}), (p_2^+, q_2^+, r_2^+, s_2^+)(w_{b_2^+})], [(p_2, q_2, r_2, s_2); (\eta_{b_2})]\rangle$

$= [\prod_{i=1}^{2}(\tilde{p}_i^-)^{w_i}, \prod_{i=1}^{2}(\tilde{q}_i^-)^{w_i}, \prod_{i=1}^{2}(\tilde{r}_i^-)^{w_i}, \prod_{i=1}^{2}(\tilde{s}_i^-)^{w_i}] \prod_{i=1}^{2}(w_{b_i^-})^{w_j}; [\prod_{i=1}^{2}(\tilde{p}_i^+)^{w_i}, \prod_{i=1}^{2}(\tilde{q}_i^+)^{w_i}, \prod_{i=1}^{2}(\tilde{r}_i^+)^{w_i}, \prod_{i=1}^{2}(\tilde{s}_i^+)^{w_i}]; \prod_{i=1}^{2}(w_{b_i^+})^{w_j}], 1 - \prod_{i=1}^{2}(1-\tilde{p}_i)^{w_i}, 1 - \prod_{i=1}^{2}(1-\tilde{q}_i)^{w_i}, 1 - \prod_{i=1}^{2}(1-\tilde{r}_i)^{w_i}, 1 - \prod_{i=1}^{2}(1-\tilde{s}_i)^{w_i}]; [1 - \prod_{i=1}^{2}(1-\eta_{b_i})^{w_j}]$.

Let us suppose that the statement is true for $n = k$, that is

$$\text{TrCFWG}(b_1, b_2, ...b_n) = \prod_{i=1}^{k} b_i^{w_j} = [\prod_{i=1}^{k}(\tilde{r}_i^-)^{w_i}, \prod_{i=1}^{k}(\tilde{s}_i^-)^{w_i}, \prod_{i=1}^{k}(\tilde{t}_i^-)^{w_i}]\prod_{i=1}^{k}(w_{b_i^-})^{w_j};$$

$$[\prod_{i=1}^{k}(\tilde{r}_i^+)^{w_i}, \prod_{i=1}^{k}(\tilde{s}_i^+)^{w_i}, \prod_{i=1}^{k}(\tilde{t}_i^+)^{w_i}]; \prod_{i=1}^{k}(w_{b_i^+})^{w_j}], 1 - \prod_{i=1}^{k}(1-\tilde{r}_i)^{w_i},$$

$$1 - \prod_{i=1}^{k}(1-\tilde{s}_i)^{w_i}, 1 - \prod_{i=1}^{k}(1-\tilde{t}_i)^{w_i}]; [1 - \prod_{i=1}^{k}(1-\eta_{b_i})^{w_j}]$$

Now when $n = k + 1$

$$\text{TrCFWG}(\tilde{b}_1, \tilde{b}_2, ..., \tilde{b}_n) = \prod_{i=1}^{k+1} \tilde{b}_i^{w_j} = \text{TrCFWG}(\tilde{b}_1, \tilde{b}_2, ..., \tilde{b}_k) \otimes \tilde{b}_{k+1}$$

$$= [\prod_{i=1}^{k}(\tilde{p}_i^-)^{w_i}, \prod_{i=1}^{k}(\tilde{q}_i^-)^{w_i}, \prod_{i=1}^{k}(\tilde{r}_i^-)^{w_i}, \prod_{i=1}^{k}(s_i^-)^{w_i}]\prod_{i=1}^{k}(w_{b_i^-})^{w_j}; [\prod_{i=1}^{k}(\tilde{p}_i^+)^{w_i}, \prod_{i=1}^{k}(\tilde{q}_i^+)^{w_i},$$

$$\prod_{i=1}^{k}(\tilde{r}_i^+)^{w_i}, \prod_{i=1}^{k}(s_i^+)^{w_i}]; \prod_{i=1}^{k}(w_{b_i^+})^{w_j}], 1 - \prod_{i=1}^{k}(1-\tilde{p}_i)^{w_i}, 1 - \prod_{i=1}^{k}(1-\tilde{q}_i)^{w_i}, 1 -$$

$$\prod_{i=1}^{k}(1-\tilde{r}_i)^{w_i}, 1 - \prod_{i=1}^{k}(1-s_i)^{w_i}]; \quad [1 - \prod_{i=1}^{k}(1-\eta_{b_i})^{w_j}] \otimes [\prod_{i=1}^{k+1}(\tilde{p}_i^-)^{w_i}, \prod_{i=1}^{k+1}(\tilde{q}_i^-)^{w_i},$$

$$\prod_{i=1}^{k+1}(\tilde{r}_i^-)^{w_i}, \prod_{i=1}^{k+1}(\tilde{s}_i^-)^{w_i}]\prod_{i=1}^{k+1}(w_{b_i^-})^{w_j}; [\prod_{i=1}^{k+1}(\tilde{p}_i^+)^{w_i}, \prod_{i=1}^{k+1}(\tilde{q}_i^+)^{w_i}, \prod_{i=1}^{k+1}(\tilde{r}_i^+)^{w_i}, \prod_{i=1}^{k+1}(\tilde{s}_i^+)^{w_i}];$$

$$\prod_{i=1}^{k+1}(w_{b_i^+})^{w_j}], 1 - \prod_{i=1}^{k+1}(1-\tilde{p}_i)^{w_i}, 1 - \prod_{i=1}^{k+1}(1-\tilde{q}_i)^{w_i}, 1 - \prod_{i=1}^{k+1}(1-\tilde{r}_i)^{w_i}, 1 - \prod_{i=1}^{k+1}(1-\tilde{s}_i)^{w_i}];$$

$$[1 - \prod_{i=1}^{k+1}(1-\eta_{\tilde{b}_i})^{w_j}]$$

Which completes the proof. □

5. Algorithm

Step 1: The decision maker give their investigation of each different produce on each criteria. The presentation of every alternative M_i with reverence to every criteria C_i.

Step 2: In this step we aggregate the decision matrix into only decision matrix by using the trapezoidal cubic fuzzy weighted geometric operator.

Step 3: In this step we gauge the value of score function by using the formulas of score function.

Step 4: The last step provide for the appraisal of score functions we discover the plan of the competitors.

6. Numerical applications

Step 1: Let $M = \{M_1, M_2, M_3\}$ be a set of Metrobus of the alternative and $R = \{R_1, R_2, R_3\}$ be the set of road of the criteria. Let D be set of decision matrix. The decision matrix estimate each alternative based on given standards

$D =$

	Road_1
M_1	$\left\langle \begin{array}{c}[(0.2,0.4,0.6,0.9)(0.8),(0.3,0.5,0.7,0.11)(0.4)],\\ (0.4,0.6,0.8,0.13)(0.6)\end{array} \right\rangle$
M_2	$\left\langle \begin{array}{c}[(0.8,0.10,0.16,0.25)(0.8);\\ (0.14,0.16,0.18,0.28)(0.4)],(0.12,0.15,0.17,0.26)(0.6)\end{array} \right\rangle$
M_3	$\left\langle \begin{array}{c}[(0.1,0.6,0.8,0.22)(0.8),\\ (0.5,0.8,0.10,0.25)(0.4)],,(0.7,0.9,0.11,0.34)(0.6)\end{array} \right\rangle$
	Road_2
M_1	$\left\langle \begin{array}{c}[(0.3,0.5,0.7,0.11,)(0.8),\\ (0.9,0.12,0.15,0.34)(0.4)],(0.5,0.11,0.17,0.25)(0.6)\end{array} \right\rangle$
M_2	$\left\langle \begin{array}{c}[(0.7,0.9,0.11,0.13)(0.8),\\ (0.9,0.13,0.15,0.20)(0.4)],(0.8,0.10,0.12,0.16)(0.6)\end{array} \right\rangle$
M_3	$\left\langle \begin{array}{c}[(0.9,0.13,0.15,0.27)(0.8),\\ (0.13,0.15,0.17,0.39)(0.4)],(0.12,0.14,0.16,0.32)(0.6)\end{array} \right\rangle$
	Road_3
M_1	$\left\langle \begin{array}{c}[(0.5,0.7,0.9,0.11)(0.8),\\ (0.8,0.12,0.17,0.25)(0.4)],(0.6,0.10,0.12,0.21)(0.6)\end{array} \right\rangle$
M_2	$\left\langle \begin{array}{c}[(0.10,0.19,0.22,0.26)(0.8),\\ (0.19,0.25,0.29,0.34)(0.4)],(0.18,0.22,0.25,0.29)(0.6)\end{array} \right\rangle$
M_3	$\left\langle \begin{array}{c}[(0.14,0.17,0.19,0.23)(0.8),\\ (0.20,0.22,0.25,0.28)(0.4)],(0.16,0.19,0.23,0.26)(0.6)\end{array} \right\rangle$

Step 2: Applying trapezoidal cubic fuzzy weighted geometric operator and $w = 0.23, 0.31, 0.46$, then

M_1	$\langle[(0.4464,0.6362,0.7995,0.3536),(0.8572)],$ $[(0.6218,0.2166,0.2870,0.2349),(0.4264)],$ $[(0.6229,0.4075,0.5872,0.2627),(0.7176)]\rangle$
M_2	$\langle[(0.5153,0.3922,0.2787,0.3202),(0.8527)],$ $[(0.3144,0.1958,0.2223,0.2928),(0.4264)],$ $[(0.5895,0.2114,0.2418,0.3135),(0.7176)]\rangle$
M_3	$\langle[(0.3656,0.1539,0.4191,0.3725),(0.8572)],$ $[(0.2602,0.3241,0.1839,0.2987),(0.4264)],$ $[(0.4998,0.7063,0.2243,0.3977),(0.7176)]\rangle$

Step 3: Find the value of score function $S(A_1) = 0.0955, S(A_2) = 0.0629$, $S(A_3) = 0.0222$.

Step 4:According to the assessment of the score function, the ranking of the candidates can be established $S_1 > S_2 > S_3$ and so the best Metrobus M_1 is S_1.

7. Conclusion

In this paper, we consider the cubic fuzzy multiple attribute group decision-making problem in which the attributes and decision makers are at changed principal identical. We have developed a trapezoidal cubic fuzzy weighted geometric operator and numerical applications. All these operators can weight together the given opinions and their orderly points instantaneously just as the widespread trapezoidal cubic fuzzy weighted geometric operator undertake. At last, a numerical example is peeled to annotate the given approach.

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Accepted: 9.04.2017

On weakly S-permutable subgroups and P-nilpotency of finite groups

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Abstract. Suppose that G is a finite group and H is a subgroup of G. H is said to be weakly s-permutable in G if there is a subnormal subgroup T of G such that $G = HT$ and $H \cap T \leq H_{sG}$, where H_{sG} is the subgroup of H generated by all those subgroups of H which are s-permutable in G. We fix in every non-cyclic Sylow subgroup P of G some subgroup D satisfying $1 < |D| < |P|$ and study the p-nilpotency of G under the assumption that every subgroup H of P with $|H| = |D|$ is weakly s-permutable in G. Some recent results and the Frobenius$'$ theorem are generalized.
Keywords: p-nilpotent group, weakly s-permutable subgroup, finite group.

1. Introduction

All groups considered in this paper are finite. We use conventional notions and notation. G always means a group, $|G|$ is the order of G, $\pi(G)$ denotes the set of all primes dividing $|G|$.

A subgroup H of G is called s-quasinormal (or s-permutable, π-quasinormal) in G provided H permutes with all Sylow subgroups of G, i.e, $HP = PH$ for any Sylow subgroup P of G. This concept was introduced by Kegel in [4] and has been studied extensively by Deskins [1] and Schmidt [8].

More recently, Skiba in [7] introduced the following concept: A subgroup H of G is said to be weakly s-permutable in G if there is a subnormal subgroup T of G such that $G = HT$ and $H \cap T \leq H_{sG}$, where H_{sG} is the subgroup of H generated by all those subgroups of H which are s-permutable in G. Miao in [6] proved the following result:

Theorem 1.1. *Let p be an odd prime divisor of $|G|$ and P be a Sylow p-subgroup of G. If $N_G(P)$ is p-nilpotent and every maximal subgroup of P is weakly s-permutable in G, then G is p-nilpotent.*

A celebrated theorem of Frobenius [3, Satz. IV.5.8] asserts that G is p-nilpotent if $N_G(H)$ is p-nilpotent for every p-subgroup H of G. In this article, we replace some of the conditions of the Frobenius$'$ theorem and Theorem 1.1, namely, H is restricted to be a p-subgroup of a fixed order, the condition of p-nilpotency of $N_G(P)$ is changed by the p-nilpotency of $N_G(H)$, and we assume

that H is a weakly s-permutable subgroup of G. The results of this article can be considered as extensions of the Frobenius' theorem and Theorem 1.1. Our main theorem is as follows:

Theorem 3.2. *Let G be a group and P a Sylow p-subgroup of G, where p is an odd prime. If P has a subgroup D with $1 < |D| < |P|$ such that all subgroups H of P with order $|H| = |D|$ are weakly s-permutable in G and $N_G(H)$ is p-nilpotent, then G is p-nilpotent.*

2. Basic definitions and preliminary results

In this section, we collect some known results that are useful later.

Lemma 2.1 ([7])**.** *Let H be a weakly s-permutable subgroup of a group G:*

(i) *If $H \leq K \leq G$, then H is weakly s-permutable in K;*

(ii) *If N is normal in G and $N \leq H \leq G$, then H/N is weakly s-permutable in G/N;*

(iii) *If H is a π-subgroup and N is a normal π'-subgroup of G, then HN/N is weakly s-permutable in G/N;*

(iv) *Suppose H is a p-group for some prime p and H is not s-permutable in G. Then G has a normal subgroup M such that $|G : M| = p$ and $G = HM$.*

Lemma 2.2 ([8])**.** *If P is an s-quasinormal p-subgroup of a group G for some prime p, then $N_G(P) \geq O^p(G)$.*

Lemma 2.3 (3, III, 5.2 and IV, 5.4)**.** *Suppose that p is a prime and G is a minimal non-p-nilpotent group, i.e., G is not a p-nilpotent group but whose proper subgroups are all p-nilpotent.*

(i) *G has a normal Sylow p-subgroup P for some prime p and $G = PQ$, where Q is a non-normal cyclic q-subgroup for some prime $q \neq p$.*

(ii) *$P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$.*

(iii) *The exponent of P is p or 4.*

Lemma 2.4 ([7])**.** *Let N be an elementary abelian normal p-subgroup of a group G. If there exists a subgroup D in N such that $1 < |D| < |N|$ and every subgroup H of N with $|H| = |D|$ is weakly s-permutable in G, then there exists a maximal subgroup M of N such that M is normal in G.*

3. Main results

In this section, we will prove our main results.

Theorem 3.1. *Let G be a group and P a Sylow p-subgroup of G, where p is an odd prime. If each maximal subgroup P_1 of P is weakly s-permutable in G and $N_G(P_1)$ is p-nilpotent, then G is p-nilpotent.*

Proof. Assume that the theorem is not true and let G be a counterexample of minimal order. We derive a contradiction in several steps.

Step 1. $O_{p'}(G) = 1$. Suppose that $O_{p'}(G) = 1$. Consider $G/O_{p'}(G)$. Let $M/O_{p'}(G)$ be a maximal subgroup of $PO_{p'}(G)/O_{p'}(G)$. Then $M = M \cap PO_{p'}(G) = (M \cap P)O_{p'}(G)$. Let $P_1 = M \cap P$. It is easy to see that P_1 is maximal in P. Let $N_{G/O_{p'}(G)}(P_1O_{p'}(G))/O_{p'}(G) = K/O_{p'}(G)$. Then $P_1O_{p'}(G) \lhd K$, and thus $K = N_K(P_1)P_1O_{p'}(G) = N_G(P_1)O_{p'}(G) \leq K$; that is,

$$N_{G/O_{p'}(G)}(P_1O_{p'}(G))/O_{p'}(G) = N_G(P_1)O_{p'}(G)/O_{p'}(G).$$

By the hypothesis, $N_G(P_1)O_{p'}(G)/O_{p'}(G)$ is p-nilpotent. Then, by Lemma 2.1, we have that $G/O_{p'}(G)$ satisfies the hypothesis of the theorem. The choice of G yields that $G/O_{p'}(G)$ is p-nilpotent, which implies that G is p-nilpotent, a contradiction.

Step 2. Let T be a subgroup of G such that $P \leq T \leq G$. Then T is p-nilpotent.

Let P_1 be a maximal subgroup of P. Obviously, $N_T(P_1) \leq N_G(P_1)$. By the hypothesis, we have $N_T(P_1)$ is p-nilpotent and by Lemma 2.1 P_1 is weakly s-permutable in T. Hence T satisfies the hypothesis of the theorem. The minimality of G forces that T is p-nilpotent.

Step 3. $O_p(G)$ is the unique minimal normal subgroup of G and $G/O_p(G)$ is p-nilpotent. Moreover, $\Phi(G) = 1$. As G is not p-nilpotent, by the Glauberman-Thompson Theorem $N_G(Z(J(P)))$ is not p-nilpotent, where $J(P)$ is the Thompson subgroup of P. Noticing that $Z(J(P))$ is a characteristic subgroup of P, we get $N_G(P) \leq N_G(Z(J(P)))$. By Step 2, we have $N_G(Z(J(P))) = G$ and so $O_p(G) \neq 1$. Let N be a minimal normal subgroup of G contained in $O_p(G)$. If $N = P$, then obviously G/N is p-nilpotent. If N is maximal in P, then by the hypothesis $G = N_G(N)$ is p-nilpotent, a contradiction. Hence we may assume that $|P : N| \geq p^2$. By Lemma 2.1, it is easy to see that G/N satisfies the hypothesis of the theorem, so the choice of G yields that G/N is p-nilpotent. The uniqueness of N and $\Phi(G) = 1$ are obvious. Now by [5, Lemma 2.6], we have $N = O_p(G)$.

Step 4. The final contradiction. By Step 3, $\Phi(G) = 1$. Therefore, G has a maximal subgroup M such that $G = MN$ and $M \cap N = 1$. Clearly, $P = N(P \cap M)$. Since $P \cap M < P$, there exists a maximal subgroup P_1 of P such

that $P \cap M \leq P_1$. By hypothesis, P_1 is weakly s-permutable in G. Then there is a subnormal subgroup T of G such that $G = P_1T$ and

$$P_1 \cap T \leq (P_1)_{sG} \leq O_p(G) = N \leq O^p(G)$$

because N is the unique minimal normal subgroup of G. Since $|G : T|$ is a power of p, $O^p(G) \leq T$. Hence,

$$P_1 \cap T \leq (P_1)_{sG} \leq O^p(G) \cap P_1 \leq T \cap P_1,$$

and so

$$P_1 \cap T = (P_1)_{sG} = O^p(G) \cap P_1.$$

Consequently, $G = PO^p(G)$ implies that $(P_1)_{sG}$ is normal in G by Lemma 2.2. By the minimality of N, we have $(P_1)_{sG} = N$ or $(P_1)_{sG} = 1$. If $(P_1)_{sG} = N$, then $N \leq P_1$ and $P = NP_1 = P_1$, a contradiction. Thus $P_1 \cap T = (P_1)_{sG} = 1$, and so $|T|_p = p$. Then T is p-nilpotent. Let $T_{p'}$ be the normal p-complement of T. Then $T_{p'}$ is subnormal in G and $T_{p'}$ is a p'-Hall subgroup of G. It follows that $T_{p'}$ is the normal p-complement of G, a contradiction. □

Theorem 3.2. *Let G be a group and P a Sylow p-subgroup of G, where p is an odd prime. If P has a subgroup D with $1 < |D| < |P|$ such that all subgroups H of P with order $|H| = |D|$ are weakly s-permutable in G and $N_G(H)$ is p-nilpotent, then G is p-nilpotent.*

Proof. Suppose that the theorem is false and let G be a counterexample of minimal order. We will derive a contradiction in several steps.

Step 1. $O_{p'}(G) = 1$. If $O_{p'}(G) \neq 1$, Lemma 2.1 guarantee that $G/O_{p'}(G)$ satisfies the hypotheses of the theorem. Thus $G/O_{p'}(G)$ is p-nilpotent by the choice of G. Then G is p-nilpotent, a contradiction.

Step 2. Let T be a subgroup of G such that $P \leq T < G$. Then T is p-nilpotent. This is proved with the same arguments shown in Step 2 of the proof of Theorem 3.1.

Step 3. $|P : D| > p$. By Theorem 3.1.

Step 4. $|D| > p$. Suppose that $|D| = p$. Since G is not p-nilpotent, G has a minimal non-p-nilpotent subgroup G_1. By Lemma 2.3 (i), $G_1 = [P_1]Q$, where $P_1 \in \mathrm{Syl}_p(G_1)$ and $Q \in \mathrm{Syl}_q(G_1)$, $p \neq q$. Let $X/\Phi(P_1)$ be a subgroup of $P_1/\Phi(P_1)$ of order p, $x \in X \backslash \Phi(P_1)$ and $L = \langle x \rangle$. Then L is of order p or 4 by Lemma 2.3 (iii). By the hypotheses, L is weakly s-permutable in G, thus in G_1 by Lemma 2.1. Then there is a subnormal subgroup T of G_1 such that $G_1 = LT$ and $L \cap T \leq L_{sG_1}$. Hence $P_1 = P_1 \cap G_1 = P_1 \cap LT = L(P_1 \cap T)$. Since $P_1/\Phi(P_1)$ is abelian, we have $(P_1 \cap T)\Phi(P_1)/\Phi(P_1)$ is normal in $G_1/\Phi(P_1)$. Since $P_1/\Phi(P_1)$

is the minimal normal subgroup of $G_1/\Phi(P_1)$, we have that $P_1 \cap T \leq \Phi(P_1)$ or $P_1 = (P_1 \cap T)\Phi(P_1) = P_1 \cap T$. If $P_1 \cap T \leq \Phi(P_1)$, then $L = P_1$ is normal in G_1. It follows that G_1 is p-nilpotent, a contraction. If $P_1 = P_1 \cap T$, then $T = G_1$ and so $L = L_{sG_1}$ is s-permutable in G_1. For any element x in P_1, now we have $\langle x\rangle Q$ is a proper subgroup of G_1, then $\langle x\rangle Q = \langle x\rangle \times Q$. This implies that $G_1 = P_1 \times Q$, a contradiction.

Step 5. $O_p(G) \neq 1$ and $G = PQ$ where $Q \in \mathrm{Syl}_q(G)$ and $q \neq p$. As G is not p-nilpotent, by the Glauberman-Thompson Theorem $N_G(Z(J(P)))$ is not p-nilpotent, where $J(P)$ is the Thompson subgroup of P. Noticing that $Z(J(P))$ is a characteristic subgroup of P, we get $N_G(P) \leq N_G(Z(J(P)))$. By Step 2, we have $N_G(Z(J(P))) = G$ and so $O_p(G) \neq 1$. Consider $G = G/O_p(G)$ and let G_1 be the inverse imagine of $N_G(Z(J(P)))$ in G. Since $O_p(G)$ is the largest normal subgroup of G contained in P, we have $N_G(P) \leq G_1 < G$. By Step 2, G_1 is p-nilpotent and by [2, Theorem 8.3.1] again, G is p-nilpotent. Then there exists a Sylow q-subgroup Q of G such that PQ is a subgroup of G for any $q \in \pi(G)$ with $q \neq p$ by [2, Theorem 6.3.5]. If $PQ < G$, then PQ is p-nilpotent by Step 2. Hence $Q \leq C_G(O_p(G)) \leq O_p(G)$ by [2, Theorem 6.3.2], a contradiction. Thus $PQ = G$.

Step 6. Let N be a minimal normal subgroup of G. Then $|N| < |D|$. If $|N| = |D|$, then by the hypothesis, $G = N_G(N)$ is p-nilpotent, a contradiction. Suppose that $|N| > |D|$. Since $N \leq O_p(G)$, N is elementary abelian. By Lemma 2.4, N has a maximal subgroup which is normal in G, contrary to the minimality of N. Hence $|N| < |D|$.

Step 7. G/N is p-nilpotent, N is the unique minimal normal subgroup of G and $\Phi(G) = 1$. By Step 6 and Lemma 2.1, it is easy to see that G/N satisfies the hypothesis of the theorem, so the choice of G yields that G/N is p-nilpotent. The uniqueness of N and $\Phi(G) = 1$ are obvious.

Step 8. The final contradiction. Since G is solvable by Step 5, there is a maximal subgroup M of G such that $|G : M|$ is a prime. If $|G : M| \neq p$, then M is p-nilpotent by Step 2 and therefore $P = M \trianglelefteq G$ by Step 1, a contradiction. Thus we may assume that $|G : M| = p$. Then it follows that $P \cap M$ is a maximal subgroup of P and also a Sylow p-subgroup of M. If $N_G(P \cap M) < G$, then $N_G(P \cap M)$ is p-nilpotent by Step 2 and so is $N_M(P \cap M)$. Since $|P : D| > p$ by Step 3, every subgroup of $P \cap M$ of order $|D|$ is weakly s-permutable in M by Lemma 2.1. Consequently, M satisfies the hypotheses of our theorem and therefore the choice of G implies that M is p-nilpotent, a contradiction. Hence $P \cap M \trianglelefteq G$ and $N = O_p(G) = P \cap M$ is a maximal subgroup of P by Step 7. This leads to $|D| < |N|$ by Theorem 3.1, in contradiction to Step 6, the final contradiction. The proof of the theorem is complete.

4. Acknowledgements

The paper is dedicated to Professor John Cossy for his 80th birthday.

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Accepted: April 04, 2017

Two-warehouse inventory model for instantaneous deteriorating items with price and time dependent demand, complete backlogging

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Abstract. This paper analyses an inventory model for deteriorating items which is stored in two different warehouses (one is the existing storage known as own warehouse (OW) and the other is hired on rental basis known as rented warehouse (RW) The RW is used for storing the excess units over the fixed capacity w of the OW and is assumed to offer better preserving facilities than the OW resulting in a lower rate of deterioration. Also, the RW is assumed to charge higher holding cost than the OW. Here the demand is considered as two factors such as selling price and time dependent demand, since, not only the selling price, but also the time is a crucial factor to enhance the demand in the market as well as affecting the overall finance. In this model, shortage is allowed in OW and the excess demand is backlogged. The purpose of this study is to find the optimal replenishment policies for minimizing the total relevant inventory costs. Furthermore, Numerical examples are provided to illustrate the proposed model.
Keywords: inventory, constant deterioration, two warehouse, price and time dependent demand.

1. Introduction

In some practical situations, when suppliers offer price discounts for bulk purchases or products are seasonal, retailers may purchase more goods than stored in their own warehouse (OW). Therefore, a rented warehouse (RW) is used to store the excess units over the fixed capacity w of the own warehouse. Usually, RW may charge a higher unit holding cost than the OW due to the additional cost of maintenance, material handling, etc. Moreover, the RW generally provides better preserving facility than the OW resulting in a lower deterioration

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rate for the goods. To reduce the inventory costs, it will be economical to consume the goods of RW earliest. Consequently, the firm stores goods in OW before RW but clears the stocks in RW before OW. Two-warehouse model was first introduced by Hartely in (1976). Bhunia and Maiti (1998) considered a deterministic model with different levels of item deterioration in both warehouses, shortages and linearly increasing demand. Yang (2006) developed the effect of inflation with deteriorating item having two separate warehouses and partial backlogging. Ghosh and Chakrabarty (2009) represented an order-level inventory model for two levels of storage. Liang and Zhou (2011) presented the two-warehouse problem with constant demand under credit period policy. Agrawal, Banerjee and Papachristos (2013) proposed an inventory model with ramp-type demand and partially backlogged for a two-warehouse system. Kaliraman, et al. (2017) dealt with exponential demand rate and deterioration rate is constant in both warehouses. Karthikeyan and Santhi (2015) developed EOQ model for constant deteriorating items with cubic demand rate and salvage Value. Rangarajan and Karthikeyan (2017) proposed an EOQ models for non-instantaneous deteriorating items with cubic demand rate under inflation and permissible delay in payments

In this paper, we extend the work of Palanivel, et al. (2016) by incorporating the two-warehouse system involving non-instantaneous deteriorating items, stock-dependent demand and Partial backlogged. Here we introduce two-warehouse inventory model for instantaneous deteriorating items in which the demand rate as a function of two factors where the first factor is of decreasing nature with increasing selling price and the second factor is of exponentially decreasing nature with increasing time. Moreover shortage is allowed and completely backlogging is used in this model. In addition, it was assumed that the RW has higher unit holding cost than the OW. The main objective of this model is to minimize the total cost. Numerical examples are given in order to illustrate the model.

2. Assumptions and notations

The mathematical models of the two warehouse inventory problems are based on the following assumptions and notations:
(1) The demand rate is taken as price and time dependent.
(2) Shortages are allowed with complete backlogging.
(3) The deterioration of items in the OW and the RW follow different constant rates and the RW offers better facilities.
(4) Holding costs in the RW are higher than those of items in the OW.
(5) Replenishment rate is infinite, lead time is assumed to be negligible.
(6) The OW has a limited capacity of w units, and the RW has unlimited capacity. For economic reasons, the items of RW are consumed first, and then the items of OW.

Notations

A	Ordering cost per order
w	Capacity of the OW
S	Maximum inventory level per cycle
Q	Order quantity per cycle
BI	Maximum inventory level during the shortage period
p	Selling price per unit of the item
h_r	Holding cost per unit time in RW
h_0	Holding cost per unit time in OW, $h_r > h_o$
C_1	Shortage cost per unit time
C	Purchasing cost per unit
α	Deterioration rate in OW, $0 \leq \alpha < 1$
β	Deterioration rate in RW, $0 \leq \beta < 1$, $\alpha > \beta$
t_w	Time at which the inventory level reaches zero in RW
t_1	Time at which the inventory level reaches zero in OW
T	Length of the inventory cycle, hence $T = t_1 + t_2$
$Q_r(t)$	Level of positive inventory in at time t, where $0 \leq t \leq t_w$
$Q_o(t)$	Level of positive inventory in at time t, where $t_w \leq t \leq t_1$
$Q_s(t)$	Level of negative inventory at time t, where $t_1 \leq t \leq T$

3. Model description and analysis

3.1 Two-warehouse model

The two-warehouse inventory system is shown in Figure 1. The inventory system starts at time $t = 0$ when a lot size of a certain number of units enters the system, i.e. w units are kept in the OW and the remaining units are stored in RW. The goods in the OW are consumed only after consuming the goods kept in the RW. In the RW, during the interval $(0, t_w)$ the inventory level is gradually decreases due to demand and deterioration. In OW the inventory

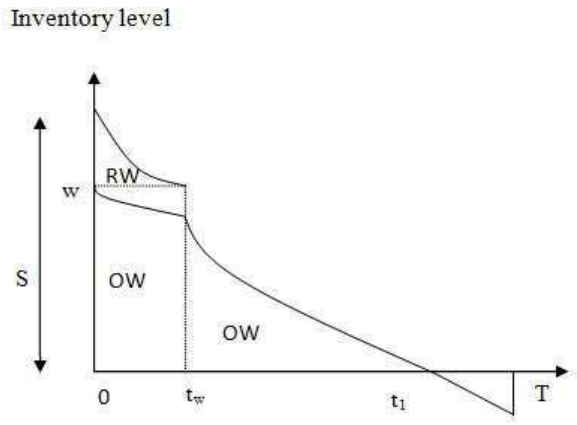


Figure 1: Graphical representation of the inventory system in RW and OW.

w decreases during $(0, t_w)$ due to deterioration only, but during (t_w, t_1) the inventory level is depleted due to both demand and deterioration. By the time t_1 both warehouses are empty. Finally, shortage occurs due to demand and complete backlogging during the time interval (t_1, T). The behavior of the inventory model is demonstrated in Figure1.

The differential equation of the inventory model is

$$\frac{dQ_r(t)}{dt} + \beta Q_r(t) = -\frac{e^{-at}}{p}, \qquad 0 < t \leq t_w \tag{1}$$

with the condition $Q_r(t_w) = 0$,

The stock OW decreases due to deterioration only. Hence, the rate of change of stock at OW at any instant $t\,(0 < t \leq t_w)$ is equal to the deterioration at that time.

$$\frac{dQ_o(t)}{dt} = -\alpha Q_o(t), \qquad 0 < t \leq t_w \tag{2}$$

with the boundary condition $Q_r(0) = w$ and $Q_r(t_w) = w$

$$\frac{dQ_o(t)}{dt} + \alpha Q_o(t) = -\frac{e^{-at}}{p}, \qquad t_w < t \leq t_1 \tag{3}$$

with the condition $Q_o(t_1) = 0$.

The solutions to the above differential equations (1)-(3) are:

$$Q_r(t) = \frac{1}{p}\left[-t + \frac{at^2}{2} - \frac{\beta t^2}{2} + \frac{a\beta t^3}{3} + t_w - \frac{at_w^2}{2} + \frac{\beta t_w^2}{2} - \frac{a\beta t_w^3}{3}\right], \tag{4}$$

$$Q_o(t) = we^{-\alpha t}, \tag{5}$$

$$Q_o(t) = \frac{1}{p}\left[-t + \frac{at^2}{2} - \frac{\alpha t^2}{2} + \frac{a\alpha t^3}{3} + t_1 - \frac{at_1^2}{2} + \frac{\alpha t_1^2}{2} - \frac{a\alpha t_1^3}{3}\right]. \tag{6}$$

The maximum inventory level S is given by

$$\begin{aligned} S &= Q_r(0) + Q_o(0) \\ &= \frac{1}{p}\left[t_w - \frac{at_w^2}{2} + \frac{bt_w^2}{2} - \frac{a\beta t_w^3}{3} + t_1 - \frac{at_1^2}{2} + \frac{\alpha t_1^2}{2} - \frac{a\alpha t_1^3}{3}\right] + w. \end{aligned} \tag{7}$$

During the interval $[t_1, T]$ shortage occurred and demand is completely backlogged. That is, the inventory level at time t is governed by the following differential equation

$$\frac{dQ_s(t)}{dt} = -\frac{e^{-at}}{p}, \qquad t_1 < t \leq T \tag{8}$$

with the condition $Q_s(t_1) = 0$, the solution for equation (8) becomes

$$Q_s(t) = \frac{1}{p}\left[(t_1 - t) + \frac{a}{2}\left(t^2 - t_1^2\right)\right], \qquad t_1 < t \leq T. \tag{9}$$

The maximum backordered inventory BI is obtained at $t = T$, then, from equation (9)

$$BI = \frac{1}{p}\left[(T - t_1) + \frac{a}{2}\left(t_1^2 - T^2\right)\right]. \tag{10}$$

Thus, the order size during total time interval $[0, T]$ is (11)

$$Q{=}S{+}BI{=}w{+}\frac{1}{p}\left[t_w{-}\frac{at_w^2}{2}{+}\frac{\beta t_w^2}{2} - \frac{a\beta t_w^3}{3} + \frac{\alpha t_1^2}{2} - \frac{a\alpha t_1^3}{3} + T - \frac{aT^2}{2}\right]. \tag{11}$$

Ordering cost per cycle is $OC = A$.

The present value of the inventory holding cost HC_r for the RW during the period $[0, T]$ is

$$HC_r = h_r\left[\int_0^{t_w} Q_r(t)\,dt\right] = \frac{h_r}{p}\left[-\frac{t_w^2}{2} + \frac{at_w^3}{3} + \frac{\beta t_w^3}{3} - \frac{a\beta t_w^4}{4}\right].$$

The present value of the inventory holding cost HC_o for the OW during the period $[0, T]$ is

$$\begin{aligned} HC_r &= h_o\left[\int_0^{t_w} Q_o(t)\,dt + \int_{t_w}^{t_1} Q_o(t)\,dt\right] = \frac{h_o w}{\alpha}\left[1 - e^{-\alpha t_w}\right] \\ &+ \frac{h_o}{p}\left[\frac{t_1^2}{2} - \frac{at_1^3}{3} + \frac{\alpha t_1^3}{3} - \frac{a\alpha t_1^4}{4} + \frac{t_w^2}{2}\right. \\ &\left. - \frac{at_w^3}{6} + \frac{\alpha t_w^3}{6} - \frac{a\alpha t_w^4}{12} - t_1 t_w + \frac{at_1^2 t_w}{2} - \frac{\alpha t_1^2 t_w}{2} + \frac{a\alpha t_1^3 t_w}{3}\right]. \end{aligned} \tag{12}$$

The present value of the deteriorating cost DC the during the period $[0, T]$ is

$$\begin{aligned} DC_r &= C_d\left[S - \int_0^{t_1} D(t,p)\,dt\right] \\ &= C_d\left[w + \frac{1}{p}\left(t_w - \frac{at_w^2}{2} + \frac{\beta t_w^2}{2} - \frac{a\beta t_w^3}{3} + t_1 - \frac{at_1^2}{2} + \frac{\alpha t_1^2}{2} - \frac{a\alpha t_1^3}{3}\right)\right. \\ &\left. + \frac{\beta}{p}\left(t_w - \frac{at_w^2}{2}\right) + \frac{\alpha}{p}\left(t_1 - \frac{at_1^2}{2} - t_w + \frac{at_w^2}{2}\right)\right]. \end{aligned} \tag{13}$$

The present value of the shortage cost SC the during the period $[0, T]$ is given by

$$SC = -C_1\left[\int_{t_1}^{T} Q_s(t)dt\right] = -\frac{C_1}{p}\left[\left(t_1 T - \frac{T^2}{2}\right) + \frac{a}{2}\left(\frac{T^3}{3} - t_1^2 T\right) + \frac{t_1^2}{2} - \frac{at_1^3}{3}\right].$$

The present value of the purchase cost PC the during the period $[0, T]$ is given by

$$PC = C.Q = C\left[w + \frac{1}{p}\left(t_w - \frac{at_w^2}{2} + \frac{\beta t_w^2}{2} - \frac{a\beta t_w^3}{3} + \frac{\alpha t_1^2}{2} - \frac{a\alpha t_1^3}{3} + T - \frac{aT^2}{2}\right)\right].$$

Total average cost per cycle = (ordering cost+ holding cost in RW + holding cost in OW + deterioration cost in RW + deterioration cost in OW + shortage cost + purchase cost)/T

$$
\begin{aligned}
TC &= \frac{1}{T}\left[OC + HC_r + HC_o + DC_r + DC_o + SC + PC\right] \\
&= \frac{1}{T}\left\{A + \frac{h_r}{p}\left[-\frac{t_w^2}{2} + \frac{at_w^3}{3} + \frac{\beta t_w^3}{3} - \frac{a\beta t_w^4}{4}\right] + \frac{h_o w}{\alpha}\left[1 - e^{-\alpha t_w}\right]\right. \\
&+ \frac{h_o}{p}\left[\frac{t_1^2}{2} - \frac{at_1^3}{3} + \frac{\alpha t_1^3}{3} - \frac{a\alpha t_1^4}{4} + \frac{t_w^2}{2} - \frac{at_w^3}{6} + \frac{\alpha t_w^3}{6} - \frac{a\alpha t_w^4}{12}\right. \\
&\left.-t_1 t_w + \frac{at_1^2 t_w}{2} - \frac{\alpha t_1^2 t_w}{2} + \frac{a\alpha t_1^3 t_w}{3}\right] \\
&+ C_d\left[w + \frac{1}{p}\left(t_w - \frac{at_w^2}{2} + \frac{\beta t_w^2}{2} - \frac{a\beta t_w^3}{3} + t_1 - \frac{at_1^2}{2} + \frac{\alpha t_1^2}{2} - \frac{a\alpha t_1^3}{3}\right)\right. \\
&\left.+\frac{\beta}{p}\left(t_w - \frac{at_w^2}{2}\right) + \frac{\alpha}{p}\left(t_1 - \frac{at_1^2}{2} - t_w + \frac{at_w^2}{2}\right)\right] \\
&- \frac{C_1}{p}\left[\left(t_1 T - \frac{T^2}{2}\right) + \frac{a}{2}\left(\frac{T^3}{3} - t_1^2 T\right) - \frac{t_1^2}{2} + \frac{at_1^3}{3}\right] \\
&+ C\left[w + \frac{1}{p}\left(t_w - \frac{at_w^2}{2} + \frac{\beta t_w^2}{2} - \frac{a\beta t_w^3}{3} + \frac{\alpha t_1^2}{2} - \frac{a\alpha t_1^3}{3} + T - \frac{aT^2}{2}\right)\right].
\end{aligned} \tag{14}
$$

The necessary conditions for the total relevant cost per time unit to be minimum are

$$
\frac{\partial TC}{\partial t_w} = 0, \qquad \text{and} \qquad \frac{\partial TC}{\partial t_1} = 0. \tag{15}
$$

Providing that the optimal values for t_w and t_1 obtained from equation (15) satisfy the sufficient condition,

$$
\frac{\partial^2 TC}{\partial t_w^2} \times \frac{\partial^2 TC}{\partial t_1^2} - \left(\frac{\partial^2 TC}{\partial t_w \partial t_1}\right)^2 > 0.
$$

3.2 Single warehouse model

Figure 2 shows the graphical representation of one warehouse inventory system. Now considering the one warehouse inventory system we derive the inventory

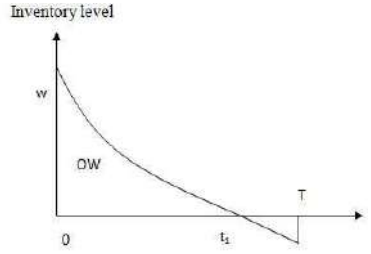


Figure 2: Graphical representation of the inventory system in OW.

level during time periods t_1 and T which are represented by differential equation

$$\frac{dQ_o(t)}{dt} + \alpha Q_o(t) = -\frac{e^{-at}}{p}, \qquad 0 < t \leq t_1. \tag{16}$$

With the boundary condition $Q_o(t_1) = 0$ and $Q_o(0) = w$

$$Q_o(t) = \frac{1}{p}\left[-t + \frac{at^2}{2} - \frac{\alpha t^2}{2} + \frac{a\alpha t^3}{3} + t_1 - \frac{at_1^2}{2} + \frac{\alpha t_1^2}{2} - \frac{a\alpha t_1^3}{3}\right]. \tag{17}$$

The maximum inventory level is given by $w = \frac{1}{p}[t_1 - \frac{at_1^2}{2} + \frac{\alpha t_1^2}{2} - \frac{a\alpha t_1^3}{3}]$. During the interval $[t_1, T]$ shortage occurred and demand is completely backlogged. That is, the inventory level at time t is governed by the following differential equation

$$\frac{dQ_s(t)}{dt} = -\frac{e^{-at}}{p}, \qquad t_1 < t \leq T \tag{18}$$

with the condition $Q_s(t_1) = 0$, the solution for equation (18) is

$$Q_s(t) = \frac{1}{p}\left[(t_1 - t) + \frac{a}{2}\left(t^2 - t_1^2\right)\right], \qquad t_1 < t \leq T. \tag{19}$$

The maximum backordered inventory BI is obtained at $t = T$, then, from equation (19)

$$BI = \frac{1}{p}\left[(T - t_1) + \frac{a}{2}\left(t_1^2 - T^2\right)\right]. \tag{20}$$

Thus, the order size during total time interval $[0, T]$ is

$$Q = w + BI = \frac{1}{p}\left[\frac{\alpha t_1^2}{2} - \frac{a\alpha t_1^3}{3} + T - \frac{aT^2}{2}\right]. \tag{21}$$

The present value of the inventory holding cost HC_o for the OW during the period $[0, T]$ is

$$HC_o = h_o\left[\int_0^{t_1} Q_o(t)\, dt\right] = \frac{h_o}{p}\left[\frac{t_1^2}{2} - \frac{at_1^3}{3} + \frac{\alpha t_1^3}{3} - \frac{a\alpha t_1^4}{4}\right].$$

The present value of the deteriorating cost DC during the period $[0, T]$ is

$$DC = C_d\left[w - \int_0^{t_1} D(t,p)\, dt\right] = C_d\left[\frac{1}{p}\left(t_1 - \frac{at_1^2}{2} + \frac{\alpha t_1^2}{2} - \frac{a\alpha t_1^3}{3}\right) + \frac{\alpha}{p}\left(t_1 - \frac{at_1^2}{2}\right)\right].$$

The present value of the shortage cost SC the during the period $[0, T]$ is given by

$$SC = -C_1\left[\int_{t_1}^{T} Q_s(t)dt\right] = -\frac{C_1}{p}\left[\left(t_1 T - \frac{T^2}{2}\right) + \frac{a}{2}\left(\frac{T^3}{3} - t_1^2 T\right) + \frac{t_1^2}{2} - \frac{at_1^3}{3}\right].$$

The present value of the purchase cost PC the during the period $[0, T]$ is given by

$$
\begin{aligned}
PC &= C.Q = C\left[\frac{1}{p}\left(\frac{\alpha t_1^2}{2} - \frac{a\alpha t_1^3}{3} + T - \frac{aT^2}{2}\right)\right], \\
TC &= \frac{1}{T}\left[OC + HC_o + DC + SC + PC\right] \\
&= \frac{1}{T}\left\{A + \frac{h_o}{p}\left[\frac{t_1^2}{2} - \frac{at_1^3}{3} + \frac{\alpha t_1^3}{3} - \frac{a\alpha t_1^4}{4}\right]\right. \\
&+ C_d\left[\frac{1}{p}\left(t_1 - \frac{at_1^2}{2} + \frac{\alpha t_1^2}{2} - \frac{a\alpha t_1^3}{3}\right) + \frac{\alpha}{p}\left(t_1 - \frac{at_1^2}{2}\right)\right] \\
&- \frac{C_1}{p}\left[\left(t_1 T - \frac{T^2}{2}\right) + \frac{a}{2}\left(\frac{T^3}{3} - t_1^2 T\right) + \frac{t_1^2}{2} - \frac{at_1^3}{3}\right] \\
&\left.+ C\left[\frac{1}{p}\left(\frac{\alpha t_1^2}{2} - \frac{a\alpha t_1^3}{3} + T - \frac{aT^2}{2}\right)\right]\right\}.
\end{aligned} \tag{22}
$$

The necessary conditions for the total relevant cost per time unit to be minimum are $\frac{dTC}{dt_1}$ and $\frac{d^2TC}{dt_1^2} > 0$ and for all $t_1 > 0$.

4. Numerical examples

Example 1 (Two-warehouse model). Let us consider an inventory system with the following data: $T = 1$ year, $A = 600$, $a = 10$, $p = 8$, $w = 200$, $C_1 = 0.8$, $C_d = 0.3$, $C = 5$, $\alpha = 0.5$, $\beta = 0.2$, $h_r = 4$, $h_0 = 2$. Then, we get the optimal values as $t_w = 0.4472,\ t_1 = 0.8656,\ Q = 2965.6660,\ TC = 1698.0054$.

Example 2 (Single warehouse model). Let us consider an inventory system with the following data:$T = 1$ year, $A = 600$, $a = 10$, $p = 8$, $w = 200$, $C_1 = 0.8$, $C_d = 0.3$, $C = 5$, $\alpha = 0.5$, $h_0 = 2$. Then, we get the optimal values as $t_1 = 0.6310,\ Q = 2032.1456,\ TC = 1724.2298$.

5. Conclusion

The two-warehouse inventory model for deteriorating items under the effect of selling price and time dependent demand with completely backlogging has been investigated. The model assumed that the capacity of OW is limited. Our aim is to find the optimum replenishment policy i.e. to minimize the total relevant cost of the inventory system. Numerical examples are presented to illustrate validity of the model. It is clear from the numerical illustration that the total cost of the Single warehouse model is higher than two warehouse model. The system has no space to store excess units and the total inventory cost is higher due to shortage cost. The model is useful for the commodities for seasonal items like food grains like paddy, wheat etc. where demand decreases during the end of the season. It is observed that the retailer avail product at a low price when initially received

to market, for making the items of the company. So demand is initially high, but as product gets its recognition in the market, its price increases and accordingly the demand decreases due to hike in the price of that product which is in good agreement with these situations of market. Therefore, the choice of the demand function $D(t, p)$ in the present model is decreasing exponentially with time and inversely with price. The proposed model can be extended in several ways. For example, we may extend the model to incorporate some more realistic features, such as quantity discount, multi-item, stock-dependent demand, time-dependent demand, etc.

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Accepted: September 19, 2017

On solving matrix games with payoffs presented by dual hesitant fuzzy sets

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Abstract. The solving methodology of matrix games, with payoffs presented by dual hesitant fuzzy sets, is investigated in this paper. Firstly, the notion of dual hesitant fuzzy sets is given. Next, the concept of solutions for the matrix games is defined on the basis of dual hesitant fuzzy sets, which thereby prove the solutions of the matrix games can be obtained through solving a pair of linear programming models. Lastly, a numerical example is given to illustrate the effectiveness of the proposed method.

Keywords: matrix games, dual hesitant fuzzy set, multi-objective programming,linear programming.

1. Introduction

Game theory is an effective tool for analyzing the interaction of a group of rational agents with strategic behaviors. There are three elements in the game model: players, strategy sets and payoff function. Game theory has become a relatively complete theoretical system, rich in content, and has been widely applied to various fields. With the development of social economics, there are four critical points of the Game Theory in which to consider: the increase of information, incomplete information, unclear problems, and slow decision making. These often lead to some problems appearing in the uncertainty information. In 1974, based on the fuzzy set theory established by US cybernetic expert professor Zadeh [1], Aubin first combined this theory into cooperative game model [2]. Since then, this theory has garnered the attention of a number of scholarly individuals in fuzzy multi-attribute decision [3], [4], [5], [6] and fuzzy game modeling. However, the traditional fuzzy sets are unable to adequately identify all

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the problems in characterization. Due to the incomplete information, the game party's judgment often carries a certain degree of hesitation in which causes a certain level of influence over game selection. Hesitant fuzzy set (HFS)£¬ originally introduced by Torra [7]£¬ is the extension of the FS. The membership functions of HFS are a set composed of several numbers in $[0, 1]$. However, the HFS cannot reflect the degree of opposition expressed by the players, and lacks regard for the importance of non-membership. The intuitionistic fuzzy set(IFS) proposed by Atanassov [8] can more accurately represent the preference information of the players for support, opposition or hesitation, but cannot describe the information of the decision makers' hesitation. Zhu et al. [9][10] studied the interrelationship among Zadeh's fuzzy sets, interval-valued fuzzy sets(IVFSs), Atanassov's intuitionistic fuzzy sets(IFSs), fuzzy multisets(FMSs), hesitant fuzzy sets(HFSs) etc., and developed dual hesitant fuzzy set(DHFS). Dual hesitant fuzzy set (DHFS) is a new extension of IFS and HFS. DHFS consist of two parts: the membership hesitancy function and the non-membership hesitancy function. The membership degrees and non-membership degrees are two sets consisting of values in $[0, 1]$ that can substantiate a more versatile form of the hesitation of decision makers. Moreover, it is more compatible with facts. DHFSs can produce a better view during the analysis by obtaining more information and increasing its reliability. Compared to the other fuzzy sets, DHFS can be regarded as a more comprehensive set, which provides a more flexible approach when the players present their judgments. DHFS has attracted the ever increasing attention of scholars, due to the advantages it provides.

A number of publications have focused on fuzzy matrix game. Li and Cheng [11] introduced a new type of matrix game with fuzzy constraint and explored a computational method for its solution. Nan et al. [12] given the definition of matrix games with payoffs of IFSs and developed a pair of linear programming models to solve the matrix games. Li et al. [13] obtained the matrix games with payoffs of Atanassov¡¯s IFSs solutions through solving a pair of nonlinear programming models, which derived from two auxiliary nonlinear bi-objective programming models. Cevikel and Ahlatçıoğlu[14] explored two-person zero-sum games with fuzzy payoffs and fuzzy goals. Li and Zhang [15] introduced the concept of crisp bi-matrix games with parameters, and obtained the (weak) Pareto Nash equilibrium strategies for the fuzzy matrix games by solving the parametric bi-matrix games. Li [16] developed a valid method for solving matrix games with payoffs presented by triangular fuzzy numbers. Aggarwal et al. [17] studied a class of linear programming problems with fuzzy goals or fuzzy constraints and used it to solve the two-person zero-sum matrix game problems with IFSs. Yang et al. [18] extended the multi-objective bi-matrix game in the intuitionistic fuzzy environment, and proposed a non-linear programming model to find the mixed Nash equilibrium solution. Nan and Li [19] investigated a method to solve matrix games in which goals of players are pressed by IFSs and payoffs are expressed with real numbers. Chandra and Aggarwal [20] provided certain appropriate modifications in Li's [16] model, and studied an

algorithm to solve matrix games with payoffs presented by general piecewise linear numbers. Zhang et al. [21] established a multi-objective bi-matrix game model based on fuzzy goals. An et.al [22] proposed a weighted mean-area ranking method of Atanassov¡¯s intuitionistic fuzzy numbers, and developed a new methodology for solving bi-matrix games with payoffs of IFNs. Selvakumari and Lavanya [23] developed intuitionistic fuzzy game problem with payoffs in the form of octagonal intuitionistic fuzzy number, and converted the octagonal intuitionistic fuzzy numbers into intuitionistic numbers, and solved the intuitionistic fuzzy game problems. Based on the framework of a cooperative fuzzy game, Qiu et al.[24] put forward the multi-objective matrix game model based on fuzzy payoffs and concluded that the equilibrium solution could be converted into the optimal solution of multicriteria linear optimization problems. Borkotokey et al. [25] developed a simultaneous resource accumulation and payoff allocation algorithm. Khanizad and Montazer [26] applied fuzzy game theory to optimize the allocation of human resources. Yu et al. [27] proposed the coincident fuzziness form for games with fuzzy coalition. However, there exists less investigation on matrix games with DHFSs.

The remainder of this paper is organized as follows: Section 2 introduces some basic definitions and operators on dual hesitant fuzzy set, and the natation on matrix game with payoffs of dual hesitant fuzzy sets. Section 3, studies a programming method for matrix game with payoffs presented by dual hesitant fuzzy sets. Section 4 presents a simple numerical example to illustrate the validity of the proposed method. Section 5, summarizes the major conclusions achieved in this work.

2. Matrix games with payoffs of dual hesitant fuzzy sets and solution concept

We briefly review the concept and the operations of DHFSs, which would be used in the following section.

2.1 The concept and the operations of DHFSs

Definition 2.1 ([9]). *Let X be a a fixed set, a dual hesitant fuzzy set $DHFS$ on X is defined as $D = \{< x, h(x), g(x) > \mid x \in X\}$ in which $h(x)$ and $g(x)$ are two sets of some values in $[0, 1]$, denoting the possible membership degrees and the possible non-membership degrees of the element $x \in X$ to the D respectively, with the conditions: $0 \leq \gamma, \eta \leq 1$; $0 \leq \gamma^+ + \eta^+ \leq 1$. where $\gamma \in h(x), \eta \in g(x), \gamma^+ = \max\{\gamma\}, \eta^+ = \max\{\eta\}$ for all $x \in X$.*

For convenience, we call the pair $d(x) = (h(x), g(x))$ a dual hesitant fuzzy element (DHFE), denoted by $d = (h, g)$.

Definition 2.2 ([10]). *Assume two DHFEs as $d_1 = (h_1, g_1), d_2 = (h_2, g_2)$, $\lambda \geq 0$, then their operational laws are listed as below:*

(1) $h_1 \oplus h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{\gamma_1 + \gamma_2 - \gamma_1\gamma_2\}$;
(2) $h_1 \otimes h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{\gamma_1\gamma_2\}$;
(3) $d_1 \oplus d_2 = \{h_1 \oplus h_2, g_1 \otimes g_2\}$;
(4) $d_1 \otimes d_2 = \{h_1 \otimes h_2, g_1 \oplus g_2\}$;
(5) $\lambda d_1 = (\bigcup_{\gamma_1 \in h_1, \eta_1 \in g_1} \{1 - (1 - \gamma_1)^\lambda\}, \bigcup_{\gamma_1 \in h_1, \eta_1 \in g_1} \{{\eta_1}^\lambda\})$;
(6) ${d_1}^\lambda = (\bigcup_{\gamma_1 \in h_1, \eta_1 \in g_1} \{{\gamma_1}^\lambda\}, \bigcup_{\gamma_1 \in h_1, \eta_1 \in g_1} \{1 - (1 - \eta_1)^\lambda\})$.

Definition 2.3 ([10]). *There is a DHFE, denoted by* $t = (\{\gamma^l | l = 1, \ldots, \#h\}, \{\eta^{l'} | l' = 1, \ldots, \#g\})$, *where* $\#h$ *and* $\#g$ *are the numbers of elements in* h *and* g *respectively, let* $\gamma^- = \min\{\gamma^l\}, \eta^- = \min\{\eta^{l'}\}$ *and* $\gamma^+ = \max\{\gamma^l\}$, $\eta^+ = \max\{\eta^{l'}\}$,$\varsigma(0 \le \varsigma \le 1)$ *be an optimized parameter, then we call* $\vartheta = \varsigma\gamma^+ + (1 - \varsigma)\gamma^-$ *an added membership degree, and* $\psi = \varsigma\eta^- + (1 - \varsigma)\eta^+$ *an added non-membership degree.*

Apparently, the DMs can add the membership degree and non-membership degree according to their risk preference. If $\varsigma = 1$, which means that the DM is risk-seeking; if $\varsigma = 0$, which indicates that the DM is risk-averse; if $\varsigma = 0.5$, which reflects that the DM is risk-neutral. In this study, we assume that the DMs are risk-seeking.

Definition 2.4 ([10]). *For two DHFEs* $t_1 = (h_1, g_1), t_2 = (h_2, g_2)$ *, if* $\#h_1 = \#h_2, \#g_1 = \#g_2$, *then they are called two normalized DHFEs.*

Remark 2.1. If the given two DHFEs are not normalized DHFEs, we can add some membership degrees and non-membership degrees to normalize them by an optimized $\varsigma(0 \le \varsigma \le 1)$.

Definition 2.5 ([10]). *Given two normalized DHFEs:*

$$t_1 = (\{\gamma_1^l | \ l = 1, \ldots, \#h_1\}, \{\eta_1^{l'} | \ l' = 1, \ldots, \#g_1\}),$$
$$t_2 = (\{\gamma_2^l | \ l = 1, \ldots, \#h_2\}, \{\eta_2^{l'} | \ l' = 1, \ldots, \#g_2\}).$$

where $\#h_1 = \#h_2, \#g_1 = \#g_2$.

If ${\gamma_1}^{\sigma(l)} \ge {\gamma_2}^{\sigma(l)}$, ${\eta_1}^{\sigma(l')} \le {\eta_2}^{\sigma(l')}$, *then* $t_1 \supseteq t_2$. *Where* ${\gamma_i}^{\sigma(l)}, {\eta_i}^{\sigma(l')}$ *are the* l*-th and* l'*-th elements in* h_i *and* g_i *respectively,* $i = 1, 2$.

It is easily reached that, if $h_1 \subseteq h_2, g_1 \supseteq g_2$, then $t_1 \supseteq t_2$.

Remark 2.2. In this study, all the DHFEs are normalized DHFEs, and all the elements are arranged in increasing order.

Definition 2.6. *Given a vector* $\mathbf{x} = (x_1, \ldots, x_m)$, *and a matrix* $B = (t_{ij})_{m \times n}$ *where* $t_{ij} = (h_{ij}, g_{ij}), \#h_{ij} = l, \#g_{ij} = l', h_{ij} = \{\gamma_{ij}^1, \ldots, \gamma_{ij}^l\}, g_{ij} = \{\eta_{ij}^1, \ldots, \eta_{ij}^{l'}\}$,

then

$$\begin{aligned}\mathbf{xB} =&(\bigcup_{\gamma_{ij}\in h_{ij}}\{1-\prod_{i=1}^{n}(1-\gamma_{ij})^{x_i}\},\ \bigcup_{\eta_{ij}\in g_{ij}}\{\prod_{i=1}^{n}(\eta_{ij})^{x_i}\})\\ =&(\{1-\prod_{i=1}^{n}(1-\gamma_{ij}^{\sigma(1)})^{x_i},\ldots,1-\prod_{i=1}^{n}(1-\gamma_{ij}^{\sigma(k)})^{x_i},\ldots,1-\prod_{i=1}^{n}(1-\gamma_{ij}^{\sigma(l)})^{x_i}\},\\ &\{\prod_{i=1}^{n}(\eta_{ij}^{\sigma(1)})^{x_i},\ldots,\prod_{i=1}^{n}(\eta_{ij}^{\sigma(k')})^{x_i},\ldots,\prod_{i=1}^{n}(\eta_{ij}^{\sigma(l')})^{x_i}\})_{1\times n}(j=1,2,\ldots,n)\end{aligned}$$

2.2 Matrix games with payoffs of dual hesitant fuzzy sets and solution concept

There are two players p_1 and p_2, assume that $S_1=\{\alpha_1,\alpha_2,\ldots,\alpha_m\}$ and $S_2=\{\beta_1,\beta_2,\ldots,\beta_n\}$ be sets of pure strategies for players p_1 and p_2, respectively. If player p_1 chooses the pure strategy $\alpha_i\in S_1$ and player p_2 chooses the pure strategy $\beta_j\in S_2$, then the player p_1 gains a payoff expressed by DHFS(h_{ij},g_{ij}), while the player p_2 loss payment value is (h_{ij},g_{ij}).

Player p_1 choose the pure strategy $\alpha_i\in S_1$ with the probability $x_i(i=1,2,\ldots,m)$, and $y_j(j=1,2,\ldots,n)$ is the probability that player p_2 chooses the pure strategy $\beta_j\in S_2$. The vectors $\mathbf{x}=(x_1,x_2,\ldots,x_m)^T$ and $\mathbf{y}=(y_1,y_2,\ldots,y_n)^T$ are mixed strategies for players p_1 and p_2 , respectively.

Let $X=\{\mathbf{x}\in R^m|\ \sum_{i=1}^{m}x_i=1,x_i\geq 0,i=1,2,\ldots,m\}$, $Y=\{\mathbf{y}\in R^n|\ \sum_{j=1}^{n}y_j=1,y_j\geq 0,j=1,2,\ldots,n\}$. then X,Y are known as the strategy spaces for players p_1 and p_2 , respectively.

If player p_1 and p_2 choose mixed strategies $\mathbf{x}\in X,\mathbf{y}\in Y$,respectively, according to Definition 2.2 and Definition 2.6, the expect payoff $E(\mathbf{x},\mathbf{y})$ can be calculated as follows:

$$\begin{aligned}E(\mathbf{x},\mathbf{y})=&\mathbf{x}^TA\mathbf{y}\\ =&(\bigcup_{\gamma_{ij}\in h_{ij}}\{1-\prod_{j=1}^{n}\prod_{i=1}^{m}(1-\gamma_{ij})^{x_iy_j}\},\ \bigcup_{\eta_{ij}\in g_{ij}}\{\prod_{j=1}^{n}\prod_{i=1}^{m}(\eta_{ij})^{x_iy_j}\})\\ =&(\{1-\prod_{j=1}^{n}\prod_{i=1}^{m}(1-\gamma_{ij}^{\sigma(1)})^{x_iy_j},\ldots,1-\prod_{j=1}^{n}\prod_{i=1}^{m}(1-\gamma_{ij}^{\sigma(k)})^{x_iy_j},\ldots,\\ &1-\prod_{j=1}^{n}\prod_{i=1}^{m}(1-\gamma_{ij}^{\sigma(l)})^{x_iy_j}\},\{\prod_{j=1}^{n}\prod_{i=1}^{m}(\eta_{ij}^{\sigma(1)})^{x_iy_j},\ldots,\prod_{j=1}^{n}\prod_{i=1}^{m}(\eta_{ij}^{\sigma(k')})^{x_iy_j},\ldots,\\ &\prod_{j=1}^{n}\prod_{i=1}^{m}(\eta_{ij}^{\sigma(l')})^{x_iy_j}\})\end{aligned}$$

Obviously, the expect payoff of player p_1 is a DHFS.

The interests of the two players is precisely opposite in one game. The payoff of each player's is not only related to the strategy he chooses, but also

depends on the strategy chosen by the opponent. Therefore, each player should consider the strategy chosen by the opponent to choose the strategy which is most beneficial to himself.

Assume that the players in the game are rational. The player p_1 will choose the strategy $\mathbf{x}$ so as to maximize the expect payoff $E(\mathbf{x},\mathbf{y}) = \mathbf{x}^T A\mathbf{y}$, while the player p_2 will choose the strategy $\mathbf{y}$ so as to minimize the expect payoff $E(\mathbf{x},\mathbf{y}) = \mathbf{x}^T A\mathbf{y}$. If the player p_1 choose the strategy $\mathbf{x}$, then he could gain the payoff $\min_{\mathbf{y}\in Y}\{\mathbf{x}^T A\mathbf{y}\}$ at least, the player p_1 hope the more the payoff he get, the better, so he would choose the strategy make the payoff is no less than $\max_{\mathbf{x}\in X}\min_{\mathbf{y}\in Y}\{\mathbf{x}^T A\mathbf{y}\}$.

Similarly, if the player p_2 choose the strategy $\mathbf{y}$, then he could gain the payoff $\max_{\mathbf{x}\in X}\{\mathbf{x}^T A\mathbf{y}\}$ at most, the player p_2 hope the less the payoff he loss, the better, so he would choose the strategy make the payoff is no more than $\min_{\mathbf{y}\in Y}\max_{\mathbf{x}\in X}\{\mathbf{x}^T A\mathbf{y}\}$.

Theorem 2.1. *For a matrix game A with payoffs of dual hesitant fuzzy sets, then*

$$\max_{\mathbf{x}\in X}\min_{\mathbf{y}\in Y}\{\mathbf{x}^T A\mathbf{y}\} \subseteq \min_{\mathbf{y}\in Y}\max_{\mathbf{x}\in X}\{\mathbf{x}^T A\mathbf{y}\}$$

Proof. According to Definition 2.2, assume to $\#h_{ij} = l, \#g_{ij} = l'$, then $E(\mathbf{x},\mathbf{y})$ can be calculated as follows:

$$\begin{aligned}
\mathbf{x}^T A\mathbf{y} &= \Big(\bigcup_{\gamma_{ij}\in h_{ij}}\{1-\prod_{j=1}^{n}\prod_{i=1}^{m}(1-\gamma_{ij})^{x_i y_j}\}, \bigcup_{\eta_{ij}\in g_{ij}}\{\prod_{j=1}^{n}\prod_{i=1}^{m}(\eta_{ij})^{x_i y_j}\}\Big)\\
&= \Big(\{1-\prod_{j=1}^{n}\prod_{i=1}^{m}(1-\gamma_{ij}^{\sigma(1)})^{x_i y_j},\ldots,1-\prod_{j=1}^{n}\prod_{i=1}^{m}(1-\gamma_{ij}^{\sigma(k)})^{x_i y_j},\ldots,1\\
&\quad -\prod_{j=1}^{n}\prod_{i=1}^{m}(1-\gamma_{ij}^{\sigma(l)})^{x_i y_j}\},\\
&\quad \{\prod_{j=1}^{n}\prod_{i=1}^{m}(\eta_{ij}^{\sigma(1)})^{x_i y_j},\ldots,\prod_{j=1}^{n}\prod_{i=1}^{m}(\eta_{ij}^{\sigma(k')})^{x_i y_j},\ldots,\prod_{j=1}^{n}\prod_{i=1}^{m}(\eta_{ij}^{\sigma(l')})^{x_i y_j}\}\Big)
\end{aligned}$$

then

$$\begin{aligned}
\max_{\mathbf{x}\in X}\min_{\mathbf{y}\in Y}\{\mathbf{x}^T A\mathbf{y}\} = \Big(&\max_{\mathbf{x}\in X}\min_{\mathbf{y}\in Y}\{\bigcup_{\gamma_{ij}\in h_{ij}}\{1-\prod_{j=1}^{n}\prod_{i=1}^{m}(1-\gamma_{ij})^{x_i y_j}\}\},\\
&\min_{\mathbf{x}\in X}\max_{\mathbf{y}\in Y}\{\bigcup_{\eta_{ij}\in g_{ij}}\{\prod_{j=1}^{n}\prod_{i=1}^{m}(\eta_{ij})^{x_i y_j}\}\}\Big)\\
\min_{\mathbf{y}\in Y}\max_{\mathbf{x}\in X}\{\mathbf{x}^T A\mathbf{y}\} = \Big(&\min_{\mathbf{y}\in Y}\max_{\mathbf{x}\in X}\{\bigcup_{\gamma_{ij}\in h_{ij}}\{1-\prod_{j=1}^{n}\prod_{i=1}^{m}(1-\gamma_{ij})^{x_i y_j}\}\},
\end{aligned}$$

$$\max_{\mathbf{y}\in Y}\min_{\mathbf{x}\in X}\{\bigcup_{\eta_{ij}\in g_{ij}}\{\prod_{j=1}^{n}\prod_{i=1}^{m}(\eta_{ij})^{x_iy_j}\}\})$$

$$\min_{\mathbf{y}\in Y}\{1-\prod_{j=1}^{n}\prod_{i=1}^{m}(1-\gamma_{ij}^{\sigma(k)})^{x_iy_j}\}\le 1-\prod_{j=1}^{n}\prod_{i=1}^{m}(1-\gamma_{ij}^{\sigma(k)})^{x_iy_j}(k=1,\dots,l)$$

for any $\mathbf{x}\in X$

$$\max_{\mathbf{x}\in X}\{1-\prod_{j=1}^{n}\prod_{i=1}^{m}(1-\gamma_{ij}^{\sigma(k)})^{x_iy_j}\}\ge 1-\prod_{j=1}^{n}\prod_{i=1}^{m}(1-\gamma_{ij}^{\sigma(k)})^{x_iy_j}(k=1,\dots,l)$$

for any $\mathbf{y}\in Y$
then

$$\max_{\mathbf{x}\in X}\{1-\prod_{j=1}^{n}\prod_{i=1}^{m}(1-\gamma_{ij}^{\sigma(k)})^{x_iy_j}\}\ge\min_{\mathbf{y}\in Y}\{1-\prod_{j=1}^{n}\prod_{i=1}^{m}(1-\gamma_{ij}^{\sigma(k)})^{x_iy_j}\}(k=1,\dots,l)$$

for any $\mathbf{x}\in X,\mathbf{y}\in Y$ The above formula is valid for any $\mathbf{y}\in Y$, then

$$\min_{\mathbf{y}\in Y}\max_{\mathbf{x}\in X}\{1-\prod_{j=1}^{n}\prod_{i=1}^{m}(1-\gamma_{ij}^{\sigma(k)})^{x_iy_j}\}\ge\min_{\mathbf{y}\in Y}\{1-\prod_{j=1}^{n}\prod_{i=1}^{m}(1-\gamma_{ij}^{\sigma(k)})^{x_iy_j}\}(k=1,\dots,l)$$

The above formula is valid for any $\mathbf{x}\in X$, then

$$\begin{aligned}&\min_{\mathbf{y}\in Y}\max_{\mathbf{x}\in X}\{1-\prod_{j=1}^{n}\prod_{i=1}^{m}(1-\gamma_{ij}^{\sigma(k)})^{x_iy_j}\}\\ &\ge\max_{\mathbf{x}\in X}\min_{\mathbf{y}\in Y}\{1-\prod_{j=1}^{n}\prod_{i=1}^{m}(1-\gamma_{ij}^{\sigma(k)})^{x_iy_j}\}(k=1,\dots,l)\end{aligned}\tag{2.1}$$

Similarly

$$\max_{\mathbf{y}\in Y}\{\prod_{j=1}^{n}\prod_{i=1}^{m}(\eta_{ij}^{\sigma(k')})^{x_iy_j}\}\ge\prod_{j=1}^{n}\prod_{i=1}^{m}(\eta_{ij}^{\sigma(k')})^{x_iy_j}\ (k'=1,\dots,l')$$

for any $\mathbf{x}\in X$

$$\prod_{j=1}^{n}\prod_{i=1}^{m}(\eta_{ij}^{\sigma(k')})^{x_iy_j}\ge\min_{\mathbf{x}\in X}\{\prod_{j=1}^{n}\prod_{i=1}^{m}(\eta_{ij}^{\sigma(k')})^{x_iy_j}\}\ (k'=1,\dots,l')$$

for any $\mathbf{y}\in Y$ then

$$\max_{\mathbf{y}\in Y}\{\prod_{j=1}^{n}\prod_{i=1}^{m}(\eta_{ij}^{\sigma(k')})^{x_iy_j}\}\ge\min_{\mathbf{x}\in X}\{\prod_{j=1}^{n}\prod_{i=1}^{m}(\eta_{ij}^{\sigma(k')})^{x_iy_j}\}\ (k'=1,\dots,l')$$

for any $\mathbf{x} \in X, \mathbf{y} \in Y$.

The above formula is valid for any $\mathbf{y} \in Y$,then

$$\max_{\mathbf{y}\in Y}\{\prod_{j=1}^{n}\prod_{i=1}^{m}(\eta_{ij}^{\sigma(k')})^{x_iy_j}\} \geq \max_{\mathbf{y}\in Y}\min_{\mathbf{x}\in X}\{\prod_{j=1}^{n}\prod_{i=1}^{m}(\eta_{ij}^{\sigma(k')})^{x_iy_j}\}\ (k'=1,\ldots,l')$$

The above formula is valid for any $\mathbf{x} \in X$,then

$$\begin{aligned}&\min_{\mathbf{x}\in X}\max_{\mathbf{y}\in Y}\{\prod_{j=1}^{n}\prod_{i=1}^{m}(\eta_{ij}^{\sigma(k')})^{x_iy_j}\}\\ &\geq \max_{\mathbf{y}\in Y}\min_{\mathbf{x}\in X}\{\prod_{j=1}^{n}\prod_{i=1}^{m}(\eta_{ij}^{\sigma(k')})^{x_iy_j}\}\ (k'=1,\ldots,l')\end{aligned} \tag{2.2}$$

The following conclusion is easily reached from Eqs.(2.1)(2.2) and Definition 2.2:

$$\max_{\mathbf{x}\in X}\min_{\mathbf{y}\in Y}(\{1-\prod_{j=1}^{n}\prod_{i=1}^{m}(1-\gamma_{ij}^{\sigma(1)})^{x_iy_j},\ldots,1-\prod_{j=1}^{n}\prod_{i=1}^{m}(1-\gamma_{ij}^{\sigma(l)})^{x_iy_j}\},$$
$$\{\prod_{j=1}^{n}\prod_{i=1}^{m}(\eta_{ij}^{\sigma(1)})^{x_iy_j},\ldots,\prod_{j=1}^{n}\prod_{i=1}^{m}(\eta_{ij}^{\sigma(l')})^{x_iy_j}\})\subseteq \min_{\mathbf{y}\in Y}\max_{\mathbf{x}\in X}(\{1-\prod_{j=1}^{n}\prod_{i=1}^{m}(1-\gamma_{ij}^{\sigma(1)})^{x_iy_j},\ldots,$$
$$1-\prod_{j=1}^{n}\prod_{i=1}^{m}(1-\gamma_{ij}^{\sigma(l)})^{x_iy_j}\},\{\prod_{j=1}^{n}\prod_{i=1}^{m}(\eta_{ij}^{\sigma(1)})^{x_iy_j},\ldots,\prod_{j=1}^{n}\prod_{i=1}^{m}(\eta_{ij}^{\sigma(l')})^{x_iy_j}\})$$

i.e. $\max_{\mathbf{x}\in X}\min_{\mathbf{y}\in Y}\{\mathbf{x}^TA\mathbf{y}\} \subseteq \min_{\mathbf{y}\in Y}\max_{\mathbf{x}\in X}\{\mathbf{x}^TA\mathbf{y}\}$. □

Theorem 2.1 shows that the minimum win of the player p_1 in the game does not exceed the maximum loss of the player p_2 , which is consistent with the conclusion of the classical matrix game.

Definition 2.7. *If there is a mixed strategy* $(\mathbf{x}^0, \mathbf{y}^0)$*, satisfied the following equation:*

$$\mathbf{x}^{0T}A\mathbf{y}^0 = \max_{\mathbf{x}\in X}\min_{\mathbf{y}\in Y}\{\mathbf{x}^TA\mathbf{y}\} = \min_{\mathbf{y}\in Y}\max_{\mathbf{x}\in X}\{\mathbf{x}^TA\mathbf{y}\}$$

Then the mixed strategy $(\mathbf{x}^0, \mathbf{y}^0)$ *is called mixed balance strategy.* $\mathbf{x}^0, \mathbf{y}^0$ *each are the optimal strategies for the two players.* $\mathbf{x}^{0T}A\mathbf{y}^0$ *is called a value of the dual hesitant fuzzy matrix game* A *.*

Theorem 2.2. *For a matrix game* A *with payoffs of dual hesitant fuzzy sets,*

$$\mathbf{x}^{0T}A\mathbf{y}^0 = \max_{\mathbf{x}\in X}\min_{\mathbf{y}\in Y}\{\mathbf{x}^TA\mathbf{y}\} = \min_{\mathbf{y}\in Y}\max_{\mathbf{x}\in X}\{\mathbf{x}^TA\mathbf{y}\}$$

if and only if $\mathbf{x}^TA\mathbf{y}^0 \subseteq \mathbf{x}^{0T}A\mathbf{y}^0 \subseteq \mathbf{x}^{0T}A\mathbf{y}$

Proof. Sufficiency: If there exist $\mathbf{x}^0, \mathbf{y}^0$, s.t. $\mathbf{x}^T A\mathbf{y}^0 \subseteq \mathbf{x}^{0T} A\mathbf{y}^0 \subseteq \mathbf{x}^{0T} A\mathbf{y}$ for any $\mathbf{x} \in X, \mathbf{y} \in Y$
then

$$1 - \prod_{j=1}^{n}\prod_{i=1}^{m}(1-\gamma_{ij}^{\sigma(k)})^{x_i y_j^0} \leq 1 - \prod_{j=1}^{n}\prod_{i=1}^{m}(1-\gamma_{ij}^{\sigma(k)})^{x_i^0 y_j^0} \ (k=1,\ldots,l) \tag{2.3}$$

$$\prod_{j=1}^{n}\prod_{i=1}^{m}(\eta_{ij}^{\sigma(k')})^{x_i y_j^0} \geq \prod_{j=1}^{n}\prod_{i=1}^{m}(\eta_{ij}^{\sigma(k')})^{x_i^0 y_j^0} \ (k'=1,\ldots,l') \tag{2.4}$$

for any $\mathbf{x} \in X, \mathbf{y} \in Y$. And

$$1 - \prod_{j=1}^{n}\prod_{i=1}^{m}(1-\gamma_{ij}^{\sigma(k)})^{x_i^0 y_j^0} \leq 1 - \prod_{j=1}^{n}\prod_{i=1}^{m}(1-\gamma_{ij}^{\sigma(k)})^{x_i^0 y_j} \ (k=1,\ldots,l) \tag{2.5}$$

$$\prod_{j=1}^{n}\prod_{i=1}^{m}(\eta_{ij}^{\sigma(k')})^{x_i^0 y_j^0} \geq \prod_{j=1}^{n}\prod_{i=1}^{m}(\eta_{ij}^{\sigma(k')})^{x_i^0 y_j} \ (k'=1,\ldots,l') \tag{2.6}$$

for any $\mathbf{x} \in X, \mathbf{y} \in Y$.

Due to (2.3) and (2.4)are valid for any $\mathbf{x} \in X, \mathbf{y} \in Y$, then

$$\max_{\mathbf{x}\in X}\{1 - \prod_{j=1}^{n}\prod_{i=1}^{m}(1-\gamma_{ij}^{\sigma(k)})^{x_i y_j^0}\} \leq 1 - \prod_{j=1}^{n}\prod_{i=1}^{m}(1-\gamma_{ij}^{\sigma(k)})^{x_i^0 y_j^0} \ (k=1,\ldots,l)$$

$$\min_{\mathbf{x}\in X}\{\prod_{j=1}^{n}\prod_{i=1}^{m}(\eta_{ij}^{\sigma(k')})^{x_i y_j^0}\} \geq \prod_{j=1}^{n}\prod_{i=1}^{m}(\eta_{ij}^{\sigma(k')})^{x_i^0 y_j^0} \ (k'=1,\ldots,l')$$

Additionally,

$$\min_{\mathbf{y}\in Y}\max_{\mathbf{x}\in X}\{1-\prod_{j=1}^{n}\prod_{i=1}^{m}(1-\gamma_{ij}^{\sigma(k)})^{x_i y_j}\} \leq \max_{\mathbf{x}\in X}\{1-\prod_{j=1}^{n}\prod_{i=1}^{m}(1-\gamma_{ij}^{\sigma(k)})^{x_i y_j^0}\} \ (k=1,\ldots,l)$$

$$\max_{\mathbf{y}\in Y}\min_{\mathbf{x}\in X}\{\prod_{j=1}^{n}\prod_{i=1}^{m}(\eta_{ij}^{\sigma(k')})^{x_i y_j}\} \geq \min_{\mathbf{x}\in X}\{\prod_{j=1}^{n}\prod_{i=1}^{m}(\eta_{ij}^{\sigma(k')})^{x_i y_j^0}\} \ (k'=1,\ldots,l')$$

for any $\mathbf{x} \in X, \mathbf{y} \in Y$, then

$$\min_{\mathbf{y}\in Y}\max_{\mathbf{x}\in X}\{1 - \prod_{j=1}^{n}\prod_{i=1}^{m}(1-\gamma_{ij}^{\sigma(k)})^{x_i y_j}\} \leq 1 - \prod_{j=1}^{n}\prod_{i=1}^{m}(1-\gamma_{ij}^{\sigma(k)})^{x_i^0 y_j^0} \ (k=1,\ldots,l)$$

$$\max_{\mathbf{y}\in Y}\min_{\mathbf{x}\in X}\{\prod_{j=1}^{n}\prod_{i=1}^{m}(\eta_{ij}^{\sigma(k')})^{x_i y_j}\} \geq \min_{\mathbf{x}\in X}\{\prod_{j=1}^{n}\prod_{i=1}^{m}(\eta_{ij}^{\sigma(k')})^{x_i^0 y_j^0}\} \ (k'=1,\ldots,l')$$

i.e.

$$\min_{\mathbf{y}\in Y}\max_{\mathbf{x}\in X}\{\mathbf{x}^T A\mathbf{y}\} \subseteq \mathbf{x}^{0T} A\mathbf{y}^0 \tag{2.7}$$

Similarity, (2.5) and (2.6)are valid for any $\mathbf{x} \in X, \mathbf{y} \in Y$, then

$$1-\prod_{j=1}^{n}\prod_{i=1}^{m}(1-\gamma_{ij}^{\sigma(k)})^{x_i^0 y_j^0} \leq \min_{\mathbf{y}\in Y}\{1-\prod_{j=1}^{n}\prod_{i=1}^{m}(1-\gamma_{ij}^{\sigma(k)})^{x_i^0 y_j}\}\ (k=1,\ldots,l)$$

$$\prod_{j=1}^{n}\prod_{i=1}^{m}(\eta_{ij}^{\sigma(k')})^{x_i^0 y_j^0} \geq \max_{\mathbf{y}\in Y}\{\prod_{j=1}^{n}\prod_{i=1}^{m}(\eta_{ij}^{\sigma(k')})^{x_i^0 y_j}\}\ (k'=1,\ldots,l')$$

Additionally,

$$\min_{\mathbf{y}\in Y}\{1-\prod_{j=1}^{n}\prod_{i=1}^{m}(1-\gamma_{ij}^{\sigma(k)})^{x_i^0 y_j}\} \leq \max_{\mathbf{x}\in X}\min_{\mathbf{y}\in Y}\{1-\prod_{j=1}^{n}\prod_{i=1}^{m}(1-\gamma_{ij}^{\sigma(k)})^{x_i y_j}\}\ (k=1,\ldots,l)$$

$$\max_{\mathbf{y}\in Y}\{\prod_{j=1}^{n}\prod_{i=1}^{m}(\eta_{ij}^{\sigma(k')})^{x_i^0 y_j}\} \geq \min_{\mathbf{x}\in X}\max_{\mathbf{y}\in Y}\{\prod_{j=1}^{n}\prod_{i=1}^{m}(\eta_{ij}^{\sigma(k')})^{x_i y_j}\}\ (k'=1,\ldots,l')$$

for any $\mathbf{x} \in X, \mathbf{y} \in Y$, then

$$1-\prod_{j=1}^{n}\prod_{i=1}^{m}(1-\gamma_{ij}^{\sigma(k)})^{x_i^0 y_j^0} \leq \max_{\mathbf{x}\in X}\min_{\mathbf{y}\in Y}\{1-\prod_{j=1}^{n}\prod_{i=1}^{m}(1-\gamma_{ij}^{\sigma(k)})^{x_i y_j}\}\ (k=1,\ldots,l)$$

$$\prod_{j=1}^{n}\prod_{i=1}^{m}(\eta_{ij}^{\sigma(k')})^{x_i^0 y_j^0} \geq \min_{\mathbf{x}\in X}\max_{\mathbf{y}\in Y}\{\prod_{j=1}^{n}\prod_{i=1}^{m}(\eta_{ij}^{\sigma(k')})^{x_i y_j}\}\ (k'=1,\ldots,l')$$

i.e.

$$\mathbf{x}^{0T} A\mathbf{y}^0 \subseteq \max_{\mathbf{x}\in X}\min_{\mathbf{y}\in Y}\{\mathbf{x}^T A\mathbf{y}\} \tag{2.8}$$

According to (2.7) and (2.8), we can get the following conclusion:

$$\min_{\mathbf{y}\in Y}\max_{\mathbf{x}\in X}\{\mathbf{x}^T A\mathbf{y}\} \subseteq \max_{\mathbf{x}\in X}\min_{\mathbf{y}\in Y}\{\mathbf{x}^T A\mathbf{y}\} \tag{2.9}$$

The following conclusion is easily reached from Eq. (2.9) and Theorem 2.1:

$$\max_{\mathbf{x}\in X}\min_{\mathbf{y}\in Y}\{\mathbf{x}^T A\mathbf{y}\} = \min_{\mathbf{y}\in Y}\max_{\mathbf{x}\in X}\{\mathbf{x}^T A\mathbf{y}\}$$

Necessity:

If $\max_{\mathbf{x}\in X}\min_{\mathbf{y}\in Y}\{\mathbf{x}^T A\mathbf{y}\} = \min_{\mathbf{y}\in Y}\max_{\mathbf{x}\in X}\{\mathbf{x}^T A\mathbf{y}\}$ there must exist $\mathbf{x}^0 \in X, \mathbf{y}^0 \in Y$, s.t. $\max_{\mathbf{x}\in X}\min_{\mathbf{y}\in Y}\{\mathbf{x}^T A\mathbf{y}\} = \min_{\mathbf{y}\in Y}\{\mathbf{x}^{0T} A\mathbf{y}\}$ and

$$\min_{\mathbf{y}\in Y}\max_{\mathbf{x}\in X}\{\mathbf{x}^T A\mathbf{y}\} = \max_{\mathbf{x}\in X}\{\mathbf{x}^T A\mathbf{y}^{\mathbf{0}}\}$$

then

$$\max_{\mathbf{x}\in X}\{\mathbf{x}^T A\mathbf{y}^{\mathbf{0}}\} = \min_{\mathbf{y}\in Y}\{\mathbf{x}^{0T} A\mathbf{y}\}$$

i.e.

$$\begin{aligned}&\max_{\mathbf{x}\in X}\{1-\prod_{j=1}^{n}\prod_{i=1}^{m}(1-\gamma_{ij}^{\sigma(k)})^{x_i y_j^0}\}\\ &=\min_{\mathbf{y}\in Y}\{1-\prod_{j=1}^{n}\prod_{i=1}^{m}(1-\gamma_{ij}^{\sigma(k)})^{x_i^0 y_j}\}\ (k=1,\ldots,l)\end{aligned}\tag{2.10}$$

$$\begin{aligned}&\min_{\mathbf{x}\in X}\{\prod_{j=1}^{n}\prod_{i=1}^{m}(\eta_{ij}^{\sigma(k')})^{x_i y_j^0}\}\\ &=\max_{\mathbf{y}\in Y}\{\prod_{j=1}^{n}\prod_{i=1}^{m}(\eta_{ij}^{\sigma(k')})^{x_i^0 y_j}\}\ (k'=1,\ldots,l')\end{aligned}\tag{2.11}$$

Additionally,

$$\begin{aligned}&\max_{\mathbf{x}\in X}\{1-\prod_{j=1}^{n}\prod_{i=1}^{m}(1-\gamma_{ij}^{\sigma(k)})^{x_i y_j^0}\}\geq 1-\prod_{j=1}^{n}\prod_{i=1}^{m}(1-\gamma_{ij}^{\sigma(k)})^{x_i^0 y_j^0}\\ &\geq\min_{\mathbf{y}\in Y}\{1-\prod_{j=1}^{n}\prod_{i=1}^{m}(1-\gamma_{ij}^{\sigma(k)})^{x_i^0 y_j}\}\ (k=1,\ldots,l)\end{aligned}\tag{2.12}$$

$$\begin{aligned}&\min_{\mathbf{x}\in X}\{\prod_{j=1}^{n}\prod_{i=1}^{m}(\eta_{ij}^{\sigma(k')})^{x_i y_j^0}\}\leq\prod_{j=1}^{n}\prod_{i=1}^{m}(\eta_{ij}^{\sigma(k')})^{x_i^0 y_j^0}\\ &\qquad\leq\max_{\mathbf{y}\in Y}\{\prod_{j=1}^{n}\prod_{i=1}^{m}(\eta_{ij}^{\sigma(k')})^{x_i^0 y_j}\}\ (k'=1,\ldots,l').\end{aligned}\tag{2.13}$$

According to (2.10) and (2.12), we can get the following conclusion:

$$\begin{aligned}&\max_{\mathbf{x}\in X}\{1-\prod_{j=1}^{n}\prod_{i=1}^{m}(1-\gamma_{ij}^{\sigma(k)})^{x_i y_j^0}\}=1-\prod_{j=1}^{n}\prod_{i=1}^{m}(1-\gamma_{ij}^{\sigma(k)})^{x_i^0 y_j^0}\\ &=\min_{\mathbf{y}\in Y}\{1-\prod_{j=1}^{n}\prod_{i=1}^{m}(1-\gamma_{ij}^{\sigma(k)})^{x_i^0 y_j}\}\ (k=1,\ldots,l).\end{aligned}\tag{2.14}$$

We can get the following equation according to (2.11) and (2.13):

$$\min_{\mathbf{x}\in X}\{\prod_{j=1}^{n}\prod_{i=1}^{m}(\eta_{ij}^{\sigma(k')})^{x_i y_j^0}\} = \prod_{j=1}^{n}\prod_{i=1}^{m}(\eta_{ij}^{\sigma(k')})^{x_i^0 y_j^0} \tag{2.15}$$
$$= \max_{\mathbf{y}\in Y}\{\prod_{j=1}^{n}\prod_{i=1}^{m}(\eta_{ij}^{\sigma(k')})^{x_i^0 y_j}\}\ (k'=1,\ldots,l')$$

Combine (2.14) and (2.15), then

$$1-\prod_{j=1}^{n}\prod_{i=1}^{m}(1-\gamma_{ij}^{\sigma(k)})^{x_i y_j^0} \le 1-\prod_{j=1}^{n}\prod_{i=1}^{m}(1-\gamma_{ij}^{\sigma(k)})^{x_i^0 y_j^0} \tag{2.16}$$
$$\le 1-\prod_{j=1}^{n}\prod_{i=1}^{m}(1-\gamma_{ij}^{\sigma(k)})^{x_i^0 y_j}\ (k=1,\ldots,l)$$

and

$$\prod_{j=1}^{n}\prod_{i=1}^{m}(\eta_{ij}^{\sigma(k')})^{x_i y_j^0} \ge \prod_{j=1}^{n}\prod_{i=1}^{m}(\eta_{ij}^{\sigma(k')})^{x_i^0 y_j^0} \ge \prod_{j=1}^{n}\prod_{i=1}^{m}(\eta_{ij}^{\sigma(k')})^{x_i^0 y_j}(k'=1,\ldots,l') \tag{2.17}$$

for any $\mathbf{x}\in X, \mathbf{y}\in Y$.

The following conclusion is easily reached from Eqs. (2.16), (2.17) and Definition 2.5 $\mathbf{x}^T A\mathbf{y}^0 \subseteq \mathbf{x}^{0T} A\mathbf{y}^0 \subseteq \mathbf{x}^{0T} A\mathbf{y}$ □

The player p_1 chooses the strategy according to maximin principle, while the player p_2 chooses his strategy according to minimax principle. Then, if there exist a pair of mixed strategies $(\mathbf{x}^0, \mathbf{y}^0)$, satisfying

$$\mathbf{x}^{0T} A\mathbf{y}^0 = \max_{\mathbf{x}\in X}\min_{\mathbf{y}\in Y}\{\mathbf{x}^T A\mathbf{y}\} = \min_{\mathbf{y}\in Y}\max_{\mathbf{x}\in X}\{\mathbf{x}^T A\mathbf{y}\}, (\mathbf{x^0}, \mathbf{y^0})$$

are called the optimal strategies for the two players, respectively. $\mathbf{x}^{0T} A\mathbf{y}^0$ is the value of the mixed equilibrium situation. However, the expected payoff $E(\mathbf{x}, \mathbf{y})$ is a DHFS with the membership degrees and non-memberships. Thus, usually there do not exist $(\mathbf{x}^0, \mathbf{y}^0)$ meet the two scales at the same time. In fact, the problems $\max_{\mathbf{x}\in X}\min_{\mathbf{y}\in Y}\{\mathbf{x}^T A\mathbf{y}\}$ and $\min_{\mathbf{y}\in Y}\max_{\mathbf{x}\in X}\{\mathbf{x}^T A\mathbf{y}\}$ are bi-objective programming problems.

Similar to the Pareto solution [28], the concept of solution of the DHFS matrix game A may be given as follows.

Definition 2.8. *Let $\mu^* = (h_1^*, g_1^*)$ and $\omega^* = (h_2^*, g_2^*)$ be two DHFSs. Assume that there exist $\mathbf{x}^* \in X, \mathbf{y}^* \in Y$. Then $(\mathbf{x}^*, \mathbf{y}^*, \mu^*, \omega^*)$ is called a solution of the DHFS matrix game A for any $\mathbf{x}\in X$ and $\mathbf{y}\in Y$ satisfying both the following conditions.*

(1) *$\mu^* \subseteq \mathbf{x}^{*T} A\mathbf{y}$ and $\mathbf{x}^T A\mathbf{y}^* \subseteq \omega^*$.*

(2) *There do not exist any $\mu \neq \mu^*$ and $\omega \neq \omega^*$ such that $\mu^* \subseteq \mu$ and $\omega \subseteq \omega^*$.*

$\mathbf{x}^*$ and $\mathbf{y}^*$ are called player p_1's and player p_2's optimal strategies, respectively; μ^* is called gain-floor for player p_1 and ω^* is called loss-ceiling for player p_2 , respectively.

3. Programming methods for matrix game with payoffs presented by dual hesitant fuzzy sets

Assume that each player chooses an optimal mixed strategy under each other's pure strategy. According to Definition 2.8, the optimal strategy $\mathbf{x}^*$ and the gain-floor $\mu^* = (h_1^*, g_1^*)$, the optimal strategy $\mathbf{y}^*$ and the loss-ceiling $\omega^* = (h_2^*, g_2^*)$ can be generated by solving the following pair of mathematical programming models.

$$\max\{(h_1, g_1)\}$$
$$s.t. \begin{cases} \bigcup\limits_{\gamma_{ij}\in h_{ij}, \eta_{ij}\in g_{ij}} (1 - \prod\limits_{i=1}^{m}(1-\gamma_{ij})^{x_i}, \prod\limits_{i=1}^{m}(\eta_{ij})^{x_i}) \supseteq (h_1, g_1) \\ (j = 1, 2, \ldots, n) 0 \le h_1^+ + g_1^+ \le 1 \\ \sum\limits_{i=1}^{m} x_i = 1 \\ h_1^- \ge 0, g_1^- \ge 0, x_i \ge 0 \ (i = 1, 2, \ldots, m) \end{cases} \tag{3.1}$$

$$\min\{(h_2, g_2)\}$$
$$s.t. \begin{cases} \bigcup\limits_{\gamma_{ij}\in h_{ij}, \eta_{ij}\in g_{ij}} (1 - \prod\limits_{j=1}^{n}(1-\gamma_{ij})^{y_j}, \prod\limits_{j=1}^{n}(\eta_{ij})^{y_j}) \subseteq (h_2, g_2) \\ (i = 1, 2, \ldots, m) 0 \le h_2^+ + g_2^+ \le 1 \\ \sum\limits_{j=1}^{n} y_j = 1 \\ h_2^- \ge 0, g_2^- \ge 0, y_j \ge 0 \ (j = 1, 2, \ldots, n) \end{cases} \tag{3.2}$$

According to Definition 2.5 , Eq. (3.1) can be converted into the following multi-objective programming model:

$$\max\{\gamma_1^k\}, \min\{\eta_1^{k'}\} (k = 1, \ldots, l.\ k' = 1, \ldots, l')$$
$$s.t. \begin{cases} \prod\limits_{i=1}^{m} (1-\gamma_{ij}^k)^{x_i} \le 1 - \gamma_1^k \ (j = 1, 2, \ldots, n.\ k = 1, \ldots, l) \\ \prod_{i=1}^{m} (\eta_{ij}^{k'})^{x_i} \le \eta_1^{k'} \ (j = 1, 2, \ldots, n.\ k' = 1, \ldots, l') \\ 0 \le \gamma_1^l + \eta_1^{l'} \le 1 \\ \sum\limits_{i=1}^{m} x_i = 1 \\ \gamma_1^1 \ge 0, \eta_1^1 \ge 0, x_i \ge 0 \ (i = 1, 2, \ldots, m) \end{cases} \tag{3.3}$$

where $l = \#h_1$, $l' = \#g_1$.

It easily seen that $\max\{\gamma_1^k\}$ is equivalent to $\min\{1-{\gamma_1}^k\}$ since $0 \le \gamma_1^k \le 1$ for any $k = 1, \ldots, l$. Moreover, $\min\{1-{\gamma_1}^k\}$ and $\min\{{\eta_1^{k'}}\}$ can be converted into $\min\{\ln(1-\gamma_1^k)\}$ and $\min\{\ln {\eta_1^{k'}}\}$ since $0 \le 1-\gamma_1^k \le 1$ and $0 \le \eta_1^{k'} \le 1$ for any $k = 1, \ldots, l$ and $k' = 1, \ldots, l'$, respectively. Using the method of multi-objective programming, Eq. (3.3) can be written as follows:

$$
\begin{aligned}
&\min\{\sum_{k=1}^{l} \ln(1-\gamma_1^k) + \sum_{k'=1}^{l'} \ln(\eta_1^{k'})\} \\
&s.t. \begin{cases}
\sum_{i=1}^{m} x_i \ln(1-\gamma_{ij}^k) \le \ln(1-\gamma_1^k)\ (j = 1, 2, \ldots, n.\ k = 1, \ldots, l) \\
\sum_{i=1}^{m} x_i \ln {\eta_{ij}}^{k'} \le \ln {\eta_1}^{k'}\ (j = 1, 2, \ldots, n.\ k' = 1, \ldots, l') \\
\gamma_1^l + \eta_1^{l'} \le 1 \\
\sum_{i=1}^{m} x_i = 1 \\
\gamma_1^1 \ge 0, \eta_1^1 \ge 0, x_i \ge 0\ (i = 1, 2, \ldots, m)
\end{cases}
\end{aligned} \tag{3.4}
$$

Let

$$
\ln(1-\gamma_1^k) = s_k, \ln \eta_1^{k'} = t_{k'} \tag{3.5}
$$

Hence, Eq. (3.4) can be written as the following linear programming model:

$$
\begin{aligned}
&\min\{\sum_{k=1}^{l} s_k + \sum_{k'=1}^{l'} t_{k'}\} \\
&s.t. \begin{cases}
\sum_{i=1}^{m} x_i \ln(1-\gamma_{ij}^k) \le s_k\ (j = 1, 2, \ldots, n.\ k = 1, \ldots, l) \\
\sum_{i=1}^{m} \ln \eta_{ij}^{k'} \le t_k'\ (j = 1, 2, \ldots, n.\ k' = 1, \ldots, l') \\
\sum_{i=1}^{m} x_i = 1 \\
x_i \ge 0\ (i = 1, 2, \ldots, m)
\end{cases}
\end{aligned} \tag{3.6}
$$

The optimal solution of Eq. (3.6) can be obtained by solving the linear programming method. Let an optimal solution of Eq. (3.6) be denoted by $(\mathbf{x}^*, \mathbf{s}^*, \mathbf{t}^*)$, then, $\mathbf{x}^*$ is an optimal strategy of multi-objective programming model (i.e. (3.1)). According to Eq. (3.5), the gain-floor $\mu^* = (h_1^*, g_1^*)$ of player p_1 can be generated.

Similarity, let $\ln(1-\gamma_2^k) = p_k$ and $\ln \eta_2^{k'} = q_{k'}$, an optimal strategy $\mathbf{y}^*$ of player p_2 and his loss-ceiling $\omega^* = (h_2^*, g_2^*)$ can be generated by solving the

following linear programming model:

$$\max\{\sum_{k=1}^{l} p_k + \sum_{k'=1}^{l'} q_{k'}\}$$

$$s.t. \begin{cases} \sum_{j=1}^{n} y_j \ln(1-\gamma_{ij}^{k}) \geq p_k \ (i=1,2,\ldots,m.\ k=1,\ldots,l) \\ \sum_{j=1}^{n} y_j \ln \eta_{ij}^{k'} \geq q_{k'} \ (i=1,2,\ldots,m.\ k'=1,\ldots,l') \\ \sum_{j=1}^{n} y_j = 1 \\ y_j \geq 0 \ (j=1,2,\ldots,n) \end{cases} \tag{3.7}$$

If $(\mathbf{y}^*, \mathbf{p}^*, \mathbf{q}^*)$ is an optimal solution of Eq.(3.7), then, $\mathbf{y}^*$ is called the player $p_2's$ optimal strategy. Let $\ln(1-\gamma_2^*) = p_k^*$ and $\ln \eta_2^{k'^*} = q_{k'}^*$, thus, the loss-ceiling $\omega = (h_2^*, g_2^*)$ of player p_2 can be obtained.

4. A numerical example

Considering the duopoly market, there are two companies p_1 and p_2 aiming to enhance the market share. They formulated the product sales plans of the next cycle respectively in order to increase themselves scales. As the market demand for the product is basically fixed, an increase in the sales volume of one company will inevitably lead to a decrease in the sales volume of another company. Both companies are considering about three strategies to increase scales of their products: α_1 (Increase advertising), α_2 ((Reduce the price), α_3 (Improve the quality). The above problem can be regarded as a matrix game presented by dual hesitant fuzzy sets. The companies p_1 and p_2 are regarded as two players respectively. Considering some experts' opinions, the payoff matrix of p_1 is given presented by dual hesitant fuzzy sets as follows:

$$A^0 = \begin{bmatrix} (\{0.70,0.75,0.80\},\{0.10,0.15\}) & (\{0.70,0.75\},\{0.20\}) & (\{0.60,0.65\},\{0.30\}) \\ (\{0.20,0.25\},\{0.65,0.70\}) & (\{0.80,0.85\},\{0.05,0.1\}) & (\{0.65,0.70\},\{0.20,0.25\}) \\ (\{0.45,0.50,0.55\},\{0.40\}) & (\{0.10,0.20\},\{0.75\}) & (\{0.85,0.90\},\{0.05\}) \end{bmatrix}$$

where dual hesitant fuzzy set $(\{0.70,0.75,0.80\},\{0.10,0.15\})$ means that when both companies choose the strategy α_1, the probability of company p_1 increasing its product scales may be 0.7, it may be 0.75 or 0.8, the likelihood of reducing product sales may be 0.1 or 0.05. Other dual hesitant fuzzy sets can be interpreted similarly. Try to design optimal strategies for two companies.

Let $\varsigma = 1$, the payoff matrix after extension can be presented as follows:

$$A = \begin{bmatrix} (\{0.70,0.75,0.80\},\{0.10,0.15\}) & (\{0.70,0.75,0.75\},\{0.20,0.20\}) & (\{0.60,0.65,0.65\},\{0.30,0.30\}) \\ (\{0.20,0.25,0.25\},\{0.65,0.70\}) & (\{0.80,0.85,0.85\},\{0.05,0.1\}) & (\{0.65,0.70,0.70\},\{0.20,0.25\}) \\ (\{0.45,0.50,0.55\},\{0.40,0.40\}) & (\{0.10,0.20,0.20\},\{0.75,0.75\}) & (\{0.85,0.90,0.90\},\{0.05,0.05\}) \end{bmatrix}$$

According to Eqs. (3.6) (3.7), the two linear programming models are constructed as follows:

$$\min\{\sum_{k=1}^{3} s_k + \sum_{k'=1}^{2} t_{k'}\} \qquad \max\{\sum_{k=1}^{3} p_k + \sum_{k'=1}^{2} q_{k'}\}$$

(4.1)

$$(a)\ s.t.\begin{cases} x_1 \ln 0.3 + x_2 \ln 0.8 + x_3 \ln 0.55 \le s_1 \\ x_1 \ln 0.25 + x_2 \ln 0.75 + x_3 \ln 0.5 \le s_2 \\ x_1 \ln 0.2 + x_2 \ln 0.75 + x_3 \ln 0.45 \le s_3 \\ x_1 \ln 0.3 + x_2 \ln 0.2 + x_3 \ln 0.9 \le s_1 \\ x_1 \ln 0.25 + x_2 \ln 0.15 + x_3 \ln 0.8 \le s_2 \\ x_1 \ln 0.25 + x_2 \ln 0.15 + x_3 \ln 0.8 \le s_3 \\ x_1 \ln 0.4 + x_2 \ln 0.35 + x_3 \ln 0.15 \le s_1 \\ x_1 \ln 0.35 + x_2 \ln 0.3 + x_3 \ln 0.1 \le s_2 \\ x_1 \ln 0.35 + x_2 \ln 0.3 + x_3 \ln 0.1 \le s_3 \\ x_1 \ln 0.1 + x_2 \ln 0.65 + x_3 \ln 0.4 \le t_1 \\ x_1 \ln 0.15 + x_2 \ln 0.7 + x_3 \ln 0.4 \le t_2 \\ x_1 \ln 0.2 + x_2 \ln 0.05 + x_3 \ln 0.75 \le t_1 \\ x_1 \ln 0.2 + x_2 \ln 0.1 + x_3 \ln 0.75 \le t_2 \\ x_1 \ln 0.3 + x_2 \ln 0.2 + x_3 \ln 0.05 \le t_1 \\ x_1 \ln 0.3 + x_2 \ln 0.25 + x_3 \ln 0.05 \le t_2 \\ x_1 + x_2 + x_3 = 1 \\ x_i \ge 0\ (i = 1, 2, 3) \end{cases} \qquad (b)\ s.t.\begin{cases} y_1 \ln 0.3 + y_2 \ln 0.3 + y_3 \ln 0.4 \ge p_1 \\ y_1 \ln 0.25 + y_2 \ln 0.35 + y_3 \ln 0.35 \ge p_2 \\ y_1 \ln 0.2 + y_2 \ln 0.25 + y_3 \ln 0.35 \ge p_3 \\ y_1 \ln 0.8 + y_2 \ln 0.2 + y_3 \ln 0.35 \ge p_1 \\ y_1 \ln 0.75 + y_2 \ln 0.15 + y_3 \ln 0.3 \ge p_2 \\ y_1 \ln 0.75 + y_2 \ln 0.15 + y_3 \ln 0.3 \ge p_3 \\ y_1 \ln 0.55 + y_2 \ln 0.9 + y_3 \ln 0.15 \ge p_1 \\ y_1 \ln 0.5 + y_2 \ln 0.8 + y_3 \ln 0.1 \ge p_2 \\ y_1 \ln 0.45 + y_2 \ln 0.8 + y_3 \ln 0.1 \ge p_3 \\ y_1 \ln 0.1 + y_2 \ln 0.2 + y_3 \ln 0.3 \ge q_1 \\ y_1 \ln 0.15 + y_2 \ln 0.2 + y_3 \ln 0.3 \ge q_2 \\ y_1 \ln 0.65 + y_2 \ln 0.05 + y_3 \ln 0.2 \ge q_1 \\ y_1 \ln 0.7 + y_2 \ln 0.1 + y_3 \ln 0.25 \ge q_2 \\ y_1 \ln 0.4 + y_2 \ln 0.75 + y_3 \ln 0.05 \ge q_1 \\ y_1 \ln 0.4 + y_2 \ln 0.75 + y_3 \ln 0.05 \ge q_2 \\ y_1 + y_2 + y_3 = 1 \\ y_i \ge 0\ (j = 1, 2, 3) \end{cases}$$

Solving Eqs. 4.1(a) and 4.1(b), optimal solutions $(\mathbf{x}^*, \mathbf{s}^*, \mathbf{t}^*)$ and $(\mathbf{y}^*, \mathbf{p}^*, \mathbf{q}^*)$ can be obtained, respectively. Where

$$\mathbf{x}^* = (0.8013, 0.0521, 0.1466), \quad \mathbf{s}^* = (-1.0640, -1.2274, -1.2416),$$
$$\mathbf{t}^* = (-1.4878, -1.4517), \quad \mathbf{y}^* = (0.3078, 0.3014, 0.3908),$$
$$\mathbf{p}^* = (-1.0915, -1.2548, -1.3235), \quad \mathbf{q}^* = (-1.6644, -1.5395)$$

According to the Eq. (3.5), the optimal strategy and the gain-floor for company p_1 can be obtained as follows $\mathbf{x}^* = (0.8013, 0.0521, 0.1466)$, $(h_1^*, g_1^*) = (\{0.6459, 0.7069, 0.7111\}, \{0.2259, 0.2342\})$ Similarly, the optimal strategy and the loss-ceiling for company p_2 can be obtained as follows $\mathbf{y}^* = (0.3078, 0.3014, 0.3908)$, $(h_2^*, g_2^*) = (\{0.6643, 0.7149, 0.7338\}, \{0.1893, 0.2145\})$.

The results show that the proposed methods in this paper are reasonable and may provide more information for players to make decision.

5. Conclusions

Dual hesitant fuzzy set is an extension of hesitant fuzzy set and intuitionistic fuzzy set, which provides decision makers more information, as well as better selections of strategies. This study explain the matrix game with payoffs presented by dual hesitant fuzzy sets, based on the dual hesitant fuzzy elements algorithm and the definition of the dual hesitant fuzzy matrix game solution, the solution

of the matrix game for players are got through solving a pair of linear programming model, which derived from some auxiliary nonlinear multi-objective programming models, and a numerical example to verify the effectiveness of the method introduced in this paper. The method proposed in this paper enriches the theory and application of dual hesitant fuzzy sets, as well as provides a new method for solving the dual hesitant fuzzy matrix games. It may also be applied to similar competitive decision and matrix game problems with dual hesitant fuzzy information. It provides a new way to solve competition decision-making and matrix game problems with complex fuzzy information. Therefore, more effective methods for solving the dual hesitant fuzzy matrix game problems will be investigated in the future.

Acknowledgment

This work was supported by the National Natural Science Foundation of China (no. 71762001).

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Accepted: 26.07.2018

On condition (P'_E)

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Abstract. In (Semigroup and Applications: Proceedings of Conference, world scientific, (1998) 72-77) Golchin and Renshaw introduced Condition (P_E) and showed that this condition implies weak flatness, but not the converse. In this paper, we introduce a generalization of Conditions (P_E), called Condition (P'_E), and will show that this condition implies principal weak flatness. Also we give a characterization of monoids by this condition over their right acts.
Keywords: condition (P'_E), condition $(W_{(WF)'})$, P'_E-left annihilating, weakly right reversible.

1. Introduction

Laan [12] introduced Condition (E'), a generalization of Condition (E). Golchin and Renshaw [3] introduced a generalization of Condition (P) called Condition (P_E). After that Golchin and Mohammadzadeh [7] provided a characterization of monoids by this Condition of their right acts. They also introduced another generalization of Condition (P) called Condition (P'), and gave a characterization of monoids by this property of their right acts [9]. In this paper, we introduce a generalization of Conditions (P_E) and (P'), called Condition (P'_E) and will give a characterization of monoids by this condition of their right acts.

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Throughout this paper S will denote a monoid. We refer the reader to [11] for basic definitions and terminologies relating to semigroups and acts over monoids, one can see and [2] for definitions and results on flatness properties which are used here.

A nonempty set A is called a right S-act, usually denoted A_S, if S acts on A unitarily from the right, that is, there exists a mapping $A \times S \to A$, $(a, s) \mapsto as$, satisfying the conditions $(as)t = a(st)$ and $a1 = a$, for all $a \in A$, and all $s, t \in S$. A right S-act A_S satisfies *Condition* (E) if for all $a \in A_S$, $s, t \in S$, $as = at$ implies that there exist $a' \in A_S$, $u \in S$ such that $a = a'u$ and $us = ut$. A_S satisfies *Condition* (E') if for all $a \in A_S$, $s, t, z \in S$, $as = at$ and $sz = tz$ imply that there exist $a' \in A_S$, $u \in S$ such that $a = a'u$ and $us = ut$. A right S-act A_S satisfies *Condition* (PWP) if for all $a, a' \in A_S$, $s \in S$, $as = a's$ implies that there exist $a'' \in A_S$, $u, v \in S$ such that $a = a''u$, $a' = a''v$ and $us = vs$. A right S-act A_S satisfies *Condition* (P) if for all $a, a' \in A_S$, $s, t \in S$, $as = a't$ implies that there exist $a'' \in A_S$ and $u, v \in S$ such that $a = a''u$, $a' = a''v$ and $us = vt$. A_S satisfies *Condition* (P') if for all $a, a' \in A_S$, $s, t, z \in S$, $as = a't$ and $sz = tz$ imply that there exist $a'' \in A_S$ and $u, v \in S$ such that $a = a''u$, $a' = a''v$ and $us = vt$. A right S-act A_S satisfies *Condition* (P_E) if for all $a, a' \in A_S$, $s, t \in S$, $as = a't$, implies that there exist $a'' \in A_S$, $u, v \in S, e, f \in E(S)$ such that $ae = a''ue$, $a'f = a''vf$, $es = s, ft = t$ and $us = vt$. It is clear that Condition (P) implies Condition (P_E).

2. General properties

Definition 2.1. *A right S-act A_S satisfies Condition (P'_E) if for all $a, a' \in A_S$, $s, t, z \in S$, $as = a't$ and $sz = tz$, imply that there exist $a'' \in A_S$, $u, v \in S, e, f \in E(S)$ such that $ae = a''ue$, $a'f = a''vf$, $es = s, ft = t$ and $us = vt$.*

It is obvious that Conditions (P_E) and (P') imply Condition (P'_E). Examples 2.3 and 3.9 show that these implications are strict.

Definition 2.2 ([1])**.** *A right S-act A_S satisfies Condition $(W_{(WF)'})$, if $as = a't$ and $sz = tz$, for $a, a' \in A_S, s, t, z \in S$, imply that there exist $a'' \in A_S, w \in Ss \cap St$, such that $as = a't = a''w$.*

Theorem 2.1. *For any right S-act A_S we have the following strict implications:*

(1) *Condition $(P'_E) \Rightarrow$ principal weak flatness;*

(2) *Condition $(P'_E) \Rightarrow$ Condition $(W_{(WF)'})$.*

Proof. (1) Suppose that A_S satisfies Condition (P'_E) and let $as = a's$, for $a, a' \in A_S, s \in S$. By assumption there exist $a'' \in A_S, u, v \in S, e, f \in E(S)$ such that $ae = a''ue, a'f = a''vf, es = fs = s$ and $us = vs$. Then we have

$$\begin{aligned} a \otimes s &= a \otimes es = ae \otimes s = a''ue \otimes s = a'' \otimes ues = a'' \otimes us \\ &= a'' \otimes vs = a'' \otimes vfs = a''vf \otimes s = a'f \otimes s = a' \otimes fs = a' \otimes s \end{aligned}$$

in $A_S \otimes {}_S Ss$. That is, A_S is principally weakly flat as required.

(2) Suppose that A_S satisfies Condition (P'_E), and let $as = a't, sz = tz$ for $a, a' \in A_S, s, t, z \in S$. By assumption there exist $a'' \in A_S, u, v \in S, e, f \in E(S)$ such that $ae = a''ue, a'f = a''vf, es = s, ft = t$ and $us = vt$. If $w = us = vt$ then $w \in Ss \cap St$, and $as = aes = a''ues = a''us = a''w$. Similarly, $a't = a''w$ and so A_S satisfies Condition $(W_{(WF)'})$ as required. □

Example 3.1 shows that these implications are strict.

Lemma 2.1. *Let S be a monoid. Then:*

(1) *if $\{A_i \mid i \in I\}$ is a chain of subacts of an act A_S and every $A_i, i \in I$ satisfies Condition (P'_E), then $\bigcup_{i\in I} A_i$ satisfies Condition (P'_E);*

(2) *$A_S = \coprod_{i\in I} A_i$ satisfies Condition (P'_E) if and only if every $A_i, i \in I$ satisfies Condition (P'_E);*

(3) *the right S-act S_S satisfies Condition (P'_E).*

Proof. It is clear from definition. □

Theorem 2.2. *Any retract of a right S-act satisfying Condition (P'_E) satisfies Condition (P'_E).*

Proof. Let A_S be a retract of B_S and suppose B_S satisfies Condition (P'_E). Let $as = a't, sz = tz$, for $a, a' \in A_S, s, t, z \in S$. Since A_S is a retract of B_S, there are homomorphisms $\varphi : A_S \to B_S$ and $\varphi' : B_S \longrightarrow A_S$, such that $\varphi'\varphi = 1_A$. Then we have $\varphi(as) = \varphi(a't)$ or $\varphi(a)s = \varphi(a')t$. Since $\varphi(a), \varphi(a') \in B_S$, by assumption there exist $b \in B_S, u, v \in S$, $e, f \in E(S)$ such that $\varphi(a)e = bue$, $\varphi(a')f = bvf$, $es = s, ft = t$ and $us = vt$. Then $\varphi'(\varphi(a)e) = \varphi'(bue) = \varphi'(b)ue$, and $\varphi'(\varphi(a')f) = \varphi'(bvf) = \varphi'(b)vf$. If $\varphi'(b) = a''$, then $ae = a''ue$, $a'f = a''vf$. It is obvious that $a'' \in A_S$ and so A_S satisfies Condition (P'_E) as required. □

Definition 2.3 ([9])**.** *Let S be a monoid and let $P \subseteq S$ be a submonoid of S. P is called weakly right reversible if*

$$(\forall s, t \in P)(\forall z \in S)(sz = tz \Rightarrow (\exists u, v \in P)(us = vt)).$$

Proposition 2.1. *For any monoid S the following statements are equivalent:*

(1) *if $A_S = \prod_{i\in I} A_i$ satisfies Condition (P'_E), then every A_i, $i \in I$ satisfies Condition (P'_E);*

(2) *the one-element right S-act Θ_S satisfies Condition (P'_E);*

(3) *the one-element right S-act Θ_S satisfies Condition $(W_{(WF)'})$;*

(4) *S is weakly right reversible;*

(5) *there exists a right S-act which contains a zero and satisfies Condition (P'_E).*

Proof. Since Θ_S is a retract of any right S-act containing a zero, (2) and (5) are equivalent by Theorem 2.2.

(1) $\Rightarrow$ (2). Since $S_S \cong S_S \times \Theta_S$, it follows from Lemma 2.1(3).

(2) $\Rightarrow$ (3). It follows From Theorem 2.1(2).

(3) $\Rightarrow$ (4). Let $sz = tz$ for $s, t, z \in S$. Since $\theta s = \theta t$, and the right S-act Θ_S satisfies Condition $(W_{(WF)'})$, there exist $u, v \in S$ such that $us = vt$, that is, S is weakly right reversible.

(4) $\Rightarrow$ (1). Suppose that $A_S = \prod_{i\in I} A_i$ satisfies Condition (P'_E), and let $a_i s = a'_i t, sz = tz$, for $a_i, a'_i \in A_i, s, t, z \in S$. Since S is weakly right reversible, there exist $u_1, v_1 \in S$ such that $u_1 s = v_1 t$. For every $j \in I \setminus \{i\}$, let a_j be a fix element of A_j. Define

$$c_j = \begin{cases} a_j u_1, & j \neq i \\ a_i, & j = i \end{cases}, \quad d_j = \begin{cases} a_j v_1, & j \neq i \\ a'_i, & j = i \end{cases}.$$

Thus $(c_j)_I s = (d_j)_I t$, and so by assumption there exist $(a''_j)_I \in \prod_{i\in I} A_i$, $u, v \in S, e, f \in E(S)$ such that $(c_j)_I e = (a''_j)_I ue, (d_j)_I f = (a''_j)_I vf, es = s, ft = t$, and $us = vt$. Hence $a_i e = a''_i ue, a'_i f = a''_i vf$, and so A_i satisfies Condition (P'_E). □

Corollary 2.1. *Let S be a commutative monoid and $\{A_i |\ i \in I\}$ be a family of right S-acts. If $A_S = \prod_{i\in I} A_i$ satisfies Condition (P'_E), then every A_i, $i \in I$ satisfies Condition (P'_E).*

Example 2.3. Let $S = \left\{ \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \middle| a, b \in \mathbb{Z}, a \neq 0 \right\}$ with matrix product as operation, then S is a right cancellative monoid, and so it is weakly right reversible, but S is not right reversible, since for every $a, b, c, d \in Z$ with $a, c \neq 0$,

$$\begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 3 & 1 \end{pmatrix} \neq \begin{pmatrix} c & 0 \\ d & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 4 & 1 \end{pmatrix}.$$

Now the one-element right S-act Θ_S satisfies Condition (P'_E) by Theorem 2.1, but it does not satisfy Condition (P_E), since otherwise it is weakly flat which is not the case.

3. Homological classifications

In this section we will classify monoids by Condition (P'_E) of their (cyclic, Rees factor) acts.

Let K be a proper right ideal of a monoid S. If x, y and z denote elements not belonging to S, define

$$A(K) = (\{x, y\} \times (S \setminus K)) \cup (\{z\} \times K)$$

and define a right S-action on $A(K)$ by

$$(x,u)s = \begin{cases}(x,us), & us \notin K\\ (z,us), & us \in K\end{cases}$$
$$(y,u)s = \begin{cases}(y,us), & us \notin K\\ (z,us), & us \in K\end{cases}$$
$$(z,u)s = (z,us).$$

$A(K)$ is a right S-act, which called the amalgamated coproduct of S_S and S_S by the core K_S, and usually denoted by $S_S \coprod^{K_S} S_S$.

Recall from [11] that a right ideal K_S of a monoid S is called *left stabilizing* if for every $k \in K_S$, there exists $l \in K_S$ such that $lk = k$. The following example shows that implications in Theorem 2.1 are strict.

Example 3.1. Let $S = \{0, 1, e, a\}$ be a monoid with the identity element 1 and zero element 0, and $e^2 = e, ae = a, a^2 = ea = 0$. Let $K = \{0, e\}$. Now the right S-act $A(K) = S_S \coprod^{K_S} S_S$ does not satisfy Condition (P'_E), since the equality $(x,a)a = (y,a)a$ implies $x = y$ which is a contradiction. It is a routine matter to verify that A_S satisfies Condition $(W_{(WF)'})$. Since K_S is a left stabilizing right ideal, A_S is principally weakly flat by ([11], III, 12.19).

Lemma 3.1. *Let K_S be a proper right ideal of a monoid S. If the right S-act $S_S \coprod^{K_S} S_S$ satisfies Condition (P'_E), then K_S is left stabilizing.*

Proof. Suppose that the right S-act $S_S \coprod^{K_S} S_S$ satisfies Condition (P'_E) and let $k \in K_S$. Since $(x,1)k = (y,1)k$, there exist $w \in \{x, y, z\}, r, u, v \in S$, and $e, f \in E(S)$ such that $(x,1)e = (w,r)ue$, $(y,1)f = (w,r)vf, ek = fk = k$ and $uk = vk$. If $e, f \notin K_S$, then $(x,e) = (w, rue)$ and $(y,f) = (w, rvf)$, which implies that $w = x = y$, a contradiction. Thus at least one of e or f belongs to K_S, and so the equality $ek = fk = k$ shows that K_S is left stabilizing. □

We recall from [11] that a monoid S is called *left PP* if every principal left ideal of S (as a left S-act) is projective, it is equivalent to say that for every $s \in S$ there exists an idempotent $e \in S$ such that $es = s$ and for all $u, v \in S$, $us = vs$ implies $ue = ve$.

Theorem 3.2 ([3])**.** *Let S be a left PP monoid and A_S be a right S-act. Then A_S is weakly flat if and only if it satisfies Condition (P_E).*

Theorem 3.3. *For any monoid S the following statements are equivalent:*

(1) *all right S-acts of the form $S_S \coprod^{K_S} S_S$ satisfy Condition (P_E);*

(2) *all right S-acts of the form $S_S \coprod^{K_S} S_S$ satisfy Condition (P'_E);*

(3) *S is a regular monoid.*

Proof. The implication (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (3). Let $s \in S$. If $sS = S$ then s is obviously regular. Assume that $sS \neq S$, then by assumption the right S-act $S_S \coprod^{sS} S_S$ satisfies Condition (P'_E) and so by Lemma 3.1, sS is left stabilizing. Thus there exists $l \in sS$ such that $ls = s$, that is s is regular as required.

(3) $\Rightarrow$ (1). Let K_S be a proper right ideal of S and $k \in K_S$. By assumption there exist $k' \in S$ such that $k = kk'k$ and so K_S is left stabilizing. Thus $S_S \coprod^{K_S} S_S$ is weakly flat by ([11], III, 12.19). Since S is regular, it is left PP and so by Theorem 3.2, $S_S \coprod^{K_S} S_S$ satisfies Condition (P_E). □

Theorem 3.4. *Let S be a monoid and ρ be a right congruence on S. Then the right S-act S/ρ satisfies Condition (P'_E) if and only if, for all $x, y, z, s, t \in S$ with $(xs)\rho(yt)$ and $sz = tz$, there exist $u, v \in S, e, f \in E(S)$, such that $(xe)\rho(ue), (yf)\rho(vf), es = s, ft = t$ and $us = vt$.*

Proof. *Necessity.* Let $(xs)\rho(yt)$ and $sz = tz$, for $x, y, z, s, t \in S$. Then $[x]_\rho s = [y]_\rho t$, and so Condition (P'_E) of S/ρ implies that there exist $u', v', z' \in S, e, f \in E(S)$, such that $[x]_\rho e = [z']_\rho u'e = [z'u']_\rho e$, $[y]_\rho f = [z']_\rho v'f = [z'v']_\rho f, es = s, ft = t$ and $u's = v't$. If $u = z'u'$ and $v = z'v'$, then $(xe)\rho(ue)$, $(yf)\rho(vf)$ and $us = vt$, as required.

Sufficiency. Let $[x]_\rho s = [y]_\rho t$ and $sz = tz$, for $x, y, z, s, t \in S$. Then $(xs)\rho(yt)$, and so by assumption there exist $u, v \in S, e, f \in E(S)$, such that $(xe)\rho(ue)$, $(yf)\rho(vf), es = s, ft = t$ and $us = vt$. $[x]_\rho e = [1]_\rho ue$ and $[y]_\rho f = [1]_\rho vf$, and so S/ρ satisfies Condition (P'_E) as required. □

Corollary 3.1. *Let S be a monoid and $u \in S$. Then the principal right ideal uS satisfies Condition (P'_E) if and only if, for all $x, y, z, s, t \in S$, $uxs = uyt$ and $sz = tz$ imply the existence of $v, r \in S, e, f \in E(S)$, such that $uxe = uve, uyf = urf, es = s, ft = t$, and $vs = rt$.*

Proof. Since $uS \cong S/ker\lambda_u$, it suffices to take $\rho = ker\lambda_u$. □

Theorem 3.5 ([9])**.** *Let S be a monoid and ρ be a right congruence on S. Then the right S-act S/ρ satisfies Condition (P') if and only if, for all $x, y, z, s, t \in S$ with $(xs)\rho(yt)$ and $sz = tz$, there exist $u, v \in S$, such that $x\rho u, y\rho v$ and $us = vt$.*

Theorem 3.6. *Let S be a monoid such that every element of $E(S) \setminus \{1\}$ is a right zero and let ρ be a right congruence on S. Then S/ρ satisfies Condition (P') if and only if it satisfies Condition (P'_E).*

Proof. It is clear that Condition (P') implies Condition (P'_E). Now suppose that S/ρ satisfies Condition (P'_E) and let $(xs)\rho(yt), sz = tz$, for $x, y, z, s, t \in S$. By assumption there exist $u, v \in S, e, f \in E(S)$ such that $(xe)\rho(ue), (yf)\rho(vf), es = s, ft = t$ and $us = vt$. If $e = f = 1$ then the result follows from Theorem 3.5. If $e = 1, f \neq 1$, then f and so t are right zeros, thus $x\rho u, y\rho y$ and $us = yt$.

If $e \neq 1, f = 1$, a similar argument as the previous case shows the result. If $e \neq 1, f \neq 1$ then e and f are both right zero and so are s and t. Then $x\rho x, y\rho y$ and $xs = yt$, thus S/ρ satisfies Condition (P') by Theorem 3.5. □

Definition 3.1. *Let S be a monoid. A right ideal K_S of S is called P'_E-left annihilating if for all $x, y, z, s, t \in S$,*

$$[(xs \neq yt) \wedge (sz = tz)] \Rightarrow [(xs \notin K_S) \vee (yt \notin K_S) \vee,$$
$$(\exists u, v \in S, e, f \in E(S) : es = s, ft = t, us = vt,$$
$$(xe \neq ue \Rightarrow xe, ue \in K_S), (yf \neq vf \Rightarrow yf, vf \in K_S))]$$

Lemma 3.2. *Let S be a monoid and K_S be a right ideal of S. If K_S is P'_E-left annihilating, then K_S is left stabilizing.*

Proof. If $K_S = S$, then obviously K_S is left stabilizing. Thus we assume that K_S is a proper P'_E-left annihilating right ideal and let $k \in K_S$. For every $l \in K_S$ either $lk = k$, and so we are done, or $lk \neq k$. Since $k, lk \in K_S$ by assumption there exist $u, v \in S, e, f \in E(S)$ such that $ek = fk = k, uk = vk$, and $le \neq ue$ implies that $le, ue \in K_S$, and $f \neq vf$ implies that $f, vf \in K_S$. If $f \neq vf$ then $f \in K_S$ and $fk = k$ implies that K_S is left stabilizing. If $f = vf$, then $le \neq ue$, otherwise

$$lk = lek = uek = uk = vk = vfk = fk = k,$$

a contradiction. So $ue \in K_S$ and

$$k = fk = vfk = vk = uk = uek,$$

thus K_S is left stabilizing as required. □

Theorem 3.7. *Let S be a monoid and K_S be a right ideal of S. Then S/K_S satisfies Condition (P'_E), if and only if S is weakly right reversible and K_S is P'_E-left annihilating.*

Proof. *Necessity.* Suppose that S/K_S satisfies Condition (P'_E), for the right ideal K_S of S, and let $sz = tz$, for $s, t, z \in S$. Let $k \in K_S$, since $[k]_{\rho_K} s = [k]_{\rho_K} t$, Condition $(W_{(WF)'})$, implies that there exist $u, v \in S$ such that $us = vt$, thus S is weakly right reversible. To show that K_S is P'_E-left annihilating, suppose that $xs \neq yt, sz = tz$, for $x, y, z, s, t \in S$. If $xs \notin K_S$ or $yt \notin K_S$ then the result follows, otherwise $xs, yt \in K_S$ and so $(xs)\rho_K(yt)$. Thus by Theorem 3.4 there exist $u, v \in S, e, f \in E(S)$ such that $(xe)\rho_K(ue), (yf)\rho_K(vf), es = s, ft = t$ and $us = vt$. Since $(xe)\rho_K(ue)$, it follows that $xe = ue$ or $xe, ue \in K_S$. A similar argument shows that $yf = vf$ or $yf, vf \in K_S$.

Sufficiency. Suppose that S is a weakly right reversible monoid and K_S is a P'_E-left annihilating right ideal of S. Then there are two cases as follow:

Case 1. $K_S = S$. Since S is weakly right reversible, $S/K_S \cong \Theta_S$ satisfies Condition (P'_E) by Theorem 2.1.

Case 2. $K_S \neq S$. Suppose that $(xs)\rho_K(yt)$ and let $sz = tz$, for $x, y, z, s, t \in S$. Then there are two possibilities that can arise:

(1) $xs = yt$. If $u = x, v = y$ and $e = f = 1$, then by Theorem 3.4, S/K_S satisfies Condition (P'_E).

(2) $xs \neq yt$. Then $xs, yt \in K_S$. Since K_S is P'_E-left annihilating, there exist $u, v \in S, e, f \in E(S)$ such that $es = s, ft = t, us = vt$, if $xe \neq ue$ then $xe, ue \in K_S$, and if $yf \neq vf$ then $yf, vf \in K_S$. If $xe \neq ue$, then by assumption $(xe)\rho_K(ue)$, otherwise $xe = ue$, and again $(xe)\rho_K(ue)$. A similar argument shows that $(yf)\rho_K(vf)$, and so by Theorem 3.4, S/K_S satisfies Condition (P'_E). □

Definition 3.2 ([9]). *Let S be a monoid and K_S be a right ideal of S. K_S is called completely left annihilating if for all $x, y, z, s, t \in S$,*

$$[(xs \neq yt) \wedge (sz = tz)] \Rightarrow [(xs \notin K_S) \vee (yt \notin K_S) \vee (x \in K_S) \vee (y \in K_S)]$$

Theorem 3.8 ([9]). *Let S be a monoid and K_S be a right ideal of S. Then S/K_S satisfies Condition (P'), if and only if $K_S = S$ and S is weakly right reversible or K_S is a completely left annihilating and left stabilizing proper right ideal.*

The following example shows that Condition (P'_E) does not imply Condition (P').

Example 3.9. Let S and K_S be as in Example 3.1. Then $0 = aa \neq 1e = e$ and $0, e \in K_S, a, 1 \notin K_S$, and so K_S is not completely left annihilating, thus S/K_S does not satisfy Condition (P') by Theorem 3.8. It can be seen that K_S is P'_E-left annihilating and S is weakly right reversible, and so S/K_S satisfies Condition (P'_E).

Definition 3.3 ([1]). *A monoid S satisfies Condition $(R_{(WF)'})$, if for all x, y, z, $s, t \in S$, $sz = tz$ implies the existence of $w \in Ss \cap St$ such that $w\rho(xs, yt)xs$.*

Theorem 3.10. *For any monoid S the following statements are equivalent:*

(1) *all right S-acts satisfy Condition (P'_E);*

(2) *all finitely generated right S-acts satisfy Condition (P'_E);*

(3) *all cyclic right S-acts satisfy Condition (P'_E);*

(4) *all monocyclic right S-acts satisfy Condition (P'_E);*

(5) *S is regular and satisfies Condition $(R_{(WF)'})$.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (5). Since by assumption all monocyclic right S-acts are principally weakly flat, S is regular by ([11], IV, 6.6). Now let $sz = tz$, and suppose that $\rho = \rho(xs, yt), x, y, z, s, t \in S$. By assumption S/ρ satisfies Condition (P'_E). Since $(xs)\rho(yt)$, by Theorem 3.4, there exist $u, v \in S, e, f \in E(S)$, such that $(xe)\rho(ue), (yf)\ \rho(vf), es = s, ft = t$ and $us = vt$. Then $xs = (xes)\rho(ues) = us$. Now $w = us = vt \in Ss \cap St$, and $w\rho(xs)$ as required.

(5) $\Rightarrow$ (3). Let ρ be a right congruence on S, and suppose that $(xs)\rho(yt), sz = tz, x, y, z, s, t \in S$. Since S satisfies Condition $(R_{(WF)'})$, there exist $u, v \in S$ such that $w = us = vt$, $xs\rho(xs, yt)us$ and $yt\rho(xs, yt)\ vt$. Since S is regular there exist $s', t' \in S$ such that $s = ss's, t = tt't$ and so $(xss')\rho(xs, yt)(uss')$, $(ytt')\rho(xs, yt)(vtt')$. If $e = ss', f = tt'$, then $xe\rho(xs, yt)ue$, $yf\rho(xs, yt)vf$, $es = s$ and $ft = t$. But $\rho(xs, yt) \subseteq \rho$, and so S/ρ satisfies Condition (P'_E) by Theorem 3.4.

(3) $\Rightarrow$ (1). Let A_S be a right S-act and $as = a't, sz = tz$, for $a, a' \in A_S, s, t, z \in S$. Since (3) $\Leftrightarrow$ (5), S is regular and so there exist s', t' such that $s = ss's$ and $t = tt't$. Then $a(ss's) = a'(tt't)$, and so $(bs')s = (bt')t$, where $b = as = a't$. By assumption bS satisfies Condition (P'_E), and so there exist $b'' \in bS \subseteq A_S, u, v \in S, e', f' \in E(S)$ such that

$$bs'e' = b''ue', bt'f' = b''vf', e's = s, f't = t, us = vt.$$

If $e = ss', f = tt'$, then

$$ae = ass' = a(ss's)s' = (ass'e')ss' = (bs'e')ss' = (b''ue')ss' = b''uss' = b''ue.$$

Similarly, $a'f = b''vf$. It is clear that $es = s$ and $ft = t$ and so A_S satisfies Condition (P'_E) as required. □

Definition 3.4 ([11])**.** *Let L denotes the set of all left ideals of a monoid S. S is called right L-reductive if for all ${}_SK \in L$ and for all $a, b \in {}_SK, a \neq b$, there exists $x \in {}_SK$ such that $ax \neq bx$.*

Theorem 3.11. *Let S be a monoid. If all right S-acts satisfy Condition (P'_E), then S is right L-reductive.*

Proof. Suppose that S is not L-reductive, thus there exist a left ideal ${}_SK$, and $a, b \in {}_SK$ such that $a \neq b$ and for all $x \in {}_SK$, $ax = bx$. By Theorem 3.10, S satisfies Condition $(R_{(WF)'})$, so there exists $w \in Sa \cap Sb$ such that $w\rho(a, b)a$. Thus by ([11], I, 4.37) either $w = a$ or there exist $u_1, ..., u_n, a_1, b_1, ..., a_n, b_n \in S$, such that $\{a_i, b_i\} = \{a, b\}$ for $i = 1, ..., n$, and

$$a = a_1u_1 \quad b_2u_2 = a_3u_3 \quad .... \quad b_nu_n = w.$$
$$b_1u_1 = a_2u_2 \quad b_3u_3 = a_4u_4 \quad ...$$

Since S is regular by Theorem 3.10, there exist $a', b' \in S$ such that $a = aa'a, b = bb'b$. Now $a \in {}_SK$, implies that $u_1a'a \in {}_SK$. Then

$$a = aa'a = a_1u_1a'a = b_1u_1a'a = a_2u_2a'a = ... = b_nu_na'a = wa'a = w.$$

Thus in both cases $a = w$, also $b \in {}_SK$ implies that $b'b \in {}_SK$, and so

$$b = bb'b = ab'b = wb'b = w.$$

Hence $a = b$, which is a contradiction. □

We recall from [11] that a *band* is an idempotent semigroup. Let A and B be nonempty sets and $S = A \times B$, define a multiplication on S by $(a, b)(c, d) = (a, d)$ for $a, c \in A, b, d \in B$. This semigroup is called a *rectangular band*. An idempotent monoid S is called *left regular* if $st = sts$, for every $s, t \in S$.

Theorem 3.12 ([11])**.** *A band is a semilattice of rectangular bands.*

Theorem 3.13. *Let S be an idempotent monoid. Then all right S-acts satisfy Condition (P'_E), if and only if S is left regular.*

Proof. *Necessity.* Since S is an idempotent monoid, it is a semilattice of rectangular bands by Theorem 3.12. Let $S = \dot{\bigcup}_{\gamma\in\Gamma} S_\gamma$ be a semilattice such that each S_γ is a rectangular band. By dual of ([11], I, 3.46) we need to show that each S_γ is a left zero band, so let $x, y \in S_\gamma$, $(\gamma \in \Gamma)$. Let $z \in S_\gamma$ be an arbitrary element, and let $\lambda \in \Gamma$ be such that $1 \in S_\lambda$, thus $1xz, 1yz \in S_{\lambda\gamma}$, and since $S_{\lambda\gamma}$ is a rectangular band we have $1xz = 1yz = 1z = z$. By assumption all right S-acts satisfy Condition (P'_E), and so by Theorem 3.10, S satisfies Condition $(R_{(WF)'})$. Thus there exists $w = ux = vy \in Sx \cap Sy$ such that $w\rho(x, y)x$, and so by ([11], I, 4.37) either $x = w = ux$ or there exist $u_1, ..., u_n, x_1, y_1, ..., x_n, y_n \in S$, such that $\{x_i, y_i\} = \{x, y\}$, for $i = 1, ..., n$ and

$$\begin{array}{ccccc} x = x_1u_1 & y_2u_2 = x_3u_3 & ... & y_nu_n = vy. \\ & y_1u_1 = x_2u_2 & y_3u_3 = x_4u_4 & ... \end{array}$$

In first case $xy = xvy = xux = xx = x$. In the second case it can be seen that $w = vy$ belongs to S_γ. Thus $xy = x(vy)y = xvy = xux = x$ as required.

Sufficiency. Since S is an idempotent monoid, obviously it is regular. Now let $x, y, z, s, t \in S$, and $sz = tz$, then $xs\rho(xs, yt)yt$ implies that $xst\rho(xs, yt)yt$. If $w = xsts = xst$ then $w \in Ss \cap St$ and $w\rho(xs, yt)xs$. Thus S satisfies Condition $(R_{(WF)'})$, and so every right S-act satisfies Condition (P'_E) by Theorem 3.10. □

Lemma 3.3. *Let S be a weakly right reversible regular monoid. Then every right Rees factor S-act satisfies Condition $(W_{(WF)'})$.*

Proof. Suppose that S is a weakly right reversible regular monoid and K_S is a right ideal of S. Then there are two cases as follow:

Case 1. $K_S = S$. Since S is weakly right reversible, $S/K_S \cong \Theta_S$ satisfies Condition $(W_{(WF)'})$ by Theorem 2.1.

Case 2. $K_S \neq S$. Since S is regular, S/K_S is principally weakly flat by ([11], IV, 6.6) and so K_S is left stabilizing. Let $[x]_{\rho_K} s = [y]_{\rho_K} t$ and $sz = tz$, for $x, y, z, s, t \in S$. Since $(xs)\rho_K(yt)$ then $xs = yt$ or $xs, yt \in K_S$. If $xs = yt$, set $u = x$ and $v = y$, then $x(\rho_K \vee ker\rho_s)u$ and $y(\rho_K \vee ker\rho_t)v$. If $xs, yt \in K_S$, then there exist $l_1, l_2 \in K_S$ such that $l_1 xs = xs$ and $l_2 yt = yt$. Thus, $(l_1 x)ker\rho_s(x)$ and $(l_2 y)ker\rho_t(y)$. Since S is weakly right reversible there exist $u', v' \in S$ such that $u's = v't$. If $u = l_1 u', v = l_1 v'$, then $(x)ker\rho_s(l_1 x)\rho_K(u)$ and so $x(\rho_K \vee ker\rho_s)u$. Similarly, $y(\rho_K \vee ker\rho_t)v$, and $us = l_1 u's = l_1 v't = vt$. So by ([11], III, 10.6) in both cases $[x]_{\rho_K} \otimes s = [u]_{\rho_K} \otimes s$, and $[y]_{\rho_K} \otimes t = [v]_{\rho_K} \otimes t$ in $S/\rho_K \otimes {}_S Ss$, and $S/\rho_K \otimes {}_S St$, respectively. Hence

$$[x]_{\rho_K} \otimes s = [u]_{\rho_K} \otimes s = [1]_{\rho_K} \otimes us = [1]_{\rho_K} \otimes vt = [v]_{\rho_K} \otimes t = [y]_{\rho_K} \otimes t,$$

in $S/\rho_K \otimes_S (Ss \cup St)$. Thus, there exist $s_1, ..., s_k, t_1, ..., t_k, b_1, ..., b_{k-1} \in S, u_1, ..., u_k \in Ss \cup St$, such that

$$\begin{array}{cc} & s_1 u_1 = s \\ [x]_{\rho_K} s_1 = [b_1]_{\rho_K} t_1 & s_2 u_2 = t_1 u_1 \\ [b_1]_{\rho_K} s_2 = [b_2]_{\rho_K} t_2 & s_3 u_3 = t_2 u_2 \\ ... & ... \\ [b_{k-1}]_{\rho_K} s_k = [y]_{\rho_K} t_k & t = t_k u_k. \end{array}$$

Let j be the first index such that $u_j \in St$. If $j = 1$, then $s = s_1 u_1 \in St$, and so $s = v_1 t$, for some $v_1 \in S$, thus Condition $(W_{(WF)'})$ holds for $w = s$ and $a'' = x$. Suppose now that $j > 1$. Then $u_{j-1} \in Ss$, and since $s_j u_j = t_{j-1} u_{j-1}$, we have $w = s_j u_j \in Ss \cap St$ and so

$$[x]_{\rho_K} s = [x]_{\rho_K} s_1 u_1 = [b_1]_{\rho_K} t_1 u_1 = ... = [b_{j-1}]_{\rho_K} s_j u_j = [b_{j-1}]_{\rho_K} w.$$

Thus Condition $(W_{(WF)'})$ holds for $a'' = b_{j-1}$. □

Theorem 3.14 ([11])**.** *Let S be a left PP monoid. A right S-act A_S is principally weakly flat if and only if for every $a, a' \in A_S, s \in S, as = a's$ implies that there exists $e \in E(S)$ such that $es = s$ and $ae = a'e$.*

Theorem 3.15. *For any monoid S the following statements are equivalent:*

(1) *all right Rees factor S-acts satisfy Condition (P'_E);*

(2) *S is a weakly right reversible regular monoid.*

Proof. (1) ⇒ (2). Suppose that all right Rees factor S-acts satisfy Condition (P'_E), then by Theorem 2.1(1) all right Rees factor S-acts are principally weakly flat and so S is regular by ([11], IV, 6.6). Also by assumption the one-element

right S-act Θ_S satisfies Condition (P'_E), and so S is weakly right reversible by Theorem 2.1.

(2) $\Rightarrow$ (1). Suppose that S is a weakly right reversible regular monoid and let K_S be a right ideal of S. Then there are two cases that can arise:

Case 1. $K_S = S$. Then $S/K_S \cong \Theta_S$ satisfies Condition (P'_E) by Theorem 2.1.

Case 2. $K_S \neq S$. Then S/K_S is principally weakly flat by ([11], IV, 6.6). Let $[x]_{\rho_K} s = [y]_{\rho_K} t$, and $sz = tz$, for $x, y, z, s, t \in S$. By Lemma 3.3, S/K_S satisfies Condition $(W_{(WF)'})$, so there exist $w, u, v \in S$ such that $[x]_{\rho_K} s = [w]_{\rho_K} us, [y]_{\rho_K} t = [w]_{\rho_K} vt$ and $us = vt$. Since S is regular it is left PP and so there exist $e, f \in E(S)$ such that $es = s, ft = t, [x]_{\rho_K} e = [w]_{\rho_K} ue$, and $[y]_{\rho_K} f = [w]_{\rho_K} vf$ by Theorem 3.14. Thus S/K_S satisfies Condition (P'_E) as required. □

In what follows we give a characterization of monoids over which all right acts satisfying Condition (P'_E) have some other flatness properties and vise versa.

Theorem 3.16. *For any monoid S the following statements are equivalent:*

(1) *all right S-acts satisfying Condition (P'_E) are free;*

(2) *all finitely generated right S-acts satisfying Condition (P'_E) are free;*

(3) *all (mono)cyclic right S-acts satisfying Condition (P'_E) are free;*

(4) *all right S-acts satisfying Condition (P'_E) are projective generators;*

(5) *all finitely generated right S-acts satisfying Condition (P'_E) are projective generators;*

(6) *all (mono)cyclic right S-acts satisfying Condition (P'_E) are projective generators;*

(7) $S = \{1\}$.

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (6), (1) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6), are obvious.

(6) $\Rightarrow$ (7). Suppose that all (mono)cyclic right S-acts satisfying Condition (P'_E) are projective generators. Then all (mono)cyclic right S-acts satisfying Condition (P) are projective generators and so $S = \{1\}$ by ([11], IV, 12.8).

(7) $\Rightarrow$ (1). It is obvious. □

Theorem 3.17 ([3])**.** *Let S be a left PP monoid and A_S be a right S-act. Then A_S is weakly flat if and only if it satisfies Condition (P_E).*

From Theorems 3.21, 3.17 and ([11], IV, 8.1) we have the following result:

Proposition 3.1. *Let S be a left PP right collapsible monoid. If S satisfies Condition*

$$(L') : \forall e, f \in E(S), \exists z \in eS \cap fS : z\lambda(e, f)e$$

then every right S-act satisfying Condition (P'_E) is flat.

We recall from [6] that a monoid S satisfies Condition (FP_2) if every left collapsible submonoid of $< E(S) >$ contains a left zero.

Theorem 3.18 ([6])**.** *Let S be a monoid with $E(S) \subseteq C(S)$ and let U be a property of S-acts implied by Condition (P). If all cyclic right S-acts with property U are regular then S is an idempotent monoid which satisfies Condition (FP_2).*

Theorem 3.19. *For any monoid S with $E(S) \subseteq C(S)$, the following statements are equivalent:*

(1) *all right S-acts satisfying Condition (P'_E) are regular;*

(2) *all finitely generated right S-acts satisfying Condition (P'_E) are regular;*

(3) *all cyclic right S-acts satisfying Condition (P'_E) are regular;*

(4) *$S = \{0, 1\}$ or $S = \{1\}$.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (4). By Theorem 3.18, S is an idempotent monoid, thus S is commutative and left PP, and so by Theorem 3.2 Condition (P_E) and weak flatness coincide. Thus by assumption all weakly flat cyclic right S-acts are regular, and so by [[6], Theorem 1.14] every element of $S \setminus \{1\}$ is a right zero. Since S is commutative every element of S different from 1 is a zero, and so $S = \{0, 1\}$ or $S = \{1\}$.

(4) $\Rightarrow$ (1). It follows from ([11], IV, 14.4). □

Theorem 3.20. *Let S be a simple monoid, then the following statements are equivalent:*

(1) *all right S-acts satisfying Condition (P'_E) are regular;*

(2) *all finitely generated right S-acts satisfying Condition (P'_E) are regular;*

(3) *all cyclic right S-acts satisfying Condition (P'_E) are regular;*

(4) *$S = \{1\}$.*

Proof. It follows from [[6], Theorem 4.1]. □

Theorem 3.21. *Let S be a monoid. If S is right collapsible, then every right S-act satisfying Condition (P'_E) satisfies Condition (P_E).*

Proof. Suppose that A_S satisfies Condition (P'_E), and let $as = a't$, for $a, a' \in A_S$, and $s, t \in S$. Since S is right collapsible, there exists $z \in S$ such that $sz = tz$. Thus by assumption there exist $a'' \in A_S$, $u, v \in S, e, f \in E(S)$ such that $ae = a''ue$, $a'f = a''vf$, $es = s, ft = t$ and $us = vt$, that is A_S satisfies Condition (P_E) as required. □

We recall from [13] that an act A_S is called *strongly torsion free* if for every $a, a' \in A_S$ and every $s \in S$, the equality $as = a's$ implies $a = a'$. By [[13], Proposition 2.1] S_S is strongly torsion free if and only if S is right cancellative, while for any monoid S, S_S satisfies Condition (P'_E), so Condition (P'_E) does not imply strong torsion freeness in general.

Proposition 3.2. *For any monoid S the following statements are equivalent:*

(1) *all right S-acts satisfying Condition (P'_E) are strongly torsion free;*

(2) *all finitely generated right S-acts satisfying Condition (P'_E) are strongly torsion free;*

(3) *all cyclic right S-acts satisfying Condition (P'_E) are strongly torsion free;*

(4) *S is right cancellative.*

Proof. It follows from [[13], Theorem 3.1]. □

Recall from [11] that a right S-act A_S is divisible if $Ac = A$ for every left cancellable element $c \in S$. By Lemma 2.1, S_S satisfies Condition (P'_E), while it is not divisible in general. Now it is natural to ask for which monoids Condition (P'_E) implies divisibility.

Proposition 3.3. *For any monoid S the following statements are equivalent:*

(1) *all right S-acts satisfying Condition (P'_E) are divisible;*

(2) *all finitely generated right S-acts satisfying Condition (P'_E) are divisible;*

(3) *all cyclic right S-acts satisfying Condition (P'_E) are divisible;*

(4) *all left cancellable elements of S are left invertible.*

Proof. Implications (1) ⇒ (2) ⇒ (3) are obvious.

(3) ⇒ (4). By Lemma 2.1(3), S_S satisfies Condition (P'_E), and so by assumption it is divisible. Thus $Sc = S$, for every left cancellable element $c \in S$. That is, there exists $x \in S$ such that $xc = 1$.

(4) ⇒ (1). It is clear from ([11], III, 2.2). □

We recall from [5] that a right S-act A_S satisfies *Condition* (EP) if for all $a \in A_S, s, t \in S$, $as = at$ implies that there exist $a' \in A_S, u, v \in S$ such that $a = a'u = a'v$ and $us = vt$. Also we recall from [4] that a right S-act A_S satisfies *Condition* $(E'P)$ if for all $a \in A_S, s, t, z \in S$, $as = at$ and $sz = tz$ imply that there exist $a' \in A_S$, and $u, v \in S$ such that $a = a'u = a'v$ and $us = vt$. It is obvious that $(E) \Rightarrow (EP) \Rightarrow (E'P)$ and $(E) \Rightarrow (E') \Rightarrow (E'P)$.

Notice that Condition (E) does not imply Condition (P'_E), otherwise Condition (E) implies principal weak flatness, which is not the case.

Theorem 3.22. *For any monoid S, the following statements are equivalent:*

(1) *S is regular;*

(2) *all right S-acts satisfying Condition $(E'P)$ satisfy Condition (P'_E);*

(3) *all right S-acts satisfying Condition (EP) satisfy Condition (P'_E);*

(4) *all right S-acts satisfying Condition (E') satisfy Condition (P'_E);*

(5) *all right S-acts satisfying Condition (E) satisfy Condition (P'_E).*

Proof. Implications (2) $\Rightarrow$ (3) $\Rightarrow$ (5), (2) $\Rightarrow$ (4) $\Rightarrow$ (5) are obvious.

(1) $\Rightarrow$ (2). Suppose that the right S-act A_S satisfies Condition $(E'P)$ and let $as = a't,\ sz = tz$, for $a, a' \in A_S,\ s, t, z \in S$. Since S is regular, there exists $s' \in S$ such that $s = ss's$ and $s' = s'ss'$, thus $a't = ass's = a'ts's$ and $ts' = ts'ss'$. Since A_S satisfies Condition $(E'P)$, there exist $a'' \in A_S,\ u, v \in S$, such that $a' = a''u = a''v$ and $ut = vts's$. If $w = vts', e = ss', f = 1$, then $ae = a(ss')e = (a'ts')e = (a''vts')e = a''we$, $a'f = a''uf$, $es = s, ft = t$ and $ws = ut$. Thus A_S satisfies Condition (P'_E) as required.

(5) $\Rightarrow$ (1). Let $s \in S$. If $sS = S$ then obviously s is regular. Otherwise sS is a proper right ideal of S and the right S-act $A(sS) = S_S \coprod^{sS} S_S$ satisfies Condition (E) by ([11], III, 14.3(3)). Thus by assumption it satisfies Condition (P'_E) and so sS is left stabilizing by Lemma 3.1. That is, there exists $t \in sS$ such that $s = ts$ which implies that s is regular. □

Remark 3.23. By Theorem 2.1, the one-element right S-act Θ_S satisfies Condition (P'_E), if and only if S is a weakly right reversible monoid, but as we know for any monoid S, Θ_S satisfies Condition (PWP).

Theorem 3.24. *Let S be a right cancellative monoid. Then every right S-act satisfying Condition (PWP) satisfies Condition (P'_E).*

Proof. Suppose that the right S-act A_S satisfies Condition (PWP) and let $as = a't,\ sz = tz$, for $a, a' \in A_S, s, t, z \in S$. Since S is right cancellative $s = t$, and so $as = a's$. Thus by assumption there exist $a'' \in A_S$ and $u, v \in S$ such that $a = a''u, a' = a''v$ and $us = vs$. We take $e = f = 1$ and then A_S satisfies Condition (P'_E) as required. □

Recall from [8] that a right S-act A_S satisfies *Condition* (PWP_E) if for all $a, a' \in A_S$, $s \in S$, $as = a's$ implies that there exist $a'' \in A_S$, and $u, v \in S, e, f \in E(S)$ such that $ae = a''ue, a'f = a''vf, es = s = fs$ and $us = vs$.

It is obvious that Condition $(P'_E) \Rightarrow$ Condition (PWP_E).

Theorem 3.25 ([8])**.** *Let S be a monoid and A_S be a right S-act. If A_S satisfies Condition (PWP_E), then it is principally weakly flat. If S is a left PP monoid, then A_S is principally weakly flat if and only if it satisfies Condition (PWP_E).*

Theorem 3.26. *For any monoid S the following statements are equivalent:*

(1) *all torsion free right S-acts satisfy Condition (P'_E);*

(2) *all torsion free right S-acts are principally weakly flat and satisfy Condition $(W_{(WF)'})$;*

(3) *all torsion free right S-acts satisfy Condition $(W_{(WF)'})$ and S is left almost regular;*

(4) *all principally weakly flat right S-acts satisfy Condition (P'_E) and S is left almost regular;*

(5) *all principally weakly flat right S-acts satisfy Condition $(W_{(WF)'})$ and S is left almost regular;*

(6) *all right S-acts satisfying Condition (PWP_E) satisfy Condition (P'_E) and S is left almost regular;*

(7) *all right S-acts satisfying Condition (PWP_E) satisfy Condition $(W_{(WF)'})$ and S is left almost regular.*

Proof. Implications (4) $\Rightarrow$ (6), (5) $\Rightarrow$ (7) follow from Theorem 3.25. Implications (1) $\Rightarrow$ (2), (4) $\Rightarrow$ (5) and (6) $\Rightarrow$ (7) follow from Theorem 2.1.

(2) $\Rightarrow$ (3). By assumption all torsion free right S-acts are principally weakly flat, and so by ([11], IV, 6.5) S is left almost regular.

(3) $\Rightarrow$ (4). Suppose that A_S is a principally weakly flat right S-act and let $as = a't, sz = tz$ for $a, a' \in A_S, s, t, z \in S$. Since by assumption A_S satisfies Condition $(W_{(WF)'})$, there exist $a'' \in A_S, u, v \in S$ such that $as = a''us, a't = a''vt$ and $us = vt$. Since also S is left almost regular it is left PP and so by Theorem 3.14 there exist $e, f \in E(S)$ such that $es = s, ft = t, ae = a''ue$, and $a'f = a''vf$. Thus A_S satisfies Condition (P'_E) as required.

(7) $\Rightarrow$ (1). Since S is left almost regular, all torsion free right S-acts are principally weakly flat by ([11], IV, 6.5). Also S is left PP and so by Theorem 3.25 all principally weakly flat right S-acts satisfy Condition (PWP_E), thus by assumption all torsion free right S-acts satisfy Condition $(W_{(WF)'})$. The result follows similar to the proof of (3) $\Rightarrow$ (4). □

By [[13], Proposition 2.1] Θ_S is strongly torsion free for any monoid S, while it satisfies Condition (P'_E) if and only if S is weakly right reversible. So strong torsion freeness does not imply Condition (P'_E) in general.

Theorem 3.27. *Let S be an idempotent monoid. Then the following statements are equivalent:*

(1) *all strongly torsion free right S-acts satisfy Condition (P'_E);*

(2) *all strongly torsion free right S-acts satisfy Condition $(W_{(WF)'})$;*

(3) *all finitely generated strongly torsion free right S-acts satisfy Condition (P'_E);*

(4) *all finitely generated strongly torsion free right S-acts satisfy Condition $(W_{(WF)'})$;*

(5) *all cyclic strongly torsion free right S-acts satisfy Condition (P'_E);*

(6) *all cyclic strongly torsion free right S-acts satisfy Condition $(W_{(WF)'})$;*

(7) *S is weakly right reversible.*

Proof. Implications (1) $\Rightarrow$ (3) $\Rightarrow$ (5), (2) $\Rightarrow$ (4) $\Rightarrow$ (6) are obvious. Implications (1) $\Rightarrow$ (2) and (5) $\Rightarrow$ (6) follow from Theorem 2.1.

(6) $\Rightarrow$ (7). The one-element right S-act Θ_S is strongly torsion free by [[13], Proposition 2.1], and so by assumption it satisfies Condition $(W_{(WF)'})$. Thus S is weakly right reversible by Theorem 2.1.

(7) $\Rightarrow$ (1). Suppose that the right S-act A_S is strongly torsion free. Then the equality $ae = (ae)e$, for $a \in A_S$ and $e^2 = e \in S$, implies that $ae = a$. Hence $aS = \{a\}$, for every $a \in A_S$. Now let $as = a't$, $sz = tz$, for $a, a' \in A_S$, and $s, t, z \in S$. Then $a = a'$. Since S is weakly right reversible, there exist $u, v \in S$ such that $us = vt$. Let $a'' = a$, then $ae = a''ue, a'f = a''vf, es = s$ and $ft = t$ for $e = f = 1$, and so A_S satisfies Condition (P'_E) as required. □

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Accepted: 28.05.2019

Solution of random ordinary differential equations using Laplace variational iteration method

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Abstract. In this article, we shall introduce the Laplace transformation method in connection with the vriational iteration method to study and solve random ordinary differential equations. The sequence of approximated closed form iterated solutions is derived based on the general Lagrange multiplier evaluated using the well-known convolution theorem of Laplace transformation method. Two illustrative examples are considered, linear and nonlinear random ordinary differential equations. The obtained results of the closed form solution have very height accuracy in comparison with the exact solution, if exist, or have a very height convergence between the iterated solutions.
Keywords: variational iteration method, Laplace transformation, Random ordinary differential equations.

1. Introduction

Random ordinary differential equations (RODE's) are considered to be an ordinary differential equation (ODE) that include a stochastic process in their vector field. It is notable that, there is a great amount of researchers and articals con-

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cerned with random differential equations that appeared in recent years, such as [1]-[5], [13]-[15], [18] and [19]. Also, since many random differential equations have no known analytical solution, so it is necessary to derive numerical methods to generate approximations to the exact solutions.

J. He in 1999 described and used a new kind of analytical technique for solving nonlinear problems called the variational iteration method (VIM) to give an approximate solution of the problem [6]-[9]. E. Hesameddini and H. Latifzadeh in 2009 [10] suggests an alternative approach to the derivation of the variational iteration formulations using the Laplace transform. S. A. Khuri and A. Sayfy in 2012 [16] introduced the Laplace variational numerical scheme based on the VIM and the Laplace transform for evaluation the approximate solution of certain classes of linear and nonlinear differential equations. Also G. C. Wu in 2012 [20] present a Laplace transform approach in the determination of the Lagrange multiplier when the VIM is applied to time fractional heat diffusion equation. G. C. Wu and D. Baleanu in 2013 [21] proposed a modification of the VIM by means of the Laplace transformation method to solve the fractional order differential equations. E. M. Hilal and T. M. Elzaki in 2014 [11] give in his study a good strategy for solving some linear and nonlinear partial differential equations in engineering and physics fields by combining Laplace transform and the modified VIM.

In this paper, we will combine between the Laplace transforms and the VIM for solving linear and nonlinear RODEs. This approach give closed form solution of the considered RODE, which are so difficult to solve analytically and also give a very high accurate results in comparison with other numerical or approximate methods for solving such type of equations.

2. Fundamental and basic concepts

In this section, some primitive and basic concepts that are necessary for this work will be introduced. These concepts include basic definitions of stochastic calculus and some basic properties of Laplace transform.

Definition 2.1 ([17]). *A probability space is a triplet* $(\Omega, \mathcal{F}, P)$*, where* Ω *is a sample space (set of all possible outcomes of random increment),* $\mathcal{F}$ *is class of all subset of* Ω *and* P *is a probability measure whose domain is* Ω *and counter domain is the interval [0,1].*

Definition 2.2 ([17]). *A random variable is a real valued function* $X(\omega), \omega \in \Omega$*, which is measurable with respect to the probability measure* P*.*

In the theory stochastic calculus and related fields, a stochastic or random process is a mathematical object which is usually defined as a family of random variables that depends also on the space and/or time variables. These processes are widely used as mathematical modeling of systems and phenomena's that appear to vary in a random manner.

Definition 2.3 ([17]). *A stochastic process is a family of random variables $X_t(\omega)$ (or briefly X_t) of two variables. Let $t \in [t_0, T] \subset [0, \infty), \omega \in \Omega$ on a common probability space $(\Omega, \mathcal{F}, P)$, which assumes real values and is P measurable as a function of ω for each fixed t. The parameter t is interpreted as time. $X_t(.)$ represents a random variable on the above probability space Ω, while $X.(\omega)$ is called a sample path or trajectory of the stochastic process.*

Among the different types of stochastic processes, which will be used in this paper, is the so called Wiener process or Brownian motion, as it is given in the next definition.

Definition 2.4 ([17]). *A stochastic process W_t, for all $t \in [0, \infty)$, is said to be a Wiener process or Brownian motion, if:*

1. *$P(\omega \in \Omega \mid W_0(\omega) = 0) = 1$.*
2. *For $0 < t_0 < t_1 < \ldots < t_n$, the increments $W_{t_1} - W_{t_0}, W_{t_2} - W_{t_1}, \ldots, W_{t_n} - W_{t_{n-1}}$ are independent.*
3. *For an arbitrary t and $h > 0$, $W_{t+h} - W_t$ has a Normal distribution with mean 0 and variance h.*

The Laplace transform is an integral transform that changes a real variable function $f(t)$ into a function $F(s)$, which is defined as follows [11]:

$$F(s) \equiv \mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt,$$

where, in general, s is a real variable. Also, after a certain problem is transformed into an algebraic equation and then solved for $F(s)$, we need to transform $F(s)$ back into $f(t)$. The solution of the original problem which is called the inverse Laplace transform of $F(s)$ is denoted by $f(t) = \mathscr{L}^{-1}\{F(s)\}$.

Among the most important properties of the Laplace transform, that will be used later with the VIM, is the convolution of two functions f_1 and f_2. Let the functions $f_1(t)$ and $f_2(t)$ be defined for all $t \geq 0$, then the convolution of the functions f_1 and f_2, denoted by $(f_1 * f_2)(t)$, and is defined by the integral, [12]:

$$(f_1 * f_2)(t) = \int_0^t f_1(t - \xi) f_2(\xi)\, d\xi.$$

Now, let $\mathscr{L}\{f_1(t)\} = F_1(s)$, $\mathscr{L}\{f_2(t)\} = F_2(s)$, then:

$$\mathscr{L}\left\{\int_0^t f_1(t - \xi) f_2(\xi)\, d\xi\right\} = F_1(s) F_2(s).$$

Conversely:

$$\mathscr{L}^{-1}\{F_1(s) F_2(s)\} = \int_0^t f_1(t - \xi) f_2(\xi)\, d\xi.$$

Also, the convolution is commutative, that is:

$$(f_1 * f_2)(t) = \int_0^t f_1(t-\xi) f_2(\xi)\, d\xi = \int_0^t f_2(t-\xi) f_1(\xi)\, d\xi = (f_2 * f_1)(t).$$

In addition, the most two important properties of the Laplace transform for solving differential equations is that it transform the differential operator $\frac{df}{dt}$ into an algebraic operator $sF(s) - f(0)$.

3. Solution of RODEs using Laplace VIM

To illustrate the idea of the Laplace VIM for solving RODE's, the following general RODE will be considered in operator form:

$$L[x(t;\omega)] + N[x(t;\omega)] = g(t;\omega), t \in [t_0, T] \tag{1}$$

where L is a linear random ordinary differential operator given by $\frac{d^n}{dt^n}$, n is a natural number, N is a nonlinear operator and $g(t;\omega)$ is a known analytical function, which contains random variables.

According to the VIM, the following correction functional related to equation (1) may be constructed:

$$x_{m+1}(t;\omega) = x_m(t;\omega) + \int_{t_0}^t \lambda(\xi,t) \left\{ L(x_m(\xi;\omega)) + N(\tilde{x}_m(\xi;\omega)) - g(\xi;\omega) \right\} d\xi, \tag{2}$$

$m = 0, 1, \ldots$, where λ is the general Lagrange multiplier which can be identified optimally via the variational theory, the subscript m denotes the m^{th} approximation and $N(\tilde{x}_m(\xi;\omega))$ is considered as a restricted variation, that is, $\delta\tilde{x}_m(\xi;\omega) = 0$.

To start the approach, take the Laplace transform of equation (1), yields to:

$$\mathscr{L}\{L[x(t;\omega)]\} + \mathscr{L}\{N[x(t;\omega)]\} = \mathscr{L}\{g(t;\omega)\} \tag{3}$$

By using the Laplace transform of the differential operator of the correction functional (2), one may get:

$$\begin{aligned} \mathscr{L}\{x_{m+1}(t;\omega)\} &= \mathscr{L}\{x_m(t;\omega)\} \\ &\quad + \mathscr{L}\left\{ \int_{t_0}^t \lambda(\xi,t) \left\{ L(x_m(\xi;\omega)) + N(\tilde{x}_m(\xi;\omega)) - g(\xi;\omega) \right\} d\xi \right\}, \end{aligned} \tag{4}$$

$m = 0, 1, \ldots$ Therefore, upon using the convolution theorem with respect to t, we have:

$$\begin{aligned} \mathscr{L}\{x_{m+1}(t;\omega)\} &= \mathscr{L}\{x_m(t;\omega)\} + \mathscr{L}\{\bar{\lambda}(t-s) * [L(x_m(t;\omega)) \\ &\quad + N(\tilde{x}_m(t;\omega)) - g(t;\omega)]\} \\ &= \mathscr{L}\{x_m(t;\omega)\} + \mathscr{L}\{\bar{\lambda}(s;t)\}\mathscr{L}\{L(x_m(t;\omega)) \\ &\quad + N(\tilde{x}_m(t;\omega)) - g(t;\omega)\} \end{aligned} \tag{5}$$

then equation (5) becomes:

$$\begin{aligned}\mathscr{L}\{x_{m+1}(t;\omega)\} &= \mathscr{L}\{x_m(t;\omega)\} + \Lambda(s)\{s^n\mathscr{L}[x_m(t;\omega)] \\ &\quad - s^{n-1}x_m(0;\omega) - \ldots - x_m^{(n-1)}(0;\omega)\} \\ &\quad + \mathscr{L}\left\{N(\tilde{x}_m(t;\omega)) - g(t;\omega)\right\},\end{aligned} \tag{6}$$

where $\Lambda(s) = \mathscr{L}\left\{\bar{\lambda}(s;t)\right\}$. The iteration formula of equation (6) may be used to suggest the main scheme involving the Lagrange multiplier.

Now, consider the term $\mathscr{L}\left\{N(\tilde{x}_m(t;\omega)) - g(t;\omega)\right\}$ in equation (6) as restricted variation, this make equation (6) stationary with respect to x_m, and hence:

$$\mathscr{L}\{\delta x_{m+1}(t;\omega)\} = \mathscr{L}\{\delta x_m(t;\omega)\} + \Lambda(s)s^n\mathscr{L}[\delta x_m(t;\omega)], \tag{7}$$

where δ is the classical first variation, the optimality condition for the extremum is $\mathscr{L}\{\delta x_{m+1}(t;\omega)\} = 0$ and hence equation (7) leads to:

$$\Lambda(s) = \frac{-1}{s^n},\ s > 0.$$

The sequential approximations are obtained by taking the inverse Laplace transform to equation (6) after substituting $\lambda(s)$ yields:

$$\begin{aligned}x_{m+1}(t;\omega) &= x_m(t;\omega) - \mathscr{L}^{-1}\left\{\frac{1}{s^n}[X_m(s;\omega) - s^{n-1}x(0;\omega) - ... - x^{(n-1)}(0;\omega)\right. \\ &\quad \left. + \mathscr{L}\left(Nx_m(t;\omega) - g(t;\omega)\right)]\right\},\end{aligned} \tag{8}$$

where $X_m(s;\omega) = \mathscr{L}\left\{x_m(t;\omega)\right\}$. Rearrange equation (8) will implies to:

$$\begin{aligned}x_{m+1}(t;\omega) &= \mathscr{L}^{-1}\left\{\frac{x(0;\omega)}{s} + ... + \frac{x^{(n-1)}(0;\omega)}{s^n}\right\} \\ &\quad - \mathscr{L}^{-1}\left\{\frac{1}{s^n}\left(\mathscr{L}\left[Nx_m(t;\omega) - g(t;\omega)\right]\right)\right\}\end{aligned} \tag{9}$$

with initial approximation to equation (9), the following equation can be considered:

$$x_0(t;\omega) = \mathscr{L}^{-1}\left\{\frac{x(0;\omega)}{s} + \frac{x'(0;\omega)}{s^2} + ... + \frac{x^{(n-1)}(0;\omega)}{s^n}\right\}. \tag{10}$$

After applying the inverse Laplace transform to equation (10), we get:

$$x_0(t;\omega) = x(0;\omega) + x'(0;\omega)t + \ldots + \frac{x^{(n-1)}(0;\omega)}{(n-1)!}t^{n-1}. \tag{11}$$

Finally, the values of $x_1(t;\omega), x_2(t;\omega), \ldots, x_m(t;\omega)$ are obtained. Therefore, the solution of equation (1) will be $x(t;\omega) = \lim_{m\to\infty} x_m(t;\omega)$.

4. Illustrative examples

In this section, for illustration purpose, the Laplace VIM introduced will be used to solve linear and nonlinear RODE's, but before indulging with these examples, it is necessary to note that the generation of different discretized Brownian motion over the unit interval [0,1] will be considered. These generations are given with total number of generation $N = 100$, 500 and 1000.

Example 4.1. Consider the linear RODE:

$$\frac{dx(t;\omega)}{dt} = -x(t;\omega) + sinW_t(\omega), t \in [0,1],$$

$x_0(t;\omega) = x_0(0,\omega) = 1$ which has the exact solution, for comparison purpose, given by [5]:

$$x(t;\omega) = x_0 e^{-(t-t_0)} + e^{-t}\int_0^t e^s \sin W_s(\omega)\, ds.$$

We can obtain the successive approximate solutions as:

$$\begin{aligned}
x_0(t;\omega) &= 1,\\
x_1(t;\omega) &= x_0(0;\omega) - tx_0(t;\omega) + t\sin W_t(\omega),\\
x_2(t;\omega) &= \mathscr{L}^{-1}\left\{\frac{x(0;\omega)}{s} - \frac{1}{s}\mathscr{L}\left[1 - t + t\sin W_t(\omega) - \sin W_t(\omega)\right]\right\}\\
&= 1 - t + \frac{t^2}{2} - \frac{t^2}{2}\sin W_t(\omega) + t\sin W_t(\omega)\\
&= 1 - t(1 - \sin W_t(\omega)) + \frac{t^2}{2}(1 - \sin W_t(\omega))\\
x_3(t;\omega) &= \mathscr{L}^{-1}\left\{\frac{x(0;\omega)}{s} - \frac{1}{s}\mathscr{L}\left[1 - t + \frac{t^2}{2} - \frac{t^2}{2}\sin W_t(\omega)\right.\right.\\
&\quad \left.\left. + t\sin W_t(\omega) - \sin W_t(\omega)\right]\right\}\\
&= 1 - t + \frac{t^2}{2} - \frac{t^3}{6} + \frac{t^3}{6}\sin W_t(\omega) - \frac{t^2}{2}\sin W_t(\omega) + t\sin W_t(\omega)\\
&= 1 - t(1 - \sin W_t(\omega)) + \frac{t^2}{2}(1 - \sin W_t(\omega)) - \frac{t^3}{6}(1 - \sin W_t(\omega)),\\
x_4(t;\omega) &= 1 - (1 - \sin W_t(\omega)) + \frac{t^2}{2}(1 - \sin W_t(\omega)) - \frac{t^3}{6}(1 - \sin W_t(\omega))\\
&\quad + \frac{t^4}{24}(1 - \sin W_t(\omega)),\\
x_5(t;\omega) &= 1 - (1 - \sin W_t(\omega)) + \frac{t^2}{2}(1 - \sin W_t(\omega)) - \frac{t^3}{6}(1 - \sin W_t(\omega))\\
&\quad + \frac{t^4}{24}(1 - \sin W_t(\omega)) - \frac{t^5}{120}(1 - \sin W_t(\omega))
\end{aligned}$$

and so on by induction:

$$x_m(t;\omega) = 1 + \sum_{i=1}^{m}\frac{(-t)^i}{i!}(1 - \sin W_t(\omega)), \quad m \in N. \tag{12}$$

The signal simulation of the generated discretized Brownian motion with total number of generations $N = 100$, 500 and 1000 over the unit interval [0,1] and the corresponding approximate solutions (12) for each generation are given in Figures (1)-(6).

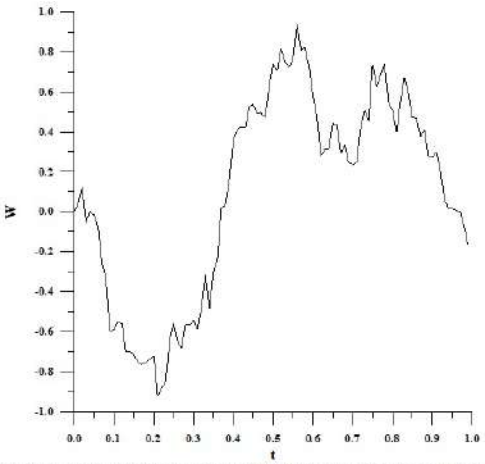


Figure 1: Discretized Brownian path for 100 generation.

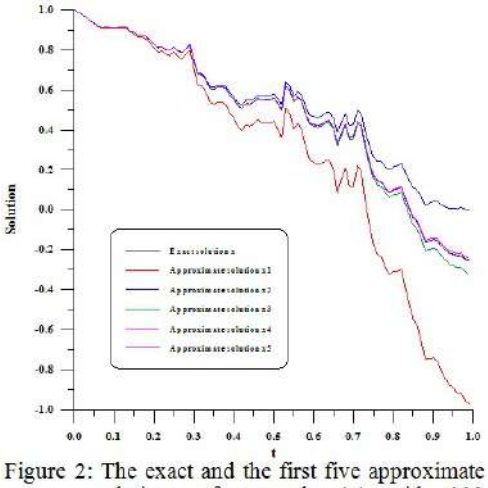


Figure 2: The exact and the first five approximate solutions of example 4.1 with 100 generation of Brownian motion.

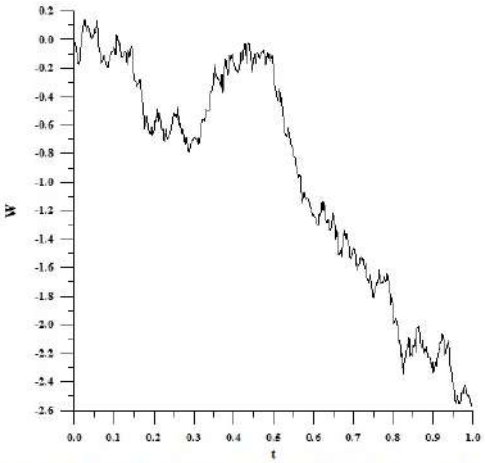


Figure 3: Discretized Brownian path for 500 generation.

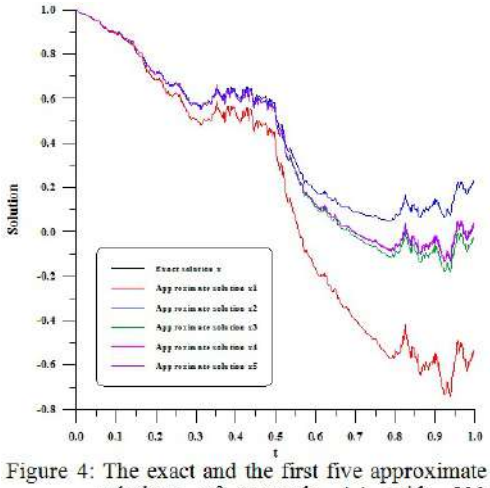


Figure 4: The exact and the first five approximate solutions of example 4.1 with 500 generation of Brownian motion.

Example 4.2. Consider the nonlinear RODE:

$$\frac{d^2x(t;\omega)}{dt^2} + W_t^2(\omega)x(t;\omega) = 0, t \in [0,1], \quad x(0,\omega) = 1, \frac{dx(0;\omega)}{dt} = 0.$$

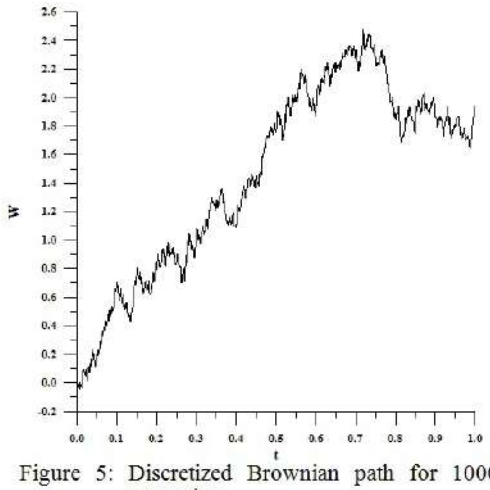


Figure 5: Discretized Brownian path for 1000 generation.

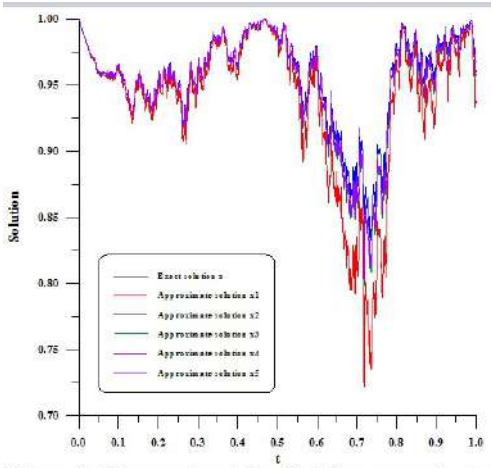


Figure 6: The exact and the first five approximate solutions of example 4.1 with 1000 generation of Brownian motion.

Starting with the initial guess solution $x_0(t;\omega) = 1$, we can find the successive approximate solutions using the proposed method in this paper, to get:

$$
\begin{aligned}
x_1(t;\omega) &= 1 - \frac{t^2}{2}W_t^2(\omega),\\
x_2(t;\omega) &= 1 - \mathscr{L}^{-1}\left\{\frac{1}{s^2}\mathscr{L}\left[W_t^2(\omega)\left(1 - \frac{t^2}{2}W_t^2(\omega)\right)\right]\right\}\\
&= 1 - \frac{t^2}{2}W_t^2(\omega) + \frac{t^4}{24}W_t^4(\omega),\\
x_3(t;\omega) &= 1 - \mathscr{L}^{-1}\left\{\frac{1}{s^2}x'(0;\omega) + \frac{1}{s^2}x(0;\omega) + \frac{1}{s^2}\mathscr{L}\left[W_t^2(\omega)x_2(t;\omega)\right]\right\}\\
&= 1 - \mathscr{L}^{-1}\left\{\frac{1}{s^2}\mathscr{L}\left[W_t^2(\omega)\left(1 - \frac{t^2}{2}W_t^2(\omega) + \frac{t^4}{24}W_t^4(\omega)\right)\right]\right\}\\
&= 1 - \frac{t^2}{2}W_t^2(\omega) + \frac{t^4}{24}W_t^4(\omega) - \frac{t^6}{6!}W_t^6(\omega)\\
x_4(t;\omega) &= 1 - \frac{t^2}{2}W_t^2(\omega) + \frac{t^4}{24}W_t^4(\omega) - \frac{t^6}{6!}W_t^6(\omega) + \frac{t^8}{8!}W_t^8(\omega)
\end{aligned}
$$

and so on upon using mathematical induction, the general solution may be given as:

$$x_m(t;\omega) = 1 + \sum_{i=1}^{m}\frac{(-1)^i t^{2i}}{(2i)!}W_t^{2i}(\omega). \tag{13}$$

Similarly, as in example 1, the signal simulation of the generated discretized Brownian motion with total number of generations $N = 100$, 500 and 1000 over the unit interval [0,1] and the corresponding four approximate solutions given by (13) for each generation are given in Figures (7)-(12).

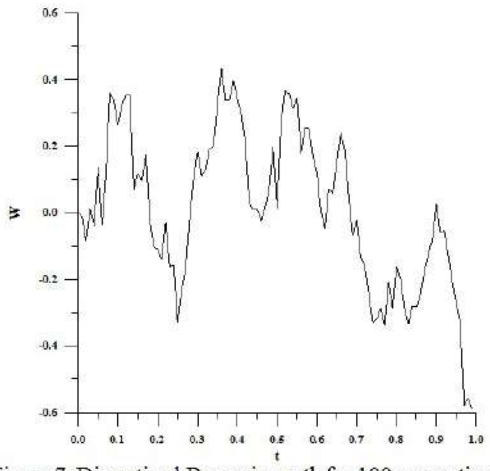


Figure 7: Discretized Brownian path for 100 generation.

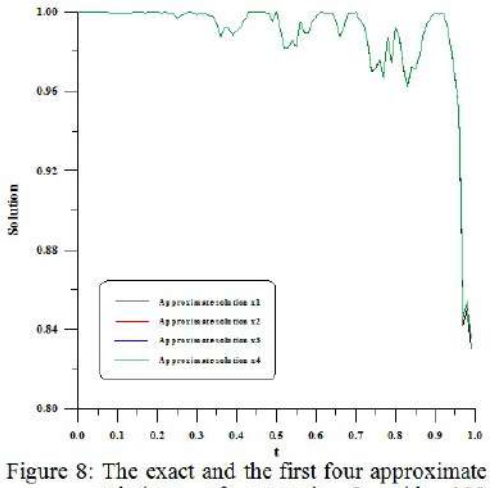


Figure 8: The exact and the first four approximate solutions of example 2 with 100 generation of Brownian motion.

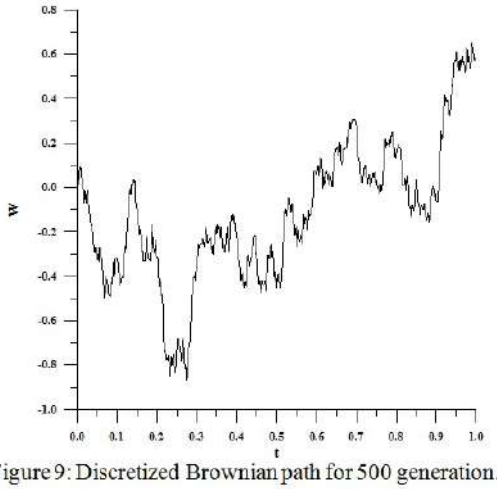


Figure 9: Discretized Brownian path for 500 generation.

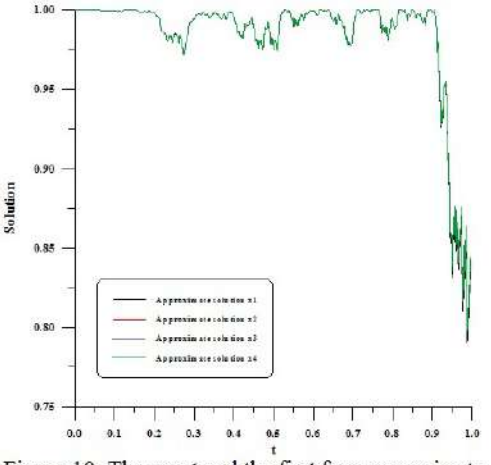


Figure 10: The exact and the first four approximate solutions of example 4.2 with 500 generation of Brownian motion.

5. Conclusions

The followed method proposed in this paper consists of a combination between the Laplace transformation method and VIM to solve linear and nonlinear RODEs, which give a closed form approximate solutions, while the VIM that gives only an approximate solutions that may be complicated in evaluation. This implies, more accurate and reliable results may be obtained for any number of iterations. Also, it is notable that the solution of the RODE's varies depending on varient number of generations of the Brownian motion and the signal of the Brownian path used in the simulation of the closed form solution.

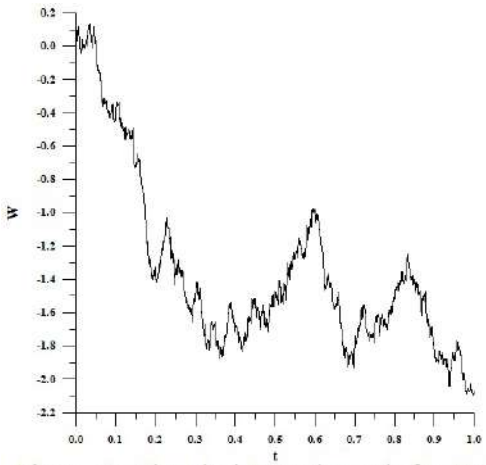


Figure 11: Discretized Brownian path for 1000 generation.

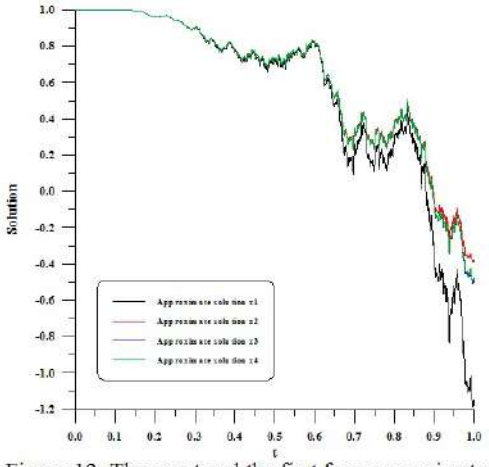


Figure 12: The exact and the first four approximate solutions of example 4.2 with 1000 generation of Brownian motion.

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Accepted: 2.11.2019

Closed multiplication modules

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Abstract. In this paper, all rings are commutative with identity, all modules are unitary, S denotes a ring and W denotes an S-module. A submodule U of an S-module W is said to be closed in W, if U has no proper essential extensions in W. In this paper we introduce the notion of multiplication module concerning closed submodule as a new generalization of multiplication module namely closed multiplication module. Some basic properties of this notion are given. Some related modules with this concept are investigated and studied too

Keywords: multiplication module, closed multiplication module, CL-duo module.

1. Introduction and preliminaries

Throughout, all rings are commutative with identity, all modules are unitary. A module W in which each submodule U of W has the form IW for some an ideal I in S is called a multiplication module. A submodule U of an S-module W is said to be closed in W, if U has no proper essential extensions in W. Some authors studied various types of generalizations of the multiplication module as small multiplication module in [1] and pure multiplication module in [4], and fully invariant multiplication module [18]. In this paper, we shall introduce a new generalization of the multiplication module concerning closed submodule, where an S-module W is called closed multiplication if each closed submodule U of W is equal to IW for some ideal I of S, and hence every multiplication module is c-multiplication. The converse is true under the class semisimple module. We studied some basic properties of this notion. Also, we investigate some connections between it and other related concepts where we proved that every simple closed module is closed multiplication but not conversely such as the Z_6 as

Z-module. Then we proved that under the class semisimple module both of the concepts closed multiplication and simple closed are coincided. Recall that "a submodule U of W is called stable (fully invariant), if U contains $f(U)$ for each S-homomorphism $f : U \to W (f : W \to W)$, and an S-module M is called fully stable (duo) if each submodule of W is stable (fully invariant)" [12]. It is clear that every fully stable module is duo but the converse may not be true. Recall that "a module W is called CL-duo if for each closed submodule U of W is fully invariant"[2]. Also we proved every closed multiplication is CL-duo module. After that, we proved that any closed submodule U of closed multiplication module W is stable. Finally, recall that "a submodule U of an S-module W is called dual stable, if $U \subseteq kerf$ for each S-homomorphism $f : W \to W/U$ [1]. We proved that every closed submodule of c-multiplication module is dual stable. We used $U \leq_c W$ to denote a closed submodule U of W and $End(W)$ to denote for the set all endomorphisms of W.

2. Closed multiplication module

We introduced the notion of a closed multiplication module as a new generalization of the multiplication module.

Definition 2.1. *An S-module W is called closed multiplication (shortly, c-multiplication) if each closed submodule U of W is equal to IW for some ideal I of S. A ring S is called c-multiplication if S is an c-multiplication S-module.*

Eamples and Remarks 2.1.

(1) Every multiplication S-module is c-multiplication. The converse may be not true in general, for example: each of the Z-modules Z_{P^∞} and Q are c-multiplication since the only closed submodules are (0) and Z_{P^∞} of Z_{P^∞} and (0) and Q are the only closed submodules of Q but it is well-known that each of them is not multiplication module.

(2) Every cyclic module over a commutative ring S is c-multiplication S-module.

(3) Let $W = Z_8 \oplus Z_2$ as Z-module, M is not c-multiplication Z-module since the submodule $U =< (\bar{2}, \bar{1}) >= \{(\bar{2}, \bar{1}), (\bar{4}, \bar{0}), (\bar{6}, \bar{1}), (\bar{0}, \bar{0})\}$ is closed submodule of W, but $U \subseteq IW$ for each ideal I of Z.

(4) It is clear that every simple module is c-multiplication, but not conversely, see examples in part(1).

(5) Let S be a ring and W be a module. Then W is c-multiplication S-module if and only if W is c-multiplication $\bar{S}$-module, where $\bar{S} = S/annW$.
Proof. It easy since each submodule is closed of S-module W if and only if it is closed of $\bar{S}$-module.

(6) One can easy to see if U is a proper closed submodule of c-multiplication module then $U = (U : W)W$. Where $(U : W) = \{ r \in S : W \subseteq U\}$ which is called residual ideal.

(7) It is known every commutative ring with identity is multiplication ring and hence c-multiplication.

(8) Every simple closed S-module is c-multiplication where a module W is called simple closed if W has the only two closed submodules are 0 and W [12]. But the converse may be not true in general, for example: Z_6 as Z-module is c-multiplication but it is not simple closed.

Proposition 2.1. *Let $f : W \to W'$ be a monomorphism and W is c-multiplication module. Then $f(W)$ is c-multiplication.*

Proof. Let $U \leq_c f(W)$. Then we can show that $f^{-1}(U) \leq_c W$. Assume $f^{-1}(U) = F$. Since f is monomorphism, $f(f^{-1}(U)) = f(F)$ is closed submodule of $f(W)$. But one sees easily that $U = f(f^{-1}(U))$, so $U \leq_c f(W)$, and hence $f(F) = U$. It follows that $f^{-1}(U) = f^{-1}(f(F)) = F$. Therefore $f^{-1}(U) \leq_c W$, so that $f^{-1}(U) = IW$ for some $I \leq S$ and hence $f(f^{-1}(U)) = f(IW) = If(W) = IW$. Thus $U = If(W)$ and so $f(W)$ is c-multiplication module. □

Proposition 2.2. *For c-multiplication module W over a ring S, the following assertions hold:*

(*i*) *Every closed submodule U of W is a fully invariant submodule of W,*

(*ii*) *If U is a submodule of W such that $U \cap IW = IU$ for every ideal I in S, then U is a c-multiplication module.*

Proof. (i) Let $L \leq_c W$, $f : W \to W$ there exists an ideal I in S such that $L = IW$, consider $f(L) = f(IW) \subseteq If(W) = IW = L$ and hence L is fully invariant submodule of W.

(ii) Let $V \leq_c U$. Since W is a c-multiplication module, there exists an ideal I of S such that $V = IW$. By assumption, $U \cap W = IU$. Therefore, $U \cap V = IU$ and so $V = IU$ hence U is c-multiplication. □

Proposition 2.3. *Every direct summand of a c-multiplication module is c-multiplication.*

Proof. Let U be a direct summand of W and $L \leq_c U$. Then $W = U \oplus V$ for some submodule V of W, $L \oplus V \leq U \oplus V = W$. But $L \leq_c U$ and $V \leq_c V$, so $L \oplus V \leq_c W$. Then $L \oplus V = IW = I(U \oplus V) = IU \oplus V$ for some ideal I of S. Hence $L = IU$, so U is c-multiplication module. □

Proposition 2.4. *Let W be a divisible module over integral domain S. Then W is c-multiplication if and only if W is simple closed.*

Proof. Let $U \leq_c W$ and $U \neq 0$. Since W is c-multiplication, so there exists an ideal I of S such that $U = IW$. But W is divisible module over integral domain hence $IW = W$ and hence $U = W$. Thus W is simple closed. Conversely, It is clear by Remark 2.1(8). □

Recall that, an S-module W is called essentially second if for all an ideal I in S, either $IW \leq_e W$ or $IW = (0)$[13].

Proposition 2.5. *Every second c-multiplication S-module is simple closed.*

Proof. Let $U \leq_c W$. Since W is c-multiplication, then $U = IW$ for some $I \leq S$. As W is an essentially second module, so either $IW \leq_e W$ or $IW = (0)$; that is either $U \leq_e W$ or $U = (0)$. Since $U \leq_e W$, then $U = W$ or $U = 0$. Therefore W is simple closed. □

In the following, we have both concepts c-multiplication and multiplication modules are equivalent under the class semisimple module.

Proposition 2.6. *Let W be a semisimple S-module. Then W is c-multiplication if and only if W is multiplication module.*

Proof. Since every submodule of W is a direct summand, hence it is closed. Thus the result holds. □

Since each module over a semisimple ring S is semisimple S-module, then we have the following:

Corollary 2.1. *Let W be a module over semisimple ring S. Then W is c-multiplication iff W is multiplication.*

Recall that "an S-module W is called extending (or CS-module), if every submodule of W is essential in a direct summand" [3]. "An S-module W is called quasi-injective if for each monomorphism $f : U \to W$, where U is a submodule of W, and for each S-homomorphism $g : U \to W$ there exists an S-homomorphism $h : W \to W$ such that $h \circ f = g$ [11].

Proposition 2.7. *Let W be a quasi-injective (or extending) module. If W is c-multiplication S-module, then every closed submodule of W is stable.*

Proof. Let $U \leq_c W$. Since W is quasi-injective (or extending) module, then U is direct summand of W. Let $f : U \to W$ be a homomorphism. There exists $g : W \to W$ defined by $g(x) = f(x)$ for all $x \in U$ and $g(x) = 0$ otherwise, W is c-multiplication so $U = IW$ for some ideal I in S, hence $f(U) = g(U) = g(IW) = Ig(W) \subseteq IW = U$, and so $f(U) \subseteq U$. Thus U is stable. □

Recall that, an S-module W is called CL-duo if for each closed submodule U of W is fully invariant[2].

Proposition 2.8. *Every c-multiplication S-module is CL-duo.*

Proof. Let $U \leq_c W$ and W be a c-multiplication module, so $U = IW$ for some $I \leq S$, $f : W \to W$. $f(U) = f(IW) = If(W) \subseteq IW = U$. Thus W is CL-duo. □

Recall that "an S-module W is called a projective module if for each epimorphism $f : A \to B$ and a homomorphism $g : W \to B$, there exists a homomorphism $h : W \to A$ satisfies $f \circ h = g$, where A and B are any S-modules" [7]. The converse of Proposition 2.8 is true under the class projective module.

Proposition 2.9. *Let W be a projective S-module. Then W is CL-duo if and only if W is c-multiplication.*

Proof. To prove this we can do analogous the same proof way in [18, Theorem 3.12]. □

Since S is projective S-module, then we obtain the following:

Corollary 2.2. *Let S be a ring. Then S is CL-duo if and only if it is c-multiplication.*

Remarks 2.1. The direct sum of two c-multiplication modules may be not c-multiplication module. For example: Consider $W = Z \oplus Z$ as Z-module. It is clear that Z is multiplication and hence it is c-multiplication but $W = Z \oplus Z$ is not c-multiplication because let $U = Z \oplus (0) \leq_c W$, but there is no ideal I in S holds $IW = U$.

Recall that "an S-module W is said to be fully invariant multiplication if each fully invariant submodule U of W there exists an ideal I of S such that $U = IW$[18].

Proposition 2.10. *Let W be a CL-duo module. If W is fully invariant multiplication S-module, then W is c-multiplication.*

Proof. Let $U \leq_c W$. Since W is CL-duo, U is fully invariant in W. But W is fully invariant multiplication, then $U = IW$ for some $I \leq S$. Thus W is c-multiplication. □

Proposition 2.11. *Let $T = W \oplus W$ and W is a c-multiplication S-module. If T is a distributive module, then T is c-multiplication.*

Proof. Let $U \leq_c T$. Since T is distributive, then $U = (U \cap W) \oplus (U \cap W)$. But $U \leq_c T$, so $U \cap \leq_c W$ and hence $U \cap W = IW$ for some ideal I in S. Thus $U = IW \oplus IW = I(W \oplus W) = IT$. Therefore T is c-multiplication module. □

Proposition 2.12. *Let $T = W \oplus W$ be a CL-duo module. If W is a c-multiplication S-module, then T is c-multiplication.*

Proof. Let $U \leq_c T$. Since T is CL-duo, so U is a fully invariant submodule of T, and so $U = (U \cap W) \oplus (U \cap W)$ by [16]. But $U \cap W$ is closed submodule of W, so $U \cap W = IW$. Hence $U = IW \oplus IW = I(W \oplus W) = IT$, then T is c-multiplication. □

Proposition 2.13. *Let W be a c-multiplication S-module such that $W = W_1 \oplus W_2$. Then the following assertions hold:*

(i) $f(W_1) \subseteq W_1$ for any homomorphism $f : W_1 \to W$.

(ii) $Hom(W_1, W_2) = Hom(W_2, W_1) = 0$.

Proof. (i) There exists an endomorphism g of the module M such that $g(x + y) = f(x)$ for all elements $x \in W_1$ and $y \in W_2$. By proposition 2.3, all submodules of the c-multiplication module W are fully invariant submodules of W. Therefore, $f(W_1) = g(W_1) \subseteq W_1$.

(ii) Let $f \in Hom(W_1, W_2)$, then by (i), $f(W_1) \subseteq W_1 \cap W_2 = 0$. □

Proposition 2.14. *Let W_1 be a c-multiplication S_1-module and W_2 be a c-multiplication S_2-module where S_1 and S_2 are commutative rings with identities). Let $W = W_1 \oplus W_2$ be a distributive S-module where $S = S_1 \oplus S_2$. Then W is c-multiplication iff W_1 and W_2 are c-multiplication modules.*

Proof. Let $U \leq_c W$, then $U = (U \cap W_1) \oplus (U \cap W_2)$. $U \cap W_1 \leq_c W_1$ and $U \cap W_2 \leq_c W_2$, so $U \cap W_1 = IW_1$ and $U \cap W_2 = JW_2$ where $I \leq S_1$ and $J \leq S_2$. As $(I \oplus J).(W_1 \oplus W_2) = IW_1 \oplus JW_2 = (U \cap W_1) \oplus (U \cap W_2) = U$; that is $U = (I \oplus J)(W)$.

Conversely, Let $U \leq_c W_1$, so $U \oplus W_2 \leq_c W_1 \oplus W_2$. Then there exists an ideal $I \oplus J$ in $S_1 \oplus S_2$ such that $U \oplus W_2 = IW_1 \oplus JW_2$. Thus $U = IW_1$, then W_1 is c-multiplication. Similarly, W_2 is c- multiplication. □

Remarks 2.2. Let W be an S-module and $U \leq W$. If W/U is c-multiplication, then W is not necessary c-multiplication module. For example, consider $W = Z \oplus Z$ and $U = 2Z \oplus 3Z$ clear that $W/U = (Z \oplus Z)/(2Z \oplus 3Z) \cong Z_2 \oplus Z_3 \cong Z_6$ which is c-multiplication but $W = Z \oplus Z$ is not c-multiplication by Remark 2.1.

Recall that "an S-module W is called antihopfian if $W/U \cong W$ for all a proper submodule U of W[8]". The following proposition state that the endomorphism of any antihopfian module is c-multiplication ring.

Proposition 2.15. *Let W be an antihopfian S-module. Then the endomorphism ring of W is c-multiplication.*

Proof. Let $End(W)$ is the endomorphism ring of c-multiplication module W. Since W is antihopfian so $End(W)$ is an integral domain by [14] and hence it is a commutative ring with identity so by Remark 2.1(7). $End(W)$ is a c-multiplication ring. □

Recall that "a module W is called a prime if $ann(x) = ann(y)$ for each nonzero element x and y in W. Equivalently, a module W is prime if for all nonzero submodule K of W, $ann(K) = ann(W)$ "[5]. Recall that "a module W of commutative ring S is called a scalar module if for all $f \in End(W)$, $f \neq 0$, there exists $0 \neq r \in S$ such that $f(m) = mr$ for each $m \in W$" [17]. Now we ready to prove the following:

Proposition 2.16. *Let W be a scalar S-module with $ann(W)$ is prime ideal in S. Then $End(W)$ is c-multiplication ring.*

Proof. By [15, Lemma 6.2], we have $End(W) \cong S/(ann(W))$. But W is scalar module and $ann(W)$ is prime, then $S/(ann(W))$ is integral domain hence $End(W)$ is commutative ring with identity so by Remark 2.1(7), it is c-multiplication ring. □

Since $ann(W)$ is prime for each prime module. Thus we can get the following corollary:

Corollary 2.3. *Let W be a prime scalar S-module. Then $End(W)$ is a c-multiplication ring.*

Recall that "a module W is called faithful if $ann(W) = 0$".

Proposition 2.17. *Let W be a faithful scalar S-module. Then S is c-multiplication ring if and only if W is c-multiplication module.*

Proof. Since W is scalar S-module, then $End(W) \cong S/(ann(W))$. But M is faithful so $ann(W) = 0$, hence $End(W) \cong S$ and hence S is c-multiplication ring if and only if W is c-multiplication module. □

Recall that "a submodule U of an S-module W is called characteristic if, for all automorphism β of W, $\beta(U) = U$"[9]. Under c-multiplication module we shall see every closed submodule is characteristic as the next proposition:

Proposition 2.18. *Let W be a c-multiplication S-module. Then every closed submodule of W is characteristic.*

Proof. Let U be a closed submodule of a c-multiplication module W, so there exists an ideal I of S such that $U = IW$. Let β be an automorphism that is β is epimorphism and so $\beta(U) = \beta(IW) = I\beta(W) = IW = U$. Thus U is characteristic submodule. □

Inaam in [10] introduced purely duo modules (briefly P-duo module), which is "a module in which every pure submodule is fully invariant". Atani in [4] introduced the concept pure multiplication where "an S-module W is called a pure multiplication provided for each proper pure submodule U of W, $U = IW$ for some ideal I of S". Also Th. Ghawi in [9] studied pure multiplication modules.

Proposition 2.19. *Let W be a semisimple S-module. Then The following statements are equivalent:*

(*i*) *W is c-multiplication;*

(*ii*) *W is multiplication;*

(*iii*) *W is CL-duo;*

(*iv*) *W is duo;*

(*v*) *W is p-duo;*

(*vi*) *W is fully stable;*

(*vii*) *W is pure multiplication;*

Proof. (i)⇔(ii) It is clear by Proposition 2.6.

(ii) ⇔(iii) ⇔(iv) ⇔(v) ⇔ (vi) follows by [2, Theorem(5.1)].

(i)⇔ (vii) Since W is semisimple S-module by hypothesis so every submodule of W is direct summand and so every submodule of W is closed and pure. Thus W is c-multiplication if and only if W is pure multiplication module. □

Recall that "a submodule U of an S-module W is called dual stable, if $U \subseteq kerf$ for each S-homomorphism $f : W \to W/U$ [1]".

Proposition 2.20. *Every closed submodule c-multiplication S-module is dual stable.*

Proof. Let $U \leq_c W$, since W is c-multiplication, hence $U = IW$ for some ideal I in S. Let $f : W \to W/U$ be an homomorphism, so $f(U) = f(IW) = If(W) \subseteq I(W/U) = IW/U = U/U = 0_{W/U}$, and so $U \subseteq kerf$. Thus U is dual stable □

Proposition 2.21. *Let S be a commutative ring and W be a c-multiplication S-module such that $W = IW$ for some ideal I in S, then $U = IU$ for each closed submodule U of W.*

Proof. Let $U \leq_c W$. Since W is c-multiplication, hence U=JW for some ideal J in S. By assumption, $W = IW$. Hence $U = JW = JIW$. But $JI = IJ$, since S is a commutative ring. Therefore $U = IJW = IU$. □

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Accepted: November 29, 2019

Certain subclass of Janowski functions associated with symmetrical functions

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Abstract. The objective of the present paper is to study results that are defined using notions of Janowski classes and (j,k)-symmetrical functions. A representation theorem, Polya-Schoenberg theorem and some interesting properties are also pointed out.

Keywords: Janowski functions, subordination, starlike functions, convex functions, k-fold symmetric domain, k-fold symmetric function, (j,k)-symmetric points.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$, and $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of all function which are univalent in $\mathcal{U}$. For f and g be analytic in $\mathcal{U}$, we say that the function f is subordinate to g in $\mathcal{U}$, if there exists an analytic function w in $\mathcal{U}$ such that $|w(z)| < 1$ with $w(0) = 0$, and $f(z) = g(w(z))$, and we denote this by $f(z) \prec g(z)$. If g is univalent in $\mathcal{U}$, then the subordination is equivalent to $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$. The convolution or Hadamard product of two analytic functions $f, g \in \mathcal{A}$ where f is defined by (1) and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, is $(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n$. Using the principle of the subordination we define the class $\mathcal{P}$ of functions with positive real part.

Definition 1.1 ([8]). *Let $\mathcal{P}$ denote the class of analytic functions of the form $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ defined on $\mathcal{U}$ and satisfying $p(0) = 1$, $\mathrm{Re}\{p(z)\} > 0$, $z \in \mathcal{U}$.*

Any function p in $\mathcal{P}$ has the representation $p(z) = \frac{1+w(z)}{1-w(z)}$, where $w \in \Omega$ and

$$\Omega = \{w \in \mathcal{A} : w(0) = 0, |w(z)| < 1\}. \tag{2}$$

The class of functions with positive real part $\mathcal{P}$ plays a crucial role in geometric function theory. Its significance can be seen from the fact that simple subclasses like class of starlike $\mathcal{S}^*$, class of convex functions $\mathcal{C}$, class of starlike functions with respect to symmetric points have been defined by using the concept of class of functions with positive real part.

In [12] let $\mathcal{P}[A,B]$, with $-1 \leq B < A \leq 1$, denote the class of analytic function p defined on $\mathcal{U}$ with the representation $p(z) = \frac{1+Aw(z)}{1+Bw(z)}$, $z \in \mathcal{U}$, where $w \in \Omega$. Remark that $p \in \mathcal{P}[A,B]$ if and only if $p(z) \prec \frac{1+Az}{1+Bz}$.

Definition 1.2. *Let k be a positive integer. A domain $\mathcal{D}$ is said to be k-fold symmetric if a rotation of $\mathcal{D}$ about the origin through an angle $\frac{2\pi}{k}$ carries $\mathcal{D}$ onto itself. A function f is said to be k-fold symmetric in $\mathcal{D}$ if for every z in $\mathcal{D}$ we have*

$$f(\varepsilon z) = \varepsilon f(z),\ z \in \mathcal{D}, \varepsilon = e^{\frac{2\pi i}{k}}.$$

The family of all k-fold symmetric functions is denoted by $\mathcal{S}^k$, and for $k = 2$ we get class of odd univalent functions.

The notion of (j,k)-symmetrical functions ($k = 2, 3, \ldots$, and $j = 0, 1, 2, \ldots$, $k-1$) is a generalization of the notion of even, odd, k-symmetrical functions and also generalizes the well-known result that each function defined on a symmetrical subset can be uniquely expressed as the sum of an even function and an odd function.

The theory of (j,k) symmetrical functions has many interesting applications, for instance in the investigation of the set of fixed points of mappings, for the estimation of the absolute value of some integrals, and for obtaining some results of the type of Cartan's uniqueness theorem for holomorphic mappings [9].

Definition 1.3. *Let $\varepsilon = e^{\frac{2\pi i}{k}}$ and $j = 0, 1, 2, \ldots, k-1$, where $k \geq 2$ is a natural number. A function $f : \mathcal{D} \mapsto \mathbb{C}$ where $\mathcal{D}$ is a k-fold symmetric set, is called (j,k)-symmetrical if*

$$f(\varepsilon z) = \varepsilon^j f(z),\ z \in \mathcal{U}.$$

We note that the family of all (j,k)-symmetric functions is denoted be $\mathcal{S}^{(j,k)}$. Also, $\mathcal{S}^{(0,2)}$, $\mathcal{S}^{(1,2)}$ and $\mathcal{S}^{(1,k)}$ the classes of even, odd and k-symmetric functions respectively. We have the following decomposition theorem:

Theorem 1.1 ([9])**.** *For every mapping $f : \mathcal{D} \mapsto \mathbb{C}$, and $\mathcal{D}$ is a k-fold symmetric set, there exists exactly the sequence of (j,k)-symmetrical functions $f_{j,k}$, such that $f(z) = \sum_{j=0}^{k-1} f_{j,k}(z)$, where*

$$f_{j,k}(z) = \frac{1}{k} \sum_{v=0}^{k-1} \varepsilon^{-vj} f\left(\varepsilon^v z\right),\ z \in \mathcal{U}, (\ k = 1, 2, \ldots,\ j = 0, 1, 2, \ldots, k-1). \tag{3}$$

Al Sarari and Latha [2] introduced and studied the classes $\mathcal{S}^{(j,k)}(A,B)$ and $\mathcal{K}^{(j,k)}(A,B)$ which are starlike and convex with respect to (j,k)-symmetric

points. For more details about the classes with (j,k)-symmetrical functions see [4, 5, 3].

Definition 1.4. *A function $f \in \mathcal{A}$ is said to belongs to the class $\mathcal{S}^{(j,k)}(A,B)$, with $-1 \leq B < A \leq 1$ if $\frac{zf'(z)}{f_{j,k}(z)} \prec \frac{1+Az}{1+Bz}$, where $f_{j,k}$ are defined by (3).*

Remark 1.1. Using the definition of the subordination, we can easily obtain that the equivalent condition for a function f belonging to the class $\mathcal{S}^{(j,k)}(A,B)$, with $-1 \leq B < A \leq 1$, is

$$\left|\frac{zf'(z)}{f_{j,k}(z)} - 1\right| < \left|A - B\frac{zf'(z)}{f_{j,k}(z)}\right|, \ z \in \mathcal{U}.$$

We note that special values of j, k, A and B yield the following classes:

For $j = 1$ the class studied by Ohsang and Yaungjae [7].

For $j = 1,\ A = 1$ *and* $B = -1,\ \mathcal{S}_k^* = \mathcal{S}_k^*(1,-1)$, the class is studied by Sakaguchi [6].

For $j = 1,\ A = 1-2\alpha (0 \leq \alpha < 1)$ *and* $B = -1, \mathcal{S}_k^*(\alpha) = \mathcal{S}_k^*(1-2\alpha,-1)(0 \leq \alpha < 1)$, the class of functions that are starlike of order α with respect to k-symmetric points, etc.

We need the following lemmas to prove our main results:

Lemma 1.1 ([13])**.** *Let $f \in \mathcal{S}^*(A,B,\alpha)$, then $f(z)$ can be written in the form*

$$f(z) = \begin{cases} z\,(1+Bw(z))^{\frac{(1-\alpha)(A-B)}{B}}, & \text{if } B \neq 0, \\ z\,\exp[(1-\alpha)Aw(z)], & \text{if } B = 0. \end{cases} \tag{4}$$

Lemma 1.2 ([11])**.** *Let ϕ be convex and g starlike Then for F analytic in $\mathcal{U}$ with $F(0) = 1$,*

$$\frac{\phi * Fg}{\phi * g}(\mathcal{U}) \subset \overline{CO}(F(\mathcal{U})),$$

where $\overline{CO}(F(\mathcal{U}))$ denotes the closed convex hull of $F(\mathcal{U})$.

Lemma 1.3 ([6])**.** *Let N be regular and D starlike in $\mathcal{U}$ and $N(0) = D(0) = 0$. Then for $-1 \leq B < A \leq 1$,*

$$\frac{N'(z)}{D'(z)} \prec \frac{1+Az}{1+Bz}$$

implies that

$$\frac{N(z)}{D(z)} \prec \frac{1+Az}{1+Bz}.$$

Lemma 1.4 ([10])**.** *If $f \in \mathcal{S}^*(A,B)$, then*

$$F(z) = \frac{m+1}{z^m}\int_0^z t^{m-1}f(t)dt \in \mathcal{S}^*(A,B),$$

where $\mathcal{S}^*(A,B)$ is defined by Janowski [12].

2. Main results

Theorem 2.1. *If $f \in \mathcal{S}^{(j,k)}(A,B)$, then*

$$f_{j,k}(z) = \begin{cases} z\,(1+Bw(z))^{\frac{A-B}{B}}, & \text{if } B \neq 0, \\ z\,\exp[Aw(z)], & \text{if } B = 0, \end{cases} \tag{5}$$

where $f_{j,k}$ are defined by (3).

Proof. Supposing that $f \in \mathcal{S}^{(j,k)}(A,B)$, we can get

$$\frac{zf'(z)}{f_{j,k}(z)} \prec \frac{1+Az}{1+Bz}. \tag{6}$$

Substituting z by $\varepsilon^v z$ in (6), it follows

$$\frac{\varepsilon^v z f'(\varepsilon^v z)}{f_{j,k}(\varepsilon^v z)} \prec \frac{1+A\varepsilon^v z}{1+B\varepsilon^v z} \prec \frac{1+Az}{1+Bz},$$

hence

$$\frac{\varepsilon^{v-vj} z f'(\varepsilon^v z)}{f'_{j,k}(z)} \prec \frac{1+Az}{1+Bz}. \tag{7}$$

Letting $v = 0, 1, 2, \ldots, k-1$ in (7) and using the fact that $\mathcal{P}[A,B]$ is a convex set, we get

$$\frac{z\frac{1}{k}\sum_{v=0}^{k-1} \varepsilon^{v-vj} f'(\varepsilon^v z)}{f_{j,k}(z)} \prec \frac{1+Az}{1+Bz},$$

or equivalently

$$\frac{zf'_{j,k}(z)}{f_{j,k}(z)} \prec \frac{1+Az}{1+Bz}, \tag{8}$$

that is $f_{j,k} \in \mathcal{S}(A,B)$, and by Lemma 1.1 for $\alpha = 0$, we finally obtain our result.

Theorem 2.2. *Let $f \in \mathcal{S}^{(j,k)}(A,B)$, with $-1 \leq B < A \leq 1$. Then,*

$$f(z) = \begin{cases} \displaystyle\int_0^z (1+Bw(r))^{\frac{(A-B)}{B}} \cdot \frac{1+Aq(r)}{1+Bq(r)}\,dr, & \text{if } B \neq 0, \\ \displaystyle\int_0^z \exp[Aw(r)] \cdot [(1+A)q(r)]\,dr, & \text{if } B = 0. \end{cases}$$

for some $w, q \in \Omega$.

Proof. Supposing that $f \in \mathcal{S}^{(j,k)}(A,B)$, we can get

$$\frac{zf'(z)}{f_{j,k}(z)} = \frac{1+Aq(z)}{1+Bq(z)}, \qquad q \in \Omega.$$

By Theorem 2.1, we get

$$f'(z) = \begin{cases} (1+Bw(z))^{\frac{(A-B)}{B}} \cdot \dfrac{1+Aq(z)}{1+Bq(z)}, & \text{if } B \neq 0, \\ \exp[Aw(z)] \cdot [(1+A)q(z)], & \text{if } B = 0. \end{cases}$$

Integrating the above equation we complete the proof.

Theorem 2.3. *Let $f \in \mathcal{S}^{(j,k)}(A,B)$ and let ϕ be convex. Then $(f * \phi) \in \mathcal{S}^{(j,k)}(A,B)$.*

Proof. To prove that $(f * \phi) \in \mathcal{S}^{(j,k)}(A,B)$ it is sufficient to show that

$$\frac{z(f*\phi)'(z)}{(f*\phi)_{j,k}(z)} \subset \overline{CO}(F(\mathcal{U})),$$

where $F(z) = \frac{zf'(z)}{f_{j,k}(z)}$.

Now,

$$\frac{z(f*\phi)'(z)}{(f*\phi)_{j,k}(z)} = \frac{zf'(z)*\phi(z)}{(f_{j,k}(z)*\phi(z)} = \frac{\phi(z) * \frac{zf'(z)}{f_{j,k}(z)} . f_{j,k}(z)}{\phi(z) * f_{j,k}(z)},$$

by using Lemma 1.2 with $f_{j,k}(z) \in \mathcal{S}(A,B), F \in \mathcal{P}[A,B]$, that complete the proof.

Example. Let $f \in \mathcal{S}^{(j,k)}(A,B)$, then so does

$$F(z) = \frac{m+1}{z^m} \int_0^z t^{m-1} f(t) dt \tag{9}$$

for $m = 1, 2, 3, ...$

Proof. By using the equation (9), we have

$$F_{j,k}(z) = \frac{m+1}{z^m} \int_0^z t^{m-1} f_{j,k}(t) dt,$$

and

$$\frac{zF'(z)}{F(z)} = -m + \frac{z^m f(z)}{\int_0^z t^{m-1} f(t)\ dt}.$$

Hence

$$\begin{aligned} \frac{zF'(z)}{F_{j,k}(z)} &= \left(-m + \frac{z^m f(z)}{\int_0^z t^{m-1} f(t)\ dt} \right) \frac{F(z)}{F_{j,k}(z)} \\ &= \frac{z^m f(z) - m\int_0^z t^{m-1} f(t)\ dt}{\int_0^z t^{m-1} f_{j,k}(t)\ dt} := \frac{N(z)}{D(z)}. \end{aligned} \tag{10}$$

By (8) then $f_{j,k}(z) \in \mathcal{S}^*(A,B)$, and by Lemma 1.4 we note that $F_{j,k}(z) \in \mathcal{S}^*(A,B)$. Differentiating (10), we have

$$\frac{N'(z)}{D'(z)} = \frac{zf'(z)}{f_{j,k}(z)} \prec \frac{1+Az}{1+Bz},$$

Lemma 1.3 gives $F(z) \in \mathcal{S}^{(j,k)}(A,B)$. □

Corollary 2.1. *Let $f \in \mathcal{S}^{(j,k)}(A,B)$. Then $F_i(z) \in \mathcal{S}^{(j,k)}(A,B)$, $(i = 1,2,3,4)$, where*

$$F_1(z) = \int_0^z \frac{f(t)}{t}dt, \qquad F_2(z) = \int_0^z \frac{f(t)-f(xt)}{t-xt}dt, |x| \le 1, x \neq 1,$$
$$F_3(z) = \frac{2}{z}\int_0^z f(t)dt, \qquad F_4(z) = \frac{m+1}{m}\int_0^z t^{m-1}f(t)dt, \Re m > 0.$$

Proof. Since

$$F_1(z) = \phi_1(z) * f(z), \;\; \phi_1(z) = \sum_0^\infty \frac{1}{n}z^n = \log(1-z)^{-1},$$
$$F_2(z) = \phi_2(z) * f(z), \;\; \phi_2(z) = \sum_0^\infty \frac{1-x^n}{n(1-x)}z^n = \frac{1}{1-x}\log(\frac{1-xz}{1-z}), |x| \le 1, x \neq 1,$$
$$F_3(z) = \phi_3(z) * f(z), \;\; \phi_3(z) = \sum_0^\infty \frac{2}{n+1}z^n = \frac{-2[z+\log(1-z)]}{z},$$
$$F_4(z) = \phi_4(z) * f(z), \;\; \phi_4(z) = \sum_0^\infty \frac{1+m}{n+m}z^n, \Re\{m\} > 0.$$

□

We note that $\phi_i, i = 1,2,3,4$ are convex. Now, using Theorem 2.3 to get $F_i(z) \in \mathcal{S}^{(j,k)}(A,B), (i = 1,2,3,4)$.

Theorem 2.4. *Let $f \in \mathcal{S}^{(j,k)}(A,B)$ and $f_\lambda(z) := (1-\lambda)z + \lambda f(z)$, with $0 \le \lambda \le 1$. Then,*

(i) $f_\lambda \in \mathcal{S}^{(j,k)}(A,0)$, *if* $B = 0$,

(ii) $f_\lambda \in \mathcal{S}^{(j,k)}(A,B)$, *for* $|z| < \frac{1}{B}\sin\left(\frac{B\pi}{2A}\right)$, *if* $B > 0$,

(iii) $f_\lambda \in \mathcal{S}^{(j,k)}(A,B)$, *for* $|z| < \frac{1}{B}\sin\left(\frac{B}{2B-A}\frac{\pi}{2}\right)$, *if* $B < 0$.

Proof. Since $f \in \mathcal{S}^{(j,k)}(A,B)$, then

$$\frac{zf'(z)}{f_{j,k}(z)} \prec \frac{1+Az}{1+Bz}, \quad \text{with} \quad f_{j,k,\lambda}(z) = \frac{1}{k}\sum_{v=0}^{k-1}\varepsilon^{-vj}f_\lambda(\varepsilon^v z).$$

Thus,

$$f_{j,k,\lambda}(z) = (1-\lambda)z + \lambda f_{j,k}(z), \quad zf'_{\lambda}(z) = (1-\lambda)z + \lambda z f'(z),$$

hence

$$\frac{zf'_{\lambda}(z)}{f_{j,k,\lambda}(z)} = \frac{(1-\lambda)\frac{z}{f_{j,k}(z)} + \lambda\frac{zf'(z)}{f_{j,k}(z)}}{(1-\lambda)\frac{z}{f_{j,k}(z)} + \lambda}. \tag{11}$$

(i) For $B = 0$ it is sufficient to show that

$$\left|\frac{(1-\lambda)\frac{z}{f_{j,k}(z)} + \lambda\frac{zf'(z)}{f_{j,k}(z)}}{(1-\lambda)\frac{z}{f_{j,k}(z)} + \lambda} - 1\right| < A,\ z \in \mathcal{U}. \tag{12}$$

We have

$$\frac{zf'(z)}{f_{j,k}(z)} \prec 1 + Az \ \Rightarrow\ \left|\frac{zf'(z)}{f_{j,k}(z)} - 1\right| < A,\ z \in \mathcal{U},$$

and according to (5) $\frac{f_{j,k}(z)}{z} \prec \exp[Az]$, hence, it follows that there exists a Schwarz function $w \in \Omega$, such that $\frac{f_{j,k}(z)}{z} = \exp[Aw(z)],\ z \in \mathcal{U}$. Thus,

$$\begin{aligned}\left|\frac{(1-\lambda)\frac{z}{f_{j,k}(z)} + \lambda\frac{zf'(z)}{f_{j,k}(z)}}{(1-\lambda)\frac{z}{f_{j,k}(z)} + \lambda} - 1\right| &= \lambda\left|\frac{\frac{zf'(z)}{f_{j,k}(z)} - 1}{(1-\lambda)\frac{z}{f_{j,k}(z)} + \lambda}\right| \\ &< \frac{A\lambda}{|(1-\lambda)\exp[-Aw(z)] + \lambda|},\ z \in \mathcal{U},\end{aligned}$$

and from $|w(z)| < 1$, $z \in \mathcal{U}$, we may easily prove that $|(1-\lambda)\exp[-Aw(z)]+\lambda| > \lambda,\ z \in \mathcal{U}$. From the above two inequalities, it follows

$$\left|\frac{(1-\lambda)\frac{z}{f_{j,k}(z)} + \lambda\frac{zf'(z)}{f_{j,k}(z)}}{(1-\lambda)\frac{z}{f_{j,k}(z)} + \lambda} - 1\right| < A,\ \in \mathcal{U},$$

and consequently, from (11) we obtain $\frac{zf'_{\lambda}(z)}{f_{j,k,\lambda}(z)} \prec 1 + Az$, i.e. $f_{\lambda} \in \mathcal{S}^{(j,k)}(A, 0)$.

(ii) For $B \neq 0$ we need to determine the value $r_* \in (0,1)$, such that

$$\left|\frac{(1-\lambda)\frac{z}{f_{j,k}(z)} + \lambda\frac{zf'(z)}{f_{j,k}(z)}}{(1-\lambda)\frac{z}{f_{j,k}(z)} + \lambda} - 1\right| < \left|A - B\frac{(1-\lambda)\frac{z}{f_{j,k}(z)} + \lambda\frac{zf'(z)}{f_{j,k}(z)}}{(1-\lambda)\frac{z}{f_{j,k}(z)} + \lambda}\right|,$$

whenever $|z| < r_*$, which is equivalent to

$$\left|\frac{zf'(z)}{f_{j,k}(z)} - 1\right| < \left|\left(A - B\right)\left(\frac{1}{\lambda} - 1\right)\frac{z}{f_{j,k}(z)} + A - B\frac{zf'(z)}{f_{j,k}(z)}\right|,\ |z| < r_*. \tag{13}$$

According to the Remark 1.1 and the definition of the subordination, we have that $f \in \mathcal{S}^{(j,k)}(A,B)$, for $|z| < r_*$, is equivalent to

$$\left|\frac{zf'(z)}{f_{j,k}(z)} - 1\right| < \left|A - B\frac{zf'(z)}{f_{j,k}(z)}\right|, \ |z| < r_*.$$

We claim that

$$\left|\arg\left(\frac{z}{f_{j,k}(z)}\right) - \arg\left(A - B\frac{zf'(z)}{f_{j,k}(z)}\right)\right| < \frac{\pi}{2}, \tag{14}$$

implies

$$\left|A - B\frac{zf'(z)}{f_{j,k}(z)}\right| < \left|\left(A-B\right)\left(\frac{1}{\lambda}-1\right)\frac{z}{f_{j,k}(z)} + A - B\frac{zf'(z)}{f_{j,k}(z)}\right|, \tag{15}$$

we need proof our claim.

For $\lambda = 0$ and $\lambda = 1$ the inequality (13) is true.

New for $0 < \lambda < 1$, Since $f \in \mathcal{S}^{(j,k)}(A,B)$, from Definition 1.4 and Theorem 2.1, it follows that there exist the functions $w, q \in \Omega$, such that

$$A - B\frac{zf'(z)}{f_{j,k}(z)} = \frac{A-B}{1+Bw(z)} \qquad \text{and} \qquad \frac{z}{f_{j,k}(z)} = (1+Bq(z))^{-\frac{A-B}{B}},$$

Using the above relations, the assumption (14) is equivalent to

$$\left|\arg\frac{(1+Bq(z))^{-\frac{A-B}{B}}(1+Bw(z))}{A-B}\right| < \frac{\pi}{2}, \ z \in \mathcal{U},$$

that is

$$\operatorname{Re}\frac{(1+Bq(z))^{-\frac{A-B}{B}}(1+Bw(z))}{A-B} > 0, \ z \in \mathcal{U},$$

or

$$\operatorname{Re}\left[(1+Bq(z))^{-\frac{A-B}{B}}(1+Bw(z))\right] > 0, \ z \in \mathcal{U}.$$

If we denote $\zeta := (1+Bq(z))^{-\frac{A-B}{B}}(1+Bw(z))$, the above inequality shows that $\operatorname{Re}\zeta > 0$, therefore

$$|(1-\lambda)\zeta + \lambda| = \sqrt{[\operatorname{Re}(1-\lambda)\zeta + \lambda]^2 + [\operatorname{Im}(1-\lambda)\zeta]^2} > \lambda,$$

whenever $0 \leq \lambda < 1$.

Dividing this inequality by $\lambda > 0$, we obtain

$$\left|\left(\frac{1}{\lambda}-1\right)(1+Bq(z))^{-\frac{A-B}{B}}(1+Bw(z)) + 1\right| > 1, \ z \in \mathcal{U},$$

that is

$$\left|\frac{A-B}{1+Bw(z)}\right| < \left|\frac{A-B}{1+Bw(z)}\right|\left|\left(\frac{1}{\lambda}-1\right)(1+Bq(z))^{-\frac{A-B}{B}}(1+Bw(z)) + 1\right|, \ z \in \mathcal{U},$$

which represents (15). Hence, we will determine now the biggest value of r_* such that (15) holds for $|z| < r_*$. Since $f \in \mathcal{S}^{(j,k)}(A,B)$, there exists a function $w \in \Omega$, such that $\frac{zf'(z)}{f_{j,k}(z)} = \frac{1+Aw(z)}{1+Bw(z)},\ z \in \mathcal{U}$, therefore

$$\left|\arg\left(A - B\frac{zf'(z)}{f_{j,k}(z)}\right)\right| = \left|\arg\frac{A-B}{1+Bw(z)}\right| \tag{16}$$
$$\leq |\arg[A-B]| + |\arg[1+Bw(z)]| \leq \arcsin(|B|r),\ |z| \leq r < 1.$$

Next, we will prove that

$$\left|\arg\left(\frac{z}{f_{j,k}(z)}\right)\right| \leq \frac{A-B}{B}\arcsin(Br),\ |z| = r. \tag{17}$$

Thus, from Theorem 2.1, $\frac{f_{j,k}(z)}{z} = (1+Bq(z))^{\frac{A-B}{B}}$, for some $q \in \Omega$, and we will split this proof in the next two cases:

Case 1. If $B > 0$, since $A - B > 0$ it follows that

$$\left|(1+Bq(z))^{\frac{A-B}{B}}\right| = \left|\exp\left[\frac{A-B}{B}\log(1+Bq(z))\right]\right|$$
$$= \exp\left[\frac{A-B}{B}\ln|1+Bq(z)|\right] = |1+Bq(z)|^{\frac{A-B}{B}}$$
$$\leq (1+Br)^{\frac{A-B}{B}},\ |z| \leq r < 1.$$

Case 2. If $B < 0$, denoting $C := -B > 0$, then $A + C > 0$, and it follows that

$$\left|(1+Bq(z))^{\frac{A-B}{B}}\right| = \left|\left[(1-Cq(z))^{-1}\right]^{\frac{A+C}{C}}\right| = \left|(1-Cq(z))^{-1}\right|^{\frac{A+C}{C}}$$
$$\leq \left(\frac{1}{1-Cr}\right)^{\frac{A+C}{C}} = (1+Br)^{\frac{A-B}{B}},\ |z| \leq r < 1.$$

Combining the above two cases, we get

$$\left|\arg\left(\frac{z}{f_{j,k}(z)}\right)\right| \leq \frac{A-B}{B}|\arg(1+Br)| \leq \frac{A-B}{B}\arcsin(Br),\ |z| \leq r < 1.$$

From (16) and (17), we easily deduce that

$$\left|\arg\left(\frac{z}{f_{j,k}(z)}\right) - \arg\left(A - B\frac{zf'(z)}{f_{j,k}(z)}\right)\right|$$
$$\leq \left|\arg\left(\frac{z}{f_{j,k}(z)}\right)\right| + \left|\arg\left(A - B\frac{zf'(z)}{f_{j,k}(z)}\right)\right|$$
$$\leq \arcsin(|B|r) + \frac{A-B}{B}\arcsin(Br) < \frac{\pi}{2},\ |z| \leq r_*,$$

where r_* is given like in the assumptions (ii) and (iii) of Theorem 1.1. □

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Accepted: November 22, 2019

Scale elasticity in the presence of undesirable and nondiscretionary factors: an application to bank branches

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Abstract. The concept of returns to scale and scale elasticity has been frequently studied in the framework of data envelopment analysis. Returns to scale is a qualitative characterization to the frontier points of the technology set and its estimation using tools such as data envelopment analysis (DEA) has attracted considerable attention among researchers. In order to quantify this characterization, we can use scale elasticity that is a quantitative measure of response of the outputs to the change of the inputs of the frontier points in a production set. In the current paper, we propose a method to calculate scale elasticity in banking sector when there are non-discretionary and undesirable factors in the process.

Keywords: Data envelopment analysis, scale elasticity, undesirable outputs, nondiscretionary factors, efficiency

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1. Introduction

Modern performance measurement techniques began with the pioneering work of Farell (1957), who was the first that introduced the relative efficiency assessment of a set of homogenous decision making units (DMUs) with multiple incommensurate inputs and single output in an empirical production possibility set. An extension to the case of multiple inputs and multiple outputs has been done by Charnes, Cooper and Rhodes (1978). They termed their mathematical programing formulation as data envelopment analysis (DEA). The initial work of Charnes et al. (1978) is given in constant return to scale environment and it has been extended to variable return to scale case by Banker, Charnes and Cooper (1984). In the last three decades, theoretical and applied contributions of DEA have been numerous. (See for instances Azadeh and Moradi (2014), Pjevcevic et al. (2018), Pitchipoo et al. (2018) and Kordrostami et al. (2019)).

One of the most frequently studied topics in the context of DEA is the estimation of returns to scale that is a qualitative characterization of boundary points of the technology set. The notion of returns to scale is well-established in DEA and DEA has been proven as an excellent tool to characterize the nature of returns to scale.

In order to quantify this characterization, we can easily use scale elasticity that is a useful characterization of production frontier. Scale elasticity is a measure of response of the outputs to the change of the inputs to each point of the technology frontier. When the production function has an analytical form, scale elasticity would also has an analytical form. However, empirical production possibility set in DEA is a polyhedral set and its boundary points make the production function that is generally a piecewise linear frontier. So, there is a need to provide a procedure to characterize the scale properties of the DEA frontier in various situations. Literature on the response of outputs to the change of inputs is divided in to two groups: qualitative and quantitative methods. In what follows, we briefly review some of the works on this subject.

Scale elasticity is a quantitative measure to reflect the response of outputs to the changes of inputs. The calculation of this measure in DEA framework has been studied by Førsund and Hjalmarsson (2004). Their contributions in their paper were the application of DEA to calculate the scale elasticity and the development of formulas for SE for radial projections of inefficient observations. Krivonozhko (2004) have studied the constructions of economic functions and proposed a DEA-based procedure to calculate marginal rate of substitution using parametric optimization methods. Førsand et al. (2007) have presented two methods of obtaining numerical values of SE by direct and indirect approaches. In direct approach, they evaluate numerical scale elasticity at frontier points of DEA technology. In the indirect approach, they have used efficiency scores and dual variables for radial projections of inefficient points to the frontier. Podinovski et al. (2009) suggested a simple approach to derive formulae for scale elasticity in the variable returns to scale technology. Krivonozhko et al.

(2014) proposed and substantiated a direct method for the returns to scale measurement in the non-radial DEA models. Krivonozhko et al. (2017) have proposed a general approach in order to measure returns to scale elasticity at projection points in the non-radial DEA models and in the radial DEA models.

In real applications, we confront cases in which both undesirable products and nondiscretionary factors are present in the production process. Scale properties of the DEA frontier in such a cases are interesting and important and we have to modify the existing approaches in such a way that to be applicable in such cases. As an example of such a production process, we consider performance evaluation in banking sector. Proportionate to the volume of the banking operations, the bank branches are classified in to different degrees and the branches in high degree do business in an environment that is more favorable than the branches in low degree, hence, the degree of the branches is used as an environmental variable that in not at the discretion of the decision maker. Moreover, as some of DEA applications on banking sector, overdue debts in bank branches are considered as an undesirable output.

As this example shows, we simultaneously have both undesirable outputs and nondiscretionary factors and the calculation of scale elasticity in such case is an interesting subject. Matin and Mirjaberi (2015) have studied the problem of scale elasticity computation in the presence of exogenously fixed inputs. In the current paper, we reformulate the scale elasticity calculation in the presence of both undesirable outputs and nondiscretionary factors. Directional distance function is used to evaluate the relative efficiency of the DMUs and to calculate the quantitative measure of scale elasticity of the efficient DMUs.

The rest of the paper is organized as follows: in the next section a directional distance model is formulated when there are both undesirable outputs and nondiscretionary factors. In Section 3, we propose a procedure to calculate scale elasticity in such a production processes. A real application in banking sector is given in Section 4. Conclusions appear in Section 5.

2. Undesirable outputs and non-discretionary factors

In this section, we consider a production process within which both undesirable and nondiscretionary factors play important role in the process. To incorporate undesirable outputs and nondiscretionary factors in to the performance analysis, we modify the traditional BCC model of Banker, Charnes and Cooper (1984). One type of nondiscretionary factors is environmental factors and from experience we know that these factors are not at the discretion of the decision maker (see Ruggiero (1996, 1998, 2004)). Clearly, the environmental factors z affect the transformation of inputs to outputs. So, in the model we will propose, the effect of the environment in which the DMUs are doing business, have to be considered.

Suppose the production process consists of J DMUs and each $DMU_j : j = 1, ..., J$ consumes M inputs $x_j = (x_{1j}, ..., x_{Mj} \geq 0)$, to generate the desirable out-

puts $v_j = (v_{1j}, ..., v_{Sj} \geq 0)$ and the undesirable outputs $w_j = (w_{1j}, ..., w_{Nj} \geq 0)$, given the nondiscretionary inputs $z_j = (z_{1j}, ..., z_{Kj} \geq 0)$. To operationalize the DEA technique in this process, the production set is defined as:

$$T(z) = \{(x, v, w) : x \text{ can produce } (v, w) \text{ in given } z\}.$$

To include the environmental effect to the performance analysis, it suffices to note that to evaluate the relative efficiency of a specific DMU, it is necessary to compare it only to those DMUs with at least as harsh an environment as the DMU faces. Axioms such as inclusion, convexity, weak disposability of outputs, monotonicity and minimal extrapolation are assumed to construct the technology set. The unique technology set under these assumptions is as follows:

$$\begin{aligned} \bar{T} = \{(x, v, w) : & \sum_{j=1}^{J} \theta_j \gamma_j v_j \geq v, \\ & \sum_{j=1}^{J} \theta_j \gamma_j w_j = w, \\ & \sum_{j=1}^{J} x_j \gamma j \leq x, \\ & \sum_{j=1}^{J} \gamma_j = 1, \\ \gamma_j = 0 \text{ if } z_j > z; \text{ else } & \gamma_j \geq 0, j = 1, ..., J \} \end{aligned} \tag{1}$$

Clearly, due to the existence of $\theta_j \gamma_j$ in the representation of technology set (1),$\bar{T}$ is nonlinear and using the manner of Kuosmanen (2005) it can be transformed to the following linear form:

$$\begin{aligned} T = \{(x, v, w) : & \sum_{j=1}^{J} \lambda_j v_j \geq v, \\ & \sum_{j=1}^{J} \lambda_j w_j = w, \\ & \sum_{j=1}^{J} (\lambda_j + \gamma_j) x_j \leq x, \\ & \sum_{j=1}^{J} (\lambda_j + \gamma_j) = 1, \\ \lambda_j + \gamma_j = 0 \text{ if } z_j > z; \text{ else } & \lambda_j + \gamma_j \geq 0, j = 1, ..., J \} \end{aligned} \tag{2}$$

We can now apply the performance measurement techniques to evaluate the DMUs. For example if we want to increase the desirable outputs and simultaneously decrease inputs and undesirable outputs, we can use the directional

distance function as follows:

$$
\begin{aligned}
& e_0^* = \text{Max } e \\
& \text{s.t.} \\
& \sum_{j=1}^{J} (\lambda + \gamma_j)\, x_{mj} \geq (1-e)\, x_{mo}, \;\; m = 1, ..., M \\
& \sum_{j=1}^{J} \lambda_j w_{nj} = (1-e)\, w_{no}, \qquad\qquad n = 1, ..., N \\
& \sum_{j=1}^{J} \lambda_j v_{sj} \geq (1+e)\, v_{so}, \qquad\qquad s = 1, ..., S \\
& \sum_{j=1}^{J} (\lambda_j + \gamma_j) = 1, \\
& \lambda_j + \gamma_j = 0 \text{ if } z_{kj} > z_{ko}, \;\; k = 1, ..., K \\
& \text{else } \lambda_j + \gamma_j \geq 0; \text{ else } \lambda_j + \gamma_j \geq 0; \;\; j = 1, ..., J
\end{aligned} \tag{3}
$$

Note that the direction we have used in model (3) is $(d_x, d_v, d_w) = (x_o, v_o, w_o)$.

Definition 1. DMU_o is said to be efficient if and only if $e_0^* = 0$.
If $DMU_o = (x_o, v_o, w_o)$ prevails as inefficient, then its efficient projection on the technology frontier is as follows:

$$
\begin{aligned}
\bar{x}_o &= (1 - e_o^*)\, x_o - s^{(x)} \\
\bar{x}_o &= (1 - e_o^*)\, w_o \\
\bar{v}_o &= (1 - e_o^*)\, v_o - s^{(v)}
\end{aligned} \tag{4}
$$

In which $s^{(x)}$ and $s^{(v)}$ are respectively the optimal slack variables corresponding to the first and third constraints of (3). In model (3), the objective function and the first four constraints are linear. However due to the existence of the conditional constraint, model (3) is not a complete linear programing problem and we cannot normally use the dual rule of linear programing problems to write the dual formulation of model (3). In what follows, we provide a pseudo-dual formulation to model (3):

$$
Min\Lambda - [\sum_{s=1}^{S} \delta_s^{(v)} v_{so} - \sum_{n=1}^{N} \delta_s^{(w)} w_{no} - \sum_{m=1}^{M} \delta_m^{(x)} x_{mo}]
$$

s.t.

$$
\sum_{s=1}^{S} \delta_s^{(v)} v_{sj} - \sum_{n=1}^{N} \delta_s^{(w)} w_{nj} - \sum_{m=1}^{M} \delta_m^{(x)} x_{mj} - \Lambda + \epsilon d_j \leq 0, \tag{5}
$$

$$\sum_{s=1}^{S} \delta_s^{(v)} v_{so} - \sum_{n=1}^{N} \delta_s^{(w)} w_{no} - \sum_{m=1}^{M} \delta_m^{(x)} x_{mo}$$
$$d_j = 1 \text{ if } z_j > z_o \text{ else } d_j = 0 \;\; j = 1, ..., J$$
$$\delta_s^{(v)}, \delta_m^{(x)} \geq 0, \text{ for all s and m}$$

in which $\epsilon > 0$ is sufficiently small. The binary variable d_j in the first constraint indicates that if DMU_j must be included in the reference set of DMU_o or not. If $d_j = 1$, the j-th constraint $\sum_{s=1}^{S} \delta_s^{(v)} v_{sj} - \sum_{n=1}^{N} \delta_n^{(w)} w_{nj} - \sum_{m=1}^{M} \delta_{mo}^{(x)} - \Lambda \leq 0$ is not binding and this guarantees that is not included in the reference set of . However, if we have, permits to be included in the reference set of if needed. If we ignore the conditional constraint in (5), the remaining is a linear programming problem, however it can easily be solved using optimization softwares such as GAMS or MATLAB in its current form.

3. Scale elasticity in the presence of undesirable and ND factors

As we stated before, scale elasticity in quantitative form, or returns to scale in qualitative form are the response of the outputs to the change of inputs in frontier points of the technology set. In the current section, we propose a procedure to calculate the scale elasticity of a frontier DMU in the presence of both nondiscretionary inputs and undesirable outputs.

Suppose there are J DMUs to be evaluated. $DMU_j : j = 1, ..., J$ uses the discretionary inputs $x_j = (x_{1j}, ..., x_{Mj}) \geq 0$ and nondiscretionary inputs $z_j = (z_{1j}, ..., z_{Kj}) \geq 0$ to produce the desirable outputs $v_j = (v_{1j}, ..., v_{Sj}) \geq 0$ and undesirable outputs $w_j = (w_{1j}, ..., w_{Nj}) \geq 0$. The response function $\beta(\alpha)$, for each $\alpha > 0$, can be calculated by the following linear programming problem:

$$\begin{aligned}
&\beta(\alpha) = Max \;\; \rho \\
&\text{s.t.} \\
&\sum_{j=1}^{J} (\lambda_j + \gamma_j) x_{mj} \leq \alpha x_{mo}, \quad m = 1, ..., M \\
&\sum_{j=1}^{J} \lambda_j w_{nj} = \rho w_{no}, \qquad n = 1, ..., N \\
&\sum_{j=1}^{J} \lambda_j v_{sj} \geq \rho v_{so}, \qquad s = 1, ..., S \\
&\sum_{j=1}^{J} (\lambda_j + \gamma_j) = 1,
\end{aligned} \tag{6}$$

$$\begin{aligned}
&\lambda_j + \gamma_j = 0 \text{ if } z_{kj} > z_{ko}: \ \ k = 1, ..., K \text{ else } \lambda_j + \gamma_j \geq 0; \\
&\text{ else } \lambda_j + \gamma_j \geq 0; j = 1, ..., J, \\
&\lambda_j, \gamma_j, \alpha, \rho \geq 0.
\end{aligned}$$

In model (6) the input bundle x_o is scaled by $\alpha > 0$ and the response of this change is calculated by ρ . It should be pointed out that $\beta(\alpha)$ is the maximum proportion of the output (v_o, w_o) for a given vector αx_o in all environments no better than z_o.

Definition 2. Scale elasticity (denoted by $\epsilon(x_o, v_o, w_o, z_o)$) is a function of inputs and outputs and at any frontier point $(\alpha x_o, \beta(\alpha) v_o, \beta(\alpha) w_o, z_o)$ we have:

$$\epsilon(x_o, v_o, w_o, z_o) = \frac{\beta'(\alpha)}{\frac{\beta(\alpha)}{\alpha}} = \frac{\alpha \beta'(\alpha)}{\beta(\alpha)} \tag{7}$$

in which $\beta'(\alpha)$ is the marginal productivity and $\frac{\beta(\alpha)}{\alpha}$ is the average productivity. (See Podinovski et al. (2009)). Note that z_o is unchanged and this means that we do not want to change the environment in which DMU_o is doing business. In other word, scaling the inputs and outputs up and down will not change the performance environment of DMU_o . To interpret $\epsilon(x_o, v_o, w_o, z_o)$, we can say that if the quantity of the input x_o is increased by $\alpha > 1$, then, the maximum quantity of the outputs (v_o, w_o) will increase by $\epsilon(x_o, v_o, w_o, z_o) \times \alpha$, while remaining in the technology set in same environment. Clearly, if $(x_o, v_o, w_o, z_o) = (\alpha x_o, \beta(\alpha) v_o, \beta(\alpha) w_o, z_o)$, then $\alpha = \beta(\alpha) = 1$ and $\epsilon(x_o, v_o, w_o, z_o) = \beta'(1)$. (Note that (x_o, v_o, w_o, z_o) is an efficient point.) In DEA technology set, the production frontier is a piecewise linear function and in the extreme efficient points, the function $\beta(\alpha)$ is not differentiable, but, its one-sided derivatives are available. So, we can define the right and left scales elasticity as follows:

$$\begin{aligned}
\epsilon_+(x_o, v_o, w_o, z_o) &= \beta'_+(1) \\
\epsilon_-(x_o, v_o, w_o, z_o) &= \beta'_-(1)
\end{aligned} \tag{8}$$

Now suppose $DMU_o : (x_o, v_o, w_o, z_o)$ is inefficient. To calculate the scale elasticity to this unit, we first need to project it to the efficient frontier and then, the scale elasticity of the projected point is considered as a scale elasticity to this inefficient unit. Clearly, different projection points lead to different scale elasticities. However, in this study, we will use the directional model (3) to evaluate the relative efficiency of a specific DMU.

Cleary, we do not have an analytical form to $\beta'_+(\alpha)$ and $\beta'_-(\alpha)$. In order to calculate the numerical values to $\beta'_+(1)$ and $\beta'_-(1)$, we can easily use the following theorem.

Theorem 1. *Suppose* Λ^{Max} *and* Λ^{Min} *are respectively the optimal values to the following programs* (9) *and* (10):

$$\Lambda^{Max} = Max\ \Lambda$$

s.t.

$$\begin{aligned}
&\sum_{m=1}^{M} \delta_m^{(x)} x_{mo} + \Lambda = 1 \\
&\sum_{s=1}^{S} \delta_s^{(v)} v_{sj} + \sum_{n=1}^{N} \delta_n^{(w)} w_{nj} - \sum_{m=1}^{M} \delta_m^{(x)} x_{mj} - \Lambda + \epsilon d_j \leq 0 \\
&\sum_{m=1}^{M} \delta_m^{(x)} x_{mj} + \Lambda + \epsilon d_j \geq 0 \\
&\sum_{s=1}^{S} \delta_s^{(v)} v_{so} + \sum_{n=1}^{N} \delta_n^{(w)} w_{no} = 1 \\
&d_j \text{ if } z_j > z_o \text{ else } d_j = 0 \ \ j = 1, ..., J \\
&\delta_s^{(v)}, \delta_m^{(x)} \geq 0, \text{ for all s and m}
\end{aligned} \tag{9}$$

$$\Lambda^{Min} = Min\ \Lambda \tag{10}$$

Subject to the same conditions as in program (9).
Then, $\beta'_+(1) = 1 - \Lambda^{Max}$ and $\beta'_-(1) = 1 - \Lambda^{Min}$

Proof. The function $\beta(\alpha)$ defined in (6) can be viewed as the function $\varphi(b)$ of the vector $b = (\alpha x_o, 0, 0, 1) \in R^{2m+n+s+2}$ on the right-hand side of (6). As Podinovski et al. (2009) stated, $\beta'_+(1)$ is the directional derivative of the function $\varphi(b)$ taken at $\hat{b} = (x_o, 0, 0, 1)$ in the direction $\hat{d} = (x_o, 0, 0, 0) \in R^{2m+n+s+2}$ that is $\beta'_+(1) = \hat{\varphi}\left(\hat{b}; \hat{d}\right)$. Similarly, we have $\beta'_-(1) = -\hat{\varphi}\left(\hat{b}; -\hat{d}\right)$. As we stated before, model (6) is not complete linear programing problem and we cannot use dual role for it. The following pseudo-dual formulation is given to model (6) with $\alpha = 1$:

$$\beta(1) = Min \sum_{m=1}^{M} \delta_m^{(x)} x_{mo} + \Lambda$$

s.t.

$$\begin{aligned}
&\sum_{s=1}^{S} \delta_s^{(v)} v_{sj} + \sum_{n=1}^{N} \delta_n^{(w)} w_{nj} - \sum_{m=1}^{M} \delta_m^{(x)} x_{mj} - \Lambda + \epsilon d_j \leq 0 \\
&\sum_{m=1}^{M} \delta_m^{(x)} x_{mj} + \Lambda \geq 0
\end{aligned}$$

$$
\begin{aligned}
(11) \qquad & \sum_{s=1}^{S} \delta_s^{(v)} v_{sj} + \sum_{n=1}^{N} \delta_n^{(w)} w_{nj} = 1 \\
& d_j \text{ if } z_j > z_o \text{ else } d_j = 0 \;\; j = 1, ..., J \\
& \delta_s^{(v)}, \delta_m^{(x)} \geq 0, \text{ for all s and m.}
\end{aligned}
$$

Since we assume a non-zero vector is a feasible direction at b,then the directional derivative of φ at $\hat{b}$ exist and $\varphi\prime\left(\hat{b}, \hat{d}\right) = Min\{w\hat{d}|w \in \Omega\}$ and $-\varphi\prime\left(\hat{b}, -\hat{d}\right) = -Max\{w\hat{d}|w \in \Omega\}$, in which Ω is the set of all optimal solutions to model (11). This leads to programs (9) and (10) which include the constraints of (11) and the first constraint of (9) that is equal to the objective function of (11). □

4. An application to bank branches

To illustrate the real applicability of the proposed methodology of calculating scale elasticity, we conduct an empirical study on Iranian banking sector in 2015. Banking in Iran is one of the most prosperous professions and a trusted industry on the Iranian stock exchange. The main function of the Iranian banks is the attracting deposits and then the dispatching of the deposits to the customers. A bank branch is seen as a transformer of resources in to services and products and clearly, this transformation is affected by a lot of variables that are controllable and non-controllable. The systematic view of this process is illustrated in Figure 1.

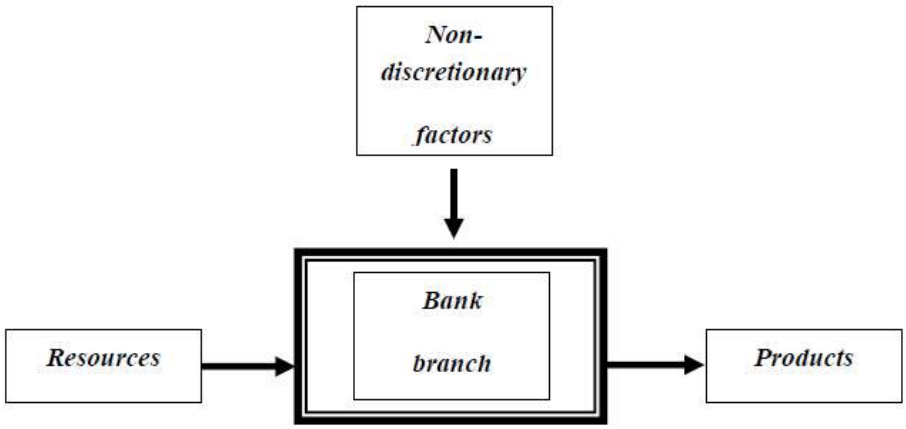


Figure 1: Systematic view of a bank branch

As Khoshandam et al. (2014) stated, bank branches in Iran are divided in to seven degrees and in this study we have focused on the branches in degree 2

and 3. It should be pointed out that the environment that the branches with degree 2 do business are more favorable than the environment of the branches with degree 3. So, the degree of the branches is considered as an environmental variable.

Data on 35 sample branches in 2015 is selected. To evaluate the relative efficiencies of the branches, we have used seven variables from data set as inputs and outputs. We have considered two discretionary inputs, four outputs and one non-discretionary input.

Inputs include: number of check accounts (x_1) and operational costs (including staff costs)(x_2) and outputs include deposits (v_1), loans (v_2), profits (v_3) and overdue debts (w_1) . Note that (w_1) (overdue debts) is the single undesirable output. As we said before, the single nondiscretionary variable that is used in our study is the degree of the branches.

In what follows, a brief description of the variables are given:

-Number of check account: A checking account is a deposit account held at bank branches that allows withdrawals and deposits.

-Operational costs: Operational costs are the expenses which are related to the operations of business, or to the operation of a device, component and piece of equipment or facility. Moreover, staff costs are included in operational costs.

-Deposits: Bank deposits consist of money placed into banking institutions for safekeeping. These deposits are made to deposits accounts such as saving accounts, checking accounts and money market accounts.

-Loans: A loan is a debt provided by a bank branch to a customer at an interest rate and evidenced by a promissory note which specifies, among other things, the principle amount of money borrowed, the interest rate the lender is charging, and date of repayment.

- Profits: Profit is an amount of money that the branches gain when they do their business.

- Overdue debts: A debt is a creditor's right to be paid by a debtor. A debt becomes an overdue debt once it is due and outstanding, provided that it is certain, liquidated, due and payable.

Table 1 shows summary statistics for the inputs and outputs of these 35 branches.

Table 1: Summary statistics for 35 bank branches

	Number of check account	Operational costs	Deposits	Loans	Profits	Overdue debts
Mean	2357.229	30.97536	59825.7	126208.8	8908.08	18438.73
STDEV	607.9711	1085246	210861.3	98499.38	3577.111	10542.38
Minimum	1500	12	250000	75	2500	12000
Maximum	3500	50	1000000	300000	1500	3500

The monetary variables are stated in current Iranian one-hundred million Rial.

The Pearson correlation coefficients of the variables (given in Table 2) show that all of the variables are almost independent and we cannot ignore none of the variables.

Table 2: Pearson correlation coefficients of the variables

	x_1	x_2	v_1	v_2	v_3	w_1
x_1	1	0.295	-	-	-	-
x_2	-	1	-	-	-	-
v_1	-	-	1	0.229	-0.501	-0.005
v_2	-	-	-	1	0.160	0369
v_3	-	-	-	-	1	0.030
w_1	-	-	-	-	-	1

To evaluate the relative efficiencies of the branches, the directional model (3) is applied to this data set. The results are listed in second column of Table 3. Of 35 branches, 28 branches are extreme efficient. So, we can directly calculate their scale elasticities. However, to other seven inefficient branches, we first constructed their efficient projection on the frontier and then, the scale elasticity to these branches are calculated. The right and left scale elasticities are calculated to these branches and the results are shown in the third and fourth columns of Table 3. Among 28 efficient branches, 19 branches have zero right scale elasticities and other nine branches have positive scale elasticities. However, all efficient branches have positive left scale elasticities. Considering this fact that the efficient projection of the inefficient branches are non-extreme points of the frontier, the right and left scale elasticity of these branches are exist, as we expected.

We have also calculated the response function $\beta(\alpha)$ to six different values of α : $\alpha = 1.05$, $\alpha = 1.1$, $\alpha = 1.2$, $\alpha = 0.95$, $\alpha = 0.9$, $\alpha = 0.8$. The results are listed in the last six columns of Table 3.

An important point to be noted is that $\epsilon_+ = 0$ means that if we scale up the inputs, no changes will result in the outputs. This is also true when $\epsilon_- = 0$. This clearly is shown in columns 5, 6 and 7 of Table 3. All response functions for $\alpha > 1$ are feasible and this means that scaling up the inputs will result a feasible point in the production set. However, when we come to calculate the response of outputs to $\alpha < 1$, we have infeasibility in some cases.

Now, we analyze a sample branch: Consider, for example, DMU_1. This branch is efficient in our directional model (3) and its right and left scale elasticities are respectively $\epsilon_+ = 0$ and $\epsilon_- = 5.2$. Clearly, with $\epsilon_+ = 0$, no changes in outputs is allowable when $\alpha > 1$ and hence, we have $\beta(\alpha) = 1$ for $\alpha = 1.05, 1.1, 1.2$.

However, for $\alpha < 1$, we have two infeasibility and one feasible response. Scaling down the inputs by 0.90 and 0.85, lead to infeasible points outside of the production technology set. However, the response for $\alpha = 0.95$ is $\beta(0.95) =$

0.7317. This means that if we scale down the inputs to %95, the outputs should be contracted by 0.7317, while remaining in the production set. The response function for $\alpha = 0.9$ and $\alpha = 0.8$ is infeasible.

Now consider the efficient branch 12 with $\epsilon_+ = 0.4375$ and $\epsilon_- = 3.2$. All responses to this unit are feasible. If the inputs are scaled up to $\alpha = 1.05$, then, the outputs are also scaled up to $\beta = 1.0219$. When $\alpha = 0.8$, the response is 0.3598. These values to α along with the response values β show that for $\alpha > 1$, the increase in inputs are faster than the increase in outputs.

Table 3: The results efficiency score and scale elasticity

Branches	Efficiency	ϵ_+	ϵ_-	$\beta\,(1.05)$	$\beta\,(1.1)$	$\beta\,(1.2)$	$\beta\,(0.95)$	$\beta\,(0.9)$	$\beta\,(0.8)$
1	0	0	5.2	1.0000	1.0000	1.0000	0.7317	Infeasible	Infeasible
2	0	0	0.4056	1.0000	1.0000	1.0000	0.9847	0.9694	0.9388
3	0.0591	0.2924	1.683	1.0071	1.0141	1.0281	0.743	0.4874	Infeasible
4	0	0.0471	24.4	1.0024	1.0047	1.0094	Infeasible	Infeasible	Infeasible
5	0	0	0.2195	1.0000	1.0000	1.0000	0.9890	0.9780	0.9561
6	0	0	7.3	1.0000	1.0000	1.0000	0.6367	0.2734	Infeasible
7	0	0	Undefined	1.0000	1.0000	1.0000	Infeasible	Infeasible	Infeasible
8	0.0202	0.0006	0.081	1.0000	1.0000	1.0000	0.9863	0.9691	0.8815
9	0	0	1.086	1.0000	1.0000	1.0000	0.9565	0.9138	0.8277
10	0	0.2577	15.1	1.0153	1.0306	1.0611	Infeasible	Infeasible	Infeasible
11	0	0	2	1.0000	1.0000	1.0000	0.8978	0.7956	Infeasible
12	0	0.4375	3.2	1.0219	1.0437	1.0875	0.8400	0.6799	0.3598
13	0	0	0.058	1.0000	1.0000	1.0000	0.9971	0.9942	0.9875
14	0	0	0.7489	1.0000	1.0000	1.0000	0.9626	0.9251	0.8502
15	0	0	Undefined	1.0000	1.0000	1.0000	0.9734	0.9468	0.8936
16	0	0.1116	2	1.0056	1.0112	1.0223	Infeasible	Infeasible	Infeasible
17	0.1526	0.3746	0.5833	1.0188	1.0376	1.0628	0.9708	0.9413	0.4565
18	0	0	1.18	1.0000	1.0000	1.0000	0.9410	0.8820	0.7640
19	0.4628	1.949	5	1.0441	1.0519	1.0675	Infeasible	Infeasible	Infeasible
20	0.0212	0	0.1611	1.0000	1.0000	1.0000	0.9754	0.8458	Infeasible
21	0.1813	0.341	0.5916	1.0141	1.0257	1.0361	0.97	0.7722	Infeasible
22	0	0	3.7	1.0000	1.0000	1.0000	0.8126	0.6253	0.2506
23	0	0.7021	6.4	1.0351	1.0675	1.1141	0.6810	0.3619	Infeasible
24	0	0	0.8078	1.0000	1.0000	1.0000	0.9596	0.9192	0.8148
25	0	0.4201	Undefined	1.0340	1.0659	1.0968	Infeasible	Infeasible	Infeasible
26	0	0	0.0443	1.0000	1.0000	1.0000	0.9978	0.9956	0.9911
27	0	0	0.441	1.0000	1.0000	1.0000	0.9780	0.9559	0.9118
28	0	0	5.6	1.0000	1.0000	1.0000	0.7212	0.3906	Infeasible
29	0.3355	0.2216	0.6474	1.0108	1.0215	1.0319	0.9676	0.9092	0.2787
30	0	0	5.3	1.0000	1.0000	1.0000	0.7326	0.4653	Infeasible
31	0	0.3566	Undefined	1.0178	1.0357	1.0713	Infeasible	Infeasible	Infeasible
32	0	0.265	1.888	1.0132	1.0265	1.0530	0.9056	0.7700	Infeasible
33	0	0.0749	24.6	1.0037	1.0075	1.0150	Infeasible	Infeasible	Infeasible
34	0	0	3.1	1.0000	1.0000	1.0000	0.8452	0.6903	0.3806
35	0	0	9.8	1.0000	1.0000	1.0000	Infeasible	Infeasible	Infeasible

5. Conclusions

One of the most frequently studied problems in the context of DEA, is determining the nature of returns to scale to the points on the production function. Returns to scale is a qualitative characterization of frontier points of production technology set and we can use scale elasticity as a quantification of scale

properties for boundary points of the technology set. Calculating the scale elasticity of efficient DMUs in DEA framework in the presence of undesirable outputs and nondiscretionary factors is the problem that we have studied in the current paper. A DEA-based procedure is proposed to calculate the scale elasticity of the DMUs. Then, the applicability of the proposed approach has been illustrated by a real case on banking industry in Iran. The advantages of the proposed procedure is to introduce a DEA-based method to investigate the scale properties of frontier points in the presence of both undesirable outputs and non-discretionary factors.

It is possible to use scale elasticities to determine optimal scale size in DEA technologies. This is our future research.

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Accepted: December 20, 2019

Note on a relative Hilali conjecture

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Abstract. In this paper, we focus on the relative Hilali conjecture proposed by T. Yamaguchi and S. Yokura, that for a continuous map between two simply connected elliptic spaces $f : X \to Y$, $\dim \operatorname{Ker} \Pi_*(f)_{\mathbb{Q}} \leq \dim \operatorname{Ker} H_*(f;\mathbb{Q}) + 1$. Our aim is to prove this conjecture for fibrations whose fibre has at most two-oddly generators. Also we show it in the case $H \to G \to G/H$, where G is a compact connected Lie group and H is a closed sub-Lie group of G.

Keywords: rational homotopy theory, elliptic space, Sullivan minimal model, cohomology, Halperin conjecture, Hilali conjecture.

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1. Introduction

In rational homotopy theory there are a lot of conjectures dealing with the question of how large the cohomology algebra of a space has to be under certain conditions. An example of those conjectures is Hilali's [7], which is based on the size of the rationnally elliptic spaces i.e., simply connected spaces satisfying dim $\pi_*(X)_{\mathbb{Q}}$ and dim $H^*(X;\mathbb{Q})$ both finite.

Conjecture 1 (Hilali)**.** *Let X be a simply connected rationnally elliptic space, then*

$$\text{(H)} \qquad dim\ \pi_*(X)_{\mathbb{Q}} \leq\ dim\ H^*(X;\mathbb{Q}).$$

Recall that the computation of Betti numbers of a simply connected elliptic space is a NP-hard (see, [6]). However, little is known about rational cohomology of function spaces. In particular, in 2018, T. Yamaguchi and S. Yokura proposed a relative version of the conjecture (H) (see, [17]).

Conjecture 2 (Yamaguchi-Yokura)**.** *Let $f : X \to Y$ be a continuous map between two elliptic spaces, then*

$$\text{(RH)} \qquad dim\ Ker\ \pi_*(f)_{\mathbb{Q}} \leq dim\ Ker\ H_*(f;\mathbb{Q}) + 1,$$

where

$$Ker\ \pi_*(f)_{\mathbb{Q}} := \oplus_{i\geq 1} Ker\ (\pi_i(f)_{\mathbb{Q}} : \pi_i(X)_{\mathbb{Q}} \to \pi_i(Y)_{\mathbb{Q}})\ \ and$$
$$Ker\ H_*(f;\mathbb{Q}) := \oplus_{i\geq 0} Ker\ (H_i(f;\mathbb{Q}) : H_i(X;\mathbb{Q}) \to H_i(Y;\mathbb{Q})).$$

Note that when $Y = \{*\}$, we get

$$Ker\ \pi_*(f)_{\mathbb{Q}} = \pi_*(X)_{\mathbb{Q}} \qquad and \qquad Ker\ H_*(f;\mathbb{Q}) \oplus \mathbb{Q} = H_*(X;\mathbb{Q}).$$

So by taking their dimension, we obtain the conjecture (H). This proves that the conjecture (RH) is a generalization of the conjecture (H).

Our goal in this paper is to prove the conjecture (RH) in some new cases.
Our main results are the following:

Theorem 3. *The conjecture (RH) holds for every fibration with fiber F such that rank $\pi_{odd}(F)_{\mathbb{Q}} \leq 2$.*

Theorem 4. *Let G be a compact connected Lie group and H be a closed subgroup of G, then the fibration $H \to G \xrightarrow{f} G/H$ satisfies the conjecture (RH).*

Now we briefly summarise the content of this paper. In section 2 we recall the necessary definitions and preliminaries concerning elliptic spaces and some of their properties. We go on to discuss the model of a rational fibration and the main technical tool that we use. Section 3 is devoted to the proof of our main

results. In the begining we prove Theorem 3 using standard familiar tools from rational homotopy theory, namely the Sullivan minimal model, the homotopy Euler characteristic, the formal dimension, Halperin conjecture... . So that, we prove Theorem 4 using Sullivan minimal model of Lie group and of homogeneous spaces. Finally, we use the conjecture (H) as a condition to prove the conjecture (RH) for fibration over product of spheres. We conclude this section by an example and suggestion for further work.

2. Preliminaries

We begin by recalling the definition of Sullivan minimal model of a simply connected space. Further details can be found in the references [4] and [5]. All spaces are assumed to be simply connected. A minimal algebra is a free graded commutative algebra ΛV, for some finite type graded vector space V, togother with a differential d of degree $+1$ that is decomposable, i.e., satisfies $d : V^i \to \left(\Lambda^{\geq 2} V\right)^{i+1}$.

We assume that the minimal algebra is simply connected, i.e., that the vector space V has no generators in degree lower than 2. If $\{v_1, ..., v_n\}$ is a graded basis for V, then we write ΛV as $\Lambda(v_1, ..., v_n)$. A basis can always be chosen so that $dv_1 = 0$ and $dv_i \in \Lambda(v_1, ..., v_{i-1})$ for $i \geq 2$.

Every simply connected space with rational cohomology of finite type has a corresponding Sullivan minimal model, which is a minimal algebra that encodes the rational homotopy type of the space. In particular, if $(\Lambda V, d)$ is the Sullivan minimal model of X, there are isomorphism's:

$$H^*(\Lambda V, d) \cong H^*(X; \mathbb{Q}) \text{ as graded commutative algebras,}$$
$$Hom(V, \mathbb{Q}) \cong \Pi_*(X)_{\mathbb{Q}} \text{ as graded vector spaces.}$$

Although our results are stated and proved in purely algebraic terms, they do admit topological interpretations via this correspondance.

Therefore, we can also characterize an elliptic space in terms of its Sullivan minimal model**.** A space X with Sullivan minimal model $(\Lambda V, d)$ is elliptic if V and $H^*(\Lambda V, d)$ are both finite dimensional**.** There is a remarkable sub-class of elliptic spaces called pure spaces.

Definition 5 (Pure space/Pure Sullivan minimal model)**.** *An elliptic Sullivan minimal model $(\Lambda V, d)$ is called pure if $dV^{even} = 0$ and $dV^{odd} \subset \Lambda V^{even}$. Also, a simply connected elliptic space X is called pure if its Sullivan minimal model is so.*

An elliptic space has a variety of very nice properties. Let us briefly show them up. By the formal dimension of an elliptic space X we denote

$$fd(X) = \max\left\{k \in \mathbb{N} \text{ such that } H^k(X; \mathbb{Q}) \neq 0\right\}.$$

Also, we know from (Theorem 32.6 of [4]), that when $v_1, ..., v_n$ are the elements of a homogeneous basis of V

$$fd(X) = \sum_{|v_i| \text{ odd}} |v_i| - \sum_{|v_i| \text{ even}} (|v_i| - 1),$$

where $|v|$ denotes the degree of v. Moreover, define the homotopy Euler characteristic by

$$\chi_\pi(X) = \sum_k (-1)^k \dim \pi_k(X)_{\mathbb{Q}};$$

and the cohomology Euler characteristic by

$$\chi_c(X) = \sum_k (-1)^k \dim H^k(X; \mathbb{Q}).$$

It is well known that:

Theorem 6 (Theorem 32.10 of [4])**.** *If X is a simply connected elliptic space, then*

$$\chi_\pi(X) \leq 0 \text{ and } \chi_c(X) \geq 0.$$

Moreover, the following conditions are equivalent:

$$\chi_\pi(X) = 0 \text{ and } \chi_c(X) > 0.$$

Then the cohomology Euler characteristic of an elliptic space is always non-negative. Now, we may take the following definition:

Definition 7 (F_0-space)**.** *An elliptic space X is said to be an F_0-space if* $\chi_c(X) > 0$.

Examples 8. There are many examples of such spaces:

- Finite products of even dimensional spheres,
- Finite products of complex projective spaces,
- Homogeneous spaces G/H, where H is a closed sub-group of maximal rank of a compact connected Lie group G.

Therefore, we remark that in terms of Sullivan minimal model, the conjecture (H) can be rewritten as follows:

Conjecture 9. *If $(\Lambda V, d)$ is a simply connected elliptic Sullivan minimal model, then*

$$\text{(H)} \qquad \dim V \leq \dim H^*(\Lambda V, d)$$

This conjecture has been established in various special cases, but in general it remains open. It is known for a pure elliptic spaces [7], or hyperelliptic spaces [2], or two-stage spaces [1] and for the other cases see ([3], [8], [9], [10], [13], [14], [18]).

Let us see another version of the conjecture (RH) in the context of fibration.

Let $F \to X \xrightarrow{f} Y$ be a fibration. Recall that the KS-model for f is a short exact sequence:

$$(\Lambda W, d_Y) \xrightarrow{j} (\Lambda W \otimes \Lambda V, D) \xrightarrow{q} (\Lambda V, d_F)$$

of DGA, in which $j : (\Lambda W, d_Y) \to (\Lambda W \otimes \Lambda V, D)$ is the inclusion and $q : (\Lambda W \otimes \Lambda V, D) \to (\Lambda V, d_F)$ is the projection onto the quotient of $(\Lambda W \otimes \Lambda V, D)$ by the ideal generated by $\Lambda^+ W$. The differential D satisfies: $D(w) = d_Y(w)$ for $w \in W$ and $D(v) - d_F(v) \in \Lambda^+ W.(\Lambda W \otimes \Lambda V)$ for $v \in V$.

Here $(\Lambda W, d_Y)$ and $(\Lambda V, d_F)$ are the Sullivan minimal models for Y and F respectively (Proposition 15.5 of [4]). The DGA $(\Lambda W \otimes \Lambda V, D)$ is a Sullivan model for the total space X but is not in general minimal. As in [17], we denote by $H^*(j, D_1) : W \to H^*(W \oplus V, D_1)$, where D_1 is the linear part of D and $H^*(j) : H^*(\Lambda W, d_Y) \to H^*(\Lambda W \otimes \Lambda V, D)$, thus

$$\text{Coker } H^*(j, D_1) \cong \text{Hom}(\text{Ker } \pi_*(f)_{\mathbb{Q}}, \mathbb{Q})$$
$$\text{Coker } H^*(j) \cong \text{Hom}(\text{Ker } H_*(f; \mathbb{Q}), \mathbb{Q}).$$

Remark 10. Note that Coker $H^*(j, D_1)$ is a subspace of V and the equality holds when D is decomposable.

Hence the conjecture (RH) can also be rewritten as:

Conjecture 11. *If $(\Lambda W, d_Y) \xrightarrow{j} (\Lambda W \otimes \Lambda V, D) \xrightarrow{q} (\Lambda V, d_F)$ is the KS-model for a rational fibration of elliptic spaces, then*

(RH) $$\textit{dim Coker } H^*(j, D_1) \leq \textit{dim Coker } H^*(j) + 1$$

Next, we give an example where the conjecture (RH) is true.

Example 12. Let us consider the fibration

$$\mathbb{S}^3 \times \mathbb{S}^4 \to X \xrightarrow{f} \mathbb{CP}^2$$

given by the KS-model

$$(\Lambda(x, y), d) \xrightarrow{j} (\Lambda(x, y) \otimes \Lambda(u, v, w), D) \to (\Lambda(u, v, w), d_F),$$

where $|x| = 2$, $|y| = 5$, $|u| = 3$, $|v| = 4$, $|w| = 7$ and the non-zero differentials are given by: $Dy = x^3$, $Du = x^2$ and $Dw = v^2 + vx^2$. Hence, we have dim Coker $H^*(j, D_1) \leq 3$ and $\mathbb{Q}\{[v], [vx]\} \subset$ Coker $H^*(j)$, then

$$\dim \text{Coker } H^*(j) + 1 \geq 3 \geq \dim \text{Coker } H^*(j, D_1).$$

Thus, f satisfies the conjecture (RH).

Until now, this conjecture is affirmed for spherical fibration, for T.N.C.Z fibration whose fibre satisfies the conjecture (H) (see, [17]) and for certain reasonable cases (see, [18]).

A fibration $F \xrightarrow{i} X \to Y$ is called totally non cohomologous to zero (abbreviated T.N.C.Z) if the induced homomorphism $H^*(i) : H^*(X;\mathbb{Q}) \to H^*(F;\mathbb{Q})$ is surjective. It is equivalent to require that the Serre spectral sequence collapses at E_2-term. In this case, there is an isomorphism: $H^*(X;\mathbb{Q}) \cong H^*(F;\mathbb{Q}) \otimes H^*(Y;\mathbb{Q})$ of $H^*(Y;\mathbb{Q})$-modules. The following conjecture plays a central role in rational homotopy theory.

Conjecture 13 (Halperin)**.** *Every rational fibration with fibre an F_0-space is T.N.C.Z.*

Examples 14. This conjecture holds for large classes of F_0-spaces:

• Homogeneous spaces of the form G/H, where G is a compact connected Lie group and H is a closed subgroup of maximal rank (see, [15]),

• If the cohomolgy of the fibre has at most 3 generators (see, e.g., [11], [16]).

We end this section by some notations and conventions. In general, we use V or W to denote a positively graded rational vector space of finite type. The cohomology of a DGA (A,d) is denoted $H^*(A,d)$ or just $H^*(A)$ and let $[x] \in H^*(A,d)$ stand for the cohomolgy class of the cocycle $x \in A$.

As an overriding hypothesis, we assume that all spaces appearing in this paper are *rational simply connected elliptic spaces*.

3. Proof of our results

In this section, we give a proof of our results using essentially the algebraic form of the conjecture (RH), the theory of Sullivan minimal model and other invariants in rational homotopy theory.

We are now ready to proceed with the proof of Theorem 3.

Proof of Theorem. 3 Let $F \to X \to Y$ be a fibration with rank $\Pi_{odd}(F) \leq 2$, then the KS-model of this fibration is given as

$$(\Lambda W, d_Y) \xrightarrow{j} (\Lambda W \otimes \Lambda V, D) \xrightarrow{q} (\Lambda V, d_F)$$

with $(\Lambda W, d_Y)$ and $(\Lambda V, d_F)$ are the Sullivan minimal models for Y and F respectively. Further, as stated in the preliminaries $Hom(W,\mathbb{Q}) \cong \pi_*(F)$, the condition rank $\Pi_{odd}(F) \leq 2$ is equivalent to dim $W^{odd} \leq 2$. Our aim is to prove that the dimension of Coker $H^*(j)$ is strictly greater than the dimension of Coker $H^*(j, D_1)$. In the remainder of this proof, we use essentially x_1, x_2 for the generators of even degree and y_1, y_2 for the generators of odd degree, then we consider the following cases:

Case 1. $(\Lambda V, d_F) = (\Lambda(y_1), 0)$, so F has the rational homotopy type of $\mathbb{S}^{|y_1|}$, hence our primary fibration becomes

$$\mathbb{S}^{|y_1|} \to X \to Y,$$

then the conjecture (RH) holds from Theorem 1.1 of [17].

Case 2. $(\Lambda V, d_F) = (\Lambda(x_1, y_1), d_F)$ or $(\Lambda V, d_F) = (\Lambda(x_1, x_2, y_1, y_2), d_F)$, this implies that $\chi_\pi(F) = 0$ and since $(\Lambda V, d_F)$ is elliptic, it follows by Theorem 6 that F is an F_0-space. It is well known that this fibration satisfies Halperin conjecture (Theorem 3 of [16]). Then by Theorem 1.5 of [17] the conjecture (RH) holds.

Case 3. $(\Lambda V, d_F) = (\Lambda(y_1, y_2), 0)$, regarding the total space, since X is elliptic we have $[\mu_X] \neq 0$, where $[\mu_X]$ denotes the fundamental class of X. Thus we have immediately by degree arguments $[\mu_X] \in \text{Coker } H^*(j)$. Finally, we deduce that

$$\begin{aligned} \dim \text{Coker } H^*(j) + 1 &\geq 2 = \dim V \\ &\geq \dim \text{Coker } H^*(j, D_1) \qquad \text{(Remark 10)}. \end{aligned}$$

Case 4. $(\Lambda V, d_F) = (\Lambda(x_1, y_1, y_2), d_F)$, if there is at least an element $\omega \in \Lambda^1(x_1, y_1, y_2)$ such that $D\omega = 0$, then $[\omega]$ and $[\mu_X]$ in Coker $H^*(j)$. This implies that

$$\begin{aligned} \dim \text{Coker } H^*(j) + 1 &\geq 3 = \dim V \\ &\geq \dim \text{Coker } H^*(j, D_1) \qquad \text{(Remark 10)}. \end{aligned}$$

Now, we assume that Dx_1, Dy_1 and Dy_2 are non-zero. We distinguish two possibilities:

The first when $|y_1| < |x_1|$. For degree reasons and by ellipticity argument we write $d_F(x_1) = d_F(y_1) = 0$ and $d_F(y_2) = x_1^m$ for some natural $m \geq 2$. Hence, we can write the Sullivan minimal model of the fiber F as $(\Lambda(y_1) \otimes \Lambda(x_1, y_2), d_F)$. So, we form the following short exact sequence

$$(\Lambda W \otimes \Lambda(y_1), d) \to (\Lambda W \otimes \Lambda(y_1, x_1, y_2), D) \to \left(\Lambda(x_1, y_2), \overline{d_F}\right),$$

where $d(w) = d_Y(w)$ for $w \in W$, $d(y_1) \in \Lambda W$ and $\left(\Lambda(x_1, y_2), \overline{d_F}\right)$ be the quotient obtained by factoring out the DG ideal generated by the generator y_1. Hence as recalled above (see Examples 14) this fibration satisfies Halperin conjecture, thus

(1) $H^*(\Lambda W \otimes \Lambda(y_1, x_1, y_2), D) \cong H^*(\Lambda W \otimes \Lambda(y_1), d) \otimes H^*\left(\Lambda(x_1, y_2), \overline{d_F}\right)$.

On the other hand, the inclusion j will be defined as the composite of:

$$(\Lambda W, d_Y) \hookrightarrow (\Lambda W \otimes \Lambda(y_1), d) \hookrightarrow (\Lambda W \otimes \Lambda(y_1, x_1, y_2), D).$$

Further, on passing to cohomology and from (1), we get

$$H^*(j): H^*(\Lambda W, d_Y) \to H^*(\Lambda W \otimes \Lambda(y_1), d) \hookrightarrow H^*(\Lambda W \otimes \Lambda(y_1), d) \otimes H^*(\Lambda(x_1, y_2), \overline{d_F}).$$

Besides, it is easy to see that $[x_1] \in H^*(\Lambda(x_1, y_2), \overline{d_F})$ and $fd(X) > fd(F)$, these imply that in particular $[x_1]$ and $[\mu_X] \in \text{Coker } H^*(j)$. So

$$\dim \text{ Coker } H^*(j) + 1 \geq \dim \text{ Coker} H^*(j, D_1).$$

Second, we suppose $|x_1| < |y_1| \leq |y_2|$. Then we may write $Dx_1 = \alpha$ for $\alpha \in \Lambda W$. If $[\alpha] = 0$, i.e., there is γ in ΛW such that $D\gamma = d_Y(\gamma) = \alpha$, we obtain $[x_1 - \gamma]$ is non-zero in Coker $H^*(j)$. Hence, as above we have

$$[x_1 - \gamma] \text{ and } [\mu_X] \in \text{Coker } H^*(j).$$

We now consider $[\alpha] \neq 0$ in $H^*(Y; \mathbb{Q})$. Therefore, it is easy to verify that

$$fd(X) > 3|x_1|. \tag{2}$$

To complete this proof we will focus on $d_F(y_1)$, for this we show the following lemmas:

Lemma 15. *The conjecture (RH) holds if $d_F(y_1) = x_1^n$ for some $n \geq 2$.*

Proof. The condition on the differential implies that $Dy_1 = x_1^n + \sum_{p=0}^{n-1} a_p x_1^p$ with $a_p \in \Lambda W$ for $0 \leq p \leq n-1$. Then, applying D again

$$0 = nD(x_1)\, x_1^{n-1} + D(a_{n-1})\, x_1^{n-1} + (n-1)\, a_{n-1} D(x_1)\, x_1^{n-2} + D(\sum_{p=0}^{n-2} a_p x_1^p).$$

We obtain $x_1 + \frac{1}{n} a_{n-1}$ is non-exact D-cycle. So from (2), we have

$$\left[x_1 + \frac{1}{n} a_{n-1}\right] \text{ and } [\mu_X] \quad \text{in Coker } H^*(j).$$

We see immediately from Remark 10 dim Coker $H^*(j) + 1 \geq 3 = \dim V \geq$ dim Coker $H^*(j, D_1)$. □

Lemma 16. *The conjecture (RH) holds also when $d_F(y_1) = 0$.*

Proof. Under our assumptions ($d_F(y_1) = 0$ and F is elliptic), we may write

$$Dy_1 = \sum_{p=0}^{r} a_p x_1^p \text{ and } Dy_2 = x_1^m + \sum_{k=0}^{m-1} b_k x_1^k + \sum_{l=0}^{m-2} c_l x_1^l y_1$$

with a_p, b_k and $c_l \in \Lambda W$ for $0 \leq p \leq r$, $0 \leq k \leq m-1$ and $0 \leq l \leq m-2$.

1. Let us deal with $r \geq 2$, applying D on Dy_1 we obtain in particular $D(a_{r-1} + ra_r x_1) = 0$. Since $|a_{r-1} + ra_r x_1| < |y_1| \leq |y_2|$, we have $[a_{r-1} + ra_r x_1]$ is non-zero in $H^*(X, \mathbb{Q})$. For degree reasons we have

$$[a_{r-1} + ra_r x_1] \text{ and } [\mu_X] \text{ in Coker } H^*(j).$$

2. When $r = 0$, this means that $Dy_1 = a_0$ with $a_0 \in \Lambda W$. We can apply again the same process on Dy_2 to deduce that $[b_{m-1} + mx_1]$ is non-zero in $H^*(X, \mathbb{Q})$. From (2) and Remark 10, we conclude that

$$\dim \text{Coker } H^*(j) + 1 \geq \dim \text{Coker } H^*(j, D_1).$$

3. For $r = 1$, we write $Dy_1 = a_1 x + a_0$ with a_0 and a_1 in ΛW. Our key observation is: since $|\alpha|$ is odd, there exists $i \geq 2$ such that $\alpha^i = 0$ and $\alpha^k \neq 0$ for $1 \leq k \leq i-1$, thus $D\left(\alpha^{i-1} x_1\right) = 0$.

3-1. Next, we can consider $\left[\alpha^{i-1} x_1\right] = 0$, i.e., there exist β and β' in ΛW so that $D(\beta y_1 + \beta') = \alpha^{i-1} x_1$, then we compute $D\left(\alpha^{i-2} x_1^2 - 2\beta y_1 - 2\beta'\right) = 0$.

- If $i = 2$, we have $D\left(x_1^2 - 2\beta y_1 - 2\beta'\right) = 0$. It is possible there is ω in $\Lambda W \otimes \Lambda(x_1, y_1)$ so that $D(y_2 + \omega) = x_1^2 - 2\beta y_1 - 2\beta'$ (otherwise the result follows from (2)).

  A direct computation shows that $D(\alpha y_2 + \alpha\omega + 2\beta x_1 y_1 + 2\beta' x_1 - x_1^3) = 0$. We claim that $[\alpha y_2 + \alpha\omega + 2\beta x_1 y_1 + 2\beta' x_1 - x_1^3]$ is non-zero in $H^*(X, \mathbb{Q})$. Indeed, suppose there are γ and γ' in $\Lambda W \otimes \Lambda(x_1, y_1)$ for which $D(\gamma y_2 + \gamma') = \alpha y_2 + \alpha\omega + 2\beta x_1 y_1 + 2\beta' x_1 - x_1^3$. This implies $D\gamma = \alpha$, which contradicts the fact that $[\alpha]$ is supposed non-zero in $H^*(Y, \mathbb{Q})$. Using (2) again, we conclude that

$$\left[\alpha y_2 + \alpha\omega + 2\beta x_1 y_1 + 2\beta' x_1 - x_1^3\right] \text{ and } [\mu_X] \text{ in Coker } H^*(j).$$

- If $i \geq 3$, we see that $D(\alpha^{i-2} x_1^2 - 2\beta y_1 - 2\beta') = 0$, then we have $[\alpha^{i-2} x_1^2 - 2\beta y_1 - 2\beta'] \neq 0$. Indeed, suppose there is φ in $\Lambda W \otimes \Lambda V$ such that $D(\varphi) = \alpha^{i-2} x_1^2 - 2\beta y_1 - 2\beta'$. This implies that $D\left(\varphi - \frac{1}{3}\alpha^{i-3} x_1^3\right) = -2\beta y_1 - 2\beta'$. It is automatic that $D(\beta y_1 + \beta') = 0$, which is a contradiction. Also, we have $D(\alpha^{i-1} x_1^2) = 0$, then we distinguish two cases:

  $[\alpha^{i-1} x_1^2] \neq 0$, thus $[\alpha^{i-2} x_1^2 - 2\beta y_1 - 2\beta']$ and $[\alpha^{i-1} x_1^2]$ in Coker $H^*(j)$, as required.

  $[\alpha^{i-1} x_1^2] = 0$, i.e., there is ψ in $\Lambda W \otimes \Lambda V$ so that $D(\psi) = \alpha^{i-1} x_1^2$. By the same manner we verify that $\psi - \frac{1}{3}\alpha^{i-2} x_1^3$ is non-exact D-cycle. Consequently, we have

$$\left[\alpha^{i-2} x_1^2 - 2\beta y_1 - 2\beta'\right] \text{ and } \left[\psi - \frac{1}{3}\alpha^{i-2} x_1^3\right] \text{ in Coker } H^*(j).$$

3-2. To finish this case, we assume that $\left[\alpha^{i-1}x_1\right]$ is non-zero. We have $D\left(\alpha^{i-1}x_1^2\right)=0$, then if $\left[\alpha^{i-1}x_1^2\right]=0$ (otherwise, we obtain $\left[\alpha^{i-1}x_1\right]$ and $[\alpha^{i-1}x_1^2]$ in Coker $H^*(j)$), we use a similar argument as above to prove that $\left[\phi-\frac{1}{3}\alpha^{i-2}x_1^3\right]$ is non-zero with ϕ in $\Lambda W\otimes\Lambda^+(x_1,y_1,y_2)$ such that $D\phi=\alpha^{i-1}x_1^2$ for $i\geq 3$. For $i=2$, we have from (2) $|\mu_X|>|\alpha x_1|$. Then $[\mu_X]$ and $[\alpha x_1]$ are in Coker $H^*(j)$, which implies that dim Coker $H^*(j)+1\geq$ dim Coker $H^*(j,D_1)$ (Remark 10).

□

Now, we go on to establish the second theorem, for this purpose let us recall the definition of homogeneous spaces.

Let G be a compact connected Lie group and H a closed sub-group of G, the quotient G/H is called homogeneous spaces. These spaces form a very well-studied and interesting class of manifolds. They appear abundantly in geometry and topology. It is a classical result that the quotient G/H is an elliptic spaces. We refer the reader to [4], page 218-221 and [5], page 26 for more details.

Proof of Theorem. 4 We denote the Sullivan minimal model of the group G by $(\Lambda W,0)=(\Lambda(y_1,...,y_q),0)$ and the Sullivan minimal model of the sub-Lie group H by $(\Lambda V,0)=(\Lambda(x_1,...,x_p),0)$, where $|x_k|$ and $|y_l|$ are odd for $1\leq k\leq p$ and $1\leq l\leq q$ with $q\geq p$ (Example 2.39 of [5]). Hence a Sullivan model for the homogeneous spaces G/H is given by (Theorem 2.71 of [5])

$$\left(\Lambda\left(s^{-1}V\right)\otimes\Lambda W,d\right).$$

Here, the notation "$s^{-1}V$" stands for the suspension of V, i.e., the degrees of V are shifted by $+1$. We denote by $i:H\to G$ the canonical inclusion, it induces a map $Bi:BH\to BG$ between the classifying spaces.

The differential d in $\left(\Lambda\left(s^{-1}V\right)\otimes\Lambda W,d\right)$ is defined by $d=0$ in $s^{-1}V$ and $d(y_l)=H^*(Bi)(s^{-1}y_l)$ for $1\leq l\leq q$, where

$$H^*(Bi):H^*(BG)\to H^*(BH).$$

Now, we can consider the KS-model of f:

$$M(f):\left(\Lambda\left(s^{-1}V\right)\otimes\Lambda W,d\right)\to(\Lambda W,0)$$

sending the elements of $s^{-1}V$ to zero. On passing to cohomology we obtain

$$H^*(M(f)):H^*\left(\Lambda\left(s^{-1}V\right)\otimes\Lambda W,d\right)\to H^*(\Lambda W,0)\cong\Lambda(y_1,...,y_q).$$

Since $\left(\Lambda\left(s^{-1}V\right)\otimes\Lambda W,d\right)$ is a pure elliptic space and from degree consideration, it follows immediately

$$\text{dim Im } H^*(M(f))\leq 2^{q-p}.$$

So, we obtain

$$\begin{aligned}\text{dim Coker } H^*(M(f)) + 1 &= \text{dim } H^*(G) - \text{dim Im } H^*(M(f)) + 1 \\ &\geq 2^q - 2^{q-p} + 1 \\ &\geq p,\end{aligned}$$

which completes the proof. □

In the remainder of this section, we focus on the case in which the base is a product of spheres, in particular we prove the following:

Proposition 17. *Let $F \to X \xrightarrow{f} Y$ be a fibration, suppose that:*

(i) X satisfies the conjecture (H),

(ii) Y has the rational homotopy type of $\mathbb{S}^n$ or $\mathbb{S}^{2n} \times \mathbb{S}^{2m+k}$ for $k = 0$ or 1,

then f satisfies the conjecture (RH).

Proof. Let

$$(\Lambda W, d_Y) \xrightarrow{j} (\Lambda W \otimes \Lambda V, D) \to (\Lambda V, d_F)$$

be the KS-model of f with $(\Lambda W, d_Y)$ and $(\Lambda V, d_F)$ are the Sullivan minimal models for Y and F respectively . With the same notation as above, we must show that

$$\text{dim Coker } H^*(j) + 1 \geq \text{dim Coker } H^*(j, D_1).$$

Let us begin by $Y \simeq_{\mathbb{Q}} \mathbb{S}^{2n}$, hence its Sullivan minimal model is

$$(\Lambda W, d_Y) = (\Lambda(x_1, y_1), d),$$

where degrees and differential are described by: $|x_1| = 2n$, $|y_1| = 4n - 1$ and $dx_1 = 0$, $dy_1 = x_1^2$. Here " $\simeq_{\mathbb{Q}}$ " means having the same rational homotopy type.

Therefore, we have

$$\text{dim Im } H^*(j) \leq 2$$

this implies that

$$\begin{aligned}\text{dim Coker } H^*(j) + 1 &= \text{dim } H^*(\Lambda W \otimes \Lambda V, D) - \text{dim Im } H^*(j) + 1 \\ &\geq \text{dim } H^*(\Lambda W \otimes \Lambda V, D) - 1 \\ &\geq \text{dim } V + \text{dim } W - 1 \quad \text{(conjecture (H))} \\ &\geq \text{dim } V + 1 \\ &\geq \text{dim Coker } H^*(j, D_1).\end{aligned}$$

Now, if $Y \simeq_{\mathbb{Q}} \mathbb{S}^{2n+1}$, the Sullivan minimal model of Y takes the form $(\Lambda(y), 0)$ with $|y| = 2n + 1$. Further, we argue exactly as above to deduce our result in this case.

Next, consider the product $Y \simeq_{\mathbb{Q}} \mathbb{S}^{2n} \times \mathbb{S}^{2m}$, then we have

$$(\Lambda W, d_Y) = (\Lambda (x_1, y_1) \otimes (x_2, y_2), d),$$

with $|x_1| = 2n$, $|x_2| = 2m$, $|y_1| = 4n-1$ and $|y_2| = 4m-1$. The differential d is defined as follows: $dx_1 = dx_2 = 0$, $dy_1 = x_1^2$ and $dy_2 = x_2^2$. Consequently

$$\dim \text{Im } H^*(j) \leq 4$$

and hence

$$\begin{aligned} \dim \text{Coker } H^*(j) + 1 &\geq \dim H^*(\Lambda W \otimes \Lambda V, D) - 3 \\ &\geq \dim V + \dim W - 3 \qquad \text{(conjecture (H))} \\ &\geq \dim V + 1. \end{aligned}$$

Finally, if $Y \simeq_{\mathbb{Q}} \mathbb{S}^{2n} \times \mathbb{S}^{2m+1}$ it suffices to write the Sullivan minimal model of Y, and by the same way as above we can conclude the result. □

Example 18. Note that under (i), the condition above (ii) is sufficient but not necessary. Indeed, consider the non-trivial fibration $F \to X \xrightarrow{f} \mathbb{S}^3 \times \mathbb{S}^5$ given by the following KS-model

$$(\Lambda (x, y), 0) \to \left(\Lambda (x, y) \otimes \Lambda \left(a, b, a', b', c\right), D\right) \to \left(\Lambda \left(a, b, a', b', c\right), \bar{D}\right)$$

with deg $x = 3$, deg $y = 5$, deg $a = 2$, deg $b = 3$, deg $a' = 6$, deg $b' = 11$, deg $c = 7$, $Dx = Dy = Da = Da' = 0$, $Db = a^2$, $Db' = a'^2$ and $Dc = aa' + xy$. A careful check reveals that D defines a differential. It is easy to see that dim Coker $H^*(j, D_1) = 5$, and since $\mathbb{Q}\{[x], [y], [a], [a'], [ax], [ay], [a'x], [a'y]\} \subset H^*(\Lambda (x, y) \otimes \Lambda (a, b, a', b', c); \mathbb{Q})$, so the total space satisfies the conjecture (H).

In addition to that, we have $\mathbb{Q}\{[a], [a'], [ax], [ay], [a'x], [a'y]\} \subset$ Coker $H^*(j)$, so

$$\dim \text{Coker } H^*(j) + 1 \geq 7 \geq \dim \text{Coker } H^*(j, D_1).$$

Thus, f satisfies the conjecture (RH).

We conclude this section with a question arising from Proposition 17 in which we proved the conjecture (RH) for a particular base. It would be interesting to have a generalization of this proposition. We suggest the following as a specific question in this area.

Question 19. *Let $F \to X \to Y$ be a fibration of simply connected elliptic spaces. If X satisfies the conjecture (H), does the fibration satisfy the conjecture (RH)?*

Acknowledgements

It is our pleasure to thank Manuel Amann for various fruitful discussions. The authors are grateful to the referee for his helpful suggestions and valuable comments.

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Accepted: November 18, 2019

Adjoints of some weighted composition operators on the Fock space

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Abstract. In this paper, we find the adjoints of weighted composition operators $C_{\psi,\varphi}$ on the Fock space, when $\psi = K_c$ for $c \in \mathbb{C}$. Then, we show that $C^*_{\psi,\varphi}C_{\psi,\varphi}$ and $C^*_{\psi,\varphi} + C_{\psi,\varphi}$ commute, when $\varphi(z) = az + b$, $a, b \in \mathbb{C}$ and $|a| = 1$.
Keywords: Fock space, weighted composition operator, adjoint.

1. Introduction

For f an entire function on $\mathbb{C}$, the Fock space $\mathcal{F}^2$ is the set

$$\left\{ f : \|f\|^2 = \frac{1}{\pi} \int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2} dA(z) < \infty \right\},$$

where dA is the Lebesgue measure on $\mathbb{C}$. The Fock space is a reproducing kernel Hilbert space with inner product

$$\langle f, g \rangle = \frac{1}{\pi} \int_{\mathbb{C}} f(z)\overline{g(z)} e^{-|z|^2} dA(z)$$

and reproducing kernel function $K_w(z) = e^{\overline{w}z}$, for any $w \in \mathbb{C}$. Not that for any $w \in \mathbb{C}$, $\|K_w\| = e^{|w|^2/2}$. It is well known that the set $\{e_m : m \geq 0\}$ is an orthonormal basis for $\mathcal{F}^2$. For entire functions ψ and φ and $f \in \mathcal{F}^2$, composition operator C_φ and weighted composition operator $C_{\psi,\varphi}$ on $\mathcal{F}^2$ are given by $C_\varphi(f) = f \circ \varphi$ and $C_{\psi,\varphi}(f) = \psi f \circ \varphi$, respectively. Moreover, we define a multiplication operator $M_\psi : \mathcal{F}^2 \to \mathcal{F}^2$ by $M_\psi(f) = \psi f$, for all $f \in \mathcal{F}^2$.

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Recently many authors have worked on the weighted composition operators on the Fock space (see [1], [2], [3], [5], [6], [7] and [4]). Carswell et al. in [1] showed that C_φ is bounded on the Fock space if and only if $\varphi(z) = az+b$, where $|a| \leq 1$ and if $|a| = 1$, then $b = 0$. Ueki in [4] characterized boundedness and compactness of $C_{\psi,\varphi}$ on the Fock space. Next in [3], Le found the easier criteria for boundedness and compactness of weighted composition operators. Note that there are some interesting papers like [1] which were written on another Fock space (see [8]), but their results hold for $\mathcal{F}^2$ by the same idea. Then we use them frequently in this paper.

Suppose that H is a Hilbert space. For bounded operators A and B on H, we use the notation $[A, B] = AB - BA$ for the commutator of A and B.

Finding adjoint of an operator is one of an important tool to investigate their properties. In the second section, we obtain the adjoints of weighted composition operators $C_{\psi,\varphi}$, when $\psi = K_c$ for $c \in \mathbb{C}$. Next in Section 3, we show that $[C^*_{\psi,\varphi}C_{\psi,\varphi}, C^*_{\psi,\varphi} + C_{\psi,\varphi}] = 0$, where $\varphi(z) = az + b$ and $|a| = 1$.

2. Adjoints of some weighted composition operators

In this section, we find the adjoint of weighted composition operator $C_{\psi,\varphi}$, where $\psi = K_c$ for $c \in \mathbb{C}$. Le in [3, Theorem 2.2] showed that if $C_{\psi,\varphi}$ is bounded on $\mathcal{F}^2$, then $\varphi(z) = \varphi(0) + \lambda z$ with $|\lambda| \leq 1$. First, we assume that $|\lambda| < 1$ and find the adjoint of $C_{\psi,\varphi}$ in Theorem 2.3. Then, we suppose that $|\lambda| = 1$ and obtain the adjoint of $C_{\psi,\varphi}$ in Theorem 2.5.

Lemma 2.1. *Suppose that $\varphi(z) = a_1z+b_1$ and $\psi(z) = a_2z+b_2$, where $|a_1| \leq 1$, $|a_2| \leq 1$ and $b_1, b_2 \in \mathbb{C}$. Let C_φ and C_ψ be bounded on $\mathcal{F}^2$. Then*

$$C^*_\varphi C_\psi = M_{e^{\overline{b_1}z}} C_{a_2\overline{a_1}z+b_2}. \tag{1}$$

Proof. By [1, Lemma 2], we see that

$$\begin{aligned} C^*_\varphi C_\psi &= M_{K_{b_1}} C_{\overline{a_1}z} C_{a_2z+b_2} = M_{K_{b_1}} C_{(a_2z+b_2)\circ(\overline{a_1}z)} \\ &= M_{K_{b_1}} C_{a_2\overline{a_1}z+b_2}. \end{aligned}$$

□

In the following proposition, we show that if $\varphi(z) = az + b$, where $|a| < 1$ and ψ is a kernel function K_c, then $C_{\psi,\varphi}$ can be written by the product of the adjoint of a composition operator and another composition operator.

Proposition 2.2. *Let $\varphi(z) = az + b$ and $\psi(z) = K_c(z)$, when $|a| < 1$ and $b, c \in \mathbb{C}$. Then there is a positive integer n such that*

$$C_{\psi,\varphi} = C^*_{\frac{\overline{a}(n+1)}{n}z+c} C_{\frac{n}{n+1}z+b}.$$

Proof. Since $|a| < 1$, there is a positive integer n such that $|a + \frac{a}{n}| < 1$. Let $a_1 = \frac{\overline{a}(n+1)}{n}$ and $a_2 = \frac{n}{n+1}$, $b_1 = c$ and $b_2 = b$. Then by Lemma 2.1, $C_{\psi,\varphi} = C^*_{a_1z+b_1} C_{a_2z+b_2}$. □

In the next theorem, we find the adjoint of weighted composition operator $C_{k_c,az+b}$, where $|a|<1$ and $b,c\in\mathbb{C}$.

Theorem 2.3. *Let $\psi(z)=K_c(z)$ and $\varphi(z)=az+b$, when $|a|<1$ and $b,c\in\mathbb{C}$. Then $C^*_{\psi,\varphi}=C_{e^{\bar{b}z},\overline{a}z+c}$.*

Proof. By Proposition 2.2, there is a positive integer n such that

$$C_{\psi,\varphi}=C^*_{\frac{\overline{a}(n+1)}{n}z+c}C_{\frac{n}{n+1}z+b}.$$

Then

$$C^*_{\psi,\varphi}=C^*_{\frac{n}{n+1}z+b}C_{\frac{\overline{a}(n+1)}{n}z+c}.$$

From Equation (1), we have $C^*_{\psi,\varphi}=M_{e^{\bar{b}z}}C_{\overline{a}z+c}$ and so the result follows. □

Lemma 2.4. *Suppose that $C_{\psi,\varphi}$ is a bounded weighted composition operator on $\mathcal{F}^2$. Then*

$$C^*_{\psi,\varphi}K_w=\overline{\psi(w)}K_{\varphi(w)}.$$

Proof. Let f be an arbitrary function in $\mathcal{F}^2$. We see that

$$\begin{aligned}\langle C^*_{\psi,\varphi}K_w,f\rangle &= \langle K_w,C_{\psi,\varphi}f\rangle=\langle K_w,\psi f\circ\varphi\rangle\\ &= \overline{\psi(w)f(\varphi(w))}=\overline{\psi(w)}\langle K_{\varphi(w)},f\rangle.\end{aligned}$$

Hence $C^*_{\psi,\varphi}K_w=\overline{\psi(w)}K_{\varphi(w)}$. □

We know that for $\varphi(z)=az+b$ such that $|a|=1$, if $C_{\psi,\varphi}$ is a bounded operator on $\mathcal{F}^2$, then $\psi(z)=\psi(0)e^{-a\bar{b}z}$ (see [3, Proposition 2.1]). In the following theorem, we find the adjoints of this case of weighted composition operators.

Theorem 2.5. *Let $\varphi(z)=az+b$ and $\psi(z)=\psi(0)e^{-a\bar{b}z}$, where $|a|=1$ and $b\in\mathbb{C}$. Then $C^*_{\psi,\varphi}=C_{\overline{\psi(0)}e^{\bar{b}z},\frac{1}{a}z-\frac{b}{a}}$.*

Proof. By Lemma 2.4, we can see that

$$C^*_{\psi,\varphi}K_w=\overline{\psi(w)}K_{\varphi(w)}=\overline{\psi(0)}e^{-\overline{a}b\overline{w}}K_{aw+b}=\overline{\psi(0)}e^{-\overline{a}b\overline{w}}e^{\overline{(aw+b)}z}$$

and

$$C_{\overline{\psi(0)}e^{\bar{b}z},\frac{1}{a}z-\frac{b}{a}}K_w=\overline{\psi(0)}e^{\bar{b}z}e^{\overline{w}(\frac{1}{a}z-\frac{b}{a})}=\overline{\psi(0)}e^{\bar{b}z}e^{\frac{\overline{w}}{a}z-\frac{b}{a}\overline{w}}=\overline{\psi(0)}e^{-\overline{a}b\overline{w}}e^{\overline{(aw+b)}z}.$$

Since the span of the kernel functions is dense in $\mathcal{F}^2$, the result follows. □

Example 2.6. (a) Let $\varphi(z)=\frac{z}{2}+i$ and $\psi(z)=e^{iz}$. Then, Theorem 2.3 implies that $C^*_{\psi,\varphi}=C_{e^{-iz},\frac{z}{2}+i}$.

(b) Let $\varphi(z)=iz-\frac{\sqrt{2}}{2}i$ and $\psi(z)=e^{\frac{\sqrt{2}}{2}z}$. We infer from Theorem 2.5 that $C^*_{\psi,\varphi}=C_{e^{\frac{\sqrt{2}}{2}iz},\bar{i}z+\frac{\sqrt{2}}{2}}$.

3. Weighted composition operators for which $C^*_{\psi,\varphi}C_{\psi,\varphi}$ and $C^*_{\psi,\varphi}+C_{\psi,\varphi}$ commute

In this section, we show that $[C^*_{\psi,\varphi}C_{\psi,\varphi}, C^*_{\psi,\varphi}+C_{\psi,\varphi}]=0$, where $\varphi(z)=az+b$ and $|a|=1$ and $C_{\psi,\varphi}$ is bounded on $\mathcal{F}^2$.

Theorem 3.1. *Let $\varphi(z)=az+b$ and $|a|=1$. Suppose that $C_{\psi,\varphi}$ is bounded on $\mathcal{F}^2$. Then $[C^*_{\psi,\varphi}C_{\psi,\varphi}, C^*_{\psi,\varphi}+C_{\psi,\varphi}]=0$.*

Proof. From [3, Proposition 2.1], $\psi(z)=\psi(0)e^{-a\bar{b}z}$. By Theorem 2.5, we see that

$$
\begin{aligned}
&[C^*_{\psi,\varphi}C_{\psi,\varphi}, C^*_{\psi,\varphi}+C_{\psi,\varphi}]\\
&= C^*_{\psi,\varphi}C_{\psi,\varphi}C^*_{\psi,\varphi}+C^*_{\psi,\varphi}C_{\psi,\varphi}C_{\psi,\varphi}-C^*_{\psi,\varphi}C^*_{\psi,\varphi}C_{\psi,\varphi}-C_{\psi,\varphi}C^*_{\psi,\varphi}C_{\psi,\varphi}\\
&= C_{\overline{\psi(0)}e^{\bar{b}z},\frac{1}{a}z-\frac{b}{a}}C_{\psi(0)e^{-a\bar{b}z},az+b}C_{\overline{\psi(0)}e^{\bar{b}z},\frac{1}{a}z-\frac{b}{a}}\\
&+ C_{\overline{\psi(0)}e^{\bar{b}z},\frac{1}{a}z-\frac{b}{a}}C_{\psi(0)e^{-a\bar{b}z},az+b}C_{\psi(0)e^{-a\bar{b}z},az+b}\\
&- C_{\overline{\psi(0)}e^{\bar{b}z},\frac{1}{a}z-\frac{b}{a}}C_{\overline{\psi(0)}e^{\bar{b}z},\frac{1}{a}z-\frac{b}{a}}C_{\psi(0)e^{-a\bar{b}z},az+b}\\
&- C_{\psi(0)e^{-a\bar{b}z},az+b}C_{\overline{\psi(0)}e^{\bar{b}z},\frac{1}{a}z-\frac{b}{a}}C_{\psi(0)e^{-a\bar{b}z},az+b}\\
&= \overline{\psi(0)}\psi(0)M_{e^{\bar{b}z}}M_{e^{-a\bar{b}(\frac{1}{a}z-\frac{b}{a})}}C_{a(\frac{1}{a}z-\frac{b}{a})+b}C_{\overline{\psi(0)}e^{\bar{b}z},\frac{1}{a}z-\frac{b}{a}}\\
&+ \overline{\psi(0)}\psi(0)M_{e^{\bar{b}z}}M_{e^{-a\bar{b}(\frac{1}{a}z-\frac{b}{a})}}C_{a(\frac{1}{a}z-\frac{b}{a})+b}C_{\psi(0)e^{-a\bar{b}z},az+b}\\
&- \overline{\psi(0)}\psi(0)C_{\overline{\psi(0)}e^{\bar{b}z},\frac{1}{a}z-\frac{b}{a}}M_{e^{\bar{b}z}}M_{e^{-a\bar{b}(\frac{1}{a}z-\frac{b}{a})}}C_{a(\frac{1}{a}z-\frac{b}{a})+b}\\
&- \overline{\psi(0)}\psi(0)C_{\psi(0)e^{-a\bar{b}z},az+b}M_{e^{\bar{b}z}}M_{e^{-a\bar{b}(\frac{1}{a}z-\frac{b}{a})}}C_{a(\frac{1}{a}z-\frac{b}{a})+b}\\
&= |\psi(0)|^2e^{|b|^2}C_zC_{\overline{\psi(0)}e^{\bar{b}z},\frac{1}{a}z-\frac{b}{a}}\\
&+ |\psi(0)|^2e^{|b|^2}C_zC_{\psi(0)e^{-a\bar{b}z},az+b}\\
&- |\psi(0)|^2e^{|b|^2}C_{\overline{\psi(0)}e^{\bar{b}z},\frac{1}{a}z-\frac{b}{a}}C_z\\
&- |\psi(0)|^2e^{|b|^2}C_{\psi(0)e^{-a\bar{b}z},az+b}C_z
\end{aligned}
$$

and so obviously $[C^*_{\psi,\varphi}C_{\psi,\varphi}, C^*_{\psi,\varphi}+C_{\psi,\varphi}]=0$. □

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Accepted: 25.11.2019

Some more algebraic hyperstructures in chemistry and biochemistry

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Abstract. In classical group theory, composition of two elements yields one element only. This theory, definitely is not applicable in the study of reactants in the blood system and chlorination where composition of two elements can yield more than one.

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Many researchers have studied a lot about finite hyperstructures. But, in this paper, it is shown that chlorination of methane is an infinite hyperstructure. This motivates reason for more studies on algebraic properties of infinite hyperstructures rather than the finite ones.

Besides, it shows that the interaction between the reactants in the blood system and chemical structure of ozone layer satisfy the axioms of some kind of hyperstructures.
Keywords: infinite hyperstructure, hypergroup, H_ν-group, ozone layer.

1. Introduction

The theory of hyperstructures began in 1934 when Marty [9], at the 8th Congress of Scandinavian Mathematicians, illustrated the definition of hypergroups and gave some applications. Since then, many researchers have been exploring the concept in different directions. Hyperstructure is a kind of generalization of the concept of algebraic structure.

Some of the researchers working on hyperstructures are Corsini [3], Davvaz [5], Vougiouklis [14], Leoreanu [8], etc. They have done very extensive studies in the theory of hyperstructures and their applications.

It is important to note that Davvaz has identified some genetic structures [6] and chemical structures [5] which can be studied as hyperstructures. As a matter of fact, Davvaz in [5] pointed out that the chain reactions are hyperstructures. More details on occurrence of hyperstructures in physical science and some algebraic properties can be found in [11] and [13] respectively.

In this paper, we first show that the chlorination of methane is an infinite hyperstructure. Then, from blood buffer solution and ozone layer, we identify some hyperstructures which are different from ones of Davvaz [5, 6]. In particular, the ozone layer loses its chemical nature when there is interference from human and ultra-violet race, which causes serious change in the weather condition as a lot of heat is generated. This is supported by the fact that the ozone layer loses its algebraic structure from these interferences.

The results presented in this paper can be applied to study some global warming conditions as it mirrors the physical situation. More studies on the algebraic properties of these hyperstructures can give insight into the physical situations which they represent.

2. Preliminaries

Definition 2.1 ([3])**.** *Let H be a non empty set. The operation $\circ : H \times H \to \mathcal{P}^*(H)$ is called a hyperoperation and $(H, \circ)$ is called a hypergroupoid, where $\mathcal{P}^*(H)$ is the collection of all non empty subsets of H.*

For $A, B \subseteq H$,

$$A \circ B = \cup\{a \circ b | a \in A, b \in B\},$$

where the notations $a \circ A$ and $A \circ a$ are used for $\{a\} \circ A$ and $A \circ \{a\}$ respectively.

Definition 2.2 ([3]). *A hypergroupoid $(H, \circ)$ is called a semihypergroup if*

$$a \circ (b \circ c) = (a \circ b) \circ c, \text{ for all } a, b, c \in H \qquad (Associativity)$$

Definition 2.3 ([3]). *A hypergroupoid $(H, \circ)$ is called a quasihypergroup if*

$$H \circ a = a \circ H = H, \text{ for all } a \in H \qquad (Reproduction\ Axiom)$$

Definition 2.4 ([3]). *A hypergroupoid $(H, \circ)$ is said to be a hypergroup if it is both a semihypergroupoid and quasihypergroupoid.*

Example 2.1 ([4]). The following are some hypergroups:

(i) Let H be a non empty set and for all $x, y \in H$, define $x \circ y = H$. Then H is a hypergroup called the total hypergroup.

(ii) If $(G, \cdot)$ is a group, H is a normal subgroup of G and for all $x, y \in G$, we define $x \circ y = xyH$, then $(G, \circ)$ is a hypergroup.

Lemma 2.1 ([4]). *A hypergroup for which the hyperproduct of any two elements has exactly one element is a group. Indeed, let $(H, \circ)$ be a hypergroup, such that for all x, y of H , we have $| x \circ y | = 1$. Then, $(H, \circ)$ is a semigroup, such that for all a, b in H, there exist x and y for which we have $a = b \circ x$ and $a = y \circ b$. It follows that $(H, \circ)$ is a group.*

Definition 2.5 ([14]). *A hypergroupoid $(H, \circ)$ is called a H_ν-group if it satisfies*

(i) $(a \circ b) \circ \ c \ \cap \ a \circ (b \circ c) \neq \emptyset, \quad \forall a, b, c \in H$ *(Weak Associativity)*

(ii) $a \circ H = H = H \circ a$ *(Reproduction Axiom)*

Example 2.2. The radiolytic processes involved in the Portland cement is already studied and established as an algebraic hyperstructure. Though details can be seen in [1, 7] but here is the summary. In the following, all elements and compounds with the negative charge $(^-)$ are radicals, H_2O is water, H_2 is hydrogen gas and OH is hydroxyl (an alcohol). Let

$$\begin{aligned} \mathcal{H} \ = \ & \{e^-_{aq}, H_2, OH^-, H, O^-, OH, H_2O, HO_2^-, H_2O_2, O_2^-, O_2^{2-}, \\ & HO_2, O_2, O_3^-, O_3, O^{2-}\} \end{aligned}$$

such that $e^-_{aq} = H_2O^-$. Let the hyperoperation $\oplus$ be the chemical reaction between these elements and compounds. Then, $(\mathcal{H}, \oplus)$ is an H_ν-semigroup.

Example 2.3 ([5]). Furthermore, $Cu(0), Cu(I), Cu(II)$ and $Cu(III)$ are the four oxidation states of copper. Its different species can react with themselves (without energy) as defined below:

(i) $Cu^{3+} + Cu^+ \mapsto 2Cu^{2+}$

(ii) $Cu^{3+} + Cu \mapsto Cu^{2+} + Cu^+$

$\circ$	Cu	Cu^+	Cu^{2+}	Cu^{3+}
Cu	Cu	Cu, Cu^+	Cu, Cu^{2+}	Cu^+, Cu^{2+}
Cu^+	Cu, Cu^+	Cu^+	Cu^+, Cu^{2+}	Cu^{2+}
Cu^{2+}	Cu, Cu^{2+}	Cu^+, Cu^{2+}	Cu^{2+}	Cu^{2+}, Cu^{3+}
Cu^{3+}	Cu^+, Cu^{2+}	Cu^{2+}	Cu^{2+}, Cu^{3+}	Cu^{3+}

Table 1: Redox (Oxidation-Reduction) reaction of Copper

$\circ$	Cu	Cu^+	Cu^{2+}
Cu	Cu	Cu, Cu^+	Cu, Cu^{2+}
Cu^+	Cu, Cu^+	Cu^+	Cu^+, Cu^{2+}
Cu^{2+}	Cu, Cu^{2+}	Cu^+, Cu^{2+}	Cu^{2+}

Table 2: Another Redox reaction of Cu

The product of these reactions are in Table 1.

It can be seen that, letting $G = \{Cu, Cu^+, Cu^{2+}, Cu^{3+}\}$, $(G, \circ)$ is weak associative and

$$Cu^+ \circ G = G \circ Cu^+ \neq G.$$

Thus, $(G, \circ)$ is an H_ν-semigroup. However, $\{Cu, Cu^+\}$, $\{Cu, Cu^{2+}\}$ $\{Cu^+, Cu^{2+}\}$ and $\{Cu^{2+}, Cu^{3+}\}$ are hypergroups with respect to $\circ$.

From Table 2, it can be seen that $(\{Cu, Cu^+, Cu^{2+}\}, \circ)$ is a hypergroup.

It should be noted that $\{Cu, Cu^+\}$, $\{Cu, Cu^{2+}\}$ and $\{Cu^+, Cu^{2+}\}$ are sub-hypergroups of $(\{Cu, Cu^+, Cu^{2+}\}, \circ)$.

Example 2.4 ([12])**.** Let

$$G = \{AA, AS, SS\}$$

and the hyperoperation $\oplus$ denotes mating. The blood genotype is a kind of hyperstructure.

In Table 3, $\{AA\} \oplus G \neq G \neq G \oplus \{AA\}$; the *reproduction axiom* is not satisfied. Also, it is weak associative. It is an H_ν-semigroup.

$\oplus$	AA	AS	SS
AA	AA	AA, AS	AS
AS	AA, AS	AA, AS, SS	AS, SS
SS	AS	AS, SS	SS

Table 3: Genotype as a Hyperstructure

$\otimes$	O	A	B	AB
O	O	O, A	O, B	A, B
A	O, A	O, A	O, A, B	A, B, AB
B	O, B	O, A, B, AB	O, B	A, B, AB
AB	A, B	A, B, AB	A, B, AB	A, B, AB

Table 4: Blood Group as a Hyperstructure

Example 2.5 ([6])**.** Considering $H = \{O, A, B, AB\}$, from Table 4, $(H, \otimes)$ is an H_ν-semigroup. It is weak associative and reproduction axiom is not satisfied. This is because

$$(O \otimes B) \otimes AB = \{O, B\} \otimes AB = \{A, B, AB\},$$

$$O \otimes (B \otimes AB) = O \otimes \{A, B, AB\} = \{O, A, B\}.$$

Hence, $(O \otimes B) \otimes AB \neq O \otimes (B \otimes AB)$. Also $(O \otimes B) \otimes AB \bigcap O \otimes (B \otimes AB) \neq \emptyset$.

But if $H_0 = \{O, A\}$ and $H_1 = \{O, B\}$, then $(H_0, \otimes)$ and $(H_1, \otimes)$ are hypergroups.

3. Some results

3.1 What is a Buffer solution?

A Buffer solution is a solution of weak acid (HA) and its conjugate base (A^-). A Buffer solution resists changes in pH (the level of basicity/acidity in a solution) and would rather maintain its stability. They will maintain nearly constant pH level when small amounts of acid or base are added to them because the weak acid can donate a proton to any HO^- added to the solution and its conjugate base can accept any H^+ that is added to the solution. This is seen in the following chemical reaction:

$$HA + HO^- \longrightarrow A^- + H_2O,$$

$$A^- + H_2O \longrightarrow HA + HO^-.$$

More details on buffer solution can be found in [10].

3.2 Buffer solution in blood

During physical exercise, the body metabolism speeds up, producing large amounts of CO_2. Hence, the increased concentration of CO_2 shifts the equilibrium between carbonic acid and bicarbonate to the right, which increases the concentration of H^+. Significant amounts of lactic acid are also produced. This further

increases the concentration of H^+. Receptors in the brain respond to the increased concentration of H^+ and triggers a reflex that increases the rate of breathing. Haemoglobin then releases more oxygen to the cells to eliminate more CO_2 by exhalation. Both processes decrease the concentration of H^+ in the blood by shifting both equilibria to the left.

Thus, any disorder that decreases the rate and depth of ventilation, such as emphysema, will decrease the pH of the blood – a condition called acidosis. In contrast, any excessive increase in the rate and depth of ventilation, as with hyperventilation due to anxiety, will increase the pH of blood – a condition called alkalosis.

Blood is the fluid that transports oxygen to all the cells of the human body. The normal pH of human blood is 7.3 to 7.4. Death will result if this pH decreases to a value which is less than 6.8 or increases to a value which is greater than 8.0 for even a few seconds. The protein in the blood called haemoglobin HbH^+ transports oxygen to the cells. When haemoglobin binds oxygen O_2, haemoglobin losses a proton, which would make the blood more acidic if it does not contain a buffer to maintain its pH value.

Carbonic acid/bicarbonic H_2CO_3/HCO_3^- acts as the acid buffer in the blood. This helps to control the pH value of the blood. An important feature of this buffer is that carbonic acid decomposes to CO_2 and H_2O.

Reaction Equations: The following is a summary of buffer reaction in blood. More information on blood buffer solutions can also be obtained from [2].

$$HbH^+ + O_2 \rightleftarrows HbO_2 + H^+,$$

$$CO_2 + H_2O \rightleftarrows H_2CO_3 \rightleftarrows HCO_3^- + H^+.$$

3.3 Blood solution as an H_ν-group

Let the set of reactants in the blood be

$$\begin{aligned} B_0 &= \{HbH^+, O_2, CO_2, H^+, H_2O, HbO_2, H_2CO_3, HCO_3^-\},\\ B_1 &= \{HbH^+, O_2, H^+, H_2O, HbO_2\},\\ B_2 &= \{HbH^+, O_2, H^+, HbO_2\},\\ B_3 &= \{CO_2, H^+, H_2O, H_2CO_3, HCO_3^-\} \end{aligned}$$

and operation $\oplus$ is the chemical reaction.

Theorem 3.1. *$(B_0, \oplus)$ is a hypergroup.*

Proof. This is obvious from Table 5. □

Theorem 3.2. *$(B_1, \oplus), (B_2, \oplus)$ and $(B_3, \oplus)$ are subhypergroups of $(B_0, \oplus)$.*

Proof. The case of $(B_1, \oplus)$ is obvious from Table 6. Since B_2 is a subhypergroup of B_1 in Table 6, it follows that B_2 is a subhpergroup of B_0. Also, the case of B_3 is obvious from Table 5. □

$\oplus$	HbH^+	O_2	CO_2	H^+	H_2O	HbO_2	H_2CO_3	HCO_3^-
HbH^+	HbH^+	HbH^+, O_2, H^+, HbO_2	HbH^+, CO_2	HbH^+, H^+	HbH^+, H_2O	HbH^+, HbO_2	HbH^+, CO_2, H^+, H_2O, H_2CO_3, HCO_3^-	HbH^+, HCO_3^-
O_2	HbH^+, O_2, H^+, HbO_2	O_2	O_2, CO_2	O_2, H^+	O_2, H_2O	O_2, HbO_2	O_2, CO_2, H^+, H_2O, H_2CO_3, HCO_3^-	O_2, HCO_3^-
CO_2	HbH^+, CO_2	O_2, CO_2	CO_2	CO_2, H^+	CO_2, H^+, H_2O, H_2CO_3, HCO_3^-	CO_2, HbO_2	CO_2, H^+, H_2O, H_2CO_3, HCO_3^-	CO_2, HCO_3^-
H^+	HbH^+, H^+	O_2, H^+	CO_2, H^+	H^+	H^+, H_2O	HbH^+, O_2, H^+, HbO_2	CO_2, H^+, H_2O, H_2CO_3, HCO_3^-	CO_2, H^+, H_2O, H_2CO_3, HCO_3^-
H_2O	HbH^+, H_2O	O_2, H_2O	CO_2, H^+, H_2O, H_2CO_3, HCO_3^-	H^+, H_2O	H_2O	H_2O, HbO_2	CO_2, H^+, H_2O, H_2CO_3, HCO_3^-	H_2O, HCO_3^-
HbO_2	HbH^+, HbO_2	O_2, HbO_2	CO_2, HbO_2	HbH^+, O_2, H^+, HbO_2	H_2O, HbO_2	HbO_2	HbH^+, O_2, CO_2, H^+, H_2O, HbO_2, H_2CO_3, HCO_3^-	HbO_2, HCO_3^-
H_2CO_3	HbH^+, CO_2, H^+, H_2O, H_2CO_3, HCO_3^-	O_2, CO_2, H^+, H_2O, H_2CO_3, HCO_3^-	CO_2, H^+, H_2O, H_2CO_3, HCO_3^-	CO_2, H^+, H_2O, H_2CO_3, HCO_3^-	CO_2, H^+, H_2O, H_2CO_3, HCO_3^-	HbH^+, O_2, CO_2, H^+, H_2O, HbO_2, H_2CO_3, HCO_3^-	CO_2, H^+, H_2O, H_2CO_3, HCO_3^-	CO_2, H^+, H_2O, H_2CO_3, HCO_3^-
HCO_3^-	HbH^+, HCO_3^-	O_2, HCO_3^-	CO_2, HCO_3^-	CO_2, H^+, H_2O, H_2CO_3, HCO_3^-	H_2O, HCO_3^-	HbO_2, HCO_3^-	CO_2, H^+, H_2O, H_2CO_3, HCO_3^-	HCO_3^-

Table 5: Haemoglobin Reaction I

$\oplus$	HbH^+	O_2	H^+	H_2O	HbO_2
HbH^+	HbH^+	HbH^+, O_2, H^+, HbO_2	HbH^+, H^+	HbH^+, H_2O	HbH^+, HbO_2
O_2	HbH^+, O_2, H^+, HbO_2	O_2	O_2, H^+	O_2, H_2O	O_2, HbO_2
H^+	HbH^+, H^+	O_2, H^+	H^+	H^+, H_2O	HbH^+, O_2, H^+, HbO_2
H_2O	HbH^+, H_2O	O_2, H_2O	H^+, H_2O	H_2O	H_2O, HbO_2
HbO_2	HbH^+, HbO_2	O_2, HbO_2	HbH^+, O_2, H^+, HbO_2	H_2O, HbO_2	HbO_2

Table 6: Haemoglobin Reaction II

3.4 Chlorination of Alkanes is an infinite hypergroup

Alkanes react with chlorine (Cl_2) to form alkyl chlorides or alkyl bromides. These reactions take place only at high temperatures or in the presence of light as shown below.

$$CH_4 + Cl_2 \longrightarrow CH_3Cl + HCl.$$

They are the only reactions that alkanes undergo without a metal catalyst being involved. This is with the exception of combustion, a reaction with oxygen that takes place at high temperatures and converts alkanes to carbon dioxide and water.

Mechanism of Mono Chlorination of Methane: During chemical reactions, bond-breaking can occur through heterolytic bond cleavage (when both electrons from the bonds stay with one of the atoms) and hemolytic bond cleavage (when each of the atoms retains one of the bonding electrons). All alkane compounds follows the mechanisms below:

Chain Initiation: The chain reaction is initiated by ultraviolet light breaking a chlorine molecule into free radicals as in

$$Cl_2 \longrightarrow 2Cl^{\bullet}.$$

Chain Propagation: These reactions keep the chain going as shown below.

$$CH_4 + Cl^{\bullet} \longrightarrow CH_3^{\bullet} + HCl,$$
$$CH_3^{\bullet} + Cl_2 \longrightarrow CH_3Cl + Cl^{\bullet}.$$

Chain Termination: These reactions remove free radicals from the system without replacing them with new ones as

$$2Cl^{\bullet} \longrightarrow Cl_2,$$
$$CH_3^{\bullet} + Cl^{\bullet} \longrightarrow CH_3Cl,$$
$$CH_3^{\bullet} + CH_3^{\bullet} \longrightarrow C_2H_6.$$

Let $A = \{Cl_2, Cl^{\bullet}, HCl, \{C_nH_{2n+2}\}, \{C_nH_{2n+1}^{\bullet}\}, \{C_nH_{2n+1}Cl\}\}$, where $n \geq 1$, then $(A, \oplus)$ is an infinite hypergroup as can be seen from Table 7.

$\oplus$	Cl_2	$Cl^\bullet$	HCl	$\{C_nH_{2n+2}\}_{n\geqslant 1}$	$\{C_nH^\bullet_{2n+1}\}_{n\geqslant 1}$	$\{C_nH_{2n+1}Cl\}_{n\geqslant 1}$
Cl_2	Cl_2, $Cl^\bullet$	Cl_2, $Cl^\bullet$	Cl_2, $Cl^\bullet$, HCl	Cl_2, $Cl^\bullet$, HCl, $\{C_nH_{2n+2}\}_{n\geqslant 2}$, $\{C_nH^\bullet_{2n+1}\}_{n\geqslant 1}$, $\{C_nH_{2n+1}Cl\}_{n\geqslant 1}$	Cl_2, $Cl^\bullet$, $\{C_nH^\bullet_{2n+1}\}_{n\geqslant 1}$, $\{C_nH_{2n+1}Cl\}_{n\geqslant 1}$	Cl_2, $Cl^\bullet$, $\{C_nH_{2n+1}Cl\}_{n\geqslant 1}$
$Cl^\bullet$	Cl_2, $Cl^\bullet$	Cl_2, $Cl^\bullet$	Cl_2, $Cl^\bullet$, HCl	Cl_2, $Cl^\bullet$, HCl, $\{C_nH^\bullet_{2n+1}\}_{n\geqslant 1}$, $\{C_nH_{2n+2}\}_{n\geqslant 2}$, $\{C_nH_{2n+1}Cl\}_{n\geqslant 1}$	Cl_2, $Cl^\bullet$, HCl, $\{C_nH^\bullet_{2n+1}\}_{n\geqslant 1}$, $\{C_nH_{2n+2}\}_{n\geqslant 2}$, $\{C_nH_{2n+1}Cl\}_{n\geqslant 1}$	Cl_2, $Cl^\bullet$, $\{C_nH_{2n+1}Cl\}_{n\geqslant 1}$
HCl	Cl_2, $Cl^\bullet$, HCl	Cl_2, $Cl^\bullet$, HCl	HCl	HCl, $\{C_nH_{2n+2}\}_{n\geqslant 2}$	HCl, $\{C_nH^\bullet_{2n+1}\}_{n\geqslant 1}$, $\{C_nH_{2n+2}\}_{n\geqslant 2}$	HCl, $\{C_nH_{2n+1}Cl\}_{n\geqslant 1}$
$\{C_nH_{2n+2}\}_{n\geqslant 1}$	Cl_2, $Cl^\bullet$, HCl, $\{C_nH_{2n+2}\}_{n\geqslant 2}$, $\{C_nH^\bullet_{2n+1}\}_{n\geqslant 1}$, $\{C_nH_{2n+1}Cl\}_{n\geqslant 1}$	Cl_2, $Cl^\bullet$, HCl, $\{C_nH^\bullet_{2n+1}\}_{n\geqslant 1}$, $\{C_nH_{2n+2}\}_{n\geqslant 2}$, $\{C_nH_{2n+1}Cl\}_{n\geqslant 1}$	HCl, $\{C_nH_{2n+2}\}_{n\geqslant 2}$	$\{C_nH_{2n+2}\}_{n\geqslant 2}$	$\{C_nH^\bullet_{2n+1}\}_{n\geqslant 1}$, $\{C_nH_{2n+2}\}_{n\geqslant 2}$	$\{C_nH_{2n+2}\}_{n\geqslant 1}$, $\{C_nH_{2n+1}Cl\}_{n\geqslant 1}$
$\{C_nH^\bullet_{2n+1}\}_{n\geqslant 1}$	Cl_2, $Cl^\bullet$, $\{C_nH^\bullet_{2n+1}\}_{n\geqslant 1}$, $\{C_nH_{2n+1}Cl\}_{n\geqslant 1}$	Cl_2, $Cl^\bullet$, HCl, $\{C_nH^\bullet_{2n+1}\}_{n\geqslant 1}$, $\{C_nH_{2n+2}\}_{n\geqslant 2}$, $\{C_nH_{2n+1}Cl\}_{n\geqslant 1}$	HCl, $\{C_nH^\bullet_{2n+1}\}_{n\geqslant 1}$, $\{C_nH_{2n+2}\}_{n\geqslant 2}$	$\{C_nH^\bullet_{2n+1}\}_{n\geqslant 1}$, $\{C_nH_{2n+2}\}_{n\geqslant 2}$	$\{C_nH_{2n+2}\}_{n\geqslant 2}$	$\{C_nH^\bullet_{2n+1}\}_{n\geqslant 1}$, $\{C_{2n}H_{2n+2}\}_{n\geqslant 2}$, $\{C_nH_{2n+1}Cl\}_{n\geqslant 1}$
$\{C_nH_{2n+1}Cl\}_{n\geqslant 1}$	Cl_2, $Cl^\bullet$, $\{C_nH_{2n+1}Cl\}_{n\geqslant 1}$	Cl_2, $Cl^\bullet$, $\{C_nH_{2n+1}Cl\}_{n\geqslant 1}$	HCl, $\{C_nH_{2n+1}Cl\}_{n\geqslant 1}$	$\{C_nH_{2n+2}\}_{n\geqslant 1}$, $\{C_nH_{2n+1}Cl\}_{n\geqslant 1}$	$\{C_nH^\bullet_{2n+1}\}_{n\geqslant 1}$, $\{C_nH_{2n+2}\}_{n\geqslant 2}$, $\{C_nH_{2n+1}Cl\}_{n\geqslant 1}$	$\{C_nH_{2n+1}Cl\}_{n\geqslant 1}$

Table 7: Chlorination of Alkanes

$\star$	O_2	O_3
O_2	O_2	O_2, O_3
O_3	O_2, O_3	O_3

Table 8: Hyperstructure in ozone layer when no external interference

4. Chemical reactions in the ozone layer as hyperstructures

The ozone layer, in the stratosphere (upper layer) of the atmosphere, acts as the sunshade which protects the Earth from many ultraviolet (UV) rays that may be harmful to it. It comprises mainly oxygen molecules O_3 and O_2. Its rapid depletion in the recent time has been the concern of many scientists, including mathematicians. Two main things have been found to possibly affect the ozone layer: the effect of the ultraviolet rays from the Sun and the growing industrial activities on Earth.

This section shows that the reaction between all molecules in the ozone layer when neither human nor ultraviolet rays interfere is a H_ν-group.

4.1 Chemistry of Ozone Layer

(i) $O_2 + UV \rightarrow O^\bullet + O^\bullet$,

(ii) $O_2 + O^\bullet \rightleftharpoons O_3$,

(iii) $O_3 + UV \rightarrow O_2 + O^\bullet$,

(iv) $Cl + O_3 \rightarrow ClO + O_2$,

(v) $ClO + O^\bullet \rightarrow Cl + O_2$,

(vi) $ClO + O_3 \rightarrow Cl + 2O_2$.

Let $S_1 = \{O_2, O_3\}$, $S_2 = \{O^\bullet, O_2, O_3\}$, and $S_3 = \{O^\bullet, O_2, O_3, Cl, ClO\}$, when neither UV nor human interfere with the ozone layer, the following table is obtained from the chemistry of ozone layer and the binary operation $\star$ (as shown in Table 8) is the chemical reaction between the molecules.

If the ozone layer of the Earth were to be shielded from the UV rays due to some other planet acting as obstacles and no human interferes, the following condition is obtainable.

Proposition 4.1. $(S_1, \star)$ *is a hypergroup when no external factor interferes.*

Proof. The proof can be obtained from Table 8. □

$\star_{uv-h}$	O_2	O_3	$O^\bullet$	Cl	ClO
O_2	$O_2, O_3, O^\bullet$	$O_2, O_3, O^\bullet$	$O_2, O_3, O^\bullet$	O_2, O_3, $O^\bullet$, Cl, ClO	O_2, O_3, $O^\bullet$, Cl, ClO
O_3	$O_2, O_3, O^\bullet$	$O_2, O_3, O^\bullet$	$O_2, O_3, O^\bullet$	O_2, O_3, $O^\bullet$, Cl, ClO	O_2, O_3, $O^\bullet$, Cl, ClO
$O^\bullet$	$O_2, O_3, O^\bullet$	$O_2, O_3, O^\bullet$	$O^\bullet$	$O^\bullet, Cl$	O_2, O_3, $O^\bullet$, Cl, ClO
Cl	O_2, O_3, $O^\bullet$, Cl, ClO	O_2, O_3, $O^\bullet$, Cl, ClO	$O^\bullet, Cl$	Cl	Cl, ClO
ClO	O_2, O_3, $O^\bullet$, Cl, ClO	O_2, O_3, $O^\bullet$, Cl, ClO	O_2, O_3, $O^\bullet$, Cl, ClO	Cl, ClO	ClO

Table 9: Hyperstructure in Ozone Layer

However, nature is not without some interactions between its various systems. The dynamism changes when either UV rays or humans interfere, which is the real case.

The case of only human interfering is rarely possible and might not be worth considering. Next is the case when the activity of the ozone layer is interfered by both human and UV rays. Take $\star_{uv-h}$ as the operation when either UV rays and/or humans interfere.

From Table 9 we can get the following propositions.

Proposition 4.2. *In the event that the UV rays interfere in ozone layer,* $(S_1, \star_{uv-h})$ *is not closed.*

In practical sense, the ozone layer has lost its chemical nature and there is serious change in the weather condition as a lot of heat is generated in this case and global warming effect is obtained.

Proposition 4.3. $(S_2, \star_{uv-h})$, *is a hypergroup.*

Proposition 4.4. $(S_3, \star_{uv-h})$, *is a hypergroup.*

Although $(S_2, \star_{uv-h})$ and $(S_3, \star_{uv-h})$ are hypergroups, there are still environmental issues here because $(S_1, \star_{uv-h}) \subset (S_2, \star_{uv-h}) \subset (S_3, \star_{uv-h})$ and there is alteration in the chemical nature of the ozone layer. It is now weak in its concentration of O_3. Note that $(S_2, \star_{uv-h})$ is a subhypergroup of $(S_3, \star_{uv-h})$.

5. Conclusions

The study has motivated reason for more studies on the theory of infinite hyperstructures. The paper has also shown that the blood buffer solution can be modelled with hypergroup theory. The paper has also established, as some earlier works have, that the study of hyperstructures is relevant to many aspects of physical, chemical and biological sciences.

Acknowledgments

This work is supported by Foundation of Chongqing Municipal Key Laboratory of Institutions of Higher Education ([2017]3), Foundation of Chongqing Development and Reform Commission (2017[1007]), and Foundation of Chongqing Three Gorges University.

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Accepted: 9.12.2019

Eventually constant intertwining linear maps between complete locally convex spaces

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Abstract. Starting from Sinclair's 1976 work [6] on automatic continuity of linear operators on Banach spaces, we prove that sequences of intertwining continuous linear maps are eventually constant with respect to the separating space of a fixed linear map. Our proof uses a gliding hump argument. We also consider aspects of continuity of linear functions between locally convex spaces and prove that such a linear function T from the locally convex space X to the locally convex space Y is continuous whenever the

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separating space $G(T)$ is the zero vector in Y and for which X and Y satisfy conditions for a closed graph theorem.
Keywords: separating space, intertwining, gliding hump.

1. Introduction

From A. Sinclair's fundamental work [6], one of the main result with respect to linear maps between Banach spaces is Lemma 1.6 ([6]; p. 11), which gives conditions under which a sequence of intertwining linear maps is eventually constant:

Lemma ([6]). *Suppose X and Y are Banach spaces and let (T_n) and (R_n) be sequences of continuous linear operators on X and Y, respectively. If S is a linear operator from X to Y that satisfies $ST_n = R_nS$ for all $n \in \mathbb{N}$, then there is an integer $N \in \mathbb{N}$ such that $\forall n \geq N$,*

$$\overline{R_1 \cdots R_n \mathcal{G}} = \overline{R_1 \cdots R_N \mathcal{G}},$$

where $\mathcal{G}$ is the separating space of S:

$$\mathcal{G} = \{y \in Y : \text{there is a sequence } (x_n) \to 0 \text{ in } X \ni (Sx_n) \to y \text{ in } Y\}.$$

The proof utilizes a gliding hump argument. See [7] for example. The importance of the result on eventually constant sequences of intertwining maps is in its application to results on automatic continuity. Some instances include results on automatic continuity in [6] for Banach spaces (see also the references therein), in [3], for a class of locally convex spaces, and within the context of generalized local linear operators, and in [8], in the context of convex bornological spaces where the results are given with regard to bounded linear maps. In this paper, we define the separating space for general locally convex spaces and prove that Sinclair's result holds in for sequences of intertwining linear maps between complete locally convex spaces. This opens the door to applications in general locally convex spaces for which only completeness is needed. Like in the proof of the Banach space result, our proof utilizes a gliding hump argument. Similar generalizations to complete locally convex spaces or related spaces have been obtained recently, such as in [1], and [5].

The notation here is as follows. Let (X, τ) be a Hausdorff locally convex topological space (lcs) over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Denote by $\{\rho_j\}_J$ a family of seminorms that generate τ on X. Let $Z_{\rho_j} = \{x \in X : \rho_j(x) = 0\}$. We denote the zero vector of X by $\hat{0}$. Observe that Z_{ρ_j} is a closed linear subspace of X and $\bigcap_J Z_{\rho_j} = \{\hat{0}\}$. We denote a linear subspace M of a linear space L by $M < L$.

Let (Y, σ) be another lcs, with $H = \{\sigma_i\}_I$ a family of seminorms generating σ. Denote by Z_{σ_i} the corresponding σ- closed linear subspace of Y at which $\sigma_i = 0$ in Y. We denote the zero of Y by $\tilde{0}$.

Let $T : (X, \tau) \to (Y, \sigma)$ be a linear map, not necessarily continuous, and notice that $T(Z_{\rho_j}) < Z_{\sigma_i}$ need not hold.

As necessary, we may also include a third lcs (W, λ), with corresponding family of seminorms $\{\lambda_k\}_K$ that generates λ.

All other unspecified notation is standard, e.g., as in [2].

The definition below, inspired by [6], page 8, is defined for locally convex spaces, using seminorms.

Definition 1.1. *For each $j \in J$ and each $i \in I$, let*

$$G_{ji}(T) = \{y \in Y : \exists \{x_n\}_{\mathbb{N}} \overset{\rho_j}{\to} \hat{0} \in X \text{ and } \{Tx_n\}_{\mathbb{N}} \overset{\sigma_i}{\to} y\}.$$

Then, define

$$G(T) = \bigcap_{J \times I} G_{ji}(T) = \{y \in Y : \exists \{x_n\}_{\mathbb{N}} \overset{\tau}{\to} \hat{0} \in X \text{ and } \{Tx_n\}_{\mathbb{N}} \overset{\sigma}{\to} y\}.$$

We have $G_{ji}(\alpha T) = \alpha G_{ji}(T)$, $\forall \alpha \in \mathbb{K}$.

2. Preliminary results

Proposition 2.1. *For each $(j, i) \in J \times I$ $G_{ji}(T)$ satisfies:*

1. *$G_{ji}(T)$ is a linear subspace of Y.*
2. *$G_{ji}(T)$ is σ_i-closed.*
3. *$G_{ji}(T)$ is σ-closed.*
4. *$Z_{\sigma_i} < G_{ji}(T)$.*
5. *$T(Z_{\rho_j}) < G_{ji}(T)$.*

Proof. 1. Let $y, y' \in G_{ji}(T)$ and $\lambda \in \mathbb{K}$. There exist sequences $\{x_{k_n}\}_{n\in\mathbb{N}} \overset{\rho_j}{\to} \hat{0}$ and $\{x'_n\}_{\mathbb{N}} \overset{\rho_j}{\to} \hat{0}$ in X such that $\{T(x_n)\}_{\mathbb{N}} \overset{\sigma_i}{\to} y$ and $\{T(x'_n)\}_{\mathbb{N}} \overset{\sigma_i}{\to} y'$. From this $\{x_n + \lambda x'_n\}_{\mathbb{N}} \overset{\rho_j}{\to} \hat{0}$ and $\{T(x_n + \lambda x'_n)\}_{\mathbb{N}} \overset{\sigma_i}{\to} y + \lambda y'$. Hence, $y + \lambda y' \in G_{ji}(T)$.

2. Let $\{y_k\}_{\mathbb{N}} \subset G_{ji}(T)$ such that $\{y_k\}_{\mathbb{N}} \overset{\sigma_i}{\to} y \in Y$. For each y_k there exists $\{x_{kn}\}_{n\in\mathbb{N}} \overset{\rho_j}{\to} \hat{0}$ and $\{T(x_{kn})\}_{n\in\mathbb{N}} \overset{\sigma_i}{\to} y_k$ as $n \to \infty$. For each $\epsilon_k = \frac{1}{k}$ let $N_k \in \mathbb{N}$ such that $\rho_j(x_{kn}) < \epsilon_k$ and $\sigma_i(T(x_{kn}) - y) < \epsilon_k, \forall n \geq N_k$. Let $\{x_k = x_{kN_k}\} \subset X$.

Claim 1. $\{x_k\}_{\mathbb{N}} \overset{\rho_j}{\to} \hat{0}$. To see this, let $\epsilon > 0$ and consider $K \in \mathbb{N}$ such that $\frac{1}{K} < \epsilon$: $\rho_j(x_{kN_k}) < \frac{1}{K} < \epsilon$ if $k \geq K$.

Claim 2. $\{T(x_k)\}_{\mathbb{N}} \overset{\sigma_i}{\to} y$. For this, let $\epsilon > 0$ and $K \in \mathbb{N}$ such that $\frac{1}{K} < \frac{\epsilon}{2}$. Then $\forall k \geq K$:

$$\sigma_i(T(x_k) - y) \leq \sigma_i(T(x_k) - y_k) + \sigma_i(y_k - y) < \frac{1}{K} + \frac{\epsilon}{2} < \epsilon.$$

Thus, $y \in G_{ji}(T)$, which proves G_{ji} is sequentially σ_i-closed, and we also conclude σ_i-closed.

3. Consider the net $\{y_\lambda\}_\Lambda \subset G_{ji}(T)$ such that $\{y_\lambda\}_\Lambda \xrightarrow{\sigma} y \in Y$. Because the topology generated by σ_i is contained in the topology σ, we have $\{y_\lambda\}_\Lambda \xrightarrow{\sigma_i} y \in Y$. By item 2, $y \in G_{ji}(T)$.

4. Let $y \in Z_{\sigma_i}$. Then $\sigma_i(y) = 0$. Take the sequence $\{\hat{0}, \hat{0}, \hat{0}, ...\}$, in X, which converges to $\hat{0}$. We show that $\{T(\hat{0}), T(\hat{0}), T(\hat{0}), ...\} = \{\tilde{0}, \tilde{0}, \tilde{0}, ...\} \xrightarrow{\sigma_i} y$. Indeed, $\sigma_i(T(\hat{0}) - y) \leq \sigma_i(\tilde{0}) + \sigma_i(y) = 0$. It follows that $y \in G_{ji}(T)$.

5. Let $x \in Z_{\rho_j}$. Then we have $\rho_j(x) = 0$. This time, we take the sequence $\{x, x, x, ...\}$ in X, which is ρ_j-convergent to $\hat{0}$. Put $y{=}T(x)$. Then $\{T(x), T(x), T(x), ...\} \xrightarrow{\sigma_i} y$. We conclude that $T(x) = y \in G_{ji}(T)$. □

We observe that, given a linear $T : X \to Y$ that need not be continuous, the subspaces $T(Z_{\rho_j})$ and Z_{σ_i} have a nontrivial intersection. This occurs despite the fact that $T(Z_{\rho_j})$ might not be a subspace of Z_{σ_i}. On the other hand, not every linear map will send the linear subspace Z_{ρ_j} of the domain into the subspace Z_{σ_i} of the codomain. However, item 5 in the above proposition is somewhat surprising because it indicates that all linear maps will send Z_{ρ_j} into $G_{ji}(T)$, even though the general size of $Z_{\sigma_i} < G_{ji}(T)$ is not much larger. We will see shortly that when these two subspaces coincide, we can conclude that T is in fact, continuous.

Corollary 2.1. *$G(T)$ is a σ-closed subspace of Y.*

Proposition 2.2. *Suppose $T : (X, \rho_j) \to (Y, \sigma_i)$ is continuous. Then $G_{ji}(T) = Z_{\sigma_i}$.*

Proof. Proposition 2.1-4 tells us that $Z_{\sigma_i} < G_{ji}(T)$ always holds. The continuity of T implies the existence of $c > 0$ such that$\sigma_i(T(x)) \leq c\rho_j(x)\ \forall x \in X$.

Let $y \in G_{ji}(T)$ and let $\{x_n\}_\mathbb{N} \xrightarrow{\rho_j} \hat{0} \in X$ such that $\{T(x_n)\}_\mathbb{N} \xrightarrow{\sigma_i} y \in Y$. $\{T(x_n)\}_\mathbb{N} \xrightarrow{\sigma_i} \tilde{0}$. These two convergences imply that $y = y - \tilde{0} \in Z_{\sigma_i}$, which proves that $G_{ji}(T) < Z_{\sigma_i}$. □

Proposition 2.3. *Let $T : (X, \rho_j) \to (Y, \sigma_i)$ be linear and such that $G_{ji}(T) = Z_{\sigma_i}$ with the spaces (X, ρ_j) and (Y, σ_i) being complete. Then T is continuous.*

Proof. We apply the closed graph theorem to the corresponding complete seminormed spaces. Let $\{x_n\}_\mathbb{N} \xrightarrow{\rho_j} x \in X$ and $\{T(x_n)\}_\mathbb{N} \xrightarrow{\sigma_i} y \in Y$. Then $\{x_n - x\}_\mathbb{N} \xrightarrow{\rho_j} \hat{0} \in X$ and $\{T(x_n - x)\}_\mathbb{N} \xrightarrow{\sigma_i} y - T(x) \in Y$. That is, $y - T(x) \in G_{ji}(T) = Z_{\sigma_i}$, from which we obtain $y \in T(x) + Z_{\sigma_i}$, and this tells us that the graph of T is closed in $(X, \rho_j) \times (Y, \sigma_i)$. By the closed graph theorem,$T : (X, \rho_j) \to (Y, \sigma_i)$ is continuous. □

Corollary 2.2. *1. If $T : (X, \tau) \to (Y, \sigma)$ is linear and continuous, then for each σ_i there is a ρ_j such that $G_{ji}(T) = Z_{\sigma_i}$.*

2. If for each σ_i the seminormed space (Y, σ_i) is complete and if there exists ρ_j such that $G_{ji}(T) = Z_{\sigma_i}$ and (X, ρ_j) is also complete, then T is continuous.

Proof. 1. The continuity of T implies that for each seminorm, σ_i from the family that generates σ there is a corresponding seminorm $\rho_j \in \tau$ such that $T : (X, \rho_j) \to (Y, \sigma_i)$ is continuous. We apply Proposition 2.3 to obtain the conclusion. For item 2, we directly apply Proposition 2.3. □

Corollary 2.3. *For any linear $T : (X, \tau) \to (Y, \sigma)$, with (X, τ) and (Y, σ) complete, one has $G(T) = \{\tilde{0}\}$*

Proof. $\{\tilde{0}\} < G(T) = \bigcap_{J\times I} G_{ji}(T) < \bigcap_{i\in I} Z_{\sigma_i} = \{\tilde{0}\}$. □

Observe that if $T : (X, \rho_{j_o}) \to (Y, \sigma_{i_o})$ is continuous, Proposition 2.3 tells us that $G_{j_o i_o} = Z_{\sigma_{i_o}}$. On the other hand,

$$Z_{\sigma_{i_o}} = G_{j_o i_o}(T) = \bigcap_{j\in J} G_{j i_o}(T) = \bigcap_{i\in I} G_{j_o i}(T) < G_{j_o i_o}(T).$$

□

We conclude that the above subspaces are equal.

3. A continuity result

The main result of this section is that, under our constructions, we can conclude the continuity of $T : (X, \tau) \to (Y, \sigma)$ whenever the closed graph theorem applies. This generalizes Sinclair [6] to large classes of locally convex spaces, such as Pták spaces, webbed spaces, and so on. See [4].

Theorem 3.1. *If $G(T) = \{\tilde{0}\}$ and the lcs (X, ρ_j) and (Y, σ_i) are complete for each $(j, i) \in J \times I$, then the graph of $T : (X, \tau) \to (Y, \sigma)$ is $(\tau \times \sigma)$- closed in $X \times Y$. In this case a linear $T : (X, \tau) \to (Y, \sigma)$ is continuous whenever the lcs (X, τ) and (Y, σ) satisfy the assumptions of a closed graph theorem.*

Proof. Let Δ be a directed set and let $\{(x_\alpha, T(x_\alpha))\}_\Delta \overset{\tau\times\sigma}{\to} (x, y) \in X \times Y$. That is, $\{x_\alpha\}_\Delta \overset{\tau}{\to} x$ and $\{T(x_\alpha)\}_\Delta \overset{\sigma}{\to} y$. Let $(j, i) \in J \times I$ be fixed, but arbitrary. Given that the topologies induced by the seminorms ρ_j y σ_i in X and Y, respectively, are weaker than τ y σ (respectively), we have that $\{x_\alpha\}_\Delta \overset{\rho_j}{\to} x$ and $\{T(x_\alpha)\}_\Delta \overset{\sigma_i}{\to} y$.

Nevertheless, we need convergence in terms of sequences. For each $n \in \mathbb{N}$, put $\epsilon_n = \frac{1}{n}$. Thus, there exist α'_n and α''_n in Δ such that $(\forall \alpha \geq \alpha'_n)\rho_j(x_\alpha - x) < \epsilon_n$, and $(\forall \alpha \geq \alpha''_n \sigma_i)\ T((x_\alpha) - y) < \epsilon_n$.

Let $\alpha_1 \in \Delta$ such that $\alpha_1 \geq \alpha'_1, \alpha''_1$. Then let $\alpha_2 \in \Delta$ such that $\alpha_2 \geq \alpha'_2, \alpha''_2, \alpha_1$. In general, we have $\alpha_n \in \Delta$ such that $\alpha_n \geq \alpha'_n, \alpha''_n, \alpha_{n-1}$.

We now have two sequences in place of subnets, namely, $\{x_{\alpha_n}\}_\mathbb{N} \overset{\rho_j}{\to} x \in X$ and $\{T(x_{\alpha_n})\}_\mathbb{N} \overset{\sigma_i}{\to} y \in Y$.

Following the proof of Proposition 2.3 we obtain $y - T(x) \in G_{ji}(T)$, though not necessarily in Z_{σ_i}. By the above, $y - T(x) \in \bigcap_{J\times I} G_{ji}(T) = G(T) = \{\tilde{0}\}$,

which implies $y = T(x)$, and therefore, (x, y) belongs to the graph of T in $X \times Y$. □

The following is an example of how this result applies to generalize a result like [6], Lemma 1.3 - (i), page 8:

Corollary 3.1. *Suppose $G(T) = \{\tilde{0}\}$ and the lcs (X, ρ_j) $(Y, \sigma_i))$ are complete for each $(j, i) \in J \times I$. If the spaces (X, τ), (Y, σ) are, respectively, barrelled and Pták spaces, then the linear map $T : (X, \tau) \to (Y, \sigma)$ is continuous.*

Proof. See [2] 14.9.1, page 327. □

4. Eventually constant intertwining linear maps

Our goal in this section is to generalize Sinclair's fundamental result [6], Lemma 1.6, regarding eventually constant intertwining linear maps between Banach spaces, to linear maps between complete, locally convex spaces. We need some preliminary results as we build from the ground up using seminorms.

Proposition 4.1. *Let $R : (Y, \sigma_i) \to (W, \lambda_k)$, be linear and continuous. If there is $j \in J$ such that $R(G_{ji}(T)) < Z_{\lambda_k}$, then $R \circ T : (X, \rho_j) \to (W, \lambda_k)$ is continuous.*

Comment: The requirement that $R(G_{ji}(T)) < Z_{\lambda_k}$ would make for a trivial result, since it would force the null sequences in the domain to satisfy that their images under the composition $R \circ T$, to converge to the zero in the codomain, which of course, would be continuity. The fact that $\{T(x_n)\}_{\mathbb{N}}$ does not always converge in Y is what makes this result interesting, though it also makes for more work to prove it.

Proof. We will use Proposition 2.3, proving that $G_{jk}(R \circ T) = Z_{\lambda_k}$. By part 4 of Proposition 2.1, $Z_{\lambda_k} < G_{jk}(R \circ T)$.

Let $w \in G_{jk}(R \circ T) < W$ and let $\{x_n\}_{\mathbb{N}} \overset{\rho_j}{\to} \hat{0} \in X$ such that $\{(R \circ T)(x_n)\}_{\mathbb{N}} \overset{\lambda_k}{\to} w \in Y$. If $\{T(x_n)\}_{\mathbb{N}} \overset{\sigma_i}{\to} \tilde{0}$, then $w \in G_{ik} = Z_{\lambda_k}$.

If $\{T(x_n)\}_{\mathbb{N}} \overset{\sigma_i}{\to} y \neq \tilde{0}$, then $y \in G_{ji}(T)$, from which we have $R(y) \in R(G_{ji}(T)) \in Z_{\lambda_k}$. Thus, $\{R(T(x_n) - y)\}_{\mathbb{N}} \overset{\lambda_k}{\to} w - R(y)$. From this, $w - R(y) \in G_{ik}(R) = Z_{\lambda_k}$ and $w \in R(y) + Z_{\lambda_k} = Z_{\lambda_k}$. Let $\eta : (Y, \sigma_i) \to (Y/G_{ji}(T), \hat{\sigma}_i)$ be the canonical epimorphism. Let $\hat{R} : (Y/G_{ji}(T), \hat{\sigma}_i) \to (W, \lambda_k)$ under the association $\hat{R}(\eta(y)) = R(y)$. This function is well defined, and by hypothesis, $R(G_{ji}(T)) < Z_{\lambda_k} = G_{ik}(R)$.

Claim 1. $\hat{R}$ is continuous; that is, $G_{ik}(\hat{R}) = Z_{\lambda_k}$.

Let $w \in G_{ik}(\hat{R})$ and let $\{\eta(y_n)\}_{\mathbb{N}} \overset{\hat{\sigma}_i}{\to} \eta(\tilde{0}) \in Y/G_{ji}(T)$ such that $\{R(y_n)\}_{\mathbb{N}} = \{\hat{R}(\eta(y_n))\}_{\mathbb{N}} \overset{\lambda_k}{\to} w \in W$. Then there exists $\{y'_n\}_{\mathbb{N}} \subset Y$ such that $\eta(y'_n) = \eta(y_n)$ y $\{y'_n\}_{\mathbb{N}} \overset{\sigma_i}{\to} \tilde{0}$. From this, we have $\{\hat{R}(\eta(y_n))\}_{\mathbb{N}} = \{\hat{R}(\eta(y'_n))\}_{\mathbb{N}} = \{R(y'_n)\}_{\mathbb{N}} \overset{\lambda_k}{\to} 0 \in W$. These two convergences imply that $w \in Z_{\lambda_k}$.

Claim 2. $\eta\circ T : (X,\rho_j) \to (Y/G_{ji}(T), \hat{\sigma}_i)$ is also continuous; that is, $G_{ji}(\eta\circ T) = Z_{\lambda_k}$.

Let $\eta(y) \in G_{ji}(\eta \circ T)$ and let $\{x_n\}_{\mathbb{N}} \overset{\rho_j}{\to} \hat{0} \in X$ such that $\{(\eta \circ T)(x_n)\}_{\mathbb{N}} \overset{\hat{\sigma}_i}{\to} \eta(y) \in Y/G_{ji}(T)$. We now have $\{\eta(T(x_n) - y)\}_{\mathbb{N}} \to \eta(\tilde{0})$, in other words, $\hat{\sigma}_i(\eta(T(x_n) - y)) \to 0$. Thus, for each $n \in \mathbb{N}$ there is $y_n \in G_{ji}(T)$ such that $\hat{\sigma}_i(\eta(T(x_n) - y)) \leq \sigma_i(T(x_n) - y - y_n) < \hat{\sigma}_i(\eta(T(x_n) - y)) + \frac{1}{n}$.
$y_n \in G_{ji}(T) \Rightarrow \eta(T(x_n) - y_n) = \eta(T(x_n))$. This means that, given $\epsilon > 0$, there exists $N \in \mathbb{N}$ for which $\forall n \geq N$:

$$\hat{\sigma}_i(\eta(T(x_n) - y)) \leq \sigma_i(T(x_n) - y - y_n) < \hat{\sigma}_i(\eta(T(x_n) - y)) + \frac{1}{n} < \frac{\epsilon}{2} + \frac{\epsilon}{2}.$$

From this, $\{T(x_n) - y_n\}_{\mathbb{N}} \overset{\sigma_i}{\to} y$. On the other hand, for each $y_n \in G_{ji}(T)$ there exists $\{x_{nm}\}_{m\in\mathbb{N}} \overset{\rho_j}{\to} \hat{0} \in X$, as $m \to \infty$, with $\{T(x_{nm})\}_{m\in\mathbb{N}} \overset{\sigma_i}{\to} y_n$. For each $n \in \mathbb{N}$ there exists m_n such that $\rho_j(x_{nm}) < \frac{1}{n}$ and

$$\sigma_i(T(x_{nm}) - y_n) < \frac{1}{n}, \forall m \geq m_n.$$

We thus have, $\{x_n - x_{nm_n}\}_{n\in\mathbb{N}} \to \hat{0}$ and

$$\begin{aligned} \sigma_i(T(x_n - x_{nm_n}) - y) &= \sigma_i(T(x_n) - T(x_{nm_n}) - y) \\ &\leq \sigma_i(T(x_n) - y_n - y) + \sigma_i(y_n - T(x_{nm_m})) \\ &\to 0. \end{aligned}$$

The above tells us that $y \in G_{ji}(T)$, and therefore, $\eta(y) = \eta(\tilde{0}) = Z_{\lambda_k}$.

Having shown that the functions $\eta \circ T$ and $\hat{R}$ are continuous, their composition $\hat{R} \circ \eta \circ T$ is of course, also continuous. Because $\hat{R} \circ \eta \circ T = R \circ T$, $R \circ T$ is continuous, which is what we intended to prove. □

Remark 4.1. If the linear function $T : (X,\rho_j) \to (Y,\sigma_i)$ were continuous, the number M_{ji} would exist, where $0 \leq M_{ji} = \inf\{\sigma_i(Tx)|\rho_j(x) \leq 1\}$ which satisfies $\sigma_i(T(x)) \leq M_{ji}\rho_j(x)$ for each $x \in X$. Analogously, if the linear function $R : (Y,\sigma_i) \to (W,\lambda_k)$ is continuous, then $0 \leq M_{ik} = \inf\{\lambda_k(R(y))|\sigma_i(y) \leq 1\}$ exists such that $\lambda_k(R(y)) \leq M_{ik}\sigma_i(y)$ for each $y \in Y$. We conclude from this that for the composition $(R \circ T) : (X,\rho_j) \to (W,\lambda_k)$ we have $\lambda_k((R \circ T)(x)) \leq M_{ik}M_{ji}\rho_j(x)$ for each $x \in X$.

We can ask the question:

What happens if $R \circ T$ is continuous, but T is not, as in the context of Proposition 4.1? We will see what this tells us in the next proposition, after the following technical lemma.

Lemma 4.1.

$$\begin{aligned} M_{ik} &= \inf\{\lambda_k(R(y))|\sigma_i(y) \leq 1\} \\ &= M_{\hat{i}k} \\ &= \inf\{\lambda_k(R(y + G_{ji}(T))|\hat{\sigma}_i(y + G_{ji}(T)) \leq 1\}. \end{aligned}$$

Proof. From Proposition 4.1 we have that the function $\hat{R} : (Y/G_{ji}(T), \hat{\sigma}_i) \to (W, \lambda_k)$, $\hat{R}(y + G_{ji}(T)) = R(y)$, is continuous. Hence, there exists a number $M_{\hat{i}k} \geq 0$ for which we have

$$\lambda_k(R(y)) = \lambda_k(\hat{R}(y + G_{ji}(T))) \leq M_{\hat{i}k}\hat{\sigma}_i(y + G_{ji}(T)) \leq M_{\hat{i}k}\sigma_i(y),$$

$\forall y \in Y$. The previous statement tells us that $M_{ik} \leq M_{\hat{i}k}$, where

$$M_{ik} = \inf\{\lambda_k(R(y))|\sigma_i(y) \leq 1\}.$$

We recall that $\hat{\sigma}_i(y + G_{ji}(T)) = \inf\{\sigma_i(y + z)|z \in G_{ji}(T)\} \leq \sigma_i(y)$. By supposing $M_{ik} < M_{\hat{i}k}$ we have that there exists $y_1 \in Y$ with $\sigma_i(y_1) \leq 1$ such that $M_{ik} \leq \lambda_k(R(y_1)) < M_{\hat{i}k}$.

Nevertheless, $\lambda_k(R(y_1)) = \lambda_k(\hat{R}(y_1+G_{ji}(T))$ and $\hat{\sigma}_i(y_1+G_{ji}(T)) \leq \sigma_i(y_1) \leq 1$, that is, $M_{\hat{i}k} \leq \lambda_k(R(y_1)) < M_{\hat{i}k}$. This contradiction shows that $M_{ik} = M_{\hat{i}k}$.□

Here is the proposition alluded to a short time ago:

Proposition 4.2. *Suppose the linear function $R : (Y, \sigma_i) \to (W, \lambda_k)$ is continuous, such that $(R \circ T) : (X, \rho_j) \to (W, \lambda_k)$is also continuous. Suppose additionally, that $R(G_{ji}(T)) < Z_{\lambda_k}$. Then there exists a number $M > 0$, independent of the the function R and of the vector space W, such that $\lambda_k((R \circ T)(x)) \leq M_{ik}M\rho_j(x)$ for each $x \in X$.*

Proof. Based on what we did in Proposition 4.1, we have that $R \circ T = \hat{R} \circ \eta \circ T$, where $\eta \circ T : (X, \rho_j) \to (Y/G_{ji}(T), \hat{\sigma}_i)$ is a linear, continuous function, as well as $\hat{R} : (Y/G_{ji}(T), \hat{\sigma}_i) \to (W, \lambda_k)$. By the continuity of $\eta \circ T$, there is a number $M_{j\hat{i}} \geq 0$ such that

$$\hat{\sigma}_i((\eta \circ T)(x)) \leq M_{j\hat{i}}\rho_j(x) \ \ (\forall x \in X).$$

Finally, $(\forall x \in X)$,

$$\lambda_k((R \circ T)(x)) = \lambda_k((\hat{R} \circ \eta \circ T)(x)) \leq M_{ik}\hat{\sigma}_i((\eta \circ T)(x)) \leq M_{ik}M_{j\hat{i}}\rho_j(x),$$

Choose $M = M_{j\hat{i}}$ and observe that this number is independent of the function R as well as the vector space W. □

Corollary 4.1. *Suppose the function $R : (Y, \sigma) \to (W, \lambda)$ is continuous, such that $(R \circ T) : (X, \tau) \to (W, \lambda)$is also continuous. Moreover, because for each $\lambda_k \in \lambda$ there exists $\sigma_i \in \sigma$ for which $R : (Y, \sigma_i) \to (W, \lambda_k)$ is continuous, we can have that for this $\sigma_i \in \sigma$ there will exist $\rho_j \in \tau$ such that $R(G_{ji}(T)) < Z_{\lambda_k}$. Then, $(R \circ T) : (X, \tau) \to (W, \lambda)$ is continuous.*

The next five results inform us about the vector subspaces of W : $G_{jk}(R \circ t)$, $G_{ik}(R)$ and $R(G_{ji}(T))$.

Corollary 4.2. *Let $R : (Y, \sigma) \to (W, \lambda)$ be linear and continuous, that is, for each $\lambda_k \in \lambda$there exists $\sigma_i \in \sigma$ such that $R : (Y, \sigma_i) \to (W, \lambda_k)$ is continuous. If for this seminorm σ_i there is $j \in J$ for which $R(G_{ji}(T)) < Z_{\lambda_k}$, then $R \circ T : (X, \tau) \to (W, \lambda)$ is continuous.*

Proposition 4.3. *Let $R : (Y, \sigma_i) \to (W, \lambda_k)$ be linear and continuous. Then $R(G_{ji}(T)) < G_{ik}(R) = Z_{\lambda_k}$ and $\overline{R(G_{ji}(T))}^{\lambda_k} < G_{ik}(R \circ T)$ for every $j \in J$.*

Proof. Consider $j \in J$ and $T : (X, \rho_j) \to (Y, \sigma_i)$, linear, but not necessarily continuous. By the continuity of R, Proposition 2.3 tells us that $G_{ik}(R) = Z_{\lambda_k}$. Let $y \in G_{ji}(T)$ and let $\{x_n\}_{\mathbb{N}} \overset{\rho_j}{\to} \hat{0} \in X$ such that $\{T(x_n)\}_{\mathbb{N}} \overset{\sigma_i}{\to} y \in Y$. Then, $\{R(T(x_n))\}_{\mathbb{N}} \overset{\lambda_k}{\to} R(y)$ and $(R \circ T)(x_n)\}_{\mathbb{N}} \overset{\lambda_k}{\to} 0 \in W$. These two convergences imply two things. The first is that $R(y) \in Z_{\lambda_k} = G_{ik}(R)$ and the other is $R(y) \in G_{jk}(R \circ T)$. From the first one we conclude that $R(G_{ji}(T)) < G_{ik}(R)$, and the second tells us that $R(G_{ji}(T)) < G_{jk}(R \circ T)$. From this, $\overline{R(G_{ji}(T))}^{\lambda_k} < \overline{G_{jk}(R \circ T)}^{\lambda_k} = Gjk(R \circ T)$. □

Corollary 4.3. *Suppose $R : (Y, \sigma_i) \to (W, \lambda_k)$ is linear and continuous, and $T : (X, \rho_j) \to (Y, \sigma_i)$ is linear, such that the composition $(R \circ T) : (X, \rho_j) \to (W, \lambda_k)$ is continuous. Then $G_{ik}(R) = Z_{\lambda_k} = G_{jk}(R \circ T)$.*

Corollary 4.4. *Let $R : (Y, \sigma) \to (W, \lambda)$ be linear and continuous. If $R \circ T : (X, \tau) \to (W, \lambda)$ is continuous, then for each $\lambda_k \in \lambda$ there exists $j \in J$ and $i \in I$ for which $\overline{R(G_{ji}(T))}^{\lambda_k} < G_{ik}(R) = Z_{\lambda_k} = G_{jk}(R \circ T)$.*

Proposition 4.4. *Let $R : (Y, \sigma_i) \to (W, \lambda_k)$ be linear and continuous. Then*

$$\overline{R(G_{ji}(T))}^{\lambda_k} = G_{jk}(R \circ T).$$

Proof. Let $y \in G_{ji}(T)$ and let $\{x_n\}_{\mathbb{N}} \overset{\rho_j}{\to} \hat{0} \in X$ such that

$$\{T(x_n)\}_{\mathbb{N}} \overset{\sigma_i}{\to} y \in Y.$$

By the continuity of R, $\{R(T(x_n))\}_{\mathbb{N}} \overset{\lambda_k}{\to} R(y)$, from which $R(y) \in G_{jk}(R \circ T)$. In other words, $R(G_{ji}(T)) < G_{jk}(R \circ T)$ and because $G_{jk}(R \circ T)$ is λ_k- closed, we have $\overline{R(G_{ji}(T))}^{\lambda_k} < G_{jk}(R \circ T)$. By Proposition 2.4, $R \circ T : (X, \rho_j) \to (W, \lambda_k)$ is continuous.

Now, we prove the other contention: By the continuity of $R \circ T$, we have $Z_{\lambda_k} = G_{jk}(R \circ T)$. Consider the canonical epimorphism $\eta : W \to (W/\overline{R(G_{ji}(T))}, \hat{\lambda}_k)$, which happens to be continuous. That continuity implies that $\eta \circ R : (Y, \sigma_i) \to (W/\overline{R(G_{ji}(T))}, \hat{\lambda}_k)$, is continuous and that also,

$$(\eta \circ R) \circ T = \eta \circ (R \circ T) : (X, \rho_j) \to (W/\overline{R(G_{ji}(T))}, \hat{\lambda}_k).$$

By Proposition 4.1 $\overline{R(G_{ji}(T))}^{\lambda_k} < G_{ik}(R) = Z_{\lambda_k}$. Let $y \in G_{jk}(R \circ T) = Z_{\lambda_k}$ and $\{x_n\}_{\mathbb{N}} \overset{\rho_j}{\to} \hat{0}$ such that $\{(R \circ T)(x_n)\}_{\mathbb{N}} \overset{\lambda_k}{\to} y$. We then have two convergences, knowing that $\{(\eta \circ R \circ T)(x_n)\}_{\mathbb{N}} \to \eta(y)$ and that $\{(\eta \circ R \circ T)(x_n)\}_{\mathbb{N}} \to \eta(\hat{0})$, from which $\eta(y) \in Z_{\hat{\lambda}_k} = \eta(0) = [0]$. This implies that $y \in \overline{R(G_{ji}(T))}^{\lambda_k}$. □

Corollary 4.5. *Let $R : (Y, \sigma_i) \to (W, \lambda_k)$ be linear and continuous, and $T : (X, \rho_j) \to (Y, \sigma_i)$ linear such that the composition $(R \circ T) : (X, \rho_j) \to (W, \lambda_k)$ is continuous. Then $\overline{R(G_{ji}(T)}^{\lambda_k} = Z_{\lambda_k}$.*

Proof. By Proposition 4.10 $\overline{R(G_{ji}(T)}^{\lambda_k} = G_{jk}(R \circ T) = Z_{\lambda_k}$, this last equality being from the continuity of $R \circ T$. □

Corollary 4.6. *Let $R : (Y, \sigma) \to (W, \lambda)$ be linear and continuous. Then $\overline{R(G(T))}^{\lambda} < G(R \circ T)$. This is Proposition 4.4, with the topologies τ and σ.*

Proof.

$$\begin{aligned}
\overline{R(G(T))}^{\lambda} &= \overline{R(\bigcap_{J\times I} G_{ji}(T))}^{\lambda} \\
&< \overline{\bigcap_{J\times I} R(G_{ji}(T))}^{\lambda} \\
&< \bigcap_{J\times I} \overline{R(G_{ji}(T))}^{\lambda} \\
&= \bigcap_{J\times K} G_{jk}(T) = G(R \circ T)
\end{aligned}$$

□

Proposition 4.5. *Suppose $S : (X, \rho_l) \to (X, \rho_j)$ and $R : (Y, \sigma_i) \to (Y, \sigma_k)$ are continuous, linear functions, and consider $T : (X, \rho_j) \to (Y, \sigma_i)$. These satisfy the following:*

1. *$G_{li}(T \circ S) < G_{ji}(T)$*
2. *$R(G_{ji}(T \circ S)) < G_{jk}(R \circ T)$*
3. *If $T \circ S = R \circ T$, then $R(G_{ji}(T)) < G_{jk}(T)$.*

Proof. 1) Let $y \in G_{li}(T \circ S)$, let $\{x_n\}_{\mathbb{N}} \overset{\rho_j}{\to} \hat{0} \in X$, and $\{(T \circ S)(x_n)\}_{\mathbb{N}} \overset{\sigma_i}{\to} y \in Y$. By the continuity of S, $\{S(x_n)\}_{\mathbb{N}} \overset{\rho_j}{\to} \hat{0}$, $\{T(S(x_n))\}_{\mathbb{N}} \overset{\sigma_i}{\to} y$. This proves that $y \in G_{ji}(T)$, which in turn implies $G_{li}(T \circ S) < G_{ji}(T)$.

2) Let $y \in G_{ji}(T \circ S)$. From this, there exists $\{x_n\}_{\mathbb{N}} \overset{\rho_l}{\to} \hat{0} \in X$ such that $\{(T \circ S)(x_n)\}_{\mathbb{N}} \overset{\sigma_i}{\to} y \in Y$. By the continuity of S, $\{S(x_n)\}_{\mathbb{N}} \overset{\rho_j}{\to} \hat{0}$. By the continuity of R, $\{(R \circ T)(S(x_n))\} = \{R(T \circ S)(x_n)\}_{\mathbb{N}} \overset{\sigma_k}{\to} R(y)$. From here, we obtain $R(y) \in G_{jk}(R \circ T)$.

3) Let $y \in G_{ji}(T)$ and let $\{x_n\}_{\mathbb{N}} \overset{\rho_j}{\to} \hat{0} \in X$ y $\{T(x_n)\}_{\mathbb{N}} \overset{\sigma_i}{\to} y \in Y$. This gives us $\{S(x_n)\}_{\mathbb{N}} \overset{\rho_j}{\to} \hat{0}$ and $\{T(S(x_n))\}_{\mathbb{N}} = \{R(T(x_n))\}_{\mathbb{N}} \overset{\sigma_k}{\to} R(y)$. The last piece tells us that $R(y) \in G_{jk}(T)$. □

Corollary 4.7. *If $S : (X, \tau) \to (X, \tau)$ and $R : (Y, \sigma) \to (Y, \sigma)$ are continuous, linear functions, then they satisfy the following:*

1. $G(T \circ S) < G(T)$
2. $R(G(T \circ S)) < G(R \circ T)$
3. *If $T \circ S = R \circ T$, then $R(G(T)) < G(T)$.*

Proposition 4.6. *Let $U < X$ and $V < Y$ be closed vector subspaces for which $T(U) < V$. Let $\hat{T} : (X/U, \hat{\rho}_j) \to (Y/V, \hat{\sigma}_i)$ be linear, such that $\hat{T} \circ \eta_U = \eta_V \circ T$. (This is possible by the hypothesis). Then $\hat{T}$ is continuous iff $G_{ji}(T) < V$.*

$\eta_U : (X, \rho_j) \to (X/U, \hat{\rho}_j)$ and $\eta_V : (Y, \sigma_i) \to (Y/V, \hat{\sigma}_i)$ are the canonical epimorphisms, which are continuous.

($\Rightarrow$): The continuity of $\hat{T}$ implies that $\hat{T} \circ \eta_U$ is also continuous. Let $y \in G_{ji}(T)$ as well as $\{x_n\}_{\mathbb{N}} \overset{\rho_j}{\to} \hat{0} \in X$, such that $\{T(x_n)\}_{\mathbb{N}} \overset{\sigma_i}{\to} y \in Y$. This shows that $\{\eta_U(x_n)\}_{\mathbb{N}} \to \eta_U(\hat{0})$ and $\{\hat{T}(\eta_U(x_n)\}_{\mathbb{N}} = \{\eta_V(T(x_n)\}_{\mathbb{N}} \to \eta_V(y)$. Meanwhile, $\{\hat{T}(\eta_U(x_n)\}_{\mathbb{N}} \to \hat{T}(\eta_U(\hat{0}) = \eta_V(T(\hat{0})) = \eta_V(\tilde{0})$. These two convergences tell us that $y \in V$.

($\Leftarrow$): Our task is to show that $G_{ji}(\hat{T}) = Z_{\hat{\sigma}_i} = \eta_V(\tilde{0})$. To this end, we start with $G_{ji}(T) < V \Rightarrow \eta_V(G_{ji}(T)) = \eta_V(\tilde{0}) < Z_{\hat{\sigma}_i}$. By Proposition 4.1, $\eta_V \circ T$ is continuous, and thus, so is $\hat{T} \circ \eta_U$. Let $\eta_V(y) \in G_{ji}(\hat{T})$ and let $\{\eta_U(x_n)\}_{\mathbb{N}} \to \eta_U(\hat{0})$ such that $\{\eta_V(T(x_n))\}_{\mathbb{N}} = \{\hat{T}(\eta_U(x_n))\}_{\mathbb{N}} \to \eta_V(y)$. Then, there exists a sequence $\{x'_n\}_{\mathbb{N}}$ in X such that $\eta_U(x'_n) = \eta_U(x_n)$ and in addition, $\{x'_n\}_{\mathbb{N}} \overset{\rho_j}{\to} \hat{0}$. Therefore, $\{\eta_U(x'_n)\}_{\mathbb{N}} \to \eta_U(\hat{0})$. We now obtain $\{\hat{T}(\eta_U(x_n))\}_{\mathbb{N}} = \{\hat{T}(\eta_U(x'_n))\}_{\mathbb{N}} = \{(\hat{T} \circ \eta_U)(x'_n)\}_{\mathbb{N}} \to \eta_V(\tilde{0})$, which implies $\eta_V(y) \in Z_{\hat{\sigma}_i}$. By Proposition 2.3, $\hat{T}$ is continuous. □

Corollary 4.8. *Let $U < X$ y $V < Y$ be closed vector subspaces such that $T(U) < V$. Let $\hat{T} : (X/U, \hat{\tau}) \to (Y/V, \hat{\sigma})$ be linear such that $\hat{T} \circ \eta_U = \eta_V \circ T$. (This is possible by the hypothesis). Then:*

(*i*) *If $\hat{T}$ is continuous, then, $G(T) < V$.*

(*ii*) *If for each $i \in I$ there exists $j \in J$ such that $G_{ji}(T) < V$, then $\hat{T}$ is continuous.*

Here $\eta_U : (X, \tau) \to (X/U, \hat{\tau})$ and $\eta_V : (Y, \sigma) \to (Y/V, \hat{\sigma})$ are the continuous canonical epimorphisms.

Definition 4.1. *Suppose $\{S_n : (X, \rho_{j_{n-1}}) \to (S, \rho_{j_n})\}_{\mathbb{N}}$ and $\{R_n : (Y, \sigma_{i_n}) \to (Y, \sigma_{i_{n-1}})\}_{\mathbb{N}}$ are continuous linear functions such that $T \circ S_n = R_n \circ T \ \forall n \in \mathbb{N}$ and $\tau_{\rho_{n-1}} \subset \tau_{\rho_n}$. Then we say that $\{S_n\}$ and $\{R_n\}$ are* **intertwining**.

The following is the main result. It generalizes Lemma 1.6 of [6]. In the proof we will employ a gliding hump argument, applied to the seminorms.

Theorem 4.1. *Let* $\{S_n : (X, \rho_{j_{n-1}}) \to (S, \rho_{j_n})\}_{\mathbb{N}}$ *and* $\{R_n : (Y, \sigma_{i_n}) \to (Y, \sigma_{i_{n-1}})\}_{\mathbb{N}}$ *be continuous linear functions that are intertwining. Then there exists* $N \in \mathbb{N}$ *such that* $\forall n \geq N$*:*

$$\overline{R_1 \circ R_2 \circ ... \circ R_n}^{\sigma_{i_o}} = \overline{R_1 \circ R_2 \circ ... \circ R_N}^{\sigma_{i_o}}.$$

Proof. Claim 1. $\tau_{\rho_{j_{n-1}}} \subset \tau_{\rho_n} \Rightarrow G_{j_{n+1}i_m}(T) < G_{j_n i_m}(T)$ $(\forall n, m \in \mathbb{N})$.

Indeed, for the continuity of the linear functions $S_n : (X; \rho_{j_{n-1}}) \to (X, \rho_{j_n})$ there are positive numbers $\{c_n\}_{\mathbb{N}}$ such that $\rho_{j_n}(S_n(x)) \leq c_n \rho_{j_{n-1}}(x)$, $\forall x \in X$. We have $y \in G_{j_{n+1}i_m}(T)$ and $\{x_n\}_{\mathbb{N}} \overset{\rho_{j_{n+1}}}{\to} \hat{0}$ such that $\{T(x_n)\}_{\mathbb{N}} \overset{\sigma_{i_m}}{\to} y$ which tells us $\{x_n\}_{\mathbb{N}} \overset{\rho_{j_n}}{\to} \hat{0}$ and $\{T(x_n)\}_{\mathbb{N}} \overset{\sigma_{i_m}}{\to} y$; that is, $y \in G_{j_n i_m}(T)$.

Claim 2. For each $n \in \mathbb{N}$, $R_{n+1}(G_{j_{n+1}i_{n+1}}(T)) < G_{j_n i_n}(T)$ and

$$\overline{R_1 \circ R_2 \circ ... \circ R_{n+1}}^{\sigma_{i_o}} = \overline{R_1 \circ R_2 \circ ... \circ R_n}^{\sigma_{i_o}}.$$

By Proposition 4.5-3, and Claim 1, $R_{n+1}(G_{j_{n+1}i_{n+1}}(T)) < G_{j_{n+1}i_n}(T) < G_{j_n i_n}(T)$. From this, $(R_1 \circ R_2 \circ ... \circ R_n)(R_{n+1}(G_{j_{n+1}i_{n+1}}(T)) < (R_1 \circ R_2 \circ ... \circ R_n)(G_{j_n i_n}(T))$, and then

$$\overline{R_1 \circ R_2 \circ ... \circ R_n \circ R_{n+1}(G_{j_{n+1}i_{n+1}}(T))}^{\sigma_{i_o}} < \overline{R_1 \circ R_2 \circ ... \circ R_n(G_{j_n i_n}(T))}^{\sigma_{i_o}}.$$

Let $W_n = \overline{R_1 \circ R_2 \circ ... \circ R_n(G_{j_n i_n}(T))}^{\sigma_{i_o}} < Y$. We have that $W_{n+1} < W_n$. For each $n \in \mathbb{N}$ we consider the canonical linear epimorphism $\eta_n : (Y, \sigma_{i_o}) \to (Y/W_n, \hat{\sigma}_{i_o})$, which is continuous.

Claim 3. $\eta_n \circ R_1 \circ ... \circ R_n \circ T : (X, \rho_{j_n}) \to (Y/W_n, \overline{\sigma}_{i_o})$ is continuous.
$((\eta_n \circ R_1 \circ ... \circ R_n) \circ T)(G_{j_n i_n}(T) = [\tilde{0}]_{W_n} = Z_{\overline{\sigma}_{i_o}}$. By Proposition 4.1, $\eta_n \circ R_1 \circ ... \circ R_n \circ T$ is continuous. From this, there exist positive numbers $\{d_n\}_{\mathbb{N}}$ such that $\hat{\sigma}^n_{i_o}(\eta_n \circ R_1 \circ ... \circ R_n \circ T(x)) \leq d_n \rho_{j_n}(x)$, $\forall x \in X$.

Claim 4. $\eta_n \circ R_1 \circ ... \circ R_{n-1} \circ T : (X, \rho_{j_{n-1}}) \to (Y/W_n, \overline{\sigma}_{i_o})$ is not continuous if $W_{n+1} \subsetneq W_n$ for infinitely many n.

If $W_{n+1} \subsetneq W_n$ for infinitely many n, then by the discontinuity of the functions of Claim 4, we can construct the following sequence:

Let $x_1 \in X$ such that $\rho_{j_1}(x_1) < \frac{1}{2}$ and $\hat{\sigma}^2_{i_o}((\eta_2 \circ R_1 \circ T)(x_1)) > 1 + d_2$. Let $x_2 \in X$ such that $\rho_{j_2}(x_2) < \frac{1}{2^2}$ and $\hat{\sigma}_{i_o}((\eta_3 \circ R_1 \circ R_2 \circ T)(x_2)) > 2 + d_3 + \hat{\sigma}^3_{i_o}((\eta_3 \circ T)(S_1(x_1))$. Let $x_3 \in X$ such that $\rho_{j_3}(x_3) < \frac{1}{2^3}$ and $\hat{\sigma}_{i_o}((\eta_4 \circ R_1 \circ R_2 \circ R_3 \circ T)(x_3)) > 3 + d_4 + \hat{\sigma}^4_{i_o}(\eta_4 \circ T)(S_1(x_1) + (S_2 \circ S_1)(x_2))$. In general, let $x_n \in X$ such that $\rho_{j_n}(x_n) < \frac{1}{2^n}$ and

$$\begin{aligned} \hat{\sigma}_{i_o}((\eta_{n+1} \circ R_1 \circ ... \circ R_n \circ T)(x_n)) &> n + d_{n+1} \\ &+ \hat{\sigma}^{n+1}_{i_o}((\eta_{n+1} \circ T)(S_1(x_1) + (S_2 \circ S_1)(x_2) \\ &+ ... + (S_{n-1} \circ ...S_1)(x_{n-1})))). \end{aligned}$$

Notice that, for $m < n \in \mathbb{N}$, $\rho_{j_m} \leq c_{m+1}...c_n\rho_{j_n}$ and $\hat{\sigma}^m_{i_o} \leq \hat{\sigma}^n_{i_o}$. For each $n \in \mathbb{N}$ let $z_n = (S_1 \circ ... \circ S_n)(x_n) \in X$.

$$\begin{aligned}\rho_{j_o}(z_n) \leq c_1\rho_{j_1}((S_2 \circ ... \circ S_n)(x_n)) &\leq c_1c_2\rho_{j_2}((S_3 \circ ... \circ S_n)(x_n) \\ &\leq c_1c_2...c_n\rho_{j_n}(x_n) \\ &< (c_1c_2...c_n)\frac{1}{2^n}\end{aligned}$$

We consider two cases.

Case 1. $c_n > 1$, for infinitely many $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ let the seminorm $\rho'_{j_n} = \frac{1}{c_1...c_n}\rho_{j_n}$. Then, $\rho'_{j_n}(S_n(x)) = \frac{1}{c_1...c_n}\rho_{j_n}(S_n(x)) \leq \frac{1}{c_1...c_n}c_n\rho_{j_n}(x) = \rho'_{j_{n-1}}(x)$, $\forall x \in X$. $\overline{A}^{\rho_{j_n}} = \overline{A}^{\rho'_{j_n}}$, $\forall A \subset X$. The seminorms ρ_{j_n} and ρ'_{j_n} are equivalent.

Case 2. $c_n \leq 1$, for infinitely many $n \in \mathbb{N}$. We will first need to prove the following:

Claim 5. $\hat{\sigma}^{n+1}_{i_o}(\eta_{n+1}(T(\sum_{k=n+1}^{\infty} z_k))) \leq \frac{1}{2^{n+1}}$.

To see this, we have

$$\begin{aligned}&\hat{\sigma}^{n+1}_{i_o}(\eta_{n+1}(T(\sum_{k=n+1}^{\infty} z_k))) \\ &= \hat{\sigma}^{n+1}_{i_o}(\eta_{n+1}(T(S_1 \circ ... \circ S_{n+1})(\sum_{k=2}^{\infty}(S_{n+2} \circ ... \circ S_{n+k})(x_{n+k})))) \\ &= \hat{\sigma}^{n+1}_{i_o}(\eta_{n+1}(R_1 \circ ... \circ R_{n+1} \circ T)(\sum_{k=2}^{\infty} z_k))) \\ &\leq d_{n+1}\rho_{j_n}(\sum_{k=2}^{\infty}(S_{n+2} \circ ... \circ S_{n+k})(x_{n+k})) \\ &\leq d_{n+1}\sum_{k=2}^{\infty} c_{n+2}...c_{n+k}\rho_{j_o}(x_{n+k}) \leq d_{n+1}\sum_{k=2}^{\infty}\frac{1}{2^{n+k}} \\ &= d_{n+1}\frac{1}{2^{n+1}}.\end{aligned}$$

Returning to the examination of Case 2, we have that $\sum_{n=1}^{\infty}\rho_{i_o}(z_n) \leq 1$. As we required (X, ρ_{i_o}) to be complete, there exists $z = \sum_{n=1}^{\infty} z_n \in X$. Otherwise, $T(z_n) = T((S_1 \circ ... \circ S_n)(x_n)) = (R_1 \circ ... \circ R_n)(T(x_n))$. Here, comes the gliding hump part. For each $n \in \mathbb{N}$, we have:

$$\begin{aligned}\sigma_{i_o}(T(z)) \geq \hat{\sigma}^{n+1}_{i_o}(\eta_{n+1}(T(z))) &\geq \hat{\sigma}^{n+1}_{i_o}(\eta_{n+1}(T(\sum_{k=1}^{n} z_k)) + \eta_{n+1}(T(\sum_{k=n+1}^{\infty} z_k)) \\ &\geq \hat{\sigma}^{n+1}_{i_o}(\sum_{k=1}^{n}\eta_{n+1}((R_1 \circ ... \circ R_k)(T(x_k)))\end{aligned}$$

$$
\begin{aligned}
&- \hat{\sigma}_{i_o}^{n+1}(\eta_{n+1}(T(\sum_{k=n+1}^{\infty} z_k))) \\
&\geq \hat{\sigma}_{i_o}^{n+1}(\eta_{n+1} \circ R_1 \circ ... \circ R_n \circ T(x_n)) \\
&- \hat{\sigma}_{i_o}^{n+1}(\sum_{k=1}^{n-1} \eta_{n+1}((R_1 \circ ... \circ R_k)(T(x_k))) \\
&- \hat{\sigma}_{i_o}^{n+1}(\eta_{n+1}(T(\sum_{k=n+1}^{\infty} z_k))) \\
&> n + d_{n+1} - \hat{\sigma}_{i_o}^{n+1}(\eta_{n+1}(T(\sum_{k=n+1}^{\infty} z_k))) \\
&\geq n + d_{n+1} - \frac{1}{2^{n+1}} d_{n+1} > n.
\end{aligned}
$$

This contradiction indicates to us that there can be only finitely many proper containments of the form $W_{n+1} < W_n$. □

The following result generalizes [6], Lemma 1.8, page 13 to general locally convex spaces. In the proof, we will have the pleasure of applying Zorn's Lemma several times.

Theorem 4.2. *Let $\{R_n : (Y,\sigma) \to (Y,\sigma)\}_{n\in\mathbb{N}}$ be a sequence of commuting continuous linear functions. Then the following properties hold:*

1. *There exists $Y_\infty < Y$ such that $R_n(Y_\infty) = Y_\infty \ \forall n \in \mathbb{N}$.*
2. *If $V < Y$ is such that $R_n(V) = V \ \forall n \in \mathbb{N}$, then $V < Y_\infty$.*
3. *There exists $Y^\infty < Y$ σ - closed such that $\overline{R_n(Y^\infty)}^\sigma = Y^\infty \ \forall n \in \mathbb{N}$.*
4. *If $W < Y$ is σ - closed and satisfies $\overline{R_n(W)}^\sigma = W \ \forall n \in \mathbb{N}$ then $W < Y^\infty$.*
5. *$\overline{R_n(Y^\infty)}^\sigma = \overline{R_n(R_m(Y^\infty))}^\sigma = Y^\infty$, $\forall n, m \in \mathbb{N}$.*
6. *$\overline{Y_\infty}^\sigma < Y^\infty$.*

Proof. 1. Let us say that $V < Y$ satisfies *property* $P(\models P(V))$, if $R_n(V) = V$ $\forall n \in \mathbb{N}$. Let $F = \{V < Y | \models P(V)\}$ be ordered by inclusion. Observe that $\{\tilde{0}\} \in F$. Also, let $C = \{V_\lambda\}_\Lambda \subset F$ be a chain. Then $V = \bigcup_\Lambda V_\lambda < Y$. On the other hand, for each $n \in \mathbb{N}$, $R_n(V) = \bigcup_\Lambda R_n(V_\lambda) = \bigcup_\Lambda V_\lambda = V$. In other words, $V \in F$ and is a maximal element of C. Zorn's Lemma now guarantees the existence of a maximal subspace in F. Denote said maximal subspace by $Y_\infty \in F$. Now, in item 2, we will prove that Y_∞ is unique.

2. We will prove that if $V \in F$, then $V + Y_\infty \in F$. To this end, we have $R_n(V + Y_\infty) = R_n(V) + R_n(Y_\infty) = V + Y_\infty$. $Y_\infty < V + Y_\infty \Rightarrow Y_\infty = V + Y_\infty \Rightarrow V < Y_\infty$.

3. For this part, let us say that $W < Y$ satisfies *property* Q ($\models Q(W)$) if $\overline{R_n(W)}^{\sigma} = W \ \forall n \in \mathbb{N}$. Now, let $G = \{W < Y | \models Q(W)\}$ be ordered by inclusion and observe that $\{\tilde{0}\} \in G$. Let $D = \{W_\lambda\}_\Lambda \subset G$be a chain. We have that $W = \overline{\bigcup_\Lambda W_\lambda} < Y$. For each $n \in \mathbb{N}$:

$$\begin{aligned} R_n(W) < \overline{R_n(\bigcup_\Lambda W_\lambda)}^{\sigma} &= \overline{\bigcup_\Lambda R_n(W_\lambda)}^{\sigma} \\ &< \overline{\bigcup_\Lambda \overline{R_n(W_\lambda)}}^{\sigma} \\ &= \overline{\bigcup_\Lambda W_\lambda}^{\sigma} \\ &= W. \end{aligned}$$

From the above we obtain $\overline{R_n(W)}^{\sigma} < W$. Let $y \in W$ and let $O \subset Y$ be a closed, absolutely convex neighborhood of $\tilde{0}$. Finally, let O'be another neighborhood of $\tilde{0}$ for which $O' + O' \subset O$. By the σ - density of the subspace $\bigcup_\Lambda W_\lambda$ in W, there exists $w \in W_\lambda$ (for some $\lambda \in \Lambda$) such that $w \in y + O'$. Meanwhile, by the σ- density of $R_n(W_\lambda)$ in W_λ, there exists $w' \in W_\lambda$ such that $R_n(w') \in w + O'$. We conclude from this that $R_n(w') \in w + O' \subset (y + O') + O' \subset y + O$. In other words, $y \in \overline{R_n(W)}^{\sigma}$. We have shown that $W \in G$ and is an upper bound for D. Here comes Zorn's Lemma again: G has a maximal element. Put $Y^\infty \in G$ as a maximal element, and in the next item we will prove that it is unique.

4. We will prove that if $W \in G$ then $\overline{W + Y^\infty}^{\sigma} \in G$: For one direction we have:

$$\begin{aligned} R_n(\overline{W + Y^\infty}) &< \overline{R_n(W) + R_n(Y^\infty)}^{\sigma} \\ &< \overline{R_n(W)}^{\sigma} + \overline{R_n(Y^\infty)}^{\sigma} \\ &< \overline{\overline{R_n(W)}^{\sigma} + \overline{Y^\infty}^{\sigma}}^{\sigma} \\ &= \overline{W + Y^\infty}^{\sigma}. \end{aligned}$$

Now, we will prove the other direction: Let $y \in \overline{W + Y^\infty}^{\sigma}$ and let $O \subset Y$ be a closed, absolutely convex zero neighborhood. Also, let V be another zero neighborhood for which $V + V \subset O$. There exist, then, $w \in W = \overline{R_n(W)}^{\sigma}$ and $z \in Y^\infty = \overline{R_n(Y^\infty)}^{\sigma}$ such that $(w + z) \in y + V$. Moreover, there exist $w' \in W$ and $z' \in Y^\infty$ such that $R_n(w') \in w + V$ and $R_n(z') \in z + V$. We have

$$R_n(w' + z') = R_n(w') + R_n(z') \in (w + V) + (z + V) = (w + z) + (V + V) \subset y + O.$$

In other words, $y \in \overline{R_n(W + Y^\infty)}^{\infty}$. Thus, we have proved that $\overline{W + Y^\infty}^{\sigma} \in G$. Now, $Y^\infty = \overline{Y^\infty}^{\sigma} < \overline{W + Y^\infty}^{\sigma} \Rightarrow Y^\infty = \overline{W + Y^\sigma}^{\sigma} \Rightarrow W < Y^\infty$.

5. Observe:

$$\overline{R_n(R_m(Y^\infty))}^{\sigma} > R_n(\overline{R_m(Y^\infty)}^{\sigma}) = R_n(Y^\infty).$$

This implies that

$$Y^\infty = \overline{R_n(Y^\infty)}^\sigma < \overline{R_n(R_m(Y^\infty))}^\sigma < \overline{R_n(Y^\infty)}^\sigma = Y^\infty.$$

On the other hand

$$Y^\infty = \overline{R_n(Y^\infty)}^\sigma = \overline{R_n(R_m(Y^\infty))}^\sigma = \overline{R_n(Y^\infty)}^\sigma.$$

6. First, let us recall that for each $A \subset Y$, $\overline{A}^\sigma = \bigcap_I \overline{A}^{\sigma_i}$.

Claim. $\overline{Y_\infty}^\sigma \in G$. To see this, let

$$y \in \overline{R_n(\overline{Y_\infty}^\sigma)}^\sigma = \bigcap_I \overline{R_n(\overline{Y_\infty})}^{\sigma_i}.$$

Thus, for each $i \in I$ as well as each $\epsilon > 0$, we have: There exists $z_i \in \overline{Y_\infty}^\sigma$ such that $\sigma_i(y - z_i) < \frac{\epsilon}{2}$ and there is $y_i \in Y_\infty$ such that $\sigma_i(y_i - z_i) < \frac{\epsilon}{2}$. Hence, $\sigma_i(y - y_i) < \epsilon$. This proves that $y \in \overline{Y_\infty}^{\sigma_i}$ for each $i \in I$ and therefore $y \in \overline{Y_\infty}^\sigma$.This last statement tells us that $\overline{R_n(\overline{Y_\infty}^\sigma)} < \overline{Y_\infty}^\sigma$. Now we prove the other direction: $R_n(Y_\infty) = Y_\infty \Rightarrow$

$$\overline{Y_\infty}^\sigma = \overline{R_n(Y_\infty)}^\sigma = \bigcap_I \overline{R_n(Y_\infty)}^{\sigma_i} < \bigcap_I \overline{R_n(\overline{Y_\infty}^\sigma)}^{\sigma_i} = \overline{R_n(\overline{Y_\infty}^\sigma)}^\sigma.$$

Putting both directions together proves that $\overline{Y_\infty}^\sigma \in G$ and as a consequence, $\overline{Y_\infty}^\sigma < Y^\infty$. □

Acknowledgements

The authors C. Bosch, C. García, C. Gómez-Wulschner, and R. Vera were partially supported by the Asociación Mexicana de Cultura, A.C.

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Accepted: 17.11.2019

Banach gamma-algebra modules and full stability

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Abstract. In this paper the full stable Banach gamma-algebra modules, fully stable Banach gamma-algebra modules relative to ideal are introduced. Some properties and characterizations of these classes of full stability are studied.

Keywords: fully stable Banach Algebra Modules, fully stable Banach gamma algebra modules, fully stable Banach gamma algebra modules relative to an ideal.

1. Introduction

In [1], "An algebra $\mathcal{A}$ is non-empty set with $(\mathcal{A}, +, \cdot)$ is a vector space over a field $\mathcal{F}$, $(\mathcal{A}, +, \circ)$ is a ring and $(\alpha a) \circ \mathrm{b} = \alpha\,(a \circ b) = a \circ (\alpha b)$ for every$\alpha \in \mathcal{F}$, for every $a, b \in \mathcal{A}$". "A ring $\mathcal{R}$ is an algebra $\langle \mathcal{R}, +, \cdot, -, 0\rangle$ where $+$ and $\cdot$ are two binary operations, $-$ is unary and 0 is nullary element satisfying, $\langle R, +, -, 0\rangle$ is an Abelian group, $\langle \mathcal{R}, .\rangle$ is a semigroup and $x.\,(y + z) = (x.y) + (x.z)$ and $(x + y)\,.z = (x.z) + (y.z)$" [2]. Following [1], let $\mathcal{A}$ be an algebra, recall that a Banach space ε is a Banach left $\mathcal{A}-$ module, and $\|a.x\| \leq \|a\|\,\|x\|\,(a \in \mathcal{A}, x \in \varepsilon)$. A map from a left Banach $\mathcal{A}-$ module $\mathcal{X}$ into a left Banach $\mathcal{A}-$ module $\mathcal{Y}$ ($\mathcal{A}$ is not necessary commutative) is said a multiplier (homomorphism) if it satisfies $\mathcal{T}\,(a.x) = a.\mathcal{T}x$ for all $a \in \mathcal{A}, x \in \mathcal{X}$ [3]. In [4], "a submodule $\mathcal{N}$ of an $\mathcal{R}-$ module $\mathcal{M}$ is said to be stable, if $f(\mathcal{N}) \subseteq \mathcal{N}$ for each $\mathcal{R}-$ homomorphism $f : \mathcal{M} \to \mathcal{N}$. $\mathcal{M}$ is called a fully stable module, each submodule of $\mathcal{M}$ is stable". Following [5], a Banach algebra module $\mathcal{M}$ is called fully stable Banach $\mathcal{A}-$ module if for every submodule $\mathcal{N}$ of $\mathcal{M}$ and for each multiplier $\theta : \mathcal{M} \to \mathcal{N}$ such that $\theta(\mathcal{N}) \subseteq \mathcal{N}$. Following [6], a left Banach $\mathcal{A}-$ module $\mathcal{X}$ is $n-$ generated for $n \in \mathcal{N}$ if there exists $x_1, \ldots, x_n \in \mathcal{X}$ such that each $x \in \mathcal{X}$ can be represented as $x = \sum_{k=1}^{n} a_k.x_k$ for some $a_1, \ldots, a_n \in \mathcal{A}$. A cyclic module is just a $1-$generated module. In [7,8,9], "Let $\mathcal{X}$ be Banach $\mathcal{A}-$ module, $\mathcal{X}$ is called fully stable Banach $\mathcal{A}-$ module relative to ideal $\mathcal{K}$ of $\mathcal{A}$, if for every submodule $\mathcal{N}$ of $\mathcal{X}$ and for each multiplier $\theta : \mathcal{N} \to \mathcal{X}$ satisfy $\theta\,(\mathcal{N}) \subseteq \mathcal{N} + \mathcal{K}\mathcal{X}$. It is clear that every fully stable Banach $\mathcal{A}-$ module is fully stable Banach $\mathcal{A}-$ module relative to ideal". "Let $\mathcal{X}$ be Banach $\mathcal{A}-$ module, $\mathcal{X}$ is called fully stable Banach $\mathcal{A}-$ module relative to ideal $\mathcal{K}$ of $\mathcal{A}$, if for every submodule $\mathcal{N}$ of $\mathcal{X}$ and for each multiplier $\theta : \mathcal{N} \to \mathcal{X}$ such that $\theta\,(\mathcal{X}) \subseteq \mathcal{N} \bigcap \mathcal{K}\mathcal{X}$. It is clear that every fully

stable Banach $\mathcal{A}-$ module is fully stable Banach $\mathcal{A}-$ module relative to ideal". Following [10], "Let Γ be a groupoid and $\mathcal{V}$ be a vector space over a field $\mathcal{F}$. Then $\mathcal{V}$ is called a $\Gamma-$ algebra over the field $\mathcal{F}$ if there exists a mapping $\mathcal{V}\times\Gamma\times\mathcal{V}\to\mathcal{V}$ (the image is denoted by $x\alpha y$ for $x,y\in\mathcal{V}$ and $\alpha\in\Gamma$) such that the following conditions hold: (1) $(x+y)\,\alpha z = x\alpha z + y\alpha z,\quad x\alpha\,(y+z) = x\alpha y + x\alpha z$, (2) $x\,(\alpha+\beta)\,y = x\alpha y + x\beta y$, (3) $(cx)\,\alpha y = c\,(x\alpha y) = x\alpha(cy)$, (4) $0\alpha y = y\alpha 0 = 0$, for all $x,y,z\in\mathcal{V},\ c\in\mathcal{V}$ and $\alpha,\beta\in\Gamma$. Moreover, a $\Gamma-$ algebra is called associative if (5) $(x\alpha y)\,\beta z = x\alpha(y\beta z)$, and unital if for every $\alpha,\beta\in\Gamma$, there is an element $1_\alpha\alpha v = v = v\alpha 1_\alpha$ for all non-zero elements of $\mathcal{V}$". In this paper, the concepts of fully stable Banach $\Gamma-$algebra modules, fully stable Banach $\Gamma-$ algebra modules relative to ideal and strongly fully stability relative to ideal for Banach $\Gamma-$ algebra modules have been introduced.

2. Fully stable Banach gamma-algebra modules (FSBGAM)

In this section, the concepts of Banach gamma-algebra modules and fully stable Banach gamma-algebra modules are introduced; some definitions and properties are submitted.

Definition 2.1. Let $\mathcal{A}$ be a $\Gamma-$algebra, a $\Gamma\mathcal{A}-$ module is a Banach space $\mathcal{V}$, together with a map $\mathcal{A}\times\Gamma\times\mathcal{V}\to\mathcal{V}$ (whose value at (a,γ,v) we will denote by $a\ \gamma\ v$) such that

(i) $(a,\gamma,v)\to a\ \gamma\ v$ is bilinear.

(ii) $(ab)\,\gamma\ v = a\,(b\ \gamma\ v)$ (Left $\mathcal{A}-$ module) for all $a,b\in\mathcal{A},\ v\in\mathcal{V},\gamma\in\Gamma$ or $(ab)\ \gamma\ v = b\,(a\ \gamma\ v)$ (Right $\mathcal{A}-$ module) for all $a,b\in\mathcal{A},\ v\in\mathcal{V},\gamma\in\Gamma$.

(iii) $\|a\gamma v\|\le\|a\|\,|\gamma|\,\|v\|$ $(a\in\mathcal{A},\ v\in\mathcal{V})\gamma\in\Gamma$.

Definition 2.2. A map from a left Banach $\Gamma\mathcal{A}-$ module $\mathcal{X}$ into a left Banach $\Gamma\mathcal{A}-$ module $\mathcal{Y}$ ($\mathcal{A}$ is not necessarily commutative) is said to be a $\Gamma-$multiplier ($\Gamma-$mult) (homomorphism) if it satisfies $\mathcal{T}\,(a.\alpha.x) = a.\alpha.\mathcal{T}x$ for all $a\in\mathcal{A},x\in\mathcal{X},\alpha\in\Gamma$.

Definition 2.3. A Banach $\Gamma\mathcal{A}-$ module $\mathcal{X}$ is called fully stable Banach $\Gamma\mathcal{A}-$ module, if every submodule $\mathcal{N}$ of $\mathcal{X}$ and for each multiplier $\theta:\mathcal{N}\to\mathcal{X}$ such that $\theta\,(\mathcal{N})\subseteq\mathcal{N}$.

Definition 2.4. Let $\mathcal{X}$ be a Banach $\Gamma\mathcal{A}$-module, $\mathcal{N}$ and $\mathcal{K}$ are subsets of $\mathcal{X}$, then

1. $\mathcal{N}_{\alpha x} = \{n_{\alpha x}|\ n\in\mathcal{N},\alpha\in\Gamma,x\in\mathcal{X}\}$, $\mathcal{K}_{\beta y} = \{k_{\beta y}|\ k\in\mathcal{K},\beta\in\Gamma,y\in\mathcal{X}\}$.
2. $ann_{\Gamma\mathcal{A}}(\mathcal{N}_{\alpha x}) = \{a.\alpha\in\Gamma\mathcal{A}, a.\alpha.n_{\alpha x}=0,\forall n_{\alpha x}\in\mathcal{N}_{\alpha x}\}$, $ann_{\Gamma\mathcal{A}}(\mathcal{K}_{\beta y})$ $=\{a.\beta\in\Gamma\mathcal{A}, a.\beta.k_{\beta y}=0,\forall k_{\beta y}\in\mathcal{K}_{\beta y}\}$.

The following proposition gives another characterization of fully stable Banach $\Gamma\mathcal{A}-$ module:

Proposition 2.5. $\mathcal{X}$ is fully stable Banach $\Gamma\mathcal{A}-$ module if and only if for each $x, y \in \mathcal{X}$ and $\mathcal{N}_{\alpha x}, \mathcal{K}_{\beta y}$ subsets of $\mathcal{X}$, $y \notin \mathcal{N}_{\alpha x}$ implies that

$$ann_{\Gamma\mathcal{A}}(\mathcal{N}_{\alpha x}) \nsubseteq ann_{\Gamma\mathcal{A}}(\mathcal{K}_{\beta y}),\ \alpha, \beta \in \Gamma.$$

Proof. Suppose that $\mathcal{X}$ is fully stable Banach $\Gamma\mathcal{A}-$ module, there exists $x, y \in \mathcal{X}$ such that $y \notin \mathcal{N}_{\alpha x}$ and $ann_{\Gamma\mathcal{A}}(\mathcal{N}_{\alpha x}) \subseteq ann_{\Gamma\mathcal{A}}(\mathcal{K}_{\beta y})$. Define $\theta : \langle \mathcal{N}_{\alpha x} \rangle \to \mathcal{X}$ by $\theta(a.\alpha.n_{\alpha x}) = a.\beta.k_{\beta y}$, for all $a \in \mathcal{A}, \alpha, \beta \in \Gamma$, if $a.\alpha.n_{\alpha x} = 0$ then $a.\alpha \in ann_{\Gamma\mathcal{A}}(\mathcal{N}_{\alpha x}) \subseteq ann_{\Gamma\mathcal{A}}(\mathcal{K}_{\beta y})$. this implies that $a.\beta.k_{\beta y} = 0$, hence θ is well-defined, it is clear that θ is a multiplier, because $\mathcal{X}$ is fully stable, there exists an element $t.\alpha \in \Gamma\mathcal{A}$ such that

$$\theta(m_{\alpha x}) = t.\alpha.m_{\alpha x} for\ each\ m_{\alpha x} \in \mathcal{N}_{\alpha x}.$$

In particular, $k_{\beta y} = \theta(n_{\alpha x}) = t.\alpha.n_{\alpha x} \in \mathcal{N}_{\alpha x}$, which is a contradiction. Conversely, assume that there is a subset $\mathcal{N}_{\alpha x}$ of $\mathcal{X}$, and a multiplier $\theta : \langle \mathcal{N}_{\alpha x} \rangle \to \mathcal{X}$ such that $\theta(\mathcal{N}_{\alpha x}) \nsubseteq \mathcal{N}_{\alpha x}$ then there exists an element $m_{\alpha x} \in \mathcal{N}_{\alpha x}$ such that $\alpha(m_{\alpha x}) \notin \mathcal{N}_{\alpha x}$. Let $s.\alpha \in ann_{\Gamma\mathcal{A}}(\mathcal{N}_{\alpha x})$ therefore, $s.\alpha.n_{\alpha x} = 0$,

$$s.\alpha.\theta(m_{\alpha x}) = \theta(s.\alpha.m_{\alpha x}) = \theta(st.\alpha.n_{\alpha x}) = \theta(st.\alpha.n_x) = \theta(0) = 0.$$

Hence, $ann_{\Gamma\mathcal{A}}(\mathcal{N}_{\alpha x}) \subseteq ann_{\Gamma\mathcal{A}}(\theta(m_{\alpha x}))$, which is a contradiction. □

Definition 2.6. A $\Gamma\mathcal{A}-$ module $\mathcal{X}$ is said to be satisfy Baer criterion if each submodule of $\mathcal{X}$ satisfies Baer criterion, that is for every submodule $\mathcal{N}$ of $\mathcal{X}$ and $\Gamma\mathcal{A}$ multiplier $\theta : \mathcal{N} \to \mathcal{X}$, there exists an element a in $\mathcal{A}$, and $\alpha \in \Gamma$ such that $\theta(n) = a\alpha n$ for all $n \in \mathcal{N}$.

Proposition 2.7. Let $\mathcal{X}$ be a Banach $\Gamma\mathcal{A}-$ module. Then Baer criterion holds for cyclic submodules of $\mathcal{X}$ if and only if

$$ann_{\Gamma\mathcal{X}}(ann_{\Gamma\mathcal{X}}(\mathcal{N}_{\alpha x})) = \mathcal{N}_{\alpha x}\ for\ each\ x \in \mathcal{X}, \alpha \in \Gamma.$$

Proof. Assume that Baer criterion holds. Let $y \in ann_{\mathcal{X}}(ann_{\Gamma\mathcal{A}}(\mathcal{N}_{\alpha x}))$. Define $\theta\langle \mathcal{N}_{\alpha x} \rangle \to \mathcal{X}$ by $(a.\alpha.n_{\alpha x}) = a.\beta.k_{\beta y}$, for all $a \in \mathcal{A}, \alpha, \beta \in \Gamma$. Let $a_1.\alpha.n_{\alpha x} = a_2.\alpha.n_{\alpha x}$, hence, $(a_1 - a_2).\alpha.n_{\alpha x} = 0$, therefore $(a_1 - a_2) \in ann_{\Gamma\mathcal{A}}(\mathcal{N}_{\alpha x})$, thus $(a_1 - a_2) \in ann_{\Gamma\mathcal{A}}(\mathcal{K}_{\beta y})$ this implies that $a_1.\beta.k_{\beta y} = a_2.\beta.k_{\beta y}$, hence, θ is well-defined, it is clear that θ is a $\Gamma-$ mult. By the assumption, there exists an element $t \in \mathcal{A}, \alpha \in \Gamma$ such that $(m_{\alpha x}) = t.\alpha.m_{\alpha x}$, for each $m_{\alpha x} \in \mathcal{N}_{\alpha x}$. In particular, $k_{\beta y} = \theta(n_{\alpha x}) = t.\alpha.n_{\alpha x} \in \mathcal{N}_{\alpha x}$, therefore, $ann_{\Gamma\mathcal{X}}(ann_{\Gamma\mathcal{A}}(\mathcal{N}_{\alpha x})) \subseteq \mathcal{N}_{\alpha x}$. Conversely, assume that $ann_{\Gamma\mathcal{X}}(ann_{\Gamma\mathcal{A}}(\mathcal{N}_{\alpha x})) = \mathcal{N}_{\alpha x}$. For each $\mathcal{N}_{\alpha x} \subseteq \mathcal{X}$. Then each $\Gamma-$mult θ: $\langle \mathcal{N}_{\alpha x} \rangle \to \mathcal{X}$. And $s.\alpha \in ann_{\Gamma\mathcal{A}}(\mathcal{N}_{\alpha x})$, we have $s.\alpha.\theta(n_{\alpha x}) = \theta(s.\alpha.n_{\alpha x}) = 0$. Thus, $\alpha.\theta(n_{\alpha x}) = \theta(\alpha.n_{\alpha x}) \in ann_{\Gamma\mathcal{X}}(ann_{\Gamma\mathcal{A}}(\mathcal{N}_{\alpha x}))$, then $\alpha.\theta(n_{\alpha x}) = t.\alpha.n_{\alpha x}$ for some $t \in \mathcal{A}$, thus Baer criterion holds. □

Corollary 2.8. *$\mathcal{X}$ is fully stable Banach $\Gamma\mathcal{A}-$ module if and only if $ann_{\Gamma\mathcal{X}}(ann_{\Gamma\mathcal{A}}(\mathcal{N}_{\alpha x})) = \mathcal{N}_{\alpha x}$ for each $x \in \mathcal{X}, \alpha \in \Gamma$.*

3. Fully stable Banach gamma-algebra modules relative to an ideal (FSBGAMRI)

In this section, the concept of fully stable Banach gamma-algebra modules relative to an ideal are introduced.

Definition 3.1. Let $\mathcal{X}$ be Banach $\Gamma\mathcal{A}-$ module, $\mathcal{X}$ is called fully stable Banach $\Gamma\mathcal{A}-$module relative to $\Gamma-$ ideal $\mathcal{J}$ of $\mathcal{A}$, if for every submodule $\mathcal{N}$ of $\mathcal{X}$ and for each multiplier $\theta : \mathcal{N} \to \mathcal{X}$ satisfy $\theta(\mathcal{N}) \subseteq \mathcal{N} \bigcap \Gamma\mathcal{J}\Gamma\mathcal{X}$. It is clear that every fully stable Banach $\Gamma\mathcal{A}-$ module is fully stable Banach $\Gamma\mathcal{A}-$module relative to an ideal.

Proposition 3.2. $\mathcal{X}$ is called fully stable Banach $\Gamma\mathcal{A}-$module if and only if for each $x, y \in \mathcal{X}$ and $\mathcal{N}_{\alpha x}, \mathcal{K}_{\beta y}$ subsets of $\mathcal{X}$, $y \notin \mathcal{N}_{\alpha x} + \Gamma\mathcal{J}\Gamma\mathcal{X}$ implies that

$$ann_{\Gamma\mathcal{A}}(\mathcal{N}_{\alpha x}) \nsubseteq ann_{\Gamma\mathcal{A}}(\mathcal{K}_{\beta y}).$$

Proof. Suppose that $\mathcal{X}$ is called fully stable Banach $\Gamma\mathcal{A}-$ module relative to $\Gamma-$ ideal $\mathcal{J}$ of $\mathcal{A}$, there exists $x, y \in \mathcal{X}$ such that $y \notin \mathcal{N}_{\alpha x} + \Gamma\mathcal{J}\Gamma\mathcal{X}$ and $ann_{\Gamma\mathcal{A}}(\mathcal{N}_{\alpha x}) \subseteq ann_{\Gamma\mathcal{A}}(\mathcal{K}_{\beta y})$. Define $\theta : \langle\mathcal{N}_{\alpha x}\rangle \to \mathcal{X}$ by $\theta(\gamma a.n_{\alpha x}) = \gamma a.k_{\beta y}$, for all $\gamma a \in \Gamma\mathcal{A}$, if $\gamma a.n_{\alpha x} = 0$ then $\gamma a \in ann_{\Gamma\mathcal{A}}(\mathcal{N}_{\alpha x}) \subseteq ann_{\Gamma\mathcal{A}}(\mathcal{K}_{\beta y})$. This implies that $\gamma a.k_{\beta y} = 0$, hence, θ is well-defined, it is clear that θ is a multiplier, because $\mathcal{X}$ is fully stable Banach $\Gamma\mathcal{A}-$module relative to a $\Gamma-$ ideal, there exists an element $\delta t \in \Gamma\mathcal{A}$ such that $\theta(m_{\alpha x}) = \delta t m_{\alpha x} + w$, for each $m_{\alpha x} \in \mathcal{N}_{\alpha x}$, $w \in \Gamma\mathcal{J}\Gamma\mathcal{X}$. In particular, $k_{\beta y} = \theta(n_{\alpha x}) = \gamma t n_{\alpha x} + w \in \mathcal{N}_{\alpha x} + \Gamma\mathcal{J}\Gamma\mathcal{X}$, which is a contradiction. Thus $\mathcal{X}$ is called fully stable Banach $\Gamma\mathcal{A}-$module relative to an ideal. Conversely, assume that there is a subset $\mathcal{N}_{\alpha x}$ of $\mathcal{X}$ and a multiplier $\theta\langle\mathcal{N}_{\alpha x}\rangle \to \mathcal{X}$ such that $\theta(\mathcal{N}_{\alpha x}) \nsubseteq \mathcal{N}_{\alpha x} + \Gamma\mathcal{J}\Gamma\mathcal{X}$ then there exists an element $m_{\alpha x} \in \mathcal{N}_{\alpha x}$ such that $\theta(m_{\alpha x}) \notin \mathcal{N}_{\alpha x} + \Gamma\mathcal{J}\Gamma\mathcal{X}$. Let $\rho s \in ann_{\Gamma\mathcal{A}}(\mathcal{K}_{\beta y})$; Therefore, $\rho s n_x = 0$,

$$\rho s\theta(m_{\alpha x}) = \theta(\rho s m_{\alpha x}) = \theta(\rho s n_x) = \theta(\gamma t s n_{\alpha x}) = \theta(0) = 0.$$

Hence, $ann_{\Gamma\mathcal{A}}(\mathcal{N}_{\alpha x}) \subseteq ann_{\Gamma\mathcal{A}}(\theta(m_{\alpha x}))$, which is a contradiction. □

Corollary 3.3. *Let $\mathcal{X}$ be a fully stable Banach $\Gamma\mathcal{A}-$ module relative to a $\Gamma-$ ideal $\mathcal{J}$ of $\Gamma\mathcal{A}$. Then for each ,$y \in \mathcal{X}$, $ann_{\Gamma\mathcal{A}}(\mathcal{N}_{\alpha x}) = ann_{\Gamma\mathcal{A}}(\mathcal{K}_{\beta y})$ implies that*

$$\mathcal{N}_{\alpha x} + \Gamma\mathcal{J}\Gamma\mathcal{X} = \mathcal{K}_{\beta y} + \Gamma\mathcal{J}\Gamma\mathcal{X}.$$

Proof. Assume that there are two elements x, y in $\mathcal{X}$ such that $ann_{\Gamma\mathcal{A}}(\mathcal{N}_{\alpha x}) = ann_{\Gamma\mathcal{A}}(\mathcal{K}_{\beta y})$ and $\mathcal{N}_{\alpha x} + \Gamma\mathcal{J}\Gamma\mathcal{X} \neq \mathcal{K}_{\beta y} + \Gamma\mathcal{J}\Gamma\mathcal{X}$. Then without loss of generality there is an element $z_{\alpha x}$ in $\mathcal{N}_{\alpha x}$ not in $\mathcal{K}_{\beta y}$. By using proposition (3.2), we have $ann_{\Gamma\mathcal{A}}(\mathcal{K}_{\beta y}) \nsubseteq ann_{\Gamma\mathcal{A}}(z_{\alpha x})$ but $ann_{\Gamma\mathcal{A}}(\mathcal{K}_{\beta y}) \subseteq ann_{\Gamma\mathcal{A}}(z_{\alpha x})$, hence $ann_{\Gamma\mathcal{A}}(\mathcal{N}_{?x}) \nsubseteq ann_{\Gamma\mathcal{A}}(\mathcal{K}_{\beta y})$, which is a contradiction. □

Definition 3.4. A Banach $\Gamma\mathcal{A}-$ module $\mathcal{X}$ is said to be satisfy Baer criterion relative to a $\Gamma-$ ideal $\mathcal{K}$ of $\Gamma\mathcal{A}$, if each submodule of $\mathcal{X}$ satisfies Baer criterion,

that is for every submodule $\mathcal{N}$ of $\mathcal{X}$ and $\Gamma\mathcal{A}$ multiplier $\theta : \mathcal{N} \to \mathcal{X}$, there exists an element αa in $\Gamma\mathcal{A}$ such that $\theta(n) - \alpha a n \in \Gamma\mathcal{A}$ for all $n \in \mathcal{N}$.
Recall that, a left Banach $\mathcal{A}-$ module $\mathcal{X}$ is $n-$ generated for $n \in \mathcal{N}$ if there exists $x_1, \ldots, x_n \in \mathcal{X}$ such that each $x \in \mathcal{X}$ can be represented as $x = \sum_{k=1}^{n} a_k.x_k$ for some $a_1, \ldots, a_n \in \mathcal{A}$. A cyclic module is just a $1-$generated module.

Definition 3.5. A left Banach $\Gamma\mathcal{A}-$ module $\mathcal{X}$ is $n-$ generated for $n \in \mathcal{N}$ if there exists $x_1, \ldots, x_n \in \mathcal{X}$ such that each $x \in \mathcal{X}$ can be represented as $x = \sum_{k=1}^{n} \alpha.x_k$ for some $\alpha_1 a_1, \ldots, \alpha_n a_n \in \Gamma\mathcal{A}$. A cyclic module is just a $1-$generated module.

In the following proposition and its corollary, another characterization of fully stable Banach $\Gamma\mathcal{A}-$ module relative to ideal is given.

Proposition 3.6. Let $\mathcal{X}$ Banach $\Gamma\mathcal{A}-$ module. Then Baer criterion holds for cyclic submodule of $\mathcal{X}$ if and only if

$$ann_{\Gamma\mathcal{X}}\left(ann_{\Gamma\mathcal{A}}\left(\mathcal{N}_{\alpha x}\right)\right) = \mathcal{N}_{\alpha x} + \Gamma\mathcal{K}\Gamma\mathcal{X} \; for \; each \; x \in \mathcal{X}.$$

Proof. Assume that Baer criterion holds. Let $y \in ann_{\Gamma\mathcal{X}}\left(ann_{\Gamma\mathcal{A}}\left(\mathcal{N}_{\alpha x}\right)\right)$. Define $\theta\left\langle\mathcal{N}_{\alpha x}\right\rangle \to \mathcal{X}$ by $\theta\left(\sigma a.n_{\alpha x}\right) = \sigma a.k_{\beta y}$, for all $\sigma a \in \Gamma\mathcal{A}$. Let $\sigma a_1.n_{\alpha x} = \sigma a_2.n_{\alpha x}$, thus $\sigma\left(a_1 - a_2\right).n_{\alpha x} = 0$, $\sigma\left(a_1 - a_2\right) \in ann_{\Gamma\mathcal{A}}\left(\mathcal{N}_{\alpha x}\right)$, hence $\sigma\left(a_1 - a_2\right) \in ann_{\Gamma\mathcal{A}}\left(\mathcal{K}_{\beta y}\right)$; therefore, $\sigma\left(a_1 - a_2\right).k_{\beta y} = 0$, then $\sigma a_1.k_{\beta y} = \sigma a_2.k_{\beta y}$, hence θ is well-defined, it is clear that θ is a $\Gamma\mathcal{A}-$ multiplier. By the assumption, there exists an element $\eta t \in \Gamma\mathcal{A}$ such that $\theta\left(m_{\alpha x}\right) - \eta t m_{\alpha x} \in \Gamma\mathcal{K}\Gamma\mathcal{X}$ for each $m_{\alpha x} \in \mathcal{N}_{\alpha x}$ which implies that, in particular, $k_{\beta y} - \eta t n_{\alpha x} = \theta\left(n_{\alpha x}\right) - \eta t n_{\alpha x} \in \Gamma\mathcal{K}\Gamma\mathcal{X}$; therefore,

$$ann_{\Gamma\mathcal{X}}\left(ann_{\Gamma\mathcal{A}}\left(\mathcal{N}_{\alpha x}\right)\right) \subseteq \mathcal{N}_{\alpha x} + \Gamma\mathcal{K}\Gamma\mathcal{X}.$$

Hence, $ann_{\Gamma\mathcal{X}}\left(ann_{\Gamma\mathcal{A}}\left(\mathcal{N}_{\alpha x}\right)\right) = \mathcal{N}_{\alpha x} + \Gamma\mathcal{K}\Gamma\mathcal{X}$. Conversely, assume that $ann_{\Gamma\mathcal{X}}\left(ann_{\Gamma\mathcal{A}}\left(\mathcal{N}_{\alpha x}\right)\right) = \mathcal{N}_{\alpha x} + \Gamma\mathcal{K}\Gamma\mathcal{X}$, for each $\mathcal{N}_{\alpha x} \subseteq \mathcal{X}$. Then for each $\Gamma-$ multiplier $\theta\left\langle\mathcal{N}_{\alpha x}\right\rangle \to \mathcal{X}$, and $\tau s \in ann_{\Gamma\mathcal{A}}\left(\mathcal{N}_{\alpha x}\right)$, we have $\tau s\theta\left(n_{\alpha x}\right) = \theta\left(\tau s n_{\alpha x}\right) = 0$. Thus, $\theta\left(n_{\alpha x}\right) \in ann_{\Gamma\mathcal{X}}\left(ann_{\Gamma\mathcal{A}}\left(\mathcal{N}_{\alpha x}\right)\right) = \mathcal{N}_{\alpha x} + \Gamma\mathcal{K}\Gamma\mathcal{X}$, then $\theta\left(n_{\alpha x}\right) - \eta t n_{\alpha x} \in \Gamma\mathcal{K}\Gamma\mathcal{X}$ for some $\eta t \in \Gamma\mathcal{A}$, thus Baer criterion holds. □

Corollary 3.7. *$\mathcal{X}$ is fully stable Banach $\Gamma\mathcal{A}-$ module relative to $\Gamma-$ ideal $\mathcal{J}$ of $\Gamma\mathcal{A}$ if and only if $ann_{\Gamma\mathcal{X}}\left(ann_{\Gamma\mathcal{A}}\left(\mathcal{N}_{\alpha x}\right)\right) = \mathcal{N}_{\alpha x} + \Gamma\mathcal{J}\Gamma\mathcal{X}$ for each $\alpha x \in \Gamma\mathcal{X}$.*

Recall that, an $\mathcal{R}-$module $\mathcal{M}$ is multiplication module if each submodule of $\mathcal{M}$ is of the form $I\mathcal{M}$ for some ideal I of $\mathcal{R}$ [9].

Definition 3.8. Banach $\Gamma\mathcal{A}-$ module $\mathcal{X}$ is called multiplication $\Gamma\mathcal{A}-$ module if each $1-$generated submodule of $\mathcal{X}$ is of the form $\Gamma\mathcal{J}\Gamma\mathcal{X}$ for some $\Gamma-$ ideal $\mathcal{J}$ of $\Gamma\mathcal{A}$.

In the following proposition we discuss the relation between full stability and full stability relative to $\Gamma-$ ideal for Banach $\Gamma\mathcal{A}-$ modules.

Proposition 3.9. Let $\mathcal{X}$ be a multiplication Banach $\Gamma\mathcal{A}-$ module. Then $\mathcal{X}$ is fully stable Banach $\Gamma\mathcal{A}-$module if and only if it is fully stable Banach $\Gamma\mathcal{A}-$module relative to $\Gamma-$ideal $\mathcal{J}$ of $\Gamma\mathcal{A}$.

Proof. Let $\mathcal{N}$ be any $1-$ generated submodule of $\mathcal{X}$ and $f : \mathcal{N} \to \mathcal{X}$ be any $\Gamma-$ homomorphism. If $\mathcal{N} = (0)$, then it is clear that $\mathcal{X}$ is fully stable. Let $\mathcal{N} \neq (0)$, and since $\mathcal{X}$ be a multiplication $\Gamma\mathcal{A}-$ module, then $\mathcal{N} = \Gamma\mathcal{J}\Gamma\mathcal{X}$, for some non-zero $\Gamma-$ ideal $\mathcal{J}$ of $\Gamma\mathcal{A}$. By hypothesis $f(\mathcal{N}) \subseteq \mathcal{N} + \Gamma\mathcal{J}\Gamma\mathcal{X} = \mathcal{N} + \mathcal{N} = \mathcal{N}$. Hence, $\mathcal{X}$ is fully stable $\Gamma\mathcal{A}-$ module relative to Γ-ideal. □

Let $\mathcal{A}$ be a unital Banach algebra, let $\mathcal{Q}$ be an $\mathcal{A}-$ module and let $\alpha > 1$. We shall then say that $\mathcal{Q}$ is $\alpha-$ injective if, whenever $i : \mathcal{M} \to \mathcal{N}$ and $\phi : \mathcal{M} \to \mathcal{Q}$ are $\mathcal{A}-$ module homomorphisms such that i is an isometry ($\mathcal{A}-$ module isomorphism is an isometry $\mathcal{A}-$ multiplier) and $\|\phi\| \leq 1$, there exists an $\mathcal{A}-$ module homomorphism $\theta : \mathcal{N} \to \mathcal{Q}$, such that $\theta oi = \phi$ and $\|\phi\| \leq \alpha$. We shall say that $\mathcal{Q}$ is uniformly injective if it is $\alpha-$injective for some α [10]. In the following definition, the concept of quasi $\alpha-$ injective has been introduced.

Definition 3.10. Let $\mathcal{A}$ be a unital Banach $\Gamma-$ algebra and let $\alpha > 1$. $\Gamma\mathcal{A}-$module $\mathcal{X}$ is called quasi $\alpha-$ injective if, $\phi : \mathcal{N} \to \mathcal{X}$ is $\Gamma\mathcal{A}-$ module homomorphism such that $\|\phi\| \leq 1$, there exists $\Gamma\mathcal{A}-$ module homomorphism $\theta : \mathcal{X} \to \mathcal{X}$, such that $\theta oi = \phi$ and $\|\phi\| \leq \alpha$ where i is an isometry from submodule $\mathcal{N}$ of $\mathcal{X}$ to $\mathcal{X}$. We shall say that $\mathcal{X}$ is quasi injective Banach $\Gamma\mathcal{A}-$module if it is quasi $\alpha-$injective for some α.

Definition 3.11. Let $\mathcal{A}$ be a unital Banach $\Gamma-$ algebra and let $\alpha > 1$. $\Gamma\mathcal{A}-$module $\mathcal{X}$ is called quasi $\alpha-$ injective relative to a $\Gamma-$ ideal $\mathcal{J}$ of $\Gamma\mathcal{A}$ if, $\phi : \mathcal{N} \to \mathcal{X}$ is $\Gamma\mathcal{A}-$ module homomorphism such that $\|\phi\| \leq 1$, there exists $\Gamma\mathcal{A}-$ module homomorphism $\theta : \mathcal{X} \to \mathcal{X}$, such that $(\theta oi)(n) - \phi(n) \in \Gamma\mathcal{J}\Gamma\mathcal{X}$ and $\|\phi\| \leq \alpha$ where i is an isometry from submodule $\mathcal{N}$ of $\mathcal{X}$. We shall say that $\mathcal{X}$ is quasi injective relative to $\Gamma-$ ideal if it is quasi ?$-$injective relative to $\Gamma-$ ideal for some?

The relation between quasi $\alpha-$ injective Banach $\Gamma\mathcal{A}-$module relative to $\Gamma-$ ideal and fully stable Banach $\Gamma\mathcal{A}-$module relative to $\Gamma-$ ideal $\mathcal{J}$ of $\Gamma\mathcal{A}$ has been given in the following proposition.

Proposition 3.12. If $\mathcal{X}$ is fully stable Banach $\Gamma\mathcal{A}-$ module relative to $\Gamma-$ ideal then $\mathcal{X}$ is quasi injective Banach $\Gamma\mathcal{A}-$ module relative to $\Gamma-$ ideal.

Proof. Let $\mathcal{N}$ be submodule of $\mathcal{X}$, let $\alpha > 1$ and $f : \mathcal{N} \to \mathcal{X}$ be any $\Gamma\mathcal{A}-$ module homomorphism such that $\|f\| \leq 1$. Since $\mathcal{X}$ is fully stable Banach $\Gamma\mathcal{A}-$ module relative to $\Gamma-$ ideal $\mathcal{J}$, then $f(\mathcal{N}) \subseteq \mathcal{N} + \Gamma\mathcal{J}\Gamma\mathcal{X}$, thus there exists $\lambda t \in \Gamma\mathcal{A}$ such that $f(n) = (\lambda t)^* n + w$. Define $g : \mathcal{X} \to \mathcal{X}$ by $g(x) = \lambda t x$, it is clear that g is well-defined and $\Gamma\mathcal{A}-$ module homomorphism. Now, $f(x) - g(x) = (\lambda t)^* x + w - (\lambda t)^* x = w \in \Gamma\mathcal{J}\Gamma\mathcal{X}$ and for each $y \in \mathcal{N}$, $(foi)(y) = f(y) - g(y) \in \Gamma\mathcal{J}\Gamma\mathcal{X}$, where i is isometry, and $\|g\| \leq \alpha$ for some α. □

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Accepted: 14.12.2019

The bifurcation analysis of an ecological model involving stage structures in both populations with toxin

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Abstract. In this paper, the sufficient conditions of the affair the local bifurcation for instance saddle node, transcritical and pitchfork for a mathematical model consisting of stage-structured in prey as well as predator and toxicity is considered to study. Lotka–Voltera type of functional response has been proposed; later the model analysed, it is found that the transcritical bifurcation occurs adjacent the equilibrium point E_0. While the saddle - node bifurcation occurs at the equilibrium points E_1 and E_2. It is worth sying; there is no circumstance occurrence of the pitchfork bifurcation to each point. Finally, some arithmetical simulations are used to illustrate the circumstance of local bifurcation of this model.

Keywords: prey-predator, toxin, local bifurcation, Sotomayors theorem.

1. Introduction

Theory of bifurcation is the mathematical analysis of the development in the topological structure or qualitative of a disposed family, for instance, the solutions of a family of differential equations and the integral trajectory of a family of vector fields. Most generally, study of dynamical systems is a bifurcation and it affairs when a slight change contrived to the parameter values of a system motive a sudden topological or qualitative difference in its behaviour, for instance, see [1], [2].

The "bifurcation" word was first introduced by Henri Poincaré in 1885 in his first work in mathematics showing such a attitude [3]. Moreover the bifurcation affair in both continuous systems (express by ordinary differential equations) [4-7] and discrete systems (express by maps) [8-10]. The bifurcation theory is branched into two main classes: global bifurcations and local bifurcations. The first type affair when larger cycles, break out with steady point. This make adjust in the topological design of the trajectories in aspect space which cannot be confining to a limited neighbourhood, as is the main case for second kind [11]. While the second type, which can be entirely analyzed through to small affairs in the local stability properties of steady points, cycles [12].

In the past decades, the above theory has developed very much through the adoption of new concepts and methods and their overview into the theory of dynamic systems. Majeed and Shawka [13] established, the conditions, which assurance the occurring of local bifurcations such for instance (saddle node, transcritical and pitchfork) of all fixed points of eco-epidemiological model involving of the prey-predator model with (SI and SIS) epidemic diseases in prey population only. Khalaf, Majeed and Naji [14] established, the conditions of the happening of local bifurcation near of the steady points of eco-epidemiological. mathematical model involving of prey-predator model with SIS infectious disease in the population of prey.

Moreover, Majeed and Ismaeel [15] studied, the conditions of the occurrence the local bifurcation for a mathematical model with stage-structured in prey in addition to predator with the existence of harvesting and toxicity with specific prominence on the Hopf bifurcation near the coexistence equilibrium point of mathematical model.

Keeping above discussion in the view, in this paper, the occurrence of the conditions of the local bifurcation like "saddle node, transcritical and pitchfork" for a mathematical model with stage-structured in prey in addition to predator and toxicity is considered to study. Lotka–Voltera type of functional response was proposed, and then study and analysis this model, it is detected that the transcritical bifurcation affairs near the equilibrium point E_0 also the happening of saddle-node bifurcation at seady points E_1 and E_2. It is worth saying; there is no likelihood occurrence of the pitchfork bifurcation at each one of point. Finally, some arithmetical simulations are used to illustrate the happening of local bifurcation of this model.

2. The mathematical model [16]

Consider the ecological model given by Majeed [16].

$$
\begin{aligned}
\frac{dX_1}{dT} &= r_1X_2\left(1-\frac{X_2}{k}\right)-\gamma_1X_1-\alpha_1X_1-\beta_1X_1X_4,\\
\frac{dX_2}{dT} &= \alpha_1X_1-\gamma_2X_2-\beta_2X_2X_4,\\
\frac{dX_3}{dT} &= r_2X_4-d_1X_3-l_1X_3-\gamma_3X_3-\alpha_2X_3,\\
\frac{dX_4}{dT} &= \alpha_2X_3+e_1\beta_1X_1X_4+e_2\beta_2X_2X_4-d_2X_4-l_2X_4-\gamma_4X_4-cX_4^2.
\end{aligned}
\tag{1}
$$

With starting conditions $X_1(0)\geq 0$, $X_2(0)\geq 0$, $X_3(0)\geq 0$ and $X_4(0)\geq 0$, where the parameters of this model illustrated in Table 1.

Table 1. The parameters of system (1).

$X_1(t)$	Immature prey population at time t
$X_2(t)$	Mature prey population at time t
$X_3(t)$	Immature predator population at time t
$X_4(t)$	Mature predator population at time t
$r_1 > 0$	Intrinsic growth rate of immature prey
$k > 0$	The carrying capacity
$\alpha_1 > 0$	The grown up rate of immature prey individuals onto mature prey.
$\gamma_i, i = 1, 2, 3, 4$	The coefficients of toxicity to the prey species and predator species respectively
$\beta_1 > 0$ and $\beta_2 > 0$	The predation rate of immature and mature prey by predator respectively
$0 < e_i < 1, i = 1, 2$	The conversion rate of food
$l_1 > 0$ and $l_2 > 0$	The natural death rates of immature and mature predator in the absence of prey respectively
$d_i, i = 1, 2$	The natural death rate for immature and mature predators
$r_2 > 0$	Intrinsic growth rate of immature predator
$c > 0$	Intra- specific competition between the mature predator

Finally, it is considered that there is no accelerated growth in the production of toxic substance of the prey species.

The dimensionless system of the previous proposed model given by , for more details see [16].

$$
\begin{aligned}
&t = r_1 T, a_1 = \frac{\gamma_1}{r_1}, a_2 = \frac{\alpha_1}{r_1}, a_3 = \frac{\beta_1 k}{r_1}, a_4 = \frac{\gamma_2}{r_1}, a_5 = \frac{\beta_2 k}{r_1},\\
&a_6 = \frac{r_2}{r_1}, a_7 = \frac{d_1}{r_1}, a_8 = \frac{l_1}{r_1}, a_9 = \frac{\gamma_3}{r_1},\\
&a_{10} = \frac{\alpha_2}{r_1}, a_{11} = \frac{e_1\beta_1 k}{r_1}, a_{12} = \frac{e_2\beta_2 k}{r_1}, a_{13} = \frac{d_2}{r_1}, a_{14} = \frac{l_2}{r_1},\\
&a_{15} = \frac{\gamma_4}{r_1}, a_{16} = \frac{ck}{r_1}, x = \frac{X_1}{k}, y = \frac{X_2}{k}, z = \frac{X_3}{k}, w = \frac{X_4}{k}.
\end{aligned}
$$

Then the non-dimensional form of system (1) can be written as:

$$
\begin{aligned}
\frac{dx}{dt} &= x\left[\frac{y(1-y)}{x} - (a_1 + a_2) - a_3 w\right] = f_1(x, y, z, w)\\
\frac{dy}{dt} &= y\left[\frac{a_2 x}{y} - a_4 - a_5 w\right] = f_2(x, y, z, w)\\
\frac{dz}{dt} &= z\left[\frac{a_6 w}{z} - (a_7 + a_8 + a_9 + a_{10})\right] = f_3(x, y, z, w)
\end{aligned} \tag{2}
$$

$$\frac{dw}{dt} = w\left[\frac{a_{10}z}{w} + a_{11}x + a_{12}y - (a_{13} + a_{14} + a_{15}) - a_{16}w\right] = f_4(x, y, z, w)$$

with $x(0) \geq 0$, $y(0) \geq 0$, $z(0) \geq 0$ and $w(0) \geq 0$.

2.1 The feasible equilibrium points of system (2)[16]

The feasible equilibrium points of system (2), are detailed below:

The equilibrium point $E_0 = (0, 0, 0, 0)$, is at all times exists, the equilibrium point $E_1 = (\bar{x}, \bar{y}, 0, 0)$ and the (coexistence) equilibrium point $E_2 = (x^*, y^*, z^*, w^*)$, for more details see [16].

3. Investigation of local bifurcation

In this section, the investigation of local bifurcation of system (2) will be examined by using Sotomayor's theorem:

Now, the general Jacobian matrix of system (2), is:

$$J = [a_{ij}]_{4\times 4}, \tag{3}$$

where

$$\begin{aligned}
a_{11} &= -(a_1 + a_2) - a_3 w, a_{12} = 1 - 2y, a_{13} = 0, a_{14} = -a_3 x, a_{21} = a_2,\\
a_{22} &= -a_4 - a_5 w, a_{23} = 0, a_{24} = -a_5 y, a_{31} = 0, a_{32} = 0,\\
a_{33} &= -(a_7 + a_8 + a_9 + a_{10}), a_{34} = a_6, a_{41} = a_{11} w, a_{42} = a_{12} w, a_{43} = a_{10},\\
a_{44} &= a_{11}x + a_{12}y - (a_{13} + a_{14} + a_{15}) - 2a_{16}w.
\end{aligned}$$

Now, we verify that for any nonzero vector $\Omega = (\Omega_1, \Omega_2, \Omega_3, \Omega_4)^T$:

$$D^2 f_\mu(X, \mu)(\Omega, \Omega) = \begin{bmatrix} -2\left(a_3\Omega_1\Omega_4 - \Omega_2^2\right) \\ -a_5\Omega_2\Omega_4 \\ 0 \\ 2\Omega_4\left(a_{11}\Omega_1 + a_{12}\Omega_2 - a_{16}\Omega_4\right) \end{bmatrix}_{4\times 1} \tag{4}$$

and

$$D^3 f_\mu(X, \mu)(\Omega, \Omega, \Omega) = [0, 0, 0, 0]^T. \tag{5}$$

Where $X = (x, y, z, w)^T$ and μ is any bifurcation parameter. In the next theorems, the conditions of accruing local bifurcation close to the equilibrium points are obtained.

Theorem 3.1. *Suppose that the next conditions are realized:*

$$K_1 \neq K_2, \tag{6}$$

where

$$\begin{aligned} K_1 &= \left[\left(2a_3\Omega_1^{[0]} + a_5\Omega_2^{[0]}\right)\Omega_4^{[0]} + a_5\Omega_2^{[0]}\Omega_4^{[0]}\right]\psi_2^{[0]} + 2a_{16}\Omega_4^{[0]^2}\psi_4^{[0]}, \\ K_2 &= 2\left[a_{11}\Omega_1^{[0]} + a_{12}\Omega_2^{[0]}\right]\Omega_4^{[0]}\psi_4^{[0]}. \end{aligned}$$

Then system (2) at the equilibrium point $E_0 = (0, 0, 0, 0)$ with the factor rate,,

$$a_6^\circ = \frac{(a_7 + a_8 + a_9 + a_{10})(a_{13} + a_{14} + a_{15})}{a_{10}}$$

has a transcritical bifurcation, saddle-node bifurcation cannot occur at $E_0 = (0, 0, 0, 0)$.

Proof. The varitional matrix of system (2) at E_0 which is given in [16]

$$J_0 = J(E_0)$$

has zero eigenvalue (say $\lambda_0 z = 0$) at $a_6 = a_6^\circ$, and the Jacobian matrix J_0 with $a_6 = a_6^\circ$ becomes:

$$J_0 = J(E_0) = \begin{bmatrix} -(a_1 + a_2) & 1 & 0 & 0 \\ a_2 & -a_4 & 0 & 0 \\ 0 & 0 & -(a_7 + a_8 + a_9 + a_{10}) & a_6^\circ \\ 0 & 0 & a_{10} & -(a_{13} + a_{14} + a_{15}) \end{bmatrix}.$$

Now, let $\Omega^{[0]} = (\Omega_1^{[0]}, \Omega_2^{[0]}, \Omega_3^{[0]}, \Omega_4^{[0]})^T$ be the eigenvector related to the eigenvalue $\lambda_{0z} = 0$. Thus, $J_0\Omega^{[0]} = 0$, which gives $\Omega_1^{[0]} = \frac{1}{a_2}\Omega_2^{[0]}$, $\Omega_3^{[0]} = \frac{(a_{13}+a_{14}+a_{15})}{a_{10}}\Omega_4^{[0]}$, with $\Omega_2^{[0]}$ and $\Omega_4^{[0]}$ any nonzero real numbers.

Let $\Psi^{[0]} = (\psi_1^{[0]}, \psi_2^{[0]}, \psi_3^{[0]}, \psi_4^{[0]})^T$ be the eigenvector related with the eigenvalue $\lambda_{0z} = 0$ of the matrix J_0^T. Then, we have $J_0^T\Psi^{[0]} = 0$. By solving this equation for $\Psi^{[0]}$, we obtain $\Psi^{[0]} = (\psi_2^{[0]}, \psi_2^{[0]}, \frac{(a_{13}+a_{14}+a_{15})}{a_{10}}\psi_4^{[0]}, \psi_4^{[0]})^T$, where $\psi_2^{[0]}$ and $\psi_4^{[0]}$ any nonzero real numbers at present, consider

$$\frac{\partial f}{\partial a_6} = f_{a_6}(X, a_6) = \left(\frac{\partial f_1}{\partial a_6}, \frac{\partial f_2}{\partial a_6}, \frac{\partial f_3}{\partial a_6}, \frac{\partial f_4}{\partial a_6}\right)^T = (0, 0, w, 0)^T.$$

So, $f_{a_6}(E_0, a_6^\circ) = (0, 0, 0, 0)^T$ and hence $\left(\Psi^{[0]}\right)^T f_{a_6}(E_0, a_6^\circ) = 0$. Therefore, according to Sotomayor's theorem the saddle-node bifurcation cannot occur, whereas the 1^{st} condition of transcritical bifurcation is satisfied.

Now, since

$$Df_{a_6}(X, a_6) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where $Df_{a_6}(X, a_6)$ signifies the derivative of $Df_{a_6}(X, a_6)$ with respect to $X = (x, y, z, w)^T$. Furthermore, it is observed that:

$$Df_{a_6}(X, a_6^\circ)\Omega^{[0]} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{a_2}\Omega_2^{[0]} \\ \Omega_2^{[0]} \\ \frac{(a_{13}+a_{14}+a_{15})}{a_{10}}\Omega_4^{[0]} \\ \Omega_4^{[0]} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \Omega_4^{[0]} \\ 0 \end{bmatrix}$$

so, $\left(\Psi^{[0]}\right)^T \left[Df_{a_6}(X, a_6^\circ)\Omega^{[0]}\right] = \Omega_4^{[0]}\psi_4^{[0]} \neq 0$.

Now, by substituting $\Omega^{[0]}$ in eq. (4), we get

$$D^2 f(E_0, a_6^\circ)(\Omega^{[0]}, \Omega^{[0]}) = \begin{bmatrix} -2\left(a_3\Omega_1^{[0]}\Omega_4^{[0]} + {\Omega_2^{[0]}}^2\right) \\ -a_5\Omega_2^{[0]}\Omega_4^{[0]} \\ 0 \\ 2\Omega_4^{[0]}\left(a_{11}\Omega_1^{[0]} + a_{12}\Omega_2^{[0]} - a_{16}\Omega_4^{[0]}\right) \end{bmatrix}_{4\times 1}$$

Hence, it is obtained that $\left(\Psi^{[0]}\right)^T D^2 f(E_0, a_6^\circ)\left(\Omega^{[0]}, \Omega^{[0]}\right) = -K_1 + K_2$, where K_1 and K_2 are mentioned in the state of the theorem. Thus, if the condition (6) is satisfied, we obtain that

$$\left(\Psi^{[0]}\right)^T D^2 f(E_0, a_6^\circ)(\Omega^{[0]}, \Omega^{[0]}) \neq 0.$$

Thus, agreeing with Sotomayor's theorem system (2) has transcritical bifurcation at E_0 at the parameter $a_6 = a_6^\circ$. □

Theorem 3.2. *Suppose that the next conditions are realized*

(7) $\bar{y} < 1/2$,

(8) $a_6 a_{10} = (a_7 + a_8 + a_9 + a_{10})(a_{13} + a_{14} + a_{15})(a_{11}\bar{x} + a_{12}\bar{y})$,

(9) $a_{16}\Omega_4^{[1]} > a_{11}\Omega_1^{[1]} + a_{12}\Omega_2^{[1]}$.

Then system (2) at the equilibrium point $E_1 = (\bar{x}, \bar{y}, 0, 0)$ with the factor

$$\bar{a_4} = \frac{a_2(1 - 2\bar{y})}{a_1 + a_2},$$

possesses saddle-node bifurcation at $E_1 = (\bar{x}, \bar{y}, 0, 0)$.

Proof. The Jacobian matrix $J(E_1)$ of system (2), which is given in [16]

$$J_1 = J(E_1) = [b_{ij}]_{4\times 4},$$

where

$$\begin{aligned} b_{11} &= -(a_1+a_2), b_{12}=1-2\bar{y}, b_{13}=0, b_{14}=-a_3\bar{x}, b_{21}=a_2, \\ b_{22} &= -a_4, b_{23}=0, b_{24}=-a_5\bar{y}, b_{31}=b_{32}=0, \\ b_{33} &= -\left(a_7+a_8+a_9+a_{10}\right), b_{34}=a_6, b_{41}=b_{42}=0, b_{43}=a_{10}, \\ b_{44} &= a_{11}\bar{x}+a_{12}\bar{y}-(a_{13}+a_{14}+a_{15}). \end{aligned}$$

at the point E_1 has zero eigenvalue (say $\lambda_{1x}=0$) at $a_4=\bar{a_4}$, and the Jacobian matrix J_1 with $a_4=\bar{a_4}$ becomes

$$\bar{J_1}=J(a_4=\bar{a_4})=\left[\bar{b_{ij}}\right]_{4\times 4},$$

where $\bar{b_{ij}}=b_{ij}$ for all $i,j=1,2,3,4$ except $\bar{b_{22}}=-\bar{a_4}$. Now, let $\Omega^{[1]}=(\Omega_1^{[1]},\Omega_2^{[1]},\Omega_3^{[1]},\Omega_4^{[1]})^T$ be the eigenvector related to the eigenvalue $\lambda_{1x}=0$. Thus, $\bar{J_1}\Omega^{[1]}=0$, which gives $\Omega^{[1]}=((\frac{-b_{12}l_1-b_{14}}{b_{11}})\Omega_4^{[1]},l_1\Omega_4^{[1]},\frac{-b_{34}}{b_{33}}\Omega_4^{[1]},\Omega_4^{[1]})^T$, where $\Omega_4^{[1]}$ any non-zero real number, and

$$l_1=\frac{b_{21}b_{14}-b_{11}b_{24}}{-\bar{a_4}b_{11}-b_{12}b_{21}}\Omega^{[1]}.$$

Let $\Psi^{[1]}=(\psi_1^{[1]},\psi_2^{[1]},\psi_3^{[1]},\psi_4^{[1]})^T$ be the eigenvector related with the eigenvalue $\lambda_{1x}=0$ of the matrix $\bar{J_1}^T$. Then, we have $\bar{J_1}^T\Psi^{[1]}=0$. By solving this equation for $\Psi^{[1]}$, we obtain

$$\Psi^{[1]}=\left(\frac{-b_{21}}{b_{11}}\psi_2^{[1]},\psi_2^{[1]},\frac{-b_{43}}{b_{33}}\psi_4^{[1]},\psi_4^{[1]}\right)^T$$

where $\psi_2^{[1]}$ and $\psi_4^{[1]}$ any nonzero real numbers. Now, consider

$$\frac{\partial f}{\partial a_4}=f_{a_4}(X,a_4)=\left(\frac{\partial f_1}{\partial a_4},\frac{\partial f_2}{\partial a_4},\frac{\partial f_3}{\partial a_4},\frac{\partial f_4}{\partial a_4}\right)^T=(0,-y,0,0)^T.$$

So, $f_{a_4}(E_1,\bar{a}_4)=(0,-\bar{y},0,0)^T$ and hence $\left(\Psi^{[1]}\right)^T f_{a_4}(E_1,\bar{a}_4)=-\bar{y}\psi_2^{[1]}\neq 0$. Therefore, according to Sotomayors theorem the first condition of saddle-node bifurcation is satisfied.

Now, by substituting $\Omega^{[1]}$ in eq.(4), we get

$$D^2f(E_1,\bar{a}_4)\left(\Omega^{[1]},\Omega^{[1]}\right)=\begin{bmatrix} -2\left(a_3\Omega_1^{[1]}\Omega_4^{[1]}+{\Omega_2^{[1]}}^2\right) \\ -a_5\Omega_2^{[1]}\Omega_4^{[1]} \\ 0 \\ 2\Omega_4^{[1]}\left(a_{11}\Omega_1^{[1]}+a_{12}\Omega_2^{[1]}-a_{16}\Omega_4^{[1]}\right) \end{bmatrix}$$

Hence, it is obtained that

$$\begin{aligned} \left(\Psi^{[1]}\right)^T D^2f(E_1,\bar{a}_4)\left(\Omega^{[1]},\Omega^{[1]}\right) &= -\frac{2b_{21}\psi_2^{[1]}}{b_{11}}\left[a_3\Omega_1^{[1]}\Omega_4^{[1]}+{\Omega_2^{[1]}}^2\right]-a_5\psi_2^{[1]}\Omega_2^{[1]}\Omega_4^{[1]} \\ &\quad +2\psi_4^{[1]}\left[a_{11}\Omega_1^{[1]}+a_{12}\Omega_2^{[1]}-a_{16}\Omega_4^{[1]}\right], \end{aligned}$$

So, by satisfying condition (9), we obtain that

$$\left(\Psi^{[1]}\right)^T D^2 f\left(E_1, \bar{a}_4\right)\left(\Omega^{[1]}, \Omega^{[1]}\right) \neq 0.$$

Thus, agreeing to Sotomayor's theorem system (2) experiences a saddle node bifurcation at the equilibrium point E_1, but not involvement a transcritical bifurcation at E_1 with the factor a_4 passes through the bifurcation value $a_4 = \bar{a}_4$. □

Theorem 3.3. *Suppose that the next conditions are realized*

$$a_{11}x^* + a_{12}y^* > a_{14} + a_{15} + 2a_{16}w^*, \tag{10}$$

$$d_{11}d_{42} < d_{12}d_{41}, \tag{11}$$

$$a_{16}\Omega_4^{[2]} > a_{11}\Omega_1^{[2]} + a_{12}\Omega_2^{[2]}. \tag{12}$$

Then, our system at the equilibrium point $E_2 = (x^*, y^*, z^*, w^*)$ with the parameter value

$$\begin{aligned} a_{13}^* = \frac{-1}{d_{33}\gamma_3}\left(d_{34}d_{43}\gamma_3 + \gamma_6\gamma_{11} - d_{33}\left(\gamma_7 + \gamma_8\right) + \gamma_9\gamma_{12} + a_{14}\right. \\ \left. + a_{15} + 2a_{16}w^* - a_{11}x^* - a_{12}y^*\right), \end{aligned}$$

possesses saddle-node bifurcation, but a transcritical bifurcation cant occur at $E_2 = (x^*, y^*, z^*, w^*)$.

Proof. The characteristic equation known by eq.(8b) in [16] processes zero eigenvalue (say $\lambda_{2w} = 0$) if and only if $F_4 = 0$ and then E_2 converts to a non-hyperbolic equilibrium point. Clearly, the Jacobian matrix of system (2) at the equilibrium point E_2 with parameter $a_{13} = a_{13}^*$ becomes

$$J_2^* = J_2\left(a_{13} = a_{13}^*\right) = \left[d_{ij}\right]_{4\times 4},$$

where

$$\begin{aligned} &d_{11} = -\left(a_1 + a_2\right) - a_3w^* < 0, d_{12} = 1 - 2y^*, d_{13} = 0, \\ &d_{14} = -a_3x^* < 0, d_{21} = a_2 > 0, \\ &d_{22} = -a_4 - a_5w^* < 0, d_{23} = 0, d_{24} = -a_5y^* < 0, d_{31} = 0, d_{32} = 0, \\ &d_{33} = -\left(a_7 + a_8 + a_9 + a_{10}\right) < 0, d_{34} = a_6 > 0, d_{41} = a_{11}w^*, \\ &d_{42} = a_{12}w^*, d_{43} = a_{10} > 0, \\ &d_{44} = a_{11}x^* + a_{12}y^* - \left(a_{13}^* + a_{14} + a_{15}\right) - 2a_{16}w^*. \end{aligned}$$

Note, that $a_{13}^* > 0$ provided that condition (10) hold.

Now, let $\Omega^{[2]} = (\Omega_1^{[2]}, \Omega_2^{[2]}, \Omega_3^{[2]}, \Omega_4^{[2]})^T$ be the eigenvector corresponding to the eigenvalue $\lambda_{2w} = 0$. Thus, $J_2^{*T}\Omega^{[2]} = 0$, which gives

$$\Omega^{[2]} = \left(l_3\Omega_4^{[2]}, l_2\Omega_4^{[2]}, l_4\Omega_4^{[2]}, \Omega_4^{[2]}\right)^T,$$

where

$$l_2 = \frac{d_{14}d_{21} - d_{11}d_{24}}{d_{11}d_{22} - d_{12}d_{21}},$$
$$l_3 = \frac{-\left(d_{12}l_2 + d_{14}\right)}{d_{11}},$$
$$l_4 = \frac{d_{41}\left(d_{12}l_2 + d_{14}\right) - d_{11}\left(d_{42}l_2 + d_{44}\right)}{d_{11}d_{43}}$$

and $\Omega_4^{[2]}$ any nonzero number.

Let $\Psi^{[2]} = (\psi_1^{[2]}, \psi_2^{[2]}, \psi_3^{[2]}, \psi_4^{[2]})^T$ be the eigenvector related with the eigenvalue $\lambda_{2w} = 0$ of the matrix $J_2^{*^T}$. Then, we have $J_2^{*^T}\psi^{[2]} = 0$. If this equation solved for $\Psi^{[2]}$, we get

$$\Psi^{[2]} = (l_6\psi_4^{[2]}, l_5\psi_4^{[2]}, \frac{-d_{43}}{d_{33}}\psi_4^{[2]}, \psi_4^{[2]})^T,$$

where

$$l_5 = \frac{d_{12}d_{41} - d_{11}d_{42}}{d_{11}d_{22} - d_{12}d_{21}}, l_6 = -\frac{(d_{21}l_5 + d_{41})}{d_{11}},$$

which are positive under condition (11) and $\psi_4^{[2]}$ any nonzero number. Now,

$$\frac{\partial f}{\partial a_{13}} = f_{a_{13}}(X, a_{13}) = \left(\frac{\partial f_1}{\partial a_{13}}, \frac{\partial f_2}{\partial a_{13}}, \frac{\partial f_3}{\partial a_{13}}, \frac{\partial f_4}{\partial a_{13}}\right)^T = (0, 0, 0, -w)^T.$$

So, $f_{a_{13}}(E_2, a_{13}^*) = (0, 0, 0, -w^*)^T$, and hence $(\Psi^{[2]})^T f_(a_{13})(E_2, a_{13}^*) = -w^*\psi_4^{[2]} \neq 0$.

Therefore, agreeing with Sotomayors theorem no a transcritical bifurcation can affair at E_2, while the 1^{st} condition of a saddle-node bifurcation is fulfilled.

Moreover, by substituting $\Omega^{[2]}$ in eq. (4), we get

$$D^2 f_{a_{13}}\left(E_2, a_{13}^*\right)\left(\Omega^{[2]}, \Omega^{[2]}\right) = \begin{bmatrix} -2\left(a_3\Omega_1^{[2]}\Omega_4^{[2]} + {\Omega_2^{[2]}}^2\right) \\ -a_5\Omega_2^{[2]}\Omega_4^{[2]} \\ 0 \\ 2\Omega_4^{[2]}\left(a_{11}\Omega_1^{[2]} + a_{12}\Omega_2^{[2]} - a_{16}\Omega_4^{[2]}\right) \end{bmatrix}.$$

Hence, it is obtained that

$$\left(\Psi^{[2]}\right)^T D^2 f_{a_{13}}\left(E_2, a_{13}^*\right)\left(\Omega^{[2]}, \Omega^{[2]}\right) = K_3 + 2\Omega_4^{[2]}\psi_4^{[2]}\left[a_{11}\Omega_1^{[1]} + a_{12}\Omega_2^{[1]} - a_{16}\Omega_4^{[1]}\right],$$

where

$$K_3 = -2\left[a_3\Omega_1^{[2]}\Omega_4^{[2]} + {\Omega_2^{[2]}}^2\right] l_6\Omega_4^{[2]} - a_5\Omega_2^{[2]}\Omega_4^{[2]}l_5\psi_4^{[2]}.$$

So, by satisfying condition (12), we obtain that

$$\left(\Psi^{[2]}\right)^T D^2 f_{a_{13}}\left(E_2, a_{13}^*\right)\left(\Omega^{[2]}, \Omega^{[2]}\right) \neq 0.$$

Thus, according to Sotomayor's theorem system (2) has a saddle-node bifurcation at $E_2 = (x^*, y^*, z^*, w^*)$, but not experience a transcritical bifurcation at E_2, with the parameter a_{13} passes through the bifurcation value $a_{13} = a_{13}^*$. □

4. Numerical analysis of system (2)

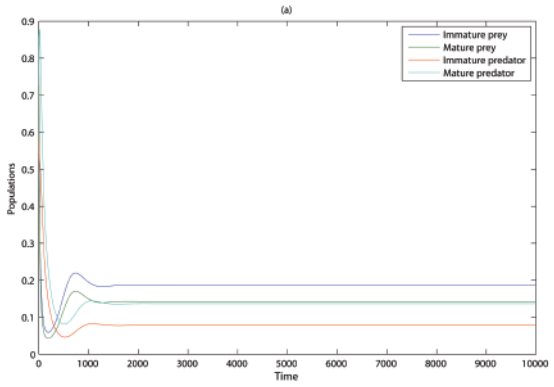
The aim of this section is to study the effect of varying the value of each parameter on the behavior of system (2) numerically, for the following set of parameter which appear in [16].

$$\begin{aligned} &a_1 = 0.1, a_2 = 0.5, a_3 = 0.4, a_4 = 0.1, \\ &a_5 = 0.4, a_6 = 0.4, a_7 = 0.1, a_8 = 0.1, \\ &a_9 = 0.1, a_{10} = 0.4, a_{11} = 0.3, a_{12} = 0.3, \\ &a_{14} = 0.1, a_{15} = 0.1, a_{16} = 0.2. \end{aligned} \tag{13}$$

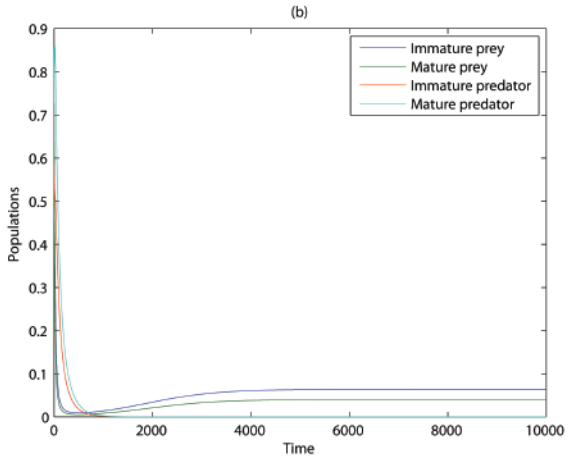
The bifurcation parameter are studied and the effects are given in the next table.

Table 2. Numerical behavior of system (2) for the data given in eq.(13) with varying one parameter at each time.

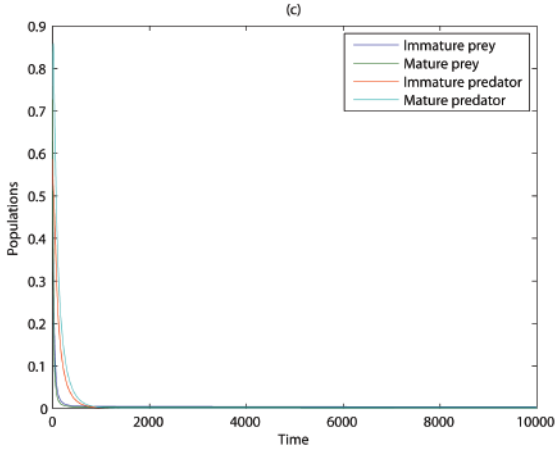
Range of parameter	Numerical behavior of system (2)	The bifurcation point
$0.1 \leq a_1 < 1$	The solution converge to E_2.	
$0.1 \leq a_2 < 1$	The solution converge to E_2.	
$0.1 \leq a_3 < 2$	The solution converge to E_2.	
$0.01 \leq a_4 < 0.74$,	The solution converge to E_2	$a_4 = 0.74$
$0.74 \leq a_4 < 0.832$	The solution converge to E_1	
$0.832 \leq a_4 < 1$	The solution converge to E_0	$a_4 = 0.832$
$0.1 \leq a_5 \leq 2$	The solution converge to E_2	
$0.1 \leq a_6 < 1$	The solution converge to E_2	
$0.1 \leq a_7 < 1$	The solution converge to E_2	
$0.1 \leq a_8 < 1$	The solution converge to E_2	
$0.1 \leq a_9 < 2$	The solution converge to E_2	
$0.1 \leq a_{10} < 1$	The solution converge to E_2	
$0.1 \leq a_i < 1$ for $i = 11, 12$	The solution converge to E_2	
$0.1 \leq a_{13} < 0.345$	The solution converge to E_2	$a_{13} = 0.345$
$0.345 \leq a_{13} < 1$	The solution converge to E_1	
$0.1 \leq a_{14} < 0.345$	The solution converge to E_2	$a_{14} = 0.345$
$0.345 \leq a_{14} < 1$	The solution converge to E_1	
$0.1 \leq a_{15} < 0.345$	The solution converge to E_2	$a_{15} = 0.345$
$0.345 \leq a_{15} < 1$	The solution converge to E_1	
$0.1 \leq a_{16} \leq 2$	The solution converge to E_2	



(a) The solution of system (2) for the numbers assumed by eq. (13) at $a_4 = 0.6$, approaches to $E_2 = (0.187, 0.143, 0.782, 0.137)$ inside of R^4_+



(b) Time series of the solution of System (2) for the values in eq. (13) at $a_4 = 0.8$, approaches to $E_1 = (0.064, 0.039, 0, 0)$ in the positive quadrant of xy-plane



(c) The solution of system (2) for the values in eq. (13) at $a_4 = 0.832$, approaches $E_0 = (0, 0, 0, 0)$.

Figure 1: $a_4 = 0.74$ and $a_4 = 0.832$ are bifurcation points for System (2).

5. Conclusions and discussion

In this work, the sufficient conditions of the happening the local bifurcation for instance saddle node, transcritical and pitchfork for a mathematical model consisting of age-structured in prey as well as predator and toxicity is suggested to study. Lotka –Voltera type of functional response was suggested, later, it is observed that from study and analysis, the transcritical bifurcation affairs near the equilibrium. point E_0. While, the happening of saddle-node bifurcation at equilibrium. Points E_1 and E_2. It is worth saying, there is no possibility happening of the pitchfork bifurcation at each point. Finally, some numerical simulation is used to illustrate the happening of local bifurcation of this model. Thus, system. (2)was resolved numerically for given set of parameters in eq. (13) and sets of starting point the result are summery in the following.

1. The parameter values, a_i, $i = 1, 2, 3, 5, 9, 7, 8, 9, 10, 11, 12$ and 16 do not have any influence on the trajectory of System (2) and the. trajectory of the system approaches to $E_2 = (x^*, y^*, z^*, w^*)$.

2. As the parameter a_4 goes to 0.74 keeping the other parameters as in eq. (4), the trajectory of system (2) approaches to E_2. But if $0.74 \leq a_4 < 0.832$, the trajectory. approaches to $E_1 = (\bar{x}, \bar{y}, 0, 0)$, also, if $0.832 \leq a_4 < 1$ the trajectory approaches E_0 thus, the parameters $a_4 = 0.74$ and $a_4 = 0.832$ are a bifurcation. points of system (2).

3. Finally, the parameters a_{13}, a_{14} and a_{15} have the similar influence on the behavior of solution of system (2), that is when the parameters increasing to 0.345, the solution of system (2) approaches to positive. equilibrium point E_2. While the same parameters increasing as in $0.345 \leq a_i < 1$, $i = 13, 14$ and 15 the solution of system (2) approaches to $E_1 = (\bar{x}, \bar{y}, 0, 0)$ thus, the parameters $a_i = 0.345$ for $i = 13, 14$ and 15 are bifurcation points of system (2).

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Accepted: 15.11.2019

Wrapped shanker distribution

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Abstract. By wrapping Shanker distribution on the real line around a circle, we drive a new circular distribution named wrapped Shanker distribution. The expressions for the densities functions, trigonometric moments, and some related parameters for the new distribution are also obtained.
Keywords: Shanker distribution, circular statistics, wrapped distribution, trigonometric moments.

1. Introduction

Directional data are used in many scientific fields, where the measurements are direction (see, for example, [6],[7]). Direction measurements could be in two or three dimensions, and in many scientific fields the two-dimensional data are known as circular data, since we can represent them as points on the circumference of a unit circle centered at the origin, such that, these points are angles with respect to the starting point and the direction of rotation (i.e., clockwise or anti-clockwise). Obtaining the best statistical inference from any data set depends on selecting the optimal probability distribution for the data set, and for circular data there are different methods to generate circular models from known probability distribution. Many authors generate circular distribution by wrapping a linear distribution around the unit circle, for some of these wrapped probability distributions (see, for example, [1], [2] and [3]).

For the wrapped Shanker distribution, we derive explicit forms for the probability density function, and the cumulative distribution function in Section 2. In Section 3 we obtain the maximum likelihood estimator for the distribution parameter λ. The characteristic function and the trigonometric moments and

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some related parameters are given in Section 4 and Section 5 respectively. Also, we discuss the location and dispersion of the wrapped Shanker distribution in Section 6. We calculate the skewness and kurtosis of the wrapped Shanker distribution in Section 7. While in Section 8 some simulations are done. Finally, some conclusions in Section 9.

2. A wrapped Shanker distribution

A circular distribution is a probability distribution whose total probability is concentrated on the circumference of a unit circle $\{(\cos\ \theta, \sin\ \theta)\}|\theta \in [0, 2\pi)\}$ (see, for example,[8],[9]). The probability density function $g(\theta)$ for a circular random variable θ in a continuous circular distribution must satisfies the following properties:

(1) $g(\theta) \geq 0$ for all θ,

(2) $\int_0^{2\pi} g(\theta)d\theta = 1$,

(3) $g(\theta) = g(\theta + 2\pi m)$ for $m \in \mathbb{Z}$ (i.e., $g(\theta)$ is periodic function).

For any linear random variable X on the real line with probability density function $f(x)$ we may define a circular random variable $\theta = X(mod\ 2\pi)$ with probability density function

$$g(\theta) = \sum_{m=-\infty}^{\infty} f(\theta + 2\pi m), \qquad 0 \leq \theta < 2\pi. \tag{1}$$

Shanker distribution is one parameter lifetime distribution which is a mixture of exponential (λ) and gamma $(2, \lambda)$ distributions introduced by [10]. Let X have a Shanker distribution with probability density function and cumulative distribution function defined as follows:

$$f(x;\lambda) = \frac{\lambda^2}{\lambda^2+1}(\lambda + x)e^{-x\lambda} \qquad ; x > 0, \quad \lambda > 0, \tag{2}$$

and

$$F(x;\lambda) = 1 - \frac{\lambda^2 + x\lambda + 1}{\lambda^2+1}e^{-x\lambda} \qquad ; x > 0, \quad \lambda > 0. \tag{3}$$

Also, Shanker distribution has a characteristic function defined as

$$\varphi_X(t) = \left(1 - \frac{\lambda it}{\lambda^2+1}\right)\left(1 - \frac{it}{\lambda}\right)^{-2} \qquad ; i = \sqrt{-1}. \tag{4}$$

Then we can transform X to a circular random variable $\theta = X(mod\ 2\pi)$, and the probability density function $g(\theta;\lambda)$ can be derived by (1) as follows

$$
\begin{aligned}
g(\theta;\lambda) &= \sum_{m=0}^{\infty} f(\theta+2m\pi) \\
&= \sum_{m=0}^{\infty} \frac{\lambda^2}{\lambda^2+1}(\lambda+\theta+2m\pi)e^{-(\theta+2m\pi)\lambda} \\
&= \frac{\lambda^2\ e^{-\theta\lambda}}{\lambda^2+1}\left[\sum_{m=0}^{\infty}(\lambda+\theta)e^{-2m\pi\lambda} + \sum_{m=0}^{\infty} 2m\pi e^{-2m\pi\lambda}\right] \qquad (5)\\
&= \frac{\lambda^2\ e^{-\theta\lambda}}{\lambda^2+1}\left[(\lambda+\theta)\sum_{m=0}^{\infty} e^{-2m\pi\lambda} + 2\pi\sum_{m=0}^{\infty} m e^{-2m\pi\lambda}\right] \\
&= \frac{\lambda^2\ e^{-\theta\lambda}}{\lambda^2+1}\left[\frac{\lambda+\theta}{1-e^{-2\pi\lambda}} + \frac{2\pi e^{-2\pi\lambda}}{(1-e^{-2\pi\lambda})^2}\right]; \qquad \theta\in[0,2\pi),\ \lambda>0.
\end{aligned}
$$

The cumulative distribution function $G(\theta;\lambda)$ for $\theta\in[0,2\pi)$ and $\lambda>0$ is given by

$$
\begin{aligned}
G(\theta;\lambda) &= \sum_{m=0}^{\infty}\left[F(\theta+2m\pi) - F(2m\pi)\right] \\
&= \sum_{m=0}^{\infty}\left[\left(1-\frac{\lambda^2+\lambda(\theta+2m\pi)+1}{\lambda^2+1}e^{-(\theta+2m\pi)\lambda}\right)\right. \\
&\qquad \left. -\left(1-\frac{\lambda^2+\lambda(2m\pi)+1}{\lambda^2+1}e^{-(2m\pi)\lambda}\right)\right] \qquad (6)\\
&= \sum_{m=0}^{\infty}\left[\frac{(\lambda^2+1+2m\pi\lambda)}{\lambda^2+1}e^{-2m\pi\lambda} - \frac{(\lambda^2+1+\theta\lambda+2m\pi\lambda)}{\lambda^2+1}e^{-\theta\lambda-2m\pi\lambda}\right] \\
&= \frac{1}{1-e^{-2\pi\lambda}}\left[1-e^{-\theta\lambda}-\frac{\theta\lambda e^{-\theta\lambda}}{\lambda^2+1}\right] + \frac{2\pi\lambda(1-e^{-\theta\lambda})e^{-2\pi\lambda}}{(\lambda^2+1)(1-e^{-2\pi\lambda})^2}.
\end{aligned}
$$

Now we are ready to defined the wrapped Shanker distribution as follows

Definition 2.1. *A random variable θ on the unit circle is said to have a wrapped Shanker distribution with parameter $\lambda>0$, denoted by $WS(\lambda)$, if the probability density function is given by*

$$
g(\theta;\lambda) = \frac{\lambda^2\ e^{-\theta\lambda}}{\lambda^2+1}\left[\frac{\lambda+\theta}{1-e^{-2\pi\lambda}} + \frac{2\pi e^{-2\pi\lambda}}{(1-e^{-2\pi\lambda})^2}\right]; \qquad \theta\in[0,2\pi).
$$

Figure 1 shows the probability density function (PDF) of the wrapped Shanker distribution for different values of λ. While, Figure 2 shows the cumulative distribution function of the wrapped Shanker distribution with different values of λ .

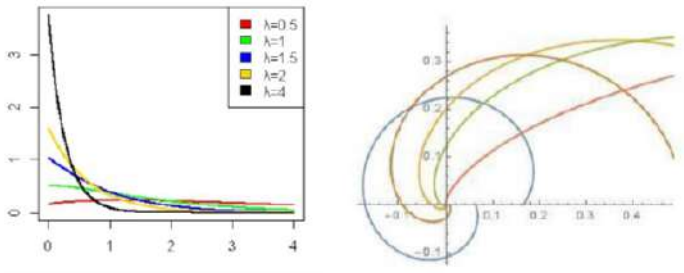


Figure 1: The Linear and Circular representation for the PDF of wrapped Shanker distribution for different values of λ

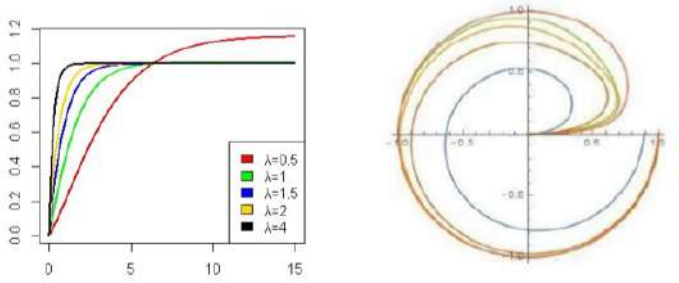


Figure 2: The Linear and Circular representation for the CDF of wrapped Shanker distribution for different values of λ

3. Maximum likelihood estimator

To estimate the distribution parameter λ we use the maximum likelihood estimation technique, the likelihood function $L(\theta_1, \theta_2, \ldots, \theta_n; \lambda)$ is given by

$$\begin{aligned} L(\theta_1, \theta_2, \ldots, \theta_n; \lambda) &= \prod_{i=1}^{n} g(\theta_i; \lambda) \\ &= \prod_{i=1}^{n} \frac{\lambda^2 \ e^{-\theta_i \lambda}}{\lambda^2 + 1} \left[\frac{\lambda + \theta_i}{1 - e^{-2\pi\lambda}} + \frac{2\pi e^{-2\pi\lambda}}{(1 - e^{-2\pi\lambda})^2} \right] \\ &= \frac{\lambda^{2n} e^{-\lambda \sum_{i=1}^{n} \theta_i}}{(\lambda^2 + 1)^n (1 - e^{-2\pi\lambda})^{2n}} \prod_{i=1}^{n} [\lambda + \theta_i + (2\pi - \lambda - \theta_i) e^{-2\pi\lambda}]. \end{aligned} \tag{7}$$

Then the log likelihood function is

$$\begin{aligned} \ln L(\theta_1, \theta_2, \ldots, \theta_n; \lambda) &= 2n \ln(\lambda) \\ &\quad - \lambda \sum_{i=1}^{n} \theta_i - n \ln(\lambda^2 + 1) - 2n \ln(1 - e^{-2\pi\lambda}) \end{aligned} \tag{8}$$

$$+\sum_{i=1}^{n}\ln[\lambda+\theta_i+(2\pi-\lambda-\theta_i)e^{-2\pi\lambda}],$$

the partial derivative of the log likelihood function with respect to λ is

$$\begin{aligned}\frac{d\ \ln L(\theta_1,\theta_2,\ldots,\theta_n;\lambda)}{d\lambda} &= \frac{2n}{\lambda}-\sum_{i=1}^{n}\theta_i-\frac{2n\lambda}{\lambda^2+1}-\frac{4n\pi e^{-2\pi\lambda}}{1-e^{-2\pi\lambda}}\\ &\quad+\sum_{i=1}^{n}\left[\frac{1-(1-2\pi(\lambda+\theta_i)+4\pi^2)e^{-2\pi\lambda}}{\lambda+\theta_i+(2\pi-\lambda-\theta_i)e^{-2\pi\lambda}}\right].\end{aligned}\tag{9}$$

Equating the partial derivative (9) to zero and using some numerical techniques we can find a solution for the estimator of the parameter λ.

4. Characteristic function

The continuous circular wrapped Shanker distribution can be described by its characteristic function as follows

$$\begin{aligned}\varphi_\theta(t) &= E\left(e^{it\theta}\right)=\int_0^{2\pi}e^{it\theta}g(\theta;\lambda)d\theta\\ &=\int_0^{2\pi}e^{it\theta}\left[\left(\frac{\lambda^2\ e^{-\theta\lambda}}{\lambda^2+1}\right)\left(\frac{\lambda+\theta}{1-e^{-2\pi\lambda}}+\frac{2\pi e^{-2\pi\lambda}}{(1-e^{-2\pi\lambda})^2}\right)\right]d\theta\\ &=\frac{\lambda^2}{(\lambda^2+1)(1-e^{-2\pi\lambda})^2}\int_0^{2\pi}e^{(it-\lambda)\theta}\left((\lambda+\theta)(1-e^{-2\pi\lambda})+2\pi e^{-2\pi\lambda}\right)d\theta\\ &=\frac{\lambda^2}{(\lambda^2+1)(1-e^{-2\pi\lambda})^2}\left[\left(\lambda(1-e^{-2\pi\lambda})+2\pi e^{-2\pi\lambda}\right)\int_0^{2\pi}e^{(it-\lambda)\theta}d\theta\right.\\ &\qquad\left.+(1-e^{-2\pi\lambda})\int_0^{2\pi}\theta e^{(it-\lambda)\theta}d\theta\right]\\ &=\frac{\lambda^2(1-e^{-2\pi\lambda})(e^{2\pi(it-\lambda)}-1)(\lambda(it-\lambda)-1)+2\pi\lambda^2(it-\lambda)e^{2\pi(it-\lambda)}}{(\lambda^2+1)(1-e^{-2\pi\lambda})^2(it-\lambda)^2}.\end{aligned}\tag{10}$$

However, we know that θ is a periodic random variable (i.e., $g(\theta;\lambda)=g(\theta+2\pi;\lambda)$), then the characteristic function $\varphi_\theta(t)$ must be defined only at integer values.

5. Trigonometric moments and some related parameters

[6] showed that the p^{th} trigonometric moment for a wrapped circular distribution is equal to the value of the characteristic function of the unwrapped random variable at the integer value p. Then the p^{th} trigonometric moment for wrapped

Shanker distribution is given by

$$\begin{aligned}\varphi_\theta(p) = \varphi_X(p) &= \left(1 - \frac{\lambda ip}{\lambda^2+1}\right)\left(1 - \frac{ip}{\lambda}\right)^{-2} \\ &= \frac{\lambda^2}{\lambda^2+1}((\lambda^2+1) - ip\lambda)(\lambda - ip)^{-2}\ ; i = \sqrt{-1},\ p = \mp 1, \mp 2, \ldots\end{aligned} \tag{11}$$

For $a, b, r \in \Re$, we know that $(a - ib)^{-r} = (a^2+b^2)^{\frac{-r}{2}} e^{ir \arctan(b/a)}$. Then we may write

$$((\lambda^2+1) - ip\lambda) = ((\lambda^2+1)^2 + (p\lambda)^2)^{\frac{1}{2}} e^{-i\ \arctan\left(\frac{p\lambda}{\lambda^2+1}\right)}, \tag{12}$$

$$(\lambda - ip)^{-2} = (\lambda^2+p^2)^{-1} e^{2i\ \arctan\left(\frac{p}{\lambda}\right)}. \tag{13}$$

Therefore, by substituting formulas (9) and (10) in equation (8) the p^{th} trigonometric moment for the wrapped Shanker distribution $\varphi_\theta(p)$ can be written as $\varphi_\theta(p) = \rho_p e^{i\mu_p}$; $\quad i = \sqrt{-1},\ p = \mp 1, \mp 2, \ldots$, where

$$\rho_p = \frac{\lambda^2\sqrt{(\lambda^2+1)^2 + p^2\lambda^2}}{(\lambda^2+1)(\lambda^2+p^2)}, \tag{14}$$

$$\mu_p = 2\ \arctan\left(\frac{p}{\lambda}\right) - \arctan\left(\frac{p\lambda}{\lambda^2+1}\right). \tag{15}$$

According to Euler's formula, there is a relation between the trigonometric functions and the complex exponential function, such that, for any real number z, we have $e^{iz} = \cos\ z + i \sin\ z$. Then we can defined the p^{th} trigonometric moment of the wrapped Shanker distribution $\varphi_\theta(p)$ in terms of the p^{th} cosine and sine moments given by $\alpha_p = E(\cos\ p\theta)$ and $\beta_p = E(\sin\ p\theta)$ as $\varphi_\theta(p) = \alpha_p + i\beta_p$. By the definition of the p^{th} trigonometric moment, we have $\varphi_\theta(p) = \rho_p e^{i\mu_p}$, $\alpha_p + i\beta_p = \rho_p(\cos\ \mu_p + i \sin\ \mu_p)$, then the non-central trigonometric moments of the wrapped Shanker distribution are given by

$$\begin{aligned}\alpha_p &= \rho_p \cos\ \mu_p \\ &= \frac{\lambda^2\sqrt{(\lambda^2+1)^2 + p^2\lambda^2}}{(\lambda^2+1)(\lambda^2+p^2)} \cos\left[2\ \arctan\left(\frac{p}{\lambda}\right) - \arctan\left(\frac{p\lambda}{\lambda^2+1}\right)\right],\end{aligned} \tag{16}$$

and

$$\begin{aligned}\beta_p &= \rho_p \sin\ \mu_p \\ &= \frac{\lambda^2\sqrt{(\lambda^2+1)^2 + p^2\lambda^2}}{(\lambda^2+1)(\lambda^2+p^2)} \sin\left[2\ \arctan\left(\frac{p}{\lambda}\right) - \arctan\left(\frac{p\lambda}{\lambda^2+1}\right)\right].\end{aligned} \tag{17}$$

Now, the central trigonometric moments of the wrapped Shanker distribution are given by

$$\begin{aligned}\bar{\alpha}_p &= \rho_p \cos(\mu_p - p\mu_1) \\ &= \frac{\lambda^2\sqrt{(\lambda^2+1)^2 + p^2\lambda^2}}{(\lambda^2+1)(\lambda^2+p^2)} \cos\left[2\ \arctan\left(\frac{p}{\lambda}\right) - \arctan\left(\frac{p\lambda}{\lambda^2+1}\right)\right.\end{aligned} \tag{18}$$

$$-2p\ \arctan\left(\frac{1}{\lambda}\right)+p\arctan\left(\frac{\lambda}{\lambda^2+1}\right)\Bigg],$$

and

$$\begin{aligned}\bar{\beta}_p &= \rho_p\sin(\mu_p-p\mu_1)\\ &= \frac{\lambda^2\sqrt{(\lambda^2+1)^2+p^2\lambda^2}}{(\lambda^2+1)(\lambda^2+p^2)}\sin\Bigg[2\arctan\left(\frac{p}{\lambda}\right)-\arctan\left(\frac{p\lambda}{\lambda^2+1}\right)\\ &\qquad -2p\arctan\left(\frac{1}{\lambda}\right)+p\arctan\left(\frac{\lambda}{\lambda^2+1}\right)\Bigg].\end{aligned}\tag{19}$$

6. Location and dispersion

The first trigonometric moment of the wrapped Shanker distribution given by $\varphi_\theta(1)=\rho_1 e^{i\mu_1}$ contains important information about the distribution. The mean direction and concentration of θ are measured by the mean resultant length ρ, and the mean direction μ given by

$$\rho=\rho_1=\frac{\lambda^2\sqrt{(\lambda^2+1)^2+\lambda^2}}{(\lambda^2+1)^2},\tag{20}$$

$$\mu=\mu_1=2\arctan\left(\frac{1}{\lambda}\right)-\arctan\left(\frac{\lambda}{\lambda^2+1}\right).\tag{21}$$

The length ρ lies between 0 and 1, and measures the dispersion around the mean. The closer ρ is to 1 the more concentration towards μ. In a circular distribution, the circular variance is calculated by $V=1-\rho$, then the circular variance for the wrapped Shanker distribution is given by

$$V=1-\rho=1-\frac{\lambda^2\sqrt{(\lambda^2+1)^2+\lambda^2}}{(\lambda^2+1)^2},\tag{22}$$

While, the circular standard deviation is calculated by $\sigma=\sqrt{-2\ln\rho}$, then the circular standard deviation for the wrapped Shanker distribution is given by

$$\sigma=\sqrt{-2\ln\rho}=\sqrt{-2\ln\left(\frac{\lambda^2\sqrt{(\lambda^2+1)^2+\lambda^2}}{(\lambda^2+1)^2}\right)}.\tag{23}$$

7. Skewness and kurtosis

For a circular model the skewness coefficient is calculated by $\zeta_1=\bar{\beta}_2V^{-3/2}$, and the kurtosis coefficient is calculated by $\zeta_2=\left(\bar{\alpha_2}-(1-V)^4\right)V^{-2}$. Then for the wrapped Shanker distribution the skewness coefficient and the kurtosis

coefficient are given respectively by

$$\begin{aligned}\zeta_1 &= \bar{\beta}_2 V^{-3/2} \\ &= \left(1-\frac{\lambda^2\sqrt{(\lambda^2+1)^2+\lambda^2}}{(\lambda^2+1)^2}\right)^{\frac{-3}{2}}\left(\frac{\lambda^2\sqrt{(\lambda^2+1)^2+4\lambda^2}}{(\lambda^2+1)(\lambda^2+4)}\sin\left[2\arctan\left(\frac{2}{\lambda}\right)\right.\right. \\ &\quad \left.\left.-\arctan\left(\frac{2\lambda}{\lambda^2+1}\right)-4\ \arctan\left(\frac{1}{\lambda}\right)+2\arctan\left(\frac{\lambda}{\lambda^2+1}\right)\right]\right), \end{aligned} \tag{24}$$

and

$$\begin{aligned}\zeta_2 &= \left(\bar{\alpha_2}-(1-V)^4\right)V^{-2} \\ &= \left(1-\frac{\lambda^2\sqrt{(\lambda^2+1)^2+\lambda^2}}{(\lambda^2+1)^2}\right)^{-2}\left(\frac{\lambda^2\sqrt{(\lambda^2+1)^2+4\lambda^2}}{(\lambda^2+1)(\lambda^2+4)}\cos\left[2\arctan\left(\frac{2}{\lambda}\right)\right.\right. \\ &\quad \left.\left.-\arctan\left(\frac{2\lambda}{\lambda^2+1}\right)-4\arctan\left(\frac{1}{\lambda}\right)+2\arctan\left(\frac{\lambda}{\lambda^2+1}\right)\right]-\right. \\ &\quad \left.\left(\frac{\lambda^2\sqrt{(\lambda^2+1)^2+\lambda^2}}{(\lambda^2+1)^2}\right)^4\right). \end{aligned} \tag{25}$$

The following Table 1 shows different parameters of the wrapped Shanker distribution for various values of λ, which are also used in Figure 1 and Figure 2.

Parameter	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 1.5$	$\lambda = 2$	$\lambda = 4$
μ	1.83379	1.10715	0.74360	0.54679	0.25887
α_1	-0.05600	0.25000	0.56122	0.73600	0.93466
α_2	-0.02976	0.04000	0.21822	0.40000	0.78177
β_1	0.20800	0.50000	0.51615	0.44800	0.24751
β_2	0.06920	0.28000	0.43865	0.50000	0.41412
ρ	0.21541	0.55902	0.76249	0.86163	0.96688
V	0.78459	0.44098	0.23751	0.138373	0.03312
σ	1.75227	1.07850	0.73644	0.54577	0.25955
$\bar{\alpha}_1$	0.21541	0.55902	0.76249	0.86163	0.96688
$\bar{\alpha}_2$	-0.00901	0.20000	0.45534	0.62786	0.88375
$\bar{\beta}_1$	0	0	0	0	0
$\bar{\beta}_2$	-0.07479	-0.20000	-0.18082	-0.12566	-0.02677
ζ_1	-0.10762	-0.68296	-1.56217	-2.44118	-4.44089
ζ_2	-0.01813	0.52628	2.07978	4.00602	8.93007

Table 1: Values of different parameters of Wrapped Shanker Distribution for various values of λ

8. Simulation

We used simulation technique to study the behaviour of wrapped Shanker distribution for different values of λ and different sample size n. Figure 3, Figure 4, and Figure 5 show the distribution for a sample size $n = 30, 50, 100$ and $\lambda = 0.5, 1, 2$ respectively.

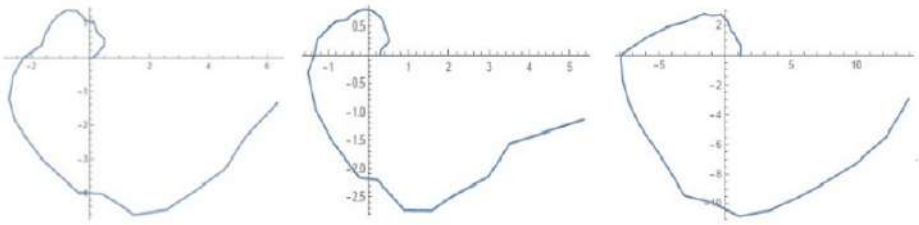


Figure 3: Wrapped Shanker distribution for $n = 30$, $\lambda = 0.5$, $\lambda = 1$, and $\lambda = 2$ respectively

Parameter	μ	ρ	V	σ	ζ_1	ζ_2
$\lambda = 0.5$	1.1524	0.2222	0.7778	99.3702	-0.1392	0.1171
$\lambda = 1$	1.2737	0.5812	0.4188	59.6901	-0.9314	0.9769
$\lambda = 2$	2.0672	0.3181	0.6819	86.7154	-0.5026	0.4019

Table 2: Simulation values of different parameters of Wrapped Shanker Distribution for various values of λ for sample size $n = 30$.

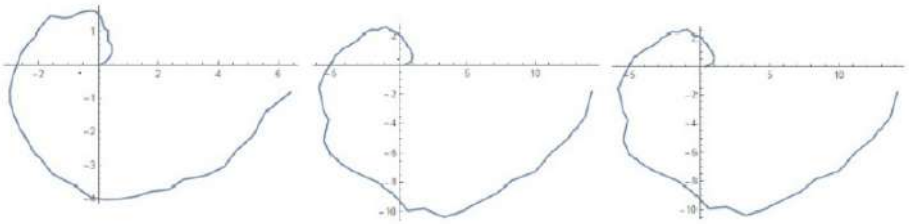


Figure 4: Wrapped Shanker distribution for $n = 50$, $\lambda = 0.5$, $\lambda = 1$, and $\lambda = 2$ respectively

Parameter	μ	ρ	V	σ	ζ_1	ζ_2
$\lambda = 0.5$	2.3342	0.1728	0.8272	107.3629	0.0725	-0.1802
$\lambda = 1$	2.3334	0.2840	0.7160	90.9095	0.1224	0.4682
$\lambda = 2$	1.5500	0.2421	0.7579	96.4978	-0.0188	0.3177

Table 3: Simulation values of different parameters of Wrapped Shanker Distribution for various values of λ for sample size $n = 50$.

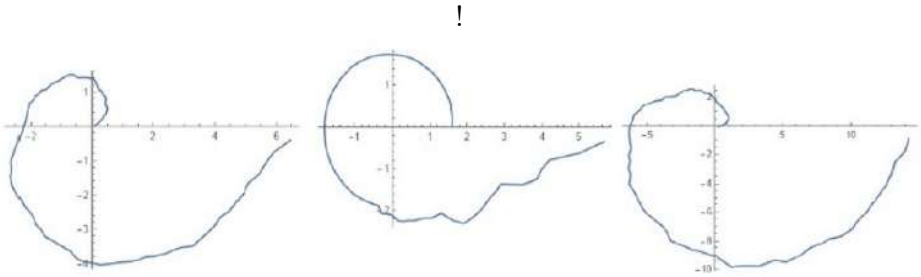


Figure 5: Wrapped Shanker distribution for $n = 100$, $\lambda = 0.5$, $\lambda = 1$, and $\lambda = 2$ respectively

Parameter	μ	ρ	V	σ	ζ_1	ζ_2
$\lambda = 0.5$	2.0016	0.1794	0.8206	106.2023	-0.1689	-0.0341
$\lambda = 1$	1.9764	0.8301	0.1699	34.9679	-2.3751	7.7983
$\lambda = 2$	1.4219	0.3088	0.6912	87.8387	-0.1648	0.2604

Table 4: Simulation values of different parameters of Wrapped Shanker Distribution for various values of λ for sample size $n = 100$.

9. Conclusions

We introduced a new circular distribution, namely, wrapped Shanker distribution. The explicit expressions for the probability density function and the cumulative density function were derived for the new distribution. Also, we obtained the characteristic function, and studied the trigonometric moments and some related parameters for the new distribution. Finally, we discussed the location, dispersion, skewness, and kurtosis of the new distribution.

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Accepted: 11.12.2019

The Lie integral method for differential equations

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Abstract. The integral theory of differential equations is a very important subject in mathematical science. In this paper, we mainly introduce the Lie integral method for some nonlinear integrable differential equations. The detailed steps and main conclusions of the one-order ordinary differential equations, the two-order ordinary differential equations and the typical partial differential equations are discussed respectively. Three examples are given to illustrate the calculation. The results show that the key of the Lie integral method is the calculation of Lie symmetry generators, which is easy to program. The purpose is to spread the idea of group analysis for differential equations.

Keywords: differential equations, Lie integral, Lie symmetry, group analysis

1. Introduction

Differential equations play an important role in mathematics, physics and engineering technology, the solutions to various differential equations have always been one of the hot topics in academia [1]. Comparing with various numerical methods, the problem of general solution of differential equations has not been solved well in the world. Some existing analytical integration methods such as constant variation method [2], first integral method [3], integral factor method [4], integral transformation method [5], special function method [6] all depend on the linearity of differential equations and the existence of constant coefficients. In order to obtain the general solution of non-linear differential equations, it is impossible to follow the simple linear superposition principle, the Lie integral method is equal in dealing with the linear and non-linear equations, and also the constant coefficients and variable coefficients. It will become the only universal and effective tool for studying the differential equations [7]. For the study of symmetry and integral method of differential equations, readers can refer to references [8-12]. The core technique of Lie integral method for differential equations is continuous group transformation.

The original equation can be classified according to its symmetry, or the group invariant solution can be constructed by its symmetry which can reduce the dimension. In this paper, the principle and application of Lie integral for differential equations are introduced in plain language.

2. Lie integral method for one-order ordinary differential equation

The one-order ordinary differential equation can be expressed as follows:

$$\dot{y} = f(x, y). \tag{1}$$

Introducing a single parameter continuous transformation group $G = \{T_a | \bar{x} = \varphi(x, y, a), \bar{y} = \psi(x, y, a)\}$, the generator of group G is:

$$X^{(0)} = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}. \tag{2}$$

Here, $(\xi(x, y) = \frac{\partial \varphi(x,y,a)}{\partial a} |_{a=0}, \eta(x, y) = \frac{\partial \psi(x,y,a)}{\partial a} |_{a=0})$ is the tangent vector of group G.

The necessary and sufficient condition for the invariance (symmetry) of differential equation (1) under group G transformation is:

$$\xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y} = \eta_x + (\eta_y - \xi_x)f - \xi_y f^2. \tag{3}$$

The group G is also called the symmetric group of equation (1), and the corresponding tangent vector is also called infinitesimal symmetry.

Introducing typical variables are:

$$t = \theta(x, y), u = \vartheta(x, y) \tag{4}$$

The θ, ϑ are determined by the following one-order partial differential equations:

$$\begin{cases} \xi\dfrac{\partial \theta}{\partial x} + \eta\dfrac{\partial \theta}{\partial y} = 1, \\ \xi\dfrac{\partial \vartheta}{\partial x} + \eta\dfrac{\partial \vartheta}{\partial y} = 0. \end{cases} \tag{5}$$

So far, the steps of Lie integral method for one-order ordinary differential equation are as follows:

- the first step is to find typical variables t, u by solving equations (3) and (5) jointly;
- the second step is to rewrite the original differential equation (1) and assume that the new variable satisfies $u = u(t)$, so the original derivative $\dot{y} = \frac{dy}{dx}$ is replaced by the new derivative $\dot{u} = \frac{du}{dt}$, then the original equation can be transformed into an integrable form: $\dot{u} = g(t, u)$;
- the third step is to integrate the new differential equation transformed in the second step, and get its solution as follows: $u = \phi(t, C)$, then the (4) is substituted into it, so the general solution of the original differential equation (1) can be obtained.

For one-order ordinary differential equations:

$$\frac{dy^\alpha}{dx} = f_\alpha(x,y)(\alpha = 1,2,...,q). \tag{6}$$

Then, a new single parameter continuous transformation group is $G = \{T_a|\bar{x} = \varphi(x,y,a), \bar{y}^\alpha = \psi^\alpha(x,y,a)\}$, the new typical variables transformation are: $t = \theta(x,y), u^\alpha = \vartheta^\alpha(x,y)$, so the equations (6) are equivalent to:

$$\frac{du^\alpha}{dt} = g_\alpha(t,u^1,u^2,...,u^{q-1})(\alpha = 1,2,...,q). \tag{7}$$

Because there is no u^q in g_α, the solution of equations (7) which contain $q-1$ number of unknown functions, as long as the $u^1(t), u^2(t), ..., u^{q-1}(t)$ are obtained, the $u^q(t)$ can be integraled directly:

$$u^q(t) = \int g_q(t,u^1(t),u^2(t),...,u^{q-1}(t))dt + c. \tag{8}$$

It is not difficult to see that the transformation $\theta(x,y), \vartheta^\alpha(x,y)$ satisfy:

$$\begin{cases} \xi\dfrac{\partial\theta}{\partial x} + \eta^\alpha\dfrac{\partial\theta}{\partial y^\alpha} = 1(\alpha = 1,2,..,q-1) \\ \xi\dfrac{\partial\vartheta^\alpha}{\partial x} + \eta^\alpha\dfrac{\partial\vartheta^\alpha}{\partial y^\alpha} = 0(\alpha = 1,2,..,q-1) \\ \xi\dfrac{\partial\vartheta^q}{\partial x} + \eta^\alpha\dfrac{\partial\vartheta^q}{\partial y^\alpha} = 1. \end{cases} \tag{9}$$

3. Lie integral method for two-order ordinary differential equation

The two-order ordinary differential equation can be expressed as follows:

$$\ddot{y} = h(x,y,\dot{y}). \tag{10}$$

The necessary and sufficient condition for the invariance (symmetry) of differential equation (10) under group G transformation is as follows:

$$\begin{cases} \eta_{xx} + (2\eta_{xy} - \xi_{xx})\dot{y} + (\eta_{yy} - 2\xi_{xy})\dot{y}^2 - \dot{y}^3\xi_{yy} - \xi h_x - \eta h_y \\ +(\eta_y - 2\xi_x - 3\dot{y}\xi_y)h - (\eta_x + (\eta_y - \xi_x)\dot{y} - \dot{y}^2\xi_y)h_{\dot{y}} = 0. \end{cases} \tag{11}$$

The equation (11) contains three variables $x, y, \dot{y}$, which is called Lie determining equation. Generally speaking, $\dot{y}$ does not appear in ξ, η, so the deterministic equation can be decomposed into several equations, which are the overdetermined partial differential equations of ξ, η. In this way, the infinitesimal symmetric solution is transformed into solving determining equation.

If the Lie bracket $[,]$ is defined in partial differential operator space, the corresponding Lie algebraic structure $\{X\}$ is obtained:

$$[X_1, X_2] = X_1X_2 - X_2X_1. \tag{12}$$

Here, X_1, X_2 are elements in partial differential operator space X.

An important discovery of Sophus Lie is that the infinitesimal generator solution of equation (11) must constitute Lie algebra structure. The Lie integral method of two-order ordinary differential equation means that any two-dimensional Lie algebra can be transformed into one of four non-similar standard forms by selecting proper base and proper variable, which are called typical variables. Furthermore, the original differential equation can be transformed into one of four standard differential equations, and its general solution can be solved maturely, then we can deduced conversely the general solution of original equation. The details are as follows:

The first step is to solve the equation (11). If the generator vectors $X_1^{(0)}, X_2^{(0)}$ are obtained, four types can be classified by introducing typical variables (4), such as Table 1:

Type	Structural characteristics	The standard form of generator
I	$\left\{X_1^{(0)}, X_2^{(0)}\right\} = 0, \xi_1\eta_2 - \eta_1\xi_2 \neq 0$	$X_1^{(0)} = \frac{\partial}{\partial t}, X_2^{(0)} = \frac{\partial}{\partial u}$
II	$\left\{X_1^{(0)}, X_2^{(0)}\right\} = 0, \xi_1\eta_2 - \eta_1\xi_2 = 0$	$X_1^{(0)} = \frac{\partial}{\partial u}, X_2^{(0)} = t\frac{\partial}{\partial u}$
III	$\left\{X_1^{(0)}, X_2^{(0)}\right\} = X_1^{(0)}, \xi_1\eta_2 - \eta_1\xi_2 \neq 0$	$X_1^{(0)} = \frac{\partial}{\partial u}, X_2^{(0)} = t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}$
IV	$\left\{X_1^{(0)}, X_2^{(0)}\right\} = X_1^{(0)}, \xi_1\eta_2 - \eta_1\xi_2 = 0$	$X_1^{(0)} = \frac{\partial}{\partial u}, X_2^{(0)} = u\frac{\partial}{\partial u}$

The second step is to solve the θ, ϑ, such as Table 2:

Type	Typical transformation satisfy
I	$X_1^{(0)}(\theta) = 1, X_2^{(0)}(\theta) = 0; X_1^{(0)}(\vartheta) = 0, X_2^{(0)}(\vartheta) = 1$
II	$X_1^{(0)}(\theta) = 0, X_2^{(0)}(\theta) = 0; X_1^{(0)}(\vartheta) = 1, X_2^{(0)}(\vartheta) = \theta$
III	$X_1^{(0)}(\theta) = 0, X_2^{(0)}(\theta) = \theta; X_1^{(0)}(\vartheta) = 1, X_2^{(0)}(\vartheta) = \vartheta$
IV	$X_1^{(0)}(\theta) = 0, X_2^{(0)}(\theta) = 0; X_1^{(0)}(\vartheta) = 1, X_2^{(0)}(\vartheta) = \vartheta$

The third step is to replace the original two-order ordinary differential equation with the typical variable, then change it into one of the four standard differential equation forms, such as Table 3.

Type	The standard form of Differential Equations
I	$\ddot{u} = h(\dot{u})$
II	$\ddot{u} = h(t)$
III	$\ddot{u} = \frac{1}{t}h(\dot{u})$
IV	$\ddot{u} = h(t)\dot{u}$

The general solution is obtained simply by integrating the standard equation. Finally, the original variable x, y is used to represent the result.

For two-order ordinary differential equations, our general method is to change them into one-order ordinary differential equations by increasing the dimension appropriately (augmented vector method), and then we can use the content of previous chapter to solve it completely. Of course, for two-order ordinary differential equations, we can also reduce and decompose them directly by introducing a multi-parameter continuous group transformation. Referring to the reference [9], for higher-order equations (more than three-order), We can deal with them by analogy. So considering the complexity of length and content, it is not necessary to elaborate here.

4. Lie integral method for partial differential equation

The main idea of Lie integral method for partial differential equation is: firstly, using the invariant condition of differential equation to obtain the allowable infinitesimal symmetry (it constitutes Lie algebraic structure), then, using similarity reduction to reduce the number of variables (group invariant solution theory), so that the nonlinear partial differential equation can be transformed into ordinary differential equation, finally, the solution of ordinary differential equation can be obtained, the solution of original partial differential equation is also obtained. Of course, the content of Lie integral method is far more than that, the other profound content will not be introduced in this chapter.

Let's look at a typical partial differential equation (nonlinear evolution equation):

$$u_t = F(t, x, u, u_x, u_{xx}), \partial F/\partial u_{xx} \neq 0. \tag{13}$$

Introducing a single parameter continuous transformation group $G = \{T_a | \bar{t} = \varphi(t, x, u, a), \bar{x} = \rho(t, x, u, a), \bar{u} = \psi(t, x, u, a)\}$, the generator of group G is:

$$X^{(0)} = \tau(t, x, u)\frac{\partial}{\partial t} + \xi(t, x, u)\frac{\partial}{\partial x} + \eta(t, x, u)\frac{\partial}{\partial u}. \tag{14}$$

Here,

$$(\tau(t, x, u) = \frac{\partial \varphi(t, x, u, a)}{\partial a}\,|_{a=0}\,, \xi(t, x, u) = \frac{\partial \rho(t, x, u, a)}{\partial a}\,|_{a=0}\,,$$
$$\eta(t, x, u) = \frac{\partial \psi(t, x, u, a)}{\partial a}\,|_{a=0}\,)$$

is the tangent vector of group G.

The necessary and sufficient condition for the invariance (symmetry) of differential equation (13) under group G transformation is:

$$\begin{aligned}
&\zeta_0 - \frac{\partial F}{\partial u_{xx}}\zeta_2 - \frac{\partial F}{\partial u_x}\zeta_1 - \frac{\partial F}{\partial u}\eta - \frac{\partial F}{\partial x}\xi - \frac{\partial F}{\partial t}\tau = 0,\\
&\zeta_0 = D_t(\eta) - u_t D_t(\tau) - u_x D_t(\xi),\\
&\zeta_1 = D_x(\eta) - u_t D_x(\tau) - u_x D_x(\xi),\\
&\zeta_2 = D_x(\zeta_1) - u_{tx} D_x(\tau) - u_{xx} D_x(\xi),\\
&D_t = \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{tt}\frac{\partial}{\partial u_t} + u_{tx}\frac{\partial}{\partial u_x},\\
&D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{tx}\frac{\partial}{\partial u_t} + u_{xx}\frac{\partial}{\partial u_x}.
\end{aligned} \tag{15}$$

The determining equation (15) is a two-order linear homogeneous partial differential equation of $\tau(t,x,u), \xi(t,x,u), \eta(t,x,u)$, and its solutions are a vector space L, in which the Lie bracket $[,]$ (commutator operations) is closed, so it is a Lie Algebra with structural constants. At the same time, the determining equation (15) is equal to six independent variables $t, x, u, u_x, u_{xx}, u_{tx}$, therefore, it can be decomposed into overdetermined equations, whose number are more than unknown functions. Generally speaking, it can have explicit solutions.

The invariant solution of a symmetric group is:

$$X^{(0)}(J) = \tau(t,x,u)\frac{\partial J}{\partial t} + \xi(t,x,u)\frac{\partial J}{\partial x} + \eta(t,x,u)\frac{\partial J}{\partial u} = 0. \tag{16}$$

Computing two similar independent variables of (16), $J_1 = \lambda(t,x), J_2 = \mu(t,x,u)$, or solving its characteristic equation:

$$\frac{dt}{\tau(t,x,u)} = \frac{dx}{\xi(t,x,u)} = \frac{du}{\eta(t,x,u)}. \tag{17}$$

Then, a variable is a special function with another variable, and the invariant solution is in the form:

$$\mu = \phi(\lambda). \tag{18}$$

Finally, the expression of u is substituted into equation (13) to obtain an ordinary differential equation with a variable λ, which reduces one number of independent variables.

5. Examples illustrate

Example 1. Using Lie integral method to solve equation: $\dot{y} = \frac{y}{x} + \frac{y^3}{x^3}$

Step 1. the allowable infinitesimal symmetry satisfy:

$$\xi\left(-\frac{y}{x^2} - 3\frac{y}{x^4}\right) + \eta\left(\frac{1}{x} + 3\frac{y^2}{x^3}\right) = \eta_x + \left(\eta_y - \xi_x\right)\left(\frac{y}{x} + \frac{y^3}{x^3}\right) - \xi_y\left(\frac{y}{x} + \frac{y^3}{x^3}\right)^2. \tag{19}$$

Obviously, the solution of equation (19) is $\xi(x,y)=x,\eta(x,y)=y$.

Step 2. a typical variable transformation satisfy:

$$\begin{cases} x\dfrac{\partial\theta}{\partial x}+y\dfrac{\partial\theta}{\partial y}=1 \\ x\dfrac{\partial\vartheta}{\partial x}+y\dfrac{\partial\vartheta}{\partial y}=0. \end{cases} \tag{20}$$

the solution of (20) is $t=\ln|x|,u=\frac{y}{x}$, rewriting the original equations with typical variables:

$$\dot{u}=u^3. \tag{21}$$

Step 3. the solution of (21) is $u=\pm\frac{1}{\sqrt{C-2t}}$, then the typical variables is substituted to obtain:

$$y=\pm\frac{x}{\sqrt{C-\ln x^2}}. \tag{22}$$

Example 2. Using Lie integral method to solve equation: $\ddot{y}=y\dot{y}^2-x\dot{y}^3$

Step 1. the allowable infinitesimal symmetry satisfy:

(23)

$$\begin{aligned}&\eta_{xx}+(2\eta_{xy}-\xi_{xx})\dot{y}+(\eta_{yy}-2\xi_{xy})\dot{y}^2-\dot{y}^3\xi_{yy}+\xi\dot{y}^3-\eta\dot{y}^2\\&+(\eta_y-2\xi_x-3\dot{y}\xi_y)(y\dot{y}^2-x\dot{y}^3)-(\eta_x+(\eta_y-\xi_x)\dot{y}-\dot{y}^2\xi_y)2y\dot{y}-3x\dot{y}^2)=0.\end{aligned}$$

Obviously, the solution of (23) is $\xi_1(x,y)=y,\eta_1(x,y)=0$, $\xi_2(x,y)=x$, $\eta_2(x,y)=0$.

Operator space satisfy $\{X_1^{(0)},X_2^{(0)}\}=X_1^{(0)},\xi_1\eta_2-\eta_1\xi_2=0$, so it is IV.

Step 2. a typical variable transformation satisfy:

$$\begin{cases} y\dfrac{\partial\theta}{\partial x}=0,x\dfrac{\partial\theta}{\partial x}=0, \\ y\dfrac{\partial\vartheta}{\partial x}=1,x\dfrac{\partial\vartheta}{\partial x}=\vartheta \end{cases} \tag{24}$$

the solution of (24) is $t=y,u=\frac{x}{y}$, rewriting the original equations with typical variables:

$$\ddot{u}=-\left(t+\frac{2}{t}\right)\dot{u}. \tag{25}$$

Step 3. the solution of (25) is $\dot{u}=\frac{C_1}{t^2}e^{-t^2/2}$, then $u=C_2+C_1\int\frac{1}{t^2}e^{-t^2/2}dt$, finally, the typical variables is substituted to obtain:

$$x=y(C_2+C_1\int\frac{1}{t^2}e^{-t^2/2}dt). \tag{26}$$

Example 3. Using Lie integral method to solve equation: $u_t - u_{xx} = 0$.

Step 1. the allowable infinitesimal symmetry satisfy:

$$\begin{aligned}
&\zeta_0 - \zeta_2 = 0,\\
&\zeta_0 = D_t(\eta) - u_t D_t(\tau) - u_x D_t(\xi),\\
&\zeta_1 = D_x(\eta) - u_t D_x(\tau) - u_x D_x(\xi),\\
&\zeta_2 = D_x(\zeta_1) - u_{tx} D_x(\tau) - u_{xx} D_x(\xi),\\
&D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{tx} \frac{\partial}{\partial u_x},\\
&D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{tx} \frac{\partial}{\partial u_t} + u_{xx} \frac{\partial}{\partial u_x}.
\end{aligned} \tag{27}$$

Obviously, the solution of (27) constitutes a six-dimensional Lie algebraic structure:

$$\begin{aligned}
&\tau_1(t,x,u) = 1, \xi_1(t,x,u) = \eta_1(t,x,u) = 0,\\
&\tau_2(t,x,u) = 0, \xi_2(t,x,u) = 1, \eta_2(t,x,u) = 0,\\
&\tau_3(t,x,u) = 2t, \xi_3(t,x,u) = x, \eta_3(t,x,u) = 0,\\
&\tau_4(t,x,u) = \xi_4(t,x,u) = 0, \eta_4(t,x,u) = u,\\
&\tau_5(t,x,u) = 0, \xi_5(t,x,u) = 2t, \eta_5(t,x,u) = -xu,\\
&\tau_6(t,x,u) = t^2, \xi_6(t,x,u) = tx, \eta_6(t,x,u) = -\frac{1}{4}(2t + x^2)u.
\end{aligned} \tag{28}$$

Step 2. choosing the fifth infinitesimal generator, one corresponding similar variables is $\lambda(t,x) = t$, another similar variable can be obtained from the characteristic equation $\frac{dx}{2t} + \frac{du}{xu} = 0$, the invariants is $J = ue^{x^2/4t}$, therefore, the group invariant solution is:

$$J = \phi(\lambda) \Rightarrow u = \phi(t)e^{-x^2/4t}. \tag{29}$$

Substituting the (29) into the original partial differential equation, the one-order ordinary differential equation is obtained:

$$\frac{d\phi}{dt} + \frac{\phi}{2t} = 0. \tag{30}$$

Step 3. the solution of (30) is $\phi(t) = C/\sqrt{t}$, finally, the typical variables is substituted to obtain:

$$u = \frac{C}{\sqrt{t}} e^{-x^2/4t}. \tag{31}$$

In this example, if the other generators are selected, and the general solutions are different, but they can be converted to each other equivalently.

6. Conclusions

The Lie symmetry plays a very important role in the study of differential equations. The new analytical solution, conservation law and reduction of equations can be obtained through Lie symmetry. In this paper, the calculation process of Lie integral method in one-order ordinary differential equations, two-order ordinary differential equations and partial differential equations are introduced in detail, the general theory of Lie symmetry and the way of exact solution for the above three kinds of equations are obtained. It should be said that the Lie integral theory of differential equations has made great achievements, but it still has a lot of work to be improved. For example, with the increasing of differential order and number of unknown functions, the calculation of symmetric group increases rapidly, although the symbolic software operation can deal with some problems efficiently, the process is still very complex. The author has limited ability to only collect some typical solutions of Lie integral. The purpose is to enable readers have a preliminary overall understanding for Lie integral theory of differential equations.

7. Acknowledgements

Natural Science Foundation of Universities in Jiangsu Province (20KJD460001).

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Accepted: July 22, 2019

Approximate methods for solving one-dimensional partial integro-differential equations of fractional order

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Abstract. In this paper, an analytic methods which are so-called ADM and HAM are applied to obtain series solutions to a class of one dimensional partial-intgro differential equations of fractional order(FPIDEs). First, existence of a unique solution under certain condition is proved. Then, the ADM and HAM are performed to get a general solution for one dimensional partial-intgro differential equations. In the next step, the convergence analysis of the proposed methods is investigated. Finally, some examples are included to demonstrate the efficiency of the proposed methods.

Keywords: Caputo fractional order derivative, partial integro-differential equations, homotopy analysis method, adomian decomposition method.

1. Introduction

Fractional calculus is a generalization of integer order integral and derivative to an arbitrary real-valued order. This idea has first emerged at the end of 17^{th} century, and developed in the area of mathematics throughout 18^{th} and 19^{th} century in the works of Liouville, Riemann, Cauchy, Abel, Grünwald and many others [2].

Many problems can be treated by fractional integro-differential equations from different sciences applications. In fact, most mathematical problems are hard to solve analytically, and therefore finding an approximate solution, by investigating several numerical methods, would be very convenient. In recent years, several numerical methods have been applied to solve fractional differential equations (FDEs) and fractional Integro-differential equations (FIDEs).

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In this paper, the semi-analytic methods which are so called HAM and ADM will be applied to solve some classes of fractional order partial integro differential equations.

In [5] study the numerical solution of nonlinear weakly singular partial integro-differential equation via operational matrices, he was analyzed and proposed an efficient matrix based on shifted Legender polynomials for the solution of non-linear volterra singular partial integro-differential equations (PIDEs).

In [6] has been studied the approximate controllability of a multi-valued fractional impulsive stochastic partial integro-differential equation with infinite delay in Hilbert spaces. In [1] was presented a finite difference scheme for nonlinear integro- differential equation.

In this paper, the proposed methods will be applied for solving a class of partial integro-differential equations of fractional order given by the following form:

$$ {}^{C}D_t^{\alpha}u(x,t) = g(x,t) + \int_a^t k(x,s)F[u(x,s)]ds \tag{1}$$

subject to:

$$ u(x,0) = u_0(x), x \in [a,b] \tag{2}$$

and

$$ {}^{C}D_t^{\alpha}u(x,t) = g(x,t) + I_x^{\beta}k(x,s)F[u(x,s)] \tag{3}$$

subject to:

$$ u(x,0) = u_0(x), x \in [a,b]. \tag{4}$$

2. Preliminaries

In this section, we presents some necessary definitions and mathematical preliminaries of the fractional calculus theory that have been needed in the construction of this paper later on.

2.1 Fractional order derivatives and integrals

In this part we shall give some basic definitions and properties of the fractional order derivative and integrals [4].

Definition 2.1. *The Riemann-Liouville (R-L) fractional integral of order* $\alpha > 0$ *is defined as follows:*

$$ I_x^{\beta}f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-\tau)^{\alpha-1}f(\tau)d\tau,\ x > 0,\ \alpha \in R^+. $$

Definition 2.2. *The Caputo fractional derivative of order $\alpha > 0$ is defined as follows:*

$$ {}^{C}D_x^{\alpha}(x) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^x \frac{f^{(m)}(\tau)}{(x-\tau)^{\alpha+1-m}} d\tau, & m-1 < \alpha < m \\ \dfrac{d^m}{dx^m} f(x), & \alpha = m \end{cases}. $$

For $\alpha > 0$, we have [3]:

1. ${}^{C}D_x^{\alpha}\left(I_x^{\alpha} f(x)\right) = f(x)$.
2. ${}^{C}D_x^{-\alpha}\left({}^{C}D_x^{\alpha} f(x)\right) = f(x) - \sum_{k=0}^{n-1} f^{(k)}(0^+)\frac{x^k}{k!}$.
3. ${}^{C}D_x^{\alpha}(c) = 0,\ c \in R$.

3. Existence and uniqueness theorems of the solutions of one-dimensional partial integro-differential equations of fractional order

The existence and uniqueness of problems (1)-(2) and (3)-(4) will be obtained in this section under certain conditions using Banach fixed point theorem.

3.1 The existence and uniqueness of the solution of problem (1)-(2)

In this subsection the existence and uniqueness of the solution of problems (1)-(2) will be investigated.

Lemma 3.1. *The function $u \in C([a,b] \times [0,T]) = X$ is a solution of problem (1)-(2) if and only if $u(x,t)$ is satisfying:*

$$ u(x,t) = u_0(x) + I_t^{\alpha} g(x,t) + \frac{1}{\Gamma(\alpha)} \int_0^t \int_a^x (t-s)^{\alpha-1} k(y,s) F[u(y,s)] dy ds. \tag{5} $$

Proof. Apply I_t^{α} on both sides of equation (1), yields:

$$ I_t^{\alpha} {}^{C}D_t^{\alpha} u(x,t) = I_t^{\alpha} g(x,t) + I_t^{\alpha} \int_0^t k(y,s) F[u(x,s)] ds $$

according to equation (2), we have:

$$ u(x,t) = u_0(x) + I_t^{\alpha} g(x,t) + \frac{1}{\Gamma(\alpha)} \int_0^t \int_a^x (t-s)^{\alpha-1} k(y,s) F[u(y,s)] dy ds $$

and hence the result is obtained. □

Theorem 3.1. *Let $T : X \longrightarrow X$ be defined as:*

$$ Tu = u_0(x) + I_t^{\alpha} g(x,t) + \frac{1}{\Gamma(\alpha)} \int_0^t \int_a^x (t-s)^{\alpha-1} k(y,s) F[u(y,s)] dy ds \tag{6} $$

such that k verified a Lipschitz condition with respect to $u(x,t)$ with a Lipschitz constant $L \geq 0$, and $\mid k(x,t) \mid \leq M$. Furthermore, if $\frac{ML(b-a)T^{\alpha}}{\Gamma(\alpha+1)} < 1$, then T has a unique solution.

Proof. Define the supermum norm, which will be needed later in the proof as:

$$\|u(x,t)\| = \sup_{\substack{x\in[a,b]\\ t\in[0,T]}} |u(x,t)|.$$

Now, to prove that T is a contractive mapping, we suppose that $u_1(x,t), u_2(x,t) \in X$, then:

$$\begin{aligned}&|Tu_1(x,t) - Tu_2(x,t)|\\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t \int_a^x (t-s)^{\alpha-1} |k(y,s)| \, |F[u_1(x,s)] - F[u_2(x,s)]| \, dyds.\end{aligned}$$

Hence:

$$\begin{aligned}\|Tu_1(x,t) - Tu_2(x,t)\| &\leq \frac{LM}{\Gamma(\alpha)} \|u_1(x,t) - u_2(x,t)\| \int_0^t \int_a^x (t-s)^{\alpha-1} dyds\\ &\leq \frac{LM}{\Gamma(\alpha)} \|u_1(x,t) - u_2(x,t)\| \frac{(x-a)t^{\alpha}}{\alpha}\\ &\leq \frac{ML(b-a)T^{\alpha}}{\Gamma(\alpha+1)} \|u_1(x,t) - u_2(x,t)\|.\end{aligned}$$

since $\frac{ML(b-a)T^{\alpha}}{\Gamma(\alpha+1)} < 1$, then T is a contractive mapping therefore, the problem (1)-(2) has a unique solution. □

3.2 Existence and uniqueness theorems of the solutions of problem (3)-(4)

In this subsection the existence and uniqueness of the solution of problem (3)-(4) will be introduced.

Lemma 3.2. *The function $u \in X$ is a solution of problem* (3)-(4) *if and only if $u(x,t)$ is satisfying:*

$$\begin{aligned}u(x,t) &= u_0(x) + I_t^{\alpha} g(x,t)\\ &\quad + \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t \int_a^x (t-s)^{\alpha-1} (x-y)^{\beta-1} k(y,s) F[u(y,s)] dyds \qquad (7)\end{aligned}$$

Proof. Apply I_t^{α} on both sides of equation (3), yields:

$$I_t^{\alpha\, C} D_t^{\alpha} u(x,t) = I_t^{\alpha} g(x,t) + I_t^{\alpha} I_x^{\beta} k(x,s) F[u(x,s)].$$

According to equation (4), we have:

$$u(x,t) = u_0(x) + I_t^{\alpha} g(x,t) + \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t \int_a^x (t-s)^{\alpha-1}(x-y)^{\beta-1} k(y,s) F[u(y,s)] dyds$$

and hence the result is obtained. □

Theorem 3.2. *Let $T : X \longrightarrow X$ be defined as:*

$$Tu = u_0(x) + I_t^{\alpha} g(x,t) + \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t \int_a^x (t-s)^{\alpha-1}(x-y)^{\beta-1} k(y,s) F[u(y,s)] dyds \tag{8}$$

such that k verified a Lipschitz condition with respect to $u(x,t)$ with a Lipschitz constant $L \geq 0$, and $\mid k(x,t) \mid \leq M$. Furthermore, if $\frac{MLT^{\alpha}(b-a)^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)} < 1$, then T has a unique solution.

Proof. Now, to prove that T is a contractive mapping, we suppose that $u_1(x,t)$, $u_2(x,t) \in X$, then:

$$|Tu_1(x,t) - Tu_2(x,t)| \leq \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t \int_a^x (t-s)^{\alpha-1}(x-y)^{\beta-1} |k(y,s)| \, |F[u_1(y,s)] - F[u_2(y,s)]| \, dyds.$$

Hence:

$$\begin{aligned} &\|Tu_1(x,t) - Tu_2(x,t)\| \\ &\leq \frac{LM}{\Gamma(\alpha)\Gamma(\beta)} \|u_1(x,t) - u_2(x,t)\| \int_0^t \int_a^x (t-s)^{\alpha-1}(x-y)^{\beta-1} dyds \\ &\leq \frac{MLT^{\alpha}(b-a)^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)} \|u_1(x,t) - u_2(x,t)\| \end{aligned}$$

since $\frac{MLT^{\alpha}(b-a)^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)} < 1$, then T is a contractive mapping therefore, the problem (3)-(4) has a unique solution. □

4. ADM for solving one-dimensional partial integro-differential equations of fractional order

In this section the implementation of the ADM for solving one dimensional FPIDEs will be presented.

4.1 ADM for solving problem (1)-(2)

To apply the ADM for solving problem (1)-(2), first operating I_t^α on both sides of equation (1) to get:

$$u(x,t) = u_0(x) + I_t^\alpha g(x,t) + I_t^\alpha \int_0^t k(x,s)F[u(x,s)]ds. \tag{9}$$

The solution will be considered to be:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t). \tag{10}$$

Approximating the term $F[u(x,t)]$ as:

$$F[u(x,t)] = \sum_{n=0}^{\infty} A_n, \tag{11}$$

where:

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F\left(\sum_{n=0}^{\infty} u_n\lambda^n\right)\right]\Bigg|_{\lambda=0}. \tag{12}$$

Substituting equations (10) and (11) into equation (5), we get:

$$\sum_{n=0}^{\infty} u_n(x,t) = u_0(x) + I_t^\alpha g(x,t) + I_t^\alpha \left[\int_0^t k(x,s)\sum_{n=0}^{\infty} A_n\, ds\right].$$

Consequently, we can write:

$$u_0(x,t) = u_0(x) + I_t^\alpha g(x,t)$$

and

$$u_{n+1}(x,t) = I_t^\alpha \left[\int_0^t k(x,s)A_n\, ds\right],\ n \geq 1.$$

If we truncate the summation into equation (10) after m-terms, so we have the m^{th} approximate solution of problem (1)-(2)

$$u_m(x,t) = \sum_{n=0}^{m} u_n(x,t).$$

4.2 ADM for solving problem (3)-(4)

In this subsection a similar manner that have been given in subsection (4.1) will be implemented in order to formulate a recurrence formula for finding the approximate solution of the problem (3)-(4) using ADM as follows:

$$u_0(x,t) = u_0(x) + I_t^\alpha g(x,t), \tag{13}$$

$$u_{n+1}(x,t) = I_t^{\alpha} I_x^{\beta} k(y,s) A_n,\ n \geq 1. \tag{14}$$

So, the m^{th} order approximate solution of the problem (3)-(4) is given by:

$$u_m(x,t) = \sum_{n=0}^{m} u_n(x,t). \tag{15}$$

5. HAM for solving one-dimensional partial integro-differential equations of fractional order

In this section the implementation of the HAM for solving one dimensional fractional partial integro-differential equations given by problems and (1)-(2) and (3)-(4) will be presented.

5.1 HAM for solving problem (1)-(2)

Rewriting equation (1) as:

$$N[u(x,t)] = 0, \tag{16}$$

where:

$$N[u(x,t)] = {}^{C}D_t^{\alpha} u(x,t) - g(x,t) - \int_a^t k(x,s) F[u(x,s)] ds. \tag{17}$$

According to the HAM, we construct the so called zero-order deformation equation as:

$$(1-q)\pounds[\phi(x,t,q) - u_0(x)] = q\hbar H(x) N[\phi(x,t,q)], \tag{18}$$

where $q \in [0,1]$ is the embedding parameter, $\hbar \neq 0$ is a non-zero auxiliary parameter, $H(x) \neq 0$ is an auxiliary function $u_0(x)$ is an initial guess of $u(x,t)$ and £is an auxiliary linear operator defined by:

$$\pounds = {}^{C}D_t^{\alpha}. \tag{19}$$

Obviously, when $q = 0$ and 1, it holds $\phi(x,t,0) = u_0(x)$ and $\phi(x,t,1) = u(x,t)$, respectively. Expanding $\phi(x,t,q)$ in a Taylor series with respect to q, we have:

$$\phi(x,t,q) = u_0(x) + \sum_{m=1}^{\infty} u_m(x,t) q^m, \tag{20}$$

where:

$$u_m(x,t) = \frac{1}{m!} \frac{\partial^m \phi(x,t,q)}{\partial q^m}\bigg|_{q=0}. \tag{21}$$

If the initial guess $u_0(x)$, the auxiliary parameter $\hbar$ and the auxiliary function $H(x)$ are so properly chosen, then the series (15) converges when $q = 1$, so we have:

$$u(x,t) = u_0(x) + \sum_{m=1}^{\infty} u_m(x,t). \tag{22}$$

Define the vector:

$$\vec{u}_n = \{u_0(x,t), u_1(x,t), \cdots, u_n(x,t)\}.$$

Differentiating equation (13) m-times with respect to the embedding parameter q and then setting $q = 0$ and finally dividing by $m!$, we have the so called m^{th} order deformation equation:

$$\pounds[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar H(x) R_m(\vec{u}_{m-1}), \tag{23}$$

where:

$$R_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\phi(x,t,q)]}{\partial q^{m-1}}\bigg|_{q=0}. \tag{24}$$

Now, letting $\hbar = -1$ and $H(x) = 1$, then the solution of the m^{th} order deformation equations (23) becomes:

$$\begin{aligned} u_m(x,t) &= \chi_m u_{m-1}(x,t) - I_t^{\alpha}[{}^{C}D_t^{\alpha} u_{m-1}(x,t) \\ &\quad - (1-\chi_m) g(x,t) - \int_0^t k(x,s) F[u(x,s)] ds] \end{aligned} \tag{25}$$

with an initial approximation $u_0(x,t) = u_0(x)$ and by means of the above iteration formula (25), we can obtain directly the other components in order one after one.

5.2 Convergence analysis

In this subsection, the convergence of the formula (25) to the exact solution $u(x,t)$, of problem (1)-(2) will be proved and before we study the convergence theorem we will consider $F[u(x,t)]$, in equation (1) to be $F[u(x,t)] = [u(x,t)]^p$, $p \geq 1$.

Theorem 5.1. *If the series $\sum_{m=0}^{\infty} u_m(x,t)$ is convergent, where $u_m(x,t)$ is produced by:*

$${}^{C}D_t^{\alpha}\left[u_m(x,t) - \chi_m u_{m-1}(x,t)\right] = \hbar H R_m(\vec{u}_{m-1}) \tag{26}$$

where:

$$\begin{aligned} R_m(\vec{u}_{m-1}) &= {}^{C}D_t^{\alpha} u_{m-1}(x,t) - (1-\chi_m) g(x,t) \\ &\quad - \int_0^t k(x,s)[u(x,s)]^p ds, \ p \geq 1 \end{aligned} \tag{27}$$

and besides $\sum_{m=0}^{\infty} {}^{C}D_t^{\alpha}\left[u_m(x,t)\right]$ *also converges, then it is the exact solution of the problem* (1)-(2).

Proof. Suppose that $\sum_{m=0}^{\infty} u_m(x,t)$ converges uniformly to $u(x,t)$, then it is clear that:

$$\lim_{m\to\infty} u_m(x,t) = 0, \text{ for all } x \text{ and } t \in R^{+} \tag{28}$$

since ${}^{C}D_t^{\alpha}$ is a linear operator, we have:

$$\begin{aligned}
&\sum_{m=1}^{n} {}^{C}D_t^{\alpha}\left[u_m(x,t) - \chi_m u_{m-1}(x,t)\right] \\
&= \sum_{m=1}^{n} \left[{}^{C}D_t^{\alpha}u_m(x,t) - \chi_m {}^{C}D_t^{\alpha}u_{m-1}(x,t)\right] \\
&= {}^{C}D_t^{\alpha}u_1(x,t) + \left({}^{C}D_t^{\alpha}u_2(x,t) - {}^{C}D_t^{\alpha}u_1(x,t)\right) + \cdots \\
&+ \left({}^{C}D_t^{\alpha}u_n(x,t) - {}^{C}D_t^{\alpha}u_{n-1}(x,t)\right) \\
&= {}^{C}D_t^{\alpha}u_n(x,t)].
\end{aligned} \tag{29}$$

Then from equation (28) and (29), we have:

$$\begin{aligned}
\sum_{m=1}^{n} {}^{C}D_t^{\alpha}\left[u_m(x,t) - \chi_m u_{m-1}(x,t)\right] &= \lim_{n\to\infty} {}^{C}D_t^{\alpha}u_n(x,t) \\
&= {}^{C}D_t^{\alpha}\left[\lim_{n\to\infty} u_n(x,t)\right] = 0.
\end{aligned}$$

Hence:

$$\hbar H \sum_{m=1}^{\infty} R_m(\vec{u}_{m-1}) = 0$$

since $\hbar$ and $H \neq 0$, then yields:

$$\sum_{m=1}^{\infty} R_m(\vec{u}_{m-1}) = 0 \tag{30}$$

and since:

$$R_m(\vec{u}_{m-1}) = {}^{C}D_t^{\alpha}u_{m-1}(x,t) - (1-\chi_m)g(x,t) - \int_0^t k(x,s)[u_{m-1}(x,s)]^p ds$$

so, we have:

$$0 = \sum_{m=1}^{\infty} \left[{}^{C}D_t^{\alpha} u_{m-1}(x,t) - (1-\chi_m) g(x,t) - \int_0^t k(x,s)[u_{m-1}(x,s)]^p ds \right]$$
$$= {}^{C}D_t^{\alpha} \sum_{m=1}^{\infty} u_{m-1}(x,t) - g(x,t)$$
$$- \sum_{m=1}^{\infty} \left[\int_0^t k(x,s) \left[\sum_{r_1=1}^{m-1} u_{m-1-r_1}(x,s) \sum_{r_2=0}^{r_1} u_{r_1-r_2}(x,s) \right.\right.$$
$$\left.\left. \sum_{r_3=0}^{r_2} u_{r_2-r_3}(x,s) \cdots \sum_{r_{p-2}=0}^{r_{p-3}} u_{r_{p-3}-r_{p-2}}(x,s) \sum_{r_{p-1}=0}^{r_{p-2}} u_{r_{p-2}-r_{p-1}}(x,s) \right] \right] ds.$$

Hence:

$$0 = {}^{C}D_t^{\alpha} \sum_{m=1}^{\infty} u_{m-1}(x,t) - g(x,t) - \int_0^t k(x,s) \left(\sum_{r_{p-1}=0}^{\infty} u_{m-1-r_1} \sum_{r_{p-2}=r_{p-1}}^{\infty} u_{r_1-r_2} \right.$$
$$\left. \sum_{r_{p-3}=r_{p-2}}^{\infty} u_{r_2-r_3} \cdots \sum_{r_2=r_3}^{\infty} u_{r_{p-3}-r_{p-2}} \sum_{r_1=r_2}^{\infty} u_{r_{p-2}-r_{p-1}} \sum_{m=r_1}^{\infty} u_{m-r_1} \right)(x,s)\, ds$$
$$0 = {}^{C}D_t^{\alpha} \sum_{m=0}^{\infty} u_m(x,t) - g(x,t) - \int_0^t k(x,s) \left(\sum_{i_1=0}^{\infty} u_{i_1}(x,s) \sum_{i_2=0}^{\infty} u_{i_2}(x,s) \right.$$
$$\left. \sum_{i_3=0}^{\infty} u_{i_3}(x,s) \cdots \sum_{i_{p-1}=0}^{\infty} u_{i_{p-1}}(x,s) \sum_{i_p=0}^{\infty} u_{i_p}(x,s) \right) ds$$

so, from equation (1), with $F[u(x,t)] = [u(x,t)]^p$, $p \geq 1$, we obtain:

$$0 = {}^{C}D_t^{\alpha} u(x,t) - g(x,t) - \int_0^t k(x,s)[u(x,s)]^p ds.$$

Moreover $\sum_{m=0}^{\infty} u_m(x,t)$ also satisfies the initial condition:

$$\sum_{m=0}^{\infty} u_m(x,0) = u_0(x,0) = u_0(x).$$

Therefore, we conclude that it's an exact solution of problem (1)-(2). □

6. HAM for solving problem (3)-(4)

The HAM can be performed for solving problem (3)-(4) in a similar manner that have been given in subsection (5.1) and therefore, we have the following

recurrence formula:

$$\begin{aligned} u_m(x,t) &= \chi_m u_{m-1}(x,t) \\ &\quad - I_t^{\alpha}\left[{}^{C}D_t^{\alpha}u_{m-1}(x,t) - (1-\chi_m)g(x,t) - k(x,s)F[u(x,s)]\right] \end{aligned} \tag{31}$$

and by means of equation (31), we can obtain the other components in order one after one.

7. Convergence analysis

In this subsection, the convergence of the formula (31) to the exact solution $u(x,t)$, of the problem (3)-(4), will be proved, similarly $F[u(x,t)]$ in equation (3) will be considered as $F[u(x,s)] = [u(x,t)]^p,\ p \geq 1$.

Theorem 7.1. *If the series $\sum_{m=0}^{\infty} u_m(x,t)$ is convergent, where $u_m(x,t)$ is produced by:*

$${}^{C}D_t^{\alpha}\left[u_m(x,t) - \chi_m u_{m-1}(x,t)\right] = \hbar H R_m(\vec{u}_{m-1}) \tag{32}$$

where:

$$R_m(\vec{u}_{m-1}) = {}^{C}D_t^{\alpha}u_{m-1}(x,t) - (1-\chi_m)g(x,t) - I_x^{\beta}k(x,s)[u_{m-1}(x,s)]^p,\ p \geq 1$$

and besides $\sum_{m=0}^{\infty} {}^{C}D_t^{\alpha}\left[u_m(x,t)\right]$ also converges, then it is the exact solution of problem (3)-(4).

Proof. The proof of Theorem 7.1 will be in similar manner to the proof of Theorem 5.1. □

8. Applications

In this section, some illustrative examples are given in order to confirm the applicability and accuracy of the HAM and ADM, for solving non- linear one dimensional partial integro-differential equation of fractional order.

Example 8.1. Consider the following linear FPIDEs:

$${}^{C}D_t^{3/4}u(x,t) = g(x,t) + \int_0^t (x-s)u(x,s)ds \tag{33}$$

subject to:

$$u_0(x,t) = 0 \tag{34}$$

where $g(x,t) = \frac{xt^{0.25}}{\Gamma(1.25)} - \frac{x^2t^2}{2} + \frac{xt^3}{3}$, and the exact solution of problem (33)-(34) is $u(x,t) = xt$.

Following Figures 1-2 represent a comparison between the approximate solution of problem (33)-(34) using HAM, ADM up to 4-terms and the exact solution.

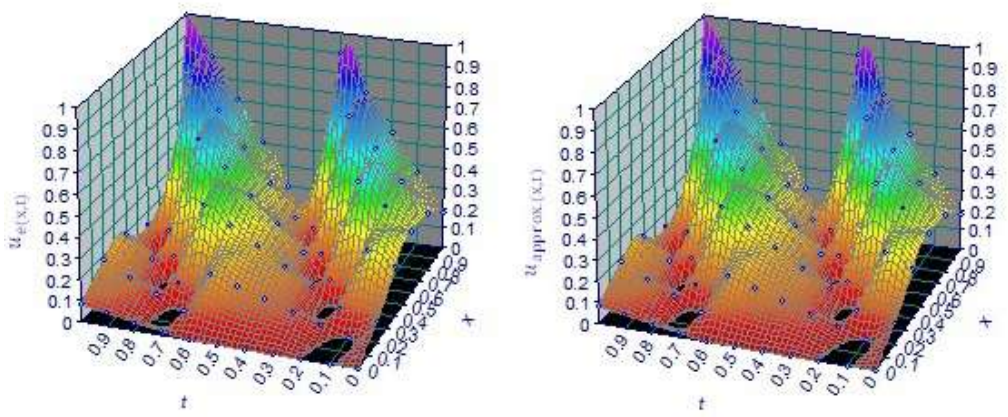


Figure 1: Comparison between the approximate solution of problem (33)-(34) using HAM up to 4-terms and the exact solution.

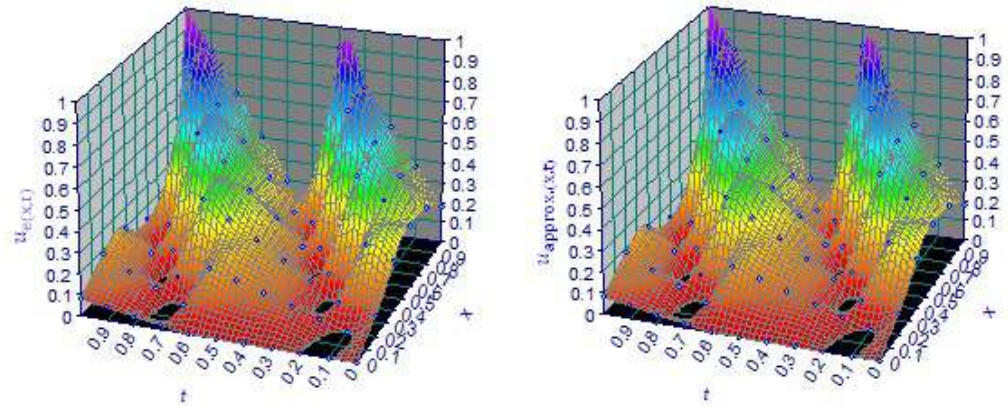


Figure 2: Comparison between the approximate solution of problem (33)-(34) using ADM up to 4-terms and the exact solution.

Example 8.2. Consider the following nonlinear FPIDEs:

$$ {}^{C}D_t^{3/4}u(x,t) = g(x,t) + \int_0^t (x-s)[u(x,s)]^2 ds \tag{35}$$

subject to:

$$ u_0(x,t) = 0 \tag{36}$$

where $g(x,t) = \frac{\Gamma(3)xt^{1.25}}{\Gamma(2.25)} - \frac{x^3t^5}{5} + \frac{x^2t^6}{6}$, and the exact solution of problem (35)-(36) is $u(x,t) = xt^2$.

Following Figures 3-4 represent a comparison between the approximate solution of problem (35)-(36) using HAM, ADM up to 4-terms and the exact solution.

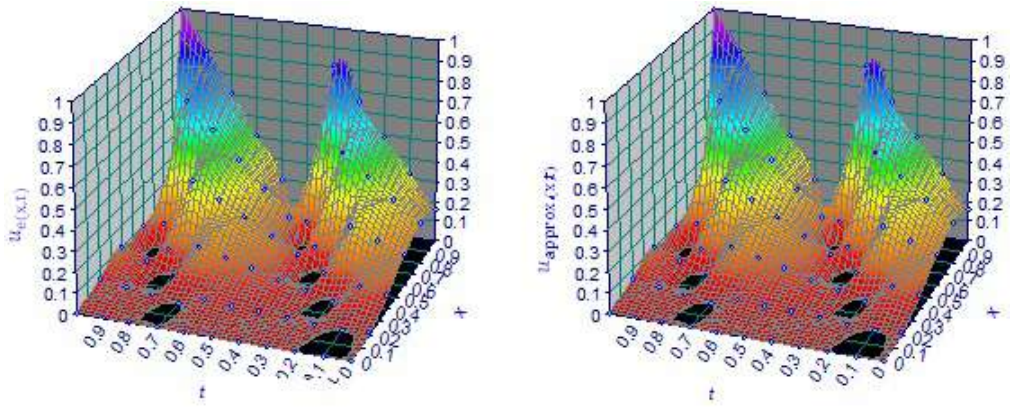


Figure 3: Comparison between the approximate solution of problem (35) -(36) using HAM up to 4-terms and the exact solution.

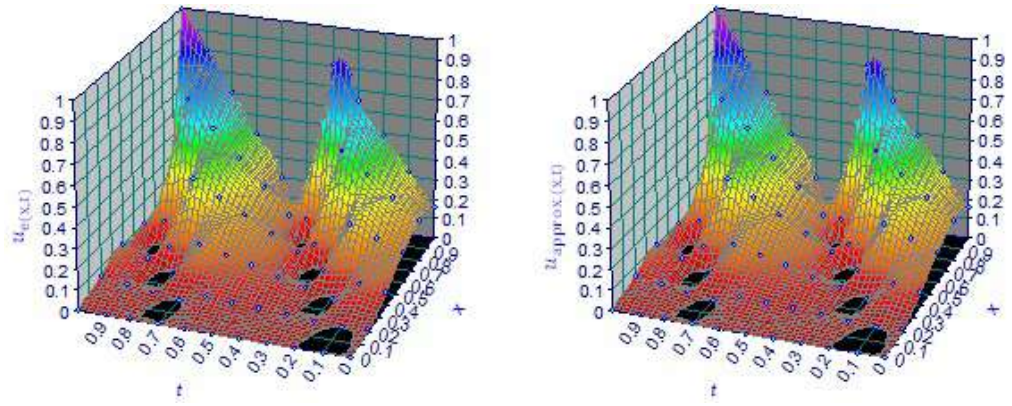


Figure 4: Comparison between the approximate solution of problem (35)-(36) using ADM up to 4-terms and the exact solution.

Example 8.3. Consider the following linear FPIDEs:

$$ {}^{C}D_t^{1/2}u(x,t) = g(x,t) + I_x^{1/3}(x-t)u(x,t) \tag{37}$$

subject to:

$$ u_0(x,t) = 0 \tag{38}$$

where $g(x,t) = \frac{\Gamma(3)xt^{3/2}}{\Gamma(5/2)} - \frac{\Gamma(3)x^{7/3}t^2}{\Gamma(10/3)} + \frac{\Gamma(2)x^{4/3}t^3}{\Gamma(7/3)}$, and the exact solution of problem (37)-(38) is $u(x,t) = xt^2$.

Following Figures 5-6 represent a comparison between the approximate solution of problem (37)-(38) using HAM, ADM up to 4-terms and the exact solution.

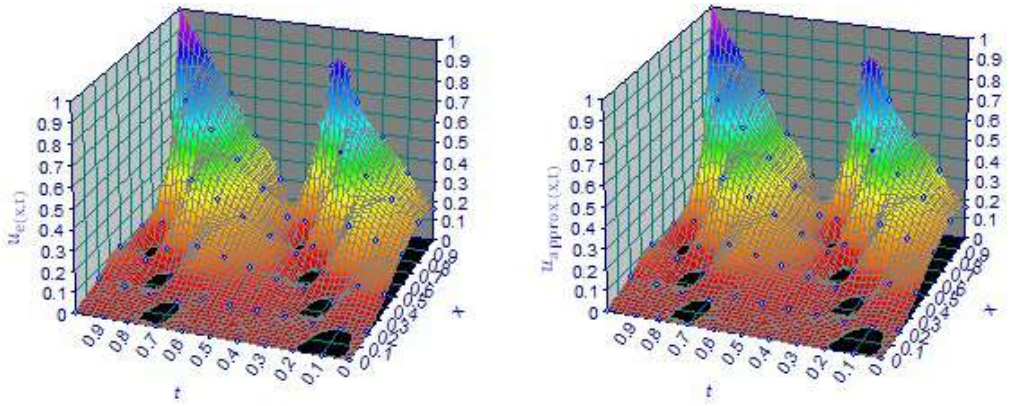


Figure 5: Comparison between the approximate solution of problem (37)-(38) using HAM up to 4-terms and the exact solution.

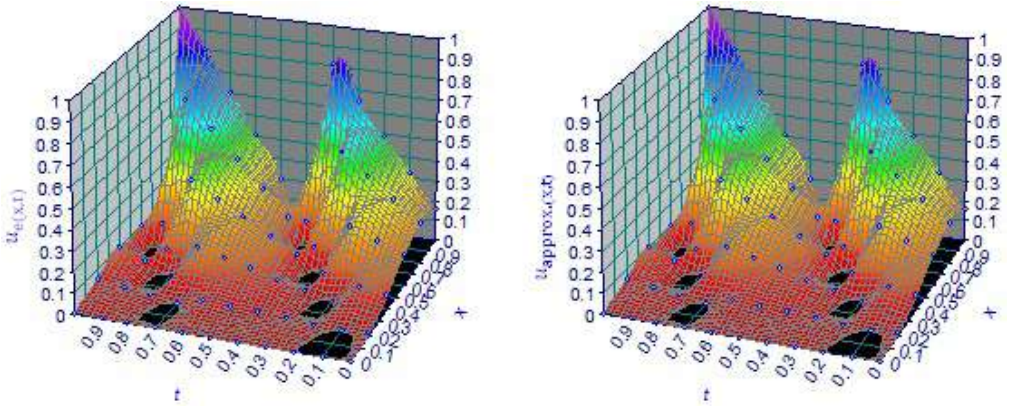


Figure 6: Comparison between the approximate solution of problem (37)-(38)) using ADM up to 4-terms and the exact solution.

Example 8.4. Consider the following nonlinear FPIDEs:

$$ {}^{C}D_t^{1/2}u(x,t) = g(x,t) + I_x^{1/3}(x-t)[u(x,t)]^2 \tag{39}$$

subject to:

$$ u_0(x,t) = 0 \tag{40}$$

where

$$g(x,t) = \frac{\Gamma(2)xt^{1/2}}{\Gamma(3/2)} - \frac{\Gamma(4)x^{10/3}t^2}{\Gamma(13/3)} + \frac{\Gamma(3)x^{7/3}t^3}{\Gamma(10/3)},$$

and the exact solution of problem (39)-(40) is $u(x,t) = xt$.

Following Figures 7-8 represent a comparison between the approximate solution of problem (39)-(40) using HAM, ADM) up to 4-terms and the exact solution.

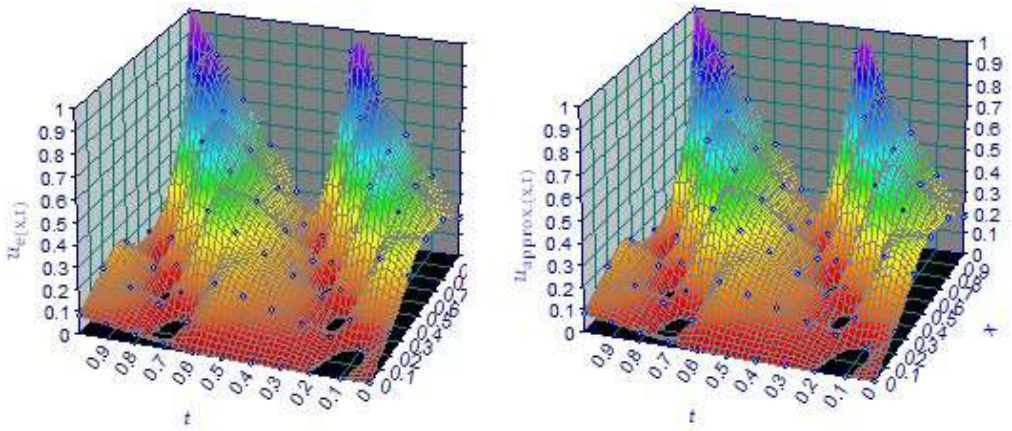


Figure 7: Comparison between the approximate solution of problem (39)-(40) using HAM up to 4-terms and the exact solution.

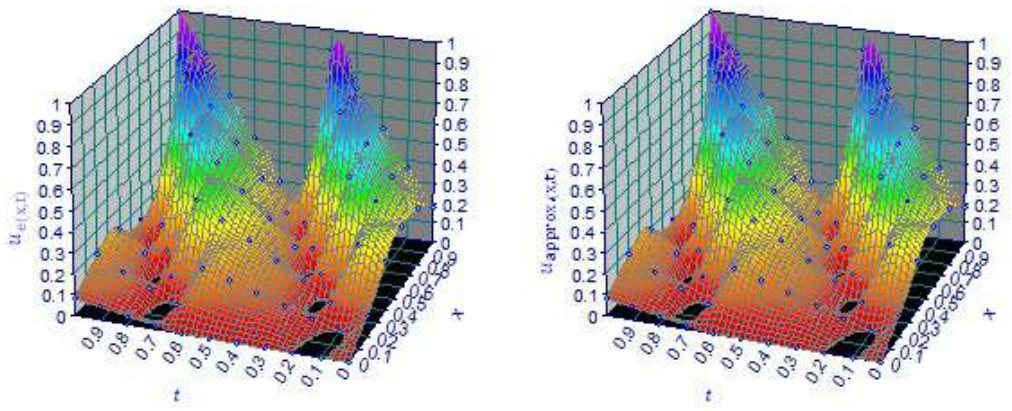


Figure 8: Comparison between the approximate solution of problem (39)-(40) using ADM up to 4-terms and the exact solution.

9. Conclusions

In this paper we have been discussed an existence and uniqueness of some classes of one dimensional partial integro-differential equations of fractional order. Then a computational methods such as ADM and HAM is employed for solving these class of equations.

The results of the numerical examples illustrate the efficiency of the present schemes for solving such kinds of equations.

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Accepted: December 06, 2019

Generalized-pre-irresolute and λ-irresolute functions in closure spaces

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Abstract. In this paper, we introduce and study two new types of maps called generalized-pre-irresolute functions and λ-irresolute functions. We discuss the relation among generalized-pre-irresolute functions λ-irresolute functions, generalized-pre-continuous functions and λ-continuous functions. And also by T_λ-spaces, $T_{p\lambda}$-spaces and T_{gp}-spaces,we find some relation between them.
Keywords: closure operator, closure space, generalized-pre-irresolute functions, λ-irresolute functions, generalized-pre-continuous functions and λ-continuous functions.

1. Introduction

In 1922, C. Kuratowski defined closure operator [18]. Let X be a set and $P(X)$ its power set. A kuratowski closure operator is a function $Cl : P(X) \to P(X)$ with the following properties:

1. $Cl(\phi) = \phi$;
2. $A \subseteq Cl(A)$, for every subset $A \subseteq X$;
3. $Cl(A \cup B) = Cl(A) \cup Cl(B)$, for any subsets $A, B \subseteq X$;
4. $Cl(Cl(A)) = Cl(A)$, for every subset $A \subseteq X$.

Then Cl, together with the universal set X, is called closure space and is denoted by (X, Cl). His closure operator makes closed sets of topological space.

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After him mathematician called this closure operator the kuratowski closure operator. In 1937, Eduard Cech [1] defined closure operator as a Cech closure operator. After that, in 1964, P.C. Hammer [14] defined closure operator as followed:

Let X be a set and $P(X)$ its power set. A function $c : P(X) \to P(X)$ defined on $P(X)$ is called a closure operator on X, if it is satisfied the following conditions:

1. $c(\phi) = \phi$;
2. $A \subseteq c(A)$, for every subset $A \subseteq X$;
3. If $A \subseteq B$, then $c(A) \subseteq c\,(B)$, for any subsets $A, B \subseteq X$.

He said the pair (X, c) is called a neighborhood space [14]. But, J. Khampakdee [16] named this space as closure space. A closure operator is said to be idempotent [17] if $cc\,(A) = c(A)$, for every subset $A \subseteq X$. It is called additive if $c(A \cup B) = c(A) \cup c(B)$ for every subsets $A, B \subseteq X$. The additive closure operator c is Cech closure operator, and the idempotent Cech closure operator is kuratowski closure operator and (X, c) is a topological space, in this case, c is denoted by Cl, where Cl is closure operator with respect kuratowski closure. In several branches of Science, the concept of closure operator and closure spaces are very advantageous material, such as Topology [3], [4], [20], [22], Computer Science Theory [21] and Biochemistry [5].

Many mathematicians studied closure operator and closure spaces (see [2], [3],[4], [6], [7], [8], [9] and [11]). Furthermore, the concept of preopen sets and precontinuous functions are introduced by Mashhure et al [19]. The concepts of precontinuity and almost - continuity (in the sense Hussain [15] are same in topological spaces, while Halgwrd M. Darwesh showed the both concepts are independent in closure spaces*:* It is well known that there are many conditions and properties that are equivalent to the concept of precontinuity and preopenness topological spaces.

In Section 3, we introduce and study new type of functions called generalized-pre-irresolute function. We discuss the relation between generalized-pre-irresolute functions and generalized-pre-continuous functions.

In Section 4, we introduce and study new type of function called λ-irresolute function. We discuss the relation among generalized-pre-irresolute functions λ-irresolute functions, generalized-pre-continuous functions and λ-continuous functions. And also by T_p-spaces, T_λ-spaces, $T_{p\lambda}$-spaces and T_{gp}-spaces, we find some relation between them.

2. Preliminaries

Let (X, c) be a closure space, a subset $A \subseteq X$ is closed [16] if $c(A) = A$. It is called open [16], if its complement in X is closed. The empty set and the *whole space* are both *open* and *closed*. Let (X, τ) be a *topological space*, a

subset $A \subseteq X$ is preopen [19] if $A \subseteq IntCl(A)$ where the interior operator $Int : P(X) \to P(X)$ corresponding to the *kuratowski closure operator* Cl on X with respect to the topology, is given by; $Int(A) = X \backslash Cl(X \backslash A)$.

Definition 2.1 ([10]). The interior operator $i : P(X) \to P(X)$ corresponding to the closure operator c on X is given by; $i(A) = X \backslash c(X \backslash A)$.

Definition 2.2 ([10]). A closure operator c on a set X is called idempotent if $cc\,(A) = c(A))$, for all $A \subseteq X$.

Definition 2.3 ([10]). A subset A of a *space* (X, c) is said to be a *preopen set*, if there exists an *open set* G such that $A \subseteq G \subseteq c(A)$. The family of all *preopen sets* is denoted by $PO\,(X, c)$. The complement of a *preopen set* is called *preclosed*. The family of all *preclosed sets* is denoted by $PC\,(X, c)$. A closure space (X, c) is said to be a T_p-space, if every preopen subset of (X, c) is open.

Theorem 2.4 ([10]). *Let c be a closure operator on X and $A \subseteq X$. If A is preopen, then $A \subseteq X \backslash c(X \backslash cA) = ic(A)$.*

Theorem 2.5 ([10]). *Let c be an idempotent closure operator on X. A subset A of X is preopen if and only if $A \subseteq X \backslash c(X \backslash cA) = ic(A)$.*

Definition 2.6 ([10]). Let (X, c_1) and (Y, c_2) be *closure spaces*. A *function* $f : (X, c_1) \to (Y, c_2)$ is said to be *precontinuous*, if the inverse image of every *open* set in (Y, c_2) under f is *preopen* in (X, c_1).

Theorem 2.7 ([10]). *Let (X, c_1) and (Y, c_2) be closure spaces. A function $f : (X, c_1) \to (Y, c_2)$ is precontinuous if and only if the inverse image of every closed set in (Y, c_2) under f is preclosed in (X, c_1).*

Definition 2.8 ([12]). Let (X, c_1) and (Y, c_2) be two *closure spaces*. A *function* $f : (X, c_1) \to (Y, c_2)$ is *preopen* if $f\,(G)$ is *preopen* in Y, for every *open subset* G of X.

Definition 2.9 ([13]). A subset B of a closure space (X, c) is called generalized-preopen, (briefly g-preopen), if there exists a preopen subset A of (X, c) such that $B \subseteq A \subseteq c(B)$.*A* subset B of X is called generalized-preclosed, briefly g-preclosed, if its complement is g-preopen.

Remark 2.10.

1. The family of all $g-$preopen (resp. $g-$preclosed) subsets of (X, c) will be denoted by $g - PO(X, c)$ (resp. $g - PC\,(X, c)$).
2. If A is preopen (resp. preclosed) in a closure space (X, c), then A is g-preopen (resp. g-preclosed).

Proposition 2.11 ([13]). *Let B be a subset of T_p- closure space (X, c). If B is g-preopen, then B is open.*

Definition 2.12 ([13]). A subset B of a closure space (X, c) is called λ-open, if there exists an open subset G of X such that $B \subseteq G$ and $c(B) = c(G)$. A subset B of X is called λ-closed, if its complement is λ-open. The family of all λ-open (λ-closed) subsets of X denoted by $\lambda O(X, c)$ ($\lambda C(X, c)$).

Remark 2.13 ([13]). If G is open (respectively, closed) in (X, c), then G is λ-open (resp. λ-closed) in (X, c).

Remark 2.14 ([13]). If B is λ-open (resp. λ-closed) in a closure space (X, c), then B is preopen (resp. preclosed) in (X, c).

Remark 2.15 ([13]). It follows from Remark 2.10, Remark 2.13 and Remark 2.14, that for a subset G of a closure space (X, c), we have the following implications:

G is open $\rightarrow$ G is λ-open $\rightarrow$ G is preopen $\rightarrow$ G is g-preopen.

Definition 2.16 ([13]). A closure space (X, c) is said to be a T_p-space, if every preopen subset of (X, c) is open.

Definition 2.17 ([13]). A closure space (X, c) is said to be:

1. a T_λ-space, if every λ-open subset of (X, c) is open.
2. a $T_{p\lambda}$-space, if every preopen subset of $(X, c)$is λ-open.
3. a T_{gp}-space, if every g-preopen subset of (X, c) is preopen.

Proposition 2.18 ([13]). *Let (X, c) be an idempotent closure space and B be a subset of X. If (X, c) is a $T_{p\lambda}$-space and B is g-preopen, then B is λ-open.*

Remark 2.19 ([13]). For a function $f : (X, c_1) \rightarrow (Y, c_2)$, where (X, c_1) and (Y, c_2) are closure spaces, the following implications hold:

f is continuous $\rightarrow$ f is λ-continuous $\rightarrow$ f is pre-continuous $\rightarrow$ f is g-pre-continuous.

3. Generalized-pre-irresolute functions in closure spaces

In this section, we introduce and study new type of functions called generalized-pre-irresolute function. We discuss the relation between generalized-pre-irresolute functions and generalized-pre-continuous functions.

Definition 3.1. A function $f : (X, c_1) \rightarrow (Y, c_2)$ is said to be generalized-pre-irresolute (briefly, g-pre-irresolute), $f^{-1}(B)$ is a g-preopen subset of (X, c_1) for every g-preopen subset B of (Y, c_2).

The proof of the following two results are easy, so they are omitted.

Proposition 3.2. *A function $f : (X, c_1) \rightarrow (Y, c_2)$ is g-pre-irresolute if and only if the inverse image of every g-preclosed subset of (Y, c_2) under f is g-preclosed in (X, c_1).*

Proposition 3.3. *Let $f : (X, c_1) \to (Y, c_2)$ and $g : (Y, c_2) \to (Z, c_3)$ be functions. If f and g are g-pre-irresolute, then gof is also g-pre-irresolute.*

Remark 3.4. The concepts of pre-irresolute functions and g-pre-irresolute functions are independent as shown by the following examples:

Example 3.5. Consider the closure operator $c : P(X) \to P(X)$ on $X = \{1, 2, 3, 4\}$ which is defined by:

$$c(A) = \begin{cases} A, & \text{if } A \in \{\phi, \{2\}, \{3\}, \{2, 3, 4\}\}, \\ \{1, 2, 3\}, & \text{if } A \in \{\{1\}, \{1, 2\}, \{1, 3\}\}, \\ \{2, 3, 4\}, & \text{if } A \in \{\{4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}. \\ X, & \text{otherwise.} \end{cases}$$

The family of $PO(X, c) = \{\phi, \{1\}, \{1, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, X\}$.

The family of $g - PO(X, c) = \{\phi, \{1\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, X\}$

Let $f : (X, c) \to (X, c)$ be a function defined by:

$$f(x) = \begin{cases} 1 & \text{if } x = 1, \\ 2 & \text{if } x = 2, \\ 3 & \text{if } x = 4, \\ 4 & \text{if } x = 3. \end{cases}$$

It easy to see that f is g-pre-irresolute, but it is not pre-irresolute because $\{1, 4\}$ is preopen in (X, c) but $f^{-1}(\{1, 4\}) = \{1, 3\}$ is not preopen set in (X, c).

Example 3.6. Let $X = \{1, 2, 3\}$ and define a closure operator $c : P(X) \to P(X)$ on X by:

$$c(A) = \begin{cases} A, & \text{if } A \in \{\phi, \{3\}\}, \\ \{1, 2\}, & \text{if } A = \{1\}, \\ \{2, 3\}, & \text{if } A = \{2\}. \\ X, & \text{otherwise.} \end{cases}$$

The family of open sets $= \{\phi, \{1, 2\}, X\}$.

$$PO(X, c) = \{\phi, \{1\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, X\},$$
$$g - PO(X, c) = \{\phi, \{1\}, \{2\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, X\}.$$

Let $f : (X, c) \to (X, c)$ be a function given by:

$$f(x) = \begin{cases} 1, & \text{if } x = 1, \\ 2, & \text{if; } x = 3, \\ 3, & \text{if } x = 2. \end{cases}$$

Clearly f is pre-irresolute, but it is not g-pre-irresolute because $\{2\}$ is g-preopen in (X, c) but $f^{-1}(\{2\}) = \{3\}$ is not g-preopen set in (X, c).

Corollary 3.7. *If $f : (X, c_1) \to (Y, c_2)$ is a pre-irresolute function, then f is pre-continuous.*

Proof. It follows from the fact every open set is preopen. □

Remark 3.8. The converse of Corollary 3.7 is not true in general. We can show it by the following example:

Example 3.9. Let $X = \{1, 2, 3\}$ and let $c_1 : P(X) \to P(X)$ be a closure operator on X which is defined in Example 3.6. And also let $c_2 : P(X) \to P(X)$ be a closure operator on X which is defined by:

$$c_2(A) = \begin{cases} A, & \text{if } A = \phi, \\ \{1, 3\}. & \text{if } A = \{1\}, \\ \{2, 3\}. & \text{if } A = \{2\}, \\ X, & \text{otherwise.} \end{cases}$$

The family of all open sets $= \{\phi, X\}$.

$$\lambda O(X, c_2) = \{\phi, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, X\} = PO(X, c_2).$$

g-$PO(X, c_2) = P(X)$.

The identity function $f : (X, c_1) \to (X, c_2)$ is pre-continuous, but it is not pre-irresolute because $\{3\}$ is preopen in (X, c_2) but $f^{-1}(\{3\}) = \{3\}$ is not pre-open set in (X, c_1).

Remark 3.10. In the following the function f is g-pre-irresolute but not λ-continuous, (because, it is not pre-continuous).

Example 3.11. The identity function $f : (X, c_1) \to (X, c_2)$, where (X, c_1) is the space of Example 3.6 and (X, c_2) is a closure space such that $c_2 : P(X) \to P(X)$ is defined by $c_2(A) = \begin{cases} A, & \text{if } A \in \{\phi, \{1\}, \{1, 3\}\}, \\ \{1, 3\}, & \text{if } A = \{3\}, \\ X, & \text{otherwise} \end{cases}$ is g-pre-continuous but not pre-continuous. Since $PO(X, c_1) = \{\phi, \{1\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, X\}$,

$$g - PO(X, c_1) = PO(X, c_1) \cup \{2\}.$$

The family of open sets of $(X, c_2) = \{\phi, \{2\}, \{2, 3\}, X\}$.

Clearly $\{2\}$ is an open set in (X, c_2) but $f^{-1}(\{2\}) = \{2\}$ is not preopen set in (X, c_1).

Corollary 3.12. *If $f : (X, c_1) \to (Y, c_2)$ is a pre-continuous function, then f is g-pre-continuous.*

Proof. It follows from the fact every pre-open set is g-preopen. □

Corollary 3.13. *If $f : (X, c_1) \to (Y, c_2)$ is a g-pre-irresolute function, then f is g-pre-continuous.*

Proof. It follows from the fact every open set is g-preopen. □

Remark 3.14. 1. Since the function of Example 3.9 is pre-continuous. So by Corollary 3.12, f is g-precontinuous, but it is not g-pre-irresolute, because the set $\{3\}$ is g-preopen in (X, c_2) but $f^{-1}(\{3\}) = \{3\}$ is not g-preopen in (X, c_1). This means that the converse of Corollary 3.13 is not true in general.

2. Since the function f in Example 3.11 is g-pre-irresolute but not λ-continuous, (because, it is not pre-continuous). However the function f of Example 3.9 is λ-continuous, but not g-pre-irresolute (because $f^{-1}(\{3\}) = \{3\}$ is not g-preopen in (X, c_1) where $\{3\}$ is g-preopen in (X, c_2). This means that, the concepts of g-pre-irresoluteness and λ-continuity of functions in closure space are independent concepts.

Remark 3.15. The concepts of g-pre-irresolute functions and λ-continuous functions are independent. By Example 3.11, f is g-pre-irresolute, but f is not λ-continuous, because $\{2\}$ is open in (X, c_2) but $f^{-1}(\{2\}) = \{2\}$ is not λ-open set in (X, c_1). By Example 3.9, the function f is λ-continuous, but not g-pre-irresolute, because $\{3\}$ is g-preopen in (X, c_2) but $f^{-1}(\{3\}) = \{3\}$ is not g-preopen set in (X, c_1).

4. λ-irresolute functions in closure spaces

In this section, we introduce and study new type of function called λ-irresolute function. We discuss the relation among generalized-pre-irresolute functions λ-irresolute functions, generalized-pre-continuous functions and λ-continuous functions. And also by T_p-space, T_λ-spaces, $T_{p\lambda}$-spaces and T_{gp}-spaces,we find some relation between them.

Definition 4.1. A function $f : (X, c_1) \to (Y, c_2)$ is said to be λ-irresolute, if $f^{-1}(B)$ is a λ-open subset of (X, c_1) for every λ-open subset B of (Y, c_2).

Proposition 4.2. *A function $f : (X, c_1) \to (Y, c_2)$ is λ-irresolute if and only if the inverse image of every λ-closed subset of (Y, c_2) under f is λ -closed in (X, c_1).*

Proof. Obvious. □

Proposition 4.3. *Let $f : (X, c_1) \to (Y, c_2)$ and $g : (Y, c_2) \to (Z, c_3)$ be two λ-irresolute functions. then gof is also λ-irresolute.*

Proof. Obvious. □

Proposition 4.4. *Every λ-irresolute function is a λ-continuous function.*

Proof. It follows from the fact that every open set is a λ-open set. □

Remark 4.5. Since the function of Example 3.9 is λ-continuous but not λ-irresolute (because, the set $\{3\}$ is λ-open in (X, c_2). This implies that, the converse of Proposition 4.4 is not true in general.

By the following examples we will show that, the concepts of g-pre-irresoluteness and λ-irresoluteness of functions in closure space are independent concepts.

Example 4.6. The identity function $f : (X, c_1) \to (X, c_2)$, where (X, c_1) is a closure space such that the closure operator $c_1 : P(X) \to P(X)$ on $X = \{1, 2, 3, 4\}$ which is defined by:

$$c_1(A) = \begin{cases} \phi, & \text{if } A = \phi, \\ \{1, 2\}, & \text{if } A = \{1\}, \\ \{2, 3\}, & \text{if } A \in \{\{2\}, \{3\}\}, \\ \{2, 4\}, & \text{if } A = \{4\}, \\ \{1, 2, 3\}, & \text{if } A \in \{\{1, 2\}, \{1, 3\}\}, \\ \{1, 2, 4\}, & \text{if } A = \{1, 4\}, \\ \{2, 3, 4\}, & \text{if } A \in \{\{2, 3\}, \{2, 4\}, \{3, 4\}, \{2, 3, 4\}\}, \\ X, & \text{otherwise.} \end{cases}$$

The family of all open sets= $\{\phi, \{1\}, X\}$. The family of $PO(X, c_1) = \{\phi, \{1\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, X\} = \lambda O(X, c_1)$.

The family of g-$PO(X, c_1) = PO(X, c_1) \cup \{\{1, 2\}, \{1, 3\}, \{1, 4\}\}$ and (X, c_2) is the same closure space of Example 3.5, is g-pre-continuous (because g-$PO(X, c_1)$ = g$-PO(X, c_2)$) but not λ-irresolute (because the set $\{1, 4\}$ is λ-open in (X, c_2) but $f^{-1}(\{1, 4\}) = \{1, 4\}$ is not λ-open in (X, c_1).

Example 4.7. Let $f : (X, c) \to (X, c)$ be a function given by:

$$f(x) = \begin{cases} 1, & \text{if } x = 3, \\ 2, & \text{if } x = 2, \\ 3, & \text{if } x = 1, \end{cases}$$

where (X, c) is the same closure space of Example such that $c : P(X) \to P(X)$ which defined by:

$$c(A) = \begin{cases} A, & \text{if } A \in \{\phi, \{3\}\}, \\ \{1, 2\}, & \text{if } A = \{1\}, \\ \{2, 3\}, & \text{if } A = \{2\}, \\ X, & \text{otherwise.} \end{cases}$$

Clearly f is λ-irresolute, but it is not g-pre-irresolute because $\{1\}$ is g-preopen in (X, c) but $f^{-1}(\{1\}) = \{3\}$ is not g-preopen set in (X, c).

Corollary 4.8. *If $f : (X, c_1) \to (Y, c_2)$ is a λ-irresolute function, then f is g-pre-continuous.*

Proof. It follows from Remark 4.5 and Remark 2.10. □

Remark 4.9. Since from Remark 4.5, we have the function f of Example 3.9 is λ-continuous, but not λ-irresolute. Then, in review of Remark 2.10 f is g-pre-continuous but not λ-irresolute. This means that, the converse of Corollary 4.8 is not true in general.

Remark 4.10. By Remark 2.10, Remark 3.4, Remark 3.8, Remark 3.14, Remark 3.15, Remark 4.5 and Remark 4.9, the interrelation among (g-pre-continuous, g-pre-irresolute, λ-continuous and λ-irresolute) functions is given by the following diagram:

Here, A B means A implies B and A B means A does not necessarily imply B .

Proposition 4.11. *Let (X, c_1) be a closure space, (Y, c_2) be a T_p-space. If $f : (X, c_1) \to (Y, c_2)$ is a g-pre-continuous function, then f is g-pre-irresolute.*

Proof. Let B be a g-preopen subset of (Y, c_2). Since (Y, c_2) is a T_p-space, by Proposition 2.11, B is open in (Y, c_2). Since f is g-pre-continuous, so $f^{-1}(B)$ is g-preopen in (X, c_1) . Therefore, f is g-pre-irresolute. □

Proposition 4.12. *Let (X, c_1) be a closure space,(Y, c_2) be a T_λ-space. If $f : (X, c_1) \to (Y, c_2)$ is a λ-continuous function, then f is λ-irresolute.*

Proof. Let B be a λ-open subset of (Y, c_2). Since (Y, c_2) is a T_λ-space, B is open in (Y, c_2). by λ-continuity of f, we get $f^{-1}(B)$ is λ-open in (X, c_1). Therefore, f is λ-irresolute. □

Proof. Let B be a λ-open subset of (Y, c_2). Since (Y, c_2) is a T_λ-space, B is open in (Y, c_2), by λ-continuity of f , we get $f^{-1}(B)$ is λ-open in (X, c_1). Therefore, f is λ-irresolute. □

Proposition 4.13. *Let (X, c_1) be a $T_{p\lambda}$-space and idempotent closure space and (Y, c_2) be a T_λ-space. If $f : (X, c_1) \to (Y, c_2)$ is a g-pre-continuous function, then f is λ-irresolute.*

Proof. Let B be a λ-open subset of (Y, c_2) . Since (Y, c_2) is a T_λ-space, B is open in (Y, c_2). by g-pre-continuity of f, we get $f^{-1}(B)$ is g-preopen in (X, c_1) . But (X, c_1) is an idempotent $T_{p\lambda}$-space, so by Proposition 2.18, $f^{-1}(B)$ is λ-open. Therefore, f is λ-irresolute. □

Proposition 4.14. *Let (X, c_1) and (Y, c_2) be closure spaces and $f : (X, c_1) \to (Y, c_2)$ be a function.*

1. If (X, c_1) is idempotent closure space and a $T_{p\lambda}$-space and f is g-pre-irresolute, then f is λ-irresolute.

2. If (Y, c_2) is idempotent closure space and a $T_{p\lambda}$-space and f is λ-irresolute, then f is g-pre-irresolute.

Proof. 1. Let B be a λ-open subset of (Y, c_2). Then, B is also g-preopen in (Y, c_2). Since f is g-pre-irresolute, so $f^{-1}(B)$ is g-preopen in (X, c_1). Since (X, c_1) is a $T_{p\lambda}$-space, by Proposition 2.18, $f^{-1}(B)$ is λ-open in (X, c_1). Therefore, f is λ-irresolute.

2. Let B be a g-preopen subset of (Y, c_2). Since (Y, c_2) is an idempotent $T_{p\lambda}$-space, by Proposition 2.18, B is λ-open in (Y, c_2). Since f is λ-irresolute, so $f^{-1}(B)$ is λ-open in (X, c_1). Consequently, $f^{-1}(B)$ is also g-preopen in (X, c_1). Therefore, f is g-pre-irresolute. □

Proposition 4.15. *Let (X, c_1) and (Y, c_2) be closure spaces and $f : (X, c_1) \to (Y, c_2)$ be a function.*

1. *If (X, c_1) is an idempotent $T_{p\lambda}$-closure space and f is g-pre-irresolute, then f is λ-continuous.*

2. *If (Y, c_2) is a T_p-space and f is λ-continuous, then f is g-pre-irresolute.*

Proof. 1. Since f is g-pre-irresolute and (X, c_1) is a $T_{p\lambda}$-space, by part (1) of Preposition 4.14, f is λ-irresolute, and by Preposition 4.4, f is λ-continuous.

2. Let B be a g-preopen subset of (Y, c_2). Since (Y, c_2) is a T_p-space, by Proposition 2.11, B is open. Since f is λ-continuous, $f^{-1}(B)$ is λ-open in (X, c_1). It follows that $f^{-1}(B)$ is also g-preopen in (X, c_1). Therefore, f is g-pre-irresolute. □

Proposition 4.16. *Let (X, c_1) and (Y, c_2) be closure spaces and $f : (X, c_1) \to (Y, c_2)$ be a preopen g-pre-irresolute and surjective function. If (X, c_1) is a T_p-space, then (Y, c_2) is a T_{gp}-space.*

Proof. Let (X, c_1) be a T_p-space and let B be a g-preopen subset of (Y, c_2). Since f is g-pre-irresolute, $f^{-1}(B)$ is g-pre-open in (X, c_1). Since (X, c_1) is a T_p-space, so by Proposition 4.4, $f^{-1}(B)$ is open in (X, c_1). Since f is a preopen surjection, then $f\left(f^{-1}(B)\right) = B$ is preopen in (Y, c_2). Therefore, (Y, c_2) is a T_{gp}-space. □

The proof of the following results are similar to the proof of Proposition 4.16.

Proposition 4.17. *Let $f : (X, c_1) \to (Y, c_2)$ be a λ-open, pre-irresolute surjective function. If (X, c_1) is a T_p-space, then (Y, c_2) is a $T_{p\lambda}$-space.*

Proposition 4.18. *Let $f : (X, c_1) \to (Y, c_2)$ be an open, λ-irresolute surjective function. If (X, c_1) is a T_λ-space, then (Y, c_2) is a T_λ-space*

Proposition 4.19. *Let $f : (X, c_1) \to (Y, c_2)$ and $g : (Y, c_2) \to (Z, c_3)$ be two functions. Then, gof is g-pre-continuous if f is g-pre-irresolute and g is g-pre-continuous.*

Proof. Obvious. □

Proposition 4.20. *Let $f : (X, c_1) \to (Y, c_2)$ and $g : (Y, c_2) \to (Z, c_3)$ be two functions. Then, gof is λ-continuous if f is λ-irresolute and g is λ-continuous.*

Proposition 4.21. *If $f : (X, c_1) \to (Y, c_2)$ and $g : (Y, c_2) \to (Z, c_3)$ are two functions, then:*

1. *If (X, c_1) is an idempotent $T_{p\lambda}$-space, f is g-pre-irresolute and g is λ-irresolute, then gof is λ-irresolute.*

2. *If (Z, c_3) is an idempotent $T_{p\lambda}$-space, f is a g-pre-irresolute and g is λ-irresolute, then gof is g-pre-irresolute.*

3. *If (Y, c_2) is an idempotent $T_{p\lambda}$-space, f is λ-irresolute and g is g-pre-irresolute, then gof is both g-pre-irresolute and λ-irresolute.*

Proof. 1. Let B be a λ-open subset of (Z, c_3). Since g is λ-irresolute, $g^{-1}(B)$ is λ-open in (Y, c_2). Consequently, $g^{-1}(B)$ is also g-preopen in (Y, c_2). Since f is g-pre-irresolute, then $f^{-1}\left(g^{-1}(B)\right) = (gof)^{-1}(B)$ is g-preopen in (X, c_1). But (X, c_1) is an idempotent $T_{p\lambda}$-space, then, by Proposition 2.18, $(gof)^{-1}(B)$ is λ-open in (X, c_1). Therefore, gof is λ-irresolute.

2. Let B be a g-preopen subset of (Z, c_3). Since (Z, c_3) is an idempotent $T_{p\lambda}$-space, so by Proposition 2.18, B is λ-open by. Since g is λ-irresolute, then $g^{-1}(B)$ is λ-open in (Y, c_2). Consequently, $g^{-1}(B)$ is also g-preopen in (Y, c_2). But f is g-pre-irresolute, hence $f^{-1}\left(g^{-1}(B)\right) = (gof)^{-1}(B)$ is g-preopen in (X, c_1). Therefore, gof is g-pre-irresolute.

3. If B is a λ-open (g-preopen) subset of (Z, c_3), then B is g-preopen in (Z, c_3). Since g is g-pre-irresolute, then $g^{-1}(B)$ is g-peropen in (Y, c_2). But, (Y, c_2) is an idempotent $T_{p\lambda}$-space, so by Proposition 2.18, $g^{-1}(B)$ is λ-open set in (Y, c_2). Since f is λ-irresolute, then $(gof)^{-1}(B) = f^{-1}\left(g^{-1}(B)\right)$ is a λ-open subset of (X, c_1). So, $(gof)^{-1}(B)$ is both λ-open and g-preopen. Therefore, gof is both λ-irresolute and g-pre-irresolute. □

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Accepted: October 05, 2019

Study on value distribution of L-functions sharing a small function

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Abstract. In this paper, we mainly concentrate on Nevanlinna theory and the properties of L-functions in the extended selberg class, we mainly study the uniqueness problems on L-functions related to Brück conjecture. This extend the result due to Q.C. Zhang.

Keywords: Nevanlinna theory, L-functions, integral function, meromorphic function, weighted sharing, differential polynomial.

1. Introduction

In this paper, a meromorphic function always mean a function which is meromorphic in the whole complex plane $\mathbb{C}$. We denote by $N_{k)}(r, \frac{1}{(f-a)})$ the counting function for zeros of $f-a$ with multiplicity $\leq k$, and by $\overline{N}_{k)}(r, \frac{1}{(f-a)})$ the corresponding one for which multiplicity is not counted. Let $N_{(k}(r, \frac{1}{(f-a)})$ be the counting function for zeros of $f-a$ with multiplicity at least k and $\overline{N}_{(k}(r, \frac{1}{(f-a)})$ the corresponding one for which multiplicity is not counted. Moreover, f and g are said to share a value c CM (counting multiplicities) if they share the value c and if the roots of the equations $f(s) = c$ and $g(s) = c$ have the same multiplicities. In terms of sharing values, two nonconstant meromorphic functions in $\mathbb{C}$ must be identically equal if they share five values IM, and one must be a Möbius transform of the other if they share four values CM.

The Riemann hypothesis as one of the millenium problems has been given a lot of attention by mathematical workers for a long time. Selberg guessed that

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the Riemann hypothesis is also true for L-functions in the selberg class. Such an L-function based on Riemann zeta function as the prototype is defined to be a Dirichlet series $L(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$ of a complex variable $s = \sigma + it$ satisfying the following axioms:

(i) Ramanujan hypothesis. $a(n) \ll n^{\epsilon}$ for every $\epsilon > 0$.

(ii) Analytic continuation. there is a non-negative integer m such that $(s-1)^m L(s)$ is an entire function of finite order.

(iii) Functional equation. L satisfies a functional equation of type

$$\Lambda_L(s) = \omega\overline{\Lambda_L(1-\overline{s})},$$

where

$$\Lambda_L(s) = L(s)Q^s \prod_{j=1}^{k} \Gamma(\lambda_j s + \nu_j)$$

with positive real numbers Q, λ_j and complex numbers ν_j, ω with $Re\nu_j \geq 0$ and $|\omega| = 1$.

(iv) Euler product. $\log L(s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s}$, where $b(n) = 0$ unless n is a positive power of a prime and $b(n) \ll n^{\theta}$ for some $\theta < \frac{1}{2}$.

Let z_0 be a zero of $f - a$ of multiplicity p and a zero of $g - a$ of multiplicity q. We denote by $\overline{N}_L(r, a; f)$ the counting function of those a-points of f and g where $p > q \geq 1$, by $N_E^{1)}(r, a; f)$ the counting function of those a-points of f and g where $p = q = 1$ and by $\overline{N}_E^{(2}(r, a; f)$ the counting function of those a-points of f and g where $p = q \geq 2$, each point in these counting functions is counted only once. In the same way we can define $\overline{N}_L(r, a; g), N_E^{1)}(r, a; g), \overline{N}_E^{(2}(r, a; g)$.

Let k be a non-negative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k + 1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.

In connection to find the relation between an entire function with its derivative when they share one value CM, in 1996, in this direction the following famous conjecture was proposed by Brück [5]:

Conjecture. Let f be a non-constant entire function such that the hyper order $\rho_2(f)$ is not a positive integer or infinite. If f and f' share a finite value a CM, then $\frac{f'-a}{f-a} = c$, where c is a nonzero constant.

Brück himself proved the conjecture for $a = 0$ and for $a = 1$, he showed that under the assumption $N(r, 0; f') = S(r, f)$ the conjecture was true.

Theorem A ([5]). *Let f be a non-constant entire function. If f and f' share the value 1 CM and if $N(r, 0; f') = S(r, f)$, then $\frac{f'-1}{f-1}$ is a nonzero constant.*

However, for entire function of finite order, Yang [30] removed the supposition $N(r, 0; f') = 0$ and obtained the following result.

Theorem B ([30]). *Let f be a non-constant entire function of finite order and let $a(\neq 0)$ be a finite constant. If $f, f^{(k)}$ share the value a CM, then $\frac{f^{(k)}-a}{f-a}$ is a nonzero constant, where $k(\geq 1)$ is an integer.*

In 2005, Zhang [33] further extended the results of Lahiri-sarkar to a small function and proved the following result for IM sharing.

Theorem C ([33]). *Let f be a non-constant meromorphic function and $k(\geq 1)$, and $l \geq 0$ be integer. Also let $a \equiv a(z)(\not\equiv 0, \infty)$ be a meromorphic small function. Suppose that $f - a$ and $(f)^{(k)} - a$ share $(0, l)$.*

If $l \geq 2$ and

$$2\overline{N}\left(r, f\right) + N_2\left(r, 0; (f)^{(k)}\right) + N_2\left(r, 0; \left(\frac{f}{a}\right)'\right) \leq (\lambda + o(1))T\left(r, (f)^{(k)}\right)$$

or, if $l = 1$

$$2\overline{N}\left(r, f\right) + N_2\left(r, 0; (f)^{(k)}\right) + 2\overline{N}\left(r, 0; \left(\frac{f}{a}\right)'\right) \leq (\lambda + o(1))T\left(r, (f)^{(k)}\right)$$

or, $l = 0$ and

$$4\overline{N}\left(r, f\right) + 3N_2\left(r, 0; (f)^{(k)}\right) + 2\overline{N}\left(r, 0; \left(\frac{f}{a}\right)'\right) \leq (\lambda + o(1))T\left(r, (f)^{(k)}\right)$$

for $r \in I$, where $0 < \lambda < 1$, and I is a set of infinite linear measure, then $\frac{f^{(k)}-a}{(f)-a} = c$, for some constant $c \in C - \{0\}$.

Now, it is natural to ask the following question which is the motivation of the paper.

Question. Can Brück type conclusion be obtained when f and $f^{(k)}$ is replaced by $[f^n]^{(k)}$ and $[L^n]^{(k)}$ in Theorem C.

Now, we have the following theorems

Theorem 1. *Let f be a nonconstant entire function in $\mathbb{C}$, let L be an non-constant L - function, and let n and k be two positive integers. If $(f^n)^{(k)}$ and $(L^n)^{(k)}$ share $(a(z), l)$, where $a(z)(\not\equiv 0, \infty)$ be a meromorphic function. Suppose one of the following conditions hold:*

I. $l \geq 2$ and $n > 2k + 4$;

II. $l = 1$ and $n > \frac{5k+9}{2}$;

III. $l = 0$ and $n > 5k + 7$.

Then, $\frac{(L^n)^{(k)}-1}{(f^n)^{(k)}-1} = C$.

2. Lemmas

Let F and G be two non-constant meromorphic functions defined in $\mathbb{C}$. We denote by H the function as follows:

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right). \tag{1}$$

Lemma 1 ([25]). *Let f be a non-constant meromorphic function and let $a_n(z)(\not\equiv 0), a_{n-1}(z), ..., a_0(z)$ be meromorphic functions such that $T(r, a_i(z)) = S(r, f)$ for $i = 0, 1, 2, ..., n$. Then*

$$T(r, a_n f^n + a_{n-1} f^{n-1} + ... + a_1 f + a_0) = nT(r, f) + S(r, f).$$

Lemma 2 ([27]). *Suppose that f is a nonconstant meromorphic function in the complex plane and k is a positive integer. Then*

$$N(r, 0; f^{(k)}) \leq N(r, 0; f) + k\overline{N}(r, \infty, f) + O(log(T(r, f)) + \log r),$$

as $r \to \infty$, outside of a possible exceptional set of finite linear measure.

Lemma 3 ([33]) *Let f be a nonconstant meromorphic function, and p, k be positive integers. Then*

$$N_p(r, 0; f^{(k)}) \leq T(r, f^{(k)}) - T(r, f) + N_{p+k}(r, 0; f) + S(r, f), \tag{2}$$

$$N_p(r, 0; f^{(k)}) \leq k\overline{N}(r, \infty, f) + N_{p+k}(r, 0; f) + S(r, f). \tag{3}$$

Lemma 4 ([4]). *Let F and G share $(1, l)$ and $\overline{N}(r, \infty; F) = \overline{N}(r, \infty; G)$ and $H \not\equiv 0$, where F, G and H are defined as earlier. Then*

$$\begin{aligned} N(r, \infty, H) \leq\ & \overline{N}(r, \infty; F) + \overline{N}(r, 0; F| \geq 2) + \overline{N}(r, 0; G| \geq 2) \\ & + \overline{N}_0(r, 0; F') + \overline{N}_0(r, 0; G') + \overline{N}_L(r, 1; F) \\ & + \overline{N}_L(r, 1; G) + S(r, F). \end{aligned}$$

Lemma 5 ([4]). *If F and G share $(1, l)$, then*

$$\begin{aligned} &\overline{N}_L(r, 1; F) \leq \frac{1}{2}\overline{N}(r, \infty; F) + \frac{1}{2}\overline{N}(r, 0; F) + S(r, F) \text{ when } l \geq 1, \\ &\overline{N}_L(r, 1; F) \leq \overline{N}(r, \infty; F) + \overline{N}(r, 0; F) + S(r, F) \text{ when } l = 0. \end{aligned}$$

Lemma 6 ([4]). *Let F and G share $(1, l)$ and $H \not\equiv 0$. Then*

$$\begin{aligned} \overline{N}(r, 1, F) + \overline{N}(r, 1, G) \leq\ & N(r, \infty; H) + \overline{N}_E^{(2}(r, 1; F) \\ & + \overline{N}_L(r, 1; F) + \overline{N}_L(r, 1; G) + \overline{N}(r, 1; G) + S(r, F). \end{aligned}$$

3. Proof of the Theorem 1

Suppose that d is the degree of L. Then $d = 2\sum_{i=1}^{k} \lambda_j$, where k and λ_j are respectively the positive integer and the positive real number in the axiom (iii) of the definition of L- function. Then, we have that

$$T(r, L) = \frac{d}{\Pi} r \log r + O(r)$$

(cf. [8] , P.150). Clearly, f and L are transcendental meromorphic functions (cf. [22] , P.43). Note that an L- function at most has one pole $z = 1$ in the complex plane.

Let $F = \frac{(f^n)^{(k)}}{a(z)}$ and $G = \frac{(L^n)^{(k)}}{a(z)}$. Then $F - 1 = \frac{(f^n)^{(k)} - a}{a}$, $G - 1 = \frac{(L^n)^{(k)} - a}{a}$. Since $(f^n)^{(k)}$ and $(L^n)^{(k)}$ share (a, l) it follows that F and G share $(1, l)$ except the zeros and poles of $a(z)$.

Now, we consider the following cases.

Case 1. Let $H \not\equiv 0$.

Subcase 1.1. Assume $l \geq 1$. Using the second fundamental Theorem and Lemmas 4, 6 we get

$$\begin{aligned}
(4)\quad T(r,F) + T(r,G) &\leq \overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + \overline{N}(r,0;F) \\
&\quad + \overline{N}(r,0;G) + N(r,H) + \overline{N}_E^{(2}(r,1;F) \\
(5)\qquad &\quad + \overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) + \overline{N}(r,1;G) \\
&\quad - \overline{N}_0(r,0;F') - \overline{N}_0(r,0;G') + S(r,F) \\
&\leq 2\overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + N2(r,0;F) + N2(r,0;G) \\
&\quad + \overline{N}_E^{(2}(r,1;F) + 2\overline{N}_L(r,1;F) \\
&\quad + 2\overline{N}_L(r,1;G) + \overline{N}(r,1;G) + S(r,F).
\end{aligned}$$

Subcase-1.1.1. Next assume $l \geq 2$. Now, by using the inequality (4) and Lemma 3, we get

$$\begin{aligned}
T(r,F) + T(r,G) &\leq 2\overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + N_2(r,0;F) + N_2(r,0;G) \\
&\quad + \overline{N}_E^{(2}(r,1;F) + 2\overline{N}_L(r,1;F) \\
&\quad + 2\overline{N}_L(r,1;G) + \overline{N}(r,1;G) + S(r,F) \\
&\leq 3\overline{N}(r,\infty;f) + N_2(r,0;F) \\
&\quad + N_2(r,0;G) + N(r,1;F) + S(r,f) \\
T(r,G) &\leq 3\overline{N}(r,\infty;f) + N_{k+2}(r,0;f^n) + k\overline{N}(r,\infty;f) \\
&\quad + N_2(r,0;G) + S(r,f) \\
&\leq 3\overline{N}(r,\infty;f^n) + N_{k+2}(r,0;f^n) + k\overline{N}(r,\infty;f^n) \\
&\quad + N_2(r,0;(L^n)^{(k)}) + S(r,f) \\
T(r,L^n) &\leq 3\overline{N}(r,\infty;f) + N_{k+2}(r,0;f^n) + k\overline{N}(r,\infty;f)
\end{aligned}$$

$$
\begin{aligned}
&+ N_{k+2}(r,0;L^n) + k\overline{N}(r,L) + S(r,f)\\
&\leq 3\overline{N}(r,\infty;f) + N_{k+2}(r,0;f^n)\\
&+ k\overline{N}(r,\infty;f) + N_{k+2}(r,0;L^n) + S(r,f)\\
&\leq (k+2)T(r,f) + (k+2)T(r,L) + O(\log r)\\
nT(r,L) &\leq (2k+4)T(r) + O(\log r),
\end{aligned}
$$

where $T(r) = \max\{T(r,f), T(r,L)\}$. In a similar way we can obtain

$$nT(r,f) \leq (2k+4)T(r) + O(\log r).$$

Combining the above two equations we see that

$$nT(r) \leq (2k+4)T(r) + O(\log r).$$

Since $n > 2k+4$, above equation leads to a contradiction.

Subcase-1.1.2. Next, we assume $l = 1$. Now inequality (4) and in view of Lemmas 3, 5 we get

$$
\begin{aligned}
T(r,F) + T(r,G) &\leq 2\overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + N_2(r,0;F) + N_2(r,0;G)\\
&+ \overline{N}_E^{(2}(r,1;F) + 2\overline{N}_L(r,1;F)\\
&+ 2\overline{N}_L(r,1;G) + \overline{N}(r,1;G) + S(r,F)\\
&\leq 2\overline{N}(r,\infty;F) + \frac{3}{2}\overline{N}(r,\infty;G) + \frac{1}{2}\overline{N}(r,0;G)\\
&+ N_2(r,0;F) + N_2(r,0;G) + \overline{N}_E^{(2}(r,1;F)\\
&+ 2\overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) + \overline{N}(r,1;G) + S(r,F)\\
&\leq 2\overline{N}(r,\infty;f) + \frac{1}{2}N_1(r,0;G) + N_2(r,0;F)\\
&+ N_2(r,0;G) + N(r,1;F) + S(r,f),\\
T(r,L^n) &\leq 2\overline{N}(r,\infty;f) + \frac{1}{2}N_1(r,0;(L^n)^{(k)})\\
&+ N_2(r,0;(f^n)^{(k)}) + N_2(r,0;(L^n)^{(k)}) + S(r,f)\\
&\leq 2\overline{N}(r,\infty;f) + \frac{1}{2}N_{k+1}(r,0;L^n)\\
&+ N_{k+2}(r,0;f^n) + k\overline{N}(r,\infty;f) + N_{k+2}(r,0;L^n)\\
&+ \frac{3}{2}k\overline{N}(r,L) + S(r,f)\\
&\leq (k+2)T(r,f) + (\frac{3k}{2} + \frac{5}{2})T(r,L) + O(\log r),\\
nT(r,L) &\leq (\frac{5k}{2} + \frac{9}{2})T(r) + O(\log r)
\end{aligned}
$$

where $T(r) = \max\{T(r,f), T(r,L)\}$. In a similar way we can obtain

$$nT(r,f) \leq (\frac{5k}{2} + \frac{9}{2})T(r) + O(\log r).$$

Combining the above two equations we see that

$$nT(r) \leq (\frac{5k+9}{2})T(r) + O(\log r).$$

Since $n > \frac{5k+9}{2}$, above equation leads to a contradiction.

Subcase-1.2. Next we assume $l = 0$. Then by using the second fundamental Theorem and Lemmas 3, 4, 5 and 6 we get

$$\begin{aligned}
T(r,F) + T(r,G) &\leq \overline{N}(r,\infty;F) + \overline{N}(r,0;F) + \overline{N}(r,1;F) \\
&\quad + \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r,1;G) \\
&\quad - N_0(r,0;F') - N_0(r,0;G') + S(r,F) + S(r,G) \\
&\leq \overline{N}(r,\infty;F) + \overline{N}(r,0;F) + \overline{N}(r,\infty;G) + \overline{N}(r,0;G) \\
&\quad + N(r,\infty;H) + \overline{N}_E^{(2}(r,1;F) \\
&\quad + \overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) + \overline{N}(r,1;G) \\
&\quad - N_0(r,0;F') - N_0(r,0;G') + S(r,f) \\
&\leq 2\overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + \frac{1}{2}\overline{N}(r,0;G) \\
&\quad + N_2(r,0;F) + N_2(r,0;G) + \overline{N}_E^{(2}(r,1;F) \\
&\quad + 2\overline{N}_L(r,1;F) + 2\overline{N}_L(r,1;G) + \overline{N}(r,1;G) + S(r,f) \\
&\leq 3\overline{N}(r,\infty;f) + N_2(r,0;F) + N_2(r,0;G) + 2\overline{N}(r,0;G) \\
&\quad + \overline{N}(r,0;F) + N(r,1;F) + S(r,f), \\
T(r,L^n) &\leq 3\overline{N}(r,\infty;f) + N_{k+2}(r,0;f^n) + k\overline{N}(r,\infty;f) + N_2(r,0;(L^n)^{(k)}) \\
&\quad + \overline{N}(r,0;(f^n)^{(k)}) + 2\overline{N}(r,0;G) + S(r,f) \\
&\leq 3\overline{N}(r,\infty;f) + N_{k+2}(r,0;f^n) + k\overline{N}(r,\infty;f) + N_{k+2}(r,0;L^n) \\
&\quad + N_{k+1}(r,0;f^n) \\
&\quad + k\overline{N}(r,f) + 2N_{k+1}(r,0;L^n) + 3k\overline{N}(r,L) + S(r,f) \\
&\leq (2k+3)T(r,f) + (3k+4)T(r,L) + O(\log r), \\
nT(r,L) &\leq (5k+7)T(r) + O(\log r),
\end{aligned}$$

where $T(r) = \max\{T(r,f), T(r,L)\}$.

In a similar way we can obtain

$$nT(r,f) \leq (5k+7)T(r) + O(\log r).$$

Combining the above two equations we see that

$$nT(r) \leq (5k+7)T(r) + O(\log r).$$

Since $n > 5k+7$, above equation leads to a contradiction

Case 2. If $H \equiv 0$, then on integration, we get

$$\frac{1}{F-1} \equiv \frac{C}{G-1} + D, \tag{6}$$

where C, D are constants and $C \neq 0$. From (3.2) it is clear that F and G share 1 CM. We first, assume that $D \neq 0$. Then, by (6) we get

$$\overline{N}(r, \infty; f) = S(r, f). \tag{7}$$

Now, we can write (6) as

$$\frac{1}{F-1} = \frac{D\left(G - 1 + \frac{C}{D}\right)}{G-1}. \tag{8}$$

Consequently,

$$\overline{N}\left(r, 1 - \frac{C}{D}; G\right) = \overline{N}(r, \infty; F) = \overline{N}(r, \infty; G) = S(r, F). \tag{9}$$

Subcase-2.1. If $\frac{C}{D} \neq 1$, by the second fundamental theorem, Lemma 3, we have

$$\begin{aligned} T(r, G) &\leq \overline{N}(r, \infty; G) + N_1(r, 0; G) + \overline{N}\left(r, 1 - \frac{C}{D}; G\right) + S(r, G) \\ &\leq \overline{N}(r, 0; G) + S(r, f) \leq N_2(r, 0; G) + S(r, f), \\ T(r, G) &\leq N_{k+2}(r, 0; L) + S(r, f), \\ T(r, L^n) &\leq (k+2)\overline{N}(r, 0; L) + S(r, f), \end{aligned}$$

i.e., $n \leq k+2$, which is a contradiction since $n > k+2$.

Subcase-2.2. If $\frac{C}{D} = 1$, we get from (6)

$$\left(F - 1 - \frac{1}{C}\right) G \equiv -\frac{1}{C} \tag{10}$$

i.e.

$$\frac{1}{L^n[(f^n)^{(k)} - (1 + \frac{1}{C})a]} \equiv -\frac{C}{a^2}\frac{[L^n]^{(k)}}{L^n}. \tag{11}$$

From (10) it follows that

$$N\left(r,0;[f^n]^{(k)}|\geq k+1\right)\leq N\left(r,0;[L^n]^{(k)}\right)$$
$$\leq N(r,0;G)\leq N(r,0;a)=S(r,f). \tag{12}$$

Applying the first fundamental theorem, (7), (11) and (12)

$$(n-k)T(r,f)+nT(r,L)=T\left(r,\frac{1}{L^n[(f^n)^{(k)}-(1+\frac{1}{C})a]}\right)$$
$$=m\left(r,\frac{(L^n)^{(k)}}{L^n}\right)+N\left(r,\frac{(L^n)^{(k)}}{L^n}\right)+S(r,f). \tag{13}$$

From (13) it follows that $(n-k)T(r,f)\leq S(r,f)$, which is impossible.

Hence, $D=0$ and $\frac{G-1}{F-1}=C$ or $\frac{(L^n)^{(k)}-1}{(f^n)^{(k)}-1}=C$.

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Accepted: December 9, 2019

Continuity of the farthest point map in vector spaces

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Abstract. In the paper, using the notion of sesquilinear form, we extend the Banach contraction principle in the vector space, involving rational square terms in the inequality. Furthermore, we discuss the sectional continuity of the farthest point map in vector space.
Keywords: farthest point, remotal set, sesquilinear form, completeness.

1. Introduction and preliminaries

The study of properties and application of fixed point of various type of contractive mapping in Hilbert spaces were obtained among others by Browder and Petryshyn [4], Hicks and Huffman, [5], Huffman [6], Koparde and Waghmode [7]. We refer to [8, 11] for more examples and properties of fixed point theorems in Hilber space. The object of the present note to present a fixed point theorem in vector space by using quadratic form. For the sake of completeness, we reproduce the following definitions and preliminary results, which will be needed in the sequel (see, [2, p. 151-153]).

Definition 1.1. *A mapping $\varphi : \mathscr{V} \times \mathscr{V} \to \mathbb{C}$ where $\mathscr{V}$ is complex vector space, is a sesquilinear form if satisfying the following two conditions:*

(a) $\varphi(\alpha x_1 + \beta x_2, y) = \alpha\varphi(x_1, y) + \beta\varphi(x_2, y)$,

(b) $\varphi(x, \alpha y_1 + \beta y_2) = \overline{\alpha}\varphi(x, y_1) + \overline{\beta}\varphi(x, y_2)$,

for any scalars α and β and any $x, x_1, x_2, y, y_1, y_2 \in \mathscr{V}$.

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Definition 1.2. *Let φ be a sesquilinear form on vector space $\mathscr{V}$,*

(a) *φ is called symmetric if $\varphi(x,y) = \overline{\varphi(y,x)}$ for all $x, y \in \mathscr{V}$.*

(b) *φ is called positive if $\varphi(x,y) \geq 0$ for all $x \in \mathscr{V}$.*

(c) *φ is called strictly if it is positive and $\varphi(x,y) > 0$ for all $x, y \in \mathscr{V}$.*

(d) *φ is called bounded if $|\varphi(x,y)| \leq M \|x\| \|y\|$ for some $M > 0$ and all $x, y \in \mathscr{V}$. Note that for a bounded sesquilinear form φ on $\mathscr{V}$ we have*

$$|\varphi(x,y)| \leq \|\varphi\| \|x\| \|y\|, \tag{1}$$

for all $x, y \in \mathscr{V}$.

We now recall that, two typical examples of sesquilinear forms are as follows:

(I) Let A and B be operators on an inner product space $\mathscr{V}$. Then $\varphi_1(x,y) = \langle Ax, y\rangle$, $\varphi_2(x,y) = \langle x, By\rangle$, and $\varphi_3(x,y) = \langle Ax, By\rangle$ are sesquilinear forms on $\mathscr{V}$.

(II) Let f and g be two linear functionals on a vector space $\mathscr{V}$. Then $\varphi(x,y) = f(x)\overline{g(y)}$ is a sesquilinear form on $\mathscr{V}$.

Remark 1.1. In general, the function $p : x \mapsto \sqrt{\varphi(x,x)}$ is a semi norm on $\mathscr{V}$; that is p satisfied the axioms of a norm except that the implication $p(x) = 0 \Rightarrow x = 0$ may not hold; see [3, p. 52].

Definition 1.3 ([2], Definition 4.3.6)**.** *Let φ be a sesquilinear form on a vector space $\mathscr{V}$. The function $\Phi : \mathscr{V} \to \mathbb{C}$ defined by $\Phi(x) = \varphi(x,x)$ is called the quadratic form associated with φ. A quadratic form Φ on a normed space $\mathscr{V}$ is called bounded if there exists a constant $K > 0$ such that*

$$|\Phi(x)| \leq K\|x\|^2$$

for all $x \in \mathscr{V}$.

We refer to [9, 10] for a recent survey and references therein.

Fixed point theory has fascinated many researchers since 1922 with the celebrated Banach fixed point theorem. There exists a vast literature on the topic field, and this is a very active field of research at present. Fixed point theorems are essential tools for proving the existence and uniqueness of the solutions to various mathematical models (integral and partial equations, variational inequalities, etc.).

The first result of the present paper is the generalization of the Banach contraction principle in vector spaces, involving reasonable square terms in the inequality. Our result mainly extend and generalize the previous result by Koparde and Waghmode [7]. The rest of the results are dedicated to generalizing the main theorem of [1].

2. Main results

Theorem 2.1. *Suppose Φ is a bounded and continuous quadratic form. Let $\mathscr{X}$ be a complete closed subset of a vector space and $T : \mathscr{X} \to \mathscr{X}$ be a self mapping satisfying the following condition*

(2)
$$\Phi(Tx - Ty) \leq a_1 \frac{\Phi(y - Ty)(1 + \Phi(x - Tx))}{1 + \Phi(x - y)} + a_2 \frac{\Phi(x - Tx)(1 + \Phi(y - Ty))}{1 + \Phi(x - y)}$$
$$+ a_3 \frac{\Phi(x - Ty)(1 + \Phi(y - Tx))}{1 + \Phi(x - y)} + a_4 \frac{\Phi(y - Tx)(1 + \Phi(x - Ty))}{1 + \Phi(x - y)} + a_5 \Phi(x - y),$$

for each $x, y \in \mathscr{X}$ and $x \neq y$, where a_1, a_2, a_3, a_4, a_5 are non-negative reals with

$$0 \leq a_1 + a_2 + a_3 + a_4 + a_5 < 1. \tag{3}$$

Then T has a unique fixed point in $\mathscr{X}$.

Proof. For some $x_0 \in \mathscr{X}$, we define a sequence $\{x_n\}$ of iterator of T as follows

$$x_1 = Tx_0,\ x_2 = Tx_1,\ x_3 = Tx_2, ..., x_{n+1} = Tx_n \tag{4}$$

for $n = 1, 2, 3, ...$.

Existence. We show that $\{x_n\}$ is a cauchy sequence in $\mathscr{X}$. For this, according to (4), we define

$$\Phi(x_{n+1} - x_n) = \Phi(Tx_n - Tx_{n-1})$$

then by using (2), we have

$$\begin{aligned}
\Phi(x_{n+1} - x_n) &\leq a_1 \frac{\Phi(x_{n-1} - Tx_{n-1})(1 + \Phi(x_n - Tx_n))}{1 + \Phi(x_n - x_{n-1})} \\
&+ a_2 \frac{\Phi(x_n - Tx_n)(1 + \Phi(x_{n-1} - Tx_{n-1}))}{1 + \Phi(x_n - x_{n-1})} \\
&+ a_3 \frac{\Phi(x_n - Tx_{n-1})(1 + \Phi(x_{n-1} - Tx_n))}{1 + \Phi(x_n - x_{n-1})} \\
&+ a_4 \frac{\Phi(x_{n-1} - Tx_n)(1 + \Phi(x_n - Tx_{n-1}))}{1 + \Phi(x_n - x_{n-1})} + a_5 \Phi(x_n - x_{n-1}).
\end{aligned} \tag{5}$$

So, (5) implies that

$$\begin{aligned}
&(1 - a_2 - 2a_4)\Phi(x_{n+1} - x_n) + (1 - a_1 - a_2)\Phi(x_{n+1} - x_n)\Phi(x_n - x_{n-1}) \\
&\leq ((a_1 + 2a_4 + a_5) + a_5 \Phi(x_n - x_{n-1}))\Phi(x_n - x_{n-1})
\end{aligned}$$

therefore,

$$\Phi(x_{n+1} - x_n) \leq p(n)\Phi(x_n - x_{n-1}), \tag{6}$$

where

$$p(n)=\frac{a_1+2a_4+a_5+a_5\Phi(x_n-x_{n-1})}{(1-a_2-2a_4)+(1-a_1-a_2)\Phi(x_n-x_{n-1})}, \tag{7}$$

for $n=1,2,3,\ldots$. Clearly $p(n)<1$, for all n as $0\leq a_1+a_2+a_3+4a_4+a_5<1$.

Repeating the same argument, we find some $S<1$, such that

$$\Phi(x_{n+1}-x_n)\leq\lambda^n\Phi(x_1-x_0),$$

where $\lambda=S^2$. Letting $n\to\infty$, we obtain $\Phi(x_{n+1}-x_n)\to 0$. It follows that $\{x_n\}$ is a Cauchy sequence in $\mathscr{X}$. So by completeness of $\mathscr{X}$ there exist a point $\mu\in\mathscr{X}$ such that $x_n\to\mu$ as $n\to\infty$. Also $\{x_{n+1}\}=\{Tx_n\}$ is a subsequence of $\{x_n\}$ converges to the same limit μ. Since T is continuous, we obtain

$$T(\mu)=T\left(\lim_{n\to\infty}x_n\right)=\lim_{n\to\infty}Tx_n=\lim_{n\to\infty}x_{n+1}=\mu.$$

Hence μ is a fixed point of T in $\mathscr{X}$.

Uniqueness. Now, we show that uniqueness of μ. If T has another fixed point γ and $\gamma\neq\mu$, then

$$\begin{aligned}
&\Phi(\mu-\gamma)=\Phi(T\mu-T\gamma)\\
&\leq a_1\frac{\Phi(\gamma-T\gamma)(1+\Phi(\mu-T\mu))}{1+\Phi(\mu-\gamma)}+a_2\frac{\Phi(\mu-T\mu)(1+\Phi(\gamma-T\gamma))}{1+\Phi(\mu-\gamma)}\\
&+a_3\frac{\Phi(\mu-T\gamma)(1+\Phi(\gamma-T\mu))}{1+\Phi(\mu-\gamma)}+a_4\frac{\Phi(\gamma-T\mu)(1+\Phi(\mu-T\gamma))}{1+\Phi(\mu-\gamma)}\\
&+a_5\Phi(\mu-\gamma),
\end{aligned} \tag{8}$$

hence (8) implies that

$$\Phi(\mu-\gamma)\leq(a_3+a_4+a_5)\Phi(\mu-\gamma).$$

This gives a contradiction for $a_3+a_4+a_5<1$. Thus μ is a unique fixed point of T in $\mathscr{X}$. □

Before proceeding, we recall further notion that will be used in the sequel. Assume that $\mathscr{V}$ is a vector space and A is a uniquely subset of $\mathscr{V}$. The set-valued map

$$Q_A(x)=\left\{q_A(x)\in A:\ \|x-q_A(x)\|=\sup_{a\in A}\|x-a\|\right\}$$

is called the farthest point map of A. If, for any $x\in\mathscr{V}$, the set $Q_A(x)$ is not empty (respectively, is singleton), then A is said to be remotal (resp. uniquely remotal). For a remotal subset A, the farthest point map Q_A is said to be sectionally continuous at x_0 if for any sequence $\{x_n\}$ with $x_n\to x$ there exist $q_A(x_n), q_A(x_0)$ such that $q_A(x_n)\to q_A(x_0)$. For any $x_0\in\mathscr{V}$, we define

$$\mathscr{V}_{x_0}=\left\{x\in\mathscr{V}:\ \|x_0-q_A(x_0)\|_\varphi\leq\|x-q_A(x)\|_\varphi\right\}.$$

Theorem 2.2. *The farthest point map q_A is continuous at x_0 if and only if the restriction of q_A to $\mathscr{V}_{x_0}$ is continuous at x_0.*

Proof. It is easy to check that if A singleton, then the result is obvious, so we assume A is not singleton. Assume that the restriction of q_A to $\mathscr{V}_{x_0}$ is continuous at x_0. Let N be a neighbourhood of $q_A(x_0)$, then there is a neighbourhood of x_0 , call G, such that $q_A(\mathscr{V}_{x_0} \cap G) \subseteq N$. Choose $\delta > 0$ such that the open ball $B_{2\delta}(x_0)$ is contained in G, where $B_{2\delta}(x_0) = \{x \in \mathscr{V} : \|x - x_0\|_\varphi < 2\delta\}$. Let $x \in B_\delta(x_0)\backslash\mathscr{V}_{x_0}$ and put $y = q_A(x) + \frac{\|x_0 - q_A(x_0)\|_\varphi}{\|x - q_A(x)\|_\varphi}(x - q_A(x))$. Therefore, $q_A(y) = q_A(x)$, $y \in \mathscr{V}_{x_0}$ and

$$\begin{aligned}
\|y - x_0\|_\varphi &\leq \|y - x\|_\varphi + \|x - x_0\|_\varphi \\
&= \|x_0 - q_A(x_0)\|_\varphi - \|x - q_A(x)\|_\varphi + \|x - x_0\|_\varphi \\
&\leq \|x - x_0\|_\varphi + \|x - x_0\|_\varphi \\
&= 2\|x - x_0\|_\varphi \\
&< 2\delta.
\end{aligned}$$

Consequently, $y \in \mathscr{V}_{x_0} \cap G, q_A(x) = q_A(y) \in N$. Thus, $q_A(B_\delta(x_0)) \subseteq N$ and q_A is continuous at x_0. □

Theorem 2.3. *Let F be the collection of all farthest point maps of remotal subsets of a vector space $\mathscr{V}$. Then every element of F is sectionally continuous at 0 if and only if every element of F is sectionally continuous on $\mathscr{V}$.*

Proof. Let $Q_r \in F$ and $x_0 \in \mathscr{V}$. We define $G = \{x_0 - t : t \in A\}$. Thus G is a remotal subset of $\mathscr{V}$ and $Q_G(x) = x_0 - Q_A(x_0 - x), x \in \mathscr{V}$. So, $Q_G \in F$ and $Q_A(x) = x_0 - Q_G(x_0 - x), x \in \mathscr{V}$. Assume that Q_G is sectionally continuous at 0. If $x_n \to x$ then, of course, $x_0 - x_n \to 0$ and hense there exist $q_G(0) \in Q_G(0), q_G(x_0 - x_n) \in Q_G(x_0 - x_n)$ such that $q_G(x_0 - x_n) \to q_G(0)$ for some $q_A(x_n) \in Q_A(x_n)$ and $q_A(x_0) \in Q_A(x_0)$. Therefore, Q_T is sectionally continuous at x_0. So, the proof is completed. □

Theorem 2.4. *Let A be a uniquely remote subset of a vector space $\mathscr{V}$ and $[c, q_A] \subseteq \mathscr{V}_c$ for some $c \in \mathscr{V}$. If q_A is continuous at c then $A = \{c\}$.*

Proof. Since q_A is continuous at c then the function $x \to \|x - q_A(x)\|_\varphi$ is differentiable at c. If $\alpha \in (0, 1)$ and $x_\alpha = (1 - \alpha)c + \alpha q_A(c)$, then

$$\lim_{\alpha \to 0} \frac{\|c - q_A(c)\|_\varphi - \|x_\alpha - q_A(x_\alpha)\|_\varphi}{\alpha} = \|c - q_A(c)\|_\varphi.$$

If $[c, q_A(c)] \subseteq \mathscr{V}_c$, then $\|c - q_A(c)\|_\varphi - \|x_\alpha - q_A(x_\alpha)\|_\varphi \leq 0$. Therefore, $\|c - q_A(c)\|_\varphi \leq 0$. Hence, A is singleton and $T = \{0\}$ since $q_A(c) \in A$. □

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Accepted: July 23, 2019

On generalized weakly conharmonically symmetric manifolds

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Abstract. The object of the present paper is to study generalized weakly conharmonically symmetric manifold and obtained some of its geometric properties. Existence of such manifold is ensured by a non-trivial example.

Keywords: weakly symmetric manifold, generalized weakly symmetric manifold, conharmonic curvature tensor.

1. Introduction

Let M be a $n(\geq 3)$-dimensional manifold equipped with a Riemannian metric g, Levi-Civita connection ∇, Riemnnan curvature R, Ricci tensor S and scalar curvature r. Generalizing Chaki pseudo symmetry, Támassy and Binh [16] introduced the notion of weakly symmetric manifold (briefly, $(w.s)_n$). The above M is called $(w.s)_n$ if

$$\nabla R = \alpha \otimes R + \beta \otimes R + \beta \otimes R + \gamma \otimes R + \gamma \otimes R.$$

For details we refer to see the paper of A. A. Shaikh and his co-authors ([3], [6], [7], [8], [9], [10], [11], [14], etc) and also the references therein.

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Generalizing $(w.s)_n$, recently, Baishya [1] introduced the notion of generalized weakly symmetric space (briefly g.w.s) defined by the following equations

$$\begin{aligned}(1)\qquad \nabla R &= \alpha_1\otimes R+\beta_1\otimes R+\beta_1\otimes R+\gamma_1\otimes R+\gamma_1\otimes R\\ &+ \alpha_2\otimes G+\beta_2\otimes G+\beta_2\otimes G+\gamma_2\otimes G+\gamma_2\otimes G,\end{aligned}$$

where G is Guassian curvature tensor and α_1, β_1, γ_1, α_2, β_2 and γ_2 are non-zero 1-forms defined by $\alpha(X)=g(X,\theta_1)$, $\beta(X)=g(X,\phi_1)$ and $\gamma(X)=g(X,\pi_1)$, $\alpha_1(X)=g(X,\theta_2)$, $\beta_1(X)=g(X,\phi_2)$ and $\gamma_1(X)=g(X,\pi_2)$ for all X. As a special subgroup of the conformal transformation group,Y. Ishii [2] introduced the notion of the conharmonic transformation and defined a curvature tensor, called, conharmonic curvature tensor $\overline{C}$.

The object of the present paper is to study the notion of g.w.s with $\overline{C}$. The manifold M is called generalized weakly conharmonically manifolds if the conharmonic curvature tensor $\overline{C}$ satisfies the following relation:

$$\begin{aligned}(2)\qquad &(\nabla_X\overline{C})(Y,Z,U,V)=\alpha_3(X)\overline{C}(Y,Z,U,V)+\beta_3(Y)\overline{C}(X,Z,U,V)\\ &+\beta_3(Z)\overline{C}(Y,X,U,V)+\gamma_3(U)\overline{C}(Y,Z,X,V)+\gamma_3(V)\overline{C}(Y,Z,U,X)\\ &+\alpha_4(X)G(Y,Z,U,V)+\beta_4(Y)G(X,Z,U,V)+\beta_4(Z)G(Y,X,U,V)\\ &+\gamma_4(U)G(Y,Z,X,V)+\gamma_4(V)G(Y,Z,U,X),\end{aligned}$$

where α_3, β_3, γ_3, α_4, β_4 and γ_4 are non-zero 1-forms defined as $\alpha_3(X)=g(X,\overline{\theta_1})$, $\beta_3(X)=g(X,\overline{\phi_1})$ and $\gamma_3(X)=g(X,\overline{\pi_1})$, $\alpha_4(X)=g(X,\overline{\theta_2})$, $\beta_4(X)=g(X,\overline{\phi_2})$ and $\gamma_4(X)=g(X,\overline{\pi_2})$. Such an n-dimensional manifold is denoted by $(GW\overline{C}S)_n$. The nature of the scalar curvature is determined and shown that the 1-forms are co-directional. It is observed that a weakly conformally symmetric space is a $(GW\overline{C}S)_n$ and vice-versa. A non-trivial example is found out to ensure the existence of such space.

The structure of the paper is as follows. Section 2 deals with preliminaries. Section 3 is concerned with some basic properties of $(GW\overline{C}S)_n$. In section 4 we investigate some spaces which are $(GW\overline{C}S)_n$. Section 5 deals with Einstein $(GW\overline{C}S)_n$. The last section is concerned with a non-trivial example of $(GW\overline{C}S)_n$.

2. Preliminaries

The conharmonic curvature tensor $(\overline{C})$ defined by Ishii [2] is given as follows

$$(3)\qquad \overline{C}=R-\frac{1}{n-2}g\wedge S,$$

where $\wedge$ is the wedge product ([12], [13]) defined by

$$\begin{aligned}(A\wedge B)(X_1,X_2,Y_1,Y_2) &= A(X_1,Y_2)B(X_2,Y_1)+A(X_2,Y_1)B(X_1,Y_2)\\ &- A(X_1,Y_1)B(X_2,Y_2)-A(X_2,Y_2)B(X_1,Y_1),\end{aligned}$$

where A and B are two (0,2) tensors and X_1, X_2, Y_1 and $Y_2 \in \chi(M)$, $\chi(M)$ being the Lie algebra of all smooth vector fields on M.

The local expression of (2) is

$$\begin{aligned}(4)\qquad \overline{C}_{mnpq,k} &= \alpha_{3k}\overline{C}_{mnpq} + \beta_{3_m}\overline{C}_{knpq} + \beta_{3_n}\overline{C}_{mkpq} + \gamma_{3_p}\overline{C}_{mnkq}\\ &+ \gamma_{3_q}\overline{C}_{mnpk} + \alpha_{4k}G_{mnpq} + \beta_{4_m}G_{knpq} + \beta_{4_n}G_{mkpq}\\ &+ \gamma_{4_p}G_{mnkq} + \gamma_{4_q}G_{mnpk}.\end{aligned}$$

If the 1-forms $\alpha_3 = \beta_3 = \gamma_3 = 0$, then the manifold reduces to a weakly conharmonically symmetric manifold [4], [5].

3. Some basic geometric properties of $(GW\overline{C}S)_n$

Now, contracting (2) we have

$$\begin{aligned}&-\frac{1}{n-2}g(Y,V)dr(X)\\ &= -\frac{1}{n-2}\alpha_3(X)rg(Y,V) - \frac{1}{n-2}rg(X,V)\beta_3(Y)\\ &-\beta_3(R(Y,X)V) - \frac{1}{n-2}[\beta_3(X)S(Y,V) + \beta_3(QX)g(Y,V)\\ &-\beta_3(Y)S(X,V) - g(X,V)\beta_3(QY)]\\ (5)\qquad &+\gamma_3(R(X,V)Y) - \frac{1}{n-2}[\gamma_3(X)S(Y,V) + \gamma_3(QX)g(Y,V)\\ &-g(Y,X)\gamma_3(QV) - \gamma_3(V)S(Y,X)] - \frac{1}{n-2}\gamma_3(V)rg(Y,X)\\ &+(n-1)[\alpha_4(X)g(Y,V) + \beta_4(Y)g(X,V) + \gamma_4(V)g(Y,X)]\\ &+[\beta_4(X)g(Y,V) - \beta_4(Y)g(X,V)] + [\gamma_4(X)g(Y,V) - \gamma_4(V)g(Y,X)].\end{aligned}$$

Contraction (5) over Y and V, we get

$$\begin{aligned}(6)\qquad &\frac{1}{n-2}ndr(X) = \frac{1}{n-2}nr\alpha_3(X) + \frac{2}{n-2}r\beta_3(X)\\ &+2n - 2r\gamma_3(X) - (n-1)[n\alpha_4(X) + 2\beta_4(X) + 2\gamma_4(X)].\end{aligned}$$

Again taking contraction of (5) over Y and X, we get

$$\begin{aligned}(7)\qquad -\frac{1}{n-2}dr(V) &= -\frac{1}{n-2}r\alpha_3(V)\\ &- \frac{1}{n-2}r\beta_3(V) - \frac{1}{n-2}r(n-1)\gamma_3(V)\\ &+ (n-1)[\alpha_4(V) + \beta_4(V) + (n-1)\gamma_4(V)].\end{aligned}$$

Contracting (5) over X and V gives us

$$\begin{aligned}(8)\qquad -\frac{1}{n-2}dr(Y) &= -\frac{1}{n-2}r\alpha_3(Y)\\ &- r\frac{(n-1)}{n-2}\beta_3(Y)-\frac{1}{n-2}r\gamma_3(Y)\\ &+ (n-1)[\alpha_4(Y)+(n-1)\beta_4(Y)+\gamma_4(Y)].\end{aligned}$$

From (6) and (7), we have

$$(9)\qquad r=(n-1)(n-2)\frac{[\beta_4(X)+(n+1)\gamma_4(X)]}{[\beta_3(X)+(n+1)\gamma_3(X)]}.$$

In view of (6) and (8) we get

$$(10)\qquad r=(n-1)(n-2)\frac{[(n+1)\beta_4(X)+\gamma_4(X)]}{[(n+1)\beta_3(X)+\gamma_3(X)]}.$$

From (7) and (8) we obtain

$$(11)\qquad r=(n-1)(n-2)\frac{[\beta_4(X)-\gamma_4(X)]}{[\beta_3(X)-\gamma_3(X)]}.$$

Also (9) and (11) yields

$$(12)\qquad \gamma_4(X)\beta_3(X)=\beta_4(X)\gamma_3(X).$$

This leads the following:

Theorem 3.1. *In a* $(GW\overline{C}S)_n$*, the* 1*-forms are related by the expression* (12)*.*

Using (12) in (11) we have

$$(13)\qquad r=(n-1)(n-2)\frac{\gamma_4(X)}{\gamma_3(X)}.$$

Again by virtue of (12), (10) takes the form

$$(14)\qquad r=(n-1)(n-2)\frac{\beta_4(X)}{\beta_3(X)}.$$

Thus, we can state the following:

Theorem 3.2. *The scalar curvature of a* $(GW\overline{C}S)_n$ *is given by* (13) *and* (14)*.*

If the scalar curvature of the manifold is constant, with the help (13) and (14) we obtain from (6) that

$$(15)\qquad r\alpha_3(X)=(n-1)(n-2)\alpha_4(X).$$

This gives the following:

Theorem 3.3. *If the scalar curvature of a $(GW\overline{C}S)_n$ is constant, then its associated 1-forms are related by* (15).

If the scalar curvature of $(GW\overline{C}S)_n$ is a non-zero constant, then we have $dr(X) = 0$ and hence from (6), (7) and (8) we obtain

$$\begin{aligned} & nr\alpha_3(X) + 2r\beta_3(X) + 2r\gamma_3(X) \\ - \ & (n-1)(n-2)\big[n\alpha_4(X) + 2\beta_4(X) + 2\gamma_4(X)\big] = 0, \end{aligned} \tag{16}$$

$$\begin{aligned} & r\alpha_3(X) + r\beta_3(X) + (n-1)r\gamma_3(X) \\ - \ & (n-1)(n-2)\big[\alpha_4(X) + \beta_4(X) + (n-1)\gamma_4(X)\big] = 0, \end{aligned} \tag{17}$$

$$\begin{aligned} & r\alpha_3(X) + r\beta_3(X) + (n-1)r\gamma_3(X) \\ - \ & (n-1)(n-2)\big[\alpha_4(X) + (n-1)\beta_4(X) + \gamma_4(X)\big] = 0. \end{aligned} \tag{18}$$

From the above relations we can state the following:

Theorem 3.4. *If the scalar curvature of a $(GW\overline{C}S)_n$ is constant, then its associated 1-forms are related by the expression (16), (17) and (18).*

Theorem 3.5. *If the scalar curvature of a $(GW\overline{C}S)_n$ is constant, then 1-forms α_3, β_3 and γ_3 are respectively co-directional with the 1-forms α_4, β_4 and γ_4.*

4. Weakly conformally symmetric space and $(GW\overline{C}S)_n$

We consider a Riemannian manifold which is weakly conformally symmetric [15], then we have

$$\begin{aligned} (\nabla_X C)(Y,Z,U,V) &= \alpha_5(X)C(Y,Z,U,V) + \beta_5(Y)C(X,Z,U,V) \\ &\quad + \beta_5(Z)C(Y,X,U,V) + \gamma_5(U)C(Y,Z,X,V) \\ &\quad + \gamma_5(V)C(Y,Z,U,X), \end{aligned} \tag{19}$$

where α_5, β_5, γ_5 are non-zero 1-forms and C is the conformal curvature tensor given by

$$C = \overline{C} + \frac{r}{(n-1)(n-2)}G.$$

Using above equation in (19) we get

$$\begin{aligned} & (\nabla_X \overline{C})(Y,Z,U,V) = \alpha_6(X)\overline{C}(Y,Z,U,V) + \beta_6(Y)\overline{C}(X,Z,U,V) \\ & + \beta_6(Z)\overline{C}(Y,X,U,V) + \gamma_6(U)\overline{C}(Y,Z,X,V) + \gamma_6(V)\overline{C}(Y,Z,U,X) \\ & + \alpha_7(X)G(Y,Z,U,V) + \beta_7(Y)G(X,Z,U,V) + \beta_7(Z)G(Y,X,U,V) \\ & + \gamma_7(U)G(Y,Z,X,V) + \gamma_7(V)G(Y,Z,U,X), \end{aligned} \tag{20}$$

where $\alpha_6 = \alpha_5$, $\beta_6 = \beta_5$, $\gamma_6 = \gamma_5$, $\alpha_7 = \frac{1}{(n-1)(n-2)}[r\alpha_5 - dr]$, $\beta_7 = \frac{r\beta_5}{(n-1)(n-2)}$ and $\gamma_7 = \frac{r\gamma_5}{(n-1)(n-2)}$ are non-zero 1-forms. Conversely, if we consider the space as a $(GW\overline{C}S)_n$, then we can easily shown that the space is weakly conformally symmetric. This leads to the following:

Theorem 4.1. *A weakly conformally symmetric space is a* $(GW\overline{C}S)_n$ *and vice-versa.*

5. Einstein $(GW\overline{C}S)_n$

Let us consider a $(GW\overline{C}S)_n$ which is Einstein. Then we have

$$S(X,Y) = \frac{r}{n}g(X,Y), \tag{21}$$

which yields $(\nabla_Z S)(X,Y) = 0$ and $dr(X) = 0$. By virtue of (21), (3) yields

$$\overline{C} = R - \frac{2r}{n(n-2)}G. \tag{22}$$

Using (22) in (2) we have

$$\begin{aligned}(\nabla_X R)(Y,U,V,W) &= \alpha_8(X)R(Y,U,V,W) + \beta_8(Y)R(X,U,V,W)\\ &+ \beta_8(U)R(Y,X,V,W) + \gamma_8(V)R(Y,U,X,W) + \gamma_8(W)R(Y,U,V,X)\\ &+ \alpha_9(X)G(Y,U,V,W) + \beta_9(Y)G(X,U,V,W) + \beta_9(U)G(Y,X,V,W)\\ &+ \gamma(V)_9G(Y,U,X,W) + \gamma_9(W)G(Y,U,V,X),\end{aligned} \tag{23}$$

where $\alpha_8 = \alpha_3$, $\beta_8 = \beta_3$, $\gamma_8 = \gamma_3$, $\alpha_9 = \alpha_4 - \frac{2r}{n(n-2)}\alpha_3$, $\beta_9 = \beta_4 - \frac{2r}{n(n-2)}\beta_3$ and $\gamma_9 = \gamma_4 - \frac{2r}{n(n-2)}\gamma_3$ are non-zero 1-forms and hence the manifold under consideration is a *g.w.s* [1].

This leads to the following:

Theorem 5.1. *An Einstein* $(GW\overline{C}S)_n$ *is a g.w.s.*

6. Existence of a $(GW\overline{C}S)_4$ space

Let $(\mathbb{R}^4, g)$ be a connected smooth 4-dimensional Riemannian space endowed with the Riemannian metric g given by

$$\begin{aligned}&ds^2 = g_{ij}dx^i dx^j = (dx^1)^2 + (x^1)^{\frac{4}{3}}(dx^2)^2 + (x^1)^{\frac{4}{3}}(dx^3)^2 + (x^1)^{\frac{4}{3}}(dx^4)^2,\\ &\text{with} \quad x^1 > 0, \quad (i,j = 1,2,3,4).\end{aligned} \tag{24}$$

$$\begin{cases}\overline{C}_{1212} = \frac{2}{9(x^1)^{\frac{2}{3}}} = \overline{C}_{1313} = \overline{C}_{1414},\\ \overline{C}_{2323} = \frac{2(x^1)^{\frac{2}{3}}}{9} = \overline{C}_{2424} = \overline{C}_{3434},\\ r = \frac{4}{3(x^1)^2}.\end{cases} \tag{25}$$

With the help of (25), we can find out

$$\begin{cases}G_{1212} = -(x^1)^{\frac{4}{3}} = G_{1313} = G_{1414},\\ G_{2323} = -(x^1)^{\frac{8}{3}} = G_{2424} = G_{3434}.\end{cases} \tag{26}$$

The non-vanishing components of covariant derivative of conharmonic curvature tensors are

$$\begin{cases} \overline{C}_{1212,1} = -\frac{4}{9(x^1)^{\frac{5}{3}}} = \overline{C}_{1313,1} = \overline{C}_{1414,1}, \\ \overline{C}_{2323,1} = -\frac{4}{9(x^1)^{\frac{1}{3}}} = \overline{C}_{2424,1} = \overline{C}_{3434,1}. \end{cases} \tag{27}$$

It is easy to check that the metric (24) is conformally flat.
We consider the 1-forms as follows:

$$\begin{cases} \alpha_3(\partial_i) = \alpha_{3i} = \begin{cases} \frac{7}{x^1}, & \text{for} \quad i = 1, \\ 0, & \text{otherwise}, \end{cases} \\ \beta_3(\partial_i) = \beta_{3i} = \begin{cases} \frac{9}{2}(x^1)^2, & \text{for} \quad i = 1, \\ 0, & \text{otherwise}, \end{cases} \\ \gamma_3(\partial_i) = \gamma_{3i} = \begin{cases} 9x^1, & \text{for} \quad i = 1, \\ 0, & \text{otherwise}, \end{cases} \end{cases} \tag{28}$$

$$\begin{cases} \alpha_4(\partial_i) = \alpha_{4i} = \begin{cases} \frac{2}{(x^1)^3}, & \text{for} \quad i = 1, \\ 0, & \text{otherwise}, \end{cases} \\ \beta_4(\partial_i) = \beta_{4i} = \begin{cases} 1, & \text{for} \quad i = 1, \\ 0, & \text{otherwise}, \end{cases} \\ \gamma_4(\partial_i) = \gamma_{4i} = \begin{cases} \frac{2}{x^1}, & \text{for} \quad i = 1, \\ 0, & \text{otherwise}, \end{cases} \end{cases} \tag{29}$$

where $\partial_i = \frac{\partial}{\partial u^i}$, u^i being the local coordinates of $\mathbb{R}^4$. By virtue of (25)-(29) it can be easily shown that the manifold considering above is a $(GW\overline{C}S)_4$ and hence we can state the following:

Theorem 6.1. *Let $M = (\mathbb{R}^4, g)$ be a Riemannian manifold equipped with the metric given by* (24). *Then M is a $(GW\overline{C}S)_4$ with non-vanishing and non-constant scalar curvature.*

7. Acknowledgement

The authors are thankful to the referee for her/his valuable suggestions.

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Accepted: December 02, 2019

Some results on uniqueness of homogeneous differential polynomials of meromorphic functions

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Abstract. In this paper, we deal with the uniqueness problem of homogeneous differential polynomials of meromorphic functions sharing one small function $a(z)$ with weight l, where l is a non negative integer, with certain essential conditions and prove some results which generalize some results due to Lahiri and Pal [9].

Keywords: meromorphic function, differential polynomial, weighted sharing, small function, uniqueness.

1. Introduction and main results

Let f be a non-constant meromorphic function defined on the open complex plane $\mathbb{C}$. We assume that the reader is familiar with the standard definitions and notations used in the Nevanlinna value distribution theory, such as $T(r,f), m(r,f), N(r,f)$ (see, [2, 16, 21]). By $S(r,f)$ we denote any quantity satisfying the condition $S(r,f) = \circ(T(r,f))$ as $r \to \infty$ possibly outside an exceptional set of finite linear measure. A meromorphic function a is called a small function with respect to f if either $a \equiv \infty$ or $T(r,a) = S(r,f)$. We denote by $S(f)$ the collection of all small functions with respect to f. Clearly $\mathbb{C} \cup \{\infty\} \subset S(f)$ and $S(f)$ is a field over the set of complex numbers. For $a \in \mathbb{C} \cup \{\infty\}$ the quantities

$$\delta(a,f) = 1 - \limsup_{r\to\infty} \frac{N(r,a;f)}{T(r,f)}$$

and

$$\Theta(a,f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r,a;f)}{T(r,f)}.$$

*. Corresponding author

are respectively called the deficiency and ramification index of a for the function f.

For any two nonconstant meromorphic functions f and g, and $a \in S(f) \cap S(g)$ we say that f and g share a IM (CM) provided that $f - a$ and $g - a$ have the same zeros ignoring (counting) multiplicities. If $\frac{1}{f}$ and $\frac{1}{g}$ share 0 IM (CM), we say that f and g share ∞ IM (CM). Let f and g share 1 IM and let z_0 be a zero of $f - 1$ of multiplicity p and a zero of $g - 1$ of multiplicity q. By $\overline{N}_L(r, 1; f)$ we denote the reduced counting function of those 1-points of f and g where $p > q \geq 1$; $\overline{N}_L(r, 1; g)$ is defined similarly.

In this paper, we use n to denote any nonnegative integer. Also $\rho(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}$ is the order of f.

In 1976, M. Ozawa [12] proved the following theorem.

Theorem 1.1. *Let f and g be two non-constant entire functions such that f, g share the value 1 CM. If $\delta(0, f) > 0$ and 0 is a Picard exceptional value of g, then either $f \equiv g$ or $f.g \equiv 1$.*

Now, a days the problem on sharing values between two nonconstant meromorphic functions f and g and their relationship is an interesting area of research of uniqueness theory of functions.

In 1976, C. C. Yang [15] posted the following question:

Question 1.1. *Suppose that f and g are two transcendental entire functions such that f and g share the value 0 CM and $f^{(1)}$, $g^{(1)}$ share the value 1 CM. What can be said about the relationship between f and g ?*

Many authors, including Shibazaki [13], Yi [17, 18], Yang and Yi [19], Hua [3], Mues and Reinders [11], Lahiri [5, 6] studied the question.

In [18] H. X. Yi gave an answer to the above question of C. C. Yang [15] and proved the following result:

Theorem 1.2 ([18])**.** *Let f and g be two nonconstant meromorphic functions. If f, g share the value 0 CM, $f^{(n)}$, $g^{(n)}$ share the value 1 CM and $2\delta(0, f) + (n + 2)\Theta(\infty, f) > n + 3$, then either $f \equiv g$ or $f^{(n)}.g^{(n)} \equiv 1$.*

Later Yi [20] proved the following improvement of Theorem 1.2:

Theorem 1.3 ([20])**.** *Let f and g be two nonconstant meromorphic functions such that $f^{(n)}$, $g^{(n)}$ share the value 1 CM and f, g share the value ∞ CM. If*

$$N(r, 0; f) + N(r, 0; g) + (n + 2)\overline{N}(r, f) < (\lambda + o(1))T(r),$$

for $r \in I$, a set of infinite linear measure and λ is a positive constant < 1, then $f^{(n)}.g^{(n)} \equiv 1$ unless $f \equiv g$.

In 1997, Yi [22] proved some results which improved Theorems 1.2-1.3.

Theorem 1.4 ([22]). *Let f and g be two nonconstant meromorphic functions such that $f^{(n)}$, $g^{(n)}$ share the value 1 CM. If*

$$2\delta(0,f)+(n+4)\Theta(\infty,f)>n+5 \text{ and}$$

$$2\delta(0,;g)+(n+4)\Theta(\infty,g)>n+5,$$

then either $f\equiv g$ or $f^{(n)}.g^{(n)}\equiv 1$.

Theorem 1.5 ([22]). *Let f and g be two nonconstant meromorphic functions such that $f^{(n)}$, $g^{(n)}$ share the value 1 IM. If*

$$5\delta(0,f)+(4n+7)\Theta(\infty,f)>4n+11 \text{ and}$$

$$5\delta(0,g)+(4n+7)\Theta(\infty,g)>4n+11,$$

then either $f\equiv g$ or $f^{(n)}.g^{(n)}\equiv 1$.

In [10] Li and Li considered the problem of replacing the derivatives by linear differential polynomials. Let f be a nonconstant meromorphic function. An expression of the form

$$L(f)=f^{(n)}+a_{k-1}f^{(n-1)}+....+a_0f,$$

where $a_0,a_1,...,a_{n-1}$ are complex constants, is called a linear differential polynomial generated by f.

Li and Li [10] proved that following theorems:

Theorem 1.6 ([10]). *Let f and g be two nonconstant entire functions. Suppose that f, g share the value 0 CM and $L(f)$, $L(g)$ share the value 1 CM and $\delta(0,f)>\frac{1}{2}$. If $\rho(f)\neq 1$, then either $f\equiv g$ or $L(f).L(g)\equiv 1$.*

Theorem 1.7 ([10]). *Let f and g be two nonconstant entire functions. Suppose that f, g share the value 0 CM and $L(f)$, $L(g)$ share the value 1 IM and $\delta(0,f)>\frac{4}{5}$. If $\rho(f)\neq 1$, then either $f\equiv g$ or $L(f).L(g)\equiv 1$.*

Definition 1.1. *Let m (≥ 1) be a positive integer, t (≥ 0) be an integer and let f be a nonconstant meromorphic function. An expression of the form*

$$P[f]=\sum_{k=1}^{m}a_k\prod_{j=0}^{t}\left(f^{(j)}\right)^{l_{kj}}, \tag{1}$$

where $a_k\in S(f)$ for $k=1,2,.....,m$ and l_{kj} $(1\leq k\leq m;\ 0\leq j\leq t)$ are nonnegative integers and $d=\sum_{j=0}^{t}l_{kj}$ for $k=1,2,.....,m$, is called a homogeneous differential polynomial of degree d generated by f. Also we denote by Q the quantity $Q=\max_{1\leq k\leq m}\sum_{j=0}^{t}jl_{kj}$.

Let f and g be two nonconstant meromorphic functions. When we consider $P[f]$ and $P[g]$ are nonconstant homogeneous differential polynomials of f and g respectively, then we understand that the coefficients $a_j \in S(f) \cap S(g)$.

Recently in 2017 Lahiri and Pal [9] extended the results of Li and Li [10] to homogeneous differential polynomial and obtained the following theorem.

Theorem 1.8. *Let f and g be two nonconstant meromorphic functions, a ($\not\equiv 0, \infty$) $\in S(f) \cap S(g)$. Suppose $P[f]$ and $P[g]$, as defined by (1) are nonconstant. If $P[f]$ and $P[g]$ share a IM and*

$$\min\left\{5\delta(0,f) + \frac{4Q+7}{d}\Theta(\infty,f), 5\delta(0,g) + \frac{4Q+7}{d}\Theta(\infty,g)\right\} > \frac{4Q+4d+7}{d}, \tag{2}$$

then either $P[f] \equiv P[g]$ or $P[f].P[g] \equiv a^2$.

Remark 1.1. If $P[f]$ and $P[g]$ share a CM, then the condition (2) of Theorem 1.8 can be replaced by the following

$$\min\left\{2\delta(0,f) + \frac{Q+4}{d}\Theta(\infty,f), 2\delta(0,g) + \frac{Q+4}{d}\Theta(\infty,g)\right\} > \frac{Q+d+4}{d}.$$

So, one may ask the following question which is the motivation of this paper.

Question 1.2. *Keeping the conclusion of the above theorem intact is it possible to relax the nature of sharing the function a between CM and IM ?*

To investigate this problem we use a new notion of scaling between CM and IM known as weighted sharing, which is introduced in [7, 8]. In the following definition we explain this notion.

Definition 1.2 ([7, 8])**.** *Let l be a nonnegative integer or infinity and $a \in S(f)$. We denote by $E_l(a, f)$ the set of all zeros of $f - a$, where a zero of multiplicity m is counted m times if $m \leq l$ and $l + 1$ times if $m > l$. If $E_l(a, f) = E_l(a, g)$, we say that f, g share the function a with weight l. We write f and g share (a, l) to mean that f and g share the function a with weight l. Since $E_l(a, f) = E_l(a, g)$ implies that $E_s(a, f) = E_s(a, g)$ for any integer s $(0 \leq s < l)$, if f, g share (a, l), then f, g share (a, s), $(0 \leq s < l)$. Moreover, we note that f and g share the function a IM or CM if and only if f and g share $(a, 0)$ or (a, ∞) respectively.*

In this paper, we try to give a positive answer to question 1.2 and obtain the following results.

Theorem 1.9. *Let f and g be two nonconstant meromorphic functions, a ($\not\equiv 0, \infty$) $\in S(f) \cap S(g)$. Suppose $P[f]$ and $P[g]$, as defined by (1) are nonconstant. If $P[f]$ and $P[g]$ share (a, l) with one of the following conditions:*

(i) $l = \infty$ *and*

$$\min\left\{2\delta(0,f)+\frac{Q+4}{d}\Theta(\infty,f), 2\delta(0,g)+\frac{Q+4}{d}\Theta(\infty,g)\right\} > \frac{Q+d+4}{d}, \tag{3}$$

(ii) $0 < l < \infty$ *and*

$$\begin{aligned}\min\Big\{&\frac{(2l+1)d}{l}\delta(0,f)+\left(\frac{Q+1+2l}{l}+Q+2\right)\Theta(\infty,f),\\ &\frac{(2l+1)d}{l}\delta(0,g)+\left(\frac{Q+1+2l}{l}+Q+2\right)\Theta(\infty,g)\Big\}\\ &> \frac{(l+1)d+Q+1}{l}+4+Q,\end{aligned} \tag{4}$$

(iii) $l = 0$ *and*

$$\min\left\{5\delta(0,f)+\frac{4Q+7}{d}\Theta(\infty,f), 5\delta(0,g)+\frac{4Q+7}{d}\Theta(\infty,g)\right\} > \frac{4Q+4d+7}{d}, \tag{5}$$

then either $P[f] \equiv P[g]$ *or* $P[f].P[g] \equiv a^2$.

Theorem 1.10. *Let* f *and* g *be two nonconstant meromorphic functions,* $a\ (\not\equiv 0, \infty) \in S(f) \cap S(g)$. *Suppose* $P[f]$ *and* $P[g]$, *as defined by (1) are nonconstant. If* f *and* g *share the value* 0 *CM and* ∞ *IM and* $P[f]$, $P[g]$ *share* (a, l) *with one of the following conditions:*

(i) $l = \infty$ *and*

$$2\delta(0,f)+\frac{Q+4}{d}\Theta(\infty,f) > \frac{Q+d+4}{d},$$

(ii) $0 < l < \infty$ *and*

$$\frac{(2l+1)d}{l}\delta(0,f)+\left(\frac{Q+1+2l}{l}+Q+2\right)\Theta(\infty,f) > \frac{(l+1)d+Q+1}{l}+4+Q,$$

(iii) $l = 0$ *and*

$$5\delta(0,f)+\frac{4Q+7}{d}\Theta(\infty,f) > \frac{4Q+4d+7}{d},$$

then either $P[f] \equiv P[g]$ *or* $P[f].P[g] \equiv a^2$.

Theorem 1.11. *Let* f *and* g *be two nonconstant entire functions,* $a\ (\not\equiv 0, \infty) \in S(f) \cap S(g)$. *Suppose* $P[f]$ *and* $P[g]$ *are nonconstant homogeneous differential polynomials of degree* d *as defined by (1). If* f *and* g *share the value* 0 *CM,* $P[f]$ *and* $P[g]$ *share* (a, l) *with one of the following conditions:*

(i) $l = \infty$ *and*

$$\delta(0,f) > \frac{1}{2},$$

(*ii*) $0 < l < \infty$ *and*

$$\delta(0, f) > \frac{l+1}{2l+1},$$

(*iii*) $l = 0$ *and*

$$\delta(0, f) > \frac{4}{5},$$

then either $P[f] \equiv P[g]$ *or* $P[f].P[g] \equiv a^2$.

Corollary 1.1. *Let* f *and* g *be two nonconstant entire functions such that* $L(f)$ *and* $L(g)$ *are nonconstant linear differential polynomials. If* f *and* g *share the value* 0 *CM,* $L(f)$ *and* $L(g)$ *share* $(1, l)$ *with one of following conditions:*

(*i*) $l = \infty$ *and*

$$\delta(0, f) > \frac{1}{2},$$

(*ii*) $0 < l < \infty$ *and*

$$\delta(0, f) > \frac{l+1}{2l+1},$$

(*iii*) $l = 0$ *and*

$$\delta(0, f) > \frac{4}{5},$$

then either $f = g$ *or* $L(f).L(g) \equiv 1$ *under any one of the following conditions:*

(*i*) $\rho(f) \neq 1$,

(*ii*) $\rho(f) = 1$ *and* (*a*) f *has at most a finite number of zeros, or*

(*b*) f *has infinitely many zeros and* f *is of minimal type.*

Suppose F and G share $(1, l)$ and let z_0 be a zero of $F - 1$ of multiplicity p and a zero of $G - 1$ of multiplicity q. We define by $\overline{N}^L_{(l+1}(r, 1; F)$ the reduced counting function of those 1-points of F and G where $p > q \geq l+1$; $\overline{N}^L_{(l+1}(r, 1; G)$ is defined similarly. Note that for $l = 0$, we have $\overline{N}^L_{(l+1}(r, 1; F) = \overline{N}_L(r, 1; F)$. Also denote by $N^{1)}_E(r, 1; F)$ the counting function of those 1-points of F and G where $p = q = 1$ and by $\overline{N}^{(2}_E(r, 1; F)$ the counting function of those 1-points of F and G where $p = q \geq 2$, where each such zero is counted only once.

2. Lemmas

Let F and G be two nonconstant meromorphic functions. We shall define by H the following function

$$H = \left(\frac{F^{(2)}}{F^{(1)}} - 2\frac{F^{(1)}}{F-1}\right) - \left(\frac{G^{(2)}}{G^{(1)}} - 2\frac{G^{(1)}}{G-1}\right).$$

Lemma 2.1 ([9]). *Let f be a nonconstant meromorphic function and $P[f]$ be defined as (1). Then*

$$\begin{aligned}(i)T(r,P) &\leq dT(r,f)+Q\overline{N}(r,\infty;f)+S(r,f).\\ (ii)N(r,0;P) &\leq T(r,P)-dT(r,f)+dN(r,0;f)+S(r,f)\\ &\leq Q\overline{N}(r,\infty;f)+dN(r,0;f)+S(r,f).\end{aligned}$$

Lemma 2.2 ([1]). *Let F and G be two nonconstant meromorphic functions sharing $(1,l)$ and $H\not\equiv 0$. Then*

$$\begin{aligned}N(r,\infty,H) &\leq \overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}_{(2}(r,0;G)\\ &+ \overline{N}_{(2}(r,0;F)+\overline{N}_{L}(r,1;F)\\ &+ \overline{N}_{L}(r,1;G)+\overline{N}_{0}(r,0;F^{(1)})+\overline{N}_{0}(r,0;G^{(1)})+S(r,F)+S(r,G).\end{aligned}$$

Lemma 2.3 ([4]). *Let f be a transcendental meromorphic function and $P[f]$ be a homogeneous differential polynomial generated by f of degree $d\geq 1$. Then*

$$dT(r,f)\leq \overline{N}(r,\infty;f)+\overline{N}(r,1;F)+N(r,0;f^{d})-N_{0}(r,0;(P(f))^{(1)})+S(r,f),$$

where $N_0(r,0;(P[f])^{(1)})$ denotes the counting function corresponding to the zeros of $(P[f])^{(1)}$ which are not the zeros of $P[f]$ and $P[f]-1$.

Lemma 2.4 ([14]). *Let f be a nonconstant meromorphic function and let*

$$p(f)=a_nf^n+a_{n-1}f^{n-1}+...+a_1f+a_0,$$

where $a_i\in S(f)$ for $i=0,\ 1,...\ n$, $a_n\ (\neq 0)$ be a polynomial of degree n. Then $T(r,p(f))=nT(r,f)+S(r,f)$.

Lemma 2.5. *If F and G be two nonconstant meromorphic functions sharing $(1,l)$ where l is positive integer and $H\not\equiv 0$. Then*

$$\begin{aligned}T(r,F) &\leq (1+\frac{1}{l})N(r,0;F)+(2+\frac{1}{l})\overline{N}(r,\infty;F)+N(r,0;G)+2\overline{N}(r,\infty;G)\\ &+ S(r,F)+S(r,G).\end{aligned}$$

Proof. Suppose $H\not\equiv 0$. Then by a simple calculation we see that

$$\begin{aligned}N_{E}^{1)}(r,1;F) &\leq N(r,0;H)\leq T(r,H)+O(1)\\ &\leq N(r,\infty;H)+S(r,F)+S(r,G).\end{aligned}\tag{6}$$

Since F and G share $(1,l)$, we have from Lemma 2.2

$$\begin{aligned}N(r,\infty;H)\leq{}& \overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}^{L}_{(l+1}(r,1;F)\\ &+\overline{N}^{L}_{(l+1}(r,1;G)+\overline{N}_{(2}(r,0;F)\\ &+\overline{N}_{(2}(r,0;G)+\overline{N}_{0}(r,0;F^{(1)})+\overline{N}_{0}(r,0;G^{(1)})+S(r,F)+S(r,G).\end{aligned}\tag{7}$$

By Nevanlinna's Second Fundamental Theorem we have

$$\begin{aligned} T(r,F)+T(r,G) &\leq \overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,1;F)+\overline{N}(r,0;G) \\ &+ \overline{N}(r,\infty;G)+\overline{N}(r,1;G)-N_0(r,0;F^{(1)}) \\ &- N_0(r,0;G^{(1)})+S(r,F)+S(r,G), \end{aligned} \tag{8}$$

where $N_0(r,0;F^{(1)})$ denotes the counting function corresponding to the zeros of $F^{(1)}$ which are not the zeros of F and $F-1$. Similarly we define $N_0(r,0;G^{(1)})$.

Since F and G share $(1,l)$, using (6) and (7) we have

$$\begin{aligned} &\overline{N}(r,1;F)+\overline{N}(r,1;G) \\ &= N_E^{1)}(r,1;F)+\overline{N}^L_{(l+1}(r,1;F)+\overline{N}^L_{(l+1}(r,1;G) \\ &+ N_E^{(2}(r,1;F)+\overline{N}(r,1;G) \\ &\leq \overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}_{(2}(r,0;F)+\overline{N}_{(2}(r,0;G) \\ &+ 2\overline{N}^L_{(l+1}(r,1;F)+2\overline{N}^L_{(l+1}(r,1;G)+N_E^{(2}(r,1;F)+\overline{N}(r,1;G) \\ &+ \overline{N}_0(r,0;F^{(1)})+\overline{N}_0(r,0;G^{(1)})+S(r,F)+S(r,G) \\ &\leq \overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}_{(2}(r,0;F)+\overline{N}_{(2}(r,0;G) \\ &+ \overline{N}^L_{(l+1}(r,1;F)+\overline{N}^L_{(l+1}(r,1;G)+N(r,1;G)+\overline{N}_0(r,0;F^{(1)}) \\ &+ \overline{N}_0(r,0;G^{(1)})+S(r,F)+S(r,G) \\ &\leq \overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}_{(2}(r,0;F)+\overline{N}_{(2}(r,0;G) \\ &+ \overline{N}^L_{(l+1}(r,1;F)+\overline{N}^L_{(l+1}(r,1;G)+T(r,G)+\overline{N}_0(r,0;F^{(1)}) \\ &+ \overline{N}_0(r,0;G^{(1)})+S(r,F)+S(r,G). \end{aligned} \tag{9}$$

Also, we have

$$\begin{aligned} \overline{N}^L_{(l+1}(r,1;F)+\overline{N}^L_{(l+1}(r,1;G) &\leq \frac{1}{l}N(r,0;F^{(1)})+S(r,F) \\ &\leq \frac{1}{l}N(r,0;F)+\frac{1}{l}\overline{N}(r,\infty;F)+S(r,F). \end{aligned} \tag{10}$$

Now, using (9), (10) in (8) we get

$$\begin{aligned} T(r,F)+T(r,G) &\leq 2\overline{N}(r,\infty;F)+2\overline{N}(r,\infty;G)+\overline{N}(r,0;F)+\overline{N}_{(2}(r,0;F) \\ &+ \overline{N}(r,0;G)+\overline{N}_{(2}(r,0;G)+\frac{1}{l}N(r,0;F)+\frac{1}{l}\overline{N}(r,\infty;F) \\ &+ T(r,G)+S(r,F)+S(r,G). \end{aligned}$$

Since $\overline{N}(r,0;F)+\overline{N}_{(2}(r,0;F)\leq N(r,0;F)$, we have

$$\begin{aligned} T(r,F) &\leq (1+\frac{1}{l})N(r,0;F)+(2+\frac{1}{l})\overline{N}(r,\infty;F)+N(r,0;G)+2\overline{N}(r,\infty;G) \\ &+ S(r,F)+S(r,G). \end{aligned}$$

□

3. Proof of main theorems

Proof of Theorem 1.9:

Proof. Let

$$F = \frac{P[f]}{a},\ G = \frac{P[g]}{a}.$$

Since $P[f]$ and $P[g]$ share (a, l), it follows that F, G share $(1, l)$ except at the zeros and poles of a.

Now, we consider the following cases:

Case 1: $l = \infty$. This case follows from Remark 1.1.

Case 2: $0 < l < \infty$.

Suppose $H \not\equiv 0$. From Lemma 2.5 and Lemma 2.1, we get

$$\begin{aligned} T(r,F) &\leq (1+\frac{1}{l})N(r,0;F) + (2+\frac{1}{l})\overline{N}(r,\infty;F) + N(r,0;G) \\ &\quad + 2\overline{N}(r,\infty;G) + S(r,F) + S(r,G). \\ &\Rightarrow dT(r,f) \leq \frac{Q+1+2l}{l}\overline{N}(r,\infty;f) + (2+Q)\overline{N}(r,\infty;g) \\ &\quad + \frac{(l+1)d}{l}N(r,0;f) + dN(r,0;g) + S(r,f) + S(r,g). \end{aligned} \tag{11}$$

Similarly,

$$\begin{aligned} dT(r,g) &\leq \frac{Q+1+2l}{l}\overline{N}(r,\infty;g) + (2+Q)\overline{N}(r,\infty;f) + \frac{(l+1)d}{l}N(r,0;g) \\ &+ dN(r,0;f) + S(r,f) + S(r,g). \end{aligned} \tag{12}$$

Combining (11) and (12), we get

$$\begin{aligned} dT(r,f) + dT(r,g) &\leq (\frac{Q+1+2l}{l} + Q + 2)\overline{N}(r,\infty;f) + \frac{(2l+1)d}{l}N(r,0;f) \\ &+ (\frac{Q+1+2l}{l} + Q + 2)\overline{N}(r,\infty;g) + \frac{(2l+1)d}{l}N(r,0;g) \\ &+ S(r,f) + S(r,g). \end{aligned}$$

$$\begin{aligned} &\left\{\frac{(2l+1)d}{l}\delta(0,f) + (\frac{Q+1+2l}{l} + Q + 2)\Theta(\infty,f)\right. \\ &\left. -\frac{(l+1)d+Q+1}{l} - 4 - Q\right\}T(r,f) \\ &+ \left\{\frac{(2l+1)d}{l}\delta(0,g) + (\frac{Q+1+2l}{l} + Q + 2)\Theta(\infty,g)\right. \\ &\left. -\frac{(l+1)d+Q+1}{l} - 4 - Q\right\}T(r,g) \\ &\leq S(r,f) + S(r,g), \end{aligned}$$

which contradict (4). Therefore $H \equiv 0$ and so integrating twice we get

$$\frac{1}{G-1} = \frac{A}{F-1} + B,$$

where A $(\neq 0)$ and B are constants.

Thus,

$$G = \frac{(B+1)F + (A-B-1)}{BF + (A-B)} \tag{13}$$

and

$$F = \frac{(B-A)G + (A-B-1)}{BG - (B+1)}. \tag{14}$$

Next, we consider following three subcases:

Subcase 2.1: $B \neq 0, -1$. Then, from (14) we have

$$\overline{N}(r, \frac{B+1}{B}; G) = \overline{N}(r, \infty; F).$$

By Nevanlinna Second Fundamental Theorem and (ii) of Lemma 2.1 we get

$$\begin{aligned} T(r,G) &\leq \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r,\frac{B+1}{B};G) + S(r,G) \\ &\leq \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r,\infty;F) + S(r,G) \\ &\leq \overline{N}(r,\infty;G) + T(r,G) - dT(r,g) + dN(r,0;g) + \overline{N}(r,\infty;F) + S(r,G), \end{aligned}$$

$$\Rightarrow dT(r,g) \leq \overline{N}(r,\infty;f) + dN(r,0;g) + \overline{N}(r,\infty;g) + S(r,f) + S(r,g). \tag{15}$$

If $A - B - 1 \neq 0$, then it follows from (13) that

$$\overline{N}(r, \frac{-A+B+1}{B+1}; F) = \overline{N}(r,0;G).$$

Using Nevanlinna Second Fundamental Theorem and Lemma 2.1

$$T(r,F) \leq \overline{N}(r,\infty;F) + \overline{N}(r,0;F) + \overline{N}(r,\frac{-A+B+1}{B+1};F) + S(r,F),$$

$$\begin{aligned} &\Rightarrow dT(r,f) \leq \overline{N}(r,\infty;f) + dN(r,0;f) \\ &+ Q\overline{N}(r,\infty;g) + dN(r,0;g) + S(r,f) + S(r,g). \end{aligned} \tag{16}$$

Adding (15) and (16), we obtain

$$\begin{aligned} T(r,f) + T(r,g) &\leq N(r,0;f) + \frac{2}{d}\overline{N}(r,\infty;f) + 2N(r,0;g) \\ &+ \frac{Q+1}{d}\overline{N}(r,\infty;g) + S(r,f) + S(r,g), \end{aligned}$$

which contradicts (4).

Therefore $A - B - 1 = 0$. Then from (13), it follows that

$$\overline{N}(r, -\frac{1}{B}; F) = \overline{N}(r, \infty; G).$$

Again by Nevanlinna Second Fundamental Theorem, we have

$$\begin{aligned} T(r,F) &\leq \overline{N}(r,\infty;F) + \overline{N}(r,0;F) + \overline{N}(r,-\frac{1}{B};F) + S(r,F) \\ &\leq \overline{N}(r,\infty;f) + T(r,F) - dT(r,f) + dN(r,0;f) \\ &\quad + \overline{N}(r,\infty;g) + S(r,f) + S(r,g) \end{aligned}$$

$$(17) \quad \Rightarrow dT(r,f) \leq \overline{N}(r,\infty;f) + dN(r,0;f) + \overline{N}(r,\infty;g) + S(r,f) + S(r,g).$$

Combining (15) and (17), we get

$$\begin{aligned} T(r,f) + T(r,g) &\leq N(r,0;f) + \frac{2}{d}\overline{N}(r,\infty;f) + N(r,0;g) \\ &+ \frac{2}{d}\overline{N}(r,\infty;g) + S(r,f) + S(r,g), \end{aligned}$$

which violates (4).

Subcase 2.2: $B = -1$. Then from (13) and (14), we get

$$G = \frac{A}{A + 1 - F}$$

and

$$F = \frac{(1+A)G - A}{G}.$$

If $A + 1 \neq 0$, then

$$\overline{N}(r, A+1; F) = \overline{N}(r, \infty; G),$$

$$\overline{N}(r, \frac{A}{A+1}; G) = \overline{N}(r, 0; F).$$

By similar argument as in Subcase 2.1 we arrive at a contradiction.

Therefore, $A + 1 = 0$, then

$$FG = 1$$

$$\Rightarrow P[f].P[g] \equiv a^2.$$

Subcase 2.3: $B = 0$. Then (13) and (14) gives $G = \frac{F+A-1}{A}$ and $F = AG + 1 - A$. If $A - 1 \neq 0$, $\overline{N}(r, 1-A; F) = \overline{N}(r, 0; G)$ and $\overline{N}(r, \frac{A-1}{A}; G) = \overline{N}(r, 0; F)$. Proceeding similarly as in Subcase 2.1 we get a contradiction.

Therefore $A - 1 = 0$ then $F \equiv G$ i.e.,

$$P[f] \equiv P[g].$$

Case 3: $l = 0$. The conclusion follows from Theorem 1.1 of [9].

This complete the proof. □

Proof of Theorem 1.10:

Proof. Let

$$F = \frac{P[f]}{a},\ G = \frac{P[g]}{a}.$$

Since $P[f]$ and $P[g]$ share (a, l), it follows that F, G share $(1, l)$ except at the zeros and poles of a. By Lemma 2.1 and Lemma 2.3, we get

$$\begin{aligned} dT(r,f) &\leq \overline{N}(r,\infty;f) + \overline{N}(r,1;F) + N(r,0;f^d) + S(r,f) \\ &= \overline{N}(r,\infty;g) + \overline{N}(r,1;G) + N(r,0;g^d) + S(r,f) \\ &\leq (1+Q+2d)T(r,g) + S(r,f) + S(r,g). \end{aligned} \tag{18}$$

Similarly,

$$dT(r,g) \leq (1+Q+2d)T(r,f) + S(r,f) + S(r,g). \tag{19}$$

From (18) and (19) we get $S(r,f) = S(r,g)$. The rest of the proof is similar to that of Theorem 1.9. □

Proof of the Corollary 1.1:

Proof. By Theorem 1.11 we get either $L(f) \equiv L(g)$ or $L(f).L(g) \equiv 1$. Let $L(f) \equiv L(g)$ so that $L(f-g) \equiv 0$. Proceeding similarly as in the proof of Corollary 1.2 of [9], we obtain $f = g$. □

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Accepted: 9.12.2019

The construction of fraction N-subgroups in near rings

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Abstract. One of the first constructions of algebra is the quotient module. In this paper, we introduce the concepts of fraction N-subgroups and pure N-subgroups in near rings. Meanwhile, some properties of them are investigated. Finally, we show that the relationship between pure N-subgroups and fraction N-subgroups.

Keywords: fraction of N-group, pure N-subgroup.

1. Introduction

In 1969, [5] Golman introduced the consept of fraction modules in rings. The fraction modules have a great deal of use in Dedekind modules, valuation modules and psedu valuation modules [6].

The concept of near rings was expressed by Dickson in 1905 [3], that followed by other mathematicians, create the concepts and ruler space on the rings in near rings with conditions. Near rings have a great deal of use in dynamic systems and automation systems [8]. We know that the quotient field has a great deal of use in the creation of the valuative rings and Dedekind domains, and also the Dedekind domain is widely used in number theory [6].

In 1971, Paul Cohen [2] did a lot of work and made some useful results due to the importance of the fraction ring. Here, too, we decided to follow the results of near rings.

The paper is organized as follows. In Section 2, we recall some facts about near ring and N-groups in near rings. In Section 3, pure N-groups are introduced and some properties of pure N-groups are studied. Finally, in Section 4, the

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fraction N-groups are introduced and found the relationship between fraction N-subgroups and pure N-subgroups.

2. Preliminarise

The following, we remind some basic definitions and properties of near rings and N-groups in near ring. We refer the reader to [8].

Definition 2.1 ([8]). *A right (left) near-ring is a non-empty set N together with two binary operations "$+$" and "$\cdot$" such that:*

(a) $(N,+)$ *is a group (not necessarily abelian),*

(b) $(N,\cdot)$ *is a semigroup,*

(c) $(n_1+n_2)n_3 = n_1n_3+n_2n_3$ $(n_1(n_2+n_3) = n_1n_2+n_1n_3), \forall n_1,n_2,n_3 \in N$.

Definition 2.2 ([8]). *Let (M,+) be a group with zero 0, N be a near ring and the map $\mu : N \times M \to M$ such that $\mu(n,m) = nm$, $\forall n \in N$, $\forall m \in M$. Then, M is called an N-group if M satisfies the following conditions:*

(i) $(n+n')m = nm+n'm$,

(ii) $(nn')m = n(n'm), \forall n,n' \in N, \forall m \in M$.

It is written ${}_NM$ for the N-group above.

Let M be a group, then ${}_Ng = \{{}_NM | N$ is a near ring$\}$.
${}_NM \in {}_Ng$ is called an unitary if $\forall m \in M$, $1 \cdot m = m$.
A subgroup Δ of an N-group M with, $N\Delta \subseteq \Delta$ is said to be an N-subgroup of M (notation: $\Delta \leq_N M$).

Definition 2.3 ([8]). *A normal subgroup I of $(N,+)$ is called an ideal of N $(I \trianglelefteq N)$ if:*

(i) $IN \subseteq I$,

(ii) $n(n'+i) - nn' \in I$, $\forall n,n' \in N, \forall i \in I$.

The normal subgroup R of $(N,+)$ with (i) is called the right ideal of N $(R \trianglelefteq_r N)$, while normal subgroup L of $(N,+)$ with (ii) is said to be the left ideal $(L \trianglelefteq_l N)$.

Definition 2.4 ([8]). *A normal subgroup Δ of N-group M is called an ideal of M $(\Delta \trianglelefteq_N M)$, if Δ satisfies in $n(m+\delta) - nm \in \Delta; \forall m \in M$, $\forall \delta \in \Delta$, $\forall n \in N$.*

Definition 2.5 ([8]). *Let N be a near ring and I be an ideal of N. Then, the addition and multiplication in ${}^N\!/_I = \{n+I \mid n \in N\}$ are defined as follows:*

$$(n+I)+(n'+I) = (n+n')+I \ , (n+I).(n'+I) = n.n'+I, \ \forall\, n,n' \in N.$$

Definition 2.6 ([8]). *An N-homomorphism is a mapping h of an N-group M to an N-group M' such that:*

(*i*) $h(m_1 + m_2) = h(m_1) + h(m_2)$,

(*ii*) $h(nm_1) = nh(m_1), \forall m_1, m_2 \in M, \forall n \in N$.

Let N, N' be near rings and if there exists a monomorphism $N \to N'$, we say that, N is embedded in N' and written $N \hookrightarrow N'$.

Definition 2.7 ([8]). *Let N be a near-ring and S be sub-semigroup of $(N, \cdot)$. Then N is said to the left ore condition, if $\forall (n, s) \in N \times S$, there exists $(n_1, s_1) \in N \times S$ such that $sn_1 = ns_1$.*

Definition 2.8 ([8]). *Let N be a near-ring and S be a subsemigroup of $(N, \cdot)$. A near-ring N_s is called a quotient left (right) of N if*

(*a*) $N_s \in \eta_1$,

(*b*) $\exists h : N \hookrightarrow N_S$,

(*c*) $\forall s \in S : h(s)$ *is invertible in* $(N_S, \cdot)$,

(*d*) $\forall q \in N_S, \exists s \in S, \exists n \in N$ *such that* $q = h(n)h(s)^{-1}$ $(q = h(s)^{-1}h(n))$, *where* η_1 *is the set of all near rings with 1.*

Theorem 2.1 ([8]). *Let N be a near ring and S be a closed subset of $(N, \cdot)$. an equivalence relation $\sim$ on $N \times S$ by $(n, s) \sim (n', s') \Leftrightarrow \exists\ (n_1\ , s_1) \in N \times S$ such that $ns_1 = n'n_1$ and $ss_1 = s'n_1$. $[n, s] = \frac{n}{s}$) denoted the equivalence relation class of (n, s) and N_S denoted the set of all equivalence relation classes. Also, addition and multiplication of these fractions as follows $(n, s)+(n', s') = (ns_1+n'n_1, ss_1)$, $(n, s) \cdot (n', s') = (n'n_1, ss_1)$; $n, n' \in N,\ s, s' \in S,\ (\exists (n_1, s_1) \in N \times S)$.*

Lemma 2.1. *Let M be an N-group, and P, P' be N-subgroups of M, then $P \cap P'$ is an N-subgroup of M ([8]).*

Throughout the paper, N will denote a right near ring with identity and M be an unital N-group.

3. Pure N-subgroups in near rings

In this section, the notion of pure N-subgroups will be introduced and some properties of them will be investigated.

Definition 3.1. *Let N be a near ring, and M be an N-group. An N-subgroup K is called a pure N-subgroup of M if $IK = K \cap IM$, for every ideal I of N.*

Example 3.1 ([4]). Consider the near ring $(N, +, \cdot)$ where $(N, +)$ is kleins, four group $\{0, a, b, c\}$ and " $\cdot$ " defined as the following:

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	0	0	a
b	0	a	b	b
c	0	a	b	c

It is easy to see that, $I_1 = \{0\}$, $I_2 = \{0, a\}$ and N are the only ideals of N, and $J_1 = \{0, a\}$, $J_2 = \{0, b\}$ and $M = N^N$ are the only N-subgroups of M. In the following paragraph it is shown that, J_2 is a pure N-subgroup of M

$$I_1J_2 = \{0\} = J_2 \cap I_1M,$$
$$I_2J_2 = \{0\} = J_2 \cap I_2M,$$
$$NJ_2 = \{0, b\} = J_2 \cap NM.$$

So that $IJ_2 = J_2 \cap IM$ is hold for every ideal I of N.

Theorem 3.1. *Let K be an N-subgroup of M and L be an N-subgroup of K, such that L and K be pure N-subgroups of M, then L is a pure N-subgroup of K.*

Proof. Assume that L and K are pure N-subgroups of M, so that $\forall I \trianglelefteq N$, $IK = K \cap IM$ and $IL = L \cap IM$ are hold. We show that L is a pure N-subgroup of K. Since L is an N-subgroup of K, $IL \subseteq L \cap IK$ for every ideal I of N. Let $\sum_{i=1}^{n} a_i k_i \in L \cap IK$. Then, $\sum_{i=1}^{n} a_i k_i \in IK$. Since K is a pure N-subgroup of M therefore, $IK = K \cap IM$ and $\sum_{i=1}^{n} a_i k_i \in (L \cap K) \cap IM = L \cap IM = IL$. As L is a pure N-subgroup of M, $\sum_{i=1}^{n} a_i k_i \in IL$. Consequently, $L \cap IK \subseteq IL$ is confirmed. □

Proposition 3.1. *Let K be a pure N-subgroup of L and L be a pure N-subgroup of M, then K is a pure N-subgroup of M.*

Proof. Assume that K is a pure N-subgroup of L and L is a pure N-subgroup of M. Therefore, we have $IL = L \cap IM$, $IK = K \cap IL$, for every ideal I of N. We show that K is a pure N-subgroup of M, for every ideal I of N. Let $x \in K \cap IM \subseteq L \cap IM = IL \Rightarrow x \in K \cap IL = IK$. Consequently, $K \cap IM \subseteq IK$, and the conversely conclusion is obvious. Hence K is a pure N-subgroup of M. □

Theorem 3.2. *Let I be an ideal of M, then P is a pure N-subgroup of M contains $I \Leftrightarrow \frac{P}{I}$ is a pure N-subgroup of $\frac{M}{I}$.*

Proof. $\Longrightarrow$ Let P be a pure N-subgroup of M contains I such that $I \trianglelefteq_N M$ therefore, $JP = P \cap JM$, for every ideal J of N. It is enough to show that $J\frac{P}{I} = \frac{P}{I} \cap J\frac{M}{I}$. Suppose $x \in \frac{P}{I} \cap J\frac{M}{I} \Rightarrow x \Rightarrow \exists p_1 \in P$ such that $x = p_1 + I = \sum_{i=1}^{n} j_i(m_i + I) = \sum_{i=1}^{n} (j_i m_i) + I \in \frac{JM}{I} \Rightarrow p_1 + I \in \frac{JM}{I} \Rightarrow p_1 \in JM$, and $p_1 \in P \Rightarrow p_1 \in P \cap JM = JP \Rightarrow p_1 \in JP \Rightarrow p_1 + I \in \frac{JP}{I} = J\frac{P}{I}$. Consequently, $\frac{P}{I} \cap J\frac{M}{I} \subseteq J\frac{P}{I}$, and the conversely conclusion is obvious.

$\Longleftarrow$ Let $\frac{P}{I}$ be a pure N-subgroup of $\frac{M}{I}$, then $J\frac{P}{I} = \frac{P}{I} \cap J\frac{M}{I}$ for every ideal J of N. Now, we show that P is a pure N-subgroup of M.

Let $p_1 \in P \cap JM \Rightarrow p_1 + I \in \frac{P}{I} \cap J\frac{M}{I} = J\frac{P}{I} \Rightarrow p_1 + I \in J\frac{P}{I} = \frac{JP}{I} \Rightarrow p_1 \in JP$. Consequently $P \cap JM \subseteq JP$, and the conversely conclusion is obvious. □

4. Fraction of N-subgroups in near rings

In this section, the notion of fraction N-subgroup will be introduced. Our purpose is to find the relationship between fraction N-subgroup and pure N-subgroup.

Definition 4.1. *Let S be a sub-semigroup of $(N, \cdot)$, then N is said to weak left ore condition, if $\forall (s, s') \in S \times S$ then there exists $(t_1, t_2) \in S \times S$ such that $st_2 = s't_1$.*

Definition 4.2. *Let M be an N-group, N be a near ring and let S be a subsemigroup of $(N, \cdot)$. An N-group M_s is called an N-group left(right) fraction of M if:*

(a) *$M_s \in {}_N g$ is unitary,*

(b) *$\exists h : M \hookrightarrow M_S$,*

(c) *$\forall\ s \in S :$ s is invertible in $(N, \cdot)$,*

(d) *$\forall\ q \in M_S, \exists s \in S, \exists m \in M$ such that $q = h(m)s^{-1}$ $(q = s^{-1}h(m))$.*

Theorem 4.1. *Let N be weak left ore condition, S be a sub-semigroup of $(N, \cdot)$ and also S be invertible in $(N, \cdot)$, then M has N-group M_s of left fractions.*

Proof. Define an equivalence relation $\sim$ on $M \times S$ by $(m, s) \sim (m', s') \Leftrightarrow \exists\ s_1\ , s_2 \in S$ such that $ms_1 = m's_2$ and $ss_1 = s's_2$.

We show that the relation $\sim$ is reflexive, symmetric and transitive:

(i) reflexive: $(m, s) \sim (m, s) \Leftrightarrow$ take $s_1 = s_2 = 1$, then $ms_1 = ms_2$, $ss_1 = ss_2$.

(ii) symmetric: if $(m, s) \sim (m', s') \Leftrightarrow (m', s') \sim (m, s)$. Let $(m, s) \sim (m', s') \Leftrightarrow \exists s_1\ , s_2 \in S$ such that $ms_1 = m's_2, ss_1 = s's_2 \Rightarrow m's_2 = ms_1, s's_2 = ss_1 \Rightarrow (m', s') \sim (m, s)$. Similarly, $(m', s') \sim (m, s) \Rightarrow (m, s) \sim (m', s')$.

(iii) transitive : if $(m, s) \sim (m', s'), (m', s') \sim (m'', s'') \Rightarrow (m, s) \sim (m'', s'')$.

Let $(m, s) \sim (m', s') \Leftrightarrow \exists s_1, s_2 \in S : ms_1 = m's_2, ss_1 = s's_2$, $(m', s') \sim (m'', s'') \Leftrightarrow \exists t_1\ , t_2 \in S : m't_1 = m''t_2, s't_1 = s''t_2$,

$$\begin{aligned} ms_1 = m's_2 &\Rightarrow m = m's_2s_1^{-1}, \\ ss_1 = s's_2 \quad &\Rightarrow s = s's_2s_1^{-1}, \\ m't_1 = m''t_2 &\Rightarrow m' = m''t_2t_1^{-1}, \\ s't_1 = s''t_2 \ &\Rightarrow s' = s''t_2t_1^{-1}. \end{aligned}$$

By considering $(t_2t_1^{-1}s_2 = z)$ we have:

$$ss_1 = s's_2 = (s''t_2t_1^{-1})s_2 = s''(t_2t_1^{-1}s_2) = s''z \Rightarrow ss_1 = s''z,$$
$$ms_1 = m's_2 = (m''t_2t_1^{-1})s_2 = m''(t_2t_1^{-1}s_2) = m''z \Rightarrow ms_1 = m''z.$$

By $ms_1 = m''z$, $ss_1 = s''z$, we have $(m, s) \sim (m'', s'')$. By using (i), (ii) and (iii) $\sim$ is equivalence relation. Now we prove that $(M_S, +)$ is a group. We define the addition of this fraction N-subgroup as follow:

$$+ : M_S \times M_S \longrightarrow M_S,$$
$$(\frac{m}{s}, \frac{m'}{s'}) \longrightarrow \frac{m}{s} + \frac{m'}{s'} = \frac{ms_1 + m's_2}{ss_1} \ \ (\exists s_1, s_2 \in S).$$

We show that this definition is well-defined.

If $(\frac{m}{s}, \frac{m'}{s'}) = (\frac{a}{t}, \frac{b'}{t'})$, then $\frac{m}{s} = \frac{a}{t} \Rightarrow \exists s_1, s_2 \in S$ such that $ms_1 = as_2, ss_1 = ts_2$, $\frac{m'}{s'} = \frac{b}{t'} \Rightarrow \exists s_3, s_4 \in S$ such that $m's_3 = bs_4, s's_3 = t's_4$, therefore, the following relations are included:

$$\begin{cases} ms_1 = as_2 \Rightarrow m = as_2s_1^{-1}, \\ ss_1 = ts_2 \Rightarrow s = ts_2s_1^{-1}, \\ m's_3 = bs_4 \Rightarrow m' = bs_4{s_3}^{-1}, \\ s's_3 = t's_4 \Rightarrow s' = t's_4{s_3}^{-1}. \end{cases}$$

Consider $(s', s) \in S \times S$. Since N is weak left ore condition, there exists $(t_1, t_2) \in S \times S$ such that $s't_2 = st_1$. Then, by considering $((s_2s_1^{-1}t_1) = x, (s_4s_3^{-1}t_2) = z)$,

$$\frac{m}{s} + \frac{m'}{s'} = \frac{mt_1 + m't_2}{st_1} = \frac{(as_2s_1^{-1})t_1 + (bs_4s_3^{-1})t_2}{(ts_2s_1^{-1})t_1}$$
$$= \frac{a(s_2s_1^{-1}t_1) + b(s_4s_3^{-1}t_2)}{t(s_2s_1^{-1}t_1)} = \frac{ax + bz}{tx}.$$

Now, we show that

$$\frac{ax + bz}{tx} = \frac{a}{t} + \frac{b}{t'}.$$

It is enough to show that $tx = t'z$, we have $tx = t(s_2s_1^{-1}t_1) = (ts_2s_1^{-1})t_1 = st_1 = s't_2 = (t's_4s_3^{-1})t_2 = t'(s_4s_3^{-1}t_2) = t'z$. So,

$$\frac{ax + bz}{tx} = \frac{ax}{tx} + \frac{bz}{tx} = \frac{ax}{tx} + \frac{bz}{t'z} = \frac{a}{t} + \frac{b}{t'}$$

thus, the above addition is well-defined.

(i) Now we prove that M_S is associative $\forall a, b, c \in M, \forall x, y, z \in S$

$$\left[\left(\frac{a}{x}\right) + \left(\frac{b}{y}\right)\right] + \left(\frac{c}{z}\right) = \left(\frac{a}{x}\right) + \left[\left(\frac{b}{y}\right) + \left(\frac{c}{z}\right)\right].$$

Consider $(y,x)\in S\times S$. Since N is weak left ore condition, there exists $(s_1,s_2)\in S\times S$ such that $ys_2=xs_1$, $[(\frac{a}{x})+(\frac{b}{y})]=\frac{as_1+bs_2}{xs_1}$. Consider $(z,xs_1)\in S\times S$. Since N is weak left ore condition, there exists $(s_3,s_4)\in S\times S$ such that $zs_4=xs_1s_3$,

$$\left(\frac{as_1+bs_2}{xs_1}\right)+\left(\frac{c}{z}\right)=\frac{(as_1+bs_2)s_3+cs_4}{xs_1s_3}$$

therefore,

$$\begin{aligned}[\left(\frac{a}{x}\right)+\left(\frac{b}{y}\right)]+\left(\frac{c}{z}\right)&=\frac{(as_1+bs_2)s_3+cs_4}{xs_1s_3}\\&=\frac{as_1s_3+bs_2s_3+cs_4}{xs_1s_3}.\end{aligned}$$

Consider $(z,y)\in S\times S$. Since N is weak left ore condition, there exists $(t_1,t_2)\in S\times S$ such that $zt_2=yt_1$

$$[(\frac{b}{y})+(\frac{c}{z})]=\frac{bt_1+ct_2}{yt_1}$$

Consider $(yt_1,x)\in S\times S$. Since N is weak left ore condition, there exists $(t_3,t_4)\in S\times S$ such that $yt_1t_4=xt_3$

$$\left(\frac{a}{x}\right)+\left(\frac{bt_1+ct_2}{yt_1}\right)=\frac{at_3+(bt_1+ct_2)t_4}{xt_3}$$

therefore, the following relations are included:

$$\begin{cases}ys_2=xs_1\Rightarrow x=ys_2{s_1}^{-1}\\zs_4=xs_1s_3\Rightarrow x=zs_4{s_3}^{-1}{s_1}^{-1}\\zt_2=yt_1\\yt_1t_4=xt_3\Rightarrow x=yt_1t_4{t_3}^{-1}=zt_2t_4{t_3}^{-1}\end{cases}$$

$$\Rightarrow\begin{cases}yt_1t_4{t_3}^{-1}=ys_2{s_1}^{-1}\Rightarrow t_1t_4{t_3}^{-1}=s_2{s_1}^{-1}\\zs_4{s_3}^{-1}{s_1}^{-1}=zt_2t_4{t_3}^{-1}\Rightarrow s_4{s_3}^{-1}{s_1}^{-1}=t_2t_4{t_3}^{-1}.\end{cases}$$

Suppose that $\frac{as_1s_3+bs_2s_3+cs_4}{xs_1s_3}=\frac{at_3+bt_1t_4+ct_2t_4}{xt_3}$ Therefore, there exists $r,r'\in S$ so that

$$\begin{aligned}&as_1s_3r+bs_2s_3r+cs_4r=at_3r'+bt_1t_4r'+ct_2t_4r',\\&xs_1s_3r=xt_3r'\end{aligned}$$

by choose $(r={s_3}^{-1}{s_1}^{-1},r'={t_3}^{-1})$ then, $[(\frac{a}{x})+(\frac{b}{y})]+(\frac{c}{z})=(\frac{a}{x})+[(\frac{b}{y})+(\frac{c}{z})]$.

(ii) For every $(\frac{m}{s})\in M_S$, there exists a (two-sided) identity element $(\frac{0}{s})\in M_S$, such that

$$\frac{m}{s}+\frac{0}{s}=\frac{0}{s}+\frac{m}{s}=\frac{m}{s}.$$

(iii) For every $(\frac{m}{s}) \in M_S$, there exists a (two-sided) inverse element $(\frac{-m}{s}) \in M_S$ such that

$$\frac{m}{s} + \frac{-m}{s} = \frac{-m}{s} + \frac{m}{s} = \frac{0}{s},$$

thus by (i), (ii), (iii); $(M_S, +)$ is a group. Next, we show that M_S is an N-group:

$$\mu : N \times M_S \to M_S, (n, \frac{m}{s}) \to \frac{n \cdot m}{s}.$$

Obviously, μ is well-defined $\forall n_1, n_2 \in N, \forall \frac{m}{s} \in M_S$:

(i) $\mu(n_1 + n_2, \frac{m}{s}) = \mu(n_1, \frac{m}{s}) + \mu(n_2, \frac{m}{s})$;

(a) $\mu(n_1 + n_2, \frac{m}{s}) = (n_1 + n_2)(\frac{m}{s}) = \frac{(n_1+n_2)m}{s} = \frac{(n_1m+n_2m)}{s}$.

(b) Consider $(s, s) \in S \times S$. Since N is weak left ore condition, there exists $(s_1, s_2) \in S \times S$ so that $ss_2 = ss_1 \Rightarrow s_1 = s_2, (\mu(n_1, \frac{m}{s}) + \mu(n_2, \frac{m}{s}) = \frac{n_1m}{s}) + (\frac{n_2m}{s}) = \frac{(n_1ms_1+n_2ms_2)}{ss_1} = \frac{(n_1ms_1+n_2ms_1)}{ss_1} = \frac{(n_1m+n_2m)s_1}{ss_1} = \frac{(n_1m+n_2m)}{s}$.

Then, by (a), (b)

$$\mu\left(n_1 + n_2, \frac{m}{s}\right) = \mu\left(n_1, \frac{m}{s}\right) + \mu(n_2, \frac{m}{s}).$$

(ii) $\mu(n_1 \cdot n_2, \frac{m}{s}) = n_1\mu(\frac{n_2 \cdot m}{s})$;

(a) $\mu(n_1 \cdot n_2, \frac{m}{s}) = (n_1 \cdot n_2)(\frac{m}{s}) = \frac{(n_1 \cdot n_2)m}{s} = \frac{n_1(n_2 \cdot m)}{s}$;

(b) $n_1\mu(\frac{n_2 \cdot m}{s}) = n_1\frac{(n_2m)}{s} = \frac{n_1(n_2 \cdot m)}{s}$.

Thus, by (i), (ii); M_S is an N-group. $M_s \in {}_Ng$ is clearly unitary. We Define $h : M \to M_S$ such that $h(m) = \frac{m}{1}$. Now, we show that h is monomorphism. Let $h(m) = h(m') \Rightarrow \frac{m}{1} = \frac{m'}{1} \Rightarrow \exists s_1, s_2 \in S$ so that $ms_1 = m's_2, 1s_1 = 1s_2 \Rightarrow s_1 = s_2 \Rightarrow m = m'$. Thus, h is injective. We prove that h is N-homomorphism. For all $m, m' \in M, n \in N$ we have:

$$h(m + m') = h(m) + h(m'), h(m + m') = \frac{m + m'}{1}. \tag{1}$$

Consider $(1, 1) \in S \times S$. Since N is weak left ore condition, there exists $(s_1, s_2) \in S \times S$, so that $1s_2 = 1s_1 \Rightarrow s_1 = s_2$,

$$\begin{aligned} h(m) + h(m') &= \frac{m}{1} + \frac{m'}{1} = \frac{ms_1 + m's_2}{1s_1} \\ &= \frac{(ms_1 + m's_1)}{1s_1} = \frac{(m + m')s_1}{1s_1} = \frac{(m + m')}{1}. \end{aligned}$$

We have $h(m + m') = h(m) + h(m')$

$$h(nm) = nh(m). \tag{2}$$

$h(nm) = \frac{nm}{1}, nh(m) = n\frac{m}{1} = \frac{nm}{1} \Rightarrow h(nm) = nh(m)$. By using (1) and (2), h is N-homomorphism. Finally, h is monomorphism. Thus, every $\frac{m}{s}$ can be written as $h(m)s^{-1}$. □

Example 4.1. Let $M(\mathbb{Z}_3) = \{f | f : \mathbb{Z}_3 \to \mathbb{Z}_3, f \text{ is function}\}$ and S be sub-semigroup of $(M(\mathbb{Z}_3), \circ)$ therefore, we have $M(\mathbb{Z}_3) = \{f_1, f_2, f_3, f_4, ..., f_{27}\}$, $S = \{f_1, f_{19}\}$ defined as the following:

$$\begin{array}{ccc} f_1 : \mathbb{Z}_3 \to \mathbb{Z}_3 & & f_{19} : \mathbb{Z}_3 \to \mathbb{Z}_3 \\ \bar{0} \to \bar{0} & & \bar{0} \to \bar{2} \\ \bar{1} \to \bar{1} & , & \bar{1} \to \bar{1} \\ \bar{2} \to \bar{2} & & \bar{2} \to \bar{0} \end{array}$$

It is clear that S is a sub-semigroup of $(M(\mathbb{Z}_3), \circ)$. We have construct the $(M(\mathbb{Z}_3))_S$ as follows

$$(M(\mathbb{Z}_3))_S = \left\{ \frac{f_1}{f_1}, \frac{f_5}{f_{19}}, \frac{f_8}{f_{19}}, \frac{f_{10}}{f_{19}}, \frac{f_{11}}{f_{19}}, \frac{f_{12}}{f_1}, \frac{f_{13}}{f_{19}}, \frac{f_{14}}{f_1}, \frac{f_{17}}{f_{19}}, \frac{f_{20}}{f_{19}}, \frac{f_{21}}{f_{19}}, \frac{f_{23}}{f_{19}}, \frac{f_{24}}{f_{19}} \right\}.$$

Theorem 4.2. *Let S be a closed subset of $(N, \cdot)$ and P, P' be N-subgroups of M. Then:*

(i) $(P + P')_S = P_S + P'_S$;

(ii) $(P \cap P')_S = P_S \cap P'_S$.

Proof. To prove part (i):

$$let \;\; P_S = \left\{ \frac{p_i}{t_i} \, | p_i \in P, t_i \in S \right\}, \;\; P'_S = \left\{ \frac{p'_i}{t'_i} \, | p'_i \in P', t'_i \in S \right\}$$

For arbitrary, $\frac{p_i}{t_i} \in P_S, \frac{p'_i}{t'_i} \in P'_S$, we have $\frac{p_i}{t_i} + \frac{p'_i}{t'_i} \in P_S + P'_S \Rightarrow \exists s_1, s_2 \in S$ such that

$$\frac{p_i}{t_i} + \frac{p'_i}{t'_i} = \frac{p_i s_1 + p'_i s_2}{t_i s_1} \in (P + P')_S$$

then $P_S + P'_S \subseteq (P + P')_S$.

Conversely, let $\frac{p_i + p'_i}{t_i} \in (P + P')_S$ and since S is unitary, hence;

$$\frac{(p_i + p'_i)}{t_i} = \frac{p_i 1 + p'_i 1}{t_i 1} = \frac{p_i}{t_i} + \frac{p'_i}{t'_i} \in P_S + P'_S \Rightarrow (P + P')_S \subseteq P_S + P'_S.$$

Now, to prove part (ii): since $(P \cap P')_S \subseteq P_S$ and $(P \cap P')_S \subseteq P'_S$, $(P \cap P')_S \subseteq P_S \cap P'_S$. Conversely, for arbitrary $\frac{x}{t_i} \in P_S, \frac{y}{t'_i} \in P'_S$ so that $\frac{x}{t_i} = \frac{y}{t'_i} \in P_S \cap P'_S \Rightarrow \exists s_1, s_2 \in S$ so that $xs_1 = ys_2$, $t_i s_1 = t'_i s_2$

$$\begin{cases} xs_1 = ys_2 \in P' \\ xs_1 = ys_2 \in P \end{cases} \Rightarrow xs_1 = ys_2 \in P \cap P'$$

$$\Rightarrow \frac{xs_1}{t_i s_1} = \frac{ys_2}{t'_i s_2} \in (P \cap P')_S \Rightarrow \frac{x}{t_i} = \frac{y}{t'_i} \in (P \cap P')_S.$$

Then, $P_S \cap P'_S \subseteq (P \cap P')_S$. □

Lemma 4.1. *Let N be a near ring and M be an N-group. If S is closed subset of $(N,\cdot)$, then M'_S is N-subgroup of M_S such that*

$$M' = \left\{ m' \in M \,\middle|\, \frac{m'}{1} \in M_S \right\}.$$

Proof. It is clear. □

Theorem 4.3. *Let N be a near ring, M be an N-group and S be sub-semigroup of $(N,\cdot)$. If J is a pure N-subgroup of M_S, then there exists M', which is a pure N-subgroup of M such that $J = M'_S$.*

Proof. By Lemma 4.1, suppose $J = M'_S$ is an N-subgroup of M_S. We show that if $J = M'_S$ is a pure N-subgroup of M_S then, M' is pure N-subgroup of M. We show $\forall I \trianglelefteq N$, $IM' = M' \cap IM$.

Let $m' \in M' \cap IM \Rightarrow \exists x_i \in I, m_i \in M$ such that $m' = \sum_{i=1}^{n} x_i m_i$.

If $m' = \sum_{i=1}^{n} x_i m_i \in M' \cap IM$, then $\frac{\sum_{i=1}^{n} x_i m_i}{1} \in (M' \cap IM)_S$. By Theorem 4.2,

$$\frac{\sum_{i=1}^{n} x_i m_i}{1} \in {M'}_S \cap (IM)_S.$$

Since ${M'}_S$ is a pure N-subgroup of M,

$$\frac{\sum_{i=1}^{n} x_i m_i}{1} \in (IM')_S \Rightarrow \sum_{i=1}^{n} x_i m_i \in IM' \Rightarrow M' \cap IM \subseteq IM'.$$

Consequently, $M' \cap IM \subseteq IM'$ and conclusion is obvious. Hence M' is a pure N-subgroup of M. □

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Accepted: 26.11.2019

A posteriori error analysis for discontinuous Galerkin methods for time discrete semilinear parabolic problems

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Abstract. *A posteriori* error estimates for time discretisations for semilinear parabolic (evolution) problems by the discontinuous Galerkin method DG(r) of arbitrary order $r \geq 0$ are derived and analysed. The semilinear evolutionary problems of the form $u' + Au = f(u)$ with A is either linear or monotone γ^2-angle bounded operator are considered. The main tool in this analysis is the time reconstruction function $\hat{U}$ of the approximate discrete solution U of the exact solution u. Two classes of nonlinearities are addressed: firstly, when the source term is globally Lipschitz continuous and secondly when the source term is locally Lipschitz continuous.

Keywords: a posteriori error analysis, finite element methods, semilinear parabolic problems, discontinuous Galerkin methods, reconstruction technique, time-stepping methods.

1. Introduction

The discontinuous Galerkin time-stepping scheme is a variational one-step implicit time marching scheme. The variational time marching schemes are dated back to the 1960s. The research paper of Argyris and Sharpf [2] in 1969 is considered the first work used the variational formulation of time integration in a systemic way. Hulme, in 1972 [23,24] addressed for the first time the variational continuous in time Galerkin (cG) finite element method for ordinary differential equations. In 1973, Reed and Hill [42] studied for the first time the discontinuous Galerkin finite element methods for first order hyperbolic problems. The first error analysis is done by Lesaint and Raviart in 1974 for the variational cG time–stepping schemes [32]. The first error analysis of linear parabolic problems using variational time-marching schemes is investigated by Jamet in 1978 [25]. Eriksson and coworkers investigated in detail the analysis of DG time-stepping schemes in [13]. Nonlinear parabolic problems are considered in [10,31,52,57] and the references therein for different numerical aspects of the finite element methods. The *p*- and *hp*-versions Galerkin time-stepping methods are considered and addressed from different numerical viewpoints in [4,5,18,28,51,53,54,55,56,62].

A posterior error analysis for transient problems recently has witnessed an increasing activity especially for the DG, Runge–Kutta and multistep methods [1,14,15,17,20,21,22,28,29,60]. The available error estimators either can be derived analytically by using stability features of the underlying problems (for this reason they are called analytical error estimators) [11,33,34,40,41,59], or computationally by using suitable computational methods (computational error estimators) [3,12,16]. The main techniques in deriving the error estimators are the energy, duality and reconstruction techniques.

We use in this research paper for deriving our *a posteriori* error estimates for higher order DG(r) methods the energy technique (analytical approach) since it is more flexible in dealing with the nonlinearities (the forcing functions) with the aid of the ideas of reconstruction and continuation argument. The basic idea behind the energy technique is deriving the error representation formula and then to test it by a quantity which hinges upon the error, this quantity depends on the underlying problem. The elliptic reconstruction technique was introduced by Makridakis and Nochetto [36] in 2003 for deriving optimal order *a posteriori* error estimates for semidiscrete linear parabolic problems. Then extended to the fully discrete setting for linear parabolic PDEs [30]. Also, Makridakis and Nochetto in 2006 [37] extended the idea of the elliptic reconstruction technique by presenting a novel time reconstruction for time semidiscrete linear and nonlinear parabolic problems. In [48] Y. Sabawi studied and investigated the *a posteriori* error bounds for DG methods for semilinear interface problems. Cangiani et al [8] examined and derived the *a posteriori* error estimates for the discontinuous Galerkin finite element methods for elliptic interface problems. Also, they considered and examined the convergence of the adaptive DG methods for elliptic interface problems in [7]. Georgoulis and coworkers [18] introduced a new space-time reconstruction for fully discrete *hp*-version Galerkin timestepping for linear evolutionary problems. Cangiani and coworkers [6] extended the work in [18] from linear to semilinear parabolic problems, and M. Sabawi [44] investigated the *a posteriori* and *a priori* error analyses for fully discrete *hp*-version Galerkin timestepping for semilinear parabolic problems. The *a posteriori* error analysis for DG time discretisation of semilinear evolutionary problems and fully discrete backward Euler-Galerkin finite element method for semilinear parabolic problems are derived in [45,46] respectively. Y. Sabawi derived *a posteriori* error bounds for DG methods in the $L_2(H^1) + L_\infty(L_2)$- and $L_2(H^1)$-norms in [47,49], respectively. The basic idea of the elliptic reconstruction technique is to construct an appropriate auxiliary continuous function satisfies a perturbed equation of the original problem. The counterpart of the time discretisation is the time reconstruction which simplifies the analysis and results in optimal order-regularity error estimators. For more details about the ideas such as the elliptic reconstruction and time reconstruction see the survey in [35]. In [37] the DG timestepping reconstruction uses the Gauss-Radua nodes, and these nodes have an appealing property that the DG method has superconvergence at these points. A review of superconvergence in DG methods and postprocessing as-

pects are considered in [43]. For more details about superconvergence and the related issues see [26,39,61].

In this work, we extend the work of Makridakis and Nochetto in [37] to the semilinear case for globally and locally Lipschitz continuous nonlinear source terms, and we mostly use the same notation. The remainder of this work is organised as follows: In Section 2, we introduce the preliminary and basic definitions, notation, the mathematical problem and the discontinuous Galerkin method. Section 3 is devoted for deriving the *a posteriori* error estimators for linear, monotone and γ^2-angle bounded operators. The conclusions are given in Section 4.

2. Problem setting and the numerical method

We consider the discretisations in time by the use of the discontinuous Galerkin time-stepping method DG(r) of any order $r \geq 0$ of the dissipative evolutionary initial value problem

$$u'(t) + Au(t) = f(u(t)),\ u(0) = u_0, \tag{1}$$

A is a γ^2-angle bounded operator in a Hilbert space H with inner product $\langle ., . \rangle$ and norm $\|.\|_H$. For simplicity in notations we omit the dependence upon t and write u instead of $u(t)$. Although the analysis will be done in an abstract setting, the typical case will be when $H = L_2, V = H_0^1$ and $V^* = H^{-1}$. The operator A is called γ^2-angle bounded operator if

$$\langle A(v) - A(w), w - z \rangle \leq \gamma^2 \langle A(v) - A(z), v - z \rangle \ \forall v, w, z \in D(A), \tag{2}$$

where $\gamma \geq \frac{1}{2}$. We restrict our analysis to the case when $\gamma = 1$. The operator A is monotone if it satisfies the monotonicity condition

$$\langle A(v) - A(w), v - w \rangle \geq 0 \ \forall v, w \in D(A). \tag{3}$$

Also, the operator A satisfies the coercivity condition if

$$\langle A(v) - A(w), v - w \rangle \geq \| v - w \|^2 \ \ \forall v, w \in D(A). \tag{4}$$

The function f is globally Lipschitz continuous if

$$|f(v) - f(w)| \leq C|v - w| \ \forall v, w \in R, \tag{5}$$

and is said to be locally Lipschitz continuous if it satisfies the following growth condition

$$|f(v) - f(w)| \leq C|v - w|(1 + |v|^a + |w|^a),\ a \geq 0, \tag{6}$$

where $|.|$ denotes the Euclidean norm on R^N.

2.1 Reconstruction function

The reconstruction function $\hat{U}$ is a continuous interpolant of degree $r+1$ (higher degree) of the discontinuous approximate solution U which is of degree r, $\hat{U}$ satisfies the perturbed ODE

$$\hat{U}' + \Pi A(U) = \Pi f(U), \tag{7}$$

where $\Pi : H \longrightarrow V_k(r)$, is a suitable projection or interpolation operator and this depends upon the nature of the problem, $V_k(r)$ is the space of discontinuous piecewise polynomials of degree $\leq r$, and $\Pi f(U)$ is a discontinuous piecewise polynomial approximation of $f(U)$. Let $0 = t_0 < t_1 < ... < t_N = T$ be a partition Υ of the time interval $[0,T]$, $I_n = (t_n, t_{n+1}]$, and $k_n = t_{n+1} - t_n$ is the variable time step. We define $C(r)$ as the space of polynomials of degree $\leq r$, both spaces $C(r)$ and $V_k(r)$ are defined over the partition Υ with values in $D(A)$, if $\upsilon \in V_k(r)$, then

$$\upsilon|_{I_n(t)} = \sum_{i=0}^{r} \chi_i t^i,\ \chi_i \in D(A),\ 0 \leq i \leq r. \tag{8}$$

The discontinuous Galerkin discretisation of (1) is: find $U \in V_k(r)$ such that $U_0 = u_0$ and

$$\int_{I_n} (\langle U', v\rangle + \langle A(U), v\rangle)dt + \langle U_n^+ - U_n, v_n^+\rangle = \int_{I_n} \langle f(U), v\rangle dt,\ \forall v \in V_k(r), \tag{9}$$

where $0 \leq n \leq N-1,\ v_n = v(t_n),\ v_n^+ = \lim_{\epsilon \to 0^+} v(t_n + \epsilon)$.
The discontinuous Galerkin discretisation (9) for (1) is a finite element discretisation in time and when use with it a suitable Radau quadrature rules it turns out to the classical implicit Runge-Kutta-Radau methods which refer to it as IRK-R(r). Let $\{\tau_i\}_{i=1}^{r+1}$ be the Radau points in the interval $[0,1]$, such that $0 < \tau_1 < ... < \tau_{r+1} = 1$ with the corresponding weights $\{w_i\}_{i=1}^{r+1}$. The Radau quadrature rule on $[0,1]$

$$\int_0^1 \psi(\tau)d\tau \cong \sum_{i=1}^{r+1} w_i \psi(\tau_i), \tag{10}$$

is exact for polynomials of degree less than or equal to $2r$. Let $\{\ell_i\}_{i=1}^{r+1} \subset C(r)$ be the Lagrange polynomials of degree r associated with $\{\tau_i\}_{i=1}^{r+1}$ and $\{\hat{\ell}_i\}_{i=0}^{r+1} \subset C(r+1)$ be the Lagrange polynomials of degree $r+1$ associated with $\{\tau_i\}_{i=0}^{r+1}$ with $\tau_0 = 0$. The Radau points in the interval $\bar{I}_n = [t_n, t_{n+1}]$ are denoted by $t_{n,i}$ and the corresponding Lagrange polynomials by $\ell_{n,i},\ \hat{\ell}_{n,j}$, respectively, and they satisfy

$$\begin{aligned} t_{n,i} &= t_n + \tau_i k_n,\ i = 0, ..., r+1\ (t_{n,0} = t_n,\ t_{n,r+1} = t_{n+1}), \\ \ell_{n,i}(t) &= \ell_i(\tau),\ \hat{\ell}_{n,i}(t) = \hat{\ell}_i(\tau), t = t_n + \tau k_n. \end{aligned} \tag{11}$$

Let the interpolation operator $I : C(0,T;D(A)) \longrightarrow V_k(r)$ be defined by

$$Iv|_{I_n(t)} = \sum_{i=1}^{r+1} \ell_{n,i}(t) v(t_{n,i}), \tag{12}$$

hence, if $v(t) \in V_k(r)$, then $v(t) = Iv(t)$, $\forall t$. Now, we define the reconstruction operator $\hat{I} : V_k(r) \longrightarrow V_k(r+1)$ by $\hat{U} = \hat{I}U \in V_k(r)$ and satisfies on I_n

$$\begin{aligned} \hat{U}_n^+ &= U_n, \\ \int_{I_n} \langle \hat{U}', v\rangle dt &= \int_{I_n} \langle U', v\rangle dt + \langle U_n^+ - U_n, v_n^+\rangle, \ \forall v \in V_k(r). \end{aligned} \tag{13}$$

The basic idea of this research is to obtain *a posteriori* error estimates of the form

$$\| u - U \| \leq \eta(u_0, U, f(U)), \tag{14}$$

where u is the exact solution of (1), U is the approximate solution of u and $\| \, . \, \|$ is the energy norm. The quantity $\eta(u_0, U, f(U))$ is of optimal order-regularity and computable in terms of the known quantities u_0. U, and $f(U)$.

2.2 The discontinuous Galerkin method

By using (13), then (9) can be written as

$$\int_{I_n} (\langle \hat{U}', v\rangle + \langle A(U), v\rangle) dt = \int_{I_n} \langle f(U), v\rangle dt, \ \forall v \in V_k(r). \tag{15}$$

Eq. (15) can be written in the pointwise form as

$$\hat{U}' + \Pi A(U) = \Pi f(U), \tag{16}$$

where the operator $\Pi : H \longrightarrow V_k(r)$ is either $\Pi = P$ the L_2-projection operator if we use DG(r) or $\Pi = I$ where I is the Lagrange interpolation operator at the Radau points if we use IRK-R(r). We will restrict our analysis to the case when $\Pi = P$, and for the other case when $\Pi = I$, we can obtain the same results with simple and straightforward modifications. Note that these methods are connected and equivalent for more details, see [37]. We derive in the next section the representation formula for DG(r) and by applying the energy technique on this formula we derive later *a posteriori* error estimators. For more details about the method see [38,50,58,63] and for a general and brief introduction to the DG method for linear problems see [19].

3. *A Posteriori* error analysis in the case of linear operators

Now, we consider the case when the operator A is linear, namely $\Pi A(U) = A(U)$ for DG(r), hence the representation formula (16) simplifies to

$$\hat{U}' + A(U) = Pf(U). \tag{17}$$

Let $A : D(A) \longrightarrow H$ is a linear operator, the energy semi-norm related to the operator A over $D(A)$ is defined by

$$\| w \| = \langle A(w), w \rangle^{1/2}, \ \forall w \in D(A), \tag{18}$$

and we define $\Re = \{w \in H : \|w\| < \infty\} \subset H$. We can now prove the upper and lower *a posteriori* error estimates and before that we first define the error measure Λ

(19)

$$\Lambda = \left\{ \max \left(\max_{0 \le t \le T} \|(u - \hat{U})(t)\|_H^2, \frac{1}{2} \int_0^T \|u - \hat{U}\|^2 dt, \frac{1}{2} \int_0^T \|u - U\|^2 dt \right) \right\}^{1/2}.$$

We use in this paper the same error measure Λ as in [37].

Proposition 3.1 (Reconstruction [37])**.** *The function $\hat{U}$ is uniquely defined by* (13)*, is globally continuous, and satisfies*

$$\hat{U}(t_{n,i}) = U(t_{n,i}), \ i = 0, ..., r+1 \ (U(t_{n,0} = U_n). \tag{20}$$

Proposition 3.2 (Properties of $\hat{U}$ [37])**.** *The following representation of $\hat{U} - U$ is valid*

$$(\hat{U} - U)|_{I_n(t)} = \hat{\ell}_{n,0}(t)(U_n - U_n^+), \ \forall t \in I_n. \tag{21}$$

In addition, if

$$\alpha_p = (\int_0^1 |\hat{\ell}_0(\tau)|^p d\tau)^{1/p}, \ \forall 1 \le p \le \infty, \tag{22}$$

then for any semi-norm $\| \, . \, \|$ in H

$$(\int_{I_n} \| \hat{U} - U \|)^{1/p} = \alpha_p k_n^{1/p} \| U_n^+ - U_n \|, \ \forall \le p \le \infty. \tag{23}$$

Theorem 3.3 (Upper Bound of Globally Lipschitz Case)**.** *Let A be a linear operator and $u_0 \in \Re$ and the function $f(u)$ is globally Lipschitz continuous, then the following upper bound estimate for DG(r) is valid for any $r \ge 0$:*

$$\max_{0 \le t \le T} \|(u - \hat{U})(t)\|_H^2 \le \xi(U) e^{C_1 T}, \tag{24}$$

where

$$\xi(U) = 2\alpha_2^2 \sum_{n=0}^{N-1} k_n^2 \|U_n^+ - U_N\|^2 + \int_0^T \|f(U) - Pf(U)\|_H^2 dt + C\alpha_2^2 \sum_{n=0}^{N-1} k_n^2 \|U_n^+ - U_N\|_H^2.$$

Proof. By subtracting (16) from (1) and using $\Pi = P$, we obtain the error representation formula

$$(u-\hat{U})' + A(u-U) = f(u) - Pf(U) \text{ (since A is linear operator).} \tag{25}$$

Multiplying the equation (25) by $(u-\hat{U})$ to have

$$\frac{1}{2}\frac{d}{dt}\|u-\hat{U}\|_H^2 - \langle A(U-u), u-\hat{U}\rangle = \langle f(u) - Pf(U), u-\hat{U}\rangle. \tag{26}$$

The nonlinear term in the righthand side of (26) is the more difficult term to deal with. To bound the righthand side term $\langle f(u) - Pf(U), u-\hat{U}\rangle$ we rewrite it in following form

$$\begin{aligned} \langle f(u) - Pf(U), u-\hat{U}\rangle &= \langle f(u) - f(U) + f(U) - Pf(U), u-\hat{U}\rangle = \\ \langle f(u) - f(U), u-\hat{U}\rangle &+ \langle f(U) - Pf(U), u-\hat{U}\rangle. \end{aligned} \tag{27}$$

Substituting (27) in (26), using (2), (4) and Young's inequality, yield

$$\begin{aligned} \frac{d}{dt}\|u-\hat{U}\|_H^2 - 2\gamma^2\ \|U-\hat{U}\|^2 &\le \|f(u)-f(U)\|_H^2 \\ &+\|f(U)-Pf(U)\|_H^2 + 2\|u-\hat{U}\|_H^2. \end{aligned} \tag{28}$$

Now, if f is globally Lipschitz continuous, namely,

$$\|f(u)-f(U)\|_H \le C\|u-U\|_H \le C\|u-\hat{U}\|_H + C\|\hat{U}-U\|_H, \tag{29}$$

inserting (29) in (28) and integrating in time to obtain

$$\begin{aligned} \|(u-\hat{U})(t)\|_H^2 &\le 2\gamma^2 \int_0^T \|\hat{U}-U\|^2 dt + \int_0^T \|f(U)-Pf(U)\|_H^2 dt \\ &+ C_1 \int_0^T \|u-\hat{U}\|_H^2 dt + C\int_0^T \|U-\hat{U}\|_H^2 dt, \end{aligned} \tag{30}$$

where $C_1 = C+2$. Using Proposition 3.1 the first and last terms on the righthand side of (30) can be expressed as follows

$$\int_0^T \|U-\hat{U}\|^2 dt = \alpha_2^2 \sum_{n=0}^{N-1} k_n^2 \|U_n^+ - U_N\|^2, \tag{31}$$

$$\int_0^T \|U-\hat{U}\|_H^2 dt = \alpha_2^2 \sum_{n=0}^{N-1} k_n^2 \|U_n^+ - U_N\|_H^2. \tag{32}$$

Consequently, by substituting (31) and (32) in (30), we obtain

$$\begin{aligned} \|(u-\hat{U})(t)\|_H^2 &\le 2\gamma^2\alpha_2^2 \sum_{n=0}^{N-1} k_n^2 \|U_n^+ - U_N\|^2 + \int_0^T \|f(U)-Pf(U)\|_H^2 dt \\ &+ C_1 \int_0^T \|u-\hat{U}\|_H^2 dt + C\alpha_2^2 \sum_{n=0}^{N-1} k_n^2 \|U_n^+ - U_N\|_H^2. \end{aligned} \tag{33}$$

Let

$$\xi(U)=2\gamma^2\alpha_2^2\sum_{n=0}^{N-1}k_n^2\|U_n^+-U_N\|^2+\int_0^T\|f(U)-Pf(U)\|_H^2dt+C\alpha_2^2\sum_{n=0}^{N-1}k_n^2\|U_n^+-U_N\|_H^2,$$

and by substituting that in (33), we have

$$\|(u-\hat{U})(t)\|_H^2\leq\xi(U)+C_1\int_0^T\|u-\hat{U}\|_H^2dt, \tag{34}$$

and by Grönwall's inequality, we obtain the required result. □

Theorem 3.4 (Upper Bound of Locally Lipschitz Case)**.** *Let A be a linear operator and $u_0\in\Re$ and the function $f(u)$ is locally Lipschitz continuous, then the following upper bound estimate for DG(r) is valid for any $r\geq 0$:*

$$\max_{0\leq t\leq T}\|(u-\hat{U})(t)\|_H^2\leq e^{\hat{C}_1\int_0^t G(U)dt}\big(\eta(U)\hat{C}_2\,(\eta(U))^{\frac{a+1}{a}}\big), \tag{35}$$

where

$$\begin{aligned}\eta(U)=2&\int_0^T\|U-\hat{U}\|^2dt+\int_0^T\|f(U)-Pf(U)\|_H^2dt\\&+C\int_0^T G(U)\|U-\hat{U}\|^2dt+C\int_0^T\|U-\hat{U}\|_H^a\|\nabla(U-\hat{U})\|_H^2dt.\end{aligned}$$

Proof. Now, we consider the case when the function f is locally Lipschitz continuous, namely, f satisfies the following growth condition. Now, we bound the nonlinear term by using the growth condition (6)

$$|\langle f(u)-f(U),u-\hat{U}\rangle|\leq C\int_\Omega|u-U|(1+|u|^a+|U|^a)|u-\hat{U}|dx, \tag{36}$$

by using the following inequality

$$|u|^a=|u-U+U|^a\leq C(|u-U|^a+|U|^a), \tag{37}$$

we can rewrite (36) as follows

$$\begin{aligned}|\langle f(u)-f(U),u-\hat{U}\rangle|\leq C&\int_\Omega(1+|U|^a)|u-U||u-\hat{U}|dx\\&+C\int_\Omega|u-U|^{a+1}|u-\hat{U}|dx.\end{aligned} \tag{38}$$

The first term on the righthand side of (38), can be bounded as

$$\begin{aligned}&\int_\Omega(1+|U|^a)|u-U||u-\hat{U}|dx\\&\leq\int_\Omega(1+|U|^a)\big(|u-\hat{U}|+|U-\hat{U}|\big)|U-\hat{U}|dx\\&\leq CG(U)\big(\|u-\hat{U}\|^2+\|U-\hat{U}\|^2\big),\end{aligned} \tag{39}$$

where $G(U) = \left(1 + \|U\|^a_{L_\infty}\right)$. The second term, we bound it by using the following inequality [9],

$$\int \rho^{\sigma+1}\delta dx \leq \frac{\sigma+1}{\sigma+2}\int \rho^{\sigma+2}dx + \frac{1}{\sigma+2}\int \delta^{\sigma+2}dx,\ \rho,\delta,\sigma > 0, \tag{40}$$

the second term can be written as

$$\int |u-U|^{a+1}|u-\hat{U}|dx \leq \int |u-\hat{U}|^{a+2}dx + \int |U-\hat{U}|^{a+1}|u-\hat{U}|dx. \tag{41}$$

Now by applying the inequality (40) on the second term in (41) and after some maths, we obtain

$$\int |u-U|^{a+1}|u-\hat{U}|dx \leq C\left(\|U-\hat{U}\|^{a+2}_{L_{a+2}} + \|u-\hat{U}\|^{a+2}_{L_{a+2}}\right). \tag{42}$$

Substituting (39) and (42) in (36), and after some mathematical manipulations, we find

$$\begin{aligned} &|\langle f(u)-f(U), u-\hat{U}\rangle| \leq CG(U)\big(\|u-\hat{U}\|^2 + \|U-\hat{U}\|^2\big) \\ &+C\|U-\hat{U}\|^a_H\|\nabla(U-\hat{U})\|^2_H + C\|u-\hat{U}\|^a_H\|\nabla(u-\hat{U})\|^2_H, \end{aligned} \tag{43}$$

and by putting (43) in (28), we have

$$\begin{aligned} &\frac{d}{dt}\|u-\hat{U}\|^2_H - 2\gamma^2\ \|U-\hat{U}\|^2 \leq \|f(U)-Pf(U)\|^2_H \\ &+ 2\|u-\hat{U}\|^2_H + CG(U)\big(\|u-\hat{U}\|^2 + \|U-\hat{U}\|^2\big) \\ &+ C\|U-\hat{U}\|^a_H\|\nabla(U-\hat{U})\|^2_H + C\|u-\hat{U}\|^a_H\|\nabla(u-\hat{U})\|^2_H, \end{aligned} \tag{44}$$

and by grouping the same and similar terms together, and integrating in time, we obtain

$$\begin{aligned} &\|(u-\hat{U})(t)\|^2_H \leq 2\gamma^2\int_0^T \|U-\hat{U}\|^2dt + \int_0^T \|f(U)-Pf(U)\|^2_H dt \\ &+\int_0^T (CG(U)+2)\|u-\hat{U}\|^2dt + C\int_0^T G(U)\|U-\hat{U}\|^2dt \\ &+C\int_0^T \|U-\hat{U}\|^a_H\|\nabla(U-\hat{U})\|^2_H dt + C\int_0^T \|u-\hat{U}\|^a_H\|\nabla(u-\hat{U})\|^2_H dt. \end{aligned} \tag{45}$$

Now, let $\eta(U)^2 = 2\gamma^2 \int_0^T \|U-\hat{U}\|^2 dt + \int_0^T \|f(U)-Pf(U)\|_H^2 dt + C\int_0^T G(U)\|U-\hat{U}\|^2 dt + C\int_0^T \|U-\hat{U}\|_H^a \|\nabla(U-\hat{U})\|_H^2 dt$ and by substituting that in (45), implies

$$\begin{aligned}
\|(u-\hat{U})(t)\|_H^2 &\leq \eta(U)^2 + \int_0^T (CG(U)+2)\|u-\hat{U}\|^2 dt \\
&+ C \sup_{t\in[0,T]} \|(u-\hat{U})(t)\|_H^a \int_0^T \|\nabla(u-\hat{U})\|_H^2 dt \\
&\leq \eta(U)^2 + \hat{C}_1 \int_0^T G(U)\|u-\hat{U}\|^2 dt \qquad (46) \\
&+ \hat{C}_2 \left(\sup_{t\in[0,T]} \|(u-\hat{U})(t)\|_H^2 + \int_0^T \|\nabla(u-\hat{U})\|_H^2 dt \right)^{\frac{a+2}{2}},
\end{aligned}$$

where, $\hat{C}_1, \hat{C}_2 > 0$. Now, consider the set

$$A = \{t \in [0,T] : \sup_{t\in[0,T]} \|(u-\hat{U})(t)\|_H^2 + \int_0^T \|\nabla(u-\hat{U})\|_H^2 dt \leq 4\delta E(t,U)\}.$$

The set A is nonempty since $u(0)-\hat{U}(0) = 0$ and by continuity of left hand side it should be closed. Let $t^* = \max_{0\leq t\leq T} A$, be such that $t^* < t^{\#}$, implies that $t^{\#} \notin A$, then, we have $\eta(t^{\#}, U) \geq \eta(t^*, U)$. Also, for simplicity of notation let $E(t,U) = e^{\hat{C}_1\left(\int_0^t G(U)dt\right)}$. When $t = t^{\#}$, from (46), we conclude that

$$\|(u-\hat{U})(t^*)\|_H^2 \leq \eta(U)^2 + \hat{C}_1 \int_0^{t^*} G(U)\|u-\hat{U}\|^2 dt + \hat{C}_2 \left(4\delta E(t^{\#},U)\right)^{\frac{a+2}{2}}. \qquad (47)$$

Using Grönwall's inequality, we have

$$\|(u-\hat{U})(t^*)\|_H^2 \leq E(t^{\#},U)\left(\delta + \hat{C}_2 \left(4\delta E(t^*,U)\right)^{\frac{a+2}{2}}\right), \qquad (48)$$

since $\delta(t^{\#}) \geq \delta(t^*)$, upon assuming δ is such that

$$\delta \leq \hat{C}_2^{\frac{-2}{a}} \left(4E(t^{\#},U)\right)^{\frac{a+2}{2}},$$

which contradicts the assumption that t^* is the maximum element of A, hence we conclude that $t^* = t^{\#}$ and $A = [0,T]$. Consequently, we get the required estimate in (35). □

Theorem 3.5 (Lower Bound)**.** *If A is a linear operator and $u_0 \in \Re$, then the following error bound is valid for DG(r), $r \geq 0$:*

$$\frac{1}{8}\alpha_2^2 \sum_{n=0}^{N-1} k_n \|U_n^+ - U_n\|^2 \leq \Lambda^2. \qquad (49)$$

Proof.

$$\left(\int_0^T \|\hat{U}-U\|_H^2 dt\right)^{1/2} = \left(\sum_{n=0}^{N-1}\int_{I_n} \|\hat{U}-U\|_H^2 dt\right)^{1/2}, \tag{50}$$

which implies that

$$\int_0^T \|\hat{U}-U\|^2 dt = \sum_{n=0}^{N-1}\int_{I_n} \|\hat{U}-U\|^2 dt, \tag{51}$$

by using (23) in Proposition 3.2, we have

$$\int_{I_n} \|\hat{U}-U\|^2 dt = \alpha_2^2 k_n \|U_n^+ - U_n\|^2, \tag{52}$$

by substituting (52) in (51), we obtain

$$\int_0^T \|\hat{U}-U\|^2 dt = \sum_{n=0}^{N-1}\int_{I_n} \|\hat{U}-U\|^2 dt = \alpha_2^2 \sum_{n=0}^{N-1} k_n \|U_n^+ - U_n\|^2. \tag{53}$$

Consequently, using (19), we have

$$\begin{aligned} \alpha_2^2 \sum_{n=0}^{N-1} k_n \|U_n^+ - U_n\|^2 &\le 2\int_0^T \|u-\hat{U}\|^2 dt + 2\int_0^T \|u-U\|^2 dt \\ &\le 4\max\left(\frac{1}{2}\int_0^T \|u-\hat{U}\|^2 dt\right) + 4\max\left(\frac{1}{2}\int_0^T \|u-U\|^2 dt\right) \\ &\le 4\Lambda^2 + 4\Lambda^2 = 8\Lambda^2. \end{aligned} \tag{54}$$

Finally, from (54), we obtain the estimate in (49). □

Theorem 3.6 (Error Estimate of Globally Lipschitz Case)**.** *Let A be a linear monotone operator and $u_0 \in D(A)$ and f is globally Lipschitz continuous, then*

$$\max_{0\le t\le T} \|(u-\hat{U})(t)\|_H \le \beta(U) e^{CT}, \tag{55}$$

where

$$\beta(U) = \alpha_1^2 \sum_{n=0}^{N-1} k_n^2 \left(\|A(U_n^+ - U_n)\|_H^2 + C\|U_n^+ - U_n\|_H^2\right) + \int_0^T \|f(U) - Pf(U)\|_H^2 dt.$$

Proof. By repeating the same first two steps in the previous theorem, we arrive at (26) which can be rewritten in the following way

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u-\hat{U}\|_H^2 + \langle A(u-\hat{U}), u-\hat{U}\rangle = \langle A(U-\hat{U}), u-\hat{U}\rangle \\ &+ \langle f(u)-f(U), u-\hat{U}\rangle + \langle f(U)-Pf(U), u-\hat{U}\rangle. \end{aligned} \tag{56}$$

Consequently, after some mathematical manipulations, we get

$$\begin{aligned}\|u-\hat{U}\|_H^2 &\leq \int_0^T \|A(U-\hat{U})\|_H^2 + \int_0^T \|f(U)-Pf(U)\|_H^2 + \int_0^T \|u-\hat{U}\|_H^2 \\ &+ 2\int_0^T |\langle f(u)-f(U), u-\hat{U}\rangle|. \end{aligned} \tag{57}$$

Now, we bound the first term in the righthand side of (57), we have

$$\int_0^T \|A(U-\hat{U})\|_H^2 dt = \sum_{n=0}^{N-1} \int_{I_n} \|A(U-\hat{U})\|_H^2 dt, \tag{58}$$

and by Proposition 3.2, we have

$$\int_{I_n} \|A(U-\hat{U})\|_H^2 dt = \alpha_1^2 k_n^2 \|A(U_n^+ - U_n)\|_H^2, \text{ (since } \hat{U}_n = U_n), \tag{59}$$

by substituting (58) and (59) in (57), we get

$$\begin{aligned}\|u-\hat{U}\|_H^2 &\leq \alpha_1^2 \sum_{n=0}^{N-1} k_n^2 \|A(U_n^+ - U_n)\|_H^2 + \int_0^T \|f(U)-Pf(U)\|_H^2 \\ &+ \int_0^T \|u-\hat{U}\|_H^2 + 2\int_0^T |\langle f(u)-f(U), u-\hat{U}\rangle|. \end{aligned} \tag{60}$$

Now, we bound the last term in the righthand side of (60) $|\langle f(u)-f(U), u-\hat{U}\rangle|$ by using Cauchy-Schwarz and Young inequalities, we have

$$\begin{aligned}&|\langle f(u)-f(U), u-\hat{U}\rangle| \\ &\leq \|f(u)-f(U)\|_H \|u-\hat{U}\|_H \leq \frac{1}{2}\|f(u)-f(U)\|_H^2 + \frac{1}{2}\|u-\hat{U}\|_H^2, \end{aligned} \tag{61}$$

and by substituting (61) in (60), we obtain

$$\begin{aligned}\|u-\hat{U}\|_H^2 &\leq \alpha_1^2 \sum_{n=0}^{N-1} k_n^2 \|A(U_n^+ - U_n)\|_H^2 + \int_0^T \|f(U)-Pf(U)\|_H^2 + \\ &+ \int_0^T \|f(u)-f(U)\|_H^2 dt + 2\int_0^T \|u-\hat{U}\|_H^2 dt, \end{aligned} \tag{62}$$

and using Lipschitz continuity of f and Proposition 3.2, to find that

$$\begin{aligned}\|u-\hat{U}\|_H^2 &\leq \alpha_1^2 \sum_{n=0}^{N-1} k_n^2 \|A(U_n^+ - U_n)\|_H^2 + C\alpha_1^2 \sum_{n=0}^{N-1} k_n^2 \|U_n^+ - U_n\|_H^2 \\ &+ \int_0^T \|f(U)-Pf(U)\|_H^2 dt + C\int_0^T \|u-\hat{U}\|_H^2 dt. \end{aligned} \tag{63}$$

Let

$$\beta(U) = \alpha_1^2 \sum_{n=0}^{N-1} k_n^2 \left(\|A(U_n^+ - U_n)\|_H^2 + C\|U_n^+ - U_n\|_H^2 \right) + \int_0^T \|f(U) - Pf(U)\|_H dt,$$

and by substituting that in (63), we get

$$\|u - \hat{U}\|_H^2 \leq \beta(U) + C \int_0^T \|u - \hat{U}\|_H^2 dt. \tag{64}$$

Finally, by using Grönwall's proposition, we have the estimate in (55). □

Theorem 3.7 (Error Estimate of Locally Lipschitz Case)**.** *Let A be a linear monotone operator and $u_0 \in D(A)$ and f is locally Lipschitz continuous, then*

$$\max_{0 \leq t \leq T} \|(u - \hat{U})(t)\|_H^2 \leq 2\lambda(U)^2 e^{\bar{C}_1 T}, \tag{65}$$

where

$$\begin{aligned} \lambda(U)^2 = \alpha_1^2 \sum_{n=0}^{N-1} k_n^2 \|A(U_n^+ - U_n)\|_H^2 + \int_0^T \|f(U) - Pf(U)\|_H^2 dt \\ + K\alpha_1^2 \sum_{n=0}^{N-1} k_n^2 \|U_n^+ - U_n\|_H^2 dt + \bar{C} \int_0^T \|U - \hat{U}\|_H^a \|\nabla(U - \hat{U})\|_H^2 dt. \end{aligned}$$

Proof. In the case of the locally Lipschitz continuous function f, we return to (60)

$$\begin{aligned} \|u - \hat{U}\|_H^2 &\leq \alpha_1^2 \sum_{n=0}^{N-1} k_n^2 \|A(U_n^+ - U_n)\|_H^2 + \int_0^T \|f(U) - Pf(U)\|_H^2 + \\ &+ 2\int_0^T \|u - \hat{U}\|_H^2 + 2\int_0^T |\langle f(u) - f(U), u - \hat{U} \rangle|. \end{aligned} \tag{66}$$

Now, we bound the nonlinear term $\int_0^T |\langle f(u) - f(U), u - \hat{U} \rangle|$ by using the growth condition (6) and following the same steps as in the Theorem 2, we arrive at

$$\begin{aligned} \|u - \hat{U}\|_H^2 \leq \alpha_1^2 \sum_{n=0}^{N-1} k_n^2 \|A(U_n^+ - U_n)\|_H^2 + \int_0^T \|f(U) - Pf(U)\|_H^2 dt + \\ + \bar{C}_1 \int_0^T \|u - \hat{U}\|_H^2 dt + K\alpha_1^2 \sum_{n=0}^{N-1} k_n^2 \|U_n^+ - U_n\|_H^2 dt \\ + \bar{C}_2 \int_0^T \|u - \hat{U}\|_H^a \|\nabla(u - \hat{U})\|_H^2 dt \\ + \bar{C} \int_0^T \|U - \hat{U}\|_H^a \|\nabla(U - \hat{U})\|_H^2 dt, \end{aligned} \tag{67}$$

where $\bar{C} = 2C\tilde{C}$, $\bar{C}_1 = 3K + 2$ and $\bar{C}_2 = 2C + \bar{C}$. Now, let

$$\lambda(U)^2 = \alpha_1^2 \sum_{n=0}^{N-1} k_n^2 \|A(U_n^+ - U_n)\|_H^2 + \int_0^T \|f(U) - Pf(U)\|_H^2 dt + K\alpha_1^2 \sum_{n=0}^{N-1} k_n^2 \|U_n^+ - U_n\|_H^2 dt + \bar{C} \int_0^T \|U - \hat{U}\|_H^a \|\nabla(U - \hat{U})\|_H^2 dt,$$

and by substituting that in (67), we have

$$\|u - \hat{U}\|_H^2 \leq \lambda(U)^2 + \bar{C}_1 \int_0^T \|u - \hat{U}\|_H^2 dt + \bar{C}_2 \left(\sup_{t\in[0,T]} \|u - \hat{U}\|_H^2 + \int_0^T \|\nabla(u - \hat{U})\|_H^2 dt \right)^{\frac{a+2}{2}}. \tag{68}$$

To bound the last term, we follow the same techniques as in Theorem 2, we end up with the estimate in (65). □

4. Conclusions

Optimal order *a posteriori* error estimators of discontinuous Galerkin method for semidiscretised in time semilinear parabolic problems have been derived and presented. These error bounds are derived for the linear, monotone and γ^2-angle bounded operators. The main tool in deriving these error estimates is the time reconstruction operator introduced in [37]. The nonlinear evolutionary problems when the nonlinearity is both globally and locally Lipschitz continuous are investigated.

Acknowledgements

The author would like to thank Prof. E. Georgoulis (University of Leicester, UK) and Assist Prof. Andrea Cangiani (University of Leicester, UK) for their insightful suggestions and ideas at the time of preparation of the manuscript of the paper.

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Accepted: 12.11.2019

Controllability of semilinear boundary control inequality problem in infinite dimensional state space

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Abstract. This study aims to propose the basic theory for controllability of mild solutions of semilinear boundary control inequality problems in an infinite dimension state space by establishing the necessary and sufficient conditions for this issue.

Keywords: boundary control inequality, controllability, mild solution, semilinear.

1. Introduction

The importance of the investigations of inequalities in dynamical differential equations lies in the fact that they include several classes of differential equations as special cases. The study of inequalities in differential equations is addressed in multiple papers, including those by Popa D. and Lungu N.[6], Jacek Szarski [9], Ladas G. and Stavroulakis [5], and Olga Salieva [8]. Additionally boundary control problems have been addressed in the study of semilinear differential problems with boundary control for achieving global absolute stability of an unstable equilibrium solution [10], and the study of approximate controllability with boundary controls for some classes of semilinear delay control system appearing in [13]. The controllability with a sufficient conditions for stochastic system including boundary control systems has been established and the results are presented using the Banach contraction mapping theorem appearing in [11].

Some basic and fundamental results concerning the strict and nonstrict classes of nonlinear first order Volterra integro-differential equations including maximal and minimal Solutions, for first order hyper integro-differential equation inequality inequalities are presented in references [2] and [3].

In this paper, we first present the basic theory of a semilinear boundary control inequality problem in an infinite dimensional state space and then prove the boundary controllability for mild solutions of semilinear boundary control inequality problems in an infinite dimensional state space. We claim that the results of this study are basic and an important contribution to the theory of semilinear boundary control inequality problems .

2. Preliminaries:

The following definitions and results are used:

Definition 2.1 ([2]). *The closed set V is a cone if it satisfies the following conditions:*

i. If $x \in V$ and $y \in V$ then $x + y \in V$.

ii. If $x \in V$ and $\lambda \in V$ then $\lambda x \in V$.

Definition 2.2 ([9]). *Given a cone V, we define on a Banach space E a partial order $\leq$ by$x \leq y \equiv x - y \in V$, the partial order defined above is called a partial order induced by V.*

Remark 2.1 ([9]). From the definition of a cone, we have

i. If $x \leq y$ and $y \leq z$ imply $x \leq z$

i. If $x \leq y$ implies that $x + z \leq y + z$ for arbitrary $z \in E$.

i. If $x \leq y$ and $\lambda \geq 0$ imply $\lambda x \leq \lambda y$

Remark 2.2 ([4]). Let $x(t)$ be a real valued function and suppose that $\dot{x} \leq Ax, x \in [a, b]$ with A being a constant then $x(t) \leq x(a)e^{A(t-a)}, a \leq x \leq b$ is called Bendixion equation and it is a differential inequality.

Theorem 2.1 ([5]). *The uniqueness of a fixed point exists for mapping T, if the contraction mapping T defined on the Banach space X.*

Definition 2.3 ([1]). *The family of bounded linear operators defined on the Banach space X is a semigroup if $T(0) = I$. I is identity operator on X, and $T(t + s) = T(t)T(s)$ for every $t, s \geq 0$.*

Definition 2.4 ([1]). *Let $T(t)$ be a semigroup then $T(t)$ is called strongly continuous and which denoted by C_0 on a Banach space X if $\lim_{t\downarrow 0} \parallel T(t)x - Ix \parallel_X = 0$.*

Definition 2.5 ([1]). *The domain of the linear operator $A : D(A) \subseteq X \longrightarrow X$ is defined as follows:*

The domain $D(A) = \{x \in X : \lim_{t\downarrow 0} \frac{T(t)x-x}{t}$ exist$\}$ and $Ax = \frac{dT(t)}{dt}|_{t=0} = \lim_{t\downarrow 0} \frac{T(t)x-x}{t}$ for $x \in D(A)$. A is the generator of the semigroup $T(t)$.

Remark 2.3 ([12]). $\parallel T(t) \parallel_{L(X)} \leq Me^{wt}$, for $M \geq 1$.
The family of linear operator $t \longrightarrow T(t)$ is differentiable which is $\frac{dT(t)}{dt} = AT(t) = T(t)A$

The following includes important facts on the semigroup theory of bounded linear operators

Theorem 2.2 ([12]). *A linear (unbounded) operator A is the generator of a strong semigroup of contraction family $\{T(t)\}_{t\geq 0}$ if and only if:*

(*i*) *The generator linear operator A is closed and densely defined, and*

(*ii*) *the resolvent set $\rho(A)$ of A contains R^+ and $\| (\lambda I - A)^{-1} \| \leq \frac{1}{\lambda}$ for every $\lambda \geq 0$*

Remark 2.4 ([12]). Let $T(t)$ be a C_0- semigroup then $\lim_{h\to 0} \frac{1}{h} \int_t^{t+h} T(\tau)xds = T(\tau)x,\ h \in (0,t), \text{for } x \in X.$

Theorem 2.3 ([9]). *Let V be a cone and let A be a generator of a positive strong semigroup $\{T(t)\}$. Assume that the function $x(t)$ is strongly continuous on $[0,a)$ and $D_+^s x(t) \leq Ax(t)$ nearly everywhere on $[0,a), x(0) \leq \theta$, then $x(t) \leq \theta$ on $[0,a)$.*

Theorem 2.4 ([9]). *Let A be generator of a positive strong semigroup $\{T(t)\}$. Let $B(t)$ be a strong continuous operator valued function. Assume that $B(t) + \beta I \geq \theta$ for $t \in [0,a)$ where β is a real number. If $x(t)$ is strongly differentiable to $\dot{x}(t)$ on $[0,a)$ and $\dot{x}(t) \leq [A + B(t)]x(t)$ on $[0,a), x(0) \leq \theta$, then $x(t) \leq \theta$ on $[0,a)$.*

Theorem 2.5 ([9]). *Let $x(t)$ be weakly continuous in $(0,b)$ and it satisfy nearly everywhere the inequality $D_+^W x(t) \leq Ax(t), A$ is the generator of semigroup $T(t)$ then $x(t) \leq T(t-s)x(s)$ for $0 < s < t < b$.*

Theorem 2.6 ([9]). *Let the function $f(t,x)$ be strongly continuous in (t,x) and bounded by $\| f(t,x) \| \leq M$. Assume that $f(t,x)$ is increasing in x. Let the transformation $F : z(t) \longrightarrow \int_0^t T(t-\tau)f(\tau, z(\tau))d\tau$ be completely continuous when considered in the space $C[0,a]$. Suppose that the function $x(t)$ is strongly differentiable on $[0,a]$ and $\dot{x}(t) \leq Ax(t) + f(t,x(t)), 0 \leq t \leq a$. A is the generator of semigroup $T(t)$, then*

$$x(t) \leq T(t).\ x(0) + \int_0^t T(t-\tau)f(\tau, z(\tau))d\tau, \quad 0 \leq t \leq a.$$

3. Main results

Let $L(E)$ be the space of all linear bounded operators on a Hilbert space E. In this section we will Consider the similinear boundary value control inequality problem in an infinite dimensional state space . We state with: define the space of linear bounded operators on E, and let V be a close and convex cone subset of E such that $X(.)$ defined on V. Let U be an admissible Hilbert control space, where we consider the semilinear boundary value control inequality problem in an infinite dimensional state space :

$$(1) \quad \begin{cases} \dfrac{d}{dt}(X(t) + g(t, X(t))) \leq LX(t) + X(t)A + BZ(t) + F(t, N(t, X(t))) \\ X(t) = Z(t) + B_2u(t), \quad t \in J = [0,b] \\ \tau(z(t)) = -B_1u(t) \\ X(0)x \leq \theta \end{cases}$$

Where $X(.)$ is an operator which is defined on a convex cone V and U is an admissible Hilbert control space. Here $B_2 : U \longrightarrow V$ is a linear strongly continuous operator, $u \in L^2(J,U)$ is a Hilbert space of admissible control functions, N is nonlinear operator defined on $J \times L(V), g$ is nonlinear operator defined on $J \times L(V)$ also τ be an operator with $D(\tau)$ and $\text{Rang}(\tau) \subseteq E.A, B$ and Z belong to $L(E)$ with $Z(t) = X(t) - B_2u(t)$ such that $\tau(Z(t)) = -B_1u(t)$ and $\tau(B_2u(t)) = B_1u(t)$, then

$$\tau(X(t)) = \tau(Z(t) + B_2u(t)) = \tau(Z(t)) + \tau(B_2u(t)) = -B_1u(t) + B_1u(t) = 0.$$

Let $B : V \longrightarrow V$ be a linear operator defined by $X(t)x \in D(B) = \{X(t)x \in D(L) : \tau X(t) = \theta\}$. Therefore

$$\begin{aligned}\frac{d}{dt}(X(t) + g(t, X(t))) &= LX(t) + X(t)A + B(X(t) - B_2u(t)) + F(t, N(t, X(t))) \\ &= LX(t) + X(t)A + BX(t) - BB_2u(t) + F(t, N(t, X(t))).\end{aligned}$$

Therefore

$$\begin{aligned}\frac{d}{dt}(X(t) + g(t, X(t))) &\leq (L + B)X(t) + X(t)A - BB_2u(t) + F(t, N(t, X(t))) \\ X(0)x &\leq \theta.\end{aligned}$$

3.1 Boundary controllability

In this section we study the basic theorem of the controllability of inequality mild solutions to the semilinear boundary control inequality problems. We consider the semilinear boundary control inequality problem in an infinite dimensional state space:

$$(2) \qquad \begin{cases} \dfrac{d}{dt}(X(t) + g(t, X(t)) \\ \qquad \leq (L + B)X(t) + X(t)A - BB_2u(t) + F(t, N(t, X(t)), \\ X(0)x \leq \theta \end{cases}$$

For $J = [0, b]$ where the linear unbounded operator $((L+B)+A)$ is the generator of C_0- composite semigroup and $B_2 : U \longrightarrow V$ is a linear strongly continuous operator, $\mathcal{T}(t) = \tilde{T}(t).T(t), \quad t \geq 0$, such that $L + B$ is the generator semigroup $\tilde{T}(t)$ and A is the generator of semigroup $T(t)$ and for the two constants $M_1 \geq 0,\ M_2 \geq 0$, therefore, we obtain

$$(3) \qquad \| T(t) \| \leq M_1 e^{w_1 t}, \quad w_1 \geq 0,\ t \geq 0, \quad \text{and}$$

$$(4) \qquad \| \tilde{T}(t) \| \leq M_2 e^{(w_2 + \|B\|)t}$$

Here, $B : U \longrightarrow E$ is a linear admissible control operator. Additionally,

$$B_r = \{X(t) \in L(V) : \|X(t)\|_{L(V)} \leq r \ \text{ for some } \ r > 0\}.$$

Our aim is to compute the inequality mild solution of problem (2). Let $X(.) \in L(V)$ the be strongly continuous solution $X(.)x \in D((L+B).+.A)$ of problem (2). Then by [12], we determine that $\mathcal{T}(t)(X(t)+g(t,X(t)))$ is differential. This implies that the $L(V)$-value function $\tilde{H}s)$ satisfy the following inequality

$$\tilde{H}(s) \leq \mathcal{T}(t-s)(X(t)+g(t,X(t)))$$

is differentiable for $0 < s < t$ as follows: $\tilde{H}(s) \leq \mathcal{T}(t-s)(X(t)+g(s,x(s)))\frac{d}{ds}\tilde{H}(s) \leq \mathcal{T}(t-s)\frac{d}{ds}(X(t)+g(t,X(t))) + (X(t)+g(t,X(t)))\frac{d}{ds}\mathcal{T}(t-s)\frac{d}{ds}\tilde{H}(s) \leq \mathcal{T}(t-s)\left[(L+B)X(t)+X(t)A - BB_2u(t)+F(t,N(t,X(t)))\right]+(X(s)+g(s,X(s)))((L+B).+.A)\mathcal{T}(t-s) = \mathcal{T}(t-s)((L+B).+.A)X(s) - \mathcal{T}(t-s)BB_2u(t) + \mathcal{T}(t-s)F(s,N(s,X(s)))-\mathcal{T}(t-s)((L+B).+.A)X(s)-g(s,X(s))((L+B).+.A)\mathcal{T}(t-s)$. Hence,

$$\frac{d}{ds}\tilde{H}(s) \leq -\mathcal{T}(t-s)BB_2u(s) + T(t-s)F(s,N(s,X(s))) - g(s,X(s)) \tag{5}$$
$$((L+B).+.A)\mathcal{T}(t-s).$$

Now, by integrating (3) from 0 to t, we have

$$\tilde{H}(s) - \tilde{H}(0) = -\int_0^t \mathcal{T}(t-s)BB_2u(s)ds + \int_0^t \mathcal{T}(t-s)F(s,N(s,X(s)))ds - \int_0^t g(s,X(s))((L+B).+.A)\mathcal{T}(t-s)ds.$$

By definition of the composite semigroup, we have $\hat{T}(0) = I$, and by definition of the above function $\tilde{H}(s)$, we obtain

$$X(t)+G(t,X(t)) - \mathcal{T}(t)\theta - \mathcal{T}(t)g(0,\theta) = -\int_0^t \mathcal{T}(t-s)BB_2u(s)ds + \int_0^t \mathcal{T}(t-s)F(s,N(s,X(s)))ds - \int_0^t g(s,X(s))((L+B).+.A)\mathcal{T}(t-s)ds.$$

Hence,

$$X(t) \leq \mathcal{T}(t)g(0,\theta) - g(t,X(t)) - \int_0^t \mathcal{T}(t-s)BB_2u(s)ds + \int_0^t \mathcal{T}(t-s). \tag{6}$$
$$F(s,N(s,X(s)))ds - \int_0^t g(s,X(s))((L+B).+.A)\mathcal{T}(t-s)ds.$$

According to equation (6), the following definition is presented.

Definition 3.1. *A continuous function $X : [0,b] \longrightarrow L(V)$ defined by*

$$X(t) \leq \int_0^t \mathcal{T}(t-s)F(s,N(s,X(s)))ds - \int_0^t \mathcal{T}(t-s)BB_2u(s)ds + \mathcal{T}(t)g(0,\theta) - g(t,X(t)) - \int_0^t g(s,X(s))((L+B).+.A)\mathcal{T}(t-s)ds.$$

is called a strongly continuous mild solution to the semilinear boundary control inequality problem.

Definition 3.2. *The system* (2) *is said to be controllable on the interval* $J = [0, b]$ *if for every operator* $X(0) = \theta, X(b) = X_1 \in L(V)$ *there exists a control* $u \in L^2(J, U)$ *such that the mild inequality solution*

$X(t) \leq \int_0^t \mathcal{T}(t-s)F(s, N(s, X(s)))ds - \int_0^t \mathcal{T}(t-s)BB_2u(s)ds + \mathcal{T}(t)g(0, \theta) - g(t, X(t)) - \int_0^t g(s, X(s))((L+B).+.A)\mathcal{T}(t-s)ds.$

Satisfies $X(b) \leq X_1$.

For problem (2) to make sense, throughout the paper, we assume the following basis hypothesis:

a) The family $\{\mathcal{T}(t)\}$ invariant is the set V.

b) b) The strong derivative $\frac{\partial \hat{T}(t-s)X}{\partial s}, s < t$ exists for $X(t) \in D(A)$ and $\frac{\partial \mathcal{T}(t-s)X}{\partial s} + \mathcal{T}(t-s)AX = 0$

c) For each $X \in E$, the function $\mathcal{T}(t-s)X$ is strongly continuous in s.

d) The nonlinear operator $N : J \times L(V) \longrightarrow L(V)$, satisfies Lipchitz condition on second argument $\|N(t, X_1) - N(t, X_2)\| \leq M_3\|X_1X_2\|$, where M_3 is a positive constant and $X_1, X_2 \in B_r$.

e) The nonlinear operator $F : J \times L(V) \longrightarrow L(V)$, is continuous and has $\|F(t, N(t, X_1)) - F(t, N(t, X_2))\| \leq M_4\|N(t, X_1) - N(t, X_2)\|$, where $M_4 = max\|F(t, N(t, \theta))\|$.

f) Let $B : V \longrightarrow V$ be a linear bounded operator.

g) The nonlinear operator $g : J \times L(V) \longrightarrow L(V)$, satisfies lipschitz condition on second argument and for all $X_1, X_2 \in B_r, L_1, L_2 > 0$.
$\|g(t, X_1) - g(t, X_2)\| \leq L_1\|X_1 - X_2\|$and $L_2 = max\|g(t, \theta)\|$.

Theorem 3.1. *Consider the inequality semilinear boundary control problem*

$$\begin{aligned} \frac{d}{dt}(X(t) + g(t, X(t))) &\leq LX(t) + X(t)A + F(t, N(t, X(t))) \\ Z(t) &= X(t) - B_2u(t) \\ \tau(Z(t)) &= -B_1u(t), \quad X(0) \leq \theta \end{aligned}$$

Furthermore, based on hypothesis (a-g), we assume the following:

h) Define a linear continuous operator W from $L^2(J, U)$ into $L(V)$ as follows

$$Wu = \int_0^t \mathcal{T}(t-s)Bu(s)ds, \text{ where } W : \frac{L^2(J, U)}{kerW} \longrightarrow V$$

and suppose there $W^{-1} : Range(W) \longrightarrow \frac{L^2(J,U)}{kerW}$ exists with a constant $k_1 > 0$ such that $\|W^{-1}\| \leq k_1$.

i) There exists a positive function $v_0 \in L(0,b)$, such that $\|((L+B).+.A)\mathcal{T}(t)\| = \|((L+B).+.A)\tilde{T}(t).T(t)\| \leq v_0(t)$, $t \in (0,b)$.

j) $k_1 > 0$ is a constant such that $\int_0^b v_0(t)dt \leq k_1$.

k) $M_1M_2e^{(w_1+w_2+M_2\|B\|)b}L_2+h_2+h_2k_1+k_bk_2\|X_1\|+M_1M_2e^{(w_1+w_2+M_2\|B\|)b}L_2+$
$L_1\|X_1\| + L_2 + h_2k_1 + \frac{M_1M_2\ e^{(w_1+w_2+M_2\|B\|)(b-r_1)}}{(w_1+w_2+M_2\|B\|)}h_3 +$
$\frac{M_1M_2\ e^{(w_1+w_2+M_2\|B\|)(b-r_1)}}{(w_1+w_2+M_2\|B\|)}h_3 \leq r$, where $h_2 = L_1r + L_2, h_3 = M_3M_4r + M_s$.

l) $b \leq \frac{\log \frac{(w_1+w_2+M_2\|B\|)}{(k_bk_2+1)(M_1M_2M_3M_4)}(\delta - L_1 + L_1k_1 + k_bk_2k_1)}{(w_1+w_2+M_2\|B\|)}$ such that
$L_1 + L_1k_1 + k_bk_2k_1 < \delta < 1$.

Then, the system (2) is boundary controllable on J.

Proof. By using definition (3.2) and equation (6) we obtain $X_1 = X(b) \leq \mathcal{T}(b)g(0,\theta) - g(b,X(b)) - \int_0^b \mathcal{T}(b-s)BB_2u(s)ds - \int_0^b g(s,X(s))((L+B).+.A)\mathcal{T}(b-s)ds + \int_0^b \mathcal{T}(b-s)F(s,N(s,X(s)))ds$.

By hypothesis (h), we obtain $X_1 \leq \mathcal{T}(b)g(0,\theta) - g(b,X(b)) - \int_0^b g(s,X(s))((L+B).+.A)\mathcal{T}(b-s)ds + \int_0^b \mathcal{T}(b-s)F(s,N(s,X(s)))ds - Wu$, therefore

$$Wu \leq \mathcal{T}(b)g(0,\theta) - g(b,X_1) - \int_0^b g(s,X(s))((L+B).+.A)\mathcal{T}(b-)ds + \int_0^b \mathcal{T}(b-s)F(s,N(s,X(s)))ds - X_1.$$

We have

$$u(t) \geq W^{-1}(Wu(t)) > W^{-1}(\mathcal{T}(b)g(0,\theta) - g(b,X_1) - \int_0^b g(s,X(s))$$

$$((L+B).+.A)\mathcal{T}(b-s)ds + \int_0^b \mathcal{T}(b-s)F(s,N(s,X(s)))ds - X_1) \tag{7}$$

Let $Q = C(J,B_r)$ (the space of all strongly continuous functions defined from J into B_r) ,by using control $u(t)$ defined by equation (7) and definition of the mild continuous inequality solution (6). We shall show that operator ϕ defined by

$$\phi X(t) \leq \mathcal{T}(b)g(0,\theta) - g(t,X(t)) - \int_0^t g(s,X(s))((L-B).+.A)\mathcal{T}(t-s)ds + \int_0^b \mathcal{T}(t-s)BB_2W^{-1}\left[\mathcal{T}(b)g(0,\theta) - g(b,X_1)\right.$$

$$
\begin{aligned}
&- \int_0^b g(s, X(s))((L-B).+.A) \\
&\mathcal{T}(b-s)ds + \int_0^b \mathcal{T}(b-s)F(s, N(s, X(s)))ds - X_1\Big] ds \\
&+ \int_0^b F(s, N(s, X(s)))\hat{T}(b-s)ds
\end{aligned} \tag{8}
$$

has a fixed point, first show ϕ map Q into itself, i.e., to show for some $r > 0$, from (d),(e) and (g) with definition of B_r, to show that $\|\phi X(t)\| \leq r$, we have

$$
\begin{aligned}
&\|\phi X(t)\| \leq \|\mathcal{T}(t)g(0,\theta)\| + \|g(t, X(t))\| \\
&+ \| \int_0^t g(s, X(s))((L+B).+.A)\mathcal{T}(t-s)ds\| \\
&+ \| \int_0^b \mathcal{T}(t-s)BW^{-1}\Big[\mathcal{T}(b)g(0,\theta) - g(b, X_1) - \int_0^b g(s, X(s)).((L+B).+.A) \\
&\mathcal{T}(b-s)ds + \int_0^b \hat{T}(b-s)F(s, N(s, X(s)))ds - X_1\Big] ds + \int_0^b F(s, N(s, X(s))) \\
&\mathcal{T}(b-s)ds\| \leq \|\mathcal{T}(t)g(0,\theta)T(t)\| \\
&+ \| \int_0^t g(s, X(s))((L+B).+.A)\hat{T}(t-s)ds\| + k_t\|W^{-1} \\
&\Big[\|\mathcal{T}(b)g(0,\theta)T(b)\| + \|g(b, X_1)\| + \| \int_0^b g(r_1, X(r_1))\|\|((L+B).+.A)\mathcal{T}(b-r_1) \\
&T(b-r_1)\|dr_1 - \int_0^b \|\mathcal{T}(b-r_1)F(r_1, N(r_1, X(r_1)))T(b-r_1)\|dr_1 + \|X_1\|\Big] ds\| + \\
&\|\mathcal{T}(b-r_1)F(r_1, N(r_1, X(r_1)))T(b-r_1)\|.
\end{aligned}
$$

From (3), (4), (d), (g), (e) and $t < b$, we obtain

$$
\begin{aligned}
&\|\phi X(t)\| \leq M_1M_2e^{(w_1+w_2M_2\|B\|)b}\|g(0,\theta)\| + \|g(t, X(t)) - g(t,\theta) + g(t,\theta) \\
&+ \int_0^t \|g(s, X(s)) - g(s,\theta) + g(s,\theta)\|\|((L-B).+.A)T(t-s)\|ds \\
&+ k_t\|W^{-1}\| \left[\|X_1\| + M_1M_2\right. \\
&e^{(w_1+w_2M_2\|B\|)b}\|g(0,\theta)\| + \|g(b, X_1) - g(b,\theta) + g(b,\theta)\| \\
&+ \int_0^t \|g(r_1, X(r_1)) - g(r_1,\theta) + \\
&g(r_1,\theta)\|\|((L-B).+.A)\mathcal{T}(b-r_1)T(b-r_1)\|dr_1 \\
&\left. + \int_0^b M_1M_2e^{(w_1+w_2M_2\|B\|)(b-r_1)}\right\|
\end{aligned}
$$

$$F(r_1, N(r_1, X(r_1))) - F(r_1, N(r_1, \theta)) + F(r_1, N(r_1, \theta))\|dr_1]\,(s)ds\|$$
$$+ \int_0^t M_1 M_2 e^{(w_1+w_2 M_2\|B\|)(b-s)}\|F(s, N(s, X(s)))$$
$$- F(s, N(s, \theta)) + F(s, N(s, \theta))\|ds,$$

hence,

$$\|\phi X(t)\| \leq M_1 M_2 e^{(w_1+w_2 M_2\|B\|)b}(L_2) + L_1 r + L_2 + \int_0^t (L_1 r + L_2)\|((L-B).+$$
$$.A)\mathcal{T}(t-s)\|ds + k_t k_2 \Big[\|X_1\| + M_1 M_2 e^{(w_1+w_2 M_2\|B\|)b}(L_2) + L_1\|X_1\|+$$
$$L_2 + \int_0^b (L_1 r + L_2)\|((L-B).+.A)\tilde{T}(b-r_1)T(b-r_1)\|dr_1+$$
$$\int_0^b M_1 M_2 e^{(w_1+w_2 M_2\|B\|)(b-r_1)}(M_3 M_4 r + M_5)\Big]\, dr_1$$
$$+ \int_0^t M_1 M_2 e^{(w_1+w_2 M_2\|B\|)(b-s)}(M_3 M_4 r + M_5)ds$$

By condition (i), we have

$$\|\phi X(t)\| \leq M_1 M_2 e^{(w_1+w_2 M_2\|B\|)b}(L_2) + (L_1 r + L_2) + (L_1 r + L_2)\int_0^t v_0(s)ds$$
$$+ k_b k_2 \Big[\|X_1\| + M_1 M_2 e^{(w_1+w_2 M_2\|B\|)b}(L_2) + L_1\|X_1\| + L_2 + \int_0^b v_0(r_1)(L_1 r + L_2)dr_1$$
$$+ \int_0^b M_1 M_2 e^{(w_1+w_2 M_2\|B\|)(b-r_1)}(M_3 M_4 r + M_5)dr_1\Big]$$
$$+ \int_0^t M_1 M_2 e^{(w_1+w_2 M_2\|B\|)(b-s)}(M_3 M_4 r + M_5)ds$$

By condition (k) and $k_t \leq k_b$ for $t < b$.

$$\|\phi X(t)\| \leq M_1 M_2 e^{(w_1+w_2 M_2\|BB^*Z\|)b}(L_2) + (L_1 r + L_2) + (L_1 r + L_2)k_1$$
$$+ k_b k_2 \Big[\|X_1\| + M_1 M_2 e^{(w_1+w_2 M_2\|BB^*Z\|)b}(L_2) + L_1\|X_1\| + L_2 + (L_1 r + L_2)k_1$$
$$+ \frac{M_1 M_2}{w_1 + w_2 + M_2\|BB^*Z\|} e^{(w_1+w_2 M_2\|BB^*Z\|)(b-r_1)}(M_3 M_4 r + M_5)\Big]$$
$$+ \frac{M_1 M_2}{w_1 + w_2 + M_2\|BB^*Z\|} e^{(w_1+w_2 M_2\|BB^*Z\|)(b-r_1)}(M_3 M_4 r + M_5) \leq r.$$

Thus, ϕ maps Q into itself, now for $X_1(t), X_2(t) \in Q$ and from definition of $\phi X(t)$ in (8), we have to show that ϕ is a contraction mapping, we get that

$$\|\phi X_1(t) - \phi x_2(t)\| = \|T(t)g(0, X_0) - g(t, X(t))$$
$$- \int_0^t g(s, X_1(s))((L+B).+.A)\mathcal{T}(t-s)ds$$

$$+\int_0^t \mathcal{T}(t-s)BB_2W^{-1}\left[\hat{T}(b)g(0,\theta)-g(b,X_1)-\int_0^b g(s,X(s))\right.$$
$$\left.((L+B).+.A)\mathcal{T}(b-s)ds+\int_0^b \hat{T}(b-s)F(s,N(s,X(s)))ds-X_1\right](s)ds$$
$$+\int_0^t \hat{T}(t-s)F(s,N(s,X_1(s)))ds-\mathcal{T}(t)g(0,X_0)-g(t,X_2(t))$$
$$-\int_0^t g(s,X_2(s))((L+B).+.A)\mathcal{T}(t-s)ds$$
$$-\int_0^t \mathcal{T}(t-s)BB_2W^{-1}\left[\mathcal{T}(b)g(0,\theta)-g(b,X_1)-\int_0^b g(s,X(s))\right.$$
$$\left.((L+B).+.A)\mathcal{T}(b-s)ds+\int_0^b \mathcal{T}(b-s)F(s,N(s,X(s)))ds-X_1\right](s)ds$$
$$-\int_0^b \mathcal{T}(b-s)F(s,N(s,X_2(s)))ds\|,$$
$$\|\phi X_1(t)-\phi X_2(t)\|\leq\|-g(t,X_1(t)+g(t,X_2(t))\|$$
$$+\|\int_0^t g(s,X_1(s)-g(s,X_2(s))((L+B).+.A)\mathcal{T}(t-s)ds\|$$
$$+\|\int_0^t \hat{T}(t-s)BB_2W^{-1}\left[\int_0^b g(r_1,X_1(r_1))-\right.$$
$$g(r_1,X_2(r_1))((L+B).+.A)\mathcal{T}(b-r_1)dr_1-\int_0^b \mathcal{T}(b-r_1)F(r_1,N(r_1,X_1(r_1)))-$$
$$\left.\mathcal{T}(b-r_1)F(r_1,N(r_1,X_2(r_1)))dr_1\right](s)ds\|\leq\|g(t,X_2(t))-g(t,X_1(t)\|$$
$$+\|\int_0^t g(s,X_1(s)-g(s,X_2(s))\|\|((L+B).+.A)\mathcal{T}(t-s)\|ds$$
$$+k_t\|W^{-1}\|\left[\int_0^b \|g(r_1,X_1(r_1))-g(r_1,X_2(r_1))\|\|((L+B).+.A)\right.$$
$$\left.\mathcal{T}(b-r_1)\|dr_1+\int_0^b \|\mathcal{T}(b-r_1)\|\|F(r_1,N(r_1,X_1(r_1)))\right.$$
$$\left.-F(r_1,N(r_1,X_2(r_1)))\|dr_1\right](s)ds\|+\int_0^t \|\mathcal{T}(t-s)\|L_1\|F(s,N(s,X_1(s)))$$
$$-F(s,N(s,X_2(s)))\|ds$$

By conditions (f) and (g), we obtain

$$\leq L_1\|X_1(t)-X_2(t)\|+\int_0^t L_1\|X_1(t)-X_2(t)v_0(s)ds$$
$$+k_1k_2\left[\int_0^b L_1\|X_1(r_1)-X_2(r_1)\right.$$

$$v_0(r_1)dr_1 + \int_0^b M_1M_2e^{(w_1+w_2+M_2\|B\|)(b-r_1)}M_3M_4\|X_1(r_1) - X_2(r_1)\|dr_1\Big]$$
$$+ \int_0^b M_1M_2e^{(w_1+w_2+M_2\|B\|)(t-s)}M_3M_4\|X_1(s) - X_2(s)\|ds$$

By condition $\int_0^t v_0(s)ds \leq k_1$, we obtain

$$\|\phi X_1(t) - \phi X_2(t)\| \leq L_1\|X_1(t) - X_2(t)\| + L_1\|X_1(s) - X_2(s)\|k_1 + k_bk_2$$
$$\Big[L_1\|X_1(r_1) - X_2(r_1)\|k_1 + \frac{M_1M_2}{(w_1 + w_2 + M_2\|B\|)}e^{(w_1+w_2+M_2\|B\|)b}M_3M_4$$
$$\|X_1(r_1)-X_2(r_1)\|] + \frac{M_1M_2}{(w_1 + w_2 + M_2\|B\|)}e^{(w_1+w_2+M_2\|B\|)b}M_3M_4\|X_1(t)-X_2(t)\|$$
$$\leq L_1\|X_1(t) - X_2(t)\| + L_1k_1\|X_1(t) - X_2(t)\| + k_bk_2k_1L_1\|X_1(t) - X_2(t)\| +$$
$$(k_bk_2 + 1)\frac{M_1M_2M_3M_4}{(w_1 + w_2 + M_2\|B\|)}e^{(w_1+w_2+M_2\|B\|)b}\|X_1(t) - X_2(t)\|$$
$$\leq (L_1 + L_1k_1 + k_bk_2k_1L_1 + (k_bk_2 + 1)\frac{M_1M_2M_3M_4}{(w_1 + w_2 + M_2\|B\|)}e^{(w_1+w_2+M_2\|B\|)b})$$
$$\cdot \|X_1(t) - X_2(t)\|$$

By condition (l), we obtain

$$\|\phi X_1(t) - \phi X_2(t)\| \leq \delta\|X_1(t) - X_2(t)\|.$$

Therefore ϕ is a contraction mapping. By theorem (2.5), there exist unique fixed point $X \in Q$ such that $\phi X(t) = X(t)$. Thus any fixed point of θ is a strongly continuous mild inequality solution of (4) on J, which satisfies $X(b) = X_1$. Hence the system is a boundary controllable on J. □

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Accepted: 15.11.2019

Some parameterized inequalities related to (s,m)-convexity and their applications

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Abstract. By introducing multi-parameters, the authors establish an integral identity for a mapping whose second derivative is integrable, present several integral inequalities of the Hadamard–Simpson type involving (s,m)-convexity, and apply these inequalities to the special means.

Keywords: Hermite-Hadamard inequality, Simpson inequality, (s,m)-convex functions.

1. Introduction

If $g:\mathcal{I}\subseteq\mathbb{R}\to\mathbb{R}$ is a convex mapping and $r_1,r_2\in\mathcal{I}$ with $r_1\neq r_2$, then one has

$$g\Big(\frac{r_1+r_2}{2}\Big)\leq\frac{1}{r_2-r_1}\int_{r_1}^{r_2}g(\tau)\mathrm{d}\tau\leq\frac{g(r_1)+g(r_2)}{2},$$

which is called as the Hermite–Hadamard's inequality in the literature.

The following inequality is named the Simpson's integral inequality:

$$\begin{aligned}&\left|\frac{1}{6}\left[g(r_1)+4g\Big(\frac{r_1+r_2}{2}\Big)+g(r_2)\right]-\frac{1}{r_2-r_1}\int_{r_1}^{r_2}g(\tau)\mathrm{d}\tau\right|\\&\leq\frac{1}{2880}\left\|g^{(4)}\right\|_\infty(r_2-r_1)^4,\end{aligned}$$

where $g:[r_1,r_2]\to\mathbb{R}$ is a four times continuously differentiable mapping on (r_1,r_2) and $\left\|g^{(4)}\right\|_\infty=\sup_{\tau\in(r_1,r_2)}\left|g^{(4)}(\tau)\right|<\infty$.

For recent results related to Hermite–Hadamard type and Simpson type inequalities, we refer to some studies by Awan et al. [1], Cristescu et al. [2], Dragomir [3], Khan et al. [8], Kórus [9], Latif [10], Mihai et al. [12], Sarikaya et al. [15], Tunç [18] and Wu et al. [19].

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In [6], Eftekhari contributed the following concept of (s,m)-convex mappings, which is a generalization of s-convexity [7] and m-convexity [17].

Definition 1.1. *For certain fixed $s \in (0,1]$ and $m \in (0,1]$, a mapping $g : \mathcal{I} \subseteq \mathbb{R}_0 \to \mathbb{R}$ is named (s,m)-convex if the following inequality*

$$g\big(\tau u + m(1-\tau)v\big) \le \tau^s g(u) + m(1-\tau)^s g(v)$$

holds for all $u, v \in \mathcal{I}$ and $\tau \in [0,1]$.

For some related results concerning (s,m)-convex mappings, we refer to studies by Du et al. [4] and [5].

In [20], by introducing a parameter, Xi and Qi established the following lemma.

Lemma 1.1. *Let $\mathcal{I} \subseteq \mathbb{R}$, $g : \mathcal{I} \to \mathbb{R}$ be a twice differentiable mapping on $\mathcal{I}^\circ$ ($\mathcal{I}^\circ$ is interior of $\mathcal{I}$) satisfying $g'' \in L^1([r_1, r_2])$, where $r_1, r_2 \in \mathcal{I}$ with $0 < r_1 < r_2$. Then, for any $\theta \in [0,1]$, the following equality holds:*

$$\begin{aligned}
&\frac{1}{r_2 - r_1}\int_{r_1}^{r_2} g(t)\mathrm{d}t - (1-\theta)g\Big(\frac{r_1+r_2}{2}\Big) - \theta\frac{g(r_1)+g(r_2)}{2} \\
&= \frac{(r_2-r_1)^2}{16}\Bigg[\int_0^1 \big(t^2 - 2\theta t\big)g''\Big(t\frac{r_1+r_2}{2} + (1-t)r_1\Big)\mathrm{d}t \\
&\quad + \int_0^1 \big(t^2 - 2\theta t\big)g''\Big(t\frac{r_1+r_2}{2} + (1-t)r_2\Big)\mathrm{d}t\Bigg].
\end{aligned}$$

Also Liu [11] considered the same Lemma 1.1 above and gave several upper bounds for mappings whose absolute values of the second derivatives are s-convex mappings.

In [16], Set et al. gave the following lemma and developed some Ostrowski type inequalities via s-convexity.

Lemma 1.2. *Let $\mathcal{I} \subseteq \mathbb{R}$, $g : \mathcal{I} \to \mathbb{R}$ be a twice differentiable mapping on $\mathcal{I}^\circ$ with $g'' \in L^1([r_1, r_2])$ where $r_1, r_2 \in \mathcal{I}$ with $r_1 < r_2$. Then for any $x \in [r_1, r_2]$, the following equality holds:*

$$\begin{aligned}
&\frac{1}{r_2 - r_1}\int_{r_1}^{r_2} g(u)\mathrm{d}u - g(x) + \left(x - \frac{r_1+r_2}{2}\right)g'(x) \\
&= \frac{(x-r_1)^3}{2(r_2-r_1)}\int_0^1 t^2 g''\big(tx + (1-t)r_1\big)\mathrm{d}t + \frac{(r_2-x)^3}{2(r_2-r_1)}\int_0^1 t^2 g''\big(tx + (1-t)r_2\big)\mathrm{d}t.
\end{aligned}$$

Inspired by the results mentioned above, especially the results developed in [16] and [20], we notice that it is possible to treat these results uniformly by introducing a multi-parameter identity. For this purpose, we will establish a general integral identity for differentiable mappings. Using this integral identity, we derive certain parameterized integral inequalities, which unify the midpoint inequalities, the Simpson inequalities, the averaged midpoint-trapezoid inequalities, and the trapezoid inequalities. This is the main contribution of this work.

2. Main results

In order to establish main results, we need the following lemma. Before we move further, for simplicity, we let $\Delta_1 = \frac{2x+(n-1)r_1}{n+1}$ and $\Delta_2 = \frac{2x+(n-1)r_2}{n+1}$.

Lemma 2.1. *Let $\mathcal{I} \subseteq \mathbb{R}$, $g : \mathcal{I} \to \mathbb{R}$ be a twice differentiable mapping on $\mathcal{I}^\circ$ satisfying $g'' \in L^1([r_1, r_2])$, where $r_1, r_2 \in \mathcal{I}$ with $0 < r_1 < r_2$. Then, for any $n \in N^*, \theta \in [0,1]$, the following equality holds:*

$$
\begin{aligned}
&\frac{1}{r_2 - r_1}\bigg[\int_{r_1}^{\Delta_1} g(t)\mathrm{d}t + \int_{\Delta_2}^{r_2} g(t)\mathrm{d}t\bigg] - \mathcal{T}(n,\theta,x;r_1,r_2) \\
&= \frac{4(x-r_1)^3}{(n+1)^3(r_2-r_1)}\int_0^1 \big(t^2 - 2\theta t\big)g''\Big(\frac{2t}{n+1}x + \frac{n+1-2t}{n+1}r_1\Big)\mathrm{d}t \\
&\quad + \frac{4(r_2-x)^3}{(n+1)^3(r_2-r_1)}\int_0^1 \big(t^2 - 2\theta t\big)g''\Big(\frac{2t}{n+1}x + \frac{n+1-2t}{n+1}r_2\Big)\mathrm{d}t,
\end{aligned} \tag{1}
$$

where

$$x = \zeta r_1 + (1-\zeta)r_2,\ \zeta \in [0,1]$$

and

$$
\begin{aligned}
&\mathcal{T}(n,\theta,x;r_1,r_2) \\
&= \frac{2}{(n+1)(r_2-r_1)}\theta\Big[(x-r_1)g(r_1) + (r_2-x)g(r_2)\Big] \\
&\quad + \frac{2}{(n+1)(r_2-r_1)}(1-\theta) \\
&\quad \times \left[(x-r_1)g\left(\frac{2x+(n-1)r_1}{n+1}\right) + (r_2-x)g\left(\frac{2x+(n-1)r_2}{n+1}\right)\right] \\
&\quad - \frac{2}{(n+1)^2(r_2-r_1)}\big(1-2\theta\big) \\
&\quad \times \left[(x-r_1)^2 g'\left(\frac{2x+(n-1)r_1}{n+1}\right) - (r_2-x)^2 g'\left(\frac{2x+(n-1)r_2}{n+1}\right)\right].
\end{aligned}
$$

Proof. Integrating by parts, we have the following identity:

$$
\begin{aligned}
I_1 &= \int_0^1 \big(t^2 - 2\theta t\big)g''\left(\frac{2t}{n+1}x + \frac{n+1-2t}{n+1}r_1\right)\mathrm{d}t \\
&= \frac{n+1}{2(x-r_1)}\big(t^2 - 2\theta t\big)g'\left(\frac{2t}{n+1}x + \frac{n+1-2t}{n+1}r_1\right)\Big|_0^1 \\
&\quad - \frac{n+1}{x-r_1}\int_0^1 \big(t-\theta\big)g'\left(\frac{2t}{n+1}x + \frac{n+1-2t}{n+1}r_1\right)\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{n+1}{2(x-r_1)}\big(1-2\theta\big)g'\bigg(\frac{2x+(n-1)r_1}{n+1}\bigg)\\
&\quad - \frac{(n+1)^2}{2(x-r_1)^2}\bigg[\big(t-\theta\big)g\bigg(\frac{2t}{n+1}x+\frac{n+1-2t}{n+1}r_1\bigg)\bigg|_0^1\\
&\quad - \int_0^1 g\bigg(\frac{2t}{n+1}x+\frac{n+1-2t}{n+1}r_1\bigg)\mathrm{d}t\bigg]\\
&= \frac{n+1}{2(x-r_1)}\big(1-2\theta\big)g'\bigg(\frac{2x+(n-1)r_1}{n+1}\bigg)\\
&\quad - \frac{(n+1)^2}{2(x-r_1)^2}\big(1-\theta\big)g\bigg(\frac{2x+(n-1)r_1}{n+1}\bigg) - \frac{(n+1)^2}{2(x-r_1)^2}\theta g(r_1)\\
&\quad + \frac{(n+1)^2}{2(x-r_1)^2}\int_0^1 g\left(\frac{2t}{n+1}x+\frac{n+1-2t}{n+1}r_1\right)\mathrm{d}t.
\end{aligned}
\tag{2}
$$

Using the change of variable $\tau = \frac{2t}{n+1}x + \frac{n+1-2t}{n+1}r_1$ for $t \in [0,1]$ and multiplying both sides of (2) by $\frac{4(x-r_1)^3}{(n+1)^3(r_2-r_1)}$, we obtain that

$$
\begin{aligned}
&\frac{4(x-r_1)^3}{(n+1)^3(r_2-r_1)}\int_0^1 \big(t^2-2\theta t\big)g''\bigg(\frac{2t}{n+1}x+\frac{n+1-2t}{n+1}r_1\bigg)\mathrm{d}t\\
&= \frac{2(x-r_1)^2}{(n+1)^2(r_2-r_1)}\big(1-2\theta\big)g'\bigg(\frac{2x+(n-1)r_1}{n+1}\bigg)\\
&\quad - \frac{2(x-r_1)}{(n+1)(r_2-r_1)}(1-\theta)g\bigg(\frac{2x+(n-1)r_1}{n+1}\bigg)\\
&\quad - \frac{2(x-r_1)}{(n+1)(r_2-r_1)}\theta g(r_1) + \frac{1}{r_2-r_1}\int_{r_1}^{\Delta_1} g(\tau)\mathrm{d}\tau.
\end{aligned}
\tag{3}
$$

Similarly, we observe that

$$
\begin{aligned}
&\frac{4(r_2-x)^3}{(n+1)^3(r_2-r_1)}\int_0^1 \big(t^2-2\theta t\big)g''\bigg(\frac{2t}{n+1}x+\frac{n+1-2t}{n+1}r_2\bigg)\mathrm{d}t\\
&= -\frac{2(r_2-x)^2}{(n+1)^2(r_2-r_1)}\big(1-2\theta\big)g'\bigg(\frac{2x+(n-1)r_2}{n+1}\bigg)\\
&\quad - \frac{2(r_2-x)}{(n+1)(r_2-r_1)}(1-\theta)g\bigg(\frac{2x+(n-1)r_2}{n+1}\bigg)\\
&\quad - \frac{2(r_2-x)}{(n+1)(r_2-r_1)}\theta g(r_2) + \frac{1}{r_2-r_1}\int_{\Delta_2}^{r_2} g(\tau)\mathrm{d}\tau.
\end{aligned}
\tag{4}
$$

Thus, adding (3) and (4), we get the required identity (1). □

Remark 2.1. Consider Lemma 2.1,

(i) Taking $x = \frac{r_1+r_2}{2}$ and $n = 1$, we have Lemma 2.1 given by Xi and Qi in [20].

(ii) Taking $\theta = 0$ and $n = 1$, we have Lemma 1 presented by Set et al. in [16]. Furthermore, if we take $x = \frac{r_1+r_2}{2}$ and use the change of the variable $u = 1-t$, for $t \in [0,1]$, then we have Lemma 1 presented by Özdemir in [13].

(iii) Taking $\theta = \frac{1}{2}, n = 1$ and $x = r_1$, we have Lemma 1 presented by Özdemir in [14].

Let us begin with the following Theorem.

Theorem 2.1. *Let $g : \mathcal{I} \subseteq \mathbb{R}_0 \to \mathbb{R}$ be a differentiable mapping on $\mathcal{I}^\circ$ satisfying $g'' \in L^1\big([r_1, r_2]\big)$. Suppose that $|g''|^q$ for $q \geq 1$ is an (s, m)-convex mapping on $\big[r_1, \frac{r_2}{m}\big]$ for certain fixed $s, m \in (0, 1]$, where $r_1, r_2, \frac{r_1}{m}, \frac{r_2}{m} \in \mathcal{I}$ along with $0 < r_1 < r_2$. If $n \in N^*, 0 \leq \theta \leq 1$ and $x = \zeta r_1 + (1-\zeta) r_2$ for $\zeta \in [0, 1]$, then we have the following results.*

(a) For $0 \leq \theta < \frac{1}{2}$, we have that

$$
\begin{aligned}
&\left| \frac{1}{r_2 - r_1} \left[\int_{r_1}^{\Delta_1} g(t)\mathrm{d}t + \int_{\Delta_2}^{r_2} g(t)\mathrm{d}t \right] - \mathcal{T}(n, \theta, x; r_1, r_2) \right| \\
&\leq \frac{4(x - r_1)^3}{(n+1)^3 (r_2 - r_1)} \left(\frac{8\theta^3 - 3\theta + 1}{3} \right)^{1 - \frac{1}{q}} \\
&\quad \times \left(\mathbb{A}_1(n, \theta; s) \big| g''(x) \big|^q + m \mathbb{A}_2(n, \theta; s) \Big| g''\Big(\frac{r_1}{m} \Big) \Big|^q \right)^{\frac{1}{q}} \\
&\quad + \frac{4(r_2 - x)^3}{(n+1)^3 (r_2 - r_1)} \left(\frac{8\theta^3 - 3\theta + 1}{3} \right)^{1 - \frac{1}{q}} \\
&\quad \times \left(\mathbb{A}_1(n, \theta; s) \big| g''(x) \big|^q + m \mathbb{A}_2(n, \theta; s) \Big| g''\Big(\frac{r_2}{m} \Big) \Big|^q \right)^{\frac{1}{q}},
\end{aligned} \tag{5}
$$

where

$$
\begin{aligned}
\mathbb{A}_1(n, \theta; s) &= \int_0^1 \big| t^2 - 2\theta t \big| \Big(\frac{2t}{n+1} \Big)^s \mathrm{d}t \\
&= \Big(\frac{2}{n+1} \Big)^s \frac{2(2\theta)^{s+3} - 2\theta(s+3) + s + 2}{(s+2)(s+3)}
\end{aligned} \tag{6}
$$

and

$$
\begin{aligned}
&\mathbb{A}_2(n, \theta; s) \\
&= \int_0^1 \big| t^2 - 2\theta t \big| \Big(\frac{n + 1 - 2t}{n+1} \Big)^s \mathrm{d}t \\
&= \frac{(n+1)^2}{8} \Bigg\{ (n+1) \frac{1}{s+3} \left[2 \Big(\frac{n + 1 - 4\theta}{n+1} \Big)^{s+3} - \Big(\frac{n-1}{n+1} \Big)^{s+3} - 1 \right] \\
&\quad - 2(n + 1 - 2\theta) \frac{1}{s+2} \left[2 \Big(\frac{n + 1 - 4\theta}{n+1} \Big)^{s+2} - \Big(\frac{n-1}{n+1} \Big)^{s+2} - 1 \right] \\
&\quad + (n + 1 - 4\theta) \frac{1}{s+1} \left[2 \Big(\frac{n + 1 - 4\theta}{n+1} \Big)^{s+1} - \Big(\frac{n-1}{n+1} \Big)^{s+1} - 1 \right] \Bigg\}.
\end{aligned} \tag{7}
$$

(*b*) *For* $\frac{1}{2} \le \theta \le 1$, *we have that*

$$
\begin{aligned}
&\left|\frac{1}{r_2-r_1}\left[\int_{r_1}^{\Delta_1} g(t)\mathrm{d}t + \int_{\Delta_2}^{r_2} g(t)\mathrm{d}t\right] - \mathcal{T}(n,\theta,x;r_1,r_2)\right| \\
&\le \frac{4(x-r_1)^3}{(n+1)^3(r_2-r_1)}\Big(\theta-\frac{1}{3}\Big)^{1-\frac{1}{q}}\Bigg(\mathbb{B}_1(n,\theta;s)\Big|g''(x)\Big|^q \\
&\quad + m\mathbb{B}_2(n,\theta;s)\Big|g''\Big(\frac{r_1}{m}\Big)\Big|^q\Bigg)^{\frac{1}{q}} \\
&\quad + \frac{4(r_2-x)^3}{(n+1)^3(r_2-r_1)}\Big(\theta-\frac{1}{3}\Big)^{1-\frac{1}{q}}\Bigg(\mathbb{B}_1(n,\theta;s)\Big|g''(x)\Big|^q \\
&\quad + m\mathbb{B}_2(n,\theta;s)\Big|g''\Big(\frac{r_2}{m}\Big)\Big|^q\Bigg)^{\frac{1}{q}},
\end{aligned} \tag{8}
$$

where

$$
\mathbb{B}_1(n,\theta;s) = \int_0^1 \big|t^2-2\theta t\big|\Big(\frac{2t}{n+1}\Big)^s\mathrm{d}t = \Big(\frac{2}{n+1}\Big)^s\left[\frac{2\theta}{s+2}-\frac{1}{s+3}\right] \tag{9}
$$

and

$$
\begin{aligned}
&\mathbb{B}_2(n,\theta;s) \\
&= \int_0^1 \big|t^2-2\theta t\big|\Big(\frac{n+1-2t}{n+1}\Big)^s\mathrm{d}t \\
&= -\frac{(n+1)^3}{8}\Bigg\{\frac{n+1-2\theta}{n+1}\frac{2}{s+2}\left[\Big(\frac{n-1}{n+1}\Big)^{s+2}-1\right] \\
&\quad - \frac{n+1-4\theta}{n+1}\frac{1}{s+1}\left[\Big(\frac{n-1}{n+1}\Big)^{s+1}-1\right] - \frac{1}{s+3}\left[\Big(\frac{n-1}{n+1}\Big)^{s+3}-1\right]\Bigg\}.
\end{aligned} \tag{10}
$$

Proof. If we use Lemma 2.1 and Hölder's inequality, then we have

$$
\begin{aligned}
&\left|\frac{1}{r_2-r_1}\left[\int_{r_1}^{\Delta_1} g(t)\mathrm{d}t + \int_{\Delta_2}^{r_2} g(t)\mathrm{d}t\right] - \mathcal{T}(n,\theta,x;r_1,r_2)\right| \\
&\le \frac{4(x-r_1)^3}{(n+1)^3(r_2-r_1)}\int_0^1 \big|t^2-2\theta t\big|\Big|g''\Big(\frac{2t}{n+1}x+\frac{n+1-2t}{n+1}r_1\Big)\Big|\mathrm{d}t \\
&\quad + \frac{4(r_2-x)^3}{(n+1)^3(r_2-r_1)}\int_0^1 \big|t^2-2\theta t\big|\Big|g''\Big(\frac{2t}{n+1}x+\frac{n+1-2t}{n+1}r_2\Big)\Big|\mathrm{d}t \\
&\le \frac{4(x-r_1)^3}{(n+1)^3(r_2-r_1)}\Big(\int_0^1 \big|t^2-2\theta t\big|\mathrm{d}t\Big)^{1-\frac{1}{q}} \\
&\quad \times \left[\int_0^1 \big|t^2-2\theta t\big|\Big|g''\Big(\frac{2t}{n+1}x+\frac{n+1-2t}{n+1}r_1\Big)\Big|^q\mathrm{d}t\right]^{\frac{1}{q}}
\end{aligned} \tag{11}
$$

$$+\frac{4(r_2-x)^3}{(n+1)^3(r_2-r_1)}\bigg(\int_0^1\big|t^2-2\theta t\big|\mathrm{d}t\bigg)^{1-\frac{1}{q}}\times\bigg[\int_0^1\big|t^2-2\theta t\big|\bigg|g''\Big(\frac{2t}{n+1}x+\frac{n+1-2t}{n+1}r_2\Big)\bigg|^q\mathrm{d}t\bigg]^{\frac{1}{q}}.$$

Since $\big|g''\big|^q$ is (s,m)-convex on $\big[r_1,\frac{r_2}{m}\big]$, for any $t\in[0,1]$, we have

$$\begin{aligned}&\bigg|g''\Big(\frac{2t}{n+1}x+\frac{n+1-2t}{n+1}r_1\Big)\bigg|^q\\&\quad\le\Big(\frac{2t}{n+1}\Big)^s\big|g''(x)\big|^q+m\bigg(\frac{n+1-2t}{n+1}\bigg)^s\Big|g''\Big(\frac{r_1}{m}\Big)\Big|^q\end{aligned}\tag{12}$$

and

$$\begin{aligned}&\bigg|g''\Big(\frac{2t}{n+1}x+\frac{n+1-2t}{n+1}r_2\Big)\bigg|^q\\&\quad\le\Big(\frac{2t}{n+1}\Big)^s\big|g''(x)\big|^q+m\bigg(\frac{n+1-2t}{n+1}\bigg)^s\Big|g''\Big(\frac{r_2}{m}\Big)\Big|^q.\end{aligned}\tag{13}$$

Using (12) and (13) in (11), we have

$$\begin{aligned}&\bigg|\frac{1}{r_2-r_1}\bigg[\int_{r_1}^{\Delta_1}g(t)\mathrm{d}t+\int_{\Delta_2}^{r_2}g(t)\mathrm{d}t\bigg]-\mathcal{T}(n,\theta,x;r_1,r_2)\bigg|\\&\le\frac{4(x-r_1)^3}{(n+1)^3(r_2-r_1)}\bigg(\int_0^1\big|t^2-2\theta t\big|\mathrm{d}t\bigg)^{1-\frac{1}{q}}\\&\quad\times\bigg(\int_0^1\big|t^2-2\theta t\big|\bigg[\Big(\frac{2t}{n+1}\Big)^s\big|g''(x)\big|^q+m\bigg(\frac{n+1-2t}{n+1}\bigg)^s\Big|g''\Big(\frac{r_1}{m}\Big)\Big|^q\bigg]\mathrm{d}t\bigg)^{\frac{1}{q}}\\&\quad+\frac{4(r_2-x)^3}{(n+1)^3(r_2-r_1)}\bigg(\int_0^1\big|t^2-2\theta t\big|\mathrm{d}t\bigg)^{1-\frac{1}{q}}\\&\quad\times\bigg(\int_0^1\big|t^2-2\theta t\big|\bigg[\Big(\frac{2t}{n+1}\Big)^s\big|g''(x)\big|^q+m\bigg(\frac{n+1-2t}{n+1}\bigg)^s\Big|g''\Big(\frac{r_2}{m}\Big)\Big|^q\bigg]\mathrm{d}t\bigg)^{\frac{1}{q}}.\end{aligned}$$

By simple computation, we see that the desired inequalities hold. The proof is thus completed. □

Corollary 2.1. *If we take $q=1$ and $\mathcal{M}=\sup_{x\in\left[r_1,\frac{r_2}{m}\right]}|g''|$ in Theorem 2.1, then we have*

$$\bigg|\frac{1}{r_2-r_1}\bigg[\int_{r_1}^{\Delta_1}g(t)\mathrm{d}t+\int_{\Delta_2}^{r_2}g(t)\mathrm{d}t\bigg]-\mathcal{T}(n,\theta,x;r_1,r_2)\bigg|\tag{14}$$

$$\leq \begin{cases} 4\mathcal{M}\frac{(x-r_1)^3+(r_2-x)^3}{(n+1)^3(r_2-r_1)}\Big[\mathbb{A}_1(n,\theta;s)+m\mathbb{A}_2(n,\theta;s)\Big], & 0\leq\theta<\frac{1}{2},\\ 4\mathcal{M}\frac{(x-r_1)^3+(r_2-x)^3}{(n+1)^3(r_2-r_1)}\Big[\mathbb{B}_1(n,\theta;s)+m\mathbb{B}_2(n,\theta;s)\Big], & \frac{1}{2}\leq\theta\leq 1, \end{cases}$$

where $\mathbb{A}_i(n,\theta;s)$ *and* $\mathbb{B}_i(n,\theta;s), i=1,2$ *are defined by (6), (7), (9) and (10) in Theorem 2.1, respectively.*

Remark 2.2. In Theorem 2.1, if we take $\theta=0$, $n=1=m$ in inequality (5), then we achieve inequality (2.9) in Theorem 6 established by Set et al. in [16].

Remark 2.3. In Theorem 2.1, if we take $n=1=m, x=\frac{r_1+r_2}{2}$ in inequalities (5) and (8), respectively, then we achieve inequalities (7) and (8) in Theorem 3 established by Liu in [11].

Theorem 2.2. *Suppose that all assumptions of Theorem 2.1 are satisfied. Then, for* $x=\frac{r_1+r_2}{2}$*, we have the following results.*

(*a*) *For* $0\leq\theta<\frac{1}{2}$*, we have that*

$$\begin{aligned} &\Big|\frac{1}{r_2-r_1}\Big[\int_{r_1}^{\Delta_1}g(t)\mathrm{d}t+\int_{\Delta_2}^{r_2}g(t)\mathrm{d}t\Big]-\mathcal{T}\Big(n,\theta,\frac{r_1+r_2}{2};r_1,r_2\Big)\Big|\\ &\leq\frac{(r_2-r_1)^2}{2(n+1)^3}\Big(\frac{8\theta^3-3\theta+1}{3}\Big)^{1-\frac{1}{q}}\\ &\quad\times\Bigg[\Big(\mathbb{A}_3(n,\theta;s)\big|g''(r_2)\big|^q+m\mathbb{A}_4(n,\theta;s)\Big|g''\Big(\frac{r_1}{m}\Big)\Big|^q\Big)^{\frac{1}{q}}\\ &\quad+\Big(\mathbb{A}_3(n,\theta;s)\big|g''(r_1)\big|^q+m\mathbb{A}_4(n,\theta;s)\Big|g''\Big(\frac{r_2}{m}\Big)\Big|^q\Big)^{\frac{1}{q}}\Bigg], \end{aligned}\tag{15}$$

where

$$\begin{aligned} \mathbb{A}_3(n,\theta;s)&=\int_0^1\big|t^2-2\theta t\big|\Big(\frac{t}{n+1}\Big)^s\mathrm{d}t\\ &=\Big(\frac{1}{n+1}\Big)^s\frac{2(2\theta)^{s+3}-2\theta(s+3)+s+2}{(s+2)(s+3)} \end{aligned}\tag{16}$$

and

$$\begin{aligned} &\mathbb{A}_4(n,\theta;s)\\ &=\int_0^1\big|t^2-2\theta t\big|\Big(\frac{n+1-t}{n+1}\Big)^s\mathrm{d}t\\ &=(-1)(n+1)^3\Bigg\{\frac{n+1-2\theta}{(s+1)(n+1)}\Big[1+\Big(\frac{n}{n+1}\Big)^{s+1}-2\Big(\frac{n+1-2\theta}{n+1}\Big)^{s+1}\Big]\\ &\quad-\frac{2(n+1-\theta)}{(s+2)(n+1)}\Big[1+\Big(\frac{n}{n+1}\Big)^{s+2}-2\Big(\frac{n+1-2\theta}{n+1}\Big)^{s+2}\Big]\\ &\quad+\frac{1}{s+3}\Big[1+\Big(\frac{n}{n+1}\Big)^{s+3}-2\Big(\frac{n+1-2\theta}{n+1}\Big)^{s+3}\Big]\Bigg\}. \end{aligned}\tag{17}$$

(*b*) *For* $\frac{1}{2} \le \theta \le 1$, *we have that*

$$
\begin{aligned}
&\left|\frac{1}{r_2-r_1}\left[\int_{r_1}^{\Delta_1} g(t)\mathrm{d}t + \int_{\Delta_2}^{r_2} g(t)\mathrm{d}t\right] - \mathcal{T}\left(n,\theta,\frac{r_1+r_2}{2};r_1,r_2\right)\right| \\
&\le \frac{(r_2-r_1)^2}{2(n+1)^3}\left(\theta-\frac{1}{3}\right)^{1-\frac{1}{q}}\left[\left(\mathbb{B}_3(n,\theta;s)\left|g''(r_2)\right|^q + m\mathbb{B}_4(n,\theta;s)\left|g''\left(\frac{r_1}{m}\right)\right|^q\right)^{\frac{1}{q}}\right. \\
&\quad \left. + \left(\mathbb{B}_3(n,\theta;s)\left|g''(r_1)\right|^q + m\mathbb{B}_4(n,\theta;s)\left|g''\left(\frac{r_2}{m}\right)\right|^q\right)^{\frac{1}{q}}\right],
\end{aligned} \tag{18}
$$

where

$$
\begin{aligned}
\mathbb{B}_3(n,\theta;s) &= \int_0^1 \left|t^2-2\theta t\right|\left(\frac{t}{n+1}\right)^s \mathrm{d}t \\
&= \left(\frac{1}{n+1}\right)^s\left[\frac{2\theta}{s+2} - \frac{1}{s+3}\right]
\end{aligned} \tag{19}
$$

and

$$
\begin{aligned}
&\mathbb{B}_4(n,\theta;s) \\
&= \int_0^1 \left|t^2-2\theta t\right|\left(\frac{n+1-t}{n+1}\right)^s \mathrm{d}t \\
&= (n+1)^3\left\{\frac{1}{s+3}\left(\frac{n}{n+1}\right)^{s+3} - \frac{2}{s+2}\left(\frac{n}{n+1}\right)^{s+2}\right. \\
&\left. + \frac{1}{s+1}\left(\frac{n}{n+1}\right)^{s+1} - \frac{1}{s+3} + \frac{2}{s+2} - \frac{1}{s+1}\right\} \\
&+ 2\theta(n+1)^2\left\{\frac{1}{s+2}\left(\frac{n}{n+1}\right)^{s+2} - \frac{1}{s+1}\left(\frac{n}{n+1}\right)^{s+1} - \frac{1}{s+2} + \frac{1}{s+1}\right\}.
\end{aligned} \tag{20}
$$

Proof. Noting that

$$
\frac{2t}{n+1}\frac{r_1+r_2}{2} + \frac{n+1-2t}{n+1}r_1 = \frac{t}{n+1}r_2 + (1-\frac{t}{n+1})r_1
$$

and

$$
\frac{2t}{n+1}\frac{r_1+r_2}{2} + \frac{n+1-2t}{n+1}r_2 = \frac{t}{n+1}r_1 + (1-\frac{t}{n+1})r_2,
$$

the proofs of (15) and (18) follow using the same approach as used in Theorem 2.1. □

Corollary 2.2. *If we take $q=1$ and $\mathcal{M}=\sup_{x\in[r_1,\frac{r_2}{m}]}|g''|$ in Theorem 2.2, then we have*

$$\begin{aligned}&\left|\frac{1}{r_2-r_1}\left[\int_{r_1}^{\Delta_1}g(t)\mathrm{d}t+\int_{\Delta_2}^{r_2}g(t)\mathrm{d}t\right]-\mathcal{T}\Big(n,\theta,\frac{r_1+r_2}{2};r_1,r_2\Big)\right|\\ &\leq\begin{cases}\mathcal{M}\frac{(r_2-r_1)^2}{(n+1)^3}\Big(\mathbb{A}_3(n,\theta;s)+m\mathbb{A}_4(n,\theta;s)\Big), & 0\leq\theta<\frac{1}{2},\\ \mathcal{M}\frac{(r_2-r_1)^2}{(n+1)^3}\Big(\mathbb{B}_3(n,\theta;s)+m\mathbb{B}_4(n,\theta;s)\Big), & \frac{1}{2}\leq\theta\leq 1,\end{cases}\end{aligned}\tag{21}$$

where $\mathbb{A}_i(n,\theta;s), i=3,4$ and $\mathbb{B}_i(n,\theta;s), i=3,4$ are defined by (16), (17), (19) and (20) in Theorem 2.2, respectively.

In the next theorem, we will use the following functions.

(I) The Beta function:

$$\beta(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}=\int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t,\quad x,y>0.$$

(II) The incomplete Beta function:

$$\beta(a;x,y)=\int_0^a t^{x-1}(1-t)^{y-1}\mathrm{d}t,\quad 0<a<1,\ x,y>0.$$

(III) The hypergeometric function:

$$\begin{aligned}&{}_2F_1(a,b;c;z)\\ &=\frac{1}{\beta(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\mathrm{d}t,\quad c>b>0,|z|<1.\end{aligned}$$

Theorem 2.3. *Let $g:\mathcal{I}\subseteq\mathbb{R}_0\to\mathbb{R}$ be a differentiable mapping on $\mathcal{I}^\circ$ satisfying $g''\in L^1\big([r_1,r_2]\big)$. Suppose that $|g''|^q$ with $q>1$ is (s,m)-convex on $\big[r_1,\frac{r_2}{m}\big]$ for certain fixed $s,m\in(0,1]$, where $r_1,r_2,\frac{r_1}{m},\frac{r_2}{m}\in\mathcal{I}$ with $0<r_1<r_2$ and $p^{-1}+q^{-1}=1$. If $n\in N^*, 0\leq\theta\leq 1$ and $x=\frac{r_1+r_2}{2}$, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{1}{r_2-r_1}\left[\int_{r_1}^{\Delta_1}g(t)\mathrm{d}t+\int_{\Delta_2}^{r_2}g(t)\mathrm{d}t\right]-\mathcal{T}\Big(n,\theta,\frac{r_1+r_2}{2};r_1,r_2\Big)\right|\\ &\leq\frac{(r_2-r_1)^2}{2(n+1)^3}\Big(\mathbb{W}(\theta,p)\Big)^{\frac{1}{p}}\Bigg\{\left[\Big(\frac{1}{n+1}\Big)^s\frac{1}{s+1}\big|g''(r_2)\big|^q\right.\\ &+\frac{m(n+1)}{s+1}\left[1-\Big(\frac{n}{n+1}\Big)^{s+1}\right]\Big|g''\Big(\frac{r_1}{m}\Big)\Big|^q\Bigg]^{\frac{1}{q}}\\ &+\left[\Big(\frac{1}{n+1}\Big)^s\frac{1}{s+1}\big|g''(r_1)\big|^q+\frac{m(n+1)}{s+1}\left[1-\Big(\frac{n}{n+1}\Big)^{s+1}\right]\Big|g''\Big(\frac{r_2}{m}\Big)\Big|^q\right]^{\frac{1}{q}}\Bigg\},\end{aligned}\tag{22}$$

where

$$\mathbb{W}(\theta,p)=\int_0^1\left|t^2-2\theta t\right|^p\mathrm{d}t$$

$$=\begin{cases}\dfrac{1}{2p+1}, & \theta=0,\\[2ex] \left[\begin{array}{l}(2\theta)^{2p+1}\beta(p+1,p+1)\\ +\dfrac{(2\theta)^p}{p+1}(1-2\theta)^{p+1}{}_2F_1\big(-p,p+1;p+2;1-\dfrac{1}{2\theta}\big)\end{array}\right], & 0<\theta<\frac{1}{2},\\[2ex] \dfrac{(2\theta)^p}{p+1}{}_2F_1\big(-p,p+1;p+2;\dfrac{1}{2\theta}\big), & \frac{1}{2}\le\theta\le 1.\end{cases}\tag{23}$$

Proof. Similar to the proof of Theorem 2.2. □

Corollary 2.3. *If we take $n=1$ and $\theta=0$ in Theorem* 2.3*, then we get a midpoint type inequality*

$$\begin{aligned}&\left|\frac{1}{r_2-r_1}\int_{r_1}^{r_2}g(t)\mathrm{d}t-g\Big(\frac{r_1+r_2}{2}\Big)\right|\\ &\le\frac{(r_2-r_1)^2}{16}\Big(\frac{1}{2p+1}\Big)^{\frac{1}{p}}\Bigg\{\left[\Big(\frac{1}{2}\Big)^s\frac{1}{s+1}\left|g''(r_2)\right|^q+\frac{2m}{s+1}\left[1-\Big(\frac{1}{2}\Big)^{s+1}\right]\left|g''\Big(\frac{r_1}{m}\Big)\right|^q\right]^{\frac{1}{q}}\\ &\quad+\left[\Big(\frac{1}{2}\Big)^s\frac{1}{s+1}\left|g''(r_1)\right|^q+\frac{2m}{s+1}\left[1-\Big(\frac{1}{2}\Big)^{s+1}\right]\left|g''\Big(\frac{r_2}{m}\Big)\right|^q\right]^{\frac{1}{q}}\Bigg\}\\ &\le\frac{(r_2-r_1)^2}{16}\Big(\frac{1}{2p+1}\Big)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\Bigg[\Big(\frac{1}{2}\Big)^{\frac{s}{q}}\Big(\left|g''(r_1)\right|+\left|g''(r_2)\right|\Big)\\ &\quad+(2m)^{\frac{1}{q}}\left(1-\Big(\frac{1}{2}\Big)^{s+1}\right)^{\frac{1}{q}}\Big(\left|g''\Big(\frac{r_1}{m}\Big)\right|+\left|g''\Big(\frac{r_2}{m}\Big)\right|\Big)\Bigg].\end{aligned}$$

Here, $0<\frac{1}{q}<1$ for $q>1$. To prove the second inequality above, we use the fact that

$$\sum_{i=1}^n(u_i+v_i)^\gamma\le\sum_{i=1}^n u_i^\gamma+\sum_{i=1}^n v_i^\gamma,$$

for $0\le\gamma\le 1$, $u_1,u_2,\ldots,u_n\ge 0$ and $v_1,v_2,\ldots,v_n\ge 0$.

Remark 2.4. For $n=1$, if we take $\theta=\frac{1}{3},\frac{1}{2},1$ in Theorem 2.3, respectively, then we have several integral inequalities, for instance, on Simpson-type, averaged midpoint-trapezoid, and trapezoid-like integral inequalities.

3. Comparison of the results

Now we compare the results of Corollary 2.1 and Corollary 2.2 with $x = \frac{r_1+r_2}{2}, m = 1$ and $s = 0.4$. Let bounds in the second inequality of (14) and (21), respectively, be denoted by $E_1(n,\theta)$, $E_2(n,\theta)$, that is

$$E_1(n,\theta) = \mathbb{B}_1(n,\theta;0.4) + \mathbb{B}_2(n,\theta;0.4)$$

and

$$E_2(n,\theta) = \mathbb{B}_3(n,\theta;0.4) + \mathbb{B}_4(n,\theta;0.4),$$

where $\mathbb{B}_i(n,\theta;s)$, $i = 1,2,3,4$ are defined by (9), (10), (19) and (20), respectively. We have omitted $4\mathcal{M}\frac{(x-r_1)^3+(r_2-x)^3}{(n+1)^3(r_2-r_1)}$ and $\mathcal{M}\frac{(r_2-r_1)^2}{(n+1)^3}$, since they are equal. Fig. 1(a) and (b) show that $E_2(n,\theta)$ is a better error bound than $E_1(n,\theta)$. In this case, the second inequality of (21) is better than the second inequality of (14).

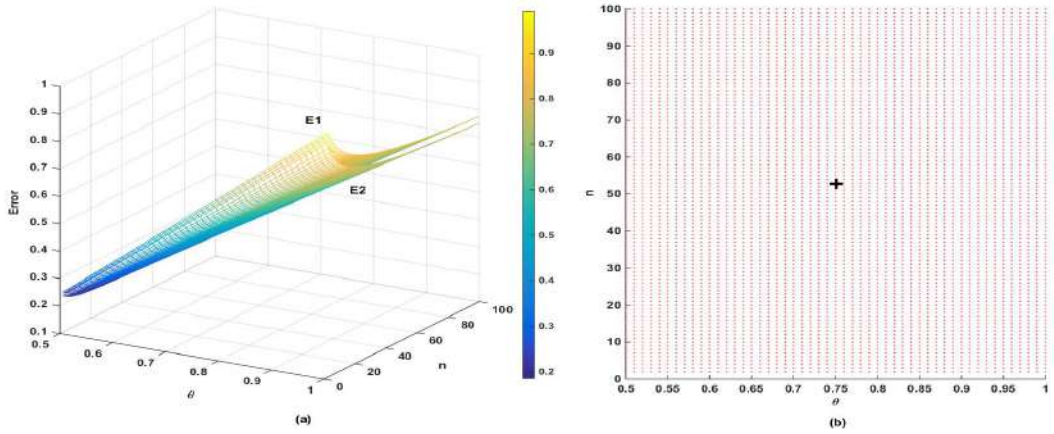


Figure 1: *
(a) Error surface of E_1 and E_2 with respect to the variables n and θ.
(b) The red part describes $E_1 - E_2 > 0$.

4. Application

For positive number $r_2 > r_1 > 0$ and $n \in \mathbb{Z} \setminus \{-1, 0\}$, we define

(I) The arithmetic mean: $\mathcal{A}(r_1,r_2) = \frac{r_1+r_2}{2}$.

(II) The generalized logarithmic mean: $\mathcal{L}_n(r_1,r_2) = \left[\frac{r_2^{n+1}-r_1^{n+1}}{(n+1)(r_2-r_1)}\right]^{\frac{1}{n}}$.

(III) The the identric mean: $\mathcal{I}(r_1,r_2) = \frac{1}{e}\left(\frac{(r_2)^{r_2}}{(r_1)^{r_1}}\right)^{\frac{1}{r_2-r_1}}$. We observe that $\ln \mathcal{I}(r_1,r_2) = \frac{1}{r_2-r_1}\int_{r_1}^{r_2} \ln u du$.

Proposition 4.1. *If we take $s = 1$ in Theorem 2.3, then we have the following inequality:*

$$
\begin{aligned}
&\Bigg| - \ln \mathcal{I}\big(r_1, r_2\big) + \frac{n-1}{n+1} \ln \mathcal{I}\big(\varphi_1, \varphi_2\big) + \frac{2\theta}{n+1} \mathcal{A}\big(\ln r_1, \ln r_2\big) \\
&\quad + \frac{2(1-\theta)}{n+1} \mathcal{A}\big(\ln \varphi_1, \ln \varphi_2\big) - \frac{(r_2 - r_1)(1-2\theta)}{2(n+1)^2} \left(\frac{1}{\varphi_1} - \frac{1}{\varphi_2} \right) \Bigg| \\
&\leq \frac{(r_2 - r_1)^2}{2(n+1)^3} \Big(\mathbb{W}(\theta, p) \Big)^{\frac{1}{p}} \Bigg\{ \left[\frac{1}{2(n+1)} \left(\frac{1}{r_2} \right)^{2q} + \frac{(2n+1)m}{2(n+1)} \left(\frac{m}{r_1} \right)^{2q} \right]^{\frac{1}{q}} \\
&\quad + \left[\frac{1}{2(n+1)} \left(\frac{1}{r_1} \right)^{2q} + \frac{(2n+1)m}{2(n+1)} \left(\frac{m}{r_2} \right)^{2q} \right]^{\frac{1}{q}} \Bigg\},
\end{aligned} \tag{24}
$$

where

$$
\varphi_1 = \frac{n r_1 + r_2}{n+1}, \tag{25}
$$

$$
\varphi_2 = \frac{r_1 + n r_2}{n+1}, \tag{26}
$$

and $\mathbb{W}(\theta, p)$ is defined by (23).

Proof. The assertion follows from applying inequality (22) with $s = 1$ to the mapping $g(x) = -\ln x$, $x \in [r_1, r_2]$ with $0 \leq r_1 < r_2$ which implies that $|g''(x)|^q = \frac{1}{x^{2q}}$ is convex for $q > 1$ on $[r_1, r_2]$. □

Proposition 4.2. *Let $0 < r_1 < r_2$, $-1 < sq \leq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$. Then, we have*

$$
\begin{aligned}
&\Big| \mathcal{A}^{s+2}(r_1, r_2) - \mathcal{L}_{s+2}^{s+2}(r_1, r_2) \Big| \\
&\leq \frac{(r_2 - r_1)^2}{16} \Big(\frac{1}{2p+1} \Big)^{\frac{1}{p}} (s+1)^{1-\frac{1}{q}} (s+2) \Bigg\{ \left[\left(\frac{1}{2} \right)^s r_2^{sq} + 2 \left[1 - \left(\frac{1}{2} \right)^{s+1} \right] r_1^{sq} \right]^{\frac{1}{q}} \\
&\quad + \left[\left(\frac{1}{2} \right)^s r_1^{sq} + 2 \left[1 - \left(\frac{1}{2} \right)^{s+1} \right] r_2^{sq} \right]^{\frac{1}{q}} \Bigg\} \\
&\leq \frac{(r_2 - r_1)^2}{8} \Big(\frac{1}{2p+1} \Big)^{\frac{1}{p}} (s+1)^{1-\frac{1}{q}} (s+2) \left[\left(\frac{1}{2} \right)^{\frac{s}{q}} + 2^{\frac{1}{q}} \left(1 - \left(\frac{1}{2} \right)^{s+1} \right)^{\frac{1}{q}} \right] \mathcal{A}\big(r_1^s, r_2^s\big).
\end{aligned} \tag{27}
$$

Proof. These assertions follow from applying the result in Corollary 2.3 with $m = 1$ to the mapping $g(x) = x^{s+2}$, $x \in [r_1, r_2]$ with $0 < r_1 < r_2$, which implies that $\big|g''(x)\big|^q = \big|(s+2)(s+1)x^s\big|^q$ is s-convex on $[r_1, r_2]$. □

Remark 4.1. If we take $s=1$ in the second inequality of Proposition 4.2, then we have

$$\begin{aligned}&\left|\mathcal{A}^3(r_1,r_2)-\mathcal{L}_3^3(r_1,r_2)\right|\\&\leq\frac{3(r_2-r_1)^2}{16}\left[\Big(\frac{4}{2p+1}\Big)^{\frac{1}{p}}+3\Big(\frac{4}{3(2p+1)}\Big)^{\frac{1}{p}}\right]\mathcal{A}\big(r_1,r_2\big).\end{aligned}$$

Note that the left-hand side of the inequality above does not depend on p, so we can conclude that

$$\begin{aligned}&\left|\mathcal{A}^3(r_1,r_2)-\mathcal{L}_3^3(r_1,r_2)\right|\\&\leq\frac{3(r_2-r_1)^2}{16}\inf_{p>1}\left[\Big(\frac{4}{2p+1}\Big)^{\frac{1}{p}}+3\Big(\frac{4}{3(2p+1)}\Big)^{\frac{1}{p}}\right]\mathcal{A}\big(r_1,r_2\big),\end{aligned}$$

for any $p>1$, $r_1<r_2$.

Let

$$f(p)=\Big(\frac{4}{2p+1}\Big)^{\frac{1}{p}}+3\Big(\frac{4}{3(2p+1)}\Big)^{\frac{1}{p}},$$

for $p\geq 1$. Then, we have that

$$f'(p)=\left(\frac{4}{2p+1}\right)^{\frac{1}{p}}\left(-\frac{1}{p^2}\right)\left[\ln\frac{4}{2p+1}+\frac{2p}{2p+1}+3^{1-\frac{1}{p}}\left(\ln\frac{4}{3(2p+1)}+\frac{2p}{2p+1}\right)\right].$$

Let

$$h_1(p)=\ln\frac{4}{2p+1}+\frac{2p}{2p+1}+3^{1-\frac{1}{p}}\left(\ln\frac{4}{3(2p+1)}+\frac{2p}{2p+1}\right).$$

Then, we have that

$$h_1'(p)=(1+3^{1-\frac{1}{p}})\left(\frac{-4p}{(2p+1)^2}\right)+\frac{3\ln 3}{p^2\cdot 3^{\frac{1}{p}}}\left[\ln\frac{4}{3(2p+1)}+\frac{2p}{2p+1}\right].$$

Let

$$h_2(p)=\ln\frac{4}{3(2p+1)}+\frac{2p}{2p+1}.$$

Then we have that

$$h_2'(p)=\frac{-4p}{(2p+1)^2}.$$

So, $h_2(p)$ decreases.

Since

$$\lim_{p\to\infty}h_2(p)=-\infty$$

and

$$\lim_{p\to 1^+} h_2(p) = \ln\frac{4}{9} + \frac{2}{3} < 0,$$

we have that $h_2(p) < 0$ for $p \geq 1$. Hence, we have that $h_1'(p) < 0$ for $p \geq 1$, so $h_1(p)$ decreases.

Since

$$\lim_{p\to\infty} h_1(p) = -\infty$$

and

$$\lim_{p\to 1^+} h_1(p) = \ln\frac{16}{27} + \frac{4}{3} > 0,$$

we have that $h_1(p) > 0$ for $(1, p_0)$ and $h_1(p) < 0$ for (p_0, ∞), where p_0 is an unique solution of equation

$$\ln\frac{4}{2p+1} + \frac{2p}{2p+1} + 3^{1-\frac{1}{p}}\left(\ln\frac{4}{3(2p+1)} + \frac{2p}{2p+1}\right) = 0.$$

Hence, the function $f(p)$ decreases on $(1, p_0)$ and increases on (p_0, ∞). For $p > 1$, we conclude that

$$\begin{aligned}&\left|\mathcal{A}^{s+2}(r_1, r_2) - \mathcal{L}_{s+2}^{s+2}(r_1, r_2)\right| \\ &\leq \frac{3(r_2 - r_1)^2}{16}\left[\Big(\frac{4}{2p_0+1}\Big)^{\frac{1}{p_0}} + 3\Big(\frac{4}{3(2p_0+1)}\Big)^{\frac{1}{p_0}}\right]\mathcal{A}\big(r_1, r_2\big).\end{aligned}$$

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Accepted: 07.06.2020

A comment on weakly πg-closed sets in topological spaces

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Abstract. In this paper, we show that the notions of πgp-closed sets and weakly πg-closed sets are equivalent.
Keywords: π-open sets, preclosed sets, πgp-closed sets, weakly πg-closed sets.

1. Introduction

Mashhour et al. [2] introduced the notion of preopen sets and obtained their properties. In 1968, Zaitsav [5] introduced the concept of π-open sets in topological spaces. Many results had been obtained by using the concept of π-open sets.

In 2004, J. H Park [3] introduced and studied the concept of πgp-closed sets. Recently, O. Ravi et al. [4] discussed and established the concept of weakly πg-closed sets in topological spaces.

In this paper, we show that the concept of πgp-closed set and a weakly πg-closed set are same.

2. Preliminaries

Definition 2.1. *Let (X, τ) be a topological space. A subset A of X is said to be preclosed* [2] *if $cl(int(A)) \subseteq A$.*

Definition 2.2 ([5])**.** *Let (X, τ) be a topological space. A subset A of X is said to be π-open if A is the finite union of regular open sets.*

Definition 2.3 ([3])**.** *Let (X, τ) be a topological space. A subset A of X is said to be πgp-closed if pcl(A)$\subseteq G$ whenever $A \subseteq U$ and U is π-open in X.*

Definition 2.4 ([4])**.** *Let (X, τ) be a topological space. A subset A of X is said to be weakly πg-closed if $cl(int(A)) \subseteq G$ whenever $A \subseteq U$ and U is π-open in X.*

Lemma 2.1 ([1])**.** *Let (X, τ) be a topological space and $A \subseteq X$. Then, $pcl(A) = A \cup cl(int(A))$.*

3. Main result

Theorem 3.1. *Let (X, τ) be a topological space and $A \subseteq X$. Then, the following are equivalent:*

(i) A is πgp-closed.

(ii) A is weakly πg-closed.

Proof. (i)⇒(ii). Let $A \subseteq X$ be a πgp-closed set and U be an π-open set containing A. By Definition 2.3, pcl(A)⊆U. Using Lemma 2.5, A∪cl(int(A))⊆U which implies cl(int(A))⊆U.

This shows that A is weakly πg-closed.

(ii)⇒(i). Let A⊆X be a weakly πg-closed set such that A⊆ V and V is π-open set in X. Then by Definition 2.4, cl(int(A))⊆V. Using Lemma 2.1, we have pcl(A)=A∪cl(int(A))⊆A∪V=V.

Therefore, pcl(A)⊆V. Hence, A is πgp-closed. □

4. Illustrative examples

In this section, we give some examples to confirm our claim.

Example 4.1. Let X = {a,b,c} and τ = {X, ϕ, {a}, {b}, {a,b}, {a,c}}, then

$$\begin{aligned} O(X) &= \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}, \\ C(X) &= \{X, \phi, \{b\}, \{c\}, \{b, c\}, \{a, c\}\}, \\ \pi O(X) &= \{X, \phi, \{b\}, \{a, c\}\}, \\ \pi GPC(X) &= P(X) = W\pi GC(X). \end{aligned}$$

Therefore, πGPC(X) = Wπ GC(X).

Example 4.2. Let X = {a,b,c} and τ = {X, ϕ, {a}, {b,c}}, then

$$\begin{aligned} O(X) &= \{X, \phi, \{a\}, \{b, c\}\}, \\ C(X) &= \{X, \phi, \{a\}, \{b, c\}\}, \\ \pi O(X) &= \{X, \phi, \{a\}, \{b, c\}\}, \pi GPC(X) = P(X) = W\pi GC(X). \end{aligned}$$

Therefore, πGPC(X) = Wπ GC(X).

Example 4.3. Let X = {a,b,c,d} and τ = {X, ϕ, {a}, {b}, {a,b}, {a,b,c}}, then O(X) = {X, ϕ, {a}, {b}, {a,b}, {a,b,c}},
C(X) = {X, ϕ, {d}, {c,d}, {a,c,d}, {b,c,d}},
πO(X) = {X, ϕ, {a}, {b}, {a,b}},
πGPC(X) = {X, ϕ, {c}, {d}, {a,c}, {a,d}, {b,c}, {b,d}, {c,d}, {a,b,c}, {a,b,d}, {a,c,d}, {b,c,d}},
Wπ GC(X) = {X, ϕ, {c}, {d}, {a,c}, {a,d}, {b,c}, {b,d}, {c,d}, {a,b,c}, {a,b,d}, {a,c,d}, {b,c,d}}.

Therefore, πGPC(X) = Wπ GC(X).

Example 4.4. Let X = {a,b,c,d} and τ = {X, ϕ, {a,b}, {c,d}}, then

$$\begin{aligned} O(X) &= \{X, \phi, \{a, b\}, \{c, d\}\}, \\ C(X) &= \{X, \phi, \{a, b\}, \{c, d\}\}, \\ \pi O(X) &= \{X, \phi, \{a, b\}, \{c, d\}\}, \\ \pi GPC(X) &= P(X) = W\pi GC(X). \end{aligned}$$

Therefore, πGPC(X) = WπGC(X).

Example 4.5. Let X = {a,b,c,d,e} and τ = {X, ϕ, {a}, {b}, {c}, {a,b}, {a,c}, {b,c}, {a,b,c}}, then O(X) = {X, ϕ, {a}, {b}, {c}, {a,b}, {a,c}, {b,c}, {a,b,c}},
C(X) = {X, ϕ, {d,e}, {a,d,e}, {b,d,e}, {c,d,e}, {a,b,d,e}, {a,c,d,e}, {b,c,d,e}},
πO(X) = {X, ϕ, {a}, {b}, {c}, {a,b}, {a,c}, {b,c}, {a,b,c}},
πGPC(X) = {X, ϕ, {d}, {e}, {a,d}, {a,e}, {b,d}, {b,e}, {c,d}, {c,e}, {d,e}, {a,b,e}, {a,b,d}, {a,c,d}, {a,c,e}, {a,d,e}, {b,c,d}, {b,c,e}, {b,d,e}, {c,d,e}, {a,b,c,d}, {a,c,d,e}, {b,c,d,e}, {a,b,d,e}, {a,b,c,e}},
WπGC(X) = {X, ϕ, {d}, {e}, {a,d}, {a,e}, {b,d}, {b,e}, {c,d}, {c,e}, {d,e}, {a,b,e}, {a,b,d}, {a,c,d}, {a,c,e}, {a,d,e}, {b,c,d}, {b,c,e}, {b,d,e}, {c,d,e}, {a,b,c,d}, {a,c,d,e}, {b,c,d,e}, {a,b,d,e}, {a,b,c,e}}.
Therefore, πGPC(X) = WπGC(X).

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Accepted: December 16, 2019

Partial dynamical systems and partial geodesic equations

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Abstract. In this paper, we generalize the Kato's approach to solve the Cauchy problem for quasi-linear symmetric hyperbolic systems by permitting the coefficients of first-order space-derivatives of the unknown depend on their first-order time-derivative. As an application, we show the existence and uniqueness of the solutions of *partial dynamical systems* and *partial geodesic equation*. These are modifications of the ordinary dynamical systems and geodesic equation in which we replace the ordinary differential operator with the partial differential operator. The motivation comes from quantum theory.

Keywords: quasi-linear hyperbolic symmetric systems, evolution equation, perturbation, dynamical systems, geodesic equation.

1. Introduction

In this paper, we give a new model of geometry in which a straight line is not a curve in the space of points on a manifold. In fact, we take a straight line as a specific set-valued curve on the manifold. More specifically, let M and N be two smooth manifolds. We set $\mathcal{M}$ to be the set of all smooth functions $P : N \to M$ and call each of them a *nonpure point* on manifold M and visualize the nonpure point $P : N \to M$ by its image $P(N) \subset M$. A *nonpure curve* over M is a map $\alpha : (a, b) \to \mathcal{M}$ or equivalently $\alpha : (a, b) \times N \to M$. In other words, a nonpure curve is an N-dependent curve $\alpha(t, x)$ on M where the variable x varies in N.

Next, we modify the ordinary geodesic equation to the following *partial* geodesic equation given in local coordinates u and x for M and N, respectively

$$\begin{cases} \dfrac{\partial^2 u_k}{\partial t^2} + \sum\limits_{i,j} a_{ij}(x,u)\dfrac{\partial u_j}{\partial t}\dfrac{\partial^2 u_k}{\partial x_i \partial t} + \sum\limits_{i,j} \Gamma^k_{ij}(x,u)\dfrac{\partial u_i}{\partial t}\dfrac{\partial u_j}{\partial t} = 0 \\ u(0,x) = u_0(x), \end{cases} \tag{1.1}$$

for given matrices $a_{ij}(x,u)$ and N-dependent Christoffel symbols $\Gamma^k_{ij}(x,u)$, [3]. The motivation for this generalization of the ordinary geodesic equation to the partial geodesic equation comes from the first author's studies on the foundation of quantum mechanics which is an ongoing project. Here, we wish to argue only on the solutions of the above system of partial geodesic equation. It can be shown that the form of this system of equations is independent of the coordinate systems.

It is well-known that the existence and uniqueness of solutions of the ordinary geodesic equation which is a second-order system of ODEs, are the consequences of the fundamental theorem of ODE, after we convert this second-order system to a first-order system. Now in our case of the partial geodesic equation which is a second-order system of PDEs, it turns out that the powerful Kato's theorem [4, 5] on the solution of quasi-linear symmetric hyperbolic systems plays the role of the fundamental theorem of ODE and our second-order PDE after converting it to a first-order PDE can be solved by the aid of this theorem.

We also apply the above idea to dynamical systems. Namely, we modify the ordinary dynamical systems to the following *partial dynamical systems*

$$\begin{cases} \dfrac{\partial u}{\partial t} + \sum\limits_{i,j} a_{ij}(x,u)\dfrac{\partial u_j}{\partial t}\dfrac{\partial u}{\partial x_i} = X(x,u), \\ \\ u(0,x) = u_0(x), \end{cases} \tag{1.2}$$

where X is an N-dependent vector field over M. It can be shown that the form of this system of equations is independent of the coordinate systems. Unfortunately, Kato's theorem can not be applied to this system, since the coefficients of the space partial derivatives depend on the time partial derivative of the unknown which is not allowed in Kato's approach. Among recent works on hyperbolic systems, we can mention [7], [8], [9], [10], [11], [12], [15], [16]. But unfortunately, they cannot solve our problem. Thus, in this paper, after giving the preliminaries in section 2, we generalize the Kato's theorem in section 3. In section 4, we apply it to the partial geodesic equation and dynamical systems. Our results may have applications in *quantum geometry*.

2. Preliminaries

In this section, following [2], [4] and [14] we bring some preliminaries. Consider $\mathbb{R}^n$ with the standard norm. $B(\mathbb{R}^n)$ denotes the Banach space of linear operators on $\mathbb{R}^n$.

i) For two topological spaces X and Y, $C(X,Y)$ is the space of all continuous functions from X to Y. In this paper, we denote $C([a,b],Y)$ by $C[a,b;Y]$.

ii) For $l \geq 0$, $C_b^l(\mathbb{R}^m,\mathbb{R}^n)$ is the space of all $\mathbb{R}^n$-valued functions on $\mathbb{R}^m$, that have continuous and bounded derivatives of order $\leq l$, with the usual supremum norm.

iii) $L^p(\mathbb{R}^m,\mathbb{R}^n)$ is the space of all $\mathbb{R}^n$-valued strongly measurable functions u on the Euclidean space $\mathbb{R}^m$ such that $\| u \|_{L^p} = (\int | u(x) |^p \, dx)^{\frac{1}{p}} < \infty$ with the usual convention for $p = \infty$. The norm on L^∞ is given by the essential supremum. We define $L^p(\mathbb{R}^m, B(\mathbb{R}^n))$ and $C_b^l(\mathbb{R}^m, B(\mathbb{R}^n))$ similarly and in the following, we write $|||$ instead of $|||_{L^2}$.

iv) For $s \in \mathbb{N} \cup \{0\}$, $H^s(\mathbb{R}^m,\mathbb{R}^n)$ is the space of all $u \in L^p(\mathbb{R}^m,\mathbb{R}^n)$ such that all distribution derivatives $D^\alpha u$ with $| \alpha | \leq s$ belong to $L^2(\mathbb{R}^m,\mathbb{R}^n)$. In this space, we have a norm

$$\| u \|_s^2 = \sum_{|\alpha| \leq s} c_\alpha \| D^\alpha u \|^2,$$

where c_α are certain positive constants. It is well-known that the space $H^s(\mathbb{R}^m, \mathbb{R}^n)$ is a Hilbert space. $H^s(\mathbb{R}^m, B(\mathbb{R}^n))$ is defined similarly, and it is a Banach space.

v) $L^p_{ul}(\mathbb{R}^m,\mathbb{R}^n)$ is the set of all $\mathbb{R}^n$-valued strongly measurable functions u on $\mathbb{R}^m$ such that

$$\| u \|_{L^p_{ul}} = \sup_{y \in \mathbb{R}^m} (\int_{|x-y|<1} | u(x) |^p \, dx)^{\frac{1}{p}} < \infty.$$

$L^p_{ul}(\mathbb{R}^m, B(\mathbb{R}^n))$ is defined similarly.

vi) $H^s_{ul}(\mathbb{R}^m,\mathbb{R}^n)$ is the set of all $u \in L^2_{ul}(\mathbb{R}^m,\mathbb{R}^n)$ such that all distribution derivatives $D^\alpha u$ with $| \alpha | \leq s$ are in L^2_{ul}. We put the following norm on this space

$$\| u \|_{s,ul} = \sup_{|\alpha| \leq s} \| D^\alpha u \|_{L^2_{ul}} .$$

By this norm, H^s_{ul} is a Banach space. We define $H^s_{ul}(\mathbb{R}^m, B(\mathbb{R}^n))$ similarly.

vii) For $f : I \to X$ a.e., we write

$$|||f|||_{\infty,X} = \operatorname{esssup}_{t \in I} \| f(t) \|_X, \qquad |||f|||_{1,X} = \int_I^* \| f(t) \|_X \, dt,$$

where $\int^*$ denotes upper-integral and X is an arbitrary Banach space.

Remark 1. (a) $H^s \subset H^s_{ul}$.

(b) $D^\alpha H^s \subset H^{s-|\alpha|}$, for $| \alpha | \leq s$.

(c) If $s \geq \frac{m}{2}$, then $H^s \subset C_b^k$ for $k = s - [\frac{m}{2}] - 1$.

(d) If $r = \min\{s, t, s + t - [\frac{m}{2}] - 1\} \geq 0$, then

$$H^s H^t \subset H^r,$$
$$H^s_{ul} H^t \subset H^r,$$
$$H^t H^s_{ul} \subset H^r.$$

(e) Each closed ball in $H^s(\mathbb{R}^m, \mathbb{R}^n)$ or $H^s(\mathbb{R}^m, B(\mathbb{R}^n))$ is a complete metric space under the metric induced by the H^0-norm. A similar result holds for H^s_{ul} with the H^0_{ul}-norm.

(f) In this paper, c denotes various constants, which need not be the same throughout.

Consider the following system,

$$[L] \begin{cases} a_0 \dfrac{\partial u}{\partial t} + \displaystyle\sum_{j=1}^{m} a_j \frac{\partial u}{\partial x_j} + a_{m+1} u = f, \\ \\ u(0, x) = u_0(x). \end{cases}$$

Here $u = u(t, x)$, $f = f(t, x)$, and $u_0 = u_0(x)$ take values in the Hilbert space $\mathbb{R}^n$, and the coefficients $a_j = a_j(t, x)$, $j = 0, 1, \ldots, m + 1$, in $B(\mathbb{R}^n)$. It is necessary to mention that $u = u(t, x)$ is a function of t with values in a function space over the space variable $x \in \mathbb{R}^m$. We use $u(t)$ instead of $u(t, .)$. In the same way for a_j and f.

Theorem 2.1 ([5]-Theorem *I*)**.** *Let s be an integer such that $s > \frac{m}{2} + 1$, and let $1 \leq s' \leq s$. Assume the following conditions hold:*

(1) $a_j \in C[0, T; H^0_{ul}(\mathbb{R}^m, B(\mathbb{R}^n))], 0 \leq j \leq m + 1,$

(2) $\| a_0(t) \|_{s,ul} \leq K_0, \| a_j(t) \|_{s,ul} \leq K, 0 \leq t \leq T, 1 \leq j \leq m + 1,$

(3) $\| a_0(t') - a_0(t) \|_{s,ul} \leq L \mid t' - t \mid, 0 \leq t, t' \leq T,$

(4) $a_j(t, x)$ *is symmetric for* $0 \leq t \leq T$, $x \in \mathbb{R}^m$, $0 \leq j \leq m$,

(5) $a_0(t, x) \geq 1$ *for all* t, x, *(uniformly positive-definite),*

(6) $f \in L^1[0, T; H^{s'}(\mathbb{R}^m, \mathbb{R}^n)] \cap C[0, T; H^{s'-1}(\mathbb{R}^m, \mathbb{R}^n)],$

(7) $u_0 \in H^{s'}(\mathbb{R}^m, \mathbb{R}^n)$.

Then there is a solution u of the system $[L]$ such that

$$u \in C[0, T; H^{s'}(\mathbb{R}^m, \mathbb{R}^n)] \cap C^1[0, T; H^{s'-1}(\mathbb{R}^m, \mathbb{R}^n)].$$

The solution u is unique in $C[0,T;H^1(\mathbb{R}^m,\mathbb{R}^n)]\cap C^1[0,T;H^0(\mathbb{R}^m,\mathbb{R}^n)]$ and

$$\| u(t)-u_{00}\|_r \le Ce^{c(L+K+KK_0^r)T}[\| u_0-u_{00}\|_r + c(1+K_0^r)(F_r^1+KT\| u_{00}\|_{r+1})], \tag{2.1}$$

and

$$\| \frac{d}{dt}u(t)\|_{r-1}\le c(1+K_0^{r-1})(\| f(t)\|_{r-1}+K\| u(t)\|_r), 0\le t\le T, \tag{2.2}$$

where $0\le r\le s'$ in (2.1) and $1\le r\le s'$ in (2.2); u_{00} is an arbitrary element in $H^{r+1}(\mathbb{R}^m,\mathbb{R}^n)$;

$$C=\| a_0(0)\|_{L_\infty}^{\frac{1}{2}}\le c\| a_0(0)\|_{s-1,ul}^{\frac{1}{2}},$$
$$F_r^1=\int_0^T \| f(t)\|_r\, dt,$$

and c is a constant depending only on m and s.

The quasi-linear version of the above theorem where the coefficients a_j depend on u has been proved in [4]. In the next section, we will generalize this theorem to the case where the coefficients a_j depend on both u and $\partial u/\partial t$. The proof of this new theorem is obtained by a suitable modification of the proof given in [4].

3. The generalized Kato's theorem

Theorem 3.1. *Consider the following system of partial differential equations*

$$\begin{cases}\dfrac{\partial u}{\partial t}+\displaystyle\sum_{j=1}^m a_j(t,x,u,\frac{\partial u}{\partial t})\frac{\partial u}{\partial x_j}=f(t,x,u),\\ u(0)=u_0.\end{cases} \tag{3.1}$$

Let Ω be the subset of $\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}^n$ consisting of all triads x, p and q such that

$$|\, p-v_0(x)\,|<\omega,$$

where $\omega>0$ and $v_0\in H^s(\mathbb{R}^m,\mathbb{R}^n)\subset C^1(\mathbb{R}^m,\mathbb{R}^n)$ are fixed. We suppose D is a ball in $H^s(\mathbb{R}^m,\mathbb{R}^n)$ with v_0 as its center and with radius r_0 and $u_0\in D$. If $s\ge\frac{m}{2}+1$, r_0 is small enough, and also the following conditions hold:

$$a_j\in C[0,T;C_b^s(\Omega,B(\mathbb{R}^n))],\qquad 1\le j\le m, \tag{3.2}$$

$$a_j(t,x,p,q)\in B(\mathbb{R}^n)\ \text{ is symmetric for } 0\le t\le T,(x,p,q)\in\Omega, \tag{3.3}$$

$$f\in[0,T;C_b^{s+1}(\Omega,\mathbb{R}^n)], \tag{3.4}$$

$$f_0\in L^\infty[0,T;H^s(\mathbb{R}^m,\mathbb{R}^n)]\cap C[0,T;H^0(\mathbb{R}^m,\mathbb{R}^n)], \tag{3.5}$$

where $f_0(t,x) = f(t,x,v_0(x))$*, then there is a unique solution* u *of* (3.1) *defined on* $[0,T']$*, where* $0 < T' \leq T$*, such that*

$$u \in C[0,T';D] \cap C^1[0,T';H^{s-1}(\mathbb{R}^m,\mathbb{R}^n)].$$

Proof. Let $R' > 0$ be the radius of the largest open ball with center u_0 contained in D and $R := \frac{R'}{2}$. By Lemma (4.2) in [4], there exists $u_{00} \in D \cap H^{s+1}(\mathbb{R}^m,\mathbb{R}^n)$ such that $\| u_0 - u_{00} \|_s \leq \frac{R}{2}$. Now, let $v \in H^s$ and $\| v - u_{00} \|_s \leq R$, since

$$\| v - u_0 \|_s \leq \| v - u_{00} \|_s + \| u_0 - u_{00} \|_s \leq R + \frac{R}{2} \leq R',$$

then $v \in D$.

Suppose that $S \subset C^1[0,T';H^s(\mathbb{R}^m,\mathbb{R}^n)]$, such that for each $v \in S$,

$$\| v(t) - u_{00} \|_s \leq R, \qquad 0 \leq t \leq T' \tag{3.6}$$

$$\| v(t) - v(t') \|_{s-1} \leq L'(t-t'), \qquad 0 \leq t' \leq t \leq T' \tag{3.7}$$

$$\| \frac{\partial v}{\partial t}(t) \|_{s-1,ul} \leq L', \qquad 0 \leq t \leq T' \tag{3.8}$$

where $T' \leq T$ and L' are positive constants which we will determine them later. We define a metric on S as follows.

$$\begin{aligned} d(a,b) &= \sup_{0\leq t\leq T'} \| a_t - b_t \|_0 + \sup_{0\leq t\leq T'} \| (\frac{\partial a}{\partial t})_t - (\frac{\partial b}{\partial t})_t \|_0 \\ &= |||a-b|||_{\infty,X} + \left|\left|\left| \frac{\partial a}{\partial t} - \frac{\partial b}{\partial t} \right|\right|\right|_{\infty,X}, \end{aligned} \tag{3.9}$$

where $X = H^s(\mathbb{R}^m,\mathbb{R}^n)$. Suppose that $\{v^n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in S. Thus, the sequences $\{v^n : [0,T'] \to X\}$ and $\{\frac{\partial v^n}{\partial t} : [0,T'] \to X\}$ are Cauchy sequences in complete metric space X and thus converge uniformly on $[0,T']$. So, by Theorem (7.17) in [13], $\{v^n\}$ converges uniformly to a function $v(t) \in X$ and $\frac{\partial v}{\partial t}(t) = \lim_{n\to\infty} \frac{\partial v^n}{\partial t}(t)$, for $t \in [0,T']$. Since the norm is a continuous function, we have

$$\begin{aligned} \| v(t) - u_{00} \|_s &= \| \lim_{n\to\infty} v^n(t) - u_{00} \|_s \\ &= \lim_{n\to\infty} \| v^n(t) - u_{00} \|_s \\ &< R, \end{aligned}$$

$$\begin{aligned} \| v(t) - v(t') \|_{s-1} &= \| \lim_{n\to\infty} (v^n(t) - v^n(t')) \|_{s-1} \\ &= \lim_{n\to\infty} \| v^n(t) - v^n(t') \|_{s-1} \\ &\leq L'(t-t'), \end{aligned}$$

$$\begin{aligned}\| \frac{\partial v}{\partial t}(t) \|_{s-1,ul} &= \| \lim_{n\to\infty} \frac{\partial v^n}{\partial t}(t) \|_{s-1,ul} \\ &= \lim_{n\to\infty} \| \frac{\partial v^n}{\partial t}(t) \|_{s-1,ul} \\ &\leq L'.\end{aligned}$$

Therefore, the conditions (3.6), (3.7) and (3.8) hold for v. Thus S is complete with the metric (3.9).

Note that, if $v \in S$, then by condition (3.6) and the fact that we showed at the beginning of the proof, we have $v_t \in D$ and

$$\| v - v_0 \|_{L_\infty} \leq c \| v - v_0 \|_s < cr_0 < \omega,$$

by choosing enough small r_0. Thus, if we set

$$a_j^v(t,x) = a_j(t,x,v(t,x),(\frac{\partial v}{\partial t})(t,x)), \qquad 1 \leq j \leq m, \tag{3.10}$$

$$f^v(t,x) = f(t,x,v(t,x)), \tag{3.11}$$

then these are well-defined. By replacing the pair $(v, \frac{\partial v}{\partial t})$ with v in the formula (5.6) of [4] and applying the formula of (5.8) inside the proof of Theorem IV in the same reference, we get for $|\alpha| \leq s$,

$$D^\alpha a_j^v(t,x) \leq \| a_j(t) \|_{C^s} \sum \Big(|D^{\beta_1} v| \dots |D^{\beta_k} v| |D^{\gamma_1} \frac{\partial v}{\partial t}| \dots |D^{\gamma_l} \frac{\partial v}{\partial t}| \Big),$$

where $C^s = C^s(\Omega, B(\mathbb{R}^n))$ and summation is taken over all integers $0 \leq k \leq |\alpha|, 0 \leq l \leq |\alpha|$ and all multi-indices $\beta_1 > 0, \dots, \beta_k > 0, \gamma_1 > 0, \dots, \gamma_l > 0$ such that $|\beta_1| + \dots |\beta_k| + |\gamma_1| + \dots |\gamma_l| \leq |\alpha|$. Thus

$$\begin{aligned}\| a_j^v(t) \|_{s,ul} &= \sup_{|\alpha|\leq s} \| D^\alpha a_j^v(t) \|_{L^2_{ul}} \\ &= \sup_{|\alpha|\leq s} (\sup_x (\int_{|y-x|<1} | D^\alpha a_j^v(t)(y) |^2 \, dy)^{\frac{1}{2}}) \\ &\leq c \| a_j(t) \|_{C^s} (1 + \| v \|_s^s \| \frac{\partial v}{\partial t} \|_{s-1}^s) \\ &\leq constant,\end{aligned} \tag{3.12}$$

for each $v \in S$. This proves that $a_j^v(t)$ is bounded in H^s_{ul}-norm for each v.

Consider the linear differential equation $[L^v]$ obtained from $[L]$ by setting $a_j = a_j^v$, $0 \leq j \leq m$, $a_{m+1} = 0$ and $f = f^v$. We claim that all conditions of Theorem 2.1 are satisfied: first, we check that the map $v \mapsto a_j^v(t)$ is uniformly Lipschitz continuous from H^0-norm to H^0_{ul}-norm, for each $t \in [0, T']$.

$$\begin{aligned}(a_j^v - a_j^w)(t) = \int_0^1 \Big(& D_p a_j(t,x,\upsilon_\lambda(t,x), \frac{\partial \upsilon_\lambda}{\partial t}(t,x))(v(t,x) - w(t,x)) \\ & + D_q a_j(t,x,\upsilon_\lambda(t,x), \frac{\partial \upsilon_\lambda}{\partial t}(t,x))(\frac{\partial v}{\partial t}(t,x) - \frac{\partial w}{\partial t}(t,x)) \Big) d\lambda,\end{aligned}$$

where $\upsilon_\lambda = \lambda v + (1-\lambda)w$. Hence, for $0 < r \leq s-2$,

$$\| (a_j^v - a_j^w)(t) \|_r \leq c\Big(\| v - w \|_r \sup_{0\leq\lambda\leq 1} \| (D_p a_j)^{\upsilon_\lambda}(t) \|_{s-2,ul} + \| \frac{\partial v}{\partial t} - \frac{\partial w}{\partial t} \|_{r-1} \sup_{0\leq\lambda\leq 1} \| (D_q a_j)^{\upsilon_\lambda}(t) \|_{s-2,ul} \Big), \tag{3.13}$$

for some constant c (see the proof of Theorem IV of [4]). Since $D_p a_j$ and $D_q a_j$ satisfy (3.2) with s replaced by $s-2$, and since (3.12) is true with s replaced by $s-1$ and by (3.8), we obtain that

$$\|(a_j^v - a_j^w)(t)\|_r \leq c\Big(\|v-w\|_r \|a_j(t)\|_{C^s} \sup_{0\leq\lambda\leq 1} (1+\|\nu_\lambda\|_{s-1}^{s-1}\|\frac{\partial \nu_\lambda}{\partial t}\|_{s-2}^{s-1}) + \|\frac{\partial v}{\partial t} - \frac{\partial w}{\partial t}\|_{r-1}\|a_j(t)\|_{C^s} \sup_{0\leq\lambda\leq 1} (1+\|\nu_\lambda\|_{s-1}^{s-1}\|\frac{\partial \nu_\lambda}{\partial t}\|_{s-2}^{s-1})\Big) \tag{3.14}$$

$$\leq d(v,w) \times constant, \tag{3.15}$$

for $0 < r \leq s-2$.

Now, we show that the function $t \mapsto a_j^v(t)$ is continuous.

$$\begin{aligned} \| a_j^v(t') - a_j^v(t) \|_{0,ul} \leq & \| a_j(t', ., v_{t'}, \left(\frac{\partial v}{\partial t}\right)_{t'}) - a_j(t', ., v_t, \left(\frac{\partial v}{\partial t})_t\right) \|_{0,ul} \\ & + \| a_j(t', ., v_t, \left(\frac{\partial v}{\partial t})_t\right) - a_j(t, ., v_t, \left(\frac{\partial v}{\partial t})_t\right) \|_{0,ul} . \end{aligned} \tag{3.16}$$

By (3.2) and (3.15), the right side of (3.16) tends to 0, when $t' \to t$.

In the same way, we observe that the function $v \mapsto f^v(t)$ is uniformly Lipschitz-continuous on D from the H^0-norm to H^0-norm, uniformly in t. On the other hand, according to (3.4),

$$\| f^v(t) \|_s = \sum_{|\alpha|\leq s} c_\alpha \| D^\alpha f^v \| .$$

So, $f^v(t)$ is bounded in H^s-norm for $v_t \in D$. Now, we show that $t \mapsto f^v(t)$ is continuous in the H^0-norm, for each $v \in D$. To this end, as in (3.14) we have

$$\| (f^v - f_0)(t') - (f^v - f_0)(t) \|_0 \leq \| f(t') - f(t) \|_{C^s} \times constant.$$

Thus $(f^v - f_0)(t)$ is continuous in t in H^0-norm by (3.4), and $f_0(t)$ is continuous by (3.5), thus $t \mapsto f^v(t)$ is continuous in the H^0-norm for each v.

So, we showed that

$$a_j^v \in C[0,T; H_{ul}^0(\mathbb{R}^m, B(\mathbb{R}^n))],$$

and by (3.12), there is a constant $\lambda > 0$ such that $\| a_j^v(t) \|_{s,ul} \leq \lambda$ and $\| f^v(t) \|_s \leq \lambda$. Finally $a_j^v(t)(x)$ is symmetric for $0 \leq t \leq T'$, $x \in \mathbb{R}^m$ and $0 \leq j \leq m$.

Therefore, all the conditions of Theorem 2.1 are satisfied and it is enough to put $K_0 = K = \lambda$ in (2.1) and (2.2). So, the system

$$\frac{\partial u}{\partial t} + \sum_{j=1}^{m} a_j^v(t,x)\frac{\partial u}{\partial x_j} = f^v(t,x),$$

has a solution u such that

$$u \in C[0,T';D] \cap C^1[0,T';H^{s-1}(\mathbb{R}^m,\mathbb{R}^n)].$$

This solution is unique in a larger class,

$$C[0,T';H^1(\mathbb{R}^m,\mathbb{R}^n)] \cap C^1[0,T';H^0(\mathbb{R}^m,\mathbb{R}^n)].$$

By Theorem 2.1, there are the estimates

$$\| u(t) - u_{00} \|_s \leq e^{\alpha(L')T'} \left[\| u_0 - u_{00} \|_s + c(1+\lambda^s)\lambda(1+ \| u_{00} \|_{s+1})T' \right], \tag{3.17}$$

$$\| \frac{du_t}{dt} \|_{s-1} \leq c(1+\lambda^{s-1})\lambda(1+ \| u_{00} \|_s + \beta(L',T')), \tag{3.18}$$

where c depends only on m and s, and

$$\alpha(L') = c(L' + \lambda + \lambda^{s+1}).$$

Here, $F_s^1 = \int_0^{T'} \| f^v(t) \|_s \, dt \leq \lambda T'$ and $\beta(L',T')$ is the right side of (3.17).

Next, we show that $u \in S$, for appropriately chosen values of L' and T'. We must show that the conditions (3.6) and (3.8) hold. It is enough to show that

$$e^{\alpha(L')T'} \left[\| u_0 - u_{00} \|_s + c(1+\lambda^s)\lambda(1+ \| u_{00} \|_{s+1})T' \right] \leq R,$$

and

$$c(1+\lambda^{s-1})\lambda(1+ \| u_{00} \|_s + \beta(L',T')) \leq L'.$$

The above inequalities reduce to the following inequalities for the case $T' = 0$,

$$\begin{aligned} &\| u_0 - u_{00} \|_s < R, \\ &c(1+\lambda^{s-1})\lambda(1+ \| u_{00} \|_s + \| u_0 - u_{00} \|_s) < L'. \end{aligned}$$

The first one obviously holds by the argument at the beginning of the proof of the current theorem. The second one holds for suitably large L'. Then the required conditions also hold for sufficiently small $T' > 0$, since the right terms of (3.17) and (3.18) are continuous in $T'(\geq 0)$ for any fixed L'. Now, we claim that the map $v \mapsto u$ denoted by ϕ is a contraction map of S into itself when

T' is chosen sufficiently small. According to (3.2), (3.4), (3.15) and Remark 1, there exists a positive constant μ such that,

$$\begin{aligned}(3.19)\quad \| a_j^v(t) - a_j^w(t) \|_{0,ul} &= \| a_j(t,.,v(t),\frac{\partial v}{\partial t}(t)) - a_j(t,.,w(t),\frac{\partial w}{\partial t}(t)) \|_{0,ul} \\ &\leq \mu(\| v(t) - w(t) \|_{0,ul} + \| \frac{\partial v}{\partial t}(t) - \frac{\partial w}{\partial t}(t) \|_{0,ul}) \\ &\leq \mu d(v,w),\end{aligned}$$

and

$$(3.20)\qquad \| f^v(t) - f^w(t) \| \leq \mu d(v,w).$$

Now, we consider the evolution equation version of (L^v), i.e.,

$$(E^v)\qquad \frac{du}{dt} + A^v(t)u = f^v(t),$$

where

$$A^v(t) = \sum_{j=1}^{m} a_j^v(t,x)\frac{\partial}{\partial x_j}.$$

By the proof of Theorem 2.1, which is given in [4], there is a unique evolution operator $U^v(t,t')$ such that the solution of (E^v) is given by

$$u(t) = U^v(t,0)u_0 + \int_0^t U^v(t,t')f^v(t')dt'.$$

Since, $A^v(t) - A^w(t) \in H^0_{ul}$ and $H^0_{ul}H^{s-1} \subset H^0$, by $s > \frac{m}{2} + 1$, we have

$$\begin{aligned}(3.21)\quad \| [A^v(t) - A^w(t)]u \|_0 &\leq c \| A^v(t) - A^w(t) \|_{0,ul} \| u \|_s \\ &\leq c\mu \| u \|_s \, d(v,w).\end{aligned}$$

To complete the proof we need the following lemma.

Lemma 3.1. *Suppose we have two evolution equations as follows*

$$(E)\qquad \frac{du}{dt} + A(t)u = f(t), \qquad u(0) = \varphi,$$

$$(E')\qquad \frac{du'}{dt} + A'(t)u' = f'(t), \qquad u(0) = \varphi',$$

in the Banach space X. Let $Y \subset X$. The solution of (E) is given by

$$u(t) = U(t,0)\varphi + \int_0^t U(t,s)f(s)ds,$$

and the solution of (E') is given by

$$u'(t) = U'(t,0)\varphi' + \int_0^t U'(t,s)f'(s)ds,$$

where $\varphi \in Y$, $f \in L(Y)$, $\varphi' \in X$ and $f' \in L(X)$. By the definition of metric in (3.9)*, we have*

$$d(u',u) \le K'[\|\varphi' - \varphi\|_X + \|\!|\!| f' - f |\!|\!|_{1,X} + \|\!|\!| (A' - A)u |\!|\!|_{1,X} + \|f' - f\|_X + \|(A' - A)u\|_X] + K''\|\varphi' - \varphi\|_X, \tag{3.22}$$

where K' and K'' are constants depending only on A'. Here $A' - A$ is regarded as a multiplication operator on $C(Y)$ to $L(X)$.

Proof of the Lemma. Since we have two systems A, U and A', U' with the same spaces X and Y, we have ([6])

$$U' - U = -U'(A' - A)U,$$

where

$$U'(A' - A)U(t,s) = \int_s^t U'(t,r)(A'(r) - A(r))U(r,s)dr.$$

Then, for $y \in Y$,

$$U'(t,r)y - U(t,r)y = -\int_r^t U'(t,s)(A'(s) - A(s))U(s,r)yds.$$

Setting $r = 0$ and $y = \varphi$ gives

$$(U' - U)(t,0)\varphi = -\int_0^t U'(t,s)(A'(s) - A(s))U(s,0)\varphi ds.$$

Setting $y = f(r)$,

$$(U' - U)(t,r)f(r) = -\int_r^t U'(t,s)(A'(s) - A(s))U(s,r)f(r)ds,$$

and by integrating in r, we have

$$\int_0^t (U' - U)(t,r)f(r)dr = -\int_0^t \int_r^t U'(t,s)(A'(s) - A(s))U(s,r)f(r)dsdr.$$

Thus, $(U' - U)f = -U'(A' - A)Uf$. So,

$$\begin{aligned} u'(t) - u(t) &= U'(t,0)\varphi' - U(t,0)\varphi + \int_0^t [U'(t,s)f'(s) - U(t,s)f(s)]ds \\ &= U'(t,0)(\varphi' - \varphi) + \int_0^t U'(t,s)(f'(s) - f(s))ds \\ &\quad + (U' - U)(t,0)\varphi + \int_0^t (U'(t,s) - U(t,s))f(s)ds. \end{aligned}$$

On the other hand,

$$\begin{aligned}\frac{d}{dt}(u'(t)-u(t)) &= \frac{d}{dt}[U'(t,0)(\varphi'-\varphi)] + \frac{d}{dt}[\int_0^t U'(t,s)(f'(s)-f(s))ds] \\ &\quad + \frac{d}{dt}(-U'(A'-A)u) \\ &= \frac{dU'}{dt}(t,0)(\varphi'-\varphi) + U'(t,t)(f'(t)-f(t)) \\ &\quad - U'(t,t)(A'(t)-A(t))u(t) \\ &= \frac{dU'}{dt}(t,0)(\varphi'-\varphi) \\ &\quad + U'(t,t)[(f'(t)-f(t)) - (A'(t)-A(t))u(t)].\end{aligned}$$

Hence

$$\begin{aligned}d(u',u) &= \left|\left|\left|u'-u\right|\right|\right|_{\infty,X} + \left|\left|\left|\frac{\partial u'}{\partial t} - \frac{\partial u}{\partial t}\right|\right|\right|_{\infty,X} \\ &\le \left|\left|\left|U'\right|\right|\right|_{\infty,X}(\|\varphi'-\varphi\|_X + \left|\left|\left|f'-f\right|\right|\right|_{1,X} + \left|\left|\left|(A'-A)u\right|\right|\right|_{1,X}) \\ &\quad + \left|\left|\left|\frac{dU'}{dt}\right|\right|\right|_{\infty,X}(\|\varphi'-\varphi\|_X) + \left|\left|\left|U'\right|\right|\right|_{\infty,X}(\|f'-f\|_X + \|(A'-A)u\|_X) \\ &\le K'[\|\varphi'-\varphi\|_X + \left|\left|\left|f'-f\right|\right|\right|_{1,X} + \left|\left|\left|(A'-A)u\right|\right|\right|_{1,X} \\ &\quad + \|f'-f\|_X + \|(A'-A)u\|_X] + K''\|\varphi'-\varphi\|_X,\end{aligned}$$

where $K' = \left|\left|\left|U'\right|\right|\right|_{\infty,X}$ and $K'' = \left|\left|\left|\frac{dU'}{dt}\right|\right|\right|_{\infty,X}$. This completes the proof of the lemma.

Now, we complete the proof of the theorem. Let ϕv and ϕw be the solutions of (E^v) and (E^w), respectively. By Lemma 3.1,

$$\begin{aligned}d(\varphi v,\varphi w) &\le \left|\left|\left|U^w\right|\right|\right|_{\infty,X}\Big(\|f^v-f^w\|_X + \|(A^v-A^w)\phi v\|_X \\ &\quad + \int_0^{T'}[\|f^v(t)-f^w(t)\|_0 + \|(A^v(t)-A^w(t))\varphi v(t)\|_0]dt\Big),\end{aligned} \tag{3.23}$$

where $U^w = \{U^w(t,t')\}$ is the evolution operator for (E^w). Since $\left|\left|\left|U'\right|\right|\right|_{\infty,X}$, $\|\varphi v(t)\|_s$ and $\left|\left|\left|\frac{dU'}{dt}\right|\right|\right|_{\infty,X}$ are uniformly bounded, and $\phi v(t) \in D$, we see from (3.19), (3.20), (3.21) and (3.23), $d(\varphi v,\varphi w) \le \textit{constant} \times d(v,w)$. We use the contraction mapping theorem to complete the proof. Since ϕ is a contraction map of S into itself, if T' is chosen sufficiently small, then there is a unique fixed point $u = \phi u \in S$. □

Remark 2. In the above theorem, if we replace $\mathbb{R}^m$ with $\overline{U}$, where U is a bounded open subset of $\mathbb{R}^m$, and if the coefficients $a_j(t,x,p,q)$ do not depend on q and if we redefine the set Ω, as all pairs (x,p) where $x \in \overline{U}$ and$|\ p-v_0(x)\ | \le \omega$, then since Ω is compact, the boundedness of a_j and f are automatic.

4. Applications

4.1 Partial dynamical systems

According to the Introduction, in this section, we prove the existence of the solution for the partial dynamical systems. Namely, we consider the following nonlinear system

$$(Q)\begin{cases}\dfrac{\partial u_k}{\partial t}(t,x)+\displaystyle\sum_{i,j}a_{ij}(x,u(t,x))\frac{\partial u_j}{\partial t}(t,x)\frac{\partial u_k}{\partial x_i}(t,x)=X_k(x,u(t,x)),\\ \\ u(0,x)=u_0(x),\end{cases}$$

where $u:[0,T]\times U\longrightarrow\mathbb{R}^n$ is a continuous function with the continuously differentiable components $u_j:[0,T]\times U\longrightarrow\mathbb{R}$ and U is a bounded open subset of $\mathbb{R}^m$. Here a_{ij} and X_k are smooth real-valued functions for all $1\leq i\leq m, 1\leq j\leq n, 1\leq k\leq n$ and $u_0(x)$ takes value in $\mathbb{R}^n$.

Now, we can rewrite (Q) as the following system which is of the type of system (3.1),

$$(Q')\begin{cases}\dfrac{\partial u_k}{\partial t}+\displaystyle\sum_{1\leq i\leq m,}a_i\frac{\partial u_k}{\partial x_i}=X_k,\\ \\ u(0,x)=u_0(x),\end{cases}$$

where $u=u(t,x)$ and

$$a_i(t,x,v,w)=(\sum_{j=1}^{n}a_{ij}(x,v)w_j)I_n, \tag{4.1}$$

and I_n is the identity matrix. Therefore, a_i are constant functions relative to t and by the Remark 2 all the conditions of the Theorem 3.1 hold and hence there is a unique solution u of (Q') defined on $[0,T']$, such that

$$u\in C[0,T';H^s(\mathbb{R}^m,\mathbb{R}^n)]\cap C^1[0,T';H^{s-1}(\mathbb{R}^m,\mathbb{R}^n)].$$

4.2 Partial geodesic equation

Consider the non-linear system

$$(Z)\begin{cases}\dfrac{\partial^2 u_k}{\partial t^2}+\displaystyle\sum_{i,j}a_{ji}(x,u)\frac{\partial u_i}{\partial t}\frac{\partial^2 u_k}{\partial x_j\partial t}+\sum_{i,j}\Gamma_{ij}^k(x,u)\frac{\partial u_i}{\partial t}\frac{\partial u_j}{\partial t}=0,\\ \\ u(0,x)=u_0(x),\end{cases}$$

where $u:[0,T]\times U\longrightarrow\mathbb{R}^n$ is a continuous function with the continuously differentiable components $u_j:[0,T]\times U\longrightarrow\mathbb{R}$ and U is a bounded open subset of $\mathbb{R}^m$. Here, a_{ji} and Γ_{ij}^k are smooth real-valued functions for all $1\leq j\leq m, 1\leq$

$i \leq n, 1 \leq k \leq n$ and $u_0(x)$ takes value in $\mathbb{R}^n$. Γ_{ij}^k are the Christoffel symbols which we have assumed to depend to variable u as well as variable x.

Now, we rewrite (Z) as the following first-order system

$$(Z') \begin{cases} \dfrac{\partial w_k}{\partial t} = w_{n+k} & (Z_1') \\ \dfrac{\partial w_{n+k}}{\partial t} + \displaystyle\sum_{i,j} a_{ji}(x, w_1, \ldots, w_n) w_{n+i} \dfrac{\partial w_{n+k}}{\partial x_j} & \\ \qquad + \sum_{i,j} \Gamma_{ij}^k(x, w_1, \ldots, w_n) w_{n+i} w_{n+j} = 0 & (Z_2'). \end{cases}$$

Namely, from a given solution $(w_1, \ldots, w_{2n})$ for the system (Z'), we get a solution $(u_1, \ldots, u_n) = (w_1, \ldots, w_n)$ for the system (Z). We rewrite (Z') as the following system

$$\frac{\partial w}{\partial t} + \sum_{j=1}^{m} a_j(x, w) \frac{\partial w}{\partial x_j} = f(x, w),$$

where

(i) $a_j : \Omega \to B(\mathbb{R}^{2n}) = M_{2n\times 2n}(\mathbb{R})$,

$$a_j(x, w) = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & (\sum_{i=1}^{n} a_{ji}(x, w_1, \ldots, w_n) w_{n+i}) I_n \end{array}\right), \quad (x, w) \in \Omega. \tag{4.2}$$

(ii) $w : [0, T] \times \mathbb{R}^m \to \mathbb{R}^{2n}$.

(iii) $f : \Omega \to \mathbb{R}^n$,

$$f(x, w) = \left(\begin{array}{c} w_{n+1} \\ \vdots \\ w_{2n} \\ \hline -(\sum_{i,j} \Gamma_{ij}^k(x, w_1, \ldots, w_n) w_{n+i} w_{n+j})_{k=1}^{n} \end{array}\right)_{2n\times 1}. \tag{4.3}$$

Since the functions a_j are constant functions relative to t and their domains with respect to the variable x are bounded open subset U, then by the Remark 2 all the conditions of the Theorem 3.1, and in particular, the Theorem 2.1 hold, so there is a unique solution w of (Z') and thus we get a unique solution for (Z).

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Accepted: April 12, 2020

Fixed point theory for study the controllability of control systems in reflexive Banach space

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Abstract. In this paper, fixed point theory and strongly continuous semigroup theory are introduced to study the sufficient conditions for controllability of nonlinear control systems in reflexive Banach spaces. Example is gave to illustrate the value of the theory.
Keywords: controllability, reflexive Banach space, Opial's condition, normal structure, semigroup theory.

1. Introduction

Many engineering and scientific systems in the control theory in infinite dimensional spaces can be formulation by partial differential equations, integral equations, or fractional differential equations. By using semigroup theory we can characterization these systems as differential equations and then study the solution of these problems. Controllability is one of most significant properties of the control system, its means that the ability to transmit the system from an arbitrary initial state to an arbitrary final state of a given set in a finite time by a convenient option of the control function, one can refer to the [1,2,3].

In this paper we introduce the sufficient conditions for controllability of the following control problem in arbitrary reflexive Banach spaces (rBs).

$$(1)\quad \dot{w}(t)+Bw=(As)(t)+g(t,w(t))+H(t,w(t))) \text{ a.e. in } J=[0,a], w(0)=w_0,$$

where $w(.)$ takes values in (rBs) Z with norm $||.||$, the control function $s(.)\in L^2(J,S)$ be a (rBs) of admissible control functions, with S a Banach space (Bs).

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Let $-B$ be a linear operator generates a strongly continuous semigroup (C_0 – semigroup) $Y(t), t \geq 0$, and $A : S \longrightarrow Z$ be a bounded linear operator. The nonlinear operators g, N and H are continuous from $J \times Z$ into Z and all satisfy Lipschitz condition on the second argument.

Fixed point theorems (FPTs) are basic mathematical tools used in studying the controllability results of nonlinear equations. Controllability of the system (1.1) with different geometric conditions on the spaces Z and S has been studied by using Banach contraction theorem, Schauder (FPT) and Kirk (FPT), see [4,5,6]. Nonexpansive mappings on a spaces has normal structure play an important role in fixed point theory, see section 2 of this paper.

Since every uniformly convex Banach space (ucBs) is (rBs) but the converse, in general, not true [7], and since a nonexpansive mapping on a (Bs) has no fixed point (FP) in general. Thus the work in this paper is extended to studied the controllability of (1.1) in the (ucBs) by using Kirk (FPT) [5]. The aim of this article is discuss study the controllability of the system (1.1) in arbitrary (rBs) by using (FPT) deals with nonexpansive mapping defined on a set has normal structure.

2. Preliminaries

In this section some well known definitions, theorem and examples that will be used in the prove of the main theorem (Theorem 3.1) have been introduced.

Definition 2.1 ([8])**.** Let X be a normed, space a self mapping T is said to be **Lipschitz continuous** if there is $\varphi \geq 0$, such that:

$$\|T(x_1) - T(x_2)\| \leq \varphi \|x_1 - x_2\|,$$

for all $x_1, x_2 \in X$. The smallest φ is the Lipschitz constant of T. If $\varphi < 1$, then T is **contraction** and if $\varphi \leq 1$, then T is **nonexpansive**.

Clearly, contraction map is nonexpansive, and isometry map is nonexpansive but not contraction.

Definition 2.2 ([9])**.** A (Bs) X is said to be satisfy **Opial's condition** if for each x in X and each sequence x_n weakly convergent to x, then:

$$\lim_{n\to\infty} \inf \|x_n - y\| > \lim_{n\to\infty} \inf \|x_n - x\|$$

holds for all $y \neq x$.

Finite dimensional (Bs), l_p spaces for $1 < p < \infty$ and L_p for $p = 2$ (Hilbert space) satisfy Opial's condition.

Definition 2.3 ([7])**.** Let X^* and X^{**} be the first and second dual spaces of a normed space X. Define a mapping $\jmath : X \to X^{**}$ by $\jmath(x) = F_x$, where $F_x = f(x)$, and $f \in X^*$. The normed space X is called **Reflexive** if the natural embedding is an onto mapping.

Clearly, (Bs) is reflexive if the natural embedding is an onto mapping from X into X^{**}.

Example 2.4 ([7])**.** The Euclidean space R^n, all finite dimensional spaces and Hilbert space are reflexive, so L_p and l_p spaces for $1 < p < \infty$ and (ucBs) are reflexive, but the space of continuous real value function on $[0, 1]$ is not reflexive.

Definition 2.5 ([7])**.** Let X be a normed space, a subset K of X is called **weakly compact**, if every sequence $\{x_n\}$ in K contains a subsequence which is converges weakly in X.

Remark 2.6. Every nonempty, bounded, closed and convex (bcc) subset of (rBs) is weakly compact [8,9].

Definition 2.7. [8] Let X be a Banach space, and $K \subseteq X$ be nonempty, (bcc). A point $x \in K$ is said to be **diametral** if $\sup\{\|x - k\| : k \in K\} = \text{diam}K$. A subset K of X has **normal structure**, if for each nonempty, convex $D \subseteq K$ with diam > 0, there exist a point $x \in D$, which is not diametral.

Example 2.8 ([8])**.** In (Bs) compact convex set has normal structure, so nonempty, (bcc) subset of (ucBs) has normal structure, also Opial's condition implies normal structure, see [10].

To have an extension of Kirk (FPT), on (ucBs), we needed some geometric conditions on the spaces in the domain of the nonexpansive maps in (rBs).

Theorem 2.9 ([8, 11])**.** Let T be nonexpansive mapping from C into C, where C is a nonempty weakly compact convex subset having normal structure in a (Bs) X, then T has a (FP) in C.

Remark 2.10 ([11])**.** In above theorem the convexity can′t be dispense see the simple example:

Let $C = [-2, -1] \cup [1, 2] \subset R$ and T is a self map on C defined by $Tx = -x$, $x \in C$, therefore T is nonexpansive, but T has no (FP) in C.

Note that, the nonexpansive map on a non convex set in (Bs) has no fixed point.

Definition 2.11 ([1])**.** Let X be a (Bs). A one parameter family $Y(t), 0 < t < \infty$ of linear bounded operators from a (Bs) X into itself, is called a **strongly continuous semigroup C_0-semigroup**, if it's satisfied the following statement conditions:

(i) $Y(0) = I$;

(ii) $Y(t + s) = Y(t)Y(s)$, for every $t, s \geq 0$ (the semigroup property);

(iii) $\lim_{t\to 0} Y(t)x = x$, for every $x \in X$.

Definition 2.12 ([11])**.** The infinitesimal generator B of the semigroup $Y(t)$ on a (Bs) X is defined by $Bx = \lim_{t\to 0^+} \frac{1}{t}(Y(t) - I)x$, for $x \in D(B)$, whenever the limit exists.

3. Controllability of nonlinear control systems

The main objective of this paper, is to study the controllability of mild solution to the initial value control system (1.1) in (rBs) by using **C_0-semigroup** and **(FPT)** (2.9).

3.1 System formulation (I)

Throughout this subsection, let Z be a (rBs) with norm $\|.\|$, and S be a (Bs). Now consider the nonlinear control system in infinite dimensional state space:

$$\begin{aligned} &\dot{w}(t) + Bw(t) = (As)(t) + g(t, w(t)) + H(t, N(t, w(t)), \\ &\text{a.e. in } J = [0, a],\ w(0) = w_0 \end{aligned} \tag{2}$$

where $w(.)$ takes values in a (rBs) Z, let $B : D(B) \subset Z \to Z$ be a linear operator generated C_0-semigroup $Y(t)$, $t \geq 0$ and $A : S \to Z$ be a bounded linear operator, such that $\|A\| \leq k$, where k is a positive constant. The control function $s(.) \in L^2(J, S)$ be a (rBs) of admissible control functions.

Here, let $Z_0 = \{w : w \in C(J, Z),\ w(0) = w_0, \|w(t)\| \leq r, \text{ for all } t \in J\}$, where r is a positive constant, be a subset of Z, since Z_0 is closed subset of a (rBs), then Z_0 is a (rBs) [7].

Also, throughout this paper, we suppose the basic hypothesis as follows:
(C_1) The linear operator $-B$ generates a C_0-semigroup $Y(t)$, $t \geq 0$, such that $\|Y(t)\| \leq l$, where l is a positive constant.
(C_2) The nonlinear operators $g : J \times Z \to Z$, and $N : J \times Z \to Z$ are continuous and satisfy Lipschitz condition on the second argument.

Let $w_1, w_2 \in Z_0$, such that:

$$\|g(t, w_1(t)) - g(t, w_2(t))\| \leq l_1 \|w_1(t) - w_2(t)\|$$

and

$$\|N(t, w_1(t)) - N(t, w_2(t))\| \leq l_2 \|w_1(t) - w_2(t)\|,$$

where l_1, l_2 are positive constants. Also, let $l_3 = \max_{t \in J} \|g(t, 0)\|$ and $k_1 = \max_{t \in J} \|w(t)\|$.

Further, the nonlinear operator $H : J \times Z \to Z$ is continuous and satisfy Lipschitz condition, such that for all $w_1, w_2 \in Z_0$, we have:

$$\begin{aligned} \|H(t, N(t, w_1(t))) - H(t, N(t, w_2(t)))\| &\leq l_4 \|N(t, w_1(t)) - N(t, w_2(t))\| \\ &\leq l_4 l_2 \|w_1(t) - w_2(t)\|, \end{aligned}$$

where l_4 are positive constant. Also, let $l_5 = \max_{t \in J} \|H(t, N(t, 0))\|$.
(C_3) The linear operator U from $L^2(J, S)$ into, defined by:

$$Us = \int_a^b Y(t - z) As(z)\, dz. \tag{3}$$

Lead to a bounded inverse operator $\tilde{U}^{-1}$ defined on $L^2(J, S)/\ker(U)$, and hence $\|\tilde{U}^{-1}\| \leq k_2$, where k_2 is a positive constant. (For a more details about there exist a bounded inverse operator of U, see [5,12].)

3.2 Controllability result of the system formulation (I)

Throughout this subsection, we want to define and find the mild solution to the problem (3.1). Let $w(.) \in Z$ be a solution of problem (3.1), from condition (C_1), we have $Y(t)$ is a C_0-semigroup generated by linear operator $-B$, for all $t \geq 0$, since $Y(t)w$ is differentiable (For more details, see [4]).

Therefore, the Z-value function $Q(z) = Y(t-z)w(z)$ is differentiable for $0 < z < t$; we can explain this as follows:

$$\begin{aligned}\frac{d}{dz}Q(z) &= Y(t-z)\frac{d}{dz}w(z) + w(z)\frac{d}{dz}Y(t-z)\\ &= Y(t-z)\left[-Bw(z) + (As)(z) + g(z,w(z)) + H(z,N(z,w(z)))\right]\\ &+ w(z)\left[BY(t-z)\right]\\ &= Y(t-z)(As)(z) + Y(t-z)g(z,w(z)) + Y(t-z)H(z,N(z,w(z))).\end{aligned}$$

Recently, we must integrating from 0 to t, yields:

$$\begin{aligned}Q(t) - Q(0) = &\int_0^t Y(t-z)As(z)dz + \int_0^t Y(t-z)g(z,w(z))dz\\ &+ \int_0^t Y(t-z)H(z,N(z,w(z)))dz.\end{aligned}$$

Since $Q(z) = Y(t-z)w(z)$, therefore:

$$\begin{aligned}w(t) = Y(t)w_0 &+ \int_0^b Y(t-z)As(z)\,dz + \int_0^t Y(t-z)g(z,w(z))\,dz\\ &+ \int_0^t Y(t-z)H(z,N(z,w(z)))\,dz.\end{aligned} \tag{4}$$

A continuous function $w(t) \in Z$ has a special from given by (3.3), is called a mild solution to the problem (3.1).

Definition 3.1 ([11])**.** The nonlinear control system (3.1) is said to be controllable on the interval $J = [0, a]$, if $\forall\ w_0,\ w_1 \in Z$, $\exists\ s(.) \in L^2(J, S)$, such that the mild solution $w(t)$ satisfying $w(a) = w_1$.

Theorem 3.2. Let Z be a (rBs), which satisfying Opial's condition and the hypothesis (C_1)-(C_3) are satisfied for the nonlinear control problem (3.1)

$$\dot{w}(t)+Bw(t) = (As)(t)+g(t,w(t))+H(t,N(t,w(t)), \text{a.e. in } J=[0,a],\ w(0)=w_0.$$

Further, suppose that: **(C_4)** Let $b_1 = all_1k_1, b_2 = all_3, b_3 = all_4l_2k_1$ and $b_4 = all_5$, such that:

$$l\left\|w_0\right\| + b_1 + b_2 + b_3 + b_4 + alkk_2\left[\left\|w_1\right\| + l\left\|w_0\right\| + b_1 + b_2 + b_3 + b_4\right] \leq r.$$

(C_5) $\varphi = all_1 + all_4l_2 + a^2l^2l_1kk_2 + a^2l^2l_4l_2kk_2$, such that $0 \leq \varphi \leq 1$
Then the nonlinear initial control system (3.1) is controllable on the interval J.

Proof. From definition (3.1), and mild solution equation (3.3), we obtain that:

$$w_1 = w(a) = Y(a)w_0 + \int_0^a Y(a-z)(As)(z) + \int_0^a Y(a-z)g(z,w(z))\,dz + \int_0^a Y(a-z)H(z,N(z,w(z)))\,dz.$$

From condition (C_3), we get:

$$w_1 = w(a) = Y(a)w_0 + Us + \int_0^a Y(a-z)g(z,w(z))\,dz + \int_0^a Y(a-z)H(z,N(z,w(z)))dz$$

and then:

$$Us = w_1 - Y(a)w_0 - \int_0^a Y(a-z)g(z,w(z))\,dz - \int_0^a Y(a-z)H(z,N(z,w(z)))dz.$$

From constriction the operator U in (C_3), since $s\,(t) = \tilde{U}^{-1}(Us\,(t))$, thus:

$$s(t) = \tilde{U}^{-1}\left(w_1 - Y(a)w_0 - \int_0^a Y(a-z)g(z,w(z))dz - \int_0^a Y(a-z)H(z,N(z,w(z)))dz\right)(t). \tag{5}$$

We can explain implication when using this control the operator defined by:

$$(\beta w)(t) = Y(a)w_0 + \int_0^t Y(t-z)(As)(z)dz + \int_0^t Y(t-z)g(z,w(z))\,dz + \int_0^t Y(t-z)H(z,N(z,w(z)))\,dz$$

has a (FP), this (FP) is then a solution of (3.3).

Note that, $(\beta w)(a) = w_1$, it know that the control s transmit the control system from the initial w_0 to w_1 in time a, provided we can obtain a (FP) of the nonlinear operator β.

Let Z be a (rBs) satisfying Opial's condition and let $Z_0 = \{w : w \in C(J,Z),\ w(0) = w_0, \|w(t)\| \leq r,\ \text{for } t \in J\}$, it's clear that Z_0 is (bcc) subset of Z [7]. Also Z_0 is weakly compact and has normal structure in (rBs) Z, see Remark 2.6 and Example 2.8.

Here, we will define a mapping $\beta : Z \to Z_0$ by:

$$(\beta w)(t) = Y(t)w_0 + \int_0^t Y(t-z)g(z,w(z))dz + \int_0^t Y(t-z)H(z,N(z,w(z)))dz + \int_0^a Y(t-\xi)A\tilde{U}^{-1}\left[w_1 - Y(a)w_0 - \int_0^a Y(a-z)g(z,w(z))\,dz - \int_0^a Y(a-z)H(z,N(z,w(z)))\,dz\right](\xi)d\xi. \tag{6}$$

Now, we want to show that the operator β is continuous and maps Z_0 into itself. Thus, for this take the norm of both sides of (3.5)

$$\begin{aligned}
\|(\beta w)(t)\| &= \Big\| Y(t)w_0 + \int_0^t Y(t-z)g(z,w(z))dz + \int_0^t Y(t-z)H(z,N(z,w(z)))dz \\
&\quad + \int_0^t Y(t-\xi)A\tilde{U}^{-1}\left[w_1 - Y(a)w_0 - \int_0^t Y(a-z)g(z,w(z))\,dz \right. \\
&\quad \left. - \int_0^a Y(a-z)H(z,N(z,w(z)))\,dz\right](\xi)d\xi \Big\|, \\
\|(\beta w)(t)\| &= \|Y(t)\|\,\|w_0\| + \left\| \int_0^t Y(t-z)\left[g(z,w(z)) - g(z,0) + g(z,0)\right]dz \right\| \\
&\quad + \left\| \int_0^t Y(t-z)\left[H(z,N(z,w(z))) - H(z,N(z,0)) + H(z,N(z,0))\right]dz \right\| \\
&\quad + \left\| \int_0^t Y(t-\xi)A\tilde{U}^{-1}\left[w_1 - Y(a)w_0 - \int_0^a Y(a-z)\left[g(z,w(z)) - g(z,0)\right.\right.\right. \\
&\quad \left.\left.\left. + g(z,0)\right]\,dz - \int_0^a Y(a-z)\left[H(z,N(z,w(z))) - H(z,N(z,0))\right.\right.\right. \\
&\quad \left.\left.\left. + \ H(z,N(z,0))\right]\,dz\right](\xi)\,d\xi \right\|, \\
\|(\beta w)(t)\| &= \|Y(t)\|\,\|w_0\| + \int_0^t \|Y(t-z)\|\left[\|g(z,w(z)) - g(z,0)\| + \|g(z,0)\|\right]dz \\
&\quad + \int_0^t \|Y(t-z)\|\left[\|H(z,N(z,w(z))) - \ H(z,N(z,0))\| \right.\right. \\
&\quad \left.\left. + \ \|H(z,N(z,0))\|\right]dz + \int_0^t \|Y(t-\xi)\|\,\|A\|\left\|\tilde{U}^{-1}\right\|\left[\|w_1\| \right.\right. \\
&\quad \left.\left. + \|Y(a)\|\,\|w_0\| + \int_0^a \|Y(a-z)\|\left[\|g(z,w(z)) - g(z,0)\| \right.\right.\right. \\
&\quad \left.\left.\left. + \ \|g(z,0)\|\right]dz + \int_0^a \|Y(a-z)\|\left[\|H(z,N(z,w(z))) \right.\right.\right. \\
&\quad \left.\left.\left. - \ H(z,N(z,0))\| + \ \|H(z,N(z,0))\|\right]\,dz\right](\xi)\,d\xi.
\end{aligned}$$

Under the conditions (C_1)-(C_3), we obtain that:

$$\begin{aligned}
\|(\beta w)(t)\| &\le l\,\|w_0\| + \int_0^t l\left[l_1\,\|w\,(z)\| + l_3\right]dz + \int_0^t l[l_4 l_2\,\|w\,(z)\| + l_5]dz \\
&\quad + \int_0^t lkk_2[\|w_1\| + l\,\|w_0\| + al\left[l_1\,\|w\,(z)\| + l_3\right] + al[l_4 l_2\,\|w\,(z)\| + l_5]](\xi)d\xi.
\end{aligned}$$

Since $w \in Z$, then by (C_2), $\|w\,(z)\| \le k_1$, and hence:

$$\begin{aligned}
\|(\beta w)(t)\| &\le l\,\|w_0\| + all_1k_1 + all_3 + all_4l_2k_1 + all_5 + alkk_2[\|w_1\| + \|w_0\| \\
&\quad + lall_1k_1 + all_3 + all_4l_2k_1 + all_5].
\end{aligned}$$

From the condition (C_4), we get that:

$$\|(\beta w)(t)\| \leq l\,\|w_0\|+b_1+b_2+b_3+b_4+alkk_2\,[\|w_1\|+l\,\|w_0\|+b_1+b_2+b_3+b_4] \leq r.$$

Since g, N and H are continuous and $\|(\beta w)(t)\| \leq r$, it follows that β is also continuous mapping from Z_0 into itself.

Now, we must prove that the operator β is nonexpansive mapping from Z_0 into itself. Let $w_1, w_2 \in Z_0$, then:

$$\begin{aligned}
&\|(\beta w_1)(t)-(\beta w_2)(t)\| \\
&= \Big\| Y(t)w_0 + \int_0^t Y(t-z)g(z,w_1(z))dz + \int_0^t Y(t-z)H(z,N(z,w_1(z)))dz \\
&+ \int_0^t Y(t-\xi)A\tilde{U}^{-1}\left[w_1 - Y(a)w_0 - \int_0^a Y(a-z)g(z,w_1(z))\,dz\right. \\
&\left. - \int_0^a Y(a-z)H(z,N(z,w_1(z)))\,dz\right](\xi)d\xi - Y(t)w_0 \\
&\int_0^t Y(t-z)g(z,w_2(z))dz + \int_0^t Y(t-z)H(z,N(z,w_2(z)))dz \\
&+ \int_0^t Y(t-\xi)A\tilde{U}^{-1}[w_2 - Y(a)w_0 - \int_0^a Y(a-z)g(z,w_2(z))\,dz \\
&- \int_0^a Y(a-z)\,H(z,N(z,w_2(z)))\,dz]\,(\xi)d\xi\Big\|,
\end{aligned}$$

$$\begin{aligned}
&\|(\beta w_1)(t)-(\beta w_2)(t)\| \\
&= \Big\| Y(t)w_0 + \int_0^t Y(t-z)g(z,w_1(z))dz + \int_0^t Y(t-z)H(z,N(z,w_1(z)))dz \\
&+ \int_0^t Y(t-\xi)A\tilde{U}^{-1}\left[w_1 - Y(a)w_0 - \int_0^a Y(a-z)g(z,w_1(z))\,dz\right. \\
&\left. - \int_0^a Y(a-z)H(z,N(z,w_1(z)))\,dz\right](\xi)d\xi - Y(t)w_0 \\
&- \int_0^t Y(t-z)g(z,w_2(z))dz - \int_0^t Y(t-z)H(z,N(z,w_2(z)))dz \\
&- \int_0^t Y(t-\xi)A\tilde{U}^{-1}[w_1 - Y(a)w_0 - \int_0^a Y(a-z)g(z,w_2(z))\,dz \\
&- \int_0^a Y(a-z)\,H(z,N(z,w_2(z)))\,dz]\,(\xi)d\xi\Big\| \\
&\|(\beta w_1)(t)-(\beta w_2)(t)\| \\
&= \Big\| \int_0^t Y(t-z)\,[g(z,w_1(z)) - g(z,w_2(z))]\,dz
\end{aligned}$$

$$+\int_0^t Y(t-z)\left[H(z,N(z,w_1(z)))-H(z,N(z,w_2(z)))\right]dz$$
$$+\int_0^t Y(t-\xi)A\tilde{U}^{-1}\left[\int_0^a Y(a-z)\left[g(z,w_1(z))-g(z,w_2(z))\right]dz\right.$$
$$\left.-\int_0^a Y(a-z)\ \left[H(z,N(z,w_1(z)))-H(z,N(z,w_2(z)))\right]dz\right](\xi)d\xi\Bigg\|$$
$$\|(\beta w_1)(t)-(\beta w_2)(t)\|$$
$$\leq\int_0^t\|Y(t-z)\|\,\|g(z,w_1(z))-g(z,w_2(z))\|\,dz$$
$$+\int_0^t\|Y(t-z)\|\,\|H(z,N(z,w_1(z)))-H(z,N(z,w_2(z)))\|\,dz$$
$$+\ w_2(z)))\|\,dz+\int_0^t\|Y(t-\xi)\|\,\|A\|\left\|\tilde{U}^{-1}\right\|\left[\int_0^a\|Y(a-z)\|\right.$$
$$\|g(z,w_1(z))-g(z,w_2(z))\|\,dz+\int_0^a\|Y(a-z)\|$$
$$\left.\|H(z,N(z,w_1(z)))-H(z,N(z,\ w_2(z)))\|\,dz\right](\xi)d\xi$$

From condition (C_1)-(C_3), we obtain that:

$$\|(\beta w_1)(t)-(\beta w_2)(t)\|$$
$$\leq\int_0^t ll_1\|w_1(z)-w_2(z)\|\,dz+\int_0^t ll_4l_2\|w_1(z)-w_2(z)\|\,dz$$
$$+\int_0^t lkk_2[all_1\|w_1(z)-w_2(z)\|+all_4l_2\|w_1(z)-w_2(z)\|]dz,$$
$$\|(\beta w_1)(t)-(\beta w_2)(t)\|$$
$$\leq[all_1+all_4l_2]\|w_1(t)-w_2(t)\|+[a^2l^2l_1kk_2+a^2l^2l_4l_2kk_2]\|w_1(t)-w_2(t)\|,$$
$$\|(\beta w_1)(t)-(\beta w_2)(t)\|$$
$$\leq\left[all_1+all_4l_2+a^2l^2l_1kk_2+a^2l^2l_4l_2kk_2\right]\|w_1(t)-w_2(t)\|.$$

By condition (C_5), we get that:

$$\|(\beta w_1)(t)-(\beta w_2)(t)\|\leq\varphi\|w_1(t)-w_2(t)\|.$$

Consequentially, β is nonexpansive mapping. Thus from Theorem (2.9), there exists a (FP) $w\in Z_0$, such that $(\beta w)(t)=w(t)$, and hence this (FP) is a solution of system (3.1) on the interval J, which satisfied $w(a)=w_1$, therefore the nonlinear control system (3.1) is controllable on J. □

3.3 System formulation (II)

Through this subsection, we given Z be a (rBs) with norm $\|.\|$, be a (Bs) and consider the initial value optimal control system in infinite dime1nsional state

space:

$$\begin{aligned} &\frac{d}{dt}w(t)+\frac{d}{dt}F(t,w(t))=Bw(t)+(As)(t)+g(t,w(t))+H(t,N(t,w(t))\\ &\text{a.e. in } J=[0,a],\ w(0)=w_0, \end{aligned} \tag{7}$$

where the linear operator B generates a (C_0-semigroup) $Y(t), t \geq 0$ and $A : S \to Z$ be a bounded linear operator, such that $\|A\| \leq k$, where k is a positive constant. Also, let the control function $s(.) \in L^1(J, S)$ is a (Bs) of admissible control functions.

We shall assume with hypothesis (C_1), (C_2) in the system formulation (I) the following condition:

(E_1) The nonlinear operator $F : J \times Z \to Z$ is continuous and satisfies Lipschitz condition on second argument, for all $w_1, w_2 \in Z_0$, we have:

$$\|F(t, w_1(t)) - F(t, w_2(t))\| \leq h_1 \|w_1(t) - w_2(t)\|,$$

where h_1 is a positive constant. Also, $h_2 = \max_{t\in J} \|F(t, 0)\|$.

Now, it easy to find the mild solution of the problem (3.6) by using the same technique in subsection (3.2) as follows:

A continuous function $w : [0, a] \to Z$ is given as follows:

$$\begin{aligned} w(t) &= Y(t)w_0 + Y(t)F(0, w_0) - F(t, w(t))\\ &+ \int_0^b Y(t-z)As(z)\,dz + \int_0^t Y(t-z)g(z, w(z))\,dz\\ &+ \int_0^t Y(t-z)H(z, N(z, w(z)))\,dz - \int_0^t F(z, w(z))BY(t-z)\,dz \end{aligned} \tag{8}$$

is called a mild solution to the optimal control problem.

Theorem 3.3. Let Z be a (rBs), which satisfying Opial's condition and the hypothesis (C_1)-(C_3) and (E_1) are satisfied for the initial value optimal control problem (3.6). Further, suppose that:

(E_2) There exists a positive function $f \in L^1(0, a)$, with $\|BY(t)\| \leq f(t)$ a.e., $t \in (0, a)$; such that $\int_0^a f(t)dt \leq c$, where c is a positive constant.

(E_3) Let $b_1 = (h_1 \|w_0\| + h_2)$, $b_2 = (h_1k_1 + h_2)$, $b_3 = al\,(l_1k_1 + l_3)$ and $b_4 = al(l_4l_2k_1 + l_5)$, such that:

$$\begin{aligned} &l\|w_0\| + lb_1 + b_2 + b_3 + b_4 + b_2c + alkk_2[\|w_1\| + l\|w_0\| + lb_1\\ &+ (h_1\|w_1\| + h_2) + b_3 + b_4 + b_2c] \leq r. \end{aligned}$$

(E_4) $q = h_1 + all_1 + all_4l_2 + h_1c + a^2l^2l_1kk_2 + a^2l^2l_4l_2kk_2 + alkk_2h_1c$, where $0 \leq q \leq 1$. Then the optimal control problem (3.6) is controllability on interval J.

Proof. By using mild solution (3.7) and definition (3.1), we obtain that:

$$\begin{aligned} w_1 = w(a) &= Y(a)w_0 + Y(a)F(0,w_0) - F(a,w(a)) + \int_0^a Y(a-z)As(z)\,dz \\ &+ \int_0^a Y(a-z)g(z,w(z))\,dz + \int_0^a Y(a-z)H(z,N(z,w(z)))\,dz \\ &- \int_0^a F(z,w(z))BY(a-z)\,dz. \end{aligned}$$

Combined with condition (C_3), we get that:

$$\begin{aligned} w_1 &= Y(a)w_0 + Y(a)F(0,w_0) - F(a,w(a)) + Us + \int_0^a Y(a-z)g(z,w(z))\,dz \\ &+ \int_0^a Y(a-z)H(z,N(z,w(z)))\,dz - \int_0^a F(z,w(z))BY(a-z)\,dz \end{aligned}$$

and then:

$$\begin{aligned} Us &= w_1 - Y(a)w_0 - Y(a)F(0,w_0) + F(a,w(a)) - \int_0^a Y(a-z)g(z,w(z))\,dz \\ &- \int_0^a Y(a-z)H(z,N(z,w(z)))\,dz + \int_0^a F(z,w(z))BY(a-z)\,dz \end{aligned}$$

From construction of $\widetilde{U}^{-1}$ in (C_3), since:

$$\begin{aligned} s(t) &= \tilde{U}^{-1}[w_1 - Y(a)w_0 - Y(a)F(0,w_0) \\ &+ F(a,w(a)) - \int_0^a Y(a-z)g(z,w(z))\,dz \\ &- \int_0^a Y(a-z)H(z,N(z,w(z)))\,dz + \int_0^a F(z,w(z))BY(a-z)\,dz](t). \end{aligned} \tag{9}$$

We can explain the implication when using this control the operator defined by:

$$\begin{aligned} \beta w(t) &= Y(t)w_0 + Y(t)F(0,w_0) - F(t,w(t)) + \int_0^b Y(t-z)(As)(z)\,dz \\ &+ \int_0^t Y(t-z)g(z,w(z))\,dz + \int_0^t Y(t-z)H(z,N(z,w(z)))\,dz \\ &- \int_0^t F(z,w(z))BY(t-z)\,dz. \end{aligned} \tag{10}$$

By using the same way in proof of Theorem (3.1) we obtain that β is nonexpansive mapping, and hence it has a (FP) $w(t)$, which is a solution of nonlinear system (3.6), and satisfying $w(a) = w_1$. Therefore the optimal control problem (3.6) is controllable in J. □

4. Application

Consider the following partial integro-differential system:

$$y_t(t,x) = y_{xx}(t,x) + (As)(t) + \omega_1(t, y_{xx}(t,x)) + \int_0^t \omega_3(t,z,y_{xx}(z,x)),$$

$$\int_0^t \omega_2(z,\tau,y_{xx}(\tau,x)d\tau)dz,\ x \in I = (0,1),\ t \in J = [0,a] \tag{11}$$

$$y(0,1) = y(1,t) = 0 \qquad \text{(initial condition)} \tag{12}$$

$$y(x,0) = y_0(x),\ x \in I \quad \text{(boundary condition)}, \tag{13}$$

where $A : S \to E$ with $S \subset J$, and $E = L^2[I,R]$ is a linear operator, such that there exists an invertibled operator U^{-1} on $L^2[J,S]/\ker(U)$, where U is defined by:

$$Us = \int_0^a Y(a-z)As(z)\,dz,$$

$Y(t)$ is a strongly continuous semigroup and $\omega_1 : J \times E \to E$, $\omega_2 : J \times J \times E \to E$, $\omega_3 : J \times J \times E \times E \to E$, are all continuously differentiable functions by positive constants. Thus the problem (4.1)-(4.3) can be brought to the form (3.1) as given in [13] by making suitable choices of B, A, g, N, H.

Let $E = L^2[I,R]$, $Bz = w_{xx}$, $A : S \to E$, and $D(B) = \{w \in E, w_{xx} \in X; w(0) = w(1) = 0\}$, be such that the condition C_3 is satisfied and let $g(t,w)(x) = \omega_1(t, w_{xx}), (t,w) \in J \times E$,

$$N(t,w)(x) = \int_0^t \omega_2(z,\tau,w_{xx}(x))d\tau$$

and

$$H(t,w,\mu)(x) = \int_0^t \omega_3(t,z,w_{xx}(x),\mu(x))dt.$$

Therefore, the system (4.1)-(4.2) becomes an abstract formulation of (3.1). See, Theorem 3 in [13] the solutions are all bounded, and then the conditions presented in the Theorem (3.1) are all satisfied. Consequential the integro-differential system (4.1)-(4.2) is controllable on J.

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Accepted: April 12, 2020

On graded J_{gr}-semiprime submodules

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Abstract. Let G be a group with identity e. Let R be a G-graded commutative ring and M a graded R-module. A proper graded submodule S of M is called *a graded semiprime submodule* if whenever $r \in h(R)$, $m \in h(M)$ and $k \in \mathbb{Z}^+$ with $r^k m \in S$, then $rm \in S$. In this paper, we introduce the concept of graded J_{gr}-semiprime submodule as a new generalization of graded semiprime submodule and we give a number of results concerning such graded modules. We say that a proper graded submodule N of M is *a graded J_{gr}-semiprime submodule of M* if whenever $r \in h(R)$, $m \in h(M)$ and $k \in \mathbb{Z}^+$ with $r^k m \in N$, then $rm \in N + J_{gr}(M)$, where $J_{gr}(M)$ is the graded Jacobson radical of M.

Keywords: graded J_{gr}-semiprime submodule, graded semiprime submodule, graded prime.

1. Introduction and preliminaries

Throughout this paper all rings are commutative with identity and all modules are unitary. Graded semiprime submodules of graded modules over graded commutative rings, have been introduced and studied in [2, 3, 4, 5, 7, 11]. Here, we introduce the concept of graded J_{gr}-semiprime submodules of graded modules as a new generalization of graded semiprime submodules and show a number of results in this class.

First, we recall some basic properties of graded rings and modules which will be used in the sequel. We refer to [6] and [8-10] for these basic properties and more information on graded rings and modules.

Let G be a multiplicative group and e denote the identity element of G. A ring R is called a graded ring (or G-graded ring) if there exist additive subgroups

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R_α of R indexed by the elements $\alpha \in G$ such that $R = \bigoplus_{\alpha\in G} R_\alpha$ and $R_\alpha R_\beta \subseteq R_{\alpha\beta}$ for all $\alpha, \beta \in G$. The elements of R_α are called homogeneous of degree α and all the homogeneous elements are denoted by $h(R)$, i.e. $h(R) = \cup_{\alpha\in G} R_\alpha$. If $r \in R$, then r can be written uniquely as $\sum_{\alpha\in G} r_\alpha$, where r_α is called a homogeneous component of r in R_α. Moreover, R_e is a subring of R and $1 \in R_e$. Let $R = \bigoplus_{\alpha\in G} R_\alpha$ be a G-graded ring. An ideal I of R is said to be a graded ideal if $I = \bigoplus_{\alpha\in G}(I \cap R_\alpha) := \bigoplus_{\alpha\in G} I_\alpha$. Let $R = \bigoplus_{\alpha\in G} R_\alpha$ be a G-graded ring. A Left R-module M is said to be *a graded R-module* (or *G-graded R-module*) if there exists a family of additive subgroups $\{M_\alpha\}_{\alpha\in G}$ of M such that $M = \bigoplus_{\alpha\in G} M_\alpha$ and $R_\alpha M_\beta \subseteq M_{\alpha\beta}$ for all $\alpha, \beta \in G$. Here, $R_\alpha M_\beta$ denotes the additive subgroup of M consisting of all finite sums of elements $r_\alpha m_\beta$ with $r_\alpha \in R_\alpha$ and $m_\beta \in M_\beta$. Also if an element of M belongs to $\cup_{\alpha\in G} M_\alpha = h(M)$, then it is called a homogeneous. Note that M_α is an R_e-module for every $\alpha \in G$. So, if $I = \bigoplus_{\alpha\in G} I_\alpha$ is a graded ideal of R, then I_α is an R_e-module for every $\alpha \in G$. Let $R = \bigoplus_{\alpha\in G} R_\alpha$ be a G-graded ring. A submodule N of M is said to be *a graded submodule of M* if $N = \bigoplus_{\alpha\in G}(N \cap M_\alpha) := \bigoplus_{\alpha\in G} N_\alpha$. In this case, N_α is called the α-component of N. In this case, N_g is called the g-component of N. Moreover, M/N becomes a G-graded R-module with g-component $(M/N)_g := (M_g + N)/N$ for $g \in G$.

Let R be a G-graded ring and M a graded R-module. A proper graded submodule S of M is said to be *a graded semiprime submodule* if whenever $r \in h(R)$, $m \in h(M)$ and $k \in \mathbb{Z}^+$ with $r^k m \in S$, then $rm \in S$ (see [5].)

A graded submodule N of M is said to be *a graded maximal* if $N \neq M$ and if there is a graded submodule L of M such that $N \subseteq L \subseteq M$, then $N = L$ or $L = M$ (see [10].)

The graded Jacobson radical of a graded module M, denoted by $J_{gr}(M)$, is defined to be the intersection of all gr-maximal submodules of M (if M has no gr-maximal submodule then we shall take, by definition, $J_{gr}(M) = M$) (see [10].)

2. Results

Definition 2.1. Let R be a G-graded ring and M a graded R-module. A proper graded submodule N of M is called *a graded J_{gr}-semiprime submodule of M* if whenever $r^k m \in N$ where $r \in h(R)$, $m \in h(M)$ and $k \in \mathbb{Z}^+$, implies $rm \in N + J_{gr}(M)$.

Theorem 2.2. *Let R be a G-graded ring, M a graded R-module and N a proper graded submodule of M. If N is a graded semiprime submodule of M, then N is a graded J_{gr}-semiprime submodule of M.*

Proof. Suppose that N is a graded semiprime submodule of M. Let $r \in h(R)$, $m \in h(M)$ and $k \in \mathbb{Z}^+$ such that $r^k m \in N$, then by our assumption we get $rm \in N$, hence $rm = rm + 0 \in N + J_{gr}(M)$. □

The next example shows that a graded J_{gr}-semiprime submodule is not necessarily graded semiprime submodule.

Example 2.3. Let $G = \mathbb{Z}_2$ and $R = \mathbb{Z}$ be a G-graded ring with $R_0 = R$ and $R_1 = \{0\}$. Let $M = \mathbb{Z}_8$ be a graded R-module with $M_0 = M$ and $M_1 = \{0\}$. Now, consider the graded submodule $N =< 4 >$ of M. Then it is a graded J_{gr}-semiprime submodule of M. However, the graded submodule N is not graded semiprime submodule of M, since $2^2 \cdot 1 = 4 \in N$ but $2 \cdot 1 = 2 \notin N$.

Example 2.4. Let $G = \mathbb{Z}_2$ and $R = \mathbb{Z}$ be a G-graded ring with $R_0 = R$ and $R_1 = \{0\}$. Let $M = \mathbb{Z}$ be a graded R-module with $M_0 = M$ and $M_1 = \{0\}$. Then $J_{gr}(M) = \{0\}$. Now, $N = 4\mathbb{Z}$ is not a graded J_{gr}-semiprime submodule of M since $2^2 \cdot 1 = 4 \in N$, but $2 \cdot 1 = 2 \notin N + \{0\} = N$.

Remark 2.5. Let R be a G-graded ring and M a graded R-module.

(i) If $J_{gr}(M) = 0$, then every graded J_{gr}-semiprime submodule of M is a graded semiprime submodule of M.

(ii) If $J_{gr}(M)$ is contained in every graded submodule of M, then every graded J_{gr}-semiprime submodule of M is a graded semiprime submodule of M.

The following Lemma is known, (see [7, Lemma 1.2)], but we write it her for the sake of references.

Lemma 2.6. *Let R be a G-graded ring and M a graded R-module. Then the the following hold:*

1. *If N is a graded submodule of M, $r \in h(R)$, $x \in h(M)$ and I is a graded ideal of R, then Rx, IN and rN are graded submodules of M.*

2. *If N and K are graded submodules of M, then $N + K$ and $N \cap K$ are also graded submodules of M.*

Theorem 2.7. *Let R be a G-graded ring, M a graded R-module and N a graded J_{gr}-semiprime submodule of M. If L is a proper graded submodule of M such that $L \nsubseteq N$ and $J_{gr}(M) = J_{gr}(L)$, then $N \cap L$ is a graded J_{gr}-semiprime submodule of L.*

Proof. $N \cap L$ is a proper graded submodule of L by Lemma 2.6 and since $L \nsubseteq N$. Now, let $r \in h(R)$, $m \in h(M) \cap L$ and $k \in \mathbb{Z}^+$ such that $r^k m \in N \cap L$, so $r^k m \in N$. This yields that $rm \in N + J_{gr}(M) = N + J_{gr}(L)$. Hence $rm \in (N + J_{gr}(L)) \cap L$. Now, by the modular law, $rm \in (N \cap L) + J_{gr}(L)$. Therefore $N \cap L$ is a graded J_{gr}-semiprime submodule of L. □

Theorem 2.8. *Let R be a G-graded ring, M a graded R-module, N and L be a graded J_{gr}-semiprime submodules of M with $L \nsubseteq N$ and either $J_{gr}(M) \subseteq L$ or $J_{gr}(M) \subseteq N$. Then $N \cap L$ is a graded J_{gr}-semiprime submodule of M.*

Proof. Since N is a proper graded submodule of M, $N \cap L$ is a proper graded submodule of M. Now, let $r \in h(R)$, $m \in h(M)$ and $k \in \mathbb{Z}^+$ such that $r^k m \in N \cap L$. Then $r^k m \in N$ and $r^k m \in L$. Since N and L are graded J_{gr}-semiprime submodules of M, we get $rm \in N + J_{gr}(M)$ and $rm \in L + J_{gr}(M)$. This implies that $rm \in (N + J_{gr}(M)) \cap (L + J_{gr}(M))$. Now, if $J_{gr}(M) \subseteq N$, then $rm \in N \cap (L + J_{gr}(M)) \subseteq (N \cap L) + J_{gr}(M)$ by the modular law. Similarly if $J_{gr}(M) \subseteq L$, we get $rm \in (N \cap L) + J_{gr}(M)$. Thus $N \cap L$ is a graded J_{gr}-semiprime submodule of M. □

Theorem 2.9. *Let R be a G-graded ring, M a graded R-module and N a proper graded submodule of M. Then the following statements are equivalent:*

(i) *N is a graded J_{gr}-semiprime submodule of M.*

(ii) *For every graded submodule K of M and every graded ideal I of M and $n \in \mathbb{Z}^+$ such that $I^n K \subseteq N$, $IK \subseteq N + J_{gr}(M)$.*

Proof. $(i) \Rightarrow (ii)$ Suppose that N is a graded J_{gr}-semiprime submodule of M. Let I be a graded ideal of R, K be a graded submodule of M and $n \in \mathbb{Z}^+$ such that $I^n K \subseteq N$. Let $i = \sum_{g \in G} i_g \in I$ and $k = \sum_{g \in G} k_g \in K$ such that $ik \in IK$, hence for any $g, h \in G$ we have $i_g^n k_h \in I^n K \subseteq N$. Then $i_g k_h \in N + J_{gr}(M)$ since N is a graded J_{gr}-semiprime submodule of M. This yields that $ik \in N + J_{gr}(M)$, and hence $IK \subseteq N + J_{gr}(M)$.

$(ii) \Rightarrow (i)$ Let $r \in h(R)$, $m \in h(M)$ and $n \in \mathbb{Z}^+$ such that $r^n m \in N$. Let $I =< r >$ be a graded ideal of R generated by r and $K = Rm$ be a graded submodule of M generated by m. Then $I^n K \subseteq N$, so by the assumption we have $IK \subseteq N + J_{gr}(M)$, hence $rm \in IK \subseteq N + J_{gr}(M)$. Therefore N is a graded J_{gr}-semiprime submodule of M. □

Recall that a proper graded submodule N of a graded R-module M is said to be *a gr-small submodule of M* (for short $N <<_g M$) if for every proper graded submodule L of M, we have $N + L \neq M$ (see [1].)

Theorem 2.10. *Let R be a G-graded ring and M a graded R-module. Then $J_{gr}(M) = \sum\{N : N \text{ is a gr-small submodule of } M\}$.*

Proof. Let $\Gamma = \sum\{N : N \text{ is a } gr\text{-small submodule of } M\}$. Suppose that S is a gr-small submodule of M, and let L be any gr-maximal submodule of M. Then $S \subseteq L$. Otherwise, we have $L + S = M$. But S is a gr-small submodule of M, so $L = M$, a contradiction. It follows that $\Gamma \subseteq J_{gr}(M)$. Now, let $m \in J_{gr}(M) \cap h(M)$. Suppose that the graded submodule Rm is not gr-small in M, so there is a proper graded submodule B of M such that $Rm + B = M$. Set $\Omega = \{V : V \text{ is a proper graded submodule of } M \text{ such that } Rm + V = M\}$. Since $B \in \Omega$, $\Omega \neq \emptyset$. By the Maximal Principle, Ω has maximal element, say V_0. We claim that V_0 is gr-maximal in M. Otherwise, there is a graded submodule C of M such that $V_0 \subsetneq C \subsetneq M$. Thus $M = Rm + V_0 \subseteq Rm + C$.

So $C \in \Omega$, which contradicts the maximality of V_0. Hence V_0 is gr-maximal in M. So $m \in J_{gr}(M) \subseteq V_0$ and $Rm \subseteq V_0$. Since $Rm + V_0 = M$, it follows that $V_0 = M$, a contradiction. Hence Rm is gr-small in M and so $J_{gr}(M) \subseteq \Gamma$. Therefore, $J_{gr}(M) = \sum\{N : N \text{ is a } gr\text{-small submodule of } M\}$. □

Let R be a G-graded ring and M, M' graded R-modules. Let $\varphi : M \to M'$ be an R-module homomorphism. Then φ is said to be a graded homomorphism if $\varphi(M_g) \subseteq M'_g$ for all $g \in G$ (see [10].)

Lemma 2.11. *Let R be a G-graded ring and M, M' be a graded R-modules.*

(i) *If K and N are graded submodules of M with $K \subseteq N$ and $N <<_g M$, then $K <<_g M$.*

(ii) *If N and L are graded submodules of M, then $N + L <<_g M$ if and only if $N <<_g M$ and $L <<_g M$.*

(iii) *If $f : M \to M'$ is a graded homomorphism and K is a graded submodule of M such that $K <<_g M$, then $f(K) <<_g M'$.*

(iv) *If $f : M \to M'$ is a graded epimorphism, with $ker(f) <<_g M$, and K is a graded submodule of M' such that $K <<_g M'$, then $f^{-1}(K) <<_g M$.*

Proof. (i) Let L be a graded submodule of M such that $L + K = M$. Then $M = L + K \subseteq L + N \subseteq M$, so $L + N = M$. Since $N <<_g M$, $L = M$. Hence $K <<_g M$.

(ii) Suppose that $N + L <<_g M$. Since $N \subseteq N + L$ and $L \subseteq N + L$, by part i, we get $N <<_g M$ and $L <<_g M$. Conversely, suppose that $N + L + K = M$, where K is a graded submodule of M. Since $N <<_g M$, $L + K = M$ and since $L <<_g M$, $K = M$. Hence $L + N <<_g M$.

(iii) Suppose that U is a graded submodule of M' such that $U + f(K) = M'$. This implies that $f^{-1}(U) + K = M$. Since $K <<_g M$, we get $f^{-1}(U) = M$, so $U = f(M)$. Hence $f(K) \subseteq f(M) = U$ and $U = M'$. Therefore $f(K) <<_g M'$.

(iv) Suppose that U is a graded submodule of M such that $f^{-1}(K)+U = M$. Then $K + f(U) = f(M) = M'$. Since $K <<_g M'$, we get $f(U) = M' = f(M)$. Claim that $U = M$, suppose not $U \neq M$. Since $ker(f) <<_g M$, $ker(f)+U \neq M$, so there exists $m \in h(M)$ such that $m \notin ker(f)+U$, but $f(U) = f(M)$, so there exists $u \in h(M) \cap U$ such that $f(u) = f(m)$, it follows that $m - u \in ker(f)$, this implies that $m = (m - u) + u \in ker(f) + U$, a contradiction, so $U = M$. Hence $f^{-1}(K) <<_g M$. □

Theorem 2.12. *Let R be a G-graded ring and M, M' be a graded R-modules.*

(i) *If $\varphi : M \to M'$ is a graded homomorphism, then $\varphi(J_{gr}(M)) \subseteq J_{gr}(M')$.*

(ii) *If $\varphi : M \to M'$ is a graded epimorphism and $ker(\varphi) <<_g M$, then $\varphi(J_{gr}(M)) = J_{gr}(M')$.*

Proof. (i) By Theorem 2.10, we have $J_{gr}(M) = \sum\{N : N <<_g M\}$. Then $J_{gr}(M) <<_g M$ by Lemma 2.11(ii). So $\varphi(J_{gr}(M)) <<_g M'$ by Lemma 2.11(iii) and hence $\varphi(J_{gr}(M)) \subseteq \sum\{N' : N' <<_g M'\} = J_{gr}(M')$.

(ii) By part (i), we have $\varphi(J_{gr}(M)) \subseteq J_{gr}(M')$. Now, similarly by Lemma 2.11(iv), we get $\varphi^{-1}(J_{gr}(M')) \subseteq J_{gr}(M)$, this implies that $J_{gr}(M') \subseteq \varphi(J_{gr}(M))$. Therefore, $\varphi(J_{gr}(M)) = J_{gr}(M')$. □

Theorem 2.13. *Let R be a G-graded ring, M and M' be a graded R-modules and N a proper graded submodule of M. Let $\varphi : M \to M'$ be a graded epimorphism with $ker(\varphi) \subseteq N$. If N is a graded J_{gr}-semiprime submodule of M, then $\varphi(N)$ is a graded J_{gr}-semiprime submodule of M'.*

Proof. Suppose that N is a graded J_{gr}-semiprime submodule of M. It is easy to see $\varphi(N)$ is a proper graded submodule of M'. Now, let $r \in h(R)$, $m' \in h(M')$ and $k \in \mathbb{Z}^+$ such that $r^k m' \in \varphi(N)$, hence there exists $m \in h(M)$ such that $\varphi(m) = m'$ since φ is a graded epimorphism. So $r^k m' = r^k \varphi(m) = \varphi(r^k m) \in \varphi(N)$, it follows that there exists $n \in N \cap h(M)$ such that $\varphi(r^k m) = \varphi(n)$ which implies that $r^k m - n \in ker(\varphi) \subseteq N$ and hence $r^k m \in N$. Since N is a graded J_{gr}-semiprime submodule of M, we get $rm \in N + J_{gr}(M)$. Then $\varphi(rm) = r\varphi(m) = rm' \in \varphi(N) + \varphi(J_{gr}(M)) \subseteq \varphi(N) + J_{gr}(M')$ by Theorem 2.12(i), so $rm' \in \varphi(N) + J_{gr}(M')$. Therefore $\varphi(N)$ is a graded J_{gr}-semiprime submodule of M'. □

Theorem 2.14. *Let R be a G-graded ring, M and M' be a graded R-modules. Let $\varphi : M \to M'$ be a graded epimorphism with $ker(\varphi) <<_g M$. If N' is a graded J_{gr}-semiprime submodule of M', then $\varphi^{-1}(N')$ is a graded J_{gr}-semiprime submodule of M.*

Proof. Suppose that N' is a graded J_{gr}-semiprime submodule of M'. It is easy to see $\varphi^{-1}(N')$ is a proper graded submodule of M. Now, let $r \in h(R)$, $m \in h(M)$ and $k \in \mathbb{Z}^+$ such that $r^k m \in \varphi^{-1}(N')$. Then $\varphi(r^k m) = r^k \varphi(m) \in N'$. Since N' is a graded J_{gr}-semiprime submodule of M' and $\varphi(m) \in h(M')$, we get $r\varphi(m) \in N' + J_{gr}(M')$. By Theorem 2.12(ii), we have $\varphi(J_{gr}(M)) = J_{gr}(M')$. This yields that $\varphi(rm) \in N' + \varphi(J_{gr}(M))$, hence $rm \in \varphi^{-1}(N') + J_{gr}(M)$. Therefore $\varphi^{-1}(N')$ is a graded J_{gr}-semiprime submodule of M. □

Definition 2.15. Let R be a G-graded ring. A graded R-module M is said to be *a gr-fully J_{gr}-semiprime module* if every proper graded submodule of M is a graded J_{gr}-semiprime submodule of M.

Recall that a graded R-module M is said to be *a gr-hollow module* if each of its proper graded submodules is gr-small (see [1].)

Theorem 2.16. *Let R be a G-graded ring, M a gr-hollow R-module and M' any graded R-module. Let $\varphi : M \to M'$ be a graded epimorphism such that $ker(\varphi)$ contained in each graded submodule of M. Then M is gr-fully J_{gr}-semiprime R-module if and only if M' is a gr-fully J_{gr}-semiprime R-module.*

Proof. Suppose that M is a gr-fully J_{gr}-semiprime R-module. Let K' be a proper graded submodule of M', it follows that $\varphi^{-1}(K')$ is a proper graded submodule of M. This yields that $\varphi^{-1}(K')$ is a graded J_{gr}-semiprime submodule of M and $ker(\varphi) \subseteq \varphi^{-1}(K')$. By Theorem 2.13, $\varphi(\varphi^{-1}(K')) = K'$ is a graded J_{gr}-semiprime submodule of M'. Therefore M' is a gr-fully J_{gr}-semiprime module. Conversely, suppose that M' is a gr-fully J_{gr}-semiprime R-module. Let K be a proper graded submodule of M, it follows that $\varphi(K)$ is a proper graded submodule of M'. This yields that $\varphi(K)$ is a graded J_{gr}-semiprime submodule of M'. Since M is gr-hollow module, $ker(\varphi) <<_g M$. By Theorem 2.14, we get $\varphi^{-1}(\varphi(K)) = K$ is a graded J_{gr}-semiprime submodule of M. Therefore M is a gr-fully J_{gr}-semiprime module. □

Definition 2.17. Let R be a G-graded ring. A graded R-module M is said to be a graded F-J_{gr}-regular if for each $m \in h(M)$ and each $r \in h(R)$, there exists $t \in h(R)$ such that $rm - rtrm \in J_{gr}(M)$.

Theorem 2.18. *Let R be a G-graded ring and M a graded R-module. If M is a graded F-J_{gr}-regular module, then M is a gr-fully J_{gr}-semiprime module.*

Proof. Let N be a proper graded submodule of M, and let $r \in h(R)$, $m \in h(M)$ and $k \in \mathbb{Z}^+$ such that $r^k m \in N$. Since M is a graded F-J_{gr}-regular module, there exists $t \in h(R)$ such that $rm = rtrm + s_1$ where $s_1 \in J_{gr}(M)$, so we have $rm = rtrm + s_1 = rt(rm) + s_1 = rt(rtrm + s_1) + s_1 = r^2t^2(rm) + s_2$, where $s_2 \in J_{gr}(M)$, then after a finite number of steps we have $rm = r^k t^{\,k-1} m + s_{k-1} \in N + J_{gr}(M)$. Thus N is a graded J_{gr}-semiprime submodule of M. Therefore M is a gr-fully J_{gr}-semiprime module. □

Definition 2.19. Let R be a G-graded ring and M a graded R-module. A proper graded submodule N of M is said to be a graded J_{gr}-pure if $IM \cap N = IN + (J_{gr}(M) \cap IM \cap N)$, for each proper graded ideal I of R.

Lemma 2.20. *Let R be a G-graded ring and M a graded R-module. If each proper graded submodule of M is a graded J_{gr}-pure, then M is a graded F-J_{gr}-regular module.*

Proof. Let $r \in h(R)$ and $m \in h(M)$. Set $I =< r >$ and $N =< r > m$. Since M is a graded J_{gr}-pure, we get $IM \cap N = IN + (J_{gr}(M) \cap IM \cap N)$. It follows that $< r > m =< r >< r > m + (J_{gr}(M) \cap < r > m) =< r^2 > m + (J_{gr}(M) \cap < r > m) \subseteq r^2 m + J_{gr}(M)$. So $rm \in < r^2 > m + J_{gr}(M)$. Then there exist $t \in h(R)$ and $s \in J_{gr}(M) \cap h(M)$ such that $rm = tr^2 m + s$. Therefore M is a graded F-J_{gr}-regular module. □

Theorem 2.21. *Let R be a G-graded principal ideal domain and M a graded R-module. Then M is a graded F-J_{gr}-regular module if and only if M is a graded gr-fully J_{gr}-semiprime module.*

Proof. ($\Rightarrow$) By Theorem 2.18.

($\Leftarrow$) Suppose that M is a graded gr-fully J_{gr}-semiprime module. Let K be a proper graded submodue of M and I a proper graded ideal of R. Then there exists $r \in h(R)$ such that $I =< r >$. Clearly $rK+(J_{gr}(M)\cap rM\cap K) \subseteq rM\cap K$. Now, let $x \in rM \cap K$, then $x = rm \in rM$, for some $m = \sum_{g\in G} m_g \in M$ and $x \in K$, so $rx = r^2m = \sum_{g\in G} r^2m_g \in rK$. It follows that $r^2m_g \in rK$, for each $g \in G$. Then $rm_g \in rK + J_{gr}(M)$ as rK is a graded J_{gr}-semiprime submodule of M. So $rm \in rK + J_{gr}(M)$. Hence $x = rm \in (rK + J_{gr}(M)) \cap (rM \cap K)$. By modular law, $x \in rK + (J_{gr}(M) \cap rM \cap K)$ since $rK \subseteq (rM \cap K)$. This yields that $IM \cap K = IK + (J_{gr}(M) \cap IM \cap K)$. Thus M is a graded F-J_{gr}-regular module by Lemma 2.20. □

Recall that a G-graded ring R is said to be *a graded regular ring* if for each $r \in h(R)$, there exists $t \in h(R)$ such that $r = rtr$ (see [10].)

Theorem 2.22. *Let R be a G-graded ring and M a graded R-module. If $R/(Ann_R(x) + J_{gr}(R))$ is a gr-regular ring for each $x \in h(M)$, then M is a gr-fully J_{gr}-semiprime module.*

Proof. Let N be a proper graded submodule of M and let $r \in h(R), m \in h(M)$ and $k \in \mathbb{Z}^+$ such that $r^km \in N$, now since $R/(Ann_R(m)+J_{gr}(R))$ is a gr-regular ring, there exists $t \in h(R)$ such that $r+(Ann_R(m)+J_{gr}(R)) = rtr+(Ann_R(m)+J_{gr}(R))$. So $r - r^2t \in Ann_R(m) + J_{gr}(R)$, then $r - r^2t - s \in Ann_R(m)$ for some $s \in J_{gr}(R)$. Hence $(r - r^2t - s)m = 0$ then $(r - r^2t)m = sm \in J_{gr}(R)M \subseteq J_{gr}(M)$. Thus $rm - r^2tm \in J_{gr}(M)$. Hence $rm = r^2tm + s_1$ where $s_1 \in J_{gr}(M)$ and then $rm = rt(r^2tm + s_1) + s_1 = r^3t^2m + s_2$, where $s_2 \in J_{gr}(M)$, then after a finite number of steps, we get $rm = r^kt^{k-1}m + s_{k-1} \in N + J_{gr}(M)$. Hence N is a graded J_{gr}-semiprime submodule of M. □

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Accepted: 28.02.2020

An improved transformation-based kernel estimator for population abundance with shoulder condition

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Abstract. The estimation of parameter $f_X(0)$, which is the probability density function at the perpendicular distance $x = 0$, is a common target in the line transect sampling to estimate the population abundance, D. The key assumption of the density shape in the line transect sampling is known as the shoulder condition ($f'_X(0) = 0$). In this paper, we propose a log-transformation application as an adaptation of the classical kernel method to estimate the population abundance in the line transect sampling. The proposed transformation produces a simple and interpretable estimator as the usual kernel estimator while holding theoretical and practical advantages. The asymptotic properties of the proposed transformation are derived. A simulation study using the half-normal detection function is also investigated and applied using various sample

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sizes. Theoretical and practical results show the superior potential properties of the proposed transform estimator over the usual kernel estimator.

Keywords: line transect, log-transformation, kernel estimator, shoulder condition, abundance, bandwidth.

1. Introduction

Several common methods to estimate population density; the line transect method is highly recommended in recent years. In this sampling technique, the study area consists of at least one strip or several non-overlapping strips of total length L. Then, the transect line(s) are located at random in each strip. Furthermore, an observer moves in each line and detects an object from the line while recording the perpendicular distance, x. It is assumed that the observer detects a random sample of n objects $x_1, x_2, ..., x_n$.

Let $g_X(x)$ be the detection function, which is the conditional probability function of detecting an object given that its vertical distance x from the transect line. The random variable X has a probability density function $f_X(x)$ for $0 \leq x \leq w$, where the functions $g_X(x)$ and $f_X(x)$ have the same shape, such that $f_X(x) = \frac{g_X(x)}{\int_0^w g_X(t)dt}$. Then, the population abundance is $D = \frac{nf_X(0)}{2L}$ and its estimation is $\hat{D} = \frac{n\hat{f}_X(0)}{2L}$ [1], where $\hat{f}_X(0)$ is an estimate of $f_X(0)$ obtained based on detected perpendicular distances $x_1, x_2, ...x_n$ (a random sample of size n), while L is the length of the transect line. If the population area A is known, the population size N is easy to estimate using the formula $\hat{N} = A\hat{D}$.

In line transect sampling, several assumptions about the probability of detecting objects (function $g_X(x)$) are made. The first assumption is $g_X(0) = 1$, which means that all objects that are placed on the transect line itself cannot be missed. The second logical assumption is $g_X(x)$ is non-increasing, i.e., the highest probability of detecting an object is for the closest object to the transect line.

Many practical cases in the literature are assuming that the shape of the detection function $g_X(x)$ have a shoulder at distance $x = 0$ (see [1-3]). Mathematically, this condition is equivalent to $f'_X(0) = 0$, which is known in the line transect literature as the shoulder condition. Several test suggestions are stated in the literature to examine the shoulder condition [4]. Generally, the shoulder condition is commonly assumed for several population densities because the probability of detected objects closer to the transect line is still certain.

As shown above, the estimation of $f_X(0)$ is important to estimate density D. Many approaches are introduced in the line transect sampling literature to estimate $f_X(0)$. The two common estimation methods are parametric and non-parametric. For the parametric method, one fits the sample density shape with an appropriate parametric distribution. Then the parameter(s) are estimated using a robust method (e.g. maximum likelihood estimation). The negative exponential [5] and the half-normal [6] are prevalent models for fitting the den-

sity of the perpendicular distances and making an estimate of $f_X(0)$. However, recent researches have focused on using the non-parametric kernel method for estimation.

2. Kernel estimation method

Kernel method is a popular non-parametric estimation method to estimate $f(x)$ based on a random sample. The advantage of the kernel method compared to the parametric method is that it allows the data to be illustrated itself. Silverman [7] introduces an extended description of this method. The classical Rosenblatt–Parzen kernel estimator is given by

$$\hat{f}_X(x) = \frac{1}{nh}\sum_{i=1}^{n} K(\frac{x - x_i}{h}), \ -\infty < x < \infty, \tag{1}$$

where $K(u)$ is the kernel function, which is usually chosen as a symmetric density function. Silverman [7] denoted that there are very few differences in efficiency to adopt between the different kernel functions. Moreover, it is common to choose K as the standard normal while the smoothing parameter h is a positive real number. Several methods can be chosen to compute the value of h; a value of h that minimizing the estimator asymptotic mean squared error is common. It is worth noting that the kernel estimator performance is sensitive to the value of the smoothing parameter h, and thus, several techniques are recommended in the literature to find the 'leading' one. Therefore, several methods for the optimal bandwidth are suggested in the subsection 'Optimal Bandwidth'.

In line transect sampling, the perpendicular distances are positive. While kernel estimator for non-negative data at equation (1) effected by the boundary at $x = 0$ which it has high negative bias. A common adaptation method of it to apply a so-called "re-normalization" by applying a reflection technique; which change each sampled data value x_i with x_i and it's reflection $-x_i$ [7-8], if $K(u)$ is assumed to be an even and symmetric function around zero; the obtained reflected estimator is:

$$\hat{f}_X(x) = \frac{1}{nh}\sum_{i=1}^{n} K(\frac{x - x_i}{h}) + K(\frac{x + x_i}{h}), \ x \geq 0. \tag{2}$$

The reflection kernel estimator of $f_X(0)$ is gained by substituting $x = 0$ in (2), which gives [9]

$$\hat{f}_X(0) = \frac{2}{nh}\sum_{i=1}^{n} K(\frac{x_i}{h}). \tag{3}$$

Assuming the shoulder condition is valid ($f'_X(0) = 0$), the bias and variance of the estimator (3) are:

$$Bias[\hat{f}_X(0)] = \ h^2 f''_X(0) \int_0^{\infty} u^2 \ K(u)du + o(h^2), \tag{4}$$

$$Var[\hat{f}_X(0)] = \frac{4}{nh} f_X(0) \int_0^\infty K^2(u)du + o(\frac{1}{nh}). \tag{5}$$

The asymptotic mean squared error (AMSE) of $\hat{f}_X(0)$ is obtained after the small-expression $o(.)$ are negligent:

$$AMSE[\hat{f}_X(0)] = \frac{4}{nh} f_X(0) \int_0^\infty K^2(u)du + h^4[f_X''(0)]^2[\int_0^\infty u^2\ K(u)du]^2. \tag{6}$$

In practical cases of the line transect sampling, the kernel method is commonly used to locate a good estimate of $f_X(0)$. An extended description of this method can found in the literature. Jones [10] proposed different boundary correction methods for the kernel estimator while Marron and Ruppert [11] suggested different transformations that reduced boundary bias in the kernel estimator. On the other hand, Chen [9] derived mean, variance, and the asymptotic mean squared error of estimator (3). Zhang, Karunamuni, and Jones [12] introduced a new transformation for kernel density estimation to adaptive the boundary and Mack [13] proposed several bias reduction techniques for $\hat{f}_X(0)$. In addition, Eidous [14] introduced some adaptations of the kernel estimator of $f_X(0)$ while Karunamuni and Zhang [15] suggested new techniques to correction kernel estimator boundary effect. Eidous [16] suggested a new kernel estimator of $f_X(0)$ for densities that do not have a shoulder and Wen and Wu [17] proposed data transformation for the kernel density estimation. Charpentier and Flachaire [18] applied the logarithmic transformation of the kernel density estimator for income data distribution. Albadareen and Ismail [19] suggested the power transformation to improve kernel estimator for population abundance. Recently, Albadareen and Ismail [20] studied several parameters of Epanechnikov kernel function and recommended one for population abundance estimation, Al-bassam and Eidous [3] modified the common kernel estimator to be more appropriates for population density estimation, and Albadareen and Ismail [21-22] applied data transformation for the kernel estimator based on generalized form of power transformation.

In this article, the non-parametric kernel method is considered when the shoulder condition is valid. A logarithmic transformation is applied to the data with the kernel estimation method, and the original density is gained using back-transformation. The proposed estimator is mathematically proven to compare its efficiency with the reflection kernel estimator. A simulation study is also carried out to compare the efficiency of the proposed estimators and the kernel estimator.

3. Kernel estimator with log-transformation

The estimation of the kernel density based on transformed data is widely considered in many types of research. The logarithmic transformation $Y = \log(X)$

was suggested by Devroye and Györfi [23], briefly discussed by Silverman [7] and proposed by Marron and Ruppert [11]. Other studies can also be found in Charpentier and Flachaire [18] who showed that the log-transformation of $Y = \log(X)$ of heavy-tailed distribution such as income distribution fit much better density estimate. Geenens and Wang [24] proposed a kernel estimators that combining the transformations with local likelihood density estimation methods. Nguyen, Jones, and McLachlan [25] introduced the R package for positive data kernel density estimation using log-transformation. Bii et al. [26] proposed a kernel estimator based on combines the transformation and the reflection methods.

The outcomes of the line transect sampling are the perpendicular distances, which are non-negative data and the estimation of interest is $f_X(x)$ at $x = 0$. Since the logarithmic transformation $Y = \log(X)$ is unsuitable when $x = 0$, we propose shifted logarithm transformation $Y = \log(X + 1)$ such that Y is a strictly increasing function. Thus, the original and proposed density functions are $f_X(x)$ and $f_Y(y)$ respectively. By applying the transformation method of the density function $f_X(x)$, we obtain $f_Y(y)$,

$$f_Y(y) = f_X(x)|\frac{dx}{dy}| = f_X(e^y - 1)e^y,\ x \geq 0. \tag{7}$$

The propose transformation is $Y = \log(X + 1)$. $f_y(0) = f_x(0)$ at $x = 0$, $y = 0$. Moreover, the population abundance D is obtained using $D = \frac{nf_X(0)}{2L}$. Therefore, it is equivalent to $D = \frac{nf_Y(0)}{2L}$.

To make a good abundance estimation of D, the estimation of $f_Y(0)$ is critical. It requires $f_Y(0)$ to be substituted with $\hat{f}_Y(0)$. In this article, the kernel method is applied where the proposed kernel estimator $\hat{f}_Y(0)$ is:

$$\hat{f}_Y(0) = \frac{2}{nh}\sum_{i=1}^{n} K(\frac{Y_i}{h})\ ,\ \ Y = \log(X + 1). \tag{8}$$

The bias and variance of $\hat{f}_Y(0)$ are:

$$Bias[\hat{f}_Y(0)] = 2hf'_Y(0)\int_0^\infty uK(u)du + O(h^2). \tag{9}$$

Assuming the shoulder condition is valid (i.e. $f'_X(0) = 0$),

$$Bias[\hat{f}_Y(0)] = 2h(f_X(0))\int_0^\infty uK(u)du + O(h^2), \tag{10}$$

$$Var[\hat{f}_Y(0)] = \frac{4}{nh}f_Y(0)\int_0^\infty K^2(u)du + o(\frac{1}{nh}) \tag{11}$$

$$= \frac{4}{nh}f_X(0)\int_0^\infty K^2(u)du + o(\frac{1}{nh}) \tag{12}$$

By ignoring the small terms $o(.)$, the asymptotic mean squared error of $\hat{f}_Y(0)$ is:

$$AMSE[\hat{f}_Y(0)] = \frac{4}{nh} f_X(0) \int_0^\infty K^2(u)du + [2h(f_X(0)) \int_0^\infty uK(u)du]^2. \quad (13)$$

By comparing estimator variances in terms (12) and (5), it was found that $Var[\hat{f}_Y(0)] = Var[\hat{f}_X(0)]$.

4. Simulation study

The theoretical characteristics of the proposed transform estimator in equations (10) and (12) are asymptotic, which implies that the sample size is assumed to grow infinitely (i.e. $n \to \infty$). In this simulation study, different sample sizes are applied ($n = 50,\ 100,\ and\ 500$). A comparison between the performance of the proposed estimator and the reflection kernel estimator is introduced. The relative bias (RB) and the relative mean error (RME) are computed using $RB = E[\hat{f}(0) - f(0)]/f(0)$ and $RME = \sqrt{MSE[\hat{f}(0)]}/f(0)$.

In this article, the case of the shoulder condition is assumed. A common density in the line transect sampling that has a shoulder at $(x = 0)$ is the half-normal model which will have applied in this study, where $f_X(x) = \frac{2e^{-x^2/2\beta^2}}{\beta\sqrt{2\pi}}$, $\beta > 0$, $0 \le x \le w$. Five detection functions are chosen under the validity of the shoulder condition. The parameter values are $\beta = 0.1,\ 0.2,\ 0.3,\ 0.4$ and 0.5, and the truncation point is $\omega = 1.5$. The detection functions cover several possible model cases for the perpendicular distances in the case of the validity of the shoulder condition, as demonstrated in Figure 1.

Optimal bandwidth

The estimator in equation (3) yields convergent AMSE when the symmetric kernel functions $K(.)$ are chosen, such as Gaussian, biweight, and Epanechnikov [27]. Therefore, the Gaussian kernel function is considered in this study.

The performance key in the kernel method is the smoothing parameter h [28]. From past studies, different approaches were suggested to choose an efficient bandwidth. Generally, a sequence of bandwidth values can be applied and their results are compared to understand the structure of the unknown probability density function because there is no 'best' method in choosing the bandwidth parameter [27]. The same approach is also suggested by Silverman [29]. Thus, different bandwidth methods are used in this simulation study, along with some comparisons between classical and proposed estimators.

For this simulation, we consider the following bandwidth methods, each with underlying estimators:

a) Silverman [7] rule of thumb: $h = 1.06\hat{\sigma}n^{(-1/5)}$, this bandwidth is derived by minimizing the approximate mean squared error (AMSE) of $\hat{f}_X(x)$.

In the line transect method, the half-normal distribution is assumed as the reference density of $f_X(x)$ to estimate σ via the maximum likelihood estimator $\hat{\sigma} = \sqrt{\sum_{i=1}^{n} x_i^2/n}$. This bandwidth is applied to the original data, which results in the estimator Est1.

b) Silverman [7] modified the rule of thumb: $h = 0.9\ \min(\frac{IQR(X)}{1.349}, \hat{\sigma})n^{(-1/5)}$, this bandwidth is similar to the previous bandwidth except that it is modified for the long-tailed and skewed distribution. $IQR(X)$ is the interquartile range of the perpendicular with distance X. This bandwidth is applied on the original data, which results in the estimator Est2.

c) Similar to (a), this bandwidth is applied to the transformed data, which results in the estimator Est3.

d) Similar to (b), this bandwidth is applied to the transformed data, which results in the estimator Est4.

Another efficient bandwidth can be obtained by minimizing the $AMSE[\hat{f}_Y(0)]$ in equation (13). The bandwidth is $h = (\frac{\int_0^\infty [K(t)]^2 dt}{2nf_X(0)(\int_0^\infty t^2 K(t)dt)^2})^{(1/3)}$, by assuming that the shoulder condition is valid and the unknown value $f_X(0)$ are estimated using a suitable reference distribution model (see [2-3,14,30,31]). This bandwidth value will be applied in parts (e) and (f).

e) Assuming that $f_X(x)$ is the negative exponential model ($f_X(x) = \frac{1}{\beta}e^{-x/\beta}$, $0 \le x \le w$) and the MLE of β, $\hat{\beta} = \bar{X}$, is used, this bandwidth is applied to the transformed data which results in the estimator Est5.

f) Assuming that $f_X(x)$ is a half-normal model ($f_X(x) = \frac{2e^{-x^2/2\beta^2}}{\beta\sqrt{2\pi}}$, $\beta > 0, 0 \le x \le w$), and the MLE of β, $\hat{\beta} = \sqrt{\sum_{i=1}^{n} x_i^2/n}$, is used, this bandwidth is applied to the transformed data which results in the estimator Est6.

Our simulation study focuses on six estimation values of $f_Y(0)$, where Est1 and Est2 are applied to the original data as stated in the literature, while Est3, Est4, Est5, and Est6 are the proposed estimators applied to the transformed data. Table 1 shows the relative bias (RB) and the relative mean error (RME) for the estimators based on the simulation study.

The simulation results show that the proposed transformed estimator (Est3) is superior, in which it has smaller absolute relative bias and relative mean error compared to the classical kernel estimators (Est1 and Est2) for each density family and all cases. Moreover, the RME of (Est3) decreases as the sample size increases, hence it is concluded that (Est3) is asymptotically consistent. Figure 2 depicted these results.

The proposed transformed estimator (Est5) also has efficient performance compared to estimators (Est1 and Est2) for all sample sizes. The proposed

transformed estimator (Est6) also works well for a small value of β (i.e. $\beta = 0.1, 0.2, and\ 0.3$) with various sample sizes.

Looking at the modified bandwidth [7], ($h = 0.9\ \min(\frac{IQR(X)}{1.349}, \hat{\sigma})n^{(-1/5)}$), the classical kernel estimator (Est2), and the proposed transformed estimator (Est4) have unsatisfactory performance for the line transect data when the shoulder condition is valid. Therefore, the bandwidth suggested by Silverman [7] ($h = 1.06\hat{\sigma}n^{(-1/5)}$) is more appropriate for the log-transformed data in this study.

5. Conclusion

In this study, an efficient and consistent estimator for wildlife population density using the line transect sampling is proposed. The application of log-transformation to the kernel estimation is proven to perform well compared to the reflection kernel estimator in the case of the shoulder condition. The asymptotic mathematical properties (bias, variance, and mean squared error (MSE)) of the proposed transform estimator were obtained. The proposed transformed estimator was shown to have smaller AMSE if the detection function has a shoulder. A simulation study using the half-normal model was employed to compare the efficiency of the proposed and the reflection kernel estimators. The simulation results showed that the proposed transformed estimators have smaller relative mean error than the classical reflection kernel estimators. Generally, the proposed transformed estimators have better performance than the classical kernel estimators in terms of absolute relative error with fixed variance, especially for small samples.

Acknowledgments

The authors would like to thank the editor and the reviewers for their valuable comments and constructive reviews to further improve this paper. The authors gratefully acknowledge the financial support granted by the Ministry of Higher Education (MOHE) and Universiti Kebangsaan Malaysia (UKM) in the form of research grants (FRGS/1/2019/STG06/UKM/01/5 and GUP-2019-031).

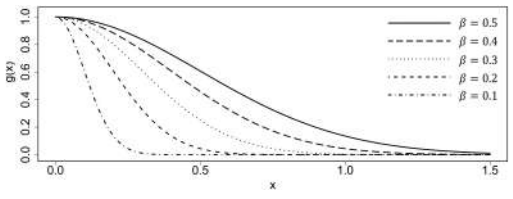


Figure 1: The detection functions of the half-normal models.

Table 1: Relative bias and relative mean error of the half-normal models.

		$n = 50$		$n = 100$		$n = 500$	
	Estimator	RB	RME	RB	RME	RB	RME
$\beta = 0.1$	Est1	-0.104	0.161	-0.079	0.122	-0.042	0.068
	Est2	-0.053	0.203	-0.036	0.154	-0.01	0.083
	Est3	-0.078	0.144	-0.055	0.106	-0.022	0.057
	Est4	-0.036	0.2	-0.021	0.151	0.002	0.083
	Est5	-0.232	0.24	-0.158	0.167	-0.051	0.065
	Est6	-0.226	0.233	-0.154	0.162	-0.049	0.063
$\beta = 0.2$	Est1	-0.106	0.156	-0.075	0.12	-0.043	0.068
	Est2	-0.053	0.189	-0.025	0.151	-0.013	0.084
	Est3	-0.054	0.124	-0.025	0.095	-0.001	0.052
	Est4	-0.019	0.183	0.007	0.15	0.012	0.084
	Est5	-0.08	0.121	-0.033	0.089	0.006	0.059
	Est6	-0.079	0.119	-0.033	0.088	0.006	0.06
$\beta = 0.3$	Est1	-0.096	0.157	-0.077	0.12	-0.042	0.068
	Est2	-0.041	0.203	-0.032	0.151	-0.01	0.083
	Est3	-0.019	0.12	-0.002	0.089	0.022	0.057
	Est4	0.011	0.201	0.016	0.15	0.027	0.088
	Est5	-0.003	0.125	0.015	0.105	0.029	0.078
	Est6	-0.005	0.124	0.014	0.106	0.028	0.079
$\beta = 0.4$	Est1	-0.092	0.151	-0.077	0.123	-0.041	0.067
	Est2	-0.03	0.194	-0.028	0.157	-0.009	0.081
	Est3	0.012	0.115	0.024	0.095	0.045	0.068
	Est4	0.04	0.197	0.036	0.16	0.041	0.091
	Est5	0.044	0.149	0.045	0.134	0.042	0.09
	Est6	0.041	0.149	0.043	0.136	0.042	0.091
$\beta = 0.5$	Est1	-0.092	0.152	-0.072	0.12	-0.042	0.068
	Est2	-0.041	0.198	-0.023	0.155	-0.012	0.082
	Est3	0.039	0.121	0.054	0.105	0.065	0.083
	Est4	0.048	0.201	0.059	0.166	0.051	0.096
	Est5	0.067	0.174	0.068	0.158	0.048	0.101
	Est6	0.063	0.175	0.066	0.159	0.047	0.102

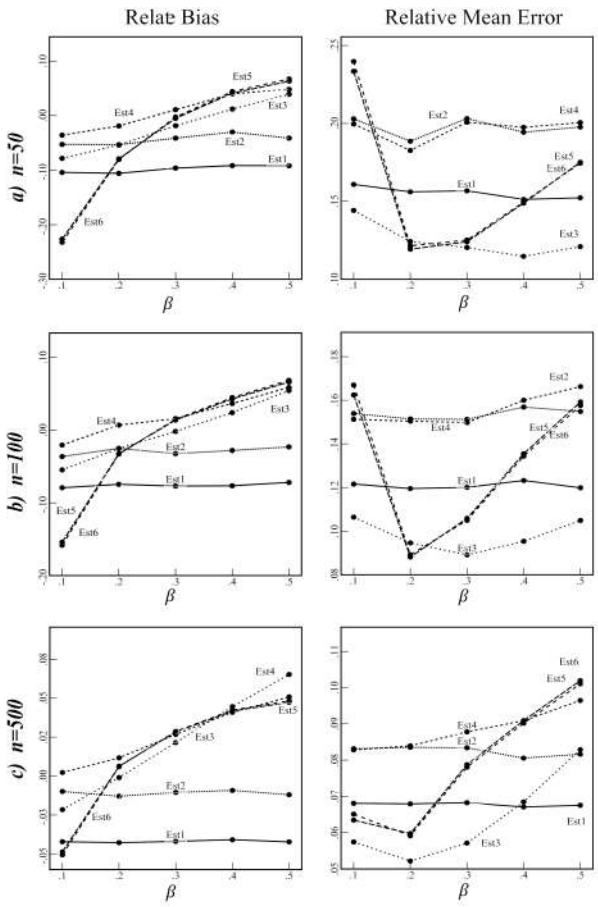


Figure 2: The detection functions of the half-normal models.

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Accepted: May 05, 2020

Multi-fuzzy hypergroups

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Abstract. In this paper, the concept of multi-fuzzy hyperstructure and multi-fuzzy hyperoperation are presented and developed. This concept generalized the concept of fuzzy hypergroup based on fuzzy space to the context of multi-fuzzy hypergroup based on multi-fuzzy space.

Keywords: multi-fuzzy space, multi-fuzzy hyperoperation, multi-fuzzy hypergroup and multi-fuzzy hyperstructure.

1. Introduction

Fuzzy set theory was pioneered by Zadeh [16], where the fuzzy set A of a universe of discourse U has only a basic component a membership function $A : U \longrightarrow [0, 1]$. In which, it is generalized classical set theory from the integer 0 and 1 to the interval $[0, 1]$ for the membership degree of objects. The idea of the concept of Atanassov's intuitionistic fuzzy set (AIFS) was introduced by Atanassov [3]. Atanassov's intuitionistic fuzzy set has been found to be greatly functional in dealing with vagueness. Atanassov achieved his concept by adding the non-membership term to the definition of fuzzy set that was given by Zadeh [15], while the fuzzy set has only basic component a membership function Sebastian and Ramakrishnan ([12, 13, 14]) generalized fuzzy sets to multi-fuzzy sets in terms of multi-dimensional membership functions and multi-level fuzziness and introduce multi-fuzzy subgroup.

Algebraic hyperstructures were introduced by Marty in [13] as a suitable generalisation of classical algebraic structures. In classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. Formally, if H is a nonempty set and $\rho^*(H)$ is the set of all nonempty subsets of H then we consider maps of the following type:

$$f_i : H \times H \longrightarrow \rho^*(H),$$

where $i = 1, 2, \ldots, n$ and n is a positive integer. The maps f_i are called (*binary*) hyperoperations.

A *hypergroup* H is a *hyperstructure* $(H, \circ)$ satisfying

(1) $x \circ (y \circ z) = (x \circ y) \circ z$ for all $x, y, z \in H$ (called semihypergroup),
(2) $\{x\} \circ H = H \circ \{x\}$ for all $x \in H$.

The concept of H_v-structures ([15]) constitute a generalization of the well-known algebraic hyperstructures (hypergroup, hyperring, hypermodule). Actually some axioms concerning the above hyperstructures such as the associative law, the distributive law and so on are replaced by their corresponding weak axioms. The study of fuzzy algebraic structures started with the introduction of the concept of fuzzy subgroup of an ordinary group by Rosenfeld ([11]). Davvaz ([15]) applied the concept of fuzzy sets to the theory of algebraic hyperstructures and defined fuzzy sub-hypergroup (respectively $H_v$subgroup) of a hypergroup (respectively H_v-group) which is a generalization of the concept of Rosenfeld's fuzzy subgroup of a group.

The relations between fuzzy sets and hyperstructures have been already considered by Corsini, Davvaz, Leoreanu, Feng and others ([28-50]). A recent book ([41]) contains a wealth of applications. By this book, Corsini and Leoreanu ([37]) presented some of the numerous applications of algebraic hyperstructures, especially those from the last fifteen years, to the following subjects: geometry, hypergraphs, binary relations, lattices, fuzzy sets and rough sets, automata, cryptography, codes, median algebras, relation algebras, artificial intelligence and probabilities.

Fathi ([13]) defined fuzzy hypergroup (fuzzy H_v-group) and fuzzy subhypergroup (fuzzy H_v-subgroup). which is a generalization of the concept of Dib's fuzzy group of the fuzzy spaces ([7]). The notion of multi-spaces was introduced by Smarandache in 1969 ([19]). Hila and Davvaz ([9]) the conception of multi-hypergroup space is a generalization of the algebraic hypergroup. By combining the above Smarandache multi-spaces with hypergroups in hyperstructure theory, a new kind of algebraic hyperstructure called multi-hypergroup space.

In this work, the notion of multi-fuzzy space is used to define multi-fuzzy hyperstructure and multi- fuzzy hyperoperation. Also, the concept of multi-fuzzy hypergroup (multi-fuzzy H_v-group) is introduced and some interesting results are derived.

2. Preliminaries

In this section, we recall the definitions and related results which are needed in this work.

Definition 2.1 ([16]). *A fuzzy set A in a universe of discourse U is characterized by a membership function $\mu_A(x)$ that takes values in the interval $[0, 1]$.*

Definition 2.2 ([3]). *An intuitionistic fuzzy set A in a non-empty set U (a universe of discourse) is an object having the form:*

$$A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in U\},$$

where the functions $\mu_A(x) : U \longrightarrow [0,1]$ *and* $\gamma_A(x) : U \longrightarrow [0,1]$, *denote the degree of membership and degree of non-membership of each element* $x \in U$ *to the set* A *respectively, and* $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ *for all* $x \in U$.

Definition 2.3 ([13]). *Let* k *be a positive integer, A multi-fuzzy set* A *in* U *is a set of ordered sequences* $A = \{u/(\mu_1(u), \mu_2(u), \ldots, \mu_i(u), \ldots, \mu_k(u)) : u \in U\}$, *where* $\mu_i \in \rho(U)$, $i = 1, 2, 3, \ldots, k$. *The function* $\mu_A = (\mu_1, \mu_2, \ldots, \mu_k)$ *is called the multi-membership function of multi-fuzzy set* A, k *is called the dimension of* A. *The set of all multi-fuzzy sets of dimension* k *in* U *is denoted by* $M^kFS(U)$. *A multi-fuzzy set of dimension 1 is a Zadeh's fuzzy set and a multi-fuzzy set of dimension 2 with* $\mu_1(u) + \mu_2(u) \leq 1$ *is an Atanassov's intuitionistic fuzzy set.*

Definition 2.4 ([7]). *A fuzzy space* $(X, I = [0,1])$ *is the set of all ordered pairs* (x, I), $x \in X$

$$(X, I) = \{(x, I) : x \in X\},$$

where $(x, I) = \{(x, r) : r \in I\}$. *The ordered pair* (x, I) *is called a fuzzy element in the fuzzy space* (X, I).

Definition 2.5 ([13]). *If* G *is an ordinary group having the ordinary binary operation* $\circ$ *and* A *is a multi-fuzzy subset of* G, *then* A *is called a multi-fuzzy subgroup of* G *if and only if*

(1) $A(x \circ y) \geq \min\{A(x), A(y)\}$,
(2) $A(x^{-1}) \geq A(x)$, *for all* $x, y \in G$.

Note. *A multi-fuzzy subgroup of dimension 1 is a Rosenfeld's fuzzy subgroup and a multi-fuzzy subgroup of dimension 2 is the Zhan's intuitionistic fuzzy subgroup.*

Definition 2.6 ([2]). *Let* X *be a non-empty set. A multi-fuzzy space* $(X, I_i^X = [0,1])$ *is the set of all ordered sequences* (x, I_i^X), $x \in X$; *where* $i \in \mathbb{N}$, *that is,* $(X, I_i^X) = \{(x, I_i^X) : x \in X\}$, *where* $(x, I_i^X) = \{(x, r_i) : r_i \in I_i^X\}$. *The ordered sequences* (x, I_i^X) *is called a multi-fuzzy element in the multi-fuzzy space* (X, I_i^X). *Therefore, the multi-fuzzy space is an (ordinary) set of ordered sequences. In each ordered sequences the first component indicates the (ordinary) element and the second component indicates a set of possible multi-membership values.*

Note. *A multi-fuzzy space of dimension 1 is a Dib's fuzzy group and a multi-fuzzy space of dimension 2 is the Fathi's intuitionistic fuzzy group.*

Definition 2.7 ([2]). *A multi-fuzzy binary operation* $\underline{F} = (F(x, y), f_{(xy)_i})$ *on the multi-fuzzy space* (X, I_i^X) *is a multi-fuzzy function from* $(X, I_i^X) \times (X, I_i^X) \longrightarrow (X, I_i^X)$ *with multi-comembership functions* $f_{(xy)_i}$ *that satisfy:*

(*i*) $f_{(xy)_i}(r_i, s_i) \neq 0$ *if* $r_i \neq 0$, $s_i \neq 0$, *where* $i \in \mathbb{N}$,
(*ii*) $f_{(xy)_i}(r_i, s_i)$ *are onto, that is,* $f_{(xy)_i}(I_i^X \times I_i^X) = I_i^X$, $x, y \in X$.

Definition 2.8 ([2]). *A multi-fuzzy algebraic system* $((X, I_i^X), \underline{F})$ *is called a multi-fuzzy group if and only if for every* $(x, I_i^X), (y, I_i^X), (z, I_i^X) \in (X, I_i^X)$ *the following conditions are satisfied:*

(*i*) *Associative:*

$$((x, I_i^X)\underline{F}(y, I_i^X))\underline{F}(z, I_i^X) = (x, I_i^X)\underline{F}((y, I_i^X)\underline{F}(z, I_i^X)),$$

that is

$$((xFy)Fz, I_i^X) = (xF(yFz), I_i^X).$$

(*ii*) *Existence of a multi-fuzzy identity element* (e, I_i^X) *for which*

$$(x, I_i^X)\underline{F}(e, I_i^X) = (e, I_i^X)\underline{F}(x, I_i^X) = (x, I_i^X),$$

that is

$$(xFe, I_i^X) = (eFx, I_i^X) = (x, I_i^X).$$

(*iii*) *Every multi-fuzzy element* (x, I_i^X) *has an inverse* (x^{-1}, I_i^X) *such that*

$$(x, I_i^X)\underline{F}(x^{-1}, I_i^X) = (x^{-1}, I_i^X)\underline{F}(x, I_i^X) = (e, I_i^X).$$

Theorem 2.1 ([2]). *Associated to each multi-fuzzy group* $((X, I_i^X), \underline{F})$ *where* $\underline{F} = (F, f_{(xy)_i})$ *is an intuitionistic fuzzy group* $((X, I, I), \boldsymbol{F})$ *which is isomorphic to the multi-fuzzy group* $((X, I_i^X), \underline{F})$ *by the correspondence* $(X, I_i^X) \longleftrightarrow (x, I, I)$.

Definition 2.9 ([21]). *Let* $\langle (H, I), \diamond \rangle$ *be a fuzzy hypergroup (fuzzy* H_v*-group) and let* $U = \{(x, u_x) : x \in U_0\}$ *be a fuzzy subspace of fuzzy space* (H, I). *Then* $(U, \diamond)$ *is called a fuzzy subhypergroup (fuzzy* H_v*-subgroup) of the fuzzy hypergroup* $\langle (H, I), \diamond \rangle$ *if* $\diamond$ *is closed on the fuzzy subspace* U *and* $(U, \diamond)$ *satisfies the conditions of hypergroup (*H_v*-group).*

Theorem 2.2 ([21]). *For each fuzzy hypergroup (fuzzy* H_v*-group) there is an associated ordinary hypergroup (*H_v*-group)* (H, Δ) *which is isomorphic to the fuzzy hypergroup (fuzzy* H_v*-group)* $\langle (H, I), \diamond \rangle$ *by the correspondence* $(x, I) \longleftrightarrow x$.

Definition 2.10 ([20]). *An intuitionistic fuzzy hypergroup is an intuitionistic fuzzy hyperstructure* $\langle (H, I, I), \underline{F} \rangle$ *satisfying the following axioms:*

(1) $((x, I, I)\underline{F}(y, I, I))\underline{F}(z, I, I) = (x, I, I)\underline{F}((y, I, I)\underline{F}(z, I, I))$, *for all* (x, I, I), (y, I, I), $(z, I, I) \in (H, I, I)$.

(2) $(x, I, I)\underline{F}(H, I, I) = (H, I, I)\underline{F}(x, I, I) = (H, I, I)$, *for all* $(x, I, I) \in (H, I, I)$.

Theorem 2.3 ([20]). *For each intuitionistic fuzzy hypergroup (intuitionistic fuzzy* H_v*-group) there are two fuzzy hypergroups (fuzzy* H_v*-group)* $\langle (H, I), F' \rangle$ *and* $\langle (H, I), F' \rangle$ *(such that* $F = (F, \underline{f}_{xy})$ *and* $F' = (F, 1 - \bar{f}_{xy})$*) which are isomorphic to the intuitionistic fuzzy hypergroup (intuitionistic fuzzy* H_v*-group)* $\langle (H, I, I), \underline{F} \rangle$ *by the correspondence* $(x, I, I) \longleftrightarrow (x, I)$, $x \in H$.

3. Multi-fuzzy hypergroup

In this work, multi-fuzzy hypergroups are defined a correspondence relation between these multi-fuzzy notions and fuzzy hypergroups is given.

Definition 3.1. *Let (H, I_i^X) be a non-empty multi-fuzzy space. A multi-fuzzy hyperstructure (multi-fuzzy hypergroupoid), denoted by $((H, I_i^X), \underline{F} = (F, f_{(xy)_i}))$ is a multi-fuzzy space together with a multi-fuzzy function having onto co-membership functions (referred as a multi-fuzzy hyperoperation) $\underline{F} : (H, I_i^X) \times (H, I_i^X) \longrightarrow \rho^*((H, I_i^X))$, where $\rho^*((H, I_i^X))$ denotes the set of all non-empty multi-fuzzy subspaces of multi-fuzzy spaces (H, I_i^X) and $\underline{F} = (F, f_{(xy)_i})$ with $F : H \times H \longrightarrow \rho^*(H)$ and $f_{(xy)_i} : I_i^X \times I_i^X \longrightarrow I_i^X$.*

A multi-fuzzy hyperoperation $\underline{F} = (F, f_{(xy)_i})$ on (H, I_i^X) is said to be *uniform* if the associated co-membership functions $f_{(xy)_i}$ are identical, that is, $f_{(xy)_i} = f_i$ for all $x, y \in H$. A uniform multi-fuzzy hyperstructure $\langle (H, I), \underline{F} \rangle$ is a multi-fuzzy hyperstructure $\langle (H, I_i^X), \underline{F} \rangle$ with uniform multi-fuzzy hyperoperation.

Example 3.1. Let $H = \{-1, 1, -i, i\}$, where $i = \sqrt{-1}$. Define the multi-fuzzy hyperoperation $\underline{F} = (F, f_{(xy)_i})$ over the multi-fuzzy space (H, I_i^X) such that $F : H \times H \longrightarrow \rho^*(H)$ with $F(x, y) = xFy = \{x, y\}$ and $f_{(xy)_i} : I_i^X \times I_i^X \longrightarrow I_i^X$ with $f_{(xy)_i}(r_i, s_i) = r_i \wedge s_i$, that is,

$$(x, I_i^X)\underline{F}(y, I_i^X) = \{(x, f_{(xy)_i}(I_i^X \times I_i^X)), (y, f_{(yx)_i}(I_i^X \times I_i^X))\},$$

for all $(x, I_i^X), (y, I_i^X) \in (H, I_i^X)$. Therefore $\langle (H, I_i^X), \underline{F} \rangle$ is a uniform multi-fuzzy hyperstructure.

After introducing multi-fuzzy hyperstructure, the notion of a multi-fuzzy hypergroup is now defined.

Definition 3.2. *A multi fuzzy hypergroup is a multi-fuzzy hyperstrucure $\langle (H, I_I^X), \underline{F} \rangle$ satisfying the following axioms:*

(i) $((x, I_i^X)\underline{F}(y, I_i^X))\underline{F}(z, I_i^X) = (x, I_i^X)\underline{F}((y, I_i^X)\underline{F}(z, I_i^X))$, *for all* (x, I_i^X), (y, I_i^X), $(z, I_i^X) \in (H, I_i^X)$,

(ii) $(x, I_i^X)\underline{F}(H, I_i^X) = (H, I_i^X)\underline{F}(x, I_i^X) = (H, I_i^X)$, *for all* $(x, I_i^X) \in (H, I_i^X)$.

Definition 3.3. *A multi-fuzzy H_v-group is a multi-fuzzy hyperstructure $\langle (H, I_i^X), \underline{F} \rangle$ satisfying the following conditions:*

(i) $((x, I_i^X)\underline{F}(y, I_i^X))\underline{F}(z, I_i^X) \cap (x, I_i^X)\underline{F}((y, I_i^X)\underline{F}(z, I_i^X)) \neq \phi$, *for all* (x, I_i^X), (y, I_i^X),

$(z, I_i^X) \in (H, I_i^X)$,

(ii) $(x, I_i^X)\underline{F}(H, I_i^X) = (H, I_i^X)\underline{F}(x, I_i^X) = (H, I_i^X)$, *for all* $(x, I_i^X) \in (H, I_i^X)$.

If $\langle (H, I_i^X), \underline{F} \rangle$ satisfies only the first condition of Definition 3.3 (Defnition 3.3), then it is called a multi-*fuzzy semihypergroup* (multi-*fuzzy* H_v-*semigroup*).

The next theorem gives a correspondence relation between multi-fuzzy hypergroup and multi-fuzzy hypergroup.

Theorem 3.1. *For each multi-fuzzy hypergroup there is an associated intuitionistic fuzzy hypergroup* $\langle (H, I, I), F\rangle$ *which is isomorphic to the multi-fuzzy hypergroup* $\langle (H, I_i^X), \underline{F}\rangle$ *by the correspondence* $(x, I_i^X) \longleftrightarrow (x, I, I)$.

Proof of Theorem 3.1. *Consider the multi-fuzzy hypergroup* $\langle (H, I_i^X), \underline{F}\rangle$. *Now using the correspondence* $(x, I_i^X) \longleftrightarrow (x, I, I)$. *We can redefine multi-fuzzy hyperopertion* $\underline{F} = (F, f_{(xy)_i})$ *with* $\underline{F} : (X, I_i^X) \times (X, I_i^X) \longrightarrow \rho^*((H, I_i^X))$ *to be* $\underline{F'} : (X, I, I) \times (X, I, I) \longrightarrow \rho^*((H, I, I))$. *That is* $\underline{F'}$ *define an intuitionistic fuzzy hyperoertion over* (H, I, I) *is an intuitionistic fuzzy hyperopertion in sense Fathi.*

4. Conclusion

In this work, we have generalized the concept of fuzzy hypergroup based on fuzzy space to the context of multi-fuzzy hypergroup based on multi-fuzzy space and multi-fuzzy H_v-group.

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Accepted: January 22, 2020

Some results on Wiener indices for a connected graph G

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Abstract. Wiener, hyper-Wiener and partial-Wiener indices are single number and defined by $W(G) = \sum_{\{u,v\}\subseteq V(G)} d(u,v|G)$, $WW(G) = \frac{1}{2}\sum_{\{u,v\}\subseteq V(G)}[d^2(u,v|G) + d(u,v|G)]$ and $W_i(G) = \sum_{\{u,v\}\subseteq V(G) d(u,v|G)\geq i} d(u,v|G), 1 \leq i \leq \delta$; respectively, where $d(u,v|G)$ is the distance between two vertices u and v in a graph G, $V(G)$ is the vertex set of G and δ is the diameter of G. In this paper, we find a relationship between some Wiener indices and boundary of hyper-Wiener index depending on the diameter and Wiener index.

Keywords: Wiener index, hyper-Wiener index, partial Wiener index.

1. Introduction

In general, the Wiener index measures how compact a molecule is for its given weight. It therefore has predictive value. Chemists and physics have found many such uses for the Wiener index. Wiener index thus has been studied in depth in the literature. We especially recommend the reference survey by Dobrynin, Entringer and Gutman [4], which summarized the uses of the Wiener index and all known efficient and elegant results on Wiener numbers of trees. The original publications of some of the best results are in [10, 11].

Problems about Wiener indices are still legion and vary greatly in scopes and direction. Explicit formulas for the Wiener indices (and its generalization to polynomials) of classes of graphs are sought and properties are derived there from. Investigators also try to determine those natural numbers which Wiener index can be attained for classes of graphs. Also, there are different kinds of especially interesting problems regarding the Wiener index for a given type of graphs [4].

We deal in this paper the graph G is an undirected connected of order n and size m without loops or multiple edges. The vertices–set and edges–set of a graph G denote by $V = V(G)$ and $E = E(G)$, respectively.

"Let u and v be any two vertices of G, the length of a shortest path between u and v is called the distance and denoted by $d(u,v|G)$". Note that $d(u,v|G) = 0$ when $u = v$, in this paper we will rule out $u = v$. The maximum distance

between two distinct vertices is the diameter of G, denote by δ. "Wiener index of a connected graph G is the sum of distances between all vertices of G, [12], that is:

$$W(G) = \sum_{\{u,v\}\subseteq V(G)} d(u,v|G). \tag{1}$$

The polynomial of this index was first put forward in 1988 by Hosoya [8], is defined by:

$$"W(G;x) = \sum_{k=1}^{\delta} C(G,k)x^k", \tag{2}$$

where $C(G,k)$ is the number of (u,v) in G at $d(u,v\,|G) = k$, $k = 0,1,2,\ldots,\delta$. In particular $C(G,0) = n$ (the order of G) and $C(G,1) = m$ (the size of G).

This polynomial also known as the Hosoya polynomial. The Wiener polynomial can be write as:

$$"W(G;x) = \sum_{\{u,v\}\subseteq V(G)} x^{d(u\,,v|G)}". \tag{3}$$

It is clear that:

$$W(G) = \frac{d}{dx}W(G;x\,)|_{x=1} = \sum_{k=1}^{\delta} k\,C(G,k) \tag{4}$$

Klein et al. [9], defined the hyper Wiener index of any connected graph G by:

$$WW(G) = \frac{1}{2}\sum_{\{u,v\}\subseteq V(G)} [d^2(u,\,v|\,G) + d(u,\,v|\,G)]. \tag{5}$$

In 2011, Fath-Tabar and Ashrafi [5] defined the "hyper-Wiener polynomial as:

$$WW(G;x) = \frac{1}{2}\sum_{\{u,v\}\subseteq V(G)} x^{d^2(u,v|G)+d(u,v|G)}. \tag{6}$$

We can write the hyper Wiener polynomial by the formula:

$$WW(G;x) = \frac{1}{2}\sum_{k=1}^{\delta} C(G,k)x^{k^2+k}. \tag{7}$$

Then

$$\begin{aligned} WW(G) &= \frac{d}{dx}WW(G;x)\Big|_{x=1} \\ &= \frac{1}{2}\sum_{k=1}^{\delta} k(k+1)C(G,k) = \frac{1}{2}\frac{d^2}{dx^2}\,(x\,.\,W(G;x))|_{x=1}. \end{aligned} \tag{8}$$

In 2002, Gutman find the inequality between of these indices, $\frac{3n}{2}W(T) - \frac{1}{12}n(n-1)^2(n+1) \leq WW(T) \leq (\frac{n}{2}-1)W(T) - \frac{1}{2}(n^2-2n+2)(n-1)$, [6]. And also many other relations, and Zhou and Gutman determined lower bounded of hyper Wiener with respect Wiener index, [13], where T is a tree with n vertices.

Now, we define " partial Wiener index of G as the sum of distances between all vertices of G that are at distance i apart,$1 \leq i \leq \delta$ ", that is:

$$W_i(G) = \sum_{\substack{\{u,v\}\subseteq V(G) \\ d(u,v|G)\geq i}} d(u,v|G), 1 \leq i \leq \delta. \tag{9}$$

Clearly $W_i(G) = W(G)$, when $i = 1$. The Wiener partial polynomial is:

$$W_i(G;x) = \sum_{k=i}^{\delta} C(G,k)x^k, 1 \leq i \leq \delta. \tag{10}$$

Also, we can write $W_i(G) = \sum_{k=i}^{\delta} k\, C(G,k), 1 \leq i \leq \delta$.

Finally our terminology and notations, in general, applications and some formulas, can be seen in [1-4,7].

2. Main results

Theorem 2.1. *For any graph G of order n, $WW(G) = \alpha W(G) + \beta\binom{n}{2} + \gamma$, where α, β, and γ are real number.*

Proof. Since

$$WW(G;x) = \frac{1}{2}\sum_{k=1}^{\delta} C(G,k)x^{k^2+k} \text{ and } W(G;x) = \sum_{k=1}^{\delta} C(G,k)x^k,$$

then $\frac{WW(G;x)}{W(G;x)} = \frac{1}{2}\sum_{i=0}^{\delta^2} m(i)\, x^i + \frac{\sum_{j=1}^{\delta-1} r(j)\, x^j}{2\, W(G;x)}$, where $\sum_{i=0}^{\delta^2} m(i)\, x^i$ is divide output and $\sum_{j=1}^{\delta-1} r(j)\, x^j$ is division remainder of $\frac{WW(G;x)}{W(G;x)}$ such that $m(i)$ and $r(j)$ are real number.

Hence, $WW(G;x) = \left[\sum_{i=0}^{\delta^2} m(i)\, x^i\right] \frac{W(G;x)}{2} + \frac{1}{2}\sum_{j=1}^{\delta-1} r(j)\, x^j$.

To obtain the formula of Theorem 2.1, differentiating the last equation with respect to x and then compensate for x value by one. Then: $WW(G) = \sum_{i=0}^{\delta^2} m(i)\ \frac{W(G)}{2} + \sum_{i=0}^{\delta^2} im(i)\ \frac{W(G;1)}{2} + \frac{1}{2}\sum_{j=1}^{\delta-1} jr(j)$. Hence,$WW(G) = \alpha W(G) + \binom{n}{2}\beta + \gamma$,where $\alpha = \sum_{i=0}^{\delta^2} m(i)\,/2$, $\beta = \sum_{i=0}^{\delta^2} im(i)/2$ and $\gamma = \sum_{j=1}^{\delta-1} r(j)\,/2$ are real numbers. □

Example 1. Let G be a graph shown in Figure 1. Then, $WW(G) = 42$, $W(G) = 27$, $\binom{6}{2} = 15$. From Theorem 2.1, we have: $\alpha = -17/2$, $\beta = 13/2$, and $\gamma = 174$.

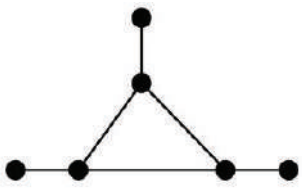


Figure 1: 3-cycle graph

Clearly the values α, β and γ are not unique. If we take $\alpha = \beta = 1$ and $\gamma = 0$ in Example 1, then the formula of Theorem 2.1 is satisfy. That is, there are many different values of α, β and γ.

Remark 1. We can determine $m(i)$ for some i, $m(\delta^2) = 1$, $m(\delta^2 - 1) = -\frac{C(G;\delta-1)}{C(G;\delta)}$, $m(\delta^2 - 2) = \left(\frac{C(G;\delta-1)}{C(G;\delta)}\right)^2 - \frac{C(G;\delta-2)}{C(G;\delta)}$, $m(\delta^2 - 3) = \left(\frac{C(G;\delta-1)}{C(G;\delta)}\right)^3 - \frac{2C(G;\delta-1)C(G;\delta-2)}{(C(G;\delta))^2} + \frac{C(G;\delta-3)}{C(G;\delta)}$.

Corollary 2.1. *If G is a connected graph with diameter two, then:*

$$WW(G) = \frac{1}{2}\left[h + \sum_{i=0}^{4}(-1)^i h^{4-i}\right] W(G) + \frac{1}{2}\binom{n}{2}\left[\sum_{i=1}^{4}(-1)^i i\, h^{4-i}\right] - \frac{1}{2}(h^4 + h)C(G,1),$$

where $h = \frac{C(G,1)}{C(G,2)}$

Proof. Clearly

$$WW(G;x) = \frac{1}{2}(C(G,1)\,x^2 + C(G,2)\,x^6) \text{ and } W(G;x) = C(G,1)\,x + C(G,2)\,x^2.$$

Using the algebraically division, we get: $m(0) = \left(\frac{C(G,1)}{C(G,2)}\right)^4 + \frac{C(G,1)}{C(G,2)}$, $m(1) = -\left(\frac{C(G,1)}{C(G,2)}\right)^3$, $m(2) = \left(\frac{C(G,1)}{C(G,2)}\right)^2$, $m(3) = -\left(\frac{C(G,1)}{C(G,2)}\right)$, $m(4) = 1$ and $r(1) = -\frac{(C(G,1))^5}{(C(G,2))^4} - \frac{(C(G,1))^2}{C(G,2)}$. □

Remark 2. If G has diameter one, then $WW(G) = W(G)$. For example: If G is a complete graph K_n.

Theorem 2.2. *$WW(G) = \frac{1}{2}W(G) + \frac{1}{2}\binom{n}{2} + t$, where t is non negative number.*

Proof. Since $WW(G) = \frac{1}{2}\frac{d^2}{dx^2}(x\,.\,W(G;x))|_{x=1}$, that is:

- for $x = 1$: "$WW(G) = \frac{1}{2}\frac{d^2}{dx^2}(x\,.\,W(G;x))$", we integrate the both sides with respect to x, we get:

- for $x = 1$: $xWW(G)+c = \frac{1}{2}\frac{d}{dx}(x\,.\,W(G;x))$, where c is an integral constant.
- for $x = 1$:$xWW(G) + c = \frac{1}{2}(x\,.\,W(G) + W(G;x))$, This implies that $WW(G) + c = \frac{1}{2}\left(W(G) + \binom{n}{2}\right)$, when $x = 1$.

Then $WW(G) = \frac{1}{2}W(G) + \frac{1}{2}\binom{n}{2} + t$, where $t = -c$, c is non positive number. □

From Theorems 2.1, 2.2, we can determined the values α, β and γ, where $\alpha = \beta = 0.5$ and $\gamma = t$, t is non negative number (see Example 2). But, if we take $\alpha = 1$ and $\beta = 0.5$, then can be γ negative or non negative. For example, if we take K_n, then $\gamma = -n(n-1)/4$, but $\gamma = (n-1)(n-4)/4$ at S_n for all $n \geq 4$. Also, if we take $\alpha = \beta = 1$, then can be γ negative or non negative. For example, if we take K_n, then $\gamma = -n(n-1)/2$, $n \geq 2$, but $\gamma = n(n-1)(n^2-2n-27)/48$ at C_n for all $n \geq 7$, n is an odd number.

Example 2. Take $\alpha = \beta = 0.5$, $\gamma = t$, t is non negative.

Table 1: General formulas of some special graphs.

Type of Graph	WW	W	t
The complete graph K_n	$n(n-1)/2$	$n(n-1)/2$	$\mathbf{0}$
The star graph S_n	$(n-1)(3n-4)/2$	$(n-1)^2$	$\mathbf{3(n-1)(n-2)/4}$
The cycle graph C_n n is odd	$n(n^2-1)(n+3)/48$	$n(n^2-1)/8$	$\mathbf{n(n-1)(n-3)(n+4)/48}$
The cycle graph C_n n is even	$n^2(n+1)(n+2)/48$	$n^3/8$	$\mathbf{n(n^3-10n+12)/48}$
The path graph P_n	$n(n^2-1)(n+2)/24$	$n(n^2-1/6$	$\mathbf{n(n-1)(n-2)(n+3)/24}$
The wheel graph W_n	$(n-1)(3n-8)/2$	$(n-1)(n-2)$	$\mathbf{3(n-1)(n-4)/4}$
The Fan graph F_n	$3(n-1)(n-2)/2$	$(n-1)(2n-3)/2$	$\mathbf{3(n-1)(n+5)/4}$

Example 3.

If $\alpha = 1$ and $\beta = 0$, then $WW(G) = W(G) + \gamma$.
For $\gamma = 0$ then $G = K_n$.
For $\gamma = 2$ then $G = K_n/\{e\}$, where e is an edge of K_n.
For $\gamma = 2j$ then $G = K_n/\{e_1, e_2, \ldots, e_j\}$, where $\{e_1, e_2, \ldots, e_j\}$ is a set of independent edges of K_n, $1 \leq j \leq \lfloor \frac{n}{2} \rfloor$.
Question: For any connected graph G, determining the values of α and β in order to make the value of γ be non positive?

Corollary 2.2. *If γ is a non positive number, then $WW(G) \leq \frac{1}{6}[n(n-1)(\alpha + 3\beta) + \alpha(n+1)]$, for any values α and β.*
Conjecture. *It is possible to determine the upper bound of R such that $WW(G) = W(G) + R$, $R \geq 0$.*

Now, we denote for the sum of the number (u, v) that are at distance k from i into δ by: $D_i(G)$, $0 \leq i \leq \delta$, i.e.

$$D_i(G) = \sum_{k=i}^{\delta} C(G,k), 0 \leq i \leq \delta. \tag{11}$$

It is clearly that: $D_0(G) = \binom{n+1}{2}$, $D_1(G) = \binom{n}{2}$, $D_2(G) = \binom{n}{2} - m$, and so on $D_n(G) = \binom{n}{2} - \sum_{i=1}^{n-1} C(G,i)$, $n \geq 2$.

We note that: $C(G,k) = D_k(G) - D_{k+1}(G)$, $0 \leq k \leq \delta - 1$.

Theorem 2.3. $WW(G) = (\delta+1)\binom{n}{2} - 2m - \sum_{j=3}^{\delta}\sum_{i=1}^{j-1} jC(G,i)$

Proof. Since

$$\begin{aligned}
\sum_{k=1}^{\delta} k(k+1)C(G,k) &= 2\sum_{k=1}^{\delta} C(G,k) + 4\sum_{k=2}^{\delta} C(G,k) \\
&\quad + 6\sum_{k=3}^{\delta} C(G,k) + \ldots + 2\delta\, C(G,\delta) \\
&= 2D_1(G) + 4D_2(G) + \sum_{j=3}^{\delta} 2jD_j(G) \\
&= 2\binom{n}{2} + 4\left\{\binom{n}{2} - m\right\} + \sum_{j=3}^{\delta} 2j\left\{\binom{n}{2} - \sum_{i=1}^{j-1} C(G,i)\right\} \\
&= 6\binom{n}{2} - 4m + 2(\delta-2)\binom{n}{2} - \sum_{j=3}^{\delta} 2j\sum_{i=1}^{j-1} C(G,i) \\
&= 2(\delta+1)\binom{n}{2} - 4m - 2\sum_{j=3}^{\delta} j\sum_{i=1}^{j-1} C(G,i),
\end{aligned}$$

therefore $WW(G) = (\delta+1)\binom{n}{2} - 2m - \sum_{j=3}^{\delta}\sum_{i=1}^{j-1} jC(G,i)$.

□

Theorem 2.4. $WW(G) = W(G) + \frac{1}{2}\sum_{j=2}^{\delta} W_j(G)$.

Proof. Since

$$\begin{aligned}
\sum_{k=1}^{\delta} k(k+1)C(G,k) &= 2\sum_{k=1}^{\delta} kC(G,k) + \sum_{k=2}^{\delta} kC(G,k) + \sum_{k=3}^{\delta} kC(G,k) + \ldots \\
&\quad + \sum_{k=\delta-1}^{\delta} kC(G,k) + \delta\, C(G,\delta) \\
&= 2\sum_{k=1}^{\delta} kC(G,k) + \sum_{j=2}^{\delta}\sum_{k=j}^{\delta} kC(G,k) = 2W(G) + \sum_{j=2}^{\delta} W_j(G),
\end{aligned}$$

therefore $WW(G) = W(G) + \frac{1}{2}\sum_{j=2}^{\delta} W_j(G)$. □

Theorem 2.5. *For any connected graph G of order n, then $WW(G) \geq \frac{5}{2}W(G) - n(n-1)$.*

Proof. Since $(k-2)^2 \geq 0$, then $k^2+k \geq 5k-4$, multiply both sides of inequality by $C(G,k)$ and take the summation from $k=1$ to $k=\delta$, we have:

$$\sum_{k=1}^{\delta} (k^2+k)\, C(G,k) \geq 5 \sum_{k=1}^{\delta} kC(G,k) - 4 \sum_{k=1}^{\delta} C(G,k).$$

This implies that $WW(G) \geq \frac{5}{2} W(G) - n(n-1)$. □

From Theorems 2.3 2.4 2.5, we obtained on the upper bounded and lower bounded of hyper–Wiener index in the following corollary:

Corollary 2.3. *For any connected graph G of order n, size m and diameter δ, $\delta \geq 2$:*

1. $\frac{5}{2} W(G) - n(n-1) \leq WW(G) \leq (\delta+1)\binom{n}{2} - 2m.$
2. $\frac{5}{2} W(G) - n(n-1) \leq WW(G) \leq \frac{1}{2}\{(\delta+1)\, W(G) - (\delta-1)m\}.$

Proof. 1. Since $\sum_{j=3}^{\delta} \sum_{i=1}^{j-1} jC(G,i) \geq 0$, then from Theorem 2.3, we have: $WW(G) \leq (\delta+1)\binom{n}{2} - 2m$.

2. Since $\sum_{j=2}^{\delta} W_j(G) \leq (\delta-1)\{W(G)-m\}$, then from Theorem 2.4, we have: $WW(G) \leq \frac{1}{2}\{(\delta+1)\, W(G) - (\delta-1)m\}$. □

Example 4. We take some special graphs with diameter 2, such as star graph, wheel graph and fan graph, see Example 2. We note that lower bounds of $WW$is very close to W but the upper bounds are equal.

The lower bound of S_n, W_n and F_n are:

$$\begin{aligned}
WW(S_n) &> \frac{5}{2} W(S_n) - n(n-1) = \frac{(n-1)(3n-5)}{2}, \\
WW(W_n) &> \frac{5}{2} W(W_n) - n(n-1) = \frac{(n-1)(3n-10)}{2}, \\
WW(F_n) &> \frac{5}{2} W(F_n) - n(n-1) > \frac{3(n-1)(n-4)}{2}.
\end{aligned}$$

But the upper bounds are:

$$\begin{aligned}
WW(S_n) &= 3\binom{n}{2} - 2m = \frac{1}{2}\{3\, W(S_n) - m\} = \frac{(n-1)(3n-4)}{2}, \\
WW(W_n) &= 3\binom{n}{2} - 2m = \frac{1}{2}\{3\, W(W_n) - m\} = \frac{(n-1)(3n-8)}{2}
\end{aligned}$$

and $WW(F_n) = 3\binom{n}{2} - 2m = \frac{1}{2}\{3\, W(F_n) - m\} = \frac{3(n-1)(n-2)}{2}$.

3. Conclusions

In this paper, we find a general formula between hyper-Wiener index and Wiener index or partial Wiener index of any connected graph G of order n and also, we find the lower bounded and upper bounded of hyper Wiener index depending on the diameter and Wiener index.

Acknowledgements

This paper was supported by the College Computer Sciences and Mathematics, University of Mosul, Republic of Iraq.

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Accepted: March 03, 2020

Optimality analysis of fuzzy critical path for projects (an applied study)

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Abstract. The Critical Path Method (CPM) is a crucial agent for the control of complex let alone the planning projects. The prominent achievement of CPM for each activity needs the existence of a valid identified time duration. However, in applicable situations this requirement is usually difficult to fulfill however a great number of activities will be fulfilled for the first time. Thus, there is always a doubt in the network planning about the time durations of activities, a matter that leads to the advancement and development of fuzzy (CPMs.). Rather than employing objective probability measures only A rational approach toward decision making should take into account human subjectivity. We used a method to schedule and control project of extending railway (Basra - Faw) using a fuzzy critical path method. Time is considered to be the major element and the real cost in every project. Thus, estimating activity time methods affect the calculations of total project time and finally the total cost. The total time of the project was calculated with the crispy time defined by $\alpha - Cut$. The researchers determined the fuzzy critical path, fuzzy critical activities, and fuzzy float for the rest of the activities. The total expected time to complete the project will be between (54.4, 68, 84) weeks, which was calculated by using the fuzzy critical path method, which is identical to the length of the first fuzzy critical path when calculating the lengths of all paths, and the estimate of expected time with this method It enables the decision-maker to follow two policies in the completion of the project that accommodate all the conditions that can be encountered without making mistakes, and it also allows him to obtain a clear value in the amount determined by the level of the cut(α).

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Keywords: fuzzy set theory, fuzzy numbers, fuzzy critical path analysis, project network.

1. Introduction

Over the past 30 years, extensively the fuzzy set theory has been studied. In fuzzy set theory pertained were most of the early interest to representing uncertainty in human cognitive processes. Gupta and Kaufmann (1988) mention that over 7,000 research papers, monographs, reports, and books on applications the fuzzy set theory, these have been in publication since 1965 **??**. It does not require any argument that most projects are difficult to be tackled and they encompass risk, due to the existence of many activities joined in a complex way. As a result, to determine the identification of the so-called critical path (CP), many network techniques which determine the project completion time have been developed. The easiest procedure suitable for modeling the fulfillment of a project, and Critical path method (CPM) has been managed to various industrial fields. The (CPM) and its development to probable environment, the Program Evaluation and Review Technique (PERT) are the mostly used tools for tackling and predicting the various short time or long time projects. There is a great gap and distance between the conjugated probability and the real implementation probabilities. The program evaluation and review technique (PERT) can be employed to deal quantitatively with imprecise data, Regardless of this, there is the criticism of PERT, Substitutional procedure for tackling the inaccurate data is to apply the concept of fuzziness, with this, the ambiguous activity times can be expressed by the fuzzy sets. The case where activity times in a project are approximately known has been studied by various studies and these activity times are more appropriately acted rather than clear numbers by fuzzy sets.

2. Fuzzy sets

Fuzzy means ambiguity and lack of clarity, and it is a kind of uncertainty, and in the science of traditional management and engineering means probabilistic uncertainty with a random characteristic, but from the point of view of fuzzy theory is the uncertainty of meanings and description of things such as: excellent, large, small, good, high ... The formation of a conceptual framework that creates a balance among the various aspects. The framework has been used in the case of normal sets, but is more common than the latter. It is the fuzzy set and, essentially, may be found to possess a much wider scope of applicability, mainly in the information processing fields and of pattern classification. Basically, such a framework introduces a normal way of tackling the problems in which the inaccuracy source is the unavailability of sharply definite criteria of class membership rather than the presence of random variables. If the function $(\mu_{\tilde{A}})$ is an affiliation function that gives to each element a degree of belonging

to the set ($\tilde{A}$), then the value of the affiliation function is (1) for the elements belonging to the subset ($\mu_{\tilde{A}}(x) = 1$) and the value of the function (0) is for the elements that do not belong to the subset ($\mu_{\tilde{A}}(x) = 0$)*i.e.* :

$$\mu_{\tilde{A}}(X) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases} \tag{1}$$

As for the fuzzy sets, its elements are not clearly defined because it allows the elements to partially belong to it and it has a affiliation function ranging from (0, 1) Similar points to the concept expressed by the fuzzy set. The fuzzy number ($\tilde{A}$) is a fuzzy set whose membership function $\mu_{\tilde{A}}(X)$ satisfies the following conditions:

i. $\mu_{\tilde{A}}(X)$ is piecewise continuous.

ii. $\mu_{\tilde{A}}(X)$ Fuzzy subset is a convex.

iii. $\mu_{\tilde{A}}(X)$ is the normality for a fuzzy subset, implying that for at least one element x_0 the membership degree must be1, i.e. $\mu_{\tilde{A}}(x_0) = 1$.

Definition. A fuzzy number with membership function in the form:

$$\mu_{\tilde{A}}(x; a, m, b) = \begin{cases} \frac{x-a}{m-a}, & x \in [a, m), \\ 1, & x = m, \\ \frac{b-x}{b-m}, & x \in (m, b], \\ 0, & x \notin [a, b], \end{cases} \tag{2}$$

$\tilde{A} = (a, m, b)$ is called a triangular fuzzy number

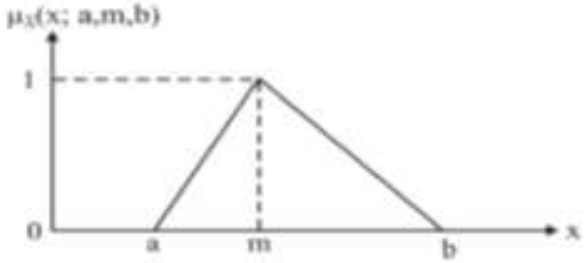


Figure 1: Triangular Fuzzy Number A

3. Basic algebraic operations

For the *union* $(A \cup B)$of fuzzy sets A,B he considered the definition:

$$\mu_{A \cup B}(X) = defMax(\mu_A(x), \mu_B(x)), \quad \forall x \in X.$$

For the intersection $(A \cap B)$ of fuzzy sets A,B the definition:

$$\mu_{A\cap B}(X) = defMin(\mu_A(x), \mu_B(x)), \forall x \in X.$$

For the fuzzy numbers addition; let $\tilde{A} = (a_1, m_1, b_1)$ and $\tilde{B} = (a_2, m_2, b_2)$ he considered the definition:

$$\tilde{A} + \tilde{B} = (a_1 + a_2, m_1 + m_2, b_1 + b_2). \tag{3}$$

For the fuzzy numbers subtraction; let $\tilde{A} = (a_1, m_1, b_1)$ and $\tilde{B} = (a_2, m_2, b_2)$ he considered the definition:

$$\tilde{A} - \tilde{B} = (a_1 - b_2, m_1 - m_2, b_1 - a_2). \tag{4}$$

4. Fuzzy critical path analysis

When estimating time for activities in the business network marred by uncertainty, we move from the statistical theory that is adopted in the style of review and evaluation PERT to fuzzy theory and follow the fuzzy critical path method for estimating the total project time where each activity is estimated for triangular fuzzy times (a, m, b), when use the fuzzy critical path analysis we need an outline theorem used in the proposed and the important properties following:

Property (I): $F\widetilde{E}S_j = Max\{F\widetilde{E}S_i + F\widetilde{A}T_{ij}\} j \neq 1, i < j,$

Property (II): $FLC_i = Min\{FLC_j - FAT_{ij}\} i \neq n, i < j,$

Property (III): $FCP(P_k) = \sum 1 \leq i < j \leq nFAT_{ij}, P_k \in P.$

5. Algorithm the fuzzy critical path analysis

1. Identify project activities.
2. For all activities establish precedence relationships.
3. With respect to each activity estimate the fuzzy time.
4. Building the project network.
5. Find all the possible paths for the network by using property (III).
6. Calculate FAT_{ij} with respect in a network to each activity.
7. Let FES_1=(0, 0, 0, 0) by using property (I), Calculate FES_j, j=2, 3, . . . , n.
8. Let $FLC_n = FES_n$ and calculate FLC_i by using the property (II).
9. Get it the fuzzy critical path.
10. Get it the range of membership that the project can be finished at a specific time.

Notations

N: All nodes of the set in a network.
A_{ij} : The activity between nodes i and j.
FAT: The time of A_{ij} for the fuzzy activity.
FES_j: The earliest fuzzy time for node j.
FLC_j: The total slack fuzzy time for A_{ij}.
P: The whole paths found in a network.
FCP(P_k): The fuzzy completion time for path P_k in a network.

Ranking of fuzzy numbers

One of the most important methods used to compare the fuzzy numbers is the rank function $R : F(R) \to R$, in which F(R) It is the fuzzy set numbers defined on the real numbers. There are several functions, the most important of which is the ranking of trigonometric fuzzy numbers:

$$R\tilde{A} = \frac{a + 2m + b}{4} \tag{5}$$

6. α-cut

When we want to display an element $(x \in X)$that typically belongs to a fuzzy set A, we may demand its membership number to be greater than some threshold $\alpha \in [0, 1]$. The ordinary set of such element is the $\alpha - Cut\ A_\alpha$ of A,$A_\alpha = \{x \in X, \mu_A(x) \geqslant \alpha\}$. On also defines *strong* $\alpha{-}Cut\ A_\alpha = \{x \in X, \mu_A(x) > \alpha\}$. When the $\tilde{A} = (a, m, b)$ is triangular fuzzy number then $\alpha{-}Cut\ \alpha \in [0, 1][A_i(\alpha), A_u(\alpha)]$

$$A_i(\alpha) = a + (m - a)\alpha \tag{6}$$

$$A_u(\alpha) = b + (m - b)\alpha \tag{7}$$

$$= b - (b - m)\alpha \tag{8}$$

7. Application

Extend railway (Basra - Faw) project: This project is among the future projects of the General Iraqi Railways Company, as the importance of the project lies in:

1. Part of the dry canal that connects Europe with East Asia through (Turkey-Zakho-Dohuk-Mosul-Erbil-Kirkuk-Baquba-Baghdad-Kut-Amara-Basra-Faw).

2. Connecting Al-Fao Port to Abu Flus Port.

The investment company that presented the study for this project is the German company (Doorsh group), as it divided the project into three sectors,

and each sector is subdivided into small projects, and thus we will get eight sub-projects distributed among three sectors and the following table represents the sectors and projects branching from them.

Table I: represents the sectors and projects branching

Sectors		Projects	Time \ Week	Sector length / km
Sector (1)	1-1	Tunnel construction work	224	10
	1-2	Earth works and railway construction	66	
Sector (2)	2-1	Ground works for the railway	68	45
	2-2	Ground works and building housing complexes	176	
	2-3	Extending a railway	60	
Sector (3)	3-1	Ground works for the railway	68	40
	3-2	Ground works and building housing complexes	350	
	3-3	Extending a railway	66	

Selection of the third sector, the first stage (ground works for the railway), which is at the time of the completion of this stage (68) weeks, as shown in the above table:

Table II: The priority of activities for Project (3-1)

No	Activities		*Prior Activity*	Time\ week		
				a	*m*	*b*
1	Clean the site and weed out	A		16	20	23
2	Tear down buildings and concrete structures	B		12.8	16	20
3	Removing old paved roads	C		12.8	16	20
4	Removal and transmission of electrical and communication cables	D	A,B,C	12.8	16	21
5	Scrape the natural ground with a thickness of 30 cm	E	A,B,C	6.4	8	10
6	Cutting, straightening and leveling	F	C	9.6	12	15
7	Dig rainwater drainage channels	G	E	4.8	6	7
8	Vertical channels for rain water drainage	H	D,F	12.8	16	19
9	Supplying and installing mesh for earthen dams	I	D,F	12.8	16	20
10	Create Earth dam on the line	J	I	12.8	16	20
11	Lighting and electrical work	K	G,H	6.4	8	10

Step (1) Construct the project network.

Step (2) The possible paths for the network by using property (III) and equation (3).

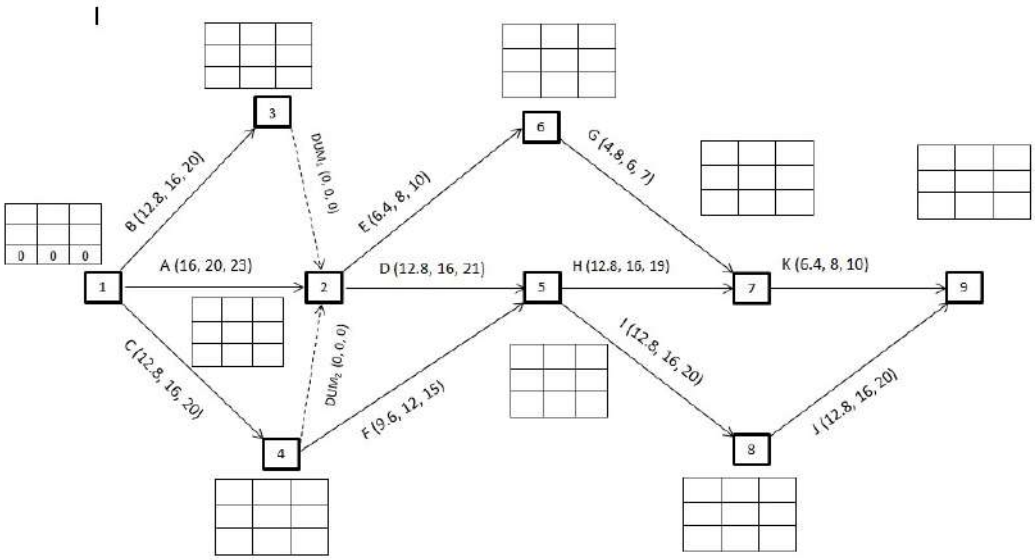


Figure 2: Network Project

$P_1 = FAT_{12}+FAT_{25}+FAT_{57}+FAT_{79} = (16, 20, 23)+(12.8, 16, 21)+(12.8, 16, 19)+(6.4, 8, 10) = (48, 60, 73)$
$P_2 = FAT_{12}+FAT_{26}+FAT_{67}+FAT_{79} = (16, 20, 23)+(6.4, 8, 10)+(4.8, 6, 7)+(6.4, 8, 10) = (33.6, 42, 50)$
$P_3 = FAT_{12}+FAT_{25}+FAT_{58}+FAT_{89} = (16, 20, 23)+(12.8, 16, 21)+(12.8, 16, 20)+(12.8, 16, 20) = (54.4, 68, 84)$
$P_4 = FAT_{13}+FAT_{32}+FAT_{25}+FAT_{57}+FAT_{79} = (12.8, 16, 20)+(0, 0, 0)+(12.8, 16, 21)+(12.8, 16, 19)+(6.4, 8, 10) = (44.8, 56, 70)$
$P_5 = FAT_{13}+FAT_{32}+FAT_{26}+FAT_{67}+FAT_{79} = (12.8, 16, 20)+(0, 0, 0)+(6.4, 8, 10)+(4.8, 6, 7)+(6.4, 8, 10) = (30.4, 38, 47)$
$P_6 = FAT_{13}+FAT_{32}+FAT_{25}+FAT_{58}+FAT_{89} = (12.8, 16, 20)+(0, 0, 0)+(12.8, 16, 21)+(12.8, 16, 20)+(12.8, 16, 20) = (51.2, 64, 81)$
$P_7 = FAT_{14}+FAT_{42}+FAT_{25}+FAT_{57}+FAT_{79} = (12.8, 16, 20)+(0, 0, 0)+(12.8, 16, 21)+(12.8, 16, 19)+(6.4, 8, 10) = (44.8, 56, 70)$
$P_8 = FAT_{14}+FAT_{42}+FAT_{25}+FAT_{58}+FAT_{89} = (12.8, 16, 20)+(0, 0, 0)+(12.8, 16, 21)+(12.8, 16, 20)+(12.8, 16, 20) = (51.2, 64, 81)$
$P_9 = FAT_{14}+FAT_{42}+FAT_{26}+FAT_{67}+FAT_{79} = (12.8, 16, 20)+(0, 0, 0)+(6.4, 8, 10)+(4.8, 6, 7)+(6.4, 8, 10) = (30.4, 38, 47)$
$P_{10} = FAT_{14}+FAT_{45}+FAT_{58}+FAT_{89} = (12.8, 16, 20)+(9.6, 12, 15)+(12.8, 16, 20)+(12.8, 16, 20) = (48, 60, 75)$
$P_{11} = FAT_{14}+FAT_{45}+FAT_{57}+FAT_{79} = (12.8, 16, 20)+(9.6, 12, 15)+(12.8, 16, 19)+(6.4, 8, 10)+(6.4, 8, 10) = (41.6, 52, 64)$

Step (3) By using property (I) and equation (3) find the earliest fuzzy time for node j.

$FES_{12} = (0,0,0),\ FES_{13} = (0,0,0),\ FES_{14} = (0,0,0)$

$$FES_2 = Max\left\{\begin{array}{c} FES_{12} + FAT_{12} = (0,0,0) + (16,20,23) \\ FES_{13} + FAT_{13} = (0,0,0) + (12.8,16,20) \\ FES_{14} + FAT_{14} = (0,0,0) + (16,20,23) \end{array}\right\} = (16,20,23)$$

$FES_3 = FES_{13} + FAT_{13} = (0,0,0) + (12.8,16,20) = (12.8,16,20)$

$FES_4 = FES_{14} + FAT_{14} = (0,0,0) + (12.8,16,20) = (12.8,16,20)$

$$FES_5 = Max\left\{\begin{array}{c} FES_2 + FAT_{25} = (16,20,23) + (12.8,16,21) \\ FES_4 + FAT_{45} = (12.8,16,20) + (9.6,12,15) \end{array}\right\} = (28.8,36,44)$$

$FES_6 = FES_2 + FAT_{26} = (16,20,23) + (6.4,18,10) = (22.4,28,33)$

$$FES_7 = Max\left\{\begin{array}{c} FES_5 + FAT_{57} = (28.8,36,44) + (12.8,16,19) \\ FES_6 + FAT_{67} = (22.4,28,33) + (4.8,6,7) \end{array}\right\} = (41.6,52,63)$$

$FES_8 = FES_5 + FAT_{58} = (28.8,36,44) + (12.8,16,20) = (41.6,52,64)$

$$FES_9 = Max\left\{\begin{array}{c} FES_7 + FAT_{79} = (41.6,52,63) + (6.4,8,10) \\ FES_8 + FAT_{89} = (41.6,52,64) + (12.8,16,20) \end{array}\right\} = (54.4,68,84)$$

Step (4) By using property (II) and equation (4) find the fuzzy latest compilation of node i:

$FLC_9 = FES_9 = (54.4,68,84)$

$FLC_8 = FLC_9 - FAT_{89} = (54.4,68,84) - (12.8,16,20) = (34.4,52,71.2)$

$FLC_7 = FLC_9 - FAT_{79} = (54.4,68,84) - (6.4,8,10) = (44.4,60,77.6)$

$FLC_6 = FLC_7 - FAT_{67} = (44.4,60,77.6) - (4.8,6,7) = (37.4,54,72.8)$

$$FLC_5 = Min\left\{\begin{array}{c} FLC_7 - FAT_{57} = (44.4,60,77.6) - (12.8,16,19) \\ FLC_8 - FAT_{58} = (34.4,52,71.2) - (12.8,16,20) \end{array}\right\} = (14.4,36,58.4)$$

$$FLC_2 = Min\left\{\begin{array}{c} FLC_6 - FAT_{26} = (37.4,54,72.8) - (6.4,8,10) \\ FLC_5 - FAT_{25} = (14.4,36,58.4) - (12.8,16,21) \end{array}\right\} = (-6.6,20,45.6)$$

$$FLC_4 = Min\left\{\begin{array}{c} FLC_2 - FAT_{42} = (-6.6,20,45.6) - (0,0,0) \\ FLC_5 - FAT_{45} = (14.4,36,58.4) - (9.6,12,15) \end{array}\right\} = (-6.6,20,45.6)$$

$FLC_3 = FLC_2 - FAT_{32} = (-6.6,20,45.6) - (0,0,0) = (-6.6,20,45.6)$

$$FLC_1 = Min\left\{\begin{array}{c} FLC_2 - FAT_{12} = (-6.6,20,45.6) - (16,20,23) \\ FLC_3 - FAT_{13} = (-6.6,20,45.6) - (12.8,16,20) \\ FLC_4 - FAT_{14} = (-6.6,20,45.6) - (12.8,16,20) \end{array}\right\} = (-29.6,0,29.6)$$

$FF_1 = FLC_1 - FES_1 = (-29.6,0,29.6) - (0,0,0) = (-29.6,0,29.6)$

$FF_2 = FLC_2 - FES_2 = (-6.6,20,45.6) - (16,20,23) = (-29.6,0,29.6)$

$FF_3 = FLC_3 - FES_3 = (-6.6,20,45.6) - (12.8,16,20) = (-26.6,4,32.8)$

$FF_4 = FLC_4 - FES_4 = (-6.6,20,45.6) - (12.8,16,20) = (-26.6,4,32.8)$

$FF_5 = FLC_5 - FES_5 = (14.4,36,58.4) - (28.8,36,44) = (-29.6,0,29.6)$

$FF_6 = FLC_6 - FES_6 = (37.4,54,72.8) - (22.4,28,33) = (4.4,26,50.4)$

$FF_7 = FLC_7 - FES_7 = (44.4,60,77.6) - (41.6,52,63) = (-18.6,8,36)$

$FF_8 = FLC_8 - FES_8 = (34.4,52,71.2) - (41.6,52,64) = (-29.6,0,29.6)$

$FF_9 = FLC_9 - FES_9 = (54.4,68,84) - (54.4,68,84) = (-29.6,0,29.6)$

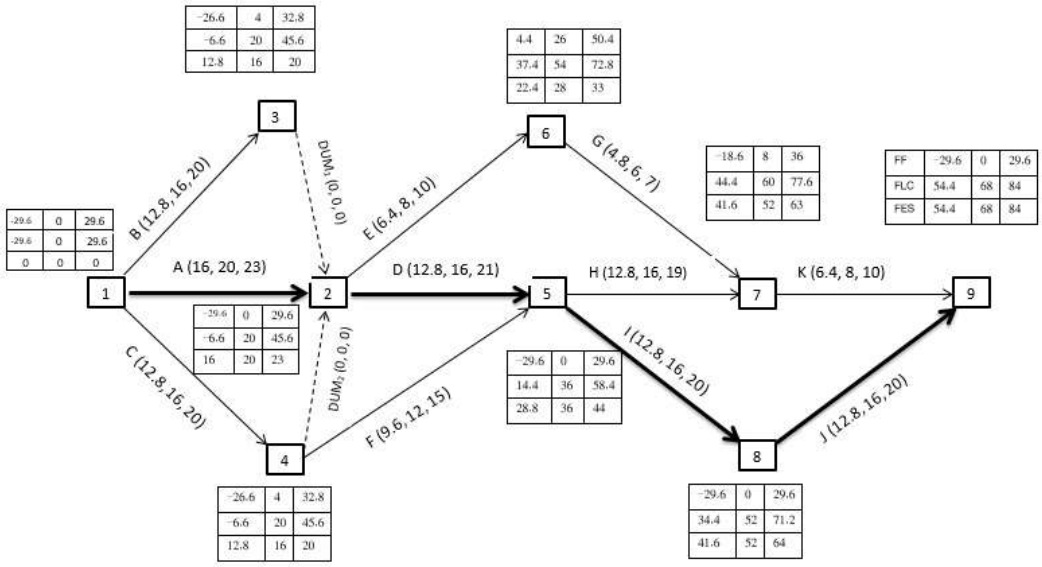


Figure 3: Network Project Includes (FES, FLC, FF) and FCP

Note that the median value for critical activities is (M=0) (FF1, FF2, FF5, FF8, FF9) and the fuzzy critical path is FCP=(A, D, I, J).

8. Calculation of relationship and $\alpha - Cut$:

$$F\tilde{A}T_{12} = (16, 20, 23),\ \mu_{D12}(X) = \begin{cases} \frac{X-16}{4}, & 16 \le X \le 20, \\ 1, & X = 20, \\ \frac{23-X}{3}, & 20 \le X \le 23. \end{cases}$$

The duration of the activity is (A = 20) which is the confirmed period and everywhere we further for it, it less the relationship, So the clear value determined by the $\alpha - Cut$ for activity (A), $\alpha - Cutof12 = [4\alpha + 16, 23 - 3\alpha]$ By fixing the value (α) of the information available to the decision maker $F\tilde{A}T_{25} =$

$$(12.8, 16, 21),\ \mu_{D25}(X) = \begin{cases} \frac{X-12.8}{3.2}, & 12.8 \le X \le 16, \\ 1, & X = 16, \\ \frac{21-X}{5}, & 16 \le X \le 21. \end{cases}$$

The duration of the activity is (D = 16) which is the confirmed period and everywhere we further for it, it less the relationship, So the clear value determined by the $\alpha - Cut$ for activity (D):

$\alpha - Cutof25 = [3.2\alpha + 12.8, 21 - 5\alpha]$. By fixing the value ($\alpha$) of the information available to the decision maker

$$F\tilde{A}T_{58} = (12.8, 16, 20),\ \mu_{D58}(X) = \begin{cases} \frac{X-12.8}{3.2}, & 12.8 \le X \le 16, \\ 1, & X = 16, \\ \frac{20-X}{4}, & 16 \le X \le 20. \end{cases}$$

The duration of the activity is (I = 16) which is the confirmed period and everywhere we further for it, it less the relationship, So the clear value determined by the $\alpha - Cut$ for activity (I):
$\alpha - Cutof58 = [3.2\alpha + 12.8, 20 - 4\alpha]$
By fixing the value (α) of the information available to the decision maker.

$$F\tilde{A}T_{89} = (12.8, 16, 20)$$
$$\mu_{D89}(X) = \begin{cases} \frac{X-12.8}{3.2}, & 12.8 \le X \le 16, \\ 1, & X = 16, \\ \frac{20-X}{4}, & 16 \le X \le 20. \end{cases}$$

The duration of the activity is (J = 16) which is the confirmed period and everywhere we further for it, it less the relationship, So the clear value determined by the $\alpha - Cut$ for activity (J):
$\alpha - Cutof58 = [3.2\alpha + 12.8, 20 - 4\alpha]$
By fixing the value (α) of the information available to the decision maker.
Project end time:

$$\mu_{D19}(X) = \begin{cases} \frac{X-54.4}{13.6} & 54.4 \le X \le 68, \\ 1 & X = 68, \\ \frac{84-X}{16} & 68 \le X \le 84 \end{cases}$$
$\alpha - CutofD_{19} = [13.6\alpha + 68, 84 - 16\alpha]$.

By fixing the value (α) of the information available to the decision maker.

9. Conclusions

1- It is very oftentimes that information available for making the decision is uncertain and vague, in analyzing the evaluation of the project. Thus, it is difficult to get the exact data of the activity assessment. Accordingly, the traditional accuracy-based/normal oriented analysis of the project seems to be of less effectivity in transmitting the information available in such an inaccurate and fuzzy decision society or environment.

2- Therefore, using the fuzzy critical path method is more realistic than jealousy in fuzzy project network.

3- The fuzzy critical path of the project is (A, D, I, j) and these activities should be observed and followed up to include the project is not delayed.

4- The total expected time to complete the project will be between (54.4, 68, 84) weeks, which was calculated by using the fuzzy critical path method,

which is identical to the length of the first fuzzy critical path when calculating the lengths of all paths, and this estimate of the expected time with this method It enables the decision-maker to follow two policies in the completion of the project that accommodate all the conditions that can be encountered without making mistakes in incomplete accounts, and it also allows him to obtain a clear value in the amount determined by the level of the cut (α).

5- By using the network analysis and fuzzy set we show that the compilation time to complete the project under study is equal to (68) weeks the same as the company determinant.

Acknowledgements

The researchers gratefully acknowledge Mustansiriyah University, College of Science/Department of Mathematics,and Mustansiriyah University, College of Administration and Economics / Department of statistics, and Ministry of Education and your esteemed journal for supporting this research.

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Accepted: April 18, 2020

Injectivity versus dense injectivity in topological spaces

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Abstract. An injective space is a topological space with a strong extension property for continuous functions with values on it. The topological characterizations by injectivity turns out to follow from the algebraic characterizations and general category theory (Escardo 1998). In this paper we study injectivity in the context of topological spaces. We show that, in the category $\mathcal{T}op_*$, dense injective spaces are precisely the injective spaces, and in Thychonoff spaces dense injective spaces are precisely compact spaces. This way we obtain enriched versions of known results about injective topological spaces and obtain new proofs and new characterizations of topological spaces by injectivity.

Keywords: injectivity, compactness, Tychonoff spaces.

1. Introduction and preliminaries

The injective objects are investigated in many branches of mathematics. The topological characterization by injectivity turns out to follow from the algebraic characterization and general category theory ([4]). In the category $\mathcal{T}op$ ($\mathcal{T}op_0$) of (T_0) topological spaces, injective objects with respect to the embeddings are characterized as retracts of products of the three-point space $S = \{0, 1, 2\}$(0 unique non-trivial open) (S the Sierpinski space for $\mathcal{T}op_0$), see [8, 6]. Dana Scott [8] characterized the continuous lattices endowed with the Scott topology precisely as the spaces that are injective over all subspace embeddings. The continuous Scott domains are the algebras of the proper filter monad [9]. For this monad, the associated embeddings are precisely the dense subspace embeddings, and hence the injective spaces over dense embeddings are characterized as the continuous Scott domains.

In this paper we study injectivity in the context of topological spaces. We show that, in the category $\mathcal{T}op_*$, of all topological spaces which one point sets are closed, dense injective spaces are precisely the injective spaces, and in Thychonoff spaces dense injective spaces are precisely compact spaces. In this way we obtain enriched versions of known results about injective topological spaces and obtain new proofs and new characterizations of topological spaces by injectivity. The main result of this article is Theorem 2.2, which says that the

concepts of injectivity and dense injectivity are equivalent in the category of topological spaces.

Consider $\mathcal{T}op$ as a category of all topoplogical spaces and continuous functions between them. We denote the open sets of a topological space X by $\mathcal{T}_X$. For a subtopological space A of a topological space B, consider $\overline{A}$ to be the elements x in B such that the intersection of A and every open subset of B containing x is non-empty. A space A is a *dense* (*closed*) subspace of a topological space B, if $\overline{A} = B(\overline{A} = A)$. Also a continuous function $f : A \to B$ is dense (closed), if $f(A)$ is a dense (closed) subspace of B. A topological space X is said to be *completely regular space*, if for every closed subset C of X and every $x \in X \setminus C$, there is a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(C) = 1$.

In mathematics, an adjunction space (or attaching space) is a common construction in topology where one topological space is attached onto another. Specifically, let X and Y be topological spaces with A a subspace of Y. Let $f : A \to X$ be a continuous function, called the attaching function. One forms the adjunction space $X \cup_f Y$ by taking the disjoint union of X and Y and identifying x with $f(x)$ for all $x \in A$. Schematically, $X \cup_f Y = (X \sqcup Y)/\{f(A) \sim A\}$.

As a set, $X \cup_f Y$ consists of the disjoint union of X and $(Y \setminus A)$. The topology, however, is specified by the quotient construction.

The attaching construction is an example of a pushout in the category of topological spaces. That is to say, the adjunction space is universal with respect to the following commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{\tau} & Y \\ f\downarrow & & \downarrow h_Y \\ X & \xrightarrow{h_X} & X \cup_f Y \end{array} \qquad (*)$$

where τ is the inclusion function and h_X, h_Y are the functions obtained by composing the quotient function with the canonical injections into the disjoint union of X and Y.

2. Injectivity

Injectivity is one of the most central notions in many branches of mathematics.

A topological space A is said to be *injective* if for any subtopological space $\tau : B \to C$, any continuous function $f : B \to A$ can be lifted to a continuous function $\bar{f} : C \to A$, one can state that the following diagram is commutative:

$$\begin{array}{ccc} B & \xrightarrow{\tau} & C \\ f\downarrow & \swarrow \bar{f} & \\ A & & \end{array}$$

A topological space X is said to be *dense injective*(*closed injective*) if it is injective with respect to all dense subspaces (closed subspaces), which means,

for each dense subspaces (closed subspaces) A of B and each continuous function $f : A \to X$, there is a continuous function $g : B \to X$ such that $g \mid_A = f$.

The following theorem which has an easy proof, concludes the equivalent conditions for injectivity.

Theorem 2.1. *A topological space E is injective if and only if it is closed injective as well as dense injective.*

Proof. We show only the unclear direction. Let $\tau : A \to B$ be an inclusion function and $f : A \to E$ a continuous function. Since A is a dense subtopological space of $\overline{A}$, there is a continuous function $f_1 : \overline{A} \to E$ which extends f and since $\overline{A}$ is closed in B, there exists $\bar{f} : B \to E$ such that $\bar{f} \mid_{\overline{A}} = f_1$. Hence we have $\bar{f}\tau = f$. □

Lemma 2.1. *In the pushout diagram $(*)$, h_X is one to one.*

Proof. As we mentioned earlier, $X \cup_f Y$ consists of the disjoint union of X and $(Y \setminus A)$. Thus $h_X : X \to X \cup_f Y$ is an embedding continuous function. □

A subtopological space A of B is a retract of B, if there is a continuous function $f : B \to A$ such that $f|_A = id_A$. A topological space A is said to be absolute (dense, closed) retract if it is a retract of each of its (dense, closed) extensions.

By Lemma 2.1, we have the following proposition.

Proposition 2.1. *A topological space A is an injective space if and only if it is an absolute retract.*

Consider $\mathcal{T}op_*$, the sub category of $\mathcal{T}op$, of all topological spaces which one point sets are closed. It is clear that every injective space is dense injective. For the converse, we have,

Theorem 2.2. *In the category $\mathcal{T}op_*$, each dense injective space X is injective.*

Proof. Let A be a closed subspace of B and $f : A \to X$ be a continuous function. Consider Σ the set of all pairs (C, f_C) of closed subspaces of B containing A and a continuous function $f_C : C \to X$ which extends f. The set Σ is a partially ordered set by the order relation $(C_1, f_{C_1}) \leq (C_2, f_{C_2})$ if and only if $C_1 \subseteq C_2$ and $f_{C_2} \mid_{C_1} = f_{C_1}$. Let $\{(C_i, f_{C_i})\}_{i \in I}$ be a chain in Σ. The function $f_C : C = \cup C_i \to X$ defined by $f_C(t) = f_{C_i}(t)(t \in C_i)$ is a continuous function. Indeed, let $V \in \mathcal{T}_X$. We have $(f_C^{-1}(V))^c = (\cup f_{C_i}^{-1}(V))^c = \cap (f_{C_i}^{-1}(V))^c$, in which $(f_C^{-1}(V))^c = C \setminus f_C^{-1}(V)$. Since for each $i \in I$, $(f_{C_i}^{-1}(V))^c$ is closed in C_i, hence $(f_{C_i}^{-1}(V))^c$ is a closed subset of B. So $(f_C^{-1}(V))^c = \cap (f_{C_i}^{-1}(V))^c$ is closed in B and hence it is closed in C. Thus $f_C^{-1}(V)$ is an open subset of C, which implies that $f_C : C \to X$ is a continuous function. Since X is dense injective, f_C can be extended to $f_{\bar{C}} : \bar{C} \to X$ which is an upper bound for the chain. By Zorn's lemma Σ has a maximal element (M, f_M). Let $M \neq B$. So there exists

$b_0 \in B \setminus M$. Consider $N = M \cup \{b_0\}$ as a subtopological space of B. Choose and fixed $x_0 \in X$ and define a continuous function $g : \{b_0\} \to X$ by $g(b_0) = x_0$. By Pasting theorem there exists a continuous function $h : N \to X$ extending f_M, which contradicts the maximality of M. Thus $B = M$ that shows X is closed injective. Now Theorem 2.1 complets the proof. □

Here, we have a relationship between injectivity and compactness. We show that every completely regular dense injective topological space is compact, but the converse is not generally true.

Theorem 2.3. *In the category of Tychonoff spaces, a topological space X is compact if and only if it is dense injective.*

Proof. Let Y be a dense extension of X. Since X is a compact space, $X = \overline{X} = Y$. Thus X is an absolute dense retract. Now the proof is complete by using Proposition 2.1.

For the converse, let X be a dense injective space and Y be a Stone-Cech compactification of X. Hence with $\overline{X} = Y$ and Proposition 2.1, there exists a continuous function $g : Y \to X$ such that $g \mid_X = id_X$. Since the image of any compact set under a continuous function is compact, $X = g(Y)$ is a compact space. □

Corollary 2.1. *Let $X \in \mathcal{T}op_*$ be a Tychonoff space. Then X is an injective space if and only if it is a compact space.*

Corollary 2.2. *Every completely regular dense injective topological space is compact.*

In the following we show that the converse of Corollary 2.2 is not generally true.

Suppose that each compact topological space is completely regular dense injective. The two-point discrete space $2 = \{0, 1\}$ and the Euclidean interval $[0, 1]$ are both completely regular, and $A = 2$ is a subspace of $B = [0, 1]$. Now let $X = 2$, and $f : A \to X$ be the identity function. Since X, being finite, is compact, we conclude that $f : A \to X$ extends to a continuous function $g : B \to X$. That is a continuous function $g : [0, 1] \to 2$ with $g(0) = 0$ and $g(1) = 1$. But this is impossible because $[0, 1]$ is connected, whereas the existence of such a $g : [0, 1] \to 2$ gives non-empty disjoint open sets $g^{-1}(0)$ and $g^{-1}(1)$, which means that $[0, 1]$ is disconnected.

Analogously to the Tychonoff's theorem, we have the following corollary.

Corollary 2.3. *If every compact topological space is dense injective, then the product of each family of compact spaces is compact in the product topology.*

Proof. Let $\{A_i\}_{i \in I}$ be a family of compact spaces. So it is a family of dense injective spaces. By [7, Theorem 33.2], $\prod A_i$ is Tychonoff and by [1, Proposition

10.40], $\prod A_i$ is a dense injective space, and then a compact space by applying Corollary 2.2. □

The following corollary is a more generality of Tietze extension theorem.

Corollary 2.4. *If every compact topological space is dense injective, then every retract of $[a,b]^I$ for the set indexed I is an injective object in the category of normal topological spaces with continuous functions.*

Proof. By Tietze extension theorem (see, [7, Theorem 35.1]), the interval $[a,b]$ is closed injective and since each closed subset of $\mathbb{R}$ is a compact space, $[a,b]$ is a dense injective space by using Corollary 2.2. From Theorem 2.1, one can deduces that $[a,b]$ is an injective space. By [1, Proposition 10.40], $[a,b]^I$ is injective where [1, Proposition 9.5] shows that every retract of $[a,b]^I$ is an injective space. □

Acknowledgment

I would like to express my appreciation to the referee for carefully reading the paper.

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Accepted: January 30, 2020

LG-fuzzy tangent spaces of C^∞ L-fuzzy manifolds with L-gradation of openness

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Abstract. In this paper, we introduce the concept of the L-fuzzy tangent spaces of an C^∞ L-fuzzy manifold with L-gradation of openness $(X, \mathfrak{T})$, which X itself is an L-fuzzy subset of a crisp set M and $\mathfrak{T}$ is an L-gradation of openness of fuzzy subsets of M which are less or equal to X and $L =< L, \leq, \bigwedge, \bigvee, ' >$ denotes a complete distributive lattice with at least two elements. Then, we define C^∞ LG-fuzzy vector fields and C^∞ LG-fuzzy one-parameter and LG-fuzzy local one-parameter group acting on an C^∞ L-fuzzy manifold with L-gradation of openness. We prove some theorems and give some examples about them.

Keywords: C^∞ L-fuzzy tangent spaces with gradation of openness $\mathfrak{T}$, C^∞ L-fuzzy vector fields, C^∞ L- fuzzy one-parameter group.

1. Introduction

Fuzzy sets theory introduced by Zadeh in 1965 [27] and the concept of fuzzy topology was first defined in 1968 by Chang [2]. In 1974, Wong [26] introduced product and quotient fuzzy topologies and studied in this spirit. Later concept of fuzzy topology redefined in somewhat different way by Lowen [16] and Hutton [11]. In 1976, Lowen [16] introduced two functions ω and ι which connect between his new structure of fuzzy topological spaces and the ordinary topological

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spaces. Then, Rodabaugh discssed a categorical accomodation of various notions of fuzzy topology [24]. In 1985, Shostak [25] introduced the fundamental concept of a fuzzy topological structure as an extension of both crisp topology. In 1992 Chattopadhyay [3] et al. introduced a new concept of gradation of openness of a fuzzy subset and later in [4] introduced a fuzzy closure operator in smooth topological spaces. In addition, they defined smooth compactness in the sence of lowen [16] and obtained its properties. In 1977, Katsaras and Liu[15] applied the concept of a fuzzy set to the elementary theory of vector spaces and topological vector spaces. Gregori [9] proved that gradation of openness will enable us to introduce fuzzy topological concepts which are generalitation of the corresponding ones in General Topology and to work with points of X instead of fuzzy points.

Stadler and Vicente [18] have introduced a new concept of fuzzy topological subspace which coincides with the usual definition in the case that $\mu = X_Y$ for $Y \subset X$. Also, Jaager [14] investigated fuzzy properties in fuzzy convergences spaces. For more details on fuzzy topological spaces and related results we refer the reader to [1, 5, 10, 11, 13, 14, 17, 21, 22, 24].

In 1993, Ferraro and Foster [6] introduced the notion of C^1 fuzzy manifold as a natural development of the notion of a fuzzy topological vector space and of a fuzzy derivative of a fuzzy continuous mapping between fuzzy topological vector spaces. In 2016, Ibedou [13] discussed graded fuzzy topological spaces.

While all of the researches about the C^1 or C^∞ fuzzy manifolds, focused on a crisp set, but in 2017 M. Mostafavi [19] investigated some properties of a novel fuzzy topological space (X, τ), where X is itself a fuzzy subset of a crisp set M. Perhaps the most important generalization of the aforementioned structures, is the consideration of lattice L beyond the unit interval $I = [0, 1]$. Let $L =< L, \leq, \bigwedge, \bigvee, ' >$ be a complete distributive lattice set with at least 2 elements; 0 is the bottom element and 1 is the top element of L. An L-fuzzy subset D of the crisp set M, in Goguen's sense [8], is a function $D : M \to L$ and is denoted by $D \in L^M$. In 2020, Mostafavi defined the concept of an LG-fuzzy topological space $(X, \mathfrak{T})$ which X is itself an L-fuzzy subset of a crisp set M and $\mathfrak{T}$ is an L-gradation of openness of L-fuzzy subsets of M which are less than or equal to X. Then she defined C^∞ L-fuzzy manifolds with L-gradation of openness and C^∞ LG-fuzzy mappings of them [20] in a way different from [22, 6, 23].

In this paper we introduce concept of the L-fuzzy tangent spaces of C^∞ L-fuzzy manifolds with gradation of openness $(X, \mathfrak{T})$, with a different perception from [6] and [23] and we give some examples and prove several theorems. The concept of fuzzy vector space and fuzzy topological vector space were introduced by Katsaras and Liu [15]. We define notion of L-fuzzy vector spaces and show that for any p of an L-fuzzy manifold with L-gradation of openness $(X, \mathfrak{T})$, L-fuzzy tangent space of X at p $(LGT_p(X),\ \eta)$, is an L-fuzzy vector space.

Finally, we define LG-fuzzy one-parameter and LG-fuzzy local one-parameter group acting on an C^∞ L-fuzzy manifolds with L-gradation of openness.

2. Preliminaries

Definition 2.1. *Let X be an L-fuzzy subset of M. Then, any L-fuzzy subset of M which is less or equal to X is called an L-fuzzy subset of X. We denote the set of all L-fuzzy subsets of X by L_X^M. If τ as a collection of L-fuzzy subsets of X, satisfies the following conditions, then (X,τ) is called an L-fuzzy topological space (Lfts):*

1) $X,\ \phi \in \tau$,

2) $\{A_i\}_{i\in I} \subseteq \tau \ \Rightarrow\ \bigcup_{i\in I} A_i \in \tau$,

3) $A, B \in \tau \ \Rightarrow\ A \cap B \in \tau$.

Example 2.1. Let $M = \mathbb{R}^n$ and $X = 1$ be a constant L-fuzzy subset of M. Let $B(a,r,b)$ be an L-fuzzy subset that is equal to zero outside or on the sphere $B(a,r)$ and equal to the function b with values in L, inside $B(a,r)$. We call the L-fuzzy topology induced by

$$\beta_{Ln} = \{B(a,r,b),\ a \in \mathbb{R}^n,\ r \in \mathbb{R}^+,\ b : B(a,r) \to L,\ \text{is a function}\}$$

the L-fuzzy Euclidean topology of dimension n and denote it by τ_{Ln}. Therefore we have the L-fuzzy Euclidean topological space $(1_{\mathbb{R}^n},\ \tau_{Ln})$.

Definition 2.2. *Let $\mathfrak{T} : L_X^M \to L$, be a mapping satisfying:*

i) $\mathfrak{T}(X) = \mathfrak{T}(\tilde{0}) = 1$,

ii) $\mathfrak{T}(A \cap B) \geq \mathfrak{T}(A) \wedge \mathfrak{T}(B)$,

iii) $\mathfrak{T}(\bigcup_{j\in J} A_j) \geq \bigwedge_{j\in J} \mathfrak{T}(A_j)$.

Then, $\mathfrak{T}$ is called an L-gradation of openness on X and $(X,\ \mathfrak{T})$ is called an LG-fuzzy topological space (Lgfts).

Let $x \in M$ and $A \in L_X^M$. When we write $x \in A$, we mean $x \in \operatorname{supp} A$.

Example 2.2. Let $M = \mathbb{R}^n$ and $X = 1$ be a constant L-fuzzy subset of M. As two useful examples, we define

$$\mathfrak{T}_{Ln} \ :\ L_X^M \to L, \qquad \mathfrak{T}_{Ln}(B) = \begin{cases} 1, & B \in \tau_{Ln}, \\ 0, & \text{elsewhere.} \end{cases} \tag{1}$$

and

$$\mathfrak{T}_{Linf} \ :\ L_X^M \to L, \qquad \mathfrak{T}_{Linf}(B) = \begin{cases} 1, & B = \tilde{0}, \\ \inf\{B(x) : x \in M\}, & \tilde{0} \neq B \in \tau_{Ln}, \\ 0, & \text{elsewhere.} \end{cases} \tag{2}$$

Definition 2.3. *Let $(X,\mathfrak{T})$ be an Lgfts. Set* $\operatorname{supp}\mathfrak{T} = \{A \in L_X^M :\ \mathfrak{T}(A) > 0\}$, *then A is called an LG-open subset of X if $A \in \operatorname{supp}\mathfrak{T}$. Further*

1) *Suppose $x \in X$ and $V \in L_X^M$. If there exists an LG-open subset U of X such that $U(x) = V(x)$ and $U \leq V$, then V is called an LG-neighborhood of x in X. We denote the set of all LG-neighborhoods of x in X by $LGN(x)$.*

2) *If for all $x, y \in X$, $x \neq y$, there exist two LG-neighborhoods $U_x \in LGN(x)$, $U_y \in LGN(y)$ such that $U_x \cap U_y = 0$. Then, $(X, \mathfrak{T})$ is called a Hausdorff Lgfts.*

3) *For each L-fuzzy subset A of X and any $U \subset \operatorname{supp} X$, we define the L-fuzzy subset $\chi_{U,A}$ of X by $\chi_{U,A}(z) = \begin{cases} A(z), & z \in U, \\ 0, & \text{elsewhere.} \end{cases}$ From now, we write χ_U instead of $\chi_{U,X}$.*

4) *A is called an LG-closed subset of X if $X - A \in \operatorname{supp} \mathfrak{T}$.*

5) *Let Z be an LG-open subset of X. Define $\mathfrak{T}_Z : L_Z^M \to L$, by $\mathfrak{T}_Z(A) = \mathfrak{T}(A)$. Then, $(Z, \mathfrak{T}_Z)$ is called an LG-fuzzy topological subspace of X (Lgtfss).*

From now, we suppose that M_1, M_2 are two crisp sets, $X \in L^{M_1}, Y \in L^{M_2}$ and $(X, \mathfrak{T}), (Y, \mathfrak{R})$ are two LG-fuzzy topological spaces.

Definition 2.4. *Let $f : M_1 \to M_2$ be a function and $f[X]$ be an L-fuzzy subset of M_2, defined by $f[X](y) = \bigvee\{X(x) \mid x \in f^{-1}(y)\}$. If we have $f[X] \leq Y$. Then, f is called an LG-related function from X to Y and the set of all these functions is denoted by $LGRf(X, Y)$.*

Further if we have $\mathfrak{R}(H) \leq \mathfrak{T}(f^{-1}[H])$, for all LG-fuzzy subset H of Y. Then, f is an L-gradation preserving LG-related function so it is called an LGP-related function or LGP-fuzzy mapping from X to Y, so $f \in LGPRf(X, Y)$.

i) f is called an one-to-one LG-related (LGP-related) function if $f|_{\operatorname{supp} X} : \operatorname{supp} X \to \operatorname{supp} Y$ is one-to-one function.

ii) f is called an onto LG-related (LGP-related) function if $f[X] = Y$.

Remark 2.1. Let $A \in \operatorname{supp} \mathfrak{T}$ and $B \in \operatorname{supp} \mathfrak{R}$. Let $f \in LGRf(X, Y)$, such that $f[A] \leq B$. Then, we have $\mathfrak{R}(H) \leq \mathfrak{T}(f^{-1}[H])$, for each LG-fuzzy subset of Y and in paticular case when $H \leq B$. Thus $\mathfrak{R}_B(H) \leq \mathfrak{T}_A(f^{-1}[H])$ for each LG-fuzzy subset H of Y which $H \leq B$. Therefore f can be cosidered as an LG-related function of two $Lgfts's$, $(A, \mathfrak{T}_A)$ and $(B, \mathfrak{R}_B)$. So we can write $f \in LGRf(A,\ B)$.

Definition 2.5. *Let $(X, \mathfrak{T})$, $(Y, \mathfrak{R})$ be two Lgfts's and $f \in LGRf(X, Y)$ then*

i) f is called LG-open if $f[A] \in \operatorname{supp} \mathfrak{R} - \{\tilde{0}, Y\}$, $\forall A \in \operatorname{supp} \mathfrak{T} - \{\tilde{0}, X\}$ and $f[X] \in \operatorname{supp} \mathfrak{R}$.

ii) f is called LG-continuous if $f^{-1}[H] \in \operatorname{supp} \mathfrak{T} - \{\tilde{0}, X\}$, $\forall H \in supp\mathfrak{R} - \{\tilde{0}, Y\}$ and $f^{-1}[Y] \in \operatorname{supp} \mathfrak{T}$.

iii) *f is called LG-homeomorphism if is one -to -one, onto, LG-continuous, LG-open and* $f^{-1} \in LGRf(Y, X)$.

iv) *f is called LGP-homeomorphism if is an bijective and f,* f^{-1} *are LGP-fuzzy mapping.*

Proposition 2.1. *Let* $(X, \mathfrak{T})$, $(Y, \mathfrak{R})$ *be two Lgfts's and A, B be LG-open subsets of them respectively. Let* $\psi : M_1 \to M_2$ *be a function. Then,* ψ *is an LGP-homeomorphism from A to B if and only if* ψ *satisfies two following conditions:*

i) $A(p) = B(\psi(p))$, *for all* $p \in A$ *or* $B(q) = A(\psi^{-1}(q))$, *for all* $q \in B$;

ii) $\mathfrak{R}(H) = \mathfrak{T}(\psi^{-1}[H])$, *for all LG-fuzzy subset H of Y which* $H \leq B$.

Definition 2.6. *Let* $\mathfrak{T}$ *be an L-gradation of openness on X. Then,* $(X, \mathfrak{T})$ *is an LG-fuzzy topological space of dimension n, if we have for any* $x \in X$, *there exist an LG-open subset A of X containing x and an LG-open subset B of* $(1_{\mathbb{R}^n}, \mathfrak{T}_{Ln})$, *together an LGP-homeomorphism* $\psi \in LGPRf(A, B)$.

The pair (A, ψ) *is called an LG-local coordinate neighborhood of each* $q \in A$ *and we assign to q the n LG-local coordinates* $x_1(q), x_2(q), \ldots, x_n(q)$, *of its image* $\psi(q)$ *in* $\mathbb{R}^n$.

Definition 2.7. *Let* $\mathfrak{A} = \{(A_i,\ \psi_i)\ i \in J\}$ *be a collection of LG-local coordinate neighborhoods. Since* ψ_i *is an LGP-homeomorphismfor all* $i \in J$, *then for all* $i, j \in J$ *whenever* $A_i \cap A_j \neq \phi$,

$$\psi_j \circ\ \psi_i^{-1} : \psi_i(\mathrm{supp}(A_i \cap A_j)) \to \psi_j(\mathrm{supp}(A_i \cap A_j))$$

is an LGP-homeomorphism, that is called an LG-transition function.

$$\psi_j \circ \psi_i^{-1}(x_1^i,\ x_2^i,\ \ldots,\ x_n^i) = (x_1^j,\ x_2^j,\ \ldots,\ x_n^j)$$

If $\psi_i \circ \psi_j^{-1}$ *and* $\psi_j \circ \psi_i^{-1}$ *changing the LG-local coordinates are infinitely differentiable or* C^∞, *we shall say that* $(A_i,\ \psi_i)$ *is* C^∞ *compatible with* $(A_j,\ \psi_j)$ *whenever* $A_i \cap A_j \neq \phi$.

Definition 2.8. *An LG-fuzzy topological space* $(X, \mathfrak{T})$ *is called an LG-fuzzy topological manifold of dimension n, if it satisfies two following conditions:*

i) *X is an LG-fuzzy topological space of dimension n,*

ii) *X is a Hausdorff Lgfts.*

Definition 2.9. *A differentiable or* C^∞ *LG-fuzzy structure on an LG-fuzzy topological manifold* $(X, \mathfrak{T})$, *is a family* $\mathfrak{A} = \{(A_\alpha,\ \psi_\alpha),\ \ \alpha \in J\}$ *of LG-local coordinate neighborhoods such that*

1) $X = \bigcup_{\alpha \in J} A_\alpha$;

2) *Each pair* $(A_\alpha,\ \psi_\alpha)$ *and* $(A_\beta,\ \psi_\beta)$ *are compatible for all* $\alpha,\ \beta \in J$.

3) *Any LG-local coordinate neighborhood* (V, φ) *that is compatible with every* $(A_\alpha,\ \psi_\alpha), \alpha \in J$ *is in* $\mathfrak{A}$ *itself.*

A C^∞ *LG-fuzzy manifold* $(X, \mathfrak{T})$ *is an LG-fuzzy topological manifold on it and with an* C^∞ *LG-fuzzy structure on it. For convenience, "LG-fuzzy manifold with LG-fuzzy structure" will mean* C^∞ *LG-fuzzy manifold with* C^∞ *LG-fuzzy structure, in the following of this paper.*

Theorem 2.1. *Let* $(X, \mathfrak{T})$ *be an LG-fuzzy manifold of dimension* n *and with structure* $\mathfrak{A} = \{(A_j,\ \psi_j), j \in J\}$. *If* $\mathfrak{T}^\triangleleft = \{\operatorname{supp} A \mid \mathfrak{T}(A) > 0\}$, *then* $(\operatorname{supp} X, \mathfrak{T}^\triangleleft)$ *is a topological manifold of dimension* n *called premanifold with the structure* $\mathfrak{A}^\triangleleft = \{(\operatorname{supp} A_j,\ \psi_j|_{\operatorname{supp} A_j}),\ j \in J\}$ *called prestructure.*

The concept of the fuzzy vector space (V, η) over a field F was defined in [15]. We extend this definition by L-fuzzificaion:

Definition 2.10. *An L-fuzzy vector space* (V, η) *or* ηV *over a field* K *is an ordinary vector space* V *over the field* F, *with a map* $\eta : V \to L$ *satisfying the following conditions for all* $a, b \in V$ *and* $r \in K$.

1) $\eta(a + b) \geq min\{\eta(a),\ \eta(b)\}$,

2) $\eta(-a) = \eta(a)$,

3) $\eta(0) = 1$,

4) $\eta(ra) \geq \eta(a)$.

Definition 2.11. *Let* $(X, \mathfrak{T})$ *be an LG-fuzzy manifold of dimension* n, $U \in \operatorname{supp} \mathfrak{T}$ *and* $V \in \operatorname{supp} \mathfrak{T}_{L1}$. *The LG-related function* $f \in LGRf(U, V)$ *is called an* C^∞ *LG-fuzzy mapping, if for every* $p \in U$,

$$\hat{f} = f \circ \psi^{-1} : \psi(\operatorname{supp}(A \cap U)) \ \to \ \operatorname{supp} V$$

is C^∞ *where* (A, ψ) *is an LG-local coordinate neighborhood of* p.

We denote the set of all C^∞ LG-fuzzy mappings from an LG-open subset U of X, containing p to $\mathbb{R}$, by $C_L^\infty(p)$. If we define $C_L^\infty(p)$ may be considered as an L-fuzzy vector space $(C_L^\infty(p), \eta)$. Let $\psi(q) = (x_1,\ \ldots,\ x_n),\ \forall q \in \operatorname{supp}(A \cap U)$. Then, $\hat{f}(x_1,\ \ldots,\ x_n) = y(q)$, and since $\hat{f}$ is C^∞, there exist all partial derivatives of any order of y.

From now on, we suppose that $M_1,\ M_2$ are two crisp sets, $X \in L^{M_1},\ Y \in L^{M_2}$ such that $(X, \mathfrak{T}),\ (Y, \mathfrak{R})$ are two LG-fuzzy manifolds of dimension n, m and LG-fuzzy structures $\mathfrak{A} = \{(A_i, \psi_i),\ i \in K\}$ *and* $\mathfrak{D} = \{(D_j, \varphi_j),\ j \in J\}$ respectively and $U \in \operatorname{supp} \mathfrak{T}$.

Definition 2.12. *An LG-fuzzy function $F \in LGRf(U,Y)$ is an C^∞ LG-fuzzy mapping if for every $p \in U$,*

$$\hat{F} = \varphi \circ F \circ \psi^{-1} : \psi(\operatorname{supp}(A \cap U)) \ \to \ \varphi(\operatorname{supp} B)$$

is C^∞ where $(A,\psi),\ (B,\varphi)$ are LG-local coordinate neighborhoods of p and $F(p)$ respectively.

$F \in LGRf(X,Y)$ is called an LGP-diffeomorphism if it is an LGP-homeomorphism and F^{-1} is C^∞.

More precisely, if $\psi(q) = (x_1,\ \ldots,\ x_n),\ \forall q \in \operatorname{supp}(A \cap U)$ and $\varphi(w) = (y_1,\ \ldots,\ y_m),\ \forall w \in B$, then

$$\hat{F}(x_1,\ \ldots,\ x_n) = (f_1(x_1,\ \ldots,\ x_n),\ \ldots,\ f_n(x_1,\ \ldots,\ x_n))$$

and each $y_i = f_i(x_1,\ \ldots,\ x_n)$ is C^∞ on $\psi(A)$.

Definition 2.13. *The rank of $F \in LGRf(X,Y)$ at p is equal to the rank at $x = \psi(q)$ of the Jacobian matrix:*

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}_x$$

Example 2.3. Let $M_1 = M_2 = \mathbb{R}^2,\ \ L = I$ and $X : M_1 \to I,\ Y : M_2 \to I$ be defined by:

$$X(x_1,x_2) = \begin{cases} 1, & \|x\| = 1, \\ 0, & \|x\| \neq 1. \end{cases} \quad \text{and} \quad Y(y_1,y_2) = \begin{cases} 1, & \|y\| = 1, \\ 0, & \|y\| \neq 1. \end{cases}$$

If we define $\mathfrak{T} : I_X^{M_1} \to I,$ *and* $\mathfrak{R} : I_Y^{M_2} \to I$, by

$$\mathfrak{T}(A) = \begin{cases} 1, & A \in \operatorname{supp} \mathfrak{T}_{I2},\ A \leq X, \\ 0, & \text{elsewhere.} \end{cases}$$

and

$$\mathfrak{R}(D) = \begin{cases} 1, & D \in \operatorname{supp} \mathfrak{T}_{I2},\ D \leq Y, \\ 0, & \text{elsewhere.} \end{cases}$$

In a similar manner to the Example 3.5 of [20], we can prove that $(X,\mathfrak{T}),\ (Y,\mathfrak{R})$ are *IG*-fuzzy manifolds. Let

$$F : M_1 \to M_2,\ \ F(x_1,\ x_2) = (x_1 - x_2,\ \sqrt{2x_1x_2}).$$

We prove that $F \in IGRf(X,Y)$ is an C^∞ *IG*-fuzzy mapping. First we show $F|\operatorname{supp} X : \operatorname{supp} X \to \operatorname{supp} Y$ is well-defined and $F[X] = Y$:

$$(x_1,\ x_2) \in S^1 \Rightarrow\ x_1^2 + x_2^2 = 1 \Rightarrow\ (x_1 - x_2)^2 + (\sqrt{2x_1x_2})^2 = 1 \Rightarrow\ F(x_1,\ x_2) \in S^1,$$

$$F[X](y_1,\ y_2) = \bigvee\{X(x_1,x_2) : (x_1,x_2) \in F^{-1}(y_1,y_2)\} = 1 = Y(y_1,y_2).$$

Let $(A_1^+,\psi_1^+),(D_2^+,\varphi_2^+)$ be IG-local coordinate neighborhoods on X, Y respectively, then we see

$$\begin{aligned}\varphi_2^+ \circ F \circ (\psi_1^+)^{-1}(y) &= \varphi_2^+ \circ F(\sqrt{1-y^2},y) = \varphi_2^+(\sqrt{1-y^2}-y,\sqrt{2y\sqrt{1-y^2}})\\ &= \sqrt{1-y^2}-y,\end{aligned}$$

is C^∞. Similarly, one can show that $\varphi_i^\pm \circ F \circ (\psi_j^\pm)^{-1}$ is C^∞ for all $i,j=1,2$.

Example 2.4. Let $F : \mathbb{R} \to \mathbb{R}^2,\ F(t) = (\cos(t-\frac{\pi}{2}),\ \sin(t-\frac{\pi}{2}))$. Then, $F \in IGRf(1_{\mathbb{R}},\ 1_{\mathbb{R}^2})$ and $rank\, F = 1$ at every point of X.

Theorem 2.2 (LG-fuzzy rank theorem)**.** *Let $F \in LGRf(U,Y)$ be an C^∞ fuzzy mapping and $rank F = k$ at every point of X. If $p \in X$, then there exist LG-local coordinate neighborhoods (A,ψ), (B,φ) such that*

$$\psi(p) = (0,\dots,0) \in \mathbb{R}^n,\ \ \varphi(F(p)) = (0,\dots,0) \in \mathbb{R}^m$$

and $\hat{F} = \varphi \circ F \circ \psi^{-1}$ is given by:

$$(*) \qquad \hat{F}(x_1,\ \dots,\ x_n) = (x_1,\ \dots,\ x_k,0,\dots,0).$$

Theorem 2.3. *Let $F : \mathbb{R}^n \to \mathbb{R}^m$, be an continuous function. Then, F is an $IG-$ fuzzy continuous function.*

Proof. Let the $L-$gradation of openness of Euclidean fuzzy topological space equals to 1 on τ_{Im}, hence $\operatorname{supp}\mathfrak{T}_{Im} = \tau_{Im}$. For each element $B(a,r,b)$ of the basis of τ_{Im}, we have $F^{-1}[B(a,r)] \in \tau_{In}$. So $F^{-1}[B(a,r)] = \bigcup_{\alpha\in J} B(p_\alpha,r_\alpha)$. Then, $F^{-1}[B(a,r,b)] = \bigcup_{\alpha\in J} B(p_\alpha,r_\alpha,boF)$, therefore $F^{-1}[B(a,r,b)] \in \mathfrak{T}_{Im}$. Hence F is an $IG-$ fuzzy continuous function. □

3. C^∞ LG-fuzzy tangent spaces of an C^∞ LG-fuzzy manifold $\mathfrak{T}$

Definition 3.1. *Let $(X,\mathfrak{T})$ be an C^∞ LG-fuzzy manifold of dimension n and $p \in X$. We define the LG-fuzzy tangent space $LGT_p(X)$ to X at p to be the set of all mappings $Z_p : C_L^\infty(p) \to \mathbb{R}$ satisfying for all $\alpha,\beta \in \mathbb{R}$ and $f,g \in C_L^\infty(p)$ the two conditions:*

i) $Z_p(\alpha f+\beta g) = \alpha(Z_p f)+\beta(Z_p g)$

ii) $Z_p(fg) = (Z_p f)g(p) + f(p)(Z_p g)$ *(leibniz rule),*

and define the LG-fuzzy vectore space operations in $LGT_p(X)$ by $(Z_p+W_p)f = Z_pf + W_pf$, $(\alpha Z_p)f = \alpha(Z_pf)$.

If we define $\eta : LGT_p(X) \to L,\ \eta(f) = A(p)$, where (A,ψ) is an LG-coordinate neighborhood of p. Then, $(LGT_p(X),\ \eta)$ is an L-fuzzy vector space.

Each $Z_p \in LGT_p(X)$ is called an LG-fuzzy tangent vector to X at p.

Example 3.1. Let $f(x,y,z) = x^2 + y^2 + z^2$ and $a = 1$, then unit sphere $S^2 = f^{-1}(a)$ in $\mathbb{R}^3$ are given. Let $p = (a,b,c)$, therefore

$$(\partial f/\partial x)(p) = 2a, (\partial f/\partial y)(p) = 2b, (\partial f/\partial z)(p) = 2c.$$

Then, equation of LG-fuzzy tangent space to unit sphere at p is $2ax+2by+2cz = 0$ or $ax + by + cz = 0$.

Theorem 3.1. *Let $F \in LGRf(X,Y)$ is an C^∞ LG-fuzzy function of two LG-fuzzy manifolds. Then, for $p \in X$,*

$$F^* : C_L^\infty(F(p)) \to C_L^\infty(p)$$

with $F^*(f) = f \circ F$ is a fuzzy homorphism and induces an L-fuzzy dual vector space homomorphism

$$F_* : LGT_p(X) \to LGT_{F(p)}(Y)$$

that defined by $F_*(Z_p)f = Z_p(F^*f)$, where $F_*(Z_p)$ is an LG-fuzzy map of $C_L^\infty(F(p))$ to R. If $F \in LGRf(X,X)$ is the fuzzy identity, F^* and F_* are the LG-fuzzy identity isomorphism. Let F, G be C^∞ LG-fuzzy functions and $H = G \circ F$ is a composition of them, then $H^* = F^* \circ G^*$ and $H_* = G_* \circ F_*$.

Proof. Let $Z_p \in LGT_p(X)$ and $f, g \in C_L^\infty(F(p))$. Linearity is obviously.

Suppose that $F_*(Z_p) : C_L^\infty(F(p)) \to R$. we should prove that it is a linear map satisfying the Leibniz rule. We have:

$$F_*(Z_p)(fg) = Z_p[(f \circ F)(g \circ F)] = Z_p(f \circ F)g(F(p)) + f(F(p))Z_p(g \circ F).$$

Then, $F_*(Z_p)(fg)$=$F_*(Z_p)(f)g(F(P))$+$f(F(p))F_*(Z_p)g$.

Therefore, $F_* : LGT_p(X) \to LGT_{F(p)}(Y)$ is well defined. Also, F_* is an L-fuzzy homomorphism:

$$\begin{aligned} F_*(\alpha Z_p + \beta W_p)f &= (\alpha Z_p + \beta W_p)(F \circ f) = \alpha Z_p(F \circ f) + \beta W_p(F \circ f) \\ &= \alpha F_*(Z_p)f + \beta F_*(W_p)f \\ &= [\alpha F_*(Z_p) + \beta F_*(W_p)]f. \end{aligned}$$

□

Corollary 3.1. *Let $F \in LGRf(X,Y)$ is an LG-fuzzy diffeomorphism of X onto an LG-open set $W \subset Y$ and $p \in X$, then $F_* : LGT_p(X) \to LGT_{F(p)}(Y)$ is an LG-fuzzy isomorphism onto.*

Proof. Let G is inverse F, therefore $G_* \circ F_* : LT_p(X) \to LGT_p(X)$ and $F_* \circ G_* : LGT_{F(p)}(Y) \to LGT_{F(p)}(Y)$ are the identity L-fuzzy isomorphism. □

Definition 3.2. *Let (A,ψ) be an LG-coordinate neighborhood on X, the LG-coordinate map ψ induces an LG-fuzzy isomorphism*

$$\psi_* : LGT_p(X) \to LGT_{\psi(p)}(1_{\mathbb{R}^n})$$

of the L-fuzzy tangent space at each $p \in A$ onto $LGT_x(1_{\mathbb{R}^n})$, $x = \psi(p)$. The LG-fuzzy images $H_{ip} = \psi_*^{-1}(\partial/\partial x_i)$, $i = 1,\dots,n$, of the fuzzy basis $\partial/\partial x_1,\dots,\partial/\partial x_n$ at each $x \in \psi(A) \subset 1_{\mathbb{R}^n}$ assign at $p = \psi^{-1}(x) \in X$ a basis $H_{1p},\dots,H_{np}$ of $LGT_p(X)$; that these bases called the L-fuzzy coordinate frames.

Thus, dim $LGT_p(X)$ = dim X.

If $\psi \in LGRf(A,B)$ where B is an LG-fuzzy open subset of $1_{\mathbb{R}^n}$, then by Proposition 2.1, we have $A(p) = B(\psi(p))$. We have:

$$\eta : LGT_p(X) \to L,\ \ \eta(X_p) = A(p)\ \ and\ \ \eta' : LGT_x(1_{\mathbb{R}^n}) \to L,\ \ \eta'(X_x) = B(x).$$

Therefore, $(LGT_p(X),\eta)$ and $(LGT_x(1_{\mathbb{R}^n}),\eta')$ are L-fuzzy vector spaces. Then,

$$\eta(H_{ip}) = A(p) = B(\psi(p)) = \eta'\left(\frac{\partial}{\partial x_i}\right)|_{\psi(p)}\ .$$

Remark 3.1. Let f be an C^∞ L-fuzzy function defined on an L-neighborhood of p, and $\hat{f} = f \circ \psi^{-1}$ its expression in local LG-coordinates relative to (A,ψ). Then,

$$H_{ip}f = \left(\psi_*^{-1}\left(\frac{\partial}{\partial x_i}\right)\right) f = \frac{\partial}{\partial x_i}\left(f \circ \psi^{-1}\right)|_{x=\psi(p)} = \frac{\partial \hat{f}}{\partial x_i}|_{\psi(p)}.$$

Let $F \in LGRf(X,Y)$ be a smooth LG-fuzzy related function, and let (A,ψ) and (B,φ) be LG-coordinate neighborhoods on X and Y respectively. Let F is given by $y_i = f_i(x_1,\dots,x_n)$, $i = 1,\dots,m.$ and p is a point with LG-coordinates $a = (a_1,\dots,a_n)$. Therefore, $F(p)$ has m LG-coordinates distincted by these functions. Other let $\partial y_j/\partial x_i$ denotes $\partial f_j/\partial x_i$.

Theorem 3.2. *Let $H_{ip} = \psi_*^{-1}(\partial/\partial x_i)$ and $\tilde{H}_{jF(p)} = \varphi_*^{-1}(\partial/\partial y_j)$, $i = 1,\dots,\ n$ and $j = 1,\dots,\ m$, be the fuzzy basis of $LGT_p(X)$ and $LGT_{F(p)}(Y)$, respectively, defined by the LG- coordinates neighborhoods. Then,*

$$F_*(H_{ip}) = \sum_{j=1}^{m}\left(\frac{\partial y_j}{\partial x_i}\right)_a \tilde{H}_{jF(p)},\ \ i = 1,\dots,\ n.$$

If $Z_p = \sum \alpha_i H_{ip}$ mapping to $F_*(Z_p) = \sum \beta_j \tilde{H}_{jF(p)}$, then

$$\beta_j = \sum_{i=1}^{n} \alpha_i \left(\frac{\partial y_j}{\partial x_i}\right)_a,\ \ j = 1,\dots,\ m.$$

Proof. We know $F_*(H_{ip}) = F_* \circ \psi_*^{-1}(\partial/\partial x_i)_{\psi(p)}$ and to calculate its components relative to $\tilde{H}_{jF(p)}$, we have to use this vector as an agent on $C^\infty(F(p))$ to the *LG*-coordinate functions y_j

$$F_*(H_{ip})y_j = \left(F_* \circ \psi_*^{-1}(\frac{\partial}{\partial x_i})_{\psi(p)}\right) y_j = \frac{\partial}{\partial x_i} y_j(F \circ \psi^{-1})(a) = \left(\frac{\partial f_j}{\partial x_i}\right)_a .$$

□

Corollary 3.2. *$dim(Image(\ F_*(LGT_p(X)) = rankF$ at p. Then, F_* is an LG-fuzzy isomorphism if this rank equals to the dimension of X; it is onto if the rank is $dimY$.*

Proof. We have from linear algebra wheras $(\partial y_j/\partial x_i)$ is quite the Jacobian of $\psi \circ F \circ \psi^{-1}$, that we apply to determine the rank and further is the matrix of the fuzzy linear transformation $F_* : Ltg_p(X) \to Ltg_p(Y)$ in the bases. □

If we use the theorem to the *L*-fuzzy maps $F = \widetilde{\psi} \circ \psi^{-1}$ and $F^{-1} = \psi \circ \widetilde{\psi}^{-1}$ that admit the alteration of *LG*-coordinates from (A, ψ) to $(\widetilde{A}, \widetilde{\psi})$ in $A \cap \widetilde{A}$ on X, then we obtain formulas for change of basis in $LGT_p(X)$ and the corresponding alteration of elements respective to these bases.

Corollary 3.3. *Let $p \in A \cap \widetilde{A}$ and let $\widetilde{H}_{ip} = \psi_*^{-1}(\partial/\partial\widetilde{x}_i)$ and $\widetilde{H}_{ip} = \widetilde{\psi}_*^{-1}(\partial/\partial x_i)$ be the basis of $LGT_p(X)$ from the two LG-coordinate systems. Then, we have*

$$H_{ip} = \sum_k \left(\frac{\partial \widetilde{x}_k}{\partial x_i}\right)_{\psi(p)} \widetilde{H}_{kp} \quad and \quad \widetilde{H}_{jp} = \sum_l \left(\frac{\partial x_l}{\partial \widetilde{x}_j}\right)_{\widetilde{\psi}(p)} H_{lp}.$$

If $Z_p = \sum \alpha_i H_{ip} = \sum \beta_j \widetilde{H}_{jp}$, then

$$\alpha_i = \sum_j \beta_j \left(\frac{\partial x_i}{\partial \widetilde{x}_j}\right) \quad and \quad \beta_j = \sum_i \alpha_i \left(\frac{\partial \widetilde{x}_j}{\partial x_i}\right).$$

where indices are from 1 to n.

Example 3.2. Let $X = \chi_{(a,b)}$ where (a, b) is an open interval of $\mathbb{R}$. Let $F \in LGRf(X, Y)$ be an C^∞ *LG*-fuzzy curve in an *LG*-fuzzy manifold Y; for the time being we abandon the need that F is an *LG*-fuzzy immersion. Let $t_0 \in X$, $a < t_0 < b$, then d/dt at t_0 is a fuzzy basis for $LGT_{t_0}(X)$. Assume that $p = F(t_0)$ and $f \in C_L^\infty(p)$, then $F_*(d/dt)$ is defined by its value on all such f:

$$F_*\left(\frac{d}{dt}\right) f = (d/dt(f \circ F))_{t_0} .$$

We say this vector the (*LG*-fuzzy tangent) speed vector to the curve at p.

In particular, let (A,ψ) is an LG-chart at p, therefore in the LG-local coordinates F is given by $\hat{F}(t)=(x_1(t),\dots,x_n(t))$. The LG-coordinate x_i is an LG-function on A and using partly untidy notation, we have $x_i(t)=(x_i\circ F)(t)$; then $F_*(d/dt)x_i=(dx_i/dt)_{t_0}$, that we indicate $\dot{x}_i(t_0)$, $i=1,\dots,n$. Thus

$$F_*\left(\frac{d}{dt}\right)=\sum_{i=1}^{n}\dot{x}_i(t_0)H_{ip}.$$

Example 3.3. Let $M=\mathbb{R}^3$, $X:\mathbb{R}^3\to L$, $X(x)=\begin{cases}1, & \|x\|=1,\\ 0, & \|x\|\neq 1.\end{cases}$ and $\mathfrak{T}:L_M^X\to L$ $\mathfrak{T}(A)=\begin{cases}1, & A\in\tau_L 3,\ A\leq X\\ 0, & \text{elsewhere.}\end{cases}$. We prove that $(X,\mathfrak{T})$ is an C^∞ LG-fuzzy 2-manifold.

Define $A_i:\mathbb{R}^3\to L$, $\psi_i:\mathbb{R}^3\to\mathbb{R}^2$ *for* $i=1,2$ by:

$$A_1(x)=\begin{cases}1, & \|x\|=1,\ x\neq(0,0,1)\\ 0, & \text{elsewhere.}\end{cases},\quad A_2(x)=\begin{cases}1, & \|x\|=1,\ x\neq(0,0,-1)\\ 0, & \text{elsewhere.}\end{cases}.$$

$$\psi_1(x)=\begin{cases}(\frac{x}{1-z},\ \frac{y}{1-z}), & x\neq(0,0,1)\\ 0, & \text{elsewhere.}\end{cases},$$

$$\psi_2(x)=\begin{cases}(\frac{x}{1+z},\ \frac{y}{1+z}), & x\neq(0,0,-1)\\ 0, & \text{elsewhere.}\end{cases}.$$

Then, (A_i,ψ_i), $i=1,2$ are two C^∞ compatible LG-coordinate neighborhoods covering X. Let $F\in LGRf(A_i,1_{\mathbb{R}^3})$ be an LG-fuzzy imbedding. We may suppose that λ is given by LG-coordinate functions

$$x=f(u,v)=2u+v,\quad y=g(u,v)=0,\quad z=h(u,v)=u-2v.$$

F is an LG-fuzzy imbedding, therefore the Jacobian matrix $\partial(f,g,h)/\partial(u,v)$ has rank 2 at each point of A_i. Then,

$$(Z_u)_0=F_*\left(\frac{\partial}{\partial u}\right)=\frac{\partial x}{\partial u}\frac{\partial}{\partial x}+\frac{\partial y}{\partial u}\frac{\partial}{\partial y}+\frac{\partial z}{\partial u}\frac{\partial}{\partial z}=2\frac{\partial}{\partial x}+\frac{\partial}{\partial z},$$

$$(Z_v)_0=F_*\left(\frac{\partial}{\partial v}\right)=\frac{\partial x}{\partial v}\frac{\partial}{\partial x}+\frac{\partial y}{\partial v}\frac{\partial}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial}{\partial z}=\frac{\partial}{\partial x}-2\frac{\partial}{\partial z}.$$

F_* has rank 2, so these are linearly LG-fuzzy independent vectors, and they extend a two-dimensional LG-fuzzy subspace of $LGT_{(x_0,y_0,z_0)}(X)$.

Example 3.4. Consider the $IG-$fuzzy path $\psi : 1_{\mathbb{R}} \longrightarrow 1_{\mathbb{R}^2}$, $\psi(t) = e^t(\cos t, \sin t)$. ψ is bijective and C^∞ from $\mathbb{R}$ to $\mathbb{R}^2$. Then, $\psi'(t) = e^t \begin{pmatrix} \cos t - \sin t \\ \cos t + \sin t \end{pmatrix}$

$$\begin{aligned} \psi_* \left(\frac{d}{dt} \right) &= \frac{dx}{dt}\frac{d}{dx} + \frac{dy}{dt}\frac{d}{dy} \\ &= e^t(\cos t - \sin t)\frac{d}{dx} + e^t(\cos t + \sin t)\frac{d}{dy}. \end{aligned}$$

Since $t = \psi^{-1}(x, y) = \frac{1}{2}\ln(x^2 + y^2)$, therefore

$$\psi_*^{-1} \left(\frac{d}{dx} \right) = \frac{dt}{dx}\frac{d}{dt} = \frac{x}{x^2 + y^2}\frac{d}{dt} = \frac{e^t \cos t}{e^t}\frac{d}{dt} = \cos t \frac{d}{dt},$$

$$\psi_*^{-1} \left(\frac{d}{dy} \right) = \frac{dt}{dy}\frac{d}{dt} = \frac{y}{x^2 + y^2}\frac{d}{dt} = \frac{e^t \sin t}{e^t}\frac{d}{dt} = \sin t \frac{d}{dt}.$$

Theorem 3.3. *Following the Example 3.3 of [20], let $J = (0,\ 2\pi)$ be an open interval about $t = t_o$ and let us consider an IG-fuzzy curve on Y through (x_0, y_0, z_0). Assume that the IG-fuzzy curve F defined by*

$$\alpha \in IGRf(\chi_J, A_i),\ \ \hat{\alpha}(t) = \sin(3t)\cos t\ i + \sin(3t))\sin t\ j$$

combined by F with the above definition, thus u, v, are IG-fuzzy functions of t with $u(t_0) = u_0$ and $v(t_0) = v_0$ and the IG-fuzzy curve is given by:

$$\begin{aligned} F(\alpha(t)) &= (x(u(t), v(t)),\ y(u(t), v(t)),\ z(u(t), v(t))) \\ &= (2\sin(3t)\cos(t) + \sin(3t)\sin t,\ 0,\ \sin(3t)\cos t - 2\sin(3t)\sin t). \end{aligned}$$

The IG-fuzzy tangent to the IG-fuzzy curve at (x_0, y_0, z_0) is defined by

$$(F \circ \alpha)_* \left(\frac{d}{dt} \right) = \dot{x}(t_0)\frac{\partial}{\partial x} + \dot{y}(t_0)\frac{\partial}{\partial y} + \dot{z}(t_0)\frac{\partial}{\partial z},$$

where

$$\dot{x}(t_0) = \left(\frac{dx}{dt} \right)_{t_0} = \frac{\partial x}{\partial u}\frac{du}{dt} + \frac{\partial x}{\partial v}\frac{dv}{dt}$$

appraised at (x_0, y_0, z_0) and $t = to$. Substituting and collecting terms, we have:

$$\begin{aligned} (F \circ \alpha)_* \left(\frac{d}{dt} \right) &= \frac{du}{dt}\ F_* \left(\frac{\partial}{\partial u} \right) + \frac{dv}{dt}\ F_* \left(\frac{\partial}{\partial v} \right) \\ &= \dot{u}(t_0)(Z_u)_0 + \dot{v}(t_0)(Z_v)_0. \end{aligned}$$

Example 3.5. Suppose that $N \subset X$ is an *LG*-fuzzy regular submanifold and (A, ψ) is an *LG*-coordinate neighborhood respective to N with the *LG*-local coordinates $(x_1, \ldots, x_m)$ and *LG*$-$ coordinate frames $(H_1, \ldots, H_m)$. Let $N \cap A$ is defined by $x_{n+l} = \ldots = x_m = 0$, prove that $H_{1p}, \ldots, H_{mp}$, is a fuzzy basis of $LGT_p(N)$, for every $p \in N \cap A$. Improve this assertion so as to include *LG*-fuzzy immersed submanifolds.

Proof. Since N is an LG-fuzzy regular submanifold, then LG-fuzzy inclusion function is an embedding: $i : N \to X$, $i_* : LGT_p(N) \to LGT_p(X)$, that i is an LG-fuzzy injective homomorphism then $LGT_p(N)$ is LG-fuzzy subspace of $LGT_p(X)$. Now, if $Z_p \in LGT_p(N)$ then $Z_p \in LGT_p(X)$,

$$Z_p = \sum_{i=1}^{m} \alpha_i H_{i_p} = \sum_{i=1}^{m} \alpha_i \psi_*^{-1}(\partial/\partial x_i),$$

$$\alpha_j = Z_p x_j, \quad Z_p x_j = \sum_{i=1}^{m} \alpha_i (\partial x_j/\partial x_i) = \alpha_j$$

that if $j > n$ then $x_j = 0$ therefore $\alpha_j = 0$ because $\forall i \neq j$, $\partial x_j/\partial x_i = 0$ therefore $Z_p = \sum_{i=1}^{n} \alpha_i H_{ip}$ then Z_p is written by form of Linear composition of L-fuzzy vectors $H_{1p}, \ldots, H_{np}$. Also these vectors are independent linear and since generators of space, then they are basis for $LGT_p(N)$. If $N \subset X$ is an LG-fuzzy immersion submanifold then $i : N \to X$ is an LG-fuzzy immersion map with $dimN = n$ that is one to one and

$$i_* : LGT_p(N) \to LGT_p(X),$$

$H_{jp} = i_{*n}^{-1}(H_{jp})$, $j = 1, 2, \ldots n$ are base for N. □

Example 3.6. Let $F \in LGRf(x, y)$ be a C^∞ LG-fuzzy function of LG-fuzzy manifolds. Then, $f \to F^*(f) = F \circ f$ defines an LG-fuzzy homomorphism $F^* : C_L^\infty(F(p)) \to C_L^\infty(p)$ also F^* satisfy in the assertion of Theorem 3.1.

Proof. Let $f, g \in C_L^\infty(F(p))$ then for all $p \in X$:

$$F^*(f + g) = (f + g) \circ F = f \circ F + g \circ F = F^*(f) + F^*(g),$$

$$F^*(f.g)(p) = (f.g) \circ F(p) = f(F(p)).g(F(p)) \Longrightarrow (f.g) \circ F = (f \circ F).(g \circ F).$$

Therefore, F^* is an LG-fuzzy homomorphism.

Now, if $F \in LGRf(X, X)$ be an LG-fuzzy identity function then $F^* f = f \circ F = f$ therefore F^* also is LG-fuzzy identity function and since F is LG-fuzzy injective, we have

$$F^* f = F^* g \implies f \circ F(p) = g \circ F(p), \ \forall p \in X.$$

Since F is LG-fuzzy identity function then $f(p) = g(p)$, $\forall p \in X$ that means $f = g$, therefore F^* is LG-fuzzy injective also, as a result is an LG-fuzzy isomorphism map. Now, if $H = G \circ F$ then $H^* f = f \circ H = f \circ (G \circ F) = F^*(f \circ G) = (F^* \circ G^*)(f)$. □

Example 3.7. Consider the Example 3.11. Then, there is an LG-fuzzy parametrized curve on X at p that LG-fuzzy velocity vector is $\alpha(Z_u)_0 + \beta(Z_v)_0$, for all $\alpha, \beta \in \mathbb{R}$.

Proof. Let we have the following *LG*-fuzzy parameter change:

$$x = u, y = v, z = h(u, v) = h(x, y)$$

We have the surface that *LG*-fuzzy vector space of each point of it satisfy in the equation $r = (u, v, h(u, v))$. We get the *LG*-fuzzy tangent space at (x_0, y_0, z_0). It means that in point (u_o, v_o). First We get *LG*-fuzzy vector vertical to the surface. $r_1 = \partial r/\partial u = (1, 0, \partial h/\partial u)$ and $r_2 = \partial r/\partial v = (0, 1, \partial h/\partial v)$ that *LG*-fuzzy vector vertical to the surface is $r_1 \times r_2 = (-\partial h/\partial u, -\partial h/\partial v, 1)$. Now, the equation of a surface passing through the point (x_0, y_0, z_0) and the vector N is vertical to it, is

$$(-\partial h/\partial x)_0(x - x_0) + (-\partial h/\partial y)_0(y - y_0) + (z - z_0) = 0,$$

that this surface includes *LG*-fuzzy vectors that connect (x_0, y_0, z_0) to (x, y, z) and *LG*-fuzzy tangent to the desired surface in point (x_0, y_0, z_0) . □

Example 3.8. Let $M = \mathbb{R}^3,\ X : \mathbb{R}^3 \to I,\ X(x) = \begin{cases} 1, & \|x\| = 1, \\ 0, & \|x\| \neq 1. \end{cases}$. Then, $\operatorname{supp} X = S^2$, the unit sphere. Set $\mathfrak{T} : I_X^M \to L$,

$$\mathfrak{T}(A) = \begin{cases} 1, & A \in \tau_{L3},\ A \leq X, \\ 0, & \text{elsewhere.} \end{cases}$$

[20] proved that $(X, \mathfrak{T})$ is an *LG*-fuzzy manifold of dimension 2. Now, we define *LG*- fuzzy tangent spaces of it.

Proof. Let $J = \{1, 2, 3\}$, and

$$A_j : M \to L,\ \forall x = (x_1,\ x_2,\ x_3),\ A_j^{\pm}(x) = \begin{cases} \pm x_j, & \pm x_j > 0,\ \|x\| = 1, \\ 0, & \text{otherwise.} \end{cases}$$

$$\forall y = (y_1, y_2),\ B(y) = \begin{cases} \sqrt{1 - y_1^2 - y_2^2}, & \|y\| < 1, \\ 0, & \text{otherwise.} \end{cases}$$

$$\psi_1^{\pm}(x_1, x_2, x_3) = (x_2,\ x_3)\ , \quad (\psi_1^{\pm})^{-1}(y_1,\ y_2) = (\pm\sqrt{1 - y_1^2 - y_2^2}\ ,\ y_1,\ y_2)$$

$$\psi_2^{\pm}(x_1, x_2, x_3) = (x_1,\ x_3)\ , \quad (\psi_2^{\pm})^{-1}(y_1,\ y_2) = (y_1, \pm\sqrt{1 - y_1^2 - y_2^2}\ ,\ y_2)$$

$$\psi_3^{\pm}(x_1, x_2, x_3) = (x_1,\ x_2)\ , \quad (\psi_3^{\pm})^{-1}(y_1,\ y_2) = (y_1,\ y_2,\ \pm\sqrt{1 - y_1^2 - y_2^2}\ )$$

The basic of *IG*- fuzzy tangent spaces $IGT_p(X)$ for $p \in \operatorname{supp} A_j^{\pm}$ is:

$$H_{j1p}^{\pm} = (\psi_j^{\pm})_*^{-1}(\partial/\partial y_1),\ H_{j2p}^{\pm} = (\psi_j^{\pm})_*^{-1}(\partial/\partial y_2)$$

This space is an *IG*-fuzzy vector space with $\eta_j^{\pm} : IGT_p(X) \longrightarrow L$ where $\eta_j^{\pm}(Z_p) = A_j^{\pm}(p) = p_j = (p_1, p_2, p_3) \in \operatorname{supp} A_j^{\pm}$. □

Example 3.9. Let $L = \mathbb{N} \cup \{\infty\}$ and $M = \mathbb{R}^{n+1}$. Define an L-fuzzy subset

$$X : M \to L, \quad X(x) = \begin{cases} n+2, & \|x\| = 1, \\ 0, & \|x\| \neq 1, \end{cases}$$

$$\mathfrak{T} : L_X^M \to L, \quad \mathfrak{T}(A) = \begin{cases} \infty, & A \in \tau_{L(n+1)},\ A \leq X, \\ 0, & \text{elsewhere.} \end{cases}$$

Then, $\operatorname{supp} X = S^n$. in [20], $(X, \mathfrak{T})$ is an LG-fuzzy manifold of dimension n. Now, we define LG- fuzzy tangent spaces of it at every point $p = (p_1, \ldots, p_{n+1}) \in \operatorname{supp} A_j^{\pm}$.

Proof. Let $J = \{1,\ \ldots,\ n+1\}$. Define $A_j : M \to L,\ \ j \in J$ by:

$$\forall\ x = (x_1,\ \ldots\ x_{n+1}), \quad A_j^{\pm}(x) = \begin{cases} j, & \pm x_j > 0,\ \|x\| = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then, we show that each $A_j^{\pm}$ is LG-diffeomorphic to the LG-open subset $B_j : \mathbb{R}^n \to L$ defined by:

$$\forall y = (y_1,\ \ldots,\ y_n), \quad B_j(y) = \begin{cases} j, & \|y\| < 1, \\ 0, & \text{otherwise.} \end{cases}$$

with $2(n+1)$ LG-maps $\psi_j^{\pm} : A_j^{\pm} \to B_j$ defined by:

$$\psi_j^{\pm}(x_1,\ \ldots,\ x_{n+1}) = (x_1,\ \ldots,\ \widehat{x_j},\ \ldots,\ x_{n+1})$$

$$(\psi_j^{\pm})^{-1}(y_1,\ \ldots,\ y_n) = (y_1,\ \ldots,\ \pm\sqrt{1 - (y_1^2 + \ \ldots\ + y_n^2)}\ ,\ \ldots,\ y_n)$$

where $\widehat{x_j}$ means omit x_j.

$$H_{kj}^{\pm} = (\psi_j^{\pm})_*^{-1}(\partial/\partial y_k)$$

where $k = 1, \ldots, n$, $j \in J$ and $\eta_j^{\pm}(Z_p) = A_j^{\pm}(p) = j$. □

Example 3.10. Let $L = \mathbb{N} \cup \{\infty\}$. Then, L is complete Lattice. Always we denote the top element of of any lattice by 1, but ∞ is the top element of this L. We define the LG-fuzzy Euclidean topological space $(1_{\mathbb{R}^n},\ \mathfrak{T}_{Lsup})$ by

$$1_{\mathbb{R}^{mn}} : \mathbb{R}^{mn} \to L, \qquad 1_{\mathbb{R}^{mn}}\big((a_1, a_2, \ldots, a_{mn})\big) = \infty,$$

$$\mathfrak{T}_{Lsup} : L_{1_{\mathbb{R}^{mn}}}^{\mathbb{R}^{mn}} \to L, \qquad \mathfrak{T}_{Lmn}(D) = \begin{cases} \infty, & D \in \tau_{Lmn}, \\ 0, & \text{elsewhere.} \end{cases}$$

Let $M = \mathcal{M}_n(\mathbb{R})$ be the set of $n \times n$ matrixes with real entires and $X \in L^M$ be defined by

$$X\big((a_{ij})\big) = 1 + \sup\big\{\lfloor\ |a_{ij}|\ \rfloor \mid 1 \leq i, j \leq n\big\},$$

where $\lfloor |x| \rfloor$ equals to the greatest integer less or equal to $|x|$. There is a bijection map ψ from M to $\mathbb{R}^{mn}$:

$$\psi(a_{ij}) = (a_{11},\ \ldots,\ a_{1n},\ \ldots,\ a_{n1},\ \ldots,\ a_{n^2})$$

Hence, using ψ and (2.1), we define

$$\mathfrak{T}\ :\ L_X^M \to L, \qquad \mathfrak{T}(A) = \begin{cases} \infty, & \psi[A] \in \tau_{_{Ln^2}}, \\ 0, & \text{elsewhere.} \end{cases}$$

In [20] showed that $(X, \mathfrak{T})$ is an C^∞ *LG*-fuzzy n^2-manifold. We can cover $(X, \mathfrak{T})$ by the single *LG*-coodinate neighborhood (X, ψ). Hence the *LG*-fuzzy tangent space of X at each point $p = (a_{ij})$ is an L-fuzzy vector space $(LGT_p(X), \eta)$ generated by the basis

$$H_{ij}(p) = \psi_*^{-1}(\partial/\partial x_{ij}) \mid_{\psi(p)}.$$

where $\eta : LGT_p(X) \to L, \quad \eta(Z_p) = X(p) = X\big((a_{ij})\big)$

4. C^∞ *LG*-fuzzy vector fields on an C^∞ *LG*-fuzzy manifold

An *LG*-fuzzy vector field Z on C^∞ *LG*-fuzzy manifold $(X, \mathfrak{T})$ is a "function" allocating to each point p of X an element Z_p, of $LGT_p(X)$. The span is the set $LGT(X)$ made up of all *LG*-fuzzy tangent vectors at all points of X, $LGT(X) = \cup_{p\in X} LGT_p(X)$, that is partitioned into disjoint subsets $\{LGT_p(X)\}$ which are indexed by the points of X. It follows that there is a natural projection $\pi : LGT(X) \to X$ with the *LG*-fuzzy vector $Z_p \in LGT(X)$ to p. The *LG*-fuzzy vector field X is $Z : X \to LGT(X)$, satisfied the condition $\pi \circ Z = I_X$, the identity on X.

Let (A, ψ) be any *LG*-coordinate neighborhood of $p \in X$, and let $H_{1p}, \ldots, H_{np}$, be the *LG*-corresponding basis of $LGT_p(x)$. So Z_p, the value of Z at p,$Z_p = \sum_{i=1}^n \alpha_i H_{ip}$. If p is varied in A, the components $\alpha_1, \ldots, \alpha_n$ are well-defined functions of p, then, be submitted by functions of the *LG*-local coordinates.

$$\alpha_i = \alpha_i(x_1, \ldots, x_n) \ \ , i = 1, \ldots, n, \quad \text{on} \ \ \psi(A) \subset \mathbb{R}^n.$$

We call that Z is of class C^r, $r \geq 0$ if these functions are of class C^r on $\operatorname{supp} A$ for each *LG*-local coordinate system (A, ψ) are functions of class C^r on the domain (A, ψ) of the coordinate.

Definition 4.1. *An LG-fuzzy vector field Z of class C^r on X is a function determining to each point p of X, a vector $Z_p \in LGT_p(x)$, whose components in the frames of any LG-local coordinates (A, ψ) are functions of class C^r on the domain (A, ψ) of the coordinates.*

Example 4.1. Let $X = 1_{\mathbb{R}^3-\{0\}}$, then the gravitational field of an object of unit crime at 0 is a C^∞ LG-fuzzy vector field whose components $\alpha_1, \alpha_2, \alpha_3$ relative to the basis $\partial/\partial x_1 = H_1$, $\partial/\partial x_2 = H_2$ and $\partial/\partial x_3 = H_3$ are

$$\alpha_i = \frac{x_i}{r_3}, \quad i = 1, 2, 3 \text{ with } r = ((x_1)^2 + (x_2)^2 + (x_3)^2)^{\frac{1}{2}}.$$

Example 4.2. Taken any LG-coordinate neighborhood (A_i, ψ_i) on an LG-fuzzy manifold X, then $(A, \mathfrak{T})$ is an LG-fuzzy manifold of the same dimension, call n. The LG-fuzzy vector fields $H_{ip} = \psi_*^{-1}(\partial/\partial x_i)$, $i = 1, \ldots, n$, have components $\alpha_j = \delta_i^j$. These are constants and hence C^∞ LG-fuzzy functions on A, so that each H_i is a C^∞ LG-fuzzy vector field on A. The set $H_1, \ldots, H_n$ is a basis of $LGT(X)$ at each $p \in A$, the LG- coordinate frames. Let Y be another C^∞ LG-fuzzy manifold, $F : X \to Y$ is an C^∞ LG-fuzzy function. We show that $f \to F^*(f) = f \circ F$ is an LG-fuzzy homomorphism from $C^\infty(F(p))$ to $C_L^\infty(p)$. Let $f, g \in C^\infty(F(p))$, then

$$\begin{aligned} F^*(f+g) &= (f+g) \circ F = f \circ F + g \circ F = F^*(f) + F^*(g), \\ F^*(fg) &= (fg) \circ F; \\ ((fg) \circ F)(p) &= f(F(p))g(F(p)) \Longrightarrow (fg) \circ F = ((f \circ F)(g \circ F). \end{aligned}$$

then F^* is an LG-fuzzy homomorphism.

5. C^∞ LG fuzzy local one parameter groups acting on an C^∞ LG-fuzzy manifold

Definition 5.1. *Let Q be a group and G be an L-fuzzy subset of Q. Then, G be an L-fuzzy subgroup of Q If:*

$$G(xy^{-1}) \geq \min\{G(x), G(y)\}, \quad G(e) = 1.$$

Definition 5.2. *Let Q be a group and G be an LG-fuzzy subgroup which is at the sametimes an C^∞ LG-fuzzy manifold with L-gradation of openness $\mathfrak{G}$. Then, $(G, \mathfrak{G})$ is an LG-fuzzy Lie group provided that $G \times G \to G$ defined by $(x, y) \to xy$ and the mapping of $G \to G$ defined by $x \to x^{-1}$ are both C^∞ LG-fuzzy mappings.*

Definition 5.3. *Let $(G, \mathfrak{G})$ be an LG-fuzzy Lie group and $(X, \mathfrak{T})$ be an C^∞ LG-fuzzy manifold. Then, G is said to LG-fuzzy act on X (on the left) if there is an C^∞ LG-fuzzy mapping $\theta : G \times X \to X$ satisfying two conditions:*

1) *If e is the identity element of G, then $\theta(e, x) = x$, for all $x \in X$*

2) *If g_1, $g_2 \in G$, then $\theta(g_1,\ \theta(g_2,\ x)) = \theta(g_1 g_2,\ x)$ for all $x \in X$.*

Let θ: $G \times X \to X$ be a C^∞ LG-fuzzy mapping which satisfies the two conditions:

(*i*) $\theta_0(p) = p$, for all $p \in X$,

(*ii*) $\theta_t \circ \theta_s(p) = \theta_{t+s}(p) = \theta_s \circ \theta_t(p)$, for all $p \in X$ and $s, t \in G$.

[We will often write $\theta(t,p)$ as $\theta_t(p)$ or $\theta_p(t)$, depending on which variable is to be emphasized.]

Definition 5.4. *Let $(1_{\mathbb{R}^n}, \mathfrak{T}_{Ln})$ be an LG-fuzzy Euclidean manifold. Then, $1_{\mathbb{R}^n}$ be an L-fuzzy subgroup of $(\mathbb{R}^n, +)$:*

$1_{\mathbb{R}^n}(x-y) = 1 \geq \min\{1_{\mathbb{R}^n}(x), 1_{\mathbb{R}^n}(y)\} = 1, \quad 1_{\mathbb{R}^n}(0) = 1.$

On the other hand $(x,y) \to x+y$, $x \to x$ are C^∞ LG- fuzzy function, therefore $(1_{\mathbb{R}^n}, \mathfrak{T}_{Ln})$ is an LG- fuzzy lie group.

Example 5.1. Let $X = 1_{R^3}$, $G = 1_R$ and $b = (b_1, b_2, b_3)$ is fixed and suppose different from 0. Then, $\theta_t(x) = (x_1 + b_1 t, x_2 + b_2 t, x_3 + b_3 t)$ defines an C^∞ *LG*-fuzzy action of G on X. Each $t \in G$ determined $\theta_t : R^3 \to R^3$, taking the point x to the point $x + tb$. A particularly simple special case is given by $b = (1,0,0)$ so that $\theta_t(x) = (x_1 + t, x_2, x_3)$.

Definition 5.5. *Assume that $\theta : G \times X \to X$ is an C^∞ LG- fuzzy action of G on X. Then, it defines on X an C^∞ LG- fuzzy vector field Z which is called infinity decimal generator of θ. For each $p \in X$ we define $Z_p : C_L^\infty(p) \to G$ by*

$$Z_p f = \lim_{\triangle t \to 0} \frac{1}{\triangle t} \left[f(\theta_{\triangle t}(p)) - f(p) \right]. \tag{3}$$

We have from (3.2) *that Z, is a vector at p, and then verify that $p \to Z_p$, defines an LG-fuzzy vector field, or we have as follows. Let (A, ψ) be an LG-coordinate neighborhood of $p \in X$ and let $J_\epsilon \times V$ be an LG-open subset of $(0,p)$ in $G \times X$, where $J_\epsilon = \{\{t \in G \mid -\epsilon < t < \epsilon\}$, and V and $\epsilon > 0$ are so chosen that $\theta(J_\epsilon \times V) \subset \operatorname{supp} A$. We have, $V = \theta_0(V)$ is contained in A and contains p. Claimed to the open set $J_\epsilon \times V$, we write θ in LG- local coordinates*

$$y_1 = g_1(t, x_1, .., x_n) \quad \dots \quad y_n = g_n(t, x_1, .., x_n)$$

or $y = g(t,x)$, where $x = (x_1, .., x_n)$ are the LG-coordinates of $q \in V$ and $y = (y_1, .., y_n)$ of $\theta_t(q)$, its image. The g_i are defined and C^∞ on $J_\epsilon \times \psi(A)$ and the range of $g(t,x)$ is in $\psi(A)$. In fact θ_0 is the identity and $\theta_{t_1+t_2} = \theta_{t_1} \circ \theta_{t_2}$ is reflected in the conditions:

$$g_i(0,x) = x_i \text{and } g_i(t_1 + t_2, x) = g_i(t_1, g(t_2, x))$$

for $i = 1, \dots, n$. Now, if $\hat{f}(x_1, .., x_n)$ is the local expression for $f \in C_L^\infty(p)$, then

$$\frac{1}{\triangle t} \left[f(\theta_{\triangle t}(p)) - f(p) \right] = \frac{1}{\triangle t} \left[\hat{f}(g(\triangle t, x)) - \hat{f}(x) \right],$$

$$Z_p f = \lim_{\triangle t \to 0} \frac{1}{\triangle t} \left[\hat{f}(g(\triangle t, x)) - \hat{f}(x) \right] = \sum_{i=1}^{n} g_i^{\cdot}(0,x) \left(\frac{\partial \hat{f}}{\partial x_i} \right)_{\psi(p)},$$

where we implies that on V, $Z_p = \sum g_i^{\cdot}(0,x) H_{ip}$ with $H_i = \overset{-1}{\psi}_*(\partial / \partial x_i)$ and $x = \psi(p)$, which shows that Z is an C^∞ LG-fuzzy vector field over V. Since every point of X lies in such an LG-neighborhood, Z is C^∞ LG-fuzzy on X.

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Accepted: March 03, 2021

Variations of diagonal cyclicity of Latin squares formed by permutation polynomials

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Abstract. Cryptographic applications of Latin squares require to study them in various aspects. The Latin squares, which are due to bivariate polynomials, show some interesting patterns of entries. In this paper, we discussed the diagonally cyclic nature of Latin squares over some small finite rings with the help of the bivariate permutation polynomials, which formed them.
Keywords: permutation polynomials, Latin squares, bivariate polynomials, ring.

1. Introduction

In modern mathematics, finite fields and finite rings are important algebraic concepts. Both theoretical and practical purposes are being studied for these algebraic structures. They are widely considered in modern applications due to their great algebraic properties and finiteness. Polynomials over these finite structures induce some more mathematical properties. A polynomial can represent every function from a finite field to itself. The functions, which are also permutations of the field, give rise to permutation polynomials, which have potential applications in cryptology and coding theory (see, e.g., [13, 14]).

In the last couple of decades, permutation polynomials have increasingly attracted the attention of various researchers. As an introduction to the subject, the inspiring survey papers were given by Lidl and Mullen [8, 7]. The structures and various properties of Latin squares and permutation polynomials are explained in detail in [10, 11, 12]. For development in diagonal Latin squares and for basic results we refer [6, 4, 5].

The combinatorial theory is one of the fastest-growing areas of modern mathematics. Combinatorial designs have wide applications in a variety of fields, including coding theory and cryptography. Many examples of combinatorial designs can be listed like linked design, balanced design, one-factorization, graph, etc. *Latin square* is one of such combinatorial concepts.

Definition 1.1. A Latin square of order (or size) n is an $n \times n$ array-based on some set S of n symbols (treatments), with the property that every row and every column contains every symbol exactly once.

In other words, every row and every column is a permutation of S. Also, it can be thought of as a two dimensional analog of a permutation. A Latin square can be viewed as a quadruple $(R, C, S; L)$, where R, C, and S are sets of cardinality n, L is a mapping $L : R \times C \to S$ such that for any $i \in R$ and $x \in S$, the equation, $L(i, j) = x$ has a unique solution $j \in C$, and for any $j \in C$, $x \in S$, the same equation has a unique solution $i \in R$. That is any two of $i \in R, j \in C, x \in S$ uniquely determine the third so that $L(i, j) = x$. i.e., the cell in row i and column j contains the symbol $L(i, j)$.

Definition 1.2. Assume $f(x) = a_n x^n + \ldots + a_1 x + a_0$ is a polynomial of degree $n \geq 1$ modulo m, where $a_n \neq 0(\mod m)$. If $f(x) = ((a_n x^n + \ldots + a_1 x + a_0) \mod m)$ forms a bijection $F : \{0, 1, \ldots, m-1\} \to \{0, 1, \ldots, m-1\}$, we say that $f(x)$ is a permutation polynomial modulo m.

Using the concept of permutation functions we can define Latin squares as follows:

Definition 1.3. A function $f : S^2 \to S$ on a finite set S of size $n > 1$ is said to be a Latin square (of order n) if for any $a \in S$ both the functions $f(a, .)$ and $f(., a)$ are permutations of S.

Rivest [1] discussed the formation of Latin squares using bivariate permutation polynomials. He gave the below theorem;

Theorem 1.1. *A bivariate polynomial $P(x, y) = \sum_{i,j} a_{ij} x^i y^j$ represents a Latin square modulo $n = 2^w$, where $w \geq 2$, if and only if the four univariate polynomials $P(x, 0)$, $P(x, 1)$, $P(0, y)$, and $P(1, y)$ are all permutation polynomials modulo n.*

The Latin square entries are determined by their row and column indices: $(i,\ j)$. It is interesting, in some cases, to look at them diagonally down as a sequence of entries. This leads us to the concept of *diagonally cyclic Latin squares.*

Definition 1.4. A Latin square in which every diagonal is cyclic is said to be a *diagonally cyclic Latin square.*

A Latin square of order n possessing a cyclic automorphism of order n is said to be *diagonally cyclic* because its entries occur in cyclic order down each broken diagonal.

Example 1.1. In the Latin square of order 5 below, the rows are the cyclic automorphism with the cycle (0, 2, 4, 1, 3), and the square is diagonally cyclic with the cycle (0, 1, 2, 3, 4).

0	2	4	1	3
4	1	3	0	2
3	0	2	4	1
2	4	1	3	0
1	3	0	2	4

Such squares are named after Parker, in recognition of his seminal contribution to the study of orthogonal Latin squares.

When the rows are not arranged in a cyclic way, such squares can not be diagonally cyclic either.

Example 1.2. The Latin square below is not diagonally cyclic.

1	2	3	4	5
2	1	5	3	4
3	4	1	5	2
4	5	2	1	3
5	3	4	2	1

Let us denote the d^{th} diagonal of a Latin square L of order n as $D[d]$. Then, $D[d]$ consists of the entries in cells (i, j) satisfying $j - i = d \bmod n$. In particular, $D[0]$ is the main diagonal. So, for d =1, 2, ..., $n-1$, we have to read the n entries diagonally down continuing in the lower triangle of the square from top to bottom after the entries in the upper triangular part. But for $d = 0$, all entries are on the main diagonal. In the example 1.2, for $d = 1$, the entry 5 should be continued after the entries, 2, 5, 5, 3 of the upper diagonal and for $d = 2$, the entries 4, 3 are to be followed in order after the entries 3, 3, 2 of the upper diagonal.

We say that $D[d]$ is *constant* if every entry on it is the same. In the above example 1.2, the main diagonal $D[0]$ is constant. If the i-th row entries are just a cyclic shift (right) by one unit of the previous row (i.e., (i-1)-th row), then $D[d]$ is *constant* for every d. This can be easily observed in the following example.

Example 1.3. $D[d]$ is constant for $d = 0, 1, 2, 3, 4$.

1	2	3	4	5
5	1	2	3	4
4	5	1	2	3
3	4	5	1	2
2	3	4	5	1

Corollary 1.1 ([9])**.** *The Latin squares formed by bivariate linear permutation polynomials will always have all the rows and the columns as cyclic shifts.*

2. Permutation polynomials and diagonally cyclic Latin squares

As we could see, the Latin square structures formed by bivariate permutation polynomials over finite rings. In the papers [2, 3], the most of the squares, the rows or the columns are cyclically arranged. So, it is interesting to think about the diagonally cyclic property of such Latin squares.

Example 2.1. The Latin square formed by a linear bivariate permutation polynomial $3x + 5y$ over Z_7 is given below:

0	3	6	2	5	1	4
5	1	4	0	3	6	2
3	6	2	5	1	4	0
1	4	0	3	6	2	5
6	2	5	1	4	0	3
4	0	3	6	2	5	1
2	5	1	4	0	3	6

This square has the cycle (0, 1, 2, 3, 4, 5, 6) in all the diagonals.

Theorem 2.1. *A Latin square formed by a linear bivariate polynomial always has the same cycle in all the diagonals $D[d]$.*

Proof. We know from corollary 1.1, that all the rows and columns of a Latin square formed by a linear bivariate the polynomial will always have rows and columns in a cyclic way. Without loss of generality, let the polynomial be $ax+by$ over Z_n. Then, the first row is ax for $x = 0, 1, ..., n-1$ and the first column is by for $y = 0, 1, ..., n-1$. Also, each entry in the square is the sum of corresponding entries in the first row and first column

$L_{i,j} = ai + bj,$

$L_{i+1,j+1} = a(i+1) + b(j+1) = ai + bj + (a+b).$

So, each entry in the cell $(i+1, j+1)$ is nothing but adding constant $(a+b)$ to the entry in the cell (i, j). Hence, the Latin square will have same cycle in all the diagonals $D[d]$.

But the above property fails for bivariate quadratic polynomials. In this case, we noted several interesting patterns, some of which are described below. □

2.1 Rings of prime power orders

The ring Z_{25} is a ring of order p^2, where $p = 5$. In the following, we can see a Latin square formed by $3x + 11y + 15xy + 5x^2 + 20y^2$ in the ring Z_{25}.

In this, the main diagonal, i.e., $d = 0$, is the cycle (0, 4, 13, 2, 21 ,20, 24, 8, 22, 16, 15, 19, 3, 17, 11, 10, 14, 23, 12, 6, 5, 9, 18, 7, 1). The same cycle repeats for $d = 5, 10, 15$ and 20. So, there are only five distinct diagonal cycles. The pattern of the cycles along the diagonals can be seen as follows:
$d = 0, 5, 10, 15, 20$ $(0, 4, 13, 2, 21, 20, 24, 8, 22, 16, 15, 19, 3, 17, 11, 10, 14, 23, 12, 6, 5, 9, 18, 7, 1)$
$d = 1, 6, 11, 16, 21$ $(8, 12, 21, 10, 4, 3, 7, 16, 5, 24, 23, 2, 11, 0, 19, 18, 22, 6, 20, 14, 13, 17, 1, 15, 9)$
$d = 2, 7, 12, 17, 22$ $(1, 5, 14, 3, 22, 21, 0, 9, 23, 17, 16, 20, 4, 18, 12, 11, 15, 24, 13, 7, 6, 10, 19, 8, 2)$
$d = 3, 8, 13, 18, 23$ $(4, 8, 17, 6, 0, 24, 3, 12, 1, 20, 19, 23, 7, 21, 15, 14, 18, 2, 16, 10, 9, 13, 22, 11, 5)$
$d = 4, 9, 14, 19, 24$ $(17, 21, 5, 19, 13, 12, 16, 0, 14, 8, 7, 11, 20, 9, 3, 2, 6, 15, 4, 23, 22, 1, 10, 24, 18)$

Remark 2.2. In the above example, all the diagonals have cycles with both odd and even entries. Here, the order of the ring is 25, which is a prime power $(25 = 5^2)$. Also the same cycles appear in $d + 5$-the diagonal for all d =0, 1, 2, 3, 4.

Next, let us consider one more example for a ring of order p^2, where $p = 3$.

Example 2.2. The polynomial $2x + 5y + 3xy + 3y^2$ represents the Latin square over Z_9 shown below;

0	2	4	6	8	1	3	5	7
8	4	0	5	1	6	2	7	3
4	3	2	1	0	8	7	6	5
6	8	1	3	5	7	0	2	4
5	1	6	2	7	3	8	4	0
1	0	8	7	6	5	4	3	2
3	5	7	0	2	4	6	8	1
2	7	3	8	4	0	5	1	6
7	6	5	4	3	2	1	0	8

In this example again, similar patterns can be seen. Here there are 3 distinct cycles seen along the diagonals, every third diagonal being similar. The 3 cycles on diagonals $D[d]$ are as follows:
$d = 0, 3, 6 \quad (0, 4, 2, 3, 7, 5, 6, 1, 8)$
$d = 1, 4, 7 \quad (2, 0, 1, 5, 3, 4, 8, 6, 7)$
$d = 2, 5, 8 \quad (4, 5, 0, 7, 8, 3, 1, 2, 6)$

Remark 2.3. In both examples above, we can observe that each diagonal $D[d]$ is a permutation of the corresponding ring. i.e., each diagonal contains all the elements exactly once.

The above remark leads us to think about whether this is a property of rings of order of a prime power. But it is not so, as we can see from the example below:

Example 2.3. Latin square formed by the polynomial $5x + y + 3xy + 3x^2 + 6y^2$ over Z_9:

0	8	4	6	5	1	3	2	7
7	0	8	4	6	5	1	3	2
8	4	6	5	1	3	2	7	0
3	2	7	0	8	4	6	5	1
1	3	2	7	0	8	4	6	5
2	7	0	8	4	6	5	1	3
6	5	1	3	2	7	0	8	4
4	6	5	1	3	2	7	0	8
5	1	3	2	7	0	8	4	6

Remark 2.4.

1. The above square contains all the diagonals with 3 continuous strings of the form (a, a, b).

2. Each diagonal has only 2 elements of the ring, one appearing 6 times and the other 3 times.

3. All the 9 diagonals are distinct.

The diagonals of this square are shown below;
$d = 0$, $(0, 0, 6, 0, 0, 6, 0, 0, 6)$
$d = 1$, $(8, 8, 5, 8, 8, 5, 8, 8, 5)$
$d = 2$, $(4, 4, 1, 4, 4, 1, 4, 4, 1)$
$d = 3$, $(6, 6, 3, 6, 6, 3, 6, 6, 3)$
$d = 4$, $(5, 5, 2, 5, 5, 2, 5, 5, 2)$
$d = 5$, $(1, 1, 7, 1, 1, 7, 1, 1, 7)$
$d = 6$, $(3, 3, 0, 3, 3, 0, 3, 3, 0)$
$d = 7$, $(2, 2, 8, 2, 2, 8, 2, 2, 8)$
$d = 8$, $(7, 7, 4, 7, 7, 4, 7, 7, 4)$
Interestingly, the mirror image of the Latin square in the example 2.3 has the same properties as that of previous two examples. The mirror image is got by just looking at the square from right to left instead of left to right. The mirror image is given below:

7	2	3	1	5	6	4	8	0
2	3	1	5	6	4	8	0	7
0	7	2	3	1	5	6	4	8
1	5	6	4	8	0	7	2	3
5	6	4	8	0	7	2	3	1
3	1	5	6	4	8	0	7	2
4	8	0	7	2	3	1	5	6
8	0	7	2	3	1	5	6	4
6	4	8	0	7	2	3	1	5

In this square, there are three distinct cycles along the diagonals. Also, we can observe that all the rows of the square are cyclic shifts of a single cycle $(7, 2, 3, 1, 5, 6, 4, 8, 0)$. But the columns have three distinct cycles.

So, the polynomial $5x + y + 3xy + 3x^2 + 6y^2$ gives a Latin square, which is not diagonally cyclic, but its mirror image is a diagonally cyclic Latin square.

We give one more example:

Latin square of order 27 formed by the polynomial, $4x + 16y + 9xy + 12x^2 + 21y^2$ over Z_{27}. Here there are 3 distinct diagonal cycles, each appearing 9 times. Also, each diagonal is a permutation of Z_{27}. We can see properties similar to that of the ring Z_9 regarding cycles on the diagonals.

2.2 Rings of order $2p$

Out of 16 possible quadratic bivariate permutation polynomials (with constant term 0) which give Latin squares over Z_6, we look at two examples shown below:

Example 2.4. The polynomial $2x + 5y + 3x^2$ over Z_6 gives the Latin square shown below:

0	5	4	3	2	1
5	4	3	2	1	0
4	3	2	1	0	5
3	2	1	0	5	4
2	1	0	5	4	3
1	0	5	4	3	2

This square is diagonally cyclic with two cycles along the diagonals. Each diagonal has a cycle of length 3 repeated twice. Also, the diagonal $D[d]$ contains even entries if d is even and odd entries if d is odd. Also, the mirror image of this square is a Latin square with all its diagonals $D[d]$ being constant.

But in the following example, where both the coefficients of x and y are even, and the coefficients of x^2 and y^2 are 3, the square has constant diagonals for all d. The d the diagonal has the constant entry $n - d$ for all d. Further, the odd diagonal has odd entries, and the even diagonal has even entries in this example too.

Example 2.5. The Latin square formed by $2x + 4y + 3x^2 + 3y^2$ over Z_6.

0	5	4	3	2	1
1	0	5	4	3	2
2	1	0	5	4	3
3	2	1	0	5	4
4	3	2	1	0	5
5	4	3	2	1	0

In the mirror image of the Latin square in this case(2.5), we can see the odd diagonal with even entries and even diagonals with odd entries. The cycles $(1, 3, 5)$ and $(0, 2, 4)$ are appearing twice alternatively in diagonals.

In the case of the ring Z_{10}, we note similar things as in the case of Z_6, which may be seen in the following two examples.

Example 2.6.

0	3	6	9	2	5	8	1	4	7
3	6	9	2	5	8	1	4	7	0
6	9	2	5	8	1	4	7	0	3
9	2	5	8	1	4	7	0	3	6
2	5	8	1	4	7	0	3	6	9
5	8	1	4	7	0	3	6	9	2
8	1	4	7	0	3	6	9	2	5
1	4	7	0	3	6	9	2	5	8
4	7	0	3	6	9	2	5	8	1
7	0	3	6	9	2	5	8	1	4

Latin square formed by $3x + 8y + 5y^2$ over Z_{10}

0	9	8	7	6	5	4	3	2	1
1	0	9	8	7	6	5	4	3	2
2	1	0	9	8	7	6	5	4	3
3	2	1	0	9	8	7	6	5	4
4	3	2	1	0	9	8	7	6	5
5	4	3	2	1	0	9	8	7	6
6	5	4	3	2	1	0	9	8	7
7	6	5	4	3	2	1	0	9	8
8	7	6	5	4	3	2	1	0	9
9	8	7	6	5	4	3	2	1	0

Latin square formed by $4x + 6y + 5x^2 + 5y^2$ over Z_{10}

Example 2.7. The following is a Latin square formed by the polynomial $6x + 10y + 7x^2 + 7y^2$ over Z_{14}.

0	13	12	11	10	9	8	7	6	5	4	3	2	1
3	2	1	0	13	12	11	10	9	8	7	6	5	4
6	5	4	3	2	1	0	13	12	11	10	9	8	7
9	8	7	6	5	4	3	2	1	0	13	12	11	10
12	11	10	9	8	7	6	5	4	3	2	1	0	13
1	0	13	12	11	10	9	8	7	6	5	4	3	2
4	3	2	1	0	13	12	11	10	9	8	7	6	5
7	6	5	4	3	2	1	0	13	12	11	10	9	8
10	9	8	7	6	5	4	3	2	1	0	13	12	11
13	12	11	10	9	8	7	6	5	4	3	2	1	0
2	1	0	13	12	11	10	9	8	7	6	5	4	3
5	4	3	2	1	0	13	12	11	10	9	8	7	6
8	7	6	5	4	3	2	1	0	13	12	11	10	9
11	10	9	8	7	6	5	4	3	2	1	0	13	12

Similarly, Latin squares formed by the polynomials $3x + 10y + 11y^2$ over Z_{22} and $16x + 5y + 13x^2$ over Z_{26}.

Remark 2.5. In the case of rings Z_6, Z_{10}, Z_{14},Z_{22} and Z_{26}, which are all of the order $2p$, p prime, we observe that one of the following holds:

1. A Latin square formed by a bivariate quadratic polynomial is such that all its diagonals are constant. So, there are n diagonal cycles, where n is the order of the ring.

2. There are two diagonal cycles in the square; all $n/2$ odd numbers on the odd diagonals, and all $n/2$ even numbers are on the even diagonals.

2.3 Some more rings

The following is an example of a Latin square formed by the polynomial $5x + 7y + 6xy + 6y^2$ in the ring Z_{12}

0	5	10	3	8	1	6	11	4	9	2	7
1	0	11	10	9	8	7	6	5	4	3	2
2	7	0	5	10	3	8	1	6	11	4	9
3	2	1	0	11	10	9	8	7	6	5	4
4	9	2	7	0	5	10	3	8	1	6	11
5	4	3	2	1	0	11	10	9	8	7	6
6	11	4	9	2	7	0	5	10	3	8	1
7	6	5	4	3	2	1	0	11	10	9	8
8	1	6	11	4	9	2	7	0	5	10	3
9	8	7	6	5	4	3	2	1	0	11	10
10	3	8	1	6	11	4	9	2	7	0	5
11	10	9	8	7	6	5	4	3	2	1	0

The following is an example of a Latin square formed by the polynomial $7x + 11y + 6xy + 6x^2$ in the ring Z_{12}

0	1	2	3	4	5	6	7	8	9	10	11
11	6	1	8	3	10	5	0	7	2	9	4
10	11	0	1	2	3	4	5	6	7	8	9
9	4	11	6	1	8	3	10	5	0	7	2
8	9	10	11	0	1	2	3	4	5	6	7
7	2	9	4	11	6	1	8	3	10	5	0
6	7	8	9	10	11	0	1	2	3	4	5
5	0	7	2	9	4	11	6	1	8	3	10
4	5	6	7	8	9	10	11	0	1	2	3
3	10	5	0	7	2	9	4	11	6	1	8
2	3	4	5	6	7	8	9	10	11	0	1
1	8	3	10	5	0	7	2	9	4	11	6

From the above two examples over the ring Z_{12}, we note the following:

Remark 2.6.

1. In the ring Z_{12}, the diagonal entries are odd for odd diagonals and even for even diagonals.
2. The entries on the even diagonals are constant in the first case whereas the odd diagonals are constant in the second case.
3. When the entries on the odd diagonals are constant odd numbers, then the entries on each even diagonal are two even numbers that keep alternating.

4. When the entries on the even diagonals are constant even numbers, then the entries on each odd diagonal are two odd numbers that keep alternating.

5. In both cases, there are 12 distinct types of cycles on 12 diagonals.

6. The mirror image of both the type of squares have the odd and even diagonals with even and odd entries respectively.

Examples of squares over Z_{20} ($where\ 20 = 2^2 \times 5$) and Z_{28} ($where\ 28 = 2^2 \times 7$) are also given.

Let us consider one more Latin square, formed by $5x+19y+6xy+12x^2+18y^2$ over Z_{24}. We note the following:

Remark 2.7.

1. For $d = 0, 4$ and 8, the diagonals $D[d]$ have transpositions (or cycles of length 2) of even numbers repeated 12 times as shown below:

$$\begin{array}{ll} d = 0 & (0, 12) \\ d = 4 & (20, 8) \\ d = 8 & (16, 4) \end{array}$$

   For $d = 12, 16$ and 20, the diagonals $D[d]$ are same as the above three diagonals in the same order.

2. For $d = 2, 6, 10, 14, 18$ and 22, the six diagonals $D[d]$ are the constant diagonals with the constants $10, 6, 2, 22, 18$ and 14 respectively.

3. The remaining 12 odd diagonals are as follows;
   For $d = 1$,the cycle of 4 primes repeated 6 times; $(17, 11, 5, 23)$.
   For $d = 3$, the cycle of 4 odd multiples of 3 repeated 6 times; $(3, 9, 15, 21)$
   For $d = 5$, the cycle $(13, 7, 1, 19)$
   For $d = 7$, the cycle $(23, 5, 11, 17)$ (i.e., the reverse of the cycle for $d = 1$
   For $d = 9$, the cycle $(9, 3, 21, 15)$ (i.e., not the same order as for $d = 3$).
   For $d = 11$, the cycle $(19, 1, 7, 13)$(i.e., the reverse of the cycle for $d = 5$
   For $d = 13$, the same cycle as for $d = 1$.
   For $d = 15$, the same cycle as for $d = 3$.
   For $d = 17$, the same cycle as for $d = 5$.
   For $d = 19$, the same cycle as for $d = 7$.
   For $d = 21$, the same cycle as for $d = 9$.
   For $d = 23$, the same cycle as for $d = 11$.

4. There are 15 distinct diagonal cycles in the square;
   12 even diagonals, out of which 3 appear twice and 6 distinct constant diagonals.
   And 12 odd diagonals, out of which 6 distinct diagonal cycles appear twice.

2.4 The ring Z_{30}

The ring Z_{30} is a ring of order which is equal to the product of 3 distinct prime numbers; 2, 3 and 5.The polynomial $2x + 7y + 15x^2$ forms a Latin square over Z_{30}. 30 being an even number, the square can be divided into four equal parts, vertically and horizontally; two diagonally opposite parts being equal. So, each diagonal has a cycle repeating at least once. A few more observations are noted:

Remark 2.8.

1. All the diagonals have a cycle of length 5, repeated 5 times.
2. The 0^{th} diagonal is the cycle of 5 numbers divisible by 6, (0, 24, 18, 12, 6) in that order.
3. The first diagonal is the cycle of 5 odd numbers, (7, 1, 25, 19, 13) in that order.
4. The second diagonal is the cycle of 5 even numbers (14, 8, 2, 26, 20) in that order.
5. The third diagonal is the cycle of 5 odd multiples of 3, (21, 15, 9, 3, 27) in that order.
6. The fourth and fifth diagonals have the cycles of the remaining 5 even and 5 odd numbers (also all are prime numbers) respectively; (28, 22, 16, 10, 4) and (5, 29, 23, 17, 11) in that order.
7. The sixth diagonal onwards, the first six diagonal cycles repeat four more times, in the same order.
8. There are 6 distinct diagonal cycles, each repeated 5 times.

2.5 The rings of order 2^w

In Z_n, where $n = 2^w$, we know that squares can be considered with four parts, diagonally opposite pairs of parts being equal. So, each diagonal cycle must have atleast a repetition of one cycle. The following is a Latin square formed by the polynomial $3x + 5y + 4xy + 2x^2 + 6y^2$ over Z_8.

0	5	6	3	4	1	2	7
3	4	1	2	7	0	5	6
2	7	0	5	6	3	4	1
5	6	3	4	1	2	7	0
4	1	2	7	0	5	6	3
7	0	5	6	3	4	1	2
6	3	4	1	2	7	0	5
1	2	7	0	5	6	3	4

Here, exactly four diagonal cycles (0, 4),(2, 6),(1, 5) and (2, 6) appearing each twice. These 4 transpositions occur four times in each of their corresponding diagonals. But over the same ring Z_8, the Latin square formed by the polynomial $x + 7y + 2xy + 4x^2$ looks slightly different as shown below:

0	5	2	7	4	1	6	3
7	6	5	4	3	2	1	0
6	7	0	1	2	3	4	5
5	0	3	6	1	4	7	2
4	1	6	3	0	5	2	7
3	2	1	0	7	6	5	4
2	3	4	5	6	7	0	1
1	4	7	2	5	0	3	6

Here, with respect to the even diagonal there is no change except the numbers on the diagonal. But the odd diagonals look quite different. The first and the fifth diagonals are (5, 5, 1, 1) and the third and seventh diagonals are (7, 7, 3, 3). If we look at the coefficients, the former polynomial has all the coefficients of xy, x^2 and y^2 to be non-zero, whereas in the latter polynomial, the coefficient of y^2 is absent. The following are two Latin squares with the coefficient of xy zero in the first and the coefficients of both x^2 and y^2 zero in the second polynomial. So, the diagonals are of the form (a, a, b, b) or (a, b) in these four cases.

0	5	6	3	4	1	2	7
3	0	1	6	7	4	5	2
6	3	4	1	2	7	0	5
1	6	7	4	5	2	3	0
4	1	2	7	0	5	6	3
7	4	5	2	3	0	1	6
2	7	0	5	6	3	4	1
5	2	3	0	1	6	7	4

Latin square formed by $3x + 7y + 2x^2 + 4y^2$

0	1	2	3	4	5	6	7
5	4	3	2	1	0	7	6
2	7	4	1	6	3	0	5
7	2	5	0	3	6	1	4
4	5	6	7	0	1	2	3
1	0	7	6	5	4	3	2
6	3	0	5	2	7	4	1
3	6	1	4	7	2	5	0

Latin square formed by $x + 5y + 6xy$

The following is a Latin square formed by the polynomial $5x + 7y + 12xy + 10x^2 + 4y^2$ over Z_{16}. In this square, both the odd and even diagonals are of the form (a, b, a, c).

There are 16 distinct cycles in 16 diagonals follows;

$d = 0$; $(0, 6, 0, 14)$ $\quad$ $d = 1$; $(15, 5, 15, 13)$

$d = 2$; $(2, 8, 2, 0)$ $\quad$ $d = 3$; $(9, 15, 9, 7)$

$d = 4$; $(4, 10, 4, 2)$ $\quad$ $d = 5$; $(3, 9, 3, 1)$

$d = 6$; $(6, 12, 6, 4)$ $\quad$ $d = 7$; $(13, 3, 13, 11)$

$d = 8$; $(8, 14, 8, 6)$ $\quad$ $d = 9$; $(7, 13, 7, 5)$

$d = 10$; $(10, 0, 10, 8)$ $\quad$ $d = 11$; $(1, 7, 1, 15$

$d = 12$; $(12, 2, 12, 10)$ $\quad$ $d = 13$; $(11, 1, 11, 9)$

$d = 14$; $(14, 4, 14, 12)$ $\quad$ $d = 15$; $(5, 11, 5, 3)$

0	15	2	9	4	3	6	13	8	7	10	1	12	11	14	5
11	6	5	8	15	10	9	12	3	14	13	0	7	2	1	4
14	5	0	15	2	9	4	3	6	13	8	7	10	1	12	11
9	12	3	14	13	0	7	2	1	4	11	6	5	8	15	10
12	11	14	5	0	15	2	9	4	3	6	13	8	7	10	1
7	2	1	4	11	6	5	8	15	10	9	12	3	14	13	0
10	1	12	11	14	5	0	15	2	9	4	3	6	13	8	7
5	8	15	10	9	12	3	14	13	0	7	2	1	4	11	6
8	7	10	1	12	11	14	5	0	15	2	9	4	3	6	13
3	14	13	0	7	2	1	4	11	6	5	8	15	10	9	12
6	13	8	7	10	1	12	11	14	5	0	15	2	9	4	3
1	4	11	6	5	8	15	10	9	12	3	14	13	0	7	2
4	3	6	13	8	7	10	1	12	11	14	5	0	15	2	9
15	10	9	12	3	14	13	0	7	2	1	4	11	6	5	8
2	9	4	3	6	13	8	7	10	1	12	11	14	5	0	15
13	0	7	2	1	4	11	6	5	8	15	10	9	12	3	14

Latin square formed by $5x + 7y + 12xy + 10x^2 + 4y^2$ over Z_{16}

3. Conclusion

The entries of Latin squares appear to be irregular, even if they are formed due to bivariate polynomials over finite rings. But some of the nice patterns through the diagonal runs in the squares may help to characterize the bivariate polynomials. Some of the latest applications of Latin squares, such as cryptography and coding theory may require to study these types of special behaviour of Latin squares constructed using polynomials.

Acknowledgment

The authors would like to thank the Editor-in-chief and the anonymous referees for their valuable comments and suggestions, which helped us to improve the quality of this manuscript.

The first author and third author acknowledges the Manipal Institute of Technology (MIT), Manipal Academy of Higher Education, India, for their kind encouragement.

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Accepted: April 13, 2020

P systems on iso-triangular arrays

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Abstract. Membrane computing is a branch of natural computing in theoretical computer science. This theoretical model was initiated by G.H. Paun in 1988 and it is called P system, inspired by the structure and functioning of the living cells. P system has become an appropriate platform for solving many computational problems arising in different fields. For certain DNA molecules, the behavior of the molecules are recombined by an operation called splicing. Splicing was introduced by Tom Head. Splicing operation on arrays inspired Helan Chandra et.al. introduced H-array splicing system , which generated 2D arrays. This Paper introduced iso-triangular domino array splicing system and iso-triangular domino array splicing P system. The computational complexity of introduced P systems is examined with few existing models.

Keywords: iso-triangular tiles, tile pasting P system, DNA computing, array splicing, two dimensional arrays.

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1. Introduction

In the domain of theoretical computer science, P system [6,7] is a new computational model. The structure has membranes; each membrane is separated by the regions. Multi-sets of objects are placed inside the regions. The evaluation rules in the respective regions are associated with the target indicators. The target indicator 'in j ' communicates the object to the region j. The target indicator 'here' retains the picture pattern in the same region. The target symbol 'out' sends the resultant picture pattern to the outer region. P system can generate the string languages. P system for string languages motivated K.G. Subramanian et al. [10,11] and introduced tile pasting P system for 2D picture languages. Followed by this computational study on P system, Robinson et al. [3,8,9] proposed the models called tile pasting system and extended tile pasting system which are generated tiling and tessellations. In the survey it is found that the family of languages generated by tile pasting system and extended tile pasting system are incomparable. The family of languages generated by tile pasting P system is properly contained in the family of languages evaluated by extended tile pasting system has been noted. The research on P system induced kalyani et. al [5] and introduced a new tile pasting system called, triangular tile pasting system. It can generate iso-triangular picture languages. In this pasting system, triangular tile pasting rules are applied to the iso-triangular tiles to get glued or pasted along corresponding edges of tiles. Recently a new computational model, called triangular tile pasting P system and iso-array rewriting P system with iso-array grammar rules have been proposed in [1,2].

Splicing system is a model to DNA molecules; here the molecules are recombined by the splicing operation which cut a DNA sequence and pastes the fragment of another DNA under the action of restricted enzymes. This concept is applied theoretically to string of symbols and to 2D arrays. It was introduced by Tom Head [4]. Extending this study to images of pictures, Helen Chandra et al. [12,13] introduced H array splicing system. By applying splicing Operation on hexagonal arrays, hexagonal array splicing system and array splicing P system are introduced to generate some class of hexagonal array languages. Sheena Christy et al. introduced iso-array splicing grammar for iso-triangular tiles and discussed some results in [14]. It is motivated to introduce an iso-triangular domino array splicing system and iso-triangular domino array splicing P system on images of iso-triangular arrays. Here iso-triangular domino arrays are considered. An algorithm can be developed for the model which is defined in this paper. This algorithm can be used to generate two dimensional picture languages.

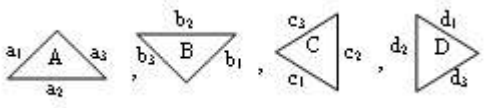
In this paper, section 2 gives the basic definition of iso-triangular tiles and pasting rules of iso triangular tiles have been recalled. In section 3 types of P systems namely triangular tile pasting P system, extended triangular tile pasting P system and Parametric triangular tile pasting P system are introduced and explained with suitable examples. The existing P system called iso-array

rewriting P system with iso-array rules [2] has been recalled. Also the computational power of the P systems is explained in this section. Section 4 denotes iso-triangular domino array splicing system and iso-triangular domino array splicing P system with examples. Section 5 explains the results of the types of tile pasting P systems and iso-triangular domino array splicing P system. It also discusses the computational powers of iso-array rewriting P system with iso-array grammar rules and iso triangular domino array splicing P system.

2. Preliminaries

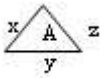
The art of tiling is a well-known theory in the applications of pattern generation. In this section we recollect the notion of iso triangular tiles and triangular tile pasting system. A tile is a topological disc with closed boundary in the XOY plane, whose edges are gluable. A tiling is a family of countable tiles with no gaps or overlaps that covers the Euclidean plane. We are concerned with labeled iso-triangular tiles, whose edges are also labeled.

Definition 2.1. *Consider the labeled iso-triangular tiles*

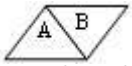


whose horizontal (vertical) and side edges are of length 1 unit and $1/\sqrt{2}$ *unit respectively.*

Definition 2.2. *Generally a pasting rule is a pair* (x, y) *of labeled tiles with distinct edges.*

For example, the iso triangular tile 

and tile 

are joined by the edges (z, t)*, which means that the edge* z *of tile* A *is glued with the edge* t *of tile* B*, we get the pattern* 

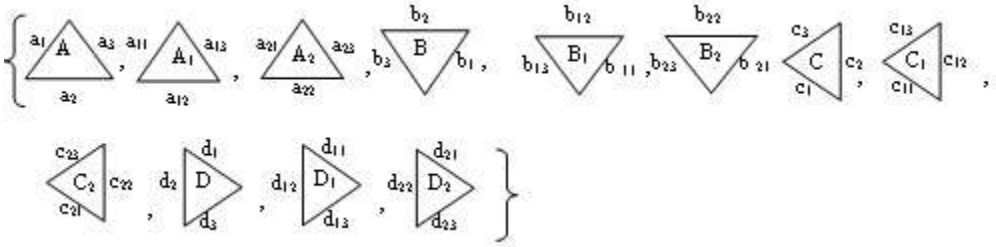
Note that the edges are of same length. The set of all edge labels is called an edge set denoted by E*. Tile pasting rules of the tiles* A, B, C, D *are given below:*

1. *Tile* A *can be glued with tile* B *by the pasting rules* $\{(a_1, b_1), (a_2, b_2), (a_3, b_3)\}$ *with tile* C *by the rule* $\{(a_3, c_1)\}$ *and with tile* D *by the rule* $\{(a_1, d_3)\}$
2. *Tile* B *can be glued with tile* A *by the pasting rules* $\{(b_1, a_1), (b_2, a_2), (b_3, a_3)\}$ *with tile* C *by the rule* $\{(b_1, c_3)\}$ *and with tile* D *by the rule* $\{(b_3, d_1)\}$
3. *Tile* C *can be glued with tile* A *by the pasting rule* $\{(c_1, a_3)\}$ *with tile* B *by* $\{(c_3, b_1)\}$ *and with the tile* D *by the pasting rules* $\{(c_1, d_1), (c_2, d_2), (c_3, d_3)\}$

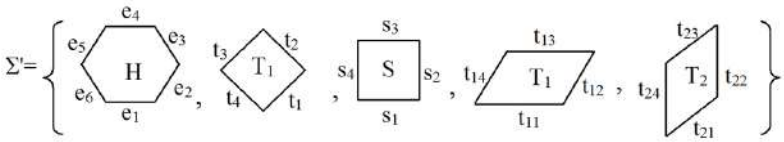
4. *Tile D can be glued with tile A by the pasting rule $\{(d_3, a_1)\}$ with tile B by $\{(d_1, b_3)\}$ and with the tile C by the pasting rules $\{(d_1, c_1)$, (d_2, c_2), $(d_3, c_3)\}$.*

Notations:

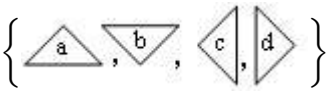
1. Iso-triangular tiles are



2. Extended Triangular tiles are



The set of labeled iso-triangular tiles

$N = \{A, A_1, B, B_1, C, C_1, D, D_1\}$ and $T =$ 

3. Types of triangular tile pasting P systems

In this section, types of triangular tile pasting P systems are defined and explained with suitable examples.

Definition 3.1. *A triangular tile pasting P system (TTPPS) is defined as 5 tuple $\Pi = (\Sigma, \mu, F_1, F_2, F_3, \ldots F_m, R_1, R_2, R_3, \ldots R_m, i_0)$, where Σ is a finite set of triangular tiles, μ is a membrane structure. In μ, the membranes are labeled in a one-one manner and the labels are $1, 2, \ldots, m$. $F_1, F_2, \ldots, F_m$ are finite sets of pictures over the iso-triangular tiles in Σ associated with m regions of the membranes. $R_1, R_2, \ldots, R_m$ are finite sets of pasting rules of type $((x_i, y_i), tar)$, $(1 \leq i \leq n)$ associated with m regions of μ and i_0 is the output membrane, which is an elementary membrane. An evaluation in TTPPS is defined in such a way that, to each picture pattern in each region of the system, a pasting rule could be applied and should be applied. The picture pattern is moved (retained) to another region (in the same region) with respect to the target indication associated with the pasting rules.*

A computation is successful only if all the pasting rules are applied. The computation is stopped, if there is no possibility of applying the pasting rules. The result of halting picture pattern is composed only by the pasting rules; the pattern is haltered in the output region of membrane i_0. The set of all picture patterns computed with the pasting rules of TTPPS is denoted by TTPPL(Π).

The set of all such languages $TTPPL(\Pi)$ is generated by the system Π is denoted by $TTPPL_m$.

Definition 3.2. *Extended triangular tile pasting P system is similar to the triangular tile pasting P system, but during the computation, to complete the picture pattern, the condition Δ over the pasting rule (a finite set of triangular pasting rules) is considered in the output region with target indications. Formally ETTPPS is defined as $\Pi = (\Sigma, \Sigma', E, \mu, F_i, R_i, \Delta, i_0)$, where*

Σ - *Finite set of labeled iso-triangular tiles.*

Σ' - *Finite set of regular or irregular polygons which are made by gluable iso-triangular tiles.*

E - *The edge set $E = \{E(\Sigma), E'(\Sigma')\}$.*

μ - *The membrane structure it has m membranes.*

R_i - *Finite sets of pasting rules associated with m regions of the membrane structure, $1 \leq i \leq m$.*

F_i - *Finite sets of pictures over the triangular tile set in $\Sigma \cup \Sigma'$ and $1 \leq i \leq m$.*

Δ - *The condition over the pasting rules to complete the picture pattern and it is defined in the output region.*

$\Delta = \{((x_i y_j, z_k), tar) \in R_i / x_i, y_j \in E(\Sigma'), z_k \in E(\Sigma)$ *and* $i, j, k \geq 1\}$. i_0 - *The output (skin) membrane. The computation is successful, if there is no possibility of applying the triangular tile pasting rules. Then the computation is stopped and the resultant picture is the halting one, which is collected in the output region. The set of all picture patterns generated by ETTPPS is called a language denoted by $ETTPPL(\Pi)$. The family of all such languages with at most m membranes is denoted by $PL_m(ETTPPS)$.*

Definition 3.3. *The geometric operations are defined in the following way:*
Translation: A translation of a tile or tiling pattern is changing the position of a tile or tiling pattern from (x, y) into (h, k) in Cartesian plane and due to target indication present in the region, the result picture moved into a region or retained in the same region. The labeled iso-triangular tile or tiling pattern T_0 is translated subject to the position (x, y) in the Cartesian plane.
Rotation: It is a mathematical operation, used to rotate a geometrical shape in the XOY plane. To rotate a tile or tiling pattern through an angle θ either in clockwise direction or anti clock wise direction with respect to an edge present in the tile / tiling pattern and generated picture enter from i^{th} region to j^{th} region. $[T_0]_i = [rot(t_i, \theta)]_j$
Replication: The operation replication is the process of multiplies the tiles in a region into k tiles and due to the target indicator in_j, the replicated tiles are

entered into the region j. An iso-triangular tile t_i or tiling pattern replicates copies which is denoted by $[(t_i)]_i \to [t_{i1}, t_{i2}t_{i3}, \ldots, t_{ik}]_j$.

Path rewriting: (vertex removal and merging of edges). The work of path rewriting a tile / tile pattern is rewritten (removing one or more vertices and merging edges of the path) by a new path of vertices without altering the shape of the tile. The sequence of vertices and edges of the form $v_i(e_iv_{i+1}e_{i+1})v_{i+2}\ldots v_{n-1}e_{i+k}v_n$ to the new sequence $v_ie_jv_{i+2}\ldots v_{n-1}e_{j+l}v_n$ for a finite sequence $k \geq l$ and maintained so for $k = l$.

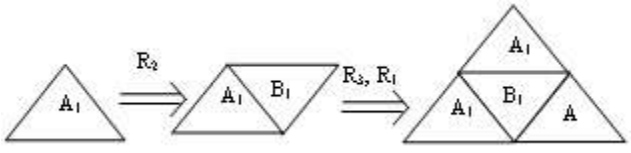
Definition 3.4. *Formally a parametric triangular tile pasting P system is defined as $\Pi = (\Sigma, \mu, F_i, R_i, i_0)$, where Σ is the finite set of labeled iso-triangular tiles, μ is the membrane structure has m membranes. R_i $(1 \leq i \leq m)$ is the finite sets of tile pasting rules associated with m regions of the membrane structure, F_i $(1 \leq i \leq m)$ is the finite sets of pictures over the tile set Σ and i_0 is the output membrane.*

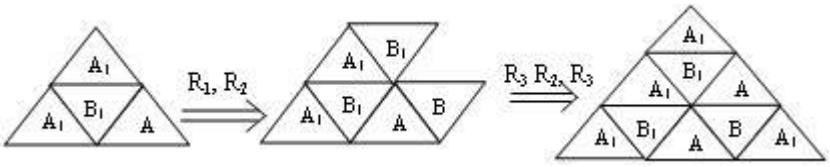
A computation in PTTPPS is defined as like as TTPPS but with geometrical operation. In this computation in each region of the system, the pasting rules of each region can be applied with the geometrical operations. Thus the picture pattern generated by the pasting rules can be placed in the region indicated by the target indicator. A computation is successful only if it stops; the halting computation is a picture such that no further rules can be applied to the existing picture pattern and which is the halting one in the membrane mentioned as output membrane. The set of all such picture pattern computed by PTTPPS is denoted by PTTPP(L) and the family of all such languages generated by Π is denoted by PL_m(PTTPPS).

Example 3.1. The triangular tile pasting P system $\Pi_1 = (\Sigma, [_1[_2]_2[_3]_3]_1, F_1, F_2, F_3, R_1, R_2, R_3, 1)$ generates a class of two dimensional iso-triangular picture language $\mathcal{L}_1$ with at most 4 distinct labeled iso- triangular tiles.
Here $\Sigma = \{A, A_1, B, B_1\}$, $F_1 = \emptyset$, $F_2 = A_1$, $F_3 = \emptyset$,
$R_1 = \{(A, (a_3, b_3), in_2), (B_1, (b_{11}, a_1), here)\}$,
$R_2 = \{(B, (b_2, a_{12}), in_3), (B, (b_1, a_{11}), in_3), (A_1, (a_{13}, b_{13}), in_3)\}$,
$R_3 = \{(B, (b_2, a_2), in_2), (B_1, (b_{12}, a_{12}), out)\}$ and 1 is the output region.

The rules of R_1, R_2 and R_3 are applied and $tar \in \{here, out\} \cup \{in_j\}$, $1 \leq j \leq m$ and then the resultant iso-triangular picture pattern is collected in the output region one finally. Here a copy of the resultant picture pattern is consider to generate the next member of the language.

Two members of the Picture language $\mathcal{L}_1$ is shown with derivation steps.





Example 3.2. The picture language $\mathcal{L}_2$ consisting of squares is constructed by the ETTPPS $\Pi_2 = (\Sigma, \Sigma', E, \mu, F_1, F_2, R_1, R_2, \Delta, 1)$, where
$\Sigma = \{A, B, C, D\}$, $\Sigma' = \{S, T_1, T_2\}$, $E = \{E(\Sigma), E(\Sigma')\}$,
$E(\Sigma) = \{a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3, d_1, d_2, d_3\}$,
$E(\Sigma') = \{s_1, s_2, s_3, s_4, t_{11}, t_{12}, t_{13}, t_{14}, t_{21}, t_{22}, t_{23}, t_{24}\}$, $\mu = [_1[_2]_2]_1$. $F_1 = S$, $F_2 = \emptyset$. $R_1 = \{((s_1, t_{13}), in), ((s_2, t_{24}), in)((a_2, t_{13}), here), ((t_{14}, t_{12}), in), ((d_2, t_{22}), here), ((t_{23}, t_{21}), in)$, $\Delta = \{((t_{12}t_{21}, a_1), here), ((t_{23}t_{12}, c_1), here), ((t_{23}t_{14}, b_3), in), ((t_{21}t_{14}, d_1), here)\}$, $R_2 = \{((s_3, t_{11}), out), ((s_4, t_{22}), out), ((b_2, t_{11}), here), ((t_{12}, t_{14}), out), ((c_2, t_{24}), here), ((t_{21}, t_{23}), out)\}$

The picture language $\mathcal{L}_2$ is shown in fig. 1

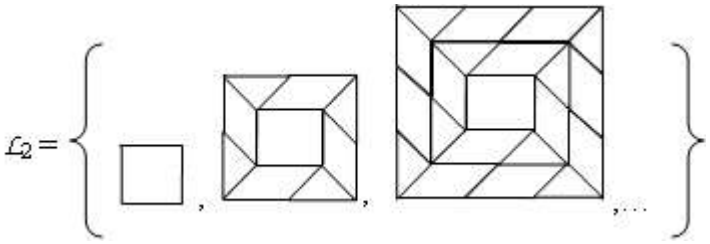
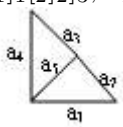


Figure 1: Language of squares

Example 3.3. The language consisting right angled triangles are generated by PTTPPS Π_3 and it is constructed with three membranes.
$\Pi_3 = (\Sigma, [_3[_1]_1[_2]_2]_3, F_1, F_2, F_3, R_1, R_2, R_3, 1)$, where $\Sigma = \{A, D\}$,

$F_1 = T_0 =$  , $F_2 = \emptyset$, $F_3 = \emptyset$.

In region one, the geometrical operation replication has been taken and the rules of R_1 is $R_1 = \{[T_0]_1 \rightarrow [T_0, T_1, T_2, T_3]_2\}$. The edges of the tile T_1, T_2, T_3 are b_i, c_i, d_i respectively, where $b_i = \phi_1(a_i)$, $c_i = \phi_2(a_i)$, $d_i = \phi_3(a_i)$. $R_2 = \{[T_0]_2 \rightarrow [T_0]_3, [T_1]_2 \rightarrow [T_1]_3, [T_2]_2 \rightarrow [T_2]_3, [T_3]_2 \rightarrow rot(T_3, 180^\circ]_3\}$, that is in R_2, the geometrical operation rotation is considered to generate the picture pattern. $R_3 = \{[T_0, T_1, T_2, rot(T_3, 180^\circ), (a_3, d_2), (a_2, d_3)(d_4, b_4), (d_1, c_1), |a_1|b_1| \rightarrow |a_1|, |a_4c_4| \rightarrow |a_4|, |b_2b_3c_2c_3| \rightarrow |a_2a_3|]in_1$. The first three members of the picture language $\mathcal{L}_3$ is shown in fig. 2.

Theorem 3.1. $\mathcal{L}(ETTPPS) \cap \mathcal{L}(TTPPS) \neq \phi$.

Proof. To explain the result, the language $\mathcal{L}_2$ generated by ETTPPS can also be generated by TTPPS. But the language $\mathcal{L}_4$ (refer L_3 in Theorem 8 of [9]).

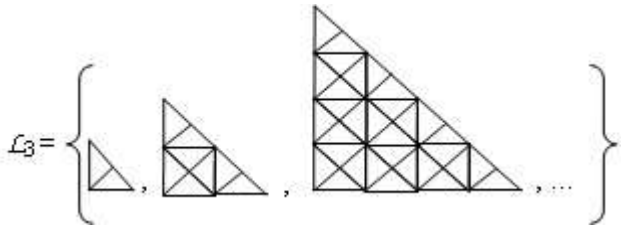


Figure 2: A set of digitized right angled triangles

The picture language $\mathcal{L}_3$ which can also be generated by the TTPPS $\Pi_4 = (\Sigma, [_1[_2]_2[_3]_3]_1, F_1, F_2, F_3, R_1, R_2, R_3, 3)$, where $\Sigma = \{A, B, C, D\}$,

$F_1 =$ 

, $F_2 = \emptyset$ and $F_3 = \emptyset$

$R_1 = \{(A, (a_1, d_1), here), (A, (a_3, c_1), here)), (C, (c_1, a_3), here), (D, (d_1, b_3), in_3)\}$,
$R_2 = \{(B, (b_1, c_3), here), (B, (b_3, c_1), here), (D, (d_1, b_3), here), (D, (d_3, a_1), in_1)\}$,
$R_3 = \{(A, (a_2, b_2), here), (B, (b_2, a_2), here), (C, (c_2, d_2), here), (D, (d_2, c_2), in_2)\}$
and 3 is the output region. But in ETTPPS, to complete the generation we must use the condition Δ over the pasting rule. So the triangular picture language generated by TTPPS cannot always be generated by ETTPPS. It proves the result. □

Theorem 3.2. *$PL_3(PTTPPS, \alpha) \subset TTPPL_3$, where α is the geometrical operation rotation on the iso-triangular tiles.*

Proof. To examine the result consider a class of language digitized right angled triangles generated by PTTPPS with the geometrical operation α. (refer example 3.3). But by parallel mechanism in TTPPS one can generate different classes of languages other than the language generated by PTTPPS. This result can be proved by an example.
The TTPPS $\Pi_5 = (\Sigma, [_1[_2]_2]_1, F_1, F_2, R_1, R_2, 1)$, where $\Sigma = \{A, B, C, D\}$,
$R_1 = \{(a_1, d_3), here), (a_3, c_1), in), (c_2, d_2), in), (c_1, a_3), out)\}$,
$R_2 = \{(d_1, b_3), here), (b_2, a_2), out), (d_3, a_1), out)(b_1, c_1), out)\}$.

$F_1 =$ 

, $F_2 = \emptyset$.

This system generates not only the language consists of right angled triangles but also the picture language of squares with output membranes one and two. □

Theorem 3.3. *The number of triangular tile types required by PTTPPS with rotation is less than the tiles required by TTPPS.*

Proof. Let the primary tiles be $P = \{t_1, t_2, t_3, t_4\}$. For each primary triangular tile $t_i \in P (1 \leq i \leq 4)$, let $\Sigma'' = \{t'_1, t'_2, t'_3, t'_4\}$ be the secondary tiles derived from the primary tiles by the gluing property as well as the geometrical operation rotation. Let $\Sigma = P \cup \Sigma''$ be the set of tiles or tiling patterns required to

generate the picture language. Using triangular tile pasting P system we can generates the picture language by the derivations. $t_0 \Rightarrow T_1 \Rightarrow T_2 \Rightarrow \cdots \Rightarrow T_i$. With the help of geometrical operation rotation in parametric triangular tile pasting P system, the picture pattern T_i can be generated by Σ''. Here it is noticed that to generate the pattern T_i only less number of iso-triangular tiles are used. Thus the operation rotation reduces the number of tile types. □

4. Iso-triangular domino array splicing P system

In this section motivation of rectangular array splicing system, an iso-triangular domino array splicing system and iso-triangular domino array splicing P system are introduced with the examples.

Definition 4.1. *An iso-triangular domino array splicing system (ITDASS) is a splicing system, it is defined as* $\Gamma = (V = (\Sigma \cup \Lambda \cup \{\#, \$\}), A, A_i, (S_i, P_i))$, *where* Σ *is the finite set of labeled iso-triangular tiles,* Λ *is an empty symbol and* $\{\#, \$\}$ *is a set of special characters involved in splicing of two iso- triangular domino arrays.* A *is an axiom of triangular picture pattern,* A_i *is the finite set of intermediate iso-triangular arrays used to generate the iso-triangular arrays,* (S_i, P_i) *is a pair of finite set of splicing rules and pasting rules in which* S_i $(1 \leq i \leq n)$ *are the iso-triangular domino array splicing rules and* P_i $(1 \leq i \leq n)$ *are the associated pasting rules over the elements of* Σ. *The iso-triangular domino array splicing rules over* Σ *are given as a scheme* $S_i = \alpha_1 \# \alpha_2 \$ \alpha_3 \# \alpha_4$ *if*

(1) *The horizontal iso-triangular domino array splicing over* V *is*

$\alpha_1 =$ 

or Λ, $\alpha_2 =$ 

or Λ, $\alpha_3 =$ 

or Λ, $\alpha_4 =$ 

or Λ.

(2) *The vertical iso-triangular domino array splicing over* V *is*

$\alpha_1 =$ 

or Λ, $\alpha_2 =$ 

or Λ, $\alpha_3 =$ 

or Λ, $\alpha_4 =$ 

or Λ.

(3) *The right iso-triangular domino array splicing over* V *is any one of the following rule*

(*a*) $\alpha_1 =$ 

or Λ, $\alpha_2 =$ 

or Λ, $\alpha_3 =$ 

or Λ, $\alpha_4 =$ 

or Λ. *(right)*

(*b*) $\alpha_1 =$ 

or Λ, $\alpha_2 =$ 

or Λ, $\alpha_3 =$ 

or Λ, $\alpha_4 =$ 

or Λ. *(right down)*

(c) $\alpha_1 =$  *or* Λ, $\alpha_2 =$  *or* Λ, $\alpha_3 =$  *or* Λ, $\alpha_4 =$  *or* Λ. *(right down)*

(d) $\alpha_1 =$  *or* Λ, $\alpha_2 =$  *or* Λ, $\alpha_3 =$  *or* Λ, $\alpha_4 =$  *or* Λ. *(right down)*

(e) $\alpha_1 =$  *or* Λ, $\alpha_2 =$  *or* Λ, $\alpha_3 =$  *or* Λ, $\alpha_4 =$  *or* Λ. *(right up)*

(4) *The left iso-triangular domino array splicing over* V *is any one of the following rule*

(f) $\alpha_1 =$  *or* Λ, $\alpha_2 =$  *or* Λ, $\alpha_3 =$  *or* Λ, $\alpha_4 =$  *or* Λ. *(left)*

(g) $\alpha_1 =$  *or* Λ, $\alpha_2 =$  *or* Λ, $\alpha_3 =$  *or* Λ, $\alpha_4 =$  *or* Λ. *(left down)*

(h) $\alpha_1 =$  *or* Λ, $\alpha_2 =$  *or* Λ, $\alpha_3 =$  *or* Λ, $\alpha_4 =$  *or* Λ. *(left up)*

(i) $\alpha_1 =$  *or* Λ, $\alpha_2 =$  *or* Λ, $\alpha_3 =$  *or* Λ, $\alpha_4 =$  *or* Λ. *(Left down)*

(j) $\alpha_1 =$  *or* Λ, $\alpha_2 =$  *or* Λ, $\alpha_3 =$  *or* Λ, $\alpha_4 =$  *or* Λ. *(left up)*

(k) $\alpha_1 =$  *or* Λ, $\alpha_2 =$  *or* Λ, $\alpha_3 =$  *or* Λ, $\alpha_4 =$  *or* Λ. *(left down)*

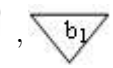
The horizontal, vertical, right, right up, right down, left, left down, left up are denoted by $\$_H$, $\$_R$, $\$_{RU}$, $\$_{RD}$, $\$_L$, $\$_{LU}$, $\$_{LD}$ *respectively. Generally iso-triangular domino array splicing is denoted by* $\$_D$. *If* X *and* Y *are two different iso-arrays then iso-triangular domino array splicing over* X *and* Y *gives the new picture* Z, *is denoted by* $(X, Y) \underset{(S,P)}{\vdash} Z$, *pasting rules* $P = (x, y)$ *where* x *and* y *are the edges of the iso-triangular tiles presented in* X *and* Y *respectively. It*

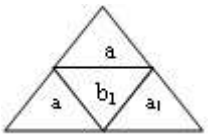
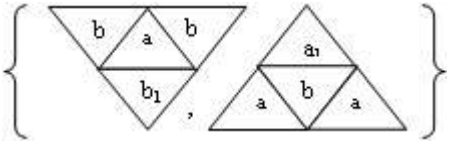
is noted that the computation is begin with the axiom of the system, then the iso-triangular domino array splicing rule is applied with the axiom and the intermediate iso-triangular arrays. The resultant iso-triangular picture has been collected and many copies have been taken as much as required to generate the next members of iso-triangular picture language.
The iso-triangular picture language generated by the system over V^{**} is denoted by $L(\Gamma) = \{M_i \in V^{**}/A \Rightarrow M_i(1 \leq i \leq n)\}$.

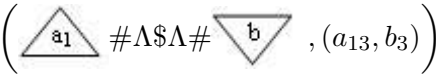
Definition 4.2. *An iso-triangular domino array splicing P system is defined as $\Pi = (V, \mu, L_i, ((S_i, P_i)_{tar}), i_0)$, where $1 \leq i \leq m$, $V = \{\Sigma \cup \Lambda \cup \{\#, \$\}\}$, Σ is the finite set of labeled iso-triangular tiles, Λ is the symbol used to denote the empty place, # and \$ are the special symbols used to splice iso-triangular dominos, which are not in Σ. μ is the membrane structure, it has m membranes. L_i is the initial iso-triangular array present in the m^{th} region. S_i is an iso-triangular domino array splicing rule over V associated with the region R_i $(1 \leq i \leq m)$. The splicing rules is of the form $S_i = \alpha_1 \# \alpha_2 \$ \alpha_3 \# \alpha_4$, where α_i's are defined in definition 4.1. P_i is the pasting rule along with the operation splicing over iso-triangular arrays and $target = \{here, out, in_j\}$ $(1 \leq j \leq m)$. Here means the resultant picture is retained in the same region, out means the picture pattern sent out from the membrane and in_j means the array can be sent into i^{th} region the membrane j. i_0 is the output membrane in which the resultant picture or iso-triangular arrays are collected.*

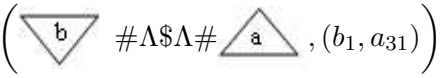
The computation starts with an initial iso-triangular array present in the first region. Note that an iso-triangular array present in the m^{th} region can be taken arbitrarily many copies. If X and Y are any two different iso-triangular arrays then the iso-triangular domino array splicing rule S_i is applied in parallel way between X and Y. The splicing rule generates a new picture W and is denoted by $(X, Y) \underset{\$_D}{\vdash} Z$. A finite sequence of iso-triangular arrays is said to be an iso-triangular array language. Here the sequence is computed by applying the iso-triangular domino splicing rules to the finite connected iso-triangular arrays L_i present in the corresponding regions. The computation of iso-triangular arrays are denoted like if $(T_1, T_2, \ldots, T_m)$ and $(T'_1, T'_2, \ldots, T'_m)$ are two sets of iso-triangular arrays over V. Then the P system gives $(T_i, T_j) \underset{\$_D}{\vdash} T'_i$. The computation stops if no splicing rule can be applied further. At last the resultant iso-triangular array is collected in the output region i_0 or sent out to the environment, it depends the target symbol associated with membrane region. $\mathcal{L}(\Pi) = \{T'_i/(T_i, T_j) \underset{\$_D}{\vdash} T'_i, 1 \leq j \leq m\}$. The family of all iso-triangular array languages is denoted $\cup \mathcal{L}_i$.

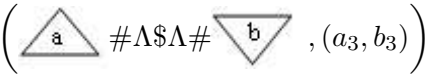
Example 4.1. The iso-triangular domino array splicing system ITDASS $= (V, A, A_i, (S_i, P_i))$, where $1 \leq i \leq 2$ generates rectangular picture language.

$V = \left\{ \right.$  ,   ,  ,  $\left. \right\} \cup \Lambda \cup \{\#, \$\}$

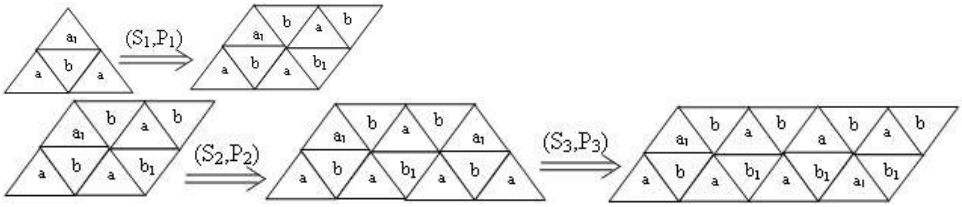
where $A =$  $A_i =$ 

$(S_1, P_1) =$ 

$(S_2, P_2) =$ 

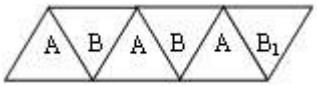
$(S_3, P_3) =$ 

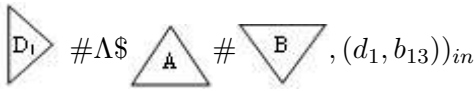
The derivation steps of two members of the picture language are shown below.

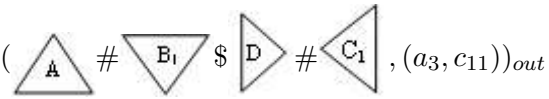


Example 4.2. A class of language consisting of staircase model of iso-triangular arrays is generated by an iso-triangular domino array splicing system Π_6.

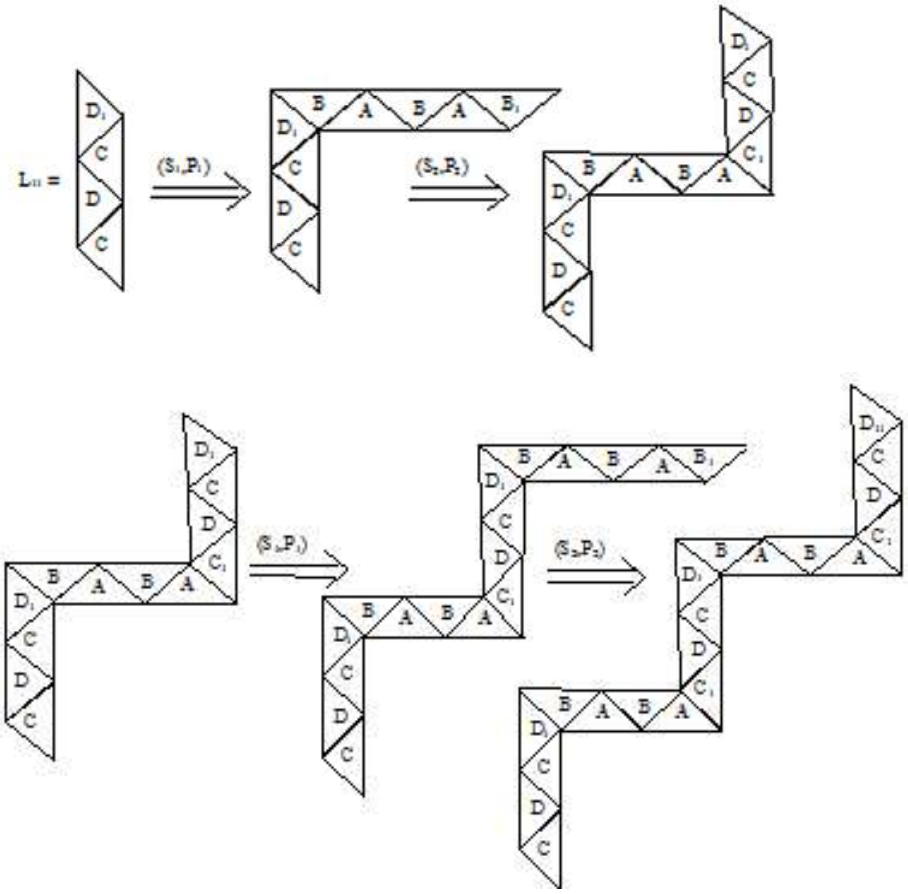
Consider an iso-triangular domino array splicing system
$\Pi_6 = (V, [_1[_2]_2]_1, L_1, L_2, S_1 = (S_i, P_i)_{tar}, 2)$ where $i = 1, 2$
$V = \{\Sigma \cup \{\Lambda\} \cup \{\#, \$\}\}$ and $\Sigma = \{A, B, C, D\}$, $L_1 = \{L_{11}, L_{12}\}$, L_2 are the initial arrays presented in the regions one and two respectively. S_i is the set of splicing rules in the regions R_i and 2 is the output membrane.

$L_{11} =$  , $L_{12} =$  , $L_2 =$ 

$(S_1, P_1) =$ 

$(S_2, P_2) =$ 

Two members of stair case model of iso-triangular arrays are generated by an iso-triangular domino triangular array splicing P system Π_6.

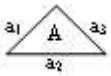


In region one, the triangular arrays L_{11} and L_{12} are spliced by the splicing rule (S_1, P_1) and the picture generated is sent to the region two and due to the target symbol 'in'. The splicing rule (S_2, P_2) is applied to the generated picture and the iso-triangular array L_2 present in the region two. The result is a picture of staircase model of iso-triangular arrays. Once the triangular array is generated, a copy has to be taken and the picture will go to the 1st membrane. Again the computation will start and the successive members of the language will be generated.

Now an algorithm can be written to generate triangular picture languages.

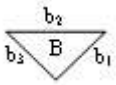
Algorithm

Step I Set the members of triangular picture language as M_i $(1 \leq i \leq n)$

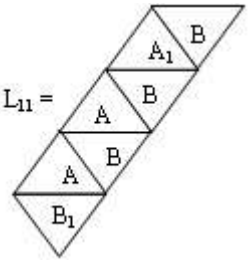
Step II Formation of iso-triangular tile $A =$ 

1. Set an edge $a_1 = x$ cm.
2. Set an edge $a_2 = 2x$ cm.
3. Set an edge $a_3 = x$ cm.
4. Join a_1 and a_2 through an angle 45°
5. Join a_2 and a_3 through an angle 45°

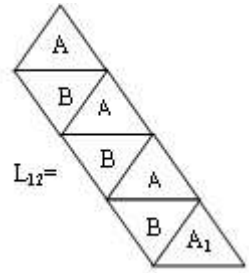
6. Label the tile as A.
7. Take a copy of the triangular tile A.
8. Rename the tile as A_1 and its edges are $|a_{11}| = |a_1|$, $|a_{12}| = |a_2|$, $|a_{13}| = |a_3|$

Step III Formation of iso-triangular tile $B =$ 

9. Rotate an iso- triangular tile A through an angle $180°$.
10. Label the tile as B and the edges of B are $|b_1| = |a_1|$, $|b_2| = |a_2|$, $|b_3| = |a_3|$
11. Take a copy of tile B and label it as B_1 and its edges are $|b_{11}| = |b_1|$, $|b_{12} = |b_2|$, $|b_{13}| = |b_3|$

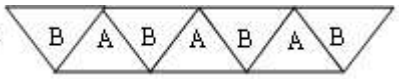
Step IV Computation of $L_{11} =$ 

12. Join the edges b_{12} of iso-triangular tile B_1 and a_2 of iso-triangular tile A and label the generated triangular array as T_{11}
13. Then join the edge a_3 of tile A in T_{11} and the edge b_3 of tile B and assign a label to the generated triangular array as T_{12}
14. Then join the edge b_2 of tile B in T_{12} and the edge a_2 of tile A and assign a label to the generated triangular array as T_{13}
15. Repeat the step 13 to the new triangular array T_{13} and label the triangular array as T_{14}
16. Then join the edge b_2 of B in T_{14} and the edge a_{12} of tile A_1 and label the triangular array as T_{15}
17. Then join the edge a_{13} of tile A_1 in T_{15} and b_3 of tile B and label the triangular array as L_{11}.

Step V Computation of $L_{12}=$ 

18. Join the edges a_{11} of iso-triangular tile A_1 and b_1 of iso-triangular tile B and label the generated triangular array as T_{21}
19. Then join the edge b_2 of tile B in T_{21} and the edge a_2 of tile A and assign a label to the generated triangular array as T_{22}
20. Then join the edge a_1 of tile A in T_{22} and the edge b_1 of tile B and assign a label to the generated triangular array as T_{23}
21. Repeat the step 19 to the triangular array T_{23} and label the generated

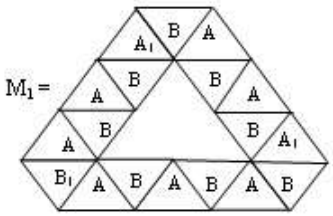
triangular array as T_{24}.
22. Repeat the step 20 to the triangular array T_{24} and label the generated triangular array as T_{25}
23. Then repeat the step 19 to the triangular array T_{25} and label the generated triangular array as L_{12}

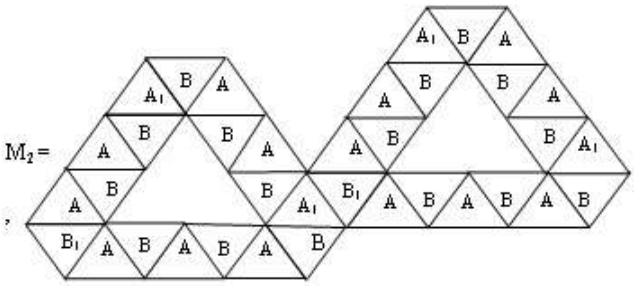
Step VI Computation of $L_2 =$ 

24. Join the edges b_1 of iso-triangular tile B and a_1 of iso-triangular tile A and label the generated triangular array as T_{31}
25. Then join the edge a_3 of tile A in T_{31} and the edge b_3 of tile A and assign a label to the generated triangular array as T_{32}
26. Repeat the step 24 to the triangular array T_{32} and label the generated triangular array as T_{33}
27. Repeat the step 25 to the triangular array T_{33} and label the generated triangular array as T_{34}
28. Repeat the step 24 to the triangular array T_{34} and label the generated triangular array as T_{35}
29. Repeat the step 25 to the triangular array T_{35} and label the generated triangular array as L_2

Step VII Computation of M_i $(i = 1)$
30. Remove the topmost tile B pasted with labeled tile A_1 in L_{11} and remove the topmost tile A pasted with tile B in L_{12}, then adjoin the triangular arrays and label the generated tile as T_1
31. Remove leftmost tile B in L_2 and adjoin the array with the left bottom tile B_1 of T_1 and then label the generated triangular array as M_i $(i = 1)$
32. Take a copy of M_1 and join the bottom tile B_1 of L_{11} with the right hand side tile A_1 of M_1 and label generated triangular tile as T_2
33. Repeat the steps 31 to the triangular array T_2 and set the label to the generated triangular array as M_2

Step VIII Computation of M_i $(1 \leq i \leq n)$
Repeat the steps 32 and 33 to get the successive members M_i $(1 \leq i \leq n)$ (Refer Example 4.3).

$M_1 =$ 

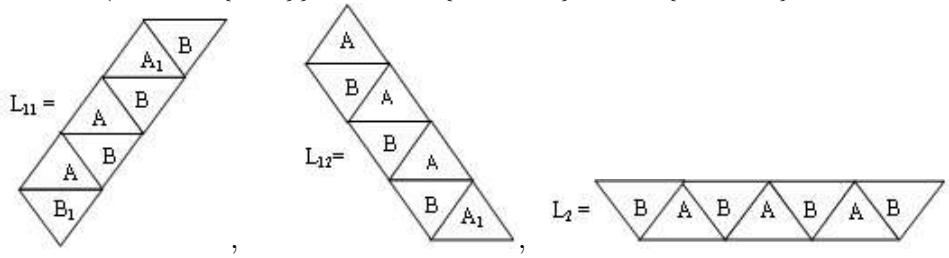
$M_2 =$ 

This algorithm can be used to generate the triangular picture language $\mathcal{L}_5$.

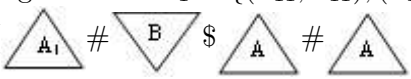
Example 4.3. An iso-triangular domino array splicing P system Π_7 generates a language consisting adjoin of hallow triangles by four iso-triangular tiles A, B, A_1, B_1

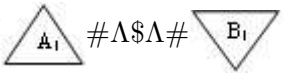
$$\Pi_7 = (V, [_1[_2]_2]_1, L_1, L_2, (S_i, P_i)_{tar}, 1)$$

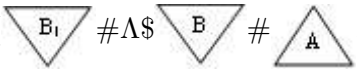
where $V = (\Sigma \cup \Lambda \cup \{\#, \$\}$ and $\Sigma = \{A, B, A_1\}$, $L_1 = \{L_{11}, L_{12}\}$,



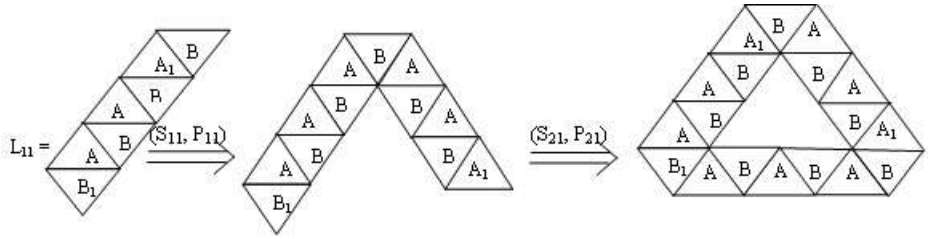
The splicing rules are $S_1 = \{(S_{11}, P_{11}), (S_{12}, P_{12})\}$, $S_2 = \{(S_{21}, P_{21})\}$ where

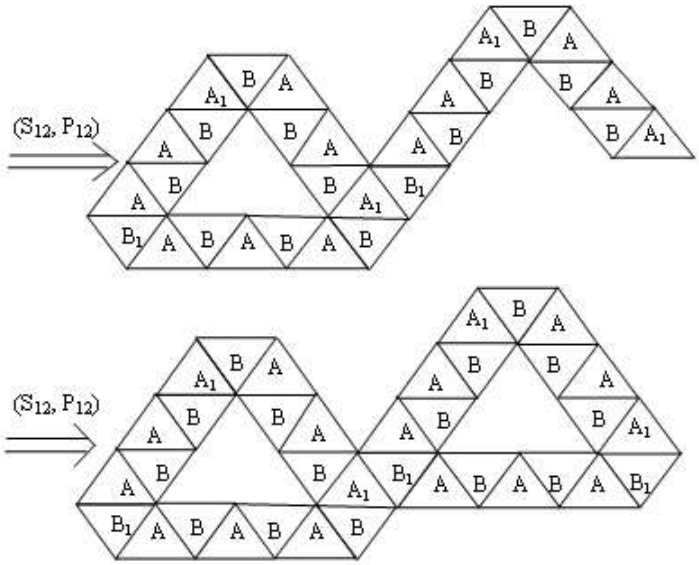
$(S_{11}, P_{11}) = ($  $, (a_{13}, b_3)_{in}$,

$(S_{12}, P_{12}) = ($  $, (a_{13}, b_3))_{here}$,

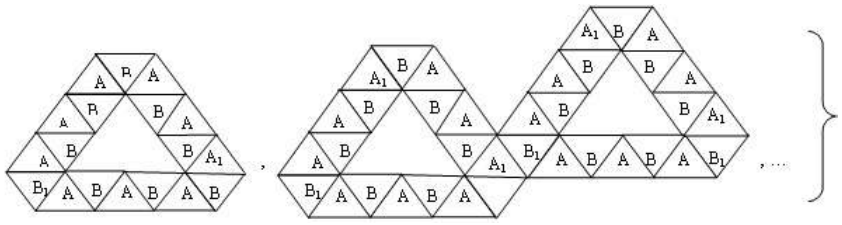
$(S_{21}, P_{21}) = ($  $, (b_1, a_1), (b_2, a_2))_{out}$

The computation begins with the triangular arrays L_{11}, L_{12} in R_1 and the iso-triangular domino array splicing rule (S_{11}, P_{11}) spliced the iso-triangulars L_{11} and L_{12}. The target symbol associated with the region one sends the picture into region two. In the region two, the iso-triangular domino array splicing rule (S_{21}, P_{21}) is applied to the result picture and L_2. The hallow iso-triangular picture is generated. Once a hallow triangular picture is generated copies of the member is taken. Again the computation starts with a copy of generated one to generate 2^{nd} member of triangular picture language. The first member is spliced by the triangular domino array splicing (S_{12}, P_{12}) in the region one. Then the splicing rule (S_{11}, P_{11}) is applied to the result triangular picture. Target symbol associated with (S_{11}, P_{11}) sends the picture into the region two. In region two, splicing is made between the resultant picture and the iso-array L_2. The the 2^{nd} member of a class of language consists of adjoin triangles made of iso-triangular tiles with hallows is generated in this way. Two members of hallow triangular picture language is derived by the following steps.



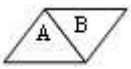
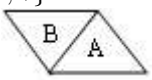


The hallow triangular picture language is shown below

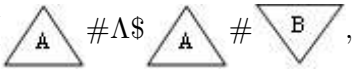
$\mathcal{L}_5 =$ 

Result 4.1. $ITDASPS \cap TTPPS \neq \emptyset$

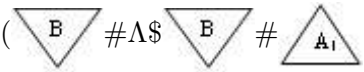
To explain the result the picture language consisting of iso-triangles generated by TTPPS Refer Example 3.1 can also be generated by $ITDASPS$ with three membranes.

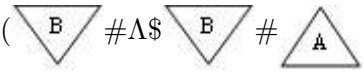
To explain the result consider the ITDASPS system $\Pi_7 = (V, [_1[_2]_2[_3]_3]_1, L_1, L_2, L_3, (S_i, P_i)_{tar}, 1)$ where $V = \Sigma \cup \{\Lambda\} \cup \{\#, \$\}$ and $\Sigma = \{A, A_1, B\}$, $L_1 =$ $\{L_{11} =$  $, L_{12} =$  $\}$, $L_2 =$, $L_3 =$  , the splicing rules are

$\{(S_1, P_1) = \{(S_{11}, P_{11}), (S_{12}, P_{12})\}, (S_2, P_2), (S_3, P_3)\}$ where

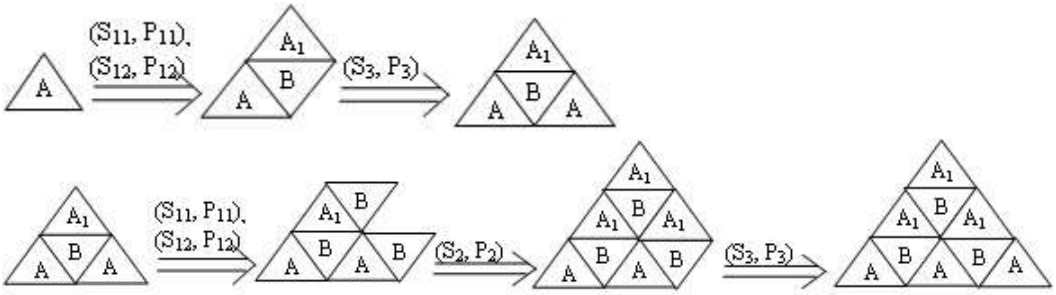
$(S_{11}, P_{11}) = ($  $, (a_3, b_3))_{in2},$

$(S_{12}, P_{12}) = ($  $, (a_{13}, b_3))_{here}$

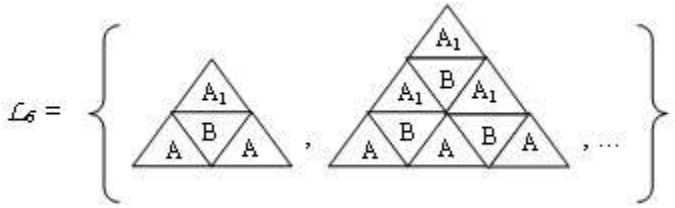
$(S_2, P_2) = ($  $, (b_2, a_{12}))_{in3}$

$(S_3, P_3) = ($  $, (b_1, a_1))_{out}$

The derivation steps of two members of the iso-triangular picture language is shown below



The iso-triangular picture language is shown below



Once a member of the language is computed in the region 3, the picture pattern sent out from the region with many copies. Again a computation is started with a copy of the member to generate the next successive member of the iso-triangular picture language.

But the triangular picture language with hallows generated by $ITDASPS$ (Π_7) cannot be generated by any TTPPS. Due to parallel mechanism the tiles A and B are glued each other we cannot generate such hallow iso triangles. The picture language shown in example 4.2 can be generated by TTPPS. It means that not all the triangular picture language generated by ITDASPS can be computed by the rules of TTPPS. Hence it proved that intersection of these two P systems are non-empty.

Result 4.2. An iso triangular domino array splicing P system is not comparable with the extended triangular tile pasting P system.

Proof. It is clear from the defns 3.2 and 4.2 □

Result 4.3. An iso triangular domino array splicing P system cannot be compared with parametric triangular tile pasting P system.

Proof. It is clear from the defn 3.3 and 4.2 □

Result 4.4. $\mathcal{L}(ITDASPS) - \mathcal{L}(IARP_m(CFPIAG)) \neq \emptyset$.

To prove this result, it is considered that the language $\mathcal{L}_5$ generated by ITDASPS cannot be generated by any ITARPS with CFIA rules. Because in the pasting rules of context free iso array grammar, no terminal symbol can be rewritten once a member is generated. It means, a triangular picture or array with terminals cannot be used to generate or compute successive members of that particular triangular picture language. But the iso-picture language $\mathcal{L}_6$ consisting of iso-triangular arrays (refer Result 4.1) can be generated by ITDASPS system. Hence the two P systems are not disjoint.

5. Conclusion

This paper introduced iso-triangular domino array splicing system, iso-triangular domino array splicing P system and explained with suitable examples. The computational power of ITDASP system is compared with the existing P systems called triangular tile pasting P system, extended triangular tile pasting system, parametric triangular tile pasting system and iso array rewriting P system with context free iso array rules. The generative powers of types of P system also examined.

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Accepted: 4.03.2020

Prime-valent one-regular graphs of order $12p$

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Abstract. A graph is *one-regular* and *arc-transitive* if its full automorphism group acts on its arcs regularly and transitively, respectively. In this paper, we classify connected one-regular graphs of prime valency and order $12p$ for each prime p. By analyzing the structure of the full automorphism group of such graphs and using the classification of arc-transitive graphs of order $2p$, we prove that there is only one such graph, that is, the cycle C_{12p} with valency two and order $12p$.
Keywords: one-regular graph, arc-transitive graph, covering graph.

1. Introduction

Throughout this paper graphs are assumed to be finite, simple, connected and undirected. For group-theoretic concepts or graph-theoretic terms not defined here we refer the reader to [16, 19] or [1, 2], respectively. Let G be a permutation group on a set Ω and $v \in \Omega$. Denote by G_v the stabilizer of v in G, that is, the subgroup of G fixing the point v. We say that G is *semiregular* on Ω if $G_v = 1$ for every $v \in \Omega$ and *regular* if G is transitive and semiregular.

For a graph X, denote by $V(X)$, $E(X)$ and $\mathrm{Aut}(X)$ its vertex set, its edge set and its full automorphism group, respectively. A graph X is said to be G-*vertex-transitive* if $G \leq \mathrm{Aut}(X)$ acts transitively on $V(X)$. X is simply called *vertex-transitive* if it is $\mathrm{Aut}(X)$-vertex-transitive. An *s-arc* in a graph is an ordered $(s+1)$-tuple $(v_0, v_1, \cdots, v_{s-1}, v_s)$ of vertices of the graph X such that v_{i-1} is adjacent to v_i for $1 \leq i \leq s$, and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s-1$. In particular, a 1-arc is just an arc and a 0-arc is a vertex. For a subgroup $G \leq \mathrm{Aut}(X)$, a graph X is said to be (G, s)-*arc-transitive* or (G, s)-*regular* if G is transitive or regular on the set of s-arcs in X, respectively. A (G, s)-arc-

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transitive graph is said to be (G,s)-*transitive* if it is not $(G,s+1)$-arc-transitive. In particular, a $(G,1)$-arc-transitive graph is called G-*symmetric*. A graph X is simply called s-*arc-transitive*, s-*regular* or s-*transitive* if it is $(\mathrm{Aut}(X),s)$-arc-transitive, $(\mathrm{Aut}(X),s)$-regular or $(\mathrm{Aut}(X),s)$-transitive, respectively.

We denote by C_n and K_n the cycle and the complete graph of order n, respectively. Denote by D_{2n} the dihedral group of order $2n$. As we all known that there is only one connected 2-valent graph of order n, that is, the cycle C_n, which is 1-regular with full automorphism group D_{2n}. Let p be a prime. Classifying s-transitive and s-regular graphs has received considerable attention. The classification of s-transitive graphs of order p and $2p$ was given in [4] and [5], respectively. Wang [18] characterized the prime-valent s-transitive graphs of order $4p$. Feng and Kwak [10] classified cubic symmetric graphs of order $4p$ or $6p$. The classification of pentavalent and heptavalent s-transitive graphs of order $12p$ was given in [14] and [11], respectively.

For 2-valent case, s-transitivity always means 1-regularity, and for cubic case, s-transitivity always means s-regularity by Miller [9]. However, for the other prime-valent case, this is not true, see for example [12] for pentavalent case and [13] for heptavalent case. Thus, characterization and classification of prime-valent s-regular graphs is very interesting and also reveals the s-regular global and local actions of the permutation groups on s-arcs of such graphs. In particular, 1-regular action is the most simple and typical situation. In this paper, we classify prime-valent one-regular graph of order $12p$ for each prime p.

2. Preliminary results

Let X be a connected G-symmetric-transitive graph with $G \leq \mathrm{Aut}(X)$, and let N be a normal subgroup of G. The *quotient graph* X_N of X relative to N is defined as the graph with vertices the orbits of N on $V(X)$ and with two orbits adjacent if there is an edge in X between those two orbits. In view of [15, Theorem 9], we have the following:

Proposition 2.1. *Let X be a connected G-symmetric graph with $G \leq \mathrm{Aut}(X)$ and prime valency $q \geq 3$, and let N be a normal subgroup of G. Then one of the following holds:*

(1) *N is transitive on $V(X)$;*

(2) *X is bipartite and N is transitive on each part of the bipartition;*

(3) *N has $r \geq 3$ orbits on $V(X)$, N acts semiregularly on $V(X)$, the quotient graph X_N is a connected q-valent G/N-symmetric graph.*

To extract a classification of connected prime-valent symmetric graphs of order $2p$ for a prime p from Cheng and Oxley [5], we introduce the graphs $G(2p,q)$. Let V and V' be two disjoint copies of $\mathbb{Z}_p$, say $V = \{0,1,\cdots,p-1\}$ and $V' = \{0',1',\cdots,(p-1)'\}$. Let q be a positive integer dividing $p-1$ and

$H(p,q)$ the unique subgroup of Z_p^* of order q. Define the graph $G(2p,q)$ to have vertex set $V \cup V'$ and edge set $\{xy' \mid x-y \in H(p,q)\}$.

Proposition 2.2. *Let X be a connected q-valent symmetric graph of order $2p$ with p,q primes. Then X is isomorphic to K_{2p} with $q = 2p-1$, $K_{p,p}$ or $G(2p,q)$ with $q \mid (p-1)$. Furthermore, if $(p,q) \neq (11,5)$ then $\mathrm{Aut}(G(2p,q)) = (\mathbb{Z}_p \rtimes \mathbb{Z}_q) \rtimes \mathbb{Z}_2$; if $(p,q) = (11,5)$ then $\mathrm{Aut}(G(2p,q)) = \mathrm{PGL}(2,11)$.*

From [8, pp.12-14] and [17, Theorem 2], we can deduce the non-abelian simple groups whose orders have at most four different prime divisors.

Proposition 2.3. *Let p and q be two odd primes, and let G be a non-abelian simple group. If the order $|G|$ divides $2^2{\cdot}3{\cdot}p{\cdot}q$, then G is isomorphic to A_5, $\mathrm{PSL}(2,11)$ or $\mathrm{PSL}(2,13)$. If the order $|G|$ has at most three different prime divisors, then G is called K_3 simple group and isomorphic to one of the following groups.*

Table 1: **Non-abelian simple $\{2,3,p\}$-groups**

Group	Order	Group	Order
A_5	$2^2 \cdot 3 \cdot 5$	$\mathrm{PSL}(2,17)$	$2^4 \cdot 3^2 \cdot 17$
A_6	$2^3 \cdot 3^2 \cdot 5$	$\mathrm{PSL}(3,3)$	$2^4 \cdot 3^3 \cdot 13$
$\mathrm{PSL}(2,7)$	$2^3 \cdot 3 \cdot 7$	$\mathrm{PSU}(3,3)$	$2^5 \cdot 3^3 \cdot 7$
$\mathrm{PSL}(2,8)$	$2^3 \cdot 3^2 \cdot 7$	$\mathrm{PSU}(4,2)$	$2^6 \cdot 3^4 \cdot 5$

3. Classification

This section is devoted to classifying prime-valent one-regular graphs of order $12p$ for each prime p. Let q be a prime. In what follows, we always denote by X the connected q-valent one-regular graph of order $12p$. Set $A = \mathrm{Aut}(X)$, $v \in V(X)$. Then the vertex stabilizer $A_v \cong \mathbb{Z}_q$ and hence $|A| = 12pq$. Clearly, if $q = 2$, then $X \cong C_{12p}$ with $A \cong D_{24p}$. Now we deal with the case $q = 3$.

Lemma 3.1. *Suppose that $q = 3$. Then there is no cubic one-regular graph of order $12p$.*

Proof. Since $q = 3$ and X has order $12p$, by Checking the information in [7] we have that there is no such graph for $p \leq 17$. Thus, in what follows, we may assume that $p > 17$. Since $|A| = 2^2{\cdot}3^2{\cdot}p$, we have that A is solvable by Proposition 2.3. Let N be a minimal normal subgroup of A. Then N is solvable and hence $N \cong \mathbb{Z}_2$, $\mathbb{Z}_2^2$, $\mathbb{Z}_3$, $\mathbb{Z}_3^2$ or $\mathbb{Z}_p$. By Proposition 2.1, the quotient graph X_N is also a cubic symmetric graph. Note that there is no regular graph of odd order and odd valency. Thus, $N \ncong \mathbb{Z}_2^2$.

Let $N \cong \mathbb{Z}_3$. Then $|X_N| = 4p$. Recall that $p > 17$. By [10, Theorem 6.2], there is no cubic symmetric graph of order $4p$ for $p > 17$, a contradiction.

Let $N \cong \mathbb{Z}_3^2$. Then X_N is a cubic symmetric graph of order $4p$. By Proposition 2.1, N is semiregular on $V(X)$. This is impossible because $N_v \cong \mathbb{Z}_3$.

Let $N \cong \mathbb{Z}_p$. Then X_N is a cubic symmetric graph of order 12. By [7] or [6], there is no cubic symmetric graph of order 12, a contradiction.

Let $N \cong \mathbb{Z}_2$. Then X_N has order $6p$ and $|A/N| = 2{\cdot}3^2{\cdot}p$. Recall that $p > 17$. By Proposition 2.3, A/N is also solvable and has a normal Sylow p-subgroup $M/N \cong \mathbb{Z}_p$. Let P be a Sylow p-subgroup of M. Then $P \cong \mathbb{Z}_p$. Since $p > 17$, by Sylow Theorem P is characteristic in M and hence normal in A. This implies that A has a normal subgroup $P \cong \mathbb{Z}_p$. By the above argument, we also have a contradiction. □

For $q = 5$ or 7, by [14, Theorem 4.1] and [11, Theorem 3.1], it is easy to see that there is no new graph. By [6], there is no prime-valent one-regular graph of order 24. Thus, we treat with the case $p \geq 3$ and $q > 7$ by proving the following lemma.

Lemma 3.2. *Let $p \geq 3$ and $q > 7$. Then there is no new graph.*

Proof. Recall that $|A| = 12pq$, $A_v \cong \mathbb{Z}_q$, $q > 7$ and $p \geq 3$. Let N be a minimal normal subgroup of A. We divide the proof into the following two cases: $p = q$ and $p \neq q$.

Case 1: Suppose that $p = q$. Then $|A| = 12p^2$ and $A_v \cong \mathbb{Z}_p$ with $p > 7$.

Since $p > 7$, we have that A is solvable by Proposition 2.3. Let P be a Sylow p-subgroup of A. Then $|P| = p^2$. Note that $p = q > 7$. Thus, by Sylow Theorem, the number of Sylow p-subgroups of A is $kp + 1 = |A : N_A(P)|$ for some integer k. Since $|A| = 12p^2$, we have that $(kp + 1) \,|\, 12$. Suppose that P is not normal in A. Then $kp + 1 > 1$ and hence $k \geq 1$. It is easy to see that the only possible is $p = 11$ and $k = 1$. Since P is not normal in A, we have that $N \cong \mathbb{Z}_2$, $\mathbb{Z}_2^2$, $\mathbb{Z}_3$ or $\mathbb{Z}_p$. By Proposition 2.1, X_N is a p-valent symmetric graph. Note that there is no regular graph of odd order and odd valency. Thus, $N \ncong \mathbb{Z}_2^2$. If $N \cong \mathbb{Z}_2$ or $\mathbb{Z}_3$, then X_N has order $6p$ or $4p$ and $|A/N| = 6p^2$ or $4p^2$. However, $p = 11$ forces that A/N has a normal Sylow p-subgroup by Sylow Theorem. It then follows that P is normal in A, contrary to our hypothesis. If $N \cong \mathbb{Z}_p$, then X_N has order 12. Since $p = 11$, we have that $X_N \cong K_{12}$ and $A/N \lesssim \mathrm{S}_{12}$ with $|A/N| = 12{\cdot}11$. However, by Magma [3], S_{12} has no transitive subgroup of order $12{\cdot}11$, a contradiction.

Thus, P is normal in A. This means that P is the only Sylow p-subgroup of A. Since $A_v \cong \mathbb{Z}_p$, we have that $A_v \leq P$, that is, $A_v = P_v \neq 1$. By Proposition 2.1, P is transitive or has two orbits on $V(X)$. Clearly, both are impossible because $|P| = p^2$ and $|V(X)| = 12p$.

Case 2: Suppose that $p \neq q$. Then $|A| = 12pq$ and $A_v \cong \mathbb{Z}_q$ with $q > 7$.

Since $|A| = 12pq$ and $A_v \cong \mathbb{Z}_q$, we have that A_v is a Sylow q-subgroup of A. It forces that the Sylow q-subgroups of A cannot be normal in A. If A is

non-solvable, then A must contain a non-solvable composition factor H, which is isomorphic to a non-abelian simple group. It forces that $|H| \,\big|\, 12pq$ and H is a K_3 or K_4 simple group. By Proposition 2.3, $H \cong \mathrm{A}_5$, $\mathrm{PSL}(2,11)$ or $\mathrm{PSL}(2,13)$.

Let $H \cong \mathrm{A}_5$. Then $p = 5$ and $A = HA_v$. Since $|V(X)| = 12{\cdot}5$, we have H is normal in A and regular on $V(X)$. Thus, X is a normal Cayley graph on the group H. If forces that $A_v \lesssim \mathrm{Aut}(H) \cong \mathrm{Aut}(\mathrm{A}_5) \cong \mathrm{S}_5$. However, this is impossible because $|A_v| = q \not| |\mathrm{S}_5|$.

Let $H \cong \mathrm{PSL}(2,11)$ or $\mathrm{PSL}(2,13)$. By the order of H, we can deduce that $A = H$. An easy calculation implies that $p = 5$ and $q = 11$ for $A \cong \mathrm{PSL}(2,11)$, $p = 7$ and $q = 13$ for $A \cong \mathrm{PSL}(2,13)$. By Magma [3], there is only one symmetric graph admitting $\mathrm{PSL}(2,11)$ and $\mathrm{PSL}(2,13)$ as an arc-transitive automorphism group, respectively. However, its full automorphism group is $\mathrm{PGL}(2,11)$ for the former, and $\mathrm{PSL}(2,13) \times \mathbb{Z}_2$ for the latter. Both cases are not one-regular, a contradiction.

Thus, we may assume A is solvable and hence N is also solvable. It follows that $N \cong \mathbb{Z}_2$, $\mathbb{Z}_2^2$, $\mathbb{Z}_3$ or $\mathbb{Z}_p$. Clearly, $N \ncong \mathbb{Z}_2^2$ because there is no regular graph of odd order and odd valency.

Suppose that $N \cong \mathbb{Z}_p$. Then X_N is a q-valent symmetric graph of order 12 and $|A/N| = 12{\cdot}q$. Recall that $q > 7$. By [6], $X_N \cong K_{12}$, $q = 11$ and $A/N \lesssim \mathrm{S}_{12}$. However, by Magma [3], S_{12} has no transitive subgroup of order $12{\cdot}11$, a contradiction.

Suppose that $N \cong \mathbb{Z}_3$. Then X_N is a q-valent symmetric graph of order $4p$ and $|A/N| = 4pq$. Let M/N be a minimal normal subgroup of A/N. Then the solvability of A implies that M/N is also solvable. Clearly, $M/N \ncong \mathbb{Z}_2^2$. Thus, $M/N \cong \mathbb{Z}_2$ or $\mathbb{Z}_p$.

Let $M/N \cong \mathbb{Z}_p$. Then X_M is a q-valent symmetric graph of order 4 with $q > 7$, a contradiction.

Let $M/N \cong \mathbb{Z}_2$. Then X_M is a q-valent symmetric graph of order $2p$. by Proposition 2.2, $X_M \cong K_{2p}$ with $q = (2p-1)$ or $G(2p,q)$ with $q \,\big|\, (p-1)$. For the former, $A/M \lesssim \mathrm{S}_{2p}$ and $|A/M| = 2pq$. By Burnside's Theorem, any 2-transitive permutation group is almost simple or affine. Since A/M is solvable, we have that A/M is affine. It forces that A/M must have a normal subgroup isomorphic to $\mathbb{Z}_p$. A similar argument as above, we also have a contradiction. For the later, $A/M \cong (\mathbb{Z}_p \rtimes \mathbb{Z}_q) \rtimes \mathbb{Z}_2$. It is easy to see that A/M has a normal Sylow p-subgroup K/M. Note that $q > 7$ and $q \,\big|\, (p-1)$. It follows that $p > 11$. By Sylow Theorem, K has a normal Sylow p-subgroup P and hence P is characteristic in K. This forces that P is normal in A. With the similar argument as above, we deduce a contradiction.

Suppose that $N \cong \mathbb{Z}_2$. Then X_N is a q-valent symmetric graph of order $6p$ and $|A/N| = 6pq$. Let M/N be a minimal normal subgroup of A/N. Then solvability of A implies that M/N is also solvable. Similarly, $M/N \ncong \mathbb{Z}_2$ and $M/N \cong \mathbb{Z}_3$ or $\mathbb{Z}_p$.

Let $M/N \cong \mathbb{Z}_3$. Then $M \cong \mathbb{Z}_2 \times \mathbb{Z}_3$ and X_M is a q-valent symmetric graph of order $2p$. By Proposition 2.2, $X_M \cong K_{2p}$ with $q = (2p-1)$ or $G(2p, q)$ with $q \,|\, (p-1)$. The same argument as above, these two cases are impossible.

Let $M/N \cong \mathbb{Z}_p$. Then $M \cong \mathbb{Z}_2 \times \mathbb{Z}_p$. Since $p \geq 5$, we have that M has a normal Sylow p-subgroup P. It then forces that P is normal in A and X_P is a q-valent symmetric graph of order 12. Again by [6], $X_P \cong K_{12}$ and $|A/P| = 12{\cdot}11$ with $q = 11$. However, by Magma [3], S_{12} has no transitive subgroup of order $12{\cdot}11$, a contradiction. □

Combining the above arguments with the cases $q = 2, 3, 5, 7$, and by Lemmas 3.1-3.2, we have the following result.

Theorem 3.1. *Let p, q be two primes and X a connected q-valent one-regular graph of order $12p$. Then $X \cong C_{12p}$ with valency 2 and $\mathrm{Aut}(X) \cong D_{24p}$.*

4. Acknowledgements

This work was supported by the National Natural Science Foundation of China (11301154).

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Accepted: March 04, 2020

A note on alternative axiomatic results and sports applications of the precore

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Abstract. Different from related results of the precore due to Hwang et al. (2015), we characterize the precore by means of a revised reduction. Furthermore, some sports applications are also proposed.
Keywords: the precore, reduction, sports applications.

1. Introduction

Consistency and its converse are important properties of solutions in the axiomatic processes of cooperative games. Consistency allows us to deduce, from the desirability of an outcome for some problem, the desirability of its restriction to each subgroup for the associated reduced game the subgroup faces; The consistency and its converse have been investigated in several classes of frameworks by applying *reduced games*. The *core* is, perhaps, the most intuitive solution concept for allocation situations. There are several different definitions of reduced games in the axiomatic processes of the core in the literature.

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Here we focus on the *precore* proposed by Hwang et al. (2015). In the framework of *multi-choice transferable-utility (TU) games*, Hwang et al. (2015) proposed the precore. Also, Hwang et al. (2015) defined an extension of the reduction due to Davis and Maschler (1965), and adopted this extended reduction to characterize the precore on the domain of all multi-choice TU games.

Here we continue and develop related results of Hwang et al. (2015). The main results of this note are as follows.

- We introduced an extension of the reduced game due to Serrano and Volij (1998), and show that the precore is the only solution satisfying one-person rationality, related consistency and its converse on the domain of all multi-choice TU games. Finally, some sports applications are proposed.

2. Preliminaries

Let U be the universe of players. For $i \in U$ and $m_i \in \mathbb{N}$, we set $M_i = \{0, 1, \cdots, m_i\}$ as the action space of player i, where action 0 means not participating, and $M_i^+ = M_i \setminus \{0\}$. For $N \subseteq U$ with $N \neq \emptyset$, let $M^N = \prod_{i \in N} M_i$ be the product set of the action spaces for players N and $L^N = \{(i, j) \mid i \in N, j \in M_i^+\}$. Denote the zero vector in $\mathbb{R}^N$ by 0_N.

A **multi-choice TU game** [1] is a pair (N, v), where N is a non-empty and finite set of players, $m = (m_i)_{i \in N}$ is a vector that describes the number of activity levels for each player, and $v : M^N \to \mathbb{R}$ is a characteristic function which assigns to each action vector $\alpha = (\alpha_i)_{i \in N} \in M^N$ the worth that the players can obtain when each player i plays at activity level $\alpha_i \in M_i$ with $v(0_N) = 0$.

Denote the class of all multi-choice TU games by Γ. Given $(N, v) \in \Gamma$, $\alpha \in M^N$ and $T \subseteq N$, let $S(\alpha) = \{i \in N | \alpha_i \neq 0\}$, $\alpha_T \in \mathbb{R}^T$ to be the restriction of α to T, $|T|$ be the number of elements in T and $e^T(N)$ be the binary vector in $\mathbb{R}^N$ whose component $e_i^T(N)$ satisfies

$$e_i^T(N) = \begin{cases} 1, & \text{if } i \in T, \\ 0, & \text{otherwise.} \end{cases}$$

Note that if no confusion can arise $e^T(N)$ will be denoted by e^T.

Given $(N, v) \in \Gamma$. A **payoff vector** of (N, v) is a vector $(x_{i,j})_{(i,j) \in L^N} \in \mathbb{R}^{L^N}$, where $x_{i,j}$ denotes the increase in payoff to player i corresponding to a change of activity from level $j-1$ to level j by this player i for all $(i, j) \in L^N$. For convenience, we define $x_{i,0} = 0$ for all $i \in N$. For $\alpha \in M^N$, we define $x(\alpha) = \sum_{i \in S(\alpha)} \sum_{j=1}^{\alpha_i} x_{i,j}$. Let $T \subseteq N \setminus \emptyset$. For convenience, we denote $x|_T = (x_{i,j})_{(i,j) \in L^T}$.

Let $(N, v) \in \Gamma$. A payoff vector x of (N, v) satisfies

- **efficiency (EFF)** if $x(m) = v(m)$.

1. As $m^U \in \mathbb{N}^U$ is fixed throughout the note, we write (N, v) rather than (N, m_N^U, v). For convenience, if no confusion can arise m_N^U will be denoted by m throughout the note.

- **individual rationality (IR)** if for all $(i,j) \in L^N$, $x(je^{\{i\}}) \geq v(je^{\{i\}})$.
- **coalitional rationality (CR)** if for all $\alpha \in M^N$, $x(\alpha) \geq v(\alpha)$.

Moreover, the set of **feasible payoff vectors** of (N,v) is denoted by

$$X^*(N,v) = \{x \in \mathbb{R}^{L^N} \mid x(m) \leq v(m)\ \},$$

whereas

$$X(N,V) = \{x \in \mathbb{R}^{L^N} \mid x \text{ satisfies EFF } \}$$

is the set of **preimputations** of (N,V). A payoff vector x is an **imputation** of (N,v) if it satisfies EFF and IR. The set of imputations of (N,v) is denoted by $I(N,v)$.

A **solution** on Γ is a function σ which associates with each game (N,v) a subset $\sigma(N,v)$ of $X^*(N,v)$.

Definition 1 ([2]). *Let $(N,v) \in \Gamma$. The* **precore** *$C(N,v)$ consists of all preimputations x of (N,v) which satisfy CR, i.e.,*

$$C_{MC}(N,v) = \{x \in X(N,v) \mid x \text{ satisfies CR } \}.$$

3. Reductions and axiomatizations

Hwang et al. (2015) extended the reduced game introduced by Davis and Maschler (1965) to multi-choice games as follows. Given $(N,v) \in \Gamma$, $S \in 2^N \setminus \{\emptyset\}$ and a payoff vector x. The **reduced game with respect to S and x** is the game (S, v_S^x) where

$$v_S^x(\alpha) = \begin{cases} 0, & \alpha = 0_S, \\ v(m) - x(m_{N\setminus S}, 0_S), & \alpha = m_S, \\ \max\limits_{\beta \in M^{N\setminus S}} \{\ v(\alpha,\beta) - x(\beta, 0_S)\ \}, & \text{otherwise.} \end{cases}$$

Consistency requires that if x is prescribed by σ for a game (N,v), then the projection of x to S should be prescribed by a solution σ for the reduced game with respect to S and x for all S. Thus, the projection of x to S should be consistent with the expectations of the members of S as reflected by their reduced game.

- **Consistency (CON)**: For all $(N,v) \in \Gamma$, for all $S \in 2^N \setminus \{\emptyset\}$, and for all $x \in \sigma(N,v)$, $(S, v_S^x) \in \Gamma$ and $x|_S \in \sigma(S, v_S^x)$.

Converse consistency requires that if the projection of an efficient payoff vector x to every proper S is consistent with the expectations of the members of S as reflected by their reduced game then x itself should be recommended for whole game.

- **Converse consistency (CCON)**: For all $(N,v) \in \Gamma$ with $|N| \geq 2$ and for all $x \in X(N,v)$, if for all $S \subset N$ such that $0 < |S| < |N|$, $(S, v_S^x) \in \Gamma$ and $x|_S \in \sigma(S, v_S^x)$, then $x \in \sigma(N,v)$.

The following axiom is a weakening of the converse consistency property, since it requires that x satisfies individually rationality as well.

- **Weak converse consistency (WCCON)**: For all $(N,v) \in \Gamma$ with $|N| \geq 2$ and for all $x \in I(N,v)$, if for all $S \subset N$ such that $0 < |S| < |N|$, $(S, v_S^x) \in \Gamma$ and $x|_S \in \sigma(S, v_S^x)$, then $x \in \sigma(N,v)$.

We will make use of the following axioms. Let $\Gamma' \subseteq \Gamma$ and σ be a solution on Γ'. σ satisfies **efficiency (EFF)** if for all $(N,v) \in \Gamma'$, $\sigma(N,v) \subseteq X(N,v)$. σ satisfies **individually rationality (IR)** if for all $(N,v) \in \Gamma'$, $\sigma(N,v) \subseteq I(N,v)$. σ satisfies **one-person rationality (OPR)** if for all $(N,v) \in \Gamma'$ with $|N| = 1$, $\sigma(N,v) = I(N,v)$.

On the domain of all multi-choice games, inspired by Serrano and Volij (1998), Hwang et al. (2015) showed that the precore is the only solution satisfying IR, OPR, CON and WCCON. Subsequently, we provide a revised reduction to characterize the precore.

Definition 2. *Let* $(N,v) \in \Gamma$, $x \in \mathbb{R}^{L^N}$ *and* $S \in 2^N \setminus \{\emptyset\}$. *The* **revised reduced game with respect to** S **and** x *is the game* $(S, v_{S,x}^R)$ *where*

$$v_{S,x}^R(\alpha) = \begin{cases} 0, & \alpha = 0_S, \\ \max\limits_{\beta \in M^{N \setminus S}} \{v(\alpha,\beta) - x(\beta, 0_S)\}, & \textit{otherwise.} \end{cases}$$

The idea of the revised reduced game was first introduced by Serrano and Volij (1998). The only difference between this and the reduced game in Section 3 is the fact that the coalition S is also allowed to imagine potential interaction with any of the subsets of $N \setminus S$. Informally, in order to reach the maximal benefit, all the coalitions formed with the members of $N \setminus S$ should be considered to cooperate with S.

"R-reduced game" instead of "reduced game", we introduce the **R-consistency (RCON)** and the **converse R-consistency (CRCON)**.

Lemma 1. *Let* $\Gamma' \subset \Gamma$. *On* Γ', *the precore* C_{MC} *satisfies RCON.*

Proof 1. We first show that the minimal consistent consistent core satisfies RCON on Γ. Let $(N,v) \in \Gamma$, $S \subseteq N$, $S \neq \emptyset$ and $x \in C_{MC}(N,v)$.

First we show that if x satisfies CR in (N,v), then $x|_S$ satisfies CR in $(S, v_{S,x}^R)$. Since $x \in C_{MC}(N,v)$, x is CR in (N,v). For all $\alpha \in M^S$,

$$\begin{aligned} & v_{S,x}^R(\alpha) - x(\alpha) \\ &= \max_{\beta \in M^{N \setminus S}} \{\ v(\alpha,\beta) - x(\beta, 0_S)\ \} - x(\alpha, 0_{N \setminus S}) \end{aligned}$$

$$= \max_{\beta \in M^{N\setminus S}} \{\ v(\alpha,\beta) - x(\beta, 0_S) - x(\alpha, 0_{N\setminus S})\ \}$$
$$= \max_{\beta \in M^{N\setminus S}} \{\ v(\alpha,\beta) - x(\alpha,\beta)\ \}.$$

Since x satisfies CR in (N,v), $v(\alpha,\beta) - x(\alpha,\beta) \leq 0$. That is, for all $\alpha \in M^N$, $x|_S(\alpha) \geq v^R_{S,x}(\alpha)$. Hence $x|_S$ satisfies CR in $(S, v^R_{S,x})$.

Next we show that if x satisfies EFF in (N,v), then $x|_S$ satisfies EFF in $(S, v^R_{S,x})$. Since $x \in C_{MC}(N,v)$, x is EFF in (N,v). By the definition of $v^R_{S,x}$ and EFF of x in (N,v),

$$\begin{aligned}
&v^R_{S,x}(m_S) - x(m_S, 0_{N\setminus\{S\}}) \\
&= \max_{\beta \in M^{N\setminus S}} \{v(m_S,\beta) - x(\beta, 0_S)\} - x(m_S, 0_{N\setminus S}) \\
&\geq v(m) - x(m_{N\setminus S}, 0_S) - x(m_S, 0_{N\setminus S}) (\text{Take } \beta = m_{N\setminus S}) \\
&= v(m) - x(m) \\
&= 0. (\text{By} EFF of x).
\end{aligned}$$

So, $v^R_{S,x}(m_S) \geq x(m_S, 0_{N\setminus\{S\}})$. By CR of $x|_S$, $v^R_{S,x}(m_S) \leq x(m_S, 0_{N\setminus\{S\}})$. That is, $v^R_{S,x}(m_S) = x(m_S, 0_{N\setminus\{S\}})$. Hence, $x|_S$ is EFF in $(S, v^R_{S,x})$.

Lemma 2. *Let $\Gamma' \subset \Gamma$. On Γ', the precore C_{MC} satisfies CRCON.*

Proof 2. Let $\Gamma' \subset \Gamma$. Let $(N,v) \in \Gamma'$ with $|N| \geq 2$ and let $x \in X(N,v)$. Suppose for all $S \subset N$ such that $0 < |S| < |N|$, $(S, v^R_{S,x}) \in \Gamma'$ and $x_S \in C_{MC}(S, v^R_{S,x})$. We will show that $x \in C_{MC}(N,v)$, i.e., for all $\alpha \in M^N$, $x(\alpha) \geq v(\alpha)$. Let $i \in N$ and $\alpha \in M^N$, $\alpha \neq 0_N$. Consider the reduced game $(\{i\}, v^R_{\{i\},x})$. Then,

$$\begin{aligned}
&\sum_{j=1}^{\alpha_i} x_{i,j} \\
&\geq v^R_{\{i\},x}(\alpha_i) && \big(\text{By } x_i \in C\big(\{i\}, v^R_{\{i\},x}\big)\big) \\
&= \max_{\beta \in M^{N\setminus\{i\}}} \{v(\alpha_i,\beta) - x(\beta,0)\} && \big(\text{By definition of } v^R_{\{i\},x}(\alpha_i)\big) \\
&\geq v(\alpha) - x(\alpha_{N\setminus\{i\}}, 0). && \big(\text{Take } \beta = \alpha_{N\setminus\{i\}}\big)
\end{aligned}$$

Hence, $x(\alpha) \geq v(\alpha)$.

Lemma 3. *Let σ be a solution on Γ. If σ satisfies OPR and RCON, then it also satisfies EFF.*

Proof 3. Let $(N,v) \in \Gamma$, $x \in \sigma(N,v)$ and $i \in N$. Assume that $|N| = 1$. The proof is trivial since OPR implies EFF in the case $|N| = 1$. Assume that $|N| > 1$. Consider the reduced game $(\{i\}, v^R_{\{i\},x})$. By RCON of σ, $x|_{\{i\}} \in \sigma(\{i\}, v^R_{\{i\},x})$.

By OPR of σ,

$$\begin{aligned}
&\sum_{j=1}^{m_i} x_{i,j} \\
&\geq v^R_{\{i\},x}(m_i) \\
&= \max_{\beta \in M^{N\setminus\{i\}}} \{v(m_i,\beta) - x(\beta,0)\} \quad \big(\text{By definition of } v^R_{\{i\},x}(m_i)\big) \\
&\geq v(m) - x(m_{N\setminus\{i\}},0). \qquad\qquad \big(\text{Take } \beta = m_{N\setminus\{i\}}\big)
\end{aligned}$$

Therefore, $x(m) \geq v(m)$. Since σ is a solution, $x \in \sigma(N,v) \subseteq X^*(N,v)$. Hence, $x(m) = v(m)$.

Theorem 1. *On Γ, the precore is the only solution satisfying OPR, RCON and CRCON.*

Proof 4. By Lemmas 1 and 2, the precore satisfies RCON and CRCON. Clearly, the precore satisfies OPR.

To prove uniqueness, assume that a solution σ satisfies OPR, RCON and CRCON. By Lemma 3, σ satisfies EFF. Let $(N,v) \in \Gamma$. The proof proceeds by induction on the number $|N|$. If $|N| = 1$ then by OPR of σ, $\sigma(N,v) = I(N,v) = C_{MC}(N,v)$. Assume that $\sigma(N,v) = C_{MC}(N,v)$ if $|N| < k$, $k \geq 2$.

The case $|N| = k$. First we show that $\sigma(N,v) \subseteq C_{MC}(N,v)$. Let $x \in \sigma(N,v)$. Since σ satisfies EFF, $x \in X(N,v)$. By RCON of σ, for all $S \subset N$ with $0 < |S| < |N|$, $x|_S \in \sigma(S, v^R_{S,x})$. By the induction hypothesis, for all $S \subset N$ with $0 < |S| < |N|$, $x|_S \in \sigma(S, v^R_{S,x}) = C_{MC}(S, v^R_{S,x})$. By CRCON of the precore, $x \in C_{MC}(N,v)$. The opposite inclusion may be shown analogously by interchanging the roles of σ and C_{MC}. Thus, $\sigma(N,v) = C_{MC}(N,v)$.

The following examples show that each of the axioms used in Theorem 1 is logically independent of the others. [2]

Example 1. Let $\sigma(N,v) = \emptyset$ for all $(N,v) \in \Gamma$. Then σ satisfies RCON and CRCON, but it violates OPR.

Example 2. For all $(N,v) \in \Gamma$, we define a solution σ on Γ to be

$$\sigma(N,v) = \begin{cases} I(N,v), & \text{if } |N| = 1 \\ X(N,v), & \text{otherwise.} \end{cases}$$

Then σ satisfies OPR and CRCON, but it violates RCON.

Example 3. For all $(N,v) \in \Gamma$, we define a solution σ on Γ to be

$$\sigma(N,v) = \begin{cases} I(N,v), & \text{if } |N| = 1 \\ \emptyset, & \text{otherwise.} \end{cases}$$

Then σ satisfies OPR and RCON, but it violates CRCON.

2. In order to show the logical independence of the used axioms $|U| \geq 2$ is needed.

4. Sports applications

One of interesting applications could be done in the setting of "sports league", such as National Basketball Association (NBA), Major League Baseball (MLB) and so on. In a sports league, each team may take several marketing strategies to manage. Besides competing in sports games, all teams should develop to raise integral utilities of whole sports league also, such as box office, affiliated product and so on. This kind of problem can be formulated as follows: Let $N = \{1, 2, \cdots, n\}$ be a set of sports teams that can be provided jointly by some organization and let $c(\alpha)$ be the profit of providing the strategical vector α in N jointly. The function c is called a profit function. Modeled in this way, a sports league can be considered as a cooperative multi-choice TU game, with c being its characteristic function. The precore provides "equilibrium strategical utilities" among all sports teams, in the sense that this organization can get profit from each combination of strategies of all sports teams under an equilibrium utility system.

Here we assume that the family of marketing strategies are only available in finite number of levels. This is the kind of situation that we want to cover in this paper: Utility problem in which strategies can be provided at a certain finite number of levels. A multi-choice TU game is exactly an appropriate game-theoretic tool for modeling this setting. Thus, in a multi-choice TU game, each component of a payoff vector represents the strategical utilities of a particular sports team. The precore of a multi-choice TU game exactly provides the strategical utilities among all sports teams. Related applications also can be found in chemical kinetics, organization theory, optimization and so on.

Due to chances of the rise of sports competence and its additional values such as commercial profits, social resources and nation images, the arrangements of sports management systems meets have developed diversely under this context. Several researches have found that the critical factor of sports management systems successfully should be that concern of "strategies and effectiveness". More precisely, the crucial notions of success in order are drafting the conditions of "efficient strategic balance for cooperation", "stable processes", and "equivalent relationship".

In order to promote the effectiveness of sports management systems, we firstly propose some relations among the game-theoretical axioms and sports management systems for sports organizations follows.

1. *One-person rationality* asserts that *the efficiency could be promoted by balanced cooperation.*: The payoff (or utility) of each sports coalition assigned by balanced cooperation may be better than the utility of each sports coalition when it works alone.

2. *R-consistency* asserts that *the operational processes should be stable and identical*: Consistency allows us to deduce, from the desirability of an outcome for some utility allocation problem, the desirability of its restric-

tion to each coalition of a sports organization for the associated reduced condition the sports coalition faces.

3. *Converse R-consistency* asserts that *the global equivalence could be promoted by all the sub-coalitions*: Converse R-consistency permits a converse operation: to deduce the desirability of an outcome for some problem from the desirability of its restriction to each sports subgroup for the associated reduced game this sports subgroup faces. Converse consistency can be seen as a property of "decentralizability".

Based on Theorem 1, the precore is the only solution satisfying these three game-theoretical axioms. Based previous relations and statements among these game-theoretical axioms and sports management systems for sports organizations, the precore could be considered as an useful theoretical technique for utilities allocation in sports management systems. The conclusion of this note also could point out that other game-theoretical methods could be considered to applied in the field of sports management.

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Accepted: March 27, 2020

Logical entropy of partitions in hyperproduct MV-algebras

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Abstract. In this paper, we introduce the algebraic structure hyperproduct MV-algebras which is a generalization of product MV-algebras. In addition, we study the logical entropy and the logical conditional entropy of partitions in a hyperproduct MV-algebra and provide their properties.
Keywords: MV-algebras, hyperproduct MV-algebras, partitions, logical entropy.

1. Introduction

The Shannon entropy is created by Claude Elwood Shannon, an American mathematician, who is well known as the founder of the information theory. This theory is found in many applications in other areas, including statistical in-

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ference, cryptography, quantum computing and so on. The basic concept of this theory is a measurable partition of a probability space which is a measurable partition $\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$ with probabilities $p_i = P(A_i)$ where $i = 1, 2, \ldots, n$. The Shannon entropy of a measurable partition $\mathcal{A}$ is the number $h^S(\mathcal{A}) = \sum_{i=1}^{n} S(p_i)$ where $S : [0,1] \to [0,\infty)$ is the Shannon entropy function defined by

$$S(X) = \begin{cases} -x \log x, & \text{if } x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

The idea of Shannon entropy was generalized in many way, e.g. Ellerman [7] introduced the logical entropy of a measeurable partition $\mathcal{A}$ which is the number $h^l(\mathcal{A}) = \sum_{i=1}^{n} l(p_i)$, where $l : [0,1] \to [0,1]$ is the logical entropy function defined by $l(x) = x(1-x)$. Furthermore, he investigated the relations between logical entropy and the Shannon entropy. The logical entropy plays an important role as a mathematical device in many problem areas, including quantum theory, information theory, computer science, statistics and many other fields. For instant, Ebrahimzadeh introduced the notion of logical entropy of partitions and logical conditional entropy of dynamical systems in quantum logics and proved some of its properties in [4] and [5]. A year later, Ebrahimzadeh, Giski and Markechová [6] defined the logical entropy and logical mutual information of finite partitions and provided its properties. Moreover, they defined the logical entropy of a dynamical system by using the concept of entropy of partitions and proved that logical entropy of dynamical systems is invariant under isomorphisms. Markechová and Riečan defined logical entropy, logical conditional entropy, logical mutual information and logical conditional mutual information of fuzzy partitions of a fuzzy probability space, and studied logical entropy of fuzzy dynamical systems in [8] and [9].

It is well known that the classical two-valued logic, there are only two possible values (TRUE and FALSE), and it can be extended to n-valued logic where n is a natural number greater than two. One of those is the Łuksiewicz many-valued logic (cf. [15]). The notion of an MV-algebra (MV := many valued) was originally presented by Chang [3]. The structure of this algebra was explored by using an algebraic counterpart of the Łuksiewicz many-valued logic. MV-algebra is mentioned by Montagna [13] and Riečan [21] that it is a generalization of the fuzzy logic and the probability logic. Consequently, the study of MV-algebras was created, for example, derivations of MV-Algebras was studied in [1] and roughness in MV-algebras was studied in [18].

Definition 1.1 ([19]). *An* MV-algebra *is an algebraic structure* $\mathcal{A} = (A, \oplus, \otimes, \perp, 0, 1)$ *where* $\oplus$ *is a commutative and associative binary operation on a nonempty set* A, $\otimes$ *is a binary operation on* A, $\perp$ *is a unary operation on* $A, 0$ *and* 1 *belong to* A *and for any* $a, b \in A$ *satisfying the following conditions:*

(1) $a \oplus 0 = a$,
(2) $a \oplus 1 = 1$,
(3) $(a^{\perp})^{\perp} = a$,

(4) $0^{\perp} = 1$,
(5) $a \oplus a^{\perp} = 1$,
(6) $(a^{\perp} \oplus b)^{\perp} \oplus b = (a \oplus b^{\perp})^{\perp} \oplus a$,
(7) $a \otimes b = (a^{\perp} \oplus b^{\perp})^{\perp}$.

Example 1.1. Let A be the unit real interval $[0,1]$ and $a, b \in A$. Define $a \oplus b = \min\{a+b, 1\}, a \otimes b = \max\{a+b-1, 0\}$ and $a^{\perp} = 1-a$. Then the system $(A, \oplus, \otimes, \perp, 0, 1)$ is an MV-algebra.

Remind that an alelian group $(G, +)$ which is also a partially ordered set such that for all $x, y, z \in G, x+z \leq y+z$ whenever $x \leq y$, is called a commutative lattice order or simply called an abelian l-group if the partial order is a lattice, i.e., for all $a, b \in G$, there exist a least upper bound and a greatest lower bound of a and b which are denoted by $a \vee b$ and $a \wedge b$, respectively. An element u in G is a strong unit of G if for all elements a of G, there is a natural number n satisfying the condition $a \leq nu$ (cf. [2]).

In [14], Mundici generalized Chang's results [3]. He proved that for every MV-algebra $\mathcal{A} = (A, \oplus, \otimes, \perp, 0, u)$, there exists an abelian l-group L with a strong unit u such that $\mathcal{A} \cong A_0(L, u)$ where $\mathcal{A}_0(L, u) := ([0, u], \oplus, \otimes, \perp, 0, u)$ and $[0, u]$ denote the set of elements in L satisfying $0 \leq a \leq u$. We say that L is an abelian l-group corresponding to $\mathcal{A}$. Hence, any MV-algebras can be represented by an abelian l-group with a strong unit u.

Example 1.2. Let $(L, +, \leq)$ be an abelian l-group, $u \in L$ be a strong unit of L such that $u > 0$ where 0 is a neutral element of L, and $[0, u]$ denote the set of elements in L satisfying $0 \leq a \leq u$. Define $a^{\perp} = u - a, a \oplus b = (a+b) \wedge u, a \otimes b = (a+b-u) \vee 0$ and $1 = u$. Then the system $\mathcal{A}_0(L, 1) := ([0,1], \oplus, \otimes, \perp, 0, 1)$ is an MV-algebra.

Obviously, $a + b \leq u$ implies $a \oplus b = a + b$. In addition, the conditions $a \otimes b = 0$ and $a + b \leq u$ are equivalent. From now on, the operation $\oplus$ in the definition is substituted by the group operation $+$ in the abelian l-group L corresponding to an MV-algebra $\mathcal{A} = (A, \oplus, \otimes, \perp, 0, 1)$ and u is a strong unit of L.

Definition 1.2 ([17]). *Let $\mathcal{A} = (A, \oplus, \otimes, \perp, 0, 1)$ be an MV-algebra. A* partition *in $\mathcal{A}$ is an n-tuple $a = (a_1, a_2, \ldots, a_n)$ of elements of A with the property $a_1 + a_2 + \ldots + a_n = u$.*

Consequently, the study of product MV-algebras was created and many researchers have been working on it ever since.

Definition 1.3 ([20]). *A* product MV-algebra *is an algebraic structure $(\mathcal{A}, \cdot) = (A, \oplus, \otimes, \cdot, \perp, 0, 1)$ where $\mathcal{A} = (A, \oplus, \otimes, \perp, 0, 1)$ is an MV-algebra and $\cdot$ is a commutative and associative binary operation on a nonempty set A satisfying the following conditions:*

(1) *for every $x \in A, u \cdot x = x$,*

(2) *if* $x, y, z \in A$ *such that* $x + y \leq u$, *then (i)* $z \cdot (x + y) = z \cdot x + z \cdot y$ *and (ii)* $z \cdot x + z \cdot y \leq u$.

A state defined on $(\mathcal{A}, \cdot)$ plays the role of a probability measure on $\mathcal{A}$. It is investigated in [21] by Riečan.

Definition 1.4 ([23])**.** *A* state *on a product MV-algebra* $(\mathcal{A}, \cdot)$ *is a mapping* $\mu : A \to [0, 1]$ *satisfying the following properties:*

(1) $\mu(u) = 1$,

(2) *if* $x, y \in A$ *such that* $x + y \leq u$, *then* $\mu(x + y) = \mu(x) + \mu(y)$.

In 2018, Markechová, Mosapour and Ebrahimzadeh [11] studied the logical entropy, the logical divergence, and the logical mutual information in a product MV-algebra. In 2019, Markechová and Riečan [10] extended the study of logical entropy of partitions in a product MV-algebra by introducing a general type of entropy of a dynamical system in product MV-algebras. Besides, the entropy of a partition and dynamical system in product MV-algebra was studied in [16] and [22].

The concept of algebraic hyperstructures was first introduced by Marty [12] in 1934 and it has been further studied in various aspects by many authors. Let H be a nonempty set. A hyperoperation $\circ$ on H is a mapping $\circ : H \times H \to P^*(H)$ where $P^*(H)$ is the set of all nonempty subsets of H. The order pair $(H, \circ)$ is called a hypergroupoid. If A and B are two nonempty subsets of H and $x \in H$, then we denote

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b, x \circ A = \{x\} \circ A \text{ and } A \circ x = A \circ \{x\}.$$

In this paper, we introduce the hyperproduct MV-algebras and generalize some results in [11] by using the notion of the hyperproduct MV-algebra. The study of the logical entropy and the logical conditional entropy of partitions in a hyperproduct MV-algebra and their properties are provided in the next section.

2. Main results

Firstly, we define hyperproduct MV-algebras as follows:

Definition 2.1. *A* hyperproduct MV-algebra *is an algebraic structure* $(A, \oplus, \otimes, \circ, \perp, 0, 1)$ *where* $\mathcal{A} = (A, \oplus, \otimes, \perp, 0, 1)$ *is an MV-algebra and* $\circ$ *is an associative and commutative binary hyperoperation on* A *satisfying the following properties:*

(1) *for every* $x, y \in A, x \circ y$ *is a finite set,*

(2) *for every* $x \in A, u \circ x = \{x\}$,

(3) *if* $x, y, z \in A$ *such that* $x + y \leq u$, *then*

(*i*) $z \circ (x + y) = z \circ x + z \circ y$ *and* (*ii*) $a \in z \circ x + z \circ y$ *implies* $a \leq u$.

Example 2.1. Let $\mathcal{A} = (A, \oplus, \otimes, \cdot, \perp, 0, 1)$ be a product MV-algebra. Define the hyperoperation $\circ$ on A by $x \circ y = \{x \cdot y\}$ for all $x, y \in A$. Then $(A, \oplus, \otimes, \circ, \perp, 0, 1)$ is a hyperproduct MV-algebra.

It can be seen that every product MV-algebra can be considered as a hyperproduct MV-algebra. This implies that hyperproduct MV-algebra is one of generalizations of product MV-algebras. Thoughout the rest of this paper, let $(\mathcal{A}, \circ)$ denote a hyperproduct MV-algebra $(A, \oplus, \otimes, \circ, \perp, 0, 1)$.

2.1 Logical entropy of hyperproduct MV-algebras

Definition 2.2. *Let $\alpha = (a_1, a_2, \ldots, a_n)$ be a partition of a hyperproduct MV-algebra $(\mathcal{A}, \circ)$ and $\mu : A \to [0, 1]$ be a state on $(\mathcal{A}, \circ)$. The* logical entropy *of α with respect to a state μ is defined by*

$$h_\mu^l(\alpha) = \sum_{i=1}^{n} \mu(a_i)(1 - \mu(a_i)).$$

Example 2.2. Let $\mu : A \to [0, 1]$ be a state on a hyperproduct MV-algebra $(\mathcal{A}, \circ)$. The following statements hold true.

(1) If we put $\alpha_1 = (u)$, then α_1 is a partition of $(\mathcal{A}, \circ)$ and $h_\mu^l(\alpha_1) = 0$.

(2) Let $a \in \mathcal{A}$ be such that $\mu(a) \in (0, 1)$. It is obvious that the order pair $\alpha_2 = (a, u - a)$ is a partition of $(\mathcal{A}, \circ)$ and $h_\mu^l(\alpha_2) = 2\mu(a)(1 - \mu(a))$.

Proposition 2.1. *Let $\alpha = (a_1, a_2, \ldots, a_n)$ be a partition of a hyperproduct MV-algebra $(\mathcal{A}, \circ)$. If $\mu : \mathcal{A} \to [0, 1]$ is a state on $(\mathcal{A}, \circ)$, then*

$$h_\mu^l(\alpha) = 1 - \sum_{i=1}^{n} (\mu(a_i))^2.$$

Proof. Since $\sum_{i=1}^{n} \mu(a_i) = \mu(\sum_{i=1}^{n}(a_i)) = \mu(u) = 1$, we have

$$\begin{aligned} h_\mu^l(\alpha) &= \sum_{i=1}^{n} \mu(a_i)(1 - \mu(a_i)) = \sum_{i=1}^{n} \mu(a_i) - (\mu(a_i))^2 \\ &= \sum_{i=1}^{n} \mu(a_i) - \sum_{i=1}^{n} (\mu(a_i))^2 = 1 - \sum_{i=1}^{n} (\mu(a_i))^2. \end{aligned}$$

□

Proposition 2.2. *Let $\alpha = (a_1, a_2, \ldots, a_n)$ be a partition of a hyperproduct MV-algebra $(\mathcal{A}, \circ)$. If $\mu : A \to [0, 1]$ is a state on $(\mathcal{A}, \circ)$ and $b \in A$, then*

$$\mu(b) = \sum_{i=1}^{n} \sum_{x \in a_i \circ b} \mu(x).$$

Proof. Let $b \in A$. Then $\{b\} = u \circ b$. Hence, $\mu(b) = \sum_{y \in \{b\}} \mu(y) = \sum_{y \in u \circ b} \mu(y) = \sum_{y \in (\sum_i^n a_i) \circ b} \mu(y) = \sum_{y \in \sum_i^n (a_i \circ b)} \mu(y) = \sum_{i=1}^n \sum_{x \in a_i \circ b} \mu(x)$. □

Let $\alpha = (a_1, a_2, \ldots, a_n)$ and $\beta = (b_1, b_2, \ldots b_m)$ be partitions of $(\mathcal{A}, \circ)$. We define the partition $\alpha \vee \beta$ of $(\mathcal{A}, \circ)$ as follows:

$$\alpha \vee \beta = (x \mid x \in a_i \circ b_j \exists i \in \{1, 2, \ldots, n\} \text{ and } \exists j \in \{1, 2, \ldots, m\}).$$

Proposition 2.3. *Let $\alpha = (a_1, a_2, \ldots, a_n)$ and $\beta = (b_1, b_2, \ldots b_m)$ be partitions of a hyperproduct MV-algebra $(\mathcal{A}, \circ)$. Then $\alpha \vee \beta$ is a partition of $(\mathcal{A}, \circ)$.*

Proof. Since $\{\sum_{i=1}^n \sum_{j=1}^m \sum_{x \in a_i \circ b_j} x\} = \sum_{i=1}^n \sum_{j=1}^m a_i \circ b_j = (\sum_{i=1}^n a_i) \circ (\sum_{j=1}^m b_j) = u \circ u = \{u\}$, we have $\sum_{i=1}^n \sum_{j=1}^m \sum_{x \in a_i \circ b_j} x = u$ which implies that $\alpha \vee \beta$ is a partition of $(\mathcal{A}, \circ)$. □

In the system of all partitions of a hyperproduct MV-algebra $(\mathcal{A}, \circ)$, we define the *refinement partial order* $\succ$ as follows:

Definition 2.3. *Let $\alpha = (a_1, a_2, \ldots, a_n)$ and $\beta = (b_1, b_2, \ldots, b_m)$ be partitions of a hyperproduct MV-algebra $(\mathcal{A}, \circ)$. We say that β is a* refinement *of α which is denoted by $\beta \succ \alpha$, if for each $i = 1, 2, \ldots, n$, there exists a subset $I(i)$ of a set $\{1, 2, \ldots, m\}$ such that $a_i = \sum_{j \in I(i)} b_j$.*

Proposition 2.4. *Let $\alpha = (a_1, a_2, \ldots, a_n)$ and $\beta = (b_1, b_2, \ldots b_m)$ be partitions of a hyperproduct MV-algebra $(\mathcal{A}, \circ)$. Then $\alpha \vee \beta \succ \alpha$.*

Proof. By the definition of hyperproducts, $\{a_i\} = a_i \circ u = a_i \circ (\sum_{j=1}^m b_j) = \sum_{j=1}^m a_i \circ b_j$, for $i = 1, 2, \ldots, n$. Hence, $a_i = \sum_{j=1}^m \sum_{x \in a_i \circ b_j} x$, so $\alpha \vee \beta \succ \alpha$. □

2.2 Logical conditional entropy of partitions in hyperproduct MV-algebras

Definition 2.4. *Let $\alpha = (a_1, a_2, \ldots, a_n)$ and $\beta = (b_1, b_2, \ldots, b_m)$ be partitions of a hyperproduct MV-algebra $(\mathcal{A}, \circ)$ and $\mu : A \to [0, 1]$ be a state on $(\mathcal{A}, \circ)$. The* logical conditional entropy *of α given β is defined by*

$$h_\mu^l(\alpha/\beta) = \sum_{i=1}^n \sum_{j=1}^m \sum_{x \in a_i \circ b_j} \mu(x)(\mu(b_j) - \mu(x)).$$

Proposition 2.5. *If $\alpha = (a_1, a_2, \ldots, a_n)$ and $\beta = (b_1, b_2, \ldots, b_m)$ are partitions of a hyperproduct MV-algebra $(\mathcal{A}, \circ)$ and $\mu : A \to [0, 1]$ is a state on $(\mathcal{A}, \circ)$, then*

$$h_\mu^l(\alpha/\beta) = \sum_{j=1}^m (\mu(b_j))^2 - \sum_{i=1}^n \sum_{j=1}^m \sum_{x \in a_i \circ b_j} (\mu(x))^2.$$

Proof. By Proposition 2.2, $\mu(b_j) = \sum_{i=1}^{n} \sum_{x \in a_i \circ b_j} \mu(x)$. Then

$$\begin{aligned} h_\mu^l(\alpha/\beta) &= \sum_{i=1}^{n}\sum_{j=1}^{m} \sum_{x \in a_i \circ b_j} \mu(x)(\mu(b_j) - \mu(x)) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} \sum_{x \in a_i \circ b_j} \mu(x)\mu(b_j) - (\mu(x))^2 \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} \sum_{x \in a_i \circ b_j} \mu(x)\mu(b_j) - \sum_{i=1}^{n}\sum_{j=1}^{m} \sum_{x \in a_i \circ b_j} (\mu(x))^2 \\ &= \sum_{j=1}^{m} (\mu(b_j))^2 - \sum_{i=1}^{n}\sum_{j=1}^{m} \sum_{x \in a_i \circ b_j} (\mu(x))^2. \end{aligned}$$

□

Note that the logical conditional entropy $h_\mu^l(\alpha/\beta)$ is always nonnegative. It can be easily verified that $h_\mu^l(\alpha/\beta) = h_\mu^l(\alpha)$ if the partition $\beta = (u)$.

Theorem 2.1. *Let α, β be two arbitrary partitions of a hyperproduct MV-algebra $(\mathcal{A}, \circ)$ and $\mu : A \to [0,1]$ be a state on $(\mathcal{A}, \circ)$. Then*

$$h_\mu^l(\alpha \vee \beta) = h_\mu^l(\alpha) + h_\mu^l(\beta/\alpha).$$

Proof. Let $\alpha = (a_1, a_2, \ldots, a_n)$ and $\beta = (b_1, b_2, \ldots, b_m)$ be partitions of $(\mathcal{A}, \circ)$. By Proposition 2.1 and Proposition 2.5,

$$\begin{aligned} h_\mu^l(\alpha) + h_\mu^l(\beta/\alpha) &= [1 - \sum_{i=1}^{n} (\mu(a_i))^2] + [\sum_{i=1}^{n} (\mu(a_i))^2 - \sum_{j=1}^{m}\sum_{i=1}^{n} \sum_{x \in b_j \circ a_i} (\mu(x))^2] \\ &= 1 - \sum_{i=1}^{n}\sum_{j=1}^{m} \sum_{x \in a_i \circ b_j} (\mu(x))^2 = h_\mu^l(\alpha \vee \beta). \end{aligned}$$

□

Corollary 2.1. *Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be partitions of a hyperproduct MV-algebra $(\mathcal{A}, \circ)$. Then*

$$h_\mu^l(\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_n) = h_\mu^l(\alpha_1) + \sum_{i=2}^{n} h_\mu^l(\alpha_i/(\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_{i-1})).$$

Proof. By Theorem 2.1,

$$\begin{aligned}
&h_\mu^l(\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_n)\\
&= h_\mu^l(\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_{n-1}) + h_\mu^l(\alpha_n/(\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_{n-1}))\\
&= [h_\mu^l(\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_{n-2}) + h_\mu^l(\alpha_{n-1}/(\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_{n-2}))]\\
&\quad + h_\mu^l(\alpha_n/(\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_{n-1}))\\
&= h_\mu^l(\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_{n-2}) + \sum_{i=n-1}^{n} h_\mu^l(\alpha_i/(\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_{i-1}))\\
&\qquad\qquad \vdots\\
&= h_\mu^l(\alpha_1) + \sum_{i=2}^{n} h_\mu^l(\alpha_i/(\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_{i-1})).
\end{aligned}$$

□

Theorem 2.2. *For arbitrary partitions α, β, γ of a hyperproduct MV-algebra $(\mathcal{A}, \circ)$ and a state $\mu : A \to [0,1]$ on $(\mathcal{A}, \circ)$,*

$$h_\mu^l((\alpha \vee \beta)/\gamma) = h_\mu^l(\alpha/\gamma) + h_\mu^l(\beta/(\alpha \vee \gamma)).$$

Proof. Let $\alpha = (a_1, a_2, \ldots, a_n)$, $\beta = (b_1, b_2, \ldots, b_m)$ and $\gamma = (c_1, c_2, \ldots, c_r)$ be partitions of $(\mathcal{A}, \circ)$. By Proposition 2.5,

$$\begin{aligned}
h_\mu^l(\alpha/\gamma) + h_\mu^l(\beta/\alpha \vee \gamma) &= [\sum_{k=1}^{r}(\mu(c_k))^2 - \sum_{i=1}^{n}\sum_{k=1}^{r}\sum_{x \in a_i \circ c_k}(\mu(x))^2]\\
&\quad + [\sum_{i=1}^{n}\sum_{k=1}^{r}\sum_{x \in a_i \circ c_k}(\mu(x))^2 - \sum_{j=1}^{m}\sum_{i=1}^{n}\sum_{k=1}^{r}\sum_{x \in b_j \circ a_i \circ c_k}(\mu(x))^2]\\
&= \sum_{k=1}^{r}(\mu(c_k))^2 - \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{r}\sum_{x \in a_i \circ b_j \circ c_k}(\mu(x))^2 = h_\mu^l((\alpha \vee \beta)/\gamma).
\end{aligned}$$

□

Corollary 2.2. *Let $\alpha_1, \alpha_2, \ldots, \alpha_n, \gamma$ be partitions of a hyperproduct MV-algebra $(\mathcal{A}, \circ)$ and $\mu : A \to [0,1]$ be a state on $(\mathcal{A}, \circ)$. Then*

$$h_\mu^l((\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_n)/\gamma) = h_\mu^l(\alpha_1/\gamma) + \sum_{i=2}^{n} h_\mu^l(\alpha_i/(\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_{i-1} \vee \gamma)).$$

Theorem 2.3. *For arbitrary partitions α, β of a hyperproduct MV-algebra $(\mathcal{A}, \circ)$ and $\mu : A \to [0,1]$ is a state on $(\mathcal{A}, \circ)$, each of the following statements hold true:*

(i) $h_\mu^l(\alpha/\beta) \leq h_\mu^l(\alpha)$,

(*ii*) $h^l_\mu(\alpha \vee \beta) \leq h^l_\mu(\alpha) + h^l_\mu(\beta)$.

Proof. Let $\alpha = (a_1, a_2, \dots a_n)$ and $\beta = (b_1, b_2, \dots, b_m)$ be partitions of $(\mathcal{A}, \circ)$.

(i) By Proposition 2.2, $\mu(a_i) = \sum_{j=1}^m \sum_{x \in a_i \circ b_j} \mu(x)$ for $i = 1, 2, \dots, n$. Then we obtain that

$$\sum_{j=1}^m \sum_{x \in a_i \circ b_j} \mu(x)(\mu(b_j) - \mu(x)) \leq [\sum_{j=1}^m \sum_{x \in a_i \circ b_j} \mu(x)][\sum_{j=1}^m \sum_{x \in a_i \circ b_j} (\mu(b_j) - \mu(x))]$$
$$= \mu(a_i)[\sum_{j=1}^m \mu(b_j) - \sum_{j=1}^m \sum_{x \in a_i \circ b_j} \mu(x)] = \mu(a_i)(1 - \mu(a_i)).$$

Hence, $h^l_\mu(\alpha/\beta) = \sum_{i=1}^n \sum_{j=1}^m \sum_{x \in a_i \circ b_j} \mu(x)(\mu(b_j) - \mu(x)) \leq \sum_{i=1}^n \mu(a_i)(1 - \mu(a_i)) = h^l_\mu(\alpha)$.

(ii) By using Theorem 2.1 and (i). □

Theorem 2.4. *For arbitrary partitions α, β of a hyperproduct MV-algebra $(\mathcal{A}, \circ)$ and a state $\mu : A \to [0, 1]$ on $(\mathcal{A}, \circ)$, each of the following statements hold true:*

(*i*) $\beta \succ \alpha$ *implies* $h^l_\mu(\beta) \geq h^l_\mu(\alpha)$,

(*ii*) $h^l_\mu(\alpha \vee \beta) \geq \max\{h^l_\mu(\alpha), h^l_\mu(\beta)\}$.

Proof. (i) Let $\alpha = (a_1, a_2, \dots, a_n)$ and $\beta = (b_1, b_2, \dots, b_m)$ be partitions of $(\mathcal{A}, \circ)$. By assumption, for each $i = 1, 2, \dots, n$, there exists a subset $I(i)$ of a set $\{1, 2, \dots, m\}$ such that $a_i = \sum_{j \in I(i)} b_j$ for each $i = 1, 2, \dots, n$. Therefore, $h^l_\mu(\alpha) = 1 - \sum_{i=1}^n (\mu(a_i))^2 = 1 - \sum_{i=1}^n (\mu(\sum_{j \in I(i)} b_j))^2 = 1 - \sum_{i=1}^n (\sum_{j \in I(i)} \mu(b_j))^2 \leq 1 - \sum_{i=1}^n \sum_{j \in I(i)} (\mu(b_j))^2 = 1 - \sum_{j=1}^m (\mu(b_j))^2 = h^l_\mu(\beta)$.

(ii) According to Proposition 2.4, it holds $\alpha \vee \beta \succ \alpha$, and $\alpha \vee \beta \succ \beta$. Then the property (ii) is a direct consequence of the property (i). □

Acknowledgment

This paper was supported by Algebra and Applications Research Unit, Prince of Songkla University.

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Accepted: June 24, 2020

On some properties of divisible and pure fuzzy multigroups

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Abstract. Many group's theoretic notions have been established in fuzzy algebra although with some modifications. The theory of fuzzy multigroups is the study of group theory in the context of fuzzy multisets. In this paper we propose and characterize the ideas of divisible and pure fuzzy multigroups with a number of results which are duly proved. It is shown that the homomorphic image and preimage of divisible and pure fuzzy multigroups are as well divisible and pure fuzzy multigroups. The relationship between divisible fuzzy multigroups and divisible groups as well as between pure fuzzy multigroups and pure groups are instituted using the idea of alpha-cuts. Finally, it is established that a fuzzy multigroup of a divisible cyclic group is constant.

Keywords: fuzzy algebra, fuzzy multigroup, divisible fuzzy multigroup, pure fuzzy multigroup.

1. Introduction

The concept of fuzziness was first studied by Zadeh [29] to curb imprecisions in real-life. After the introduction of fuzzy sets, Rosenfeld [22] orchestrated the study of fuzziness in algebra called fuzzy algebra. Myriad of researchers have extended some group's theoretic notions to fuzzy sets and in so doing, established the theory of fuzzy groups [3, 19, 20]. By a way of generalization, Yager [27] proposed fuzzy multiset as a fuzzy set that allowed the repetitions of the membership functions of elements of a universe of discourse. The idea of fuzzy multisets have been extensively studied and applied to many real-life problems as can be seen in [2, 16, 17, 18, 25].

In a way of application, Shinoj et al. [23] applied group theory to fuzzy multisets to introduce a generalized fuzzy groups called fuzzy multigroups and deduced some related results. As a follow up, Ejegwa[4] studied the analog of subgroups in fuzzy multigroup context and introduced centers and centralizers in fuzzy multigroup context. The idea of commutative fuzzy multigroups have been studied and a number of results were presented [1, 5]. Some group's analog notions like normal subgroups, characteristic subgroups, Frattini subgroups, cosets, quotient groups and homomorphism have been instituted in fuzzy multigroup environment with a number of results [6, 7, 11, 13, 14, 15]. The concept of direct product of fuzzy multigroups and its generalization were discussed in [8, 12]. To enhance a nexus between fuzzy multigroup and group theories, the concept of alpha-cuts of fuzzy multigroups was proposed [9] and the idea of its homomorphism was studied in [10].

Although several concepts of group theory have been extended to fuzzy multigroups via fuzzy multisets, some notions have not been investigated in fuzzy multigroup context. This paper seeks to strengthen fuzzy multigroup theory by characterizing the constructs of divisible and pure fuzzy multigroups which are the applications of divisible and pure groups in fuzzy multisets. The notions of divisible and pure groups have been hitherto studied in fuzzy group and intuitionistic fuzzy group contexts [24, 26, 28]. In this voyage, we first and foremost defined a special fuzzy multisets of a group to enhance the introduction of divisible and pure fuzzy multigroups. We study the homomorphic image

and preimage of divisible and pure fuzzy multigroups and show that they are as well divisible and pure fuzzy multigroups. The relationship between divisible and pure fuzzy multigroups with divisible and pure groups are instituted using the idea of alpha-cuts. The rest of the paper is delineated as follows; Section 2 presents the ideas of fuzzy multisets, fuzzy multigroups and some existing results. Section 3 discusses and characterizes divisible fuzzy multigroups and Section 4 explicates pure fuzzy multigroups with related results. Finally, Section 5 summarizes and gives scope of future studies.

2. Preliminaries

Throughout this paper X denotes non-empty set and G denotes an additive group with identity element 0.

Definition 2.1 ([27]). *Assume X is a non-empty set. Then, a fuzzy bag/multiset $\mathtt{A}$ drwan from X is an object of the form*

$$\mathtt{A} = \{\langle \frac{CM_{\mathtt{A}}(x)}{x} \rangle \mid x \in X\}$$

characterized by a count membership function $CM_{\mathtt{A}}$ such that

$$CM_{\mathtt{A}} \colon X \to Q,$$

where Q is the set of all crisp bags or multisets from the unit interval $I = [0, 1]$ and

$$CM_{\mathtt{A}}(x) = \{\mu^1_{\mathtt{A}}(x), \mu^2_{\mathtt{A}}(x), \ldots, \mu^n_{\mathtt{A}}(x), \ldots\},$$

such that $\mu^1_{\mathtt{A}}(x) \geq \mu^2_{\mathtt{A}}(x) \geq \ldots \geq \mu^n_{\mathtt{A}}(x) \geq \ldots$, whereas in a finite case, we write

$$CM_{\mathtt{A}}(x) = \{\mu^1_{\mathtt{A}}(x), \mu^2_{\mathtt{A}}(x), \ldots, \mu^n_{\mathtt{A}}(x)\},$$

for $\mu^1_{\mathtt{A}}(x) \geq \mu^2_{\mathtt{A}}(x) \geq \ldots \geq \mu^n_{\mathtt{A}}(x)$.

A fuzzy multiset can also be characterized by a high-order function. In particular, a fuzzy multiset $\mathtt{A}$ can be characterized by a function

$$CM_{\mathtt{A}} \colon X \to N^I \; or \; CM_{\mathtt{A}} \colon X \to [0, 1] \to N,$$

where $I = [0, 1]$ and $N = \mathbb{N} \cup \{0\}$.

Definition 2.2 ([27]). *Let $\mathtt{A}$, $\mathtt{B}$ be fuzzy multisets of X. Then for any $x \in X$, we have*

(*i*) $\mathtt{A} = \mathtt{B} \iff CM_{\mathtt{A}}(x) = CM_{\mathtt{B}}(x)$,

(*ii*) $\mathtt{A} \subseteq \mathtt{B} \iff CM_{\mathtt{A}}(x) \leq CM_{\mathtt{B}}(x)$,

(*iii*) $\mathtt{A} \cap \mathtt{B} \implies CM_{\mathtt{A} \cap \mathtt{B}}(x) = CM_{\mathtt{A}}(x) \wedge CM_{\mathtt{B}}(x)$,

(*iv*) $\mathtt{A} \cup \mathtt{B} \Longrightarrow CM_{\mathtt{A}\cup\mathtt{B}}(x) = CM_{\mathtt{A}}(x) \vee CM_{\mathtt{B}}(x)$,

(*v*) $\mathtt{A} \oplus \mathtt{B} \Longrightarrow CM_{\mathtt{A}\oplus\mathtt{B}}(x) = CM_{\mathtt{A}}(x) \oplus CM_{\mathtt{B}}(x)$,

where $\wedge$ *and* $\vee$ *denote minimum and maximum respectively.*

Suppose $\{\mathtt{A}_i\}_{i\in I}$ is an arbitrary family of fuzzy multisets of X. Then

$$\bigcap_{i\in I} \mathtt{A}_i = \bigwedge_{i\in I} CM_{\mathtt{A}_i}(x),\ \forall x \in X \text{ and } \bigcup_{i\in I} \mathtt{A}_i = \bigvee_{i\in I} CM_{\mathtt{A}_i}(x),\ \forall x \in X.$$

Definition 2.3 ([23]). *A fuzzy multiset* $\mathtt{A}$ *of* G *is called a fuzzy multigroup if and only if*

(*i*) $CM_{\mathtt{A}}(x+y) \geq CM_{\mathtt{A}}(x) \wedge CM_{\mathtt{A}}(y),\ \forall x, y \in G$,

(*ii*) $CM_{\mathtt{A}}(-x) \geq CM_{\mathtt{A}}(x),\ \forall x \in G$.

Certainly, $CM_{\mathtt{A}}(-x) = CM_{\mathtt{A}}(x),\ \forall x \in G$, *since*

$$CM_{\mathtt{A}}(x) = CM_{\mathtt{A}}(-(-x)) \geq CM_{\mathtt{A}}(-x).$$

Conditions (*i*) *and* (*ii*) *can be combined as* $CM_{\mathtt{A}}(x-y) \geq CM_{\mathtt{A}}(x) \wedge CM_{\mathtt{A}}(y)$ *for all* $x, y \in G$.

A fuzzy multigroup $\mathtt{A}$ *of* G *is said to be constant or has a constant count membership function if* $CM_{\mathtt{A}}(x) = CM_{\mathtt{A}}(y),\ \forall x, y \in G$. *In fact, in a fuzzy multigroup* $\mathtt{A}$ *of* G, $CM_{\mathtt{A}}(0) \geq CM_{\mathtt{A}}(x),\ \forall x \in G$.

Definition 2.4 ([5]). *A fuzzy multigroup* $\mathtt{A}$ *of* G *is said to be commutative if* $CM_{\mathtt{A}}(x+y) = CM_{\mathtt{A}}(y+x),\ \forall x, y \in G$.

Proposition 2.1 ([4]). *If* $\mathtt{A}$ *and* $\mathtt{B}$ *are fuzzy multigroups of* G, *then* (*i*) $\mathtt{A}\cap\mathtt{B}$ *and* $\mathtt{A} \oplus \mathtt{B}$ *are fuzzy multigroups of* G, *(ii)* $\mathtt{A} \cup \mathtt{B}$ *is a fuzzy multigroup of* G *provided* $\mathtt{A} \subseteq \mathtt{B}$.

Theorem 2.1 ([10]). *Let* $\mathtt{A}$ *be a fuzzy multigroup of* G. *Then, the sets* $\mathtt{A}_\alpha$ *and* $\mathtt{A}^\alpha$ *defined by* $\mathtt{A}_\alpha = \{x \in G \mid CM_{\mathtt{A}}(x) \geq \alpha\}$ *and* $\mathtt{A}^\alpha = \{x \in G \mid CM_{\mathtt{A}}(x) \leq \alpha\}$ *where* $\alpha \in [0,1]$, *are subgroups of* G *for* $\alpha \leq CM_{\mathtt{A}}(0)$ *and* $\alpha \geq CM_{\mathtt{A}}(0)$, *respectively.*

Theorem 2.2 ([10]). *Let* $\mathtt{A}$ *be a fuzzy multiset of* G *for* $CM_{\mathtt{A}}(0) = 1$ *and* $\alpha \in [0,1]$.

(*i*) *If every* $\mathtt{A}_\alpha$ *is a subgroup of* G, *then* $\mathtt{A}$ *is a fuzzy multigroup of* G.

(*ii*) *If every* $\mathtt{A}^\alpha$ *is a subgroup of* G, *then* $\mathtt{A}$ *is a fuzzy multigroup of* G.

Definition 2.5 ([7]). *Let* $f : G \to G'$ *be a homomorphism of groups. Suppose* $\mathtt{A}$ *and* $\mathtt{B}$ *are fuzzy multigroups of* G *and* G', *respectively. Then,* f *induces a homomorphism from* $\mathtt{A}$ *to* $\mathtt{B}$ *which satisfies*

(*i*) $CM_{\mathtt{A}}(f^{-1}(y_1y_2)) \geq CM_{\mathtt{A}}(f^{-1}(y_1)) \wedge CM_{\mathtt{A}}(f^{-1}(y_2))\ \forall y_1, y_2 \in G'$,

(*ii*) $CM_{\mathtt{B}}(f(x_1x_2)) \geq CM_{\mathtt{B}}(f(x_1)) \wedge CM_{\mathtt{B}}(f(x_2))\ \forall x_1, x_2 \in G$,

where

(*i*) *the image of* $\mathtt{A}$ *under* f*, denoted by* $f(\mathtt{A})$*, is a fuzzy multiset of* G' *defined by*

$$CM_{f(\mathtt{A})}(y) = \begin{cases} \bigvee_{x \in f^{-1}(y)} CM_{\mathtt{A}}(x), & f^{-1}(y) \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

for each $y \in G'$.

(*ii*) *the preimage of* $\mathtt{B}$ *under* f*, denoted by* $f^{-1}(\mathtt{B})$*, is a fuzzy multiset of* G *defined by* $CM_{f^{-1}(\mathtt{B})}(x) = CM_{\mathtt{B}}(f(x))$, $\forall x \in G$.

Proposition 2.2 ([7])**.** *Let* $f\colon G \to G'$ *be a homomorphism of groups,* $\mathtt{A}$ *and* $\mathtt{B}$ *be fuzzy multigroups of* G *and* G'*, respectively. Then* $f(\mathtt{A})$ *is fuzzy multigroup of* G' *and if* f *is an isomorphism, then* $f^{-1}(\mathtt{B})$ *is a fuzzy multigroup of* G.

3. Divisible fuzzy multigroups

In a classical sense, a divisible group is an abelian group where every element is an *nth* multiple for each positive integer n. An abelian group G is divisible if for every positive integer n and any $x \in G$, there exists $y \in G$ such that $ny = x$. Equivalently, an abelian group G is divisible if and only if $nG = G$, where n is any positive integer and $nG = \{nx | x \in G\}$.

Definition 3.1. Suppose $\mathtt{A}$ is a fuzzy multiset of G. Then $n\mathtt{A}$, where n is a positive integer is defined by

$$CM_{n\mathtt{A}}(x) = \begin{cases} \bigvee_{x=ny} CM_{\mathtt{A}}(y), & x \in nG \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 3.1. *If* $\mathtt{A}$ *and* $\mathtt{B}$ *are fuzzy multisets of* G*, then*

(*i*) $\mathtt{A} \subseteq \mathtt{B} \Longrightarrow n\mathtt{A} \subseteq n\mathtt{B}$,

(*ii*) $n(\mathtt{A} \cap \mathtt{B}) = n\mathtt{A} \cap n\mathtt{B}$,

(*iii*) $n(\mathtt{A} \cup \mathtt{B}) = n\mathtt{A} \cup n\mathtt{B}$,

(*iv*) $n(\mathtt{A} \oplus \mathtt{B}) = n\mathtt{A} \oplus n\mathtt{B}$.

Proof. The proof of (i) is straightforward. Now, we prove (ii) as follows. Suppose $x \notin nG$, then $CM_{n(\mathtt{A}\cap\mathtt{B})}(x) = 0 = CM_{n\mathtt{A}\cap n\mathtt{B}}(x)$. Again, assume $x \in nG$,

then we have

$$\begin{aligned} CM_{n\mathtt{A}\cap n\mathtt{B}}(x) &= CM_{n\mathtt{A}}(x) \wedge CM_{n\mathtt{B}}(x) \\ &= \bigvee_{x=ny} CM_{\mathtt{A}}(y) \wedge \bigvee_{x=ny} CM_{\mathtt{B}}(y) \\ &= \bigvee_{x=ny} [CM_{\mathtt{A}}(y) \wedge CM_{\mathtt{B}}(y)] \\ &= \bigvee_{x=ny} CM_{\mathtt{A}\cap\mathtt{B}}(y) \\ &= CM_{n(\mathtt{A}\cap\mathtt{B})}(x). \end{aligned}$$

Hence, $n(\mathtt{A} \cap \mathtt{B}) = n\mathtt{A} \cap n\mathtt{B}$.

The proofs of (iii) and (iv) follow directly from Definitions 2.2, 3.1 and (ii). □

Theorem 3.1. *Let $f\colon G \to G'$ be a homomorphism such that $\mathtt{A}$ is fuzzy multiset of G. Then for any $n \in \mathbb{N}$, $f(n\mathtt{A}) = nf(\mathtt{A})$.*

Proof of Theorem 3.1. Suppose $x \notin nG$, then $CM_{f(n\mathtt{A})}(x) = 0 = CM_{nf(\mathtt{A})}(x)$. Again, suppose we have $x \in nG$. If $z \in G'$ and $z = f(x)$, then

$$\begin{aligned} CM_{f(n\mathtt{A})}(z) &= \bigvee_{z=f(x)} CM_{n\mathtt{A}}(x) = \bigvee_{z=f(x)} \bigvee_{x=nw} CM_{\mathtt{A}}(w) \\ &= \bigvee_{z=f(x),\, x=nw} CM_{\mathtt{A}}(w) = \bigvee_{z=f(nw)} CM_{\mathtt{A}}(w) \\ &= \bigvee_{z=nf(w)} CM_{\mathtt{A}}(w) = \bigvee_{z=ny} \bigvee_{y=f(w)} CM_{\mathtt{A}}(w) \\ &= \bigvee_{z=f(nw)} CM_{f(\mathtt{A})}(y) = CM_{nf(\mathtt{A})}(z). \end{aligned}$$

Hence, $f(n\mathtt{A}) = nf(\mathtt{A})$.

Theorem 3.2. *Suppose f is a homomorphism from G to G' such that $\mathtt{B}$ is a fuzzy multiset of G'. Then for any $n \in \mathbb{N}$, $nf^{-1}(\mathtt{B}) \subseteq f^{-1}(n\mathtt{B})$. Moreover if f is an isomorphism, $nf^{-1}(\mathtt{B}) = f^{-1}(n\mathtt{B})$.*

Proof of Theorem 3.2. Let $x \in G$. If $x \notin nG$, then $CM_{nf^{-1}(\mathtt{B})}(x) = 0 \leq CM_{f^{-1}(n\mathtt{B})}(x)$. If $x \in nG$, then $f(x) \in nG'$. Thus

$$\begin{aligned} CM_{nf^{-1}(\mathtt{B})}(x) &= \bigvee_{x=nw} CM_{f^{-1}(\mathtt{B})}(w) = \bigvee_{x=nw} CM_{\mathtt{B}}(f(w)) \\ &\leq \bigvee_{f(x)=f(nw)} CM_{\mathtt{B}}(f(w)) \leq \bigvee_{f(x)=ny} CM_{\mathtt{B}}(y) \\ &= CM_{\mathtt{nB}}(ny) = CM_{\mathtt{nB}}(f(x)) \\ &= CM_{f^{-1}\mathtt{B}}(x). \end{aligned}$$

Hence, $nf^{-1}(\mathtt{B}) \subseteq f^{-1}(n\mathtt{B})$.

Suppose f is an isomorphism, then by using the logic in Theorem 3.1, we have

$$CM_{f^{-1}(n\mathtt{B})}(x) = CM_{f^{-1}(n(f(f^{-1}(\mathtt{B}))))}(x) = CM_{f^{-1}(f(n(f^{-1}(\mathtt{B}))))}(x) \leq CM_{n(f^{-1}(\mathtt{B}))}(x),$$

so $f^{-1}(n\mathtt{B}) \subseteq nf^{-1}(\mathtt{B})$. Therefore, $nf^{-1}(\mathtt{B}) = f^{-1}(n\mathtt{B})$.

Definition 3.2. Let G be an abelian group. A fuzzy multigroup $\mathtt{A}$ of G is called divisible if $CM_{n\mathtt{A}}(x) = CM_{\mathtt{A}}(x)$, $\forall x \in G$ or simply, $n\mathtt{A} = \mathtt{A}$ for every positive integer n.

Example 3.1. A fuzzy multigroup of (i) an additive group of rational numbers $\mathbb{Q}$ is a divisible fuzzy multigroup, (ii) a group of complex roots of unity of degrees p^k, $k = 1, 2, \ldots, n$, where p is a prime number is a divisible fuzzy multigroup.

Remark 3.1. (i) Every fuzzy multigroup of a divisible group is a divisible commutative fuzzy multigroup. (ii) If $G = \mathbb{Q}$, then every divisible fuzzy multigroup of G has a constant count membership function over $\mathbb{Q} - \{0\}$.

Proposition 3.2. *Let $f\colon G \to G'$ be a homomorphism of groups, and $\mathtt{A}$ be a divisible fuzzy multigroup of G. Then, the homomorphic image of $\mathtt{A}$ is a divisible fuzzy multigroup of G'.*

Proof. By Theorem 2.2, we see that $f(\mathtt{A})$ is a fuzzy multigroup of G'. Thus, it follows that $nf(\mathtt{A}) = f(n\mathtt{A}) = f(\mathtt{A})$, for every $n \in \mathbb{N}$ (Theorem 3.1). Hence, $f(\mathtt{A})$ is divisible. □

Proposition 3.3. *Suppose $f\colon G \to G'$ is an isomorphism of groups, and $\mathtt{B}$ is a divisible fuzzy multigroup of G'. Then, the homomorphic preimage of $\mathtt{B}$ is a divisible fuzzy multigroup of G.*

Proof. By synthesizing Theorems 2.2 and 3.2, it follows that $f^{-1}(\mathtt{B})$ is a divisible fuzzy multigroup of G. □

Definition 3.3. A fuzzy multigroup $\mathtt{A}$ of G is called p-divisible if and only if $CM_{p^k\mathtt{A}}(x) = CM_{\mathtt{A}}(x)$, $\forall x \in G$ where $k \in \mathbb{N}$ and p is a prime.

Proposition 3.4. *Suppose $\mathtt{A}$ is a fuzzy multigroup of G. Then $\mathtt{A}$ is divisible if and only if it is p-divisible for every prime p.*

Proof. Suppose that $\mathtt{A}$ is divisible fuzzy multigroup of G. Certainly, it is p-divisible since $p^k \in \mathbb{N}$.

Conversely, assume $\mathtt{A}$ is a p-divisible fuzzy multigroup of G for every prime p. Then for every $n \in \mathbb{N}$, we get $n = p_1 p_2 \ldots p_m$ where p_i (for $i = 1, 2, \ldots, m$) is prime. Thus, $n\mathtt{A} = (p_1 p_2 \ldots p_m)\mathtt{A} = \beta\mathtt{A} = \mathtt{A}$ since $n = \beta$. So, $\mathtt{A}$ is divisible. □

Theorem 3.3. *Let $\mathtt{A}$ be a divisible fuzzy multigroup of G. Then the following are divisible subgroups of G:*

(*i*) $\mathtt{A}_\alpha$, $\alpha \in [0,1]$ *for* $\alpha \leq CM_{\mathtt{A}}(0)$.

(*ii*) $\mathtt{A}^\alpha$, $\alpha \in [0,1]$ *for* $\alpha \geq CM_{\mathtt{A}}(0)$.

Proof of Theorem 3.3. (i) By Theorem 2.1, $\mathtt{A}_\alpha$ is a subgroup of G. Let $x \in \mathtt{A}_\alpha$, $\alpha \in [0,1]$ and $n \in \mathbb{N}$. Since $\bigvee_{ny=x} CM_{\mathtt{A}}(y) = CM_{\mathtt{A}}(x) \geq \alpha$, then it happens that $y \in \mathtt{A}_\alpha$ with $ny = x$. Hence, $\mathtt{A}_\alpha$ is a divisible subgroup of G.

(ii) Similarly, $\mathtt{A}^\alpha$ is a subgroup of G by Theorem 2.1. If $x \in \mathtt{A}^\alpha$, $\alpha \in [0,1]$ and $n \in \mathbb{N}$. Then since $\bigvee_{ny=x} CM_{\mathtt{A}}(y) = CM_{\mathtt{A}}(x) \leq \alpha$, it follows that $y \in \mathtt{A}^\alpha$ such that $ny = x$, and the result follows.

Theorem 3.4. *If* $\mathtt{A}$ *is a fuzzy multiset of* G *such that* $CM_{\mathtt{A}}(0) = 1$ *and every* $\mathtt{A}_\alpha$, $\alpha \in [0,1]$, *is a divisible subgroup of* G. *Then* $\mathtt{A}$ *is a divisible fuzzy multigroup of* G.

Proof of Theorem 3.4. From the given hypotheses , it follows that $\mathtt{A}$ is a fuzzy multigroup of G by Theorem 2.2. Let $x \in G$ and $CM_{\mathtt{A}}(x) = \alpha$. Since $\mathtt{A}_\alpha$ is a divisible subgroup of G for every $n \in \mathbb{N}$, $\exists\ y \in \mathtt{A}_\alpha$ such that $ny = x$, hence $CM_{\mathtt{A}}(y) \geq CM_{\mathtt{A}}(x) = \alpha$. But $CM_{\mathtt{A}}(y) \leq CM_{\mathtt{A}}(x)$ since $\mathtt{A}$ is a fuzzy multigroup of G. Hence, $\bigvee_{ny=x} CM_{\mathtt{A}}(y) = CM_{n\mathtt{A}}(x)$, for $x \in nG$ and so, $n\mathtt{A} = \mathtt{A}$.

Corollary 3.1. *Let* $\mathtt{A}$ *be a fuzzy multiset of* G *such that* $CM_{\mathtt{A}}(0) = 1$ *and every* $\mathtt{A}^\alpha$, $\alpha \in [0,1]$, *is a divisible subgroup of* G. *Then* $\mathtt{A}$ *is a divisible fuzzy multigroup of* G.

Proof. Combining Theorems 2.2 and 3.4, the result holds. □

Theorem 3.5. *Let* $\{\mathtt{A}_i\}_{i\in I}$ *be a family of divisible fuzzy multigroups of* G. *Then* $\bigcap_{i\in I} \mathtt{A}_i$ *is a divisible fuzzy multigroup of* G.

Proof of Theorem 3.5. By Proposition 2.1, $\bigcap_{i\in I} \mathtt{A}_i$ is a fuzzy multigroup of G. Assume that every $\mathtt{A}_i$ is divisible, then for $x \in nG$ we have

$$\begin{aligned} CM_{n(\bigcap_{i\in I}\mathtt{A}_i)}(x) &= \bigvee_{x=ny}\bigwedge_{i\in I} CM_{\mathtt{A}_i}(y) = \bigwedge_{i\in I}\bigvee_{x=ny} CM_{\mathtt{A}_i}(y) \\ &= \bigwedge_{i\in I} CM_{n\mathtt{A}_i}(x) = CM_{\bigcap_{i\in I}(n\mathtt{A}_i)}(x) \\ &= CM_{\bigcap_{i\in I}\mathtt{A}_i}(x). \end{aligned}$$

If $x \notin nG$, then $CM_{\bigcap_{i\in I}\mathtt{A}_i}(x) = 0 = CM_{n(\bigcap_{i\in I}\mathtt{A}_i)}(x)$. Hence, $\bigcap_{i\in I}\mathtt{A}_i$ is divisible.

Theorem 3.6. *Suppose* $\{\mathtt{A}_i\}_{i\in I}$ *is a family of divisible fuzzy multigroups of* G. *Then*

(*i*) $\bigcup_{i\in I}\mathtt{A}_i$ *is a divisible fuzzy multigroup of* G *if* $\mathtt{A}_1 \subseteq \mathtt{A}_2 \subseteq \ldots \subseteq \mathtt{A}_n$, $i = 1, 2, \ldots, n$.

(*ii*) $\Sigma_{i\in I}\mathtt{A}_i$ *is a divisible fuzzy multigroup of* G.

Proof of Theorem 3.6. By using Proposition 2.1 and following the same arguments in Theorem 3.5, the proofs of (i) and (ii) are established.

Theorem 3.7. *If* $\mathtt{A}$ *is a fuzzy multigroup of a divisible cyclic group* G*, then* $\mathtt{A}$ *is constant.*

Proof of Theorem 3.7. Let $G =< a >$ and $x \in G$. Then $x = pa$, $p \in \mathbb{N}$. Suppose that $CM_{\mathtt{A}}(pa) > CM_{\mathtt{A}}((p+1)a)$. Because G is divisible, there exists $y \in G$ such that $py = a$. But also there exists $q \in \mathbb{N}$ such that $y = qa$, and so

$$\begin{aligned} CM_{\mathtt{A}}(pa) &> CM_{\mathtt{A}}((p+1)a) = CM_{\mathtt{A}}((p+1)py) \\ &= CM_{\mathtt{A}}((p+1)pqa) = CM_{\mathtt{A}}((p+1)q(pa)) \\ &\geq CM_{\mathtt{A}}(pa) \wedge CM_{\mathtt{A}}(pa) \wedge \ldots \wedge CM_{\mathtt{A}}(pa) \\ &= CM_{\mathtt{A}}(pa), \end{aligned}$$

which is a contradiction.

Also, if $CM_{\mathtt{A}}((p+1)a) > CM_{\mathtt{A}}(pa)$. Then there exists $z \in G$ such that $(p+1)z = a$, $z = ra$, $r \in \mathbb{N}$ since G is divisible. Thus

$$\begin{aligned} CM_{\mathtt{A}}((p+1)a) &> CM_{\mathtt{A}}(pa) = CM_{\mathtt{A}}(p(p+1)ra) = CM_{\mathtt{A}}(pr(p+1)a) \\ &\geq CM_{\mathtt{A}}((p+1)a) \wedge CM_{\mathtt{A}}((p+1)a) \wedge \ldots \wedge CM_{\mathtt{A}}((p+1)a) \\ &= CM_{\mathtt{A}}((p+1)a), \end{aligned}$$

which is also a contradiction. Hence, $CM_{\mathtt{A}}((p+1)a) = CM_{\mathtt{A}}(pa)$. Since x is arbitrary, if $x = a$, then $CM_{\mathtt{A}}(a) = CM_{\mathtt{A}}(2a) = CM_{\mathtt{A}}(3a) = \ldots$, and so $\mathtt{A}$ is a constant fuzzy multigroup of a divisible cyclic group G.

4. Pure fuzzy multigroups

Recall that a subgroup H of a group G is called pure if $nH = H \cap nG$, for every positive integer n. Now, we extend the concept to fuzzy multigroup of G as follows.

Definition 4.1. Let $\mathtt{A}$ and $\mathtt{B}$ be fuzzy multigroups of G such that $\mathtt{A} \subseteq \mathtt{B}$. Then $\mathtt{A}$ is a pure fuzzy multigroup of G if $n\mathtt{A} = \mathtt{A} \cap n\mathtt{B}$ for every $n \in \mathbb{N}$, that is $CM_{n\mathtt{A}}(x) = CM_{\mathtt{A}}(x) \wedge CM_{n\mathtt{B}}(x)$ $\forall x \in G$.

Remark 4.1. Let $\mathtt{A}$, $\mathtt{B}$ and $\mathtt{C}$ be fuzzy multigroups of G such that $\mathtt{A}$ and $\mathtt{B}$ are contained in $\mathtt{C}$. If $\mathtt{A}$ and $\mathtt{B}$ are pure, then $\mathtt{A} \cap \mathtt{B}$ and $\mathtt{A} \cup \mathtt{B}$ are pure fuzzy multigroups of G since $n(\mathtt{A} \cap \mathtt{B}) = (\mathtt{A} \cap \mathtt{B}) \cap n\mathtt{C}$ and $n(\mathtt{A} \cup \mathtt{B}) = (\mathtt{A} \cup \mathtt{B}) \cap n\mathtt{C}$.

Definition 4.2. Let $\mathtt{A}$ and $\mathtt{B}$ be fuzzy multigroups of G such that $\mathtt{A} \subseteq \mathtt{B}$. Then $\mathtt{A}$ is p-pure if $p^r\mathtt{A} = \mathtt{A} \cap p^r\mathtt{B}$ for every $r \in \mathbb{N}$ and p is prime, that is $CM_{p^r\mathtt{A}}(x) = CM_{\mathtt{A}}(x) \wedge CM_{p^r\mathtt{B}}(x)$ $\forall x \in G$.

Remark 4.2. Every p-pure fuzzy multigroup of a p-divisible group is p-divisible.

Proposition 4.1. *Let $f: G \to G'$ be an isomorphism of divisible groups, $\mathtt{A}$, $\mathtt{C}$ be fuzzy multigroups of G and $\mathtt{B}$, $\mathtt{D}$ be fuzzy multigroups of G' such that $\mathtt{A} \subseteq \mathtt{C}$ and $\mathtt{B} \subseteq \mathtt{D}$. If $\mathtt{A}$ and $\mathtt{B}$ are pure fuzzy multigroups of G and G', respectively then*

(i) $f(\mathtt{A})$ is a pure fuzzy multigroup of G'.

(ii) $f^{-1}(\mathtt{B})$ is a pure fuzzy multigroup of G.

Proof. Let $x, y \in G$ and $w, z \in G'$ such that $f(x, y) = w, z$. Since G, G' are divisible groups for every $n \in \mathbb{N}$, $\exists\ y \in G$ and $z \in G'$ such that $ny = x$, $nz = w$. From Proposition 2.2, it follows that $f(\mathtt{A})$ and $f^{-1}(\mathtt{B})$ are fuzzy multigroups of G and G', respectively. Assume that $\mathtt{C}$ and $\mathtt{D}$ are constant, then by Theorems 3.1 and 3.2, we deduce that, for any $w \in G'$

$$\begin{aligned} CM_{f(n\mathtt{A})}(w) &= CM_{nf(\mathtt{A})}(w) = CM_{f(\mathtt{A})}(w) \wedge CM_{nf(\mathtt{C})}(w) \\ &= CM_{f(\mathtt{A})}(w) \wedge \bigvee_{w=nz} CM_{f(\mathtt{C})}(z) \\ &= CM_{f(\mathtt{A})}(w) \end{aligned}$$

and

$$\begin{aligned} CM_{f^{-1}(n\mathtt{B})}(x) &= CM_{nf^{-1}(\mathtt{B})}(x) = CM_{f^{-1}(\mathtt{B})}(x) \wedge CM_{nf^{-1}(\mathtt{D})}(x) \\ &= CM_{f^{-1}(\mathtt{B})}(x) \wedge \bigvee_{x=ny} CM_{f^{-1}(\mathtt{D})}(y) \\ &= CM_{f^{-1}(\mathtt{B})}(x). \end{aligned}$$

Hence, the results. □

Theorem 4.1. *Suppose $\{\mathtt{A}_i\}_{i\in I}$ and $\{\mathtt{B}_j\}_{j\in J}$ are families of fuzzy multigroups of G such that $\{\mathtt{A}_i\}_{i\in I} \subseteq \{\mathtt{B}_j\}_{j\in J}$ and $\{\mathtt{B}_j\}_{j\in J}$ is constant. If $\{\mathtt{A}_i\}_{i\in I}$ is pure, then $\bigcap_{i\in I} \mathtt{A}_i$ is a pure fuzzy multigroup of G.*

Proof of Theorem 4.1. Certainly, $\bigcap_{i\in I} \mathtt{A}_i$ is a fuzzy multigroup of G by Proposition 2.1. If every $\mathtt{A}_i$ is pure, then we consider the following cases.

Case I: Suppose $x \in nG$, we have

$$\begin{aligned} CM_{n(\bigcap_{i\in I} \mathtt{A}_i)}(x) &= CM_{(\bigcap_{i\in I} \mathtt{A}_i)\cap n(\bigcap_{j\in J} \mathtt{B}_j)}(x) \\ &= \bigwedge_{i\in I} CM_{\mathtt{A}_i}(x) \wedge \bigvee_{x=ny} \bigwedge_{j\in J} CM_{\mathtt{B}_j}(y) \\ &= \bigwedge_{i\in I} CM_{\mathtt{A}_i}(x) \wedge \bigwedge_{j\in J} \bigvee_{x=ny} CM_{\mathtt{B}_j}(y) \\ &= \bigwedge_{i\in I} CM_{\mathtt{A}_i}(x) = CM_{\bigcap_{i\in I} \mathtt{A}_i}(x). \end{aligned}$$

Case II: Suppose $x \notin nG$, then we have

$$CM_{n(\bigcap_{i\in I} \mathtt{A}_i)}(x) = CM_{(\bigcap_{i\in I} \mathtt{A}_i)\cap n(\bigcap_{j\in J} \mathtt{B}_j)}(x) = 0 = CM_{\bigcap_{i\in I} \mathtt{A}_i}(x).$$

Hence, $\bigcap_{i\in I} \mathtt{A}_i$ is a pure fuzzy multigroup of G.

Theorem 4.2. *Let* $\{\mathtt{A}_i\}_{i\in I}$ *and* $\{\mathtt{B}_j\}_{j\in J}$ *be families of fuzzy multigroups of* G *such that* $\{\mathtt{B}_j\}_{j\in J}$ *is constant and contains* $\{\mathtt{A}_i\}_{i\in I}$. *If* $\{\mathtt{A}_i\}_{i\in I}$ *is pure, then*

(*i*) $\bigcup_{i\in I}\mathtt{A}_i$ *is a pure fuzzy multigroup of* G *for* $\mathtt{A}_1 \subseteq \mathtt{A}_2 \subseteq \ldots \subseteq \mathtt{A}_n$, $i = 1, 2, \ldots, n$.

(*ii*) $\Sigma_{i\in I}\mathtt{A}_i$ *is a pure fuzzy multigroup of* G.

Proof of Theorem 4.2. By synthesizing Proposition 2.1 and following the same logic in Theorem 4.1, the proofs of (i) and (ii) are established.

Theorem 4.3. *Let* $\mathtt{A}$ *and* $\mathtt{B}$ *be fuzzy multigroups of* G *such that* $\mathtt{A} \subseteq \mathtt{B}$ *and* $\mathtt{B}$ *is constant. Then* $\mathtt{A}$ *is pure if and only if* $\mathtt{A}$ *is a divisible fuzzy multigroup of* G.

Proof of Theorem 4.3. *Suppose* $\mathtt{A}$ *is a pure fuzzy multigroup of* G. *Then* $n\mathtt{A} = \mathtt{A} \cap n\mathtt{B}$ *for* $n \in \mathbb{N}$. *Certainly,* $\mathtt{A} \cap n\mathtt{B} = \mathtt{A}$ *because*

$$\begin{aligned} CM_{\mathtt{A}}(x) \wedge CM_{n\mathtt{B}}(x) &= CM_{\mathtt{A}}(x) \wedge \bigvee_{ny=x} CM_{\mathtt{B}}(y) \\ &= CM_{\mathtt{A}}(x)\ \forall x \in X. \end{aligned}$$

Thus, $n\mathtt{A} = \mathtt{A}$ *for* $n \in \mathbb{N}$, *and hence* $\mathtt{A}$ *is a divisible fuzzy multigroup of* G.

Conversely, assume $\mathtt{A}$ *is a divisible fuzzy multigroup of* G. *Then* $n\mathtt{A} = \mathtt{A}$ *for* $n \in \mathbb{N}$. *Since* $\mathtt{A} \cap n\mathtt{B} = \mathtt{A}$, *we have* $n\mathtt{A} = \mathtt{A} \cap n\mathtt{B}$ $n \in \mathbb{N}$, *and so* $\mathtt{A}$ *is a pure fuzzy multigroup of* G.

On the strength of Theorem 4.3, we state the following results without proofs because their proofs are similar to their equivalent proofs in Section 3.

Proposition 4.2. *Let* $\mathtt{A}$ *and* $\mathtt{B}$ *be fuzzy multigroups of* G *such that* $\mathtt{A} \subseteq \mathtt{B}$ *and* $\mathtt{B}$ *is constant. Then* $\mathtt{A}$ *is pure if and only if it is* p*-pure for every prime* p.

Proof. Similar to Proposition 3.4. □

Proposition 4.3. *Let* $\mathtt{A}$ *and* $\mathtt{B}$ *be fuzzy multigroups of* G *such that* $\mathtt{A} \subseteq \mathtt{B}$ *and* $\mathtt{B}$ *is constant. If* $\mathtt{A}$ *is a pure fuzzy multigroup of* G, *then* $\mathtt{A}_\alpha$, $\alpha \in [0, 1]$ *is a pure subgroup of* G *for* $\alpha \leq CM_{\mathtt{A}}(0)$ *and* $\mathtt{A}^\alpha$, $\alpha \in [0, 1]$ *is a pure subgroup of* G *for* $\alpha \geq CM_{\mathtt{A}}(0)$.

Proof. Similar to Theorem 3.3. □

Proposition 4.4. *Let* $\mathtt{A}$ *and* $\mathtt{B}$ *be fuzzy multisets of* G *such that* $\mathtt{A} \subseteq \mathtt{B}$ *and* $\mathtt{B}$ *is constant. If* $\mathtt{A}$ *is a fuzzy multiset of* G *such that* $CM_{\mathtt{A}}(0) = 1$ *and every* $\mathtt{A}_\alpha$, $\alpha \in [0, 1]$, *is a pure subgroup of* G. *Then* $\mathtt{A}$ *is a pure fuzzy multigroup of* G.

Proof. Similar to Theorem 3.4. □

Corollary 4.1. *Let* $\mathtt{A}$ *and* $\mathtt{B}$ *be fuzzy multisets of* G *such that* $\mathtt{A} \subseteq \mathtt{B}$ *and* $\mathtt{B}$ *is constant. Let* $\mathtt{A}$ *be a fuzzy multiset of* G *such that* $CM_{\mathtt{A}}(0) = 1$ *and every* $\mathtt{A}^\alpha$, $\alpha \in [0, 1]$, *is a pure subgroup of* G. *Then* $\mathtt{A}$ *is a pure fuzzy multigroup of* G.

Proof. Similar to Corollary 3.1. □

5. Conclusion

The notion of fuzzy multigroups is the application of group theory to fuzzy multisets. Several analog concepts of group theory have been established in fuzzy multigroup environment. In this paper the concepts of divisible and pure fuzzy multigroups were established and characterized with some duly proved related results. The nexus between divisible fuzzy multigroups and divisible groups as well as between pure fuzzy multigroups and pure groups were instituted with the aid of alpha-cuts. More properties of divisible and pure fuzzy multigroups could be constructed in future research.

Acknowledgements

This work is supported by Foundation of Chongqing Municipal Key Laboratory of Institutions of Higher Education ([2017]3), Foundation of Chongqing Development and Reform Commission (2017[1007]), and Foundation of Chongqing Three Gorges University.

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Accepted: June 01, 2020

A new multi-step pseudo-spectral method for the approximate solution of inverse reaction-diffusion equation

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Abstract. In this paper, we propose a novel multi-step pseudo-spectral method for numerical solution of an inverse reaction-diffusion equation. This method is based on the approximation of the original problem with a sequence of sub-problems. We utilize the collocation method based on a new set of nonclassical basis functions using Chebyshev-Gauss-Radau quadrature points. By using the Pseudo-spectral method, inverse reaction diffusion equation with initial and boundary conditions, would reduce to a system of nonlinear algebraic equations. Due to the ill-posed system of nonlinear algebraic equations, the Tikhonov regularization scheme is employed to obtain a numerical stable solution. The proof the stability and convergence of the method is presented. Some numerical examples are given to show the accuracy of the proposed method and results are compared with those found in literature.

Keywords: reaction-diffusion equation, pseudo-spectral method, multi-step method, inverse problems, regularization.

1. Introduction

Reaction-diffusion systems are mathematical models that explain how the concentration of one or more substances distributed in the space changes under the influence of the two processes. These processes are: local chemical reactions in which the substances are transformed into each other and diffusion that causes the substances to spread out over a surface in space [1]. This description implies that reaction-diffusion systems are naturally applied in chemistry as well

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as biology, geology, physics and ecology [2]. Mathematically, reaction-diffusion systems take the form of the semi-linear parabolic partial differential equations, which are represented in the following general form:

$$u_t = \nabla(\mathbf{D}(u;x,t)\nabla u) + f(u;\nabla u,x,t), \tag{1}$$

where $u = u(x,t)$ represents the concentration of one substance, $\mathbf{D}$ is a diffusion coefficient, and f is the reaction term.
In this paper, we consider the reaction-diffusion problem in the initial and boundary value problem of the following form:

$$\begin{cases} u_t(x,t) = a(t)u_{xx}(x,t) + p(t)u(x,t) + g(x,t), & 0 < x < 1, 0 < t \le T, \\ u(x,0) = u_0(x), & 0 \le x \le 1, \\ u(0,t) = g_0(t), & 0 < t \le T, \\ u(1,t) = g_1(t), & 0 < t \le T, \end{cases} \tag{2}$$

where $T \ge 0$ is a constant value and p, g, u_0, g_0 and g_1 are known as functions. A direct problem aims to find a solution that satisfies given partial differential equation with initial and boundary conditions. Moreover, a direct problem aims to find a solution $u(x,t)$ that satisfies Eq. (2). Problems involving the determination of unknown coefficients in partial differential equations by some additional condition are well known in the mathematical literature as "The Inverse Coefficient Problems". The additional condition may be given to the whole domain, on the boundary of the domain, or at the final time [3]. In some problems, the original partial differential equation, initial conditions, and boundary conditions are not sufficient to obtain the solution and some conditions are required. Such problems are called the "Inverse problems" [4].

Here, we assume that the reaction term $a(t)$ is unknown. Therefore, by solving (1) it is aimed to obtain $u(x,t)$ and $a(t)$ at the same time. Accordingly, this problem is an inverse problem, and to solve that problem, we need an additional condition, which is

$$u(x^*,t) = E(t), \qquad 0 < x^* < 1, \quad 0 \le t \le T, \tag{3}$$

where $E(t)$ is a known function. Using Eq. (3) and Eq. (2) is written as follows:

$$\begin{cases} u_t(x,t) = a(t)u_{xx}(x,t) + p(t)u(x,t) + g(x,t), & 0 < x < 1,\ 0 < t \le T, \\ u(x,0) = u_0(x), & 0 \le x \le 1, \\ u(0,t) = g_0(t), & 0 < t \le T, \\ u(1,t) = g_1(t), & 0 < t \le T, \\ u(x^*,t) = E(t), & 0 < x^* < 1,\ 0 \le t \le T. \end{cases} \tag{4}$$

Without the extra measurement (3), the problem (2) is under-determinated and may have an infinite number of solutions. On the other hand, if more than one additional condition is imposed, the solution may not exist.

The existence and uniqueness of the solution of this problem and more applications are discussed in [5, 6]. However, the theory of the numerical solution of this problem is not satisfactory. Cannon [7] and Dehghan [8], reduced the problem to a non-linear integral equation for the coefficient $a(t)$. This approach works well for parabolic equations in one space variable but cannot be easily extended to problem with higher-dimensions because it depends on the explicit form of the fundamental solution of the heat operator. In [9], a backward Euler finite difference scheme is discussed and has been shown that this scheme is stable in the maximum norm and error estimates for u and a and some experimental numerical results are given. The authors of [10], found a numerical solution of the problem for a connected domain in $\mathbb{R}^n$. In [11], this problem has been studied from a different point of view. The authors first transformed a large class of parabolic inverse problems into a non-classical parabolic equation with coefficients, including trace type functional on the solution and with derivatives subjected to some initial and boundary conditions. For the resulted non-classical problem, they introduced a variational form by defining a new function and then employed both continuous and discrete Galerkin procedures to the non-classical problem. Dehghan [12], used several explicit and implicit finite difference methods to solve this problem. In [13], an efficient pseudo-spectral Legendre method is developed to solve the problem given in Eq. (4). The authors of [14], applied Adomian decomposition method to find the solution of this problem. In [15], the numerical solution is also considered by using Chebyshev cardinal functions. The method consists of expanding the required approximate solution as the elements of Chebyshev cardinal functions. This problem is solved by a high-order compact finite difference method in [16]. The authors of [17], applied the collocation method based on the radial basis function (RBF) to find the solution of this problem. They made an effort to solve inverse problem in a class of reaction-diffusion equation using (RBF) as a truly meshless method. The interested reader can see [18, 19, 20] for more research works on this problem. To the best of our knowledge, the above mentioned numerical methods are mainly suited for the small interval, say $T \leq 1$. Besides, it is assumed that the solution is continuous on $[0, T]$. In this study we solve the problem for $T \gg 1$. Moreover the proposed method leads to a suitable approximation for the problems with noncontinuous solution.

The reminder of the present work is organized as follows. In the next section, we review some properties of Chebyshev polynomials and explain the function approximation. In Section 3, inspired by the method introduced by Shizgal [21] for solving advection- reaction-diffusion problem, we derive a new spectral approximation. In this method, the domain of the problem is partitioned into a uniform mesh, followed by obtaining the solution on each part, using collocation method. In the next section, we review some properties of Chebyshev polynomials and function approximation. In section 3, we present the computational method for approximating the solution of inverse reaction- diffusion problem.

The convergence analysis of the present method is derived in section 4. Section 5 devoted to numerical finding and in section 6 we give conclusion.

2. Review of Chebyshev polynomials and function approximation

2.1 Chebyshev polynomials

In this section, we give some notation about most commonly used set of the orthogonal polynomials, Chebyshev polynomials [22, 23] that can be determined with the aid of the following recursive relation,

$$\begin{aligned} &T_0(x) = 1, \\ &T_1(x) = x, \\ &T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad n \geq 1. \end{aligned} \tag{5}$$

These polynomials are orthogonal on $[-1, 1]$ with respect to the weight function $w(x) = \frac{1}{\sqrt{1-x^2}}$, that is

$$\int_{-1}^{1} T_i(x)T_j(x)w(x)dx = \pi\gamma_i\delta_{ij}, \tag{6}$$

where

$$\gamma_i = \begin{cases} 1, & i = 0, \\ \dfrac{1}{2}, & i > 0, \end{cases} \tag{7}$$

and δ_{ij} is the Kronocker delta. A function $f \in C[-1, 1]$ may be represented by Chebyshev polynomials series as [24, 25]

$$f(x) = \sum_{i=0}^{\infty} f_i T_i(x). \tag{8}$$

If the infinite series in Eq. (8) is truncated, then it can be written as

$$f(x) \simeq \sum_{i=0}^{N} f_i T_i(x) = \mathbf{F}^t \mathbf{T}_N(x), \tag{9}$$

where

$$\mathbf{T}_N(x) = [T_0(x), T_1(x), \ldots, T_N(x)], \quad \mathbf{F} = [f_0, f_1, \ldots, f_N], \tag{10}$$

and

$$f_i = \frac{< f_i, T_i >_{w(x)}}{< T_i, T_i >_{w(x)}}, \qquad i = 0, 1 \ldots, N, \tag{11}$$

where $<,>$ denotes the inner product with respect to suitable weight function $w(x)$. Moreover, if $f \in C^k[-1,1]$, $k \geq 1$, then

$$f^{(k)}(x) \simeq \mathbf{F}^t \mathbf{T}_N^{(k)}(x) = \mathbf{F}^t (P_N^{-1})^k \mathbf{T}_N(x), \tag{12}$$

where $f^{(k)}(x)$ is the k^{th} derivative of $f(x)$ and, P_N is the $(N+1)(N+1)$ operational matrix [26]

$$P_N = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ -\frac{1}{4} & 0 & \frac{1}{4} & \cdots & 0 & 0 \\ -\frac{1}{3} & -\frac{1}{2} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{(-i)^{N-1}}{1-(N-1)^2} & 0 & 0 & \cdots & 0 & \frac{1}{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{(-1)^N}{1-N^2} & 0 & 0 & \cdots & -\frac{1}{2N-2} & 0 \end{pmatrix}.$$

2.2 Two dimensional Chebyshev polynomials

In this section, by using Chebyshev polynomials, we define a basis set for $L^2([-1,1]\times[-1,1])$. Let

$$T_{ij}(x) = T_i(x)T_j(x), \qquad i = 0,1,\ldots, \quad j = 0,1,\ldots. \tag{13}$$

Then, the the two-dimensional Chebyshev basis vector are as follows:

$$\begin{aligned} \mathbf{T}(x,t) &= [T_0(x)T_0(t),\ldots,T_0(x)T_M(t),T_1(x)T_0(t),\ldots,T_1(x)T_M(t),\ldots, \\ &\quad T_N(x)T_M(t)] \\ &= (\mathbf{T}_N(x)\otimes\mathbf{T}_M(x))^t, \end{aligned} \tag{14}$$

in which $\mathbf{T}_N(x) = [T_0(x), T_1(x), \ldots, T_N(x)]$ and $\mathbf{T}_M(x) = [T_0(x), T_1(x), \ldots, T_M(x)]$ and $\otimes$ is the Kroncher tensor product. The orthogonality property for these polynomials with $w(x,t) = w(x)w(t)$ on the interval $[-1,1]\times[-1,1]$ is stated as

$$< T_{ij}(x,t), T_{kl}(x,t) >_{w(x,t)} = \int_{-1}^{1}\int_{-1}^{1} T_{ij}(x,t)T_{kl}(x,t)w(x,t)dxdt = \pi^2\gamma_i\gamma_k\delta_{ij}\delta_{kl}.$$

Similarly to the previous statement, a continuous function $f(x,t)$ on $[-1,1]\times[-1,1]$ can be expanded by the two-dimensional Chebyshev polynomials as the following equation [27]:

$$f(x,t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} f_{ij}T_{ij}(x,t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} f_{ij}T_i(x)T_j(t), \tag{15}$$

$$f_{ij} = \frac{< f(x,t), T_{ij}(x,t) >_{w(x,t)}}{< T_{ij}(x,t), T_{ij}(x,t) >_{w(x,t)}}, \qquad i = 0,1,\ldots, \qquad j = 0,1,\ldots, \tag{16}$$

where $< . >$ denotes the inner product with respect to the suitable weight function $w(x,t)$. In practice, only the finite terms of the above series are considered, so we have

$$f(x,t) \simeq \sum_{i=0}^{N}\sum_{j=0}^{M} f_{ij}T_{ij}(x,t). \tag{17}$$

2.3 Shifted Chebyshev polynomials

Let the function $h(x)$ be defined on an arbitrary interval $[a,b]$. Then, it can be approximated by making transformation from $[a,b]$ to $[-1,1]$ as:

$$h(x) \simeq \sum_{i=0}^{N} h_i T_i(\frac{2}{b-a}(x-a)-1) = \mathbf{H}^t\mathbf{T}_{N,[a,b]}(x), \tag{18}$$

where $\mathbf{H} = [h_0, h_1, \ldots, h_N]$ and $\mathbf{T}_{N,[a,b]}$ is Chebyshev polynomials vector that have been shifted from $[-1,1]$ to $[a,b]$. Moreover, for an arbitrary N and M, a continuous function of two variables $u : [a,b]\times[c,d] \longrightarrow \mathbb{R}$ can be approximated by

$$\begin{aligned} u(x,t) &\simeq \sum_{n=0}^{N}\sum_{m=0}^{M} u_{nm}T_{nm}(\frac{2}{b-a}(x-a)+1, \frac{2}{d-c}(t-c)+1) \\ &= \sum_{n=0}^{N}\sum_{m=0}^{M} u_{nm}T_n(\frac{2}{b-a}(x-a)+1)T_m(\frac{2}{d-c}(t-c)+1) \\ &= \mathbf{T}^t_{N,[a,b]}(x)U\mathbf{T}_{M,[c,d]}(x), \end{aligned} \tag{19}$$

where $U = [u_{nm}]$, $n = 0,1,\ldots,N$ and $m = 0,1,\ldots,M$.

3. Pseudo-spectral method

In this section, we derive a pseudo-spectral interpolation approximation for functions of two variables and then implement it in a multi-step scheme to approximate the solution of the problem (4). The advantage of this direct interpolation is that it is stable for large mode M and N.

3.1 Function approximation using direct interpolation

The Chebyshev-Gauss-Radau collocation points, $\{x_i\}_{i=0}^{N}$, $\{t_j\}_{j=0}^{M}$ and weights, $\{w_i\}_{i=0}^{N}$, $\{v_j\}_{j=0}^{M}$ provide the approximate quadrature

$$\int_{-1}^{1}\int_{-1}^{1} w(x)v(t)u(x,t)dxdt \simeq \sum_{i=0}^{N}\sum_{j=0}^{M} w_i v_j u(x_i,t_j), \tag{20}$$

where

$$w_i = \begin{cases} \dfrac{\pi}{2N}, & i = 0, N, \\ \dfrac{\pi}{N}, & i = 1, 2, \ldots, N-1, \end{cases}, \quad v_j = \begin{cases} \dfrac{\pi}{2M}, & j = 0, M, \\ \dfrac{\pi}{M}, & j = 1, 2, \ldots, M-1. \end{cases}$$

Further, the solution of Eq. (15), $u \in L^2([0,1] \times [0,1])$, can be expanded in a Chebyshev series. Hence,

$$u(x,t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} u_{nm} T_{nm}(x,t), \tag{21}$$

where

$$u_{nm} = \frac{\gamma_n \gamma_m}{\pi^2} \int_{-1}^{1} \int_{-1}^{1} w(x) v(t) u(x,t) dx dt. \tag{22}$$

By substitution Eq. (20) into Eq. (22), truncated at N and M, we have

$$u_{nm} = \frac{\gamma_n \gamma_m}{\pi^2} \sum_{i=0}^{N} \sum_{j=0}^{M} w_i v_j u(x_i, t_j) T_{nm}(x_i, t_j), \tag{23}$$

Now, substituting Eq. (23) into Eq. (21) yields

$$\begin{aligned} u(x,t) &\simeq u_{NM}(x,t) \\ &= \sum_{n=0}^{N} \sum_{m=0}^{M} \sum_{i=0}^{N} \sum_{j=0}^{M} \frac{\gamma_n \gamma_m}{\pi^2} w_i v_j u(x_i, t_j) T_{nm}(x_i, t_j) T_{nm}(x,t). \end{aligned} \tag{24}$$

Due to the values of w_i and v_j, we rewrite Eq. (24) as follows:

$$u(x,t) = \sum_{n=0}^{N} \sum_{m=0}^{M} \sum_{i=0}^{N} \sum_{j=0}^{M} \frac{4\gamma_n \gamma_m}{NM b_i c_j} b_n c_m u(x_i, t_j) T_{nm}(x_i, t_j) T_{nm}(x,t), \tag{25}$$

where

$$b_i = \begin{cases} \frac{1}{2}, & i = 0, N, \\ 1, & i = 1, 2, \ldots, N-1, \end{cases}, \quad c_j = \begin{cases} \frac{1}{2}, & j = 0, M, \\ 1, & j = 1, 2, \ldots, M-1, \end{cases}.$$

With Eq. (25) we obtain the interpolation approximation

$$u_{NM}(x,t) \simeq \sum_{i=0}^{N} \sum_{j=0}^{M} I_{ij}(x,t) u(x_i, t_j), \tag{26}$$

where the interpolating polynomials, $I_{ij}(x,t)$, are defined by

$$I_{ij}(x,t) = \sum_{n=0}^{N} \sum_{m=0}^{M} \frac{4\gamma_n \gamma_m}{NM b_i c_j} b_n c_m T_{nm}(x_i, t_j) T_{nm}(x,t). \tag{27}$$

It can be easily checked that $I_{ij}(x_k,t_l)=\delta_{ik}\delta_{jl}$, because

$$\begin{aligned} I_{ij}(x_k,t_l) &= \sum_{n=0}^{N}\sum_{m=0}^{M}\frac{4\gamma_n\gamma_m}{NMb_ic_j}b_nc_mT_{nm}(x_i,t_j)T_{nm}(x_k,t_l) \\ &= \frac{2}{Nb_i}\sum_{n=0}^{N}b_nT_n(x_i)T_n(x_k)(\frac{2}{Mc_j}\sum_{m=0}^{M}T_m(t_j)T_m(t_l)), \end{aligned} \tag{28}$$

and according to [28] $\frac{2}{Mc_j}\sum_{m=0}^{M}T_m(t_j)T_m(t_l)=\delta_{jl}$, therefore

$$I_{ij}(x_k,t_l)=\delta_{ik}\delta_{jl}. \tag{29}$$

Now, by applying Eq. (27) and considering that $z_{ij}=\frac{4}{NMb_ic_j}$, we can write the matrix form of Eq. (26) as follows:

$$u_{NM}(x,t)=\mathbf{U}_{NM}^{T}.\mathbf{I}_{NM}, \tag{30}$$

where $\mathbf{I}_{NM}=[I_{nm}]$ and $\mathbf{U}_{NM}=[u_{nm}]$ are $(N+1)(M+1)$ matrices and

$$I_{nm}=z_{nm}[B^tT_{N,[a,b]}(x)\otimes C^tT_{M,[c,d]}(t)]^t[T_{N,[a,b]}(x_n)\otimes T_{M,[c,d]}(t_m)], \tag{31}$$

and $B=[b_0,b_1,\ldots,b_N]$, $C=[c_0,c_1,\ldots,c_M]$.

Next, by using Eq. (31) we have

$$\frac{\partial}{\partial t}I_{nm}(x,t)=z_{nm}[B^tT_{N,[a,b]}(x)\otimes C^t\frac{\partial}{\partial t}T_{M,[c,d]}(t)]^t[T_{N,[a,b]}(x_n)\otimes T_{M,[c,d]}(t_m)]^T$$

$$=\frac{2z_{nm}}{d-c}[B^tT_{N,[a,b]}(x)\otimes C^tP^{-1}MT_{M,[c,d]}(t)]^t[T_{N,[a,b]}(x_n)\otimes T_{M,[c,d]}(t_m)], \tag{32}$$

and

$$\frac{\partial^2}{\partial x^2}I_{nm}(x,t)=z_{nm}[B^t\frac{\partial^2}{\partial x^2}T_{N,[a,b]}(x)\otimes C^tT_{M,[c,d]}(t)]^t[T_{N,[a,b]}(x_n)\otimes T_{M,[c,d]}(t_m)]$$

$$=\frac{4z_{nm}}{(d-c)^2}[B^t(P_N^{-1})^2T_{N,[a,b]}(x)\otimes C^tT_{M,[c,d]}(t)]^t.[T_{N,[a,b]}(x_n)\otimes T_{M,[c,d]}(t_m)]. \tag{33}$$

Finally, Eqs. (32)-(33) together with Eq. (30), yield

$$u_{NM,t}(x,t)=\frac{\partial}{\partial t}(U_{NM}^tI_{NM}(x,t))=U_{NM}^tI_{NM,t}(x,t), \tag{34}$$

and

$$u_{NM,xx}(x,t)=\frac{\partial^2}{\partial x^2}(U_{NM}^tI_{NM}(x,t))=U_{NM}^tI_{NM,xx}(x,t), \tag{35}$$

where

$$I_{NM,t}(x,t)=\left[\frac{\partial}{\partial t}I_{nm}(x,t)\right]_{n=0,m=0}^{N,M},\qquad I_{NM,xx}(x,t)=\left[\frac{\partial^2}{\partial x^2}I_{nm}(x,t)\right]_{n=0,m=0}^{N,M}.$$

3.2 The multi-step collocation method

In the previous section, with the aid of Chebyshev polynomials and Chebyshev-Gauss-Lobatto quadrature points, a direct interpolation scheme for approximating $u(x,t)$ with reference to the pseudo-spectral method were represented. In this subsection, a multi-step collocation method is presented for approximating the solution of the problem (4). In the proposed method, we divide the interval of the problem into a set of sub-intervals and then approximate the solution of the sub-problem for each sub-intervals. An important point in this approach is that, since the original problem is solved independently on each sub-interval, it would be a useful approximation for large T, i.e., on large domains. As stated before, we consider a step size and then divide the interval $(0,T]$ to a sequence of subintervals with length equal to the considered step size. For this purpose, we first assume $\frac{T}{l}$ as the step size of the interval $(0,T]$, and l as a positive integer number. Then we divide interval $(0,T]$ into l sub-interval

$$R_k = \left(\frac{(k-1)T}{l}, \frac{kT}{l}\right], \qquad k = 1,2,\ldots,l. \tag{36}$$

Furthermore, we define the restriction of $u(x,t)$ on each R_k by a piecewise function as follows:

$$u(x,t)|_{R_k} = u^k(x,t), \qquad k = 1,2,\ldots,l, \qquad x \in (0,1), \qquad t \in R_k, \tag{37}$$

where $u^k(x,t)$, is solution of Eq. (4) on the area $\Omega_k = \{(x,t)|\ x \in [0,1],\ t \in R_k\}$. Now, we can rewrite Eq.(4) on Ω_k as follows:

$$\begin{cases} u_t^k(x,t) = a(t)u_{xx}^k(x,t) + p(t)u^k(x,t) + g(x,t), & x \in (0,1),\ t \in R_k, \\ u^k(x,0) = u_k(x), & x \in [0,1], \\ u^k(0,t) = g_0(t), & t \in R_k, \\ u^k(1,t) = g_1(t), & t \in R_k, \\ u^k(x^*,t) = E(t), & x^* \in (0,1), t \in R_k \cup \{0\}. \end{cases} \tag{38}$$

Here, we suppose that $u^k(x,t)$, $k = 0,1,\ldots,l$, approximates the solution of the problem (4) on Ω_k. So, we divide the problem (4) into l independent problems, and then we solve them consecutively. It is worth mentioning that, when we solve the problem on the Ω_k area, it means that, we have already solved it on the previous area Ω_{k-1}. So, the approximation of $u^{k-1}(x, \frac{(k-1)T}{l})$ in k^{th} step is known.

Here, we use Chebyshev-Gauss-Radau collocation points and pseudo-spectral method for approximating of $u(x,t)$. According to [28], we represent Chebyshev polynomials series of $a(t)$ on the interval R_k as follows:

$$a^k(t) \simeq \sum_{j=0}^{M} a_j I_j(t) = \mathbf{I}_{M,R_k}^T(t)\mathbf{A}^k. \tag{39}$$

where $\mathbf{I}_{M,R_k}(t) = [I_{l,R_k}]$ and $I_{l,R_k}(t) = \sum_{l=0}^{M} T_l(t)T_l(t_l)$.

To solve the sub-problem (38), first by using Eqs. (34)-(35) and Eq. (30) we rewrite this problem as the following matrix form

$$
\begin{cases}
(U^k_{NM})^t.\mathbf{I}_{NM,t}(x_i,t_j) = \mathbf{I}^t_{M,R_k}(t_j)\mathbf{A}^k(U^k_{NM})^t\mathbf{I}_{NM,xx}(x_i,t_j) \\
+P(t_j)(U^k_{NM})^t\mathbf{I}_{NM}(x_i,t_j)+g(x_i,t_j), i=1,2,\ldots,N-1,\ j=1,\ldots,M, \\
(U^k_{NM})^t.\mathbf{I}_{NM}(x_i,0) = u^k(x_i), \qquad i=1,\ldots,N, \\
(U^k_{NM})^t.\mathbf{I}_{NM}(0,t_j) = g_0(t_j), \qquad j=1,\ldots,M, \\
(U^k_{NM})^t.\mathbf{I}_{NM}(1,t_j) = g_1(t_j), \qquad j=1,\ldots,M, \\
(U^k_{NM})^t.\mathbf{I}_{NM}(x^*,t_j) = E(t_j), \qquad j=0,1,\ldots,M.
\end{cases} \tag{40}
$$

Next, let $d = (M+1)(N+3)$ is the dimension of the above system of nonlinear algebraic equations (NAEs). The NAEs (40) can be represented in the general form

$$
\begin{cases}
\mathbb{F}(.) : \mathbb{R}^d \longrightarrow \mathbb{R}^d, \\
\mathbb{F}(\mathbf{Z}) = 0,
\end{cases} \tag{41}
$$

where $\mathbb{F} = (F_1, F_2, \ldots, F_d)$, $\mathbf{Z} = (z_1, z_2, \ldots, z_d)$ and $F_i(z_1, z_2, \ldots, z_d) = 0$, $i = 1, 2, \ldots, d$. By applying a proper method for solving the nonlinear system (41), we can obtain $(M+1)(N+3)$ unknown coefficients u_{ij} and a_j, $i = 1, \ldots, N$, $j = 0, 1, \ldots, M$, and consequently $\mathbf{U}^T_{NM}.\mathbf{I}_{NM}$ and $\mathbf{I}^t_{M,R_k}(t)\mathbf{A}^k$ as the approximate solutions of the inverse problem (4) on Ω_k.

Since the original inverse reaction diffusion (4) is ill-posed, the NAEs (41) my be unstable. In other words, for the ill-posed problem or some special cases, the inverse of Jacobian matrix of $\mathbb{F}$ does not exist or cannot be found in the numerical sense. So the optimal searching direction used in the iteration process or Newton's method is not proposed [14]. So in order to stabilize the solution, we use the Tikhonov regularization scheme.

4. Convergence analysis

The propose of the present section is to obtain an error bound for the best approximation of the function u based on interpolation functions and to describe the convergence behavior of the proposed numerical method.

We suppose

$$
u(x,t) = \sum_{k=1}^{l} u^k(x,t), \tag{42}
$$

where $u^k = u\chi_{\Omega_k}$.

Recall that $(\mathbf{U}^k)^T\mathbf{I}_{NM}(x,t)$ is interpolation function of $u^k(x,t)$ on the Ω_k, hence, it is the best approximation of $u^k(x,t)$ [25].

Theorem 4.1. *Suppose that $u \in C^r([0,1]\times[0,T])$ then $\boldsymbol{U}^T\boldsymbol{I}_{NM}(x,t)$ approximates $u(x,t)$ with the following error bound*

$$\|u(x,t)-\mathbf{U}^T\mathbf{I}_{NM}(x,t)\|_2 \le (\frac{\lambda\sqrt{T}}{r!})(1+\frac{T}{l})^r, \qquad \lambda=\max_{\substack{x\in[0,T]\\0\le j\le r}}|\frac{\partial^r u(x,t)}{\partial x^j\partial t^{r-j}}|. \tag{43}$$

Proof. Let $p(x,t)$ be the Taylor expansion for the function $u^k(x,t)$, it is known as [23]:

$$|\ u^k(x,t)-p(x,t)\ |\le \frac{\lambda}{r!}(x+(t-\frac{(k-1)T}{l}))^r,\ k=1,2,\ldots,l. \tag{44}$$

Since, $(\mathbf{U}^k)^T\mathbf{I}_{NM}(x,t)$ is the best approximation of $u^k(x,t)$, hence by using Eq. (44) we obtain

$$\begin{aligned}\|u^k(x,t)-(\mathbf{U}^k)^T\mathbf{I}_{NM}(x,t)\|_2^2 &\le \|\ u^k(x,t)-\tilde{u}^k(x,t)\ \|_2^2\\ &\le \int_0^1\int_{\frac{(k-1)T}{l}}^{\frac{kT}{l}}(\frac{\lambda}{r!})^2(x+(t-\frac{(k-1)T}{l}))^{2r}dxdt\\ &\le (\frac{\lambda}{r!})^2\int_0^1\int_{\frac{(k-1)T}{l}}^{\frac{kT}{l}}(1+\frac{T}{l})^{2r}dxdt\\ &= (\frac{\lambda}{r!})^2\frac{T}{l}(1+\frac{T}{l})^{2r}.\end{aligned} \tag{45}$$

Now,

$$\begin{aligned}\|u(x,t)-u_{NM}(x,t)\|_2^2 &\le \sum_{k=1}^{l}\|u^k(x,t)-(\mathbf{U}^k)^T\mathbf{I}_{NM}(x,t)\|_2^2\\ &\le (\frac{\lambda}{r!})^2T(1+\frac{T}{l})^{2r}\end{aligned} \tag{46}$$

and, we have

$$\|u(x,t)-u_{NM}(x,t)\| \le (\frac{\lambda\sqrt{T}}{r!})(1+\frac{T}{l})^r. \tag{47}$$

□

5. Numerical examples

In this section, the theoretical considerations introduced in the previous sections will be illustrated with some examples. In the process of computation, all the symbolic and numerical computations were performed by using the package of Maple 16. We tested the accuracy and the stability of the method which was presented in this paper by performing the mentioned method for different

values of N and M. To study the convergence behavior of the multi-step pseudo-spectral method, we apply the root mean square (RMS) that is

$$RMS(u) = \sqrt{\frac{1}{NM}\sum_{i=0}^{N}\sum_{j=0}^{M}|u(x_i,t_j)-u^k(x_i,t_j)|^2}.$$

Example 5.1. As the first example, we consider the inverse reaction- diffusion equation that can be found in [17]:

$$\begin{cases} u_t(x,t) = a(t)u_{xx}(x,t), & 0<x<1, \quad 0<t\le T,\\ u(x,0) = e^{\frac{x}{2}}, & 0\le x\le 1,\\ u(0,t) = \frac{1+2t^3}{1+t^3}+\sin(\frac{t}{2}), & 0<t\le T,\\ u(1,t) = \sqrt{e}(\frac{1+2t^3}{1+t^3}+\sin(\frac{t}{2})), & 0<t\le T,\\ u(x^*,t) = 1.13315(\frac{1+2t^3}{1+t^3}+\sin(\frac{t}{2})), & 0<x^*<1, \quad 0\le t\le T. \end{cases} \tag{48}$$

The exact solution to this problem is $u(x,t) = e^{\frac{x}{2}}(\frac{1+2t^3}{1+t^3}+\sin(\frac{t}{2}))$, $a(t) = \frac{2[6t^2+(1+t^3)^2\cos(\frac{t}{2})]}{(1+t^3)[1+2t^2+(1+t^3)\sin(\frac{t}{2})]}$. We solved this example by using the method described in this paper for $N = M = 9, 10$ with the number of sub-interval $l = 10$, $x^* = 0.25$ and $T = 1$. Table 1 shows the absolute value of errors, using the proposed method and compares the result with the results obtained by using the technique in [17]. Table 2 shows the RMS errors for various values of M and N. In addition, the graphs of the error functions are plotted in Figs. 1-4 with $N = M = 10$.

Table 1: Numerical results of Example 4.1

	Method [17]		Presented Method	
t	$N=9, M=9$	$N=10, M=10$	$N=9, M=9$	$N=10, M=10$
0.1	4.9E-6	1.0E-7	6.1E-13	3.7E-17
0.3	1.3E-6	1.0E-7	8.3E-12	2.2E-18
0.5	2.2E-6	1.9E-7	5.0E-12	7.1E-15
0.7	2.6E-6	2.1E-7	8.1E-11	8.3E-14
0.9	2.6E-6	2.1E-7	2.6E-10	2.9E-13

Table 2: The RMS error for $u(x,t)$ and $a(t)$ for Example 4.1

	Method [17]		Presented Method	
x	$N=5, M=5$	$N=8, M=8$	$N=5, M=5$	$N=8, M=8$
RMS(u)	5.2E-4	6.2E-5	8.4E-7	9.4E-11
RMS(a)	3.9E-2	4.2E-3	3.3E-5	2.1E-10

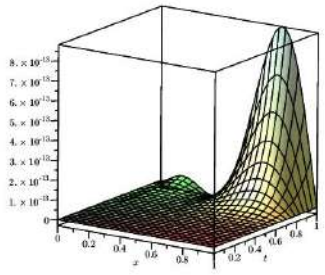


Figure 1: Graph of $u(x,t) - u_{NM}(x,t)$ for $N = M = 10$

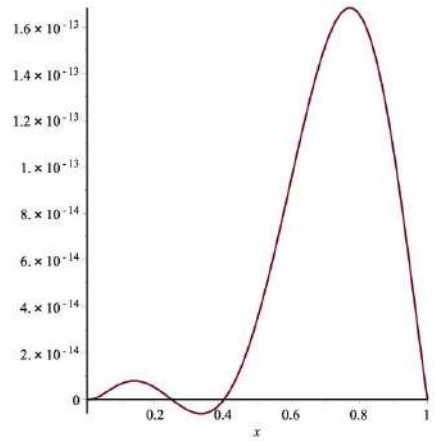


Figure 2: Graph of $u(x,t) - u_{NM}(x,t)$ for $t = 0.9$ and $N = M = 10$

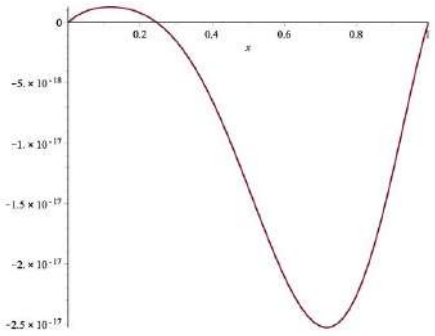


Figure 3: Graph of $u(x,t) - u_{NM}(x,t)$ for $t = 0.1$ and $N = M = 10$

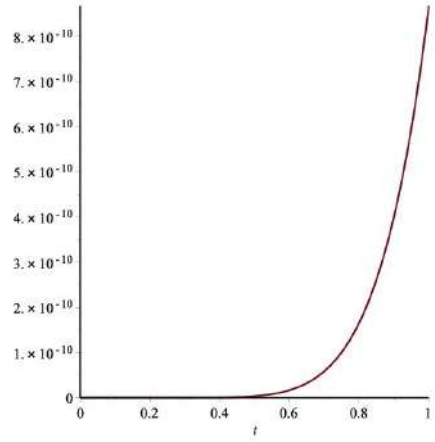


Figure 4: Graph of $a(t) - a_{app}(t)$ for $N = M = 10$

Example 5.2. In this example, we solve the problem (4) with $T = 3$ and $x^* = \frac{2}{7}$ and the initial boundary and the extra measurement condition as following:

$$
(49)\qquad \begin{cases} u_t(x,t) = a(t)u_{xx}(x,t) + (-2t^2-1)u(x) + 2t(x+t)e^{(t^2+1)x}, \\ \qquad\qquad 0 < x < 1, \quad 0 < t \le T, \\ u(x,0) = e^x, \qquad 0 \le x \le 1, \\ u(0,t) = 1, \qquad 0 < t \le T, \\ u(1,t) = e, \qquad 0 < t \le T, \\ u(x^*,t) = e^{\frac{2(t^2+1)}{7}}, \\ \qquad\qquad 0 < x^* < 1, \quad 0 \le t \le T. \end{cases}
$$

Table 3: Maximum error of numerical solution to $u(x,t)$ for Example 4.2

Sub-interval on t axes	$N=7, M=7$	$N=10, M=10$
$[0.0, 0.3]$	1.0E-11	1.0E-14
$[0.6, 0.9]$	2.2E-11	1.3E-13
$[0.9, 1.2]$	2.6E-11	2.3E-13
$[1.2, 1.5]$	2.6E-11	2.4E-13
$[1.5, 1.8]$	2.6E-10	2.4E-12
$[1.8, 2.1]$	1.5E-10	1.7E-12
$[2.1, 2.4]$	1.9E-10	3.5E-12
$[2.4, 2.7]$	1.1E-9	3.7E-11
$[2.7, 3.0]$	3.6E-9	3.8E-11

The exact solution of (49) is $u(x,t) = e^{(t^2+1)x}$ and $a(t) = \frac{1}{(1+t^2)^2}$. The propose of this example is to show that the approximate solution have a good degree of accuracy even for large values of T. The data in Table 3 shows the maximum errors between the calculated and the exact solution of $u(x,t)$ with $M = N = 7, 10$ and $l = 10$. The graphs of the error functions are plotted in Figs. 5-8 with $N = M = 10$. It is seen that the approximate solutions have a good degree of accuracy even for large values of T.

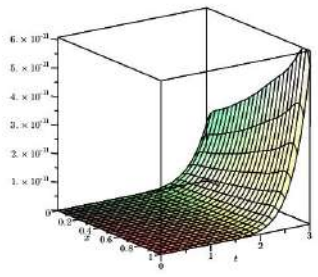


Figure 5: Graph of $u(x,t) - u_{approx}(x,t)$ for $N = M = 10$

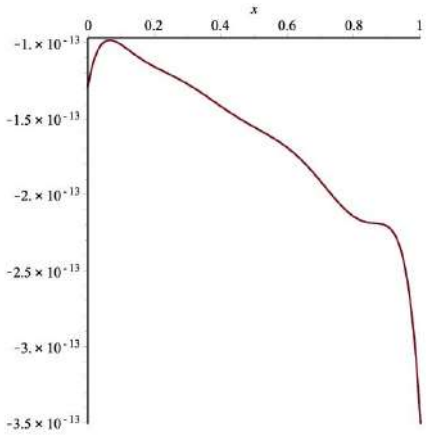


Figure 6: Graph of $u(x,t) - u_{NM}(x,t)$ for $t = 1$ and $N = M = 10$

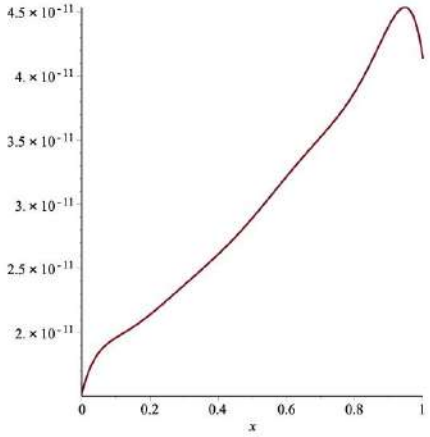


Figure 7: Graph of $u(x,t) - u_{NM}(x,t)$ for $t = 2.9$ and $N = M = 10$

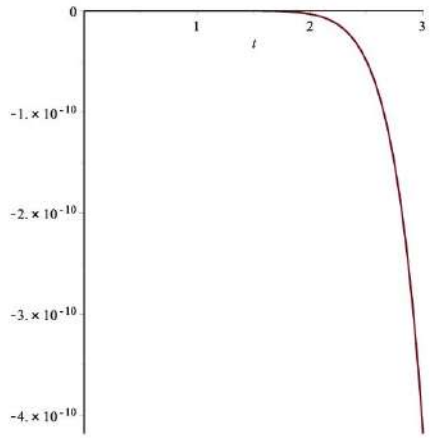


Figure 8: Graph of $a(i) - a_{app}(i)$ for $N = M = 10$

6. Conclusion

The main objective of this work is to construct a new approximation to the solution of the inverse reaction-diffusion equations system. For this purpose, the pseudo-spectral multi-step method with two-dimensional Chebyshev interpolation, are implemented. The main advantage of our method is that, since the main problem is solved independently on the subintervals, it provides reasonable approximations for large T. In addition, the proposed method is numerically stable. Numerical findings show that there is a good agreement between the obtained results and the exact values that demonstrates the validity and the accuracy of the proposed method for such problems and gives the method a wider applicability.

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Accepted: March 15, 2020

On an analytical and numerical solutions within the conformable fractional derivative for Fitzhugh-Nagumo fractional equation

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Abstract. In this article, a fractional Ricatti expansion method in order to solve a nonlinear fractional Fitzhugh-Nagumo equation in the frame of conformable derivatives, we also apply it to the nonlinear fractional Newell-Whitehead-Segel and Zeldovich equations which are a particular case of fractional Fitzhugh-Nagumo equation. In order to illustrate the accuracy and validity of this method, some numerical solution are given.

Keywords: conformable fractional derivative, fractional Ricatti expansion method, nonlinear fractional Fitzhugh-Nagumo equation.

1. Introduction

In 2014, R. Khalil et al was introduced the conformable fractional derivative (CFD) [14] in this work, we apply the fractional Ricatti expansion method using the CFD in order to solve Fitzhugh-Nagumo (FN) equation which is an important nonlinear reaction-diffusion equation, it arises in the field of the transmission of nerve impulses [11], [12], and also dealt with in the transmission of thermal energy in thermodynamics and circuit theory.

The fractional Fitzhugh-Nagumo (FN) equation [9], [16] has the form :

$$D_t^{\alpha} u(x,t) - D_x^{2\alpha} u(x,t) = u(u-\lambda)(1-u), \tag{1}$$

where λ indicates an arbitrary constant.

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If $\lambda = -1$ we have the fractional Newell-Whitehead-Segel (NWS) equation [9], [16] which is represented as:

$$D_t^{\alpha} u(x,t) - D_x^{2\alpha} u(x,t) = u - u^3. \tag{2}$$

If $\lambda = 0$ we obtain the fractional Zeldovich equation [9], [16] written as:

$$D_t^{\alpha} u(x,t) - D_x^{2\alpha} u(x,t) = u^2 - u^3. \tag{3}$$

Many works have been published about the solution of the nonlinear fractional differential equations, such as the fractional sub-equation method [2], [3], the first integral method [4], [7], variational iteration method [5], [6] and the (G'/G)-expansion method [8], [17], [9]. Most of these methods studied the exact solution of fractional equations using Jumarie's modified Riemann-Liouville derivative and the chain rule, for fractional derivative case, in his paper, Cheng-shi Liu [15] gave two counter examples and stated the opinion that Jumarie's with the two basic formulae of derivative are incorrect, also Xiaohua Liu [10] gave one counterexample to show that formula of the the chain rule is incorrect.

Recently, Xiaohua Liu [10] introduced the fractional Ricatti expansion method which that transforms fractional partial differential equations into ordinary fractional differential equations (OFDE), by using the fractional conformable derivative, without using the Jumarie's modified Riemann-Liouville derivative and the chain rule method. The aim of this work is to apply the Ricatti expansion method to the fractional Fitzhugh-Nagumo (FN) equation.

This paper is organized as follows. In section 2, we give some definition of conformable fractional derivative. In section 3, we present the fractional Ricatti expansion method. In section 4, we applied the proposed method to solve a space-time fractional Fitzhugh-Nagumo equation, Newell-Whitehead-Segel(NWS) and Zeldovich equation. Finally, in section 5, the main conclusion is drawn.

2. Basic tools

We enlist some definitions and basic results (see, for more details, [1], [13]).

A real function $f(x), x > 0$ is said to be in the space $C_{\alpha,\alpha\in R}$ if there exists a real number $p(> \alpha)$, such that $f(x) = x^p f_1(x)$ where $f_1(x) = C\,[0,\infty)$. Clearly $C_\alpha \subset C_\beta$ if $\beta \leq \alpha$.

A function $f(x), x > 0$ is said to be in space C_α^m, $m \in N \cup \{0\}$, if $f_{(m)} \in C_\alpha$.

The left sided Riemann-Liouville fractional integral of order $\mu > 0$, of a function $f \in C_\alpha, \alpha \geq 1$ is defined as:

$$I^{\mu} f(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1} f(\tau) d\tau, \mu > 0, t > 0, \tag{4}$$

$$I^0 f(t) = f(t), \tag{5}$$

The left sided Riemann-Liouville fractional derivative of f, $f \in C_{-1}^{m}$, $m \in N \cup \{0\}$ of order $\alpha > 0$, is defined as:

$$D^{\mu} f(t) = \frac{d^m}{dt^m} I^{m-\mu} f(t), m-1 < \mu \leq m, m \in N. \tag{6}$$

The right sided Caputo fractional derivative of f, $f \in C_{-1}^{m}$, $m \in N \cup \{0\}$ of order $\alpha > 0$, is defined as:

$${}^{C}D_{0^+}^{\alpha} f(t) = \begin{cases} \left[I_{0^+}^{m-\mu} f^{(m)}(t) \right], & m-1 < \mu \leq m, m \in N, \\ \dfrac{d^m}{dt^m} f(t) \mu = m. \end{cases} \tag{7}$$

Note that

(i) $I^{\mu} t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\mu+1)} t^{\gamma+\mu}, \qquad \mu > 0, \gamma > -1, t > 0.$

(ii) $I_{0^+}^{\mu}\, {}^{C}D_{0^+}^{\mu} f(t) = f(t) - \sum_{k=0}^{m-1} f^{(k)}(0_+) \frac{t^k}{k!}$, $m-1 < \mu \leq m$, $m \in N$.

(iii) ${}^{C}D_{0^+}^{\mu} f(t) = D^{\mu} \left(f(t) - \sum_{k=0}^{m-1} f^{(k)}(0_+) \frac{t^k}{k!} \right)$, $m-1 < \mu \leq m$, $m \in N$.

(iv) $D^{\beta} I^{\alpha} f(t) = \begin{cases} I^{\alpha-\beta} f(t), & \text{if } \alpha > \beta, \\ f(t), & \text{if } \alpha = \beta, . \\ D^{\beta-\alpha} f(t), & \text{if } \alpha < \beta, \end{cases}$

(v) $D_{0^+}^{\mu} D^m f(t) = D^{\mu+m} f(t)$, $m = 0, 1, 2, ..., n-1 < \mu < n$.

For a given function $f : [0, \alpha\infty) \to \Re$, "the conformable fractional derivative" of f with α order is denoted by

$$T_{\alpha} f(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon t^{1-\alpha}) - f(t)}{\epsilon}, \tag{8}$$

for $t > 0$, and $\alpha \in (0, 1]$.

2.1 Theorem [10]

Let $\alpha \in (0, 1]$ and f, g $\alpha-$differentiable at $t > 0$, then

1. $T_{\alpha}(af + bg) = aT_{\alpha}(f) + bT_{\alpha}(g)$, for all $a, b \in \mathbb{R}$.
2. $T_{\alpha}(t^p) = pt^{p-\alpha}$ for all $p \in \mathbb{R}$.
3. $T_{\alpha}(\lambda) = 0$ for all constant functions $f(t) = \lambda$.
4. $T_{\alpha}(fg) = fT_{\alpha}(g) + gT_{\alpha}(f)$.
5. $T_{\alpha}(\frac{f}{g}) = \frac{gT_{\alpha}(f) - fT_{\alpha}(g)}{g^2}$.
6. if in addition, f is differentiable, then $T_{\alpha}(f)(t) = t^{1-\alpha} \frac{df}{dt}(t)$.

Conformable fractional differential operator satisfies some critical fundamental properties like the chain rule, Taylor series expansion and Laplace transform.

The above properties play an important role in the fractional Ricatti expansion method.

2.2 Proposition [10]

For $\alpha \in (0,1]$ f, g a conformable fractional derivative, $f(g)$ is differential at point g, g is continuous at point x, then the chain rule

$$T_\alpha f(g)(x) = \frac{df}{dg} T_\alpha g(x) \tag{9}$$

holds, where "$\frac{d}{dg}$" denote the derivative of integer-order with respect to g.

3. Analysis of the method

In this section, we focus the general description of the fractional Ricatti expansion method. We consider the following type of nonlinear FDE

$$P(u, T_\alpha u(x), T_\alpha u(t), T_{2\alpha} u(x), T_{2\alpha} u(t), ...) = 0, \tag{10}$$

$T_\alpha u(x), T_\alpha u(t)$ are conformable fractional derivatives of $u = u(x,t)$ (see [14]) where $T_\alpha u(x) = \frac{D^\alpha u(x)}{Dx^\alpha}$, u is an unknown function, and P is a polynomial.
Step 1. Using the traveling wave transformation:

$$u(x,t) = u(\xi), \xi = x + ct, \tag{11}$$

where c is a constant to be determined later.

Substituting the (11) in (10), we transform (10) to a nonlinear fractional ordinary differential equation of u with the same order:

$$P(u, T_\alpha u(\xi), c^\alpha T_\alpha u(\xi), T_{2\alpha} u(\xi), c^{2\alpha} T_{2\alpha} u(\xi), ...) = 0. \tag{12}$$

Step 2. Suppose that $u(\xi)$ can be expressed by a finite power series of $F(\xi)$ as follows:

$$u(\xi) = \sum_{i=0}^{n} a_i F^i(\xi), a_n \neq 0, \tag{13}$$

where $a_i (i = 0, 1, ..., n)$ are undetermined constants, n is a positive integer determined by balancing the order between the highest order derivative term and nonlinear term order in (12), and $F(\xi)$ satisfyes the following fractional Riccati equation [18]:

$$T_\alpha F(\xi) = m + F^2(\xi), \tag{14}$$

m is a parameter, it is easy to proof [10] that the solutions of (14) are for $m < 0$

$$F(\xi) = -\sqrt{-m} \tanh(\frac{2-\alpha}{\alpha} \sqrt{-m} \xi^{\frac{\alpha}{2-\alpha}}), \tag{15}$$

$$F(\xi) = -\sqrt{-m} \coth(\frac{2-\alpha}{\alpha} \sqrt{-m} \xi^{\frac{\alpha}{2-\alpha}}) \tag{16}$$

with the chain rule of comfortable fractional derivative (9) and

$$D_x^\alpha f(g(x)) = f'_g[g(x)]D_x^\alpha g(x) = D_g^\alpha f(g(x))[g'^\alpha.$$

Step 3. Substituting (13) into (12), using (14) and collecting the terms with the same order of $F(\xi)$ setting each coefficient of the polynomial to zero yields a system of algebraic equations for $a_0, a_2, a_3..., a_n$ and c.
Step 4. Solving the equations system in Step 3, and using the solutions of (14), we found a variety of exact solutions of (10).

3.1 Fractional Ricatti expansion method for finding the solutions of fractional Fitzhugh-Nagumo

Now we consider the nonlinear fractional Fitzhugh-Nagumo equation of the form

$$D_t^\alpha u(x,t) - D_x^{2\alpha} u(x,t) = u(u-\lambda)(1-u), \tag{17}$$

where λ indicates an arbitrary constant.

Using the traveling wave transformation (11) where c is a nonzero arbitrary constant, we substitute (11) in (17) we can transforme it an ordinary fractional differential equation (OFDE)

$$D_\xi^\alpha u(\xi) - D_\xi^{2\alpha} u(\xi) = u(u-\lambda)(1-u) = -u^3 + (1+\lambda)u^2 - \lambda u. \tag{18}$$

Suppose that $u(\xi)$ can be expressed in the following form

$$u(\xi) = \sum_{i=0}^{n} a_i F^i(\xi), a_n \neq 0. \tag{19}$$

By balancing the order between the highest order derivative term and the nonlinear term order in (18), we can obtain for $n = 1$ the equation (20)

$$u(\xi) = a_0 + a_1 F^1(\xi), \tag{20}$$

where $F(\xi)$ are solutions of fractional Ricatti equation (14).

Substituting (20) into (18) and collecting all the terms with the same power together we obtain

$$\begin{aligned}
&cma_1 + a_0^3 - (1+\lambda)a_0^2 + \lambda a_0 = 0,\\
&3a_0^2 - 2m - 2(1+\lambda)a_0 + \lambda = 0,\\
&c + 3a_0a_1 - (1+\lambda)a_1 = 0,\\
&a_1^3 - 2a_1 = 0,
\end{aligned}$$

using the symbolic computation in order to solve this system we get:

$$a_0 = \frac{(1+\lambda) \pm \sqrt{\lambda^2 - \lambda + 1 - 6m\lambda}}{3}, \quad a_1 = \pm\sqrt{2}, \quad c = -3a_0a_1 + (1+\lambda)a_1).$$

Suppose that $m < 0$, and $0 < \alpha \leq 1$, we obtain the equation the following solitary wave solutions

$$u_1(x,t) = (\frac{(1+\lambda) \pm \sqrt{\lambda^2 - \lambda + 1 - 6m}}{3}) \mp \sqrt{-2m}[\tanh(\frac{2-\alpha}{\alpha}\sqrt{-m}(x+ct)^{\frac{\alpha}{2-\alpha}})] \tag{21}$$

$$u_2(x,t) = (\frac{(1+\lambda) \pm \sqrt{\lambda^2 - \lambda + 1 - 6m}}{3}) \mp \sqrt{-2m}[\coth(\frac{2-\alpha}{\alpha}\sqrt{-m}(x+ct)^{\frac{\alpha}{2-\alpha}})]. \tag{22}$$

4. Example

This section focuses on giving examples to give a more insight on the use of the fractional Ricatti expansion method

4.1 Example 1

Consider the fractional Newell-Whitehead-Segel with $\lambda = -1$ fractional Fitzhugh-Nagumo becomes fractional Newell-Whitehead-Segel in the form

$$D_t^{\alpha} u(x,t) - D_x^{2\alpha} u(x,t) = u - u^3. \tag{23}$$

After some algebraic manipulations, we obtain $a_0 = \pm\frac{1}{2}, a_1 = \pm\sqrt{2}, m = -\frac{1}{8}$ and $c = \pm\frac{3\sqrt{2}}{2}$, then the following solutions of equation (23) are

$$u_3(x,t) = (\frac{1}{2}) + \frac{1}{2}[\tanh(\frac{2-\alpha}{\alpha}\sqrt{\frac{1}{8}}(x + (\frac{-3\sqrt{2}}{2})t)^{\frac{\alpha}{2-\alpha}})], \tag{24}$$

$$u_4(x,t) = (\frac{1}{2}) + \frac{1}{2}[\coth(\frac{2-\alpha}{\alpha}\sqrt{\frac{1}{8}}(x + (\frac{-3\sqrt{2}}{2})t)^{\frac{\alpha}{2-\alpha}})], \tag{25}$$

$$u_5(x,t) = (-\frac{1}{2}) - \frac{1}{2}[\tanh(\frac{2-\alpha}{\alpha}\sqrt{\frac{1}{8}}(x + (\frac{-3\sqrt{2}}{2})t)^{\frac{\alpha}{2-\alpha}})], \tag{26}$$

$$u_6(x,t) = (-\frac{1}{2}) - \frac{1}{2}[\coth(\frac{2-\alpha}{\alpha}\sqrt{\frac{1}{8}}(x + (\frac{-3\sqrt{2}}{2})t)^{\frac{\alpha}{2-\alpha}})], \tag{27}$$

$$u_7(x,t) = (-\frac{1}{2}) + \frac{1}{2}[\tanh(\frac{2-\alpha}{\alpha}\sqrt{\frac{1}{8}}(x + (\frac{3\sqrt{2}}{2})t)^{\frac{\alpha}{2-\alpha}})], \tag{28}$$

$$u_8(x,t) = (-\frac{1}{2}) + \frac{1}{2}[\coth(\frac{2-\alpha}{\alpha}\sqrt{\frac{1}{8}}(x + (\frac{3\sqrt{2}}{2})t)^{\frac{\alpha}{2-\alpha}})], \tag{29}$$

$$u_9(x,t) = (\frac{1}{2}) - \frac{1}{2}[\tanh(\frac{2-\alpha}{\alpha}\sqrt{\frac{1}{8}}(x + (\frac{3\sqrt{2}}{2})t)^{\frac{\alpha}{2-\alpha}})], \tag{30}$$

$$u_{10}(x,t) = (-\frac{1}{2}) + \frac{1}{2}[\coth(\frac{2-\alpha}{\alpha}\sqrt{\frac{1}{8}}(x + (\frac{3\sqrt{2}}{2})t)^{\frac{\alpha}{2-\alpha}})]. \tag{31}$$

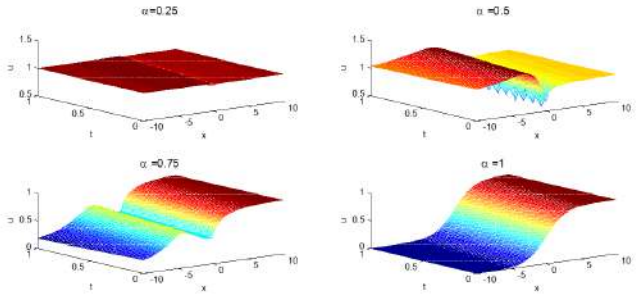


Figure 1: The solution $u_3(x,t)$ if $\lambda = -1, a_0 = 1/2, a_1 = \sqrt{2}$

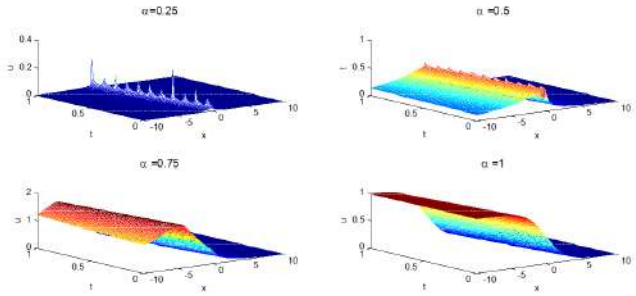


Figure 2: The solution $u_9(x,t)$ if $\lambda = -1, a_0 = 1/2, a_1 = -\sqrt{2}$

$u_3(x,t)$ is described for different values of $\alpha = 0,25; \alpha = 0,5; \alpha = 0,75$ and $\alpha = 1$. In a finite domain see Figure 1, and in Figure 2, we can see that when the fractional order α is close to 1, the solution curve is very smooth than when α is close to 0.

4.2 Example 2

Consider the fractional Zeldovich equation

If $\lambda = 0$ fractional Fitzhugh-Nagumo becomes fractional Zeldovich with the form

$$D_t^{\alpha} u(x,t) - D_x^{2\alpha} u(x,t) = u^2 - u^3. \tag{32}$$

By doing some algebraic manipulations , we obtain respectively,

$$\begin{cases} cma_1 + a_0^3 - a_0^2 = 0, \\ 3a_0^2 - 2m - 3a_0^2 = 0, \\ c + 3a_0 a_1 - a_1 = 0, \\ a_1^3 - 2a_1 = 0. \end{cases} \tag{33}$$

We solve the algebraic system (33), then the solutions are given by: $a_0 = \pm\frac{1}{3}, a_1 = \pm\sqrt{2}, m = \frac{-2a_0+3a_0^2}{2}$ and $c = a_1 - a_0 a_1$. So, we have the following solutions of equation (32), which are

$$u_{11}(x,t) = (\frac{1}{3}) \pm \sqrt{2}\sqrt{\frac{1}{6}}[\tanh(\frac{2-\alpha}{\alpha}\sqrt{\frac{1}{6}}(x+ct)^{\frac{\alpha}{2-\alpha}})], \tag{34}$$

$$u_{12}(x,t) = (\frac{1}{3}) \pm \sqrt{2}\sqrt{\frac{1}{6}}[\coth(\frac{2-\alpha}{\alpha}\sqrt{\frac{1}{6}}(x+ct)^{\frac{\alpha}{2-\alpha}})]. \tag{35}$$

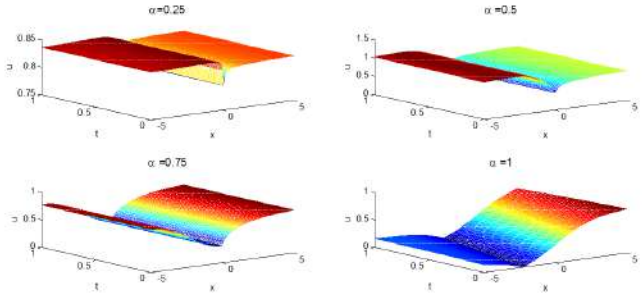


Figure 3: The solution $u_{11}(x,t)$ if $\lambda = 0, a_0 = 1/3, a_1 = \sqrt{2}$

From Figure 3 we can see, with the decrease of the order α, the wave solitary gets more acute than in $\alpha = 1$.

In Figure 4 one can see, that with the increasing of the order of α, it gets more frequent that the periodic wave reaches its local climax.

5. Conclusion

In this paper, we have a introduced the new fractional Riccati expansion method within the conformable fractional derivative for solving the nonlinear fractional

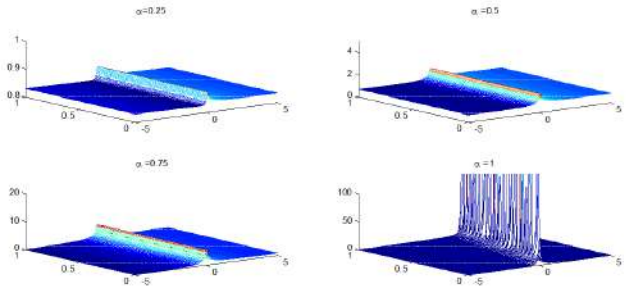


Figure 4: The solution $u_{12}(x,t)$ if $\lambda = 0, a_0 = 1/3, a_1 = \sqrt{2}$

Fitzhugh-Nagumo equation, fractional Newell-Whitehead-Segel equation and fractional Zeldovich equation, without using Jumarie's modified Riemann-Liouville derivative. This method is also applicable to many other nonlinear fractional equations which arise in application fields. Matlab has been used for computations and programming in this paper.

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Accepted: May 05, 2020

A note on restriction semigroups

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Abstract. It is proved that if S is a weakly reductive semigroup, then S is a (proper; Clifford; E-reflexive) restriction semigroup if and only if $\Omega(S)$ is a (proper; Clifford; E-reflexive) restriction semigroup and $\Pi(S)$ is a $(2,1,1)$-subsemigroup of $\Omega(S)$.
Keywords: restriction semigroup, translational hull, weakly reductive semigroup

1. Introduction

Let S be a semigroup. A mapping $\lambda : S \to S$ is called a *left translation* of S if for any $a, b \in S$, $\lambda(ab) = (\lambda a)b$. Similarly, a mapping $\rho : S \to S$ is a *right translation* if for any $a, b \in S$, $(ab)\rho = a(b\rho)$. A left translation λ and a right translation ρ of S are said to be *linked* if for any $a, b \in S$, $a(\lambda b) = (a\rho)b$. In this case, we call the pair (λ, ρ) a *bitranslation* of S. The set $\Lambda(S)$ of all left translations and the set $I(S)$ of all right translations of S form semigroups under the composition of mappings. The *translational hull* of S is the subsemigroup $\Omega(S)$ of $\Lambda(S) \times I(S)$ which consists of all bitranslations (λ, ρ). It is well known that $\Omega(S)$ forms a submonoid of $\Lambda(S) \times I(S)$ under the multiplication:

$$(\lambda, \rho)(\lambda', \rho') = (\lambda\lambda', \rho\rho'),$$

where $\lambda\lambda'$ and $\rho\rho'$ respectively represent the composition operation of λ' , λ and of ρ, ρ'. For any $a \in S$, we denote by λ_a [resp. ρ_a] the *inner* left [resp. right] translation,which is defined as $\lambda_a(x) = ax$ [resp. $(x)\rho_a = xa$]. It is easy to check that the pair $\pi_a = (\lambda_a, \rho_a)$ is the *inner bitranslation* induced by a. It is easy to check that the set $\Pi(S) = \{\pi_a : a \in S\}$ is a subsemigroup of $\Omega(S)$. Indeed, $\Pi(S)$ is still an ideal of $\Omega(S)$ (for details, see [10, p.30]).

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The translational hull of a semigroup, introduced by Petrich in 1970, plays an important role in the theory of semigroups. There are many authors having been researching translational hulls of semigroups. It was proved that if S is an inverse semigroup (an adequate semigroup; an ample semigroup, respectively), then so is $\Omega(S)$ (see [1, 2]). After then, there are several classes of semigroups whose translational hulls were researched; for example, strongly right (left) type-A semigroups (see [6]), E-reflexive type-A semigroups (see [7]), inverse wpp semigroups (see [8]), type B semigroups (see [9]), etc.

Restriction semigroups become a larger class of semigroups, including some important class of semigroups; such as, adequate semigroups, especially, inverse semigroups, as its proper subclasses. So, it is a natural topic to study characterizations of translational hulls of restriction semigroups. This is the main aim of this note.

2. Results

To begin with, we recall several concepts.

A *left restriction semigroup* is defined to be an algebra of type $(2, 1)$, more precisely, an algebra $S = (S, \cdot, ^+)$ where $(S, \cdot)$ is a semigroup and $^+$ is a unary operator such that the following identities are satisfied:

$$x^+x = x,\ x^+y^+ = y^+x^+,\ (x^+y)^+ = x^+y^+,\ xy^+ = (xy)^+x. \tag{2.1}$$

A *right restriction semigroup* is dually defined, that is, it is an algebra $(S, \cdot, *)$ satisfying the duals of the identities (2.1). If $S = (S, \cdot, +, *)$ is an algebra of type $(2, 1, 1)$ where $S = (S, \cdot, +)$ is a left restriction semigroup and $S = (S, \cdot, *)$ is a right restriction semigroup and the identities

$$(x^+)^* = x^+,\ (x^*)^+ = x^* \tag{2.2}$$

hold, then it is called a *restriction semigroup*. By definition, the defining properties of a restriction semigroup are left-right dual. Therefore in the sequel dual definitions and statements will not be explicitly formulated. Gould [5] pointed out that in a restriction semigroup, the following identities are satisfied:

$$(xy)^* = (x^*y)^*,\ (xy)^+ = (xy^+)^+. \tag{2.3}$$

Let S be a restriction semigroup. By (2.2), we have $\{x^+ : x \in S\} = \{x^* : x \in S\}$. This set is called the *set of projections of* S and denoted by $P(S)$. Again by (2.1) and its dual, $P(S)$ is a $(2, 1, 1)$-subsemigroup of S which is indeed a semilattice.

(i) If all projections of S are central, then S is called a *Clifford restriction semigroup*.

(ii) If for all $e \in P(S)$ and $x, y \in S$, $exy \in P(S)$ implies $eyx \in P(S)$, then we call S an *E-reflexive restriction semigroup*.

On S, define: for any $a, b \in S$

$$\begin{aligned} a\sigma b &\Leftrightarrow (\exists e \in P(S))ea = eb; \\ &\Leftrightarrow (\exists f \in P(S))af = bf. \end{aligned}$$

Evidently, σ is an equivalence on S. A restriction semigroup S is *proper* provided for any $a, b \in S$,

(i) if $a^+ = b^+$ and $a\sigma b$, then $a = b$; and

(ii) if $a^* = b^*$ and $a\sigma b$, then $a = b$.

Among restriction semigroups, the notion of subalgebra is understood in type $(2, 1, 1)$, which is emphasised by using the expression $(2, 1, 1)$-subsemigroup.

Recall that a semigroup S is *weakly reductive* if for any $a, b \in S$, $xa = xb$ and $ax = bx$ for all $x \in S$ implies that $a = b$. We have that any restriction semigroup is weakly reductive. Indeed, let $a, b \in S$ and assume that $xa = xb$ and $ax = bx$ for all $x \in S$, we have $a = a^+b$, so that $a^+ = (a^+b)^+ = a^+b^+$; similarly, $b^+ = b^+a^+$, thus $a^+ = b^+$ since $P(S)$ is a semilattice, it follows that $a = a^+b = b^+b = b$, as required.

We now describe our main result:

Theorem 2.1. *If S is a weakly reductive semigroup, then S is a (proper; Clifford; E-reflexive) restriction semigroup if and only if $\Omega(S)$ is a (proper; Clifford; E-reflexive) restriction semigroup and $\Pi(S)$ is a $(2, 1, 1)$-subsemigroup of $\Omega(S)$.*

3. Proofs

To begin with, we prove the following lemma.

Lemma 3.1. *If S is a restriction semigroup, then $S \cong \Pi(S)$.*

Proof. It is obvious that the mapping $\pi : S \to \Pi(S);\ a \mapsto (\lambda_a, \rho_a)$ is a surjective homomorphism. In condition of weakly reductivity, π is still injective. Therefore π is an isomorphism, as required. □

The following lemma is well known; for example, see [2, 7].

Lemma 3.2. *Let S be a restriction semigroup. Then the following statements are equivalent:*

(i) $(\forall e \in P(S))\ \lambda_1 e = \lambda_2 e$ *(respectively, $e\rho_1 = e\rho_2$);*

(ii) $\lambda_1 = \lambda_2$ *(respectively, $\rho_1 = \rho_2$).*

Lemma 3.3. *Let S be a restriction semigroup. If $(\lambda_i, \rho_i) \in \Omega(S)$ with $i = 1, 2$, then the following statements are equivalent:*

(i) $(\lambda_1, \rho_1) = (\lambda_2, \rho_2)$;

(ii) $\lambda_1 = \lambda_2$;

(iii) $\rho_1 = \rho_2$.

Proof. We need only to verify (ii)⇒(i) because (i)⇒(ii) is obvious, and (i)⇔(ii) is dual to (i)⇔(iii). Suppose that $\lambda_1 = \lambda_2$, then $\lambda_1 e = \lambda_2 e$ for any $e \in P(S)$, and so $e\rho_1 = (e\rho_1)(e\rho_1)^* = e[\lambda_1(e\rho_1)^*] = e[\lambda_2(e\rho_1)^*] = (e\rho_2)(e\rho_1)^*$; similarly, $e\rho_2 = (e\rho_1)(e\rho_2)^*$. Further,

$$e\rho_1 = (e\rho_2)(e\rho_1)^* = (e\rho_1)(e\rho_2)^*(e\rho_1)^* = (e\rho_1)(e\rho_1)^*(e\rho_2)^* = (e\rho_1)(e\rho_2)^* = e\rho_2.$$

It follows that for any $a \in S$, $a\rho_1 = a(a^*\rho_1) = a(a^*\rho_2) = a\rho_2$; that is, $\rho_1 = \rho_2$. □

Let S be a restriction semigroup. For $(\lambda, \rho) \in \Omega(S)$, we define $\lambda^+, \lambda^*, \rho^+$ and ρ^* as follows:

$$\lambda^+ a = (a^+\rho)^+ a; \qquad \lambda^* a = (\lambda a^+)^* a;$$
$$a\rho^+ = a(a^*\rho)^+; \qquad a\rho^* = a(\lambda a^*)^*.$$

Lemma 3.4. *Let S be a restriction semigroup with the set of projections $P(S)$. Then for all $e \in P(S)$ and $a \in S$,*

(i) $\lambda^* e = e\rho^* = (\lambda e)^*$;

(ii) $\lambda^+ e = e\rho^+ = (e\rho)^+$;

(iii) $\lambda^* e,\ e\rho^*, \lambda^+ e, e\rho^+ \in P(S)$;

(iv) $(\lambda a)^* = (\lambda^* a)^*,\ (a\rho)^+ = (a\rho^+)^+$.

Proof. (i) By definition, $\lambda^* e = (\lambda e)^* e = e(\lambda e)^* = e\rho^*$, and notice that $(\lambda e)^* e = ((\lambda e)e)^* = (\lambda e)^*$, we have $\lambda^* e = e\rho^* = (\lambda e)^*$.

(ii) Dual to (i).

(iii) Obvious.

(iv) Clearly, $\lambda a = \lambda(a^+ a) = (\lambda a^+)a$, then $(\lambda a)^* = [(\lambda a^+)a]^* = [(\lambda a^+)^* a]^* = (\lambda^* a)^*$. Thus, $(\lambda a)^* = (\lambda^* a)^*$, similarly, $(a\rho)^+ = (a\rho^+)^+$. □

Define $(\lambda, \rho)^+ = (\lambda^+, \rho^+)$ and $(\lambda, \rho)^* = (\lambda^*, \rho^*)$.

Similar to [2], we may verify the following lemma and we omit the detail.

Lemma 3.5. *Let $(\lambda, \rho) \in \Omega(S)$. If S is a restriction semigroup, then $(\lambda, \rho)^+$, $(\lambda, \rho)^* \in \Omega(S)$.*

Lemma 3.6. *Let $(\lambda, \rho), (\lambda_1, \rho_1),\ (\lambda_2, \rho_2) \in \Omega(S)$. If S is a restriction semigroup, then:*

(i) $(\lambda, \rho)^+(\lambda, \rho) = (\lambda, \rho)$;

(ii) $(\lambda_1,\rho_1)^+(\lambda_2,\rho_2)^+ = (\lambda_2,\rho_2)^+(\lambda_1,\rho_1)^+$;

(iii) $[(\lambda_1,\rho_1)^+(\lambda_2,\rho_2)]^+ = (\lambda_1,\rho_1)^+(\lambda_2,\rho_2)^+$;

(iv) $(\lambda_1,\rho_1)(\lambda_2,\rho_2)^+ = [(\lambda_1,\rho_1)(\lambda_2,\rho_2)]^+(\lambda_1,\rho_1)$;

(v) $[(\lambda,\rho)^+]^* = (\lambda,\rho)^+$.

Proof. (i) Compute $(\lambda,\rho)^+(\lambda,\rho) = (\lambda^+,\rho^+)(\lambda,\rho) = (\lambda^+\lambda,\rho^+\rho)$. For any $e \in P(S)$, we have $e\rho^+\rho = (e\cdot e\rho^+)\rho = (e\rho^+)\cdot e\rho = (e\rho)^+e\rho = e\rho$, and so $\rho^+\rho = \rho$. Therefore, $(\lambda,\rho)^+(\lambda,\rho) = (\lambda,\rho)$.

(ii) Compute

$$\lambda_1^+\lambda_2^+e = \lambda_1^+(\lambda_2^+e\cdot e) = (\lambda_1^+e)\cdot(\lambda_2^+e) = (\lambda_2^+e)\cdot(\lambda_1^+e) = \lambda_2^+\lambda_1^+e.$$

Then, $\lambda_1^+\lambda_2^+ = \lambda_2^+\lambda_1^+$. Together with

$$(\lambda_1,\rho_1)^+(\lambda_2,\rho_2)^+ = (\lambda_1^+\lambda_2^+,\rho_1^+\rho_2^+)$$

and

$$(\lambda_2,\rho_2)^+(\lambda_1,\rho_1)^+ = (\lambda_2^+\lambda_1^+,\rho_2^+\rho_1^+),$$

we have $(\lambda_1^+\lambda_2^+,\rho_1^+\rho_2^+) = (\lambda_2^+\lambda_1^+,\rho_2^+\rho_1^+)$ and result (ii).

(iii) Obviously, $[(\lambda_1,\rho_1)^+(\lambda_2,\rho_2)]^+ = ((\lambda_1^+\lambda_2)^+,(\rho_1^+\rho_2)^+)$. By computing, we have

$$\begin{aligned}(\lambda_1^+\lambda_2)^+e &= (e\rho_1^+\rho_2)^+ = [e(e\rho_1)^+\rho_2]^+\\ &= [(e\rho_1)^+\cdot e\rho_2]^+ = (e\rho_1)^+\cdot(e\rho_2)^+\\ &= (\lambda_1^+e)\cdot(\lambda_2^+e) = \lambda_1^+(e\cdot\lambda_2^+e)\\ &= (\lambda_1^+\lambda_2^+)e.\end{aligned}$$

Thus $(\lambda_1^+\lambda_2)^+ = (\lambda_1^+\lambda_2^+)$. Moreover,$[(\lambda_1^+\lambda_2)^+,(\rho_1^+\rho_2)^+] = (\lambda_1^+\lambda_2^+,\rho_1^+\rho_2^+)$.

(iv) Because $(\lambda_1,\rho_1)(\lambda_2,\rho_2)^+ = (\lambda_1\lambda_2^+,\rho_1\rho_2^+)$ and $[(\lambda_1,\rho_1)(\lambda_2,\rho_2)]^+(\lambda_1,\rho_1) = ((\lambda_1\lambda_2)^+\lambda_1,(\rho_1\rho_2)^+\rho_1)$, we only need to show that $\rho_1\rho_2^+ = (\rho_1\rho_2)^+\rho_1$. For any $e \in P(S)$,

$$\begin{aligned}e\rho_1\rho_2^+ &= (e\rho_1)(e\rho_1)^*\rho_2^+\\ &= (e\rho_1)[(e\rho_1)^*\rho_2^+]\\ &= [(e\rho_1)(e\rho_1)^*\rho_2^+]^+(e\rho_1)\\ &= [(e\rho_1)\rho_2^+]^+(e\rho_1) \quad (\text{by } (a\rho)^+ = (a\rho^+)^+)\\ &= (e\rho_1\rho_2)^+(e\rho_1) \quad (\text{by } (e\rho)^+ = e\rho^+)\\ &= e\cdot e(\rho_1\rho_2)^+\rho_1\\ &= e(\rho_1\rho_2)^+\rho_1.\end{aligned}$$

Thus, $\rho_1\rho_2^+ = (\rho_1\rho_2)^+\rho_1$ and results (iv).

(v) Note that $(\lambda^+)^*e = (\lambda^+e)^* = [(e\rho)^+]^* = (e\rho)^+ = \lambda^+e$. We obtain that $(\lambda^+)^* = \lambda^+$. Therefore $[(\lambda,\rho)^+]^* = (\lambda,\rho)^+$. □

Dual to Lemma 3.6, we have

Lemma 3.7. *Let* $(\lambda,\rho),(\lambda_1,\rho_1),(\lambda_2,\rho_2)\in\Omega(S)$. *If* S *is a restriction semigroup, then:*

(i) $(\lambda,\rho)(\lambda,\rho)^*=(\lambda,\rho)$;

(ii) $(\lambda_1,\rho_1)^*(\lambda_2,\rho_2)^*=(\lambda_2,\rho_2)^*(\lambda_1,\rho_1)^*$;

(iii) $[(\lambda_1,\rho_1)(\lambda_2,\rho_2))^*]^*=(\lambda_1,\rho_1)^*(\lambda_2,\rho_2)^*$;

(iv) $(\lambda_1,\rho_1)^*(\lambda_2,\rho_2)=(\lambda_2,\rho_2)[(\lambda_1,\rho_1)(\lambda_2,\rho_2)]^*$;

(v) $[(\lambda,\rho)^*]^+=(\lambda,\rho)^*$.

Lemmas 3.6 and 3.7 tell us that for a restriction semigroup S, $\Omega(S)$ is still a restriction semigroup. The following lemma gives a structure of the equivalence σ on $\Omega(S)$.

Lemma 3.8. *Let* $(\lambda_1,\rho_1),(\lambda_2,\rho_2)\in\Omega(S)$. *If* S *is a proper restriction semigroup, then the following statements are equivalent:*

(i) $(\lambda_1,\rho_1)\sigma(\lambda_2,\rho_2)$;

(ii) *For all* $e\in P(S),(\lambda_1 e)\sigma(\lambda_2 e)$;

(iii) *For all* $e\in P(S),(e\rho_1)\sigma(e\rho_1)$.

Proof. By symmetry, we need only to show (i)$\Leftrightarrow$(ii). If $(\lambda_1,\rho_1)\sigma(\lambda_2,\rho_2)$, then there exists $(\lambda^*,\rho^*)\in P(\Omega(S))$ such that $(\lambda_1,\rho_1)(\lambda^*,\rho^*)=(\lambda_2,\rho_2)(\lambda^*,\rho^*)$; that is, $(\lambda_1\lambda^*,\rho_1\rho^*)=(\lambda_2\lambda^*,\rho_2\rho^*)$. Then $\lambda_1\lambda^*=\lambda_2\lambda^*$ and for all $e\in P(S)$, $\lambda_1\lambda^* e=\lambda_2\lambda^* e$. Notice that $\lambda^* e\in P(S)$, we have $\lambda_1(\lambda^* e\cdot e)=\lambda_2(\lambda^* e\cdot e)$ and $\lambda_1 e\cdot\lambda^* e=\lambda_2 e\cdot\lambda^* e$. Therefore $(\lambda_1 e)\sigma(\lambda_2 e)$, as required.

Conversely, if for all $e\in P(S),(\lambda_1 e)\sigma(\lambda_2 e)$, then there exists $f\in P(S)$ such that $f\lambda_1 e=f\lambda_2 e$, and so $\lambda_f\lambda_1 e=\lambda_f\lambda_2 e$. By Lemma 3.2, $\lambda_f\lambda_1=\lambda_f\lambda_2$. Thus,

$$(\lambda_f\lambda_1,\rho_f\rho_1)=(\lambda_f\lambda_2,\rho_f\rho_2)$$

That is,

$$(\lambda_f,\rho_f)(\lambda_1,\rho_1)=(\lambda_f,\rho_f)(\lambda_2,\rho_2)$$

By the definition of σ, $(\lambda_1,\rho_1)\sigma(\lambda_2,\rho_2)$, as required. □

Lemma 3.9. *Let* S *be (proper; Clifford; E-reflexive) restriction semigroup. If* T *is a* $(2,1,1)$*-subsemigroup of* S, *then* T *is still a (proper; Clifford; E-reflexive) restriction semigroup.*

Proof. Because T is a $(2,1,1)$-subsemigroup of S, we have that for all $x \in T$, $x^+, x^* \in P(T) \subseteq P(S)$. By definition, the cases for (proper; Clifford) restriction semigroups are obvious. It remains to verify the case for E-reflexive restriction semigroups. To the end, we let $e \in P(T)$, $x, y \in T$ and $exy \in P(T)$. Then $exy \in P(S)$ and as S is E-reflexive, we have $eyx \in P(S)$. So that $eyx = (eyx)^+ = e(yx)^+$ since $yx \in T$ giving $(yx)^+ \in T$. It results that T is an E-reflexive restriction semigroup. □

Lemma 3.10. *Let S be a restriction semigroup. If a is an element in S, then $(\lambda_a, \rho_a)^+ = (\lambda_{a^+}, \rho_{a^+})$ and $(\lambda_a, \rho_a)^* = (\lambda_{a^*}, \rho_{a^*})$. Moreover, $\Pi(S)$ is a $(2,1,1)$-subsemigroup of $\Omega(S)$.*

Proof. For $a \in S$ and $e \in P(S)$, by Lemma 3.4, we have

$$\lambda_a^+ e = (e\rho_a)^+ = (ea)^+ = ea^+ = a^+ e = \lambda_{a^+} e,$$

so that by Lemma 3.2, $\lambda_a^+ = \lambda_{a^+}$. Further, by Lemma 3.3, $(\lambda_a, \rho_a)^+ = (\lambda_{a^+}, \rho_{a^+})$; similarly, $(\lambda_a, \rho_a)^* = (\lambda_{a^*}, \rho_{a^*})$. Now by definition, $\Pi(S)$ is a $(2,1,1)$-subsemigroup since $\Pi(S)$ is a subsemigroup of $\Omega(S)$. The proof is completed. □

Proof of Theorem 2.1: (2.1.1) *The case for restriction semigroups.* By Lemmas 3.6 and 3.10, and Corollary 3.7, we need only to verify the converse part. To the end, we assume that $\Omega(S)$ is a restriction semigroup and $\Pi(S)$ is a $(2,1,1)$-subsemigroup of $\Omega(S)$. By Lemma 3.1, it suffices to show that $\Pi(S)$ is a restriction semigroup. But $\Pi(S)$ is a $(2,1,1)$-subsemigroup of $\Omega(S)$, now by Lemma 3.9, $\Pi(S)$ is a restriction semigroup, as required.

(2.1.2) *The case for proper restriction semigroups.* Assume that S is a proper restriction semigroup. By Lemma 3.10, it suffices to verify that $\Omega(S)$ is a proper restriction semigroup. Let $(\lambda_1, \rho_1), (\lambda_2, \rho_2) \in \Omega(S)$ be such that $(\lambda_1, \rho_1)^* = (\lambda_2, \rho_2)^*$ and $(\lambda_1, \rho_1)\sigma(\lambda_2, \rho_2)$. Then by Lemmas 3.4 and 3.8, $(\lambda_1 e)^* = (\lambda_2 e)^*$ and $(\lambda_1 e)\sigma(\lambda_2 e)$ for all $e \in P(S)$, thus $\lambda_1 e = \lambda_2 e$ since S is a proper restriction semigroup, so that by Lemma 3.2, $\lambda_1 = \lambda_2$. From Lemma 3.3, it follows that $(\lambda_1, \rho_1) = (\lambda_2, \rho_2)$. Obviously, $(\lambda_1, \rho_1)^+ = (\lambda_2, \rho_2)^+$. Together with $(\lambda_1, \rho_1)\sigma(\lambda_2, \rho_2)$, this shows that $(\lambda_1, \rho_1) = (\lambda_2, \rho_2)$. Therefore $\Omega(S)$ is a proper restriction semigroup.

For the converse, by Lemma 3.9, $\Pi(S)$ is a proper restriction semigroup, and by Lemma 3.1, S is a proper restriction semigroup. The proof is completed.

(2.1.3) *The case for Clifford restriction semigroups.* If $(\lambda_1, \rho_1) \in \Omega(S)$ and $(\lambda, \rho)^+ \in P(\Omega(S))$, then

$$\lambda_1\lambda^+ e = \lambda_1 e \cdot \lambda^+ e = \lambda^+(e \cdot \lambda_1 e) = \lambda^+(\lambda_1 e \cdot e) = \lambda^+\lambda_1 e$$

and so $\lambda_1\lambda^+ = \lambda^+\lambda_1$, it follows that $(\lambda, \rho)^+(\lambda_1, \rho_1) = (\lambda_1, \rho_1)(\lambda, \rho)^+$. Therefore $\Omega(S)$ is a Clifford restriction semigroup. Together with Lemma 3.10, we have proved the direct part.

The converse part is immediate from Lemmas 3.1 and 3.9.

(2.1.4) *The case for E-reflexive restriction semigroups.* By Lemmas 3.1 and 3.9, it suffices to show the direct part. Now let S be an E-reflexive restriction semigroup. We notice that in [9, Theorem 4.6], the proof that $\Omega(S)$ is an E-reflexive type B semigroup used only the identities (2.1) and their duals. So, these arguments in [9, Theorem 4.6] have proved that $\Omega(S)$ is of the property that for all $e \in P(\Omega(S))$ and $x, y \in \Omega(S)$, $exy \in P(\Omega(S))$ implies that $eyx \in P(\Omega(S))$. So, $\Omega(S)$ is indeed an E-reflexive restriction semigroup. □

Acknowledgements

The authors are thankful to the referees for their valuable comments. And this research is jointly supported by the National Natural Science Foundation of China (grant: 11761034; 11661042).

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Accepted: June 11, 2020

The convergent sequence to the weighted operator

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Abstract. This paper focuses on the periodic weighted operator and explains its spectrum. The resolvent of the operator sequence of the weighted operator is discussed, and proof is given to indicate that this operator sequence is restricted to the space of all real valued Lebesgue measurable functions that are square integrable on real numbers. The sequence of this operator's convergence to the resolvent of the second derivative operator with the individual boundary conditions is then demonstrated. Consequently, this convergence is used to analyze the results of weighted operator model in the image processing.

Keywords: weighted operator, spectral theory, differential operator, image processing, total variation, partial differential equations.

1. Introduction

The investigations of Sturm and Liouville on the eigenvalues of certain deferential equations of second order with given boundary conditions, now referred to as Sturm-Liouville theory mark the beginning of the history of periodic spectral theory. A solution to the inverse problem (including characterization) for the vector-valued Sturm-Liouville operator on a finite interval with Dirichlet conditions was published recently in [11]. Since the periodic case is more complicated, many works have concentrated only on the direct spectral problem for periodic systems.

Lyapunov, who proved that the spectrum of the weighted periodic operator has band structure and studied the needed properties of the Lyapunov function

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[21] was the first to investigate the spectral properties of the weighted periodic operator, with the results for the more general case including 2×2 systems being reproved later by Krein [20]. Firsova [7] and Marchenko and Ostrovski [23] were responsible for simultaneously introducing the global quasimomentum into the spectral theory of the Hill operator. Two-sided estimates for various parameters of the Hill operator and as well as for the Dirac operator were obtained in [8, 9, 12, 13, 15, 17, 19] as well as the sharp two-sided estimates for gap lengths and potential [14].

The spectral properties of the Hill operator and the weighted operator show a sizeable difference, whereby the gap length of the weighted operator can go to infinity at high energy, while the gap length of the Hill operator tends to 0 at high energy [18]. The fact of the weighted operator being raised in different situations is the physical motivation for this. Examples include the photonic crystal problem, the propagation of one-dimensional waves in periodic homogeneous media with velocity and a hydrogen atom in an external, homogeneous, time-periodic magnetic field [10]. Reference to the results on the weighted operator and its history which the authors are made in [4], albeit with no claim to completeness. In particular, mention is made of many applications to physics [4]. Likewise, the weighted operator has a lot applications in the image processing, as introduced in [25, 26].

In order to investigate the spectrum of an ordinary differential equation with periodic coefficients the construction of its Bloch solutions is required. The transform matrix corresponding to a basis in the space of the solutions is introduced. Therefore, first of all, a general tool is briefly described, in the hope that the transformation operation is very natural for the spectral analysis of difference equations with periodic coefficients. Subsequently, the spectrum of an operator is investigated by this tool.

The main result of the current paper is contained in Theorem (3), describing the resolvent convergence of the periodic weighted operator sequence. We show that the resolvent of the operator $\mathcal{H}$ is restricted to set of all Lebesgue measurable functions. Also, this resolvent coincides with another resolvent which defined on the differential operator under a certian boundary conditions.

A brief outline of the contents of the paper follows: Section (2) re-states some basic concepts which are necessary for our future work. In Section (3) a periodic weighted operator $\mathcal{W}_\omega$ is studied, the spectrum of this operator being subsequently investigated. It is shown that the spectrum of $\mathcal{W}_\omega$ is absolutely continuous and consists of intervals $\mathrm{I}_n, \mathrm{I}_{n+1}$ separated by gaps $\mathcal{G}_n = (\pi n_L, \pi n_R)$; with the lengths $\mid \mathcal{G}_n \mid$, Thorem (2). Then, Section (4) comprises an investigation of the resolvent convergence of this operator sequence. In particular, our main result (Theorem (3)) is proven. Finally, Section (5) shows some applications of the weighted operator that can be used in the future work.

2. Preliminaries

At the beginning let us briefly recall some of the results that given in the previous work. We consider here the second derivative operator $\mathscr{L} = -\frac{d^2}{dy^2}$, acting in the Hilbert space $L_2(\mathbb{R})$ defined on the functions from $W_2^2(\mathbb{R} \setminus \{n\}_{n\in\mathbb{Z}})$ satisfying the boundary conditions,

$$\begin{pmatrix} u_R(n) \\ u'_R(n) \end{pmatrix} = \exp(i\theta)\, \mathcal{A} \begin{pmatrix} u_L(n) \\ u'_L(n) \end{pmatrix},\ n \in \mathbb{Z},$$

where $\mathcal{A} = \begin{bmatrix} \alpha & \beta \\ \gamma & \mu \end{bmatrix} \in \mathrm{SL}(2,\mathbb{R})$ (the unimodule group) such that $\alpha, \beta, \gamma, \mu \in \mathbb{R}$ and let $0 \leq \theta < 2\pi$. In addition, the adjoint of this operator coincides with a self-adjoint extension of the operator $\mathscr{L} = -\frac{d^2}{dy^2}$, restricted to the set of functions satisfying the boundary conditions at the origin, and fulfilling the condition $\alpha\mu - \beta\gamma = 1$ [2]. Furthermore, there is an increase in the widths of the forbidden gaps and the gap-to-band ratio is not small when the operator $\mathscr{L}$ is discribed [2].

Investigating the spectrum of an ordinary differential equation with periodic coefficients necessitates the construction of its Bloch solutions, introducing the transfer matrix $\mathbb{T}_{\mathscr{L}}$ corresponding to a basis in the space of the solutions. Consequently, the transfer matrix for the operator $\mathscr{L}$ is defined as follows [1]:

$$\mathbb{T}_{\mathscr{L}} = \begin{bmatrix} \cos\kappa & -\frac{1}{\kappa}\sin\kappa \\ \kappa\sin\kappa & \cos\kappa \end{bmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \mu \end{pmatrix}, \text{ where } \kappa = \sqrt{\lambda}.$$

And since $\det(\mathbb{T}_{\mathscr{L}}) = 1$, therefore, the specific determinant of this matrix is given by

$$\det(\mathbb{T}_{\mathscr{L}} - \lambda \mathrm{I}) = \lambda^2 - \lambda \mathrm{Tr}(\mathbb{T}_{\mathscr{L}}) + 1.$$

Thus, the operator's spectrum coincides with the set of λ where the spectrum of this operator is calculated as zeros of the following inequality [22],

$$|\ \mathrm{Tr}(\mathbb{T}_{\mathscr{L}})\ | \leq 2,$$

then

$$|\ (\alpha+\mu)\cos k + (\frac{\beta}{\kappa} - \gamma\kappa)\sin\kappa\ | \leq 2. \tag{1}$$

Let now introducing the function f to describe the spectrum of $\mathscr{L}$,

$$|\ f(\kappa)\ | = |\ (\alpha+\mu)\cos\kappa + (\frac{\beta}{\kappa} - \gamma\kappa)\sin\kappa\ | \leq 2. \tag{2}$$

The following theorem gives an explicit description of the spectrum of this operator $\mathscr{L}$ [6].

Theorem 1 ([6]). *Assume that α, γ, β, and μ are an arbitrary, then there are infinite numbers of bands of the operator $\mathscr{L}$, which has a continuous spectrum.*

3. The operator's spectrum

First, consideration is given to the periodic weighted operator, defined as follows [16],

$$\mathcal{W}\zeta = -\frac{1}{p}\frac{d}{dy}(p\frac{d}{dy}\zeta), \tag{3}$$

where ζ is an eigenfunction, $p > 0$, in $L_2(\mathbb{R})$. This operator has a continuous spectrum which contains an infinite number of bands separated by gaps $\mathcal{G}_n = (\pi n_L, \pi n_R)$ where the lenght of gaps $|\mathcal{G}_n| \geq 0$, and located in the intervals $\mathrm{I}_n = [\pi^2(n-1)^2, \pi^2 n^2]$, [16]. The corresponding intervals I_n, I_{n+1} of the gaps are merge when $\mid \mathcal{G}_n \mid = 0$. It should be noted that there exists the following asymptotic:

$$\sqrt{\pi \pm n} = (\lambda n)(1 + \mathrm{I}(1)) \,\mathrm{as}\, n \to \infty,$$

which provides a rough asymptotic of gap lengths [5]. Now we define a periodic weighted operator as follows:

$$\mathcal{W}_\omega\zeta = -\frac{1}{p_\omega(y)}\frac{d}{dy}(p_\omega(y)\frac{d}{dy}\zeta), \tag{4}$$

where

$$p_\omega(y) = 1 + \sum_{n=-\infty}^{\infty} \eta\frac{1}{\omega}\delta_\omega(y-n),\ \eta \in \mathbb{R}^+, \tag{5}$$

is called the density function which defined by using the delta function

$$\delta_\omega(y) = \begin{cases} 1, & y \in [0, \omega) \\ 0, & y \notin [0, \omega). \end{cases} \tag{6}$$

Thus, since p is chosen, if $\omega \to 0$ we get

$$\begin{aligned} \lim_{\omega\to 0} p_\omega(y) &= \lim_{\omega\to 0}[1 + \sum_{n=-\infty}^{\infty} \eta\frac{1}{\omega}\delta_\omega(y-n)] \\ &= 1 + \lim_{\omega\to 0}\sum_{n=-\infty}^{\infty} \eta\frac{1}{\omega}\delta_\omega(y-n) \\ &= 1 + \sum_{n\in\mathbb{Z}} \frac{\eta}{\omega}\delta(y-n). \end{aligned}$$

Since $\sum \frac{\eta}{\omega}\delta(y-n)$ represents the delta function, then the density function p_ω is convergent to $\sum_{n\in\mathbb{Z}} \delta_\omega + 1$ when $\omega \to 0$.

Using the relation (2) the following Theorem can easily be shown.

Theorem 2. *Assume that $\alpha = 1,\ \gamma = h,\ \beta = 0,\ \mu = 1$. Then there are infinite numbers of bands of the periodic operator $\mathcal{W}_\omega$ as $\omega \to 0$ which has an absolutely continuous spectrum.*

Proof. From the equation (5), $p_\omega(y)$ is dis-continuous at $y = n$ and $y = n + \omega$, then each functions take from the domain of the operator $\mathcal{W}_\omega$ is satisfying the boundary conditions as follow

$$\begin{aligned}
&\zeta(n_R) \ = \ \zeta'(n_L),\\
&(1+\eta\frac{1}{\omega})\zeta'(n_R) = \zeta'(n_L),\\
&\zeta((n+\omega)_R) = \zeta((n+\omega)_L),\\
&\zeta'((n+\omega)_R) = (1+\eta\frac{1}{\omega})\zeta'((n+\omega)_L).
\end{aligned}$$

Thus, the functions ζ and $p_\omega\zeta$ are continuous under these conditions. Therefore, there are four matrices for the transfer matrix of the operator $\mathcal{W}_\omega$. The first two transfer matrices defined on the differential operator on the intervals $(0^+, \omega^-)$ and $(\omega^+, 1^-)$, and another two matrices corresponded to discontinuous at $y = 0$ and $y = \omega$, i.e.

$$\begin{aligned}
\mathbb{T}_{\mathcal{W}_\omega} &= \mathbb{T}_{\frac{d^2}{dy^2}}(\omega^+, 1^-)\begin{bmatrix}1 & 0\\ 0 & 1+\frac{\eta}{\omega}\end{bmatrix}\mathbb{T}_{\frac{d^2}{dy2}}(0^+, 1^-)\begin{bmatrix}1 & 0\\ 0 & 1-\frac{\eta}{\omega+\eta}\end{bmatrix}\\
&= \begin{bmatrix}\cos\kappa - \frac{1}{\omega}\sin(1-\omega)\kappa\sin\omega\kappa & \frac{1}{\kappa}\sin\kappa - \frac{1}{\kappa(\omega+\eta)}\cos(1-\omega)\kappa\sin\omega\kappa\\ -\kappa\sin\kappa - \frac{\kappa}{\omega}\cos(1-\omega)\kappa\sin\omega\kappa & \cos\kappa + \frac{1}{\omega+\eta}\sin(1-\omega)\kappa\sin\omega\kappa\end{bmatrix}.
\end{aligned}$$

Now, since $\det(\mathbb{T}_{\mathcal{W}_\omega}) = 1$, then the spectrum of the operator can be determined by the trace of the transfer matrix $|\ \mathrm{Tr}(\mathbb{T}_{\mathcal{W}_\omega})\ |$[22]; i.e.:

$$|\ \mathbb{T}_{\mathcal{W}_\omega}\ | \leq 2.$$

Thus,

$$|\ \mathbb{T}_{\mathcal{W}_\omega}\ | = |\ 2\cos\kappa - \frac{\eta}{\omega(\omega+1)}\sin((1-\omega)\kappa)\sin\omega\kappa\ | \leq 2,$$

when the limit $\omega \to 0$. Implies that the last equation can be transformed into the equation;

$$|\ 2\cos\kappa - \eta\kappa\sin\kappa\ | \leq 2.$$

Hence, this equation is coincided with the equation (2) when the four parameters take the following values;

$$\alpha = 1,\ \gamma = h,\ \beta = 0,\ \mu = 1.$$

As a result, we conclude that there is an infinite number of band of the operator $\mathscr{L}$, which has an absolutely continuous spectrum in which the operator $\mathcal{W}_\omega$ is convergent when $\omega \to 0$ depending on these parameters. □

4. The resolvent of the operator sequence

The main goal in this section is to discuss the resolvent of the operator sequence of the operator $\mathcal{W}_\omega$. At first, let us extend the Hilbert space as $\mathbb{H} = L_2(\mathbb{R}) \oplus \ell_2$ and construct a self-adjoint operator $\mathcal{H}$ which acting on this space.

Now the self-adjoint operator $\mathcal{H}$ is defined by the following formula

$$\mathcal{H}\begin{bmatrix}\zeta\\ \xi\end{bmatrix} = \begin{bmatrix}-\frac{d^2}{dy^2}\zeta\\ \{[\xi']\}/\sqrt{\eta}\end{bmatrix} \tag{7}$$

on functions $(\zeta, \xi) \in W_2^2(\mathbb{R} \setminus \mathbb{C})$ satisfying the boundary conditions

$$\begin{matrix}\zeta(n_R) = \zeta(n_L)\\ \xi_n = -\sqrt{\eta}\{\zeta\}_n\end{matrix}\ , \ n \in \mathbb{Z}, \tag{8}$$

where $-\frac{d^2}{dy^2}\zeta$ and $\{\xi'\}$ denote the vectors from ℓ_2 with coordinates $-\frac{d^2}{dy^2}\zeta(n_R) - \frac{d^2}{dy^2}\zeta(n_L)$ and $\{\xi\}$, respectively. Since the operator $\mathcal{H}$ is symmetric and the range of $\mathbb{H}$ and $\mathcal{H} - \lambda$ is coincide when λ is arbitrary, $\Im\lambda \neq 0$, therefore, the operator $\mathcal{H}$ is a self-adjoint [3].

The following theorem is the principal result of this paper, containing the resolvents of $\mathcal{W}_\omega$ which diverges form any resolvent of operator that acting on Hilbert space $L_2(\mathbb{R})$.

Theorem 3. *The operator $\mathcal{H}$ which limited to the space of $L_2(\mathbb{R})$ and the differential operator $(-\frac{d^2}{dy^2})$ have an identical resolvent under certian individual boundary conditions when y=n;*

$$\begin{bmatrix}\zeta(n_R)\\ \zeta'(n_R)\end{bmatrix} = \begin{bmatrix}1 & 0\\ -\eta\kappa^2 & 1\end{bmatrix}\begin{bmatrix}\zeta(n_L)\\ \zeta'(n_L)\end{bmatrix}. \tag{9}$$

Proof. Firstly, we calculate the restricted resolvent of the operator $\mathcal{H}$, since from (7),

$$\mathcal{H}\begin{bmatrix}\zeta\\ \xi\end{bmatrix} = \begin{bmatrix}-\frac{d^2}{dy^2}\zeta\\ \{[\xi']\}/\sqrt{\eta}\end{bmatrix},$$

then for any function $F \in L_2(\mathbb{R})$ we get

$$(\mathcal{H} - \lambda)\begin{bmatrix}\zeta\\ \xi\end{bmatrix} = \mathcal{H}\begin{bmatrix}\zeta\\ \xi\end{bmatrix} - \lambda\begin{bmatrix}\zeta\\ \xi\end{bmatrix} = \begin{bmatrix}F\\ 0\end{bmatrix}$$
$$= \begin{cases}-\dfrac{d^2}{dy^2}\zeta(y) - \lambda\zeta(y) = F(y),\ y \notin \mathbb{Z}\\ \dfrac{1}{\sqrt{\eta}}\{[\zeta']\} - \lambda\xi = 0\end{cases}$$

implies that $\{[\zeta']\} = \sqrt{\eta}\lambda\xi$, but from the second part of the equation (8); $\xi_n = -\sqrt{\eta}\{\zeta\}_n$ then $\{[\zeta']\} = -\eta\lambda\{\xi\}$ and since $\kappa = \sqrt{\lambda}$ hence, we get an individual boundary condition for the component $\{\zeta\}$;

$$\{[\zeta']\} = -\eta\kappa^2\{\xi\}$$

which is exactly the second part of the equation (9). □

5. Applications of weighted operator

There are a lot of weighted operator applications in different sciences such as in image processing. Many problems appeared in image processing like image inpainting, deblurring, segmentation, and denoising which can solve them by modeling weighted operator [24]. The benefit of this study (i.e. the convergence of weighted operator) is used in analysis the results of weighted operator model in image processing. This analysis will help to modify this model and to enhance its results in the image processing. Two types of image processing models which are statistical, and partial differential equations. The weighted operator is used in statistical model for different problems in image processing [27]. While the differential operators are used in PDE models in image processing [28, 29, 30, 31]. The convergence of

6. Conclusions

We have introduced the spectum of the weighted operator, which it has a different situations in physics such as the photonic crystal problem and a hydrogen atom. We have used the concept of a transformation matrix instead Bloch solutions. We have provided here the spectrum of a weighted operator $\mathcal{W}_\omega$ is absolutely continuous and consists of intervals $(\mathrm{I}_n, \mathrm{I}_{n+1})$ separated by gaps $\mathcal{G}_n = (\pi n_L, \pi n_R)$; with the lengths $\mid \mathcal{G}_n \mid$. Moreover, the resolvent convergence of the periodic weighted operator sequence is described. We have proved that there exist a corresponding resolvent between the weighted operator $\mathcal{W}_\omega$ and differential operator with the individual boundary conditions. In the future work, the weighted operator will use for solving different image processing problems, the convergence of other types of operators will study and we will apply them in image processing.

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Accepted: April 15, 2020

Pathological diagnosis and clustering of single-valued neutrosophic sets based on the new similarity measure

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Abstract. Similarity measure is an effective tool for decision-making and clustering analysis. Most of the existing similarity measures only to consider the similarity between the corresponding positions of the data. On this basis, we not only consider the relationship between the corresponding positions of the data, but also consider the influence of the overall difference between the data onto the similarity in single-valued neutrosophic environment. Furthermore, similarity and distance measures with weights are given and their properties are investigated. Three examples, which include pathologic diagnosis, clustering and decision making are given to demonstrate the effectiveness of methods. It can be seen that compared with the existing methods, our methods have the characteristics of high accuracy and wide application range.

Keywords: single-valued neutrosophic set, similarity measure, distance measure, pathologic diagnosis, clustering, decision making.

1. Introduction

To handle uncertain information, Zadeh [1] presented the fuzzy set (FS), which is composed by membership function. For the voting system, there will be those who agree, disagree or abstain. In this case, it is impossible to express infor-

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mation on FS. Therefore, intuitionistic fuzzy set (IFS) which contained membership function and non-membership function is proposed by Atanassov [2]. Then, Smarandache [3] developed neutrosophic set (NS) from the philosophical standpoint, which is used to deal with indeterminate and inconsistent information. For example, in Schrödinger's Cat theory, quantum states are basically in multiple positions at the same time. They are expressed by the NS, which can represent a phenomenon that belongs both here and there at the same time. In order to simplify calculation and application convenience, Wang et al. [4] defined the special case of NS, which is called the single-valued neutrosophic set (SVNS). With the development of fuzzy information, IFS and SVNS are widely used in various fields, such as decision making [5, 6, 7, 8, 9], clustering [10, 11, 12, 13, 14] and medical diagnosis [15, 16].

Similarity measure is widely used in fuzzy information system. Different similarity measures have been studied under FS or IFS environment [17, 18, 19, 20, 21, 22, 23]. Moreover, similarity measure and correlation coefficient of SVNS have also been widely investigated [24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37]. Huang [27] introduced a new distance measure between two SVNSs for the forward and backward differences. Ye [28] developed two distance-based similarity measure of SVNS for clustering. Using the idea of priority solution, a new ranking method was proposed and applied to multi-attribute decision making by Biswas [29]. Peng [30] proposed three methods of multi-attribute decision making by defining new similarity measure and distance measure. Ye [31] introduced an improved correlation coefficient of SVNSs for multiple attribute decision making. Combined dice with cosine similarity measure, Pramanik et al. [32] defined new vector similarity measure for SVNS. Ye [33] gave a new definition of correlation coefficient of SVNS and applied it to multi-attribute decision making. Majumdar and Samanta [34] developed the distance measure and similarity measure between two SVNSs.

Pathology, as an important subject in medical science, provides an effective bridge between basic medicine and clinical medicine, and plays an important role in the theoretical research, diagnosis, treatment and prevention of diseases ([38, 39]). The basic task of surgical pathologists in the diagnosis of living tissue examination is to confirm the disease and its nature, that is, according to the organs and materials examined, first of all, to determine whether there are pathological changes, and the specific nature and type of changes, so as to help the clinical treatment and prognosis to make a judgment. At present, it is the gold standard of medical diagnosis. Classical pathological diagnosis, that is, HE slice diagnosis is mainly simple morphology, usually pathological diagnosticians directly observe and confirm the pathological tissue under naked eyes and microscopy, so as to make diagnosis according to its pathological characteristics. The intuitive features and advantages of this diagnostic method can not be replaced by other methods. However, unlike the usual medical tests, it does not reflect the indicators with precise values, but relies on the subjective experience of the examiner and lacks objective indicators. In this way, in the process of patho-

logical diagnosis, the data are highly uncertain. In the same microscopic cell morphology, different examiners sometimes to make different diagnosis, or even completely opposite diagnosis. Take soft tissue tumors as an example, the same structure can be found in different types of other tumors, such as bundle structure can be found in neoplastic fibrous tissue proliferation, fibrosarcoma, malignant peripheral nerve sheath tumor, synovial sarcoma, palisade structure occurs in neurilemmoma, malignant peripheral nerve sheath tumor, leiomyosarcoma; Pseudochrysanthemum-like and pseudochrysanthemum-like structures may appears in neuroblastoma, schwannoma, malignant peripheral neurilemmoma, etc. Different tumors often present many similar pathological structures. Fuzzy theory can effectively deal with such uncertain information ([40, 41, 42]).

In the existing literature of fuzzy similarity measurement, most of them only consider the similarity between corresponding elements, but ignore the influence of the whole data on the similarity. In order to solve this problem, a new similarity measure is presented under the single-valued neutrosophic environment in this paper, and applied to the diagnosis of soft tissue tumor and other specific examples. The main contributions of this paper are: (1) A new similarity measure for SVNS is proposed. Unlike previous definitions, we consider the effects of the differences in individual and global factors on the similarity between two SVNSs. (2) We use the defined similarity measure in pathological diagnosis. An example shows the effectiveness of the method. (3) By comparing with the existing methods of [11, 27, 28, 29, 30, 31, 32, 33], we can see that our method has higher accuracy in clustering and has a wide range of application for decision-making.

The specific framework of the article is as follows. In Section 2, we give some basic definitions. In Section 3, the new similarity measure and distance measure between two SVNSs are given. Then their properties are studied. In Section 4, three examples are given. The example of pathological diagnosis demonstrates the effectiveness of our method. The example of clustering shows the high accuracy of our method. And the example of decision making illustrates the widespread application of our method. Finally, Section 5 summarizes the paper.

2. Preliminaries

To facilitate reading and understanding, this section first reviews the concepts of NS, SVNS, entropy measure, the normalized Hamming distance and normalized Euclidian distance between two SVNSs. Finally, the abbreviations and symbols are given.

Definition 2.1 ([3]). *Let X be a space of points (objects), with a generic element in X denoted by x. A NS A in X is characterized by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$, and a falsity-membership function $F_A(x)$. The functions $T_A(x)$, $I_A(x)$, and $F_A(x)$ are real standard of non-standard subsets of $]^-0, 1^+[$. That is, $T_A x : x \to]^-0, 1^+[$, $I_A x : x \to]^-0, 1^+[$*

and $T_Ax : x \to]^-0, 1^+[$. *Thus, there is no restriction on the sum of* $T_A(x)$, $I_A(x)$ *and* $F_A(x)$, *so* $^-0 \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3^+$.

Definition 2.2 ([4]). *Let* X *be a space of points (objects) with generic elements in* X *denoted by* x. *A SVNS* A *in* X *is characterized by truth-membership function* $T_A(x)$, *indeterminacy-membership function* $I_A(x)$ *and falsity-membership function* $F_A(x)$. *For each point* x *in* X, $T_A(x), I_A(x), F_A(x) \in [0, 1]$. *Therefore, a SVNS* A *can be denoted by* $A = \{< x, T_A(x), I_A(x), F_A(x) > | x \in X\}$.

The following expressions are defined by Wang et al. in [4], for SVNSs A, B.

(1) $A \subseteq B$ if and only if $T_A(x) \leq T_B(x)$, $I_A(x) \geq I_B(x)$, $F_A(x) \geq F_B(x)$, for any x in X.

(2) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

(3) $A^c = \{< x, F_A(x), 1 - I_A(x), T_A(x) > | x \in X\}$, where A^c denotes the complement of A.

Definition 2.3 ([27]). *Let* E *be a set-to-point mapping:* $E : SVNSs \to [0, 1]$, *then* E *is an entropy measure if it satisfies the following conditions:*

$(E1)$ $E(A) = 0$ *(minimum) if and only if* A *or* A^c *is a crisp set;*

$(E2)$ $E(A) = 1$ *(maximum) if and only if* $A = A^c$, *i.e.,* $T_A(x_i) = F_A(x_i)$, $I_A(x_i) = 0.5$, *for all* $x_i \in X$;

$(E3)$ $E(A) \leq E(B)$ *if* A *is less fuzzy than* B, *i.e., if* $T_A(x_i) \leq T_B(x_i), F_B(x_i) \leq F_A(x_i)$ *for* $T_B(x_i) \leq F_B(x_i)$ *and* $I_A(x_i) = I_B(x_i) = 0.5$, *or* $T_A(x_i) \geq T_B(x_i)$, $F_B(x_i) \geq F_A(x_i)$ *for* $T_B(x_i) \geq F_B(x_i)$ *and* $I_A(x_i) = I_B(x_i) = 0.5$;

$(E4)$ $E(A) = E(A^c)$.

Definition 2.4 ([34]). *Let* A, B *be two SVNSs in* $X = \{x_1, x_2, ..., x_n\}$.

(1) *The normalized Hamming distance between* A *and* B *is defined as follows:*

$$l_N(A, B) = \frac{1}{3n} \sum_{i=1}^{n} \{|T_A(x_i) - T_B(x_i)| + |I_A(x_i) - I_B(x_i)| + |F_A(x_i) - F_B(x_i)|\}.$$

(2) *The normalized Euclidian distance between* A *and* B *is defined as follows:*

$$q_N(A, B) = \Big(\frac{1}{3n} \sum_{i=1}^{n} \{(T_A(x_i) - T_B(x_i))^2 + (I_A(x_i) - I_B(x_i))^2 + (F_A(x_i) - F_B(x_i))^2\}\Big)^{\frac{1}{2}}.$$

Example 1. Let A, B, C be three SVNSs, which denoted by

$$A = \{< x_1, 0.4, 0.6, 0.8 >, < x_2, 0.2, 0.2, 0.3 >\},$$
$$B = \{< x_1, 0.5, 0.1, 0.2 >, < x_2, 0.8, 0.1, 0.1 >\},$$
$$C = \{< x_1, 0.3, 0.0, 0.6 >, < x_2, 0.4, 0.9, 0.6 >\}.$$

Then, we can calculate that

$$l_N(A, B) = \frac{1}{6}(0.1 + 0.5 + 0.6 + 0.6 + 0.1 + 0.2) = 0.35,$$

$$l_N(A,C) = \frac{1}{6}(0.1+0.6+0.2+0.2+0.7+0.3) = 0.35,$$
$$q_N(A,B) = \sqrt{\frac{1}{6}(0.01+0.25+0.36+0.36+0.01+0.04)} = \sqrt{\frac{1.03}{6}},$$
$$q_N(A,C) = \sqrt{\frac{1}{6}(0.01+0.36+0.04+0.04+0.49+0.09)} = \sqrt{\frac{1.03}{6}}.$$

We can see that l_N and q_N cannot distinguish the distance between A and B, A and C.

Abbreviations and symbols:

Fuzzy Set $\rightarrow$ FS;

Intuitionistic Fuzzy Set $\rightarrow$ IFS;

Neutrosophic Set $\rightarrow$ NS;

Single-valued Neutrosophic Set $\rightarrow$ SVNS;

X represents a space of objects;

A, B, C represent SVNSs;

$\vartheta(\vartheta^*)$ represents the single-valued neutrosophic (weighted) similarity measure;

$d(d^*)$ represents the single-valued neutrosophic (weighted) distance measure;

$I_s(I_d)$ represents the index of similarity (distance).

3. Distance measure and similarity measure

In order to improve the shortcomings of existing distance measures and consider the global impact on individual differences between set elements, we define a new similarity measure as follows.

Definition 3.1. *For two SVNSs A and B in a universe of discourse $X = \{x_1, x_2, \ldots, x_n\}$, which are denoted by $A = \{< x_i, T_A(x_i), I_A(x_i), F_A(x_i) > |x_i \in X\}$ and $B = \{< x_i, T_B(x_i), I_B(x_i), F_B(x_i) > |x_i \in X\}$. The single-valued neutrosophic similarity measure is defined by*

$$\vartheta(A,B) = \frac{1}{3n}\sum_{i=1}^{n}\Big[\alpha_i(1-\Delta T_i) + \beta_i(1-\Delta I_i) + \gamma_i(1-\Delta F_i)\Big] \tag{1}$$

where $\Delta F_{\max} = \max_i\Big\{\Delta F_i\Big\}$

$$\alpha_i = \frac{3-\Delta T_i - \Delta T_{\max}}{3+\Delta T_{\min} - \Delta T_{\max}}, \beta_i = \frac{3-\Delta I_i - \Delta I_{\max}}{3+\Delta I_{\min} - \Delta I_{\max}}, \gamma_i = \frac{3-\Delta F_i - \Delta F_{\max}}{3+\Delta F_{\min} - \Delta F_{\max}},$$
$$\Delta T_i = \Big|T_A(x_i) - T_B(x_i)\Big|, \Delta I_i = \Big|I_A(x_i) - I_B(x_i)\Big|, \Delta F_i = \Big|F_A(x_i) - F_B(x_i)\Big|,$$
$$\Delta T_{\min} = \min_i\Big\{\Delta T_i\Big\}, \Delta T_{\max} = \max_i\Big\{\Delta T_i\Big\},$$
$$\Delta I_{\min} = \min_i\Big\{\Delta I_i\Big\}, \Delta I_{\max} = \max_i\Big\{\Delta I_i\Big\}, \Delta F_{\min} = \min_i\Big\{\Delta F_i\Big\}.$$

Definition 3.2. *Let $X = \{x_1, x_2, \ldots, x_n\}$, A and B be two SVNSs, which are denoted by $A = \{< x_i, T_A(x_i), I_A(x_i), F_A(x_i) > |x_i \in X\}$ and $B = \{< x_i, T_B(x_i), I_B(x_i), F_B(x_i) > |x_i \in X\}$.*

$$\vartheta^*(A,B) = \frac{1}{3}\sum_{i=1}^{n}\Big\{\omega_i\Big[\alpha_i(1-\Delta T_i) + \beta_i(1-\Delta I_i) + \gamma_i(1-\Delta F_i)\Big]\Big\}, \tag{2}$$

is called a single-valued neutrosophic weighted similarity measure, where ω_i $(i = 1, 2, \ldots, n)$ are the weight of the element $x_i (i = 1, 2, \ldots, n)$ with $\omega_i \geq 0$ and $\sum_{i=1}^{n} \omega_i = 1$, the definition of $\alpha_i, \beta_i, \gamma_i, \Delta T_i, \Delta I_i$ and ΔF_i are the same as Equation (1).

Theorem 3.1. *Let A, B, C be three SVNSs in X, the similarity measure ϑ in Equation* (1) *(or ϑ^* in Equation* (2)*) has the following conditions:*

(S1) $\vartheta(A,B) = \vartheta(B,A)$;

(S2) $0 \leq \vartheta(A,B) \leq 1$;

(S3) $\vartheta(A,B) = 1$ if and only if $A = B$;

(S4) If $A \subseteq B \subseteq C$, that is $T_A(x_i) \leq T_B(x_i) \leq T_C(x_i), I_A(x_i) \geq I_B(x_i) \geq I_C(x_i), F_A(x_i) \geq F_B(x_i) \geq F_C(x_i)$, for any x_i in X, then $\vartheta(A,C) \leq \min\{\vartheta(A,B), \vartheta(B,C)\}$.

Proof of Theorem 3.1. We only prove the case of ϑ.

(S1) Obviously.

(S2) With two SVNSs $A = \{< x_i, T_A(x_i), I_A(x_i), F_A(x_i) > |x_i \in X\}$ and $B = \{< x_i, T_B(x_i), I_B(x_i), F_B(x_i) > |x_i \in X\}$, we have $0 \leq \Delta T_i \leq 1$, $0 \leq \Delta I_i \leq 1$, $0 \leq \Delta F_i \leq 1$, which imply that $0 < \alpha_i \leq 1$, $0 < \beta_i \leq 1$ and $0 < \gamma_i \leq 1$. Then, the inequation

$$0 \leq \alpha_i(1-\Delta T_i) + \beta_i(1-\Delta I_i) + \gamma_i(1-\Delta F_i) \leq 3$$

holds. Therefore, we can prove that $0 \leq \vartheta(A,B) \leq 1$.

(S3) "$\Rightarrow$" While $\vartheta(A,B) = 1$, we have $\alpha_i(1-\Delta T_i)+\beta_i(1-\Delta I_i)+\gamma_i(1-\Delta F_i) = 3$ for any $i = 1, 2, \ldots, n$. Due to $\alpha_i \leq 1$, $\beta_i \leq 1$, $\gamma_i \leq 1$,$1-\Delta T_i \leq 1$, $1-\Delta I_i \leq 1$ and $1-\Delta F_i \leq 1$, we can get $\Delta T_i = \Delta I_i = \Delta F_i = 0$ for all i. This implies that $A = B$.

"$\Leftarrow$" Suppose $A = B$, that is $T_A(x_i) = T_B(x_i)$, $I_A(x_i) = I_B(x_i)$ and $F_A(x_i) = F_B(x_i)$. Further, it holds that $\Delta T_i = \Delta I_i = \Delta F_i = 0$. Then $\alpha_i = \beta_i = \gamma_i = 1$ for all i. So we have $\alpha_i(1-\Delta T_i) + \beta_i(1-\Delta I_i) + \gamma_i(1-\Delta F_i) = 3$. Therefore, $\vartheta(A,B) = 1$.

(S4) Suppose $A \subseteq B \subseteq C$, that is $T_A(x_i) \leq T_B(x_i) \leq T_C(x_i)$, $I_A(x_i) \geq I_B(x_i) \geq I_C(x_i)$, $F_A(x_i) \geq F_B(x_i) \geq F_C(x_i)$. On one hand, we have the following inequations for all $i = 1, 2, \ldots, n$,

$$\begin{aligned}
&\Delta T_i^{AC} \geq \Delta T_i^{AB} \Rightarrow \Delta T_{\min}^{AC} \geq \Delta T_{\min}^{AB}, \quad \Delta T_{\max}^{AC} \geq \Delta T_{\max}^{AB},\\
&\Delta I_i^{AC} \geq \Delta I_i^{AB} \Rightarrow \Delta I_{\min}^{AC} \geq \Delta I_{\min}^{AB}, \quad \Delta I_{\max}^{AC} \geq \Delta I_{\max}^{AB},\\
&\Delta F_i^{AC} \geq \Delta F_i^{AB} \Rightarrow \Delta F_{\min}^{AC} \geq \Delta F_{\min}^{AB}, \quad \Delta F_{\max}^{AC} \geq \Delta F_{\max}^{AB}.
\end{aligned}$$

This implies that $1-\Delta T_i^{AC} \leq 1-\Delta T_i^{BC}$, $1-\Delta I_i^{AC} \leq 1-\Delta I_i^{BC}$ and $1-\Delta F_i^{AC} \leq 1-\Delta F_i^{BC}$.

On the other hand, let $F(x,y,z)=\frac{3-x-y}{3+z-y}$. Then calculate the partial derivative of F on x,y,z, respectively, we have

$$F_x=\frac{-1}{3+z-y}<0,\ F_y=\frac{-z-x}{(3+z-y)^2}\leq 0, F_z=\frac{-3+x+y}{(3+z-y)^2}<0.$$

We know that F is monotonously decreasing on x,y,z. Thus, $\alpha_i^{AC}\leq\alpha_i^{AB}$, $\beta_i^{AC}\leq\beta_i^{AB}$ and $\gamma_i^{AC}\leq\gamma_i^{AB}$.

By Equation (1), we can get $\vartheta(A,C)\leq\vartheta(A,B)$. Similarly, we can prove $\vartheta(A,C)\leq\vartheta(B,C)$. Finally, $\vartheta(A,C)\leq\min\{\vartheta(A,B),\vartheta(B,C)\}$ holds for $A\subseteq B\subseteq C$.

Theorem 3.2. *Let A be a SVNS, then $\vartheta(A,A^c)$ is an entropy measure for A.*

Proof of Theorem 3.2. Let $E(A)=\vartheta(A,A^c)$ and need to verify it satisfies the four conditions in Definition 2.3. Suppose $A=\{<x_i,T_A(x_i),I_A(x_i),F_A(x_i)>|x_i\in X\}$ and $B=\{<x_i,T_B(x_i),I_B(x_i),F_B(x_i)>|x_i\in X\}$.

(E1) "$\Rightarrow$" When $\vartheta(A,A^c)=0$, due to $\alpha_i>0$, $\beta_i>0$, $\gamma_i>0$, we have $1-\Delta T_i^{AA^c}=1-\Delta I_i^{AA^c}=1-\Delta F_i^{AA^c}=0$ for any $i=1,2,\ldots,n$. Then, $\Delta T_i^{AA^c}=\Delta I_i^{AA^c}=\Delta F_i^{AA^c}=1$ which implies that $T_A(x_i)=1,I_A(x_i)=F_A(x_i)=0$ or $T_A(x_i)=I_A(x_i)=0,F_A(x_i)=1$ or $T_A(x_i)=I_A(x_i)=1,F_A(x_i)=0$ or $T_A(x_i)=0,I_A(x_i)=F_A(x_i)=1$ must hold for all $x_i\in X$. This shows that A or A^c is a crisp set.

"$\Leftarrow$" Assume that A is a crisp set, i.e. $T_A(x_i)=1,I_A(x_i)=F_A(x_i)=0$ or $F_A(x_i)=1,T_A(x_i)=I_A(x_i)=0$ for all $x_i\in X$. No matter in which case, we have $\vartheta(A,A^c)=0$. Analogously we can prove that $\vartheta(A,A^c)=0$, when A^c is a crisp set.

(E2) By the condition (3) in Theorem 3.1, it holds that $\vartheta(A,A^c)=1\Leftrightarrow A=A^c$.

(E3) Suppose that $T_A(x_i)\leq T_B(x_i),F_B(x_i)\leq F_A(x_i)$ for $T_B(x_i)\leq F_B(x_i)$ and $I_A(x_i)=I_B(x_i)=0.5$, we have the following results,

$$\begin{aligned}
&\Delta T_i^{AA^c}\geq\Delta T_i^{BB^c}\Rightarrow\alpha_i^{AA^c}\leq\alpha_i^{BB^c},\quad 1-T_i^{AA^c}\leq 1-\Delta T_i^{BB^c},\\
&\Delta F_i^{AA^c}\geq\Delta F_i^{BB^c}\Rightarrow\gamma_i^{AA^c}\leq\gamma_i^{BB^c},\quad 1-F_i^{AA^c}\leq 1-\Delta F_i^{BB^c},\\
&\Delta I_i^{AA^c}=\Delta I_i^{BB^c}=0\Rightarrow\beta_i^{AA^c}=\beta_i^{BB^c}=1,\quad 1-\Delta I_i^{AA^c}=1-\Delta I_i^{BB^c}=1.
\end{aligned}$$

This shows that $E(A)=\vartheta(A,A^c)\leq\vartheta(B,B^c)=E(B)$. Analogously we can prove that $\vartheta(A,A^c)\leq\vartheta(B,B^c)$ when $T_A(x_i)\geq T_B(x_i),F_B(x_i)\geq F_A(x_i)$ for $T_B(x_i)\geq F_B(x_i)$ and $I_A(x_i)=I_B(x_i)=0.5$.

(E4) By the condition (1) in Theorem 3.1, we have $E(A)=\vartheta(A,A^c)=\vartheta(A^c,A)=\vartheta(A^c,A^{cc})=E(A^c)$.

In some cases, we should consider not only the similarity between two SVNSs A and B, but also the similarity between A and B^c. The index of similarity for SVNSs A and B is defined in Definition 3.3.

Definition 3.3. *Let A and B be two SVNSs, then*

$$I_s(A,B) = \frac{\vartheta(A,B)}{\vartheta(A,B^c)}$$

is called the index of similarity.

Proposition 3.1. *The index of similarity $I_s(A,B)$ for two SVNSs A and B has the following properties:*

$(IS1)$ $I_s(A,B) = 0$ means A and B are completely dissimilar;

$(IS2)$ $I_s(A,B) \to +\infty$ means A and B^c are completely dissimilar;

$(IS3)$ $I_s(A,B) = 1$, i.e. $\vartheta(A,B) = \vartheta(A,B^c)$, this means the similarity between A and B, A and B^c are the same;

$(IS4)$ $I_s(A,B) < 1$ means compare with B^c, A is less similar to B;

$(IS5)$ $I_s(A,B) > 1$ means compare with B^c, A is more similar to B.

Definition 3.4. *Let $X = \{x_1, x_2, \ldots, x_n\}$, A and B be two SVNSs, which are denoted by $A = \{< x_i, T_A(x_i), I_A(x_i), F_A(x_i) > | x_i \in X\}$ and $B = \{< x_i, T_B(x_i), I_B(x_i), F_B(x_i) > | x_i \in X\}$. The single-valued neutrosophic distance measure are defined by*

$$d(A,B) = 1 - \vartheta(A,B), \tag{3}$$

where $\vartheta(A,B)$ is defined as Equation (1).

Remark. Using Equation 3 to calculate the distance between SVNSs A and B, A and C which are given in Example 1.

$$\begin{aligned} d(A,B) &= 1 - \frac{1}{6}\Big(\frac{23}{25}\times\frac{9}{10}+\frac{20}{26}\times\frac{5}{10}+\frac{18}{26}\times\frac{4}{10}+\frac{18}{25}\times\frac{4}{10}+\frac{24}{26}\times\frac{9}{10}+\frac{22}{26}\times\frac{8}{10}\Big) \\ &= 0.4525, \\ d(A,C) &= 1 - \frac{1}{6}\Big(\frac{27}{29}\times\frac{9}{10}+\frac{17}{29}\times\frac{4}{10}+\frac{25}{29}\times\frac{8}{10}+\frac{26}{29}\times\frac{8}{10}+\frac{16}{29}\times\frac{3}{10}+\frac{24}{29}\times\frac{7}{10}\Big) \\ &= 0.4623. \end{aligned}$$

We can see that A is closer to B.

Definition 3.5. *Let $X = \{x_1, x_2, \ldots, x_n\}$, A and B be two SVNSs, which are denoted by $A = \{< x_i,\ T_A(x_i), I_A(x_i), F_A(x_i) > | x_i \in X\}$ and $B = \{< x_i, T_B(x_i), I_B(x_i), F_B(x_i) > | x_i \in X\}$. The single-valued neutrosophic weighted distance measure are defined by*

$$d^*(A,B) = 1 - \vartheta^*(A,B), \tag{4}$$

where $\vartheta^*(A,B)$ is defined as Equation (2).

Theorem 3.3. *Let A, B, C be three SVNSs in X, the distance measure d in Equation* (3) (*or d^* in Equation* (4)) *satisfies the following properties:*

$(D1)$ $d(A, B) = d(B, A)$;

$(D2)$ $0 \leq d(A, B) \leq 1$;

$(D3)$ $d(A, B) = 0$ *if and only if* $A = B$;

$(D4)$ *If* $A \subseteq B \subseteq C$, *that is* $T_A(x_i) \leq T_B(x_i) \leq T_C(x_i)$, $I_A(x_i) \geq I_B(x_i) \geq I_C(x_i), F_A(x_i) \geq F_B(x_i) \geq F_C(x_i)$ *for any* x_i *in* X, *then* $d(A, C) \geq \max\{d(A, B), d(B, C)\}$.

Definition 3.6. *For two SVNSs A and B,*

$$I_d(A, B) = \frac{d(A, B)}{d(A, B^c)}$$

is called the index of distance.

Proposition 3.2. *Let A and B be two SVNSs, then the index of distance $I_d(A, B)$ has the following properties:*

$(ID1)$ $I_d(A, B) = 0$ *if and only if* $A = B$;

$(ID2)$ $I_d(A, B) \to +\infty$, *i.e.* $A = B^c$, *this means A and B^c are completely different;*

$(ID3)$ $I_d(A, B) = 1$, *i.e.* $d(A, B) = d(A, B^c)$, *this means the distance between A and B, A and B^c are the same;*

$(ID4)$ $I_d(A, B) < 1$ *means compare with B^c, A is less different to B;*

$(ID5)$ $I_d(A, B) > 1$ *means compare with B^c, A is more different to B;*

$(ID6)$ *If* $I_d(A, B) = \frac{0}{0}$, *i.e.* $A = B = B^c$, *then the entropy measure of A and B reaches its maximum value.*

4. Application examples

4.1 Pathological diagnosis

There are many uncertainties of pathological diagnosis. We will give the general steps and an example of pathological diagnosis under the SVNS environment.

For the set of pathological diagnosis results $A = \{A_1, A_2, \ldots, A_n\}$, the set of patients $B = \{B_1, B_2, \ldots, B_m\}$, and the set of microscopic shape results $C = \{C_1, C_2, \ldots, C_t\}$, the diagnosis steps are as follows:

Step 1. According to the doctors' experiences, each microscopic shape result of each pathological diagnosis result is described by a single-valued neutrosophic number. Then we have a matrix $M = (M_{ki})_{t\times n} = (M_1, M_2, \ldots, M_n)$;

Step 2. For every patient $B_j (j = 1, 2, \ldots, m)$, the microscopic shape result by examiner is also described by a single-valued neutrosophic number. Then we have a matrix $N = (N_{kj})_{t\times m} = (N_1, N_2, \ldots, N_m)$;

Step 3. Use the proposed similarity measure (1) to calculate $\vartheta(M_i, N_j), (i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m)$. Then we have the final matrix $D = D(ij)_{n\times m} = (D_1, D_2, \ldots, D_m)$;

Step 4. For every patient $B_j (j = 1, 2, \ldots, m)$, select the diagnosis result $D_j^* = \max_i D_{ij}$, $(i = 1, 2, \ldots, n; j = 1, 2, \ldots, m)$.

Example 2. To make a proper pathological diagnosis, we have the following sets. The set of pathological diagnosis results is

$$\begin{aligned} A &= \{A_1, A_2, A_3, A_4, A_5\} \\ &= \{\text{Fibrosarcoma, Malignant fibrous histiocytoma,} \\ &\quad \text{Spindle celltype synovial sarcoma,} \\ &\quad \text{Schwannoma, Malignant peripheral nerve sheath tumor}\}. \end{aligned}$$

The set of patients is $B = \{B_1, B_2, B_3, B_4, B_5\} = \{Au, Hung, Jo, Keo, Pao\}$. The set of microscopic shape results is

$$\begin{aligned} C &= \{C_1, C_2, C_3, C_4, C_5, C_6, C_7\} \\ &= \{\text{Palisade structure, Fishbone like structure, Spoke structure,} \\ &\quad \text{Fascicular structure,} \\ &\quad \text{Multiple nuclear fission, Pleomorphic, Fish-meat like}\}. \end{aligned}$$

The data are given in Table 1 and Table 2.

Table 1: The characteristics of each microscopic shape of the pathological diagnosis

	A_1	A_2	A_3	A_4	A_5
C_1	$<0.1, 0.2, 0.8>$	$<0.1, 0.3, 0.8>$	$<0.4, 0.3, 0.5>$	$<0.9, 0.1, 0.1>$	$<0.3, 0.2, 0.8>$
C_2	$<0.8, 0.1, 0.2>$	$<0.5, 0.1, 0.4>$	$<0.7, 0.1, 0.2>$	$<0.3, 0.2, 0.6>$	$<0.4, 0.1, 0.5>$
C_3	$<0.1, 0.3, 0.7>$	$<0.9, 0.1, 0.0>$	$<0.3, 0.2, 0.6>$	$<0.1, 0.2, 0.9>$	$<0.2, 0.1, 0.7>$
C_4	$<0.9, 0.1, 0.2>$	$<0.8, 0.1, 0.3>$	$<0.5, 0.1, 0.3>$	$<0.6, 0.3, 0.3>$	$<0.8, 0.2, 0.2>$
C_5	$<0.8, 0.2, 0.3>$	$<0.7, 0.2, 0.2>$	$<0.9, 0.1, 0.2>$	$<0.3, 0.2, 0.7>$	$<0.8, 0.2, 0.2>$
C_6	$<0.3, 0.1, 0.6>$	$<0.8, 0.3, 0.1>$	$<0.2, 0.2, 0.8>$	$<0.2, 0.1, 0.8>$	$<0.7, 0.1, 0.2>$
C_7	$<0.7, 0.3, 0.2>$	$<0.1, 0.1, 0.9>$	$<0.9, 0.1, 0.1>$	$<0.1, 0.3, 0.8>$	$<0.8, 0.2, 0.1>$

Table 2: The characteristics of each microscopic shape of the considered patients

	B_1	B_2	B_3	B_4	B_5
C_1	$<0.7, 0.2, 0.2>$	$<0.3, 0.2, 0.6>$	$<0.4, 0.2, 0.6>$	$<0.1, 0.1, 0.8>$	$<0.3, 0.1, 0.7>$
C_2	$<0.1, 0.3, 0.6>$	$<0.6, 0.2, 0.3>$	$<0.5, 0.2, 0.3>$	$<0.7, 0.2, 0.2>$	$<0.5, 0.2, 0.3>$
C_3	$<0.3, 0.2, 0.6>$	$<0.3, 0.2, 0.8>$	$<0.1, 0.3, 0.7>$	$<0.3, 0.3, 0.4>$	$<0.7, 0.1, 0.2>$
C_4	$<0.8, 0.1, 0.1>$	$<0.5, 0.2, 0.4>$	$<0.2, 0.2, 0.6>$	$<0.7, 0.3, 0.1>$	$<0.5, 0.2, 0.2>$
C_5	$<0.2, 0.2, 0.7>$	$<0.9, 0.1, 0.1>$	$<0.8, 0.2, 0.1>$	$<0.8, 0.2, 0.1>$	$<0.7, 0.3, 0.2>$
C_6	$<0.4, 0.2, 0.4>$	$<0.7, 0.1, 0.5>$	$<0.4, 0.2, 0.7>$	$<0.5, 0.3, 0.3>$	$<0.9, 0.2, 0.2>$
C_7	$<0.2, 0.1, 0.7>$	$<0.4, 0.2, 0.4>$	$<0.6, 0.1, 0.3>$	$<0.8, 0.1, 0.2>$	$<0.5, 0.1, 0.7>$

Then we use the proposed similarity measure (1) and have the pathological diagnosis results of the considered patients in Table 3.

Table 3: Similarity measure of pathological diagnosis results of each patient

	B_1(Au)	B_2(Hung)	B_3(Jo)	B_4(Keo)	B_5(Pao)
A_1 (Fibrosarcoma)	0.6524	0.7747	0.8131	**0.8467**	0.6857
A_2 (Malignant fibrous histiocytoma)	0.6533	0.6855	0.6524	0.7342	**0.8400**
A_3 (Spindle celltype synovial sarcoma)	0.6786	0.8157	**0.8177**	0.8044	0.6955
A_4 (Schwannoma)	**0.8070**	0.6724	0.6626	0.5805	0.6110
A_5 (Malignant peripheral nerve sheath tumor)	0.6927	**0.8199**	0.7884	0.8336	0.7968

Finally, we have the proper pathological diagnosis as follows:

Au got schwannoma, Hung got malignant peripheral nerve sheath tumor, Jo got spindle celltype synovial sarcoma, Keo got fibrosarcoma, Pao got malignant fibrous histiocytoma.

4.2 Clustering

Under the single-valued neutrosophic data environment, we apply the proposed similarity measure to cluster by using the method of [28]. The method is described as follows:

Step 1. Using the Equations (1) (or (2)), we can calculate a similarity matrix $C = (s_{ij})_{m\times m}$, where $s_{ij} = s_{ji} = \vartheta(A_i, A_j)$ for $i, j = 1, 2, \ldots, m$.

Step 2. Repeat the following operations

$$C \to C^2 \to C^4 \to \ldots \to C^{2^k}$$

until $C^{2^k} = C^{2^{k+1}}$, where $C^2 = C \circ C = (s'_{ij})_{m\times m} = \max_k\{\min(s_{ik}, s_{kj})\}_{m\times m}(i, j = 1, 2, \ldots, m)$. Then, we have an equivalent matrix C^{2^k}.

Step 3. Calculate the α-cutting matrix $\bar{C}_\alpha = (\bar{s}^\alpha_{ij})_{m\times m}$ of $\bar{C} = C^{2^k}$ by

$$\bar{s}^\alpha_{ij} = \begin{cases} 0, & \bar{s}_{ij} < \alpha; \\ 1, & \bar{s}_{ij} \geq \alpha, \end{cases} \tag{5}$$

for $j, j = 1, 2, \ldots, m$ and α is the confidence level with $\alpha \in [0, 1]$.

Step 4. Choose different values of α, then we can classify A_i. If $\bar{s}^\alpha_{ij} = \bar{s}^\alpha_{kj}$ for all $j = 1, 2, \ldots m$, then we can put A_i and A_k to the same class.

In the following, we consider a numerical example which is given by Ye [28].

Example 3 [28] A car market is going to classify five different cars of A_m $(m = 1, 2, 3, 4, 5)$. Every car has six evaluation factors (attributes): (i) x_1, fuel consumption; (ii) x_2, coefficient of friction; (iii) x_3, price; (iv) x_4, comfortable degree; (v) x_5, design; (vi) x_6, security coefficient. The characteristics of each car under the six attributes are represented by the form of SVNSs, and then the single-valued neutrosophic data are in Table 4:

Table 4: The characteristics of each car under the attributes are expressed by SVNSs

	x_1	x_2	x_3	x_4	x_5	x_6
A_1	$<0.3,0.2,0.5>$	$<0.6,0.3,0.1>$	$<0.4,0.3,0.3>$	$<0.8,0.1,0.1>$	$<0.1,0.3,0.6>$	$<0.5,0.2,0.4>$
A_2	$<0.6,0.3,0.3>$	$<0.5,0.4,0.2>$	$<0.6,0.2,0.1>$	$<0.7,0.2,0.1>$	$<0.3,0.1,0.6>$	$<0.4,0.3,0.3>$
A_3	$<0.4,0.2,0.4>$	$<0.8,0.2,0.1>$	$<0.5,0.3,0.1>$	$<0.6,0.1,0.2>$	$<0.4,0.1,0.5>$	$<0.3,0.2,0.2>$
A_4	$<0.2,0.4,0.4>$	$<0.4,0.5,0.1>$	$<0.9,0.2,0.0>$	$<0.8,0.2,0.1>$	$<0.2,0.3,0.5>$	$<0.7,0.3,0.1>$
A_5	$<0.5,0.3,0.2>$	$<0.3,0.2,0.6>$	$<0.6,0.1,0.3>$	$<0.7,0.1,0.1>$	$<0.6,0.2,0.2>$	$<0.5,0.2,0.3>$

Firstly, we use the similarity measure formula (1) to calculate the similarity measures between each pair of SVNSs A_m $(m = 1, 2, \ldots, 5)$. For example, we use ΔT_1^{12} to express $\Delta T_1^{A_1,A_2}$, then

$$\begin{aligned}
&\Delta T_1^{12} = 0.3,\ \Delta T_2^{12} = 0.1,\ \Delta T_3^{12} = 0.2,\ \Delta T_4^{12} = 0.1,\\
&\Delta T_5^{12} = 0.2,\ \Delta T_6^{12} = 0.1, \Delta I_1^{12} = 0.1,\ \Delta I_2^{12} = 0.1,\\
&\Delta I_3^{12} = 0.1,\ \Delta I_4^{12} = 0.1,\ \Delta I_5^{12} = 0.2,\ \Delta I_6^{12} = 0.1,\\
&\Delta F_1^{12} = 0.2,\ \Delta F_2^{12} = 0.1,\ \Delta F_3^{12} = 0.2,\ \Delta F_4^{12} = 0,\\
&\Delta F_5^{12} = 0,\ \Delta F_6^{12} = 0.1, \Delta T_{\min}^{12} = 0.1,\ \Delta T_{\max}^{12} = 0.3,\\
&\Delta I_{\min}^{12} = 0.1,\ \Delta I_{\max}^{12} = 0.2,\ \Delta F_{\min}^{12} = 0,\ \Delta F_{\max}^{12} = 0.2.
\end{aligned}$$

By Definition 3.1, we have $\alpha_1^{12} = \frac{3-0.3-0.3}{3+0.1-0.3} = \frac{24}{28}$, analogously,

$$\begin{aligned}
&\alpha_2^{12} = \frac{26}{28},\ \alpha_3^{12} = \frac{25}{28},\ \alpha_4^{12} = \frac{26}{28},\ \alpha_5^{12} = \frac{25}{28},\ \alpha_6^{12} = \frac{26}{28},\\
&\beta_1^{12} = \frac{27}{29},\ \beta_2^{12} = \frac{27}{29},\ \beta_3^{12} = \frac{27}{29},\ \beta_4^{12} = \frac{27}{29},\ \beta_5^{12} = \frac{26}{29},\ \beta_6^{12} = \frac{27}{29},\\
&\gamma_1^{12} = \frac{26}{28},\ \gamma_2^{12} = \frac{27}{28},\ \gamma_3^{12} = \frac{26}{28},\ \gamma_4^{12} = 1,\ \gamma_5^{12} = 1, \gamma_6^{12} = \frac{27}{28}.
\end{aligned}$$

Furthermore, we can get $\vartheta(A_1, A_2) = \frac{1}{18}\{\frac{24}{28}\frac{7}{10}+\frac{26}{28}\frac{9}{10}+\frac{25}{28}\frac{8}{10}+\frac{26}{28}\frac{9}{10}+\frac{25}{28}\frac{8}{10}+\frac{26}{28}\frac{9}{10}+\frac{27}{29}\frac{9}{10}+\frac{27}{29}\frac{9}{10}+\frac{27}{29}\frac{9}{10}+\frac{27}{29}\frac{9}{10}+\frac{26}{29}\frac{8}{10}+\frac{27}{29}\frac{9}{10}+\frac{26}{28}\frac{8}{10}+\frac{27}{28}\frac{9}{10}+\frac{26}{28}\frac{8}{10}+1+1+\frac{27}{28}\frac{9}{10}\} = 0.8147$.

The similarity measures between each A_i and A_j $(i, j = 1, 2, 3, 4, 5, i \neq j)$ are shown as follows:

$$\begin{aligned}
&\vartheta(A_1, A_2) = 0.8147, \quad \vartheta(A_1, A_3) = 0.8464,\\
&\vartheta(A_1, A_4) = 0.8148, \quad \vartheta(A_1, A_5) = 0.7824,\\
&\vartheta(A_2, A_3) = 0.8453, \quad \vartheta(A_2, A_4) = 0.8264,\\
&\vartheta(A_2, A_5) = 0.8294, \quad \vartheta(A_3, A_4) = 0.7563,\\
&\vartheta(A_3, A_5) = 0.7680, \quad \vartheta(A_4, A_5) = 0.7190.
\end{aligned}$$

Then, we have a similarity matrix as follows:

$$C = \begin{bmatrix} 1 & 0.8147 & 0.8464 & 0.8148 & 0.7824 \\ 0.8147 & 1 & 0.8453 & 0.8264 & 0.8294 \\ 0.8464 & 0.8453 & 1 & 0.7563 & 0.7680 \\ 0.8148 & 0.8264 & 0.7563 & 1 & 0.7190 \\ 0.7824 & 0.8294 & 0.7680 & 0.7190 & 1 \end{bmatrix}.$$

Then, by Step 2 in Chapter 4.2, we have

$$C^2 = \begin{bmatrix} 1 & 0.8453 & 0.8464 & 0.8148 & 0.8147 \\ 0.8453 & 1 & 0.8453 & 0.8264 & 0.8294 \\ 0.8464 & 0.8453 & 1 & 0.8264 & 0.8294 \\ 0.8148 & 0.8264 & 0.8264 & 1 & 0.8264 \\ 0.8147 & 0.8294 & 0.8294 & 0.8264 & 1 \end{bmatrix}.$$

Due to $C^2 \nsubseteq C$, i.e., C is not an equivalent matrix, we keep calculating.

$$C^4 = \begin{bmatrix} 1 & 0.8453 & 0.8464 & 0.8264 & 0.8294 \\ 0.8453 & 1 & 0.8453 & 0.8264 & 0.8294 \\ 0.8464 & 0.8453 & 1 & 0.8264 & 0.8294 \\ 0.8264 & 0.8264 & 0.8264 & 1 & 0.8264 \\ 0.8294 & 0.8294 & 0.8294 & 0.8264 & 1 \end{bmatrix}.$$

$$C^8 = \begin{bmatrix} 1 & 0.8453 & 0.8464 & 0.8264 & 0.8294 \\ 0.8453 & 1 & 0.8453 & 0.8264 & 0.8294 \\ 0.8464 & 0.8453 & 1 & 0.8264 & 0.8294 \\ 0.8264 & 0.8264 & 0.8264 & 1 & 0.8264 \\ 0.8294 & 0.8294 & 0.8294 & 0.8264 & 1 \end{bmatrix}.$$

It is clearly that $C^4 \nsubseteq C^2$, but $C^8 = C^4$. That is, C^4 is an equivalent matrix, denoted by $\bar{C}$.

Finally, we calculate the α-cutting matrix $\bar{C}_\alpha$ by setting different values of α.

(I) Let $0 \leq \alpha \leq 0.8264$,

$$\bar{C}_\alpha = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

By Step 4 in Section 4.2, A_m $(m = 1, 2, \ldots, 5)$ can be divided into one category $\{A_1, A_2, A_3, A_4, A_5\}$.

(II) Let $0.8264 < \alpha \leq 0.8294$,

$$\bar{C}_\alpha = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix}.$$

Then, the cars A_m $(m = 1, 2, \ldots, 5)$ can be divided into two categories $\{A_1, A_2, A_3, A_5\}$, $\{A_4\}$.

(III) Let $0.8294 < \alpha \leq 0.8453$,

$$\bar{C}_\alpha = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then, the cars A_m $(m = 1, 2, \ldots, 5)$ can be divided into three categories $\{A_1, A_2, A_3\}$, $\{A_4\}$, $\{A_5\}$.

(IV) Let $0.8453 < \alpha \leq 0.8464$,

$$\bar{C}_\alpha = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then the cars A_m $(m = 1, 2, \ldots, 5)$ can be divided into four categories $\{A_1, A_3\}$, $\{A_2\}$, $\{A_4\}$, $\{A_5\}$.

(V) Let $0.8464 < \alpha \leq 1$,

$$\bar{C}_\alpha = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then the cars A_m $(m = 1, 2, \ldots, 5)$ can be divided into five categories $\{A_1\}$, $\{A_2\}$, $\{A_3\}$, $\{A_4\}$, $\{A_5\}$.

Comparative analysis of existing methods. Refs. [27, 28, 11] considered the same problem as Example 2, the results are shown in Table 5. For convenience, we use the corresponding subscript m to represent A_m, $(m = 1, 2, 3, 4, 5)$.

Table 5: Clustering results of different Refs.

	Method from [27]	Method from [28]	Method from [11]	our method
1 category	$\{1, 2, 3, 4, 5\}$	$\{1, 2, 3, 4, 5\}$	$\{1, 2, 3, 4, 5\}$	$\{1, 2, 3, 4, 5\}$
2 categories	$\{1, 2, 3, 5\}, \{4\}$	$\{1, 2, 3, 4\}, \{5\}$	failed	$\{1, 2, 3, 5\}, \{4\}$
3 categories	$\{1, 2, 3\}, \{4\}, \{5\}$	$\{1, 2, 3\}, \{4\}, \{5\}$	$\{1, 2, 3\}, \{4\}, \{5\}$	$\{1, 2, 3\}, \{4\}, \{5\}$
4 categories	$\{1\}, \{2, 3\}, \{4\}, \{5\}$	$\{1\}, \{2, 3\}, \{4\}, \{5\}$	failed	$\{1, 3\}, \{2\}, \{4\}, \{5\}$
5 categories	$\{1\}, \{2\}, \{3\}, \{4\}, \{5\}$	$\{1\}, \{2\}, \{3\}, \{4\}, \{5\}$	$\{1\}, \{2\}, \{3\}, \{4\}, \{5\}$	$\{1\}, \{2\}, \{3\}, \{4\}, \{5\}$

By comparison, we can see the superiority of our results as following.

(1) Ref. [11] can only classify objectives into one category, three categories and five categories, but can not classify them into two categories and four categories. However, our approach divides objectives into one to five categories. This means that our method has better accuracy in some clustering problems.

(2) Our clustering results are also different from the results of [27, 28]. In [27], most classifications are the same as our results except dividing the cars into four categories $\{A_1\}$, $\{A_2, A_3\}$, $\{A_4\}$, $\{A_5\}$. In [28], the difference only appeared in dividing the cars into two categories $\{A_1, A_2, A_3, A_4\}$, $\{A_5\}$. The reason is that the similarity measure or the distance measure in [27, 28] only consider about the difference between each corresponding data, but ignore the impact on the overall data gap between each data.

4.3 Decision making

Example 3 ([8])**.** A city is going to build a municipal library. How to determine what kind of air-conditioning system should be installed in the library is one of the problems facing the city development commissioner. The contractor provides five feasible alternatives A_i $(i = 1, 2, 3, 4, 5)$, which might be adapted to the library. Supposing that three attributes C_1 (economic), C_2(functional), and C_3(operational), the weight vector of the attribute C_j $(j = 1, 2, 3)$ is $w = (0.5, 0.2, 0.3)^T$. Meanwhile, the attributes are all benefit attribute. Assume that the alternatives A_i $(i = 1, 2, 3, 4, 5)$ with respect to the attribute C_j $(j = 1, 2, 3)$ are represented by the single-valued neutrosophic number, as shown in Table 6.

Table 6: The alternatives with respect to the attribute denoted by SVNSs

	C_1	C_2	C_3
A_1	$< 0.0, 0.0, 0.0 >$	$< 1.0, 0.0, 0.0 >$	$< 0.0, 0.0, 1.0 >$
A_2	$< 0.0, 0.1, 0.1 >$	$< 0.9, 0.1, 0.2 >$	$< 0.4, 0.2, 0.2 >$
A_3	$< 0.1, 0.1, 0.1 >$	$< 0.9, 0.2, 0.3 >$	$< 0.5, 0.3, 0.2 >$
A_4	$< 0.3, 0.0, 0.1 >$	$< 1.0, 0.1, 0.2 >$	$< 0.4, 0.3, 0.2 >$
A_5	$< 0.0, 0.1, 0.0 >$	$< 0.9, 0.1, 0.2 >$	$< 0.4, 0.3, 0.2 >$

Table 7: A comparison studies with some exiting methods of Example 3

Method	The final ranking	The optimal alternative
Method from [29]	$A4 > A_1 > A_2 > A_5 > A_3$	A_4
Method from [30]	$A_4 > A_1 > A_5 > A_2 > A_3$	A_4
Method from [32]	N/A	failed
Method from [33]	N/A	failed
Method from [31]	$A_4 > A_2 > A_5 > A_1 > A_3$	A_4
Our method	$A_4 > A_1 > A_5 > A_2 > A_3$	A_4

By applying Equation (2), the similarity between an alternative A_i $(i = 1, 2, 3, 4, 5)$ and the ideal alternative $A^* = (< 1, 0, 0 >, < 1, 0, 0 >, < 1, 0, 0 >)$ is

as follows:

$$\vartheta^*(A_1, A^*)=\frac{1}{3}\Big[0.5\times(0+1+1)+0.2\times(1+1+1)+0.3\times(0+1+0)\Big]=0.6333,$$

$$\vartheta^*(A_2, A^*)=\frac{1}{3}\Big[0.5\times(0+\frac{243}{290}+\frac{243}{290})+0.2\times(\frac{171}{210}+\frac{243}{290}+\frac{208}{290})$$
$$+0.3\times(\frac{56}{210}+\frac{208}{290}+\frac{208}{290})\Big]=0.6074,$$

$$\vartheta^*(A_3, A^*)=0.5976, \qquad \vartheta*(A_4, A^*)=0.6769, \qquad \vartheta*(A_5, A^*)=0.6264$$

The results shows that the ranking order of the five alternatives is $A_4 > A_1 > A_5 > A_2 > A_3$. Therefore, the best supplier is A_4.

Comparative analysis of existing methods.

In order to further verify the practicability of the proposed new similarity measure, we use some exiting methods to calculate Example 3. The results can be seen in Table 7. By comparison, the practicability and superiority over our method is summarized as follows.

(1) The optimal alternative obtained by our method is the same as that in Refs. [29, 30, 31].

(2) Due to the division by zero problem, the ranking method of [32, 33] cannot apply to Example 3. But our method can be used to calculate it.

(3) The final ranking of the five alternatives obtained by our method is the same as using the Multi-Attributive Border Approximation area Comparison (MABAC) proposed by Peng [30]. However, the computational complexity of our method is much smaller than that in [30].

(4) Stability analysis. If the range of the weight vectors $w=(w_1, w_2, w_3)^T$ of the attribute is changed into $w_1 \in [0.4, 0.6], w_2 \in [0.1, 0.3]$, the optimal alternative calculated by our method is till A_4. This shows that our method is robust and stable.

5. Conclusion

Similarity measure (or distant measure) is an important tool for fuzzy decision-making and clustering. Under the single-valued neutrosophic set environment, the proposed similarity measure not only considers the relationship between the corresponding factors, but also considers the influence of the overall factors of the similarity. Then we use the new similarity measure in pathological diagnosis, clustering and decision-making to demonstrate its effectiveness. Comparing with the existing methods, it can be found that our method has the characteristics of high accuracy and wide application range.

Acknowledgment

This work is supported by National Natural Science Foundation Project (11701089, 11871259), the Key Program of the Natural Science Foundation of Fujian Province (2020J02043), Fujian Natural Science Foundation (2018J01422,2020J01801).

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Accepted: March 11, 2020

A modified Newton's method for solving functions of one variable

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Abstract. In this research, we developed the Newton method to diminish the number of iterations to arrive at the optimal solution for the problem, depending on the Taylor series to approximate the derivatives applying the minimum value of the problem. We compare the method presented with the traditional Newton method in requisites of the number of iteration and the time of execution. Shown numerical results effectiveness for the proposed algorithm after applying it to a group of unimodal functions.
Keywords: modified Newton's method, iteration methods, test functions.

1. Introduction

Solving nonlinear problems of one variable is an important part of iteration methods. For details see [10].

The oldest one and the most famed problems in iteration methods, for finding the minimum value of problems of the form :

$$Minimize\,\{\varpi(\upsilon), \upsilon \in R, \varpi : R \to R\}\,, \tag{1}$$

where $\varpi(\upsilon)$ is a smooth function. For more details see [4,5].

This problems may not be easy to solve. Therefore, we consider briefly a numerical method for its solution. The most widely and the best known used to find the value minimum of optimization methods is the kind Newton's algorithm which converges quadratically. For more details can be found in [12].

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The idea is Newton's formula for approximating the objective function locally by a quadratic model corresponding to the model at a point. The exact approximate function is then improved. For more details see [13]. So, the iteration of Newton's formula is given by:

$$\upsilon_{k+1} = \upsilon_k - \frac{\varpi'(\upsilon_k)}{\varpi''(\upsilon_k)} \tag{2}$$

which is a Newton method.

The secant formula can be a vision as a simplification of Newton's formula that evades calculating the second derivation is obtained as follows:

$$\upsilon_{k+1} = \upsilon_k - \frac{\varpi'(\upsilon_k)(\upsilon_k - \upsilon_{k-1})}{\varpi'(\upsilon_k) - \varpi'(\upsilon_{k-1})}. \tag{3}$$

For more details can be found in [3].

Lately, many studies of Secant-type methods, has become focused on solving nonlinear optimization issues, such that $\varpi''(\upsilon_k)$ this is not required (e.g. [3,5-9 and 14]).

This paper aims to derive a new secant method for improving the iteration of Newton's formula based on avoiding from the evaluation of $\varpi''(\upsilon_k)$.

2. A modified Newton's method

The idea of using Taylor expansion is the key element for derivation the new Newton's methods, we give [11]:

$$\varpi(\upsilon) = \varpi(\upsilon_k) + \varpi'(\upsilon_k)(\upsilon - \upsilon_k) + \frac{1}{2}\varpi''(\upsilon_k)(\upsilon - \upsilon_k)^2. \tag{4}$$

The gradient of $\varpi(\upsilon)$ for υ is given by:

$$\varpi'(\upsilon) = \varpi'(\upsilon_k) + \varpi''(\upsilon_k)(\upsilon - \upsilon_k) = 0. \tag{5}$$

Putting Eq. (5) in Eq. (4) we get:

$$2(\varpi(\upsilon_k) - \varpi(\upsilon)) = \varpi''(\upsilon_k)(\upsilon - \upsilon_k)^2. \tag{6}$$

This yields:

$$\varpi''(\upsilon_k) = \frac{2 * (\varpi(\upsilon_k) - \varpi(\upsilon))}{(\upsilon - \upsilon_k)^2}. \tag{7}$$

Substituting υ_{k-1} in to υ in above equation and can be rewritten as :

$$\varpi''(\upsilon_k) = \frac{(\varpi(\upsilon_k) - \varpi(\upsilon_{k-1}))}{(\upsilon_{k-1} - \upsilon_k)^2}. \tag{8}$$

Putting Eq. (8) into Eq. (2), we get:

$$\upsilon_{k+1} = \upsilon_k - \frac{\varpi'(\upsilon_k)(\upsilon_{k-1} - \upsilon_k)^2}{2 * (\varpi(\upsilon_k) - \varpi(\upsilon_{k-1}))}, \tag{9}$$

which is a modified Newton's formula.

The modified of Newton's formula above is presented in the following algorithm:
Stage i. Givenε, υ_0 and the function$\varpi(\upsilon_0)$. $\upsilon_1 = \upsilon_0 + 0.1$, set $k = 0$.
Stage ii. Set $k = k + 1$.
Stage iii. Compute $\upsilon_{k+1} = \upsilon_k - [\varpi'(\upsilon_k)(\upsilon_{k-1} - \upsilon_k)^2/2 * (\varpi(\upsilon_k) - \varpi(\upsilon_{k-1}))]$.
Stage iv. Stop when the absolute value of the derivative of the function of the new iteration is sufficiently small i.e. $|\varpi'(\upsilon_k)| \leq \varepsilon$.

3. Analysis of convergence

In this section, we prove that modified Newton's formula (9) has the likelihood of local convergence.

Theorem 3.1. *Let $\varpi : I \to R$ bea sufficiently differentiable function and $\upsilon^* \in I$ be a zero of ϖ, where I is an open interval then the proposed method has quadratic convergence. If υ_0 sufficiently close to υ^*.*

Proof. The modified Newton's formula is as:

$$\upsilon_{k+1} = \upsilon_k - \frac{\varpi'(\upsilon_k)(\upsilon_{k-1} - \upsilon_k)^2}{2 * (\varpi(\upsilon_k) - \varpi(\upsilon_{k-1}))}. \tag{10}$$

By subtracting υ^* from both sides of above equation, we get :

$$e_{k+1} = e_k - \frac{\varpi'(\upsilon_k)(e_{k-1} - e_k)^2}{2 * (\varpi(\upsilon_k) - \varpi(\upsilon_{k-1}))}, \tag{11}$$

where $e_k = \upsilon_{k+1} - \upsilon^*$.

By expanding Taylor's series:

$$\begin{aligned} \varpi(\upsilon^*) = \varpi(\upsilon_k) + \varpi'(\upsilon_k)(\upsilon^* - \upsilon_k) + \frac{1}{2!}\varpi''(\upsilon_k)(\upsilon^* - \upsilon_k)^2 \\ + \frac{1}{3!}\varpi'''(c)(\upsilon^* - \upsilon_k)^3, \end{aligned} \tag{12}$$

where c lies between υ^* and υ_k, the derivative of equation (12), we get:

$$\varpi'(\upsilon^*) = \varpi'(\upsilon_k) + \varpi''(\upsilon_k)(\upsilon^* - \upsilon_k) + \frac{1}{2}\varpi'''(c)(\upsilon^* - \upsilon_k)^2 = 0. \tag{13}$$

From (13), we have:

$$-\varpi'(\upsilon_k) = -e_k\varpi''(\upsilon_k) + \frac{e_k^2}{2}\varpi'''(c). \tag{14}$$

Putting equation (14) in to equation (10) we get:

$$e_{k+1} = e_k^2 \frac{\varpi'''(c)}{\varpi''(\upsilon_k)}, \tag{15}$$

which implies that the order of convergence is quadratic. The proving is complete.

4. Application examples

The Newton and modified Newton's methods were applied to single-variable functions using the software Matlab. We use accuracy is $\varepsilon = 10^{-10}$, for computer programs. Comparison between the methods based on the execution time and iteration number for the various functions.

Example 1. Problem $\varpi(\upsilon) = \cos(\upsilon) + (\upsilon - 2)^2$, initial approximation $\upsilon_0 = 2$.

Methods	Number of iterations	υ_{k+1}	Execution time
Newton method	**4**	**2.3542**	**0.5790**
New method	**2**	**2.0997**	**0.2040**

Example 2. Problem $\varpi(\upsilon) = e^{\upsilon} - 3\upsilon^2$, initial approximation $\upsilon_0 = 0.25$.

Methods	Number of iterations	υ_{k+1}	Execution time
Newton method	**4**	**0.2045**	**0.2660**
New method	**2**	**0.3498**	**0.2030**

Example 3. Problem $\varpi(\upsilon) = e^{-\upsilon} + \upsilon^2$, initial approximation $\upsilon_0 = 1$.

Methods	Number of iterations	υ_{k+1}	Execution time
Newton method	**5**	**0.3517**	**0.3130**
New method	**3**	**1.0984**	**0.2340**

Example 4. Problem $\varpi(\upsilon) = -\upsilon e^{-\upsilon}$, initial approximation $\upsilon_0 = 0$.

Methods	Number of iterations	υ_{k+1}	Execution time
Newton method	**7**	**1**	**0.3900**
New method	**2**	**0.0996**	**0.2040**

Example 5. Problem $\varpi(\upsilon) = 0.65 - 0.75/(1 + \upsilon^2) - 0.65\upsilon \tan^{-1}(1/\upsilon)$, initial approximation $\upsilon_0 = 0.1$.

Methods	Number of iterations	υ_{k+1}	Execution time
Newton method	**6**	**0.4809**	**0.7970**
New method	**2**	**0.1998**	**0.2960**

Example 6. Problem $\varpi(\upsilon) = \upsilon/\log(\upsilon)$, initial approximation $\upsilon_0 = 0.1$.

Methods	Number of iterations	υ_{k+1}	Execution time
Newton method	——	**Fail**	——
New method	**2**	**0.1996**	**0.2190**

Example 7. Problem $\varpi(\upsilon) = \upsilon^4 + 2\upsilon^2 - \upsilon - 3$, initial approximation $\upsilon_0 = 1$.

Methods	Number of iterations	υ_{k+1}	Execution time
Newton method	**7**	**0.2367**	**0.6870**
New method	**4**	**1.0672**	**0.3130**

5. Conclusion

All methods in some cases may fail to converge to the minimum and it is not necessarily of may converge to global minimum also.

Based on our findings, we now conclude that the new method is the most effective of the Newton methods we studied in this study. We need to evaluate the first derivative at each step and we will likely have difficulty obtaining the global minimum. In the worst case, it is not much worse than its affinity. Can interpret the method performance in multi-dimensional case, as such in [1,2].

Acknowledgement

The authors are very grateful to the University of Mosul/College of Computer Sciences and Mathematics for their provided facilities, which helped to improve the quality of this work.

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Accepted: February 02, 2020

New step sizes of the gradient methods for unconstrained optimization problem

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Abstract. In this work, we derive a new formula for step-length in the frame of gradient descent methods. The idea of method is based on the quadratic model for using a new approximation of the Hessian of the minimizing function. The rate of convergence is linear and it belongs to the same class of gradient descent methods. Numerical experiment shows that new method is very promising compared to the Barzilai-Borwein approach.

Keywords: modified Newton's method, iteration methods.

1. Introduction

Unconstrained optimization methods are useful in finding the optimal solution for continuous and differentiable functions.

An unconstrained optimization problems can be stated as follows:

$$\min \Psi(x), x \in R^n. \tag{1}$$

The method of steepest descent is generally the most simplest method for unconstrained optimization problem. More details can be found in[10].

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In the steepest descent method, at each iterate point x_k, the search direction is chosen as the steepest descent direction:

$$d_k = -\nabla\Psi(x_k). \tag{2}$$

The iterative procedure of this method can be stated as follows:

$$x_{k+1} = x_k - \alpha_k \nabla\Psi(x_k). \tag{3}$$

where α_k is step-size, to be computed by line search procedure along the search direction d_k. For more details see [9].

Now, we remind to several modified gradient methods with some modified step-sizes, due to its simplicity and numerical efficiency.

One of the famous methods is BB methods [5] suggested by Barzilai and Borwein in 1988, the step-sizes is given by:

$$\alpha_k^{BB1} = \frac{\|s_k\|^2}{y_k^T s_k}\ ,\ \alpha_k^{BB2} = \frac{y_k^T s_k}{\|y_k\|^2}, \tag{4}$$

where $s_k = x_{k+1} - x_k = -\alpha_k \nabla\Psi(x_{k+1})$ and $y_k = \nabla\Psi(x_{k+1}) - \nabla\Psi(x_k)$.

In 2002, Dai et al. [7] developed some new step-sizes for BB-like methods, given by:

$$\alpha_k^{DYY1} = \frac{s_k^T s_k}{2(\Psi_k - \Psi_{k+1} + g_{k+1}^T s_k)}\ ,\ \alpha_k^{DYY2} = \frac{s_k^T s_k}{6(\Psi_k - \Psi_{k+1}) + 4g_{k+1}^T 2s_k} \tag{5}$$

However, gradient methods are worthy in terms of the numerical performances. See also [2,6,8,11,12,13] for more details.

It remains to examine how to design more effective and efficient gradient methods for unrestricted improvement. Some new step sizes are described and tested.

2. Deriving new step sizes

We derive some new optimal step sizes of gradient methods based on the second-order Taylor's series approximation of the function. The model function is defined by:

$$\Psi(x_k) = \Psi(x_{k+1}) - g_{k+1}^T s_k + \frac{1}{2} s_k^T Q s_k. \tag{6}$$

Obviously, the approximate optimal step size associated to $\Psi(x_k)$ is:

$$\alpha_k = \frac{-g_k^T d_k}{d_k^T Q d_k}. \tag{7}$$

Let $\tau_k I$ be an approximation to the Hessian. Now, from the exact search property the following condition must be fulfilled:

$$g_{k+1}^T s_k = 0. \tag{8}$$

It is clear that the selection of τ_k is key to the approximate effect of the Hessian matrix. We determine τ_k by putting (8) in (6) we get:

$$2(\Psi(x_k) - \Psi(x_{k+1})) = \tau_k s_k^T s_k. \tag{9}$$

which implies that:

$$\tau_k = \frac{2(\Psi(x_k) - \Psi(x_{k+1}))}{s_k^T s_k} \tag{10}$$

We obtain the approximation optimal step-size:

$$\alpha_k = \tau_k^{-1} = \frac{s_k^T s_k}{2(\Psi(x_k) - \Psi(x_{k+1})}. \tag{11}$$

Imposing $g_{k+1}^T d_k = 0$ and using (7) which implies that:

$$s_k^T Q s = 2\Psi(x_k) - \Psi(x_{k+1}) - \alpha_k g_k^T d_k/2 \tag{12}$$

which also implies a new approximation model:

$$\tau_k s_k^T s_k = \Psi(x_k) - \Psi(x_{k+1}) - \alpha_k g_k^T d_k/2 \tag{13}$$

which implies that:

$$\tau_k = \frac{\Psi(x_k) - \Psi(x_{k+1}) - g_k^T s_k/2}{s_k^T s_k}. \tag{14}$$

Similarly, we obtain the approximation optimal step-size:

$$\alpha_k = \tau_k^{-1} = \frac{s_k^T s_k}{\Psi(x_k) - \Psi(x_{k+1} - g_k^T s_k/2}. \tag{15}$$

As a result, we adopt a new algorithm and called Algorithm BFS.

Algorithm 1 Algorithm BFS

1: Select $x_o \in R$ and compute $d_k = -\nabla\Psi(x_k)$. Set $k = 0$.
2: Test a criterion for stopping the iterations. If the test is satisfied, then stop,
3: If $\Psi(x_k) - \Psi(x_{k+1}) < 0.5$, then put $\Psi(x_k) - \Psi(x_{k+1}) = 0.5$ and compute α_k the scalars as in (11) and (15), using these scalars.
4: Update the variables: $x_{k+1} = x_k - \alpha_k \nabla\Psi(x_k)$ set and go to step 2.

3. Global convergence

Now we turn to the global convergence of the BFS under the following assumption:

Let Ψ is strongly convex function and let the level set $\Phi = \{x \in R^n : \Psi(x) \leq \Psi(x_o)\}$ is closed. Strong convexity of Ψ on Φ involves the existence of the constant m and M such that:

$$mI \leq \nabla^2\Psi(x) \leq MI, \tag{16}$$

for all $x \in \Phi$. A consequence of strong convexity of Ψ on Φ is that we can bound Ψ^* as:

$$\Psi(x) - \frac{1}{2m}\|\nabla^2\Psi(x)\|_2^2 \leq \Psi(x^*) \leq \Psi(x) - \frac{1}{2M}\|\nabla^2\Psi(x)\|_2^2. \tag{17}$$

More details, can be found in [5,6].

Theorem 3.1. *The New Algorithm with backtracking is linearly convergent and*

$$\Psi(x_k) - \Psi^* \leq \left\{\prod_{i=1}^{k-1} c_i\right\}\{\Psi(x_o) - \Psi^*\}, \tag{18}$$

where $c_1 = 1 - \min\{m, ms^{p_k}\} < 1$ *and* $p_k \geq 1$ *is an integer,* ($p_1 = 1, 2, 3, \cdots$ *given by the backtracking procedure*).

Proof. By (10), we write:

$$\Psi(x_{k+1}) = \Psi(x_k) - 1/2(\tau_k s_k^T s_k). \tag{19}$$

Now, at the point $x_{k+1} = x_k - \alpha_k\nabla\Psi(x_k)$ we have:

$$\begin{aligned} \Psi(x_{k+1}) &= \Psi(x_k) - 1/2(\tau_k\alpha_k^2\nabla\Psi(x_k)^T\nabla\Psi(x_k)), \\ \Psi(x_{k+1}) &= \Psi(x_k) - 1/2(\alpha_k\nabla\Psi(x_k)^T\nabla\Psi(x_k)). \end{aligned} \tag{20}$$

Hence

$$\Psi(x_{k+1}) = \Psi(x_k) - \frac{1}{2}\alpha_k\|\nabla\Psi(x_k)\|_2^2. \tag{21}$$

Using backtracking procedure terminates either with $\alpha_k = 1$ or with $\alpha_k = s^{p_k}$, where p_k is an integer. Therefore:

$$\Psi(x_{k+1}) = \Psi(x_k) - \min\{\frac{1}{2}, \frac{1}{2}s^{p_k}\}\|\nabla\Psi(x_k)\|_2^2. \tag{22}$$

Having in view that for strongly convex functions $\|\nabla\Psi(x_k)\|_2^2 \geq 2m(\Psi(x_k) - \Psi^*)$ it follows that:

$$\Psi(x_k) - \Psi^* \leq c_k(\Psi(x_k) - \Psi^*), \tag{23}$$

where $c_k = 1 - \min\{m, ms^{p_k}\}$.

On the other hand, from (15), we get:

$$\tau_k s_k^T s_k = \Psi(x_k) - \Psi(x_{k+1}) - \alpha_k g_k^T d_k/2. \tag{24}$$

Also, at the point $x_{k+1} = x_k - \alpha_k \nabla f(x_k)$ and $d_k = -\nabla\Psi(x_k)$ we have:

$$\tau_k \alpha_k^2 \nabla\Psi(x_k)^T \nabla\Psi(x_k) = \Psi(x_k) - \Psi(x_{k+1}) - \alpha_\nabla \Psi(x_k)^T \nabla\Psi(x_k)/2. \tag{25}$$

From the above equation, we get:

$$\begin{gathered} \alpha_k \|\nabla\Psi(x_k)\|_2^2 = \Psi(x_k) - \Psi(x_{k+1}) + \alpha_k \|\nabla\Psi(x_k)\|_2^2/2, \\ \Psi(x_k) - \Psi(x_{k+1}) = (\alpha_k/2)\|\nabla\Psi(x_k)\|_2^2, \end{gathered} \tag{26}$$

which implies that:

$$\Psi(x_{k+1}) = \Psi(x_k) - \frac{1}{2}\alpha_k \|\nabla\Psi(x_k)\|_2^2. \tag{27}$$

Similarly in case above, we obtain:

$$\Psi(x_{k+1}) - \Psi^* \leq c_k(\Psi(x_k) - \Psi^*), \tag{28}$$

where $c_k = 1 - \min\{m, ms^{p_k}\}$. Since $c_k < 1$ the sequence $\{\Psi(x_k)\}$ is linearly convergent, as a geometric series, to Ψ^*. □

4. Numerical results

We implemented the BFS and BB methods to a great number of standard test functions include unconstrained problems which are mainly from [1], to check the numerical performance. Different test functions have been used in different research such as [3, 4].

The iteration is stopped if the inequality $\| \Psi(x) \| \leq 10^{-6}$ or $\frac{\Psi(x_{k+1}) - \Psi(x)}{1 + \Psi(x_k} \leq 10^{-6}$ is satisfied. We choose the following parameters for BFS and BB: $\alpha = 0.00001$ and $s = 0.8$, then the number of iterations (NI) and the the number of function evaluations (NF) corresponding to the BFS and Barzilai- Borwein algorithms are given in Table 1.

		BB algorithm		BFS with (11)		BFS with (15)	
P. No.	**n**	**NI**	**NF**	**NI**	**NF**	**NI**	**NF**
Extended Tridiagonal 1	100	45	215	16	98	7	28
	1000	45	215	20	136	7	28
Generalized Tridiagonal 2	100	36	124	28	210	31	135
	1000	90	516	44	365	24	77
Generalized PSC1	100	794	4808	14	103	12	49
	1000	803	4745	30	306	21	50
Extended BD1	100	21	69	20	135	20	73
	1000	21	69	27	182	25	104
Quadratic QF1	100	149	649	30	208	31	132
	1000	1364	7145	94	695	90	372
Extended Quadratic Penalty QP1	100	13	58	20	173	11	52
	1000	17	73	15	126	6	49
Extended Tridiagonal 2	100	42	146	9	41	10	34
	1000	37	125	13	84	14	52
DQDRTIC (CUTE)	100	144	717	41	309	42	178
	1000	175	947	44	431	44	193
DIXMAANC (CUTE)	100	9	33	18	142	10	35
	1000	10	35	26	244	30	41
DIXMAANE (CUTE)	100	70	276	28	209	11	32
	1000	335	1618	46	407	16	42
Broyden Tridiagonal	100	132	545	23	179	33	137
	1000	104	569	44	304	40	175
Almost Perturbed Quadratic	100	153	661	25	186	28	124
	1000	1463	7534	75	608	72	303
Tridiagonal Perturbed Quadratic	100	179	813	22	163	28	131
	1000	1347	7030	69	543	72	297
ENGVAL1 (CUTE)	100	30	100	25	198	19	71
	1000	1131	7616	906	6916	27	93
Extended Wood	100	1097	7416	212	1806	21	81
	1000	1131	7616	906	6916	27	93
Total		**9888**	**54968**	**2017**	**15793**	**789**	**3219**

Table 1: Comparing different conjugate gradient methods with different test functions.

5. Conclusion

This paper presents a new deriving method for stepsizes selection by using quadratic model function and considered are modified of the Barzilai and Borwein stepsizes. Based on the numerical results show that the the new algorithms compares good and constructive with the Barzilai-Borwein approach.

Acknowledgement

The authors are very grateful to the University of Mosul / College of Computers Sciences and Mathematics for their provided facilities, which helped to improve the quality of this work.

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Accepted: April 04, 2020

Analytical investigation of transversely vibrating nonlinear beams using the iterative method

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Abstract. An iterative method is presented to obtain approximate frequencies and corresponding periodic solutions of transversely vibrating nonlinear beams, namely, the cubic-quintic Duffing oscillator. The main purpose of this study is to determine analytical results with high accuracy. In this paper, the truncated terms of the Fourier series have been used and utilized in all steps of iteration. To evaluate the correctness of the present method, the approximated results are compared with the existing results and corresponding numerical results that are considered to be exact. It is shown that the obtained approximate solutions are valid for whole ranges of amplitude of the oscillations and can meet the exact solutions with a high level of accuracy. The error analysis has also carried out and shown acceptable results for the presented iterative method. Effectiveness of the mentioned method found from comparison with the other existing methods and has revealed this method's novelty, reliability and wider applicability to other types of nonlinear engineering problem.

Keywords: vibrating nonlinear beams, quintic nonlinearity, iterative method, analytical solution, duffing equation.

1. Introduction

Mathematical modelling of many nonlinear vibration systems can be described by nonlinear differential equations and their solutions are vital subject for the interest of their nonlinear behaviour [1, 2, 3]. Nonlinear vibration analysis of beam structures is important because of the mechanical and industrial applications of beams. Beams are essential in the structural systems and are used worldwide. Their applications include helicopter rotor blades, airplane wings, space craft antennae, flexible satellites, high rise buildings, robot arms, drill strings, long span bridges and vibratory drilling. The mathematical modelling of the vibrating beams was recently governed in terms of the partial differential equation of motion by many researchers [4, 5, 6, 7]. As the amplitude of the oscillation increases, beams are subjected to nonlinear vibrations which often lead to material fatigue and structural damage. These effects become more important around the natural frequencies of the system. Nonlinear vibration of beams

is of considerable importance for researchers and has been studied, significantly these days. So, it is very significant to provide an insightful analysis towards understanding the nonlinear vibration characteristics of these structures.

Numerical methods, such as the Runge-Kutta fourth-order method are frequently used to obtain approximate solutions for nonlinear systems. However, numerical methods for stiff differential equations, chaotic differential equations, and hyperchaotic differential equations do not always yield accurate results. In the application of numerical analysis, it is a big challenge still today. As an alternative, researchers have shown interest for using analytical approximation techniques to solve nonlinear oscillatory problems. Traditional perturbation methods [8, 9] are the most widely used analytical methods for solving nonlinear equations, which are the most versatile tools available for nonlinear analysis of engineering problems. However, traditional perturbation methods have several limitations depending on the degree of nonlinearity of the equations, especially for strongly nonlinear differential equations [10].

In recent times, the transversely vibrating nonlinear beams have gained huge importance in the engineering discipline. Many practical engineering vibration systems, such as the simply supported beam have been studied by many researchers [11, 12]. The mathematical modelling of uniform and simply supported beam equation transformed into the cubic-quintic Duffing oscillator. In general, finding an exact solution for nonlinear equations is extremely difficult and this perception has led to intensive research over many decades. In the presence of fifth-order nonlinearity, the efficient and high accuracy analytical and numerical methods become highly demanding. Many researchers are investing their time and effort to solve the cubic-quintic Duffing oscillator using several analytical methods. Beléndez et al. [13], Zakeri and Yomba [14] and Zúñiga [15] have obtained an exact solution to the cubic-quintic Duffing oscillator. But the solution procedure is a tremendously difficult task, as it contains a set of complicated algebraic equations with Jacobian elliptic functions which are not easiy to solved. Higher-order approximate frequencies and corresponding periodic solutions have been determined by Chowdhury et al. [16] and Lai et al. [17] using the harmonic balance method and the Newton-harmonic balancing approach. However, the mathematical manipulation is extremely laborious especially for obtaining second- and higher-order approximations. Khan et al. [18] has implemented the coupling of homotopy and a variational approach to attain an approximate solution. One of the limitations of this method is that its trial function does not satisfy the initial conditions. Razzak [19] has applied an analytical approximate technique to acquire the approximate frequencies and the corresponding periodic solutions. This technique removes the limitations of Khan et al. [18] method. The homotopy analysis method [20] and modified homotopy perturbation method [21] have been investigated to obtain approximate solutions of the nonlinear oscillators with cubic and quintic nonlinearities. Khan et al. [22] and Ganji et al. [23] have implemented the parameters approach and He's energy balance method to the cubic-quintic Duffing oscillators. However,

only the first-order approximation has been considered, which does not lead to sufficient accuracy. Recently, He [24] has introduced a simplest approach to nonlinear oscillators with cubic and quintic nonlinearities. This method can be applied to find approximate frequencies of the cubic-quintic Duffing oscillator. However, for large amplitude of the oscillations, the method attains percentage errors around 25% and 4% respectively, which are sufficiently high.

The main aim of the presented study is to apply the iterative method in order to obtain high accuracy approximate frequencies and corresponding periodic solutions of the cubic-quintic Duffing oscillator. The iterative method is essential when the solutions with higher accuracy are required. In the presented method, only linear non-homogeneous differential equations are required to be solved at every step of the iteration. This is important for obtaining higher-order iterations of the solutions. The higher-order iterations (up to a third-order iteration) have been obtained to the cubic-quintic Duffing oscillator. A comparison has been done between the exact solutions and the existing results obtained by using the already published methods. Among of them, the improved energy balance and global residue harmonic balance method [7], the energy balance method [10], the iteration approach [11], the coupled homotopy-variational formulation [18], the parameters approaches [22], He's energy balance method [23] and the simplest approach [24]. It has been shown that the obtained results are highly accurate. High accuracy results, simple solution procedure as compared to other existing methods and computational efficiency are the advantages of the presented method.

The rest of this paper is organized as follows: In Section 2, we offer a detailed description of a uniform and simply supported beam both geometrically and mathematically. In Section 3, we present the outline of the present iterative method based on the Mickens iterative method. In Section 4, we review the Runge-Kutta fourth-order method (RK4). In Section 5, we implement the presented method to the strongly nonlinear cubic-quintic Duffing oscillator. In Section 6, results and discussions are presented in detail. We analyse the convergence and consistency of the present iterative method in Section 7. Finally, in Section 8, concluding remarks are given.

2. Governing differential equation of the problem

Figure-1 represents the schematic model of a simply supported beam of length l, a moment of inertia I, mass per unit length m and a modulus of elasticity E, which is axially compressed by a loading P.

If w is the transverse deflection, then the differential equation governing the equilibrium in the deformed situation is derived [4, 7, 10] as

$$\frac{d^2}{dx^2}\left[\frac{EIw''(x,t)}{1+w'^2(x,t)^{3/2}}\right] + Pw''(x,t)\left[1+\frac{3}{2}w'^2(x,t)\right] + m\ddot{w}(x,t) = 0. \tag{2.1}$$

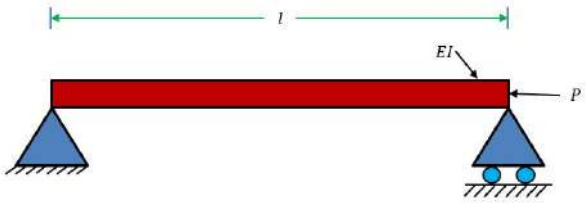


Figure 1: Geometric structure of a uniform and simply supported beam.

We take account the approximation

$$\frac{w''(x,t)}{1+w'^2(x,t)^{3/2}} \cong w''(x,t)[1-\frac{3}{2}w'^2(x,t)+\frac{15}{8}w'^4(x,t)]. \tag{2.2}$$

Subject to the boundary conditions

$$\begin{aligned} w(0,t) = w''(0,t) = 0, \\ w(l,t) = w''(l,t) = 0. \end{aligned} \tag{2.3}$$

Assuming

$$w(x,t) = \varphi(x)\bar{q}(t), \tag{2.4}$$

where $\varphi(x) = \sin(\frac{\pi x}{l})$ is the first eigenmode of the simply supported beam. Substituting $\varphi(x)$ into Eq. (2.1), and integrating the result from 0 to 1, and by introducing the following new variables as

$$\tau = \sqrt{\frac{EI}{ml^2}}t, \quad q = \frac{\bar{q}}{l}, \tag{2.5}$$

the nonlinear differential equation for the time function $q(\tau)$ can be obtained [4] as follows

$$\ddot{q}(\tau) + aq(\tau) + bq(\tau)^3 + cq(\tau)^5 = 0, \tag{2.6}$$

where

$$a = \pi^4 - \frac{Pl^2\pi^2}{EI}, \quad b = \frac{3}{8}\pi^6 - \frac{3}{8}\frac{Pl^2\pi^4}{EI}, \quad c = \frac{15}{64}\pi^8. \tag{2.7}$$

Two important physical parameters are needed in every oscillatory system, the frequency ω and the amplitude of the oscillation A_0. So, let us consider such initial conditions as

$$q(0) = A_0, \quad \dot{q}(0) = 0. \tag{2.8}$$

3. The methodology

Consider a second-order nonlinear differential equation is as the following

$$\ddot{q} + f(q, \ddot{q}) = 0, \quad q(0) = A_0, \quad \dot{q}(0) = 0, \tag{3.1}$$

where over dots denote differentiation with respect to time t. Now consider the angular frequency ω of this system. Then adding $\omega^2 q$ to both sides of Eq. (3.1), the following equation can be obtained as

$$\ddot{q} + \omega^2 q = \omega^2 q - f(q, \ddot{q}) \equiv G(q, \ddot{q}), \tag{3.2}$$

where $\omega^2 q$, the constant is currently unknown.

Now, we formulate the iteration scheme of Eq. (3.2) in the following way

$$\ddot{q}_{k+1} + \omega_k^2 q_{k+1} = G(q_k, \ddot{q}_k); \quad k = 0, 1, 2, 3, \cdots, \tag{3.3}$$

together with

$$q_0(t) = A_0 \cos(\omega_0 t). \tag{3.4}$$

Herein q_{k+1} satisfies the initial conditions

$$q_{k+1}(0) = A_0, \quad \dot{q}_{k+1}(0) = 0. \tag{3.5}$$

At each step of the iteration, ω_k is determined by the requirement that the secular terms [8] may not occur in the full solution of $q_{k+1}(t)$. This procedure gives the sequence of solutions: $q_0(t), q_1(t), q_2(t), \cdots$. The method can be proceeded to any order of approximation; but due to growing algebraic complexity the solution is confined to a reasonable order usually the third. Although, Eq. (3.1) is of odd parity, the solution will only contain odd multiples of the angular frequency [25].

4. Review of the Runge-Kutta method (RK4)

In numerical analysis, the Runge-Kutta methods are an important family of implicit and explicit iterative methods for the approximate solutions of the differential equations subject to the initial conditions. These techniques were developed around 1900 by the German mathematicians C. Runge and M.W. Kutta. One of the Runge-Kutta methods commonly used is the Runge-Kutta fourth-order method, often referred as "RK4". This method applies the following iterative formulas for the second-order differential equation [26] as

$$\begin{aligned} \dot{q}(i+1) &= \dot{q}_i + \frac{\Delta t}{6}(s_1 + 2s_2 + 2s_3 + s_4), \\ q(i+1) &= q_i + \Delta t(\dot{q}_i + \frac{\Delta t}{6}(s_1 + s_2 + s_3)), \end{aligned} \tag{4.1}$$

where Δt is the time increment and s_1, s_2, s_3 and s_4 are to be determined as follows

$$\begin{aligned} s_1 &= f(t_i, q_i, \dot{q}_i), \\ s_2 &= f(t_i + \frac{\Delta t}{2}, q_i + \frac{\Delta t}{2}\dot{q}_i, \dot{q}_i + \frac{\Delta t}{2}s_1), \\ s_3 &= f(t_i + \frac{\Delta t}{2}, q_i + \frac{\Delta t}{2}\dot{q}_i, \frac{1}{4}\Delta t^2 s_1, \dot{q}_i + \frac{\Delta t}{2}s_2), \\ s_4 &= f(t_i + \Delta t, q_i + \Delta t\dot{q}_i, \frac{1}{2}\Delta t^2 s_2, \dot{q}_i + \Delta t s_3). \end{aligned} \tag{4.2}$$

In order to start the numerical solution, the first values of the displacement function and its first-order derivative are obtained from initial conditions. Then, by choosing a small value for the time increment Δt, the displacement function and its first-order derivative are determined at the new position using Eq. (4.1). The procedure will be continued until one of the end criteria is satisfied. Note that the above formulas are valid for both scalar- and vector-valued functions so that q can be a vector and f an operator.

5. Implementation of the presented iterative method

By adding $\omega^2 q$ to both sides of Eq. (2.6), it can be transformed into

$$\ddot{q} + \omega^2 q = \omega^2 q - aq - bq^3 - cq^5. \tag{5.1}$$

Now, we formulate the iteration scheme according to Eq. (2.3) as in the following

$$\ddot{q}_{k+1} + \omega_k^2 q_{k+1} = \omega_k^2 q_k - aq_k - bq_k^3 - cq_k^5, \tag{5.2}$$

together with the initial approximation

$$q_0(t) = A_0 \cos(\omega_0 t). \tag{5.3}$$

First, we take $k = 0$ for Eq. (5.2), it can be supposed as

$$\ddot{q}_1 + \omega_0^2 q_1 = \omega_0^2 q_0 - aq_0 - bq_0^3 - cq_0^5. \tag{5.4}$$

Substituting Eq. (5.3) into Eq. (5.4), then expanding Fourier cosine series of the right-hand side of Eq. (5.4), it can be reduced as

$$\begin{aligned} \ddot{q}_1 + \omega_0^2 q_1 &= (A_0\omega_0^2 - aA_0 - \frac{3bA_0^3}{4} - \frac{5cA_0^5}{8})\cos(\omega_0 t) \\ &\quad - (\frac{bA_0^3}{4} + \frac{5cA_0^5}{16})\cos(3\omega_0 t) - (\frac{cA_0^5}{16})\cos(5\omega_0 t). \end{aligned} \tag{5.5}$$

No secular term in the solution for $q_1(t)$ requires that the coefficient of $\cos(\omega_0 t)$ term equal to zero from the right-hand side of Eq. (5.5). Thus, we have

$$\omega_0 = \sqrt{a + \frac{3bA_0^2}{4} + \frac{5cA_0^4}{8}}. \tag{5.6}$$

Therefore, the solution of Eq. (5.5) satisfying the initial condition $q_1(0) = A_0$, can be obtained as

$$q_1(t) = d_1 \cos(\omega_0 t) + d_3 \cos(3\omega_0 t) + d_5 \cos(5\omega_0 t), \tag{5.7}$$

where

$$\begin{aligned} d_1 &= A_0 - (\frac{bA_0^3}{32\omega_0^2} + \frac{cA_0^5}{24\omega_0^2}), \\ d_3 &= (\frac{4bA_0^3 + 5cA_0^5}{128\omega_0^2}), \\ d_5 &= (\frac{cA_0^5}{384\omega_0^2}). \end{aligned}$$

This is the first-order approximate solution of Eq. (2.6) where ω_0 is given by Eq. (5.6) and the related ω_1 is to be determined further.

Taking $k = 1$ for Eq. (5.2), it can be transformed into

$$\ddot{q}_2 + \omega_1^2 q_2 = \omega_1^2 q_1 - a q_1 - b q_1^3 - c q_1^5. \tag{5.8}$$

The angular frequency ω_1 will be determined from Eq. (5.8). Substituting q_1 from Eq. (5.7) into the right-hand side of Eq. (5.8), it can be obtained as

$$\begin{aligned} \ddot{q}_2 + \omega_1^2 q_2 = {} & (d_1\omega_1^2 - a d_1 - \frac{3bd_1^3}{4} - \frac{5cd_1^5}{8} - \frac{3bd_1^2 d_3}{4} + \cdots)\cos(\omega_1 t) \\ & + (d_3\omega_1^2 - a d_3 - \frac{bd_1^3}{4} - \frac{5cd_1^5}{16} - \frac{3bd_1^2 d_3}{2} + \cdots)\cos(3\omega_1 t) \\ & + (d_5\omega_1^2 - a d_5 - \frac{cd_1^5}{16} - \frac{3bd_1^2 d_3}{4} - \frac{5cd_1^4 d_3}{4} + \cdots)\cos(5\omega_1 t) \\ & + (-\frac{5cd_1^4 d_3}{16} - \frac{3bd_1 d_3^2}{4} - \frac{15cd_1^3 d_3^2}{8} + \cdots)\cos(7\omega_1 t) \\ & + (-\frac{5cd_1^3 d_3^2}{8} - \frac{bd_3^3}{4} - \frac{5cd_1^2 d_3^3}{4} + \cdots)\cos(9\omega_1 t) + \cdots . \end{aligned} \tag{5.9}$$

Avoiding secular terms in the solution of Eq. (5.9), it can be determined as

$$\omega_1 = \sqrt{a + \frac{3bd_1^2}{4} + \frac{5cd_1^4}{8} + \frac{3bd_1 d_3}{4} + \frac{3bd_3^2}{2} + \frac{3bd_3^2 d_5}{4d_1} + \frac{5cd_3^4 d_5}{4d_1} + \cdots}. \tag{5.10}$$

Hence, the solution of Eq. (5.9) and satisfying the initial condition $q_2(0) = A_0$, can be obtained as

$$\begin{aligned} q_2(t) &= e_1\cos(\omega_1 t) + e_3\cos(3\omega_1 t) + e_5\cos(5\omega_1 t) \\ &\quad + e_7\cos(7\omega_1 t) + e_9\cos(9\omega_1 t) + \cdots, \end{aligned} \tag{5.11}$$

where

$$\begin{aligned} e_1 &= A_0 - \frac{960960a(3d_3+d_5) + 5c(192192d_1^5 + 379665d_3^5) - 960960(3d_3+d_5)\omega_1^2 + \cdots}{23063040\omega_1^2}, \\ e_3 &= \frac{5cd_1^5 + 16ad_3 + 30cd_1^4d_3 + 30cd_1^3d_3^2 + 60cd_1^2d_3^3 + 10cd_3^5 - 16d_3\omega_1^2 + \cdots}{128\omega_1^2}, \\ e_5 &= \frac{c(d_1^5 + 10d_1^4(2d_3+3d_5) + 10d_1^3d_3(3d_3+4d_5)) - 16d_5\omega_1^2 + \cdots}{384\omega_1^2}, \\ e_7 &= \frac{12b(d_1^2d_5 + d_3d_5^2 + d_1d_3(d_3+2d_5)) + 5c(d_1^4(d_3+4d_5)) + \cdots}{768\omega_1^2}, \\ e_9 &= \frac{4b(d_3^3 + 6d_1d_3d_5 + 3d_1d_5^2) + 5c(d_1^4d_5 + 2d_1^2d_3(2d_3^2 + 3d_3d_5 + 3d_5^2)) + \cdots}{1280\omega_1^2}, \\ &\cdots \end{aligned}$$

Again taking $k = 2$ for Eq. (5.2), it can be written as

$$\ddot{q}_3 + \omega_2^2 q_3 = \omega_2^2 q_2 - aq_2 - bq_2^3 - cq_2^5. \tag{5.12}$$

The angular frequency ω_2 will be determined from Eq. (5.12). Substituting q_2 from Eq. (5.11) into the right-hand side of Eq. (5.12), it can be obtained

$$\begin{aligned} \ddot{q}_3 + \omega_2^2 q_3 &= \Big(\frac{16e_1\omega_2^2 - 16ae_1 - 12be_1^3 - 10ce_1^5 + \cdots}{16}\Big)\cos(\omega_2 t) \\ &\quad + \Big(\frac{16e_3\omega_2^2 - 5ce_1^5 - 20ce_1^3e_{11}^2 - 16ae_3 - 30ce_1^4e_3 + \cdots}{16}\Big)\cos(3\omega_2 t) \\ &\quad + \Big(e_5\omega_2^2 - ae_5 - \frac{3b}{4}(2e_{11}^2e_5 + e_5^3) + \cdots\Big)\cos(5\omega_2 t) \\ &\quad + \Big(e_7\omega_2^2 - ae_7 - \frac{3b}{4}(2e_1e_{11}(e_3+e_5)) + \cdots\Big)\cos(7\omega_2 t) \\ &\quad + \Big(\frac{16e_9\omega_2^2 - 20ce_1^4e_{11} - 30ce_1^2e_{11}^3 - 60ce_1^3e_{11}e_3 + \cdots}{16}\Big)\cos(9\omega_2 t) \\ &\quad + \Big(\frac{16e_{11}\omega_2^2 - 16ae_{11} - 24be_1^2e_{11} - 30ce_1^4e_{11} - 12be_{11}^3 + \cdots}{16}\Big)\cos(11\omega_2 t) \\ &\quad + \Big(\frac{3b}{4}(e_1^2e_{11} - e_3e_5^2) - \frac{5c}{16}(6e_3^3e_5^2) + \cdots\Big)\cos(13\omega_2 t) + \cdots. \end{aligned} \tag{5.13}$$

Avoiding secular terms in the solution of Eq. (5.13), it can be determined as

$$\omega_2 = \sqrt{a + \frac{3be_1^2}{4} + \frac{5ce_1^4}{8} + \frac{3be_{11}^2}{2} + \frac{5ce_{11}e_3^4}{16e_1} + \frac{15ce_{11}^2e_3e_5}{2} + \cdots}. \tag{5.14}$$

Hence, the solution of Eq. (5.13) and satisfying the initial condition $q_3(0) = A_0$, can be obtained as

$$\begin{aligned} q_3(t) &= f_1\cos(\omega_2 t) + f_3\cos(3\omega_2 t) + f_5\cos(5\omega_2 t) \\ &\quad + f_7\cos(7\omega_2 t) + f_9\cos(9\omega_2 t) + \cdots, \end{aligned} \tag{5.15}$$

where

$$\begin{aligned}
f_1 &= A_0 - \frac{\begin{array}{c}6720ce_1^5 + 1344ae_{11} + 12b(420e_1^3 + 84e_{11}^3) \\ -672(2e_{11} + 30e_3 + 10e_5)\omega_2^2 + \cdots\end{array}}{161280\omega_2^2}, \\
f_3 &= \frac{5ce_1^5 + 20ce_1^3e_{11}^2 + 16ae_3 + 30ce_1^4e_3 + 120ce_1^2e_{11}^2e_3 + 30ce_{11}^4e_3 - 16e_3\omega_2^2 + \cdots}{128\omega_2^2}, \\
f_5 &= \frac{4(4ae_5 + 3b(2e_{11}^2e_5 + 2e_3^2e_5 + e_5^3)) + c(e_1^5 + 5e_1^4(4e_3 + 6e_5)) - 16e_5\omega_2^2 + \cdots}{384\omega_2^2}, \\
f_7 &= \frac{\begin{array}{c}4(4ae_7 + 3b(2e_1e_{11}(e_3 + e_5) + 2e_{11}^2e_7)) \\ +5c(e_3^4e_5 + 6e_3^3e_5^2 + 4e_3e_5^4) - 16e_7\omega_2^2 + \cdots\end{array}}{768\omega_2^2}, \\
f_9 &= \frac{20ce_1^4e_{11} + 30ce_1^2e_{11}^3 + 60ce_1^3e_{11}e_3 + 4b(e_3^3 + 6e_3e_5(e_{11} + e_7)) - 16e_9\omega_2^2 + \cdots}{1280\omega_2^2}, \\
f_{11} &= \frac{16ae_{11} + 24be_1^2e_{11} + 30ce_1^4e_{11} + 12be_{11}^3 + 60ce_1^2e_{11}^3 - 16e_{11}\omega_2^2 + \cdots}{1920\omega_2^2}, \\
f_{13} &= \frac{12b(e_1^2e_{11} + e_3e_5^2 + e_3^2e_7) + 5c(6e_3^3e_5^2 + 4e_3e_5^4 + 4e_3^4e_7 + 12e_3^2e_5^2e_7 + e_5^4e_7}{2688\omega_2^2}, \\
&\cdots
\end{aligned}$$

In the similar way, the fourth order approximate angular frequency is obtained as

$$\omega_3 = \sqrt{\begin{array}{c}a + \frac{3bf_1^2}{4} + \frac{5cf_1^4}{8} + \frac{3bf_{11}^2}{2} + \frac{5cf_{11}f_3^4}{16f_1} \\ + \frac{3bf_3^2f_5}{4f_1} + \frac{15cf_{11}f_5^4}{4f_1} + \frac{15cf_9^4}{8} + \cdots\end{array}}. \tag{5.16}$$

6. Results and discussions

In this section, we demonstrate and verify the accuracy of the presented iterative method. The complicated elliptical integral is formerly revealed in Eq. (6.1) and then compared with the results obtained from this study. The exact frequency ω_{ex} is calculated as in Lai et al. [17] by imposing the mentioned initial conditions as follows

$$\omega_{ex}(A_0) = \frac{\pi z_1}{2\int_0^{\frac{\pi}{2}}(1 + z_2\sin^2 t + z_3\sin^4 t)^{-\frac{1}{2}}dt}, \tag{6.1}$$

where

$$z_1 = \sqrt{a + \frac{bA_0^2}{2} + \frac{cA_0^4}{3}},$$
$$z_2 = \frac{3bA_0^2 + 2cA_0^4}{6a + 3bA_0^2 + 2cA_0^4},$$
$$z_3 = \frac{2cA_0^4}{6a + 3bA_0^2 + 2cA_0^4}.$$

The approximated frequencies are illustrated in Table 1 to Table 7 considering various parameters of the system. As can be seen, the results of the third-order are in excellent agreement with the exact solution. Using the third-order approximation for the large amplitude of the oscillations, the maximum percentage error is around 0.0194%. This is much lower than the errors found using many existing methods including the improved energy balance and global residue harmonic balance method [7], the energy balance method [10], the iteration approach [11], the coupled homotopy-variational formulation [18], the parameters approaches [22], He's energy balance method [23] and the simplest approach [24]. Hence, it is particularly noted that an excellent agreement with the exact solutions for the cubic-quintic Duffing oscillator is provided. The presented iterative method is convergent and the obtained results are consistent and can also easily be generalized to the other nonlinear engineering problems. The correlation of the percentage error $E(\%)$ is determined from the following relation

$$E(\%) = \mid \frac{\omega_k - \omega_{ex}}{\omega_{ex}} \mid \times 100; \quad k = 0, 1, 2, 3, \cdots, \tag{6.2}$$

where ω_{ex} is obtained from Eq. (6.1).

To further illustrate and verify the correctness of the approximated solutions, a comparison of the time history oscillatory displacement for various system parameters with Runge-Kutta fourth-order solutions are plotted in Figure 2 to Figure 6. These Runge-Kutta fourth-order solutions are considered to be exact. As can be seen in Figure 2 to Figure 6, it is found that the presented iterative method is in excellent agreement with the numerical one with increasing approximation order. The presented method is simple, highly efficient and is valid for a wide range of amplitude of the oscillations. The mentioned method is swiftly convergent and has sufficient ability in solving strongly nonlinear problems such as the strongly nonlinear cubic-quintic Duffing oscillator.

7. Convergence and consistency analysis

Every iteration method obtains a sequence of approximate solutions q_k (as well as angular frequencies ω_k). It is the basic idea of iterative methods and has a convergence property

$$q_{ex} = \lim_{k\to\infty} q_k \quad and \quad \omega_{ex} = \lim_{k\to\infty} \omega_k, \tag{7.1}$$

where q_{ex} and ω_{ex} represent the exact solution and angular frequency respectively. The presented iterative method has revealed that the obtained solution gives better accuracy in every iteration step than previous one and finally $\mid \omega_3 - \omega_{ex} \mid < \varepsilon$, where ε is the small positive number and the initial amplitude of the oscillation A_0 is considered to be unity. Hence, it is shown that the presented iterative method is convergent.

An iterative method is said to be consistent if it satisfies the followings

$$\lim_{k\to\infty} \mid q_k - q_{ex} \mid = 0 \quad \text{and} \quad \lim_{k\to\infty} \mid \omega_k - \omega_{ex} \mid = 0. \tag{7.2}$$

The presented iterative method has already been defined in Eq. (3.3). The initial approximation as well as the initial condition is defined in Eq. (3.4)-(3.5). In the consistence analysis of the present study, it is observed that

$$\lim_{k\to\infty} \mid q_k - q_{ex} \mid = 0 \quad as \quad \mid q_3 - q_{ex} \mid \approx 0. \tag{7.3}$$

and

$$\lim_{k\to\infty} \mid \omega_k - \omega_{ex} \mid = 0 \quad or \quad \mid \omega_3 - \omega_{ex} \mid \approx 0. \tag{7.4}$$

Hence, the consistency of the presented iterative method is attained.

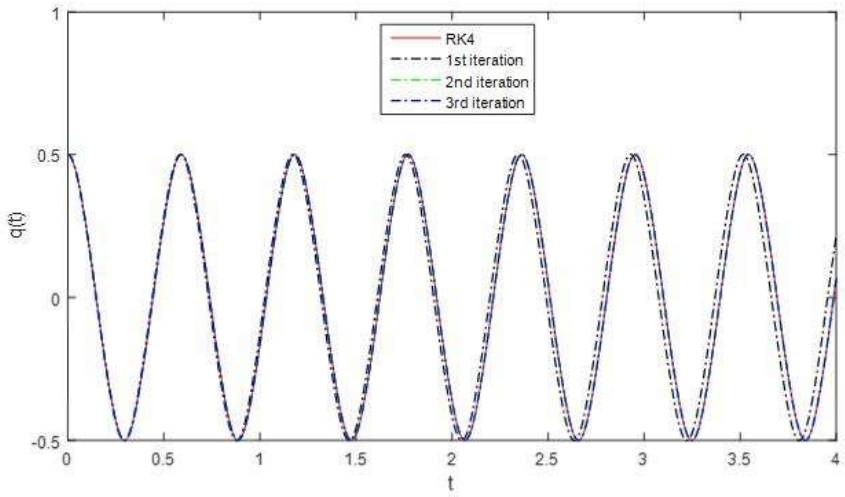


Figure 2: Comparison between approximated solutions and corresponding numerical one for $a = 96.4221$, $b = -364.1738$, $c = 2223.874$ and $A_0 = 0.5$.

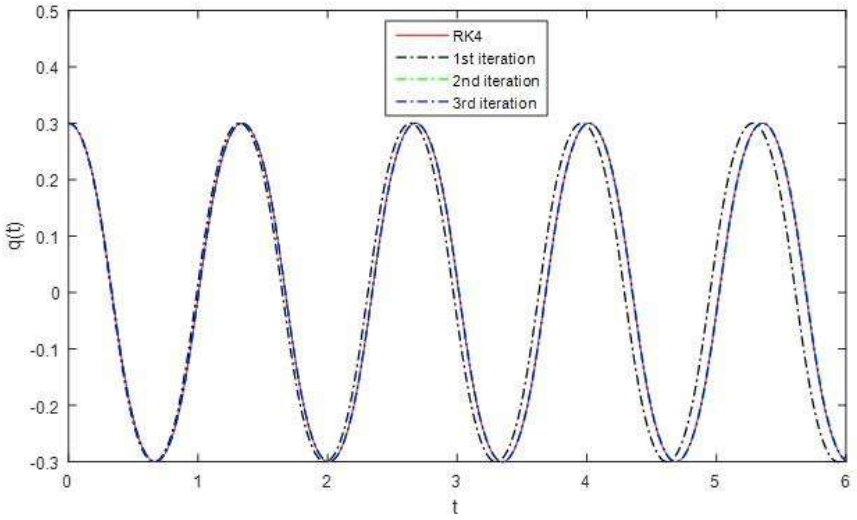


Figure 3: Comparison between approximated solutions and corresponding numerical one for $a = 48.0611$, $b = -543.1630$, $c = 2223.874$ and $A_0 = 0.3$.

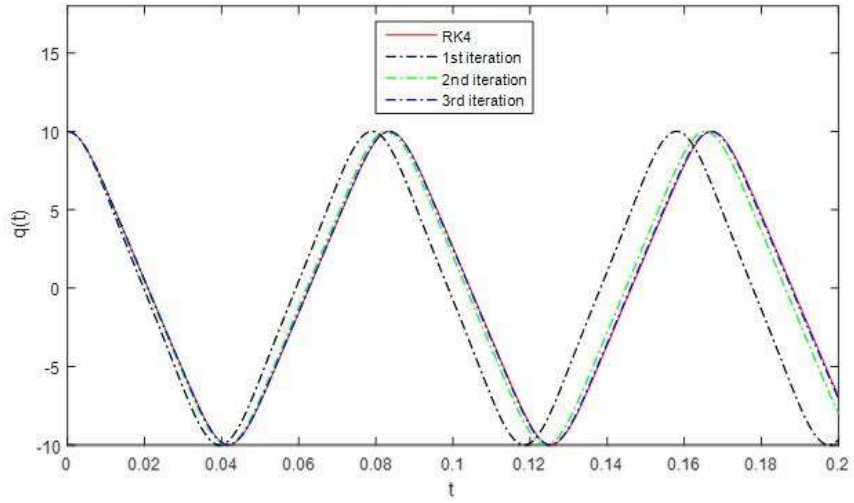


Figure 4: Comparison between approximated solutions and corresponding numerical one for $a = b = c = 1$ and $A_0 = 10$.

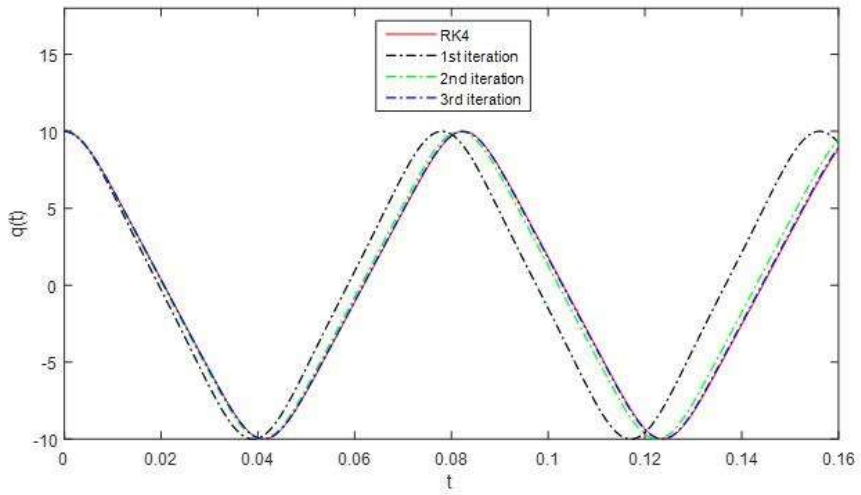


Figure 5: Comparison between approximated solutions and corresponding numerical one for $a = 5$, $b = 3$, $c = 1$ and $A_0 = 10$.

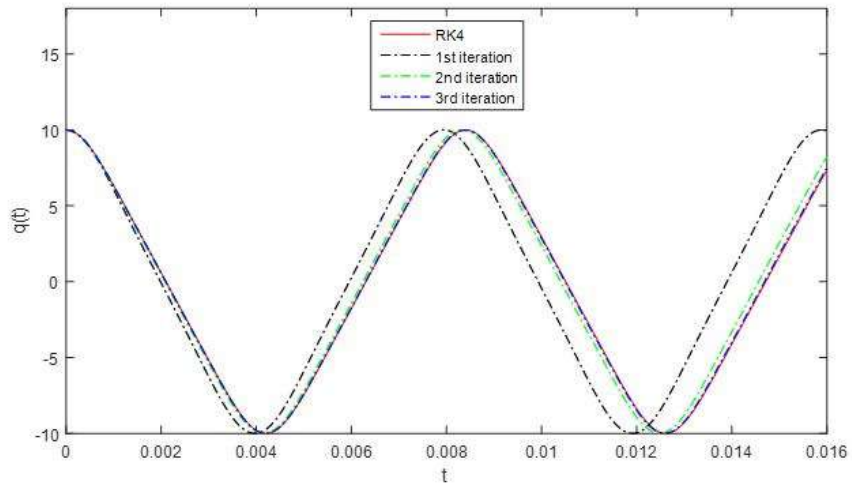


Figure 6: Comparison between approximated solutions and corresponding numerical one for $a = 1$, $b = 10$, $c = 100$ and $A_0 = 10$.

Table 1: Comparison of the approximated frequencies with existing and exact ones when $a = 96.4221$, $b = -364.1738$ and $c = 2223.874$:

A_0	$\omega_0^{[this\ study]}$ Er(%)	$\omega_1^{[this\ study]}$ Er(%)	$\omega_2^{[this\ study]}$ Er(%)	$\omega_3^{[this\ study]}$ Er(%)	$\omega_{[exact]}^{[7]}$	$\omega_{[1GRHBM]}^{[7]}$ Er(%)	$\omega_{[2GRHBM]}^{[7]}$ Er(%)	$\omega_{[1IEBM]}^{[7]}$ Er(%)	$\omega_{[2IEBM]}^{[7]}$ Er(%)
0.1	9.686577	9.686433	9.686433	9.686433	9.691432	9.6866	9.6864	9.6861	9.686232
	(0.0500)	(0.0515)	(0.0515)	(0.0515)		(0.0498)	(0.0519)	(0.0550)	(0.0536)
0.3	9.115850	9.114520	9.114537	9.114537	9.119202	9.1159	9.1142	9.0746	9.07529
	(0.0367)	(0.0513)	(0.0511)	(0.0511)		(0.0362)	(0.0548)	(0.4890)	(0.4814)
0.5	10.72425	10.64281	10.63867	10.63853	10.64332	10.7243	10.6620	10.4508	10.40287
	(0.7603)	(0.0047)	(0.0436)	(0.0450)		(0.7608)	(0.1755)	(1.8088)	(2.2591)
1	34.83120	33.23287	32.95017	32.90614	32.91108	34.8312	33.2868	33.4746	31.84017
	(5.8342)	(0.9777)	(0.1187)	(0.0150)		(5.8342)	(1.1416)	(1.7122)	(3.2539)
3	331.9967	316.6345	313.8708	313.4287	313.4685	331.9968	316.8293	320.4939	314.1535
	(5.9107)	(1.0099)	(0.1283)	(0.0126)		(5.9107)	(1.0721)	(2.2411)	(0.2185)
5	928.4228	885.8144	878.1894	876.9763	877.0938	928.4229	886.3199	896.6915	871.0218
	(5.8521)	(0.9942)	(0.1249)	(0.0133)		(5.8521)	(1.0518)	(2.2343)	(0.6922)
10	3724.512	3554.232	3523.834	3519.010	3519.494	3724.51	3556.199	3597.969	3530.220
	(5.8252)	(0.9870)	(0.1233)	(0.0137)		(5.8252)	(1.042)	(2.2297)	(0.3047)

Table 2: Comparison of the approximated frequencies with existing and exact ones when $a = 48.0611$, $b = -543.1630$ and $c = 2223.874$:

A_0	$\omega_0^{[this\ study]}$ Er(%)	$\omega_1^{[this\ study]}$ Er(%)	$\omega_2^{[this\ study]}$ Er(%)	$\omega_3^{[this\ study]}$ Er(%)	$\omega_{[exact]}^{[7]}$	$\omega_{[1GRHBM]}^{[7]}$ Er(%)	$\omega_{[2GRHBM]}^{[7]}$ Er(%)	$\omega_{[1IEBM]}^{[7]}$ Er(%)	$\omega_{[2IEBM]}^{[7]}$ Er(%)
0.1	6.642768	6.641706	6.641713	6.641713	6.645130	6.642766	6.641704	6.642068	6.642430
	(0.0355)	(0.0515)	(0.0514)	(0.0514)		(0.0355)	(0.0515)	(0.0460)	(0.0406)
0.3	4.759827	4.688124	4.692825	4.692591	4.695290	4.759825	4.685973	4.680318	4.692297
	(1.3745)	(0.1526)	(0.0524)	(0.0574)		(1.3745)	(0.1984)	(0.3188)	(0.0637)
0.5	5.752227	5.611415	5.590434	5.589052	5.591064	5.752226	5.675542	5.224631	5.205666
	(2.8825)	(0.3639)	(0.0112)	(0.0359)		(2.8825)	(1.5109)	(6.5539)	(6.8931)
1	32.10311	30.36541	30.01963	29.95832	29.95680	32.10312	30.42508	30.62595	28.98268
	(7.1646)	(1.3639)	(0.2097)	(0.0050)		(7.1646)	(1.5631)	(2.2337)	(3.2517)
3	330.0989	314.6965	311.9102	311.4618	311.4990	330.0990	314.8994	318.5275	310.8915
	(5.9710)	(1.0264)	(0.1320)	(0.0119)		(5.9710)	(1.0916)	(2.2563)	(0.1950)
5	926.5875	883.9465	876.3019	875.0834	875.1985	926.5877	884.4604	894.7912	875.7032
	(5.8717)	(0.9995)	(0.1260)	(0.0131)		(5.8717)	(1.0582)	(2.2386)	(0.0576)
10	3722.703	3552.393	3521.977	3517.148	3517.629	3722.704	3554.369	3596.097	3519.662
	(5.8298)	(0.9882)	(0.1236)	(0.0136)		(5.8298)	(1.0444)	(2.2307)	(0.0577)

Table 3: Comparison of the approximated frequencies with existing and exact ones when $a = -1.2870$, $b = -725.8050$ and $c = 2223.874$:

A_0	$\omega_0^{[this\ study]}$ Er(%)	$\omega_1^{[this\ study]}$ Er(%)	$\omega_2^{[this\ study]}$ Er(%)	$\omega_3^{[this\ study]}$ Er(%)	$\omega_{[exact]}^{[7]}$	$\omega_{[1GRHBM]}^{[7]}$ Er(%)	$\omega_{[2GRHBM]}^{[7]}$ Er(%)	$\omega_{[1IEBM]}^{[7]}$ Er(%)	$\omega_{[2IEBM]}^{[7]}$ Er(%)
1	29.05650 (9.4801)	27.10725 (2.1357)	26.65234 (0.4216)	26.55607 (0.0589)	26.54042	29.05651 (9.4801)	27.16772 (2.3635)	27.41568 (3.2978)	26.16390 (1.4186)
2	141.6334 (6.3468)	134.6869 (1.1310)	133.3880 (0.1557)	133.1715 (0.0068)	133.1806	141.6335 (6.3468)	134.8016 (1.2171)	136.2992 (2.3416)	132.9295 (0.1885)
5	924.7111 (5.8918)	882.0363 (1.0050)	874.3715 (0.1272)	873.1473 (0.0129)	873.2600	924.7112 (5.8918)	882.5588 (1.0648)	892.8479 (2.2430)	878.0963 (0.5538)
10	3720.856 (5.8346)	3550.516 (0.9895)	3520.081 (0.1238)	3515.247 (0.0136)	3515.726	3720.857 (5.8346)	3552.500 (1.0459)	3594.185 (2.2316)	3535.100 (0.5510)
20	14905.35 (5.8206)	14224.34 (0.9857)	14102.82 (0.1230)	14083.55 (0.0137)	14085.49	14905.36 (5.8206)	14232.18 (1.0414)	14399.44 (2.2288)	13935.22 (1.0668)
50	93196.81 (5.8167)	88941.10 (0.9847)	88181.97 (0.1228)	88061.62 (0.0138)	88073.80	93196.83 (5.8167)	88989.86 (1.0401)	90036.19 (2.2281)	87949.99 (0.1405)
100	372809.1 (5.8161)	355786.6 (0.9845)	352750.3 (0.1227)	352269.0 (0.0138)	352317.8	372809.2 (5.8161)	355981.6 (1.0399)	360167.4 (2.2279)	352172.2 (0.0413)

Table 4: Comparison of the approximated frequencies with existing and exact ones when $a = b = c = 1$:

A_0	$\omega_0^{[this\ study]}$ Er(%)	$\omega_1^{[this\ study]}$ Er(%)	$\omega_2^{[this\ study]}$ Er(%)	$\omega_3^{[this\ study]}$ Er(%)	$\omega_{[exact]}^{[24]}$	$\omega_{[1SA]}^{[24]}$ Er(%)	$\omega_{[2SA]}^{[24]}$ Er(%)	$\omega_{[PA]}^{[22]}$ Er(%)	$\omega_{[EBM]}^{[10]}$ Er(%)
0.1	1.003774	1.003772	1.003772	1.003772	1.003770	1.003759	1.004167	1.00378	1.003772
	(0.0003)	(0.0001)	(0.0001)	(0.0001)		(0.0010)	(0.0396)	(0.0002)	(0.0001)
0.3	1.035645	1.035539	1.035538	1.035538	1.03554	1.034423	1.038699	1.03577	1.035482
	(0.0101)	(0.0000)	(0.0001)	(0.0001)		(0.1078)	(0.3051)	(0.0226)	(0.0056)
0.5	1.107502	1.106560	1.106544	1.106544	1.10654	1.098649	1.113492	1.10835	1.106326
	(0.0869)	(0.0018)	(0.0003)	(0.0003)		(0.7131)	(0.6283)	(0.1635)	(0.0193)
1	1.541103	1.524791	1.523653	1.523589	1.52359	1.436140	1.371781	1.54847	1.527525
	(1.1494)	(0.0788)	(0.0041)	(0.0000)		(5.7396)	(9.9639)	(1.6332)	(0.2582)
3	7.640353	7.328877	7.277352	7.269826	7.26863	5.750000	7.100317	7.72648	7.416198
	(5.1140)	(0.8288)	(0.1199)	(0.0164)		(20.8928)	(2.3156)	(6.2989)	(2.0302)
5	20.25771	19.36516	19.20965	19.18561	19.1815	14.66501	18.58432	20.5097	19.60442
	(5.6106)	(0.9574)	(0.1467)	(0.0214)		(23.5455)	(3.1129)	(6.9244)	(2.2048)
8	51.07837	48.77665	48.36961	48.30567	48.2946	36.45545	46.63866	51.7378	49.37948
	(5.7641)	(0.9981)	(0.1553)	(0.0229)		(24.5163)	(2.8406)	(7.1296)	(2.2463)
10	79.53615	75.93462	75.29569	75.19499	75.1774	56.57738	72.54194	80.572	76.87218
	(5.7979)	(1.0072)	(0.1573)	(0.0233)		(24.7415)	(3.5057)	(7.1758)	(2.2543)
20	316.7033	302.2723	299.7017	299.2948	299.223	224.2788	288.4204	320.877	305.9972
	(5.8419)	(1.0190)	(0.1599)	(0.0239)		(25.0462)	(3.6102)	(7.2367)	(2.2639)
50	1976.898	1886.664	1870.572	1868.022	1867.57	1398.213	1799.592	2003.04	1909.897
	(5.8540)	(1.0224)	(0.1607)	(0.0242)		(25.1319)	(3.6400)	(7.2536)	(2.2664)
70	3874.264	3697.399	3665.855	3660.855	3659.98	2739.854	3526.648	3925.51	3742.927
	(5.8547)	(1.0223)	(0.1605)	(0.0239)		(25.1401)	(3.6429)	(7.2552)	(2.2663)
100	7906.168	7545.210	7480.831	7470.626	7468.83	5590.840	7196.640	8010.77	7638.117
	(5.8555)	(1.0226)	(0.1606)	(0.02404)		(25.1443)	(3.6443)	(7.2560)	(2.2665)
300	71151.72	67903.04	67323.58	67231.73	67215.57	50312.20	64765.15	72093.2	68739.12
	(5.8560)	(1.0227)	(0.1606)	(0.02404)		(25.1479)	(3.6456)	(7.2567)	(2.2666)
500	197642.8	188618.7	187009.1	186753.9	186709.0	139754.9	179902.1	200258.0	190941.1
	(5.8560)	(1.0228)	(0.1607)	(0.0240)		(25.1482)	(3.6457)	(7.2568)	(2.2666)
700	387379.4	369692.1	366537.3	366037.2	365949.2	273918.9	352607.7	392506.0	374244.1
	(5.8560)	(1.0227)	(0.1607)	(0.0240)		(25.1483)	(3.6457)	(7.2568)	(2.2666)
1000	790569.8	754473.3	748034.9	747014.3	746834.6	559017.6	719606.9	801031.0	763763.1
	(5.8560)	(1.0228)	(0.1607)	(0.0240)		(25.1484)	(3.6458)	(7.2568)	(2.2667)

Table 5: Comparison of the approximated frequencies with existing and exact ones when $a = b = c = 1$:

A_0	$\omega_0^{[this\ study]}$ Er(%)	$\omega_1^{[this\ study]}$ Er(%)	$\omega_2^{[this\ study]}$ Er(%)	$\omega_3^{[this\ study]}$ Er(%)	$\omega_{[exact]}^{[11]}$	$\omega_{[1IP]}^{[11]}$ Er(%)	$\omega_{[2IP]}^{[11]}$ Er(%)	$\omega_{[CHVF]}^{[18]}$ Er(%)	$\omega_{[HEBM]}^{[23]}$ Er(%)
0.1	1.003774 (0.0003)	1.003772 (0.0001)	1.003772 (0.0001)	1.003772 (0.0001)	1.003777	1.003772 (0.0001)	1.003772 (0.0001)	1.003100 (0.0674)	1.003773 (0.0003)
0.5	1.107502 (0.0869)	1.106560 (0.0018)	1.106544 (0.0003)	1.106544 (0.0003)	1.106548	1.106575 (0.0024)	1.106545 (0.0002)	1.087705 (1.7028)	1.106356 (0.0173)
1	1.541103 (1.1494)	1.524791 (0.0788)	1.523653 (0.0041)	1.523589 (0.0000)	1.523591	1.525073 (0.0972)	1.523748 (0.0103)	1.445657 (5.1151)	1.527525 (0.2582)
5	20.25771 (5.6106)	19.36516 (0.9574)	19.20965 (0.1467)	19.18561 (0.0214)	19.18157	19.37354 (1.0008)	19.22145 (0.2079)	17.82767 (7.0583)	19.60888 (2.2277)
10	79.53615 (5.7979)	75.93462 (1.0072)	75.29569 (0.1573)	75.19499 (0.0233)	75.17762	75.97375 (1.0589)	75.34539 (0.2231)	69.87608 (7.0520)	76.88958 (2.2772)
50	1976.898 (5.8540)	1886.664 (1.0224)	1870.572 (0.1607)	1868.022 (0.0242)	1867.579	1887.694 (1.0770)	1871.836 (0.2279)	1735.910 (7.050)	1910.332 (2.2892)
100	7906.168 (5.8555)	7545.210 (1.0226)	7480.831 (0.1606)	7470.626 (0.0240)	7468.852	7549.340 (1.0776)	7485.889 (0.2281)	6942.282 (7.0502)	7639.855 (2.2895)
500	197642.8 (5.8560)	188618.7 (1.0228)	187009.1 (0.1607)	186753.9 (0.0240)	186709.5	188721.9 (1.0778)	187135.5 (0.2281)	173546.2 (7.0501)	190984.5 (2.2896)
1000	790569.8 (5.8560)	754473.3 (1.0228)	748034.9 (0.1607)	747014.3 (0.0240)	746836.9	754886.5 (1.0778)	748540.9 (0.2281)	694183.4 (7.0502)	763936.8 (2.2896)

Table 6: Comparison of the approximated frequencies with existing and exact ones when $a = 5$, $b = 3$ and $c = 1$:

A_0	$\omega_0^{[this\ study]}$ Er(%)	$\omega_1^{[this\ study]}$ Er(%)	$\omega_2^{[this\ study]}$ Er(%)	$\omega_3^{[this\ study]}$ Er(%)	$\omega_{[exact]}^{[11]}$	$\omega_{[1IP]}^{[11]}$ Er(%)	$\omega_{[2IP]}^{[11]}$ Er(%)	$\omega_{[CHVF]}^{[18]}$ Er(%)	$\omega_{[HEBM]}^{[23]}$ Er(%)
0.1	2.241107	2.241106	2.241106	2.241106	2.241115	2.241106	2.241106	2.240210	2.241102
	(0.0003)	(0.0004)	(0.0004)	(0.0004)		(0.0004)	(0.0004)	(0.0403)	(0.0005)
0.5	2.366762	2.366153	2.366148	2.366148	2.366157	2.366156	2.366148	2.343456	2.366246
	(0.0255)	(0.0001)	(0.0003)	(0.0003)		(0.0000)	(0.0003)	(0.9594)	(0.0037)
1	2.806243	2.796635	2.796279	2.796269	2.796279	2.796695	2.796295	2.756650	2.798963
	(0.3563)	(0.0127)	(0.0000)	(0.0000)		(0.0148)	(0.0005)	(1.4172)	(0.0959)
5	21.25735	20.38589	20.24106	20.21979	20.21645	20.39110	20.25141	18.78950	20.64011
	(5.1487)	(0.8381)	(0.1217)	(0.0165)		(0.8639)	(0.1729)	(7.0583)	(2.0956)
10	80.49844	76.91371	76.28465	76.18665	76.17001	76.94867	76.33263	70.79627	77.88483
	(5.6825)	(0.9763)	(0.1505)	(0.0218)		(1.0222)	(0.2134)	(7.0549)	(2.2513)
50	1977.847	1887.628	1871.546	1868.999	1868.556	1888.654	1872.808	1736.815	1911.314
	(5.8489)	(1.0206)	(0.1600)	(0.0237)		(1.0755)	(0.2275)	(7.0504)	(2.2882)
100	7907.117	7546.174	7481.804	7471.602	7469.829	7550.299	7486.861	6943.188	7640.837
	(5.8540)	(1.0220)	(0.1603)	(0.0237)		(1.0772)	(0.2280)	(7.0502)	(2.2893)
500	197643.7	188619.6	187010.0	186754.9	186710.5	188722.9	187136.5	173547.1	190985.5
	(5.8557)	(1.0224)	(0.1604)	(0.0237)		(1.0778)	(0.2281)	(7.0501)	(2.2896)
1000	790570.8	754474.3	748035.9	747015.3	746837.9	754887.4	748541.8	694184.3	763937.8
	(5.8557)	(1.0225)	(0.1604)	(0.0237)		(1.0778)	(0.2281)	(7.0502)	(2.2896)

Table 7: Comparison of the approximated frequencies with existing and exact ones when $a = 1$, $b = 10$ and $c = 100$:

A_0	$\omega_0^{[this\ study]}$ Er(%)	$\omega_1^{[this\ study]}$ Er(%)	$\omega_2^{[this\ study]}$ Er(%)	$\omega_3^{[this\ study]}$ Er(%)	$\omega_{[exact]}^{[11]}$	$\omega_{[1IP]}^{[11]}$ Er(%)	$\omega_{[2IP]}^{[11]}$ Er(%)	$\omega_{[CHVF]}^{[18]}$ Er(%)	$\omega_{[HEBM]}^{[23]}$ Er(%)
0.1	1.039832 (0.0125)	1.039699 (0.0001)	1.039698 (0.0002)	1.039698 (0.0002)	1.039701	1.039699 (0.0001)	1.039697 (0.0003)	1.032599 (0.6830)	1.039642 (0.0056)
0.5	2.604083 (3.1441)	2.534428 (0.3852)	2.525728 (0.0406)	2.524796 (0.0037)	2.524702	2.535046 (0.4097)	2.526416 (0.0678)	2.354203 (6.7532)	2.554014 (1.1610)
1	8.426150 (5.1944)	8.078080 (0.8490)	8.019990 (0.1238)	8.011422 (0.0168)	8.010069	8.080690 (0.8816)	8.024286 (0.1774)	7.444004 (7.0669)	8.176911 (2.0829)
5	198.1186 (5.8328)	189.1022 (1.0163)	187.4974 (0.1590)	187.2436 (0.0235)	187.1996	189.2033 (1.0703)	187.6230 (0.2261)	174.0004 (7.0508)	191.4770 (2.2849)
10	791.0442 (5.8500)	754.9552 (1.0209)	748.5216 (0.1600)	747.5022 (0.0236)	747.3252	755.3661 (1.0759)	749.0266 (0.2276)	694.6360 (7.0503)	764.4279 (2.2885)
50	19764.71 (5.8555)	18862.30 (1.0224)	18701.35 (0.1604)	18675.83 (0.0237)	18671.40	18872.63 (1.0777)	18713.99 (0.2281)	17355.02 (7.0502)	19098.90 (2.2896)
100	79057.41 (5.8557)	75447.77 (1.0224)	74803.93 (0.1604)	74701.87 (0.0237)	74684.13	75489.08 (1.0778)	74854.52 (0.2281)	69418.75 (7.0502)	76394.13 (2.2896)
500	1976424. (5.8557)	1886182. (1.0225)	1870086. (0.1604)	1867535. (0.0237)	1867091.	1887215. (1.0778)	1871351. (0.2281)	1735457. (7.0502)	1909841. (2.2896)
1000	7905694. (5.8557)	7544729. (1.0225)	7480345. (0.1604)	7470138. (0.0237)	7468365.	7548860. (1.0778)	7485404. (0.2281)	6941830. (7.0502)	7639364. (2.2896)

8. Conclusion

An iterative method has been investigated to the strongly nonlinear cubic-quintic Duffing oscillator. We have obtained highly accurate frequencies as well as periodic solutions of the strongly nonlinear cubic-quintic Duffing oscillator. The presented iterative method is a powerful and highly efficient mathematical tool for solving strongly nonlinear differential equations. The mentioned method is also convergent and the approximated solutions are consistent. As compared with existing results and exact solutions, the excellent agreement has been found. The exactness of the results shows that the presented iterative method can be easily and efficiently investigated for the analysis of strongly nonlinear vibration problems with high accuracy. Hence, it has been proven that the presented method is very effective and an efficient alternative to the previously existing methods for solving strongly nonlinear oscillatory systems.

9. Acknowledgement

The work was supported by The World Academy of Science (TWAS), Research Grant Agreement No. 18-075 RG/MATHS/AS_I-FR3240303632.

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Accepted: July 09, 2020

The theta-complete graph Ramsey number $r(\theta_k, K_5); k = 7, 8, 9$

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Abstract. Finding the Ramsey number is an important problem of the well-known family of the combinatorial problems in Ramsey theory. In this work, we investigate the Ramsey number $r(\theta_s, K_5)$ for $s = 7, 8, 9$ where θ_s is the set of theta graphs of order s and K_5 is a complete graph of order 5. Our result closed the problem of finding $R(\theta_s, K_5)$ for each $s \geq 6$.

Keywords: Ramsey number, theta graph, complete graph.

1. Introduction

All graphs we consider are undirected, finite and simple. Let G be a graph with the *vertex set* $V(G)$ and the *edge set* $E(G)$. A subset $S \subset V(G)$ is an *independent set* if no two vertices of S are adjacent. The size of the largest

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independent set of the graph G is called *independent number* which is denoted by $\alpha(G)$. For a given vertex u of a graph G, the number of incident edges with u is called the *degree* of u, denoted by $d(u)$. $N(u)$ stands for the set of all vertices adjacent to u, and $N[u] = N(u) \cup \{u\}$. For a set of vertices $\{u_1, u_2, \ldots, u_t\}$, $N[u_1, u_2, \ldots, u_t] = N[u_1] \cup N[u_2] \cup \ldots \cup N[u_t]$ and the *induced subgraph* $G\langle u_1, u_2, \ldots, u_t\rangle_G$ of G consists of the vertex set $\{u_1, u_2, \ldots, u_t\}$ and all edges of G connecting two vertices in $\{u_1, u_2, \ldots, u_t\}$. $\Delta(G)$ and $\delta(G)$ stand for the *maximum* and *minimum* degree in the graph G, respectively. We denote C_s, P_s and K_s to be cycle, path and complete graph of order s, respectively. The *theta graph* of order s is a cycle C_s and an edge joining two non-adjacent vertices in C_s.

For a given set of graphs $\mathcal{S}$ and a graph H, the *Ramsey number* $R(\mathcal{S}, H)$ is defined to be the smallest positive integer N such that for every graph G with at least N vertices, either G contains a graph in $\mathcal{S}$ as a subgraph or the complement of G (i.,e. $\overline{G}$) contains H as a subgraph. If $\mathcal{S}$ contains only one graph, say H_1, we simply write $R(H_1, H)$.

Ramsey numbers for graphs of small orders have been studied for long time (see, [5, 8, 9, 10, 12, 20, 22]) and it was noted that computing the exact Ramsey numbers of such graphs become more challenging and their difficulties increase with the increasing of the number of edges of the given graphs, but it is more interesting for spare graphs.

One of the Ramsey numbers problems is the Erdös et al. [10] conjecture which states that $r(C_s, K_t) = (s-1)(t-1)+1$, for all $s \geq t \geq 3$ except $r(C_3, K_3) = 6$. The conjecture attracted the attention of many authors and confirmed only on few cases (see, [1, 3, 4, 7, 13, 14, 15, 16, 21, 24, 25, 26, 27, 28]).

In this paper, we end the remaining open values of the Ramsey numbers of theta graphs verses complete graphs of order 5 that initiated in [14] see Theorem 1.2. In fact, we prove that $r(\theta_7, K_5) = 25, r(\theta_8, K_5) = 29$, and $r(\theta_9, K_5) = 33$.

Let us present some results concerning the Ramsey number of the theta graphs verses complete graph and complete deleting edge. Chvátal and Harary [8], determined that $r(\theta_4, K_4) = 11$. Bolze and Harborth [5] and Faudree et al. [11] proved that $r(\theta_4, K_5) = 16$ and $r(\theta_4, K_5 - e) = 13$, respectively. McNamara [19] showed that $r(\theta_4, K_6) = 21$ and McNamara and Radziszowski [20] obtained the following two results: $r(\theta_4, K_6 - e) = 17$ and $r(\theta_4, K_7 - e) = 28$. An upper bound for $r(\theta_4, K_7)$ and the exact number for $r(\theta_4, K_8)$ were investigated by Boza [6], in fact, he obtained that $r(\theta_4, K_7) \leq 31$ and $r(\theta_4, K_8) = 42$. Bataineh et al. [2], established the following result:

Theorem 1.1. *For $t = 2, 3, 4$ and $s > t$, $r(\theta_s, K_t) = (s-1)(t-1)+1$.*

Finally, Jaradat et. al, [14] proved the following result:

Theorem 1.2. *The Ramsey number $r(\theta_s, K_5) = 4s - 3$ for $s = 6$ and $s \geq 10$.*

For more results on Ramsey numbers, we direct the reader to the well known updated survey by Radziszowski [22].

2. Main results

The purpose of this section is to investigate the Ramsey number of theta graphs of order $7, 8$, and 9 versus complete graphs of order 5 which are the remaining open value in Theorem 1.2. One can see that $G = (m-1)K_{n-1}$, contains neither θ_n nor m-element independent set. Hence, we obtained that $r(\theta_s, K_t) \geq (s-1)(t-1)+1$. Now, we present some results that will be used in the sequel. Clancy [9] proved that

$$r(C_4, K_5) = 14. \tag{1}$$

Hendrey [12], Jayawardene and Rousseau [17] determined that

$$r(C_5, K_5) = 17. \tag{2}$$

Further, Jayawardene and Rousseau [18] showed that

$$r(C_6, K_5) = 21. \tag{3}$$

Finally, for any graph H, it is clear that

$$r(H, K_0) = 1. \tag{4}$$

To achieve our goal, we first prove the following sequence of six Lemmas:

Lemma 2.1. *Let G be a graph of order more than or equal to $4(s-1)+1$, $s = 7, 8, 9$, that contains neither θ_s nor 5-element independent set. Then $\delta(G) \geq s-1$.*

Proof. Assume that G contains a vertex of degree less than $s-1$, say u. Then $\mid V(G - N[u]) \mid \geq 3(s-1)+1$. Since $r(\theta_s, K_4) = 3(s-1)+1$, $G - N[u]$ contains an independent set consisting of 4 vertices. This set with the vertex u is a 5-element independent set, a contradiction. □

Lemma 2.2. *Let G be a graph of order more than or equal to $4(s-1)+1$, $s = 7, 8, 9$, and contains neither θ_s nor 5-element independent set. If $\{u_1, u_2, \ldots, u_t\}$, $t \leq 4$, is an independent set of vertices, then $\mid N(u_1) \cup N(u_2) \cup \ldots \cup N(u_t) \mid \geq t(s-2)+1$.*

Proof. Suppose that $\mid N(u_1) \cup N(u_2) \cup \ldots \cup N(u_t) \mid < t(s-2)+1$. Then $\mid V(G - N[u_1, u_2, \ldots, u_t]) \mid \geq (4-t)(s-1)+1$. Since, by (4) and Theorems 1.1, 1.2, $r(\theta_s, K_{5-t}) = (4-t)(s-1)+1$, $G - N[u_1, u_2, \ldots, u_t]$ contains an independent set consisting of $5-t$ vertices. This set with the vertices u_1, u_2, …, u_t is a 5-element independent set, a contradiction. □

Lemma 2.3. *Let G be a graph of order $4(s-1)+1$, $s = 7, 8, 9$. If G contains neither θ_s nor 5-element independent set, then G contains no K_{s-1}.*

Proof. Suppose that G contains K_{s-1}. Let $W = \{u_1, u_2, \ldots, u_{s-1}\}$ be the vertices of K_{s-1}, $R = G - W$ and $W_i = N(u_i) \cap V(R)$, $i = 1, 2, \ldots, s-1$. Since $\delta(G) \geq s-1$, $\mid W_i \mid \geq 1$. Also, since G does not contain θ_s, $W_i \cap W_j = \emptyset$, $1 \leq i < j \leq s-1$, and $xy \notin E(G)$ for any $x \in W_i$ and $y \in W_j$, $1 \leq i < j \leq s-1$. Note that, $\{v_1, v_2, \ldots, v_{s-1}\}$ is an independent set where $v_i \in W_i$, $i = 1, 2, \ldots, s-1$. Since $s - 1 \geq 6$, we conclude that G contains an independent set with at least 6 vertices, a contradiction. □

Lemma 2.4. *Let G be a graph of order $4(s-1)+1$, $s = 7, 8, 9$. If G contains neither θ_s nor 5- element independent set, then G contains no $K_1 + P_{s-2}$.*

Proof. Suppose that G contains $K_1 + P_{s-2}$. Let $W = \{u_1, u_2, \ldots, u_{s-2}\}$ and u be the vertices of P_{s-2} and K_1, respectively. Let $R = G - (W \cup \{u\})$ and $W_i = N(u_i) \cap V(R)$, $i = 1, 2, \ldots, s-2$. Since $\delta(G) \geq s-1$, $\mid W_i \mid \geq 1$, say $v_i \in W_i$, $i = 1, 2, \ldots, s-2$. Note that no vertex of $\{v_1, v_2, v_3, v_{s-2}\}$ is adjacent to two vertices of $\{u_1, u_2, u_3, u_{s-2}\}$ as otherwise θ_s is produced. Also, there is no edge between any two vertices of $\{v_1, v_2, v_3, v_{s-2}\}$ as otherwise θ_s is produced. Note that $uv_{s-2} \notin E(G)$ and $u_i v_{s-2} \notin E(G)$, $i = 1, 2, \ldots, u_{s-3}$ as otherwise θ_s is produced. Since G contains no θ_s, $v_i w \notin E(G)$ for any $w \in N(v_{s-2}) \cap V(R)$, $i = 1, 2, 3$. Therefore, $\mid N(v_{s-2}) \cap V(R) \mid \geq s-2$. Since by Lemma 2.3 G contains no K_{s-1}, as a result $N[v_{s-2}]$ contains two independent vertices, say $\{w_1, w_2\}$. Hence, $\{v_1, v_2, v_3, w_1, w_2\}$ is a 5−element independent set, a contradiction. □

Lemma 2.5. *Let G be a graph of order $4(s-1)+1$, $s = 7, 8, 9$. If G contains neither θ_s nor 5-element independent set, then G contains no $K_1 + C_{s-3}$.*

Proof. Suppose that G contains $K_1 + C_{s-3}$. Let $W = \{u_1, u_2, \ldots, u_{s-3}\}$ and u be the vertices of C_{s-2} and K_1, respectively. Let $R = G - (W \cup \{u\})$ and $W_i = N(u_i) \cap V(R)$, $i = 1, 2, \ldots, s-3$. Since $\delta(G) \geq s-1$, $\mid W_i \mid \geq 2$, $i = 1, 2, \ldots, s-3$. By Lemma 2.4, $uv \notin E(G)$ for any $v \in W_i$, $i = 1, 2, \ldots, s-3$. Since G contains no θ_s, as a result for any $x \in W_i$ and $y \in W_j$, $1 \leq i < j \leq s-3$, we have $xy \notin E(G)$. Now we consider two cases according to $W_i \cap W_j$:

Case 2.0.1. $W_i \cap W_j = \emptyset$, $1 \leq i < j \leq 4$.

In this case, $\{u, v_1, v_2, v_3, v_4\}$ is a 5-element independent set where $v_i \in W_i$, $i = 1, 2, 3, 4$. This is a contradiction.

Case 2.0.2. $W_i \cap W_j \neq \emptyset$, for some $1 \leq i < j \leq 4$.

Let $v \in W_a \cap W_b$ where $1 \leq a < b \leq 4$. since $W_i \geq 2$, then $\mid W_a \cup W_b \mid \geq 2$. Let $w \in W_a \cup W_b$. Moreover, since G contains no θ_s, $(W_i - \{v\}) \cap (W_j - \{v\}) = \emptyset$, $1 \leq i < j \leq 4$ and $\{i, j\} \neq \{a, b\}$. Therefore, $\{v, w, f, g, u\}$ is a 5-element independent set where $f \in W_c$ and $g \in W_d$, $1 \leq c < d \leq 4$ and $\{c, d\} \neq \{a, b\}$, a contradiction.

□

Lemma 2.6. *Let G be a graph of order equal $4(s-1)+1$, $s=7,8,9$. If G contains neither θ_s nor 5-element independent set, then $\langle N(N(u))-\{u\}\rangle_G$ does not contain C_{s-3}.*

Proof. Suppose that G contains C_{s-3}. By Lemma 2.5, the vertices of C_{s-3} are not adjacent to the same vertex in $N(u)$. Therefore, there are two adjacent vertices in the vertex set of C_{s-3} that adjacent to two different vertices in the vertex set of $N(u)$. The vertex set $V(C_{s-3})$, two vertices of the vertex set of $N(u)$ and u generate θ_s, a contradiction. □

Theorem 2.1. $r(\theta_s, K_5) = 4(s-1)+1$, $s=7,8,9$.

Proof. By the inequality, $r(\theta_n, K_m) \geq (n-1)(m-1)+1$, it suffices to prove that $r(\theta_s, K_5) \leq 4(s-1)+1$. Let G be a graph on $4(s-1)+1$ vertices. Suppose that G contains no θ_s as a subgraph and $\alpha(G) \leq 4$, then $\delta(G) \geq s-1$. Let u be a vertex of $v(G)$, $H = \langle N(u)\rangle_G$ and S be the remaining vertices. We consider a number of cases according to $\alpha(H)$.

Case 2.1.1. $\alpha(H) = 1$. So, obviously G contains θ_s as a subgraph, a contradiction.

Case 2.1.2. $\alpha(H) = 2$. Then H is a union of two complete disjoint components, say $H = K_r \cup K_t$, with $\mid V(K_r \cup K_t) \mid \geq s-1$. Since G does not contain θ_s, $N(V(K_r)) \cap N(V(K_t)) \cap S = \emptyset$ and $xy \notin E(G)$ for any $x \in V(K_r)$ and $y \in V(K_t)$. Since G does not contain θ_s, as a result $\langle N(V(K_r)) \cap S\rangle_G$ and $\langle N(V(K_t)) \cap S\rangle_G$ are not complete. Therefore, $\{u_1, u_2, v_1, v_2, u\}$ is a 5-element independent set, where $\{u_1, u_2\} \in V(K_r)$ and $\{v_1, v_2\} \in V(K_t)$, a contradiction.

Case 2.1.3. $3 \leq \alpha(H) \leq 4$, say $\{u_1, \ldots, u_t\}$, $3 \leq t \leq 4$. Then by Lemma 2.2, $\mid V(N(\{u_1, \ldots, u_t\}) - \{u\}) \mid \geq 3(s-2)$. By Lemma 2.6, $\langle N(\{u_1, \ldots, u_t\}) - \{u\}\rangle_G$ does not contain C_{s-3}. Therefore, since by (1), (2) and (3), $r(C_{s-3}, K_5) \leq 3(s-2)$, then $\langle N(\{u_1, \ldots, u_t\}) - \{u\}\rangle_G$ contains a 5-element independent set, a contradiction.

□

The following result follows from Theorems 1.2 and 2.1.

Theorem 2.2. *The Ramsey number $r(\theta_s, K_5) = 4(s-1)+1$, for each $s \geq 6$.*

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Accepted: March 03, 2020

A note on G_δ-diagonals in semitopological groups

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Abstract. In this note, we prove that:

(1) if G is a sof-countable semitopological group, then the diagonal Δ of G can be represented as the intersection of a countable family of sequentially open neighborhoods of Δ in $G \times G$, which generalizes the corresponding result in [3];

(2) if G is a sof-countable paratopological group, then G has a $G_{s\delta}$-diagonal of infinite rank; (3) let G be a regular semitopological group and H a closed invariant subgroup of G with a quasi-G_δ-diagonal, if G/H has a quasi-G_δ-diagonal, then G has a quasi-G_δ-diagonal.

Keywords: G_δ-diagonal, semitopological groups, paratopological groups, sn-network, so-network.

1. Introduction

Recall that a right (resp., left) topological group is a group with a topology such that the right (resp., left) action of a is a continuous mapping, for each $a \in G$. A semitopological group is a left topological group which is also a right topological group. A paratopological group is a group with a topology such that multiplication on the group is jointly continuous. Every topological group is a paratopological group and every paratopological group is a semitopological group.

A space is called submetrizable if there is a one-to-one continuous mappings from it onto a metrizable space. In view of submetrizability is influenced by its diagonal, diagonals have been largely studied in the past few years, for example, in [1, 2, 3, 10, 11, 12]. In 2004, P. Li, L.H. Xie, S. Lin [8] prove that $\Delta(G) \leq Hs(G) \cdot \psi(G)$ for every Hausdorff semitopological group G, and furthermore, they show that every Hausdorff paracompact semitopological group G with $Hs(G) \cdot \psi(G) \leq \omega$ is submetrizable.

In 2007, A.V. Arhangel'skiĭ, A. Bella in [3] obtain that every first-countable paratopological group has a G_δ-diagonal of infinite rank by introducing T-linked local π-base. In 2019, P. Li, L. Mou, F. Wang, L. Xie in [7] consider some three-space properties of semitopological and paratopological groups. They point out that having a G_δ-diagonal is a three-space property for semitopological groups.

In this paper, we mainly discuss diagonal by mans of sequentially open subsets in a semitopological group and prove that having a quasi G_δ-diagonal is a three-space property for semitopological groups. The readers may consult [4, 5] for notation and terminology not explicitly given here. All spaces are assumed to be T_2 if it is not stated explicitly.

2. Preliminaries

Let X be a space. For $P \subset X$, the set P is a sequentially neighborhood of x in X if every sequence converging to x is eventually in P. P is a sequentially open subset of X if P is a sequentially neighborhood of x in X, for each $x \in P$. X is said to be a sequential space [6] if each sequentially open subset is open in X.

It is easy to check the validity of the next results.

Proposition 2.1. *Every sequentially open subgroup of a right (or left) topological group G is sequentially closed in G.*

Proof. Let H be a sequentially open subgroup in G. If H is not sequentially closed, there exists a sequence $\{h_n\}$ converging to $h \notin H$ where $h_n \in H$, for each $n \in N$. Since G is a right topological group, the sequence $\{h_n h^{-1}\}$ converges to e. It follows from H is sequentially open that there is $m \in N$ such that $h_n h^{-1} \in H$, for each $n > m$. Therefore, $h \in H$ by H is a group, which is a contraction. The same holds true in the case when G is a left topological group. □

Lemma 2.1. *Let G be a right topological group. If U is a sequentially open neighborhood of e, then Ux is a sequentially open neighborhood of x, for each $x \in G$.*

Proof. For each $y \in Ux$, if there is a sequence $\{x_n : n \in \omega\}$ converging to x in X, then the sequence $\{x_n x^{-1} : n \in \omega\}$ converges to e. Since U is sequentially open and $e \in U$, there exists $m \in N$ such that $x_n x^{-1} \in U$, for each $n > m$. Thus $x_n \in Ux$, for each $n > m$. It follows from the definition of sequentially open sets that Ux is sequentially open. □

It is easy to see that if G is a left topological group, then xU is a sequentially open neighborhood of x, for each $x \in G$ where U is a sequentially open neighborhood of e.

Proposition 2.2. *Suppose that a subgroup H of a right (or left) topological group G contains a non-empty sequentially open subset of G. Then H is sequentially open in G.*

Proof. Let U be a sequentially open subset of G with $U \subset H$. For each $h \in H$, Uh is sequentially open in G by Lemma 2.1. Therefore $H = \bigcup_{h \in H} Uh$ is sequentially open. □

Clearly, the following lemma is true.

Lemma 2.2. *Let G be a semitopological group with a neutral element e. If U is a sequentially open neighborhood of e, then $U \times U$ is a sequentially open neighborhood of (x, x) in $G \times G$, for each $x \in G$.*

Definition 2.1 ([9])**.** *Let $\mathcal{P} = \bigcup_{x\in X} \mathcal{P}_x$ be a cover of a space X such that for each $x \in X$:*

(a) if $U, V \in \mathcal{P}_x$, then $W \subset U \cap V$ for some $W \in \mathcal{P}_x$;

(b) the family $\mathcal{P}_x$ is a network of x in X, i.e., $x \in \cap\mathcal{P}_x$, and if $x \in U$ with U open in X, then $P \subset U$ for some $P \in \mathcal{P}_x$.

(1) The family $\mathcal{P}$ is called an sn-network (sequential-neighborhood network) for X if each element of $\mathcal{P}_x$ is a sequential neighborhood of x in X, for each $x \in X$. X is called snf-countable, if X has an sn-network $\mathcal{P}$ such that each $\mathcal{P}_x$ is countable. A regular space X is called sn-metrizable if X has an σ-locally finite sn-network.

(2) The family $\mathcal{P}$ is called an so-network (sequentially-open network) for X if each element of $\mathcal{P}_x$ is a sequentially open neighborhood of x in X, for each $x \in X$. X is called sof-countable, if X has an so-network $\mathcal{P}$ such that each $\mathcal{P}_x$ is countable. A regular space X is called so-metrizable if X has an σ-locally finite so-network.

It is clear that every sof-countable space is snf-countable, and a space is first-countable if and only if it is sequential and sof-countable.

Definition 2.2. *The sequentially diagonal of a space X denoted by $\Delta_s(X)$ is the minimum cardinal number κ such that the diagonal $\Delta = \{(x, x) : x \in X\}$ can be represented as the intersection of a family $\mathcal{P}$ of sequentially open neighborhoods of Δ in $X \times X$ and $|\mathcal{P}| \leq \kappa$.*

A space X is called have a $G_{s\delta}$-diagonal, if $\Delta_s(X) \leq \omega$. It is easy to see that if X is a sequentially space with a $G_{s\delta}$-diagonal, then X has a G_δ-diagonal.

Definition 2.3. *For a Hausdorff semitopological group G with the neutral element e the sequentially Hausdorff number of G denoted by $Hss(G)$ is the minimum cardinal number κ such that for every sequentially open neighborhood U of e in G, there exists a family $\mathcal{V}$ of sequentially open neighborhoods of e such that $\bigcap_{V\in\mathcal{V}} VV^{-1} \subset U$ and $|\mathcal{V}| \leq \kappa$.*

Note that $\bigcap_{P\in\mathcal{P}_e} PP^{-1} = \{e\}$, one can easily check the following.

Proposition 2.3. *Let $\mathcal{P} = \bigcup_{g\in G} \mathcal{P}_g$ be an so-network of a Hausdorff semitopological group G with $|\mathcal{P}_g| \leq \kappa$, for each $g \in G$ where κ is an infinite cardinal. Then $Hss(G) \leq \kappa$.*

It follows from proposition 2.3 that every sof-countable semitopological group G satisfies $Hss(G) \leq \omega$.

Definition 2.4. *$\psi_s(x, X)$, $\psi_s(X)$ denote the sequentially pseudocharacter of a space X at x and the sequentially pseudocharacter of a space X defined, respectively, as follows*

$\psi_s(x, X) = \min\{|\mathcal{U}| : \cap\mathcal{U} = \{x\}$ and $\mathcal{U}$ is a family of sequentially open subsets of $X\}$.

$\psi_s(X) = \sup\{\psi(x, X) : x \in X\}$.

3. Main results

Using Lemmas 2.1 and 2.2, we can modify P. Li, L.H. Xie and S. Lin's proof [8] for extending the result to the class of semitopological groups which are sequentially spaces.

Theorem 3.1. *Let G be a Hausdorff semitopological group. Then $\Delta_s(G) \leq \psi_s(G) \cdot Hss(G)$.*

Proof. Assume that $\psi_s(G) \cdot Hss(G) = \kappa$. Let $\bigcap_{\alpha<\kappa} P_\alpha = \{e\}$, where P_α is a sequentially open neighborhood of e, for each $\alpha < \kappa$. It follows from $Hss(G) \leq \kappa$ that there exists a family γ_α of sequentially open neighborhoods of e such that $\bigcap_{Q\in\gamma_\alpha} QQ^{-1} \subset P_\alpha$, for each $\alpha \leq \kappa$. Put $\gamma = \bigcup_{\alpha<\kappa} \gamma_\alpha = \{Q_i : i \leq \kappa\}$. Next, we shall verify that $\Delta_s(G) \leq \kappa$. Put $W_i = \bigcup_{x\in G}(xQ_i \times xQ_i)$, for each $i \leq \kappa$. It is only need to notice that $\Delta_s(G) = \bigcap_{i\leq\kappa} W_i$. If not, there are two distinct elements $x, y \in G$ such that for each $i \leq \kappa$, $(x, y) \in W_i = \bigcup_{x\in G}(xQ_i \times xQ_i)$. Thus, there is $x_i \in G$ such that $(x, y) \in x_iQ_i \times x_iQ_i$, from which we can conclude that $x \in x_iQ_i$ and $y \in x_iQ_i$. Thus, $y^{-1}x \in Q_i^{-1}Q_i$, for each $i \leq \kappa$. Indeed, $\bigcap_{i\leq\kappa} Q_i^{-1}Q_i = \{e\}$, a contradiction. □

Making use of Proposition 2.3, it is easy to deduce the corollary below.

Corollary 3.1. *If G is a Hausdorff sof-countable semitopological group, then $\Delta_s(G) \leq \omega$.*

Since every first-countable topological space is sof-countable, it is easy to see that the next corollary proved in [3] is true.

Corollary 3.2. *If G is a Hausdorff first-countable semitopological group, then G has a G_δ-diagonal.*

A. V. Arhangel'skii and A. Bella [3] prove that if G is a paratopological group with a countable T-linked local π-base at the neutral element, then G has a G_δ-diagonal of infinite rank.

Assume that $\mathcal{U}$ is a collection of a space X, we put $st^1(\mathcal{U}, x) = st(\mathcal{U}, x) = \cup\{U : U \in \mathcal{U}$ and $x \in U\}$ and recursively define $st^{n+1}(\mathcal{U}, x) = st(\mathcal{U}, st^n(\mathcal{U}, x))$. Inspired by the definition of a space X with G_δ-diagonal of infinite rank defined in [3], a space X with a $G_{s\delta}$-diagonal of infinite rank is defined as follows: if, for each $n \in \omega$ there exists a countable collection $\{\mathcal{U}_k : k < \omega\}$ of sequentially open covering of X such that $\cap\{st^n(\mathcal{U}_k, x) : k < \omega\} = \{x\}$, for each $x \in X$.

Theorem 3.2. *Let G be a Hausdorff sof-countable paratopological group. Then G has a $G_{s\delta}$-diagonal of infinite rank.*

Proof. Let $\mathcal{P} = \bigcup_{x\in X} \mathcal{P}_x$ be a so-network for G, where each $\mathcal{P}_x$ is countable. For each $P \in \mathcal{P}_e$, let $\mathcal{U}(P) = \{xP \cap Px : x \in G\} = \{P[x] : x \in G\}$. It is clear that every $\mathcal{U}(P)$ is a cover of sequentially open subsets of G. We will prove that the collection $\Gamma = \{\mathcal{U}(P) : P \in \mathcal{P}_e\}$ witnesses the infinite rank of the diagonal. Indeed, we have to check that there exists some $P \in \mathcal{P}_e$ such that $z \notin st^n(\mathcal{U}(P), y)$ for a fixed integer n and a pair of distinct points $y, z \in G$. It is easy to find $P \in \mathcal{P}_e$ such that $P^n y \cap zP^n = \emptyset$. Next, we shall show that $z \notin st^n(\mathcal{U}(P), y)$ for this P. If not, there are $x_i, i = 1, 2, \cdots, n-1$ such that $y \in P[x_1], z \in P[x_n]$ and $P[x_i] \cap P[x_{i+1}] \neq \emptyset$ for $1 \leq i \leq n-1$. Thus there exist points $p_1, \cdots, p_n, q_0, \cdots, q_{n-1} \in P$ such that $y = x_1q_0, z = p_nx_n$ and $p_ix_i = x_{i+1}q_i$. It follows from the equality $x_i = p_i^{-1}x_{i+1}q_i$, it is easy to conclude that $y = p_1^{-1}p_2^{-1}\cdots p_n^{-1}zq_{n-1}q_{n-2}\cdots q_1q_0$. Then $p_np_{n-1}\cdots p_2p_1y = zq_{n-1}q_{n-2}\cdots q_1q_0$. Hence, $P^ny \cap zP^n \neq \emptyset$, a contradiction. □

A.V. Arhangel'skii, A. Bella show that a topological space X with a G_δ-diagonal of rank 3 has also a regular G_δ-diagonal [3].

In view of Theorem 3.2 and every first-countable space is sof-countable, it is obvious that the following statements are true.

Corollary 3.3 ([3])**.** *If G is a first-countable paratopological group. Then, G has a G_δ-diagonal of infinite rank.*

Corollary 3.4 ([3])**.** *Any first countable paratopological group has a regular G_δ-diagonal.*

A space X is said to have a quasi-G_δ-diagonal if , there is a sequence $\{\mathcal{U}_n\}$, where each $\mathcal{U}_n$ is a family of open subsets of X, such that, for every $x \in X$, $\{x\} = \bigcap_{n\in N}\{st(x, \{\mathcal{U}_n\}) : x \in \cup\{\mathcal{U}_n\}\}$. The sequence $\{\mathcal{U}_n\}$ is called a quasi-G_δ-diagonal sequence.

In [7], it is proved that having a G_δ-diagonal is a three-space property for semitopological groups. It is nature to consider three-space property of having a quasi-G_δ-diagonal in semitopological groups.

Theorem 3.3. *Let G be a regular semitopological group and H a closed invariant subgroup of G with a quasi-G_δ-diagonal. If G/H has a quasi-G_δ-diagonal, then G has a quasi-G_δ-diagonal.*

Proof. Let $\pi : G \to G/H$ be the canonical quotient mapping. Let $\{x_\beta H : \beta \in \Lambda\}$ be distinct left cosets of G. Assume that $\{\mathcal{C}_n\}$ is a quasi-G_δ-diagonal sequence for H. For every $n \in N$ and $C \in \mathcal{C}_n$, there is an open set $D(n, C)$ in G such that $D(n, C) \cap H = C$. Let $\mathcal{V}_n = \{x_\alpha D(n, C) : C \in \mathcal{C}_n, \alpha \in \Lambda\}$. Let $\{\mathcal{U}_n\}$ be a quasi-G_δ-diagonal sequence for G/H. Then $\{\pi(e)\} = \bigcap_{n\in N}\{st(\pi(e), \mathcal{U}_n) : \pi(e) \in \cup\mathcal{U}_n\}$. Put $\pi^{-1}(\mathcal{U}_n) = \{\pi^{-1}(U) : U \in \mathcal{U}_n\}$. Then $H = \bigcap_{n\in N}\{st(H, \pi^{-1}(\mathcal{U}_n)) : H \subset \cup\pi^{-1}(\mathcal{U}_n)\}$. Indeed, it is obvious that $H \subset \bigcap_{n\in N}\{st(H, \pi^{-1}(\mathcal{U}_n)) : H \subset$

$\cup\pi^{-1}(\mathcal{U}_n)\}$. If $x \in \bigcap_{n\in N}\{st(H, \pi^{-1}(\mathcal{U}_n)) : H \subset \cup\pi^{-1}(\mathcal{U}_n)\}$, there are $h_n \in H$ and $U_n \in \mathcal{U}_n$ such that $h_n \in \pi^{-1}(U_n)$ and $x \in \pi^{-1}(U_n)$ whenever $H \subset \cup\pi^{-1}(\mathcal{U}_n)$. It follows from $\pi(h_n) = \pi(e) \in \cup\mathcal{U}_n$ that $\pi(x) = \pi(e)$. Thus, $x \in H$.

Since $\{\mathcal{U}_n\}$ is a quasi-G_δ-diagonal sequence for G/H,

$$xH = \bigcap_{n\in N}\{st(xH, \pi^{-1}(\mathcal{U}_n)) : xH \subset \cup\pi^{-1}(\mathcal{U}_n)\},$$

for each $x \in G$. Let $\mathcal{W}_n = \pi^{-1}(\mathcal{U}_n) \cap \mathcal{V}_n = \{W : W = \pi^{-1}(U) \cap V, U \in \mathcal{U}_n, V \in \mathcal{V}_n, W \neq \emptyset\}$. Next, we shall prove that $\{\mathcal{W}_n\}_{n\in N}$ is a quasi-G_δ-diagonal sequence for G. For each $x \in G$, there is a unique $\alpha \in \Lambda$ such that $x \in x_\alpha H$. We claim that $\{x\} = \bigcap_{n\in N}\{st(x, \mathcal{W}_n) : x \in \cup\mathcal{W}_n\}$. In fact, take a $y \in \bigcap_{n\in N}\{st(x, \mathcal{W}_n) : x \in \cup\mathcal{W}_n\}$.

Case 1. $y \notin x_\alpha H$. In this case, $y \notin \bigcap_{n\in N}\{st(x_\alpha H, \pi^{-1}(\mathcal{U}_n)) : x_\alpha H \subset \cup\pi^{-1}(\mathcal{U}_n)\} = x_\alpha H$. Because $\{st(x, \mathcal{W}_n) : x \in \cup\mathcal{W}_n\} \subset \{st(x_\alpha H, \pi^{-1}(\mathcal{U}_n)) : x_\alpha H \subset \cup\pi^{-1}(\mathcal{U}_n)\}$, we have $y \notin \bigcap_{n\in N}\{st(x, \mathcal{W}_n) : x \in \cup\mathcal{W}_n\}$.

Case 2. $y \in x_\alpha H$ and $y \neq x$. Since $\{st(x, \mathcal{W}_n) : x \in \cup\mathcal{W}_n\} \subset \{st(x, \mathcal{V}_n) : x \in \cup\mathcal{V}_n\}$ and $y \notin \bigcap_{n\in N}\{st(x, \mathcal{V}_n) : x \in \cup\mathcal{V}_n\}$, we have $y \notin \bigcap_{n\in N}\{st(x, \mathcal{W}_n) : x \in \cup\mathcal{W}_n\}$.

We have completed our proof. □

Acknowledgments

This research is supported by the NSFC (No.11801254,12071199), the Institute of Meteorological Big Data-Digital Fujian and Fujian Key Laboratory of Date Science and Statistics, the Key Program of the Natural Science Foundation of Fujian Province (No. 2020J02043), the FJNSF(No. 2020J01797).

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Accepted: April 21, 2020

On solvability of finite groups with some nonnormal nonabelian subgroups

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Abstract. For a finite group G, the symbol $\mathcal{M}(G)$ denotes the number of conjugacy classes of maximal subgroups of G, and $\mathcal{T}(G)$ denotes the number of conjugacy classes of non-normal non-abelian subgroups of G. In this paper, we prove the finite groups with $\mathcal{T}(G) < \mathcal{M}(G)$ are solvable.

Keywords: maximal subgroup, nonnormal subgroup, nonabelian subgroup.

1. Introduction

All groups considered in this paper are finite. Throughout the following, G always denotes a finite group. For G, we use the symbol $\mathcal{T}(G)$ to denote the number of conjugacy classes of non-normal non-abelian subgroups of G and $\pi(G)$ to denote the set of the prime divisors of $|G|$. Obviously, $\mathcal{T}(G) = 0$ if and only if G is a Meta-Hamilton group (i.e. G is a non-abelian group whose every non-abelian subgroup is normal). Nagrebeckii [5] characterized the structure of the finite non-nilpotent meta-Hamilton group. An [1] gave a complete classification of finite meta-Hamilton p-groups. Recently, Meng et al. [3] proved the groups with $\mathcal{T}(G) \leq |\pi(G)| - 1$ are solvable and the only non-abelian simple group with $\mathcal{T}(G) = |\pi(G)|$ is isomorphic to A_5.

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In the following, we use the symbol $\mathcal{M}(G)$ to denote the number of conjugacy classes of maximal subgroups of G.

Example 1.1. Let $G = A_5$. Then G has three conjugacy classes of maximal subgroups, and three conjugacy classes of non-normal non-abelian subgroups, that is, $\mathcal{T}(G) = \mathcal{M}(G)$.

In the light of Example 1.1, we obtain the main results of this paper as follows:

Theorem 1.1. *If all non-normal non-abelian maximal subgroups of G are conjugate, then G is a solvable group.*

Theorem 1.2. (1) *Let G be a group with $\mathcal{T}(G) < \mathcal{M}(G)$. Then G is a solvable group.*

(2) *Suppose G is a non-abelian simple group. Then $\mathcal{T}(G) = \mathcal{M}(G)$ if and only if $G \cong A_5$.*

Theorem 1.3. (1) *Let G be a group with $\mathcal{T}(G) \leq 2$. Then G is a solvable group.*

(2) *Suppose G is a non-abelian simple group. Then $\mathcal{T}(G) = 3$ if and only if $G \cong A_5$.*

The notation is standard and follows that of Huppert [2].

2. Some lemmas

Lemma 2.1 ([6])**.** *Let $P \in \mathrm{Syl}_p(G)$. If $H \geq N_G(P)$, then $N_G(H) = H$.*

Lemma 2.2 ([6])**.** *If G has an abelian maximal subgroup, then G is solvable.*

Lemma 2.3 ([7])**.** *Let G be an unsolvable group. If every second maximal subgroup of G is nilpotent, then $G \cong A_5$ or $SL(2,5)$.*

Lemma 2.4 ([4])**.** *If G has at most two conjugacy classes of maximal subgroups, then G is solvable.*

3. Proofs

Proof of Theorem 1.1. Let $P \in \mathrm{Syl}_p(G)$ for some $p \in \pi(G)$. Assume that P is normal in G. Consider the group G/P. Let M_1/P and M_2/P are non-normal non-abelian maximal subgroups of G/P. Then M_1 and M_2 are non-normal non-abelian maximal subgroups of G. By the hypothesis, there exists $g \in G$, such that $M_1^g = M_2$. Thus, $(M_1/P)^{\bar{g}} = M_2/P$, that is, all non-normal non-abelian maximal subgroups of G/P are conjugate. By induction, G/P is solvable, and so G is solvable.

Now, assume that all Sylow subgroups of G are not normal in G. Let $\pi(G) = \{p_1, \cdots, p_n\}$, and $P_i \in \mathrm{Syl}_{p_i}(G)$ where $1 \leq i \leq n$. Since P_i is not

normal in G, we have that the normalizer $N_G(P_i)$ is a proper subgroup of G. Let M_i be a maximal subgroup of G that contains $N_G(P_i)$. By Lemma 2.1, M_i is self-normal in G. If all $M_1, \cdots, M_n$ are not abelian, then $M_1, \cdots, M_n$ are conjugate in G. In particular, $|M_1| = \cdots = |M_n|$. This implies that $|G| = |M_1|$ and $G = M_1$, a contradiction. Thus we must have that some M_i is abelian. By Lemma 2.2, G is solvable. This completes the proof. □

Proof of Theorem 1.2. (1) Let $\{M_1, M_2, \cdots, M_n\}$ be representative system of conjugacy classes of all maximal subgroups of G. If some M_i is abelian, then, by Lemma 2.2, G is solvable and we are done. Thus we may assume that all $M_1, M_2, \cdots, M_n$ are not abelian.

Assume that all $M_1, M_2, \cdots, M_n$ are not normal in G. Thin implies that $\mathcal{T}(G) \geq \mathcal{M}(G)$, which is impossible.

Assume that all $M_1, M_2, \cdots, M_n$ are normal in G. This implies that G is nilpotent, and so G is solvable.

Assume that $M_1, \cdots, M_s$ are normal in G and $M_{s+1}, \cdots, M_n$ are not normal in G, where $1 < s < n$. Then $\mathcal{T}(G) \geq n - s$.

Assume that all $M_1, \cdots, M_s$ are not solvable. By Theorem 1.1, every M_i at least has one non-normal non-abelian maximal subgroup H_i. First, we know that all $H_1, \cdots, H_s$ are not normal in G. Second, $H_i \neq H_j$ when $i \neq j$, otherwise $H_i = H_j = M_i \cap M_j$ is normal in G, which is imposable. Now, we claim that H_i is not conjugate to H_j when $i \neq j$. In fact, if there exists $g \in G$ such that $H_i^g = H_j$, then $H_j = H_i^g \leq M_i^g = M_i$. By maximality of H_i in M_i, we have that $\langle H_i, H_j \rangle = M_i$. Similarly, $\langle H_i, H_j \rangle = M_j$. Thus $M_i = M_j$, a contradiction.

Furthermore, H_i is not conjugate to M_j, where $1 \leq i \leq s$, $s+1 \leq j \leq n$. In fact, if there exists $g \in G$ such that $H_i^g = M_j$, then $M_j = H_i^g \leq M_i^g = M_i$, a contradiction.

Now we know that $H_1, \cdots, H_s$, $M_{s+1}, \cdots, M_n$ are some representative elements of conjugacy classes of non-normal non-abelian subgroups of G. Then $\mathcal{T}(G) \geq n$, which contradicts the hypothesis.

Thus we must have that some M_i is solvable. By induction, G/M_i is solvable, so G is solvable.

(2) By Example 1.1, it is clear that if G is isomorphic to A_5, then $\mathcal{T}(G) = \mathcal{M}(G)$.

Conversely, assume that G is a non-abelian simple group with $\mathcal{T}(G) = \mathcal{M}(G)$. Let $\{M_1, M_2, \cdots, M_n\}$ be representative system of conjugacy classes of all maximal subgroups of G. Since G is a non-abelian simple group, by Lemma 2.2, we have that all $M_1, M_2, \cdots, M_n$ are non-abelian non-normal. Thus, by the hypothesis, every proper subgroup of M_i is abelian or normal for all $1 \leq i \leq n$. Since G is a non-abelian simple group, it follows that every subgroup of M_i is abelian. Thus G is isomorphic to A_5 by Lemma 2.3, as desired. □

Proof of Theorem 1.3. (1) If G is unsolvable, then $\mathcal{T}(G) \geq \mathcal{M}(G)$ by Theorem 1.2(1). Then $\mathcal{M}(G) \leq 2$. It follows from the Lemma 2.4 that G is solvable, a contradiction. Thus, G is solvable, as desired.

(2) By Example 1.1, it is clear that if G is isomorphic to A_5, then $\mathcal{T}(G) = 3$.

Conversely, assume that G is a non-abelian simple group with $\mathcal{T}(G) = 3$. By Theorem 1.2(1) and Lemma 2.4, we have that $\mathcal{T}(G) = \mathcal{M}(G)$. By Theorem 1.2(2), G is isomorphic to A_5, as desired. □

Acknowledgements

The authors wish to express sincerest thanks to the referee for many helpful comments on this paper.

W. Kang is supported by Guangxi Natural Science Foundation Program (2020GXNSFAA238045) and the Guangxi Basic Ability Promotion Project for Young and Middle-aged Teachers (2021KY1597); J. Lu is supported by National Natural Science Foundation of China (11861015), Training Program for 1,000 Young and Middle-aged Cadre Teachers in Universities of Guangxi Province and Guangxi Basic Ability Promotion Project for Young and Middle-aged Teachers (2020KY02019); W. Meng is supported by National Natural Science Foundation of China (11761079) and Guangxi Colleges and Universities Key Laboratory of Data Analysis and Computation.

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Accepted: May 28,2020

Rough structure on groups

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Abstract. In this paper, we introduce the notion of rough mapping between two approximation spaces, then rough structure on a group. Many properties have been investigated and examples have been given to illustrate our study. In accordance with our definitions, we show that left and right translations are rough isomorphisms. As a second topic treated is the behavior of the family of lower and upper approximations of a set and of a subgroup with respect to a family of increasing equivalence relations defined on the approximation space.

Keywords: rough sets, rough isomorphism, rough groups, rough subgroups, finer approximation.

1. Introduction

Rough sets and their various aspects represent an efficient mathematical approach to study data presenting some degree of uncertainty. They are used in many branches of science as imagery in medicine, crystallography in chemistry. All the concepts of rough set theory raised from the fabulous work initiated in 1982 by Pawlak (see [10]). During the last decades, mathematical approaches have been combined with the rough set theory. Algebraic structures of rough sets have been well defined and have been the subject of many studies. For example, Bonikowaski [2], Iwinski [3], and Pomykala and Pomykala [11]. In their work Biswas and Nanda [1], have extended the rough set concepts to groups. This work has been improved by Miao et al. [9], where they studied many properties in a general frame. For their contribution, Kuroki and Wang [6] have been focused on rough groups endowed with the relation modulo a normal subgroup. In this paper we introduce a notion of rough subgroup, that is a subset of a group G such that its lower and upper approximations are subgroups of G. We also define the notion of a rough mapping and show that the left and right translations are rough mappings. We have also been concerned by the best approximation of a rough subgroup modulo an increasing family of equivalence relations. Some examples have been introduced to illustrate our approach.

2. Preliminaries

In the present section, we list some classical and well known definitions and results related to rough sets and rough groups.

Definition 2.1 ([1, 12])**.** *A finite and non-empty set U endowed with an equivalence relation $\mathcal{R}$ is called an approximation space. In the sequel $\overline{x}$ will denote the equivalence class modulo the relation $\mathcal{R}$ of the element $x \in U$.*

Definition 2.2 ([1, 12])**.** *Let $(U, \mathcal{R})$ be an approximation space and X be a subset of U. The sets*

1. $\underline{X} = \{x | \overline{x} \subset X\}$,
2. $\overline{X} = \{x | \overline{x} \cap X \neq \emptyset\}$,

are called respectively, the lower approximation and the upper approximation of X in $(U, \mathcal{R})$.

Proposition 2.1 ([1, 12])**.** *Let $X, Y \subset U$, where U is a universe endowed with the equivalence relation $\mathcal{R}$. Then, the lower and the upper approximations have the following properties:*

1. $\underline{X} \subset X \subset \overline{X}$,
2. $\underline{\emptyset} = \overline{\emptyset} = \emptyset, \underline{U} = \overline{U} = U$,
3. $\underline{X} \cap \underline{Y} = \underline{X \cap Y}$,
4. $\overline{X} \cap \overline{Y} \supset \overline{X \cap Y}$,
5. $\underline{X} \cup \underline{Y} \subset \underline{X \cup Y}$,
6. $\overline{X} \cup \overline{Y} = \overline{X \cup Y}$,
7. $X \subset Y \Longrightarrow \underline{X} \subset \underline{Y}$ *and* $\overline{X} \subset \overline{Y}$.

3. Main results

Definition 3.1. *If $(U, \mathcal{R}), (U', \mathcal{R}')$ are two approximation spaces. A mapping $\psi : U \to U'$ is called semi-surjective on the equivalence classes if:*

$$y \in \psi(U) \Longrightarrow \forall y' \in \overline{y}, y' \in \psi(U).$$

Definition 3.2. *Let $(U, \mathcal{R}), (U', \mathcal{R}')$ be two approximation spaces and $\psi : U \to U'$ be a mapping. The relations $\mathcal{R}, \mathcal{R}'$ are said to be $\psi-$congruent if:*

$$x \mathcal{R} x' \Longleftrightarrow \psi(x) \mathcal{R}' \psi(x').$$

Proposition 3.1. *Let $(U,\mathcal{R}),(U',\mathcal{R}')$ be two approximation spaces and $\psi : U \to U'$ be an injective mapping. If ψ is semi-surjective on the equivalence classes and $\mathcal{R},\mathcal{R}'$ are $\psi-$congruent, then for any subset $H \subset U$, we have*

1. $\psi(\underline{H}) = \underline{\psi(H)}$,
2. $\psi(\overline{H}) = \overline{\psi(H)}$.

Proof. 1. (a) If $y \in \psi(\underline{H})$, then $\exists x \in \underline{H}$ such that $y = \psi(x)$. We have to show that $\overline{y} \subset \psi(H)$.
Let $y' \in \overline{y}$ since $y = \psi(x)$ and by the hypothesis, $\exists x' \in \overline{x}$ such that $y' = \psi(x')$. As $\overline{x} \subset H$, we get $y' = \psi(x') \in \psi(H)$ and consequently $\overline{y} \subset \psi(H)$.

(b) Now if $y \in \underline{\psi(H)}$ then $\overline{y} \subset \psi(H)$. As $y \in \overline{y} \subset \psi(H)$, there exists $x \in H$ such that $y = \psi(x)$. Let $x' \in \overline{x}$ then as $x'\mathcal{R}x$ we also have $\psi(x')\mathcal{R}'\psi(x) = y$ and $\psi(x') \in \overline{y} \subset \psi(H)$. As ψ is injective, we deduce that $x' \in H$. Finally $\overline{x} \subset H$ with $y = \psi(x)$.

2. (a) If $y \in \psi(\overline{H})$ then $\exists x \in \overline{H}$ such that $y = \psi(x)$. This last assumption is equivalent to

$$\exists x \in X | \overline{x} \cap H \neq \emptyset \text{and} \psi(x) = y.$$

We have to show that $\exists y' \in \overline{y} \text{and} \exists x' \in H \text{such that} y' = \psi(x')$.
As $\overline{x} \cap H \neq \emptyset$, then there exists $x' \in H$ such that $x' \in \overline{x}$. Let $y' \in \psi(x') \in \psi(H)$. Since $x'\mathcal{R}x$ then $y' = \psi(x')\mathcal{R}'\psi(x) = y$ and consequently $y' \in \overline{y}$ and then $\overline{y} \cap \psi(H) \neq \emptyset$.

(b) Assume now that $y \in \overline{\psi(H)}$, then $\overline{y} \cap \psi(H) \neq \emptyset$, that is, there exists $y' \in \overline{y}, \exists x' \in H$ such that $y' = \psi(x')$. We should show that $\exists x \in \overline{H}$ such that $\psi(x) = y$.
Since ψ is semi-surjective on the equivalence classes and $\mathcal{R},\mathcal{R}'$ are $\psi-$congruent; there exists $x \in \overline{x'}$ such that $y = \psi(x)$. On the other hand as $x' \in H, \overline{x} = \overline{x'}$ then $x' \in H \cap \overline{x'} = H \cap \overline{x}$ so $x \in \overline{H}$ and as $\psi(x) = y$ the conclusion follows.

□

Example 3.1. Let $U = U' = \mathbb{Z}$ and $\mathcal{R},\mathcal{R}'$ be respectively the congruences $\equiv [2]$ and $\equiv [6]$ on $\mathbb{Z}$. The mapping

$$\psi : \mathbb{Z} \to \mathbb{Z}, n \longmapsto 3n$$

is injective, semi-surjective and $\mathcal{R},\mathcal{R}'$-congruent.
If $H = 3\mathbb{Z}$, then $\psi(\underline{H}) = \underline{\psi(H)} = \psi(\overline{H}) = \overline{\psi(H)} = \overline{0}$.

Proposition 3.2. *Let $(U,\mathcal{R}),(U',\mathcal{R}')$ be two approximation spaces, $\psi : U \to U'$ be an injective mapping and H be a subset of U.*

$$\underline{H} \subsetneq H \subsetneq \overline{H} \Longleftrightarrow \psi(\underline{H}) \subsetneq \psi(H) \subsetneq \psi(\overline{H}).$$

Proof. 1. $\Longrightarrow$:

Suppose that $\underline{H} \subsetneq H \subsetneq \overline{H}$ but $\psi(\underline{H}) = \psi(H) = \psi(\overline{H})$. As the mapping ψ is injective, we have $\underline{H} = \psi^{-1}(\psi(\underline{H})) = \psi^{-1}(\psi(H)) = H$ and $H = \psi^{-1}(\psi(\overline{H})) = \overline{H}$. We get a contradiction.

2. $\Longleftarrow$:

By contradiction, the proof is straightforward.

□

Corollary 3.1. *Let $(U, \mathcal{R}), (U', \mathcal{R}')$ be two approximation spaces, $\psi : U \to U'$ be an injective mapping and H be a subset of U.*

1. *If $x \in H$ is such $\overline{x} \subsetneq H$, then there exists $y \in H$ such that $\overline{y} \subsetneq \psi(H)$.*
2. *If $x \in \overline{H}$ is such $x \notin H$, then there exists $y \in \psi(\overline{H})$ such that $y \notin \psi(H)$.*

Definition 3.3. *Let $(U, \mathcal{R}), (Y, \mathcal{R}')$ be two approximation spaces. $\psi : U \to Y$. f is called rough mapping if*

1. *$\forall K \subset Y; \psi^{-1}(\underline{K}) = \underline{\psi^{-1}(K)}$,*
2. *$\forall K \subset Y; \psi^{-1}(\overline{K}) = \overline{\psi^{-1}(K)}$.*

Examples of rough mappings are given by Proposition 3.6.

Proposition 3.3. *If $(U, \mathcal{R}), (U', \mathcal{R}')$ and $(U'', \mathcal{R}'')$ are three approximation spaces and $\psi : U \to U', \phi : U' \to U''$ are two rough mapping, then $g \circ f$ is also a rough mapping.*

Proof. Let H be a subset of U", then

$$(\phi \circ \psi)^{-1}(\underline{H}) = \psi^{-1}(\phi^{-1}(\underline{H})) = \psi^{-1}(\underline{\phi^{-1}(H)}) = \underline{\psi^{-1}(\phi^{-1}(H))} = \underline{(\phi \circ \psi)^{-1}(H)}.$$

If H be a subset of U", then $(\phi \circ \psi)^{-1}(\overline{H}) = \psi^{-1}(\phi^{-1}(\overline{H})) = \psi^{-1}(\overline{\phi^{-1}(H)}) = \overline{\psi^{-1}(\phi^{-1}(H))} = \overline{(\phi \circ \psi)^{-1}(H)}$. □

Definition 3.4. *Let $(U, \mathcal{R}), (U', \mathcal{R}')$ be two approximation spaces. $\psi : U \to U'$. ψ is called rough isomorphism if*

1. *ψ is bijective,*
2. *ψ and ψ^{-1} are rough mappings.*

Proposition 3.4. *Let $(U, \mathcal{R})$ and $(U', \mathcal{R}')$ be two approximation spaces. If $\psi : U \to U'$ is surjective and $\mathcal{R}, \mathcal{R}'$ are $\psi-$congruent, then ψ is a rough mapping.*

Proof. Let H be a subset of U'.

1. Let us prove that $\psi^{-1}(\underline{H}) = \underline{\psi^{-1}(H)}$.

(a) Let $x \in \psi^{-1}(\underline{H})$, then there exists $y \in \underline{H}$ such that $y = \psi(x)$. Let x' be an element of $\overline{x}$. As $x'\mathcal{R}x$, we have $\psi(x')\mathcal{R}'\psi(x) = y$. But $\overline{y} \subset H$, so $x' \in \psi^{-1}(H)$ and then $\overline{x} \subset \psi^{-1}(H)$. The inclusion $\psi^{-1}(\underline{H}) \subset \underline{\psi^{-1}(H)}$ follows.

(b) Let $x \in \underline{\psi^{-1}(H)}$, then $x \in \overline{x} \subset \psi^{-1}(H)$. So there exists $y \in H$ such that $y = \psi(x)$. Let us prove that $\overline{y} \subset H$. Let $y' \in \overline{y}$. As ψ is surjective there exists $x' \in U$ such that $\psi(x') = y'$. As $\mathcal{R}, \mathcal{R}'$ are $\psi-$congruent, $y\mathcal{R}'y'$, necessary $x'\mathcal{R}x$. As $\overline{x} \subset \psi^{-1}(H)$, then $x' \in \psi^{-1}(H)$ and finally $y' \in H$, which implies that $\overline{y} \subset H$.

2. Let us prove that $\psi^{-1}(\overline{H}) = \overline{\psi^{-1}(H)}$.

(a) Let $x \in \psi^{-1}(\overline{H})$, so there exists $y \in \overline{H}$ such that $y = \psi(x)$. Let $y' \in \overline{y} \cap H$, as ψ is surjective there exists $x' \in U$ such that $\psi(x') = y'$. But since $y' \in H$; $x' \in \psi^{-1}(H)$ and then $\overline{x} \cap \psi^{-1}(H) \neq \emptyset$.

(b) Now suppose that $x \in \overline{\psi^{-1}(H)}$. We can deduce from $\overline{x} \cap \psi^{-1}(H) \neq \emptyset$ that $\exists x' \in \overline{x}$ such that $y' = \psi(x') \in \psi(H)$. Let $y = \psi(x)$. As $x'\mathcal{R}x$, we have $y'\mathcal{R}'y$ and since $y' = \psi(x') \in H$, we finally have $\overline{y} \cap H \neq \emptyset$ and $x \in \psi^{-1}(\overline{H})$.

□

Definition 3.5. *A group $(G, .)$ endowed with an equivalence relation $\mathcal{R}$ is called a rough group if the following mappings are rough mappings*

$$\psi : G \to G; x \longmapsto x^{-1},$$
$$\phi : G \times G \to G; (x, y) \longmapsto x.y$$

The group $G \times G$ is endowed with the relation product.

Proposition 3.5. *The group $(\mathbb{Z}, +)$ endowed with the equivalence relation $\equiv [n]$ for some $n \in \mathbb{N}$ is a rough group.*

Proof. Let H be a subset of $\mathbb{Z}$.

1. It is easy to show that $x \in \psi^{-1}(\underline{H}) \Longleftrightarrow -x+n\mathbb{Z} \subset H$ and $x \in \underline{\psi^{-1}(H)} \Longleftrightarrow -x + n\mathbb{Z} \subset H$. So
$$\psi^{-1}(\underline{H}) = \underline{\psi^{-1}(H)}.$$

2. It is also easy to show that $x \in \psi^{-1}(\overline{H}) \Longleftrightarrow \exists k \in \mathbb{Z}$, such that $-x + kn \in H$ and $x \in \overline{\psi^{-1}(H)} \Longleftrightarrow \exists k \in \mathbb{Z}$such that $-x + kn \in H$. So
$$\psi^{-1}(\overline{H}) = \overline{\psi^{-1}(H)}.$$

□

Proposition 3.6. *If $(G,.)$ is a group endowed with a congruence relation $\mathcal{R}$, then the left translation;*

$$\forall a \in G; \mathcal{T}_{a,l} : G :\to G, x \longmapsto \mathcal{T}_{a,l}(x) = ax,$$

and the right translation;

$$\forall a \in G; \mathcal{T}_{a,r} : G :\to G, x \longmapsto \mathcal{T}_{a,r}(x) = xa,$$

are rough mappings.

Proof. Let H be a subset of G. We will just give the proof for $\mathcal{T}_{a,l}$, the proof for $\mathcal{T}_{a,r}$ is similar.

1. (a) We have $x \in \mathcal{T}_{a,l}^{-1}(\underline{H}) \Longleftrightarrow \mathcal{T}_{a,l}(x) = \overline{ax} \subset H$. Now let $x' \in \overline{x}$, then $ax'\mathcal{R}ax$. If we set $y' = ax', y = ax$, then we have $y' \in \overline{y} \subset H$ with $y' = \mathcal{T}_{a,l}(x')$. Finally $x' \in \mathcal{T}_{a,l}^{-1}(H)$ implies $\overline{x} \subset \mathcal{T}_{a,l}^{-1}(H)$ and the conclusion follows.
   (b) Conversely let $x \in \underline{\mathcal{T}_{a,l}^{-1}(H)}$, then $\overline{x} \subset \mathcal{T}_{a,l}^{-1}(H)$. So for any $x' \in \overline{x}$, we have $ax' \in H$. If $y = ax$, we have to show that $\forall y' \in \overline{y}; y' \in H$. The element $x' = a^{-1}y' \in G$ verifies $x'\mathcal{R}(a^{-1}y) = x$, so $y' = ax' \in H$ and $\overline{y} \subset H$.

2. (a) Now, if $x \in \mathcal{T}_{a,l}^{-1}(\overline{H})$, then $\overline{y} \cap H \neq \emptyset$, where $y = ax$. Now let $y' \in \overline{y}$ and $x' = a^{-1}y'$ Since $y'\mathcal{R}y$; then $x' = a^{-1}y'\mathcal{R}a^{-1}y = x$. But $T_{a,l}(x') = y' \in H$. Consequently; we have found $x' \in \overline{x}$ such that $T_{a,l}(x') \in H$ and $\overline{x} \cap \mathcal{T}_{a,l}^{-1}(H) \neq \emptyset$.
   (b) Let $x \in \overline{\mathcal{T}_{a,l}^{-1}(H)}$, then there exists $x' \in x$ such that $ax' \in H$. If $y' = ax'$ and since $x'\mathcal{R}x$, we have $y'\mathcal{R}y$. Consequently, there exists $y' \in \overline{y} = \overline{ax}$ such that $y' \in H$ which implies that $x \in \mathcal{T}_{a,l}^{-1}(\overline{H})$

□

Proposition 3.7. *If $(G,.)$ is a group endowed with a congruence relation $\mathcal{R}$, then the left and the right translations are rough isomorphisms. That is the lower and the upper approximations of a subset of a group are invariant by the right and the left translations.*

Proof. According to the above proposition and to the fact that the right and the left translations are bijective, it suffices to show that $\mathcal{T}_{a,l}(\underline{H}) = \underline{\mathcal{T}_{a,l}(H)}$ and $\mathcal{T}_{a,l}(\overline{H}) = \overline{\mathcal{T}_{a,l}(H)}$.

Let $y \in \mathcal{T}_{a,l}(\underline{H})$, then there exists $x \in \underline{H}$ such that $y = ax$. For $y' \in \overline{y}$; $x' = a^{-1}y' \in \overline{x} \subset H$. So $y' = \mathcal{T}_{a,l}(x') \in \mathcal{T}_{a,l}(H)$ and $\overline{y} \subset \mathcal{T}_{a,l}(H)$.

Now let $y \in \underline{\mathcal{T}_{a,l}(H)}$, then $y \in \overline{y} \subset \mathcal{T}_{a,l}(H) \Longrightarrow \exists x \in H$ such that $y = ax$. Let $x' \in \overline{x}$ and $y' \overline{= ax'}$. We then have $y' \in \overline{y} \subset \mathcal{T}_{a,l}(H)$ and since $\mathcal{T}_{a,l}$ is injective we get $x' \in H$ and finally $\overline{x} \subset H$.

By similar arguments, we can prove the others assumptions. □

Lemma 3.1. *Let H be a subgroup of G and $\mathcal{R}$ be the left equivalence relation modulo the subgroup H on G, then $\underline{H} = H = \overline{H}$.*

Proof. The proof is straightforward. □

Proposition 3.8. *Let H be a subgroup of G and $\mathcal{R}$ be the left equivalence relation modulo the subgroup H on G. Any non empty subset K of H is such that:*

$$\underline{K} = \emptyset \text{ and } \overline{K} = H \text{ or } \underline{K} = H = \overline{K}.$$

Proof. Suppose that $\underline{K} \neq \emptyset$, then there exists $x \in G$ such that $\overline{x} \subset K \subset H$. But the only equivalence class contained in H is H itself, so $H = \overline{x} \subset K \subset H$. The conclusion follows from the lemma.

Now suppose that $\underline{K} = \emptyset$ and let $x \in \overline{K}$, then $\overline{x} \cap K \neq \emptyset$ and then $\overline{x} \cap H \neq \emptyset$. It follows that $\overline{x} = H$ and $x \in H$ which implies that $\overline{K} \subset H$. □

Definition 3.6. *Let H be a subgroup of G, K be a subset of G and $\mathcal{R}$ be the left equivalence relation modulo the subgroup H on G. We say that K is $H-$closed or crisp or exact with respect to $\mathcal{R}$ if:*

$$\forall x \in K, \overline{x} \subset K.$$

Proposition 3.9. *Let G be a group, H be a subgroup and K a superset of H. If $\mathcal{R}$ is the left equivalence relation modulo the subgroup H on G, then $\underline{K} = K = \overline{K}$ (i.e. any superset of H is a crisp set).*

Proof.

1. Let $x \in K$ as K is $H-$closed, $\overline{x} \subset K$ and then $x \in \underline{K}$.
2. Let $x \in \overline{K}$, then there exists $x' \in \overline{x}$ which is also in K. Consequently $\overline{x'} \subset K$ but $\overline{x'} = \overline{x}$ so $x \in \overline{x} \subset K$ and the conclusion follows.

□

The following definition introduces another notion of rough subgroups of a group G.

Definition 3.7. *Let $(G, .)$ be a group (respectively a semigroup) and $(G, \mathcal{R})$ be an approximation space. A subset H of universe G is called a rough subgroup (respectively quasi-rough subgroup) if $\underline{H}$ and $\overline{H}$ are subgroups of G (respectively $\underline{H}$ or $\overline{H}$ are subsemigroups of G).*

Remark 3.1. Any rough subgroup H is not empty, it contains at least the identity e of G.

Example 3.2.

1. The following example illustrate that the concept defined above is well defined.

   Let $G = (\mathbb{Z}/8\mathbb{Z}, +)$, H be the set $H = \{0, 4, 6\}$ and the relation $\mathcal{R}$ defined on G by

   $$\mathcal{R} = \{(0,0), ..., (7,7), (0,4), (4,0), (2,6), (6,2)\}.$$

   It is clear that $\mathcal{R}$ is an equivalence relation. Indeed we have $\underline{H} = \{0, 4\}$ and $\overline{H} = \{0, 2, 4, 6\}$ and the two sets are subgroups of G.

2. Let G be the additive group $\mathbb{Z}$, the relation $\mathcal{R}$ be the equivalence relation modulo the subgroup $8\mathbb{Z}$. If $H = \overline{0} \cup \{3, 5\}$, then $\underline{H} = \overline{0} = 8\mathbb{Z}$ is a subgroup so H is a quasi-rough subgroup of $\mathbb{Z}$. Indeed H is not a rough subgroup since $\overline{H} = \overline{0} \cup \overline{3} \cup \overline{5}$ is not a subgroup of $\mathbb{Z}$.

3. More generally if the group $\mathbb{Z}$ is endowed with any equivalence relation modulo a subgroup K of $\mathbb{Z}$ and H is a finite subset of $\mathbb{Z}$, then

   (a) $\underline{H} = \emptyset$, indeed suppose that H is not empty then there exists $n \in \mathbb{Z}$ such that $n \in H$ and then $\overline{n} \cap H \neq \emptyset$. Now if $m \in \overline{n}, n \in \overline{m} \cap H$, which implies that $m \in \underline{H}$ and consequently any element of $\overline{n}$ is also in $\underline{H}$. As $\overline{n}$ is infinite and $\underline{H}$ is a subset of H, H is necessarily infinite contradiction.

   (b) $\overline{H} = \bigcup_{n \in H} \overline{n}$.

      i. Indeed, let $n \in H$ and $m \in \overline{n}$ then $n \in \overline{m} \cap H \neq \emptyset$ so $m \in \overline{H}$ and then $\overline{n} \subset \overline{H}$ and $\overline{H} \supset \bigcup_{n \in H} \overline{n}$.

      ii. Let $n \in \overline{H}$, then $\overline{n} \cap H \neq \emptyset$, then there exists $m \in \overline{n}$ such that $m \in H$. But $n \in \overline{m}$ and finally $n \in \overline{m} \subset \bigcup_{a \in H} \overline{a}$ since $m \in H$ and consequently $\overline{H} \subset \bigcup_{n \in H} \overline{n}$.

Remarks 3.1. 1. When the universe is infinite the notion of approximation can be superfluous, since as in the following example, two different sets can have the same approximations and then they are indistinguishable.

Let G be the group $\mathbb{Z}, \mathcal{R}$ be the congruence modulo 5. If $H_1 = \{0, 1, 2, 3, 4\}$, $H_2 = \{3, 4, 5, 6, 7\}$, then

$$\underline{H_1} = \underline{H_2} = \emptyset, \overline{H_1} = \overline{H_2} = \mathbb{Z}.$$

The sets are different but have the same lower and upper approximations. We think that this observation is one of the reasons that can justify the term of roughness approximation.

2. Let H be a strictly rough subgroup (i.e. H is not a subgroup and $\underline{H} \subsetneq H \subsetneq \overline{H}$), then

   (a) The elements of $H \setminus \underline{H}$ are those presenting some anomalous. More precisely if $x \in H \setminus \underline{H}$ then $x^{-1} \notin \underline{H}$, otherwise as $\underline{H}$ is s subgroup x should be in $\underline{H}$ contradiction. In addition for any $y \in \underline{H}$, $x.y \notin \underline{H}$ and $y.x \notin \underline{H}$.

   (b) Also the elements of $\overline{H} \setminus H$ are those presenting some anomalous. More exactly If $x \in \overline{H} \setminus H$, there exists an element $y \in H$ such that $xy, yx \notin H$. Indeed, let e be the neutral element of the group G and suppose that for all $y \in H, xy \in H$, then as $H \supset \underline{H} \ni e = x^{-1}x, x = x(x^{-1}x) \in H$, contradiction. The same argument can show that $yx \notin H$.

Definition 3.8. *Let G be a group and $(G,\mathcal{R})$ be an approximation space. A subset H of universe G is called closed with respect to $\mathcal{R}$ if $x \in H \Longrightarrow \overline{x} \subset H$.*

Proposition 3.10. *Let $(G,.)$ be a group and $(G,\mathcal{R})$ be an approximation space. A rough subgroup H of universe G is subgroup if it is closed with respect to $\mathcal{R}$.*

Proof. It suffices to note that H closed with respect to $\mathcal{R}$, implies $\underline{H} = H = \overline{H}$ and the result follows. □

Proposition 3.11. *Let $(G,.)$ be a group, K a subgroup of G and $\mathcal{R}$ be the left relation modulo the subgroup K. A rough subgroup H of universe G is subgroup of H if and only if it is closed with respect to $\mathcal{R}$.*

Proof.
1. Suppose that H is a subgroup. Since $\underline{H}$ is a subgroup, then $e \in \underline{H}$ which implies $K = \overline{e} \subset H$. Let $x \in H$, then $\overline{x} = xK \subset H$ since H is a subgroup and finally H is closed with respect to $\mathcal{R}$.
2. The converse follows from the above proposition.

□

Proposition 3.12. *Let $(G,.)$ be a group, K a subgroup of G and $\mathcal{R}$ be the left relation modulo the subgroup K. If H is a subgroup of G and $e \in \underline{H}$, then $\underline{H} = H$.*

Proof. It suffices to show that $H \subset \underline{H}$.

Suppose that H is a subgroup and $x \in H$. Since $e \in \underline{H}$ then $K = \overline{e} \subset H$. As H is a subgroup, the closure with respect to the product of G, implies that for all $k \in K, xk \in H$. So $\overline{x} = xK \subset H$ and $x \in \underline{H}$.

□

Proposition 3.13. *If H, K are rough subgroups of G such that*

$$(A) : (\overline{x} \cap H \neq \emptyset \ \text{ and } \ \overline{x} \cap K \neq \emptyset) \Longrightarrow (\overline{x} \cap (H \cap K) \neq \emptyset),$$

then $H \cap K$ is also a rough subgroup of G.

Proof. Let us prove that the condition (A) implies that $\overline{H} \cap \overline{K} = \overline{H \cap K}$. So suppose that $x \in \overline{H} \cap \overline{K}$ then $\overline{x} \cap H \neq \emptyset$ and $\overline{x} \cap K \neq \emptyset$. By our assumption we then have $\overline{x} \cap (H \cap K) \neq \emptyset$ and finally $x \in \overline{H \cap K}$. From the fourth result of Proposition 2.4, we get the equality of the sets.

On the other hand $\underline{H \cap K} = \underline{H} \cap \underline{K} \subset H \cap K \subset \overline{H} \cap \overline{K} = \overline{H \cap K}$. As the intersection of subgroups is a subgroup, we can conclude that $\underline{H} \cap \underline{K}$ and $\overline{H} \cap \overline{K}$ are subgroups and the result follows.

□

Proposition 3.14. *Let H be a rough subgroup of G. If K is a subgroup closed with respect to $\mathcal{R}$, then $H \cap K$ is a rough subgroup of G.*

Proof. From the equality $\underline{K} = K = \overline{K}$, it follows that $\underline{H} \cap \underline{K}$ and $\overline{H} \cap \overline{K}$ are subgroups of G. First note that $\underline{H \cap K} = \underline{H} \cap \underline{K} \subset H \cap K \subset \overline{H} \cap \overline{K} \supset \overline{H \cap K}$ (by Proposition 2.4).

Now suppose that $x \in \overline{H} \cap \overline{K}$ then $\overline{x} \cap H \neq \emptyset$ and $\overline{x} \cap K \neq \emptyset$. From $\overline{x} \cap K \neq \emptyset$, we deduce that $\overline{x} \subset K$ and from $\overline{x} \cap H \neq \emptyset$, we finally deduce that $\overline{x} \cap (H \cap K) \neq \emptyset$ and then $x \in \overline{H \cap K}$.

□

Proposition 3.15. *Let G be a group of order a prime p. If H is a rough subgroup and not a subgroup then $\underline{H} = \{e\}$ and $\overline{H} = G$.*

Proof. Since $|G| = p$ is prime the only subgroups of G are the trivial subgroups $\{e\}$ and G. Since H is a rough subgroup but not a subgroup, the only possible configuration is $\{e\} \subsetneq H \subsetneq G$, so $\underline{H} = \{e\}$ and $\overline{H} = G$. □

Definition 3.9. *A relation $\mathcal{R}$ is less than a relation $\mathcal{R}'$ denoted by $\mathcal{R} \leq \mathcal{R}'$ if*

$$(x, y) \in \mathcal{R} \Longrightarrow (x, y) \in \mathcal{R}'.$$

Proposition 3.16. *Let G be a group, $\mathcal{R}, \mathcal{R}'$ two equivalence relations on G such that $\mathcal{R} \leq \mathcal{R}'$. If H is a rough subgroup with respect to both $\mathcal{R}$ and $\mathcal{R}'$, then*

$$\underline{\underline{H}} \subset \underline{H} \subset H \subset \overline{H} \subset \overline{\overline{H}},$$

where $\underline{H}, \overline{H}$ are the subgroups defined by $\mathcal{R}$ and $\underline{\underline{H}}, \overline{\overline{H}}$ are those defined with respect to $\mathcal{R}'$.

Proof. The proposition follows from the inclusion $\overline{x} \subset \widehat{x} where\ \overline{x}$ is the class of x modulo $\mathcal{R}$ and $\widehat{x}$ is the class of x modulo $\mathcal{R}'$. More exactly suppose that $x \in \underline{\underline{H}}$ then $\widehat{x} \subset H$. As $\overline{x} \subset \widehat{x}$, we can conclude that $x \in \underline{H}$.

If $x \in \overline{H}$ then $\overline{x} \cap H \neq \emptyset$, but as $\overline{x} \subset \widehat{x}$ we deduce that $\widehat{x} \cap H \neq \emptyset$ and then $x \in \overline{\overline{H}}$. □

Corollary 3.2. *Let $(\mathcal{R}_i)_{i \in I}$ be a decreasing sequence of equivalence relations, H a rough subgroup of a group G with respect to each equivalence relation $\mathcal{R}_i$. If we set $(\underline{H_i}, \overline{H_i})$ the couple of subgroups associated to H with respect to $\mathcal{R}_i$. The sequence $(\underline{H_i})_{i \in I}$ is increasing and the sequence $(\overline{H_i})_{i \in I}$ is decreasing. For any $x \in G$, the family $(\overline{x}^{\mathcal{R}_i})_{i \in I}$ is decreasing, where $\overline{x}^{\mathcal{R}_i}$ is the equivalence class of x with respect to $\mathcal{R}_i$.*

Proof. It suffices to apply the above proposition. □

At this stage we remark that if we approximate the rough subgroup H by the sequence of lower and upper subgroups $\underline{H_i}, \overline{H_i}$ associated to a decreasing sequence of equivalence relations, we obtain more information on H (i.e. the equivalence classes will form a decreasing sequence of sets and then give more precision on the boundary of H). As we will prove in the following proposition, $(\bigcup_i \underline{H_i}, \bigcap_i \overline{H_i})$ will be the best approximation in this meaning.

Lemma 3.2. *Any decreasing sequence $(\mathcal{R}_i)_{i\in I}$ of equivalence relations on a finite group G stops. Moreover if H a rough subgroup of a group G with respect to each equivalence relation $\mathcal{R}_i$. If we set $(\underline{H_i}, \overline{H_i})$ the couple of subgroups associated to H with respect to $\mathcal{R}_i$, then there exists $n \in \mathbb{N}^*$ such that $(\underline{H_n}, \overline{H_n})$ represents the best approximation of H with respect to the family $(\underline{H_i}, \overline{H_i})_{i\in I}$.*

Proof. In the sequel by the notation $\mathcal{R} < \mathcal{R}'$ we mean that $\mathcal{R} \leq \mathcal{R}'$ and there exists $(x, y) \in \mathcal{R}'$ such that $(x, y) \notin \mathcal{R}$. In this case we have $\overline{x}^{\mathcal{R}} \subsetneq \overline{x}^{\mathcal{R}'}$ between the equivalence classes.

Now if we suppose that the sequence $(\mathcal{R}_i)_{i\in I}$ doesn't stop, the decreasing sequence $\overline{x}^{\mathcal{R}_i}$ is infinite which is in contradiction with G is a finite group. It is easy to finish the proof. □

Lemma 3.3. *Let $(\mathcal{R}_i)_{i\in I}$ be a decreasing sequence of equivalence relations, H a rough subgroup of a group G with respect to each equivalence relation $\mathcal{R}_i$. If we set $(\underline{H_i}, \overline{H_i})$ the couple of subgroups associated to H with respect to $\mathcal{R}_i$. The relations $\mathcal{R}$ and $\mathcal{R}'$ defined by $(x\mathcal{R}y \iff \exists i \in I | x\mathcal{R}_i y)$ and $(x\mathcal{R}'y \iff \forall i \in I | x\mathcal{R}_i y)$ are equivalence relations on G.*

Proof.
1. It is easy to prove that $\mathcal{R}$ is both reflexive and symmetric on G. For the transitivity, if $x\mathcal{R}_i y$ and $y\mathcal{R}_j z$, then for $k = min\{i, j\}$ we have $x\mathcal{R}_k z$ and the first assumption follows.
2. For the relation $\mathcal{R}'$ it is easy to check that it is an equivalence relation.

□

Proposition 3.17. *Let $\mathcal{R}_1, \mathcal{R}_2$ be two equivalence relations on a finite group G and H be a rough subgroup with respect to each relation. We then have the following assumptions.*

1. *$\mathcal{R} = \mathcal{R}_1 \cap \mathcal{R}_2$ is an equivalence relation on G.*
2. *If $\underline{H}, \underline{H_1}, \underline{H_2}$ (resp. $\overline{H}, \overline{H_1}, \overline{H_2}$) represent the lower (resp. the upper) approximations of the set H with respect to $\mathcal{R}, \mathcal{R}_1$ and $\mathcal{R}_2$, then*

$$\underline{H} \supset \underline{H_1} \cap \underline{H_2}$$

and

$$\overline{H} \subset \overline{H_1} \cap \overline{H_2}.$$

Proof.
1. The first point is straightforward, however, it is clear that for any $x \in G$; $\overline{x} = \overline{x}^1 \cap \overline{x}^2$, where $\overline{x}, \overline{x}^1$ and $\overline{x}^2$ denote the equivalence classes with respect to $\mathcal{R}, \mathcal{R}_1$ and $\mathcal{R}_2$ respectively.
2. Let $x \in \underline{H_1} \cap \underline{H_2}$, then $\overline{x}^1 \subset H$ and $\overline{x}^2 \subset H$ and consequently their intersection is also in H, that is $\overline{x}^1 \cap \overline{x}^2 \subset H$ and $\underline{H_1} \cap \underline{H_2} \subset \underline{H}$.

   Since $x \in \overline{H} \iff \overline{x} \subset H$ and since $\overline{x} = \overline{x}^1 \cap \overline{x}^2$, then there exists $x' \in \overline{x}$ such that $x' \in \overline{x}^1, x' \in H$ so $x' \in \overline{x}^1 \cap H \neq \emptyset$. But $x' \in \overline{x}^2$, so $x' \in \overline{x}^2 \cap H \neq \emptyset$ and then $(\overline{x} = \overline{x}^1 \cap \overline{x}^2) \cap H \neq \emptyset$ and x is in $\overline{H}$

□

4. Conclusion

In this paper, we have showed that under certain conditions on a mapping and relations, the direct image of the lower (resp. upper) approximation of a set is the lower (resp. upper) approximation of the direct image of the set. We have also showed that the roughness property is stable by a one-to-one mapping. As the roughness is a structural property of a set, we have introduced the notion of rough mapping, rough isomorphism and then a new notion of rough group motivated by an example. We have proved that the left and right translations on a group are rough isomorphisms. We have studied the roughness in a group when the relation is the left or the right relation modulo a subgroup H. As a consequence, we have found that the lower and the upper approximation of a subset of H are trivial. As a second item treated, we have been concerned by subsets of a group for which, lower and upper approximations are subgroups, we called them rough subgroups. We have investigated many properties of such subsets. The most important is the behavior of the sequence of these lower and upper approximations when we introduce a family of increasing equivalence relations on the group. Our future objective will be focused on families of equivalence relations on a set (on a group) to refine the family of lower and upper approximations.

Acknowledgements

The author is grateful to the referee for his valuable comments and suggestions that make this article more readable.

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Accepted: May 2, 2020

Rings whose elements are a sum of a unit regular and nilpotent

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Abstract. The concept of nil clean rings has been extended in several ways. This paper continues investigation new generalizations of such rings. We define a U-nil clean ring as the one in which every element is a sum of a nilpotent and a unit regular. A U-nil clean ring is actually an important example of the class of NR-clean rings introduced in [14]. We introduce many examples and fundamental properties of these rings and investigate the behavior of such properties under various ring extensions. For a ring R and a group G, we determine some necessary conditions on R and G such that the group ring RG is U-nil clean.

Keywords: clean ring, nil clean ring, NR-clean ring, U-nil clean ring.

1. Introduction

All rings R in this paper are assumed to be associative with 1 unless otherwise indicated. The symbols $U(R)$, $Id(R)$, $N(R)$, $J(R)$ and $ur(R)$ will stand for the group of units, the set of idempotents, the set of nilpotents, the Jacobson radical and the set of unit regular elements respectively.

In 1977, Nicholson [18] introduced the concept of a clean ring as a ring in which every element is a sum of an idempotent and a unit. He proved that this class of rings contains unit-regular rings and semiperfect rings. Moreover, Xiao and Tong [16] studied n-clean rings as a generalization of clean rings. A ring R is called n-clean if every element of R is the sum of an idempotent and n units.

Over the last twenty years there has been an explosion of interest in the class of clean rings as well as in their many generalizations and variations. In [7], Diesel introduced an interesting subclass of clean rings, the class of nil-clean rings. A ring R is called nil-clean if for every element $a \in R$, there are an idempotent e and a nilpotent b in R such that $a = e + b$.

An element r of a ring is called regular (in the sense of von Neumann) if there exists an element x such that $r = rxr$. Moreover, if every element in a ring is regular, then we call it a regular ring. The class of regular rings includes a special type of regular elements called unit regular elements. An element r in a ring R is called unit regular if $r = rur$ for some $u \in U(R)$. Equivalently, r is a unit regular if and only if $r = eu$ for some $e \in Id(R)$

and $u \in U(R)$. A ring R is called unit regular if $R = ur(R)$. Unit regular rings are examples of clean rings and play an important role in the structure theory of regular rings. In fact, they are precisely those regular rings over which the finitely generated left projective modules satisfy the cancellation law. For various properties and results of regular, unit regular, clean and nil clean rings one may see [1, 9, 18, 19, 3, 11, 7].

Hongbo and Weting [12] defined a ring to be G-clean if every element is the sum of a unit and a unit regular. More generally, Ling and Long, [16] defined R to be a UR-ring if every element is a sum of a regular element and a unit. The class of G-clean rings lies properly between the classes of clean rings and UR-rings. As an interesting variant of a UR-rings, an NR-clean ring was defined by Khashan [14] to be a ring in which every element is a sum of a regular element and nilpotent. Many properties and examples of NR-clean rings has been investigated in [14]. Moreover, the behavior of such properties under various ring extensions has been explored.

This article is devoted to study an important sub class of both NR-clean and G-clean rings. A ring R is called a U-nil clean if every element in R is a sum of a unit regular element and a nilpotent. The aim of this paper is to study the properties of this class of rings and clarify its relation with other related types of rings.

The paper is organized as follows: in section 2, we present the main definitions of the used concepts. We moreover, clarify many basic properties concerning U-nil clean rings. We first give an example to prove that the class of U-nil clean rings is properly contained in the class of NR-clean rings. We also prove that the Jacobson radical of any abelian U-nil clean ring is nil. Then we study U-nil cleanliness in some ring extensions. In particular, we prove that If I is a nil ideal of a ring R, then R is U-nil clean if and only if R/I is U-nil clean. This useful characterization is used to prove that U-nil clean property passes to a ring R from certain homomorphic images of R. Among many other properties, we prove that the U-nil clean property of R passes to the triangular matrix ring, formal triangular matrix ring, a quotient of a skew polynomial ring, and finite direct product of rings. Finally we define uniquely U-nil clean rings and characterized them as the the class of strongly regular rings.

For a ring R and a group G, we set and explore in the third section some conditions under which the group ring RG is U-nil clean. We prove that if RG is U-nil clean, then always R is so. Then we introduced several examples of groups G and rings R such that the converse is true.

2. Some properties of U-NIL clean rings

In this section, we introduce various examples and properties concerning U-nil clean rings. Many of the studied results are motivated by analogous results for NR-clean rings as found in [14].

Definition 2.1. *Let R be a ring. An element $r \in R$ is called U-nil clean if there is a unit regular element a and a nilpotent element b such that $r = a + b$. A ring R is called a U-nil clean ring if every element in R is U-nil clean.*

Clearly, the classes of nil clean rings and unit regular rings are contained in the class of U-nil clean rings. The containment is proper as for example for any prime integer p and $n \geq 2$, any element in $\mathbb{Z}_{p^n}$is either a nilpotent or a unit regular. Thus, $\mathbb{Z}_{p^n}$ is an U-nil clean ring which is clearly not a unit regular. Moreover, any field of more than two elements is a U-nil clean ring that is not nil clean.

Proposition 2.1. *Let R be a U-nil clean ring. Then R is a G-clean ring (and so is a UR-ring). If moreover, R is abelian, then it is a 2-clean ring.*

Proof. Let $r \in R$ and choose $a \in ur(R)$, $b \in N(R)$ such that $r-1 = a+b$. Then $r = a+(b+1)$ is a sum of a unit and unit regular in R. Therefore, R is a G-clean ring. Moreover, suppose R is abelian and choose $e \in Id(R)$ and $u \in U(R)$ such that $a = ue$. Then $v = (ue + e - 1) \in U(R)$ with $v^{-1} = u^{-1}e + e - 1$. Therefore, $r = a + b + 1 = (1 - e) + v + (b + 1)$ is a sum of an idempotent and two units in R. It follows that R is a 2-clean ring. □

Obviously, every U-nil clean ring is NR-clean. However, the converse is not true in general. For example, consider the ring R constructed by Bergman in [10] as follows: Let $A = K[[x]]$ where K is a field with $char(K) \neq 2$ and let Q be the field of fractions of A. Define

$R = \{r \in End(A_K) : \exists q \in Q$ and $\exists n \geq 1$ with $r(a) = q(a)$ for all $a \in x^n A\}$. Since it is well known that R is regular, then clearly, it is an NR-clean ring. On the other hand, R is not U-nil clean since otherwise it will be G-clean by Proposition 2.1 which is a contradiction.

The converse of Proposition 2.1 also need not be true. For example, consider the infinite direct product $R = \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_8 \times ....$ Then R is a G-clean ring by Proposition (2.6) in [12]. But R is not U-nil clean since for example the element $(\overline{0}, \overline{2}, \overline{2}, \overline{2}, ...)$ is not U-nil clean in R.

It must be noted that the implication in Proposition 2.1 is a global ring property. However, it is interesting to investigate whether this is true for a single element or not.

Next, we introduce some basic properties of U-nil clean rings.

Proposition 2.2. *Let R be a ring.*

1. *R is U-nil clean if and only if for any $r \in R$, there are $a \in ur(R)$ and $b \in N(R)$ such that $r = a - b$.*
2. *If R is a U-nil clean, then any homomorphic image of R is U-nil clean.*
3. *If I is a nil ideal of R and R/I is U-nil clean, then R is U-nil clean.*

4. If $N(R)$ is an ideal of R, then R is U-nil clean if and only if $R/N(R)$ is a unit regular ring.

Proof. 1) Suppose R is U-nil clean. Then for any $r \in R$, there are $a \in ur(R)$ and $b \in N(R)$ such that $-r = a + b$. Hence, $r = (-a) - b$ where clearly $-a \in ur(R)$. For the converse, let $r \in R$ and choose $a \in ur(R)$, $b \in N(R)$ such that $-r = a - b$. Then $r = (-a) + b$ is a U-nil clean decomposition of r.

2) Let $f : R \to S$ be a ring epimorphism where R is a U-nil clean ring. Let $s \in S$ and choose $r \in R$ such that $f(r) = s$. Then there are $a \in ur(R)$ and $b \in N(R)$ such that $r = a + b$. Therefore, $s = f(r) = f(a) + f(b)$ where clearly $f(a) \in ur(S)$ and $f(b) \in N(S)$. Thus, S is a U-nil clean.

3) Suppose R/I is U-nil clean where I is nil. Then idempotents lift modulo I and $I \subseteq J(R)$. Also, clearly units lift modulo I as $I \subseteq J(R)$. For simplicity, we write $\overline{R} = R/I$ and $\overline{r} = r + I \in \overline{R}$ for $r \in R$. Let $\overline{a} = \overline{eu}$ be any unit regular element in $\overline{R}$ where $\overline{u} \in U(\overline{R})$ and $\overline{e} \in Id(\overline{R})$. Then we may assume that $e \in Id(R)$ and $u \in U(R)$. It follows that $\overline{a} = \overline{eu}$ where $eu \in ur(R)$. Thus, unit regular elements of R are also lift modulo I. Now, let $r \in R$ and choose $\overline{a} \in ur(R)$, $\overline{b} \in N(\overline{R})$ such that $\overline{r} = \overline{a} + \overline{b}$. Then $\overline{r - a} = \overline{b}$ where we may assume $a \in ur(R)$. Since I is nil and $r - a$ is nilpotent modulo I, then $r - a$ is nilpotent. Therefore, r is a U-nil clean element and R is U-nil clean.

4) Suppose R is U-nil clean and let $r + N(R) \in R/N(R)$. Write $r = eu + b$ for some $e \in Id(R)$, $u \in U(R)$ and $b \in N(R)$. Then $r + N(R) = eu + N(R) = (e + N(R))(u + N(R))$ where $e + N(R) \in Id(R/N(R))$ and $u + N(R) \in U(R/N(R))$. Conversely, if $R/N(R)$ is a unit regular ring, then it is obviously U-nil clean. Since $N(R)$ is nil, then R is U-nil clean by (3). □

We point out that the condition "I is nil" in part 3 of Proposition 2.2 can not be omitted. For example, while the field $\mathbb{Z}/3\mathbb{Z}$ is U-nil clean, the domain $\mathbb{Z}$ is not so.

Proposition 2.3. *Let R be an abelian (all idempotent of R are central) U-nil clean ring. Then the Jacobson radical $J(R)$ of R is nil.*

Proof. Let $r \in J(R)$ and choose $a = eu \in ur(R)$ and $b \in N(R)$ where $e \in Id(R)$ and $u \in U(R)$ such that $r = a - b$. Let n be the smallest positive integer such that $b^n = 0$. By binomial expansion, we have $0 = (a - r)^n = a^n + \sum_{i=1}^{n} (-1)^i \begin{pmatrix} n \\ i \end{pmatrix} a^{n-i} r^i = a^n + rx$ for some $x \in R$. Since R is abelian, we get $eu^n = e^n u^n = (eu)^n = a^n = -rx \in J(R)$ since $J(R)$ is an ideal of R. Hence, $e = eu^n u^{-n} \in J(R)$ and so $e = 0$. It follows that $r = -b \in N(R)$ and so $J(R)$ is nil. □

We recall that a ring R is called left perfect if $R/J(R)$ is semisimple (and so unit regular) and $J(R)$ is left T-nilpotent. Since any T-nilpotent ideal in R is nil, then we conclude that a left perfect ring is a U-nil clean ring.

Since for a ring R, the prime radical $P(R)$ of R is nil, then we have

Corollary 2.1. *A ring R is U-nil clean if and only if $R/P(R)$ is U-nil clean.*

Remark 2.1. Let R be a ring and consider the ring

$$S = \left\{ \begin{bmatrix} r & r_{1,2} & \cdots & r_{1,n} \\ 0 & r & \cdots & r_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r \end{bmatrix} : r, r_{i,j} \in R \ (i \lneqq j) \right\}.$$

Then one can easily justify that $I = \left\{ \begin{bmatrix} 0 & r_{1,2} & \cdots & r_{1,n} \\ 0 & 0 & \cdots & r_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} : r_{i,j} \in R \ (i \lneqq j) \right\}$ is an ideal of S satisfying $I^n = 0$. Moreover, we can see that $S/I \cong R$. Therefore, we conclude by Proposition 2.2 that R is U-nil clean if and only if S is U-nil clean.

Corollary 2.2. *Let R be a ring. Then R is U-nil clean if and only if $R[x]/(x^n)$ is U-nil clean for all $n \geq 2$.*

Proof. It is well known that

$$R[x]/(x^n) \cong \left\{ \begin{bmatrix} r_1 & r_2 & r_3 & \cdots & r_{n-1} & r_n \\ 0 & r_1 & r_2 & \cdots & r_{n-2} & r_{n-1} \\ 0 & 0 & r_1 & \ddots & r_{n-3} & r_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & r_1 & r_2 \\ 0 & 0 & 0 & \cdots & 0 & r_1 \end{bmatrix} : r_{i,j} \in R \ (i \lneqq j) \right\}.$$

Hence, the result is obtained by Remark 2.1. □

For a ring R and an endomorphism σ of R, a skew polynomial ring $R[x, \sigma]$ is defined to be the set of finite sums $\sum_{i=0}^{n} a_i x^i$ where $a_i \in R$ for each i in which addition is defined as usual and multiplication is defined subject to the relation $xa = \sigma(a)x$ for all $a \in R$. It was proved in the literature that many properties can transfer between a ring R and the quotient ring $R[x, \sigma]/(x^n)$ where $n \in \mathbb{N}$. In the following proposition, we conclude that being U-nil clean is one of such properties.

Proposition 2.4. *Let R be a ring, σ an endomorphism of R and $n \in \mathbb{N}$. Then R is an U-nil clean ring if and only if $R[x, \sigma]/(x^n)$ is U-nil clean.*

Proof. Let $f : R[x, \sigma]/(x^n) \to R$ defined by $f(a_0 + a_1 x + ... + a_{n-1}x^{n-1} + (x^n)) = a_0$. Then clearly f is a ring epimorphism with kernel the ideal I of $R[x, \sigma]/(x^n)$ generated by x. Moreover, it is clear that I is a nil ideal. Now, the result follows directly by Proposition 2.2. □

Proposition 2.5. *Let $R_1, R_2, ..., R_k$ be rings. Then $R = \prod_{i=1}^{k} R_i$ is a U-nil clean ring if and only if R_i is U-nil clean for all $i \in \{1, 2, ..., k\}$.*

Proof. $\Rightarrow$) : For each $i \in \{1, 2, ..., k\}$, R_i is a homomorphic image of R. Thus, R_i is U-nil clean by Proposition 2.2.

$\Leftarrow$) : Let $r = (r_1, r_2, ..., r_k) \in R$. For each i, choose $a_i \in ur(R_i)$ and $b_i \in N(R_i)$ such that $r_i = a_i + b_i$. Then clearly $a = (a_1, a_2, ..., a_k) \in ur(R)$, $b = (b_1, b_2, ..., b_k) \in N(R)$ and $r = a + b$. It follows that R is U-nil clean. □

We note that for $k \geq 1$, the rings $\mathbb{Z}_{2^k}$ are U-nil clean. But the infinite direct product $\prod_{k\in\mathbb{N}} \mathbb{Z}_{2^k}$ is not U-nil clean since for example the element $(\overline{0}, \overline{2}, \overline{2}, \overline{2}, ...)$ is not U-nil clean in $\prod_{k\in\mathbb{N}} \mathbb{Z}_{2^k}$. Thus the converse part of Proposition 2.5 is true only in the finite product case.

Let R be a commutative with identity. It is well known that

$$N(R[x]) = \left\{ \sum a_n x^n \in R[x] : a_n \in N(R) \text{ for all } n \right\}.$$

Moreover, if R is Noetherian, then

$$N(R[[x]]) = \left\{ \sum a_n x^n \in R[[x]] : a_n \in N(R) \text{ for all } n \right\}.$$

Lemma 2.1. *[2] Let R be a commutative with identity. Then*

$$ur(R[x]) = \left\{ \begin{array}{c} \sum a_n x^n \in R[x] : a_0 = ue,\ a_n \in e(N(R)) \\ \text{for every } n \geq 1 \text{ for some } u \in U(R) \text{ and } e \in Id(R) \end{array} \right\}$$

and

$$ur(R[[x]]) = \left\{ \begin{array}{c} \sum a_n x^n \in R[[x]] : a_0 = ue,\ a_n \in \langle e \rangle \\ \text{for every } n \geq 1 \text{ for some } u \in U(R) \text{ and } e \in Id(R) \end{array} \right\}.$$

Proposition 2.6. *Let R be a non zero commutative with identity 1. Then $R[x]$ is never a U-nil clean ring. If moreover, R is Noetherian, then also $R[[x]]$ is never U-nil clean.*

Proof. Suppose $x = \sum a_n x^n + \sum b_m x^m$ where $\sum a_n x^n \in ur(R[x])$ and $\sum b_m x^m \in N(R[x])$. Then by Lemma 2.1 $1 = eb + b_1$ for some $b, b_1 \in N(R)$ and so $eb = 1 - b_1 \in U(R)$. it follows that $b \in U(R)$ which is a contradiction. Now, suppose R is Noetherian and there are $\sum a_n x^n \in ur(R[[x]])$ and $\sum b_m x^m \in N(R[[x]])$ such that $x = \sum a_n x^n + \sum b_m x^m$. Then there are $u \in U(R)$ and $e \in Id(R)$ such that $ue + b_0 = 0$ and $re + b_1 = 1$ where $r \in R$ and $b_1 \in N(R)$. Therefore, $re = 1 - b \in U(R)$ and so $e \in U(R)$. Thus, $e = 1$ and $b_0 = -u \in U(R)$ which is again a contradiction. □

We recall that for a ring R, the $n \times n$ upper triangular matrix ring over R is denoted by $T_n(R)$. In the following Proposition, we prove that the U-nil clean property of a ring R passes to $T_n(R)$.

Proposition 2.7. *For any ring R, we have R is U-nil clean if and only if $T_n(R)$ is U-nil clean.*

Proof. Let I be the ideal of $T_n(R)$ consists of all matrices with zeros along the main diagonal. Then it is clear that I is nil. Moreover, we have $T_n(R)/I \simeq \prod_{i=1}^{n} R_i$ where $R_i = R$ for $i \in \{1, 2, ..., n\}$. Since a finite direct product of U-nil clean rings is U-nil clean by Proposition 2.5, then the result follows by Proposition 2.2. □

Theorem 2.1. *Let M be an (R, S)-module where R and S are two rings . Then the formal triangular matrix ring $T = \begin{bmatrix} R & 0 \\ M & S \end{bmatrix}$ is U-nil clean if and only if R and S are U-nil clean.*

Proof. Suppose that R and S are U-nil clean rings and let $A = \begin{bmatrix} r & 0 \\ m & s \end{bmatrix} \in T$. Then $A = \begin{bmatrix} a+b & 0 \\ m & a'+b' \end{bmatrix}$ where $a \in ur(R)$, $a' \in ur(S)$, $b \in N(R)$ and $b' \in N(S)$. Thus, $A = \begin{bmatrix} a & 0 \\ 0 & a' \end{bmatrix} + \begin{bmatrix} b & 0 \\ m & b' \end{bmatrix}$ where clearly $\begin{bmatrix} a & 0 \\ 0 & a' \end{bmatrix} \in ur(T)$ and $\begin{bmatrix} b & 0 \\ m & b' \end{bmatrix} \in N(T)$. It follows that T is U-nil clean. The converse is true by Proposition 2.2 and Proposition 2.5 since $R \times S$ is a homomorphic image of T. □

Let R be a ring and M be an R-module. The trivial ring extension of R by M is the ring $R \ltimes M$ whose underlying group is $R \oplus M$ with multiplication given by $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2)$. It is well known that $R \ltimes M/(0 \oplus M) \simeq R$ where $0 \oplus M$ is a nil ideal of $R \ltimes M$. By Proposition 2.2, we conclude that R is a U-nil clean ring if and only if $R \ltimes M$ is U-nil clean.

Let R be a ring and let V be an (R, R)-bimodule which is a general ring(with or without identity) such that for all $u, v \in V$ and $r \in R$, we have $(uv)r = u(vr)$, $(ur)v = u(rv)$ and $(ru)v = r(uv)$. The Dorroh extension $I(R, V)$ of R by V is defined as the additive abelian group $I(R, V) = R \oplus V$ with multiplication $(r, u)(a, v) = (ra, rv + ua + uv)$.

Proposition 2.8. *Consider the Dorroh extension $I(R, V)$ of R by V. If $I(R, V)$ is U-nil clean, then so is R. If moreover, each element in V is nilpotent, then the converse holds.*

Proof. If $I(R, V)$ is U-nil clean, then so is R as it is a homomorphic image of $I(R, V)$ under the epimorphism $\varphi : I(R, V) \to R$ defined by $\varphi(r, v) = r$. For the converse, suppose that every element in V is nilpotent and R is U-nil clean. Let $(r, v) \in I(R, V)$ and write $r = a+b$ where $a \in ur(R)$ and $b \in N(R)$. Then $a = eu$ for some $u \in U(R)$ and $e \in Id(R)$ and so $(r, v) = (eu + b, v) = (eu, 0) + (b, v)$. Now, clearly $(eu, 0) = (e, 0)(u, 0)$ is a product of an idempotent and a unit

in $I(R,V)$. Moreover, since v is nilpotent in V, then $(b,v) \in N(I(R,V))$. Therefore, $I(R,V)$ is U-nil clean as required. □

It can be easily verified that if U is a subring of V, then the set

$$I(R,U) = \{(r,u) \in I(R,V) : u \in U\}$$

is a subring of $I(R,V)$. Hence, the following corollary of Proposition 2.8 is straightforward.

Corollary 2.3. *Let R and V be as in Proposition 2.8. If the nilradical $N(V)$ of V is a subring of V, then $I(R,N(V))$ is U-nil clean if and only if R is U-nil clean.*

Let R be a ring with identity and S be a subring of R containing the identity of R. Consider the set

$$T[R,S] = \{(r_1, r_2, ..., r_n, s, s, ...) : r_i \in R \text{ for } 1 \leq i \leq n,\ s \in S \text{ and } n \in \mathbb{N}\}.$$

Then $T[R,S]$ is clearly a ring with identity $(1,1,...,1,1,1,...)$ under componentwise addition and multiplication. One can easily verify that

$$ur(T[R,S]) = \{(r_1, r_2, ..., r_n, s, s, ...) : r_i \in ur(R) \text{ for } 1 \leq i \leq n,\ s \in ur(S)\}$$

and

$$N(T[R,S]) = \{(r_1, r_2, ..., r_n, s, s, ...) : r_i \in N(R) \text{ for } 1 \leq i \leq n,\ s \in N(S)\}.$$

Therefore, the proof of the following proposition can be directly achieved.

Proposition 2.9. *Let R be a ring with identity and S be a subring of R containing the identity of R. Then R and S are U-nil clean if and only if $T[R,S]$ is U-nil clean.*

Let R be a ring and consider the set $S = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a,b,c,d \in R \right\}$ of all 2×2 matrices over R. Define on S the usual matrix addition and the multiplication given as $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} = \begin{bmatrix} aa' & ab'+bd' \\ ca'+dc' & dd' \end{bmatrix}$. Then clearly R is a ring under these operations. This ring is usually denoted by $K_0(R)$, see [15].

Proposition 2.10. *Let R be a ring. Then R is U-nil clean if and only if $K_0(R)$ is U-nil clean.*

Proof. Suppose $K_0(R)$ is U-nil clean. Then R is so since clearly the mapping $\varphi : K_0(R) \to R$ defined by $\varphi(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = a$ is a ring epimorphism. Conversely,

suppose R is U-nil clean and let $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in K_0(R)$. If we write $a = r_1 + b_1$ and $d = r_2 + b_2$ for $r_1, r_2 \in ur(R)$ and $b_1, b_2 \in N(R)$, then $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix} + \begin{bmatrix} b_1 & b \\ c & b_2 \end{bmatrix}$. It is clear that $\begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix} \in ur(K_0(R))$. Moreover, the multiplication operation of $K_0(R)$ guarantee by simple computations that $\begin{bmatrix} b_1 & b \\ c & b_2 \end{bmatrix} \in N(K_0(R))$. Therefore, $K_0(R)$ is a U-nil clean ring. □

Proposition 2.11. *Let R be a U-nil clean ring and let $r \in R$. If Rr contains no non trivial idempotents, then r is either a unit or a sum of two units.*

Proof. Let $r \in R$ and Suppose Rr contains no non trivial idempotents of R. Then $r - 1 = a + b$ for some $a \in ur(R)$ and $b \in N(R)$. Thus, $r = eu + (b + 1)$ where $u, b+1 \in U(R)$ and $e \in Id(R)$. Now, $er = eu + e(b+1)$ and so $(1-e)r = (1-e)(b+1)$. It follows that $(b+1)^{-1}(1-e)(b+1) = (b+1)^{-1}(1-e)r \in Rr$. Moreover, $(b+1)^{-1}(1-e)(b+1)$ is clearly an idempotent in R and so by assumption it is trivial. If $(b+1)^{-1}(1-e)(b+1) = 1$, then $1 - e = 1$ and so $e = 0$. In this case, $r = b + 1$ is a unit in R. If $(b+1)^{-1}(1-e)(b+1) = 0$, then $e = 1$ and $r = u + (b + 1)$ is a sum of two units. □

A ring R is called uniquely U-nil clean if for every $r \in R$, there is a unique $b \in N(R)$ such that $r - b$ is unit regular. A ring in which for every $r \in R$, there is $x \in R$ such that $r^2x = r$ is called strongly regular. It is well known that any strongly regular ring is unit regular. Thus, the following characterization of uniquely U-nil clean rings follows by Theorem 4 in [14].

Proposition 2.12. *The following are equivalent for a ring R.*

(1) *R is a uniquely U-nil clean ring.*
(2) *R is a uniquely NR-clean ring.*
(3) *R is a strongly regular ring.*

3. U-nil clean group rings

In this section, we prove that the U-nil clean property passes from the group ring to the underlying one. Moreover, we establish several conditions under which the converse is true.

Let R be a ring and let G be a group. We denote by RG the group ring of G over R. The ring epimorphism $\varepsilon : RG \to R$ defined by $\varepsilon(\sum_{g\in G} r_g g) = \sum_{g\in G} r_g$ is called the augmentation mapping. The kernel of ε is called the augmentation ideal of RG, and is denoted by $\Delta(G)$. it is well known that $\Delta(G)$ is generated by the set $\{1 - g : g \in G\}$.

Proposition 3.1. *Let R be a ring and G be a group. If RG is U-nil clean, then so is R. If moreover $\Delta(G)$ is nil, then the converse is true.*

Proof. The proof follows since $RG/\Delta(G) \cong R$ and by using (2) and (3) of Proposition 2.2. □

Next, we determine some other special cases where the group ring RG is U-nil clean.

Proposition 3.2. *Let R be a ring with $2 \in U(R)$ and consider the cyclic group $C_2 = \{1, g\}$. Then R is U-nil clean if and only if RC_2 is U-nil clean.*

Proof. If $2 \in U(R)$, then $RG \cong R \times R$ under the isomorphism $f : RG \to R \times R$ defined by $f(a_1 + a_2 g) = (a_1 + a_2, a_1 - a_2)$. Hence, if R is U-nil clean, then so is RG by Proposition 2.5. The converse follows by Proposition 3.1. □

Lemma 3.1 ([21])**.** *Let R be a ring where $2 \in U(R)$. Then $RC_4 \cong R \times R \times R[x]/(x^2+1)$.*

Proposition 3.3. *Let R be a ring with $2 \in U(R)$. Then RC_4 is U-nil clean if and only if $R[x]/(x^2+1)$ is U-nil clean.*

Proof. If we write $C_4 = \{1, g, g^2, g^3\}$, then as proved in [21], $e = (1+g^2)/2$ is a central idempotent in RC_4 and $RC_4(1-e) \cong R[x]/(x^2+1)$. Now, if $R[x]/(x^2+1)$ is U-nil clean, then R is also U-nil clean by Proposition 3.1 and so RC_4 is U-nil clean by Proposition 2.5 and Lemma 3.1. The converse is clear again by Proposition 2.5 and Lemma 3.1. □

Lemma 3.2 ([21])**.** *Let R be a ring with $3 \in U(R)$ and write $C_3 = \{1, g, g^2\}$. Then $RC_3 \cong R \times \{a_0 + a_1 g - (a_0 + a_1)g^2 : a_0, a_1 \in R\}$.*

Proposition 3.4. *Let R be a ring with $3 \in U(R)$ and write $C_3 = \{1, g, g^2\}$. Then RC_3 is U-nil clean if and only if $\{a_0 + a_1 g - (a_0 + a_1)g^2 : a_0, a_1 \in R\}$ is U-nil clean.*

Proof. Let $e = (1+g+g^2)/3$. Then one can see that e is a central idempotent with $RC_3(1-e) \cong \{a_0 + a_1 g - (a_0 + a_1)g^2 : a_0, a_1 \in R\}$. Now, the result follows by Proposition 3.1 and Lemma 3.2. □

Lemma 3.3 ([21])**.** *If R is a ring where $R \subseteq \mathbb{C}$ and $3 \in U(R)$, then $RC_3 \cong R \times R[x]/(x^2+x+1)$.*

Proposition 3.5. *Let R be a ring with $R \subseteq \mathbb{C}$ and $3 \in U(R)$. Then RC_3 is U-nil clean if and only if $R[x]/(x^2+x+1)$ is U-nil clean.*

In the following lemma, we give a useful characterization of U-nil clean rings that will be used later on.

Lemma 3.4. *Let R be an abelian ring and $e \in Id(R)$. Then R is U-nil clean if and only if eR and $(1-e)R$ are U-nil clean.*

Proof. Suppose R is U-nil clean and let $ea \in eR$. Then $a = fu + b$ where $f \in Id(R)$, $u \in U(R)$ and $b \in N(R)$. Thus, $ea = e(fu) + eb = (ef)(eu) + eb$ since R is abelian. Also, R being abelian implies that $ef \in Id(eR)$, $eu \in U(eR)$ and $eb \in N(eR)$ and then eR is U-nil clean. Similarly, we can prove that $(1-e)R$ is also U-nil clean. Conversely, assume that eR and $(1-e)R$ are U-nil clean. Since R is abelian,we may apply pierce decomposition to write $R = eR \oplus (1-e)R \cong \begin{bmatrix} eR & 0 \\ 0 & (1-e)R \end{bmatrix}$. Let $\begin{bmatrix} er_1 & 0 \\ 0 & (1-e)r_2 \end{bmatrix} \in R$. Choose $ef_1 \in Id(eR)$, $eu_1 \in U(eR)$ and $eb_1 \in N(eR)$ (where $f_1 \in Id(R)$, $u_1 \in U(R)$ and $b_1 \in N(R)$) such that $er_1 = (ef_1)(eu_1) + eb_1$. Similarly, write $(1-e)r_2 = (1-e)f_2(1-e)u_2 + (1-e)b_2$ where $f_2 \in Id(R)$, $u_2 \in U(R)$ and $b_2 \in N(R)$. Then

$$\begin{bmatrix} er_1 & 0 \\ 0 & (1-e)r_2 \end{bmatrix} = \begin{bmatrix} ef_1 & 0 \\ 0 & (1-e)f_2 \end{bmatrix} \begin{bmatrix} eu_1 & 0 \\ 0 & (1-e)u_2 \end{bmatrix} + \begin{bmatrix} eb_1 & 0 \\ 0 & (1-e)b_2 \end{bmatrix}$$

is clearly a U-nil clean decomposition of $\begin{bmatrix} er_1 & 0 \\ 0 & (1-e)r_2 \end{bmatrix}$. Therefore, R is a U-nil clean ring. □

Idempotent elements $e_1, e_2, ..., e_n$ of a ring R are called orthogonal if $e_i e_j = 0$ for all $i \neq j$. The following corollary of Lemma 3.4 can be proved by using induction.

Corollary 3.1. *Let $e_1, e_2, ..., e_n$ be orthogonal idempotents in an abelian ring R such that $e_1 + e_2 + ... + e_n = 1$. Then R is U-nil clean if and only if $e_i R$ is U-nil clean for each i.*

We recall that for an idempotent e in a ring R, if eRe is a local ring, then e is called local. If the identity 1 in R is a finite sum of orthogonal local ring, then R is called semiperfect.

Proposition 3.6. *Let R be a semiperfect ring and let G be a group. If RG is abelian and $(eR)G$ is U-nil clean for each local idempotent e in R, then RG is U-nil clean.*

Proof. As R is semiperfect, then we can find orthogonal local idempotents $e_1, e_2, ..., e_n$ such that $1 = e_1 + e_2 + ... + e_n$. Now, for each i consider the mapping $\varphi_i : (e_i R)G \to e_i(RG)$ defined by $\varphi(\sum_{g \in G}(e_i r_g)g) = \sum_{g \in G} e_i(r_g g)$. Then one can easily check that φ_i is a ring isomorphism for each i. Therefore, $e_i(RG)$ is U-nil clean for each i and so RG is U-nil clean by Corollary 3.1. □

Proposition 3.7. *Let R be a ring and G be a finite p-group where $p \in N(R)$. If R is an U-nil clean ring, then so is RG.*

Proof. According to Proposition 3.1, it is enough to prove that $\Delta(G)$ is nil in RG. We will use mathematical induction on $|G|$. Since G is a p-group,

then it has a nontrivial center. Choose an element x in the center of G such that $|x| = p$ and consider the group $K = G/H$ where $H = \langle x \rangle$. By induction hypothesis, $\Delta(K)$ is nil in RK of order p. Now, define $\psi : RG \to RK$ by $\psi(\sum_{g\in G} r_g g) = \sum_{g\in G} r_g(gH)$. Then clearly ψ is a ring homomorphism with $\ker(\psi) = (1-x)RG$. Since $x^p = 1$ and x is central in G, then $(1-x)^p \in p(RG)$. But p is nilpotent in R which implies that $1-x$ is nilpotent in RG. Since $\Delta(K)$ is nil and clearly $\psi(\Delta(G)) \subseteq \Delta(K)$, then $\psi(\Delta(G))$ is also nil. Now, if $a \in \Delta(G)$, then $\psi(a)$ is nil in RK and so $\psi(a^n) = (\psi(a))^n = 0$ for some $n \in \mathbb{N}$. It follows that $a^n \in \ker(\psi)$ is nilpotent and so $\Delta(G)$ is nil in RG. □

We recall that a group G is called locally finite if every finitely generated subgroup is finite. For a prime p, the set G_p consists of all elements of G whose orders a power of p. A group G is called p-torsion free if G_p is trivial. Moreover, if G

is abelian, then G_p is always a subgroup of G.

For a ring and a group of certain conditions, Gilmer, [8] characterized reduced group rings as follows.

Lemma 3.5 ([8])**.** *Let R be an integral domain of characteristic p and G be a group. Then RG is reduced if and only if G is p-torsion free.*

The following characterization of regular group rings has been introduced by Auslander [3].

Lemma 3.6 ([3])**.** *Let R be a non zero ring and let G be any group. Then RG is a regular ring if and only if*

(1) *R is regular.*
(2) *G is locally finite.*
(3) *The order of any finite subgroup of G is a unit in R.*

Theorem 3.1. *Let R be a commutative ring with identity of characteristic p (prime) having $\mathbb{Z}_p$ as a homomorphic image. If G is an abelian p-group, where G/G_p is finite, then RG is not a U-nil clean ring.*

Proof. $\Rightarrow$) : Suppose on contrary that RG is U-nil clean. Then R is so by Proposition 3.1. Now, let $\varphi : RG \to R(G/G_p)$ be defined as $\sum_{g\in G} r_g g \to \sum_{g\in G} r_g(g+G_p)$. Then clearly φ is a ring epimorphism and so $R(G/G_p)$ is U-nil clean. Since $\mathbb{Z}_p$ is a homomorphic image of R, then $\mathbb{Z}_p(G/G_p)$ is a homomorphic image of $R(G/G_p)$ and so again $\mathbb{Z}_p(G/G_p)$ is U-nil clean. Moreover, as G/G_p is p-torsion free, then $\mathbb{Z}_p(G/G_2)$ is also reduced by Lemma 3.5. Thus, $\mathbb{Z}_p(G/G_p)$ is unit regular. It follows by Lemma 3.6 that the order of any finite subgroup of G/G_p is a unit in $\mathbb{Z}_p$. Let H/G_p be a subgroup of G/G_p where H is a subgroup of G containing G_p. Since H is a p-group, then the order of H/G_p is a power of p which is not a unit in $\mathbb{Z}_p$. This contradiction implies that RG is not U-nil clean. □

Acknowledgement

This work has been carried out during sabbatical leave granted to the author from Al Al-Bayt University during the academic year 2018/2019.

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Accepted: March 29, 2020

On generalized fuzzy soft compact spaces

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Abstract. In the present paper, we continue studying generalized fuzzy soft topological spaces. We first introduce generalized fuzzy soft $p-$cover and utilize it to define a new type of generalized fuzzy soft compact topological spaces so-called a generalized fuzzy soft p^*-compact topological spaces which is a generalization of compactness in fuzzy soft topological spaces in [27]. In fact, a generalized fuzzy soft p^*-compact topological space is more general than generalized fuzzy soft compact spaces in [17]. In general, we investigate some basic results, relations and properties of generalized fuzzy soft p^*-compact space and provide some illustrative examples.

Keywords: generalized fuzzy soft set, generalized fuzzy soft topology, generalized fuzzy soft $p-$cover, generalized fuzzy soft $p-$compact, generalized fuzzy soft p^*-compact.

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1. Introduction

In 1999, Molodtsov [23] introduced the concept of soft sets as a new approach to model uncertain objects. Shabir and Naz [32] introduced and studied the topological structures of soft sets. Moreover, many authors studied soft topology and its applications [1, 6, 7, 8, 9, 10, 14, 21, 28, 29, 30, 31]. The concept of fuzzy soft set introduced by Maji et. al. [19]. Later, Tanay et al. [33] and Roy et. al. [26] introduced fuzzy soft topological space independently. In 2010, Majumdar and Samanta [20] introduced the notion of generalized fuzzy soft set as a generalization of fuzzy soft sets and some of its basic properties. Chakraborty et. al. [11] gave the topological structure of generalized fuzzy soft sets.

The concept of compactness is one of the most important concepts in topological spaces. In fuzzy topology, compactness was first introduced by Chang [12] and in soft topology, compactness was introduced by Zorlutuna et. al. [31]. Then, Al-shami et al studied soft compactness in [2, 3, 4, 5] and revised the relationships etween soft compact set and soft Hausdorff in [6, 7, 8]. In fuzzy soft topology, compactness was introduced by Osmanoglu et al. [25] and Gain et al. [13] which is extension of Chang's fuzzy compactness [12]. Also, Mishra et. al. [22] introduced the compactness in fuzzy soft topology as extension of Lown's fuzzy compactness [18].

In this paper, we introduce and study a new type of cover and compactness in generalized fuzzy soft topology so-called generalized fuzzy soft $p-$cover and generalized fuzzy soft p^*-compact which is a generalization of compactness in [27], development the compactness in [13, 25], and is more general than that which are presented by Khedr et. al. [17] and give some basic definitions, results, relations and theorems related to the p^*-compactness are studied.

2. Preliminaries

In this section, we present the basic definitions and results of soft set theory which will be needed in the sequel.

Definition 2.1 ([34]). *Let X be a non-empty set. A fuzzy set A in X is defined by a membership function $\mu_A : X \to [0,1]$ whose value $\mu_A(x)$ represents the 'grade of membership' of x in A for x in X. The set of all fuzzy sets in a set X is denotes by I^X, where I is the closed unit interval $[0,1]$. The support of $A \in I^X$ is the crisp set $S(A) = \{x \in X : \mu(x) > 0\}$.*

Definition 2.2 ([19, 26]). *Let X be an initial universe set and E be a set of parameters. Let $A \subseteq E$. A fuzzy soft set f_A over X is a mapping from E to I^X, i.e., $f_A : E \longrightarrow I^X$, where $f_A(e) \neq \bar{0}$ if $e \in A \subset E$, and $f_A(e) = \bar{0}$ if $e \notin A$, where $\bar{0}$ denotes empty fuzzy set in X. The support of f_A denoted by $S(f_A(e))$ is the set, $S(f_A(e)) = \{x \in X : f_A(e)(x) > 0\}$.*

Definition 2.3 ([20]). *Let X be a universal set of elements and E be a universal set of parameters for X. Let $F : E \longrightarrow I^X$ and μ be a fuzzy subset of E, i.e., $\mu : E \longrightarrow I$. Let F_μ be the mapping $F_\mu : E \longrightarrow I^X \times I$ defined as follows: $F_\mu(e) = (F(e), \mu(e))$, where $F(e) \in I^X$ and $\mu(e) \in I$. Then F_μ is called a generalised fuzzy soft set (GFSS in short) over (X, E). The family of all generalized fuzzy soft sets (GFSSs in short) over (X, E) is denotes by $GFSS(X, E)$.*

Definition 2.4 ([17]). *Let $F_\mu \in GFSS(X, E)$. The support of F_μ denoted by $S(F_\mu(e))$ is the set $S(F_\mu(e)) = \{x \in X : F(e)(x) > 0$ and $\mu(e) > 0, e \in E\}$.*

Definition 2.5 ([20]). *Let F_μ and G_δ be two GFSSs over (X, E). Then we have:*

1) *F_μ is called a generalized null fuzzy soft set, denoted by $\widetilde{0}_\theta$, if $\widetilde{0}_\theta : E \longrightarrow I^X \times I$ such that $\widetilde{0}_\theta(e) = (\widetilde{0}(e), \theta(e))$ where $\widetilde{0}(e) = \overline{0} \ \ \forall e \in E$ and $\theta(e) = 0$ $\forall e \in E$ (Where $\overline{0}(x) = 0, \ \forall x \in X$).*

2) *F_μ is called a generalized absolute fuzzy soft set, denoted by $\widetilde{1}_\triangle$, if $\widetilde{1}_\triangle : E \longrightarrow I^X \times I$, where $\widetilde{1}_\triangle(e) = (\widetilde{1}(e), \triangle\,(e))$ is defined by $\widetilde{1}(e) = \overline{1}, \forall e \in E$ and $\triangle\,(e) = 1, \forall e \in E$ (Where $\overline{1}(x) = 1, \forall x \in X$).*

3) *F_μ is a generalized fuzzy soft subset of G_δ denoted by $F_\mu \sqsubseteq G_\delta$ if μ is a fuzzy subset of δ and $F(e)$ is also a fuzzy subset of $G(e)$, $\forall e \in E$.*

4) *The generalized fuzzy soft complement of F_μ, denoted by F^c_μ, is defined by F^c_μ=G_δ, where $\delta(e) = \mu^c(e)$ and $G(e) = F^c(e), \forall e \in E$. Obviously $(F^c_\mu)^c = F_\mu$.*

Definition 2.6 ([11]). *Let F_μ and G_δ be two GFSSs over (X, E).*

1) *The union of F_μ and G_δ, denoted by $F_\mu \sqcup G_\delta$, is The GFSS H_ν, defined as $H_\nu : E \longrightarrow I^X \times I$ such that $H_\nu(e) = (H(e), \nu(e))$, where $H(e) = F(e) \vee G(e)$ and $\nu(e) = \mu(e) \vee \delta(e), \forall e \in E$.*

2) *The Intersection of F_μ and G_δ, denoted by $F_\mu \sqcap G_\delta$, is the GFSS M_σ, defined as $M_\sigma : E \longrightarrow I^X \times I$ such that $M_\sigma(e) = (M(e), \sigma(e))$, where $M(e) = F(e) \wedge G(e)$ and $\sigma(e) = \mu(e) \wedge \delta(e)$, $\forall e \in E$.*

Definition 2.7 ([15]). *The generalized fuzzy soft set $F_\mu \in GFSS(X, E)$ is called a generalized fuzzy soft point (GFS point for short) over (X, E) if there exist $e \in E$ and $x \in X$ such that*

(1) *$F(e)(x) = \alpha (0 < \alpha \leq 1)$ and $F(e)(y) = 0$ for all $y \in X - \{x\}$,*

(2) *$\mu(e) = \lambda (0 < \lambda \leq 1)$ and $\mu(e') = 0$ for all $e' \in E - \{e\}$. We denote this generalized fuzzy soft point $F_\mu = (e_\lambda, x_\alpha)$.*

(e, x) and (λ, α) are called respectively, the support and the value of (e_λ, x_α). The class of all GFS points in (X, E), denoted by $GFSP(X, E)$.

We say that $(e_\lambda, x_\alpha) \widetilde{\in} F_\mu$ read as (e_λ, x_α) belongs to the GFSS F_μ if for the element $e \in E$, $\alpha \leq F(e)(x)$ and $\lambda \leq \mu(e)$.

The complement of a GFS point (e_λ, x_α) is a GFSS denoted by $(e_\lambda, x_\alpha)^c$, is defined by $(e_\lambda, x_\alpha)^c = G_\delta$, where $G(e) = 1 - F(e)(x)$ and $\delta(e) = 1 - \mu(e)$, $\forall x \in X, e \in E$.

Definition 2.8 ([24]). *Let $F_\mu, G_\delta \in GFSS(X,E)$ over (X,E). F_μ is said to be a generalised soft quasi-coincident with [GFS quasi−coincident in short] G_δ, denoted by $F_\mu q G_\delta$, if there exist $e \in E$ and $x \in X$ such that $F(e)(x)+G(e)(x) > 1$ and $\mu(e)+\delta(e) > 1$.*

If F_μ is not GFS quasi−coincident with G_δ, then we write $F_\mu \bar{q} G_\delta$ i.e, For every $e \in E$ and $x \in X$, $F(e)(x)+G(e)(x) \leq 1$ or for every $e \in E$ and $x \in X$, $\mu(e)+\delta(e) \leq 1$.

A GFS point (e_λ, x_α) is said to be GFS quasi−coincident with F_μ, denoted by $(x_\alpha, e_\lambda) q F_\mu$, if and only if there exists an element $e \in E$ such that $\alpha + F(e)(x) > 1$ and $\lambda + \mu(e) > 1$.

Theorem 2.9 ([16, 24]). *Let $F_\mu, G_\delta \in GFSS(X,E)$ and $(e_\lambda, x_\alpha) \in GFSP(X,E)$. Then:*

(1) $F_\mu \bar{q} G_\delta \Leftrightarrow F_\mu \sqsubseteq G_\delta^c$,
(2) $F_\mu \sqcap G_\delta = \widetilde{0}_\theta \Rightarrow F_\mu \bar{q} G_\delta$,
(3) $F_\mu q G_\delta \Rightarrow F_\mu \sqcap G_\delta \neq \widetilde{0}_\theta$,
(4) $F_\mu \bar{q} F_\mu^c$,
(5) $(x_\alpha, e_\lambda) \bar{q} F_\mu \Leftrightarrow (x_\alpha, e_\lambda) \widetilde{\in} F_\mu^c$.

Definition 2.10 ([11]). *Let T be a collection of generalized fuzzy soft sets over (X,E). Then T is said to be a generalized fuzzy soft topology (GFST, in short) over (X,E) if the following conditions are satisfied:*

(1) *$\widetilde{0}_\theta$ and $\widetilde{1}_\triangle$ are in T.*
(2) *Arbitrary unions of members of T belong to T.*
(3) *Finite intersections of members of T belong to T.*

The triple (X,T,E) is called a generalized fuzzy soft topological space (GFST-space, in short) over (X,E).

The member of T are called generalized fuzzy soft open set [GFS open for short] in (X,T,E) and their generalized fuzzy soft complements are called GFS closed sets in (X,T,E). The family of all GFSclosed sets in (X,T,E) is denoted by T^c.

Definition 2.11 ([15]). *Let (X,T,E) be a GFST-space and $Y \subseteq X$. Let H_ν^Y be a GFSS over (Y,E) where $H_\nu^Y : E \longrightarrow I^X \times I$ such that $\forall e \in E, H_\nu^Y(e) = (H^Y(e), \nu(e))$,*

$$H^Y(e)(x) = \begin{cases} 1, & x \in Y \\ 0, & x \notin Y \end{cases}, \nu(e) = 1$$

i.e., $H^Y(e) = Y$, $\forall e \in E$, $\nu(e) = 1$

Let $T_Y = \{H_\nu^Y \sqcap G_\delta : G_\delta \in T\}$, then T_Y is a GFS topology over (X,E) and $H_\nu^Y \in T$ is called a GFS subspace of (Y, T_Y, E). If $H_\nu^Y \in T$ (resp. $H_\nu^Y \in T^c$) then (Y, T_Y, E) is called GFS open (resp. closed) subspace of (X,T,E).

Definition 2.12 ([11, 15]). *Let (X,T,E) be a GFST−space. A GFSS F_μ in $GFSS(X,E)$ is called a generalized fuzzy soft neighborhood (briefly, GFS−nbd)*

of H_ν [resp. (e_λ, x_α)] if there exists $G_\delta \in T$ such that $H_\nu \sqsubseteq G_\delta \sqsubseteq F_\mu$ [resp. $(e_\lambda, x_\alpha)\widetilde{\in} \sqsubseteq G_\delta \sqsubseteq F_\mu$]. The family of all $GFS-$nbds of H_ν [resp. (e_λ, x_α)], is denoted by $N(H_\nu)$ [resp. $N(e_\lambda, x_\alpha)$].

Notation. The notation O_{H_ν} [resp. $O_{(e_\lambda, x_\alpha)}$] refers to a GFS open set contains H_ν [resp. (e_λ, x_α)] and called a $GFS-$nbd of H_ν [resp. (e_λ, x_α)].

Definition 2.13 ([17]). *Let $F_\mu \in GFSS(X, E)$. The $GFSS$ F_μ is called the $\alpha - (X, E)-$universal $GFSS$, denoted by $\widetilde{\alpha}_{(X,E)}$, if $F_\mu(e) = (\widetilde{\alpha}_E, \alpha_X)$, where $\widetilde{\alpha}_E$ the constant fuzzy soft set on (X, E) and α_X constant fuzzy set on X, i.e., $\widetilde{\alpha}_E(e) = \alpha_X$ and $\alpha_X(x) = \alpha$ for each $e \in E$. Clearly, $(\widetilde{\alpha}_{(X,E)})^c = (\widetilde{(1-\alpha)}_E, (1-\alpha)_X)$.*

Definition 2.14 ([16]). *A $GFST-$space (X, T, E) is said to be:*

(1) *Generalized fuzzy soft quasi R_0-space ($GFS\ Q - R_0-$space for short) if for every $(e_\lambda, x_\alpha), (e'_\gamma, y_\beta) \in GFSP(X, E)$ with $(e_\lambda, x_\alpha)\bar{q}cl(e'_\gamma, y_\beta) \Longrightarrow cl(e_\lambda, x_\alpha)$ $\bar{q}(e'_\gamma, y_\beta)$.*

(2) *Generalized fuzzy soft quasi R_1-space ($GFS\ Q - R_1-$space for short) if for every $(e_\lambda, x_\alpha), (e'_\gamma, y_\beta) \in GFSP(X, E)$ with $(e_\lambda, x_\alpha)\bar{q}cl(e'_\gamma, y_\beta)$ implies $\exists O_{(e_\lambda, x_\alpha)} \in N_{(e_\lambda, x_\alpha)}$ and $O_{(e'_\gamma, y_\beta)} \in N_{(e'_\gamma, y_\beta)}$ such that $O_{(e_\lambda, x_\alpha)}\bar{q}O_{(e'_\gamma, y_\beta)}$.*

Definition 2.15 ([16]). *A $GFST-$space (X, T, E) is said to be:*

(1) Generalized fuzzy soft quasi T_1-space ($GFS\ Q - T_1-$space for short) if for every $(e_\lambda, x_\alpha), (e'_\gamma, y_\beta) \in GFSP(X, E)$ with $(e_\lambda, x_\alpha)\bar{q}(e'_\gamma, y_\beta)$ implies there exist $O_{(e_\lambda, x_\alpha)} \in N_{(e_\lambda, x_\alpha)}$ such that $O_{(e_\lambda, x_\alpha)}\bar{q}(e'_\gamma, y_\beta)$ and there exist $O_{(e'_\gamma, y_\beta)} \in N_{(e'_\gamma, y_\beta)}$ such that $O_{(e'_\gamma, y_\beta)}\bar{q}(e_\lambda, x_\alpha)$.

(2) *Generalized fuzzy soft quasi T_2-space ($GFS\ Q - T_2-$space for short) if for every $(e_\lambda, x_\alpha), (e'_\gamma, y_\beta) \in GFSP(X, E)$ with $(e_\lambda, x_\alpha)\bar{q}(e'_\gamma, y_\beta)$ implies there exist $O_{(e_\lambda, x_\alpha)} \in N_{(e_\lambda, x_\alpha)}$ and $O_{(e'_\gamma, y_\beta)} \in N_q(e'_\gamma, y_\beta)$ such that $O_{(e_\lambda, x_\alpha)}\bar{q}O_{(e'_\gamma, y_\beta)}$.*

(3) *Generalized fuzzy soft quasi regular$-$space ($GFS\ Q$ regular$-$space for short) if for every $(e_\lambda, x_\alpha) \in GFSP(X, E)$ and $G_\delta \in T^c$ with $(e_\lambda, x_\alpha)\bar{q}G_\delta$ implies $\exists O_{(e_\lambda, x_\alpha)} \in N_{(e_\lambda, x_\alpha)}$ and $O_{G_\delta} \in N_{G_\delta}$ such that $O_{(e_\lambda, x_\alpha)}\bar{q}O_{G_\delta}$.*

(4) *Generalized fuzzy soft quasi normal$-$space ($GFS\ Q$ normal$-$space for short) if for every $F_\mu, G_\delta \in T^c$ with $F_\mu\bar{q}G_\delta$ implies $\exists O_{F_\mu} \in N_{(F_\mu)}$ and $O_{G_\delta} \in N_{(G_\delta)}$ such that $O_{F_\mu}\bar{q}O_{G_\delta}$.*

(5) *Generalized fuzzy soft quasi T_3-space ($GFS\ Q - T_3-$space for short) if $GFS\ Q$ regular and $GFS\ Q - T_1-$space.*

(6) *Generalized fuzzy soft quasi T_4-space ($GFS\ Q - T_4-$space for short) if $GFS\ Q$ normal and $GFS\ Q - T_1-$space.*

Definition 2.16 ([17]). *A family β of $GFSSs$ is a generalized fuzzy soft cover (GFS cover for short) of a $GFSS$ F_μ if $F_\mu \sqsubseteq \sqcup\{(F_\mu)_i : i \in I, (F_\mu)_i \in \beta\}$. It is a GFS open cover if each member of β is a GFS open set. A subcover of β is a subfamily of β which is also a cover.*

Definition 2.17 ([17]). *Let (X,T,E) be $GFST-$space and $F_\mu \in GFSS(X,E)$. A $GFSS$ F_μ is called generalized fuzzy soft compact ($GFS-$compact for short) if each GFS open cover of F_μ has a finite GFS open subcover. A $GFST-$space (X,T,E) is called $GFS-$compact if each GFS open cover of $\widetilde{1}_\triangle$ has a finite GFS open subcover.*

3. Generalized fuzzy soft p^*-compact topological spaces

Definition 3.1. *A family $\psi = \{(F_\mu)_i : i \in I, (F_\mu)_i \in GFSS(X,E)\}$ is called a generalized fuzzy soft cover $p-$cover (GFS $p-$cover for short) of a $GFSS$ G_δ if for all $(e_\lambda, x_\alpha)\widetilde{\in}G_\delta$ there exists $i_0 \in I$ such that $(e_\lambda, x_\alpha)\widetilde{\in}(F_\mu)_{i_0}$. It is a GFS open $p-$cover if every member of ψ is a GFS open set. A GFS $p-$ subcover of ψ is a subfamily of ψ which is also a GFS $p-$cover.*

Remark 3.2. Every GFS $p-$cover is a GFS cover in the sense of Definition 2.16. But the converse may not be true in general as shown by the following example.

Example 3.3. Let X be an infinite set and $\psi = \{(F_\mu)_n : n \in N\}$ be a family of $GFSSs$ over (X,E) defined $F_n(e)(x) = 1 - \frac{1}{n}, \mu_n(e) = 1 - \frac{1}{n}$ where $e \in E, x \in X$ and $n \in N$. Then ψ is a GFS cover of $\widetilde{1}_\triangle$. But ψ is not a GFS $p-$cover of $\widetilde{1}_\triangle$. For the GFS point $(e_1, x_1)\widetilde{\in}\widetilde{1}_\triangle$ there no exists any element in ψ containing $(e_1, x_1) = \{(e = \{\frac{x}{1}\}, 1)\}$.

Definition 3.4. *Let (X,T,E) be $GFST-$space and $F_\mu \in GFSS(X,E)$. A $GFSS$ F_μ is called a generalized fuzzy soft $p-$compact (GFS $p-$compact for short) if every GFS open $p-$cover of F_μ has a finite GFS open $p-$subcover. A $GFST-$space (X,T,E) is called a GFS $p-$compact if each GFS open $p-$cover of $\widetilde{1}_\triangle$ has a finite GFS open $p-$subcover.*

Definition 3.5. *Let $F_\mu \in GFSS(X,E)$. The soft support of F_μ, denoted by $Ssup(F_\mu)$, is a soft set given by, $Ssup(F_\mu) = \{(e, S(F(e)) : e \in E\}$, where $S(F(e))$ is the support of fuzzy set $F(e)$, which is given by the set $F(e) = \{x \in X : F(e)(x) > 0\} \subseteq X$.*

Remark 3.6. Every finite $GFSS$ F_μ (i.e., the support of $F(e), e \in E$ is finite) is GFS $p-$compact set. Also (X,T,E) is GFS $p-$compact if X is finite.

Definition 3.7. *A $GFST-$space (X,T,E) is called a GFS p^*-compact if every GFS closed set over (X,E) is a GFS $p-$compact set.*

Theorem 3.8. *Let X be an infinite set. A cofinite $GFST-$space (X,T_∞,E) is GFS p^*-compact space, where $T_\infty = \{\widetilde{0}_\theta, F_\mu \in GFSS(X,E) : S(F^c(e))$ is a finite subset of X and $e \in E\}$.*

Proof. Let F_μ be GFS close set in (X,T_∞,E), then F_μ is finite or $F_\mu = \widetilde{1}_\triangle$. Now we have two cases: If F_μ is finite, then the result holds. If $F_\mu = \widetilde{1}_\triangle$.

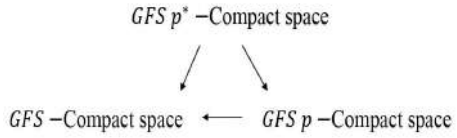
Suppose ψ is a GFS open $p-$cover of $F_\mu = \widetilde{1}_\triangle$. Choose $(e_1, x_1)\widetilde{\in}\widetilde{1}_\triangle$, then there is $O_{(e_1,x_1)} \in \psi$ and so, $O_{(e_1,x_1)^c}$ is finite. Now take $G_\delta = \{(F(e), \mu(e)) : F(e) = y_1^i, e \in E$ and $y^i \in Ssup(O^c_{(e_1,x_1)}), i = 1, 2, ...n\}$ which is finite, thus for all $(e_1, y_1^i)\widetilde{\in}G_\delta$ there exist $O_{(e_1,y_1^i)} \in \psi, i = 1, 2, ..., n$ and so the family $\{O_{(e_1,y_1^i)} : i = 1, 2, ..., n\}\sqcup\{O_{(e_1,x_1)}\}$ is a finite GFS open $p-$subcover of $F_\mu = \widetilde{1}_\triangle$, then $\widetilde{1}_\triangle$ is GFS $p-$compact. Hence (X, T_∞, E) is GFS p^*-compact. □

Remark 3.9. Every GFS p^*-compact space is GFS compact in the sense of Definition 2.17. But the converse is not necessary true, as shown by the following example.

Example 3.10. Let X be an infinite set and $T = \{\widetilde{1}_\triangle, F_\mu \in GFSS(X, E) : F_\mu \sqsubseteq (0.5)_{(X,E)}\}$, then it is easy to check that (X, T, E) is a $GFS-$compact space. but it is not GFS p^*-compact. Indeed the $GFSS$ $F_\mu = \widetilde{0.5}_{(X,E)} \in T^c$ and the family $\psi = \{(e_{0.5}, x_{0.5}) : x \in X, e \in E\}$ is a GFS open $p-$cover of F_μ which has no a finite GFS open $p-$subcover. Hence the result holds.

Remark 3.11. Every GFS p^*-compact space is GFS $p-$compact. But the converse may not be true in general, this fact can be shown by the pervious example.

From Remark 3.2, Remark 3.9, and Remark 3.11, relations between GFS-compact, GFS $p-$compact and GFS p^*-compact spaces can be described by the following diagram:



Proposition 3.12. *Let (X, T_1, E) and (X, T_2, E) be two $GFST-$space and $T_1 \sqsubseteq T_2$, then (X, T_1, E) is GFS $p-$compact (GFS p^*-compact) if (Y, T_2, E) is GFS $p-$ compact (GFS p^*-compact).*

Proof. For the first case, let $\{(F_\mu)_i : (F_\mu)_i \in T_1, i \in J\}$ be a GFS open $p-$cover of $\widetilde{1}_\triangle$. Since $T_1 \sqsubseteq T_2$, then $\{(F_\mu)_i : (F_\mu)_i \in T_2, i \in J\}$ is a GFS open $p-$cover of $\widetilde{1}_\triangle$. But (X, T_2, E) is GFS $p-$compact. So that, for all $(e_\lambda, x_\alpha)\widetilde{\in}\widetilde{1}_\triangle$ there exists $i = 1, 2, 3, ..., n : i \in J$ such that $(e_\lambda, x_\alpha)\widetilde{\in}(F_\mu)_i$ then $\{(F_\mu)_i : i = 1, 2, 3, ..., n : i \in J\}$ is a finite GFS open $p-$subcover of $\widetilde{1}_\triangle$. Hence (X, T_1, E) a is GFS $p-$compact space. The proof of the rest case is obvious. □

Remark 3.13. If X be a finite set, then (X, T, E) is GFS p^*-compact. But the convers may not be true as shown by the following example.

Example 3.14. Let X be an infinite set and $T_c = \{\widetilde{\alpha}_{(X,E)} : \alpha \in [0, 1]\}$, then (X, T_c, E) is GFS p^*-compact space.

Theorem 3.15. *Let (X,T,E) be the discrete $GFST-$space, then (X,T,E) is GFS p^*-compact if and only if X is finite.*

Proof. Let (X,T,E) be the discrete GFS p^*-compact space. Suppose that X an infinite set. Since (X,T,E) is GFS p^*-compact, then for every GFS closed set over (X,E) is a GFS $p-$compact set. Let F_μ be a GFS closed set, then for all GFS open $p-$cover of F_μ has a finite GFS open $p-$subcover. Take $\psi=\{(e_\lambda,x_\alpha):x\in X,e\in E\}$, then ψ is a GFS open $p-$cover of F_μ which has no a finite GFS open $p-$subcover. This is contradiction. Hence X is a finite set. Conversely, the proof follows direct from Remark 3.13. □

Definition 3.16. *Let $\zeta=\{(F_\mu)_i:i\in J\}$ be a family of $GFSSs$ and $G_\delta\in GFSS(X,E)$. Then:*
1) ζ is said to be have GFS $q-$intersection with respect to (w.r.t., for short) G_δ if and only if there exists $(e_\lambda,x_\alpha)\widetilde{\in}G_\delta$ such that $(e_\lambda,x_\alpha)\bar{q}(F_\mu)_i$ for all $i\in J$.
2) ζ is called has the GFS finite intersection property ($GFSFIP$, for short) w.r.t. G_δ if and only if every finite subfamily of ζ has GFS $q-$intersection w.r.t. G_δ.

Theorem 3.17. *Let (X,T,E) be a $GFST-$space. A $GFSS$ G_δ is GFS $p-$compac if and only if each family of GFS closed sets over (X,E) having the GFSFIP w.r.t. G_δ has GFS $q-$intersection w.r.t. G_δ.*

Proof. Let $G_\delta\in GFSS(X,E)$ be GFS $p-$compact and let $\zeta=\{(F_\mu)_i:i\in J\}$ be the family of GFS closed sets over (X,E) which has the $GFSFIP$ w.r.t. G_δ. Now suppose ζ has no GFS $q-$intersection w.r.t. G_δ. Then for all $(e_\lambda,x_\alpha)\widetilde{\in}G_\delta,\exists i\in J$ such that $(e_\lambda,x_\alpha)\bar{q}(F_\mu)_i$ and so, $\zeta^c=\{(F_\mu)_i^c:i\in J\}$ is a GFS open $p-$cover of G_δ. Since G_δ is GFS $p-$compac, then there is a finite GFS open $p-$subcover of ζ^c say, $\{(F_\mu)_s^c:s=1,2,...,n\in J\}$. So, $\{(F_\mu)_s:s=1,2,...,n\in J\}$ has no GFS $q-$intersection w.r.t. G_δ. Contradiction that ζ has the $GFSFIP$ w.r.t.G_δ. Hence, we obtain the result.

Conversely, let the family $\zeta=\{O^i_{(e_\lambda,x_\alpha)}:i\in J\}$ be a GFS open $p-$cover of G_δ. Then $\zeta^c=\{(O^i_{(e_\lambda,x_\alpha)})^c:i\in J\}$ has no GFS $q-$intersection w.r.t. G_δ. Thus ζ^c has no $GFSFIP$ w.r.t. G_δ. So there are $i_1,i_2,...,i_n\in J$ such that $\{(O^{i_s}_{(e_\lambda,x_\alpha)})^c:i_1,i_2,...,i_n\in J\}$ has no GFS $q-$intersection w.r.t. G_δ. Then $\{(O^{i_s}_{(e_\lambda,x_\alpha)}):i_1,i_2,...,i_n\in J\}$ is a finite GFS open $p-$subcover of G_δ. Hence G_δ is GFS $p-$compact. □

Theorem 3.18. *Every GFS closed subspace (X,T_Y,E) of a GFS p^*-compact space (X,T,E) is a GFS p^*-compact space.*

Proof. Obvious. □

4. Generalized fuzzy soft p^*-compactness and generalized fuzzy soft quasi separation axioms

Theorem 4.1. *Let (X,T,E) be a GFS $Q-T_3-$space and $G_\delta \in GFSS(X,E)$ be a GFS $p-$compact set, then for all $F_\mu \in T^c$ with $F_\mu \bar{q} G_\delta$ there are $O_{F_\mu}, O_{G_\delta} \in T$ such that $O_{F_\mu} \bar{q} O_{G_\delta}$.*

Proof. Let (X,T,E) be a GFS $Q-T_3-$space, $F_\mu \in T^c$ and $G_\delta \in GFSS(X,E)$ be GFS $p-$compac, then for every $(e_\lambda, x_\alpha)\widetilde{\in} G_\delta$ there exist $O_{(e_\lambda,x_\alpha)}, O_{F_\mu} \in T$ such that $O_{(e_\lambda,x_\alpha)} \bar{q} O_{F_\mu}$. Clearly, $\{(e_\lambda, x_\alpha) : (e_\lambda, x_\alpha)\widetilde{\in} G_\delta\}$ is a GFS open $p-$cover of G_δ. Since G_δ is GFS $p-$compac, then there exists a finite GFS open $p-$subcover of G_δ. say, $\{O^i_{(e_\lambda,x_\alpha)} : i = 1,2,...,n\}$. One readily verifies that $O_{G_\delta} = \sqcup_{i=1}^n O^i_{(e_\lambda,x_\alpha)}$ and $O_{F_\mu} = \sqcap_{i=1}^n O^i_{F_\mu}$ have the required property. □

Theorem 4.2. *Let (X,T,E) be a GFS $Q-T_2-$space, $(e_\lambda, x_\alpha) \in GFSP(X,E)$ and G_δ be GFS $p-$compact with $(e_\lambda, x_\alpha)\bar{q} G_\delta$, then there are $O_{(e_\lambda,x_\alpha)} \in T$ and $O_{G_\delta} \in T$ such that $O_{(e_\lambda,x_\alpha)} \bar{q} O_{G_\delta}$. Moreover, if F_μ and G_δ are be GFS $p-$compact with $F_\mu \bar{q} G_\delta$, then there are $O_{F_\mu} \in T$ and $O_{G_\delta} \in T$ such that $O_{F_\mu} \bar{q} O_{G_\delta}$.*

Proof. It is similar to that of the above theorem. □

Theorem 4.3. *Every GFS $p-$compact set in GFS $Q-T_2-$space is a GFS closed set.*

Proof. Let G_δ be a GFS $p-$compact set in GFS $Q-T_2-$space (X,T,E), suppose $(e_\lambda, x_\alpha)\widetilde{\in} G^c_\delta$, this implies $(e_\lambda, x_\alpha)\bar{q} G_\delta$ then from the above theorem we have for every $(e_\lambda, x_\alpha)\widetilde{\in} G_\delta$ there exist $O_{(e_\lambda,x_\alpha)} \in T$ and $O_{G_\delta} \in T$ such that $O_{(e_\lambda,x_\alpha)} \bar{q} O_{G_\delta}$ i.e., $O_{(e_\lambda,x_\alpha)} \bar{q} G_\delta$ that is, for all $(e_\lambda, x_\alpha)\widetilde{\in} G^c_\delta$ there exists $O_{(e_\lambda,x_\alpha)} \in T$ such that $O_{(e_\lambda,x_\alpha)} \sqsubseteq G^c_\delta$. Therefore, G_δ is GFS open. Hence the result holds. □

Theorem 4.4. *If (X,T,E) is GFS p^*-compact GFS $Q-T_2-$space, then (X,T,E) is a GFS $Q-T_4-$space.*

Proof. Let (X,T,E) be a GFS p^*-compact GFS $Q-T_2-$space. Let $G_\delta, F_\mu \in T^c$ with $F_\mu \bar{q} G_\delta$. Since (X,T,E) is GFS p^*-compact, then G_δ, F_μ are GFS p-compact and so, from Theorem 4.2, there exist $O_{F_\mu} \in T$ and $O_{G_\delta} \in T$ such that $O_{F_\mu} \bar{q} O_{G_\delta}$. Hence (X,T,E) is a GFS $Q-T_4-$space. □

Corollary 4.5. *If (X,T,E) is GFS p^*-compact GFS $Q-T_2-$space, then (X,T,E) is a GFS $Q-T_3-$space.*

Theorem 4.6. *Let (X,T,E) be a GFS $Q-R_1-$space. Then (X,T,E) is a GFS $Q-T_2-$space if and only if every GFS $p-$compact set is GFS closed.*

Proof. The necessity follows from Theorem 4.3

Conversely, let every GFS $p-$compact set is GFS closed, first we prove that (X,T,E) is a GFS $Q-T_1-$space. Let $(e_\lambda, x_\alpha), (e'_\gamma, y_\beta) \in GFSP(X,E)$

with $(e_\lambda, x_\alpha)\bar{q}(e'_\gamma, y_\beta)$, then $(e_\lambda, x_\alpha) \sqsubseteq (e'_\gamma, y_\beta)^c = O_{(e_\lambda, x_\alpha)} \in T$ and $(e'_\gamma, y_\beta) \sqsubseteq (e_\lambda, x_\alpha)^c = O_{(e'_\gamma, y_\beta)} \in T$ (since $(e_\lambda, x_\alpha), (e'_\gamma, y_\beta)$ are closed).

So, that there exist $O_{(e_\lambda, x_\alpha)}$ and $O_{(e'_\gamma, y_\beta)}$ such that $(e_\lambda, x_\alpha)\bar{q}(e_\lambda, x_\alpha)^c = O_{(e'_\gamma, y_\beta)}$ and $(e'_\gamma, y_\beta)\bar{q}(e'_\gamma, y_\beta)^c = O_{(e_\lambda, x_\alpha)}$. Hence (X, T, E) is a $GFS\ Q - T_1$. Now, (X, T, E) is $GFS\ Q - R_1$ and $GFS\ Q - T_1-$space, $(e_\lambda, x_\alpha), (e'_\gamma, y_\beta) \in GFSP(X, E)$ with $(e_\lambda, x_\alpha)\bar{q}(e'_\gamma, y_\beta)$ by $GFS\ Q-T_1$ implies that $(e_\lambda, x_\alpha)\bar{q}cl(e'_\gamma, y_\beta)$ [as $(e'_\gamma, y_\beta) = cl(e'_\gamma, y_\beta)$ see[16]. Theorem 4.3] and by $GFS\ Q - R_1$, there exist $O_{(e_\lambda, x_\alpha)}, O_{(e'_\gamma, y_\beta)} \in T$ such that $O_{(e_\lambda, x_\alpha)}\bar{q}O_{(e'_\gamma, y_\beta)}$. Hence (X, T, E) is a $GFS\ Q - T_2$. □

Theorem 4.7. *Every $GFS\ p^*-$compact $GFS\ Q - R_1-$space (X, T, E) is a $GFS\ Q$ regular ($GFS\ Q$ normal)$-$space.*

Proof. We prove the theorem for $GFS\ Q$ regular, the proof of the rest case is similar.

Let (X, T, E) be a $GFS\ p^*-$compact $GFS\ Q - R_1-$space and $F_\mu \in T^c$ with $(e_\lambda, x_\alpha)\bar{q}F_\mu$. Then for all GFS point $(e'_\gamma, y_\beta)\widetilde{\in}F_\mu$ we have, $(e_\lambda, x_\alpha)\bar{q}cl(e'_\gamma, y_\beta)$. Since (X, T, E) is $GFS\ Q - R_1$, then there exist $O_{(e_\lambda, x_\alpha)} \in T$ and $O_{(e'_\gamma, y_\beta)} \in T$ such that $O_{(e_\lambda, x_\alpha)}\bar{q}O_{(e'_\gamma, y_\beta)}$. Then the family $\{O_{(e'_\gamma, y_\beta)} : (e'_\gamma, y_\beta)\widetilde{\in}F_\mu\}$ is a GFS open $p-$cover of F_μ. Since (X, T, E) is $GFS\ p^*-$compact, then F_μ is $GFS\ p-$compact, and so there exists $\{O^i_{(e'_\gamma, y_\beta)} : (e'_\gamma, y_\beta)\widetilde{\in}F_\mu, i = 1, 2, ..., n\}$ is a finite GFS open $p-$subcover of F_μ. Now take $O^*_{(e_\lambda, x_\alpha)} = \sqcap^n_{i=1} O^i_{(e_\lambda, x_\alpha)}$ and $O_{F_\mu} = \sqcup^n_{i=1} O^i_{(e'_\gamma, y_\beta)}$, then $O^*_{(e_\lambda, x_\alpha)}, O_{F_\mu} \in T$ and $O^*_{(e_\lambda, x_\alpha)}\bar{q}O_{F_\mu}$. Hence (X, T, E) is a $GFS\ Q$ regular$-$space. □

Corollary 4.8. *Let (X, T, E) be $GFS\ p^*-$compact, then the following statements are equivalent:*

1) *(X, T, E) is $GFS\ Q - R_1$,*
2) *(X, T, E) is $GFS\ Q$ regular,*
3) *(X, T, E) is $GFS\ Q - R_0$ and $GFS\ Q$ normal.*

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Accepted: June 14, 2020

Ricci solitons on (LCS)-manifolds under D-homothetic deformation

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Abstract. The object of the present paper is to study Ricci solitons in LCS-manifolds under D-homothetic deformation. We examine the behavior of Ricci solitons when potential vector field is orthogonal to Reeb vector field and pointwise collinear with the Reeb vector field. Also, we have show the Ricci solitons in D-homothetically transformed LCS-manifolds are expanding, shrinking and steady.

Keywords: Ricci soliton, Einstein manifold, η-Einstein manifold, LCS-manifold, D-homothetic deformation gradient Ricci soliton.

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1. Introduction

An n-dimensional Lorentzian manifold M is a smooth connected paracompact Hausdroff manifold with a Lorentzian metric of type $(0, 2)$ such that for each point $pinM$, the tensor $g_p : T_pM \times T_pM \to R$ is a non-degenerate inner product of signature $(-, +, \cdots, +)$, where T_pM denotes the tangent vector space of M at p and R is the real number space. A non-zero vector $V \epsilon T_pM$ is said to be timelike (resp. nonspacelike, null, spacelike) if it satisfies $g_p(V, V) < 0$ (resp. $\leq 0, = 0, > 0$) ([2, 5]).

Ricci soliton is a Riemannian metric g on a manifold M together with a vector field V such that,

$$L_V g + 2S + 2\lambda g = 0, \tag{1}$$

where S is a Ricci tensor of M and L_V indicates the "Lie derivative" operator with the vector field V. Ricci soliton is a generalization of Einstein metric. The Ricci soliton is noted shrinking, steady and expanding when λ is negative, zero and positive respectively. If the vector field V is the gradient of a potential function f (denoted by Df, where D indicates gradient operator), then g is called a gradient Ricci soliton and the equation (1) becomes:

$$\nabla\nabla f = Ric + \lambda g. \tag{2}$$

Let M be the Lorentzian manifold admitting a unit timelike concircular vector field ξ, called the characteristic vector field of the manifold. Then we have:

$$g(\xi, \xi) = -1. \tag{3}$$

Since ξ is an unit concircular vector field, there exists a non-zero 1-form η such that

$$g(Z_1, \xi) = \eta(Z_1), \tag{4}$$

the equation of the following form holds:

$$(\nabla_{Z_1}\eta)(Z_2) = \alpha\{g(Z_1, Z_2) + \eta(Z_1)\eta(Z_2)\}, \quad (\alpha \neq 0), \tag{5}$$

for all vector fields Z_1, Z_2 where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g and α is a non-zero scalar function satisfying

$$\nabla_{Z_1}\alpha = (Z_1\alpha) = \alpha(Z_1) = \rho\eta(Z_1), \tag{6}$$

ρ being a certain scalar function.

If we put

$$\phi Z_1 = \frac{1}{\alpha}\nabla_{Z_1}\xi, \tag{7}$$

then from (5) and (7) we have

$$\phi Z_1 = Z_1 + \eta(Z_1)\xi, \tag{8}$$

from which it follows that ϕ is a symmetric $(1,1)$ tensor. From (5) and (7) we have

$$\phi^2 Z_1 = Z_1 + \eta(Z_1)\xi. \tag{9}$$

Hence M is a manifold with a Lorentzian almost paracontact structure (ϕ, ξ, η, g) introduced by Matsumoto [3], Mihai and Rosca [4]. Thus the Lorentzian manifold M together with the unit timelike vector field ξ, its associated 1-form η and $(1,1)$ tensor field ϕ is said to be a Lorentzian almost paracontact manifold with a structure of the concircular type and such a manifold is said to be a $(LCS)_n$-manifold ([6, 11]).

Somashekhara et al. [9], studied some results on invariant sub-manifolds of LP-Sasakian manifolds endowed with semi-symmetric metric conection. Also, they have obtained a condition for totally geodesic by using certain geometrical conditions. In [10], the authors studied the C-Bochner curvature tensor under D-homothetic deformation in LP-Sasakian manifolds.

2. Preliminaries

A differentiable manifold M of $(2n+1)$-dimension is called $(LCS)^n$-manifold if it admits a $(1,1)$ tensor field ϕ, a contravariant vector field ξ, a covariant vector field η and a Lorentzian metric g which satisfy

$$\nabla_{Z_1}\xi = \alpha\phi Z_1, \tag{10}$$

$$\eta(\xi) = -1, \phi\xi = 0, \eta(\phi\xi) = 0, g(Z_1, \xi) = \eta(Z_1), \tag{11}$$

$$g(\phi Z_1, \phi Z_2) = g(Z_1, Z_2) + \eta(Z_1)\eta(Z_2), \tag{12}$$

$$\phi^2 Z_1 = Z_1 + \eta(Z_1)\xi, \tag{13}$$

for all $Z_1, Z_2 \in TM$. Also in a $(LCS)_n$-manifold M the following relations are satisfied ([7]):

$$R(Z_1, Z_2)\xi = (\alpha^2 - \rho)[\eta(Z_2)Z_1 - \eta(Z_1)Z_2], \tag{14}$$

$$S(Z_1, \xi) = (n-1)(\alpha^2 - \rho)\eta(Z_1), i.e, Q\xi = (n-1)(\alpha^2 - \rho)\xi, \tag{15}$$

$$(\nabla_{Z_1}\phi)(Z_2) = \alpha\{g(Z_1, Z_2)\xi + 2\eta(Z_1)\eta(Z_2)\xi + \eta(Z_2)Z_1\}. \tag{16}$$

3. Ricci solitons in (LCS)-manifolds under D-homothetic deformation

Let (M, ϕ, ξ, η, g) be a Lorentzian concircular structure manifold (LCS-manifold), where g is a Ricci soliton. A D-homothetic deformation ([12]) on M is given by

$$\phi^* = \phi, \quad \zeta^* = \frac{1}{\alpha}\zeta, \quad \eta^* = a\eta, \quad g^* = bg - (a^2 - b)\eta \otimes \eta, \tag{17}$$

where a is a positive constant. Then the D-homothetically changed structure $(M, \phi^*, \xi^*, \eta^*, g^*)$ is also a LCS-structure ([8]). Throughout this paper the quantity with $*$ denote the quantities in $(M, \phi^*, \xi^*, \eta^*, g^*)$ and quantities without $*$ are from (M, ϕ, ξ, η, g).

The relation between the connections ∇ and ∇^* is given by ([1]):

$$\nabla^*_{Z_1} Z_2 = \nabla_{Z_1} Z_2 - \frac{(a^2 - b)\alpha}{a^2} \{g(Z_1, Z_2) + \eta(Z_1)\eta(Z_2)\} \zeta, \tag{18}$$

for any vector fields Z_1, Z_2 on M.

Using (18), we now calculate the Riemann curvature tensor R^* of $(M, \phi^*, \xi^*, \eta^*, g^*)$ as follows:

$$\begin{aligned} R^*(Z_1, Z_2)Z_3 = R(Z_1, Z_2)Z_3 - \frac{(a^2 - b)\alpha^2}{a^2} [\{g(Z_2, Z_3)Z_1 - g(Z_1, Z_3)Z_2\} \\ + \{\eta(Z_2)Z_1 - \eta(Z_1)Z_2\} \eta(Z_3)], \end{aligned} \tag{19}$$

for any vector fields Z_1, Z_2, Z_3 on M.

On contracting (19), we obtain the Ricci tensor S^* of D-homothetically deformed LCS manifold as:

$$\begin{aligned} &S^*(Z_2, Z_3) \\ &= S(Z_2, Z_3) - \frac{(a^2 - b)[(n-3)\alpha^2 b + 2a^2\rho]}{a^2 b}[g(Z_2, Z_3) + \eta(Z_2)\eta(Z_3)]. \end{aligned} \tag{20}$$

Contracting (20) with respect to Z_2 and Z_3, we get

$$r^* = r - \frac{(a^2 - b)[(n-3)\alpha^2 b + 2a^2\rho](n-1)}{a^2 b}, \tag{21}$$

where r^* and r are the scalar curvatures of (M, g) and (M, g^*) respectively.

Using (17) and (20), we can write

$$\begin{aligned} &(L^*_V g^*)(Z_1, Z_2) + 2S^*(Z_1, Z_2) + 2\lambda g^*(Z_1, Z_2) \\ &= g^*(\nabla^*_{Z_1} V, Z_2) + g^*(Z_1, \nabla^*_{Z_2} V) \\ &+ 2S(Z_1, Z_2) - \frac{2(a^2 - b)[(n-3)\alpha^2 b + 2a^2\rho]}{a^2 b}[g(\phi Z_1, \phi Z_2)]. \end{aligned} \tag{22}$$

Making use of (18) in (22), we obtain

$$
\begin{aligned}
(23)\qquad & (L_V^* g^*)(Z_1, Z_2) + 2S^*(Z_1, Z_2) + 2\lambda g^*(Z_1, Z_2) \\
& = b[g(\nabla_{Z_1} V, Z_2) + g(Z_1, \nabla_{Z_2} V)] \\
& - \frac{(a^2 - b)\alpha}{a^2}(2b - a^2)[g(\phi Z_1, \phi Z_2)\eta(Z_2) + g(\phi Z_2, \phi V)\eta(Z_1)] \\
& - (a^2 - b)[\eta(\nabla_{Z_1} Z_2)\eta(Z_2) + \eta(\nabla_{Z_2} V)\eta(Z_1)] + 2S(Z_1, Z_2) \\
& - \frac{2(a^2 - b)[(n-3)\alpha^2 b + 2a^2\rho]}{a^2 b}[g(\phi Z_1, \phi Z_2)] \\
& + 2\lambda[bg(Z_1, Z_2) - (a^2 - b)\eta(Z_1)\eta(Z_2)].
\end{aligned}
$$

If $V \perp \xi$, it provides $\eta(\nabla_{Z_1} V) = -g(\phi V, \phi Z_1)$. Hence, the equation (23) can be written as

$$
\begin{aligned}
(24)\qquad & (L_V^* g^*)(Z_1, Z_2) + 2S^*(Z_1, Z_2) + 2\lambda g^*(Z_1, Z_2) \\
& = b[g(\nabla_{Z_1} V, Z_2) + g(Z_1, \nabla_{Z_2} V) + 2S(Z_1, Z_2) + 2\lambda g(Z_1, Z_2)] \\
& + 2(1 - b)S(Z_1, Z_2) - \frac{2(a^2 - b)[(n-3)\alpha^2 b + 2a^2\rho]}{a^2 b}[g(\phi Z_1, \phi Z_2)] \\
& - 2\lambda(a^2 - b)\eta(Z_1)\eta(Z_2)].
\end{aligned}
$$

Since g is a Ricci soliton, the above equation takes the form

$$
\begin{aligned}
(25)\qquad & (L_V^* g^*)(Z_1, Z_2) + 2S^*(Z_1, Z_2) + 2\lambda g^*(Z_1, Z_2) \\
& = 2(1 - b)S(Z_1, Z_2) - \frac{2(a^2 - b)[(n-3)\alpha^2 b + 2a^2\rho]}{a^2 b}[g(\phi Z_1, \phi Z_2)] \\
& - 2\lambda(a^2 - b)\eta(Z_1)\eta(Z_2)].
\end{aligned}
$$

Thus g^* is a Ricci soliton if and only if

$$
\begin{aligned}
(26)\qquad S(Z_1, Z_2) = & \frac{(a^2 - b)[(n-3)\alpha^2 b + 2a^2\rho]}{(1 - b)a^2 b}[g(Z_1, Z_2)] \\
& + \left\{ \frac{(a^2 - b)[(n-3)\alpha^2 b + 2a^2\rho]}{(1 - b)a^2 b} + \frac{\lambda(a^2 - b)}{1 - b} \right\} \eta(Z_1)\eta(Z_2).
\end{aligned}
$$

By taking $Z_1 = Z_2 = \xi$ in (26), we get

$$
(27)\qquad \lambda = \frac{(1 - b)(n - 1)(\alpha^2 - \rho)}{a^2 - b} + \frac{2[(n-3)\alpha^2 b + 2\alpha^2\rho]}{a^2 b}.
$$

Thus, we have the following:

Theorem 3.1. *Let $(M, \phi^*, \xi^*, \eta^*, g^*)$ be obtained by D-homothetic deformation of a LCS-manifold (M, ϕ, ξ, η, g) with the potential vector field V orthogonal to Reeb vector field. If (M, g) is η-Einstein then (M, g^*) is a Ricci soliton and the Ricci soliton is expanding, shrinking and steady according to the equation* (27) *as $\lambda > 0, \lambda < 0$ and $\lambda = 0$.*

Now, let V be pointwise collinear with the Reeb vector field ξ^* i.e. $V = \beta\xi^*$, where β is a function on M. Then using (22), we can write

$$g[\nabla^*_{Z_1}(\beta\xi^*), Z_2] + g[\nabla^*_{Z_2}(\beta\xi^*), Z_1] + 2S^*(Z_1, Z_2) + 2\lambda g^*(Z_1, Z_2) = 0. \tag{28}$$

By using (17), (18), (20) in (28), we obtain:

$$\begin{aligned}\frac{1}{\alpha}[(Z_2\beta)\eta(Z_1) + (Z_1\beta)\eta(Z_2)] + 2S(Z_1, Z_2) - \\ \frac{2(a^2-b)[(n-3)\alpha^2 b + 2a^2\rho]}{a^2 b} g(\phi Z_1, \phi Z_2) + \\ 2\lambda[bg(Z_1, Z_2) - (a^2-b)\eta(Z_1)\eta(Z_2)] = 0.\end{aligned} \tag{29}$$

By setting $Z_2 = \xi$ in (29) and using (15), we obtain

$$Z_1\beta = [2\alpha(n-1)(\alpha^2-\rho) + 2\lambda\alpha a^2 + \xi\beta]\eta(Z_1). \tag{30}$$

Again replacing $Z_1 = \xi$ in (30)

$$\xi\beta = -\alpha(n-1)(\alpha^2-\rho) - \lambda\alpha(2b-a^2). \tag{31}$$

Substituting this in (30), we have

$$Z_1\beta = [-\alpha(n-1)(\alpha^2-\rho) - \lambda\alpha(2b-a^2)]\eta(Z_1). \tag{32}$$

By applying d on (32), we get

$$[-\alpha(n-1)(\alpha^2-\rho) - \lambda\alpha(2b-a^2)]d\eta = 0. \tag{33}$$

Since $d\eta \neq 0$,from (33), we have

$$[-\alpha(n-1)(\alpha^2-\rho) - \lambda\alpha(2b-a^2)] = 0. \tag{34}$$

Substituting (34) in (32), we conclude that β is a constant. Hence it is verified from (29)

$$\begin{aligned}S(Z_1, Z_2) = &\left\{\frac{(a^2-b)[(n-3)\alpha^2 b + 2a^2\rho]}{a^2 b} - \lambda b\right\} g(Z_1, Z_2) \\ &+ \left\{\frac{(a^2-b)[(n-3)\alpha^2 b + 2a^2\rho]}{a^2 b} + \lambda(a^2-b)\right\}\eta(Z_1)\eta(Z_2).\end{aligned} \tag{35}$$

Further from (34), we have $\lambda = \dfrac{-(n-1)(\alpha^2-\rho)}{(2b-a^2)}$.

Theorem 3.2. *Under D-homothetic deformation, LCS-manifold admitting a Ricci soliton (g^*, V, λ) with potential vector field V is pointwise collinear with ξ^*, the manifold is an η-Einstein manifold and the following conditions hold (i) Expanding for $\lambda > 0$ (ii) Shrinking for $\lambda < 0$ (iii) Steady for $\lambda = 0$.*

An interesting invariant of a concircular transformation is the concircular curvature tensor. The concircular curvature tensor [13] C^* under D-homothetic deformation ∇^* is defined by

$$C^*(Z_1,Z_2)Z_3 = R^*(Z_1,Z_2)Z_3 - \frac{r^*}{2n(2n+1)}[g^*(Z_2,Z_3)Z_1 - g^*(Z_1,Z_3)Z_2], \tag{36}$$

for all vector fields Z_1, Z_2, Z_3 on M.

Suppose M is concirularly flat. Then from (36), we have

$$R^*(Z_1,Z_2)Z_3 = \frac{r^*}{2n(2n+1)}[g^*(Z_2,Z_3)Z_1 - g^*(Z_1,Z_3)Z_2]. \tag{37}$$

Using (17), (19) and (21) in (37), we obtain

$$\begin{aligned} R(Z_1,Z_2)Z_3 &= \left[\frac{(a^2-b)\alpha^2}{a^2} + \frac{a^2br - (a^2-b)[(n-3)\alpha^2b + 2a^2\rho](n-1)}{a^2 2n(2n+1)}\right]A \\ &+ \left[\frac{(a^2-b)\alpha^2}{a^2} - \frac{a^2br - (a^2-b)[(n-3)\alpha^2b + 2a^2\rho](n-1)}{a^2b2n(2n+1)}\right]B, \end{aligned} \tag{38}$$

where $A = [g(Z_2,Z_3)Z_1 - g(Z_1,Z_3)Z_2]$ and $B = [\eta(Z_2)\eta(Z_3)Z_1 - \eta(Z_1)\eta(Z_3)Z_2]$. Contracting (38) with respect to U, taking $Z_1 = U = e_i$ and summing over $i = 1, 2, \cdots, (2n+1)$ we get

$$S(Z_2,Z_3) = [A_1]\, g(Z_2,Z_3) + [A_2]\, \eta(Z_2)\eta(Z_3), \tag{39}$$

where

$$A_1 = \left[\frac{(a^2-b)\alpha^2(n-1)}{a^2} + \frac{a^2br(n-1) - (a^2-b)[(n-3)\alpha^2b + 2a^2\rho](n-1)^2}{a^2 2n(2n+1)}\right]$$

and

$$A_2 = \left[\frac{(a^2-b)\alpha^2(n-1)}{a^2} - \frac{a^2br(n-1) - (a^2-b)[(n-3)\alpha^2b + 2a^2\rho](n-1)^2}{a^2b2n(2n+1)}\right].$$

Suppose g^* is a Ricci soliton. Then by comparing (39) and (26), we have

$$\lambda = \frac{-(b+1)2n[a^2bn(\alpha^2-\rho) - (a^2-b)[(2n-2)\alpha^2b + 2a^2\rho]}{a^4b}. \tag{40}$$

This leads to the following:

Theorem 3.3. *Under D-homothetic deformation of concircularly flat LCS-manifold (M,ϕ,ξ,η,g), where potential vector field V is orthogonal to Reeb vector field, g^* in (M,g^*) is a Ricci soliton and the Ricci soliton is expanding, shrinking and steady according to the equation* (40) *as $\lambda > 0, \lambda < 0$ and $\lambda = 0$.*

Taking $Z_3 = \xi$ in (36) and using (19) and (14), we get

$$(C^*(Z_1, Z_2)Z_3) = \left[(\alpha^2 - \rho) + \frac{(a^2 - 2b)r^*}{2n(2n+1)}\right][\eta(Z_2)Z_1 - \eta(Z_1)Z_2]. \tag{41}$$

Similarly using (14), (17),(19) in (36)

$$\eta(C^*(Z_1, Z_2)Z_3) = \left[1 + \frac{(a^2 - b)\alpha^2 + r^*a^2b}{2n(2n+1)}\right] A_3, \tag{42}$$

where $A_3 = [g(Z_1, Z_3)\eta(Z_2) - g(Z_2, Z_3)\eta(Z_1)Z_2]$.

Suppose $C^* \cdot S^* = 0$. Then we have

$$S^*(C^*(Z_1, Z_2)Z_3, U) + S^*(Z, C^*(Z_1, Z_2)U) = 0. \tag{43}$$

Taking $U = \xi$ in (43) and using (20), we obtain

$$\begin{aligned}(n-1)(\alpha^2-\rho)&\left\{1+\frac{(\alpha^2-b)\alpha^2+a^2br^*}{2n(2n+1)}\right\}[g(Z_1, Z_3)\eta(Z_2)-g(Z_2, Z_3)\eta(Z_1)] \\ &+\left\{(\alpha^2 - \rho) + \frac{(\alpha^2 - 2b)r^*}{2n(2n+1)}\right\}[\eta(Z_2)S^*(Z_3, Z_1) - \eta(Z_1)S^*(Z_3, Z_2)] = 0.\end{aligned} \tag{44}$$

Again by setting $Z_1 = \xi$ in (44) and making use of (20) and (42), we obtain

$$S(Z_2, Z_3) = \frac{(a^2 - b)}{a^2b}\{(n-3)(b + 2a^2)\}[g(Z_2, Z_3) + \eta(Z_2)\eta(Z_3)] = 0. \tag{45}$$

Theorem 3.4. *Under D-homothetic deformation of LCS-manifold is fulfilled* $C^* \cdot S^* = 0$*, then* (M, g^*, λ) *is an* η*-Einstein manifold.*

4. Gradient Ricci solitons in (LCS)-manifolds under *D*-homothetic deformation

We now consider the Lie derivative of g^* along V and using by hypothesis (1), we get

$$(L_V g^*)(Z_1, Z_2) + 2S^*(Z_1, Z_2) + 2\lambda g^*(Z_1, Z_2) = 0. \tag{46}$$

By using (17) and (20) in (46), we have

$$\begin{aligned}&b(L_V g)(Z_1, Z_2) - (a^2 - b)[(L_V\eta)(Z_1)\eta(Z_2) + \eta(Z_1)(L_V\eta)(Z_2)] \\ &+ 2S(Z_1, Z_2) - [A_4]\, g(Z_1, Z_2) - [A_5]\, \eta(Z_1)\eta(Z_2) \\ &+ 2\lambda[bg(Z_1, Z_2) - (a^2 - b)\eta(Z_1)\eta()Z_2] = 0,\end{aligned} \tag{47}$$

where

$$A_4 = \left[\frac{2(a^2 - b)\{(n-3)\alpha^2b + 2a^2\rho\}}{a^2b}\right],$$

$$A_5 = \left[\frac{2(a^2 - b)\{(n-3)\alpha^2b + 2a^2\rho\}}{a^2b}\right].$$

Employing $(L_V\eta)(Z_1) = -2[2n(\alpha^2-\rho)+\lambda]\eta(Z_1)$, then (47) takes the form

$$
\begin{aligned}
&b(L_Vg)(Z_1,Z_2) + 4(a^2-b)[2n(\alpha^2-\rho)+\lambda]\eta(Z_1)\eta(Z_2) + 2S(Z_1,Z_2)\\
&= \left[\frac{2(a^2-b)\{(n-3)\alpha^2b+2a^2\rho\}}{a^2b} - 2\lambda b\right] g(Z_1,Z_2)\\
&+ \left[\frac{2(a^2-b)\{(n-3)\alpha^2b+2a^2\rho\}}{a^2b} + 2\lambda(a^2-b)\right].\eta(Z_1)\eta(Z_2)
\end{aligned} \tag{48}
$$

Consider $(M,\phi^*,\xi^*,\eta^*,g^*)$ a $(2n+1)$-dimensional contact metric manifold. Let us take $V = Df = \operatorname{grad} f$ in (48) and using (2)

(49)

$$
\begin{aligned}
bg[\nabla_{Z_1}Df, Z_2] &= g(QZ_1,Z_2) - \left[\frac{(a^2-b)\{(n-3)\alpha^2b+2a^2\rho\}}{a^2b} - \lambda b\right] g(Z_1,Z_2)\\
&+ (a^2-b)\left[4n(\alpha^2-\rho) - \frac{(n-3)\alpha^2b+2a^2\rho}{a^2b}\right]\eta(Z_1)\xi.
\end{aligned}
$$

$$
\begin{aligned}
\nabla_{Z_1}Df &= \frac{1}{b}\left[QZ_1 - \left[\frac{(a^2-b)\{(n-3)\alpha^2b+2a^2\rho\}}{a^2b} - \lambda b\right] Z_1\right.\\
&\left.+(a^2-b)\left[4n(\alpha^2-\rho) - \frac{(n-3)\alpha^2b+2a^2\rho}{a^2b}\right]\eta(Z_1)\xi\right].
\end{aligned} \tag{50}
$$

For all vector fields $Z_1 \in TM$, where D denotes gradient operator of g. Now, from (50), it follows that

$$
\begin{aligned}
R(Z_1,Z_2)Df &= \frac{1}{b}\left[(\nabla_{Z_1}Q)Z_2 - (\nabla_{Z_2}Q)Z_1\right]\\
&+ \frac{\alpha(a^2-b)}{b}\left[4n(\alpha^2-\rho) - \frac{(n-3)\alpha^2b+2a^2\rho}{a^2b}\right][\eta(Z_2)\phi Z_1 - \eta(Z_1)\phi Z_2]\\
&- \frac{2(a^2-b)}{b}\left[2n(\alpha^2-\rho) - \frac{(n-3)\alpha^2b+2a^2\rho}{a^2b}\right] g(\phi Z_1,Z_2)\xi.
\end{aligned} \tag{51}
$$

Putting $Z_1 = \xi$ in (51) and taking inner product with ξ, we get

$$
g[R(Z_1,Z_2)Df,\xi] = \frac{1}{b}g\left([(\nabla_\xi Q)Z_2 - (\nabla_{Z_2}Q)\xi],\xi\right). \tag{52}
$$

In a LCS-manifold, $g\left([(\nabla_\xi Q)Z_2 - (\nabla_{Z_2}Q)\xi],\xi\right) = 0$.

$$
g[R(Z_1,Z_2)Df,\xi] = 0. \tag{53}
$$

Then, we get:

$$
g[R(Z_1,Z_2)Df,\xi] = g[(Df - (\xi f)\xi), Z_2]. \tag{54}
$$

From (53) and (54), it follows

$$Df = (\xi f)\xi). \tag{55}$$

Using (55) in (50), we get

$$\begin{aligned} S(Z_1, Z_2) &- \left[\frac{(a^2-b)\{(n-3)\alpha^2 b + 2a^2\rho\}}{a^2 b} - \lambda b\right] g(Z_1, Z_2) \\ &+ (a^2-b)\left[4n(\alpha^2-\rho) - \frac{(n-3)\alpha^2 b + 2a^2\rho}{a^2 b}\right]\eta(Z_1)\eta(Z_2) \\ &= b[Z_1(\xi f)\eta(Z_2) + 2(\xi f)g(\phi Z_1, Z_2)]. \end{aligned} \tag{56}$$

Symmetrizing this with respect to Z_1 and Z_2, we obtain

$$\begin{aligned} 2S(Z_1, Z_2) &- 2\left[\frac{(a^2-b)\{(n-3)\alpha^2 b + 2a^2\rho\}}{a^2 b} - \lambda b\right] g(Z_1, Z_2) \\ &+ 2(a^2-b)\left[4n(\alpha^2-\rho) - \frac{(n-3)\alpha^2 b + 2a^2\rho}{a^2 b}\right]\eta(Z_1)\eta(Z_2) \\ &= b[Z_1(\xi f)\eta(Z_2) + Z_2(\xi f)\eta(Z_1)]. \end{aligned} \tag{57}$$

Putting $Z_2 = \xi$ and $a^2 = b$ in (57), we get

$$Z_1(\xi f) = -2\left(4n\frac{(\alpha^2-\rho)}{a^2} + \lambda\right)\eta(Z_1). \tag{58}$$

Now, from (57) and (58), we get

$$\begin{aligned} S(Z_1, Z_2) &= \left[\frac{(a^2-b)\{(n-3)\alpha^2 b + 2a^2\rho\}}{a^2 b} - \lambda b\right] g(Z_1, Z_2) \\ &- \left[2b\{\frac{2n(\alpha^2-\rho)}{a^2} + \lambda\} + (a^2-b)\{2n(\alpha^2-\rho)\right. \\ &\left. - \frac{(n-3)\alpha^2 b + 2a^2\rho}{a^2 b}\}\right]\eta(Z_1)\eta(Z_2). \end{aligned} \tag{59}$$

Employing (59) in (50), we obtain

$$\nabla_{Z_1} Df = -2\left[\frac{2n(\alpha^2-\rho)}{a^2} + \lambda\right]\eta(Z_1)\xi. \tag{60}$$

Using (60), we compute $R(Z_1, Z_2)Df$ and obtain

$$g[R(Z_1, Z_2)Df, \xi] = 4\left[\frac{2n(\alpha^2-\rho)}{a^2} + \lambda\right] d\eta(Z_1, Z_2). \tag{61}$$

Using (53), and noting the face that $d\eta \neq 0$, we get

$$\frac{2n(\alpha^2-\rho)}{a^2} + \lambda = 0. \tag{62}$$

Therefore, from (58), we have $Z_1(\xi f) = 0, Z_1 \in TM$. That is, $\xi f = c$, where c is a constant. Thus (55) gives $df = c\eta$. Exterior differentiation of the above yields $cd\eta = 0$. Hence $c = 0$. Thus, f is a constant. Consequently, the equation (50) reduces to

$$S(Z_1, Z_2) = \left[\frac{(a^2 - b)\{(n-3)\alpha^2 b + 2a^2\rho\}}{a^2 b} - \lambda b\right] g(Z_1, Z_2) - (a^2 - b)\left[2n(\alpha^2 - \rho) - \frac{(n-3)\alpha^2 b + 2a^2\rho}{a^2 b}\right] \eta(Z_1)\eta(Z_2). \tag{63}$$

Also, $\lambda = \dfrac{-2n(\alpha^2 - \rho)}{a^2}$.

Theorem 4.1. *Under D-homothetic deformation of LCS-manifold is a gradient Ricci soliton, then it is an η-Einstein manifold and the following holds:*

(*i*) *Expanding for $\lambda > 0$;*

(*ii*) *Shrinking for $\lambda < 0$;*

(*iii*) *Steady for $\lambda = 0$.*

Acknowledgment

The authors would like to thank the anonymous referee for his comments that helped us improve this article.

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Accepted: March 23, 2020

Ricci solitons and generalized weak symmetries under D-homothetically deformed LP-Sasakian manifolds

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Abstract. The object of the present paper is to investigate the nature of Ricci solitons and generalized weak symmetries under D-homothetically deformed LP-Sasakian manifolds.

Keywords: LP-Sasakian manifold, D-homothetic deformation, Ricci solitons, generalized weakly symmetric manifold.

1. Introduction

In this paper, we consider an almost contact manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ that consists of a (1,1)-tensor field ϕ, a vector field ξ and a 1-form η called respectively the structure endomorphism, the characteristic vector field and the contact form. Let ∇ and ∇^* be Levi-Civita and D-homothetic deformation connections. Let R, S, Q, r, $\bar{R}$ and R^*, S^*,Q^*, r^*,$\bar{R}^*$ respectively stand for Riemannian curvature tensor, Ricci tensor, Ricci operator, scalar curvature, Riemannian curvature tensor of type $(0,4)$ with respect to ∇ and ∇^* respectively. In [1], the author introduced a new type of space called generalized weakly symmetric manifold. In

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this paper, we study some results on D-homothetic deformation of LP-Sasakian manifold.

A Riemannian manifold is said to be generalized weakly symmetric if it satisfies the condition (See [1])

$$\begin{aligned}(\nabla_{X_1}\bar{R})(X_2,Y_1,Y_2,Y_3) &= \alpha_1(X_1)\bar{R}(X_2,Y_1,Y_2,Y_3)+\beta_1(X_2)\bar{R}(X_1,Y_1,Y_2,Y_3)\\ &+\beta_1(Y_1)\bar{R}(X_2,X_1,Y_2,Y_3)+\delta_1(Y_2)\bar{R}(X_2,Y_1,X_1,Y_3)\\ &+\delta_1(Y_3)\bar{R}(X_2,Y_1,Y_2,X_1)+\alpha_2(X_1)\bar{G}(X_2,Y_1,Y_2,Y_3)\\ &+\beta_2(X_2)\bar{G}(X_1,Y_1,Y_2,Y_3)+\beta_2(Y_1)\bar{G}(X_2,X_1,Y_2,Y_3)\\ &+\delta_2(Y_2)\bar{G}(X_2,Y_1,X_1,Y_3)+\delta_2(Y_3)\bar{G}(X_2,Y_1,Y_2,X_1),\end{aligned} \tag{1}$$

where

$$\bar{G}^*(X_2,Y_1,Y_2,Y_3)=g^*(Y_1,Y_2)g^*(X_2,Y_3)-g^*(X_2,Y_2)g^*(Y_1,Y_3),$$

and α_i^* ,β_i^* and δ_i^* are the 1-forms defined by $\alpha_i^*(X_1)=g^*(X_1,\sigma_i)$, $\beta_i^*(X_1)=g^*(X_1,\vartheta_i)$ and $\delta_i^*(X_1)=g^*(X_1,\varsigma_i)$ for $i=1,2,3,\ldots$.

Further, a Riemannian manifold is said to be generalized weakly Ricci symmetric if it satisfies the condition

$$\begin{aligned}(\nabla_{X_1}S)(X_2,X_3) =& A_1(X_1)S(X_2,X_3)+B_1(X_2)S(X_1,X_3)\\ &+C_1(X_3)S(X_2,X_1)+A_2(X_1)g(X_2,X_3)\\ &+B_2(X_2)g(X_1,X_3)+C_2(X_3)g(X_2,X_1),\end{aligned} \tag{2}$$

where A_i, B_i and C_i are the 1-forms defined by $A_i(X_1)=g(X_1,\theta_i)$, $B_i(X_1)=g(X_1,\varrho_i)$ and $C_i(X_1)=g(X_1,\psi_i)$ for $i=1,2,3,\ldots$.

A Riemannian manifold admits a Ricci soliton (see [5]) if it satisfies

$$\frac{1}{2}L_Vg(X_1,X_2)+S(X_1,X_2)+\lambda g(X_1,X_2)=0, \tag{3}$$

where L_V denotes the Lie derivative along V and λ is a real number.

In [3], the authors notice that an η-Ricci soliton (V,λ,μ) is a generalization of Ricci soliton, which was further enhanced by Calin and Crasmareanu [2] on Hopf hypersurface in complex space forms and is defined as:

$$\frac{1}{2}L_Vg(X_1,X_2)+S(X_1,X_2)+\lambda g(X_1,X_2)+\mu\eta(X_1)\eta(X_2)=0, \tag{4}$$

where L_V denotes the Lie derivative along V, λ and μ are real numbers.

Somashekhara et al. [8], studied some results on invariant sub-manifolds of LP-Sasakian manifolds endowed with semi-symmetric metric conection. Also, they have obtained a condition for totally geodesic by using certain geometrical conditions. In [9], we studied the C-Bochner curvature tensor under D-homothetic deformation in LP-Sasakian manifolds.

2. Preliminaries

Let M^{2n+1} be a differentiable manifold endowed with a $(1,1)$ tensor field ϕ, a contravariant vector field ξ, a covariant vector field η and a Lorentzian metric field g, then from [6, 7] the following relations holds:

$$\phi^2 = X_1 + \eta(X_1)\xi, \quad \eta(\xi) = -1, \quad \phi\xi = 0, \quad \eta o\phi = 0, \tag{5}$$

$$g(X_1,\xi) = \eta(X_1),\ g(\phi X_1, \phi X_2) = g(X_1, X_2) + \eta(X_1)\eta(X_2),\ \nabla_{X_1}\xi = \phi X_1, \tag{6}$$

$$(\nabla_{X_1}\phi)X_2 = g(X_1, X_2)\xi + \eta(X_1)X_2 + 2\eta(X_1)\eta(X_2)\xi, \tag{7}$$

for all vector fields $X_1, X_2, X_3 \in T_pM$. Here ∇ denotes the operator of covariant differentiation with respect to Lorentzian metric g. For LP-Sasakian manifold, the following relations hold:

$$\begin{aligned} &g(R(X_1, X_2)X_3, \xi) \\ &= \eta(R(X_1, X_2)X_3) = g(X_2, X_3)\eta(X_1) - g(X_1, X_3)\eta(X_2), \end{aligned} \tag{8}$$

$$R(\xi, X_1)X_2 = g(X_1, X_2)\xi - \eta(X_2)X_1, \tag{9}$$

$$R(X_1, X_2)\xi = \eta(X_1)X_2 - \eta(X_2)X_1, \tag{10}$$

$$R(\xi, X_1)\xi = X_1 + \eta(X_1)\xi, \tag{11}$$

$$S(X_1, \xi) = 2n\eta(X_1), \tag{12}$$

for all vector fields X_1, X_2, X_3, where S is the Ricci tensor and R is the Riemannian curvature tensor.

If a contact metric manifold M^{2n+1} with the almost contact metric structure (ϕ, ξ, η, g) is transformed into $(\phi^*, \xi^*, \eta^*, g^*)$ given by

$$\phi^* = \phi,\ \xi^* = \frac{1}{p}\xi,\ \eta^* = p\eta,\ g^* = pg + p(p-1)\eta(X_1)\eta(X_2), \tag{13}$$

where p is a positive constant, then the transformation is called a D-homothetic deformation.

The relation between the Levi-Civita connections ∇ of g and ∇^* of g^* is given by

$$\nabla^*_{X_1}X_2 = \nabla_{X_1}X_2 + \frac{p-1}{p}g(\phi X_1, \phi X_2)\xi, \tag{14}$$

for any vector fields X_1, X_2 on M^{2n+1}.

The D-homothetic deformation of LP-Sasakian manifolds (M^{2n+1}, g), the following relations hold (See [4]):

$$
\begin{aligned}
R^*(X_1,X_2)X_3 &= R(X_1,X_2)X_3 + (1-a)[g(X_1,X_3)\eta(X_2)\xi \\
&- g(X_2,X_3)\eta(X_1)\xi + 2\eta(X_2)\eta(X_3)X_1 \\
&- 2\eta(X_1)\eta(X_3)X_2 + g(\phi X_1,X_3)\phi X_2 \\
&- g(\phi X_2,X_3)\phi X_1] + \frac{a-1}{a} \\
&[g(\phi X_3,X_2)\phi X_1 - g(\phi X_3,X_1)\phi X_2] \\
&+ (1-a)^2[\eta(X_1)\eta(X_3)X_2 - \eta(X_2)\eta(X_3)X_1] \\
&- \frac{(1-a)^2}{a}[g(\phi X_3,X_1)\phi X_2 - g(\phi X_3,X_2)\phi X_1],
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
S^*(X_1,X_2) &= S(X_1,X_2) - [2(1-a) + (1-a)^2]g(X_1,X_2) \\
&\quad - [2(1-a)(n-1) + (1-a)^2]\eta(X_1)\eta(X_2),
\end{aligned} \tag{16}
$$

$$Q^*X_1 = QX_1 - [2(1-a) + (1-a)^2]X_1 - [2(1-a)(n-1) + (1-a)^2]\eta(X_1)\xi, \tag{17}$$

for all vector fields X_1, X_2, X_3 on M^{2n+1}.

Further, the D-homothetic deformation of LP-Sasakian structure (ϕ, ξ, η, g) on M^{2n+1} is transformed into $(\phi^*, \xi^*, \eta^*, g^*)$, then the following relations hold:

$$(\phi^*)^2 = X_1 + \eta(X_1)\xi, \tag{18}$$

$$\eta^*(\xi^*) = -1, \tag{19}$$

$$\phi^*\xi^* = 0, \tag{20}$$

$$\eta^* o \phi^* = 0, \tag{21}$$

$$g^*(\phi^*X_1, \phi^*X_2) = g^*(X_1,X_2) + \eta^*(X_1)\eta^*(X_2), \tag{22}$$

$$g^*(X_1, \xi^*) = \eta^*(X_1), \tag{23}$$

$$\nabla^*_{X_1}\xi^* = \frac{1}{p}\phi^*X_1, \tag{24}$$

$$(\nabla^*_{X_1}\eta^*)X_2 = \frac{1}{p}(g^*(\phi^*X_1, X_2)), \tag{25}$$

$$S^*(X_1,\xi^*) = \frac{2n}{p^2}\eta^*(X_1), \tag{26}$$

$$\eta^*(R^*(X_1,X_2)X_3) = \frac{1}{p^2}\left[\eta^*(X_1)g^*(X_2,X_3) - \eta^*(X_2)g^*(X_1,X_3)\right], \tag{27}$$

$$\eta^*(R^*(X_1,X_2)\xi) = 0, \tag{28}$$

$$(R^*(\xi^*,X_1)X_2) = \frac{1}{p^2}\left[g^*(X_1,X_2)\xi^* - \eta^*(X_2)X_1\right], \tag{29}$$

$$(R^*(X_1,X_2)\xi^*) = \frac{1}{p^2}\left[\eta^*(X_2)X_1 - \eta^*(X_1)X_2\right]. \tag{30}$$

From the equations (24), (25) and (26), we deduce a equation

$$(\nabla^*_{X_2}S^*)(X_1,\xi^*) = \frac{1}{p}\left(\frac{2n}{p^2}g(X_1,\phi^*X_2) - S^*(X_1,\phi^*X_2)\right), \tag{31}$$

for any vector fields X_1 and X_2 on M^{2n+1}.

3. Generalized weakly symmetric LP-Sasakian manifolds under D-homothetic deformation

The D-homothetically deformed LP-Sasakian manifold of dimension M^{2n+1} is said to be generalized weakly symmetric if it satisfies:

$$\begin{aligned}(\nabla^*_{X_1}\bar{R}^*)(X_2,Y_1,Y_2,Y_3) &= \alpha_1^*(X_1)\bar{R}(X_2,Y_1,Y_2,Y_3)\\ &+ \beta_1^*(X_2)\bar{R}^*(X_1,Y_1,Y_2,Y_3) + \beta_1^*(Y_1)\bar{R}^*(X_2,X_1,Y_2,Y_3)\\ &+ \delta_1^*(Y_2)\bar{R}^*(X_2,Y_1,X_1,Y_3) + \delta_1^*(Y_3)\bar{R}^*(X_2,Y_1,Y_2,X_1)\\ &+ \alpha_2^*(X_1)\bar{G}^*(X_2,Y_1,Y_2,Y_3) + \beta_2^*(X_2)\bar{G}^*(X_1,Y_1,Y_2,Y_3)\\ &+ \beta_2^*(Y_1)\bar{G}^*(X_2,X_1,Y_2,Y_3) + \delta_2^*(Y_2)\bar{G}^*(X_2,Y_1,X_1,Y_3)\\ &+ \delta_2^*(Y_3)\bar{G}^*(X_2,Y_1,Y_2,X_1),\end{aligned} \tag{32}$$

where

$$\bar{G}(X_2,Y_1,Y_2,Y_3) = g(Y_1,Y_2)g(X_2,Y_3) - g(X_2,Y_2)g(Y_1,Y_3),$$

and α_i, β_i and δ_i are the 1-forms defined by $\alpha_i(X_1) = g(X_1,\sigma_i)$, $\beta_i(X_1) = g(X_1,\vartheta_i)$ and $\delta_i(X_1) = g(X_1,\varsigma_i)$ for $i = 1,2,3,\ldots$. Now, contracting $X_2 = Y_3 = e_i$, then the equation (32) reduces to

$$\begin{aligned}(\nabla^*_{X_1}S^*)(Y_1,Y_2) &= \alpha_1^*(X_1)S^*(Y_1,Y_2)\\ &+ \beta_1^*(Y_1)S^*(X_1,Y_2) + \beta_1^*(R^*(X_1,Y_1)Y_2)\\ &+ \delta_1^*(R^*(X_1,Y_2)Y_1) + \delta_1^*(Y_2)S^*(X_1,Y_1)\\ &+ \beta_2^*(X_1)g^*(Y_1,Y_2) - \beta_2^*(Y_1)g^*(X_1,Y_2)\\ &+ \delta_2^*(X_1)g^*g(Y_1,Y_2) - \delta_2^*(Y_2)g^*(X_1,Y_1)\\ &+ 2n(\alpha_2^*(X)g^*(Y_1,Y_2) + \beta_2^*(Y_1)g^*(X_1,Y_2)\\ &+ \delta_2^*(Y_2)g^*(X_1,Y_1)),\end{aligned} \tag{33}$$

substituting $Y_2 = \xi^*$ in (33) and using (26), (29) and (30), we obtain

$$\begin{aligned}(\nabla^*_{X_1}S^*)(Y_1,\xi^*) &= \left(-\frac{1}{p^2}\delta_1^*(\xi^*) - \delta_2^*(\xi^*) + 2n\delta_2^*(\xi^*)\right)g^*(X_1,Y_1)\\ &+ \Big(\frac{2n}{p^2}\alpha_1^*(X_1) + \frac{1}{p^2}\beta_1^*(X_1) + \frac{1}{p^2}\delta_1^*(X_1)\\ &+ \beta_2^*(X_1) + \delta_2^*(X_1) + 2n\alpha_2^*(X_1)\Big)\eta^*(Y_1)\\ &+ \left(\frac{(2n-1)}{p^2}\beta_1^*(Y_1) + (2n-1)\beta_2^*(Y_1)\right)\eta^*(Y_1) + \delta_1^*(\xi^*)S^*(X_1,Y_1).\end{aligned} \tag{34}$$

In view of (31), the equation (34) reduces to

$$\begin{aligned}(35)\quad &\frac{2n}{p}g^*(X_1,\phi^*Y_1)-pS^*(X_1,\phi^*Y_1)=g^*(X_1,Y_1)\left(p^2(2n-1)\delta_2^*(\xi^*)-\delta_1^*(\xi^*)\right)\\ &+\left(2n\alpha_1^*(X_1)+\beta_1^*(X_1)+\delta_1^*(X_1)+p^2\left(2n\alpha_2^*(X_1)+\beta_2^*(X_1)+\delta_2^*(X_1)\right)\right)\eta^*(Y_1)\\ &+(2n-1)\left(\beta_1^*(Y_1)+p^2\beta_2^*(Y_1)\right)+p^2\delta_1^*(\xi^*)S^*(X_1,Y_1).\end{aligned}$$

Now, putting $X_1=Y_1=\xi^*$ in (35), we have

$$(36)\qquad \delta_1^*(\xi^*)+\alpha_1^*(\xi^*)+\beta_1(\xi^*)=-p^2\left(\alpha_2^*(\xi^*)+\beta_2^*(\xi^*)+\delta_2^*(\xi^*)\right),$$

again replacing $Y_1=\xi^*$ in (35) and using (36), we have

$$\begin{aligned}(37)\quad &2n\alpha_1^*(X_1)+\beta_1^*(X_1)+\delta_2^*(X_1)+p^2\left(2n\alpha_2^*(X_1)+\beta_2^*(X_1)+\delta_2^*(X_1)\right)\\ &=-(2n-1)(p^2\alpha_2^*(\xi^*)+\alpha_1^*(\xi^*))\eta^*(X_1),\end{aligned}$$

again replacing $X_1=\xi^*$ in (35) and using (36), we have

$$(38)\qquad -(\beta_1(\xi^*)+p^2\beta_2^*(\xi^*))\eta^*(Y_1)=\beta_1^*(Y_1)+p^2\beta_2^*(Y_1),$$

using (37) and (38) in (35), we obtain

$$\begin{aligned}(39)\quad &\frac{2n}{p}g^*(X_1,\phi^*Y_1)-pS^*(X_1,\phi^*Y_1)-p^2\delta_1^*(\xi^*)S^*(X_1,Y_1)-g^*(X_1,Y_1)\\ &\left(p^2(2n-1)\delta_2^*(\xi^*)-\delta_1^*(\xi^*)\right)=(2n-1)(\delta_1^*(\xi^*)+p^2\delta_2^*(\xi^*))\eta^*(X_1)\eta^*(X_2),\end{aligned}$$

putting $Y_1=\phi Y_1$ in (39), we obtain

$$\begin{aligned}(40)\quad S^*(X_1,\phi^*Y_1)=&\frac{2n}{p^3}\frac{1}{\delta_1^*(\xi^*)}g^*(X_1,Y_1)-\frac{1}{p}\frac{1}{\delta_1^*(\xi^*)}S^*(X_1,Y_1)\\ &-\left((2n-1)\frac{\delta_2^*(\xi^*)}{\delta_1^*(\xi^*)}-\frac{1}{p^2}\right)g^*(X_1,\phi^*Y_1),\end{aligned}$$

substituting (40) in (39), we obtain

$$\begin{aligned}(41)\quad &S^*(X_1,Y_1)(1-p^2\delta_1^*(\xi^*))\\ &=\left(\frac{2n}{p^2}+p^2(2n-1)\delta_1^*(\xi^*)\delta_2^*(\xi^*)-(\delta_1^*(\xi^*))^2\right)g^*(X_1,Y_1)\\ &-\left(\frac{2n}{p}\delta_1^*(\xi^*)+(2n-1)p\delta_2^*(\xi^*)-\frac{\delta_1^*(\xi^*)}{p}\right)g^*(X_1,\phi^*Y_1)\\ &+(2n-1)(\delta_1^*(\xi^*)+p^2\delta_2^*(\xi^*))\eta^*(X_1)\eta^*(Y_1).\end{aligned}$$

Hence, we can state the following theorem:

Theorem 3.1. *A generalized weakly symmetric LP-Sasakian manifolds under D-homothetic deformation is an η-Einstein-like Ricci tensor.*

4. Generalized weakly Ricci symmetric LP-Sasakian manifolds under D-homothetic deformation

The D-homothetically deformed LP-Sasakian structure $\phi^*, \xi^*, \eta^*, g^*$ of a manifold M^{2n+1} is said to be generalized weakly Ricci symmetric if satisfies the condition,

$$\begin{aligned}&(\nabla^*_{X_1}S^*)(X_2,X_3)\\&= A_1^*(X_1)S^*(X_2,X_3)+B_1^*(X_2)S^*(X_1,X_3)+C_1^*(X_3)S^*(X_2,X_1)\\&+A_2^*(X_1)g^*(X_2,X_3)+B_2^*(X_2)g^*(X_1,X_3)+C_2^*(X_3)g^*(X_2,X_1),\end{aligned}\tag{42}$$

where A_i^*, B_i^* and C_i^* are the non-zero 1-forms defined by: $A_i^*(X_1)=g^*(X_1,\theta_i)$, $B_i^*(X_1)=g^*(X_1,\varrho_i)$ and $C_i^*(X_1)=g^*(X_1,\psi_i)$ for $i=1,2,3,\ldots$. On substituting $X_3=\xi^*$, we obtain

$$\begin{aligned}&(\nabla^*_{X_1}S^*)(X_2,\xi^*)\\&= A_1^*(X_1)S^*(X_2,\xi^*)+B_1^*(X_2)S^*(X_1,\xi^*)+C_1^*(\xi^*)S^*(X_2,X_1)\\&+A_2^*(X_1)g^*(X_2,\xi^*)+B_2^*(X_2)g^*(X_1,\xi^*)+C_2^*(\xi^*)g^*(X_2,X_1).\end{aligned}\tag{43}$$

Using (26) and (31) in (43), we get

$$\begin{aligned}&\frac{1}{p}\left(\frac{2n}{p^2}g(X_1,\phi^*X_2)-S^*(X_1,\phi^*X_2)\right)=\left(\frac{2n}{p^2}A_1^*(X_1)+A_2^*(X_1)\right)\eta^*(X_2)\\&+\left(\frac{2n}{p^2}B_1^*(X_2)+B_2^*(X_2)\right)\eta^*(X_1)+C_1^*(\xi^*)S^*(X_1,X_2)+C_2^*(\xi^*)g^*(X_1,X_2).\end{aligned}\tag{44}$$

Putting $X_1=X_2=\xi^*$ in (44) and using (19), (20) and (26), we obtain

$$2n[A_1^*(\xi^*)+B_1^*(\xi^*)+C_1^*(\xi^*)]=-p^2[A_2^*(\xi^*)+B_2^*(\xi^*)+C_2^*(\xi^*)].\tag{45}$$

Putting $X_2=\xi^*$ in (44) and using (19), (20) and (26), we obtain

$$\begin{aligned}&2n[A_1^*(X_1)-(B_1^*(\xi^*)+C_1^*(\xi^*))\eta^*(X_1)]\\&=-p^2[A_2^*(X_1)-(B_2^*(\xi^*)+C_1^*(\xi^*))\eta^*(X_1)].\end{aligned}\tag{46}$$

Putting $X_1=\xi^*$ in (44) and using (19), (20) and (26), we obtain

$$\begin{aligned}&2n[(A_1^*(\xi^*)+C_1^*(\xi^*))\eta^*(X_2)-(B_1^*(X_2)]\\&=-p^2[(A_2^*(\xi^*)+C_2^*(\xi^*))\eta^*(X_2)-(B_2^*(X_2)].\end{aligned}\tag{47}$$

Using (45) in (46) and (47), we obtain

$$2nA_1^*(X_1)+p^2A_2^*(X_1)=-[p^2A_2^*(\xi^*)+2nA_1^*(\xi^*)]\eta^*(X_1),\tag{48}$$

and

$$(2nB_1^*(\xi^*)+p^2(B_2^*(\xi^*))\eta^*(X_2)=-[p^2B_2^*(X_2+2nB_1^*(X_2))].\tag{49}$$

Using (45), (48) and (49) in (44), we obtain

$$\begin{aligned}&\frac{2n}{p}g^*(X_1,\phi^*X_2)-p^2S^*(X_1,\phi^*X_2)-p^2C_1^*(\xi^*)S^*(X_1,X_2)\\&-p^2C_2^*(\xi^*)g^*(X_1,X_2)=[p^2C_2^*(\xi^*)+2nC_1^*(\xi^*)]\eta^*(X_1)\eta^*(X_2).\end{aligned}\tag{50}$$

Putting $X_2=\phi X_2$ in (50), we obtain

$$\begin{aligned}&\frac{2n}{p^3}\frac{(1-p)}{C_1^*(\xi^*)}\eta^*(X_1)\eta^*(X_2)+\frac{2n}{p^3C_1^*(\xi^*)}g^*(X_1,X_2)-\frac{C_2^*(\xi^*)}{C_1^*(\xi^*)}g^*(X_1,\phi^*X_2)\\&-\frac{S^*(X_1,X_2)}{C_1^*(\xi^*)}=S^*(X_1,\phi^*X_2).\end{aligned}\tag{51}$$

Substituting (51) in (50), we obtain

$$\begin{aligned}S^*(X_1,X_2)p^2\left(C_1^*(\xi^*)-\frac{1}{C_1^*(\xi^*)}\right)&=\left(\frac{2n}{p}+p^2\frac{C_2^*(\xi^*)}{C_1^*(\xi^*)}\right)g^*(X_1,\phi^*X_2)\\&-\left(\frac{2n}{pC_1^*(\xi^*)}-p^2C_2^*(\xi^*)\right)g^*(X_1,X_2)-\eta^*(X_1)\eta^*(X_2)\\&\left(\frac{2n(1-p)}{pC_1^*(\xi^*)}+p^2C_2^*(\xi^*)+2nC_1^*(\xi^*)\right).\end{aligned}\tag{52}$$

Hence, we can state the following theorem:

Theorem 4.1. *A generalized weakly Ricci symmetric LP-Sasakian manifolds under D-homothetic deformation is an η-Einstein-like Ricci tensor.*

5. Ricci solitons under D-homothetically deformed LP-Sasakian manifolds with potential vector field $V=\xi^*$

Assume that in a D-homothetically deformed LP-Sasakian manifold structure $(\phi^*,\xi^*,\eta^*,g^*)$ of a manifold M^{2n+1} the pair (ξ^*,λ) defines a Ricci soliton, that is

$$\frac{1}{2}L_{\xi^*}g^*(X_1,X_2)+S^*(X_1,X_2)+\lambda g^*(X_1,X_2)=0,\tag{53}$$

for λ is a real number.

Using the definition of Lie derivative and the equation (24), we get:

$$L_{\xi^*}g^*(X_1,X_2)=\frac{2}{p}g^*(\phi^*X_1,X_2).\tag{54}$$

Substituting (54) in (53), we obtain

$$S^*(X_1,X_2)=-\frac{1}{p}g^*(\phi^*X_1,X_2)-\lambda g^*(X_1,X_2).\tag{55}$$

Substituting $X_2 = \xi^*$ and using (19), (21), (26), the equation (55) reduces to

$$\lambda = \frac{2n}{p^2}. \tag{56}$$

In view of (56) and (55), we get:

$$S*(X_1, X_2) = -\frac{1}{p}g^*(\phi^* X_1, X_2) - \frac{2n}{p^2}g^*(X_1, X_2). \tag{57}$$

Hence, we can state the following theorem:

Theorem 5.1. *Ricci solitons under D-homothetically deformed LP-Sasakian manifolds with potential vector field $V = \xi^*$ is an Einstein-like Ricci tensor.*

6. Ricci solitons under D-homothetically deformed LP-Sasakian manifolds with potential vector field collinear with $V = \xi^*$

Assume that in a D-homothetically deformed LP-Sasakian manifold structure $(\phi^*, \xi^*, \eta^*, g^*)$ of a manifold M^{2n+1} the pair (V, λ) defines a Ricci soliton, that is

$$\frac{1}{2}L_{\xi^*}g^*(X_1, X_2) + S^*(X_1, X_2) + \lambda g^*(X_1, X_2) = 0, \tag{58}$$

for λ is a real number and $V = f\xi^*$, where f is a smooth function on M^{2n+1}.

Using the definition of Lie derivative and the equation (24), we get:

$$L_{f\xi^*}g^*(X_1, X_2) = 2\frac{f}{p}g^*(\phi^* X_1, X_2) + (X_1 f)\eta^*(X_2) + (X_2 f)\eta^*(X_1). \tag{59}$$

In view of (58) and (59), we obtain

$$\begin{aligned} S^*(X_1, X_2) = &-\frac{f}{p}g^*(\phi^* X_1, X_2) - \frac{1}{2}[(X_1 f)\eta^*(X_2) \\ &+ (X_2 f)\eta^*(X_1)] - \lambda g^*(X_1, X_2). \end{aligned} \tag{60}$$

Now, putting $X_2 = \xi^*$ in (60), we obtain

$$X_1 f = \left(\frac{4n}{p^2} + \xi^* f + 2\lambda\right)\eta^*(X_1), \tag{61}$$

again put $X_1 = \xi^*$ in (61), we obtain

$$\xi^* f = -\left(\frac{2n}{p^2} + \lambda\right). \tag{62}$$

In view of (62), the equation (61) reduces to

$$X_1 f = \left(\frac{2n}{p^2} + \lambda\right)\eta^*(X_1), \tag{63}$$

substituting (63) in (60), we obtain

$$S^*(X_1, X_2) = -\frac{f}{p}g^*(\phi^* X_1, X_2) - \lambda g^*(X_1, X_2) - \left(\frac{2n}{p^2} + \lambda\right)\eta^*(X_1)\eta^*(X_2). \tag{64}$$

Hence, we can state the following theorem:

Theorem 6.1. *Ricci solitons under D-homothetically deformed LP-Sasakian manifolds with potential vector field collinear with $V = \xi^*$ is an η-Einstein-like Ricci tensor.*

7. η^*-Ricci solitons under D-homothetically deformed LP-Sasakian manifolds with potential vector field $V = \xi^*$

Assume that in a D-homothetically deformed LP-Sasakian manifold structure $(\phi^*, \xi^*, \eta^*, g^*)$ of a manifold M^{2n+1} the triple (ξ^*, λ, μ) defines a η^*-Ricci soliton, that is given by

$$\frac{1}{2}L_{\xi^*}g^*(X_1, X_2) + S^*(X_1, X_2) + \lambda g^*(X_1, X_2) + \mu\eta^*(X_1)\eta^*(X_2) = 0, \tag{65}$$

for λ and μ are real numbers.

Using the definition of Lie derivative and (24)

$$L_{\xi^*}g^*(X_1, X_2) = \frac{2}{p}g^*(\phi^* X_1, X_2), \tag{66}$$

substituting (66) in (65), we obtain

$$S^*(X_1, X_2) = -\frac{1}{p}g^*(\phi^* X_1, X_2) - \lambda g^*(X_1, X_2) - \mu\eta^*(X_1)\eta^*(X_2), \tag{67}$$

substituting $X_2 = \xi^*$, using (19), (21) and (26), equation (67) reduces to

$$\lambda = \mu - \frac{2n}{p^2}, \tag{68}$$

in view of (67) and (68), we obtain

$$S^*(X_1, X_2) = -\frac{1}{p}g^*(\phi^* X_1, X_2) - \left(\mu - \frac{2n}{p^2}\right)g^*(X_1, X_2). \tag{69}$$

Hence, we can state the following theorem:

Theorem 7.1. *η^*-Ricci solitons under D-homothetically deformed LP-Sasakian manifolds with potential vector field $V = \xi^*$ is an Einstein-like Ricci tensor.*

8. η^*-Ricci solitons under D-homothetically deformed LP-Sasakian manifolds with potential vector field collinear with $V = \xi^*$

Assume that in a D-homothetically deformed LP-Sasakian manifold structure $(\phi^*, \xi^*, \eta^*, g^*)$ of a manifold M^{2n+1} the triple (V, λ, μ) defines a η^*-Ricci Soliton, that is given by

$$\frac{1}{2}L_V g^*(X_1, X_2) + S^*(X_1, X_2) + \lambda g^*(X_1, X_2) + \mu\eta^*(X_1)\eta^*(X_2) = 0, \tag{70}$$

for λ and μ are real number and $V = f\xi^*$, where f is a smooth function on M^{2n+1}. Using the definition of Lie derivative and the equation (24), we get:

$$L_{f\xi^*} g^*(X_1, X_2) = 2\frac{f}{p}g^*(\phi^* X_1, X_2) + (X_1 f)\eta^*(X_2) + (X_2 f)\eta^*(X_1). \tag{71}$$

In view of (70) and (71), we get:

$$\begin{aligned} S^*(X_1, X_2) =& -\frac{f}{p}g^*(\phi^* X_1, X_2) - \frac{1}{2}[(X_1 f)\eta^*(X_2) + (X_2 f)\eta^*(X_1)] - \lambda g^*(X_1, X_2) \\ & - \mu\eta^*(X_1)\eta^*(X_2). \end{aligned} \tag{72}$$

Now, putting $X_2 = \xi^*$ in (72), we obtain

$$X_1 f = \left(\frac{4n}{p^2} + \xi^* f + 2\lambda - 2\mu\right)\eta^*(X_1). \tag{73}$$

Again, put $X_1 = \xi^*$ in (73), we obtain

$$\xi^* f = \left(-\frac{2n}{p^2} + \mu - \lambda\right). \tag{74}$$

In view of (74) and (73), we obtain

$$X_1 f = \left(\frac{2n}{p^2} - \mu + \lambda\right)\eta^*(X_1), \tag{75}$$

substituting (75) in (72), we obtain

$$\begin{aligned} & S^*(X_1, X_2) \\ & \quad = -\frac{f}{g}g^*(\phi^* X_1, X_2) - \lambda g^*(X_1, X_2) - \left(\frac{2n}{p^2} + \lambda - 2\mu\right)\eta^*(X_1)\eta^*(X_2). \end{aligned} \tag{76}$$

Hence we can state the following theorem:

Theorem 8.1. *η^*-Ricci solitons under D-homothetically deformed LP-Sasakian manifolds with potential vector field collinear with $V = \xi^*$ is an η-Einstein-like Ricci tensor.*

Acknowledgment

The authors thank the referee for his/her many valuable suggestions which enhanced the quality of presentation of this paper.

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Accepted: January 02, 2020

Relative study on means in linear path and two positive arguments in a curved path

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Abstract. Inspired by the Ky Fan type inequalities established for a symmetric triangular wave form over the interval $[0, 1]$, a similar symmetric function is introduced over $[0, 1]$. A relative study of arithmetic, geometric and power means between symmetric linear path and this curved path is carried out. Also some important inequality chains among these means are established.

Keywords: greek means, inequalities, Ky Fan type inequalities, concavity, symmetric function.

1. Introduction

A systematic study of the means and its inequalities has been carried out in classical book Inequalities by Hardy et al. [1]. Though this book brought some orderliness in presenting the work in the field of inequalities, it made no attempt at completeness; rather it contained fundamental contributions of each of three front rank mathematicians in the field. Such collections of results were continued in the book Analytic Inequalities by Dragoslav Mitrinović [2].

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In [5], the authors defined the homogeneous functions $Gn_{\mu,r}(a,b)$ and $gn_{\mu,r}(a,b)$. Further, they have studied properties, relations with means, applications to inequalities and partial derivatives. Also given some applications of these means related to Farey fractions and Ky Fan type inequalities.

By the motivation work done in [5], Lokesha et al. [6] obtained new inequalities involving Greek means and other means. Further, deduced the best possible values of various means with $Gn_{\mu,r}(a,b)$ and $gn_{\mu,r}(a,b)$. Also, studied the partial derivatives of important means and the value of α of second order partial derivatives.

Some of remarkable inequalities proved by Ky Fan came to light through the publication of these results in 1961, in the acclaimed book Inequalities by Beckenbach and Bellman [4]. A large collection of inequalities can be found in the book 'A Dictionary of Inequalities' by Bullen [3] and also some convexity results on various important means and their applications to mean inequalities were found in [7, 8, 9, 10, 11, 12, 13, 14, 15].

For given n arbitrary non-negative real numbers $a_1, a_2, \ldots, a_n \in (0, \frac{1}{2}]$, the unweighted arithmetic, geometric and power means represented respectively by A_n, G_n and M_r are given by:

$$A_n = \frac{1}{n}\sum_{i=1}^{n} a_i;\ G_n = \prod_{i=1}^{n} \sqrt[n]{a_i};\ \text{and}\ M_r(a,b) = \begin{cases} \sum_{i=1}^{n} \left(\frac{a_i^r}{n}\right)^{\frac{1}{r}}, & \text{if } r \neq 0 \\ G_n, & \text{if } r = 0. \end{cases}$$

Also, the arithmetic and geometric means of the set of elements $1-a_1, 1-a_2, \cdots, 1-a_n$ represented by A'_n and G'_n respectively are given by:

$$A'_n = \frac{1}{n}\sum_{i=1}^{n}(1-a_i), \qquad G'_n = \prod_{i=1}^{n} \sqrt[n]{(1-a_i)}.$$

The set of arbitrary non-negative real numbers $a_i \in (0, \frac{1}{2}]$ and $a'_i = (1-a_i) \in [\frac{1}{2}, 1)$ is represented as a function in the form given by

$$f(a) = \begin{cases} a, & 0 \leq a < \frac{1}{2}, \\ 1-a, & \frac{1}{2} \leq a < 1 \end{cases}.$$

On similar lines a new function is introduced in the form

$$f^*(a) = \begin{cases} 2a^2, & 0 < a \leq \frac{1}{2}, \\ 2(1-a)^2, & \frac{1}{2} \leq a < 1. \end{cases} \tag{1}$$

The corresponding arithmetic and geometric means are

$$A_n^* = \frac{1}{n}\sum_{i=1}^{n} 2a_i^2; \quad G_n^* = \prod_{i=1}^{n} \sqrt[n]{2^n(a_i^2)},$$

$$(A_n^*)' = \frac{1}{n}\sum_{i=1}^{n} 2(1-a_i)^2; \quad (G_n^*)' = \prod_{i=1}^{n} \sqrt[n]{2^n(1-a_i)^2}.$$

For two positive arguments a and b, these take the form:

$$A^* = \frac{2a^2 + 2b^2}{2}, \quad G^* = \sqrt{[2a^2]\,[2b^2]},$$
$$(A^*)' = \frac{2(1-a)^2 + 2(1-b)^2}{2}, \quad (G^*)' = \sqrt{[2(1-a)^2]\,[2(1-b)^2]}.$$

The functions $f(a)$ and $f^*(a)$ are shown graphically in Fig. 1

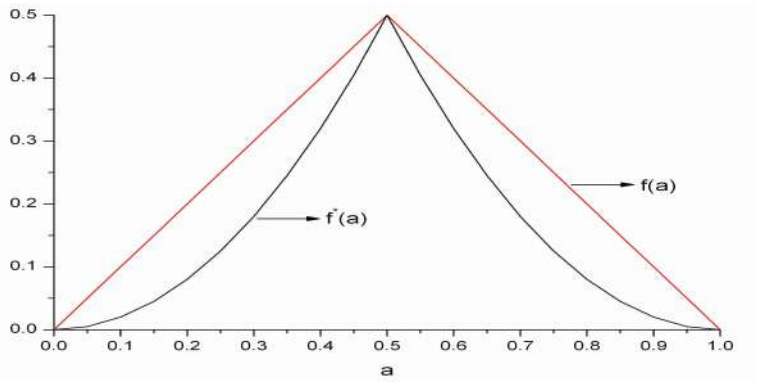


Figure 1: Graph of $f(a)$ and $f^*(a)$ over $[0, 1]$

2. Propositions

Some of the propositions related to the means $A, G, A^*, G^*, (A^*)'$ and $(G^*)'$ have been established in this section.

Proposition 2.1. *For* $0 \le a, b \le \frac{1}{2}$, *then* $(A^*)' + A^* = 2[1 - 2A + 2M_2^2] > 0$.

Proof. For $0 \le a, b \le \frac{1}{2}$, consider

$$\begin{aligned}(A^*)' + A^* &= \frac{2(1-a)^2 + 2(1-b)^2}{2} + \frac{2a^2 + 2b^2}{2} \\ &= 2 - 2(a+b) + 2(a^2 + b^2) \\ &= 2[1 - 2A + 2M_2^2] > 0.\end{aligned}$$

□

Proposition 2.2. *For* $0 \le a, b \le \frac{1}{2}$, *then* $(G^*)' + G^* = 2[1 - 2A + 2G^2] > 0$.

Proof. For $0 \le a, b \le \frac{1}{2}$, consider

$$\begin{aligned}(G^*)' + G^* &= \sqrt{[2(1-a)^2]\,[2(1-b)^2]} + \sqrt{[2a^2]\,[2b^2]} \\ &= 2 - 2(a+b) + 4ab \\ &= [2 - 4A + 4G^2] > 0.\end{aligned}$$

□

Proposition 2.3. *For* $0 \leq a, b \leq \frac{1}{2}$, *then* $(A^*)' + A^* > (G^*)' + G^*$.

Proof. Proof follows from Propositions 2.1 and 2.2. □

Proposition 2.4. *For* $0 \leq a, b \leq \frac{1}{2}$, *then* $\frac{A^*}{(A^*)'} > \frac{G^*}{(G^*)'}$.

Proof. For $0 \leq a, b \leq \frac{1}{2}$, consider

$$\frac{A^*}{(A^*)'} - \frac{G^*}{(G^*)'} = \frac{a^2 + b^2}{(1-a)^2 + (1-b)^2} - \frac{ab}{(1-a)(1-b)}$$

simplifying,

$$= \frac{\left[M_2^2 - 2G^2\right][1 - 2A]}{\left[1 + M_2^2 - 2A\right][1 - 2A + G^2]} > 0.$$

Hence the proof of Proposition 2.4. □

Interestingly, the above result is similar to Ky Fan type inequality for two positive arguments.

3. Main results

Some interesting inequality chains involving A^*, $(A^*)'$, G^*, $(G^*)'$ and the means A, $(A)'$, G, G' are established.

Theorem 3.1. *For* $a, b \in \left(0, \frac{1}{2}\right]$, *the following inequality chains hold:*

1. $(A^*)' > (G^*)' > A' > G' > A > G^*$;
2. $(A^*)' > (G^*)' > A' > G' > A > A^* > G > G^*$, *for* $a = \frac{1}{2}; 0 < b < 0.1478$;
3. $(A^*)' > (G^*)' > A' > G' > A > G > A^* > G^*$, *for* $a = \frac{1}{2}; 0.1478 < b < 0.5$.

Proof. Case [1]. For $0 \leq a, b \leq \frac{1}{2}$ then $a < a'$ and $b < b'$. Consider

$$\begin{aligned} A' - A &= \frac{1 - a + 1 - b}{2} - \frac{(a+b)}{2} \\ &= 1 - (a+b) \\ &= 1 - 2A > 0. \end{aligned}$$

As $0 < A < \frac{1}{2}$, it implies that $1 - 2A > 0$. This proves that $A' > A$.

Similar arguments give

$$G' > G, \quad A > G, \quad A' > G', \quad G' > A.$$

Together it gives the inequality

$$A' > G' > A > G.$$

Case [2]. $(A^*)' > A'$. Consider

$$\begin{aligned}(A^*)' - A' &= \frac{2(1-a)^2 + 2(1-b)^2}{2} - \frac{(1-a+1-b)}{2} \\ &= \left[(1-a)^2 - \frac{(1-a)}{2}\right] + \left[(1-b)^2 - \frac{(1-b)}{2}\right] \\ &= \left(\frac{1-a}{2}\right)[2(1-a)-1] + \left(\frac{1-b}{2}\right)[2(1-b)-1] \\ &= \left(\frac{1-a}{2}\right)[1-2a] + \left(\frac{1-b}{2}\right)[1-2b] > 0\end{aligned}$$

which proves $(A^*)' > A'$.
Case [3]. $A > A^*$. Consider

$$\begin{aligned}A - A^* &= \frac{a+b}{2} - \frac{(2a^2+2b^2)}{2} \\ &= \frac{a+b}{2} - a^2 - b^2 \\ &= \frac{a-2a^2}{2} + \frac{b-2b^2}{2} \\ &= \frac{a}{2}[1-2a] + \frac{b}{2}[1-2b] \geq 0.\end{aligned}$$

This gives $A > A^*$. By combining the Cases [1], [2] and [3] give

$$(A^*)' > A' > A > A^*. \tag{2}$$

Case [4]. $(G^*)' > G^*$. Consider

$$\begin{aligned}(G^*)' - G^* &= 2(1-a)(1-b) - 2ab \\ &= 2\,[1-(a+b)] \\ &= 2\,[1-2A] > 0.\end{aligned}$$

Therefore, $(G^*)' > G^*$.
Case [5]. $(G^*)' > G'$. Consider

$$\begin{aligned}(G^*)' - G' &= 2(1-a)(1-b) - \sqrt{(1-a)(1-b)} \\ &= \sqrt{(1-a)}\sqrt{(1-b)}\left[2\sqrt{(1-a)}\sqrt{(1-b)} - 1\right]\end{aligned}$$

since $0 < a < 1$ or $-1 < -a, -b < 0$ or $0 < 1-a, 1-b < 1$. Therefore, $(G^*)' > G'$.
Case [6]. $G^* < G$. Consider

$$G^* - G = 2ab - \sqrt{ab} = \sqrt{ab}(2\sqrt{ab} - 1) = G(2G-1) \leq 0.$$

Combining the Cases [4], [5] and [6] give

$$(G^*)' > G' > G > G^*. \tag{3}$$

Case [7]. $(A^*)' > (G^*)'$. Consider

$$\begin{aligned}(A^*)' - (G^*)' &= (1-a)^2 + (1-b)^2 - 2(1-a)(1-b)\\ &= (1-a)^2 - (1-a)(1-b) + (1-b)^2 - (1-a)(1-b)\\ &= (1-a)[b-a] + (1-b)[a-b] = (b-a)^2 > 0.\end{aligned}$$

Case [8]. $A^* > G^*$
Consider

$$\begin{aligned}A^* - G^* &= a^2 + b^2 - 2ab\\ &= a^2 - ab + b^2 - ab\\ &= (b-a)^2 > 0.\end{aligned}$$

Hence $A^* > G^*$.
Case [9]. $(A^*)' \geq (G^*)' \geq A^* > G^*$. As $(G^*)' \in \left[\frac{1}{2}, 1\right)$ and $A^* \in \left(0, \frac{1}{2}\right]$, the result

$$(G^*)' > A^* \tag{4}$$

is obvious. Combining the Cases [7] to [9] give

$$(A^*)' \geq (G^*)' \geq A^* > G^*. \tag{5}$$

Case [10]. $(G^*)' > A'$. Consider

$$\begin{aligned}(G^*)' - A' &= 2(1-a)(1-b) - \left[\frac{1-a+1-b}{2}\right]\\ &= (1-a)\left[1-b-\frac{1}{2}\right] + (1-b)\left[1-a-\frac{1}{2}\right]\\ &= (1-a)\left[\frac{1-2b}{2}\right] + (1-b)\left[\frac{1-2a}{2}\right] > 0.\end{aligned}$$

Hence $(G^*)' > A'$.
Case [11]. Consider

$$A^* - G = a^2 + b^2 - \sqrt{ab}.$$

Taking $a = \frac{1}{2}$ and $b = x$, the above expression becomes

$$A^* - G = \frac{1}{4} + x^2 - \sqrt{\frac{x}{2}}.$$

Let $f(x) = \frac{1}{4} + x^2 - \sqrt{\frac{x}{2}}$, which takes the values $f(0.1) = 0.034$ and $f(0.2) = -0.03$.

It is therefore evident that $f(x) = 0$ has one real root in the interval $(0.1, 0.2)$. The root of this equation estimated using Newton-Raphson method is found to be $x = 0.1477$. From this, one can conclude that

$$\begin{aligned} &\text{for } \ 0 < x < 0.1478, A^* < G \quad \text{and} \\ &\text{for } \ 0.1478 < x < 0.5, A^* > G. \end{aligned}$$

Analysing all the above cases, the conclusion leads to the proof of Theorem 3.1 which gives some important inequality chains. □

4. Application to mean inequality

As an application of Theorem 3.1 the following mean inequality is established.

Theorem 4.1. *For $0 \leq a, b \leq \frac{1}{2}$, the following inequality holds*

$$2G^2 < 2M_2^2 < G < A < \frac{1+2G^2}{3} < \frac{1+2M_2^2}{3}.$$

Proof. For $0 \leq a, b \leq \frac{1}{2}$, then $a < a'$, $b < b'$. It's known that $(A^*)' > (G^*)' > A'$
Case [1]. Consider

$$\begin{aligned} (A^*)' > A' &\Rightarrow (1-a)^2 + (1-b)^2 > \frac{2-(a+b)}{2} \\ &\Rightarrow 2 - 2(a+b) + (a^2+b^2) > \frac{2-2A}{2} \\ &\Rightarrow 2 - 4A + 2M_2^2 > 1 - A \\ &\Rightarrow 1 + 2M_2^2 > 3A \end{aligned}$$

or

$$A < \frac{1+2M_2^2}{3}. \tag{6}$$

Case [2]. Consider

$$\begin{aligned} (G^*)' > A' &\Rightarrow \sqrt{2(1-a)^2.2(1-b)^2} > \frac{1-a+1-b}{2} \\ &\Rightarrow 2(1-a)(1-b) > \frac{2-2A}{2} \\ &\Rightarrow 2[1 - 2A + G^2 > 1 - A \\ &\Rightarrow 1 + 2G^2 > 3A \end{aligned}$$

or

$$A < \frac{1+2G^2}{3}. \tag{7}$$

But

$$M_2^2 > G^2. \tag{8}$$

Combining inequalities (6), (7) and (8) gives

$$A < \frac{1+2G^2}{3} < \frac{1+2M_2^2}{3}. \tag{9}$$

Case [3]. Consider

$$A > A^* \Rightarrow A > \left(a^2+b^2\right),$$

$$A > 2M_2^2. \tag{10}$$

Case [4]. Consider

$$A > G^* \Rightarrow A > 2ab$$

$$A > 2G^2. \tag{11}$$

Combining inequalities (10) and (11) gives

$$A > 2M_2^2 > 2G^2 \tag{12}$$

Case [5]. Consider

$$G > A^* \Rightarrow G > \left(a^2+b^2\right)$$

$$G > 2M_2^2. \tag{13}$$

Case [6]. Consider

$$G > G^* \Rightarrow G > 2ab,$$

$$G > G^2. \tag{14}$$

Combining inequalities (13) and (14) gives

$$G > 2M_2^2 > 2G^2. \tag{15}$$

Finally combining the inequalities (9), (12) and (15) gives the inequality chain

$$2G^2 < 2M_2^2 < G < A < \frac{1+2G^2}{3} < \frac{1+2M_2^2}{3}. \tag{16}$$

□

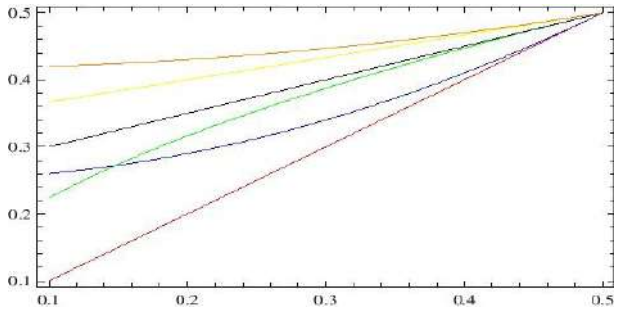


Figure 2: Graph representing inequality chain (16)

.

5. Conclusion

It is interesting to analyse the relationship between means of two different symmetric curves over the interval $[0, 1]$. This type of work can be carried out with more such functions and some generalizations can be arrived at. Some inequality chains have also been discussed.

Acknowledgment

The authors would like to extend their gratitude to the referee for the valuable suggestions.

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Accepted: January 01, 2020

On certain Somos's theta function identities of level 14 and their application to colored partition

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Abstract. M. Somos discovered around 6200 theta function identities using PARI/GP scripts without offering the proof. In this paper, we give a new proof of Somos's theta function identities of level 14 using modular equation of degree 7 and further we extract some interesting combinatorial interpretations of colored partitions.
Keywords: theta functions, modular equation, partitions.

1. Introduction

All through this paper, we have assumed $|q| < 1$ and used the standard notation

$$(x;q)_\infty := \prod_{m=0}^{\infty}(1-aq^m).$$

The general theta function $\mathfrak{f}(x,y)$ as stated by S. Ramanujan is stated below

$$\mathfrak{f}(x,y) = \sum_{n=-\infty}^{\infty} x^{n(n+1)/2}y^{n(n-1)/2}, \qquad |xy| < 1.$$

The Jacobi's triple product identity $\mathfrak{f}(x,y)$ [1, p. 36] is

$$\mathfrak{f}(x,y) = (-x,-y,xy;xy)_\infty.$$

The important special cases of $\mathfrak{f}(x,y)$ [1, p. 36] are as follows:

$$\varphi(q) := \mathfrak{f}(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2(q^2;q^2)_\infty,$$

$$\psi(q) := \mathfrak{f}(q,q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty},$$

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$$f(-q) := \mathfrak{f}(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{(3n^2-n)/2} = (q;q)_\infty.$$

After Ramanujan, define

$$\chi(q) := (-q;q^2)_\infty.$$

Observe that, if $q = e^{2\pi i\tau}$, $f(-q) = e^{-\pi i\tau/12}\eta(\tau)$, where $\eta(\tau)$ is the Dedekind η-function defined as

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n), \quad Im(\tau) > 0,$$

where τ is a complex number. For convenience we write $f(-q^n) = f_n$. A theta function identity which relates f_1, f_2, f_n and f_{2n} is known as the theta function identity of level $2n$. Recently Michael Somos [3] discovered around 6200 Dedekind η-function identities of various levels by employing PARI/GP scripts by using computer and offered no proof of these identities. Recently B. Yuttanan [10], K. R. Vasuki and R. G. Veeresha [8], B. R. Srivatsa Kumar et. al. [5, 6, 7] have obtained the proofs for the levels 4, 6, 8, 10, 12 and 14. Vasuki and Veeresha [8] proved all the twenty four identities of level 14 and in [4], Srivatsa Kumar and Veeresha obtained partition identities for the same. Also M. Rogers and B. Yuttanan [2] proved several modular equations which Somos discovered through computational searches and as an application of this they established new relation between lattice sums and hypergeometric functions. R. G. Veeresha [9] in his Ph. D. thesis deduced the proof of two Somos's 14^{th} level theta function identities and they are as follows:

$$f_1^4 + 7qf_7^4 = \chi(-q)\chi(-q^7)[\psi^4(q) + 7q^3\psi^4(q^7)] \tag{1}$$

and

$$\varphi^4(-q) + 7\varphi^4(-q^7) = 8\chi(-q)\chi(-q^7)[f_2^4 + 7q^2f_{14}^4]. \tag{2}$$

Motivated by the above, in the present work we are giving a new proof for these theta function identities by using 7^{th} degree modular equation and as an application we are extracting the colored partitions for the same. Prior to attempting to prove Somos's identities, we have chosen initially to review some modular equations and theta-function identities which will be required in future. An equation of the form

$$n\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\alpha\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right)} = \frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\beta\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\beta\right)},$$

is called an modular equation of degree n where β has degree n over α, and

$${}_2F_1(\alpha,\beta;\gamma;x) := \sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!}x^n, \qquad |x| < 1,$$

denotes an ordinary hyper-geometric series with

$$(x)_n := x(x+1)(x+2)...(x+n-1),$$

and the multiplier

$$m := \frac{z_1}{z_n},$$

where $z_1 = {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right)$ and $z_n = {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\beta\right)$. For $y = \pi\ {}_2F_1(\frac{1}{2},\frac{1}{2};1;1-x)/\ {}_2F_1(\frac{1}{2},\frac{1}{2};1;x)$, then from [1, pp. 122−124, Entry 10(i) and Entry 12(v)], we have

$$\varphi(e^{-y}) = \sqrt{z} \tag{3}$$

and

$$\chi(e^{-y}) = 2^{1/6}\left\{x(1-x)e^{-y}\right\}^{-1/24}, \tag{4}$$

where $q = e^{-y}$. Also one can deduce the following identities easily:

$$\varphi(q) = \frac{f_2^5}{f_1^2 f_4^2}, \qquad \varphi(-q) = \frac{f_1^2}{f_2}, \qquad \psi(q) = \frac{f_2^2}{f_1}, \qquad \psi(-q) = \frac{f_1 f_4}{f_2},$$

$$f(q) = \frac{f_2^3}{f_1 f_4}, \qquad \chi(q) = \frac{f_2^2}{f_1 f_4} \qquad \text{and} \qquad \chi(-q) = \frac{f_1}{f_2}. \tag{5}$$

2. Proof of (1.1) and (1.2)

Proof of (1.1). If β is of seventh degree over α and m is the multiplier then from [1, Entry 19(ix), p.314-324], we have

$$P := \{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/8} \qquad \text{and} \qquad Q := \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/6}$$

then

$$Q + \frac{1}{Q} + 7 = 2\sqrt{2}\left(P + \frac{1}{P}\right).$$

On transforming the above modular equation using (4), we obtain

$$\frac{a^4}{b^4} + \frac{b^4}{a^4} + 7 = a^3b^3 + \frac{8}{a^3b^3}, \tag{6}$$

where $q^{-1/24}\chi(q) =: u(q) = u$ and $q^{-7/24}\chi(q^7) =: v(q) = v$. Multiplying (6) by $u^{-7}v^{-11}(u^4+v^4)$, we obtain

$$\frac{u^5}{v^{11}} - \frac{u^4}{v^4} + \frac{8u}{v^7} - \frac{8}{u^2v^{10}} + \frac{8}{u^3v^3} - 1 - \frac{8}{u^6v^6} + \frac{v}{u^7} = 0,$$

which is equivalent to

$$\left(1 - \frac{v}{u^7}\right)\left(1 - \frac{8u}{v^7}\right) + \left(\frac{u^5}{v^{11}} - \frac{u^4}{v^4}\right)\left(\frac{8v}{u^7} - 1\right) = 0. \tag{7}$$

Also, from [1, Entry 19(ii), p.314-324], we have

$$m=\frac{1-4\left(\frac{\beta^7-(1-\beta^7)}{\alpha(1-\alpha)}\right)^{1/24}}{((1-\alpha)(1-\beta))^{1/8}-(\alpha\beta)^{1/8}} \quad \text{and} \quad \frac{7}{m}=\frac{1-4\left(\frac{\alpha^7-(1-\alpha^7)}{\beta(1-\beta)}\right)^{1/24}}{(\alpha\beta)^{1/8}-((1-\alpha)(1-\beta))^{1/8}}$$

which implies

$$\frac{m^2}{7}=\frac{1-4\left(\frac{\beta^7-(1-\beta^7)}{\alpha(1-\alpha)}\right)^{1/24}}{4\left(\frac{\alpha^7-(1-\alpha^7)}{\beta(1-\beta)}\right)^{1/24}-1}.$$

By employing (3) and (4), transforming the above into theta function identity, we deduce

$$\frac{\varphi^4(q)}{7\varphi^4(q^7)}=\frac{1-\frac{8u}{v^7}}{\frac{8v}{u^7}-1}. \tag{8}$$

Employing (8) in (7), we obtain

$$1-7\frac{u^4}{v^4}\frac{\varphi^4(q^7)}{\varphi^4(q)}-\frac{v}{u^7}+7\frac{u^5}{v^{11}}\frac{\varphi^4(q^7)}{\varphi^4(q)}=0. \tag{9}$$

On using (5), one can easily deduce the following:

$$\frac{\varphi(q)}{\varphi(q^7)}=\frac{u^2f_2}{q^{1/2}v^2f_{14}}. \tag{10}$$

Using (10) in (9), we find that

$$1-7q^2\frac{v^4}{u^4}\left(\frac{f_{14}}{f_2}\right)^4-\frac{v}{u^7}+\frac{7q^2}{u^3v^3}\left(\frac{f_{14}}{f_2}\right)^4=0.$$

Letting q to $-q$ in the above and then multiplying throughout by $f_1^7f_2f_7^3f_{14}$, we obtain

$$f_1^7f_2f_7^3f_{14}+7qf_1^3f_2f_7^7f_{14}-f_2^8f_7^4-7q^3f_1^4f_{14}^8=0.$$

Dividing the above throughout by $f_1^3f_2f_7^3f_{14}$ and then using (5), we obtain the desired result. □

Proof of (1.2). It is easy to see that

$$\left(1-\frac{8v}{u^7}\right)\left(1-\frac{8u}{v^7}\right)+\left(\frac{8v}{u^7}-1\right)-\frac{8u}{v^7}\left(\frac{8v}{u^7}-1\right)=0.$$

Using (8) in the above, we deduce

$$7\frac{\varphi^4(q^7)}{\varphi^4(q)}+1-\frac{8v}{u^7}-56\frac{u}{v^7}\frac{\varphi^4(q^7)}{\varphi^4(q)}=0.$$

Employing (10) in the above, we have

$$1+7q^2\frac{v^8}{u^8}\left(\frac{f_{14}}{f_2}\right)^4-8\frac{v}{u^7}-56q^2\frac{v}{u^7}\left(\frac{f_{14}}{f_2}\right)^4=0.$$

Letting q to $-q$ in the above and then multiplying throughout by $f_1^8f_{14}^4$, we obtain

$$f_1^8f_{14}^4+7f_7^8f_2^4-8f_1f_2^7f_7f_{14}^3-56q^2f_1f_7f_2^3f_{14}^7=0.$$

Dividing the above throughout by $f_1f_7f_2^3f_{14}^3$ and then simplifying using (5), we obtain the result. □

3. Applications to colored partitions

The Somos's identities have an application in colored partitions. In this section we show this by giving combinatorial interpretations for (1) and (2). For convenience, we use the notation

$$(x_1,x_2,\ldots,x_n;q)_\infty=\prod_{i=1}^{n}(x_i;q)_\infty,$$

and define

$$(q^{\pm a};q^b)_\infty:=(q^a,q^{b-a};q^b)_\infty,$$

where $a<b$, and $a,b\in\mathbb{Z}^+$.

As an example of this, $(q^{\pm3};q^8)_\infty$ implies $(q^3,q^5;q^8)_\infty=(q^3;q^8)_\infty(q^5;q^8)_\infty$. Now we define colored partition as defined as in the literature.

"A positive integer n has l colors if there are l copies of n available colors and all of them are viewed as distinct objects. Partitions of a positive integer into parts with colors are called colored partitions".

Example. If 3 colors are assigned to 1, then all possible colored partitions of 2 are 2, 1_r+1_r, 1_g+1_g, 1_y+1_y, 1_r+1_g, 1_r+1_y, and 1_g+1_y, here we used the notations r for 'red', g for 'green' and y for 'yellow' to distinguish three colors of 1. Also, the the total partitions of n is given by the generating function

$$\frac{1}{(q^\alpha;q^\beta)_\infty^\gamma}=\sum_{n=0}^{\infty}p(n)q^n, \tag{11}$$

where each and every parts are congruent to α *modulo* β with γ number of colors with $p(0)=1$.

Theorem 3.1. *Let $\alpha(m)$ represent the partitions of m being divided into parts congruent to $\pm2,\pm4,\pm6$ modulo 14 with six colors and $+7$ modulo 14 with four colors. If $\beta(m)$ is chosen to represent the partitions of m into parts congruent to $\pm1,\pm3,\pm5,+7$ modulo 14 with four colors and $\pm2,\pm4,\pm6$ modulo 14 with eleven colors. If $\gamma(m)$ indicates the partitions of m being split into parts congruent to*

$\pm1, \pm2, \pm3, \pm4, \pm5, \pm6$ *modulo 14 with seven colors and* $+7$ *modulo 14 with ten colors. If* $\delta(m)$ *is taken to represent the partitions of* m *into several parts congruent to* $\pm1, \pm3, \pm5$ *modulo 14 with three colors and* $\pm2, \pm4, \pm6$ *modulo 14 with eleven colors and* $+7$ *modulo 14 with ten colors. Then for any positive integer* $m \geq 3$*, the following relation holds:*

$$\alpha(m) + 7\beta(m-1) = \gamma(m) + 7\delta(m-3).$$

Proof. On rewriting (1), in terms of f_n by using (5) and then simplifying to common base q^{14}, we deduce

$$\begin{aligned}&\frac{1}{(q_7^{\pm2}, q_7^{\pm4}, q_7^{\pm6}, q_4^{+7}; q^{14})_\infty} + \frac{7q}{(q_4^{\pm1}, q_{11}^{\pm2}, q_4^{\pm3}, q_{11}^{\pm4}, q_4^{\pm5}, q_{11}^{\pm6}, q_4^{+7}; q^{14})_\infty}\\ &= \frac{1}{(q_7^{\pm1}, q_7^{\pm2}, q_7^{\pm3}, q_7^{\pm4}, q_7^{\pm5}, q_7^{\pm6}, q_{10}^{+7}; q^{14})_\infty}\\ &+ \frac{7q^3}{(q_3^{\pm1}, q_{11}^{\pm2}, q_3^{\pm3}, q_{11}^{\pm4}, q_3^{\pm5}, q_{11}^{\pm6}, q_{10}^{+7}; q^{14})_\infty}.\end{aligned}$$

Using (11), in the above quotients

$$\sum_{m=0}^{\infty} \alpha(m)q^m + 7q \sum_{m=0}^{\infty} \beta(m)q^m = \sum_{m=0}^{\infty} \gamma(m)q^m + 7q^3 \sum_{m=0}^{\infty} \delta(m)q^m,$$

where $\alpha(m), \beta(m), \gamma(m)$ and $\delta(m)$ are the generating functions. On extracting terms of q^m in the above, we obtain the result.

Now, we verify the above theorem for $m = 3$ in the following table:

$\alpha(3) = 0$:	
$\beta(3) = 21$:	$1_r + 1_r$, $1_g + 1_g$, $1_y + 1_y$, $1_b + 1_b$, $1_r + 1_g$, $1_r + 1_y$, $1_r + 1_b$, $1_g + 1_y$, $1_g + 1_b$, $1_y + 1_b$, $2_r, 2_g, 2_y, 2_o, 2_b, 2_p, 2_w, 2_v, 2_i, 2_{bl}, 2_{gr}$.
$\gamma(3) = 140$:	$1_r + 1_r + 1_r, 1_g + 1_g + 1_g$, 5 further partitions with same color $1_r + 1_r + 1_g, 1_r + 1_r + 1_y$, 75 additional partitions with different colors, $2_y + 1_y, 2_p + 1_p$, 5 further partitions with same color, $2_y + 1_p, 2_y + 1_b$, 40 additional partitions with different colors, $3_r, 3_g, 3_y, 3_o, 3_w, 3_v, 3_b$.
$\delta(0) = 1$:	

□

Theorem 3.2. *Let* $\alpha(m)$ *is taken to represent the partitions of* m *into several parts congruent to* $\pm1, \pm3, \pm5$ *modulo 14 with eight colors and* $\pm2, \pm4, \pm6$ *modulo 14 with four colors. If* $\beta(m)$ *represent the partitions of* m *being divided into parts congruent to* $\pm1, \pm5$ *modulo 14 with seven colors and* ±3 *modulo 14 with seven colors and* $+7$ *modulo 14 with six colors. If* $\gamma(m)$ *is chosen to represent the partitions of* m *into parts congruent to* $\pm1, \pm3, \pm5$ *modulo 14 with seven*

colors and $\pm 2, \pm 4, \pm 6$ *modulo 14 with four colors and* $+7$ *modulo 14 with six colors. Then for any positive integer* $m \geq 2$*, the following relation holds:*

$$7\alpha(m) - 8\beta(m) = 56\gamma(m-2).$$

Proof. On transcribing (2) on the same lines of Theorem 3.1, we obtain

$$1 + \frac{7}{(q_8^{\pm 1}, q_4^{\pm 2}, q_8^{\pm 3}, q_4^{\pm 4}, q_8^{\pm 5}, q_4^{\pm 6}; q^{14})_\infty} - \frac{8}{(q_7^{\pm 1}, q_7^{\pm 3}, q_7^{\pm 5}, q_6^{+7}; q^{14})_\infty} = \frac{56q^2}{(q_7^{\pm 1}, q_4^{\pm 2}, q_7^{\pm 3}, q_4^{\pm 4}, q_7^{\pm 5}, q_4^{\pm 6}, q_6^{+7}; q^{14})_\infty}.$$

Using (11), in the above quotients

$$1 + \sum_{m=0}^{\infty} \alpha(m)q^m - 8\sum_{m=0}^{\infty} \beta(m)q^m = 56q^2 \sum_{m=0}^{\infty} \gamma(m)q^m,$$

where $\alpha(m), \beta(m)$ and $\gamma(m)$ are the generating functions. On extracting the terms of q^m in the above, we obtain the result.

Now, we verify the above theorem for $m = 2$ in the following table:

$\alpha(2) = 40$:	$1_r + 1_r$, $1_g + 1_g$, $1_y + 1_y$, $1_o + 1_o$, $1_b + 1_b$,$1_p + 1_p$, $1_w + 1_w$, $1_r + 1_b$, $1_r + 1_p$, $1_r + 1_w$, $1_r + 1_v$, $1_r + 1_g$, $1_v + 1_v$, $1_r + 1_y$, $1_g + 1_b$, $1_r + 1_o$, $1_g + 1_p$, $1_g + 1_w$, $1_g + 1_v$, $1_g + 1_y$, $1_g + 1_o$,$1_y + 1_o$, $1_y + 1_b$,$1_y + 1_p$, $1_y + 1_w$, $1_o + 1_b$, $1_y + 1_v$, $1_o + 1_p$, $1_b + 1_p$, $1_o + 1_w$,$1_o + 1_v$, $1_b + 1_w$, $1_b + 1_v$, $1_p + 1_w$, $1_p + 1_v$,$1_w + 1_v$, $2_r, 2_g, 2_y, 2_o$.
$\beta(2) = 28$:	$1_r + 1_r$, $1_g + 1_g$, $1_y + 1_y$, $1_o + 1_o$, $1_b + 1_b$,$1_p + 1_p$, $1_w + 1_w$, $1_r + 1_g$, $1_r + 1_p$, $1_r + 1_w$, $1_g + 1_y$, $1_r + 1_y$, $1_r + 1_o$, $1_r + 1_b$,$1_g + 1_o$, $1_g + 1_b$, $1_g + 1_p$,$1_g + 1_w$, $1_y + 1_o$, $1_y + 1_b$, $1_y + 1_p$,$1_y + 1_w$, $1_o + 1_b$,$1_o + 1_p$, $1_o + 1_w$, $1_b + 1_p$, $1_b + 1_w$, $1_p + 1_w$.
$\gamma(0) = 1$:	

□

Acknowledgment

The second author is thankful to DST-SERB, India for support under the research project [EMR/2016/001601].

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Accepted: April 25, 2020

Updating reducts in fuzzy β-covering via matrix approaches while adding and deleting a covering element

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Abstract. Since many real data sets may vary dynamically, reduct changes dynamically as attribute set vary over time. How to update reduct based on previous information is an important task that can help to improve the efficiency of knowledge discovery. This paper focuses on the change of reduct when an element adding into or deleting from a fuzzy β-covering. In order to derive the reduct algorithms for computing reduct in a fuzzy β-covering, the relation character matrix is defined. Then, by the properties of relation character matrix, the corresponding algorithms for computing new reduct are derived when an element is added into or deleted from the fuzzy β-covering, respectively. Finally, the effectiveness of the proposed algorithms are validated by numerical experiments.

Keywords: fuzzy β-covering, relation character matrix, reduct update.

1. Introduction

The rough set theory [10, 11] was proposed by Pawlak in 1982 and is a mathematical tool for dealing with uncertain, inaccurate and incomplete information. However, Rough set theory can only process discrete data and has great limitations when dealing with real-valued data sets. In real life, the attribute values in the database can be either symbolic or real-valued, which can be handled

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effectively by fuzzy set theory[18], and it is helpful to overcome the limitation. Therefore, Zadeh generalized rough set theory to fuzzy set theory in 1965.

The covering rough set theory[19] is a generalization of the Pawlak rough set, which was proposed by Zakowski in the 1980s, and is a powerful mathematical tool in the field of knowledge discovery and rule acquisition. It not only enriches the rough set theory, but also extends its application to practical problems. Hence, scholars extend the model of covering rough set to fuzzy covering rough set.

Many scholars have constructed the fuzzy rough set model based on the concept of fuzzy covering [12, 5, 15, 8, 4], which can be regarded as a bridge between the covering rough set theory and the fuzzy rough set theory. Ma [9] proposed the concept of fuzzy β-covering, in which the parameter β is in the interval (0,1). He defines a more general fuzzy covering rough set model by combining the concepts of fuzzy β-covering and fuzzy β-neighborhood, and discusses the properties of the model and its relationship with other rough sets models. Based on the viewpoints of Ma, Yang and Hu [16] presents the concept of fuzzy β-minimal description in fuzzy covering-based rough sets and gives some new characterizations of fuzzy covering-based rough sets in terms of the notion of fuzzy β-minimal description. Yang and Hu [17] also constructed three fuzzy β-covering fuzzy rough set models. D'eer et al. [3, 2] discussed fuzzy neighborhoods based on fuzzy covering rough sets.

Attribute reduction is an important process of knowledge discovery and an important means of seeking the simplest rules and the largest generalization rules. At present, it must be noted that in real life databases, data sets usually change at an unprecedented rate over time, and it is time-consuming to repeatedly run knowledge acquisition process to obtain reduction. So how to get useful information or knowledge from these data sets is a tricky issue. In the view of the characteristics of dynamic changes in most databases, many scholars devoted themselves to the study of dynamic knowledge updating in the rough sets and its extension models [6, 7, 20, 13, 1, 14]. However, for fuzzy β-covering, few authors consider how to calculate its reduction and how to update the reduction of a fuzzy β-covering. In [17], Yang introduces the definition of union reducible element, and derive an algorithm to compute the reduct of a fuzzy β-covering. But the algorithm is not efficient enought, so we introduce the relation matrix to design the reduction algorithm to improve his algorithm.

It must be pointed out that the dynamic changes of data in the actual database are the main characteristics of most databases. A simple way to handle dynamic data changes is to recalculate from scratch. However, repeatedly running the knowledge acquisition process in many practical applications is very time consuming and extremely inefficient. Therefore, how to obtain useful information or knowledge from these data sets is a tricky issue, and it is necessary to develop technology to update useful information. The dynamic update method usually uses the previous calculation results instead of recalculating the entire data set, and has been applied in the knowledge acquisition and decision-making

of many information systems. However, for fuzzy β-covering, few authors consider how to update the reduction of a fuzzy β-covering. The purpose of this work is to investigate knowledge reduction of the fuzzy-covering approximation spaces while adding and deleting a covering element. Hence, we derive the related algorithms to updating reducts in fuzzy β-covering while adding and deleting a covering element.

The outline of the paper is organized as follows. Some basic concepts and notations in fuzzy β-covering approximation space are reviewed in Section 2. In Section 3, the reduct algorithms are designed for adding and deleting an element in the fuzzy β-covering. In Section 4, some experiments are conducted to prove the effectiveness and validity of the proposed algorithms. Section 5 summarizes the paper and discusses the prospects for the further work.

2. Preliminaries

In this section, we review several basic concepts of fuzzy β-covering, as well as the algorithm to compute the reduct in it.

Definition 2.1 ([9]). *Let U be an arbitrary universal set, and $\mathcal{F}(U)$ be the fuzzy power set of U. For each $\beta \in (0,1]$, we call $\widehat{C} = \{C_1, C_2, \ldots, C_m\}$, with $C_i \in \mathcal{F}(U)$ for any $i \in \{1,2,\ldots,m\}$, a fuzzy β-covering of $U = \{x_1, x_2, \ldots, x_n\}$, if $(\bigcup_{i=1}^{m} C_i)(x) \geq \beta$ for each $x \in U$. We also call $(U, \widehat{C})$ a fuzzy β-covering approximation space.*

In fact, the concept of fuzzy covering in [5, 15, 8] is a special case of fuzzy β-covering with $\beta = 1$. Yang gives the following definition of the reducible element. Thereby we can compute the reduct of a fuzzy β-covering by deleting all reducible elements.

Definition 2.2 ([17]). *Let $(U, \widehat{C})$ be a fuzzy β-covering of U and $C \in \widehat{C}$, If C is the union of some fuzzy sets in $\widehat{C} - \{C\}$, we say C is a reducible element of $\widehat{C}$, otherwise, C is an irreducible element.*

In a fuzzy β-covering approximation space $(U, \widehat{C})$, where $\widehat{C} = \{C_1, C_2, \ldots, C_m\}$. Yang gives the following properties of the reducible element.

Lemma 2.1 ([17]). *Let $(U, \widehat{C})$ be a fuzzy β-covering of U and C be a reducible element of $\widehat{C}$, then $\widehat{C} - \{C\}$ is still a fuzzy β-covering of U.*

Lemma 2.1 guarantees that after deleting a reducible element from a fuzzy β-covering, it is still a fuzzy β-covering.

Lemma 2.2 ([17]). *Let $(U, \widehat{C})$ be a fuzzy β-covering of U and C be a reducible element of $\widehat{C}$, then C_1 is a reducible element of $\widehat{C}$ if and only if C_1 is a reducible element of $\widehat{C} - \{C\}$.*

Lemma 2.2 shows that deleting a reducible element in a fuzzy β-covering will not affect other reducible elements, they are still reducible elements. We have the following definition of reduct by the definition of reducible elements.

Definition 2.3 ([17]). *Let $(U, \widehat{C})$ be a fuzzy β-covering of U and $\widehat{D}$ is a subset of $\widehat{C}$. If $\widehat{C} - \widehat{D}$ is the set of all reducible element of $\widehat{C}$, then $\widehat{D}$ is called the reduct of $\widehat{C}$, and is denoted as $\Gamma(\widehat{C})$.*

The matrix representation of $\widehat{C}$ is presented for the convenience of calculation.

Definition 2.4 ([9]). *Let $U = \{x_1, x_2, \ldots, x_n\}$ be a finite universe and $\widehat{C} = \{C_1, C_2, \ldots, C_m\}$ be a fuzzy β-covering of U, We call $P^{\widehat{C}} = (p_{ij}^{\widehat{C}})_{n\times m}$ a matrix representation of $\widehat{C}$, where*

$$p_{ij}^{\widehat{C}} = C_j(x_i), \qquad \forall i \in \{1, 2, \ldots, n\}, \forall j \in \{1, 2, \ldots, m\}.$$

Example 2.1. Let $(U, \widehat{C})$ be a fuzzy β-covering approximation space of U, where $U = \{x_1, x_2, x_3, x_4\}$, $\widehat{C} = \{C_1, C_2, C_3, C_4\}$. According to Definition 2.4, the matrix representation of $\widehat{C}$ is as follow,

$$P^{\widehat{C}} = \begin{bmatrix} 0.3 & 0.2 & 0.2 & 0.1 \\ 0.4 & 0.5 & 0.3 & 0.5 \\ 0.7 & 0.3 & 0.2 & 0.3 \\ 0.6 & 0.7 & 0.4 & 0.7 \end{bmatrix}.$$

In the following, we present a relation character matrix, through which we improve the algorithm for computing the reduct of a fuzzy β-covering in [17]

Definition 2.5. *Let $U = \{x_1, x_2, \ldots, x_n\}$ be a finite universe and $\widehat{C} = \{C_1, C_2, \ldots, C_m\}$ be a fuzzy β-covering of U, the relation character matrix on $\widehat{C}$ is denoted by $Q^{\widehat{C}} = (q_{ij}^{\widehat{C}})_{m\times m}$, where*

$$q_{ij}^{\widehat{C}} = \begin{cases} 1, & C_i \subseteq C_j \wedge i \neq j, \\ 0, & C_i \not\subseteq C_j \vee i = j. \end{cases} \qquad i, j \in \{1, 2, \ldots, m\}.$$

Example 2.2 (Continuation of Example 2.1). The relation character matrix on $\widehat{C}$ is as follow,

$$Q^{\widehat{C}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

By Definition 2.4 and Definition 2.5 we can change the algorithm for computing reduct of a fuzzy β-covering in [17] into a matrix representation. The total time complexity of Algorithm 1 is $O(m^2|U|)$. Steps 1-8 are to ensure $\widehat{C}$ is a fuzzy β-covering with time complexity $O(m|U|)$, Steps 9-17 are to compute $\Gamma(\widehat{C})$ with time complexity $O(m^2|U|)$.

Algorithm 1 Algorithm to compute the reduct of a fuzzy β-covering

Require: (1)$\widehat{C} = \{C_1, C_2, \ldots, C_m\}$, (2)$U = \{x_1, x_2, \ldots, x_n\}$, (3)$\beta \in (0, 1]$.
Ensure: $\Gamma(\widehat{C})$,$Q^{\widehat{C}}$.

```
for i = 1 → n do
    C(x_i) ⟵ 0
    for j = 1 → m do
        C(x_i) ⟵ C(x_i) ∨ C_j(x_i)
    end for
    if C(x_i) < β then Return Ĉ is not a fuzzy β-covering
    end if
end for
Γ(Ĉ) ⟵ Ĉ
for k = 1 → m do
    T ⟵ ∅
    for l = 1 → m do
        if C_k ⊆ C_l ∧ k ≠ l then q_kl ⟵ 1
        end if
        if q_kl^Ĉ = 1 then T ⟵ T ∪ C_k
        end if
    end for
    if T = C_l then Γ(Ĉ) ⟵ Γ(Ĉ) − {C_l}
    end if
end for
Return Γ(Ĉ)
```

3. Matrix-based approaches for updating reducts in fuzzy β-covering while adding or deleting a covering element

For convenience, the covering element represent the member of the fuzzy β-covering in the whole paper.

3.1 Matrix-based approach for updating reducts in fuzzy β-covering while adding a covering elements into the fuzzy β-covering

In this subsection, we present matrix based dynamic approaches for updating reducts in a fuzzy β-covering while adding a covering element into the fuzzy β-covering. We denote $(U, \widehat{C})$ a fuzzy β-covering approximation space of U, where $\widehat{C} = \{C_1, C_2, \ldots, C_m\}$ and $U = \{x_1, x_2, \ldots, x_n\}$. Denote $(U, \widehat{\mathcal{G}})$ a fuzzy approximation space of U, where $\widehat{\mathcal{G}} = \{C_1, C_2, \ldots, C_m, C_{m+1}\}$ and $U = \{x_1, x_2, \ldots, x_n\}$.

Theorem 3.1. *If $(U, \widehat{C})$ be a fuzzy β-covering approximation space of U, where $\widehat{C} = \{C_1, C_2, \ldots, C_m\}$ and $U = \{x_1, x_2, \ldots, x_n\}$, then $(U, \widehat{\mathcal{G}})$ is still a fuzzy β-covering approximation space of U, where $\widehat{\mathcal{G}} = \{C_1, C_2, \ldots, C_m, C_{m+1}\}$.*

Proof. If $(U, \widehat{C})$ be a fuzzy β-covering approximation space of U, then $(\bigcup_{i=1}^{m} C_i)(x) \geq \beta$ for each $x \in U$. Hence there exist $j_k \in \{1, 2, \ldots, m\}$, such that $C_j(x) \geq \beta$ for each $x \in U$. For C_{m+1}, if $C_{m+1}(x) \geq \beta$ for each $x_k \in U$, then $(\bigcup_{i=1}^{m+1} C_i)(x) \geq \beta$ for each $x \in U$. If there exist x_i such that $C_{m+1}(x_i) < \beta$, because there exist $j \in \{1, 2, \ldots, m\}$ for each $x \in U$ such that $C_j(x) \geq \beta$, then $(\bigcup_{i=1}^{m+1} C_i)(x) \geq \beta$. Hence $(U, \widehat{\mathcal{G}})$ is still a fuzzy β-covering approximation space of U. □

The above theorem guarantees that adding a covering element in a fuzzy β-covering, it is still a fuzzy β-covering. Therefore, the step of judging whether it is a fuzzy β-covering can be omitted in the algorithm when calculating the new reduct after adding a covering element into the fuzzy β-covering.

Theorem 3.2. *Let $(U, \widehat{C})$ be a fuzzy β-covering approximation space of U and $(U, \widehat{\mathcal{G}})$ is also a fuzzy β-covering approximation space of U. If C is a reducible element of $\widehat{C}$, then C is still a reducible element of $\widehat{\mathcal{G}}$.*

Proof. If C is a reducible element of $\widehat{C}$, then there exist $C_{i1}, C_{i2}, \ldots, C_{it} \in \widehat{C} - \{C\} (1 < t \leq m)$ such that $C = \cup_{k=1}^{t} C_{ik}$. Since we have $C_{i1}, C_{i2}, \ldots, C_{it} \in \widehat{C} - \{C\} \subset \widehat{\mathcal{G}} - \{C\} (1 < t \leq m)$ such that $C = \cup_{k=1}^{t} C_{ik}$. Therefore C is a reducible element of $\widehat{\mathcal{G}}$. □

The above theorem shows that C is a reducible element in a fuzzy β-covering, when the fuzzy β-covering is added a covering element, C is still a reducible element in the fuzzy β-covering. Therefore, we only need to decide whether the irreducible element is reducible or not in the new fuzzy β-covering. And partial calculations are reduced in the reduct process of a fuzzy β-covering with the adding process of covering element.

Example 3.1 (Continuation of Example 2.1)**.** If C_5 is added into $\widehat{C}$ and $P^{\widehat{C}}$ changes as follow.

$$P^{\widehat{\mathcal{G}}} = \begin{bmatrix} 0.3 & 0.2 & 0.2 & 0.1 & 0.2 \\ 0.4 & 0.5 & 0.3 & 0.5 & 0.3 \\ 0.7 & 0.3 & 0.2 & 0.3 & 0.4 \\ 0.6 & 0.7 & 0.4 & 0.7 & 0.5 \end{bmatrix}.$$

By Definition 2.2, we have $C_2 = C_3 \cup C_4$, then C_2 is a reducible element of $\widehat{C}$. If C_5 is added into $\widehat{C}$, we still have $C_2 = C_3 \cup C_4$, then C_2 is still a reducible element of $\widehat{\mathcal{G}}$.

According to Theorem 3.2, we have the following properties.

Proposition 3.1. *Let $(U, \widehat{C})$ be a fuzzy β-covering approximation space of U and $(U, \widehat{\mathcal{G}})$ is also a fuzzy β-covering approximation space of U. $\Gamma(\widehat{C})$ is the reduct of $\widehat{C}$. If C_{m+1} is a reducible element of $\widehat{\mathcal{G}}$, then $\Gamma(\widehat{C})$ is still the reduct of $\widehat{\mathcal{G}}$.*

Proof. For $\Gamma(\widehat{C})$, we have that $\widehat{C} - \Gamma(\widehat{C})$ is the set of all reducible elements of $\widehat{C}$. By Theorem3.1, we have $\widehat{C} - \Gamma(\widehat{C})$ is still the set of reducible of $\widehat{\mathcal{G}}$. If C_{m+1} is a reducible element of $\widehat{\mathcal{G}}$, then $(\widehat{C} - \Gamma(\widehat{C})) \cup \{C_{m+1}\} = \widehat{\mathcal{G}} - \Gamma(\widehat{C})$ is the set of all reducible elements of $\widehat{\mathcal{G}}$. Hence $\Gamma(\widehat{C})$ is the reduct of $\widehat{\mathcal{G}}$. □

The above properties indicate that if the added covering element in the fuzzy β-covering is a reducible element, there is no need to recalculate the reduction.

Example 3.2 (Continuation of Example 2.1)**.** If $C_5 = \frac{0.3}{x_1} + \frac{0.5}{x_2} + \frac{0.7}{x_3} + \frac{0.7}{x_4}$ is added into $\widehat{C}$. We have $C_5 = C_1 \cup C_4$, then C_5 is a reducible element of $\widehat{\mathcal{G}}$. Because C_2 is a reducible element of $\widehat{C}$, then $\Gamma(\widehat{C}) = \{C_1, C_3, C_4\}$. According to Theorem3.2, C_2 is also a reducible element of $\widehat{\mathcal{G}}$. Hence $\Gamma(\widehat{C}) = \{C_1, C_3, C_4\} = \Gamma(\widehat{\mathcal{G}})$ is still the reduct of $\widehat{\mathcal{G}}$.

According to Proposition 3.1 and Theorem 3.2, we only need to study the changes of irreducible elements in fuzzy β-covering when the deleted covering element is a irreducible element. According to Theorem 3.2, we know that a reducible element is still reducible after adding an element to a fuzzy β-covering. Therefore, we can directly observe the impact of the added element is an irreducible element to other irreducible elements after deleting the reducible elements. This can reduce the dimension of the relation character matrix during calculation.

Theorem 3.3. *Let $(U, \widehat{C})$ be a fuzzy β-covering approximation space of U, where $\widehat{C} = \{C_1, C_2, \ldots, C_m\}$. And $(U, \widehat{\mathcal{G}})$ is still a fuzzy β-covering approximation space of U, where $\widehat{\mathcal{G}} = \{C_1, C_2, \ldots, C_m, C_{m+1}\}$. $\Gamma(\widehat{C}) = \{C_{j1}, C_{j2}, \ldots, C_{jk}\}$ $(1 \leq k \leq m)$ and $|\Gamma(\widehat{C})| = k$. C_{m+1} is an irreducible element of $\widehat{\mathcal{G}}$. C_{jp} $(p \in \{1, 2, \ldots, k\})$ is an irreducible element of the $\widehat{C}$, the following result holds.*

$$q_{(k+1)jp}^{\Gamma(\widehat{C})\cup C_{m+1}} = 0 \Rightarrow C_{jp} \textit{ is still an irreducible element of the } \widehat{\mathcal{G}}, \\ (p \in \{1, 2, \ldots, k\}, 1 \leq k \leq m).$$

Proof. Suppose C_{jp} is a reducible element of the $\widehat{\mathcal{G}}$, then there exist $C_{i1}, C_{i2}, \ldots, C_{it} \in \widehat{\mathcal{G}}$ $(1 < t \leq m+1)$ such that $C_{jp} = \cup_{l=1}^{t} C_{il}$. For $q_{(k+1)jp}^{\Gamma(\widehat{C})\cup C_{m+1}} = 0$, according to Definition2.5, we have $C_{k+1} \nsubseteq C_{jp}$. Hence, we have $C_{k+1} \notin \{C_{i1}, C_{i2}, \ldots, C_{it}\}$, then $C_{i1}, C_{i2}, \ldots, C_{it} \in \widehat{C}$ $(1 < t \leq m)$ such that $C_{jp} = \cup_{l=1}^{t} C_{il}$. Therefore C_{jp} is a reducible element of the $\widehat{C}$. This is contradictory to C_{jp} is an irreducible element of the $\widehat{C}$. Hence, if $q_{(k+1)jp}^{\Gamma(\widehat{C})\cup C_{m+1}} = 0$, then C_{jp} $(p \in \{1, 2, \ldots, k\})$ is still an irreducible element of $\widehat{\mathcal{G}}$. □

According to Theorem 3.2, we know that adding a covering element in the fuzzy β-covering does not affect the reducible element, so we just need to study the effect of adding a covering element on the irreducible element. Theorem 3.3 considers the influence of irreducible elements by relation character matrix when adding a covering element after the reducible element is removed. If the added covering element is not included in the irreducible C_{jp}, then C_{jp} is still an irreducible element in fuzzy β-covering.

Example 3.3 (Continuation of Example 3.1)**.** We know C_2 is a reducible element of $\widehat{C}$ and it also is a reducible element of $\widehat{\mathcal{G}}$. Hence, we have

$$P^{\Gamma(\widehat{C})\cup C_5} = \begin{bmatrix} 0.3 & 0.2 & 0.1 & 0.2 \\ 0.4 & 0.3 & 0.5 & 0.3 \\ 0.7 & 0.2 & 0.3 & 0.4 \\ 0.6 & 0.4 & 0.7 & 0.5 \end{bmatrix}.$$

According to Theorem 3.3, the containing relation character matrix on $\Gamma(\widehat{C})\cup C_5$ can be calculated as follows,

$$Q^{\Gamma(\widehat{C})\cup C_5} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

By Example 3.1, we have $\Gamma(\widehat{C}) = \{C_1, C_3, C_4\}$. Hence $|\Gamma(\widehat{C})| = 3$. We have $q_{4,j2}^{\widehat{\mathcal{G}}} = 0$ and $q_{4,j3}^{\widehat{\mathcal{G}}} = 0$, therefore $C_{j2} = C_3$, $C_{j3} = C_4$ are still irreducible element of the $\widehat{\mathcal{G}}$.

By Theorem 3.3 and Theorem 3.2 we can propose Algorithm 2 for updating reduct in a fuzzy β-covering while adding a covering element into the fuzzy β-covering. If $N^+ = \sum_{j=1}^{k} q_{(k+1)j}^{\Gamma(\widehat{C})\cup\{C_{m+1}\}}$, the total time complexity of Algorithm 2 is $O(N^+|U|)$. Steps 1-5 are to decide whether C_{m+1} is a reducible element or not with time complexity $O(m|U|)$, Steps 6-20 are to ensure whether all element in $\Gamma(\widehat{C})$ are reducible or not with time complexity $O(N^+m|U|)$.

Algorithm 2 Algorithm to update the reduct of a fuzzy β-covering while adding a covering element into the fuzzy β-covering

Require: $(1)\widehat{\mathcal{G}} = \{C_1, C_2, \ldots, C_m, C_{m+1}\}$, $(2)U = \{x_1, x_2, \ldots, x_n\}$, $(3)\beta \in (0, 1]$, $(4)Q$, $(5)\Gamma(\widehat{C})$, $(6)|\ \Gamma(\widehat{C})\ | = k$.

Ensure: $\Gamma(\widehat{\mathcal{G}})$.

$T \longleftarrow \emptyset$
for $i = 1 \to k$ **do**
  if $q_{i(k+1)}^{\widehat{\mathcal{G}}} = 1$ **then** $T \longleftarrow T \cup C_i$
  end if
end for
if $T = C_{k+1}$ **then** $\Gamma(\widehat{\mathcal{G}}) \longleftarrow \Gamma(\widehat{C})$
else
  $\Gamma(\widehat{\mathcal{G}}) \longleftarrow \Gamma(\widehat{C}) \cup C_{m+1}$
  for $j = 1 \to k$ **do**
    if $q_{(k+1)j}^{\Gamma(\widehat{C}) \cup \{C_{m+1}\}} = 1$ **then**
      $T \longleftarrow \emptyset$
      for $l = 1 \to k$ **do**
        if $q_{lj}^{\Gamma(\widehat{C}) \cup \{C_{m+1}\}} = 1$ **then** $T \longleftarrow T \cup C_l$
        end if
      end for
      if $T = C_j$ **then** $\Gamma(\widehat{\mathcal{G}}) \longleftarrow \Gamma(\widehat{\mathcal{G}}) - \{C_j\}$
      end if
    end if
  end for
end if
Return $\Gamma(\widehat{\mathcal{G}})$

3.2 Matrix-based approach for updating reducts in fuzzy β-covering while deleting a covering from fuzzy β-covering

In this subsection, we present matrix based dynamic approaches for updating reducts in a fuzzy β-covering while deleting a covering element from the fuzzy β-covering. We denote $(U, \widehat{C})$ a fuzzy β-covering approximation space of U where $\widehat{C} = \{C_1, C_2, \ldots, C_m\}$ and $U = \{x_1, x_2, \ldots, x_n\}$. Denote $(U, \widehat{\mathcal{G}})$ a fuzzy approximation space of U, where $\widehat{\mathcal{G}} = \{C_1, C_2, \ldots, C_{\lambda-1}, C_{\lambda+1}, \ldots, C_m\}$ and $U = \{x_1, x_2, \ldots, x_n\}$.

The following is the correlation theorem for determining whether the covering to be a fuzzy β-covering after deleting a covering element in fuzzy β-covering or not.

Theorem 3.4. *Let $(U, \widehat{C})$ be a fuzzy β-covering approximation space of U, for any $x \in U$, if $C_\lambda(x) < \beta$, then $\widehat{\mathcal{G}}$ is also a fuzzy β-covering. i.e, $(U, \widehat{\mathcal{G}})$ is still a fuzzy β-covering approximation space of U.*

Proof. Suppose $\widehat{C} = \{C_1, C_2, \ldots, C_m\}$, if $\widehat{C}$ is a fuzzy β-covering, for any $x \in U$, we have that $(\bigcup_{i=1}^{m} C_i)(x) \geq \beta$. i.e, there exist $j_k \in \{1, 2, \ldots, m\}$, such that $C_j(x_k) \geq \beta$ for any $x \in U$. Since $C_\lambda(x) < \beta$ for any $x \in U$, therefore $\lambda \neq j$. It is clear that $((\bigcup_{i=1}^{\lambda-1} C_i) \cup (\bigcup_{i=\lambda+1}^{m} C_i))(x) \geq \beta$ for any $x \in U$. Hence $\widehat{\mathcal{G}}$ is a fuzzy β-covering. □

Theorem 3.4 shows that when the value of the covering element deleted in the fuzzy β-covering is less than β, it is still a fuzzy β-covering.

Example 3.4. Let $(U, \widehat{C})$ be a fuzzy β-covering approximation space of U. If $\beta = 0.4$ for any $x \in U$, we have $(\bigcup_{i=1}^{5} C_i)(x) \geq 0.4$. According to Definition 2.4, the matrix representation of $\widehat{C}$ is as follow, where $U = \{x_1, x_2, x_3, x_4\}$, $\widehat{C} = \{C_1, C_2, C_3, C_4, C_5\}$.

$$P^{\widehat{C}} = \begin{bmatrix} 0.3 & 0.2 & 0.2 & 0.1 & 0.4 \\ 0.4 & 0.5 & 0.3 & 0.5 & 0.6 \\ 0.7 & 0.3 & 0.2 & 0.3 & 0.3 \\ 0.6 & 0.7 & 0.1 & 0.7 & 0.2 \end{bmatrix}.$$

For any $x_i \in U$,we have that $C_3(x_i) < 0.4$. Hence when $\lambda = 3$, we have that $\widehat{\mathcal{G}} = \{C_1, C_2, C_4, C_5\}$ is still a fuzzy β-covering.

Theorem 3.5. *Let $(U, \widehat{C})$ be a fuzzy β-covering approximation space of U. If there exist $x_k \in U$ such that $C_\lambda(x_k) \geq \beta$, then $\widehat{C} - C_\lambda$ is a fuzzy β-covering if and only if for $x_k \in U$, there exist $l \in \{1, 2, \ldots, \lambda-1, \lambda+1, \ldots, m\}$, such that $C_l(x_k) \geq \beta$.*

Proof. Suppose $\widehat{C} = \{C_1, C_2, \ldots, C_m\}$, if there exist $x_k \in U$ such that $C_\lambda(x_k) \geq \beta$. Because $\widehat{C} - C_\lambda$ is still a fuzzy β-covering if and only if we have $((\bigcup_{i=1}^{\lambda-1} C_i) \cup (\bigcup_{i=\lambda+1}^{m} C_i))(x) \geq \beta$ for any $x \in U$ if and only if for $x_k \in U$, there exist $l \in \{1, 2, \ldots, \lambda-1, \lambda+1, \ldots, m\}$ such that $C_l(x_k) \geq \beta$. □

Given that Theorem 3.5 states that when the value of the covering element that was deleted in the fuzzy β-covering is greater than or equal to β, it is still a fuzzy β-covering that requires certain conditions. Then in the algorithm process, according to Theorem 3.4 and Theorem 3.5 we can determine whether it is a fuzzy β-covering or not.

Example 3.5 (Continuation of Example 3.4)**.** If $\lambda = 2$, $\beta = 0.4$, we have $C_2(x_2) \geq 0.4$ and $C_2(x_4) \geq 0.4$. There exist $i = 5$ such that $C_5(x_2) \geq 0.4$, and exist $i = 4$ such that $C_4(x_4) \geq 0.4$. Hence $\widehat{C} - C_2$ is still a fuzzy β-covering.

Theorem 3.6. *Let $(U, \widehat{C})$ be a fuzzy β-covering of U and $(U, \widehat{\mathcal{G}})$ be a fuzzy β-covering of U. If C_λ be a reducible element of the $\widehat{C}$, then $\Gamma(\widehat{C})$ is also the reduct of $\widehat{\mathcal{G}}$.*

Proof. It is similar to Proposition 3.1. □

The above proposition indicates that when the deleted element in the fuzzy β-covering is a reducible element, it has no effect on the reduct, so that there is no need to recalculate the reduct.

Example 3.6 (Continuation of Example 3.4)**.** By Definition 2.2, we have $C_2 = C_3 \cup C_4$, then C_2 is a reducible element of $\widehat{C}$ and $\Gamma(\widehat{C}) = \{C_1, C_3, C_4, C_5\}$. For $\lambda = 2$, we have $\widehat{\mathcal{G}} = \widehat{C} - \{C_2\}$. Therefore $\Gamma(\widehat{C})$ is still the reduct of $\widehat{\mathcal{G}}$.

Now we only need to study the changes of irreducible elements and reducible elements in fuzzy β-covering when the deleted covering element is an irreducible element.

Theorem 3.7. *Let $(U, \widehat{C})$ be a fuzzy β-covering of U, and $(U, \widehat{\mathcal{G}})$ is also a fuzzy β-covering of U, where $\widehat{\mathcal{G}} = \{C_1, \ldots, C_{\lambda-1}, C_{\lambda+1}, \ldots, C_m\}$. C_λ is an irreducible element of $\widehat{C}$. If $C_l (l = 1, 2, \ldots, \lambda - 1, \lambda + 1, \ldots, m)$ is an irreducible element of $\widehat{C}$, then C_l is still an irreducible element of $\widehat{\mathcal{G}}$.*

Proof. Suppose $\widehat{C} = \{C_1, C_2, \ldots, C_m\}$, if C_l is a reducible element of $\widehat{\mathcal{G}}$, then there exist $C_{j_1}, C_{j_2}, \ldots, C_{j_t} \in \widehat{\mathcal{G}} (1 < t \leq m, t \neq \lambda)$ such that $C_l = \cup_{k=1}^{t} C_{j_k}$. We have that $\widehat{\mathcal{G}} \subset \widehat{C}$, So there exist $C_{j_1}, C_{j_2}, \ldots, C_{j_t} \in \widehat{C} (1 < t \leq m, t \neq \lambda)$ such that $C_l = \cup_{k=1}^{t} C_{j_k}$. Therefore C_l is also a reducible element of $\widehat{C}$. This is contradictory to $C_l (l = 1, 2, \ldots, \lambda - 1, \lambda + 1, \ldots, m)$ is an irreducible element of $\widehat{C}$. □

If the covering element deleted in the fuzzy β-covering is an irreducible element, it has no effect on the remaining irreducible elements in the fuzzy β-covering. However, it has an influence on the reducible elements in fuzzy β-covering, and it may make the reducible element into the irreducible element.

Example 3.7 (Continuation of Example 3.4)**.** For $\lambda = 3$, C_3 is an irreducible element of $\widehat{C}$. And we have C_1, C_4, C_5 are irreducible elements of $\widehat{C}$, then C_1, C_4, C_5 are still irreducible elements of $\widehat{\mathcal{G}} = \widehat{C} - \{C_3\}$.

In the rest part of this subsection, we only need to study whether the reducible elements in fuzzy β-covering are still reducible elements. First, we give the definition of the partial relation character matrix, which shows the relation between irreducible elements and all the covering elements in $\widehat{\mathcal{G}}$.

Definition 3.1. *Let $(U,\widehat{C})$ be a fuzzy β-covering approximation space of U, where $\widehat{C}=\{C_1,C_2,\ldots,C_m\}$. And $(U,\widehat{\mathcal{G}})$ is still a fuzzy β-covering of U, where $\widehat{\mathcal{G}}=\{C_1,\ldots,C_{\lambda-1},C_{\lambda+1},\ldots,C_m\}$. C_λ is an irreducible element of $\widehat{\mathcal{G}}$. $\Gamma(\widehat{C})$ is the reduct of $\widehat{C}$. The relation character matrix on $\widehat{\mathcal{G}}\times(\widehat{\mathcal{G}}-\Gamma(\widehat{C}))$ is denoted by $Q=(q_{kt}^{\widehat{\mathcal{G}}\times(\widehat{\mathcal{G}}-\Gamma(\widehat{C}))})_{(m-1)\times r}$, where $m-1=|\widehat{\mathcal{G}}|$, $r=|\widehat{\mathcal{G}}-\Gamma(\widehat{C})|$*

$$q_{kt}^{\widehat{\mathcal{G}}\times(\widehat{\mathcal{G}}-\Gamma(\widehat{C}))}=\begin{cases}1, & C_k\subseteq C_t\wedge k\neq t\\ 0, & C_k\not\subseteq C_t\vee k=t\end{cases}\qquad (C_k\in\widehat{\mathcal{G}},C_t\in\widehat{\mathcal{G}}-\Gamma(\widehat{C})).$$

Theorem 3.8. *Let $(U,\widehat{C})$ be a fuzzy β-covering approximation space of U and $(U,\widehat{\mathcal{G}})$ be a fuzzy β-covering of U. For C_λ is an irreducible element of $\widehat{\mathcal{G}}$ and C_j is a reducible of $\widehat{C}$. Denote $A_j=\{i\mid q_{ij}^{\widehat{\mathcal{G}}\times(\widehat{\mathcal{G}}-\Gamma(\widehat{C}))}=1\}$, if $C_j=\bigcup_{i\in A_j}C_i$, then C_j is still a reducible element of $\widehat{\mathcal{G}}$. Otherwise, C_j is an irreducible element of $\widehat{\mathcal{G}}$.*

Proof. It is obvious by Definition 2.2 and Theorem 3.1 □

Example 3.8 (Continuation of Example 3.6)**.** We know C_2 is a reducible element of $\widehat{C}$ and $\Gamma(\widehat{C})=\{C_1,C_3,C_4,C_5\}$. If $\lambda=3$, then $\widehat{\mathcal{G}}=\{C_1,C_2,C_4,C_5\}$ and $\widehat{\mathcal{G}}-\Gamma(\widehat{C})=\{C_2\}$. According to Definition 3.1, We have

$$Q^{\widehat{\mathcal{G}}\times(\widehat{\mathcal{G}}-\Gamma(\widehat{C}))}=\begin{bmatrix}0\\0\\1\\0\end{bmatrix}.$$

Hence C_3 is not a reducible of $\widehat{\mathcal{G}}$.

By Theorem 3.6 and Theorem 3.8 we can propose Algorithm 3 for updating reduct in a fuzzy β-covering while deleting a covering element from the fuzzy β-covering. If $N^-=\sum_{i=1}^{|\widehat{\mathcal{G}}|}\sum_{j=1}^{|\widehat{\mathcal{G}}-\Gamma(\widehat{C})|}q_{ij}^{\widehat{\mathcal{G}}\times(\widehat{\mathcal{G}}-\Gamma(\widehat{C}))}$, the total time complexity of Algorithm 3 is $\Theta(N^-|U|)$. Steps 3-10 are to ensure whether $\widehat{\mathcal{G}}$ is a fuzzy β-covering or not with time complexity $\Theta(|\widehat{\mathcal{G}}||U|)$, Steps 11-18 are to decide whether all reducible elements in $\widehat{C}$ is still reducible or not with time complexity $\Theta(N^-|U|)$.

4. Construct data sets and experiments

In this section, we use two types of data sets to evaluate the effectiveness and efficiency of our algorithms.

Algorithm 3 Algorithm to update the reduct of a fuzzy β-covering while deleting a covering element from the fuzzy β-covering

Require: (1)$U = \{x_1, x_2, \ldots, x_n\}$, (2)$\widehat{C} = \{C_1, C_2, \ldots, C_m\}$, (3)$\widehat{\mathcal{G}} = \{C_1, C_2, \ldots, C_{\lambda-1}, C_{\lambda+1}, \ldots, C_m\}$, (4)$\beta \in (0, 1]$, (5)$\Gamma(\widehat{C})$, (6)$Q^{\widehat{\mathcal{G}} \times (\widehat{\mathcal{G}} - \Gamma(\widehat{C}))}$.
Ensure: $\Gamma(\widehat{\mathcal{G}})$.

if $C_\lambda \in \widehat{C} - \Gamma(\widehat{C})$ **then** $\Gamma(\widehat{\mathcal{G}}) \longleftarrow \Gamma(\widehat{C})$
else
  for $l = 1 \to n$ **do**
    $C(x_l) \longleftarrow 0$
    for $k = 1 \to |\widehat{\mathcal{G}}|$ **do**
      $C(x_l) \longleftarrow C(x_l) \vee C_k(x_l)$
    end for
    if $C(x_l) < \beta$ **then** Return $\widehat{\mathcal{G}}$ is not a fuzzy β-covering
    end if
  end for
  $\Gamma(\widehat{\mathcal{G}}) \longleftarrow \widehat{\mathcal{G}}$
  for $j = 1 \to |\widehat{\mathcal{G}}|$ **do**
    $T \longleftarrow \emptyset$
    for $i = 1 \to |\widehat{\mathcal{G}} - \Gamma(\widehat{C})|$ **do**
      if $q_{ji}^{\widehat{\mathcal{G}} \times (\widehat{\mathcal{G}} - \Gamma(\widehat{C}))} = 1$ **then** $T \longleftarrow T \cup C_j$
      end if
    end for
  end for
  if $T = C_i$ **then** $\Gamma(\widehat{\mathcal{G}}) \longleftarrow \Gamma(\widehat{\mathcal{G}}) - \{C_i\}$
  end if
end if
Return $\Gamma(\widehat{\mathcal{G}})$

4.1 Data sets were constructed

In this section, we constructs some fuzzy β-coverings and run several experiments on them. The procedure of constructing the matrices is as follows: we construct some elements of a partial matrix, the elements of the partial matrix is randomly chosen from $\{0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9\}$. The rest elements can be obtained by randomly union 5% column of the partial matrix together.Several experiments was conducted and the results are shown in Fig 1 to Fig 7. We compared the computational time of Algorithm 1(SA), 2(IA) and 3(DA) with different universe size, different covering size and different reducible element percentage. All the experiments were carried out on a personal computer with 64-bit windows 10, AMD Ryzen 5 2500U with Radeon Vega Mobile Gfx 2.00GHz, and 8GB memory. The program language was Matlab r2015b.

For comparing the computational time of the three algorithms with different percentage of reducible element. We set the size of U is 1000 and the size of covering is 1000. The results of experiments are shown in the Fig 1. By Fig 1 we can see that with percentage of reducible element increasing gradually, the computational time of the three algorithms have completely different performance. In the subgraph 1 of Fig 1, the computational time of SA hardly change with increasing percentage. However, the computational time of IA decreases gradually with the increase of percentage. And it is easy to see IA is more efficient than SA. In the subgraph 2 of Fig 1, we can see the computational time of SA and DA hardly change with increasing percentage. But DA is much more efficient than SA.

For comparing the computational time of the three algorithms with different size of U. We set the size of covering is 500 and the percentage of reducible element is 5%,50% and 95%, respectively. The results of experiments are shown in Fig 2, Fig 3 and Fig 4, respectively. It can be seen from Fig 2, Fig 3 and Fig 4 that the calculation time of DA and IA is much less than SA. So the efficiency of DA and IA is much higher than SA. By Fig 2 to Fig 4, we can observe that SA is getting less and less efficient with the percentage of reducible element increasing gradually. The efficience of IA in the percentage of reducible element is 95% is higher than it in the percentage of reducible element is 50%. For DA, the most efficiency is it in the percentage of reducible element is 5%. The computational time of DA in the percentage of reducible element is 50% is similar to it in the percentage of reducible element is 95%.

For comparing the computational time of the three algorithms with different size of covering. We set the size of U is 500 and the percentage of reducible element is 5%, 50% and 95%, respectively. The results of experiments are shown in Fig 5, Fig 6 and Fig 7, respectively. In the Fig 5, Fig 6 and Fig 7. we can see that the computational time of IA, DA and SA is gradually increasing with the size of the covering increasing gradually. The efficiency of DA and IA is significantly more higher than SA. DA is the most efficient of the three. And the efficiency of IA is the second highest, while that of SA is the lowest.

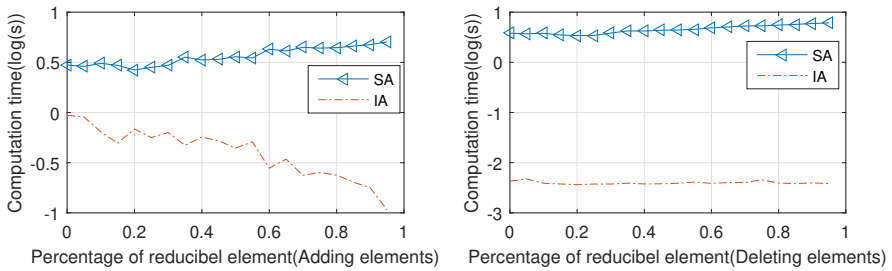


Figure 1: Comparison of Algorithm 1,2 and 3

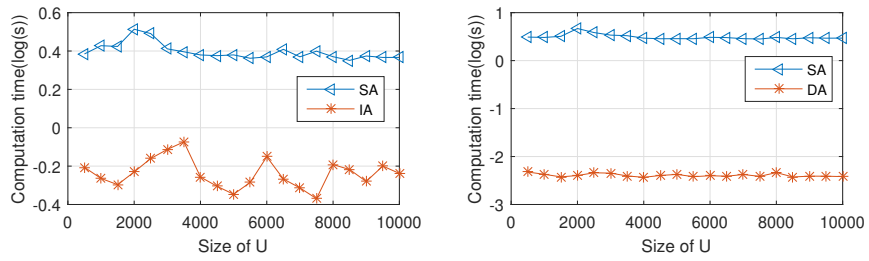


Figure 2: Comparison of Algorithm 1,2 and 3 with the percentage of reducible element is 5%

By Fig 5 to Fig 7, when the percentage of reducible element is 5%, 50% and 95%,respectively, the computational time of SA and IA is almost the same. getting more and more efficient. However, when the percentage of reducible element is 95%, IA is the most efficiency. The second fast is it in the percentage of reducible element is 50%. When the percentage of reducible element is 5%, IA is the slowest.

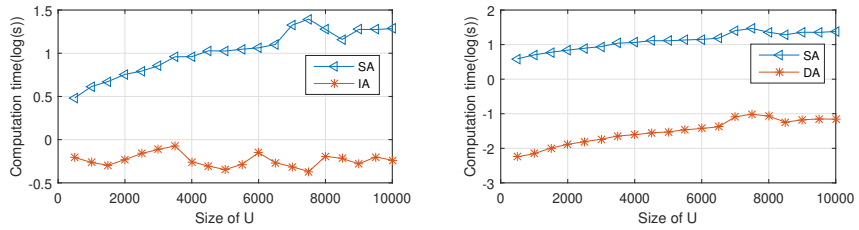


Figure 3: Comparison of Algorithm 1,2 and 3 with the percentage of reducible element is 50%

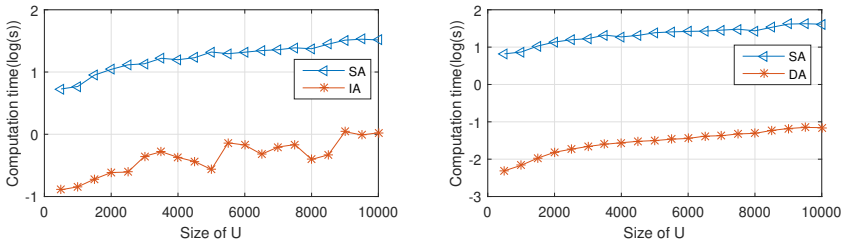


Figure 4: Comparison of Algorithm 1,2 and 3 with the percentage of reducible element is 95%

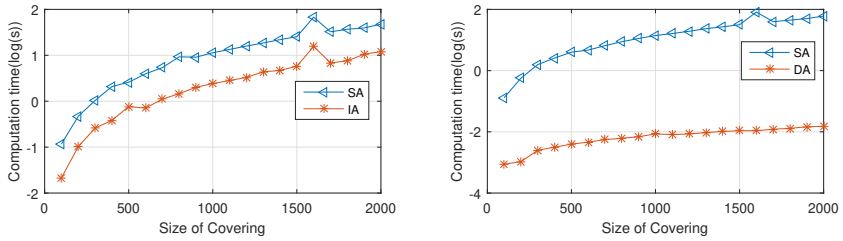


Figure 5: Comparison of Algorithm 1,2 and 3 with the percentage of reducible element is 5%

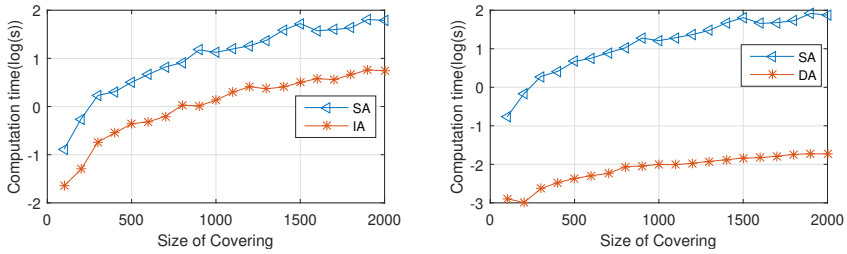


Figure 6: Comparison of Algorithm 1,2 and 3 with the percentage of reducible element is 50%

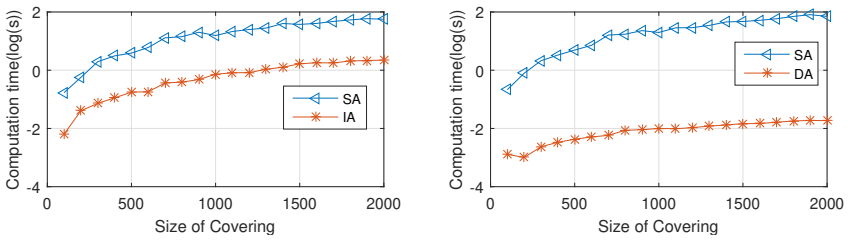


Figure 7: Comparison of Algorithm 1,2 and 3 with the percentage of reducible element is 95%

4.2 Data sets were chosen from UCI machine learning repository.

In this section, We obtain six data sets from UCI machine learning repository for experiments to verify the effectiveness of the algorithms. The details of the data sets are listed in Table 1.

In the ig 8, We compares the calculation time of the static algorithm and the dynamic algorithm of computing reducts while a covering element is added and deleted when the cardinal of data sets increases in fuzzy β-covering. First, we randomly divided every data set U into ten subsets $\{U_1, U_2, \ldots, U_{10}\}$. And then U_1 was chosen as the first temporary target concept X. Finally, we calculated the reduct in fuzzy β-covering by the static and the incremental algorithm ten times and compared the averages. We made $U_1 \cup U_2$ the second temporary target concept and repeat the whole process. From Fig 8, we can see that the incremental algorithm is more effective than the static algorithm when calculating the reducts update while adding and deleting a covering element.

In the Fig 9, We compares the computational time between static and incremental algorithms for updating reduct while a covering element is added and deleted when fuzzy β-covering with different β. First of all, we set $\beta = 0$ and make it increases with a step 0.1. We repeat each step of experiment ten times to evaluate the efficiency of algorithm 2 and algorithm 3. According to Fig 9, we can see that in most cases the calculation time of the incremental algorithm is less than that of the static algorithm.

In the Fig 10, We compares the computational time between static and incremental algorithms for updating reduct while a covering element is added and deleted when the cardinal of covering increases. First of all, we randomly chose some of covering elements to construct the reducible element by union them together. we construct 0.5 times of the cardinal of each data set and add the constructed covering elements into the data set per step and repeat ten times to evaluate the efficiency of algorithm 2 and algorithm 3. According to Fig 10, we can observe that the incremental algorithms are more efficient than the static algorithm when computation the reducts update while adding and deleting a covering element.

Table 1: Details of data sets

No.	Data sets	Samples	Attributes
1	First order theorem proving Data Set	3059	57
2	Ionosphere	351	34
3	Phishing Websites Features	11055	31
4	Polish companies bankruptcy Data Set	7027	65
5	Semeion Hand written Digit Data Set	1593	266
6	Soybean Large	307	36

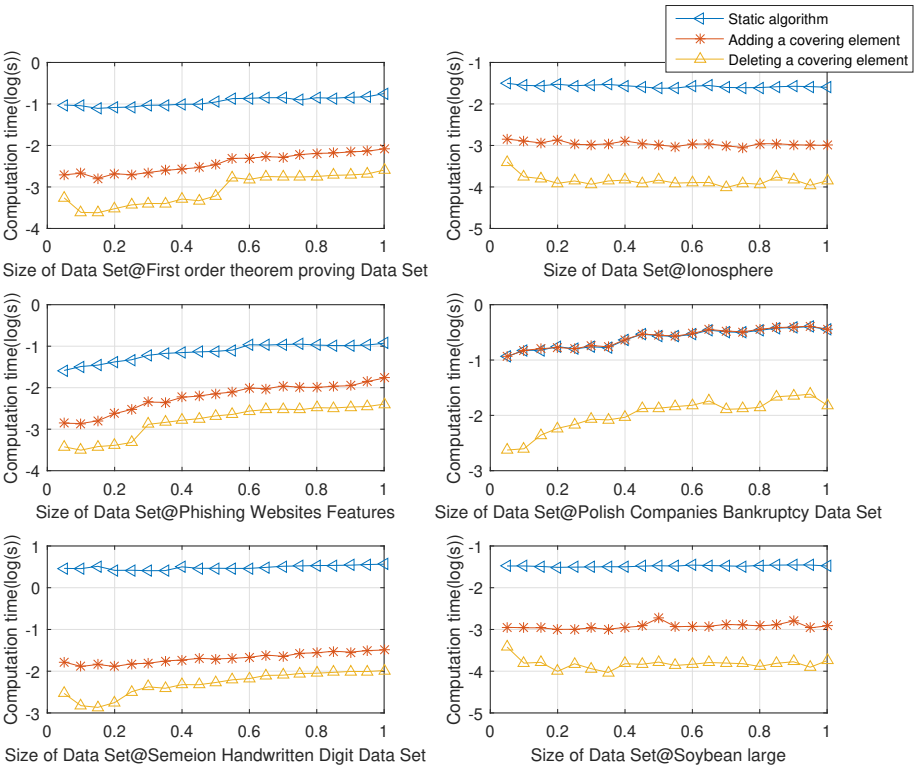


Figure 8: Comparison of Algorithm 1,2 and 3

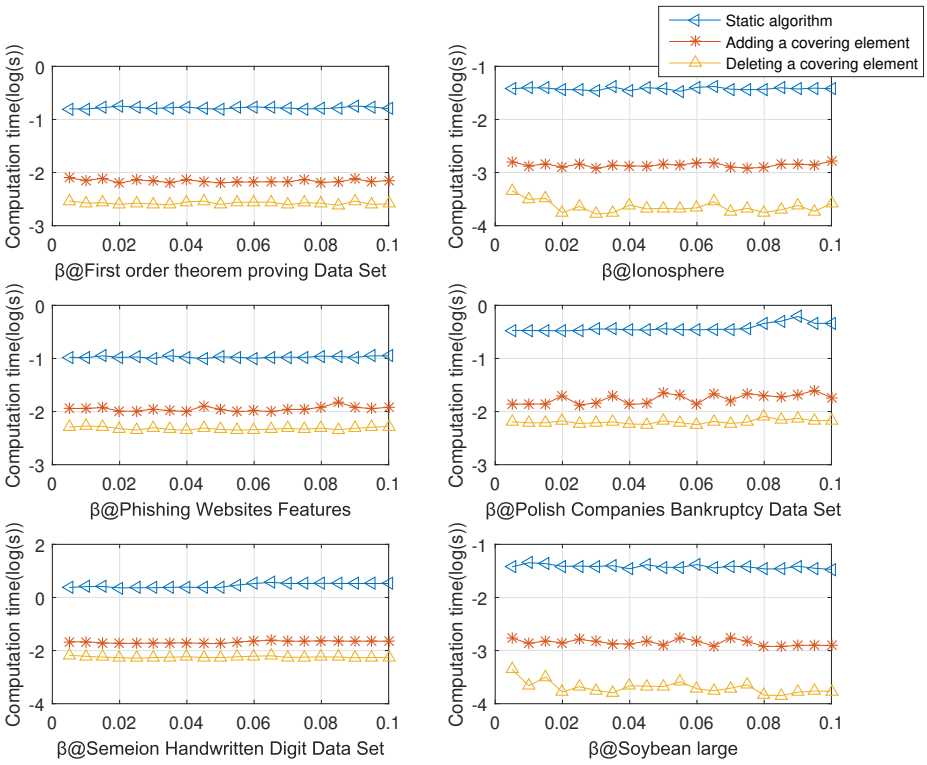


Figure 9: Comparison of Algorithm 1,2 and 3

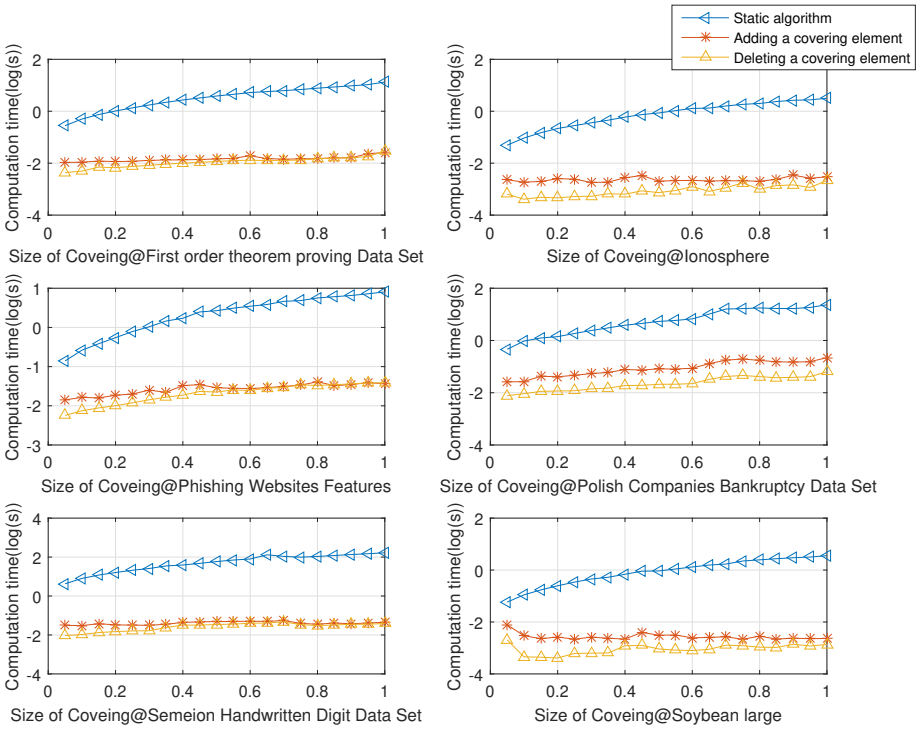


Figure 10: Comparison of Algorithm 1,2 and 3

5. Conclusion

Fuzzy β-covering approximation space is an important topic in recent years. Knowledge reduct of dynamic fuzzy β-covering approximation space is a significant challenge. In this paper, firstly, we define relation character matrix according to the inclusion relationship between fuzzy β-covering elements, and improves the algorithm for computing reduct of a fuzzy β-covering according to this matrix. Secondly, The related theorem and example are given to illustrate how to calculate the reduct of fuzzy β-covering approximation space when adding and deleting a covering element. And we have presented the related algorithms for computing reduct of fuzzy β-covering by using the relation character matrix. Finally, we have performed the experiments to illustrate that the proposed algorithms are effective and feasible for reduct when adding or deleting a covering element in fuzzy β-covering. Our future work will focus on the following issues. The first issue is to further investigate the variation of reduct of the fuzzy β-covering and the algorithm of relevant calculation reduct when multiple covering elements are added and deleted in fuzzy β-covering. Second, we can propose related algorithms for reduct of fuzzy β-covering when the covering elements are varying with time.

Acknowledgements

This work is supported by National Natural Science Foundation of China (No.11871259).

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Accepted: March 03, 2020

A note on unification of the arithmetic-geometric mean and Hölder inequalities for unitarily invariant norms

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Abstract. Let $A, X, B \in M_n$, $\frac{1}{p}+\frac{1}{q}=1, p, q>1, \alpha \in [0,1]$. In this note, we show that if $r < \max\left\{\frac{1}{p}, \frac{1}{q}\right\}$, then the following inequality

$$\left\| |A^*XB|^{2r} \right\| \leq \| |T_X(\alpha)|^{rp} \|^{1/p} \| |T_X(1-\alpha)|^{rq} \|^{1/q}$$

is note always true, where

$$T_X(\alpha) = \alpha AA^*X + (1-\alpha) XBB^*.$$

Keywords: unitarily invariant norms, arithmetic-geometric mean inequality, Hölder inequality.

1. Introduction

Let M_n be the space of $n \times n$ complex matrices. Suppose that $s_1(A) \geq \cdots \geq s_n(A) \geq 0$ are the singular values of $A \in M_n$. In this note, the symbol $\|\cdot\|$ denotes any unitarily invariant norm on M_n. Suppose that all the eigenvalues of $A \in M_n$ are real, then we label them as $\lambda_1(A) \geq \cdots \geq \lambda_n(A)$. The absolute value of matrix A is defined by $|A| = (A^*A)^{1/2}$.

Let $A, X, B \in M_n$, $\frac{1}{p}+\frac{1}{q}=1, p, q>1, \alpha \in [0,1]$. Recently, Zou proved in [1] that if $r \geq \max\{\frac{1}{p}, \frac{1}{q}\}$, then

$$\left\| |A^*XB|^{2r} \right\| \leq \| |T_X(\alpha)|^{rp} \|^{1/p} \| |T_X(1-\alpha)|^{rq} \|^{1/q}, \tag{1}$$

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where

$$T_X(\alpha) = \alpha AA^*X + (1-\alpha)XBB^*.$$

It is natural to raise the question that if $r < \max\{\frac{1}{p}, \frac{1}{q}\}$, then inequality (1) holds or not?

In this note, we present some numerical examples to show that inequality (1) is not always true when $r < \max\{\frac{1}{p}, \frac{1}{q}\}$.

2. Numerical examples

Let $p = q = 2$ and suppose that $r = 1/2 - \varepsilon, 0 < \varepsilon \leq 1/2$. If inequality (1) holds when $r < \max\{\frac{1}{p}, \frac{1}{q}\}$, then we have

$$\begin{aligned} \left\| |A^*XB|^{1-2\varepsilon} \right\| &\leq \left\| |\alpha AA^*X + (1-\alpha)XBB^*|^{1-2\varepsilon} \right\|^{1/2} \\ &\times \left\| |(1-\alpha)AA^*X + \alpha XBB^*|^{1-2\varepsilon} \right\|^{1/2}. \end{aligned} \tag{2.1}$$

Further more, setting $\alpha = 1/2$, inequality (2.1) simplify as follows

$$\left\| |A^*XB|^{1-2\varepsilon} \right\| \leq \left\| \left| \frac{AA^*X + XBB^*}{2} \right|^{1-2\varepsilon} \right\|. \tag{2.2}$$

By Fan's dominance principle [2, p. 93], we know that inequality (2.2) is equivalent to

$$\sum_{j=1}^{k} s_j \left(|A^*XB|^{1-2\varepsilon} \right) \leq \sum_{j=1}^{k} s_j \left(\left| \frac{AA^*X + XBB^*}{2} \right|^{1-2\varepsilon} \right), k = 1, \cdots, n. \tag{2.3}$$

Note that

$$|A^*XB| = \left(B^*X^*AA^*XB \right)^{1/2}$$

and

$$\left| \frac{AA^*X + XBB^*}{2} \right| = \left(\left(\frac{AA^*X + XBB^*}{2} \right)^* \left(\frac{AA^*X + XBB^*}{2} \right) \right)^{1/2}.$$

And so inequality (2.3) can be rewritten as

$$\sum_{j=1}^{k} s_j \left((B^*X^*AA^*XB)^{\eta} \right) \leq \sum_{j=1}^{k} s_j \left((K^*K)^{\eta} \right), , \tag{2.4}$$

where

$$\eta = (1-2\varepsilon)/2, k = 1, \cdots, n, K = \frac{AA^*X + XBB^*}{2}.$$

Inequality (2.4) is equivalent to

$$\sum_{j=1}^{k} \lambda_j^{\eta} \left(AA^*XBB^*X^*\right) \leq \sum_{j=1}^{k} \lambda_j^{\eta} \left(K^*K\right). \tag{2.5}$$

In the next, we will give some examples to show that inequality (2.5) is not always true.

Example 2.1. Let $\eta = 0.49$ and

$$A = \begin{bmatrix} 0.7282 & 0.2137 \\ 0.3630 & 0.9287 \end{bmatrix}, X = \begin{bmatrix} 0.2614 & 0.9143 \\ 0.2580 & 0.5004 \end{bmatrix}, B = \begin{bmatrix} 0.4441 & 0.8519 \\ 0.8839 & 0.0340 \end{bmatrix}.$$

By using Matlab software to calculate, we have

$$\lambda_1^{0.49}\left(AA^*XBB^{^*}X^*\right) = 1.1890, \lambda_2^{0.49}\left(AA^*XBB^{^*}X^*\right) = 0.0415,$$

$$\lambda_1^{0.49}\left(\left(\frac{AA^*X + XBB^*}{2}\right)^* \left(\frac{AA^*X + XBB^*}{2}\right)\right) = 1.2212$$

and

$$\lambda_2^{0.49}\left(\left(\frac{AA^*X + XBB^*}{2}\right)^* \left(\frac{AA^*X + XBB^*}{2}\right)\right) = 0.0079.$$

And so

$$\begin{aligned} \sum_{j=1}^{2} \lambda_j^{0.49}\left(AA^*XBB^{^*}X^*\right) &= 1.2305 \\ &> \sum_{j=1}^{2} \lambda_j^{0.49}\left(\left(\frac{AA^*X+XBB^*}{2}\right)^* \left(\frac{AA^*X+XBB^*}{2}\right)\right) \\ &= 1.2296. \end{aligned}$$

Hence inequality (1) is violated for these matrices and the trace norm when $r < \max\{\frac{1}{p}, \frac{1}{q}\}$.

Example 2.2. Let $\eta = 0.25$ and

$$A = \begin{bmatrix} 0.5723 & 0.1881 & 0.8595 \\ 0.3113 & 0.5687 & 0.1940 \\ 0.2338 & 0.7657 & 0.8249 \end{bmatrix}, X = \begin{bmatrix} 0.4784 & 0.4485 & 0.6560 \\ 0.8802 & 0.9207 & 0.6685 \\ 0.3575 & 0.5462 & 0.7431 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.2674 & 0.9221 & 0.0265 \\ 0.8070 & 0.4631 & 0.5350 \\ 0.2967 & 0.8386 & 0.9840 \end{bmatrix}.$$

By using Matlab software to calculate, we have

$$\lambda_1^{0.25}\left(AA^*XBB^{^*}X^*\right) = 2.2330, \lambda_2^{0.25}\left(AA^*XBB^{^*}X^*\right) = 0.2648,$$

$$\lambda_1^{0.25}\left(\left(\frac{AA^*X+XBB^*}{2}\right)^*\left(\frac{AA^*X+XBB^*}{2}\right)\right)=2.2724$$

and

$$\lambda_2^{0.25}\left(\left(\frac{AA^*X+XBB^*}{2}\right)^*\left(\frac{AA^*X+XBB^*}{2}\right)\right)=0.2140.$$

and so

$$\begin{aligned}\sum_{j=1}^{2}\lambda_j^{0.25}\left(AA^*XBB^{*}X^*\right)&=2.4978\\ &>\sum_{j=1}^{2}\lambda_j^{0.25}\left(\left(\frac{AA^*X+XBB^*}{2}\right)^*\left(\frac{AA^*X+XBB^*}{2}\right)\right)\\ &=2.4864.\end{aligned}$$

Hence inequality (1) is violated for these matrices and the Ky Fan 2-norm when $r<\max\{\frac{1}{p},\frac{1}{q}\}$.

Acknowledgements

This research was supported by the Science and Technology Research Program of Chongqing Municipal Education Commission (KJQN201800811 and KJQN202000816).

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Accepted: May 16, 2020

More generalized forms of fuzzy subgroups of groups

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Abstract. In this paper, we generalize the notion of belongs to relation ($\in$) by $\in_\gamma^h$ relation of a fuzzy point with a fuzzy subset. We introduce the concepts of (α, β)-fuzzy subgroups of a group by using $\in_\gamma^h$ relation and q_δ^k relation between a fuzzy point and a fuzzy subset. We also introduce the notion of $(\in_\gamma^h, \in_\gamma^h \vee q_\delta^k)$-fuzzy subgroup of a group which is a generalization of twelve different types of fuzzy subgroups. Further, we introduce the notions of q_δ^k-level subset, $\in_\gamma^h \vee q_\delta^k$-level subset, (δ, k)-lower and upper parts, (γ, h)-lower and upper parts of fuzzy subsets and investigate some related properties. Finally, we discuss the homomorphic image and pre-image of $(\in_\gamma^h, \in_\gamma^h \vee q_\delta^k)$-fuzzy subgroups.

Keywords: $(\in_\gamma^h, \in_\gamma^h \vee q_\delta^k)$-fuzzy subgroup, $(q_\delta^k, \in_\gamma^h \vee q_\delta^k)$-fuzzy subgroup, $\in_\gamma^h \vee q_\delta^k$-level subset, lower and upper parts of fuzzy subsets.

1. Introduction

Introduction of the notion of a fuzzy set by Zadeh [1] in 1965 paved the way of the foundation of fuzzy set theory and its applications. Extensive application of fuzzy set theory have been found in various fields such as computer science, management science, artificial intelligence, expert systems, operation research, robotics, pattern recognition and other fields. Since then, it is natural to investigate a possible connection between fuzzy sets and algebraic systems. Rosenfeld [2] initiated the fuzzification of algebraic structures and introduced the notion of fuzzy subgroup. Murali [11] proposed a definition of a fuzzy point belonging to fuzzy subset under a natural equivalence in fuzzy subset. The idea of fuzzy point and its "belonginess" and "quasicoincidence" with a fuzzy

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subset, which are discussed by Pu and Liu [3], played an important role to develop different types of fuzzy subgroups. Bhakat and Das [6, 7] introduced the concepts of (α, β)-fuzzy subgroups by using the "belongs to" relation $(\in)$ and "quasi-coincident with" relation (q) between a fuzzy point and a fuzzy subset, and introduced the concept of an $(\in, \in \vee q)$-fuzzy subgroup. It is particularly important to know that $(\in, \in \vee q)$-fuzzy subgroup is a useful generalization of Rosenfeld's fuzzy subgroup. In [8], Bhakat and Das defined the notions of $(\in, \in \vee q)$-fuzzy subring and $(\in, \in \vee q)$-fuzzy ideal of a ring. In [9], Bhakat introduced the notions of $(\in, \in \vee q)$-fuzzy normal and quasinormal subgroups of groups and discussed some of their properties. In[10], Yuan et al. defined the notion of a fuzzy subgroup with thresholds. Narayanan and Manikantan [12] further extended the notion of $(\in, \in \vee q)$-fuzzy subgroups to near-rings and introduced $(\in, \in \vee q)$-fuzzy subnear-ring, $(\in, \in \vee q)$-fuzzy ideal and $(\in, \in \vee q)$-fuzzy quasi ideal of a near-ring. Dheena and Coumaressane [13] introduced the notions of $(\in, \in \vee q)$-fuzzy subnear-ring (ideal) of a near-ring which is a generalization of an $(\in, \in \vee q)$-fuzzy subnear-ring (ideal). In [14], Jun generalized the concept of quasi-coincident (q) by quasi-k-coincident (q_k) of a fuzzy point with a fuzzy subset and introduced the notions of $(\in, q_k)$-fuzzy subalgebra and $(\in, \in \vee q_k)$-fuzzy subalgebra in a BCK/BCI-algebra. Jun et al. discussed the generalization of Dudek et al. [32] results on (α, β)-fuzzy ideals of hemirings and introduced several $(\alpha, \in \vee q_m)$-fuzzy ideals of hemirings in [15]. Tang and Xie [24] introduced the concepts of $(\in, \in \vee q_k)$-fuzzy ideals and $(\in, \in \vee q_k)$-fuzzy bi-ideal of an ordered semigroup and investigated their related properties. Shabir et al. [16] discussed the characterizations of regular semigroups by (α, β)-fuzzy ideals. In [17], Jun et al. introduced the notion of $(\in, \in \vee q_k)$-fuzzy subgroups, $(\in, \in \vee q_k)$-fuzzy (normal) subgroup which is a generalization of an $(\in, \in \vee q)$-fuzzy (normal) subgroup. Khan et al. [19, 20] generalized the concept of (α, β)-fuzzy (generalized) bi-ideal and defined $(\in, \in \vee q_k)$-fuzzy (generalized) bi-ideal in an ordered semigroup. As a general form of "quasi-coincident with" relation (q), jun et al. [28] defined the concept of "δ-quasi-coincident with" relation (q_0^δ) and introduced the notions of $(\in, \in \vee q_0^\delta)$-fuzzy subgroups, q_0^δ-level sets and $\in \vee q_0^\delta$-level sets. They [29] introduced the notions of $(\tilde{\alpha}, \tilde{\beta})$-fuzzy subsemigroup in semigroups and provided characterizations of $(\in, \in \vee q_0^\delta)$-fuzzy subsemigroups. Muhiuddin et al. [25] introduced the concepts of $(\in, \in \vee q_0^\delta)$-fuzzy subalgebras in BCK/BCI-algebras and obtained related results. In [30], Jun and Song defined the notion of $(\tilde{\alpha}, \tilde{\beta})$-fuzzy bi-ideal in a semigroup and discussed the characterizations of $(\in, \in \vee q_0^\delta)$-fuzzy generalized bi-ideal. Jun and Xin considered more general form of $(\in, \in \vee q_k)$-fuzzy subalgebra of BCK/BCI-algebras in [31]. In [26], Kang introduced the notions of $(q_k^\delta, \in \vee q_k^\delta)$-fuzzy subsemigroup, $(\in, q_k^\delta)$-fuzzy subsemigroup and $(\in, \in \vee q_k^\delta)$-fuzzy subsemigroup in a semigroup. Recently, Khan et al. [33] introduced the concept of $(\in, \in \vee q_k^\delta)$-fuzzy bi-((1, 2)-, quasi-) ideal in semigroup. In [22], Zhan and Yin introduced the notions of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subnear-rings, ideals and (strong) prime $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals of a near-ring and provided some new characterizations. Yin et al. [21]

defined the notions of (α, β)-fuzzy left (right, interior, bi-) ideals of an ordered semigroup, where $\alpha, \beta \in \{\in_\gamma, q_\delta, \in_\gamma \wedge q_\delta, \in_\gamma \vee q_\delta\}$ and $\alpha \neq \in_\gamma \wedge q_\delta$, and characterized regular ordered semigroups in terms of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right, interior, bi-) ideals. Shabir and Ali [23] generalized the notion of $(\in, \in \vee q)$-fuzzy left (right, quasi-, bi-) ideal and introduced $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right, interior, bi-) ideal of semigroups. In [27], Yousafzai and Yaqoob introduced the notions of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right, interior, bi-) ideal of an ordered Abel Grassman's groupoid (AG-groupoid). Due to the motivation and inspiration of the above concepts, in this paper, we generalize the notion of belongs to relation $(\in)$ by $\in_\gamma^h$ relation of a fuzzy point with a fuzzy subset. We introduce the concepts of (α, β)-fuzzy subgroups of a group by using $\in_\gamma^h$ relation and q_δ^k relation between a fuzzy point and a fuzzy subset. We also introduce the notion of $(\in_\gamma^h, \in_\gamma^h \vee q_\delta^k)$-fuzzy subgroup of a group which is a generalization of twelve different types of fuzzy subgroups. Further, we introduce the notions of q_δ^k-level subset, $\in_\gamma^h \vee q_\delta^k$-level subset, (δ, k)-lower and upper parts, (γ, h)-lower and upper parts of fuzzy subsets and investigate some related properties. Finally, we discuss the homomorphic image and pre-image of $(\in_\gamma^h, \in_\gamma^h \vee q_\delta^k)$-fuzzy subgroups.

2. Preliminaries

A fuzzy subset μ of a non-empty set X is a mapping from X to $[0, 1]$.
For any two fuzzy subsets μ and η in a set X, we define $\mu \cap \eta$ and $\mu \cup \eta$ in X as follows:

(i) $(\mu \cap \eta)(x) = \min\{\mu(x), \eta(x)\} = \mu(x) \wedge \eta(x)$ and

(ii) $(\mu \cup \eta)(x) = \max\{\mu(x), \eta(x)\} = \mu(x) \vee \eta(x)$ for all $x \in X$.

The product $\mu \circ \eta$ of two fuzzy subsets μ and η of a group G is defined as

$$(\mu \circ \eta)(x) = \begin{cases} \sup\limits_{x=yz} \{\min\{\mu(y), \eta(z)\}\} & \text{if } x = yz \text{ and } y, z \in G \\ 0 & \text{otherwise,} \end{cases}$$

for every $x \in G$.

Let μ be a fuzzy subset of a set X and $t \in [0, 1]$. Then the set $\mu_t = \{x \in X : \mu(x) \geq t\}$ is called the *level subset of* μ in X.

A fuzzy subset μ of a set X is of the form

$$\mu(y) = \begin{cases} t(\neq 0), & \text{if } y = x, \\ 0, & \text{if } y \neq x. \end{cases}$$

is said to be a *fuzzy point* with support x and value t and is denoted by x_t.
For a fuzzy point x_t and a fuzzy subset μ in a set X, we say that

- $x_t \in \mu$ (resp. $x_t \, q \, \mu$) (see [3]) if $\mu(x) \geq t$ (resp. $\mu(x) + t > 1$). In this case, x_t is said to *belong to* (resp. *be quasi-coincident with*) a fuzzy subset μ.
- $x_t \in \vee q \, \mu$ (resp. $x_t \in \wedge q \, \mu$) (see [3]) if $x_t \in \mu$ or $x_t \, q \, \mu$ (resp. $x_t \in \mu$ and $x_t \, q \, \mu$).
- $x_t \in_\gamma \mu$ (see [18]) if $\mu(x) \geq t > \gamma$, where $\gamma \in [0, 1)$.
- $x_t \, q_k \, \mu$ (see [13]) if $\mu(x) + t > 1 - k$, where $k \in [0, 1)$. In this case, x_t is said to be *k-quasi-coincident with* a fuzzy subset μ.
- $x_t \, q_\delta \, \mu$ (see [28]) if $\mu(x) + t > \delta$, where $\delta \in (0, 1]$. In this case, x_t is said to be *δ-quasi-coincident with* a fuzzy subset μ.

Definition 2.1 ([2]). *A fuzzy subset μ of a group G is called a fuzzy subgroup of G if it satisfies:*

$(a1)$ $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$ *for all* $x, y \in G$,

$(a2)$ $\mu(x^{-1}) \geq \mu(x)$ *for all* $x \in G$.

Equivalently

$(a3)$ $\mu(xy^{-1}) \geq \min\{\mu(x), \mu(y)\}$ *for all* $x, y \in G$.

The condition (a1) is equivalent to the condition

$(a1')$ $x_s \in \mu, y_t \in \mu \Rightarrow (xy)_{s \wedge t} \in \mu$ *(see [6]).*

The condition (a2) is equivalent to the condition

$(a2')$ $x_t \in \mu \Rightarrow (x^{-1})_t \in \mu$ *(see [7]).*

A fuzzy subset μ of a group G with conditions (a1′) and (a2′) is an $(\in, \in)$-fuzzy subgroup of G. Hence a fuzzy subgroup coincide with an $(\in, \in)$-fuzzy subgroup.

Definition 2.2 ([7]). *A fuzzy subset μ of a group G is called an $(\in, \in \vee q)$-fuzzy subgroup of G if for all $x, y \in G$ and $s, t \in (0, 1]$,*

$(a4)$ $x_s \in \mu, y_t \in \mu \Rightarrow (xy)_{s \wedge t} \in \vee q \, \mu$,

$(a5)$ $x_t \in \mu \Rightarrow (x^{-1})_t \in \vee q \, \mu$.

Definition 2.3 ([17]). *A fuzzy subset μ of a group G is called an $(\in, \in \vee q_k)$-fuzzy subgroup of G if for all $x, y \in G$, $s, t \in (0, 1]$ and $k \in [0, 1)$,*

$(a6)$ $x_s \in \mu, y_t \in \mu \Rightarrow (xy)_{s \wedge t} \in \vee q_k \, \mu$,

$(a7)$ $x_t \in \mu \Rightarrow (x^{-1})_t \in \vee q_k \, \mu$.

Definition 2.4 ([28]). *A fuzzy subset μ of a group G is called an $(\in, \in \vee q_\delta)$-fuzzy subgroup of G if for all $x, y \in G$ and $s, t, \delta \in (0, 1]$,*

$(a8)$ $x_s \in \mu, y_t \in \mu \Rightarrow (xy)_{s \wedge t} \in \vee q_\delta \, \mu$,

$(a9)$ $x_t \in \mu \Rightarrow (x^{-1})_t \in \vee q_\delta \, \mu$.

3. (α, β)-fuzzy subgroups

In this section, we generalize the notion of belongs to relation $(\in)$ by $\in_\gamma^h$ relation of a fuzzy point with a fuzzy subset. We introduce the concepts of (α, β)-fuzzy subgroups of a group by using $\in_\gamma^h$ relation and q_δ^k relation between a fuzzy point and a fuzzy subset, where α and β denote any one of $\in_\gamma^h, q_\delta^k, \in_\gamma^h \vee q_\delta^k$ and $\in_\gamma^h \wedge q_\delta^k$ but $\alpha \neq \in_\gamma^h \wedge q_\delta^k$.

Definition 3.1. *Let $\gamma, h \in [0,1)$ be such that $h \leq \gamma$ and $\gamma + h \in [0,1)$. For a fuzzy point x_t and a fuzzy subset μ in a set X, we say that $x_t \in_\gamma^h \mu$ if $\mu(x) \geq t > \gamma + h$.*

Obivously, $x_t \in \mu$ implies $x_t \in_\gamma^h \mu$ and $x_t \in_\gamma \mu$ implies $x_t \in_\gamma^h \mu$.

Remark 3.1. (i) If $\gamma = h = 0$, then $x_t \in_\gamma^h \mu = x_t \in \mu$ and (ii) if $\gamma \neq 0$ and $h = 0$, then $x_t \in_\gamma^h \mu = x_t \in_\gamma \mu$.

Definition 3.2 ([26])**.** *Let $0 \leq k < \delta \leq 1$. For a fuzzy point x_t and a fuzzy subset μ in a set X, we say that $x_t \, q_\delta^k \, \mu$ if $\mu(x) + t > \delta - k$.*

Obviously, $x_t \, q \, \mu$ implies $x_t \, q_\delta^k \, \mu$, $x_t \, q_k \, \mu$ implies $x_t \, q_\delta^k \, \mu$ and $x_t \, q_\delta \, \mu$ implies $x_t \, q_\delta^k \, \mu$.

Remark 3.2. (i) If $\delta = 1$ and $k = 0$, then $x_t \, q_\delta^k \, \mu = x_t \, q \, \mu$, (ii) if $\delta = 1$, then $x_t \, q_\delta^k \, \mu = x_t \, q_k \, \mu$ and (iii) if $k = 0$, then $x_t \, q_\delta^k \, \mu = x_t \, q_\delta \, \mu$.

To say that $x_t \in_\gamma^h \vee q_\delta^k \, \mu$ (resp. $x_t \in_\gamma^h \wedge q_\delta^k \, \mu$) means that $x_t \in_\gamma^h \mu$ or $x_t \, q_\delta^k \, \mu$ (resp. $x_t \in_\gamma^h \mu$ and $x_t \, q_\delta^k \, \mu$).

For $\alpha \in \{\in_\gamma^h, q_\delta^k, \in_\gamma^h \vee q_\delta^k, \in_\gamma^h \wedge q_\delta^k\}$, we say that $x_t \overline{\alpha} \mu$ if $x_t \alpha \mu$ does not hold. Throughout this paper, let $0 \leq h \leq \gamma < 1$ be such that $\gamma + h \in [0,1)$ and let $0 \leq k < \delta \leq 1$ be such that $\delta - k \in (0,1]$, and $\gamma + h < \frac{\delta - k}{2}$, G a group, α and β denote any one of $\in_\gamma^h, q_\delta^k, \in_\gamma^h \vee q_\delta^k$ and $\in_\gamma^h \wedge q_\delta^k$ but $\alpha \neq \in_\gamma^h \wedge q_\delta^k$.

Definition 3.3. *A fuzzy subset μ of G is called an (α, β)-fuzzy subgroup of G, where $\alpha \neq \in_\gamma^h \wedge q_\delta^k$, if for all $x, y \in G$ and $s, t \in (\gamma + h, 1]$,*

(b1) $x_s \alpha \mu$, $y_t \alpha \mu \Rightarrow (xy)_{s \wedge t} \beta \mu$,

(b2) $x_t \alpha \mu \Rightarrow (x^{-1})_t \beta \mu$.

Let μ be a fuzzy subset of G such that $\mu(x) \leq \frac{\delta - k}{2}$ for all $x \in G$. Suppose $x \in G$ and $t \in (0,1]$ be such that $x_t \in_\gamma^h \wedge q_\delta^k \, \mu$. Then $\mu(x) \geq t > \gamma + h$ and $\mu(x) + t > \delta - k$ implies $\delta - k < \mu(x) + t \leq \mu(x) + \mu(x) = 2\mu(x)$. That is $\mu(x) > \frac{\delta - k}{2}$. This shows that $\{x_t : x_t \in_\gamma^h \wedge q_\delta^k \, \mu\} = \emptyset$. So we omit $\alpha = \in_\gamma^h \wedge q_\delta^k$ case in the above definition.

Example 3.1. Consider the Klein's four group $G = \{e, a, b, c\}$ with the following table:

$\cdot$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Define a fuzzy subset μ of G by $\mu(e) = 0.4$, $\mu(a) = 0.7$ and $\mu(b) = 0.3 = \mu(c)$. Take $\gamma = 0.18$, $h = 0.02$, $\delta = 0.8$ and $k = 0.1$. We can verify that μ is an $(\in^h_\gamma, \in^h_\gamma \vee q^k_\delta)$-fuzzy subgroup of G. But

(i) μ is not an $(\in^h_\gamma, \in^h_\gamma)$-fuzzy subgroup of G, since $a_{0.6} \in^h_\gamma \mu$ and $(aa)_{0.6}\, \overline{\in^h_\gamma}\, \mu$.

(ii) μ is not an $(\in^h_\gamma, q^k_\delta)$-fuzzy subgroup of G, since $a_{0.3} \in^h_\gamma \mu, b_{0.3} \in^h_\gamma \mu$ and $(ab)_{0.3}\, \overline{q^k_\delta}\, \mu$.

(iii) μ is not an $(\in^h_\gamma, \in^h_\gamma \wedge q^k_\delta)$-fuzzy subgroup of G, since $a_{0.6} \in^h_\gamma \mu$ and $(aa)_{0.6}\, \overline{\in^h_\gamma \wedge q^k_\delta}\, \mu$.

(iv) μ is not a $(q^k_\delta, \in^h_\gamma)$-fuzzy subgroup of G, since $a_{0.6}\, q^k_\delta\, \mu$ and $(aa)_{0.6}\, \overline{\in^h_\gamma}\, \mu$.

(v) μ is not a (q^k_δ, q^k_δ)-fuzzy subgroup of G, since $a_{0.1}\, q^k_\delta\, \mu$ and $(aa)_{0.1}\, \overline{q^k_\delta}\, \mu$.

(vi) μ is not a $(q^k_\delta, \in^h_\gamma \wedge q^k_\delta)$-fuzzy subgroup of G, since $a_{0.1}\, q^k_\delta\, \mu$ and $(aa)_{0.1}\, \overline{\in^h_\gamma \wedge q^k_\delta}\, \mu$.

(vii) μ is not an $(\in^h_\gamma \vee q^k_\delta, \in^h_\gamma)$-fuzzy subgroup of G, since $a_{0.6} \in^h_\gamma \vee q^k_\delta\, \mu$ and $(aa)_{0.6}\, \overline{\in^h_\gamma}\, \mu$.

(viii) μ is not an $(\in^h_\gamma \vee q^k_\delta, q^k_\delta)$-fuzzy subgroup of G, since $a_{0.6} \in^h_\gamma \vee q^k_\delta\, \mu$, $b_{0.3} \in^h_\gamma \vee q^k_\delta\, \mu$ and $(ab)_{0.3}\, \overline{q^k_\delta}\, \mu$.

(ix) μ is not an $(\in^h_\gamma \vee q^k_\delta, \in^h_\gamma \wedge q^k_\delta)$-fuzzy subgroup of G, since $a_{0.6} \in^h_\gamma \vee q^k_\delta\, \mu$ and $(aa)_{0.6}\, \overline{\in^h_\gamma \wedge q^k_\delta}\, \mu$.

Theorem 3.1. *Every fuzzy subgroup μ of G is an $(\in^h_\gamma, \in^h_\gamma)$-fuzzy subgroup of G.*

Proof. Let μ be a fuzzy subgroup of G. Let $x, y \in G$ and $s, t \in (\gamma + h, 1)$ be such that $x_s, y_t \in^h_\gamma \mu$. Then $\mu(x) \geq s > \gamma + h$ and $\mu(y) \geq t > \gamma + h$. Since μ is a fuzzy subgroup of G, we have $\mu(xy) \geq \min\{\mu(x), \mu(y)\} \geq \min\{s, t\} > \gamma + h$ and so $(xy)_{s \wedge t} \in^h_\gamma \mu$. Also $\mu(x^{-1}) \geq \mu(x) \geq s > \gamma + h$. This implies that $(x^{-1})_s \in^h_\gamma \mu$. Hence μ is an $(\in^h_\gamma, \in^h_\gamma)$-fuzzy subgroup of G. □

Theorem 3.2. *If μ is a non zero $(q^k_\delta, \in^h_\gamma)$-fuzzy subgroup of G, then the set $G_0 = \{x \in G | \mu(x) > 0\}$ is a subgroup of G.*

Proof. Let $x, y \in G_0$. Then $\mu(x) > 0$ and $\mu(y) > 0$. Hence $\mu(x) + \delta - k > \delta - k$ and $\mu(y) + \delta - k > \delta - k$, that is, $x_{\delta-k}\, q_\delta^k\, \mu$ and $y_{\delta-k}\, q_\delta^k\, \mu$. It follows from the condition (b1) of Definition 3.3 that $(xy)_{\delta-k} \in_\gamma^h \mu$. This implies that $\mu(xy) \geq \delta - k > \gamma + h \geq 0$. Thus $xy \in G_0$. Also by the condition (b2) of Definition 3.3 that $x_{\delta-k}\, q_\delta^k\, \mu$ implies $(x^{-1})_{\delta-k} \in_\gamma^h \mu$. That is $\mu(x^{-1}) \geq \delta - k > \gamma + h \geq 0$. So $x^{-1} \in G_0$. Hence G_0 is a subgroup of G. □

Theorem 3.3. *If μ is a non zero (q_δ^k, q_δ^k)-fuzzy subgroup of G, then the set $G_0 = \{x \in G | \mu(x) > 0\}$ is a subgroup of G.*

Proof. Let $x, y \in G_0$. Then $\mu(x) > 0$ and $\mu(y) > 0$. Hence $\mu(x) + \delta - k > \delta - k$ and $\mu(y) + \delta - k > \delta - k$, that is, $x_{\delta-k}\, q_\delta^k\, \mu$ and $y_{\delta-k}\, q_\delta^k\, \mu$. It follows from the condition (b1) of Definition 3.3 that $(xy)_{\delta-k}\, q_\delta^k\, \mu$. This implies that $\mu(xy) + \delta - k > \delta - k$ and so $\mu(xy) > 0$. Thus $xy \in G_0$. Also by the condition (b2) of Definition 3.3 that $(x)_{\delta-k}\, q_\delta^k\, \mu$ implies $(x^{-1})_{\delta-k}\, q_\delta^k\, \mu$. That is $\mu(x^{-1}) + \delta - k > \delta - k$ and so $\mu(x^{-1}) > 0$. Thus $x^{-1} \in G_0$. Hence G_0 is a subgroup of G. □

Theorem 3.4. *Let $\delta - k = \gamma + h + 1$ and μ be an (α, β)-fuzzy subgroup of G. Then the set $\mu_\gamma^h = \{x \in G |\, \mu(x) > \gamma + h\}$ is a subgroup of G.*

Proof. Let $\alpha \in \{\in_\gamma^h, \in_\gamma^h \vee q_\delta^k\}$ and $x, y \in \mu_\gamma^h$. Then $\mu(x) > \gamma + h$ and $\mu(y) > \gamma + h$. This implies that $x_{\mu(x)} \in_\gamma^h \mu$ and $y_{\mu(y)} \in_\gamma^h \mu$, and so $x_{\mu(x)} \alpha \mu$ and $y_{\mu(y)} \alpha \mu$. Suppose that $\mu(xy) \leq \gamma + h$, then $\mu(xy) \leq \gamma + h < \min\{\mu(x), \mu(y)\}$ and so $(xy)_{\mu(x)\wedge\mu(y)} \overline{\in_\gamma^h} \mu$. Since $\gamma + h + 1 = \delta - k$, we have $\mu(xy) + \min\{\mu(x), \mu(y)\} \leq \gamma + h + \min\{\mu(x), \mu(y)\} \leq \gamma + h + 1 = \delta - k$ and so $(xy)_{\mu(x)\wedge\mu(y)} \overline{q_\delta^k} \mu$. Thus $(xy)_{\mu(x)\wedge\mu(y)} \overline{\in_\gamma^h} \mu$ and $(xy)_{\mu(x)\wedge\mu(y)} \overline{q_\delta^k} \mu$. This shows that $(xy)_{\mu(x)\wedge\mu(y)} \overline{\beta} \mu$ for every $\beta \in \{\in_\gamma^h, q_\delta^k, \in_\gamma^h \vee q_\delta^k, \in_\gamma^h \wedge q_\delta^k\}$ which contradicts that μ is an (α, β)-fuzzy subgroup of G.

Let $\alpha = q_\delta^k$ and $x, y \in \mu_\gamma^h$. Since $\gamma + h + 1 = \delta - k$, we have $x_1 q_\delta^k \mu$ and $y_1 q_\delta^k \mu$. This implies that $x_1 \alpha \mu$ and $y_1 \alpha \mu$. Suppose that $\mu(xy) \leq \gamma + h$, then $\mu(xy) \leq \gamma + h < \min\{\mu(x), \mu(y)\}$ and $\mu(xy) + \min\{\mu(x), \mu(y)\} \leq \gamma + h + \min\{\mu(x), \mu(y)\} \leq \gamma + h + 1 = \delta - k$ which implies that $(xy)_{\mu(x)\wedge\mu(y)} \overline{\in_\gamma^h} \mu$ and $(xy)_{\mu(x)\wedge\mu(y)} \overline{q_\delta^k} \mu$. This shows that $(xy)_{\mu(x)\wedge\mu(y)} \overline{\beta} \mu$ for every $\beta \in \{\in_\gamma^h, q_\delta^k, \in_\gamma^h \vee q_\delta^k, \in_\gamma^h \wedge q_\delta^k\}$ which contradicts that μ is an (α, β)-fuzzy subgroup of G. Hence $xy \in \mu_\gamma^h$ for all $x, y \in \mu_\gamma^h$. Similarly we can show that $x^{-1} \in \mu_\gamma^h$ for all $x \in \mu_\gamma^h$. Therefore μ_γ^h is a subgroup of G. □

Theorem 3.5. *Let $\delta - k = \gamma + h + 1$ and let A be a nonempty subset of G. Then A is a subgroup of G if and only if the fuzzy subset μ of G defined by*

$$\mu(x) = \begin{cases} \eta, & \text{if } x \in A, \\ \mathcal{E}, & \text{if } x \notin A, \end{cases}$$

where $\eta \geq \frac{\delta - k}{2}$ and $\mathcal{E} \leq \gamma + h$, is an $(\alpha, \in_\gamma^h \vee q_\delta^k)$-fuzzy subgroup of G.

Proof. Let A be a subgroup of G.

(i) Let $\alpha = \in_\gamma^h$ and $x, y \in G$. Choose $s, t \in (\gamma + h, 1]$ such that $x_s \in_\gamma^h \mu$ and $y_t \in_\gamma^h \mu$. Then $\mu(x) \geq s > \gamma + h \geq \mathcal{E}$ and $\mu(y) \geq t > \gamma + h \geq \mathcal{E}$, and so $\mu(x) = \eta = \mu(y)$. Therefore $x, y \in A$ and so $xy \in A$. Hence $\mu(xy) = \eta$. If $s \leq \frac{\delta-k}{2}$ or $t \leq \frac{\delta-k}{2}$, then $\mu(xy) = \eta \geq \frac{\delta-k}{2} \geq \min\{s, t\} > \gamma + h$ and so $(xy)_{s\wedge t} \in_\gamma^h \mu$. If $s > \frac{\delta-k}{2}$ and $t > \frac{\delta-k}{2}$, then $\mu(xy) + \min\{s, t\} = \eta + \min\{s, t\} > \frac{\delta-k}{2} + \frac{\delta-k}{2} = \delta - k$. So $(xy)_{s\wedge t}\, q_\delta^k\, \mu$. Thus $(xy)_{s\wedge t} \in_\gamma^h \vee\, q_\delta^k\, \mu$. Since $x_s \in_\gamma^h \mu$, we have $x \in A$ and so $x^{-1} \in A$. Hence $\mu(x^{-1}) = \eta$. If $s \leq \frac{\delta-k}{2}$, then $\mu(x^{-1}) = \eta \geq \frac{\delta-k}{2} \geq s > \gamma + h$ and so $(x^{-1})_s \in_\gamma^h \mu$. If $s > \frac{\delta-k}{2}$, then $\mu(x^{-1}) + s = \eta + s > \frac{\delta-k}{2} + \frac{\delta-k}{2} = \delta - k$. So $(x^{-1})_s\, q_\delta^k\, \mu$. Thus $(x^{-1})_s \in_\gamma^h \vee\, q_\delta^k\, \mu$. Hence μ is an $(\in_\gamma^h, \in_\gamma^h \vee\, q_\delta^k)$-fuzzy subgroup of G.

(ii) Let $\alpha = \ q_\delta^k$ and $x, y \in G$. Choose $s, t \in (\gamma + h, 1]$ such that $x_s\, q_\delta^k\, \mu$ and $y_t\, q_\delta^k\, \mu$. Then $\mu(x) + s > \delta - k$ and $\mu(y) + t > \delta - k$. This implies that $\mu(x) > \delta - k - s \geq \delta - k - 1 = \gamma + h \geq \mathcal{E}$ and $\mu(y) > \delta - k - t \geq \delta - k - 1 = \gamma + h \geq \mathcal{E}$. Thus $x, y \in A$ and so $xy \in A$. Hence $\mu(xy) = \eta$. If $s \leq \frac{\delta-k}{2}$ or $t \leq \frac{\delta-k}{2}$, then $\mu(xy) = \eta \geq \frac{\delta-k}{2} \geq \min\{s, t\} > \gamma + h$ and so $(xy)_{s\wedge t} \in_\gamma^h \mu$. If $s > \frac{\delta-k}{2}$ and $t > \frac{\delta-k}{2}$, then $\mu(xy) + \min\{s, t\} = \eta + \min\{s, t\} > \frac{\delta-k}{2} + \frac{\delta-k}{2} = \delta - k$, and so $(xy)_{s\wedge t}\, q_\delta^k\, \mu$. Therefore $(xy)_{s\wedge t} \in_\gamma^h \vee\, q_\delta^k\, \mu$. Since $x_s\, q_\delta^k\, \mu$, we have $x \in A$ and so $x^{-1} \in A$. Hence $\mu(x^{-1}) = \eta$. If $s \leq \frac{\delta-k}{2}$, then $\mu(x^{-1}) = \eta \geq \frac{\delta-k}{2} \geq s > \gamma + h$ and so $(x^{-1})_s \in_\gamma^h \mu$. If $s > \frac{\delta-k}{2}$, then $\mu(x^{-1}) + s = \eta + s > \frac{\delta-k}{2} + \frac{\delta-k}{2} = \delta - k$. So $(x^{-1})_s\, q_\delta^k\, \mu$. Therefore $(x^{-1})_s \in_\gamma^h \vee\, q_\delta^k\, \mu$. Thus μ is a $(q_\delta^k, \in_\gamma^h \vee\, q_\delta^k)$-fuzzy subgroup of G.

(iii) Let $\alpha = \in_\gamma^h \vee\, q_\delta^k$ and $x, y \in G$. Choose $s, t \in (\gamma + h, 1]$ such that $x_s \in_\gamma^h \mu$ and $y_t\, q_\delta^k\, \mu$. Then $\mu(x) \geq s > \gamma + h \geq \mathcal{E}$ and $\mu(y) + t > \delta - k$. It implies that $\mu(y) > \delta - k - t \geq \delta - k - 1 = \gamma + h \geq \mathcal{E}$. Thus $x, y \in A$ and so $xy \in A$. Analogous to (i) and (ii), we get $(xy)_{s\wedge t} \in_\gamma^h \vee\, q_\delta^k\, \mu$ and $(x^{-1})_s \in_\gamma^h \vee\, q_\delta^k\, \mu$. Hence μ is an $(\in_\gamma^h \vee\, q_\delta^k,\ \in_\gamma^h \vee\, q_\delta^k)$-fuzzy subgroup of G.

Conversely, assume that μ is an $(\alpha, \in_\gamma^h \vee\, q_\delta^k)$-fuzzy subgroup of G. For any $x \in A$, $\mu(x) = \eta \geq \frac{\delta-k}{2} > \gamma + h$, we have $A = \mu_\gamma^h$. It follows from Theorem 3.4 that A is a subgroup of G. □

Corollary 3.1. *Let $\delta - k = \gamma + h + 1$ and let A be a nonempty subset of G. Then A is a subgroup of G if and only if the characteristic function χ_A of A is an $(\alpha, \in_\gamma^h \vee\, q_\delta^k)$-fuzzy subgroup of G.*

Remark 3.3. In Theorem 3.5 and Corollary 3.1, the condition $\delta - k = \gamma + h + 1$ is not necessary for the case of $(\in_\gamma^h, \in_\gamma^h \vee\, q_\delta^k)$-fuzzy subgroup of G.

Theorem 3.6. *Every $(\in_\gamma^h, \in_\gamma^h)$-fuzzy subgroup μ of G is an $(\in_\gamma^h, \in_\gamma^h \vee\, q_\delta^k)$-fuzzy subgroup of G.*

Proof. Let μ be an $(\in_\gamma^h, \in_\gamma^h)$-fuzzy subgroup of G. Let $x, y \in G$ and $s, t \in (\gamma + h, 1]$ be such that $x_s \in_\gamma^h \mu$ and $y_t \in_\gamma^h \mu$. Then $(xy)_{s\wedge t} \in_\gamma^h \mu$ and so

$(xy)_{s\wedge t} \in^h_\gamma \vee q^k_\delta \mu$. Since $x_t \in^h_\gamma \mu$ implies $(x^{-1})_t \in^h_\gamma \mu$, we have $(x^{-1})_t \in^h_\gamma \vee q^k_\delta \mu$. Hence μ is an $(\in^h_\gamma, \in^h_\gamma \vee q^k_\delta)$-fuzzy subgroup of G. □

Combining Theorem 3.1 and Theorem 3.6, we have the following Theorem.

Theorem 3.7. *Every fuzzy subgroup μ of G is an $(\in^h_\gamma, \in^h_\gamma \vee q^k_\delta)$-fuzzy subgroup of G.*

Theorem 3.8. *Every $(\in^h_\gamma \vee q^k_\delta, \in^h_\gamma \vee q^k_\delta)$-fuzzy subgroup μ of G is an $(\in^h_\gamma, \in^h_\gamma \vee q^k_\delta)$-fuzzy subgroup of G.*

Proof. Let μ be an $(\in^h_\gamma \vee q^k_\delta, \in^h_\gamma \vee q^k_\delta)$-fuzzy subgroup of G. Then from $x_t, y_r \in^h_\gamma \mu$, it follows that $x_t, y_r \in^h_\gamma \vee q^k_\delta \mu$, implying $(xy)_{t\wedge r} \in^h_\gamma \vee q^k_\delta \mu$. Now from $x_t \in^h_\gamma \mu$, it follows that $x_t \in^h_\gamma \vee q^k_\delta \mu$, implying $(x^{-1})_t \in^h_\gamma \vee q^k_\delta \mu$. Hence μ is an $(\in^h_\gamma, \in^h_\gamma \vee q^k_\delta)$-fuzzy subgroup of G. □

Theorem 3.9. *Every $(q^k_\delta, \in^h_\gamma \vee q^k_\delta)$-fuzzy subgroup μ of G is an $(\in^h_\gamma, \in^h_\gamma \vee q^k_\delta)$-fuzzy subgroup of G.*

Proof. Let μ be an $(q^k_\delta, \in^h_\gamma \vee q^k_\delta)$-fuzzy subgroup of G. Let $x, y \in G$ and $s, t \in (\gamma + h, 1]$ be such that $x_s \in^h_\gamma \mu$ and $y_t \in^h_\gamma \mu$. Then $\mu(x) \geq s > \gamma + h$ and $\mu(y) \geq t > \gamma + h$. Suppose $(xy)_{s\wedge t} \overline{\in^h_\gamma \vee q^k_\delta} \mu$. Then $\mu(xy) \leq \gamma + h < \min\{s, t\}$ and $\mu(xy) + \min\{s, t\} \leq \delta - k$. This implies that $\mu(xy) + \mu(xy) < \mu(xy) + \min\{s, t\} \leq \delta - k$. So $\mu(xy) < \frac{\delta-k}{2}$. Since $\mu(xy) \leq \gamma + h < \min\{\mu(x), \mu(y), \frac{\delta-k}{2}\}$, we have $\max\{\mu(xy), \gamma + h\} < \min\{\mu(x), \mu(y), \frac{\delta-k}{2}\}$. Now choose $r \in (\gamma + h, 1]$ such that $\delta - k - \max\{\mu(xy), \gamma + h\} > r \geq \delta - k - \min\{\mu(x), \mu(y), \frac{\delta-k}{2}\}$. This implies that $\delta - k - \mu(xy) \geq \delta - k - \max\{\mu(xy), \gamma + h\} > r \geq \max\{\delta - k - \mu(x), \delta - k - \mu(y), \frac{\delta-k}{2}\}$. It follows that $\mu(x) + r \geq \delta - k$, $\mu(y) + r \geq \delta - k$, $\mu(xy) + r < \delta - k$ and $\mu(xy) \leq \gamma + h < \frac{\delta-k}{2} \leq r$. Hence $x_r q^k_\delta \mu$, $y_r q^k_\delta \mu$ but $(xy)_r \overline{\in^h_\gamma \vee q^k_\delta} \mu$ which is a contradiction. Hence $(xy)_{s\wedge t} \in^h_\gamma \vee q^k_\delta \mu$. Similarly we can show that $x_s \in^h_\gamma \mu$ implies $(x^{-1})_s \in^h_\gamma \vee q^k_\delta \mu$. Hence μ is an $(\in^h_\gamma, \in^h_\gamma \vee q^k_\delta)$-fuzzy subgroup of G. □

From the above discussion we can easily observe that every (α, β)-fuzzy subgroup of G is an $(\alpha, \in^h_\gamma \vee q^k_\delta)$-fuzzy subgroup of G. Hence in the theory of (α, β)-fuzzy subgroups, $(\in^h_\gamma, \in^h_\gamma \vee q^k_\delta)$-fuzzy subgroups play a vital role.

4. $(\in^h_\gamma, \in^h_\gamma \vee q^k_\delta)$-fuzzy subgroup

In this section, we introduce the notion of $(\in^h_\gamma, \in^h_\gamma \vee q^k_\delta)$-fuzzy subgroup of a group which is a generalization of fuzzy subgroups of the types $(\in, \in)$, $(\in, \in \vee q)$, $(\in, \in \vee q_k)$, $(\in, \in \vee q_\delta)$, $(\in, \in \vee q^k_\delta)$, $(\in_\gamma, \in_\gamma \vee q)$, $(\in_\gamma, \in_\gamma \vee q_k)$, $(\in_\gamma, \in_\gamma \vee q_\delta)$, $(\in_\gamma, \in_\gamma \vee q^k_\delta)$, $(\in^h_\gamma, \in^h_\gamma \vee q)$, $(\in^h_\gamma, \in^h_\gamma \vee q_k)$ and $(\in^h_\gamma, \in^h_\gamma \vee q_\delta)$. We also introduce the notions of q^k_δ-level subset and $\in^h_\gamma \vee q^k_\delta$-level subset of fuzzy subsets and investigate some related properties.

Definition 4.1. *A fuzzy subset μ of G is called an $(\in^h_\gamma, \in^h_\gamma \vee q^k_\delta)$-fuzzy subgroup of G if for all $x, y \in G$ and $s, t \in (\gamma + h, 1]$,*

$(c1)$ $x_s \in^h_\gamma \mu$, $y_t \in^h_\gamma \mu \Rightarrow (xy)_{s\wedge t} \in^h_\gamma \vee q^k_\delta \mu$,

$(c2)$ $x_s \in^h_\gamma \mu \Rightarrow (x^{-1})_s \in^h_\gamma \vee q^k_\delta \mu$.

Equivalently

$(c3)$ $x_s \in^h_\gamma \mu$, $y_t \in^h_\gamma \mu \Rightarrow (xy^{-1})_{s\wedge t} \in^h_\gamma \vee q^k_\delta \mu$.

Lemma 4.1. *The conditions (c1), (c2) and (c3) in Definition 4.1 are equivalent to*

$(c1')$ $\max\{\mu(xy), \gamma + h\} \geq \min\{\mu(x), \mu(y), \frac{\delta-k}{2}\}$ *for all* $x, y \in G$,

$(c2')$ $\max\{\mu(x^{-1}), \gamma + h\} \geq \min\{\mu(x), \frac{\delta-k}{2}\}$ *for all* $x \in G$ *and*

$(c3')$ $\max\{\mu(xy^{-1}), \gamma + h\} \geq \min\{\mu(x), \mu(y), \frac{\delta-k}{2}\}$ *for all* $x, y \in G$

respectively.

Proof. (c1) $\Rightarrow$ (c1′): Let $x, y \in G$. Choose $t \in (\gamma+h, 1]$ such that $\max\{\mu(xy), \gamma+h\} < t \leq \min\{\mu(x), \mu(y), \frac{\delta-k}{2}\}$. Then $\mu(x) \geq t > \gamma + h$, $\mu(y) \geq t > \gamma + h$ and $\mu(xy) < t$. Since $t \leq \frac{\delta-k}{2}$, we have $\mu(xy)+t < 2t \leq \delta-k$. That is, $(xy)_t \overline{\in^h_\gamma} \mu$ and $(xy)_t \overline{q^k_\delta} \mu$. It follows that $(xy)_t \overline{\in^h_\gamma \vee q^k_\delta} \mu$, which is a contradiction. Therefore $\max\{\mu(xy), \gamma + h\} \geq \min\{\mu(x), \mu(y), \frac{\delta-k}{2}\}$.

(c1′) $\Rightarrow$ (c1): Let $x, y \in G$ and $s, t \in (\gamma + h, 1]$ be such that $x_s \in^h_\gamma \mu$, $y_t \in^h_\gamma \mu$ but $(xy)_{\min\{s,t\}} \overline{\in^h_\gamma \vee q^k_\delta} \mu$. Then $\mu(x) \geq s > \gamma + h$, $\mu(y) \geq t > \gamma + h$, $\mu(xy) \leq \gamma + h < \min\{s, t\}$ and $\mu(xy) + \min\{s, t\} \leq \delta - k$. This implies that $\mu(xy) \leq \gamma + h < \min\{\mu(x), \mu(y)\}$ and $\mu(xy) < \frac{\delta-k}{2}$. Thus $\max\{\mu(xy), \gamma + h\} < \min\{\mu(x), \mu(y), \frac{\delta-k}{2}\}$, which is a contradiction. Therefore $x_s \in^h_\gamma \mu$, $y_t \in^h_\gamma \mu$ implies $(xy)_{\min\{s,t\}} \in^h_\gamma \vee q^k_\delta \mu$.

(c2) $\Rightarrow$ (c2′): Let $x \in G$. Choose $t \in (\gamma+h, 1]$ such that $\max\{\mu(x^{-1}), \gamma+h\} < t \leq \min\{\mu(x), \frac{\delta-k}{2}\}$. Then $\mu(x) \geq t > \gamma + h$ and $\mu(x^{-1}) < t$. Since $t \leq \frac{\delta-k}{2}$, we have $\mu(x^{-1}) + t < 2t \leq \delta - k$. That is, $(x^{-1})_t \overline{\in^h_\gamma} \mu$ and $(x^{-1})_t \overline{q^k_\delta} \mu$. It follows that $(x^{-1})_t \overline{\in^h_\gamma \vee q^k_\delta} \mu$, which is a contradiction. Therefore $\max\{\mu(x^{-1}), \gamma+h\} \geq \min\{\mu(x), \frac{\delta-k}{2}\}$.

(c2′) $\Rightarrow$ (c2): Let $x \in G$ and $s \in (\gamma + h, 1]$ be such that $x_s \in^h_\gamma \mu$ but $(x^{-1})_s \overline{\in^h_\gamma \vee q^k_\delta} \mu$. Then $\mu(x) \geq s > \gamma + h$, $\mu(x^{-1}) \leq \gamma + h < s$ and $\mu(x^{-1}) + s \leq \delta - k$. This implies that $\mu(x^{-1}) \leq \gamma + h < \mu(x)$ and $\mu(x^{-1}) < \frac{\delta-k}{2}$. Thus $\max\{\mu(x^{-1}), \gamma + h\} < \min\{\mu(x), \frac{\delta-k}{2}\}$, which is a contradiction. Therefore $x_s \in^h_\gamma \mu$ implies $(x^{-1})_s \in^h_\gamma \vee q^k_\delta \mu$.

(c3) $\Rightarrow$ (c3′): Let $x, y \in G$. Choose $t \in (\gamma+h, 1]$ such that $\max\{\mu(xy^{-1}), \gamma+h\} < t \leq \min\{\mu(x), \mu(y), \frac{\delta-k}{2}\}$. Then $\mu(x) \geq t > \gamma + h$, $\mu(y) \geq t > \gamma + h$

and $\mu(xy^{-1}) < t$. Since $t \leq \frac{\delta-k}{2}$, we have $\mu(xy^{-1}) + t < 2t \leq \delta - k$. That is, $(xy^{-1})_t \overline{\in^h_\gamma}\, \mu$ and $(xy^{-1})_t \overline{q^k_\delta}\, \mu$. It follows that $(xy^{-1})_t \overline{\in^h_\gamma \vee q^k_\delta}\, \mu$, which is a contradiction. Therefore $\max\{\mu(xy^{-1}), \gamma + h\} \geq \min\{\mu(x), \mu(y), \frac{\delta-k}{2}\}$.

(c3′) $\Rightarrow$ (c3): Let $x, y \in G$ and $s, t \in (\gamma + h, 1]$ be such that $x_s \in^h_\gamma \mu$, $y_t \in^h_\gamma \mu$ but $(xy^{-1})_{\min\{s,t\}}\ \overline{\in^h_\gamma \vee q^k_\delta}\, \mu$. Then $\mu(x) \geq s > \gamma + h$, $\mu(y) \geq t > \gamma + h$, $\mu(xy^{-1}) \leq \gamma + h < \min\{s, t\}$ and $\mu(xy^{-1}) + \min\{s, t\} \leq \delta - k$. This implies that $\mu(xy^{-1}) \leq \gamma + h < \min\{\mu(x), \mu(y)\}$ and $\mu(xy^{-1}) < \frac{\delta-k}{2}$. Thus $\max\{\mu(xy^{-1}), \gamma + h\} < \min\{\mu(x), \mu(y), \frac{\delta-k}{2}\}$, which is a contradiction. Therefore $x_s \in^h_\gamma \mu$, $y_t \in^h_\gamma \mu$ implies $(xy^{-1})_{\min\{s,t\}}\ \in^h_\gamma \vee q^k_\delta \mu$. □

Theorem 4.1. *A fuzzy subset μ of G is an $(\in^h_\gamma, \in^h_\gamma \vee q^k_\delta)$-fuzzy subgroup of G if and only if*

(c1′) $\max\{\mu(xy), \gamma + h\} \geq \min\{\mu(x), \mu(y), \frac{\delta-k}{2}\}$ *for all $x, y \in G$.*

(c2′) $\max\{\mu(x^{-1}), \gamma + h\} \geq \min\{\mu(x), \frac{\delta-k}{2}\}$ *for all $x \in G$.*

Equivalently

(c3′) $\max\{\mu(xy^{-1}), \gamma + h\} \geq \min\{\mu(x), \mu(y), \frac{\delta-k}{2}\}$ *for all $x, y \in G$.*

Proof. The proof follows from Lemma 4.1. □

If we take $\gamma = 0$, $h = 0$, $\delta = 1$ and $k = 0$ in Theorem 4.1, we have the following corollary.

Corollary 4.1 ([7], Remark 3.4 (II))**.** *A fuzzy subset μ of G is an $(\in, \in \vee q)$-fuzzy subgroup of G if and only if $\mu(xy^{-1}) \geq \min\{\mu(x), \mu(y), 0.5\}$ for all $x, y \in G$.*

If we take $\gamma = 0$, $h = 0$ and $\delta = 1$ in Theorem 4.1, we have the following corollary.

Corollary 4.2 ([13], Theorem 3.4)**.** *A fuzzy subset μ of G is an $(\in, \in \vee q_k)$-fuzzy subgroup of G if and only if $\mu(xy^{-1}) \geq \min\{\mu(x), \mu(y), \frac{1-k}{2}\}$ for all $x, y \in G$.*

If we take $\gamma = 0$, $h = 0$ and $k = 0$ in Theorem 4.1, we have the following corollary.

Corollary 4.3 ([28], Theorem 3.6)**.** *A fuzzy subset μ of G is an $(\in, \in \vee q_\delta)$-fuzzy subgroup of G if and only if $\mu(xy^{-1}) \geq \min\{\mu(x), \mu(y), \frac{\delta}{2}\}$ for all $x, y \in G$.*

Remark 4.1. For any $(\in^h_\gamma, \in^h_\gamma \vee q^k_\delta)$-fuzzy subgroup μ of G, we can conclude that

(i) if $\gamma = 0$, $h = 0$ and $\mu(x) \leq \frac{\delta-k}{2}$ for all $x \in G$, then μ is a fuzzy subgroup of G.

(ii) if $\gamma = 0$, $h = 0$, $\delta = 1$ and $k = 0$, then μ is an $(\in, \in \vee q)$-fuzzy subgroup of G.

(iii) if $\gamma = 0$, $h = 0$ and $\delta = 1$, then μ is an $(\in, \in \vee q_k)$-fuzzy subgroup of G.

(iv) if $\gamma = 0$, $h = 0$ and $k = 0$, then μ is an $(\in, \in \vee q_\delta)$-fuzzy subgroup of G.

(v) if $\gamma = 0$ and $h = 0$, then μ is an $(\in, \in \vee q_\delta^k)$-fuzzy subgroup of G.

(vi) if $h = 0$, $\delta = 1$ and $k = 0$, then μ is an $(\in_\gamma, \in_\gamma \vee q)$-fuzzy subgroup of G.

(vii) if $h = 0$ and $\delta = 1$, then μ is an $(\in_\gamma, \in_\gamma \vee q_k)$-fuzzy subgroup of G.

(viii) if $h = 0$ and $k = 0$, then μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subgroup of G.

(ix) if $h = 0$, then μ is an $(\in_\gamma, \in_\gamma \vee q_\delta^k)$-fuzzy subgroup of G.

(x) if $\delta = 1$ and $k = 0$, then μ is an $(\in_\gamma^h, \in_\gamma^h \vee q)$-fuzzy subgroup of G.

(xi) if $\delta = 1$, then μ is an $(\in_\gamma^h, \in_\gamma^h \vee q_k)$-fuzzy subgroup of G.

(xii) if $k = 0$, then μ is an $(\in_\gamma^h, \in_\gamma^h \vee q_\delta)$-fuzzy subgroup of G.

Remark 4.2. Every fuzzy subgroup, $(\in, \in \vee q)$-fuzzy subgroup, $(\in, \in \vee q_k)$-fuzzy subgroup, $(\in, \in \vee q_\delta)$-fuzzy subgroup, $(\in_\gamma, \in_\gamma \vee q)$-fuzzy subgroup, $(\in_\gamma, \in_\gamma \vee q_k)$-fuzzy subgroup, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subgroup, $(\in_\gamma, \in_\gamma \vee q_\delta^k)$-fuzzy subgroup, $(\in_\gamma^h, \in_\gamma^h \vee q)$-fuzzy subgroup, $(\in_\gamma^h, \in_\gamma^h \vee q_k)$-fuzzy subgroup, $(\in_\gamma^h, \in_\gamma^h \vee q_\delta)$-fuzzy subgroup of G is an $(\in_\gamma^h, \in_\gamma^h \vee q_\delta^k)$-fuzzy subgroup of a group G, but the converses are not true in general as shown in the following example.

Example 4.1. Consider the Klein four group G as in Example 3.1. Define a fuzzy subset μ of G by $\mu(e) = 0.4$, $\mu(a) = 0.7$ and $\mu(b) = 0.3 = \mu(c)$. Take $\gamma = 0.3$, $h = 0.2$, $\delta = 0.95$ and $k = 0.05$. We can easily verify that μ is an $(\in_\gamma^h, \in_\gamma^h \vee q_\delta^k)$-fuzzy subgroup of G. But

(i) μ is not a fuzzy subgroup of G, since $a_{0.41} \in \mu$ and $(aa)_{0.41} \overline{\in} \mu$.

(ii) μ is not an $(\in, \in \vee q)$-fuzzy subgroup of G, since $a_{0.41} \in \mu$ and $(aa)_{0.41} \overline{\in \vee q}\, \mu$.

(iii) μ is not an $(\in, \in \vee q_k)$-fuzzy subgroup of G, since $a_{0.42} \in \mu$ and $(aa)_{0.42} \overline{\in \vee q_k}\, \mu$.

(iv) μ is not an $(\in, \in \vee q_\delta)$-fuzzy subgroup of G, since $a_{0.45} \in \mu$ and $(aa)_{0.45} \overline{\in \vee q_\delta}\, \mu$.

(v) μ is not an $(\in, \in \vee q_\delta^k)$-fuzzy subgroup of G, since $a_{0.45} \in \mu$ and $(aa)_{0.45} \overline{\in \vee q_\delta^k}\, \mu$.

(vi) μ is not an $(\in_\gamma, \in_\gamma \vee q)$-fuzzy subgroup of G, since $a_{0.45} \in_\gamma \mu$ and $(aa)_{0.45} \overline{\in_\gamma \vee q}\, \mu$.

(vii) μ is not an $(\in_\gamma, \in_\gamma \vee q_k)$-fuzzy subgroup of G, since $a_{0.45} \in_\gamma \mu$ and $(aa)_{0.45} \overline{\in_\gamma \vee q_k}\, \mu$.

(viii) μ is not an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subgroup of G, since $a_{0.45} \in_\gamma \mu$ and $(aa)_{0.45} \overline{\in_\gamma \vee q_\delta}\, \mu$.

(ix) μ is not an $(\in_\gamma, \in_\gamma \vee q_\delta^k)$-fuzzy subgroup of G, since $a_{0.45} \in_\gamma \mu$ and $(aa)_{0.45} \overline{\in_\gamma \vee q_\delta^k}\, \mu$.

(x) μ is not an $(\in_\gamma^h, \in_\gamma^h \vee q)$-fuzzy subgroup of G, since $a_{0.51} \in_\gamma^h \mu$ and $(aa)_{0.51} \overline{\in_\gamma^h \vee q}\, \mu$.

(xi) μ is not an $(\in_\gamma^h, \in_\gamma^h \vee q_k)$-fuzzy subgroup of G, since $a_{0.51} \in_\gamma^h \mu$ and $(aa)_{0.51} \overline{\in_\gamma^h \vee q_k}\, \mu$.

(xii) μ is not an $(\in_\gamma^h, \in_\gamma^h \vee q_\delta)$-fuzzy subgroup of G, since $a_{0.51} \in_\gamma^h \mu$ and $(aa)_{0.51} \overline{\in_\gamma^h \vee q_\delta}\, \mu$.

Definition 4.2. *A fuzzy subset μ of G is called an $(\in_\gamma^h, \in_\gamma^h)$-fuzzy subgroup of G if for all $x, y \in G$ and $s, t \in (\gamma + h, 1]$,*

$(c4)$ $x_s \in_\gamma^h \mu$, $y_t \in_\gamma^h \mu \Rightarrow (xy)_{s \wedge t} \in_\gamma^h \mu$,

$(c5)$ $x_t \in_\gamma^h \mu \Rightarrow (x^{-1})_t \in_\gamma^h \mu$.

Equivalently

$(c6)$ $x_s \in_\gamma^h \mu$, $y_t \in_\gamma^h \mu \Rightarrow (xy^{-1})_{s \wedge t} \in_\gamma^h \mu$.

Lemma 4.2. *The conditions (c4), (c5) and (c6) are equivalent to*

$(c4')$ $\max\{\mu(xy), \gamma + h\} \geq \min\{\mu(x), \mu(y)\}$ *for all* $x, y \in G$,

$(c5')$ $\max\{\mu(x^{-1}), \gamma + h\} \geq \mu(x)$ *for all* $x \in G$ *and*

$(c6')$ $\max\{\mu(xy^{-1}), \gamma + h\} \geq \min\{\mu(x), \mu(y)\}$ *for all* $x, y \in G$, *respectively.*

Proof. The proof is similar to Lemma 4.1. □

Theorem 4.2. *A fuzzy subset μ of G is an $(\in_\gamma^h, \in_\gamma^h)$-fuzzy subgroup of G if and only if*

$(c4')$ $\max\{\mu(xy), \gamma + h\} \geq \min\{\mu(x), \mu(y)\}$ *for all* $x, y \in G$,

$(c5')$ $\max\{\mu(x^{-1}), \gamma + h\} \geq \mu(x)$ *for all* $x \in G$.

Equivalently

$(c6')$ $\max\{\mu(xy^{-1}), \gamma + h\} \geq \min\{\mu(x), \mu(y)\}$ *for all* $x, y \in G$.

Proof. The proof follows from Lemma 4.2. □

Lemma 4.3. *Let μ be an $(\in_\gamma^h, \in_\gamma^h \vee q_\delta^k)$-fuzzy subgroup of G. If $\mu(x) \geq \frac{\delta - k}{2}$ for some $x \in G$, then $\mu(e) \geq \frac{\delta - k}{2}$.*

Proof. Let μ be an $(\in_\gamma^h, \in_\gamma^h \vee q_\delta^k)$-fuzzy subgroup of G. Assume that x is an element of G such that $\mu(x) \geq \frac{\delta-k}{2}$. Then $\max\{\mu(e), \gamma+h\} = \max\{\mu(xx^{-1}), \gamma+h\} \geq \min\{\mu(x), \mu(x), \frac{\delta-k}{2}\} \geq \frac{\delta-k}{2}$. Hence $\mu(e) \geq \frac{\delta-k}{2}$. □

In the following theorem, we provide a condition for an $(\in_\gamma^h, \in_\gamma^h \vee q_\delta^k)$-fuzzy subgroup to be an $(\in_\gamma^h, \in_\gamma^h)$-fuzzy subgroup.

Theorem 4.3. *Every $(\in_\gamma^h, \in_\gamma^h \vee q_\delta^k)$-fuzzy subgroup μ of G with $\mu(e) < \frac{\delta-k}{2}$ is an $(\in_\gamma^h, \in_\gamma^h)$-fuzzy subgroup of G.*

Proof. Let μ be an $(\in_\gamma^h, \in_\gamma^h \vee q_\delta^k)$-fuzzy subgroup of G. If $\mu(e) < \frac{\delta-k}{2}$, then $\mu(x) < \frac{\delta-k}{2}$ for all $x \in G$ by Lemma 4.3. It follows from Theorem 4.1 that $\max\{\mu(xy^{-1}), \gamma+h\} \geq \min\{\mu(x), \mu(y), \frac{\delta-k}{2}\} > \min\{\mu(x), \mu(y)\}$ for all $x, y \in G$. Thus μ is an $(\in_\gamma^h, \in_\gamma^h)$-fuzzy subgroup of G. □

Proposition 4.1. *Let μ be an $(\in_\gamma^h, \in_\gamma^h \vee q_\delta^k)$-fuzzy subgroup of G such that $\mu(a) \geq \frac{\delta-k}{2}$ for some $a(\neq e) \in G$. If G has no proper subgroups, then $\mu(x) \geq \frac{\delta-k}{2}$ for all $x \in G$.*

Proof. Assume that G has no proper subgroups. Then its only subgroups are $\{e\}$ and G. Thus G is a cyclic group of prime order. Therefore $G = (a)$ for $a\,(\neq e) \in G$, and so for any $x \in G$ there exists positive integer m such that $x = a^m$. Now $\max\{\mu(a^2), \gamma + h\} \geq \min\{\mu(a), \frac{\delta-k}{2}\} = \frac{\delta-k}{2}$, $\max\{\mu(a^3), \gamma + h\} \geq \min\{\mu(a^2), \mu(a), \frac{\delta-k}{2}\} = \frac{\delta-k}{2}$, and so on. In general, $\max\{\mu(a^n), \gamma + h\} \geq \frac{\delta-k}{2}$ for every positive integer n. Hence $\mu(x) \geq \frac{\delta-k}{2}$ for all $x \in G$. □

Theorem 4.4. *Let $\{\mu_i | i \in \Lambda\}$ be a family of $(\in_\gamma^h, \in_\gamma^h \vee q_\delta^k)$-fuzzy subgroups of G. Then $\mu = \bigcap_{i\in\Lambda} \mu_i$, is an $(\in_\gamma^h, \in_\gamma^h \vee q_\delta^k)$-fuzzy subgroup of G, where $(\bigcap_{i\in\Lambda} \mu_i)(x) = \inf_{i\in\Lambda}(\mu_i(x))$.*

Proof. Let $x, y \in G$. Then

$$\begin{aligned}
\max\{\mu(xy^{-1}), \gamma + h\} &= \max\{(\bigcap_{i\in\Lambda}\mu_i)(xy^{-1}), \gamma + h\} \\
&= \max\{\inf_{i\in\Lambda}(\mu_i(xy^{-1})), \gamma + h\} \\
&= \inf_{i\in\Lambda}\{\max\{\mu_i(xy^{-1}), \gamma + h\}\} \\
&\geq \inf_{i\in\Lambda}\{\min\{\mu_i(x), \mu_i(y), \tfrac{\delta-k}{2}\}\} \\
&= \min\{\inf_{i\in\Lambda}(\mu_i(x)), \inf_{i\in\Lambda}(\mu_i(y)), \tfrac{\delta-k}{2}\} \\
&= \min\{(\bigcap_{i\in\Lambda}\mu_i)(x), (\bigcap_{i\in\Lambda}\mu_i)(y), \tfrac{\delta-k}{2}\} \\
&= \min\{\mu(x), \mu(y), \tfrac{\delta-k}{2}\}.
\end{aligned}$$

Thus $\mu = \bigcap_{i\in\Lambda} \mu_i$ is an $(\in_\gamma^h, \in_\gamma^h \vee q_\delta^k)$-fuzzy subgroup of G. □

Remark 4.3. Let $\{\mu_i | i \in \Lambda\}$ be a family of $(\in^h_\gamma, \in^h_\gamma \vee q^k_\delta)$-fuzzy subgroups of G. Then $\bigcup_{i\in\Lambda} \mu_i$ need not be an $(\in^h_\gamma, \in^h_\gamma \vee q^k_\delta)$-fuzzy subgroup of G as shown in the following example.

Example 4.2. Consider the Klein four group G as in Example 3.1. Define fuzzy subsets μ and σ of G by $\mu(e) = 0.7$, $\mu(a) = 0.5$, $\mu(b) = 0.4 = \mu(c)$, $\sigma(e) = 0.8$, $\sigma(a) = 0.3 = \sigma(c)$ and $\sigma(b) = 0.6$. Let $\eta = \mu \cup \sigma$. Then $\eta(e) = 0.8$, $\eta(a) = 0.5$, $\eta(b) = 0.6$ and $\eta(c) = 0.4$. Choose $\gamma = 0.2$, $h = 0.05$, $\delta = 0.95$ and $k = 0.1$. We can verify that μ and σ are $(\in^h_\gamma, \in^h_\gamma \vee q^k_\delta)$-fuzzy subgroups of G. But $\eta = \mu \cup \sigma$ is not an $(\in^h_\gamma, \in^h_\gamma \vee q^k_\delta)$-fuzzy subgroup of G, since $a_{0.45} \in^h_\gamma \eta$, $b_{0.6} \in^h_\gamma \eta$ and $(ab)_{0.45\wedge 0.6} \overline{\in^h_\gamma \vee q^k_\delta}\, \eta$.

We provide a condition for a finite union of $(\in^h_\gamma, \in^h_\gamma \vee q^k_\delta)$-fuzzy subgroups to be an $(\in^h_\gamma, \in^h_\gamma \vee q^k_\delta)$-fuzzy subgroup in the following theorem.

Theorem 4.5. *Let $\{\mu_i | i \in \Lambda\}$ be a family of $(\in^h_\gamma, \in^h_\gamma \vee q^k_\delta)$-fuzzy subgroups of G such that $\mu_i \subseteq \mu_j$ or $\mu_j \subseteq \mu_i$ for all $i, j \in \Lambda$. Then $\mu = \bigcup_{i\in\Lambda} \mu_i$ is an $(\in^h_\gamma, \in^h_\gamma \vee q^k_\delta)$-fuzzy subgroup of G, where $(\bigcup_{i\in\Lambda} \mu_i)(x) = \sup_{i\in\Lambda}(\mu_i(x))$.*

Proof. Let $x, y \in G$. Then

$$\begin{aligned}
\max\{\mu(xy^{-1}), \gamma + h\} &= \max\{(\bigcup_{i\in\Lambda} \mu_i)(xy^{-1}), \gamma + h\} \\
&= \max\{\sup_{i\in\Lambda}(\mu_i(xy^{-1}), \gamma + h\} \\
&= \sup_{i\in\Lambda}\{\max\{\mu_i(xy^{-1}), \gamma + h\}\} \\
&\geq \sup_{i\in\Lambda}\{\min\{\mu_i(x), \mu_i(y), \tfrac{\delta-k}{2}\}\} \\
&= \min\{\sup_{i\in\Lambda}(\mu_i(x)), \sup_{i\in\Lambda}(\mu_i(y)), \tfrac{\delta-k}{2}\} \qquad \text{(I)} \\
&= \min\{(\bigcup_{i\in\Lambda} \mu_i)(x), (\bigcup_{i\in\Lambda} \mu_i)(y), \tfrac{\delta-k}{2}\} \\
&= \min\{\mu(x), \mu(y), \tfrac{\delta-k}{2}\}.
\end{aligned}$$

Hence, $\mu = \bigcup_{i\in\Lambda} \mu_i$ is an $(\in^h_\gamma, \in^h_\gamma \vee q^k_\delta)$-fuzzy subgroup of G.

In the following we prove that equation (I) holds.

Obviously,

$$\sup_{i\in\Lambda}\{\min\{\mu_i(x), \mu_i(y), \tfrac{\delta-k}{2}\}\} \leq \min\{\sup_{i\in\Lambda}(\mu_i(x)), \sup_{i\in\Lambda}(\mu_i(y)), \tfrac{\delta-k}{2}\}.$$

Suppose that

$$\sup_{i\in\Lambda}\{\min\{\mu_i(x), \mu_i(y), \tfrac{\delta-k}{2}\}\} < \min\{\sup_{i\in\Lambda}(\mu_i(x)), \sup_{i\in\Lambda}(\mu_i(y)), \tfrac{\delta-k}{2}\}.$$

Then, there exists t such that

$$\sup_{i\in\Lambda}\{\min\{\mu_i(x),\mu_i(y),\tfrac{\delta-k}{2}\}\} < t < \min\{\sup_{i\in\Lambda}(\mu_i(x)),\sup_{i\in\Lambda}(\mu_i(y)),\tfrac{\delta-k}{2}\}.$$

Since $\mu_i \subseteq \mu_j$ or $\mu_j \subseteq \mu_i$ for all $i, j \in \Lambda$, there exists $n \in \Lambda$ such that

$$t < \min\{\mu_n(x),\mu_n(y),\tfrac{\delta-k}{2}\}.$$

On the other hand, $\min\{\mu_i(x),\mu_i(y),\frac{\delta-k}{2}\} < t$ for all $i \in \Lambda$, which is a contradiction. Hence,

$$\sup_{i\in\Lambda}\{\min\{\mu_i(x),\mu_i(y),\tfrac{\delta-k}{2}\}\} = \min\{\sup_{i\in\Lambda}(\mu_i(x)),\sup_{i\in\Lambda}(\mu_i(y)),\tfrac{\delta-k}{2}\}.$$

□

Definition 4.3. *For a subset A of G, a fuzzy subset $(\chi_A)^{\delta-k}_{\gamma+h}$ in G defined by*

$$(\chi_A)^{\delta-k}_{\gamma+h}(x) = \begin{cases} \delta-k, & \text{if } x \in A, \\ \gamma+h, & \text{otherwise,} \end{cases}$$

where $x \in G$, is called a $(\gamma+h, \delta-k)$-characteristic fuzzy subset of A in G.

Theorem 4.6. *For any subset A of G and the $(\gamma+h, \delta-k)$-characteristic fuzzy subset $(\chi_A)^{\delta-k}_{\gamma+h}$ of A in G, the following are equivalent:*

(i) A is a subgroup of G,

(ii) $(\chi_A)^{\delta-k}_{\gamma+h}$ is an $(\in^h_\gamma, \in^h_\gamma \vee q^k_\delta)$-fuzzy subgroup of G.

Proof. The proof follows from Theorem 3.5. □

If we take $\gamma = 0$, $h = 0$, $\delta = 1$ and $k = 0$ in Theorem 4.6, we have the following corollary.

Corollary 4.4 ([6], Theorem 4.1)**.** *For any subset A of G and the characteristic fuzzy subset χ_A of A in G, the following are equivalent:*

(i) A is a subgroup of G,

(ii) χ_A is an $(\in, \in \vee q)$-fuzzy subgroup of G.

If we take $\gamma = 0$, $h = 0$ and $\delta = 1$ in Theorem 4.6, we have the following corollary.

Corollary 4.5 ([13], Theorem 3.17)**.** *For any subset A of G and the k-characteristic fuzzy subset χ^k_A of A in G, the following are equivalent:*

(i) A is a subgroup of G,

(ii) χ^k_A is an $(\in, \in \vee q_k)$-fuzzy subgroup of G.

If we take $\gamma = 0$, $h = 0$ and $k = 0$ in Theorem 4.6, we have the following corollary.

Corollary 4.6 ([28], Theorem 4.5)**.** *For any subset A of G and the δ-characteristic fuzzy subset χ_A^δ of A in G, the following are equivalent:*

(i) A is a subgroup of G,

(ii) χ_A^δ is an $(\in, \in \vee q_\delta)$-fuzzy subgroup of G.

If we take $\gamma = 0$ and $h = 0$ in Theorem 4.6, we have the following corollary.

Corollary 4.7. *For any subset A of G and the $(\delta - k)$-characteristic fuzzy subset $(\chi_A)_\delta^k$ of A in G, the following are equivalent:*

(i) A is a subgroup of G,

(ii) $(\chi_A)_\delta^k$ is an $(\in, \in \vee q_\delta^k)$-fuzzy subgroup of G.

Definition 4.4. *Let μ be a fuzzy subset of a set X and $t \in (0, 1]$. Then the set $[\mu_t]_\gamma^h = \{x \in X | x_t \in_\gamma^h \mu\} = \{x \in X | \mu(x) \geq t > \gamma + h\} = \mu_t$ is called the $\in_\gamma^h$-level subset of μ in X.*

Theorem 4.7. *A fuzzy subset μ of G is an $(\in_\gamma^h, \in_\gamma^h \vee q_\delta^k)$-fuzzy subgroup of G if and only if the nonempty level subset μ_t of μ is a subgroup of G for all $t \in (\gamma + h, \frac{\delta - k}{2}]$.*

Proof. Assume that μ is an $(\in_\gamma^h, \in_\gamma^h \vee q_\delta^k)$-fuzzy subgroup of G. Let $t \in (\gamma + h, \frac{\delta-k}{2}]$ and $x, y \in \mu_t$. Then $\mu(x) \geq t > \gamma + h$ and $\mu(y) \geq t > \gamma + h$. That is $x_t \in_\gamma^h \mu$ and $y_t \in_\gamma^h \mu$. It follows that $\max\{\mu(xy^{-1}), \gamma + h\} \geq \min\{\mu(x), \mu(y), \frac{\delta-k}{2}\} \geq \min\{t, \frac{\delta-k}{2}\} = t > \gamma + h$ and so $(xy^{-1})_t \in_\gamma^h \mu$. Hence $xy^{-1} \in \mu_t$. Therefore μ_t is a subgroup of G. Conversely, let μ be a fuzzy subset of G such that μ_t is a subgroup of G for all $t \in (\gamma + h, \frac{\delta-k}{2}]$. Suppose that there exists elements $a, b \in G$ such that $\max\{\mu(ab^{-1}), \gamma + h\} < t \leq \min\{\mu(a), \mu(b), \frac{\delta-k}{2}\}$. Then $\mu(a) \geq t > \gamma + h$, $\mu(b) \geq t > \gamma + h$ and $\mu(ab^{-1}) < t$. It follows that $a, b \in \mu_t$ and $ab^{-1} \notin \mu_t$, which is a contradiction. Therefore $\max\{\mu(xy^{-1}), \gamma + h\} \geq \min\{\mu(x), \mu(y), \frac{\delta-k}{2}\}$ for all $x, y \in G$. Hence μ is an $(\in_\gamma^h, \in_\gamma^h \vee q_\delta^k)$-fuzzy subgroup of G. □

If we take $\gamma = 0$, $h = 0$, $\delta = 1$ and $k = 0$ in Theorem 4.7, we have the following corollary.

Corollary 4.8 ([7], Theorem 3.7)**.** *A fuzzy subset μ of G is an $(\in, \in \vee q)$-fuzzy subgroup of G if and only if the nonempty level subset μ_t of μ is a subgroup of G for all $t \in (0, 0.5]$.*

Taking $\gamma = 0$, $h = 0$ and $\delta = 1$ in Theorem 4.7, we have the following corollary.

Corollary 4.9 ([17], Theorem 4.10)**.** *A fuzzy subset μ of G is an $(\in, \in \vee q_k)$-fuzzy subgroup of G if and only if the nonempty level subset μ_t of μ is a subgroup of G for all $t \in (0, \frac{1-k}{2}]$.*

If we take $\gamma = 0$, $h = 0$ and $k = 0$ in Theorem 4.7, we have the following corollary.

Corollary 4.10 ([28], Theorem 3.13)**.** *A fuzzy subset μ of G is an $(\in, \in \vee q_\delta)$-fuzzy subgroup of G if and only if the nonempty level subset μ_t of μ is a subgroup of G for all $t \in (0, \frac{\delta}{2}]$.*

Definition 4.5. *Let μ be a fuzzy subset of a set X and $t \in (0, 1]$. Then the set $[\mu_t]_\delta^k = \{x \in X | x_t\, q_\delta^k\, \mu\} = \{x \in X |\, \mu(x) + t > \delta - k\}$ is called the q_δ^k-level subset of μ in X.*

- Choose $\delta = 1$ and $k = 0$ in the above definition, we get the q-level subset $[\mu_t] = \{x \in X | x_t\, q\, \mu\} = \{x \in X |\, \mu(x) + t > 1\}$ of μ in X (see [3]).
- Choose $\delta = 1$ in the above definition, we get the q_k-level subset $[\mu_t]_k = \{x \in X | x_t\, q_k\, \mu\} = \{x \in X |\, \mu(x) + t > 1 - k\}$ of μ in X (see [13]).
- Choose $k = 0$ in the above definition, we get the q_δ-level subset $[\mu_t]_\delta = \{x \in X | x_t\, q_\delta\, \mu\} = \{x \in X |\, \mu(x) + t > \delta\}$ of μ in X (see [28]).

Theorem 4.8. *Let μ be a fuzzy subset of G and $\delta - k = \gamma + h + 1$. Then μ is an $(\in_\gamma^h, \in_\gamma^h \vee q_\delta^k)$-fuzzy subgroup of G if and only if the nonempty q_δ^k-level subset $[\mu_t]_\delta^k$ of μ is a subgroup of G for all $t \in (\frac{\delta-k}{2}, 1]$.*

Proof. Assume that μ is an $(\in_\gamma^h, \in_\gamma^h \vee q_\delta^k)$-fuzzy subgroup of G. Let $t \in (\frac{\delta-k}{2}, 1]$ and $x, y \in [\mu_t]_\delta^k$. Then $\mu(x) + t > \delta - k$ and $\mu(y) + t > \delta - k$. That is $\mu(x) > \delta - k - t > \delta - k - 1 = \gamma + h$ and $\mu(y) > \delta - k - t > \delta - k - 1 = \gamma + h$. Since $\max\{\mu(xy^{-1}), \gamma + h\} \geq \min\{\mu(x), \mu(y), \frac{\delta-k}{2}\}$, we have $\mu(xy^{-1}) \geq \min\{\mu(x), \mu(y), \frac{\delta-k}{2}\} > \min\{\delta - k - t, \delta - k - t, \frac{\delta-k}{2}\}$. Since $\frac{\delta-k}{2} < t \leq 1$, we have $-t < -(\frac{\delta-k}{2})$ and so $\delta - k - t < \frac{\delta-k}{2}$. This implies that $\mu(xy^{-1}) > \delta - k - t$ and so $\mu(xy^{-1}) + t > \delta - k$. Thus $xy^{-1} \in [\mu_t]_\delta^k$. Hence $[\mu_t]_\delta^k$ is a subgroup of G.

Conversely, assume that $[\mu_t]_\delta^k$ is a subgroup of G for all $t \in (\frac{\delta-k}{2}, 1]$. Suppose $x, y \in G$ be such that

$$\begin{aligned}
&\max\{\mu(xy^{-1}), \gamma + h\} < \min\{\mu(x), \mu(y), \tfrac{\delta-k}{2}\} \\
&\implies \delta - k - \min\{\mu(x), \mu(y), \tfrac{\delta-k}{2}\} < \delta - k - \max\{\mu(xy^{-1}), \gamma + h\} \\
&\implies \max\{\delta - k - \mu(x), \delta - k - \mu(y), \tfrac{\delta-k}{2}\} \\
&< \min\{\delta - k - \mu(xy^{-1}), \delta - k - (\gamma + h)\}.
\end{aligned}$$

Since $\delta - k - (\gamma + h) = 1$, we have $\max\{\delta - k - \mu(x), \delta - k - \mu(y), \frac{\delta-k}{2}\} < (\delta - k - \mu(xy^{-1}))$. Take $t \in (\gamma + h, \frac{\delta-k}{2}]$ such that $\max\{\delta - k - \mu(x), \delta - k -$

$\mu(y), \frac{\delta-k}{2}\} < t \leq (\delta - k - \mu(xy^{-1}))$. Then $\delta - k - \mu(x) < t$, $\delta - k - \mu(y) < t$ and $t \leq \delta - k - \mu(xy^{-1})$ implies $\mu(x) + t > \delta - k$, $\mu(y) + t > \delta - k$ and $\mu(xy^{-1}) + t \leq \delta - k$, that is $x, y \in [\mu_t]^k_\delta$ but $xy^{-1} \notin [\mu_t]^k_\delta$ which is a contradiction. Thus $\max\{\mu(xy^{-1}), \gamma + h\} \geq \min\{\mu(x), \mu(y), \frac{\delta-k}{2}\}$. Hence μ is an $(\in^h_\gamma, \in^h_\gamma \vee q^k_\delta)$-fuzzy subgroup of G. □

Definition 4.6. *Let μ be a fuzzy subset of a set X and $t \in (0, 1]$. Then the set $\Omega(\mu; t) = \{x \in X | x_t \in^h_\gamma \vee q^k_\delta \mu\} = \{x \in X | \mu(x) \geq t > \gamma + h \text{ or } \mu(x) + t > \delta - k\}$ is called the $\in^h_\gamma \vee q^k_\delta$-level subset of μ in X.*

Note that $\Omega(\mu; t) = [\mu_t]^h_\gamma \cup [\mu_t]^k_\delta = \mu_t \cup [\mu_t]^k_\delta$.

- Take $\gamma = 0$, $h = 0$, $\delta = 1$ and $k = 0$ in the above definition, we get the $\in \vee q$-level subset $\Delta(\mu; t) = \{x \in X | x_t \in \vee q\, \mu\} = \{x \in X | \mu(x) \geq t \text{ or } \mu(x) + t > 1\}$ of μ in X (see [17]).
- Take $\gamma = 0$, $h = 0$ and $\delta = 1$ in the above definition, we get the $\in \vee q_k$-level subset $\Delta_k(\mu; t) = \{x \in X | x_t \in \vee q_k\, \mu\} = \{x \in X | \mu(x) \geq t \text{ or } \mu(x) + t > 1 - k\}$ of μ in X (see [17]).
- Take $\gamma = 0$, $h = 0$ and $k = 0$ in the above definition, we get the $\in \vee q_\delta$-level subset $\Delta_\delta(\mu; t) = \{x \in X | x_t \in \vee q_\delta\, \mu\} = \{x \in X | \mu(x) \geq t \text{ or } \mu(x) + t > \delta\}$ of μ in X (see [28]).

Theorem 4.9. *Let μ be a fuzzy subset of G and $\delta - k = 1 + \gamma + h$. Then μ is an $(\in^h_\gamma, \in^h_\gamma \vee q^k_\delta)$-fuzzy subgroup of G if and only if the nonempty $\in^h_\gamma \vee q^k_\delta$-level subset $\Omega(\mu; t)$ of μ is a subgroup of G for all $t \in (\gamma + h, 1]$.*

Proof. Suppose μ is an $(\in^h_\gamma, \in^h_\gamma \vee q^k_\delta)$-fuzzy subgroup of G and $x, y \in \Omega(\mu; t)$. Then $x_t \in^h_\gamma \vee q^k_\delta \mu$ and $y_t \in^h_\gamma \vee q^k_\delta \mu$, that is $\mu(x) \geq t > \gamma + h$ or $\mu(x) + t > \delta - k$ and $\mu(y) \geq t > \gamma + h$ or $\mu(y) + t > \delta - k$. Thus $\mu(x) \geq t > \gamma + h$ or $\mu(x) > \delta - k - t \geq \delta - k - 1 = \gamma + h$ and $\mu(y) \geq t > \gamma + h$ or $\mu(y) > \delta - k - t \geq \delta - k - 1 = \gamma + h$. If $t \in (\gamma + h, \frac{\delta-k}{2}]$, then $\gamma + h < t \leq \frac{\delta-k}{2}$. This implies that $\delta - k - t \geq \frac{\delta-k}{2} \geq t$. Hence it follows from the above that $\mu(x) \geq t$ and $\mu(y) \geq t$. By hypothesis $\max\{\mu(xy^{-1}), \gamma + h\} \geq \min\{\mu(x), \mu(y), \frac{\delta-k}{2}\}$ implies $\mu(xy^{-1}) \geq \min\{\mu(x), \mu(y), \frac{\delta-k}{2}\} \geq \min\{t, t, t\} = t > \gamma + h$ and so $(xy^{-1})_t \in^h_\gamma \mu$. Thus $xy^{-1} \in \Omega(\mu; t)$. If $t \in (\frac{\delta-k}{2}, 1]$, then $\frac{\delta-k}{2} < t \leq 1$. Since $-t < -\frac{\delta-k}{2}$, we have $\delta - k - t < \frac{\delta-k}{2} < t$. Hence $\mu(x) > \delta - k - t$ and $\mu(y) > \delta - k - t$. By hypothesis $\max\{\mu(xy^{-1}), \gamma + h\} \geq \min\{\mu(x), \mu(y), \frac{\delta-k}{2}\}$ implies $\mu(xy^{-1}) \geq \min\{\mu(x), \mu(y), \frac{\delta-k}{2}\} > \min\{\delta - k - t, \delta - k - t, \delta - k - t\} = \delta - k - t$ and so $\mu(xy^{-1}) + t > \delta - k$. That is $(xy^{-1})_t q^k_\delta \mu$. This implies that $xy^{-1} \in \Omega(\mu; t)$. Hence $\Omega(\mu; t)$ is a subgroup of G.

Conversely, assume that $\Omega(\mu; t)$ is a subgroup of G for all $t \in (\gamma + h, 1]$. Let $x, y \in G$ be such that $\max\{\mu(xy^{-1}), \gamma + h\} < \min\{\mu(x), \mu(y), \frac{\delta-k}{2}\}$. Choose $t \in (\gamma + h, 1]$ such that $\max\{\mu(xy^{-1}), \gamma + h\} < t \leq \min\{\mu(x), \mu(y), \frac{\delta-k}{2}\}$. Then $x_t \in^h_\gamma$

μ, $y_t \in^h_\gamma \mu$ but $(xy^{-1})_t \overline{\in^h_\gamma \vee q^k_\delta}\mu$. That is $x, y \in \Omega(\mu; t)$ and $xy^{-1} \notin \Omega(\mu; t)$. This contradicts our hypothesis. Hence $\max\{\mu(xy^{-1}), \gamma + h\} \geq \min\{\mu(x), \mu(y), \frac{\delta-k}{2}\}$. Therefore μ is an $(\in^h_\gamma, \in^h_\gamma \vee q^k_\delta)$-fuzzy subgroup of G. □

5. (δ, k)-lower and upper parts of fuzzy subsets

In this section, we introduce the notions of (δ, k)-lower and upper parts of fuzzy subsets and investigate some related properties.

Definition 5.1. *Let μ be a fuzzy subset of a set X. Then we define the (δ, k)-upper part μ^+ and the (δ, k)-lower part μ^- of μ as follows:*
$\mu^+(x) = \max\{\mu(x), \frac{\delta-k}{2}\}$ *and* $\mu^-(x) = \min\{\mu(x), \frac{\delta-k}{2}\}$ *for all* $x \in X$.

Lemma 5.1. *Let μ and η be two fuzzy subsets of a set X. Then the following holds:*
(i) $(\mu \cap \eta)^- = \mu^- \cap \eta^-$,
(ii) $(\mu \cup \eta)^- = \mu^- \cup \eta^-$,
(iii) $(\mu \cap \eta)^+ = \mu^+ \cap \eta^+$,
(iv) $(\mu \cup \eta)^+ = \mu^+ \cup \eta^+$.

Proof. The proofs are straightforward, so we omit the proofs. □

Lemma 5.2. *Let μ and η be two fuzzy subsets of a group G. Then the following holds:*
(i) $(\mu \circ \eta)^- = \mu^- \circ \eta^-$, *(ii)* $(\mu \circ \eta)^+ \geq \mu^+ \circ \eta^+$. *If every element x of G is expressible as $x = ab$, then $(\mu \circ \eta)^+ = \mu^+ \circ \eta^+$.*

Proof. The proof of (i) is straightforward.

(ii) If x is not expressible as $x = ab$ for some $a, b \in G$, then $(\mu \circ \eta)^+(x) = \max\{(\mu \circ \eta)(x), \frac{\delta-k}{2}\} = \frac{\delta-k}{2} > 0 = (\mu^+ \circ \eta^+)(x)$. But if x is expressible as $x = ab$ for some $a, b \in G$, then

$$\begin{aligned}
(\mu \circ \eta)^+(x) &= \max\{(\mu \circ \eta)(x), \tfrac{\delta-k}{2}\} \\
&= \max\{\sup_{x=ab}\{\min\{\mu(a), \eta(b)\}\}, \tfrac{\delta-k}{2}\} \\
&= \sup_{x=ab}\{\max\{\min\{\mu(a), \eta(b)\}, \tfrac{\delta-k}{2}\}\} \\
&= \sup_{x=ab}\{\min\{\max\{\mu(a), \tfrac{\delta-k}{2}\}, \max\{\eta(b), \tfrac{\delta-k}{2}\}\}\} \\
&= \sup_{x=ab}\{\min\{\mu^+(a), \eta^+(b)\}\} \\
&= (\mu^+ \circ \eta^+)(x).
\end{aligned}$$

□

Theorem 5.1. *Let μ be an $(\in^h_\gamma, \in^h_\gamma \vee q^k_\delta)$-fuzzy subgroup of G. Then μ^- is an $(\in^h_\gamma, \in^h_\gamma)$-fuzzy subgroup of G.*

Proof. For any $x, y \in G$, we have

$$\begin{aligned}
\max\{\mu^-(xy^{-1}), \gamma + h\} &= \max\{\min\{\mu(xy^{-1}), \tfrac{\delta-k}{2}\}, \gamma + h\} \\
&= \min\{\max\{\mu(xy^{-1}), \gamma + h\}, \max\{\tfrac{\delta-k}{2}, \gamma + h\}\} \\
&\geq \min\{\min\{\mu(x), \mu(y), \tfrac{\delta-k}{2}\}, \tfrac{\delta-k}{2}\} \\
&= \min\{\min\{\mu(x), \tfrac{\delta-k}{2}\}, \min\{\mu(y), \tfrac{\delta-k}{2}\}\} \\
&= \min\{\mu^-(x), \mu^-(y)\}.
\end{aligned}$$

Thus μ^- is an $(\in^h_\gamma, \in^h_\gamma)$-fuzzy subgroup of G. □

Combining Theorem 5.1 and Theorem 3.6, we have the following theorem.

Theorem 5.2. *Let μ be an $(\in^h_\gamma, \in^h_\gamma \vee q^k_\delta)$-fuzzy subgroup of G. Then μ^- is an $(\in^h_\gamma, \in^h_\gamma \vee q^k_\delta)$-fuzzy subgroup of G.*

Combining Theorem 5.2 and Theorem 4.8, we have the following corollary.

Corollary 5.1. *Let μ be a fuzzy subset of G and $\delta - k = 1 + \gamma + h$. If μ is an $(\in^h_\gamma, \in^h_\gamma \vee q^k_\delta)$-fuzzy subgroup of G, then the nonempty level subset $[\mu^-_t]^k_\delta$ of μ^- is a subgroup of G for all $t \in (\frac{\delta-k}{2}, 1]$.*

6. (γ, h)- Lower and Upper parts of fuzzy subsets

In this section, we introduce (γ, h)-lower and upper parts of fuzzy subsets and obtain related results.

Definition 6.1. *Let μ be a fuzzy subset of a set X. Then we define (γ, h)-upper part $\mu^\oplus$ and (γ, h)-lower part $\mu^\ominus$ of μ as follows:*
$\mu^\oplus(x) = \max\{\mu(x), \gamma + h\}$ *and* $\mu^\ominus(x) = \min\{\mu(x), \gamma + h\}$ *for all $x \in X$.*

Lemma 6.1. *Let μ and η be two fuzzy subsets of a set X. Then the following holds:*
(i) $(\mu \cap \eta)^\ominus = \mu^\ominus \cap \eta^\ominus$,
(ii) $(\mu \cup \eta)^\ominus = \mu^\ominus \cup \eta^\ominus$,
(iii) $(\mu \cap \eta)^\oplus = \mu^\oplus \cap \eta^\oplus$,
(iv) $(\mu \cup \eta)^\oplus = \mu^\oplus \cup \eta^\oplus$.

Proof. The proofs are straightforward, so we omit the proofs. □

Lemma 6.2. *Let μ and η be two fuzzy subsets of a group G. Then the following holds:*
(i) $(\mu \circ \eta)^\ominus = \mu^\ominus \circ \eta^\ominus$,
(ii) $(\mu \circ \eta)^\oplus \geq \mu^\oplus \circ \eta^\oplus$.
If every element x of G is expressible as $x = ab$, then $(\mu \circ \eta)^\oplus = \mu^\oplus \circ \eta^\oplus$.

Proof. The proof of (i) is straightforward.

(ii) If x is not expressible as $x = ab$ for some $a, b \in G$, then $(\mu \circ \eta)^{\oplus}(x) = \max\{(\mu \circ \eta)(x), \gamma + h\} = \gamma + h \geq 0 = (\mu^{\oplus} \circ \eta^{\oplus})(x)$. But if x is expressible as $x = ab$ for some $a, b \in G$, then

$$\begin{aligned}(\mu \circ \eta)^{\oplus}(x) &= \max\{(\mu \circ \eta)(x), \gamma + h\} \\ &= \max\{\sup_{x=ab}\{\min\{\mu(a), \eta(b)\}\}, \gamma + h\} \\ &= \sup_{x=ab}\{\min\{\max\{\mu(a), \gamma + h\}, \max\{\eta(b), \gamma + h\}\}\} \\ &= \sup_{x=ab}\{\min\{\mu^{\oplus}(a), \eta^{\oplus}(b)\}\} \\ &= (\mu^{\oplus} \circ \eta^{\oplus})(x).\end{aligned}$$

□

Theorem 6.1. *Let μ is an $(\in^h_\gamma, \in^h_\gamma \vee q^k_\delta)$-fuzzy subgroup of G. Then $\mu^{\oplus}$ is an $(\in^h_\gamma, \in^h_\gamma \vee q^k_\delta)$-fuzzy subgroup of G.*

Proof. Let $x, y \in G$. Then

$$\begin{aligned}\max\{\mu^{\oplus}(xy^{-1}), \gamma + h\} &= \max\{\max\{\mu(xy^{-1}), \gamma + h\}, \gamma + h\} \\ &\geq \max\{\min\{\mu(x), \mu(y), \tfrac{\delta - k}{2}\}, \gamma + h\} \\ &= \min\{\max\{\mu(x), \gamma + h\}, \max\{\mu(y), \gamma + h\}, \\ &\quad \max\{\tfrac{\delta - k}{2}, \gamma + h\}\} \\ &= \min\{\mu^{\oplus}(x), \mu^{\oplus}(y), \tfrac{\delta - k}{2}\}.\end{aligned}$$

Hence $\mu^{\oplus}$ is an $(\in^h_\gamma, \in^h_\gamma \vee q^k_\delta)$-fuzzy subgroup of G. □

7. Relationship between lower and upper parts of fuzzy subsets

In this section, we define different fuzzy subsets and study their relation between lower and upper parts of fuzzy subsets.

Definition 7.1. *Let μ be a fuzzy subset of a set X. Then we define the fuzzy subsets $\mu^{\diamond}$, $\mu^{\sim}$, $\mu^{\dagger}$, μ^{*} and μ^{**} of μ as follows:*

$\mu^{\diamond}(x) = \max\{\min\{\mu(x), \frac{\delta - k}{2}\}, \gamma + h\}$, $\mu^{\sim}(x) = \max\{\min\{\mu(x), \gamma + h\}, \frac{\delta - k}{2}\}$, $\mu^{\dagger}(x) = \min\{\min\{\mu(x), \frac{\delta - k}{2}\}, \gamma + h\}$, $\mu^{*}(x) = \max\{\max\{\mu(x), \frac{\delta - k}{2}\}, \gamma + h\}$ *and* $\mu^{**}(x) = \min\{\max\{\mu(x), \frac{\delta - k}{2}\}, \gamma + h\}$ *for all $x \in X$.*

Lemma 7.1. *If $\mu^{\diamond}$, $\mu^{\sim}$, $\mu^{\dagger}$, μ^{*} and μ^{**} are fuzzy subsets of a set X, then the following holds:*

(i) $\mu^{\diamond} = (\mu^{-})^{\oplus} = (\mu^{\oplus})^{-}$,
(ii) $\mu^{\sim} = (\mu^{\ominus})^{+}$,
(iii) $\mu^{\dagger} = (\mu^{-})^{\ominus} = (\mu^{\ominus})^{-} = (\mu^{\ominus})^{\ominus}$,
(iv) $\mu^{*} = (\mu^{+})^{\oplus} = (\mu^{\oplus})^{+}$, *(v)* $\mu^{**} = (\mu^{\ominus})^{\oplus}$.

Proof. The proofs are straightforward and so we omit the proofs. □

Remark 7.1. (i) If $\mu(x) \leq \frac{\delta-k}{2}$, then $\mu^{\diamond} = \mu^{\oplus}$ and $\mu^{\dagger} = \mu^{\ominus}$. (ii) If $\mu(x) \geq \frac{\delta-k}{2}$, then $\mu^{*} = \mu^{\oplus}$ and $\mu^{**} = \mu^{\ominus}$. (iii) If $\mu(x) \leq \gamma + h$, then $\mu^{\sim} = \mu^{+}$. (iv) If $\min\{\mu(x), \frac{\delta-k}{2}\} \geq \gamma + h$, then $\mu^{\diamond} = \mu^{-}$. (v) If $\max\{\mu(x), \frac{\delta-k}{2}\} \geq \gamma + h$, then $\mu^{*} = \mu^{+}$.

Lemma 7.2. *Let μ and η be fuzzy subsets of a set X. Then the following holds:*
(i) $(\mu \cap \eta)^{\diamond} = \mu^{\diamond} \cap \eta^{\diamond}$,
(ii) $(\mu \cup \eta)^{\diamond} = \mu^{\diamond} \cup \eta^{\diamond}$,
(iii) $(\mu \cap \eta)^{\sim} = \mu^{\sim} \cap \eta^{\sim}$,
(iv) $(\mu \cup \eta)^{\sim} = \mu^{\sim} \cup \eta^{\sim}$,
(v) $(\mu \cap \eta)^{\dagger} = \mu^{\dagger} \cap \eta^{\dagger}$,
(vi) $(\mu \cup \eta)^{\dagger} = \mu^{\dagger} \cup \eta^{\dagger}$,
(vii) $(\mu \cap \eta)^{*} = \mu^{*} \cap \eta^{*}$,
(viii) $(\mu \cup \eta)^{*} = \mu^{*} \cup \eta^{*}$,
(ix) $(\mu \cap \eta)^{**} = \mu^{**} \cap \eta^{**}$,
(x) $(\mu \cup \eta)^{**} = \mu^{**} \cup \eta^{**}$.

Proof. The proofs are straightforward and so we omit the proofs. □

Lemma 7.3. *Let μ and η be two fuzzy subsets of a group G. Then the following holds:*
(i) $(\mu \circ \eta)^{\diamond} \geq \mu^{\diamond} \circ \eta^{\diamond}$,
(ii) $(\mu \circ \eta)^{\sim} \geq \mu^{\sim} \circ \eta^{\sim}$,
(iii) $(\mu \circ \eta)^{\dagger} = \mu^{\dagger} \circ \eta^{\dagger}$,
(iv) $(\mu \circ \eta)^{*} \geq \mu^{*} \circ \eta^{*}$,
(v) $(\mu \circ \eta)^{**} \geq \mu^{**} \circ \eta^{**}$. *If every element x of G is expressible as $x = ab$, then $(\mu \circ \eta)^{\diamond} = \mu^{\diamond} \circ \eta^{\diamond}$, $(\mu \circ \eta)^{\sim} = \mu^{\sim} \circ \eta^{\sim}$, $(\mu \circ \eta)^{*} = \mu^{*} \circ \eta^{*}$, $(\mu \circ \eta)^{**} = \mu^{**} \circ \eta^{**}$.*

Proof. The proofs are straightforward and so we omit the proofs. □

Lemma 7.4. *If μ is an $(\in_{\gamma}^{h}, \in_{\gamma}^{h} \vee q_{\delta}^{k})$-fuzzy subgroup of G, then $\mu^{\diamond}$ (resp. $\mu^{\sim}$, μ^{*}) is an $(\in_{\gamma}^{h}, \in_{\gamma}^{h} \vee q_{\delta}^{k})$-fuzzy subgroup of G.*

Proof. Let $x, y \in G$. Then

$$
\begin{aligned}
&\max\{\mu^{\diamond}(xy^{-1}), \gamma + h\} \\
&= \max\{\min\{\max\{\mu(xy^{-1}), \gamma + h\}, \tfrac{\delta-k}{2}\}, \gamma + h\} \\
&\geq \max\{\min\{\min\{\mu(x), \mu(y), \tfrac{\delta-k}{2}\}, \tfrac{\delta-k}{2}\}, \gamma + h\} \\
&= \max\{\min\{\min\{\mu(x), \tfrac{\delta-k}{2}\}, \min\{\mu(y), \tfrac{\delta-k}{2}\}, \tfrac{\delta-k}{2}\}, \gamma + h\} \\
&= \min\{\max\{\min\{\mu(x), \tfrac{\delta-k}{2}\}, \gamma + h\}, \\
&\max\{\min\{\mu(y), \tfrac{\delta-k}{2}\}, \gamma + h\}, \max\{\tfrac{\delta-k}{2}, \gamma + h\}\}
\end{aligned}
$$

$$
\begin{aligned}
&= \min\{\min\{\max\{\mu(x), \gamma + h\}, \tfrac{\delta-k}{2}\},\\
&\min\{\max\{\mu(y), \gamma + h\}, \tfrac{\delta-k}{2}\}, \min\{\max\{\tfrac{\delta-k}{2}, \gamma + h\}, \tfrac{\delta-k}{2}\}\}\\
&\geq \min\{\mu^{\diamond}(x), \mu^{\diamond}(y), \tfrac{\delta-k}{2}\}.
\end{aligned}
$$

Hence $\mu^{\diamond}$ is an $(\in^h_\gamma, \in^h_\gamma \vee q^k_\delta)$-fuzzy subgroup of G.

Now, we prove that $\mu^{\sim}$ is an $(\in^h_\gamma, \in^h_\gamma \vee q^k_\delta)$-fuzzy subgroup of G.

$$
\begin{aligned}
\max\{\mu^{\sim}(xy^{-1}), \gamma + h\} &= \max\{\max\{\min\{\mu(xy^{-1}), \gamma + h\}, \tfrac{\delta-k}{2}\}, \gamma + h\}\\
&= \max\{\min\{\max\{\mu(xy^{-1}), \gamma + h\}, \gamma + h\}, \tfrac{\delta-k}{2}\}\\
&\geq \max\{\min\{\min\{\mu(x), \mu(y), \tfrac{\delta-k}{2}\}, \gamma + h\}, \tfrac{\delta-k}{2}\}\\
&= \max\{\min\{\min\{\mu(x), \gamma + h\}, \min\{\mu(y), \gamma + h\},\\
&\quad \min\{\tfrac{\delta-k}{2}, \gamma + h\}, \tfrac{\delta-k}{2}\}, \tfrac{\delta-k}{2}\}\\
&= \min\{\max\{\min\{\mu(x), \gamma + h\}, \tfrac{\delta-k}{2}\},\\
&\quad \max\{\min\{\mu(y), \gamma + h\}, \tfrac{\delta-k}{2}\}, \tfrac{\delta-k}{2}\}\\
&= \min\{\mu^{\sim}(x), \mu^{\sim}(y), \tfrac{\delta-k}{2}\}.
\end{aligned}
$$

Hence $\mu^{\sim}$ is an $(\in^h_\gamma, \in^h_\gamma \vee q^k_\delta)$-fuzzy subgroup of G.

Now, we prove that μ^* is an $(\in^h_\gamma, \in^h_\gamma \vee q^k_\delta)$-fuzzy subgroup of G.

$$
\begin{aligned}
&\max\{\mu^*(xy^{-1}), \gamma + h\}\\
&= \max\{\max\{\max\{\mu(xy^{-1}), \tfrac{\delta-k}{2}\}, \gamma + h\}, \gamma + h\}\\
&= \max\{\max\{\max\{\mu(xy^{-1}), \gamma + h\}, \max\{\tfrac{\delta-k}{2}, \gamma + h\}\}, \gamma + h\}\\
&\geq \max\{\max\{\min\{\mu(x), \mu(y), \tfrac{\delta-k}{2}\}, \tfrac{\delta-k}{2}\}, \gamma + h\}\\
&= \min\{\max\{\max\{\mu(x), \tfrac{\delta-k}{2}\}, \max\{\mu(y), \tfrac{\delta-k}{2}\}, \gamma + h\}, \tfrac{\delta-k}{2}\}\\
&= \min\{\max\{\max\{\mu(x), \tfrac{\delta-k}{2}\}, \gamma + h\},\\
&\max\{\max\{\mu(y), \tfrac{\delta-k}{2}\}, \gamma + h\}, \tfrac{\delta-k}{2}\}\\
&= \min\{\mu^*(x), \mu^*(y), \tfrac{\delta-k}{2}\}.
\end{aligned}
$$

Hence μ^* is an $(\in^h_\gamma, \in^h_\gamma \vee q^k_\delta)$-fuzzy subgroup of G. □

8. Homomorphic image and pre-image of $(\in^h_\gamma, \in^h_\gamma \vee q^k_\delta)$-fuzzy subgroups

In this section, we discuss the homomorphic image and pre-image of $(\in^h_\gamma, \in^h_\gamma \vee q^k_\delta)$-fuzzy subgroups.

Definition 8.1 ([2]). *Let f be a mapping from a set X into a set Y. If μ is a fuzzy subset of Y, then the pre-image of μ under f, denoted by $f^{-1}(\mu)$, is a fuzzy subset of X defined by $f^{-1}(\mu)(x) = \mu(f(x))$ for every $x \in X$.*

Theorem 8.1. *Let $f : G_1 \to G_2$ be a homomorphism between groups G_1 and G_2. If μ is an $(\in^h_\gamma, \in^h_\gamma \vee q^k_\delta)$-fuzzy subgroup of G_2, then $f^{-1}(\mu)$ is an $(\in^h_\gamma, \in^h_\gamma \vee q^k_\delta)$-fuzzy subgroup of G_1.*

Proof. Let $x, y \in G_1$. Then

$$\begin{aligned}
\max\{f^{-1}(\mu)(xy^{-1}), \gamma + h\} &= \max\{\mu(f(xy^{-1})), \gamma + h\} \\
&= \max\{\mu(f(x)f(y^{-1})), \gamma + h\} \\
&= \max\{\mu(f(x)(f(y))^{-1}), \gamma + h\} \\
&\geq \min\{\mu(f(x)), \mu(f(y)), \tfrac{\delta - k}{2}\} \\
&= \min\{f^{-1}(\mu)(x), f^{-1}(\mu)(y), \tfrac{\delta - k}{2}\}.
\end{aligned}$$

Hence $f^{-1}(\mu)$ is an $(\in^h_\gamma, \in^h_\gamma \vee q^k_\delta)$-fuzzy subgroup of G_1. □

To prove the converse of Theorem 8.1, we need to strengthen the condition on f.

Theorem 8.2. *Let $f : G_1 \to G_2$ be an onto homomorphism between groups G_1 and G_2, and let μ be a fuzzy subset of G_2. If $f^{-1}(\mu)$ is an $(\in^h_\gamma, \in^h_\gamma \vee q^k_\delta)$-fuzzy subgroup of G_1, then μ is an $(\in^h_\gamma, \in^h_\gamma \vee q^k_\delta)$-fuzzy subgroup of G_2.*

Proof. Let $a, b \in G_2$. Then $f(x) = a$ and $f(y) = b$ for some $x, y \in G_1$. Since $f^{-1}(\mu)$ is an $(\in^h_\gamma, \in^h_\gamma \vee q^k_\delta)$- fuzzy subgroup of G_1, we have

$$\begin{aligned}
\max\{\mu(ab^{-1}), \gamma + h\} &= \max\{\mu(f(x)(f(y))^{-1}), \gamma + h\} \\
&= \max\{\mu(f(x)f(y^{-1})), \gamma + h\} \\
&= \max\{\mu(f(xy^{-1})), \gamma + h\} = \max\{f^{-1}(\mu)(xy^{-1}), \gamma + h\} \\
&\geq \min\{f^{-1}(\mu)(x), f^{-1}(\mu)(y), \tfrac{\delta - k}{2}\} \\
&= \min\{\mu(f(x)), \mu(f(y)), \tfrac{\delta - k}{2}\} \\
&= \min\{\mu(a), \mu(b), \tfrac{\delta - k}{2}\}.
\end{aligned}$$

Hence μ is an $(\in^h_\gamma, \in^h_\gamma \vee q^k_\delta)$-fuzzy subgroup of G_2. □

Definition 8.2 ([2])**.** *Let f be a mapping from a set X into a set Y and let μ be a fuzzy subset of X. Then the image of μ under f, denoted by $f(\mu)$, is the fuzzy subset of Y defined by*

$$f(\mu)(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \mu(x), & \text{if } f^{-1}(y) \neq \phi, \\ 0, & \text{otherwise} \end{cases}$$

for every $y \in Y$, where $f^{-1}(y) = \{x \in X | f(x) = y\}$.

Definition 8.3 ([2])**.** *A fuzzy subset μ of a set X is said to have sup property if for any subset $A \subseteq X$, there exists $a_0 \in A$ such that $\mu(a_0) = \sup_{a \in A} \mu(a)$.*

Theorem 8.3. *Let $f : G_1 \to G_2$ be a homomorphism between groups G_1 and G_2. If μ is an $(\in^h_\gamma, \in^h_\gamma \vee q^k_\delta)$-fuzzy subgroup of G_1 with sup property, then $f(\mu)$ is an $(\in^h_\gamma, \in^h_\gamma \vee q^k_\delta)$-fuzzy subgroup of G_2.*

Proof. Let $x, y \in G_2$. Suppose either $x \notin f(G_1)$ or $y \notin f(G_1)$. Then $\min\{f(\mu)(x), f(\mu)(y), \frac{\delta-k}{2}\} = 0 \leq \max\{f(\mu)(xy^{-1}), \gamma + h\}$. Suppose $x \in f(G_1)$ and $y \in f(G_1)$ there exists $a, b \in G_1$ such that $f(a) = x$ and $f(b) = y$. Since $x, y \in f(G_1)$, we have $a_0 \in f^{-1}[f(a)]$ and $b_0 \in f^{-1}[f(b)]$ such that $\mu(a_0) = \sup_{a_1 \in f^{-1}[f(a)]} \mu(a_1)$ and $\mu(b_0) = \sup_{b_1 \in f^{-1}[f(b)]} \mu(b_1)$ respectively. Then

$$\begin{aligned}
\max\{f(\mu)(xy^{-1}), \gamma + h\} &= \max\{f(\mu)(f(ab^{-1}), \gamma + h\} \\
&= \max\{\sup_{z \in f^{-1}[f(ab^{-1})]} \mu(z), \gamma + h\} \\
&\geq \max\{\mu(a_0 b_0^{-1}), \gamma + h\} \\
&\geq \min\{\mu(a_0), \mu(b_0), \tfrac{\delta-k}{2}\} \\
&= \min\{\sup_{a_1 \in f^{-1}[f(a)]} \mu(a_1), \sup_{b_1 \in f^{-1}[f(b)]} \mu(b_1), \tfrac{\delta-k}{2}\} \\
&= \min\{f(\mu)(f(a)), f(\mu)(f(b)), \tfrac{\delta-k}{2}\} \\
&= \min\{f(\mu)(x), f(\mu)(y), \tfrac{\delta-k}{2}\}.
\end{aligned}$$

Hence $f(\mu)$ is an $(\in^h_\gamma, \in^h_\gamma \vee q^k_\delta)$-fuzzy subgroup of G_2. □

9. Conclusion

In this paper, we have generalized the notion of belongs to relation ($\in$) by $\in^h_\gamma$ relation of a fuzzy point with a fuzzy subset. We have introduced the concepts of (α, β)-fuzzy subgroups of a group by using $\in^h_\gamma$ relation and q^k_δ relation between a fuzzy point and a fuzzy subgroup. We have also introduced the notion of $(\in^h_\gamma, \in^h_\gamma \vee q^k_\delta)$-fuzzy subgroup of a group which is a generalization of fuzzy subgroups of the types $(\in, \in)$, $(\in, \in \vee q)$, $(\in, \in \vee q_k)$, $(\in, \in \vee q_\delta)$, $(\in, \in \vee q^k_\delta)$, $(\in_\gamma, \in_\gamma \vee q)$, $(\in_\gamma, \in_\gamma \vee q_k)$, $(\in_\gamma, \in_\gamma \vee q_\delta)$, $(\in_\gamma, \in_\gamma \vee q^k_\delta)$, $(\in^h_\gamma, \in^h_\gamma \vee q)$, $(\in^h_\gamma, \in^h_\gamma \vee q_k)$ and $(\in^h_\gamma, \in^h_\gamma \vee q_\delta)$. Further, we have introduced the notions of q^k_δ-level subset, $\in^h_\gamma \vee q^k_\delta$-level subset, (δ, k)-lower and upper parts, (γ, h)-lower and upper parts of fuzzy subsets and investigated some related properties. Finally, we have discussed the homomorphic image and pre-image of $(\in^h_\gamma, \in^h_\gamma \vee q^k_\delta)$-fuzzy subgroups. In our further directions of research, we will apply these results to other fuzzy algebraic substructures of semigroups, near-rings, rings etc.

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Accepted: June 05, 2020

Epidemia di Coronavirus Covid19 causata da SARS-COV-2 in Cina*

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Abstract. A logistical mathematical model allowed analyzing daily data on the evolution of the Covid19 coronavirus epidemic in China. The model made it possible to anticipate some epidemiological data, not easily identifiable by other means. Three forecasts are particularly relevant: the final cumulative value of the infected people equal to 81750, the time when the daily infection reaches the peak, estimated at 06/02/2020, the practical duration of the epidemic estimated at 100 days from the beginning of the infection. The trend of daily contagion data is compatible with a time decreasing average life span of the virus.
Keywords: epidemia, modello matematico, SARS Covid19.

1. Introduzione

I coronavirus (CoV) sono una vasta famiglia di virus che causano malattie che vanno dal comune raffreddore a patologie molto gravi, come la sindrome respiratoria mediorientale (MERS-CoV) e la sindrome respiratoria acuta grave (SARS-CoV). Queste due virosi possono configurare una comparazione significativa ai fini della valutazione di una nuova epidemia di coronavirus, quale stiamo vivendo in questi primi mesi del 2020. Per questo risulta utile ricordarne brevemente la storia. Tra il novembre 2002 e il luglio 2003, un focolaio di SARS nel sud della Cina causó 8098 casi, e 774 decessi in 17 paesi, con un tasso di mortalitá del 9,6% [1]. L'epidemia si concentró nella Cina continentale e a Hong Kong per esaurirsi completamente nel 2004. Nel 2017, gli scienziati cinesi dopo anni di ricerche in tutta la Cina, rintracciarono un ceppo virale di Coronavirus in pipistrelli a ferro di cavallo in una grotta nella provincia dello Yunnan, la cui sequenza genomica risultava pressoché sovrapponibile a quella dell'agente responsabile dell'epidemia. La sindrome respiratoria del Medio Oriente (MERS), nota anche come influenza del cammello, é una infezione respiratoria virale causata

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dal coronavirus della MERS, [2]. I sintomi possono variare da lievi a gravi. Includono febbre, tosse, diarrea e dispnea. La malattia é in genere piú grave nei soggetti con altri problemi di salute. La mortalitá é circa un terzo dei casi diagnosticati. Poco meno di 2000 casi sono stati segnalati fino all'aprile 2017. Il primo caso identificato si é verificato nel 2012 in Arabia Saudita e la maggior parte dei casi si é verificata nella penisola arabica [3]. Un ceppo di MERS-CoV noto come HCoV-EMC / 2012 isolato da un paziente a Londra nel 2012 ha evidenziato una corrispondenza del 100% con un Coronavirus dei pipistrelli tombali egiziani. Un grave focolaio si é verificato in Corea del Sud nel 2015. Nel 2018 é stato segnalato un focolaio di MERS in Arabia Saudita e in paesi limitrofi, nel 2018 i casi segnalati sono stati solo sporadici. Anche in questo caso, quindi, l'agente infettivo deriva dai pipistrelli, il cammello sembra essere un ospite intermedio che ha facilitato la trasmissione all'uomo, [3]. La nuova epidemia di Coronavirus che é iniziata a Wuhan, in Cina l'8 dicembre 2019, ha infettato piú persone della SARS-CoV e della MERS-CoV combinate [4]. Al 3 febbraio 2020 sono stati confermati 17.489 casi in Cina e in altri 24 paesi, di cui 362 pazienti sono deceduti. Ció ha indotto il governo cinese ha limitare la movimentazione di oltre 60 milioni di persone, imponendo l'obbligo dell'uso di maschere sanitarie in pubblico e il potenziamento delle strutture di assistenza sanitaria, con l'incremento di 1200 posti letto nelle strutture ospedaliere in soli 10 giorni. Al momento di massimo contagio (04/02/2020), sono stati raggiunti 3886 nuovi casi al giorno. Il 30 gennaio 2020 l'Organizzazione Mondiale della Sanitá ha dichiarato l'epidemia un'emergenza sanitaria a livello internazionale qualificandola evento straordinario destinato a coinvolgere la salute pubblica in molti Stati, che richiederá una risposta internazionale coordinata e un sostegno straordinario all'attivitá di ricerca finalizzata a sostenere le scelte dei responsabili politici. Dal punto di vista epidemiologico, la velocitá di trasmissione del virus deve essere prevista con modelli matematici attendibili. Le modalitá di trasmissione devono essere qualificate in termini sia qualitativi che quantitativi cosí da adeguare azioni efficaci. É inoltre essenziale sequenziare il genoma del virus e identificare il serbatoio animale intermedio al fine di valutare mutanti specifici dei vari contesti geografici e interrompere il ciclo di trasmissione iniziale. Queste azioni in Cina sono state attuate con sollecitudine e hanno condotto alla consapevolezza che l'epidemia é stata causata da un nuovo beta-Coronavirus, denominato 2019-nCoV e successivamente Covid19, geneticamente correlato alla SARS-CoV e ad altri Coronavirus endemici nella popolazione dei pipistrelli, [5]. I casi iniziali sono emersi in ambienti contigui ad un mercato del pesce a Wuhan, dove venivano venduti animali vivi. Ció ha suggerito un possibile serbatoio animale e ha confermato la trasmissione zoonotica iniziale, poi sostituita in netta prevalenza da quella inter-umana, [6]. Nella tabella di figura 1 sono riportati alcuni dati consuntivi sulla sindrome da Coronavirus. Ció che risulta evidente é la sostanziale evoluzione della casistica verso uno scenario pandemico, con il coinvolgimento *focale* di nuovi paesi. In altre parole la propagazione si manifesta sia per contiguitá nei centri di propagazione primaria, ovvero la Cina, sia con

la comparsa sporadica di nuovo centri di nucleazione, caratteristica tipica della propagazione in contesti geografici globalizzati caratterizzati da trasferimenti di uomini e beni rapidi, incontrollati e generalizzati. La mortalitá é molto variabile, come normalmente é osservato all'inizio delle pandemie e dipende dall'anzianitá delle coorti epidemiologiche. Non é ancora possibile qualificare l'andamento in funzione delle classi di etá in modo definitivo, tuttavia i dati pubblicati dal *Centre for Disease Control and Prevention* aiutano a inquadrare il contesto [4]. Per le grandi pandemie storiche come la Spagnola l'andamento era a U, ovvero la mortalitá era massima nei giovani e nei vecchi e piú ridotta nelle coorti di mezza etá. Nella SARS l'andamento era a forma di W. Al momento per il Covid19 si osserva solo la prevalenza nelle persone piú anziane: si passa dallo 0.2% sotto i 40 anni al 3.6% per i sessantenni fino al 14.8% per gli ottantenni [4]. I bambini sembrano meno suscettibili, (sotto i dieci anni la mortalitá si azzera), o semplicemente sono piú protetti. É possibile una suscettivitá di popolazione, forse associata a specifici aplotipi che configurano la variabilitá genetica dei recettori cellulari a cui il virus si lega (probabilmente l'*angiotensin-converting enzyme 2*) con la sua proteina S di superficie (*spike protein*). Ció potrebbe forse giustificare anche la maggiore mortalitá complessiva negli uomini (2.8%) rispetto alle donne (1.7%). Sicuramente lo stato di salute e la presenza di patologie importanti é il fattore di rischio di massima rilevanza. Le patologie cardiovascolari decuplicano il rischio, e l'ipertensione, il diabete e le patologie croniche polmonari lo incrementano di sei, sette volte. Cosí un soggetto tra i quaranta e i sessanta anni sano ha un rischio di morte medio intorno allo 0.9%, un cardiopatico nella stessa fascia di etá vede incrementato il rischio fino allo 10.5% [4]. Come altri Coronavirus umani causa di malattie del tratto respiratorio superiore negli uomini, anche il Covid19, é un virus ad RNA a singolo filamento di polaritá positiva, con un genoma di 27-31 kb, il piú grande per un virus ad RNA. Per contrastarne la propagazione e approntare farmaci efficaci alla prevenzione e alla terapia é necessaria la messa a punto di una diagnostica molecolare rapida dei virus, e l'individuazione di nuove strategie terapeutiche e bersagli molecolari da utilizzare sia per diagnosi differenziali che per terapie specifiche. Pertanto gli studi di patogenesi sono orientati all'identificazione del tropismo virale, della modulazione della risposta immune sia innata che acquisita, e, quindi, ma purtroppo in tempi piú lunghi, alla realizzazione di un vaccino efficace.

2. Analisi della serie storica dei dati del contagio da Covid19 in Cina

Come tutte le attivitá umane, anche la storia del Coronavirus Covid19 puó essere descritta secondo diverse prospettive, in funzione degli obiettivi che si vogliono raggiungere. In questo caso, focalizzeremo la nostra attenzione sui dati storici riguardanti l'evoluzione del contagio in Cina, con l'obiettivo di prevederne durata e numero totale di contagiati. Di fronte all'alternativa se attendere la conclusione dell'epidemia in Cina per avere tutti i dati, in modo da fare una analisi retrospettiva oppure pubblicare adesso puntando su un'analisi predittiva,

	Wuhan coronavirus (Covid19)	**Middle East respiratory syndrome(MERS)**	**Severe acute respiratory syndrome (SARS)**	**Sindromi influenzali comuni attribuite a Coronavirus**
Origine	Primi casi nel Dicembre 2019 nella città di Wuhan, Cina	Primi casi 2012 Arabia Saudita	Primi casi 2002 Sud della Cina	Quattro ceppi di CoV sono ritenuti causa del 15-30% dei comuni raffreddori
Trasmissione	Contatto inter-umano o toccando superfici contaminate dal virione.	Da cammelli infetti forse consumando il loro latte o la carne. Contatto inter-umano o toccando superfici contaminate dal virione	Contatto inter-umano o toccando superfici contaminate dal virione.	Contatto inter-umano o toccando superfici contaminate dal virione.
Casistica	87137 casi 2977 decessi (79968 in Cina) 44 paesi (Consuntivo del 1/3/2020)	2.494 casi confermati; 858 decessi (al 30 novembre 2019). Tasso di mortalità del 34%.	8.098 casi; 774 morti. Tasso di mortalità di circa il 10%.	Milioni ogni anno. Generalmente non letale con eccezioni associate a condizioni critiche
Localizzazione	26 febbraio 2020. *US CDCP* soddisfatti due dei tre criteri che qualificano una pandemia	penisola Arabica, l'80% in Arabia Saudita. Due dozzine di paesi, inclusi Stati Uniti, in netto calo dal 2016.	Dal 2004 non sono stati segnalati nuovi casi. L'87% dei casi precedenti in Cina e Hong Kong.	-
Valore Riproduttivo (R0)	Stimato 4.08 (febbraio 2020)	Inferiore a 1 (2019)	Non inferiore a 3 (2019)	-

Figure 1: Inquadramento generale dell'assetto (fine febbraio 2020) [7], [8], [9]. R0 : numero di riproduzione di base: quante persone ogni persona infetta infetterá

si é optato per questa seconda scelta. Consapevoli del valore del tempo, si é ritenuta preferibile una risposta incompleta ma, per tempo, piuttosto che una risposta completa ma tardiva. I dati della tabella di figura 2 raccolgono il numero cumulativo dei nuovi casi di contagio confermati, in Cina, ed é evidenziata anche l'etá del contagio, in giorni, fatto zero il 08/12/2019, giorno di identificazione del Coronavirus Covid19 da parte del dr. Li Wenliang [9]. Tutti questi dati sono stati esaminati alla luce di un modello matematico, capace di descrivere

l'evoluzione teorica del contagio stesso. La figura 2 riassume graficamente l'esito di questo processo.

Data	Casi e % di variazione	Età,gg
16-01-2020	45 (--------)	38
17-01-2020	62 (+38%)	39
18-01-2020	121 (+95%)	40
19-01-2020	198 (+64%)	41
20-01-2020	291 (+47%)	42
21-01-2020	440 (+51%)	43
22-01-2020	571 (+30%)	44
23-01-2020	830 (+45%)	45
24-01-2020	1 287 (+55%)	46
25-01-2020	1 975 (+53%)	47
26-01-2020	2 744 (+39%)	48
27-01-2020	4 515 (+64%)	49
28-01-2020	5 974 (+32%)	50
29-01-2020	7 711 (+29%)	51
30-01-2020	9 692 (+26%)	52
31-01-2020	11 791 (+22%)	53
01-02-2020	14 380 (+22%)	54
02-02-2020	17 205 (+20%)	55
03-02-2020	20 438 (+19%)	56
04-02-2020	24 324 (+19%)	57
05-02-2020	28 018 (+15%)	58
06-02-2020	31 161 (+11%)	59
07-02-2020	34 546 (+11%)	60
08-02-2020	37 198 (+7,7%)	61
09-02-2020	40 171 (+8,0%)	62
10-02-2020	42 638 (+6,1%)	63
11-02-2020	44 653 (+4,7%)	64
12-02-2020	46 472 (+4,1%) 58 761 (---)	65
13-02-2020	48 467 (+4,3%) 63 851 (+8,7%)	66
14-02-2020	49 970 (+3,1%) 66 492 (+4,1%)	67
15-02-2020	51 091 (+2,2%) 68 500 (+3,0%)	68
16-02-2020	70 548 (+3,0%)	69
17-02-2020	72 436 (+2,7%)	70
18-02-2020	73 906 (+2,0%)	71
19-02-2020	74 576 (+1,0%)	72
20-02-2020	75 465 (+1,2%)	73
21-02-2020	76 288 (+1,1%)	74
22-02-2020	76 936 (+0,85%)	75
23-02-2020	77 150 (+0,28%)	76
24-02-2020	77 658 (+0,66%)	77
25-02-2020	78 064 (+0,52%)	78
26-02-2020	78 497 (+0,55%)	79
27-02-2020	78 824 (+0,42%)	80
28-02-2020	79 251 (+0,54%)	81
29-02-2020	79 824 (+0,72%)	82
01-03-2020	80 026 (+0,25%)	83
02-03-2020	80 151 (+0,16%)	83
03-03-2020	80 270 (+0,15%)	84
04-03-2020	80 409 (+0,17%)	85
05-03-2020	80 552 (+0,18%)	86
06-03-2020	80 651 (+0,12%)	87
07-03-2020	80 695 (+0,05%)	88
08-03-2020	80 735 (+0,05%)	89
09-03-2020	80 754 (+0,02%)	90
10-03-2020		
11-03-2020		
12-03-2020		
13-03-2020		
14-03-2020		
15-03-2020		
16-03-2020		
17-03-2020		

Figure 2: Serie storica del contagio da Coronavirus Covid19, [10] NHC-China-1 dal 16/01/2020 al 15/02/2020; NHC-China-2 dal 12/01/2020 al 09/03/2020. (Dal 12 al 15 febbraio 2020) i casi confermati includono persone diagnosticate clinicamente nella provincia di Hubei, senza effettuare il test per il virus. Le coppie NHC-China-1 e 2 dal 13 al 15 febbraio identificano il fattore correttivo lineare di compatibilitá che genera la serie NHC-China-1 corretta).

La linea tratteggiata fornita dal modello di calcolo corrisponde all'equazione 4, ed ha la forma di una funzione di tipo logistico: all'inizio cresce velocemente, quindi segue un andamento quasi lineare per poi gradualmente rallentare portandosi definitivamente a un plateau. La descrizione dei dati storici di evoluzione del contagio per mezzo di una curva teorica, ha il vantaggio di poter dedurre da essa alcuni dati epidemiologici, altrimenti non facilmente individuabili per altra via. Fra questi ne segnaliamo tre, particolarmente rilevanti:

1. Il valore asintotico $N_\infty = 81750$ della curva logistica, ottenuto come best fitting dei dati storici della tabella di figura 2. Tale numero corrisponde al valore cumulativo finale del totale dei contagiati.

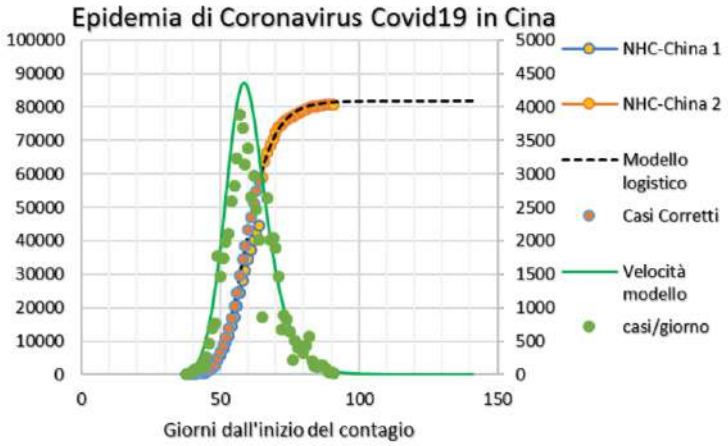


Figure 3: Contagio del Coronavirus Covid19 in Cina: dati storici a confronto con la previsione logistica (linea tratteggiata: tempo di semi-vita iniziale = 44 giorni con costante di tempo di decadimento di 21 giorni; previsione del numero di contagiati totale=81 750). In verde é rappresentata la velocitá calcolata di contagi, nr. casi/giorno: il picco teorico é stato raggiunto il 06/02/2020.

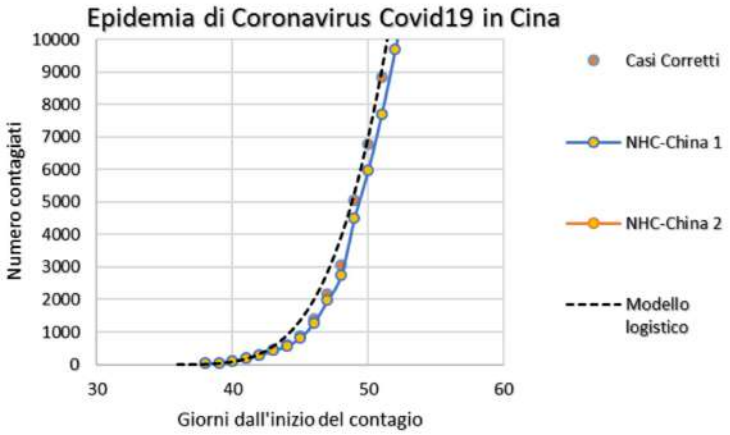


Figure 4: Contagio del Coronavirus Covid19 in Cina, Comportamento iniziale del modello.

2. Il momento in cui si raggiunge il massimo della velocitá di crescita del contagio: il cinquantanovesimo giorno dall'inizio del contagio, corrispondente al 06/02/2020. Questo dato é di interesse psicologico, strategico ed

economico per i responsabili dalla sanitá pubblica e, piú in generale, per l'Organizzazione Mondiale della Sanitá. Il risultato si ottiene calcolando la derivata della curva di crescita ed é dato dall'equazione 3.

3. La durata pratica dell'epidemia. Estendendo opportunamente la scala dei tempi della curva logistica (linea tratteggiata) allo scopo di evidenziare l'evoluzione del fenomeno, si deduce l'esaurimento pratico dell'epidemia entro 100 giorni dall'inizio del contagio. La velocitá (linea verde) é crescente sino al cinquantanovesimo giorno dall'inizio del contagio; a questa data (06/02/2020), raggiunge il massimo teorico di circa 4400 casi/giorno, dopo di che inizia un declino inarrestabile, sino alla conclusione del contagio.

Si é constatato, a posteriori, che i dati numerici di contagio sono meglio rappresentati dalla equazione 4 se la costante cinetica k é una funzione crescente del tempo, ció equivale a dire che la costante di tempo di vita media del contagio diminuisce gradualmente nel tempo, equazione 5.

3. Modello di sviluppo del contagio mondiale da Coronavirus Covid19

La descrizione dello sviluppo del contagio nel mondo del Coronavirus Covid19, potrebbe essere fatta facendo ricorso ai classici modelli di dinamica delle popolazioni, tuttavia, in questo lavoro si é preferito costruire un modello di sviluppo del contagio mediante adattamento giudizioso della teoria di crescita dei cristalli di Avrami, [11],[12],[13]. Il nostro obiettivo é determinare la crescita del numero di contagiati, come frazione del numero finale dei contagiati N_∞ il quale si puó esprimere mediante la relazione $N_\infty = \epsilon\, P$ ovvero, come prodotto della percentuale di contagiati ϵ per la popolazione esposta P. Il modello si occupa, quindi, non di tutta la popolazione esposta ma solo di quella frazione che si ammalerá. Ogni processo di crescita, in questo caso l'epidemia, puó essere descritto come il risultato di tre fenomeni: la nascita dei focolai di infezione nella popolazione esposta (cinetica di nucleazione ad opera di soggetti infetti), la successiva crescita intorno a tali nuclei (cinetica di crescita vera e propria) e l'arresto della crescita quando il sistema in accrescimento viene in contatto con un altro concorrente, interferendo con esso. Nelle aree di contatto la crescita libera é impedita dall'interferenza con i competitori che implica che non é piú disponibile ulteriore area di espansione dell'infezione, essendo giá stata occupata dal concorrente che si avvicina. Questo terzo passo é comunemente indicato con il termine tecnico di *impingement* e genera un rallentamento, come nel caso della crescita dei cristalli, perché, in altre parole tutto il cristallizzabile é giá cristallizzato. Questo argomento si traspone, esattamente, nel caso dell'epidemia, dove i giá contagiati non possono essere ulteriormente contagiati. Alle condizioni speciali che entrambe le cinetiche di nucleazione e crescita soddisfino alle stesse regole di infezione ("ipotesi iso-cinetica") é possibile raggruppare i tre fenomeni

in un'unica equazione per la crescita cumulativa. Per comoditá di trattazione analitica delle equazioni, faremo uso del modello di crescita cumulativa sviluppato da M.C. Tobin, [14]. Nel nostro modello, la quota del numero totale di contagiati é indicata dal simbolo χ. Questa variabile adimensionale χ esprime il rapporto fra il numero attuale di contagiati Na e il numero di contagiati alla fine dell'epidemia N_∞ (1). Il parametro χ puó, pertanto, assumere valori tra zero, all'inizio del contagio, e uno, quando si raggiunge la saturazione del 100% del numero finale accessibile dei contagi. La cinetica di crescita del contagio é descritta da un'equazione differenziale ordinaria del primo ordine nella variabile χ (2) con la condizione iniziale, $\chi(0) = 0$, cioé, numero di contagiati nullo all'inizio dell'espansione del virus (dicembre 2019). L'equazione 2 descrive l'andamento della velocitá assoluta di crescita dei contagiati. La costante cinetica k ha le dimensioni dell'inverso di un tempo ed é un descrittore del tempo caratteristico di evoluzione temporale del fenomeno di contagio. Il fattore di forma $f(\alpha, \chi)$ assume la forma analitica rappresentata dall'equazione 3. Questa funzione ci dice che il contagio cresce proporzionalmente alla dimensione stessa del contagio, con un andamento sub-lineare dato dal fattore $\chi^{1-\frac{1}{\alpha}}$; ci dice anche che la crescita non procede all'infinito ma si estingue gradualmente, proporzionalmente allo spazio residuo di contagio, con un andamento super-lineare dato dal fattore $(1-\chi)^{1+\frac{1}{\alpha}}$. Il numero reale positivo $\alpha = n + 1$ si riferisce alla dimensione geometrica spaziale del fenomeno in espansione, ($n = 1$ significa sviluppo di tipo direzionale, per contagio uno a uno; $n = 2$ significa sviluppo areale del contagio). La cinetica di crescita del contagio, descritta dall'equazione 2, puó essere integrata analiticamente, dando luogo alla soluzione integrale, che fornisce l'andamento del numero assoluto $N_\alpha(t)$ di contagiati nel tempo. L'equazione 4 ha la forma di una funzione di tipo logistico: all'inizio cresce velocemente, con l'andamento della funzione potenza $(K\,t)^\alpha$ per poi gradualmente rallentare portandosi definitivamente a un plateau. Da questa soluzione dell'equazione analitica si vede immediatamente che quando il tempo corrente t é tale che $K\,t = 1$, allora la dimensione del contagio raggiunge il valore $\chi = \frac{1}{2}$, cioé il 50% del valore massimo accessibile all'equilibrio. Pertanto la costante cumulativa K corrisponde all'inverso del tempo di semi-vita del contagio.

$$\chi = \frac{N_a}{N\infty}, \tag{1}$$

$$\frac{d\chi}{dt} = k\, f(\alpha, \chi), \tag{2}$$

$$f(\alpha, \chi) = \alpha\chi^{1-\frac{1}{\alpha}}(1-\chi)^{1+\frac{1}{\alpha}}, \tag{3}$$

$$N_\alpha = N_\infty \frac{(K\,t)^\alpha}{1 + (K\,t)^\alpha}, \tag{4}$$

N_∞	$\tau_0 = \frac{1}{k_0}$	T	$\alpha = n + 1$
Numero finale di contagi	giorni	giorni	adimensionale
81750	44	21	3

Table 1: Parametri identificativi del contagio da Coronavirus Covid19 in Cina

$$\tau(t) = \frac{1}{k} = \tau_0 \exp -\frac{t}{T}, \tag{5}$$

$$K(t) = \frac{1}{\tau_0} \frac{\exp(\frac{t}{T}) - 1}{(\frac{t}{T})}. \tag{6}$$

La curva rappresentata dall'equazione 4 puó essere utilizzata per descrivere i dati sperimentali di uno specifico contagio: per far questo é necessario conoscere i valori dei tre parametri liberi (N_∞, K, α). Nel nostro caso, utilizzeremo la serie storica del contagio da Coronavirus Covid19 provenienti dai rapporti giornalieri della Commissione Nazionale di Sanitá della Repubblica Popolare Cinese, [10], riportati nella tabella di figura 2. Questi dati, uniti al momento in cui ha avuto inizio il contagio, (dicembre 2019) coprono compiutamente un arco di tempo di oltre 70 giorni dall'inizio del contagio e permettono di identificare, con sufficiente affidabilitá, i tre parametri liberi N_∞, K, α con i quali si descrive analiticamente la migliore approssimazione ai dati sperimentali disponibili al momento della pubblicazione di questo lavoro, tabella 1.

Si é constatato, a posteriori, che i dati numerici di contagio sono meglio rappresentati dalla equazione 4 se la costante cinetica K é una funzione crescente del tempo, ció equivale a dire che la costante di tempo di vita media del contagio diminuisce gradualmente nel tempo, equazione 5, come rappresentato dalla figura 5 Il fatto che il tempo di vita media del contagio cambi nel tempo non é inatteso, ma questa analisi permette di azzardare, per la prima volta, dei valori numerici, attraverso i quali avviare una valutazione quantitativa del fenomeno. Nel nostro caso, il modello logistico, descrive i dati di contagio dopo un periodo di incubazione di circa 36 giorni, durante il quale i singoli focolai hanno potuto svilupparsi. Da quel momento in avanti inizia la fase esplosiva descritta dal modello.

Il tempo di vita media iniziale del contagio risulta, a posteriori, di 44 giorni, come rappresentato dalla figura 5. In corrispondenza, anche la costante integrale K diventa funzione crescente del tempo con l'andamento dato dall'equazione 6. Il parametro K é il valor medio della costante cinetica k sino al tempo t. Solo nel caso in cui la costante cinetica k sia effettivamente costante nel tempo, le due variabili coincidono. I relativi parametri identificativi sono raccolti in tabella 1.

L'andamento della curva logistica di crescita del numero dei contagiati in Cina é rappresentato in figura 2 dalla curva in tratteggio nero mentre la velocitá

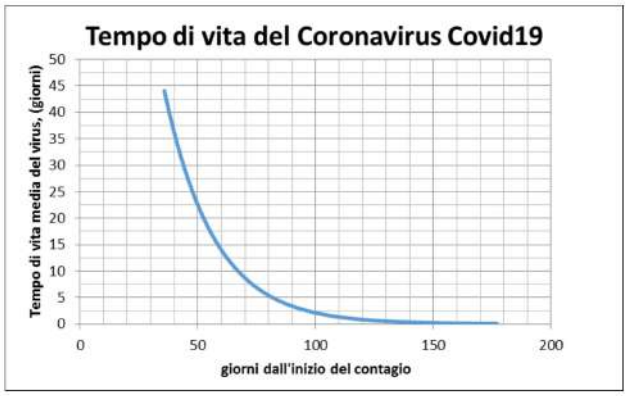


Figure 5: Identificazione della curva di evoluzione esponenziale del tempo di vita media del contagio

assoluta di variazione del contagio, (numero di contagiati al giorno) é data dalla curva continua in colore verde.

4. Osservazioni critiche al modello epidemiologico del Coronavirus Covid19

L'ipotesi fondamentale di questo lavoro é che il modello di crescita del contagio da Coronavirus possa essere descritto dall'equazione differenziale 1, in cui il secondo membro si scrive come prodotto di una costante cinetica k e di un fattore di forma $f(\alpha, \chi)$ funzione solo del valore della variabile χ. É opportuno soffermarsi e fare alcune considerazioni su questa scelta, per apprezzarne appieno la portata e le conseguenze.

1. Conseguenze matematiche del modello: A causa di questa scelta, l'equazione differenziale é indipendente dalla storia passata ma dipende solo dallo stato corrente $f(\alpha, \chi)$; in altri termini, lo stato del sistema in qualunque tempo futuro é univocamente determinato dal suo stato in qualche assegnato tempo pregresso. In molte situazioni reali questa é una manifesta assurditá e, probabilmente, anche la nostra situazione é fra queste. Tuttavia, é un'utile, anche se palese, assurditá. Il difetto di molti modelli matematici non sta tanto in questo tipo di ipotesi, quanto nella mancanza di consapevolezza delle implicazioni di questa ipotesi, [15]. Un modello matematico di un processo reale va utilizzato fin tanto che fornisce previsioni in ragionevole accordo con le osservazioni. Se si pone a confronto la complessitá del fenomeno osservato con la semplicitá di questo modello matematico non é sorprendente che si potrá essere forzati a modificarne la formulazione, di tanto in tanto, allo scopo di ottenere piú accurati risultati. É notevole,

tuttavia, come una profonda comprensione di molti processi reali possa essere conseguita per mezzo di ipotesi assai rudimentali.

2. Conseguenze fisiche del modello: Per ipotesi, il fattore di forma assume l'espressione data dall'equazione 2. Il senso di questa funzione é che la velocitá di crescita del contagio é nulla non solo all'origine del processo, quando $\chi = 0$, ma anche quando $(1-\chi) = 0$, cioé alla fine del processo. Ció implica che c'é una fase di crescita, si raggiunge un massimo e poi c'é una decrescita nella velocitá di variazione del numero dei contagi. Da un punto di vista fisico, il fattore di forma rende ragione dell'ipotesi che un sistema finito non puó crescere all'infinito ma tende a stabilizzarsi raggiungendo un valore di saturazione N_∞.

3. Capacitá predittiva del modello. La speciale scelta del fattore di forma permette di integrare analiticamente l'equazione differenziale in modo da soddisfare la condizione iniziale, $\chi(0) = 0$, ottenendo cosí l'espressione analitica dell'andamento cumulativo del contagio da Coronavirus, riportata in equazione 4. Questa é una funzione di tipo logistico, a tre parametri liberi (N_∞, K, α) che sono stati identificati, per approssimazioni successive, sulla base della serie storica di dati di contagio che copre oltre 70 giorni cioé oltre due terzi della durata pratica di questo ciclo epidemiologico, (circa 100 giorni), come si evince a posteriori dal diagramma di figura 2. Cosí facendo si viene a mitigare il grosso limite evidenziato al punto 1, attenuando la portata delle conseguenze matematiche sopra delineate. Il parametro k ha il carattere di una costante cinetica e determina l'intensitá della velocitá di variazione del contagio. Esso incorpora gli effetti degli sviluppi diagnostici e terapeutici apportati nel tempo ai processi di contenimento della diffusione del contagio ed é plausibile che tali sviluppi continueranno nel tempo. Il parametro N_∞ costituisce il numero finale dei contagiati che si puó esprimere mediante la relazione $N_\infty = \epsilon P$ ovvero, come prodotto della percentuale di contagiati ϵ per la popolazione esposta P. La percentuale di contagiati é tanto piú bassa quanto piú efficaci sono le misure di profilassi adottate mentre la popolazione esposta é tanto piú bassa quanto piú energiche ed efficaci sono le misure di limitazione di mobilitá delle persone e di quarantena adottate dall'Autoritá cinese. Il 22 gennaio 2020 il governo cinese ha messo in quarantena la cittá di Wuhan (con una popolazione di 11 milioni di persone), estendendo successivamente il provvedimento a quasi tutta la provincia di Hubei, raggiungendo il numero di 60 milioni di persone poste in quarantena, nel tentativo di contenere l'epidemia virale. Attualmente si tratta della piú rilevante misura di quarantena mai disposta nella storia umana, [10]. Se, a fine contagio, il numero totale di contagiati N_∞ risulterá confermato, la percentuale N di contagiati risulterá pari a: $\epsilon = 0.14\%$. Questo risultato é di grande interesse operativo, per tutte le autoritá sanitarie del mondo. Questa bassa percentuale se confrontata con l'analogo parametro che caratterizza le ricorrenti influenze annuali,

(che, in Italia, si attesta nell'intervallo 3,0%-7,0%), fornisce una misura dell'efficacia delle misure di profilassi adottate dall'Autoritá Cinese per l'epidemia di Coronavirus Covid19 causata da SARS-COV-2.

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On fixed point results for family of multivalued mapping with application

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Abstract. The purpose of this article is to find out some fixed point results for the family of multifunctions on proximinal sets in complete dislocated metric space. We also presented an example which shows he novelty of our results. An application to the system of nonlinear integral equations has been presented. Our results combine, extend and infer several comparable results in the literature.

Keywords: fixed point, proximinal sets, family of multifunctions, dislocated metric space.

1. Introduction

Fixed point theory has a wide range of applications in the different fields of analysis. The most important tool in fixed point theory is Banach contraction principle [5]. Many authors obtained fixed point results in various metric spaces under certain contractive conditions [1]-[22]. A new contraction called $F-$contraction was establish by Wardowski [22] which generalized the several well known results in different aspects. Afterwards, Secelean [17] proved fixed point results which contain $F-$contractions by the systems of iterated function. Piri et al. [12] proved a fixed point result for $F-$Suzuki contractions on the self map in a complete metric space. The notion of generalized set-valued $F-$contraction mappings was introduced by Acar et al. [3]. The extention of setvalued $F-$contraction was given by Acar et al. [2] and in complete metric space they obtained several fixed point results. There are many interesting results on $F-$contractions which can be seen in [13, 14, 15, 16, ?].

In this article, we obtain some fixed point results for the family of multifunctions on proximinal sets in complete dislocated metric space.

Definition 1.1 ([7]). *Let W be a nonempty set. A mapping $d_l : W \times W \to [0, \infty)$ is called a dislocated metric (or simply d_l-metric) if the followings hold, for any $u, v, w \in W$:*

(i) If $d_l(u, v) = 0$, then $u = v$;

(ii) $d_l(u, v) = d_l(v, u)$;

(iii) $d_l(u, v) \leq d_l(u, w) + d_l(w, v)$.

The pair (W, d_l) is called dislocated metric space (D.M.S) or d_l metric space. It can be seen for $d_l(u, v) = 0$, then by above condition (i), $u = v$. But if $u = v$, then $d_l(u, v)$ may not be 0. For example, if $W = \mathbb{R}^+ \cup \{0\}$, then $d_l(u, v) = u + v$ defines a d_l metric on W.

Definition 1.2 ([20]). *Let M be any nonempty subset of D.M.S in W and let $u \in W$. An element $m_0 \in M$ is called a best approximation in M if*

$$d_l(u, M) = d_l(u, m_0), \text{ where } d_l(u, M) = \inf_{m \in M} d_l(u, m).$$

If every $u \in W$ has at least one best approximation in M, then M is called a proximinal set. Here $P(W)$ denotes the collection of all closed proximinal subsets of W.

Definition 1.3 ([20]). *The function* $H_{d_l} : P(W) \times P(W) \to \mathbb{R}^+$, *defined by*

$$H_{d_l}(C, D) = \max\{\sup_{c \in C} d_l(c, D), \ \sup_{d \in D} d_l(C, d)\}$$

is called d_l *Hausdorff metric on* $P(W)$.

Definition 1.4 ([22]). *Suppose* (W, d) *is a metric space and* $H : W \to W$ *is a self mapping. Then* H *is called* F *contraction if there exists* $\tau > 0$ *such that*

$$\forall j, k \in W, \ d(Hj, Hk) > 0 \Rightarrow \tau + F\left(d(Hj, Hk)\right) \leq F\left(d(j, k)\right), \tag{1.1}$$

where F *from* $\mathbb{R}_+$ *to* $\mathbb{R}$ *is a mapping satisfying the followings:*

(C_1) F *is strictly increasing, that is for every* $j, k \in \mathbb{R}_+$ *such that* $j < k$, $F(j) < F(k)$;

(C_2) *For every sequence* $\{\alpha_m\}_{m=1}^{\infty}$ *of positive numbers,* $\lim_{m\to\infty} \alpha_m = 0$ *if and only if* $\lim_{m\to\infty} F(\alpha_m) = -\infty$;

(C_3) *There exists* $r \in (0, 1)$ *such that* $\lim_{\alpha\to 0^+} \alpha^r F(\alpha) = 0$.

We denote by $\triangle_F$, *the set of all functions satisfying the conditions* $(C_1) - (C_3)$.

Definition 1.5 ([22]). *The family of* $\mathcal{F}$ *is not empty.*

(1) $F(u) = \ln(u); u > 0$.
(2) $F(u) = u + \ln(u); u > 0$.
(3) $F(u) = \frac{-1}{\sqrt{u}}; u > 0$.

Lemma 1.1 ([20]). *Let* (W, d_l) *be a D.M.S. Let* $(P(W), H_{d_l})$ *be a dislocated Hausdorff metric space on* $P(W)$. *For all* $C, D \in P(W)$ *and for any* $c \in C$ *such that* $d_l(c, D) = d_l(c, d_c)$ *where* $d_c \in D$. *Then* $H_{d_l}(C, D) \geq d_l(c, d_c)$ *holds.*

2. Main result

Let (W, d_l) be a D.M.S, $u_0 \in W$. Consider a family of multivalued mappings $\{S_\beta : \beta \in \Omega\} : W \to P(W)$. Let $u_1 \in S_a u_0$ for some $a \in \Omega$, be an element such that $d_l(u_0, S_a u_0) = d_l(u_0, u_1)$. Let $u_2 \in S_b u_1$ be such that $d_l(u_1, S_b u_1) = d_l(u_1, u_2)$. In the same way, we can construct a sequence u_n of points in W such that $u_{n+1} \in S_\gamma u_n$, $d_l(u_n, S_\gamma u_n) = d_l(u_n, u_{n+1})$, This iterative sequence is denoted by $\{WS_\beta(x_n) : \beta \in \Omega\}$ and called $\{WS_\beta(x_n) : \beta \in \Omega\}$ is a sequence generated by u_0 in W.

We begin with the following definition.

Definition 2.1. *Let* (W, d_l) *be a complete D.M.S. Consider a family of multivalued mappings* $\{S_\beta : \beta \in \Omega\} : W \to P(W)$ *and* $\{WS_\beta(u_n) : \beta \in \Omega\}$ *be a sequence in* W *generated by* u_0. *The family* $\{S_\beta : \beta \in \Omega\}$ *is called to be Ciric type rational multivalued* $F-$*contraction, if for all* $u, v \in \{WS_\beta(u_n)\}$, *we have*

$$\tau + F(H_{d_l}(S_i u, S_j v)) \leq F(D_l(u, v)) \tag{2.1}$$

where $F \in \triangle_F$ *and* $\tau > 0$, *and*

$$D_l(u,v) = \max\left\{d_l(u,v), \frac{d_l(u,S_iu).d_l(v,S_jv)}{1+d_l(u,v)}, d_l(u,S_iu), d_l(v,S_jv)\right\}. \tag{2.2}$$

Theorem 2.1. *Let* (W,d_l) *be a complete D.M.S and* $\{S_\beta : \beta \in \Omega\}$ *be the family of Ciric type rational multivalued* $F-$*contraction. Then* $\{WS_\beta(u_n)\} \to u \in W$. *Moreover, if (2.1) also holds for* x, *then there is a common fixed point for the family of multivalued mappings* $\{S_\beta : \beta \in \Omega\}$ *in* $\{WS_\beta(u_n) : \beta \in \Omega\}$.

Proof. Suppose $\{WS_\beta(u_n) : \beta \in \Omega\}$ is a complete sequence in W generated by u_0. If $u_0 = u_1$, then u_0 is a common fixed point of S_a for all $a \in \Omega$. Let $D_l(u_0,u_1)) > 0$ for all $u_0, u_1 \in \{WS_\beta(u_n)\}$ with $u_0 \neq u_1$.

Then, from (2.1), and Lemma 1.1 we have

$$F(d_l(u_n,u_{n+1})) \leq F(H_{d_l}(S_iu_{n-1},S_ju_n)) \leq F(D_l(u_{n-1},u_n)) - \tau,$$

for all $i \in \mathbb{N} \cup \{0\}$, and

$$\begin{aligned} D_l(u_{n-1},u_n) &= \max\left\{ \begin{array}{c} d_l(u_{n-1},u_n), \dfrac{d_l(u_{n-1},S_iu_{n-1}).d_l(u_n,S_ju_n)}{1+d_l\left(u_{n-1},u_n\right)}, \\ d_l\left(u_{n-1},S_iu_{n-1}\right), d_l\left(u_n,S_ju_n\right) \end{array} \right\} \\ &= \max\left\{ \begin{array}{c} d_l(u_{n-1},u_n), \dfrac{d_l\left(u_{n-1},u_n\right).d_l(u_n,u_{n+1})}{1+d_l\left(u_{n-1},u_n\right)}, \\ d_l\left(u_{n-1},u_n\right), d_l\left(u_n,u_{n+1}\right) \end{array} \right\} \\ &= \max\{d_l\left(u_{n-1},u_n\right), d_l\left(u_n,u_{n+1}\right)\}. \end{aligned}$$

If, $D_l(u_{n-1},u_n) = d_l\left(u_n,u_{n+1}\right)$, then

$$F(d_l\left(u_n,u_{n+1}\right)) \leq F(d_l\left(u_n,u_{n+1}\right)) - \tau.$$

By (F1) it is a contradiction. So,

$$F(d_l\left(u_n,u_{n+1}\right)) \leq F(d_l\left(u_{n-1},u_n\right)) - \tau, \tag{2.3}$$

for every $i \in \mathbb{N} \cup \{0\}$. Likewise, we get

$$F(d_l\left(u_{n-1},u_n\right)) \leq F(d_l(u_{n-2},u_{n-1})) - \tau, \tag{2.4}$$

for every $i \in \mathbb{N} \cup \{0\}$. By (2.4) in (2.3), we get

$$F(d_l\left(u_n,u_{n+1}\right)) \leq F(d_l(u_{n-2},u_{n-1})) - 2\tau.$$

In the same way, we have

$$F(d_l\left(u_n,u_{n+1}\right)) \leq F(d_l(u_0,u_1)) - n\tau. \tag{2.5}$$

Taking limit $n \to \infty$, we get

$$\lim_{n\to\infty} F(d_l(u_n, u_{n+1})) = -\infty. \tag{2.6}$$

Since $F \in \triangle_F$,

$$\lim_{n\to\infty} d_l(u_n, u_{n+1}) = 0. \tag{2.7}$$

By (2.5), for every $n \in \mathbb{N}$ we get

$$\begin{aligned} &(d_l(u_n, u_{n+1}))^k((F(d_l(u_n, u_{n+1})) - F(d_l(u_0, u_1))) \\ &\leq -(d_l(u_n, u_{n+1}))^k n\tau \leq 0. \end{aligned} \tag{2.8}$$

Taking (2.6), (2.7) and applying limit $n \to \infty$ in (2.8), we get

$$\lim_{n\to\infty} (n(d_l(u_n, u_{n+1}))^k) = 0. \tag{2.9}$$

As (2.9) holds, there exists $n_1 \in \mathbb{N}$ such that $n(d_l(u_n, u_{n+1}))^k \leq 1$, for every $n \geq n_1$ or,

$$d_l(u_n, u_{n+1}) \leq \frac{1}{n^{\frac{1}{k}}}, \text{ for every } n \geq n_1. \tag{2.10}$$

By (2.10) and for $m > n > n_1$, we get

$$\begin{aligned} d_l(u_n, u_m) &\leq d_l(u_n, u_{n+1}) + d_l(u_{n+1}, u_{n+2}) + \cdots + d_l(u_{m-1}, u_m) \\ &= \sum_{i=n}^{m-1} d_l(u_i, u_{i+1}) \leq \sum_{i=n}^{\infty} d_l(u_i, u_{i+1}) \leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}. \end{aligned}$$

The convergence of the series $\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}$ entails $\lim_{n,m\to\infty} d_l(u_n, u_m) = 0$. Hence, $\{WS_\beta(u_n)\}$ is a Cauchy sequence in (W, d_l). Since (W, d_l) is a complete D.M.S, there exists $u \in W$ such that $\{WS_\beta(u_n)\} \to u$ that is

$$\lim_{n\to\infty} d_l(u_n, u) = 0. \tag{2.11}$$

Now, by Lemma 1.1, we have

$$\tau + F(d_l(u_{n+1}, S_j u)) \leq \tau + F(Hd_l(S_\gamma u_n, S_j u))). \tag{2.12}$$

As (2.1) holds for u, then we have

$$\tau + F(d_l(u_{n+1}, S_j u)) \leq F(D_l(x_n, u)), \tag{2.13}$$

where,

$$\begin{aligned} D_l(u_n, x) &= \max \left\{ \begin{array}{c} d_l(u_n, x), \frac{d_l(u_n, S_\gamma u_n).d_l(x, S_j x)}{1 + d_l(x_n, u)} , \\ d_l(u_n, S_\gamma u_n), d_l(x, S_j x) \end{array} \right\} \\ &= \max \left\{ \begin{array}{c} d_l(u_n, x), \frac{d_l(u_n, u_{n+1}).d_l(x, S_j x)}{1 + d_l(u_n, x)} , \\ d_l(u_n, u_{n+1}), d_l(x, S_j x) \end{array} \right\}. \end{aligned}$$

Applying limit $n \to \infty$, and by (2.11), we have

$$\lim_{n\to\infty} D_l(u_n, x) = d_l(x, S_j x). \tag{2.14}$$

Since F is strictly increasing, (2.13) implies

$$d_l(u_{n+1}, S_j x) < D_l(u_n, x).$$

Applying limit $n \to \infty$, and by (2.14), we have

$$d_l(x, S_j x) < d_l(x, S_j x).$$

It is a contradiction, hence $d_l(x, S_j x) = 0$ which implies that $x \in S_j x$. Hence $x \in \cap\{S_\beta : \beta \in \Omega\}$.

Now,

$$d_l(x, x) \le d_l(x, S_j x) + d_l(S_j x, x) \le 0.$$

This implies that $d_l(x, x) = 0$. □

Example 2.1. Let $W = [0, 10] \cap Q$ and $d_l(u, v) = u + v$. Then (W, d_l) is a complete D.M.S. Consider a family of multivalued mappings $\{S_\beta : \beta \in \Omega = a, 1, 2, \cdots\} : W \to P(W)$ and $\{WS_\beta(u_n) : \beta \in \Omega\}$ be a sequence in W generated by u_0. Define the mappings $S_n, S_a : W \to P(W)$ as follows:

$$S_n(u) = \left[\frac{1}{5n}u, \frac{2}{5n}u\right] \text{ for } n = 2, 3, \cdots, 14 \text{ and } S_a(u) = \left[\frac{1}{7}u, \frac{2}{7}u\right] \text{ for all } u \in W.$$

Define the function $F : \mathbb{R}^+ \to \mathbb{R}$ by $F(u) = \ln(u)$ for all $u \in \mathbb{R}^+, \tau > 0$, $\tau = \ln(1.4)$ and $u, v \in W$. Suppose that, $u_0 = 1$, then $d_l(1, S_a 1) = d_l\left(1, \left[\frac{1}{7}, \frac{2}{7}\right]\right) = 1 + \frac{1}{7} = \frac{8}{7}$. So $u_1 = \frac{1}{7}$. Now, $d_l(u_1, S_1 x_1) = d_l\left(\frac{1}{7}, S_1 \frac{1}{7}\right) = d_l\left(1, \left[\frac{1}{35}, \frac{2}{35}\right]\right) = \frac{6}{35}$.So $u_2 = \frac{1}{35}$. Now, $d_l(u_2, S_2 x_2) = d_l\left(\frac{1}{35}, S_2 \frac{1}{35}\right) = d_l\left(\frac{1}{35}, \left[\frac{1}{175}, \frac{2}{175}\right]\right) = \frac{6}{175}$.So $u_3 = \frac{1}{175}$. We define the sequence $\{WS_\beta(u_n)\} = \{1, \frac{1}{7}, \frac{1}{35}, \frac{1}{175}, \cdots\}$ in W generated by $u_0 = 1$. We have

$$\begin{aligned}
H_{d_l}(S_n u, S_a v) &= \max\left\{\sup_{x\in S_n u} d_l(x, S_a v), \sup_{y\in S_a v} d_l(S_n u, y)\right\} \\
&= \max\left\{\sup_{x\in S_n u} d_l\left(x, \left[\frac{v}{7}, \frac{2v}{7}\right]\right), \sup_{y\in S_a v} d_l\left(\left[\frac{u}{5n}, \frac{2u}{5n}\right], y\right)\right\} \\
&= \max\left\{d_l\left(\frac{2u}{5n}, \frac{v}{7}\right), d_l\left(\frac{u}{5n}, \frac{2v}{7}\right)\right\} \\
&= \max\left\{\frac{2u}{5n} + \frac{v}{7}, \frac{u}{5n} + \frac{2v}{7}\right\}
\end{aligned}$$

where

$$\begin{aligned} D_l(u,v) &= \max\left\{ d_l(u,v), \frac{d_l(u,\left[\frac{u}{5n},\frac{2u}{5n}\right]).d_l(y,\left[\frac{v}{7},\frac{2v}{7}\right]\ )}{1+d_l(u,v)}, \right. \\ &\qquad \left. d_l(u,\left[\frac{u}{5n},\frac{2u}{5n}\right]), d_l(v,\left[\frac{v}{5},\frac{2v}{5}\right]) \right\} \\ &= \max\left\{ d_l(u,v), \frac{d_l(u,\frac{u}{5n}).d_l(v,\frac{v}{7}\ )}{1+d_l(u,v)}, d_l(u,\frac{u}{5n}), d_l(v,\frac{v}{7}) \right\} \\ &= \max\left\{ u+v, \frac{(5nu+u)8v}{35n(1+u+v)}, \frac{5nu+u}{5n}, \frac{8v}{7} \right\} = u+v. \end{aligned}$$

Case i. If, $\max\left\{\frac{2u}{5n}+\frac{v}{7}, \frac{u}{5n}+\frac{2v}{7}\right\} = \frac{u}{5n}+\frac{2v}{7}$, and $\tau = \ln(1.4)$, then we have

$$\begin{aligned} \frac{7u}{n}+10v &\leq 25u+25v, \\ \frac{7}{5}(\frac{u}{5n}+\frac{2v}{7}) &\leq u+v, \\ \ln(1.4)+\ln(\frac{u}{5n}+\frac{2v}{7}) &\leq \ln(u+v), \end{aligned}$$

which implies that

$$\tau + F(H_{d_l}(S_n u, S_a v) \leq F(D_l(u,v)).$$

Case ii. Similarly, if $\max\left\{\frac{2u}{5n}+\frac{v}{7}, \frac{u}{5n}+\frac{2v}{7}\right\} = \frac{2u}{5n}+\frac{v}{7}$, and $\tau = \ln(1.4)$, then we have

$$\begin{aligned} \frac{14u}{n}+5v &\leq 25u+25v, \\ \frac{7}{5}(\frac{2u}{5n}+\frac{v}{7}) &\leq u+v, \\ \ln(1.4)+\ln(\frac{2u}{5n}+\frac{v}{7}) &\leq \ln(u+v). \end{aligned}$$

Hence,

$$\tau + F(H_{d_l}(S_n u, S_a v) \leq F(D_l(u,v).$$

Since all the conditions of Theorem 2.1 are fulfilled, hence $\{S_\beta : \beta \in \Omega\}$ have a common fixed point.

Let (W, d_l) be a D.M.S, $u_0 \in W$ and S be a multivalued mapping from W to $P(W)$. Let $u_1 \in Su_0$ be such that $d_l(u_0, Su_0) = d_l(u_0, u_1)$. Let $u_2 \in Su_1$ be such that $d_l(u_1, Su_1) = d_l(u_1, u_2)$. In the same way, we can construct a sequence u_n of points in W such that $u_{n+1} \in Su_n$, $d_l(u_n, Su_n) = d_l(u_n, u_{n+1})$. This iterative sequence is denoted by $\{WS(u_n)\}$ and called $\{WS(u_n)\}$ is a sequence generated by u_0 in W.

Corollary 2.1. *Let (W, d_l) be a complete D.M.S and $S : W \to P(W)$ be a multivalued mapping such that*

$$\tau + F(H_{d_l}(Su, Sv)) \leq F(D_l(u, v)), \tag{2.17}$$

for all $u, v \in \{SS(u_n)\}$, where $F \in \triangle_F$ and $\tau > 0$, and

$$D_l(u, v) = \max\left\{d_l(u, v), \frac{d_l(u, Su).d_l(v, Sv)}{1 + d_l(u, v)}, d_l(u, Su), d_l(v, Sv)\right\}.$$

Then $\{SS(u_n)\} \to x \in W$. Furthermore, if (2.17) also holds, then S has a fixed point x in W with $d_l(x, x) = 0$.

Theorem 2.2. *Let (W, d_l) be a complete D.M.S and $\{S_\beta : \beta \in \Omega\}$ be a family of Ciric type rational multivalued $F-$contraction. If $F \in \triangle_F$ and $\tau \in \mathbb{R}^+$ such that*

$$\tau + F(H_{d_l}(S_i u, S_j v)) \leq F\left(\begin{array}{c} b_1 d_l(u, v) + b_2 d_l(u, S_i u) + b_3 d_l(v, S_j v) \\ +b_4 \dfrac{d_l^2(u, S_i u).d_l(v, S_j v)}{1 + d_l^2(u, v)} \end{array}\right), \tag{2.18}$$

for all $u, v \in \{WS_\beta(u_n)\}$, with $u \neq v$ where $b_1, b_2, b_3, b_4 > 0$, $b_1+b_2+b_3+b_4 = 1$ and $b_3 + b_4 \neq 1$. Then $\{WS_\beta(u_n)\} \to x \in W$. Furthermore, if (2.18) also holds for x, then there is a common fixed point for the family of multivalued mappings $\{S_\beta : \beta \in \Omega\}$ in $\{WS_\beta(u_n) : \beta \in \Omega\}$.

Proof. As $u_1 \in S_i u_0$ and $u_2 \in S_j u_1$, by using Lemma 1.1

$$\begin{aligned}
\tau + F(d_l(u_1, u_2)) &= \tau + F(d_l(S_i u_0, S_j u_1)) \leq \tau + F(H_{d_l}(S_i u_0, S_j u_1)) \\
&\leq F\left(\begin{array}{c} b_1 d_l(u_0, u_1) + b_2 d_l(u_0, u_1) + b_3 d_l(u_1, Tu_1) \\ +b_4 \dfrac{d_l^2(u_0, S_i u_0).d_l(u_1, S_j u_1)}{1 + d_l^2(u_0, u_1)} \end{array}\right) \\
&\leq F\left(\begin{array}{c} b_1 d_l(u_0, u_1) + b_2 d_l(u_0, u_1) + b_3 d_l(u_1, u_2) \\ +b_4 d_l(u_1, S_j u_1)\left(\dfrac{d_l^2(u_0, u_1)}{1 + d_l^2(u_0, u_1)}\right) \end{array}\right) \\
&\leq F((b_1 + b_2)d_l(u_0, u_1) + (b_3 + b_4)d_l(u_1, u_2)).
\end{aligned}$$

Since F is strictly increasing, we get

$$\begin{aligned}
d_l(u_1, u_2) &< (b_1 + b_2)d_l(u_0, u_1) + (b_3 + b_4)d_l(u_1, u_2) \\
&< \left(\frac{b_1 + b_2}{1 - b_3 - b_4}\right) d_l(u_0, u_1).
\end{aligned}$$

From $b_1 + b_2 + b_3 + b_4 = 1$ and $b_3 + b_4 \neq 1$, we deduce $1 - b_3 - b_4 > 0$ and so

$$d_l(u_1, u_2) < d_l(u_0, u_1).$$

Consequently

$$F(d_l(u_1,u_2)) \leq F(d_l(u_0,u_1)) - \tau.$$

As we have $u_{2i+1} \in S_i u_{2i}$ and $u_{2i+2} \in S_j u_{2i+1}$ then from (2.18), and Lemma 1.1 we have

$$\begin{aligned}
&\tau + F(d_l(u_{2i+1},u_{2i+2})) \\
&= \tau + F(d_l(u_{2i+1},S_j u_{2i+1})) \leq \tau + F(H_{d_l}(S_i u_{2i},S_j u_{2i+1})) \\
&\leq F(b_1 d_l(u_{2i},u_{2i+1}) + b_2 d_l(u_{2i},S_i u_{2i}) \\
&+ b_3 d_l(u_{2i+1},S_j u_{2i+1}) + b_4 \frac{d_l^2(u_{2i},S_i u_{2i}).d_l(u_{2i+1},S_j u_{2i+1})}{1+d_l^2(u_{2i},u_{2i+1})} \\
&\leq F(b_1 d_l(u_{2i},u_{2i+1}) + b_2 d_l(u_{2i},u_{2i+1}) + b_3 d_l(u_{2i+1},u_{2i+2}) \\
&+ \frac{b_4 d_l(u_{2i+1},u_{2i+2}).d_l^2(u_{2i},u_{2i+1})}{1+d_l^2(u_{2i},u_{2i+1})}) \\
&\leq F(b_1 d_l(u_{2i},u_{2i+1}) + b_2 d_l(u_{2i},u_{2i+1}) + b_3 d_l(u_{2i+1},u_{2i+2}) + b_4 d_l(u_{2i+1},u_{2i+2})).
\end{aligned}$$

Since F is strictly increasing, and $b_1+b_2+b_3+b_4 = 1$ where $b_3+b_4 \neq 1$, we deduce $1-b_3-b_4 > 0$ so we obtain

$$\begin{aligned}
d_l(u_{2i+1},u_{2i+2}) &< b_1 d_l(u_{2i},u_{2i+1}) + b_2 d_l(u_{2i},u_{2i+1}) + b_3 d_l(u_{2i+1},u_{2i+2}) \\
&\quad + b_4 d_l(u_{2i+1},u_{2i+2})) \\
&< (b_1+b_2) d_l(u_{2i},u_{2i+1}) + (b_3+b_4) d_l(u_{2i+1},u_{2i+2}) \\
d_l(u_{2i+1},u_{2i+2}) &< \left(\frac{b_1+b_2}{1-b_3-b_4}\right) d_l(u_{2i},u_{2i+1}) = d_l(u_{2i},u_{2i+1}).
\end{aligned}$$

This implies that,

$$F(d_l(u_{2i+1},u_{2i+2})) \leq F(d_l(u_{2i},u_{2i+1})) - \tau.$$

Following similar arguments as given in Theorem 2.2, we have $\{WS_\beta(u_n)\} \to x$ that is

$$\lim_{n\to\infty} d_l(u_n,x) = 0. \tag{2.19}$$

Now, by Lemma 1.1, we have

$$\tau + F(d_l(u_{2n+1},S_j x)) \leq \tau + F(H d_l(S_i u_{2n},S_j x)).$$

By using inequality (2.18), we have

$$\begin{aligned}
\tau + F(d_l(u_{2n+1},S_j x)) &\leq F(b_1 d_l(u_{2n},x) + b_2 d_l(u_{2n},S_i u_{2n}) + b_3 d_l(x,S_j x) \\
&\quad + b_4 \frac{d_l^2(u_{2n},S_i u_{2n}).d_l(x,S_j x)}{1+d_l^2(u_{2n},x)}) \\
&\leq F(b_1 d_l(u_{2n},x) + b_2 d_l(u_{2n},u_{2n+1}) + b_3 d_l(x,S_j x) \\
&\quad + b_4 \frac{d_l^2(u_{2n},u_{2n+1}).d_l(x,S_j x)}{1+d_l^2(u_{2n},x)}).
\end{aligned}$$

As F is strictly increasing, we get

$$\begin{aligned} d_l(u_{2n+1}, S_j x) &< b_1 d_l(u_{2n}, x) + b_2 d_l(u_{2n}, u_{2n+1}) + b_3 d_l(x, S_j x) \\ &\quad + b_4 \frac{d_l^2(u_{2n}, u_{2n+1}).d_l(x, S_j x)}{1 + d_l^2(u_{2n}, x)}. \end{aligned}$$

Applying limit $n \to \infty$, and using (2.19), we have

$$d_l(x, S_j x) < b_3 d_l(x, S_j x).$$

It is a contradiction, hence $d_l(x, S_j x) = 0$ or $x \in S_j x$. Similarly by Lemma 1.1, using (2.18),(2.19), and the inequality

$$\tau + F(d_l(u_{2n+2}, S_i x)) \leq \tau + F(H_{d_l}(S_j u_{2n+1}, S_i x))$$

we can show that $d_l(x, S_i x) = 0$ or $x \in S_i x$. Hence the S_i and S_j have a common fixed point x in (W, d_l). Now,

$$d_l(x, x) \leq d_l(x, S_j x) + d_l(S_j x, x) \leq 0.$$

This implies that $d_l(x, x) = 0$. □

Corollary 2.2. *Let (W, d_l) be a complete D.M.S and $S : W \to P(W)$ be the multivalued mappings. If $F \in \triangle_F$ and $\tau \in \mathbb{R}_+$ such that*

$$\begin{aligned} &\tau + F(H_{d_l}(Su, Sv)) \\ &\leq F\left(b_1 d_l(u, v) + b_2 d_l(u, Su) + b_3 d_l(v, Sv) + b_4 \frac{d_l^2(u, Su).d_l(v, Sv)}{1 + d_l^2(u, v)}\right), \end{aligned} \tag{2.20}$$

for all $u, v \in \{SS(u_n)\}$, with $u \neq v$ where $b_1, b_2, b_3, b_4 > 0$, $b_1 + b_2 + b_3 + b_4 = 1$ and $b_3 + b_4 \neq 1$, then $\{SS(u_n)\} \to u \in W$. Furthermore, if for $x \in W$ (2.20) holds, then $x \in W$ is a fixed point for S and $d_l(x, x) = 0$.

3. Single valued result with application to system of integral equations

Definition 3.1. *Let (W, d_l) be a complete D.M.S and $\{S_\beta : \beta \in \Omega\}$ be a family of mapping from W to W and $\{WS_\beta(x_n) : \beta \in \Omega\}$ be a sequence in W generated by x_0. The family $\{S_\beta : \beta \in \Omega\}$ is called to be Ciric type rational $F-$contraction, if for all $x, y \in \{WS_\beta(x_n)\}$, we have*

$$\tau + F(d_l(S_i x, S_j v)) \leq F(D_l(x, v)) \tag{3.1}$$

where $F \in \triangle_F$ and $\tau > 0$, and

$$D_l(x, v) = \max\left\{d_l(x, v), \frac{d_l(x, S_i x).d_l(y, S_j y)}{1 + d_l(x, y)}, d_l(x, S_i x), d_l(y, S_j y)\right\}. \tag{3.2}$$

Theorem 3.1. *Let (W, d_l) be a complete D.M.S and $\{S_\beta : \beta \in \Omega\}$ be a family of self mapping of Ciric type rational $F-$contraction. Then $\{S_\beta : \beta \in \Omega\}$ have a common fixed point x in W and $d_l(x, x) = 0$.*

Proof. The proof of this theorem is same as given for Theorem 2.1 We have to prove the uniquness only. Let v be another common fixed point of S. Suppose $d_l(S_i x, S_j y) > 0$ then we have

$$\begin{aligned}
\tau + F(d_l(S_i x, S_j y)) &\leq F(D_l(x, y)) \\
&= F(\max\left\{ d_l(x, y), \frac{d_l\,(x, S_i x)\,.d_l\,(y, S_j y)}{1 + d_l\,(x, y)}, \right. \\
&\quad \left. d_l(x, S_i x), d_l(y, S_j y)\right\} \\
&= F(\max\{d_l(x, y)\}), \\
F(d_l(S_i x, S_j y)) &< d_l(x, y), \\
d_l(x, y) &< d_l(x, y)
\end{aligned}$$

which is a contradiction so $d_l(S_i x, S_j y) = 0$. Hence $x = y$.

In this section, we discuss the application of fixed point Theorem 3.1 in form of unique solution of a family of Volterra type integral equations given as:

$$p(a) = \int_0^a T_i(a, z, u(z))dz + f(a), \tag{3.3}$$

for all $a \in [0, 1]$, $i \in \Omega$ and T_i be the mapping on $[0, 1] \times [0, 1] \times C([0, 1], \mathbb{R}_+) \to \mathbb{R}$. We find the solution of (3.3). Let $W = C([0, 1])$ be the set of all real continuous functions on $[0, 1]$. For $u \in C([0, 1], \mathbb{R}_+)$, we define:

$$\|p\|_\tau = \max\left\{ \sup_{a\in[0,1]} |p(a)|\, e^{-\tau a} \right\},$$

where $\tau > 0$. Then,

$$d_\tau(p, q) = \max \sum \left\{ \sup_{a\in[0,1]} \| \, |p(a) + q(a)|\, e^{-\tau a} \|_\tau \right\},$$

for all $p, q \in C([0, 1], \mathbb{R}_+)$. With these setting, $C([0, 1], \mathbb{R}, \|\cdot\|_\tau)$ becomes complete D.M.S. □

Now we prove the following theorem to ensure the existence of solution of integral equations.

Theorem 3.2. *Assume the following conditions are satisfied:*

(i) $T_i : [0, 1] \times [0, 1] \times \mathbb{R}_+ \to \mathbb{R}_+$ *and* $f, g : [0, 1] \to \mathbb{R}$ *are continuous;*

(ii) Define

$$S_\alpha u(a) = \int_0^a T_i(a, z, u(z))dz + f(a).$$

Suppose there exists $\tau > 1$, *such that*

$$|T_i(a,z,u) + T_j(a,z,v)| \leq \tau e^{-\tau}[N(u,v)]$$

for all $a, z \in [0,1]$ *and* $u, v \in C([0,1],\mathbb{R})$, *where*

$$N(u,v) = \max \left\{ \begin{array}{c} |u(a)+v(a)|, \dfrac{|u(a)+S_\alpha u(a)|\,.\,|v(a)+S_\beta v(a)|}{1+|u(a)+v(a)|}, \\ |u(a)+S_\alpha u(a)|, |v(a)+S_\beta v(a)| \end{array} \right\}.$$

Then integral equation (3.3) *has a solution.*

Proof. By assumption (ii)

$$\begin{aligned} |S_\alpha u(a) + S_\beta v(a)| &= \int_0^a |T_i(a,z,u(z)) + T_j(a,z,v(z))|\, dz \\ &\leq \int_0^a \tau e^{-\tau}([N(u,v)]e^{-\tau z})e^{\tau z} dz \\ &\leq \int_0^a \tau e^{-\tau}\|N(u,v)\|_\tau e^{\tau z} dz \\ &\leq \tau e^{-\tau}\|N(u,v)\|_\tau \int_0^a e^{\tau z} dz \\ &\leq \tau e^{-\tau}\|N(u,v)\|_\tau \frac{1}{\tau} e^{\tau a} \\ &\leq e^{-\tau}\|N(u,v)\|_\tau e^{\tau a}. \end{aligned}$$

This implies

$$|S_\alpha u(a) + S_\beta v(a)|\, e^{-\tau a} \leq e^{-\tau}\|N(u,v)\|_\tau.$$

That is

$$\|S_\alpha u(a) + S_\beta v(a)\|_\tau \leq e^{-\tau}\|N(u,v)\|_\tau,$$

which further implies

$$\tau + \ln \|S_\alpha u(a) + S_\beta v(a)\|_\tau \leq \ln \|N(u,v)\|_\tau.$$

Since all the conditions of Theorem (3.1) are fullfilled. So, integral equation given in (3.3) has a unique common solution □

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Accepted: April 06, 2020

Characterizations of FI-lattices associated to implicative derivations

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Abstract. In this paper, we discuss related properties of implicative derivations and give some characterizations of them in FI-lattices. Then we show that every implicative derivation on regular FI-lattice is principle. Finally we prove that every Boolean algebra is isomorphic to the the algebra of all implicative derivations.
Keywords: algebras of fuzzy logic, FI-lattice, implicative derivation, Boolean algebra.

1. Introduction

In the past decades, fuzzy logical algebras and their axiomatization have become important topics in theoretical research and in the applications of fuzzy logic. It is well known that the fuzzy implication connective plays a crucial role in fuzzy logic and reasoning. Various authors studied fuzzy implications from different

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perspectives. Seeking the common properties of some important fuzzy implications used in fuzzy logic is meaningful. Consequentially, Wu [9] introduced a class of fuzzy logical algebras called fuzzy implication algebras, FI-algebras for short, in 1990. After then, various interesting results of FI-algebras, regular FI-algebras [6], commutative FI-algebras [10] were reported. In 2011, Pei [8] introduced a new type of FI-algebras, called fuzzy implication lattices, FI-lattices for short, and studied some results.

The notion of derivations, introduced from the analytic theory, is helpful for studying algebraic structures and properties in algebraic systems. In 1957, Posner [7] introduced the notion of derivations in a prime ring and character the commutative prime rings by them. Inspired by derivations on rings, Jun et al [2, 5] applied the notion of derivations to BCI-algebras and gave some characterizations of p-semisimple BCI-algebras. In 2008, Xin [12, 13] introduced the concept of derivations in a lattice and also characterized modular lattices and distributive lattices by isotone derivations. Subsequently, Alshehri et al [1, 3] applied the notions of derivations to a MV-algebra and proved that an additive derivation of a linearly ordered MV-algebra is isotone. In 2016, He [4] investigated the notions of multiplicative derivation and implicative derivation in residuated lattices, and characterized Heyting algebras in terms of the above derivations. In 2019, Wang [11] introduced the notion of implicative derivations on FI-lattices and studied some of their basic properties.

In this paper, we further study the implicative derivations of FI-lattices and obtain some new results. Indeed, we will obtain the following main results:

(1) Every implicative derivation is principle on regular FI-lattices. (See **Theorem 3.9**).

(2) The set of all implicative derivations on a distributive DFI-lattice also forms a distributive DFI-lattice (See **Theorem 4.2**).

(3) Every Boolean algebra is isomorphic to the algebra of all implicative derivations on Boolean algebras (See **Theorem 4.4**).

The paper is organized as follows: In Section 2, we review some basic definitions and results about FI-lattices. In Section 3, we review the notion of multiplicative derivations on FI-lattices and give some characterizations of them. In Section 4, we characterize the algebraic structure of kinds of FI-lattices by the set of all implicative derivations.

2. Preliminaries

In this section, we summarize some definitions and results about FI-lattices, which will be used in the following sections.

Definition 2.1 ([8]). *An algebra* $(L, \wedge, \vee, \to, 0, 1)$ *of type* $(2, 2, 2, 0, 0)$ *is called a* FI-lattice *if it satisfies the following conditions:*

(1) $(L, \wedge, \vee, 0, 1)$ *is a bounded lattice,*

(2) $p \to (q \to w) = q \to (p \to w)$,

(3) $(p \to q) \to ((q \to w) \to (p \to w)) = 1$,
(4) $p \to p = 1$,
(5) *if* $p \to p = 1$ *and* $q \to p = 1$, *then* $p = q$,
(6) $0 \to p = 1$,
(7) $p \to (p \vee w) = (p \to q) \vee (p \to w)$,
for any $p, q, w \in L$.

In what follows, by L we denote the universe of a FI-lattice $(L, \wedge, \vee, \to, 0, 1)$. On each L, we define

$$p \leq q \text{ iff } p \to q = 1.$$

Then $\leq$ is a partial order relation on L and for all $p \in L, 0 \leq p \leq 1$.

A FI-lattice L is called a DFI-lattice if it satisfies the D condition:

$$\text{(D)} \quad p \to (q \wedge w) = (p \to q) \wedge (p \to w).$$

A FI-lattice L is called a distributive FI-lattice if it satisfies the left self-distributivity law with respect to $\to$:

$$\text{(SD)} \quad p \to (q \to w) = (p \to q) \to (p \to w).$$

A FI-lattice L is said to be regular if it satisfies the double negation property:

$$\text{(DNP)} \quad \neg\neg p = p.$$

Notice that every regular FI-lattice is a regular residuated lattice, in which

$$p \odot q = \neg(p \to \neg q).$$

Proposition 2.2 ([6, 8, 9, 10])**.** *In any FI-lattice, the following properties hold: for all* $p, q, w \in L$,
(1) *if* $p \leq q$, *then* $q \to w \leq p \to w$, $w \to p \leq w \to q$,
(2) $p \to (q \to p) = 1$,
(3) $1 \to p = p$,
(4) $((p \to q) \to q) \to q = p \to q$,
(5) $(p \vee q) \to w = (p \to w) \wedge (q \to w)$.

3. Some further results on derivations of FI-lattices

In this section, we further study implicative derivations in FI-lattices and give some characterizations of them.

Definition 3.1 ([11])**.** *Let* L *be a FI-lattice. A map* $I : L \longrightarrow L$ *is called an* implicative derivation *on* L *if it satisfies the following condition:*

$$I(p \to q) = (I(p) \to q) \vee (p \to I(q)),$$

for any $p, q \in L$.

We will denote by $I(L)$ be the set of all implicative derivations of L.

Now, we present some examples for implicative derivations on FI-lattices.

Example 3.2. Define a map on L by $1_I : L \to L$ $1_I(p) = 1$ for all $p \in L$. Then 1_I is an implicative derivation on L, which is called the one implicative derivation. Moreover, we define a map $I_1 : L \to L$ by $I_1(p) = p$ for all $p \in L$. Then I_1 is also an implicative derivation on L, which is called the identity implicative derivation.

Example 3.3. Let $L = \{0, p, q, w, 1\}$ be a chain. Defining operation $\to$ is as follows:

$\to$	0	p	q	w	1
0	1	1	1	1	1
p	q	1	1	1	1
q	p	p	1	1	1
w	0	p	q	1	1
1	0	p	q	w	1

Then $(L, \wedge, \vee, \to, 0, 1)$ is a FI-lattice. Now, we define $I : L \to L$ as follow:

$$I(x) = \begin{cases} 0, & x = 0, \\ p, & x = p, \\ q, & x = q, \\ 1, & x = w, 1 \end{cases},$$

It is verified that I is an implicative derivation on L.

Proposition 3.4. *Let L be a FI-lattice and I be an implicative derivation on L. Then we have: for any $p, q \in L$,*

(1) $I(1) = 1$,
(2) $x \leq I(p)$,
(3) $I(p) \wedge I(q) \leq I(p \to q)$,
(4) $I(p) \to q \leq p \to I(q)$,
(5) $I(p \to q) = p \to I(q)$,
(6) $I(p \to q) \geq I(p) \to I(q)$.

Proof. For (1) – (3) see ([11], Proposition 2.1), we only to show (4), (5) and (6).

(4) It follows from (2) and Proposition 2.2(1).

(5) From (4), we have

$$\begin{aligned} I(p \to q) &= (I(p) \to q) \vee (p \to I(q)) \\ &= p \to I(q), \end{aligned}$$

for any $p, q \in L$.

(6) It follows from (2) and Proposition 2.2(1). □

Theorem 3.5 ([11]). *Let L be a FI-lattice and $I : L \to L$ be a map on L. Then the following statements are equivalent:*

(1) *I is an implicative derivation on L,*

(2) *$I(p \to q) = p \to I(q)$ for any $p, q \in L$.*

Remark 3.6. It is verified that the map $I_a : L \to L$, which defined by

$$I_a(x) = a \to p$$

for any $p \in L$, is also an implicative derivation on L, which is called the principle implicative derivation. Indeed,

$$\begin{aligned} I_a(p \to q) &= a \to (p \to q) \\ &= p \to (a \to q) \\ &= p \to I_a(q), \end{aligned}$$

for any $p, q \in L$.

Inspired by Remark 3.6, it is natural to ask that **whether any implicative derivation I can be represent as the form of I_a.**

The following example shows that the above assertion not hold in general FI-lattices.

Example 3.7. Let $L = \{0, p, q, 1\}$ be a chain. Defining operation $\to$ is as follows:

$\to$	0	p	q	1
0	1	1	1	1
p	0	1	1	1
q	0	p	1	1
1	0	p	q	1

Then $(L, \wedge, \vee, \to, 0, 1)$ is a FI-lattice. Now, we define $I : L \to L$ as follow:

$$I(x) = \begin{cases} p, & x = 0 \\ q, & x = q \\ 1, & x = p, 1 \end{cases},$$

It is verified that I is an implicative derivation on L. But

$$I_0(0) = 1 \neq p,\ I_p(0) = I_q(0) = I_1(0) = 0 \neq p.$$

So I is not represent as the form of I_a, for any $a \in L$.

However, we give the positive answer under certain conditions in regular FI-lattices.

Theorem 3.8. *Let L be a regular FI-lattice and I be an implicative derivation on L. Then the following statements are equivalent:,*

(1) *I is an implicative derivation on L,*

(2) *$I(p) = \neg I(0) \to p$ for any $p \in L$.*

Proof. (1) For any $p \in L$, by Theorem 3.5, we have

$$\begin{aligned} I(p) &= I(\neg\neg p) \\ &= I(\neg p \to 0) \\ &= \neg p \to I(0) \\ &= \neg I(0) \to p, \end{aligned}$$

for any $p \in L$.

(2) The proof is similar to that of Remark 3.6. □

Remark 3.9. Theorem 3.8 shows that every implicative derivation I is completely determined by the element $\neg I(0)$ on regular FI-lattices.

4. Characterizations of FI-lattices based on implicative derivations

In this section, we focus on algebraic structure of the set of all derivations and give some representations of subclasses of FI-lattices by them.

Theorem 4.1. *Let L be a DFI-lattice. Then $(I(L), \cap, \cup, I_1, 1_I)$ is a bounded lattice, where*

$$\begin{aligned} (I_i \cap I_j)(p) &= I_i(p) \wedge I_j(p), \\ (I_i \cup I_j)(p) &= I_i(p) \vee I_j(p), \end{aligned}$$

for any $I_i, I_j \in I(L)$, and $p \in L$.

Proof. For any $I_i, I_j \in I(L)$, and $p \in L$, we have

$$\begin{aligned} (I_i \cap I_j)(p \to q) &= I_i(p \to q) \wedge I_j(p \to q) \\ &= (p \to I_i(q)) \wedge (p \to I_j(q)) \\ &= p \to (I_i(q) \wedge I_j(q)) \\ &= p \to (I_i \cap I_j)(q), \end{aligned}$$

and

$$\begin{aligned} (I_i \cup I_j)(p \to q) &= I_i(p \to q) \vee I_j(p \to q) \\ &= (p \to I_i(q)) \vee (p \to I_j(q)) \\ &= p \to (I_i(q) \vee I_j(q)) \\ &= p \to (I_i \cup I_j)(q), \end{aligned}$$

which implies $I_i \cap I_j, I_i \cup I_j \in I(L)$.

Also, for any $I_i \in I(L)$ and $p \in L$, we have

$$(I_i \cap 1_I)(p) = I_i(p) \wedge 1_I(p) = I_i(p) \wedge 1 = I_i(p),$$
$$(I_i \cup 1_I)(p) = I_i(p) \vee 1_I(p) = I_i(p) \vee 1 = I_1(p)$$

which implies $I_i \cap 1_I = 1_I, I_i \cup 1_I = 1_I$.

Then $(I(L), \cap, \cup, I_1, 1_I)$ is a bounded lattice. □

Theorem 4.2. *Let L be a distributive DFI-lattice. Then $(I(L), \cap, \cup, \Rightarrow, I_1, 1_I)$ is also a distributive DFI-lattice , where*

$$(I_i \cap I_j)(p) = I_i(p) \wedge I_j(p),$$
$$(I_i \cup I_j)(p) = I_i(p) \vee I_j(p),$$
$$(I_i \Rightarrow I_j)(p) = I_i(p) \to I_j(p)$$

for any $I_i, I_j \in I(L)$, and $p \in L$.

Proof. Theorem 4.1 shows that $(I(L), \cap, \cup, I_1, 1_I)$ is a bounded lattice if L is a DFI-lattice. Now, we prove that $(I(L), \cap, \cup, I_1, 1_I)$ is a distributive DFI-lattice if L is a distributive DFI-lattice. Indeed, for any $I_i, I_j \in I(L)$ and $p \in L$, we have

$$\begin{aligned}(I_i \Rightarrow I_j)(p \to q) &= I_i(p \to q) \to I_j(p \to q)\\ &= (p \to I_i(q)) \to (p \to I_j(q))\\ &= p \to (I_i(q) \to I_j(q))\\ &= p \to (I_i \Rightarrow I_j)(q),\end{aligned}$$

which implies $I_i \Rightarrow I_j \in I(L)$.

So the operations $\cap, \cup, \Rightarrow$ are well defined if L is a distributive DFI-lattice.

Therefore $(I(L), \cap, \cup, \Rightarrow, I_1, 1_I)$ is also a distributive DFI-lattice. □

As a consequence of Theorems 4.1 and 4.2, we have the following result.

Theorem 4.3. *Let $(L, \wedge, \vee, \neg, 0, 1)$ be a Boolean algebra (or a regular distributive DFI-lattice). Then $(I(L), \cap, \cup, \star, I_1, 1_I)$ is also a Boolean algebra , where*

$$(I_i \cap I_j)(p) = I_i(p) \wedge I_j(p),$$
$$(I_i \cup I_j)(p) = I_i(p) \vee I_j(p),$$
$$(I_i)^\star(p) = I_i(p) \Rightarrow I_1(p),$$

for any $I_i, I_j \in I(L)$, and $p \in L$.

Proof. It follows from Theorem 4.1 that $(I(L), \cap, \cup, I_1, 1_I)$ is a distributive DFI-lattice if L is a distributive DFI-lattice. Now, we further prove that is a Boolean algebra if L is a Boolean algebra. Indeed, for any $I_i \in I(L)$, and $p \in L$,

we have

$$\begin{aligned}
(I_i \cap (I_i)^\star)(p) &= I_i(p) \wedge (I_i(p))^\star \\
&= I_i(p) \wedge (I_i(p) \to I_1(p)) \\
&= I_i(p) \wedge I_1(p) \\
&= I_1(p), \\
(I_i \cup (I_i)^\star)(p) &= I_i(p) \vee (I_i(p) \to I_1(p)) \\
&= I_i(p) \vee I_i(p))^\star \vee I_1(p) \\
&= 1 \vee I_1(p) \\
&= 1 \\
&= 1_I(p),
\end{aligned}$$

that is, $I_i \cap (I_i)^\star = I_1$ and $I_i \cup (I_i)^\star = 1_I$. □

As an application of Theorems 4.1-4.3, we give a representation of Boolean algebra by derivations.

Theorem 4.4. *Every Boolean algebra L is isomorphic to $(I(L), \cap, \cup, \star, I_1, 1_I)$.*

Proof. Let $\chi : L \to I(L)$ be defined by

$$\chi(a)(p) = a \vee p,$$

for any $a, p \in L$. Then it follows from Theorem 3.8 that χ is well defined.

(1) If $\chi(a) = \chi(b)$, then $\chi(a)(p) = \chi(b)(p)$, and hence $a \vee p = b \vee p$ for all $p \in L$. Now, if $p = a$, then $a = a \vee a = a \vee b$, that is, $b \leq a$. If $p = b$, then $a \vee b = b \vee b = b$, and hence $a \vee b = b$, that is, $a \leq b$. So $a = b$, which shows that χ is an injective function.

(2) For any $I \in I(L)$, there exists a $I(0) \in L$ such that $I = \chi((\neg I(0)))$, which implies that χ is a surjection function. Indeed, by Theorem 3.8(2), we have $I(p) = \neg p \to I(0) = p \vee I(0) = \chi(\neg(I(0)))(p)$, for any $p \in L$.

(3) For any $a, b \in L$, we have

$$\begin{aligned}
\chi(a \wedge b)(p) &= (a \wedge b) \vee p \\
&= (a \vee p) \wedge (b \vee p) \\
&= (\chi(a) \cap \chi(b))(p), \\
\chi(a \vee b)(p) &= (a \vee b) \wedge p \\
&= (a \wedge p) \vee (b \wedge p) \\
&= (\chi(a) \cup \chi(b))(p), \\
\chi(\neg a)(p) &= \neg a \vee p \\
&= a \to p \\
&= (a \to p) \wedge (p \to p) \\
&= (a \vee p) \to p \\
&= (\chi(a)(p))^\star.
\end{aligned}$$

which implies that χ is an isomorphism.

Therefore $(L, \wedge, \vee, \neg, 0, 1)$ is isomorphic to $(I(L), \cap, \cup, \star, I_1, 1_I)$. □

5. Conclusions

The notion of derivations is helpful for studying structures and properties in algebraic systems. In this paper, we review the implicative derivations on FI-lattices and give some characterizations of them. We also character Boolean algebras by implicative derivations. In our further work, we will introduce the derivations on FI-algebras and obtain some more general results of derivations on fuzzy logical algebras.

Acknowledgments

This study was funded by a grant of National Natural Science Foundation of China (12001423,11961016,12001413,11926503) and the Natural Science Basic Research Plan in Shaanxi Province of China (2020JQ-762,2019JQ-869, 2021JQ-579) and Natural Science Foundation of Education Committee of Shannxi Province (20JK0626).

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Accepted: June 19, 2020

On purity and divisibility of S-acts

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Abstract. In this paper we study purity of S-acts where S is a special type of monoid of the form $S = G \dot{\cup} I$, in which G is a group and I is an ideal of S. Here every S-act can be naturally considered as an I^1-act. So the question is that what is the relationship between purity and divisibility of I^1-acts and purity and divisibility of S-acts?

We respond to this question and show that divisibility of an S-act A is extendable from I^1-acts to S-acts. Also, we show that every consistent system of the third type of equations has a solution in A whenever every consistent system of the third type of equations over I^1-act A has a solution in A.

Keywords: S-act, purity, divisibility.

1. Introduction and preliminaries

The notion of acts of a semigroup or a monoid over a set is one of the very useful notions in many branches of mathematics as well as in computer science which captures the interest of many mathematicians and computer scientists, [2, 5, 6, 7, 8, 9, 11].

Nowadays purity plays a role in different branches of mathematics such as Module Theory, Model Theory, and Category Theory, see [1, 17, 3]. Especially this notion has been studied on G-acts, acts over a monoid or a group G by some authors including Banaschewski, Gould, and Normak [3, 16, 9, 10].

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Divisible act first introduced by Feller and Gantos [6], and later studied by many other authors, see [2, 5, 6, 12, 18].

A. Golchin and J. Renshaw in [7, 8], have studied on acts over a monoid of the form $S = G\dot{\cup}I$, in which G is a group and I is an ideal of S. They show that flatness for these kinds of actions is extendable from I^1-act to S-acts. Meaning that, an S-act A is flat if it is flat as an I^1-act. Thus it is a natural question to ask that: which algebraic properties are extendable from I^1-acts to S-acts, when $S = G\dot{\cup}I$?

Here we answer to this question concerning purity and divisibility. Indeed, we show that divisible property is extendable from I^1-acts to S-acts in general. Also, we show that every consistent system of the third type of equations has a solution in A whenever every consistent system of the third type of equations over I^1-act A has a solution in A. To do so, first we briefly recall some necessary notions.

Given a monoid S, a (right) S-act is a set A together with an action $A\times S \to A$, mapping each (a, s) to as, such that $(i)\,(as)t = a(st)$ and $(ii)\,a1 = a$, for every $a \in A$, $s, t \in S$, in which 1 is the identity element of the monoid S. One can naturally consider the monoid S as an S-act. A subset B of an S-act A is called an *S-subact* of A, denoted by $B \leq A$, whenever $bs \in B$, for every $b \in B$ and $s \in S$. Especially, considering naturally S as an S-act, the S-subacts of S are exactly the right ideals of S. A map $f : A \to B$ between two S-acts A and B is called an *S-homomorphism* or briefly a *homomorphism* if, for each $a \in A$, $s \in S, f(as) = f(a)s$. The usual definitions for *S-monomorphisms*, *S-epimorphisms* and *S-isomorphisms* hold. We denote the *category* of all S-acts and S-homomorphisms between them by **Act-S**.

A non-empty subset $K \subseteq S$ is called an ideal if $KS \subseteq K$ and $SK \subseteq K$ and K is said to be *finitely generated* ideal if $K = \bigcup_{i=1}^{n} s_iS = \bigcup_{i=1}^{n} Ss_i$, for some $n \in \mathbb{N}$ and $s_i \in S$. Also, K is said to be *principal* if $K = sS = Ss$, for some $s \in S$.

An S-act A is called *injective* if for every S-monomorphism $i : B \rightarrowtail C$ and every S-homomorphism $f : B \to A$, there exists an S-homomorphism $\bar{f} : C \to A$ with $\bar{f}i = f$. Also an S-act A is called (*principally, finitely generated) weakly injective* if for every (principal, finitely generated) ideal K of S and any S-homomorphism $f : K \to A$, there exists an S-homomorphism extension $\bar{f}{:}S \to A$ of f, that is, $\bar{f}|_K = f$.

An element θ in an S-act A with $\theta s = \theta$, for all $s \in S$, is called a *zero* or a *fixed element* of A. An element $e \in S$ is called *idempotent* if $e^2 = e$. An element $s \in S$ is called *left cancellable* if $sr = st$, for $r, s \in S$, implies $r = t$. An element $z \in S$ is called *left zero*, if $zs = z$, for all $s \in S$.

The notions of purity and equational compactness of universal algebras have been studied by Banaschewski and Nelson [4]. Also, Banaschewski deals with these notions in the special case of G-sets for a group G [3]. Indeed, a *free extension* of an algebra A by a set X is an algebra E containing (a copy of) A as a subalgebra, and (a copy of) X as a subset, such that any given homomorphism

$h : A \to C$ and map $u : X \to |C|$, from X to the underlying set of an algebra C, uniquely extend to a homomorphism $f : E \to C$. The free extension $A[X]$ of an algebra A by a set X exists (uniquely up to isomorphism). It is in fact $A \coprod F(X)$, where $F(X)$ is the free algebra over X. In the case of S-acts, $A[X] = A \bigcup (X \times S)$, where the coproduct is merely the disjoint union.

Recall that, any pair $(u, v) \in A[X]^2$ is called an *equation* over A, and is written as $u = v$. The formal definition of a *solution* of an equation $u = v$ over A in an extension B of A is a homomorphism $f : A[X] \to B$ such that $f|_A$ is the inclusion $A \subseteq B$, and $f(u) = f(v)$.

An algebra A is called *pure* in an extension B of A if any system of finitely many equations over A has a solution in A whenever it has a solution in B.

Throughout this paper we take S to be a monoid of the form $S = G \dot{\cup} I$, where G is a group and I is an ideal of S and $I^1 = I \dot{\cup} \{1\}$. Indeed, the group actions, because of the properties of the group, give very special properties. For instance, the group actions with a zero element are injective, Theorem 3.1.9 of [13], or the group actions are absolutely 1-pure, Corollary 3.11 of [9]. Here we are going to extend these properties from the group actions to the acts over a monoid of the form $S = G \dot{\cup} I$, where G is a group and I is an ideal of S and $I^1 = I \dot{\cup} \{1\}$. It is worth noting that, in these kinds of monoids, since I^1 is a submonoid of S, every S-act can be considered as an I^1-act. And also we should note that, for the semigroup $S = G \dot{\cup} I$, the ideal I is a maximal ideal and all the proper ideals of S are included in I. These kinds of monoids are not a few and a large number of monoids can be considered as this form, see the following examples.

Examples:

(i) Every field F is of the form $F = F \backslash \{0\} \cup \{0\}$ with the usual multiplication.

(ii) Every commutative monoid M is of the form $M = G \dot{\cup} I$, in which G is the subset of invertible elements of S and $I = S \setminus G$.

(iii) The non-commutative monoid $S = (M_n(\mathbb{R}), .)$, consisting of the $n \times n$ square matrices with the real number entries, together with the usual multiplication is of the form $S = G \dot{\cup} I$, in which G is the subset of invertible matrices of S and $I = S \setminus G$.

2. When 1-purity is extendable

Concerning the category of S-acts, it is shown that the equations over an S-act A is of one the following three types:

$$xs = yt, \quad xs = xt, \quad xs = a,$$

where $s, t \in S, a \in A$, see [9, 14, 16]. This section is concerned with the equations of the form $xs = a$. Indeed, we show that every consistent system of equations in the form of $xs = a$ over an S-act A has a solution in A whenever every consistent system of equations in the form of $xi = a$ over I^1-act A has a solution in A. But first we give the following definition.

Definition 2.1. (*i*) *A system of equations* $\sum$ *over an* S*-act* A *is said to be consistent if it has a solution in some extension of* A.

(*ii*) *An* S*-act* A *is called absolutely pure if every finite consistent system of equations over* A *has a solution in* A.

(*iii*) *An* S*-subact* A *of an* S*-act* B *is called* 1*-pure in* B *if every finite system of equations in one variable over* A*, that has a solution in* B *has a solution in* A.

(*iv*) *An act* A *is called absolutely* 1*-pure if every finite consistent system of equations in one variable over* A *has a solution in* A.

Lemma 2.1. *Let* S *be a group and let* A *be an* S*-act with a unique zero element* θ*. Then,* A *is absolutely pure.*

Proof. By Theorem 3.1.9 of [13], we know that an S-act with a zero element over a group is injective. Now since every injective S-act is absolutely pure, by Proposition 3.6.7 of [13], we get the result. □

Lemma 2.2 ([9])**.** *Let* A *be an* S*-act and let* $\sum = \{xs_i = a_i \mid s_i \in S, a_i \in A, 1 \leq i \leq n\}$ *be a finite system over* A*. Then the following conditions are equivalent:*

1. $\sum$ *is consistent.*
2. *For all elements* h, k *of* S *if* $s_ih = s_jk$ *then* $a_ih = a_jk$.

Theorem 2.1. *Let* $S = G\dot{\cup}I$ *be a monoid and* A *be an* S*-act. Then, every finite consistent system* $\{xi_k = a_k, i_k \in I^1, a_k \in A, 1 \leq k \leq n\}$ *has a solution in* A *if and only if every finite consistent system* $\{xs_k = a_k, s_k \in S, a_k \in A, 1 \leq k \leq n\}$ *has a solution in* A.

Proof. The sufficiency part is clear. For the necessity part, first we show that A is finitely generated weakly injective I^1-act. To do so, let $\sum' = \{xi_k = a_k, i_k \in I^1, a_k \in A, 1 \leq k \leq m\}$ be a finite consistent system which has a solution in A. Suppose that J is an ideal of I^1 generated by $X' = \{i_1, ..., i_m\}$, and $f : J \to A$ is an I^1-homomorphism. For every $s, s' \in I^1$ and $i_k, i_{k'} \in X'$, suppose that $i_ks = i_{k'}s'$. Then, we have $f(i_ks) = f(i_{k'}s')$ and so $f(i_k)s = f(i_{k'})s'$. Thus, by Lemma 2.2, the system $\sum_1 = \{xi_k = f(i_k), i_k \in I^1, 1 \leq k \leq m\}$ is consistent. Therefore, by the hypothesis, $\sum_1$ has a solution a in A. That is, $ai_k = f(i_k)$, for every $i_k \in X'$. Since J is generated by X', for every $j \in J$, there exists $t \in I^1$ such that $j = i_kt$. Now, by the hypothesis, we have:

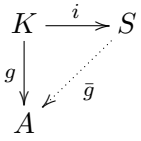
$$f(j) = f(i_kt) = f(i_k)t = (ai_k)t = a(i_kt) = aj.$$

That is A is finitely generated weakly injective as an I^1-act, see Lemma 3.5.2 of [13]. Hence, A is finitely generated weakly injective as an S-act, by Theorem 2.4 of [15].

Now let $\sum = \{xs_i = a_i, s_i \in S, a_i \in A, 1 \leq i \leq n\}$ be a finite consistent system, and K be the generated ideal by $X = \{s_1, ..., s_n\}$ of S. Consider the map $g : K \to A$, defined by $g(s_i t) = a_i t$, for any $s_i \in X, t \in S$. The defined g is well defined because if $s_i t = s_j t'$ then since $\sum$ is consistent system, by Lemma 2.2, for any $t, t' \in S$, we have $a_i t = a_j t'$. Also g is an S-homomorphism, since for any $s, t \in S$ and $s_i \in X$ we have:

$$g((s_i t)s) = g(s_i(ts)) = a_i(ts) = (a_i t)s = g(s_i t)s.$$

So, there exists an S-homomorphism $\bar{g} : S \to A$, shuch that the following diagram is commutative:



Now, for any $s_i \in X$, we have $\bar{g}(1)s_i = \bar{g}(s_i) = g(s_i) = a_i$. So, $\bar{g}(1) \in A$ is a solution of $\sum$.

□

Lemma 2.3. *Given a soluble system of equations over A in the form $\sum = \{xs = a_s, s \in S, a \in A\}$, has a solution in A if and only if the associated map $f : S \to A$, $f(s) = a_s$, is an S-homomorphism.*

Proof. Let $\sum$ be soluble on A and $a' \in A$ be a solution of $\sum$. Then, for each equation $xs = a_s$ in $\sum$, we have $a's = a_s$. Suppose that $s, t \in S$ and $s = t$ then, we have $a's = a't$ and hence $a_s = a_t$, that is, f is well-defined.

Also, we have:

$$f(st) = a_{st} = a'(st) = (a's)t = a_s t = f(s)t,$$

for every $s, t \in S$. Therefore, f is an S-homomorphism.

Conversely, suppose that $f : S \to A$ is an S-homomorphism. Then, $f(1)$ will be the desired solution. Because $f(1)s = f(s) = a_s$, for every $s \in S$. □

In Theorem 2.2 we use Lemma 2.3 and show that every consistent system of the third type of equations has a solution in A whenever every consistent system of the third type of equations over I^1-act A has a solution in A.

Theorem 2.2. *Let $S = G\dot{\cup}I$ be a monoid and A be an S-act. Then, every consistent system $\{xi = a_i, i \in I^1, a_i \in A\}$, has a solution in A if and only if every consistent system $\{xs = a_s, s \in S, a_s \in A\}$, has a solution in A.*

Proof. *Necessity.* By Lemma 2.3 for a given system $\sum = \{xs = a_s, s \in S, a_s \in A\}$, it is enough to show that the associated map $f : S \to A$, $f(s) = a_s$, is an S-homomorphism. By the hypothesis, the subsystem $\sum_1 = \{xi = a_i, i \in I^1, a_i \in A\} \subseteq \sum$, has a solution in A. So, the associated map $f|_{I^1} = f_1 : I^1 \to A$,

$f_1(i) = a_i$, is an I^1-homomorphism, by Lemma 2.3. Also, we define the map $f|_G = f_2 : G \to A$ by $f_2(g) = f_1(1)g$. For every $g_1, g_2 \in G$ we have $f_2(g_1.g_2) = f_1(1)g_1.g_2 = (f_1(1)g_1)g_2 = f_2(g_1)g_2$. So f_2 is a G-homomorphism.

Now, patch f_1, f_2 together and we obtain $f : S \to A$ as follows:

$$f(s) = \begin{cases} f_1(s), & s \in I \\ f_2(s), & s \in G. \end{cases}$$

Obviously, f is well-defined. To verify that f is an S-homomorphism it is enough to investigate the following two cases.

1. If $s \in I, t \in G$, then $f(st) = f_1(st) = f_1(1)st = (f_1(1)s)t = f_1(s)t = f(s)t$.

2. If $s \in G, t \in I$, then $f(st) = f_1(st) = f_1(1)st = (f_1(1)s)t = f_2(s)t = f(s)t$.

Sufficiency. is obvious. □

Definition 2.2 ([13])**.** *An idempotent $e \in S$ is called right special (right fg-special) if for any (finitely generated) right congruence ρ of S there exists an element $k \in eS$ such that*

(1) $(ke)\rho e$,

(2) $u\rho v, \quad u, v \in S$, *implies* $(ku)\rho(kv)$.

V. A. R. Gould in 1987 showed that if all S-acts are absolutely 1-pure, then all S-acts are absolutely pure, see Proposition 3.3 of [10]. Here using this Proposition we give a stronger form of this Proposition when S is of the form $S = G\dot{\cup}I$. Indeed, we show that absolutely 1-purity of all I^1-acts implies that all S-acts are absolutely pure when $S = G\dot{\cup}I$, see the following theorem.

Theorem 2.3. *Let $S = G\dot{\cup}I$ be a monoid whose idempotents are central. If all I^1-acts are absolutely 1-pure, then all S-acts are absolutely pure.*

Proof. Suppose that all I^1-acts are absolutely 1-pure. Then, S is absolutely 1-pure as an I^1-act. So, S has local left zeros by Lemma 3.6.6 of [13]. Also, every finitely generated ideal of I^1 is generated by right fg-special idempotent by Proposition 3.3 of [10]. Since I^1 contained all ideals and all idempotents of S, and since idempotents of S are central, every finitely generated ideal of S is generated by a central idempotent. Also since every central idempotent is right fg-special, every finitely generated right ideal of S generated by right fg-special. Hence, all S-act are absolutely pure by Proposition 3.3 of [10]. □

The above theorem gives a useful criterion to find the absolutely pure S-acts, where $S = G\dot{\cup}I$. Specially, if $S = G\dot{\cup}\{0\}$, then clearly idempotent elements are central. Now, the above theorem ensures that all S-acts are absolutely pure if all $\{0, 1\}$-acts are absolutely 1-pure. See the following example.

Example 2.1. (1) All the $\mathbb{Q}$-acts are absolutely pure, in which $\mathbb{Q}$ is the rational numbers with the usual multiplication. Indeed, one can consider $\mathbb{Q} = (\mathbb{Q} - \{0\}\dot{\cup}\{0\}$, , where $G = \mathbb{Q} - \{0\}$ is a group, $I = \{0\}$ is an ideal of S, and $I^1 = \{0, 1\}$. It worths noting that for any $a \in A$, every one variable system of equations over A, is in the form of $\sum = \{x.0 = \theta, x.1 = a, 0, 1 \in I^1\}$, which $a \in A$ is a solution of $\sum$. That is, A is absolutely 1-pure as an I^1-act and hence all I^1-acts are absolutely 1-pure.

(2) Analogously, one can see that all the $\mathbb{R}$-acts are absolutely pure, in which $\mathbb{R}$ is the monoid of the real numbers with the usual multiplication.

3. Divisibility is extendable

Recall that an element $a \in A$ is called *divisible by* $s \in S$ if there exists $b \in B$ such that $bs = a$. An S-act A is said to be divisible if $Ac = A$, for any left cancellable element $c \in S$. Also, it is known that every absolutely 1-pure S-act is divisible, Proposition 3.6.15 and Corollary 3.3.3 of [13]. In this short section, we focus on the divisible S-acts and we show that, for the monoid $S = G\dot{\cup}I$, an S-act A is divisible if it is divisible as an I^1-act. We then give a criterion to recognize divisible S-acts.

Theorem 3.1. *Let $S = G\dot{\cup}I$ be a monoid. Then A is divisible as an S-act whenever it is divisible as an I^1-act.*

Proof. Given a left cancellable element $c \in S$ and $a \in A$, we have to find $b \in A$ with $bc = a$. But since S is of the form $S = G\dot{\cup}I$, two cases may occur for c. (1) If $c \in I$ then, by the hypothesis, we get the result. (2) If $c \in G$ then, for every $a \in A$, we have $a = (ac^{-1})c$. That is, A is divisible as an S-act. □

Corollary 3.1. *Let $S = G\dot{\cup}I$ be a monoid. Then divisibility of all I^1-acts implies the divisibility of all S-acts.*

By the above theorem we get a useful criterion to check divisibility of S-acts. See the following examples.

Example 3.1. (1) Let $S = G\dot{\cup}\{0\}$ be a monoid. Then, an S-act is divisible, if it is divisible as an $\{0, 1\}$-act.

(2) Let $S = (\mathbb{Q}, \cdot)$ be a monoid of all rational numbers with the usual multiplication and A be an I^1-act. Consider $S = (\mathbb{Q} - \{0\})\dot{\cup}\{0\}$, where $G = \mathbb{Q} - \{0\}$ is a group and $I = \{0\}$ is an ideal of S. Since the only left cancellable element in the monoid $I^1 = \{0, 1\}$ is 1, implies that A is divisible as an I^1-act, and so all I^1-acts are divisible. Hence, by the above theorem all $\mathbb{Q}$-acts are divisible.

(3) Analogously, one can see that all $\mathbb{R}$-acts are divisible, in which $\mathbb{R}$ is the monoid of all real numbers with usual multiplication, $(\mathbb{R}, .)$.

Theorem 3.2. *Let $S = G\dot{\cup}I$ be a monoid and I does not contain any left cancellable element of S. Then every S-act is divisible.*

Proof. Given an S-act A, consider a left cancellable element $c \in S$ and $a \in A$. By the hypothesis, since I does not contain any left cancellable element of S, $c \in G$. So, we have $a = a(c^{-1}c) = (ac^{-1})c \in Ac$. Therefore, $A \subseteq Ac$. Also clearly $Ac \subseteq A$, and hence $A = Ac$. That is, A is divisible as an S-act. □

Definition 3.1 ([13])**.** *An element $s \in S$ is called right almost regular if there exist elements $r, r_1, ..., r_n, s_1, ..., s_n \in S$ and left cancellable elements $c_1, ..., c_n \in S$ such that $r_1 s = c_1 s_1$, $r_2 s_1 = c_2 s_2$, ..., $r_n s_{n-1} = c_n s_n$, $s = srs_n$. If all elements of S are almost regular then S is called a (right) almost regular monoid.*

Corollary 3.2. *If I^1 is an almost regular monoid and I does not contain any left cancellable element of S. Then, every S-act is principally weakly injective.*

Proof. The result follows directly by Theorem 3.2, and Theorem 4.1.5 of [13]. □

Example 3.2. For all the monoids in the form of $S = G\dot{\cup}\{0\}$, (for example, all fields with the usual multiplication, $(F, \cdot)$), clearly $I = \{0\}$, does not contain any left cancellable element of S and $I^1 = \{0, 1\}$ is an almost regular monoid. So, by the above theorem every F-act is principally weakly injective and divisible, in which F is the field with usual multiplication, $(F, \cdot)$.

By Proposition 3.3.2 of [13], we know that every principally weakly injective act is divisible. But there exist examples (see, example 3.3.11 of [13]), which show that the converse is not true. In the following theorem we show that if I^1 is a left cancellable principal right ideal monoid then divisibility as an I^1-act leads to weakly injectivity as an S-act.

Theorem 3.3. *Let $S = G\dot{\cup}I$, A be an S-act and I^1 be a left cancellable principal ideal monoid. Then, divisibility of A as an I^1-act implies weakly injectivity of A as an S-act.*

Proof. First we show that A is weakly injective as an I^1-act. To do so, suppose that K is an ideal of I^1 and $f : K \to A$ is an I^1-homomorphism. By the hypothesis, there exists $i \in I^1$ such that $K = iI^1$. So, for every $x \in K$, there exists $j \in I^1$ such that $x = ij$. Since $f(i) \in A$ and A is divisible as an I^1-act, and I^1 is a left cancellable, there exists $a \in A$ such that $f(i) = ai$. Now, we have:

$$f(x) = f(ij) = f(i)j = (ai)j = a(ij) = ax,$$

for every $x \in K$. This means that A is weakly injective as an I^1-act, by Lemma 3.5.2 of [13]. Hence, A is weakly injective as an S-act, by Theorem 2.4 of [15]. □

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Accepted: June 14, 2020

A note on extended beta, Gauss and confluent hypergeometric functions

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Abstract. The main object of this article is to present a new extension of beta function by making use of the generalized Mittag-Leffler function. Some integral representations and summation formulae for this function are also given in a systematic way. Next, we define a new type of beta distribution as an application of our extended beta function. Moreover, we consider a further extension of Gauss and confluent hypergeometric functions by introducing our new beta function.

Keywords: Beta function, extended beta function, Gauss hypergeometric function, extended Gauss hypergeometric function, confluent hypergeometric function, extended confluent hypergeometric function, Mittag-Leffler function.

1. Introduction

Numerous fundamental functions in applied sciences are interpret via improper integrals. These fundamental functions are primarily called special functions. Such functions play a prominent role in several respective fields of science and engineering. Therefore, many researchers have presented (from time to time) various extensions and associated properties of such type of functions. In particular, a number of authors have introduced and investigated a list of generalized beta functions by modifying the integral representation of classical beta function (see, for example [2], [3], [5], [6], [8], [9], [10] and the references cited therein). In order to extend the literature mentioned above, in this article, we present a new class of beta function by making use of generalized Mittag-Leffler function.

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Throughout in this paper, let $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ be the sets of natural numbers, real numbers and complex numbers, respectively, and let

$$\mathbb{N} := \{1, 2, 3, ...\},\ \mathbb{N}_0 := \{0, 1, 2, 3, ...\} = \mathbb{N} \cup \{0\}.$$

For our purpose, we begin by recalling here the definitions of some known functions and their generalizations.

The Leonhard Euler's classical beta function $B(m, n)$ is a function of two complex variables m and n and is defined by the following integral representation (see [12], see also [13]):

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx, \tag{1}$$

$$(\Re(m) > 0,\ \Re(n) > 0).$$

Its relation with well known classical gamma function (see [12], see also [13])

$$\Gamma(m) = \int_0^\infty x^{m-1}e^{-x}dx\,(\Re(m) > 0),$$

is given by

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\ (\Re(m) > 0,\ \Re(n) > 0). \tag{2}$$

The classical Gauss hypergeometric function and confluent hypergeometric function are defined, respectively by (see [12], see also [13])

$${}_2F_1(a_1, a_2; a_3; t) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n}{(a_3)_n} \frac{t^n}{n!}, \tag{3}$$

$$(a_1, a_2, a_3 \in \mathbb{C} \text{and } a_3 \neq 0, -1, -2....)$$

and

$${}_1F_1(a_2; a_3; t) = \sum_{n=0}^{\infty} \frac{(a_2)_n}{(a_3)_n} \frac{t^n}{n!}, \tag{4}$$

$$(a_2, a_3 \in \mathbb{C} \text{and } a_3 \neq 0, -1, -2....).$$

Leonhard Euler gave the following integral representations of Gauss hypergeometric function and confluent hypergeometric function:

$$F(a_1, a_2; a_3; t) = \frac{1}{B(a_2, a_3 - a_2)} \int_0^1 x^{a_2-1}(1-x)^{a_3-a_2-1}(1-tx)^{-a_1}dx, \tag{5}$$

$$(\Re(a_3) > \Re(a_2) > 0; |arg(1-t)| \leq \pi - \epsilon (0 < \epsilon < \pi)).$$

Or equivalently

$$F(a_1, a_2; a_3; t) = \frac{1}{B(a_1, a_3 - a_1)} \int_0^1 x^{a_1-1}(1-x)^{a_3-a_1-1}(1-tx)^{-a_2} dx,$$

$$(\Re(a_3) > \Re(a_1) > 0; |arg(1-t)| \leq \pi - \epsilon (0 < \epsilon < \pi))$$

and

$$(6) \quad {}_1F_1(a_2; a_3; t) = \frac{1}{B(a_2, a_3 - a_2)} \int_0^1 x^{a_2-1}(1-x)^{a_3-a_2-1} \exp(tx)\, dx,$$

$$(\Re(a_3) > \Re(a_2) > 0).$$

By using the series expansions of $(1-tx)^{-a_1}$ and $\exp(tx)$ in (5) and (6) respectively, we obtain

$$(7) \quad F(a_1, a_2; a_3; t) = \sum_{n=0}^{\infty} (a_1)_n \frac{B(a_2 + n, a_3 - a_2)}{B(a_2, a_3 - a_2)} \frac{t^n}{n!},$$

$$(\Re(a_3) > \Re(a_2) > 0, |t| < 1)$$

and

$$(8) \quad {}_1F_1(a_2; a_3; t) = \Phi(a_2; a_3; t) = \sum_{n=0}^{\infty} \frac{B(a_2 + n, a_3 - a_2)}{B(a_2, a_3 - a_2)} \frac{t^n}{n!},$$

$$(\Re(a_3) > \Re(a_2) > 0, |t| < 1).$$

In 1997, Chaudhry et al. [3] presented the following extension of the Euler's beta function by introducing the exponential factor $\exp[\frac{-p}{x(1-x)}]$ in the integrand of the integral given in (1):

$$(9) \quad B_p(m, n) = B(m, n, p) = \int_0^1 x^{m-1}(1-x)^{n-1} \exp\left[\frac{-p}{x(1-x)}\right] dx,$$

$$(\Re(m) > 0, \Re(n) > 0, \Re(p) \geq 0).$$

It is easily seen that the cases $p = 0$ of (9) yields the classical beta function, i.e. $B_0(m, n)$=$B(m, n)$.

Afterwards, Chaudhry et al. [4] used extended beta function $B_p(m, n)$ to extend the Gauss and confluent hypergeometric functions as follows:

$$(10) \quad F_p\ (a_1, a_2; a_3; t) = \sum_{n=0}^{\infty} (a_1)_n \frac{B_p(a_2 + n, a_3 - a_2)}{B(a_2, a_3 - a_2)}\ \frac{t^n}{n!},$$

$$(\Re(a_3) > \Re(a_2) > 0, p \geq 0, |t| < 1)$$

and

$$\Phi_p(a_2;a_3;t)=\sum_{n=0}^{\infty}\frac{B_p(a_2+n,a_3-a_2)}{B(a_2,a_3-a_2)}\ \frac{t^n}{n!}, \tag{11}$$

$$(\Re(a_3)>\Re(a_2)>0, p\geq 0).$$

By using the integral representation of the extended beta function (9) in (10) and (11), Chaudhry et al. [4] obtained the following integral representations of the extended Gauss and confluent hypergeometric functions:

$$\begin{aligned} F_p(a_1,a_2;a_3;t)&=\frac{1}{B(a_2,a_3-a_2)}\\ &\times\int_0^1 x^{a_2-1}\ (1-x)^{a_3-a_2-1}(1-tx)^{-a_1}\ \exp\left[\frac{-p}{x(1-x)}\right]\ dx, \qquad (12)\\ &(\Re(a_3)>\Re(a_2)>0, p\geq 0,\ |arg\ (1-t)|<\pi\ ) \end{aligned}$$

and

$$\begin{aligned} \Phi_p(a_2;a_3;t)&=\frac{1}{B(a_2,a_3-a_2)}\int_0^1 x^{a_2-1}\ (1-x)^{a_3-a_2-1}\ (1-tx)^{-a_1}\\ &\times\exp\left[tx-\frac{p}{x(1-x)}\right]dx, \qquad (13)\\ &(\Re(a_3)>\Re(a_2)>0, p\geq 0). \end{aligned}$$

Obviously for $p=0$, (10)-(13) gives the series and integral representations of the classical Gauss and confluent hypergeometric functions.

Recently, Shadab et al. [14] defined the following extension of beta function:

$$B_p^{\alpha}(m,n)=\int_0^1 x^{m-1}\ (1-x)^{n-1}\ E_{\alpha}\left[-\frac{p}{x(1-x)}\right]dx, \tag{14}$$

$$(\Re(m)>0,\ \Re(n)>0, \Re(\alpha)\geq 0\ \Re(p)\geq 0),$$

where $E_{\alpha}(t)$ is the Mittag-leffler function defined by (see [7])

$$E_{\alpha}(t)=\sum_{k=0}^{\infty}\frac{t^k}{\Gamma(1+\alpha k)}.$$

More recently, Al-Gonah and Mohammed [1] introduced and investigated a further extension of beta function as follows:

$$B_p^{\alpha,\beta,\gamma}(m,n)=\int_0^1 x^{m-1}\ (1-x)^{n-1}\ E_{\alpha,\beta}^{\gamma}\left[-\frac{p}{x(1-x)}\right]dx, \tag{15}$$

$$(\Re(m)>0,\ \Re(n)>0, \Re(\alpha)\geq 0, \Re(\beta)\geq 0, \Re(\gamma)\geq 0\ \Re(p)\geq 0),$$

where $E^{\gamma}_{\alpha,\beta,}(t)$ is the generalized Mittag-leffler function defined by (see [11])

$$E^{\gamma}_{\alpha,\beta,}(t) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k+\beta)} \frac{t^k}{k!}.$$

For $\beta = \gamma = 1$, (15) reduces to (14).

The main object of this paper is to present a further generalization of beta function by making use of the generalized Mittag-Leffler function defined by Salim [15].

2. A new extended beta function

In this section, we define the following new extended beta function by making use of the generalized Mittag-Leffler function $E^{\gamma,\delta}_{\alpha,\beta}(t)$:

$$B^{\gamma,\delta}_{\alpha,\beta,p}(m,n) = \int_0^1 x^{m-1}\ (1-x)^{n-1}\ E^{\gamma,\delta}_{\alpha,\beta}\left[-\frac{p}{x(1-x)}\right] dx, \tag{16}$$

$$(\Re(m) > 0,\ \Re(n) > 0, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\delta) > 0,\ p \geq 0),$$

where $E^{\gamma,\delta}_{\alpha,\beta,}(t)$ is the generalized Mittag-leffler function defined by (see [11])

$$E^{\gamma,\delta}_{\alpha,\beta,}(t) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k+\beta)} \frac{t^k}{(\delta)_k}.$$

If we set $\delta = 1$ in (16) then we get the generalized beta function $B_p^{(\alpha,\beta,\gamma)}(m,n)$ defined by Al-Gonah and Mohammed [1], which further for $\beta = \gamma = 1$ gives the extended beta function $B^p_\alpha(m,n)$ defined by Shadab et al. [14]. Obviously, for $p = 0$, we get the classical beta function (1). Thus, we have

$$B^{\gamma,1}_{\alpha,\beta,p}(m,n) = B_p^{(\alpha,\beta,\gamma)}(m,n),\ B^{1,1}_{\alpha,1,p}(m,n) = B^p_\alpha(m,n). \tag{17}$$

Integral representation of $B^{\gamma,\delta}_{\alpha,\beta,p}(m,n)$:

Theorem 2.1. *For $\Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\delta) > 0$, and $p \geq 0$, we have the following integral representations of $B^{\gamma,\delta}_{\alpha,\beta,p}(m,n)$:*

$$B^{\gamma,\delta}_{\alpha,\beta,p}(m,n) = 2\int_0^{\frac{\pi}{2}} \cos^{2m-1} t\ \sin^{2n-1} t\ E^{\gamma,\delta}_{\alpha,\beta}(-p\sec^2 t\ \csc^2 t)dt, \tag{18}$$

$$B^{\gamma,\delta}_{\alpha,\beta,p}(m,n) = \int_0^{\infty} \frac{w^{m-1}}{(1+w)^{m+n}}\ E^{\gamma,\delta}_{\alpha,\beta}\left[-p\left(2+w+\frac{1}{w}\right)\right] dw, \tag{19}$$

$$B^{\gamma,\delta}_{\alpha,\beta,p}(m,n) = 2^{1-m-n}\int_{-1}^{1} (1+w)^{m-1}\ (1-w)^{n-1}\ E^{\gamma,\delta}_{\alpha,\beta}\left[-\frac{4p}{(1-w^2)}\right] dw. \tag{20}$$

Proof. On putting $x = \cos^2 t$, $x = \frac{w}{1+w}$ and $x = \frac{1+w}{2}$ in (16) gives, respectively, the integral representations (18)–(20). □

3. Properties of extended beta function $B^{\gamma,\delta}_{\alpha,\beta,p}(m,n)$

In this section, we derive some interesting properties of our introduced beta function $B^{\gamma,\delta}_{\alpha,\beta,p}(m,n)$.

Theorem 3.1. *The following identity holds true for extended beta function* $B^{\gamma,\delta}_{\alpha,\beta,p}(m,n)$*:*

$$B^{\gamma,\delta}_{\alpha,\beta,p}(m,n) = \sum_{s=0}^{l} \binom{l}{s} B^{\gamma,\delta}_{\alpha,\beta,p}(m+s, n+l-k), \text{ where } l \in \mathbb{N}_0. \tag{21}$$

Proof. From (16), we have

$$B^{\gamma,\delta}_{\alpha,\beta,p}(m,n) = \int_0^1 x^{m-1}\ (1-x)^{n-1}\{x+(1-x)\}\ E^{\gamma,\delta}_{\alpha,\beta}\left[-\frac{p}{x(1-x)}\right] dx,$$

$$B^{\gamma,\delta}_{\alpha,\beta,p}(m,n) = B^{\gamma,\delta}_{\alpha,\beta,p}(m+1,n) + B^{\gamma,\delta}_{\alpha,\beta,p}(m,n+1). \tag{22}$$

Further, applying the same argument in the right side of (22), we get

$$B^{\gamma,\delta}_{\alpha,\beta,p}(m,n) = B^{\gamma,\delta}_{\alpha,\beta,p}(m+2,n) + 2B^{\gamma,\delta}_{\alpha,\beta,p}(m+1,n+1) + B^{\gamma,\delta}_{\alpha,\beta,p}(m,n+2),$$

continuing this process, by induction, we obtain the desired result. □

Theorem 3.2. *The following identity holds true for extended beta function* $B^{\gamma,\delta}_{\alpha,\beta,p}(m,n)$*:*

$$B^{\gamma,\delta}_{\alpha,\beta,p}(m,1-n) = \sum_{s=0}^{\infty} \frac{(n)_s}{s!} B^{\gamma,\delta}_{\alpha,\beta,p}(m+s,1). \tag{23}$$

Proof. On using (16) in the left side of (23), we have

$$\begin{aligned} B^{\gamma,\delta}_{\alpha,\beta,p}(m,1-n) &= \int_0^1 x^{m-1}\ (1-x)^{-n}\ E^{\gamma,\delta}_{\alpha,\beta}\left[-\frac{p}{x(1-x)}\right] dx \\ &= \int_0^1 x^{m-1}\ \sum_{s=0}^{\infty} \frac{(n)_s\ x^s}{s!}\ E^{\gamma,\delta}_{\alpha,\beta}\left[-\frac{p}{x(1-x)}\right] dx. \end{aligned}$$

Now, by interchanging the order of integration and summation in the above expression and then upon using (16), we arrive at our claimed result (23). □

Theorem 3.3. *The following summation formula holds true for extended beta function* $B^{\gamma,\delta}_{\alpha,\beta,p}(m,n)$*:*

$$B^{\gamma,\delta}_{\alpha,\beta,p}(m,n) = \sum_{s=0}^{\infty} B^{\gamma,\delta}_{\alpha,\beta,p}(m+s,n+1). \tag{24}$$

Proof. The above theorem can be easily established with the help of (16) by writing $(1-x)^{n-1} = (1-x)^n \sum_{s=0}^{\infty} x^s$. Therefore, we are omitting the details. □

4. The extended beta distribution

In this section, we define the following extended beta distribution by making use of our introduced beta function:

$$f(x)=\begin{cases}\dfrac{1}{B^{\gamma,\delta}_{\alpha,\beta,p}(a,b)}\ x^{a-1}\ (1-x)^{b-1}E^{\gamma,\delta}_{\alpha,\beta}\left[-\dfrac{p}{x(1-x)}\right], & (0<x<1)\\ 0, & \text{otherwise}\end{cases}, \tag{25}$$

$$(\Re(\alpha)>0,\Re(\beta)>0,\Re(\gamma)>0,\Re(\delta)>0,\ p\geq 0).$$

Also, we have discussed here some analytical properties of our extended beta distribution (25).

For $n\in\mathbb{R}$, the n^{th} moment of X is given by

$$E(X^n)=\frac{B^{\gamma,\delta}_{\alpha,\beta,p}(a+n,b)}{B^{\gamma,\delta}_{\alpha,\beta,p}(a,b)}, \tag{26}$$

$$(a,\ b\in\mathbb{R},\ p\geq 0,\ \Re(\alpha)>0,\Re(\beta)>0,\Re(\gamma)>0,\Re(\delta)>0).$$

The particular case of (26) for $n=1$ gives the mean of our extended beta distribution

$$E(X)=\frac{B^{\gamma,\delta}_{\alpha,\beta,p}(a+1,b)}{B^{\gamma,\delta}_{\alpha,\beta,p}(a,b)}. \tag{27}$$

The variance for our introduced distribution is defined by

$$\begin{aligned}Var(X)&=E(X^2)-[E(X)]^2,\\ Var(X)&=\frac{B^{\gamma,\delta}_{\alpha,\beta,p}(a+2,b)B^{\gamma,\delta}_{\alpha,\beta,p}(a,b)-[B^{\gamma,\delta}_{\alpha,\beta,p}(a+1,b)]^2}{[B^{\gamma,\delta}_{\alpha,\beta,p}(a,b)]^2}.\end{aligned} \tag{28}$$

The coefficient of variation for this distribution (which is defined as the ratio of the standard deviation and mean) can be expressed as

$$C.V=\sqrt{\frac{B^{\gamma,\delta}_{\alpha,\beta,p}(a+2,b)\ B^{\gamma,\delta}_{\alpha,\beta,p}(a,b)}{[B^{\gamma,\delta}_{\alpha,\beta,p}(a+1,b)]^2}-1}. \tag{29}$$

The moment generating function (m.g.f) about origin of this distribution is given by

$$\begin{aligned}M_X(t)&=\sum_{r=0}^{\infty}\frac{t^r}{r!}E(X^r),\\ M_X(t)&=\frac{1}{B^{\gamma,\delta}_{\alpha,\beta,p}(a,b)}\sum_{r=0}^{\infty}B^{\gamma,\delta}_{\alpha,\beta,p}(a+r,b)\frac{t^r}{r!}.\end{aligned} \tag{30}$$

The characteristic function for our new extended distribution can be calculated as follows:

$$E(e^{itx}) = \sum_{r=0}^{\infty} \frac{i^r t^r}{r!} E(X^r),$$

$$E(e^{itx}) = \frac{1}{B^{\gamma,\delta}_{\alpha,\beta,p}(a,b)} \sum_{r=0}^{\infty} B^{\gamma,\delta}_{\alpha,\beta,p}(a+r,b) \frac{i^r t^r}{r!}. \tag{31}$$

The cumulative distribution function of our extended beta distribution (25) can be expressed as

$$F(x) = P[X < x] = \int_0^x f(x)dx,$$

$$F(x) = \frac{B^{\gamma,\delta,x}_{\alpha,\beta,p}(a,b)}{B^{\gamma,\delta}_{\alpha,\beta,p}(a,b)}, \tag{32}$$

where $B^{\gamma,\delta,x}_{\alpha,\beta,p}(a,b)$ denotes the incomplete extended beta function defined by

$$B^{\gamma,\delta,x}_{\alpha,\beta,p}(a,b) = \int_0^x x^{a-1}\ (1-x)^{b-1}\ E^{\gamma,\delta}_{\alpha,\beta}\left[-\frac{p}{x(1-x)}\right] dx.$$

The reliability function (which is simply the complement of the cumulative distribution function) of our proposed distribution is given by

$$R(x) = P[X \geq x] = 1 - F(x) = \int_x^{\infty} f(x)dx,$$

$$R(x) = \frac{B^{\gamma,\delta,x}_{\alpha,\beta,p}(a,b)}{B^{\gamma,\delta}_{\alpha,\beta,p}(a,b)}, \tag{33}$$

where $B^{\gamma,\delta,x}_{\alpha,\beta,p}(a,b)$ is the incomplete extended beta function defined by

$$B^{\gamma,\delta,x}_{\alpha,\beta,p}(a,b) = \int_x^{\infty} x^{a-1}\ (1-x)^{b-1}\ E^{\gamma,\delta}_{\alpha,\beta}\left[-\frac{p}{x(1-x)}\right] dx.$$

5. Extended Gauss and confluent hypergeometric functions and their associated properties

In this section, we define the following extensions of Gauss and confluent hypergeometric functions by making use of our introduced extended beta function $B^{\gamma,\delta}_{\alpha,\beta,p}(a,b)$:

$$F^{\gamma,\delta}_{\alpha,\beta,p}(a_1,a_2;a_3;t) = \sum_{l=0}^{\infty} \frac{(a_1)_l\ B^{\gamma,\delta}_{\alpha,\beta,p}(a_2+l,a_3-a_2)}{B(a_2,a_3-a_2)} \frac{t^l}{l!}, \tag{34}$$

$$(p \geq 0,\ |t| < 1,\ \Re(a_3) > \Re(a_2) > 0, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\delta) > 0)$$

and

$$\Phi_{\alpha,\beta,p}^{\gamma,\delta}(a_2; a_3; t) = \sum_{l=0}^{\infty} \frac{B_{\alpha,\beta,p}^{\gamma,\delta}(a_2 + l, a_3 - a_2)}{B(a_2, a_3 - a_2)} \frac{t^l}{l!}, \tag{35}$$

$$(p \geq 0,\ \Re(a_3) > \Re(a_2) > 0, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\delta) > 0).$$

Theorem 5.1. *The following integral representations for our extended Gauss and confluent hypergeometric functions holds true:*

$$F_{\alpha,\beta,p}^{\gamma,\delta}(a_1, a_2; a_3; t) = \frac{1}{B(a_2, a_3 - a_2)} \times \int_0^1 x^{a_2 - 1}\ (1 - x)^{a_3 - a_2 - 1}\ (1 - tx)^{-a_1} E_{\alpha,\beta}^{\gamma,\delta}\left[-\frac{p}{x(1-x)}\right] dx, \tag{36}$$

$$(p \geq 0,\ |\arg(1 - t)| < \pi,\ \Re(a_3) > \Re(a_2) > 0)$$

and

$$\Phi_{\alpha,\beta,p}^{\gamma,\delta}(a_2; a_3; t) = \frac{1}{B(a_2, a_3 - a_2)} \times \int_0^1 x^{a_2 - 1}\ (1 - x)^{a_3 - a_2 - 1}\ e^{tx} E_{\alpha,\beta}^{\gamma,\delta}\left[-\frac{p}{x(1-x)}\right] dx, \tag{37}$$

$$(p \geq 0,\ \Re(a_3) > \Re(a_2) > 0).$$

Proof. The above integral representations can be easily established by using the integral representation of our extended beta function (16) in the right hand sides of (34) and (35), respectively. □

Theorem 5.2. *The following differential formulae for our extended Gauss and confluent hypergeometric functions holds true:*

$$\frac{d^k}{dt^k}\left\{F_{\alpha,\beta,p}^{\gamma,\delta}(a_1, a_2; a_3; t)\right\} = \frac{(a_1)_k (a_2)_k}{(a_3)_k} F_{\alpha,\beta,p}^{\gamma,\delta}(a_1 + k, a_2 + k; a_3 + k; t), \tag{38}$$

$$(p \geq 0, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\delta) > 0, k \in \mathbb{N}_0)$$

and

$$\frac{d^k}{dt^k}\left\{\Phi_{\alpha,\beta,p}^{\gamma,\delta}(a_2; a_3; t)\right\} = \frac{(a_2)_k}{(a_3)_k} \Phi_{\alpha,\beta,p}^{\gamma,\delta}(a_2 + k; a_3 + k; t), \tag{39}$$

$$(p \geq 0, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\delta) > 0, k \in \mathbb{N}_0).$$

Proof. On differentiating (34) with respect to t, we get

$$\frac{d}{dt}\left\{F_{\alpha,\beta,p}^{\gamma,\delta}(a_1, a_2; a_3; t)\right\} = \sum_{l=1}^{\infty} \frac{(a_1)_l\ B_{\alpha,\beta,p}^{\gamma,\delta}(a_2 + l, a_3 - a_2)}{B(a_2, a_3 - a_2)} \frac{t^{l-1}}{(l-1)!}.$$

On replacing l by $l+1$, we have

$$\frac{d}{dt}\left\{F^{\gamma,\delta}_{\alpha,\beta,p}(a_1,a_2;a_3;t)\right\}=\sum_{l=0}^{\infty}\frac{(a_1)_{l+1}\ B^{\gamma,\delta}_{\alpha,\beta,p}(a_2+l+1,a_3-a_2)}{B(a_2,a_3-a_2)}\frac{t^l}{l!}.$$

Now, by using $B(a_2,a_3-a_2)=\frac{a_3}{a_2}B(a_2+1,a_3-a_2)$ and $(a_1)_{l+1}=a_1(a_1+1)_l$, we get

$$\begin{aligned}\frac{d}{dt}\left\{F^{\gamma,\delta}_{\alpha,\beta,p}(a_1,a_2;a_3;t)\right\}&=\frac{a_1a_2}{a_3}\sum_{l=0}^{\infty}\frac{(a_1+1)_l\ B^{\gamma,\delta}_{\alpha,\beta,p}(a_2+l+1,a_3-a_2)}{B(a_2+1,a_3-a_2)}\frac{t^l}{l!}\\ &=\frac{a_1a_2}{a_3}F^{\gamma,\delta}_{\alpha,\beta,p}(a_1+1,a_2+1;a_3+1;t).\end{aligned}\tag{40}$$

Again differentiating (40) with respect to t, we get

$$\frac{d^2}{dt^2}\left\{F^{\gamma,\delta}_{\alpha,\beta,p}(a_1,a_2;a_3;t)\right\}=\frac{a_1(a_1+1)a_2(a_2+1)}{a_3(a_3+1)}F^{\gamma,\delta}_{\alpha,\beta,p}(a_1+2,a_2+2;a_3+2;t).$$

Continuing this process, by induction we obtain our desired result (38). Similarly we can establish the result (39). □

Theorem 5.3. *The following transformation formulae for our extended Gauss and confluent hypergeometric functions holds true:*

$$F^{\gamma,\delta}_{\alpha,\beta,p}(a_1,a_2;a_3;t)=(1-t)^{-a_1}F^{\gamma,\delta}_{\alpha,\beta,p}\left(a_1,a_3-a_2;a_2;-\frac{t}{(1-t)}\right),\tag{41}$$

$$(p\geq 0,\Re(\alpha)>0,\Re(\beta)>0,\Re(\gamma)>0,\Re(\delta)>0)$$

and

$$\Phi^{\gamma,\delta}_{\alpha,\beta,p}(a_2;a_3;t)=e^t\Phi^{\gamma,\delta}_{\alpha,\beta,p}(a_3-a_2;a_3;-t),\tag{42}$$

$$(p\geq 0,\Re(\alpha)>0,\Re(\beta)>0,\Re(\gamma)>0,\Re(\delta)>0.$$

Proof. On replacing x by$1-x$ in (36) and then by using $[1-t(1-x)]^{-a_1}=(1-t)^{-a_1}[1+\frac{t}{1-t}x]^{-a_1}$, we get

$$\begin{aligned}F^{\gamma,\delta}_{\alpha,\beta,p}(a_1,a_2;a_3;t)&=\frac{(1-t)^{-a_1}}{B(a_2,a_3-a_2)}\\ &\times\int_0^1 x^{a_3-a_2-1}\ (1-x)^{a_2-1}\ \left(1+\frac{t}{1-t}x\right)^{-a_1}E^{\gamma,\delta}_{\alpha,\beta}\left[-\frac{p}{x(1-x)}\right]dx,\end{aligned}$$

which further on using (36), gives our claimed result (41). In a similar way, we can establish (42). □

Theorem 5.4. *The following summation formula for our extended Gauss hypergeometric function holds true:*

$$F^{\gamma,\delta}_{\alpha,\beta,p}(a_1,a_2;a_3;1)=\frac{B^{\gamma,\delta}_{\alpha,\beta,p}(a_2,a_3-a_1-a_2)}{B(a_2,a_3-a_2)}, \tag{43}$$

$$(p\geq 0,\Re(\alpha)>0,\Re(\beta)>0,\Re(\gamma)>0,\Re(\delta)>0,\Re(a_3-a_1-a_2)>0).$$

Proof. On setting $t=1$ in (36) and then by using (16), we arrive at our required result (43). □

Theorem 5.5. *The following generating function for our extended Gauss hypergeometric function holds true:*

$$\sum_{l=0}^{\infty}(a_1)_l\ F^{\gamma,\delta}_{\alpha,\beta,p}(a_1+l,a_2;a_3;t)\frac{x^l}{l!}=(1-x)^{-a_1}F^{\gamma,\delta}_{\alpha,\beta,p}\left(a_1,a_2;a_3;\frac{t}{1-x}\right), \tag{44}$$

$$(p\geq 0,\Re(\alpha)>0,\Re(\beta)>0,\Re(\gamma)>0,\Re(\delta)>0,|x|<1).$$

Proof. On using (34) on the left hand side of (44), we have

$$\begin{aligned}&\sum_{l=0}^{\infty}(a_1)_l\ F^{\gamma,\delta}_{\alpha,\beta,p}(a_1+l,a_2;a_3;t)\frac{x^l}{l!}\\&=\sum_{l=0}^{\infty}(a_1)_l\left[\sum_{m=0}^{\infty}\frac{(a_1+l)_m\ B^{\gamma,\delta}_{\alpha,\beta,p}(a_2+m,a_3-a_2)}{B(a_2,a_3-a_2)}\frac{t^m}{m!}\right]\frac{x^l}{l!}.\end{aligned}$$

Now, by using the result $(a)_m(a+m)_l=(a)_l(a+l)_m$, in the above expression, we get

$$\begin{aligned}&\sum_{l=0}^{\infty}(a_1)_l\ F^{\gamma,\delta}_{\alpha,\beta,p}(a_1+l,a_2;a_3;t)\frac{x^l}{l!}\\&=\sum_{m=0}^{\infty}(a_1)_m\frac{B^{\gamma,\delta}_{\alpha,\beta,p}(a_2+m,a_3-a_2)}{B(a_2,a_3-a_2)}\left[\sum_{l=0}^{\infty}(a_1+m)_l\frac{x^l}{l!}\right]\frac{t^m}{m!}.\\&=\sum_{m=0}^{\infty}(a_1)_m\frac{B^{\gamma,\delta}_{\alpha,\beta,p}(a_2+m,a_3-a_2)}{B(a_2,a_3-a_2)}(1-x)^{-(a+m)}\frac{t^m}{m!},\end{aligned}$$

which, by using (34), yields our claimed result (44).

□

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Accepted: February 02, 2020

Coupled fixed point theorems for mappings satisfying Geraghty type contractive conditions

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Abstract. In this paper we establish coupled common fixed point theorems for a pair of weakly increasing mappings satisfying Geraghty type contractive conditions involving a control function, in partially ordered metric spaces. An example is given to illustrate the theorem.

Keywords: coupled fixed point, coupled common fixed point, weakly increasing mappings, partially ordered set.

1. Introduction

The fundamental theorem in metric fixed point theory, the Banach contraction principle, is one of the important theorems which has gone through a large number of extensions and generalizations in mathematics. One of the most significant generalizations was done by Rakotch [11]. Followed by his work, several articles have been published by many authors [5, 7, 8, 12]. In 1973 Geraghty [4] established a variant of Rakotch's theorem by introducing a new class of functions, $S = \{\alpha : [0,\infty) \to [0,1) \mid \alpha(t_n) \to 1 \text{ implies } t_n \to 0\}$. Further a lot of generalizations of fixed point theorems have been done, by using a control function called altering distance function, which is defined as:

Definition 1.1 ([14])**.** *Define* $\Psi = \{\psi : [0,\infty) \to [0,\infty) | \psi$ *is continuous, non decreasing and* $\psi(t) = 0$*, if and only if* $t = 0\}$*.*

A mapping $\psi \in \Psi$ *is called an altering distance function.*

Several authors have extended myriads of existing fixed point results, by considering a partial order, to partially ordered metric spaces [15, 16, 17]. In

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2010, Amini- Harandi et al. [6], extended the Geraghty's fixed point theorem to partially ordered metric spaces.

The concept of coupled fixed point as an extension of fixed point to two dimensional space was introduced by Guo and Lakshmikantham [10]. Further, a lot of articles in coupled fixed point have been published by many authors. Followed by the famous results on coupled fixed points by Gnana Bhaskar and Lakshmikantham [9], Ciric and Lakshmikantham [13] generalized the concept of coupled fixed point, by introducing the concepts of coupled coincidence point and coupled common fixed point. In [13], they introduced a new mapping called mixed g- monotone mapping and generalized the results of Gnana Bhaskar and Lakshmikantham. The authors have established an existence theorem of coupled coincidence point and a uniqueness theorem of coupled common fixed point, for a mapping g and mixed g- monotone mappings in partially ordered complete metric spaces.

In this paper we establish three coupled common fixed point results, for which two different contractive type conditions involving a control function are considered.

Some useful definitions and results are given below:

Definition 1.2 ([9]). *An element $(x, y) \in X \times X$ is said to be a coupled fixed point of the mapping $F : X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.*

Definition 1.3 ([13]). *An element $(x, y) \in X \times X$ is said to be a coupled coincidence point of the mappings $F : X \times X \to X$ and $g : X \to X$ if $F(x, y) = g(x)$ and $F(y, x) = g(y)$.*

Definition 1.4 ([13]). *An element $(x, y) \in X \times X$ is said to be a coupled common fixed point of the mappings $F : X \times X \to X$ and $g : X \to X$ if $F(x, y) = g(x) = x$ and $F(y, x) = g(y) = y$.*

Definition 1.5 ([1, 2]). *Let $(X, \leq)$ be a partially ordered set. Let $f, g : X \to X$ and $F, G : X \times X \to X$ be mappings on X.*

1. *f, g are said to be weakly increasing if $f(x) \leq g(f(x))$ and $g(x) \leq f(g(x))$ for all $x \in X$;*

2. *F, G are said to be weakly increasing if $F(x, y) \leq G(F(x, y), F(y, x))$ and $G(x, y) \leq F(G(x, y), G(y, x))$ for all $x, y \in X$.*

Motivated by the above definitions, we introduce the following definition.

Definition 1.6. *A pair of mappings $F : X \times X \to X$ and $g : X \to X$ are said to be weakly increasing if $F(x, y) \leq g(F(x, y))$ and $g(x) \leq F(g(x), g(y))$ for all $x, y \in X$.*

Lemma 1.1 ([3]). *If ψ is an altering distance function and $\phi : [0, \infty) \to [0, \infty)$ is a continuous function with condition $\psi(t) > \phi(t)$ for all $t > 0$, then $\phi(0) = 0$.*

2. Main results

In this section we prove three coupled common fixed point theorems of pair of weakly increasing mappings $F : X \times X \to X$ and $g : X \to X$.

Theorem 2.1. *Let $(X, d, \leq)$ be a partially ordered complete metric space.*

Let $F : X \times X \to X$ and $g : X \to X$ are weakly increasing mappings such that

$$\psi[d(F(x,y), g(z))] \leq \alpha(d(x,z))\beta(d(x,z)), \tag{1}$$

for all comparable $x, g(z) \in X$, where $\alpha \in S$, $\psi \in \Psi$ and $\beta : [0,\infty) \to [0,\infty)$ is a continuous function with a condition $\psi(t) > \beta(t),\ \forall\ t > 0$.

Suppose F or g is continuous, then there exist a coupled common fixed point of F and g.

Proof. Let x_0, y_0 be any two elements in X.

Define $x_1 = F(x_0, y_0)$, $y_1 = F(y_0, x_0)$ and $x_2 = g(x_1),\ y_2 = g(y_1)$. Continuing like this we get two sequences $\{x_n\}$ and $\{y_n\}$ such that $x_{2n+1} = F(x_{2n}, y_{2n})$, $x_{2n+2} = g(x_{2n+1})$ and $y_{2n+1} = F(y_{2n}, x_{2n}), y_{2n+2} = g(y_{2n+1}), \forall\ n \in \mathbb{N} \cup \{0\}$.

Since, F and g are weakly increasing we have

$$\begin{aligned}
x_1 &= F(x_0, y_0) \leq g(F(x_0, y_0)) = g(x_1) = x_2,\\
y_1 &= F(y_0, x_0) \leq g(F(y_0, x_0)) = g(y_1) = y_2,\\
x_2 &= g(x_1) \leq F(g(x_1), g(y_1)) = F(x_2, y_2) = x_3,\\
y_2 &= g(y_1) \leq F(g(y_1), g(x_1)) = F(y_2, x_2) = y_3.
\end{aligned}$$

Repeating this process, we get two increasing sequences $\{x_n\}$ and $\{y_n\}$ in X.

Next, we prove that sequences $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X.

Case 1. If $x_{2n+1} = x_{2n}$ and $y_{2m+1} = y_{2m}$ for some $n, m \in \mathbb{N} \cup \{0\}$.

Consider,

$$\begin{aligned}
&\psi[d(x_{2n+1}, x_{2n+2})] + \psi[d(y_{2m+1}, y_{2m+2})]\\
&= \psi[d(F(x_{2n}, y_{2n}), g(x_{2n+1}))] + \psi[d(F(y_{2m}, x_{2m}), g(y_{2m+1}))]\\
&\leq \alpha(d(x_{2n}, x_{2n+1}))\beta(d(x_{2n}, x_{2n+1})) + \alpha(d(y_{2m}, y_{2m+1}))\beta(d(y_{2m}, y_{2m+1})) = 0,
\end{aligned}$$

which gives $x_{2n+1} = x_{2n+2}$ and $y_{2m+1} = y_{2m+2}$.

Continuing like this we get $x_{2n} = x_p$ and $y_{2m} = y_q$ for all $p \geq 2n$ and $q \geq 2m$. Thus, the sequences $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X.

Case 2. Either $x_{2n+1} \neq x_{2n}$ or $y_{2n+1} \neq y_{2n}$, $\forall\ n \in \mathbb{N} \cup \{0\}$. Without loss of generality assume $x_{2n+1} \neq x_{2n},\ \forall\ n \in \mathbb{N} \cup \{0\}$.

Consider for $n \in \mathbb{N}$,

$$\begin{aligned} \psi[d(x_{2n+1}, x_{2n+2})] &= \psi[d(F(x_{2n}, y_{2n}), g(x_{2n+1}))] \\ &\leq \alpha(d(x_{2n}, x_{2n+1}))\beta(d(x_{2n}, x_{2n+1})) \\ &< \alpha(d(x_{2n}, x_{2n+1}))\psi[d(x_{2n}, x_{2n+1})] \\ &< \psi[d(x_{2n}, x_{2n+1})], \end{aligned} \tag{2}$$

which gives $d(x_{2n+1}, x_{2n+2}) < d(x_{2n}, x_{2n+1})$.

In the same way we get $d(x_{2n}, x_{2n+1}) < d(x_{2n-1}, x_{2n})$, for $n \in \mathbb{N}$. Thus, $\{d(x_n, x_{n+1})\}$ is a decreasing sequence of non negative reals, and hence there exist $s \geq 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = s$.

Suppose $s > 0$. By (2) we get $\frac{\psi[d(x_{2n+1}, x_{2n+2})]}{\psi[d(x_{2n}, x_{2n+1})]} < \alpha(d(x_{2n}, x_{2n+1}))$.

Now, taking the limit as $n \to \infty$ on both sides and since $\alpha \in S$ we get $1 = \lim_{n\to\infty} \alpha(d(x_{2n}, x_{2n+1}))$. And hence $\lim_{n\to\infty} d(x_{2n}, x_{2n-1}) = 0$, a contradiction. Thus, $s = 0$.

Now, to prove that the sequence $\{x_n\}$ is a Cauchy sequence in X, it is enough to prove that the subsequence $\{x_{2n}\}$ is a Cauchy sequence in X.

Suppose to the contrary assume that $\{x_{2n}\}$ is not a Cauchy sequence in X. Then, there exist $\epsilon > 0$ and two subsequences of integers $\{n(k)\}, \{m(k)\}$ with $n(k) > m(k) \geq k$ such that

$$d(x_{2m(k)}, x_{2n(k)}) \geq \epsilon \ \forall k \in \mathbb{N} \tag{3}$$

and by choosing $n(k)$ to be the smallest number exceeding $m(k)$ for which (3) holds we get,

$$d(x_{2m(k)}, x_{2n(k)-2}) < \epsilon.$$

We have from (3)

$$\begin{aligned} \epsilon &\leq d(x_{2m(k)}, x_{2n(k)}) \\ &\leq d(x_{2m(k)}, x_{2n(k)-2}) + d(x_{2n(k)-2}, x_{2n(k)-1}) + d(x_{2n(k)-1}, x_{2n(k)}) \\ &< \epsilon + d(x_{2n(k)-2}, x_{2n(k)-1}) + d(x_{2n(k)-1}, x_{2n(k)}) \end{aligned}$$

as $k \to \infty$ we get

$$\lim_{k\to\infty} d(x_{2m(k)}, x_{2n(k)}) = \epsilon. \tag{4}$$

Now, consider

$$\begin{aligned} d(x_{2n(k)}, x_{2m(k)}) &\leq d(x_{2n(k)}, x_{2m(k)-1}) + d(x_{2m(k)-1}, x_{2m(k)}) \\ &\leq d(x_{2n(k)}, x_{2m(k)}) + d(x_{2m(k)}, x_{2m(k)-1}) + d(x_{2m(k)-1}, x_{2m(k)}). \end{aligned}$$

Using (4) we get as $k \to \infty$, $\lim_{k\to\infty} d(x_{2n(k)}, x_{2m(k)-1}) = \epsilon$.

Also,

$$\begin{aligned} d(x_{2n(k)}, x_{2m(k)}) &\leq d(x_{2n(k)}, x_{2n(k)+1}) + d(x_{2n(k)+1}, x_{2m(k)}) \\ &\leq d(x_{2n(k)}, x_{2n(k)+1}) + d(x_{2n(k)+1}, x_{2n(k)}) + d(x_{2n(k)}, x_{2m(k)}). \end{aligned}$$

Using (4) we get as $k \to \infty$

$$\lim_{k\to\infty} d(x_{2n(k)+1}, x_{2m(k)}) = \epsilon.$$

Now, we have $\psi[d(x_{2n(k)+1}, x_{2m(k)})] = \psi[d(F(x_{2n(k)}, y_{2n(k)}), g(x_{2m(k)-1}))] \leq \alpha(d(x_{2n(k)}, x_{2m(k)-1}))\beta(d(x_{2n(k)}, x_{2m(k)-1})) < \beta(d(x_{2n(k)}, x_{2m(k)-1}))$ as $k \to \infty$ we get $\psi[\epsilon] < \beta(\epsilon)$, a contradiction. Thus, $\{x_n\}$ is a Cauchy sequence in X.

Similarly, we can prove that $\{y_n\}$ is a Cauchy sequence in X. Since X is a complete metric space, there exist $x, y \in X$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$.

Now, suppose that F is a continuous function. Then, we have $x = \lim_{n\to\infty} x_{2n+1}$ $= \lim_{n\to\infty} F(x_{2n}, y_{2n}) = F(x, y)$ and $y = \lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} F(y_{2n}, x_{2n}) = F(y, x)$.

Since F and g are weakly increasing $x = F(x, y) \leq g(F(x, y)) = g(x)$ and $y = F(y, x) \leq g(F(y, x)) = g(y)$.

Now, consider

$$\begin{aligned} \psi[d(x, g(x))] + \psi[d(y, g(y))] &= \psi[d(F(x, y), g(x))] + \psi[d(F(y, x), g(y))] \\ &\leq \alpha(d(x, x))\beta(d(x, x)) + \alpha(d(y, y))\beta(d(y, y)) = 0. \end{aligned}$$

Thus, we get $g(x) = x$ and $g(y) = y$. Next, suppose that g is continuous. By the continuity of g and contractive condition (1), we can easily prove that (x, y) is a coupled common fixed point of F and g.

Hence the proof. □

Example 2.1. Let $X = [0, 1]$ with usual metric.

Define $F : X \times X \to X$ and $g : X \to X$ as

$$F(x, y) = \begin{cases} \dfrac{x}{2}, & \text{if } x \leq \frac{1}{2} \\ \dfrac{y}{2}, & \text{if } x > \frac{1}{2} \end{cases} \quad \text{and} \quad g(x) = \frac{x}{2}.$$

Define a partial order $\preceq$ on X as

$$\preceq := \Big\{(x, x), \Big(\frac{x}{2}, \frac{x}{2^n}\Big) \Big| \ x \in X, \ n \in \mathbb{N}\Big\}.$$

The pair of mappings (F, g) is weakly increasing and satisfy the contraction type condition for $\psi(t) = t$, $\alpha(t) = \frac{1}{\sqrt{2}}$, $\beta(t) = \frac{t}{\sqrt{2}}$, $\forall t \in [0, \infty)$. Here, $(0, 0)$ is a coupled common fixed point of F and g.

Corollary 2.1. *Let $(X, d, \leq)$ be a partially ordered complete metric space.*

Let $F : X \times X \to X$ be a mappings such that

$$\psi[d(F(x,y),z)] \leq \alpha(d(x,z))\beta(d(x,z)), \tag{5}$$

for all comparable $x, z \in X$, where $\alpha \in S$, $\psi \in \Psi$ and $\beta : [0,\infty) \to [0,\infty)$ is a continuous function with a condition $\psi(t) > \beta(t),\ \forall\ t > 0$.

Suppose F is continuous and if for all $x, y \in X$, $x \leq F(x,y)$, then there exist a coupled fixed point of F.

Proof. By taking g to be the identity function on X in Theorem 2.1, we get the result. □

Theorem 2.2. *Let $(X, d, \leq)$ be a partially ordered complete metric space satisfies the property that if $\{x_n\}$ is an increasing sequence in X such that $x_n \to x$ as $n \to \infty$, then $x_n \leq x,\ \forall\ n \in \mathbb{N}$.*

Let $F : X \times X \to X$ and $g : X \to X$ are weakly increasing mappings such that

$$\psi[d(F(x,y),g(z))] \leq \alpha(d(x,z))\beta(d(x,z)),$$

for all comparable $x, g(z) \in X$, where $\alpha \in S$, $\psi \in \Psi$ and $\beta : [0,\infty) \to [0,\infty)$ is a continuous function with a condition $\psi(t) > \beta(t),\ \forall\ t > 0$. Then, F and g have a coupled common fixed point.

Proof. Following as in Theorem 2.1, we get two increasing sequences $\{x_n\}$ and $\{y_n\}$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, where $x_{2n+1} = F(x_{2n}, y_{2n})$, $x_{2n+2} = g(x_{2n+1})$ and $y_{2n+1} = F(y_{2n}, x_{2n}),\ y_{2n+2} = g(y_{2n+1}),\ \forall\ n \in \mathbb{N} \cup \{0\}$.

By the hypothesis, we have $x_n \leq x$ and $y_n \leq y,\ \forall\ n \in \mathbb{N}$, so $g(x_{2n-1}) = x_{2n} \leq x$ and $g(y_{2n-1}) = y_{2n} \leq y,\ \forall\ n \in \mathbb{N}$.

Consider

$$\begin{aligned}
&\psi[d(F(x,y),x_{2n})] + \psi[d(F(y,x),y_{2n})] \\
&= \psi[d(F(x,y),g(x_{2n-1}))] + \psi[d(F(y,x),g(y_{2n-1}))] \\
&\leq \alpha(d(x,x_{2n-1}))\beta(d(x,x_{2n-1})) + \alpha(d(y,y_{2n-1}))\beta(d(y,y_{2n-1}))
\end{aligned}$$

as $n \to \infty$ we get $\psi[d(F(x,y),x)] + \psi[d(F(y,x),y)] = 0$, i.e. $F(x,y) = x$ and $F(y,x) = y$.

Since F and g are weakly increasing we get $x \leq g(x)$ and $y \leq g(y)$.

Next, consider

$$\begin{aligned}
&\psi[d(F(x,y),g(x))] + \psi[d(F(y,x),g(y))] \\
&\leq \alpha(d(x,x))\beta(d(x,x)) + \alpha(d(y,y))\beta(d(y,y)) = 0,
\end{aligned}$$

i.e. $F(x,y) = g(x)$ and $F(y,x) = g(y)$. Thus, (x,y) is a coupled common fixed point of F and g. □

Remark 2.1. In Theorem 2.2, by taking g to be the identity map we get a coupled fixed point result.

Theorem 2.3. *Let $(X, d, \leq)$ be a partially ordered complete metric space.*

Let $F : X \times X \to X$ and $g : X \to X$ are weakly increasing mappings such that

$$\psi[d(F(x,y), g(z))] \leq \alpha(d(x, g(z)))\beta(d(x, g(z))), \tag{6}$$

for all comparable $x, g(z) \in X$, where $\alpha \in S$, $\psi \in \Psi$ and $\beta : [0,\infty) \to [0,\infty)$ is a continuous function with a condition $\psi(t) > \beta(t), \forall\ t > 0$. Then, F and g have a coupled common fixed point.

Proof. By considering two arbitrary elements $x_0, y_0 \in X$ and following as in Theorem 2.1, we get two increasing sequences $\{x_n\}$ and $\{y_n\}$, defined as $x_{2n+1} = F(x_{2n}, y_{2n})$, $x_{2n+2} = g(x_{2n+1})$ and $y_{2n+1} = F(y_{2n}, x_{2n})$, $y_{2n+2} = g(y_{2n+1})$, $\forall\ n \in \mathbb{N} \cup \{0\}$. Consider $\forall\ n \in \mathbb{N}$, $\psi[d(x_{2n+1}, x_{2n})] = \psi[d(F(x_{2n}, y_{2n}), g(x_{2n-1}))] \leq \alpha(d(x_{2n}, x_{2n}))\beta(d(x_{2n}, x_{2n}))$, i.e. $x_{2n+1} = x_{2n}$.

Suppose $x_{2n+1} \neq x_{2n+2}$. Consider

$$\begin{aligned}
\psi[d(x_{2n+1}, x_{2n+2})] &= \psi[d(F(x_{2n}, y_{2n}), g(x_{2n+1}))] \\
&\leq \alpha(d(x_{2n}, x_{2n+2}))\beta(d(x_{2n}, x_{2n+2})) \\
&= \alpha(d(x_{2n+1}, x_{2n+2}))\beta(d(x_{2n+1}, x_{2n+2})) \\
&< \beta(d(x_{2n+1}, x_{2n+2})) < \psi[d(x_{2n+1}, x_{2n+2})],
\end{aligned}$$

a contradiction. Thus, $x_n = x_{n+1}$, for all $n \in \{2, 3, ...\}$.

Similarly, we get $y_n = y_{n+1}$, for all $n \in \{2, 3, ...\}$, i.e. $x_2 = x_n$ and $y_2 = y_n$ for all $n \in \{2, 3, ...\}$. Hence the result. □

Remark 2.2. In Theorem 2.3, the contractive condition (6) is satisfied for all comparable $x, g(z) \in X$. So by (6) we can see that $\forall\ x \in g(X)$, and $\forall\ y \in X$, $F(x, y) = x$. Thus, $g(X) \times g(X)$ is a subset of the set of all coupled fixed points of F.

Acknowledgments

The first author would like to thank University Grant Commission (UGC) for the financial support.

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Accepted: April 02, 2020

Equitable coloring on subdivision-vertex join and subdivision-edge join of graphs

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Abstract. Adding a new vertex to any edge of a graph G gives a subdivision of G denoted by $S(G)$. Let G_1 and G_2 be two disjoint graphs. The subdivision-vertex join of G_1 and G_2, denoted by $G_1\dot{V}G_2$, is the graph obtained from $S(G_1)$ and G_2 by joining every vertex of $V(G_1)$ with every vertex of $V(G_2)$. The subdivision-edge join of G_1 and G_2, denoted by $G_1\underline{V}G_2$, is the graph obtained from $S(G_1)$ and G_2 by joining every vertex of $I(G_1)$ with every vertex of $V(G_2)$, where $I(G_1)$ is the set of inserted vertices of $S(G_1)$. In this paper we determine the equitable chromatic number of subdivision-vertex join and subdivision-edge join of path graph with path graph, complete graph and star graph.

Keywords: equitable coloring, subdivision graph, subdivision vertex join, subdivision edge join.

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1. Introduction

All graphs considered in this paper are connected, finite and simple, i.e. undirected, loopless and without multiple edges, unless otherwise stated.

If the set of vertices of a graph can be partitioned into k (possibly empty) classes $V_1, V_2, \ldots, V_k$ such that each V_i is an independent set and the condition $||V_i| - |V_j|| \leq 1$ holds for every pair (i, j), then G is said to be *equitably* k *− colorable*. The smallest integer k for which G is equitably k-colorable is known as the *equitable chromatic number* of G and it is denoted by $\chi_=(G)$ [2, 7, 11]. Since equitable coloring is a proper coloring with an additional constraint, $\chi(G) \leq \chi_=(G)$ for any graph G.

This model of coloring has many practical applications. Every time when we have to divide a system with binary conflict relations into equal or almost equal conflict-free subsystems we can model this situation by means of equitable graph coloring. One motivation for equitable graph coloring stems from scheduling problems. In this application the vertices of a graph represent a collection of jobs to be performed simultaneously. Due to load balancing considerations, it is desirable to perform equal or nearly-equal number of jobs in each time slot, and this balancing is exactly what an equitable coloring achieves. Furmanczyk [1] mentioned a specific application of this type of scheduling problem, namely assigning university course to time slots in a way that avoids scheduling incompatible pairs of courses at the same time and spreads the courses evenly among the available time slots, since then the usage of additional resources (e.g rooms) is maximal [3].

The notion of equitable colorability was introduced by Meyer [11]. However, an earlier work of Hajnal and Szemeredi [4] showed that a graph G with degree $\Delta(G)$ is equitably k -colorable if $k \geq \Delta(G) + 1$. The degree of a vertex in G is the number of vertices adjacent to it. The maximum degree over all vertices in G is denoted by $\Delta(G)$.

In [5] two new graph operations based on *subdivision graphs*: *subdivision − vertex join* and *subdivision − edge join*, are introduced.

Definition 1.1. *The subdivision graph $S(G)$ of the graph G is obtained from G by inserting a new vertex of degree 2 on each edge of G. For $k \geq 1$, the k-th subdivision graph $S_k(G)$ is obtained from G by inserting k new vertices of degree 2 on each edge of G.*

Definition 1.2. *The subdivision-vertex join of two vertex disjoint graphs G_1 and G_2 denoted by $G_1 \dot{\vee} G_2$, is the graph obtained from $S(G_1)$ and G_2 by joining each vertex of $V(G_1)$ with every vertex of $V(G_2)$.*

Definition 1.3. *The subdivision-edge join of two vertex disjoint graphs G_1 and G_2 denoted by $G_1 \underline{\vee} G_2$, is the graph obtained from $S(G_1)$ and G_2 by joining each vertex of $E(G_1)$ with every vertex of $V(G_2)$.*

The adjacency spectrum of the graphs $G_1 \dot{V} G_2$ and $G_1 \underline{V} G_2$ were obtained in [5, 9], and the distance spectrum of the graphs $G_1 \dot{V} G_2$ and $G_1 \underline{V} G_2$ were

obtained in [6]. Further works on their L-spectra and Q-spectra are given in [10]. Two new graph operations on subdivision graphs namely subdivision-vertex neighbourhood corona and subdivision-edge corona are defined in [8].

In this paper we focus on the equitable coloring of the graphs $G_1\dot{V}G_2$ and $G_1\underline{V}G_2$ where G_1 is a path graph and G_2 is path graph, complete graph and star graph.

2. Subdivision-vertex join and subdivision-edge join of a path graph with a path graph

Let G_1 and G_2 be two graphs with m and n vertices, respectively. Let the vertex sets of the graphs G_1 and G_2 be denoted by $v_i : 1 \leq i \leq m$ and $u_j : 1 \leq j \leq n$, respectively. By the definition of Subdivision graph, the vertices of the graph G_1 are subdivided exactly once. Let these subdivided vertices be denoted by $v_{i,i+1} : 1 \leq i \leq m-1$.

By the definition of Subdivision vertex join, each vertex of the graph G_1 (i.e.,)$v_i : 1 \leq i \leq m$ is joined to every vertices of the graph G_2 (i.e.,)$u_j : 1 \leq j \leq n$ and by the definition of Subdivision edge join, each subdivided vertex of the graph G_1 (i.e.,)$v_{i,i+1} : 1 \leq i \leq m-1$ is joined to every vertices of the graph G_2 (i.e.,)$u_j : 1 \leq j \leq n$.

The total number of vertices of both subdivision vertex join and subdivision edge join of a path with a path graph are $2m + n - 1$.

Theorem 2.1. *The equitable chromatic number of subdivision vertex join of a path graph P_m with a path graph P_n is given by the following formulas:*

1. $\chi_{=}(P_m\dot{V}P_n) = 3$ *for* $m = n$.
2. $\chi_{=}(P_m\dot{V}P_n) = 3$ *for* $m > n$ *and* $|m-n| = 1$.
3. $\chi_{=}(P_m\dot{V}P_n) = 4$ *for* $m > n$ *and* $|m-n| > 1$.
4. $\chi_{=}(P_m\dot{V}P_n) = 3$ *for* $m < n$ *and* $|m-n| \leq 3$.
5. $\chi_{=}(P_m\dot{V}P_n) = \lceil \frac{2m+n}{m+1} \rceil$ *for* $m < n$ *and* $|m-n| > 3$.

Proof. Let $V(P_m\dot{V}P_n) = \{v_i : 1 \leq i \leq m\} \cup \{u_j : 1 \leq j \leq n\} \cup \{v_{i,i+1} : 1 \leq i \leq m-1\}$. Since each $\{v_i : 1 \leq i \leq m\}$ is joined to all $\{u_j : 1 \leq j \leq n\}$, a minimum of two colors are needed. Also, since no two adjacent vertices of the graph G_2 can have the same color, an additional color is needed.

Hence, $\chi_{=}(P_m\dot{V}P_n) \geq 3$. In the following cases, the assigning of colors for $m = n, m > n$ and $m < n$ are done.

Case 1. If $m = n$. Since a minimum of three colors is required, let C_1, C_2 and C_3 be the color classes of the colors 1,2 and 3, respectively.

Now,

$$C_1 = \{v_i : 1 \leq i \leq m\},$$
$$C_2 = \{v_{1,2}, v_{3,4}, v_{5,6}, \ldots, u_2, u_4, u_6, \ldots\},$$
$$C_3 = \{v_{2,3}, v_{4,5}, v_{6,7}, \ldots, u_1, u_3, u_5, \ldots\}.$$

Thus, C_1 will contain m vertices, C_2 and C_3 will contain either m or $m-1$ vertices in such a way that the number of vertices in any two color classes differ by atmost one and hence is a minimal equitable coloring.

Hence, $\chi_=(P_m \dot{V} P_n) = 3$.

Case 2. If $m > n$.

(i) $|m-n| = 1$. Let color 1 be assigned to all the vertices $v_i : 1 \leq i \leq m$. Let colors 2 and 3 be assigned, alternatively to the vertices $v_{i,i+1} : 1 \leq i \leq m-1$. If m is odd, colors $2, 3, 2, 3, \ldots$ are assigned consecutively to the vertices $u_1, u_2, u_3, u_4, \ldots$, respectively. If m is even, colors $3, 2, 3, 2, \ldots$ are assigned consecutively to the vertices $u_1, u_2, u_3, u_4, \ldots$, respectively.

Thus, color 1 is assigned m times, colors 2 and 3 are assigned $m-1$ times each satisfying equitable 3-coloring and hence is a minimal equitable coloring.

Therefore, $\chi_=(P_m \dot{V} P_n) = 3$.

(ii) $|m-n| > 1$. Suppose $\chi_=(P_m \dot{V} P_n) = 3$. If color 1 is assigned to all $v_i : 1 \leq i \leq m$, then color 1 occurs m times. The remaining number of vertices are $m-1+n$. When these vertices are divided by the remaining two colors, atleast one of the two colors occurs greater than or equal to $m-2$ times which does not satisfy equitable coloring. This is a contradiction to $\chi_=(P_m \dot{V} P_n) = 3$. Hence let an additional color 4 be used.

Let color 1 be assigned to the vertices from v_1 to $v_{\lceil \frac{2m+n-1}{4} \rceil}$ and color 2 be assigned to the remaining vertices from $v_{\lceil \frac{2m+n-1}{4} \rceil + 1}$ to v_m. Here color 2 is assigned $\lfloor \frac{2m-n+1}{4} \rfloor$ times. For the vertices $v_{i,i+1} : 1 \leq i \leq m-1$, color 2 is assigned continuously to the first $\lceil \frac{n-1}{2} \rceil$ vertices. Hence in total, the color 2 is assigned either $\lfloor \frac{2m+n-1}{4} \rfloor$ or $\lceil \frac{2m+n-1}{4} \rceil$ times. Remaining $v_{i,i+1}$ vertices are colored alternatively by Colors 3 and 4, both colors occuring either $\lceil \frac{2m-n-1}{4} \rceil$ or $\lfloor \frac{2m-n-1}{4} \rfloor$ times each. Color 3 and 4 are assigned alternatively to the vertices $u_j : 1 \leq j \leq n$, either $\lceil \frac{n}{2} \rceil$ or $\lfloor \frac{n}{2} \rfloor$ times. Hence in total colors 3 and 4 are assigned either $\lceil \frac{2m+n-1}{4} \rceil$ or $\lfloor \frac{2m+n-1}{4} \rfloor$ times each.

Thus, all the four colors are assigned either $\lceil \frac{2m+n-1}{4} \rceil$ or $\lfloor \frac{2m+n-1}{4} \rfloor$ times satisfying equitable 4-coloring.

Therefore, $\chi_=(P_m \dot{V} P_n) = 4$.

Case 3. If $m < n$.

(i) $|m-n| \leq 3$. Let color 1 be assigned to $v_i : 1 \leq i \leq m$. Hence color 1 is assigned m times. Let the colors 2,3,2,3,...,2,3,... be consecutively assigned to the vertices $v_{1,2}, v_{2,3}, v_{3,4}, v_{4,5}, \ldots, u_1, u_2, \ldots$, respectively such that each color appears either m times or $m-1$ times. Thus all the three colors occurs either

m times or $m+1$ times satisfying equitable 3-coloring and hence is a minimum equitable coloring. Therefore, $\chi_=(P_m \dot{V} P_n) = 3$.

(ii) $|m-n| > 3$. Color 1 is assigned to all the vertices $v_i : 1 \leq i \leq m$. Hence color 1 occurs m times. To satisfy equitability the remaining colors must occur either m times or $m \pm 1$ times. Since m vertices are assigned color 1, the remaining number of vertices are $m-1+n$. The number of colors assigned to these vertices are found by $\left\lceil \frac{m-1+n}{m+1} \right\rceil$ which satisfies the equitability. Hence, the total number of colors needed is $\left\lceil 1 + \frac{m-1+n}{m+1} \right\rceil = \left\lceil \frac{2m+n}{m+1} \right\rceil$. Therefore, $\chi_=(P_m \dot{V} P_n) = \left\lceil \frac{2m+n}{m+1} \right\rceil$. □

Theorem 2.2. *The equitable chromatic number of subdivision edge join of a path graph with a path graph is given by the following formulas:*

1. *For $m \geq n$,*
   - *(i) $\chi_=(P_m \underline{V} P_n) = 3$ if $0 \leq |m-n| \leq 4$;*
   - *(ii) $\chi_=(P_m \underline{V} P_n) = 4$ if $|m-n| > 4$.*
2. *For $m < n$, $\chi_=(P_m \underline{V} P_n) = \left\lceil \frac{2m+n}{n} \right\rceil$.*

Proof. Let $V(P_m \underline{V} P_n) = \{v_i : 1 \leq i \leq m\} \cup \{u_j : 1 \leq j \leq n\} \cup \{v_{i,i+1} : 1 \leq i \leq m-1\}$. Since each $\{v_{i,i+1} : 1 \leq i \leq m-1\}$ is joined to all $\{u_j : 1 \leq j \leq n\}$, a minimum of two colors is needed. Also since no two adjacent vertices of $\{u_j : 1 \leq j \leq n\}$ can contain the same color, an additional color is needed. Hence $\chi_=(P_m \underline{V} P_n) \geq 3$.

Case 1. For $m \geq n$. We assign the colors by the following cases:
(i) If $0 \leq |m-n| \leq 4$. As discussed, the subdivision edge join must contain at least three colors say 1,2 and 3. Let C_1, C_2 and C_3 be the respective color classes. Let

$$C_1 = \{v_{i,i+1} : 1 \leq i \leq m-1\},$$
$$C_2 = \{v_1, v_3, v_5, \ldots, u_2, u_4, \ldots\},$$
$$C_3 = \{v_2, v_4, v_6, \ldots, u_1, u_3, \ldots\}$$

such that, the number of vertices in any two color classes differ by atmost one and hence is a minimal equitable coloring of $P_m \underline{V} P_n$. Therefore, $\chi_=(P_m \underline{V} P_n) = 3$.
(ii) If $|m-n| > 4$. Suppose $\chi_=(P_m \underline{V} P_n) = 3$. If color 1 is assigned to all $v_{i,i+1} : 1 \leq i \leq m-1$, then color 1 occurs $m-1$ times. The remaining number of vertices are $m+n$. When these $m+n$ vertices are divided by the remaining two colors, atleast one of these two colors occurs greater than or equal to $m-3$ times which does not satisfy equitable coloring. This is a contradiction to $\chi_=(P_m \underline{V} P_n) = 3$. So let an additional color 4 be used.

Let color 1 be assigned to the first $\left\lceil \frac{2m+n-1}{4} \right\rceil$ vertices of $v_{i,i+1} : 1 \leq i \leq \left\lceil \frac{2m+n-1}{4} \right\rceil$ and the remaining vertices received color 2. Hence color 1 is assigned

$\lceil \frac{2m+n-1}{4} \rceil$ times and color 2 is assigned $\lfloor \frac{2m-n-3}{4} \rfloor$ times. Now color 2 is assigned continuously to the first $\lceil \frac{n+1}{2} \rceil$ vertices of $v_i : 1 \leq i \leq m$. Thus, in total color 2 is assigned either $\lceil \frac{2m+n-1}{4} \rceil$ or $\lfloor \frac{2m+n-1}{4} \rfloor$ times. For the remaining vertices of v_i, colors 2 and 3 are assigned, alternatively with both the colors occuring either $\lceil \frac{2m-n-1}{4} \rceil$ or $\lfloor \frac{2m-n-1}{4} \rfloor$ times each. The vertices $u_j : 1 \leq j \leq n$ are assigned colors 3 and 4, alternatively each occuring either $\lceil \frac{n}{2} \rceil$ or $\lfloor \frac{n}{2} \rfloor$ times.

Thus, all the four colors are assigned either $\lceil \frac{2m+n-1}{4} \rceil$ or $\lfloor \frac{2m+n-1}{4} \rfloor$ times satisfying equitable 4-coloring. Therefore, $\chi_=(P_m \underline{V} P_n) = 4$.

Case 2. For $m < n$. Color 1 is assigned to all the vertices $v_{i,i+1} : 1 \leq i \leq m-1$. Hence color 1 occurs $m-1$ times. To satisfy equitability, the remaining colors must occur m times, $m-1$ times or $m-2$ times. Since $m-1$ vertices are assigned color 1, the remaining number of vertices are $m+n$. The number of colors assigned to these vertices are found by $\lceil \frac{m+n}{m} \rceil$ which satisfies equitability. Hence the total number of colors used is $1 + \lceil \frac{m+n}{m} \rceil = \lceil \frac{2m+n}{m} \rceil$. The colors $2, 3, 4, \ldots, \lceil \frac{2m+n}{m} \rceil$ are assigned consecutively to the vertices $v_1, v_2, v_3, \ldots, v_m, u_1, u_2, \ldots$. Each of these colors are assigned $m-1, m$ times or $m-2, m-1$ times satisfying equitable coloring. Therefore, $\chi_=(P_m \underline{V} P_n) = \lceil \frac{2m+n}{m} \rceil$. □

3. Subdivision-vertex join and subdivision-edge join of a path graph with a complete graph

Let G_1 denote a path graph P_m with m vertices and G_2 denote a complete graph K_n with n vertices. The vertex sets of the graphs G_1 and G_2 are denoted by $v_i : 1 \leq i \leq m$ and $u_j : 1 \leq j \leq n$, respectively. The subdivided vertices of the graph G_1 are denoted by $v_{i,i+1} : 1 \leq i \leq m-1$.

The total number of vertices of both subdivision vertex join and edge join of path with complete graph are $2m + n - 1$.

Theorem 3.1. *The equitable chromatic number of subdivision vertex join of a path graph with a complete graph is*

$$\chi_=(P_m \dot{V} K_n) = \begin{cases} n + \lceil \frac{m}{2} \rceil, & \text{for } m \leq n \\ n + \lceil \frac{m}{1+\lceil \frac{m}{n} \rceil} \rceil, & \text{for } m > n. \end{cases}$$

Proof. Let $V(P_m \dot{V} K_n) = \{v_i : 1 \leq i \leq m\} \cup \{u_j : 1 \leq j \leq n\} \cup \{v_{i,i+1} : 1 \leq i \leq m-1\}$. Since G_2 is a complete graph with n vertices, a minimum of n colors is needed. Since each vertex of the graph G_1 is joined with every vertices of the graph G_2, hence $\chi_=(P_m \dot{V} K_n) > n$.

Let us consider the following two cases:

Case 1. For $m \leq n$. By the definition of subdivision vertex join, the n colors assigned to the vertices of the graph G_2. i.e., $u_j : 1 \leq j \leq n$ cannot be assigned to any $v_i : 1 \leq i \leq m$. The subdivided vertices $v_{i,i+1} : 1 \leq i \leq m-1$ can be assigned the colors $1, 2, 3, \ldots, n$. Since $m \leq n$, the colors $1, 2, \ldots, n$ will

occur maximum two times. To satisfy the equitability, the remaining m vertices should be colored in such a way that each color occurs at the maximum of two times. Thus, the number of colors needed for these m vertices are found by $\lceil \frac{m}{2} \rceil$. Therefore in total $n + \lceil \frac{m}{2} \rceil$ colors are needed. The assigning of colors as follows:

Let $u_1, u_2, \ldots, u_n$ be assigned the colors $1, 2, \ldots, n$, respectively. Similarly the vertices $v_{i,i+1} : 1 \leq i \leq m-1$ are assigned the colors $1, 2, \ldots, n$ consecutively. Now the remaining m vertices are assigned $\lceil \frac{m}{2} \rceil$ colors in such a way that the vertices $v_1, v_2, v_3, v_4, \ldots$, are assigned the colors $n+1, n+1, n+2, n+2, \ldots$, respectively. Each of these colors occur either one or two times satisfying equitable coloring. Therefore, $\chi_{=}(P_m \dot{V} K_n) = n + \lceil \frac{m}{2} \rceil$.

Case 2. For $m > n$. The complete graph K_n is assigned with n colors. These n colors are assigned to the subdivided vertices $v_{i,i+1} : 1 \leq i \leq m-1$. Since $m > n$, the colors $1, 2, \ldots, n$ are assigned atmost $\lceil \frac{m}{n} \rceil$ times to $v_{i,i+1} : 1 \leq i \leq m-1$. Thus in total n colors are assigned $1 + \lceil \frac{m}{n} \rceil$ times. To find the number of colors needed for the remaining m vertices, we use $\left\lceil \frac{m}{1+\lceil \frac{m}{n} \rceil} \right\rceil$. Thus the total number of colors required satisfying equitable coloring is $n + \left\lceil \frac{m}{1+\lceil \frac{m}{n} \rceil} \right\rceil$. The assigning of these colors are as follows:

Colors $1, 2, \ldots, n$ are assigned to the vertices $u_1, u_2, \ldots, u_n$, respectively. The colors $1, 2, \ldots, n, 1, 2, \ldots$ are assigned consecutively to the vertices $v_{i,i+1} : 1 \leq i \leq m-1$. Now, the remaining vertices $v_i : 1 \leq i \leq m$ are assigned the colors $n+1, n+2, \ldots, n + \left\lceil \frac{m}{1+\lceil \frac{m}{n} \rceil} \right\rceil, n+1, n+2, \ldots$ consecutively.

Each of these colors occur either $\lceil \frac{m}{n} \rceil, \lceil \frac{m}{n} \rceil + 1$ times or $\lceil \frac{m}{n} \rceil, \lceil \frac{m}{n} \rceil - 1$ times satisfying equitable coloring.

Therefore, $\chi_{=}(P_m \dot{V} K_n) = n + \left\lceil \frac{m}{1+\lceil \frac{m}{n} \rceil} \right\rceil$ □

Theorem 3.2. *The equitable chromatic number of subdivision edge join of a path graph P_m with a complete graph K_n is given by the following formulas:*

1. *For $m = n$, $\chi_{=}(P_m \underline{V} K_n) = n + \lceil \frac{m-1}{3} \rceil$;*
2. *For $m < n$, $\chi_{=}(P_m \underline{V} K_n) = n + \lfloor \frac{m}{2} \rfloor$;*
3. *For $m > n$, $\chi_{=}(P_m \underline{V} K_n) \leq n + \left\lceil \frac{m-1}{1+\lceil \frac{m}{n} \rceil} \right\rceil$.*

Proof. Let $V(P_m \underline{V} K_n) = \{v_i : 1 \leq i \leq m\} \cup \{u_j : 1 \leq j \leq n\} \cup \{v_{i,i+1} : 1 \leq i \leq m-1\}$. Since n colors should be assigned to K_n and none of these colors can be used for $v_{i,i+1} : 1 \leq i \leq m-1$, it implies $\chi_{=}(P_m \underline{V} K_n) > n$.

Let us consider the following three cases:

Case 1. If $m = n$. The n colors assigned to $u_j : 1 \leq j \leq n$ cannot be assigned to any of the subdivided vertices $v_{i,i+1} : 1 \leq i \leq m-1$. Since $m = n$, the colors

$1, 2, \ldots, n$ are assigned to the vertices $v_1, v_2, \ldots, v_n$, respectively. Hence 1 to n colors are assigned two times. So the remaining $m-1$ vertices are assigned colors in such a way that each color occurs at the maximum three times.

The number of colors assigned to these $m-1$ vertices are found by $\lceil \frac{m-1}{3} \rceil$. Hence the total number of colors needed is $n + \lceil \frac{m-1}{3} \rceil$. The assigning of colors is as follows:

Colors $1, 2, \ldots, n$ are assigned to the vertices $v_1, v_2, \ldots, v_m$ and the vertices $u_1, u_2, \ldots, u_n$. The $\lceil \frac{m-1}{3} \rceil$ colors consecutive to n. (i.e.,) $n+1, n+2, \ldots$ are assigned to the vertices $v_{i,i+1} : 1 \leq i \leq m-1$. Each of these colors are assigned either two or three times satisfying equitable coloring.

Hence, $\chi_=(P_m \underline{V} K_n) = n + \lceil \frac{m-1}{3} \rceil$.

Case 2. If $m < n$. By the definition of subdivision edge join, the n colors assigned to the vertices of the graph G_2. (i.e) $u_j : 1 \leq j \leq n$ cannot be assigned to any $v_{i,i+1} : 1 \leq i \leq m-1$. The vertices $v_i : 1 \leq i \leq m$ can be assigned the colors $1, 2, 3, \ldots, n$. Since $m < n$, the colors $1, 2, 3, \ldots, n$ will occur at the maximum two times. To satisfy the equitability, the remaining $m-1$ vertices should be colored in such a way that each color occurs at the maximum two times. Thus the number of colors needed for these $m-1$ vertices are found by $\lfloor \frac{m}{2} \rfloor$. Therefore, in total $n + \lfloor \frac{m}{2} \rfloor$ colors are needed. The assigning of colors as follows:

Let $u_1, u_2, \ldots, u_n$ be assigned the colors $1, 2, \ldots, n$, respectively. Similarly the vertices $v_i : 1 \leq i \leq n$ are assigned the colors$1, 2, \ldots, n$ consecutively. Now, the remaining $v_{i,i+1} : 1 \leq i \leq n$ are assigned $\lfloor \frac{m}{2} \rfloor$ colors in such a way that the vertices $v_{1,2}, v_{2,3}, v_{3,4}, \ldots$ are assigned the colors $n+1, n+1, n+2, \ldots$, respectively. Each of these colors occur one or two times satisfying equitable coloring.

Hence, $\chi_=(P_m \underline{V} K_n) = n + \lfloor \frac{m}{2} \rfloor$.

Case 3. If $m > n$. The colors $1, 2, \ldots, n$ are assigned to the vertices $u_j : 1 \leq j \leq n$. These colors are also assigned to the vertices $v_i : 1 \leq i \leq m$. Since $m > n$, the colors $1, 2, \ldots, n$ are assigned atmost $\lceil \frac{m}{n} \rceil$ times to $v_i : 1 \leq i \leq m$. Thus, in total n colors are assigned atmost $1 + \lceil \frac{m}{n} \rceil$ times. To find the number of colors needed for the remaining $m-1$ vertices, we use $\left\lceil \frac{m-1}{1+\lceil \frac{m}{n} \rceil} \right\rceil$.

The colors $1, 2, \ldots, n$ are assigned to the vertices $u_1, u_2, \ldots, u_n$, respectively. To the vertices $v_i : 1 \leq i \leq m$ the colors $1, 2, \ldots, n$ are assigned consecutively. The remaining vertices $v_{i,i+1} : 1 \leq i \leq m-1$ are assigned the colors $n+1, n+2, \ldots, n + \left\lceil \frac{m-1}{1+\lceil \frac{m}{n} \rceil} \right\rceil, n+1, n+2, \ldots$ consecutively.

Each of these colors appears $\lceil \frac{m}{n} \rceil, \lceil \frac{m}{n} \rceil + 1$ times or $\lceil \frac{m}{n} \rceil, \lceil \frac{m}{n} \rceil - 1$ times satisfying equitable coloring. An easy check shows that without loss of generality, in few cases the equitability is satisfied with one color lesser than $n + \left\lceil \frac{m-1}{1+\lceil \frac{m}{n} \rceil} \right\rceil$. In this case each color appears $\lceil \frac{m}{2} \rceil + 1$ *or* $\lceil \frac{m}{2} \rceil + 2$ times.

Therefore, $\chi_=(P_m \underline{V} K_n) \leq n + \left\lceil \frac{m-1}{1+\lceil \frac{m}{n} \rceil} \right\rceil$. □

4. Subdivision-vertex join and subdivision-edge join of a path graph with a star graph

Let G_1 denote a path graph P_m with m vertices and G_2 denote a star graph S_n with n vertices. Let $v_i : 1 \leq i \leq m$ and $v_{i,i+1} : 1 \leq i \leq m-1$ denote the vertices and sub-divided vertices of the graph G_1, respectively. Let $u_j : 1 \leq j \leq n$, be the pendant vertices of the star graph S_n and u be the vertex of S_n adjacent to $u_j : 1 \leq j \leq n$. Obviously, $deg(u) = n$.

The total number of vertices of both subdivision vertex join and edge join of path with star graph are $2m + n$.

Theorem 4.1. *The equitable chromatic number of subdivision vertex join of a path graph P_m with a star graph S_n is given by the following formulas:*

1. *For $m \geq n$,*

$$\chi_=(P_m \dot{V} S_n) = \begin{cases} 3, & \text{if } 0 \leq |m-n| \leq 2 \\ 4, & \text{otherwise.} \end{cases}$$

2. *For $m < n$,*

$$\chi_=(P_m \dot{V} S_n) = \begin{cases} 3, & \text{if } |m-n| = 1, \\ 2 + \left\lceil \frac{n}{m+1} \right\rceil, & \text{if } m = 2, 3, \ldots, (2 + \lceil \frac{n+1}{3} \rceil), \\ 3 + \left\lceil \frac{n}{\lceil \frac{2m}{3} \rceil} \right\rceil, & \text{if } m = (3 + \lceil \frac{n+1}{3} \rceil), (4 + \lceil \frac{n+1}{3} \rceil), \ldots. \end{cases}$$

Proof. Let $V(P_m \dot{V} S_n) = \{v_i : 1 \leq i \leq m\} \cup \{v_{i,i+1} : 1 \leq i \leq m-1\} \cup \{u\} \cup \{u_j : 1 \leq j \leq n\}$. Since G_2 is a star graph with $1, n$ vertices, a minimum of two colors is needed. Since each vertex of the graph G_1 is joined with every vertices of the graph G_2 , $\chi_=(P_m \dot{V} S_n) \geq 3$. Let us consider the following two cases:

Case 1. For $m \geq n$.
(i) If $0 \leq |m-n| \leq 2$

As discussed, the subdivision vertex join must contain atleast three colors say $1, 2, 3$. The assigning of colors is done in the following manner. Let color 1 be assigned to all the vertices $v_i : 1 \leq i \leq m$ and color 2 be assigned to the vertex u. For $|m-n| = 0$ *and* 1, color 2 is assigned to all the vertices $v_{i,i+1} : 1 \leq i \leq m-1$. The remaining n vertices are assigned color 3. For $|m-n| = 2$, color 2 is assigned to the vertices $v_{i,i+1} : 1 \leq i \leq m-2$ and color 3 to the vertex $v_{m-1,m}$. The vertices $u_j : 1 \leq j \leq n$ are assigned color 3. In both the cases, color 1 is assigned m times, colors 2 and 3 are assigned m or $m-1$ times satisfying equitable 3-coloring.

Therefore, $\chi_=(P_m \dot{V} S_n) = 3$.

(ii) If $|m-n|>2$. By default a minimum of three colors is required. Suppose $\chi_{=}(P_m\dot{V}S_n)=3$. If color 1 is assigned to all $v_i:1\leq i\leq m$, then color 1 occurs m times. The remaining number of vertices are $m+n$. When these $m+n$ vertices are divided by colors 2 and 3, atleast one of these two colors appears greater than or equal to $m-2$ times which does not satisfy equitable coloring. Hence $\chi_{=}(P_m\dot{V}S_n)\neq 3$. So an additional color 4 is used.

The four colors are assigned in the following manner. Color 1 is assigned to the first $\lceil\frac{2m+n}{4}\rceil$ vertices of $v_i:1\leq i\leq m$ and its remaining vertices are assigned color 2.Thus color 2 occurs $\lfloor\frac{2m-n}{4}\rfloor$ times. If $\lfloor\frac{2m-n}{4}\rfloor=\lceil\frac{2m+n}{4}\rceil$ *or* $\lfloor\frac{2m+n}{4}\rfloor$ then the remaining n vertices are assigned colors 3 and 4 such that each color appears $\lceil\frac{2m+n}{4}\rceil$ or $\lfloor\frac{2m+n}{4}\rfloor$ times. If $\lfloor\frac{2m-n}{4}\rfloor\neq\lceil\frac{2m+n}{4}\rceil$ *or* $\lfloor\frac{2m+n}{4}\rfloor$ then color 2 is again assigned to the consecutive vertices of $v_{i,i+1}:1\leq i\leq m-1$ such that in total color 2 occurs either $\lceil\frac{2m+n}{4}\rceil$ or $\lfloor\frac{2m+n}{4}\rfloor$ times. Now the vertex u of the star graph is assigned color 3 and its adjacent vertices $u_j:1\leq j\leq n$ are all assigned color 4. The remaining vertices of $v_{i,i+1}:1\leq i\leq m-1$ are assigned color 3 consecutively such that in total color 3 appears either $\lceil\frac{2m+n}{4}\rceil$ or $\lfloor\frac{2m+n}{4}\rfloor$ times and left out vertices (if any) are assigned color 4. Thus, all the four colors appear either $\lceil\frac{2m+n}{4}\rceil$ or $\lfloor\frac{2m+n}{4}\rfloor$ times satisfying equitable 4-coloring. Therefore, $\chi_{=}(P_m\dot{V}S_n)=4$.

Case 2. For $m<n$. If $|m-n|=1$, as discussed earlier color 1 is assigned to all $v_i:1\leq i\leq m$, color 2 to the vertex u and all $v_{i,i+1}:1\leq i\leq m-1$ and color 3 is assigned to all $u_j:1\leq j\leq n$ satisfying equitable coloring.

If $|m-n|\neq 1$, the assigning of colors is done in two different ways as follows:
(i) For $m=2,3,\ldots,(2+\lceil\frac{n+1}{3}\rceil)$.

In this case let color 1 be assigned to all $v_i:1\leq i\leq m$. Let color 2 be assigned to the vertex u and all $v_{i,i+1}:1\leq i\leq m-1$. Thus colors 1 and 2 occurs m times each. The number of colors used to color n vertices are found by $\left\lceil\frac{n}{m+1}\right\rceil$ (colors 1 and 2 being forbidden). These $\left\lceil\frac{n}{m+1}\right\rceil$ colors are colored consecutively to $u_1,u_2,\ldots,u_n$ such that each color appears $m-1,m$ or $m,m+1$ times. Thus in total $2+\left\lceil\frac{n}{m+1}\right\rceil$ colors are used such that each color classes differ in size by at most 1 satisfying equitable coloring.

Therefore, $\chi_{=}(P_m\dot{V}S_n)=2+\left\lceil\frac{n}{m+1}\right\rceil$.
(ii) For $m=(3+\lceil\frac{n+1}{3}\rceil),(4+\lceil\frac{n+1}{3}\rceil),\ldots$

If colors 1 and 2 are assigned as discussed earlier and the remaining n vertices are assigned $\left\lceil\frac{n}{m+1}\right\rceil$ colors, the colors assigned to n vertices will occur less than $m-2$ times which does not satisfy equitable coloring. In this case in addition to colors 1 and 2, color 3 is assigned to few vertices of $v_{i,i+1}:1\leq i\leq m-1$. Let color 1 be assigned to $v_i:1\leq i\leq\lceil\frac{2m}{3}\rceil$ and color 2 be assigned to the remaining vertices of $v_i:\lceil\frac{2m}{3}+1\rceil\leq i\leq m$. Color 2 is assigned $\lfloor\frac{m}{3}\rfloor$ times. Now Color 2 is again assigned consecutively to the first $\lfloor\frac{m}{3}\rfloor$ vertices of $v_{i,i+1}:1\leq i\leq m-1$ such that in total color 2 appears $\lceil\frac{2m}{3}\rceil$ or $\lfloor\frac{2m}{3}\rfloor$ times. Similarly color 3 is assigned to the vertex u and the remaining vertices of $v_{i,i+1}:1\leq i\leq m-1$

such that the total number of times color 3 appears will be $\lceil \frac{2m}{3} \rceil$ or $\lfloor \frac{2m}{3} \rfloor$ times. The left out vertices (if any) of $v_{i,i+1} : 1 \leq i \leq m-1$ are assigned the next consecutive color. Since each of the three colors occur $\lceil \frac{2m}{3} \rceil$ or $\lfloor \frac{2m}{3} \rfloor$ times the remaining n vertices are assigned the colors $\left\lceil \frac{n}{\lceil \frac{2m}{3} \rceil} \right\rceil$. Each of the colors assigned will occur either $\lceil \frac{2m}{3} \rceil$ or $\lfloor \frac{2m}{3} \rfloor$ times satisfying equitable coloring.

Therefore, $\chi_=(P_m \dot{V} S_n) = 3 + \left\lceil \frac{n}{\lceil \frac{2m}{3} \rceil} \right\rceil$. □

Theorem 4.2. *The equitable chromatic number of subdivision edge join of a path graph P_m with a star graph S_n is given by the following formulas:*

1. *For $m = n$,*

$$\chi_=(P_m \underline{V} S_n) = \begin{cases} 4, & \text{if } m \leq 7, \\ 5, & \text{if } m > 7. \end{cases}$$

2. *For $m > n$,*

$$\chi_=(P_m \underline{V} S_n) = \begin{cases} 3, & |m-n| \leq 5, \\ 4, & \text{if } |m-n| > 5 \text{ and } n \leq \left\lceil \frac{2m+n}{4} \right\rceil, \\ 5, & \text{if } |m-n| > 5 \text{ and } n > \left\lceil \frac{2m+n}{4} \right\rceil. \end{cases}$$

3. *For $m < n$,*

$$\chi_=(P_m \underline{V} S_n) = \begin{cases} 2 + \lceil \frac{n+1}{m} \rceil & \text{if } m = 2, 3, \ldots, (2 + \lceil \frac{n+2}{2} \rceil), \\ 3 + \left\lceil \frac{n}{\lceil \frac{2m+1}{3} \rceil} \right\rceil & \text{if } m = (3 + \lceil \frac{n+2}{2} \rceil), (4 + \lceil \frac{n+2}{2} \rceil), \ldots \end{cases}$$

Proof. Let $V(P_m \underline{V} S_n) = \{v_i : 1 \leq i \leq m\} \cup \{v_{i,i+1} : 1 \leq i \leq m-1\} \cup \{u\} \cup \{u_j : 1 \leq j \leq n\}$. Since G_2 is a star graph with $1, n$ vertices, a minimum of two colors is needed. Since each sub-divided vertex of the graph G_1 is joined with every vertices of the graph G_2 , $\chi_=(P_m \underline{V} S_n) \geq 3$. Let us consider the following cases:

Case 1. For $m = n$. Let color 1 be assigned to all $v_{i,i+1} : 1 \leq i \leq m-1$. Color 1 appears $m-1$ times. To satisfy equitability, the colors assigned to the remaining $m+n+1$ vertices must occur $m-2, m-1$ *or* m times. Suppose $\chi_=(P_m \underline{V} S_n) = 3$. Then color 2 must be assigned to $v_i : 1 \leq i \leq m$ and the vertex u. Hence color 2 appears $m+1$ times. This implies $\chi_=(P_m \underline{V} S_n) \neq 3$. The assigning of colors is done by two different ways:
(i) If $m \leq 7$. Color 1 is assigned to all $v_{i,i+1} : 1 \leq i \leq m-1$ and color 2 is assigned to the vertex u. The remaining $m+n$ vertices are divided by three colors 2,3 and 4 such that each color appears $m-2, m-1$ *or* m times satisfying proper vertex coloring and equitable 4-coloring.

Therefore, $\chi_=(P_m \underline{V} S_n) = 4$.

(ii) If $m > 7$. In this case when color 1 is assigned $m-1$ times, the remaining vertices cannot be colored by three colors satisfying equitability.

Hence, $\chi_{=}(P_m \underline{V} S_n) \neq 4$. So an additional color 5 is used. Now, color 1 is assigned to first $\lceil \frac{2m+n}{5} \rceil$ vertices of $v_{i,i+1} : 1 \leq i \leq m-1$ and its remaining vertices are assigned color 2. Color 3 is assigned to the vertex u. Now the remaining m and n vertices are assigned colors $2, 3, 4$ *and* 5 such that each color appears either $\lceil \frac{2m+n}{5} \rceil$ or $\lfloor \frac{2m+n}{5} \rfloor$ times equitability (colors 2 and 3 being forbidden to n vertices).

Therefore, $\chi_{=}(P_m \underline{V} S_n) = 5$.

Case 2. For $m > n$.

(i) $|m-n| \leq 5$. In this case when $2m+n$ vertices are divided by three color classes at least one class will contain $m-1$ vertices and the remaining two classes will cotain $m-2, m-1$ *or* m vertices. Hence the difference between each color classes differ in size by atmost one satisfying equitability.

Let color 1 be assigned to all $v_{i,i+1} : 1 \leq i \leq m-1$ and color 2 to the vertex u. The remaining m and n vertices are assigned colors 2 and 3 such that each color appears either $m-2, m-1$ *or* m times satisfying equitability (color 2 forbidden to n vertices).

Therefore, $\chi_{=}(P_m \underline{V} S_n) = 3$.

(ii) $|m-n| > 5$ *and* $n \leq \lceil \frac{2m+n}{4} \rceil$

Here if $m-1$ vertices are assigned color 1, then colors 2 and 3 cannot be assigned to the reamining vertices such that it satisfies equitability. Hence $\chi_{=}(P_m \underline{V} S_n) > 3$.

Let color 1 be assigned to the first $\lceil \frac{2m+n}{4} \rceil$ vertices of $v_{i,i+1} : 1 \leq i \leq m-1$. When the remaining vertices are divided into three color classes each class will contain either $\lceil \frac{2m+n}{4} \rceil$ or $\lfloor \frac{2m+n}{4} \rfloor$ vertices satisfying equitability. Hence the remaining vertices of $v_i : \lceil \frac{2m+n}{4} \rceil + 1 \leq m$ are assigned color 2 and the vertex u is assigned color 3. The m and n vertices are assigned colors 2, 3 and 4 such that each color appears $\lceil \frac{2m+n}{4} \rceil$ or $\lfloor \frac{2m+n}{4} \rfloor$ which satisfies equitable 4-coloring (color 2 and 3 are forbidden to the n vertices). Therefore, $\chi_{=}(P_m \underline{V} S_n) = 4$.

(iii) $|m-n| > 5$ *and* $n > \lceil \frac{2m+n}{4} \rceil$. As discussed in case(ii) four colors be used. Since colors 1,2 and 3 are forbidden to the n vertices, color 4 has to be given to $u_j : 1 \leq j \leq n$. When $2m+n$ is divided by four classes atleast one class will contain vertices less than n. Hence, color 4 cannot be assigned to all $u_j : 1 \leq j \leq n$ and so an additional color must be used. Let it be color 5.

Now, $2m+n$ vertices are divided by five color classes. Color 1 is assigned to first $\lceil \frac{2m+n}{5} \rceil$ vertices of $v_i : 1 \leq i \leq m$ and its remaining vertices are assigned color 2. Let the vertex u be assigned color 3. Now as in the previous cases, colors 2,3,4 and 5 are assigned to the remaining verices, each color appearing $\lceil \frac{2m+n}{5} \rceil$ or $\lfloor \frac{2m+n}{5} \rfloor$ times satisfying equitability (colors 2 and 3 being forbidden to n vertices).

Therefore, $\chi_{=}(P_m \underline{V} S_n) = 5$.

Case 3. For $m < n$.

(i) If $m = 2, 3, \ldots, (2 + \lceil \frac{n+2}{2} \rceil)$. In this case let color 1 be assigned to all $v_{i,i+1} : 1 \leq i \leq m-1$ and color 2 be assigned to the vertex u. Hence two colors are used. To find the remaining number of colors, we use $\lceil \frac{n+1}{m} \rceil$. Thus in total $2 + \lceil \frac{n+1}{m} \rceil$. Thus color 2 and the remaining $\lceil \frac{n+1}{m} \rceil$ colors are assigned to the vertices $v_i : 1 \leq i \leq m$ and $u_j : 1 \leq j \leq n$ in such a way that each colors are assigned consecutively and appears $m-1$ *or* m times satisfying equitability (color 2 being forbidden to the n vertices).

Therefore, $\chi_{=}(P_m \underline{V} S_n) = 2 + \lceil \frac{n+1}{m} \rceil$.

(ii) If $m = (3 + \lceil \frac{n+2}{2} \rceil), (4 + \lceil \frac{n+2}{2} \rceil), \ldots$

As in case(i) if $2 + \lceil \frac{n+1}{3} \rceil$ are used then atleast one color class will appear less than $m-2$ times which is a contradiction to color 1 appearing $m-1$ times. Hence the assigning of colors is done as follows:

Color 1 is assigned to the first $\lceil \frac{2m+1}{3} \rceil$ vertices of $v_{i,i+1} : 1 \leq i \leq m-1$ and its remaining vertices by color 2. Hence color 2 cannot be assigned to the vertex u and so an additional color 3 is assigned to the vertex u. Thus three colors are used. The remaining number of colors are found by $\left\lceil \frac{n}{\lceil \frac{2m+1}{3} \rceil} \right\rceil$. Thus colors 2,3 and the remaining $\left\lceil \frac{n}{\lceil \frac{2m+1}{3} \rceil} \right\rceil$ colors are assigned to the vertices v_j, u_j so that each color is assigned consecutively and occurs either $\lceil \frac{2m+1}{3} \rceil$ or $\lfloor \frac{2m+1}{3} \rfloor$ times satisfying equitability.

Therefore, $\chi_{=}(P_m \underline{V} S_n) = 3 + \left\lceil \frac{n}{\lceil \frac{2m+1}{3} \rceil} \right\rceil$. □

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Accepted: May 31, 2020

Entire solution of certain type of delay-differential equations

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Abstract. In this article, we shall study the conditions regarding the existence of entire solutions of certain type of delay-differential equations. Our results are supplements to some results obtained recently by W. Lu et al., and references therein.
Keywords: Nevanlinna theory, entire solutions, delay-differential polynomial.

1. Introduction

Let f denote a transcendental function entire or meromorphic function. Assuming the reader is familiar with the basics of Nevanlinna's value distribution theory, we shall adopt the standard notations associated with the theory, such as $T(r, f)$, $N(r, f)$ and $m(r, f)$ (see [1, 2]).

Among many fascinating implementations of the Nevanlinna theory, there are research on the growth and existence of entire and meromorphic solutions

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of discrete types of non-linear differential equations, and one can find prototype for such equations ([3, 4, 5, 6]).

Recently, the Nevanlinna theory has been applied to study types of non-linear difference equations (see, e.g., [7, 8]). Now, we shall utilize Clunie type of theorems for delay -differential polynomials (difference-differential polynomials) to study some non-linear difference equations of more general forms and to obtain some improvements of [7, 9, 10].

2. Notation and main results

Given a meromorphic function f, recall that $\alpha \not\equiv 0, \infty$ is a small function with respect to f, if $T(r, \alpha) = S(r, f)$, where $S(r, f)$ denotes any quantity satisfying $S(r, f) = O(T(r, f))$ as $r \to \infty$, possibly outside a set of r of finite linear measure.

Definition 2.1. *Delay-differential equations are differential-difference equations in which the derivative and shifts are taken with respect to same variable.*

For the benefit of the readers, we shall give some related results.

Recently W. Lu, Lin Wu et al., continue discussing several known results for difference or differential equations obtained the following result.

Theorem I. *Let $L(z, f)$ denote a difference-differential polynomial of f of degree one with small functions as its coefficients such that $L(z, 0) \equiv 0$, and let $\rho_1, \rho_2, \alpha_1, \alpha_2$ be non zero constants such that $\alpha_1 \neq \alpha_2$. If f is an entire solution with $\rho_2 < 1$ to the following equation*

$$f^3 + L(z, f) = \rho_1 e^{\alpha_1 z} + \rho_1 e^{\alpha_2 z} \tag{2.1}$$

then, one of the following relations holds:

1. *$f(z) = k_1 e^{\frac{\alpha_1 z}{3}} + k_2 e^{\frac{\alpha_2 z}{3}}$, where k_1 and k_2 are two non zero constants satisfying $k_1^3 = \rho_1$ $k_2^3 = \rho_2$ and $\alpha_1 + \alpha_2 = 0$*
2. *$f^3(z) = (\rho_1 - k_1) e^{\alpha_1 z}$, and $L(z, f) = k_1 e^{\alpha_1 z} + \rho_1 e^{\alpha_2 z}$, where k_1 is a constant.*
3. *$f^3(z) = (\rho_2 - k_2) e^{\alpha_2 z}$, and $L(z, f) = \rho_2 e^{\alpha_1 z} + k_1 e^{\alpha_2 z}$, where k_2 is a constant.*

In the present paper we include several known results for difference or differential equations obtained earlier as its special case. In fact, we consider a slightly more general form of (2.1) by replacing f^3 by f^n in Theorem I and prove the following results.

Theorem II. *Let $L(z, f)$ denote a delay differential of f of degree one with small functions as its coefficients such that $L(z, 0) \equiv 0$.*

Let $n \geq 4$ be an integer, and $\rho_1, \rho_2, \alpha_1, \alpha_2$ be non zero constants such that $\alpha_1 \neq \alpha_2$. If there exists some entire solution f of finite order to (2.2) below.

$$f^n + L(z,f) = \rho_1 e^{\alpha_1 z} + \rho_2 e^{\alpha_2 z}. \tag{2.2}$$

Then, one of the following relations holds:

1 $f(z) = k_1 e^{\frac{\alpha_1 z}{n}}$ *and* $k_1(e^{\frac{\alpha_1}{n}} - 1) = \rho_2, \alpha_1 = n\alpha_2$;

2 $f(z) = k_2 e^{\frac{\alpha_2 z}{n}}$ *and* $k_2(e^{\frac{\alpha_2}{n}} - 1) = \rho_1, \alpha_2 = n\alpha_1$, *where* k_1, k_2 *are constants satisfying* $k_2^n = \rho_2$.

3. Preliminaries

In order to prove our conclusions, we need some lemmas.

Lemma 3.1 ([7]). *Let f be a transcendental meromorphic solution of finite order ρ of a delay equation of the form*

$$H(z,f)P(z,f) = Q(z,f),$$

where $H(z,f), P(z,f), Q(z,f)$ are delay polynomials in f such that the total degree of $H(z,f)$ in f and its shifts is n, and that the corresponding total degree of Q(z,f) is $\leq n$. If $H(z,f)$ contains just one term of a maximal total degree, then for any $\epsilon > 0$

$$m(r, P(z,f)) = O(r^{\rho-1+\epsilon}) + S(r,f),$$

possibly outside of an exponential set of finite logarithmic measure.

Remark 3.1. The following result is a clunie type Lemma 3.1 ([11]) for the delay-differential polynomial of a meromorphic function f. It can be proved by applying Lemma 3.1 ([7]) with a suitable reasoning as in [10] and stated as follows.

Let $f(z)$ be a meromorphic function of finite order, and let $P(z,f), Q(z,f)$ be two delay differential polynomials of f. If

$$f^n P(z,f) = Q(z,f)$$

holds and if the total degree of $Q(z,f)$ in f and its derivatives and their shifts is $\leq n$, then $m(r, P(z,f)) = S(r,f)$.

Lemma 3.2 ([12]). *Suppose that m, n are positive integer satisfying $\frac{1}{m} + \frac{1}{n} < 1$. Then there exist no transcendental entire solutions of f and g satisfying the equation $af^n + bg^m = 1$, with a, b being small function of f and g respectively.*

Lemma 3.3 ([13]). *Assume that $c \in \mathbb{C}$ is a non zero constant, α is a non constant meromorphic function. Then the differential equation $f^2 + (c(f^{(n)}))^2 = \alpha$ has no transcendental meromorphic solutions satisfying $T(r,\alpha) = S(r,f)$.*

Lemma 3.4 ([14, 15]). *Let f be a transcendental meromorphic function of finite order ρ, then for any complex numbers d_1, d_2 and for each*

$$\epsilon > 0, m(r, \frac{f(z+d_1)}{f(z+d_2)}) = O(r^{\rho-1+\epsilon}).$$

References [16] and [17] further pointed out the following.

Remark 3.2. If f is non constant finite order meromorphic function and $c \in \mathbb{C}$ then

$$m(r, \frac{f(z+d)}{f(z)}) = S(r, f) \tag{3.1}$$

outside of a possible exceptional set with finite logarithmic measure.

4. Proof of Theorem II

Suppose that f is a transcendental entire function of finite order to (2.2)

$$f^n + L(z, f) = \rho_1 e^{\alpha_1 z} + \rho_2 e^{\alpha_2 z}.$$

By differentiating both sides of (2.2), we have

$$nf^{n-1}f' + L'(z, f) = \alpha_1\rho_1 e^{\alpha_1 z} + \alpha_2\rho_2 e^{\alpha_2 z}. \tag{4.1}$$

From (2.2) and (4.1), we obtain

$$\alpha_2 f^n + \alpha_2 L(z, f) - nf^{n-1}f' - L'(z, f) = (\alpha_2 - \alpha_1)\rho_1 e^{\alpha_1 z}, \tag{4.2}$$

$$\alpha_1 f^n + \alpha_1 L(z, f) - nf^{n-1}f' - L'(z, f) = (\alpha_1 - \alpha_2)\rho_2 e^{\alpha_2 z}. \tag{4.3}$$

Differential (4.2) yields

$$\begin{aligned} &n\alpha_2 f^{n-1}f' + \alpha_2 L'(z, f) - n(n-1)f^{n-2}(f')^2 - nf^{n-1}f'' - L'(z, f) \\ &\quad = \alpha_1(\alpha_2 - \alpha_1)\rho_1 e^{\alpha_1 z}. \end{aligned} \tag{4.4}$$

It follows from (4.2) and (4.4) that

$$f^{n-2}\varphi = U(z, f), \tag{4.5}$$

where

$$\begin{aligned} \varphi &= \alpha_1\alpha_2 f^2 - n(\alpha_1 + {}_2)f'f + n(n-1)(f')^2 + nf''f, \\ U(z, f) &= -\alpha_1\alpha_2 L(z, f) + (\alpha_1 + \alpha_2)L'(z, f) - L''(z, f). \end{aligned}$$

Next, we shall prove $\varphi \equiv 0$. In fact, since U is a delay-differential polynomial in f, and its degree at most 1. By (4.5) and Remark 3.1 after Lemma 3.1, we

have $m(r,\varphi) = S(r,f)$, and $T(r,\varphi) = S(r,f)$, on the other hand, we can rewrite (4.5) as

$$f^{n-3}f\varphi = U(z,f),$$

which implies that

$$m(r,f\varphi) = S(r,f)$$

and $T(r,f\varphi) = S(r,f)$.

If $\varphi \not\equiv 0$, then $T(r,f) = T(r,\frac{f\varphi}{\varphi}) = S(r,f)$ and this is impossible. Hence, $\varphi \equiv 0$, and $T \equiv 0$ i.e,

$$-\alpha_1\alpha_2 L(z,f) + (\alpha_1+\alpha_2)L^{'}(z,f) - L^{''}(z,f) \equiv 0. \tag{4.6}$$

If $L(z,f) \equiv 0$, then (2.2) can be rewritten as $\frac{1}{\rho_1}(fe^{\frac{-\alpha_1 z}{n}})^n - \frac{\rho_2}{\rho_1}(e^{\frac{1}{3}(\alpha_2-\alpha_1)z})^3 = 1$, which is impossible by Lemma 3.2. Thus, $L(z,f) \not\equiv 0$. It is easily seen from $\alpha_1 \neq \alpha_2$ that

$$\alpha_1 L(z,f) - L^{'}(z,f) \equiv 0$$

and

$$\alpha_2 L(z,f) - L^{'}(z,f) \equiv 0,$$

cannot hold simultaneously.

First of all, we assume that $\alpha_2 L(z,f) - L^{'}(z,f) \not\equiv 0$ then (4.6) given $\alpha_2 L(z,f) - L^{'}(z,f) = Ae^{\alpha_1 z}$, where A is a non zero constant, substituting the above expression into (4.2), we obtain

$$f^{n-1}(\alpha_2 f - nf^{'}) = \frac{(\alpha_2-\alpha_1)P_1 - A}{A}[\alpha_2 L(z,f) - L^{'}(z,f)].$$

Again, Lemma 3.1 shows that $\alpha_2 f - nf^{'} \equiv 0$, $(\alpha_2-\alpha_1)\rho_1 = A$. So

$$f^n = Be^{\alpha_2 z}, \tag{4.7}$$

where B is constant.

Substituting (4.7) in (2.2), we find

$$(1-\frac{P_2}{B})f^n = \frac{\rho_1(\alpha_2 L(z,f) - L^{'}(z,f))}{A} - L(z,f).$$

If $B \neq P_2$, by Lemma 3.1, we get $T(r,f) = m(r,f) = S(r,f)$, which is absurd, so $B = P_2$, and $f = k_2 e^{\frac{\alpha_2 z}{n}} = k_2 e^{\alpha_1 z}$, $k_2^n = B = \rho_2$, $k_2(e^{\frac{\alpha_2}{n}} - 1) = \rho_1$ If $\alpha_1 L(z,f) - L^{'}(z,f) \not\equiv 0$, by (4.3) and using similar arguments as above, we can derive $f = k_1 e^{\frac{\alpha_1 z}{n}} = k_1 e^{\alpha_2 z}$, $k_1^n = \rho_1$, $k_1(e^{\frac{\alpha_1}{n}} - 1) = \rho_2$.

This conclude the verification of Theorem II.

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Accepted: January 14, 2020

Categorical representation of generalized topologies, generalized neighbourhood systems and generalized interior and closure operators

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Abstract. We define a new operator namely strong generalized closure operator on a generalized topological space and investigate some of its properties. Further we give a categorical representation of concepts namely generalized topology, strong generalized interior operator, strong generalized closure operator and complete generalized neighbourhood system and study the relationship between these concepts by means of natural transformations and adjunctions. Also we establish an adjunction between the functors Γ and GT to prove that the functor Γ is left adjoint of the functor GT.

Keywords: generalized topology, strongly generalized interior operator, strongly generalized closure operator, complete generalized neighbourhood system, natural transformation, category, functor, adjunction.

1. Introduction

Cśaszár [4], [5] introduced the concepts of generalized open sets and generalized topologies and defined basic operators on generalized topological spaces. In the last few years various authors have investigated their fundamental properties and their relationship with generalized systems. In particular Min [9] has defined the concept of strongly generalized interior operators and obtained a bijection between the generalized topologies and the strongly generalized interior operators. On the other hand many concepts of Category Theory especially the concept of naturality gives clarity to the meaning of isomorphism at the deeper level. We [11] have proved that the concepts of generalized topologies, strongly generalized interior operators and monotonic maps extend to functors on the category of SETS, such that the isomorphism between the generalized topologies and the strongly generalized interior operators is actually natural isomorphism between them in the sense of Category Theory.

Further, Shen [12] studied complete generalized neighbourhood systems and established many properties related to generalized systems. He proved that for a generalized interior operator I, $I = I_{\psi_I}$ and for an ascending generalized neighbourhood system ψ, $\psi = \psi_{I_\psi}$. Also he proved that for a strong generalized

interior operator I, ψ_I is complete generalized neighbourhood system and for a complete generalized neighbourhood system ψ, I_ψ is strong generalized interior operator. This study was continued by Cao [2] to find many other properties of these systems. He established a relationship among generalized topologies, complete generalized neighbourhood systems and strong generalized interior operators as the following commutative diagram:

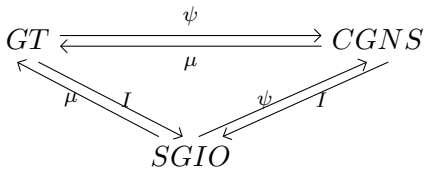


Diagram 1

The purpose of this paper is to verify whether the Diagram 1 gives rise to natural isomorphisms between the functors GT, SGI and $CGNS$ extended by generalized topologies, strong generalized interior operators and complete generalized neighbourhood systems respectively, on some suitably chosen categories. Also strongly generalized closure operator is defined and its various properties are studied in this paper. Finally an adjunction is established between the functors Γ and GT. The idea is to find new results with the application of categorical methods like functors and natural transformations and adjunctions.

We now recall some concepts and results:

Let X be a set and denote $\exp X$ the power set of X. A collection μ of subsets of a non-empty set X is called a generalized topology ([4]) on X if μ contains the empty set and is closed under arbitrary unions. The members of μ are called μ-open sets and their complements are called μ-closed sets. The collection of all generalized topologies on X is denoted by $GT(X)$.

A mapping $\psi : X \to \exp(\exp X)$ is called a generalized neighbourhood system ([4]) on X if for each $x \in X$, $V \in \psi(x)$ implies $x \in V$. The interior and closure ([4]) of $A \subset X$ on ψ are defined as follows:

$i_\psi(A) = \{x \in A : \text{there exists } V \in \psi(x) \text{ such that } V \subset A\}$;

$c_\psi(A) = \{x \in X : V \cap A \neq \emptyset, \text{ for all } V \in \psi(x)\}$.

A generalized neighbourhood system ψ on X is called ascending ([4]) if $V \in \psi(x)$ implies $U \in \psi(x)$ for each $x \in X$ and $V \subset U$. If, in addition, for each $x \in X$ and $A \in \psi(x)$, there is a set O such that $x \in O \subset A$ and $y \in O$ implies the existence of a set $B \in \psi(y)$ with $B \subset O$, then ψ is called complete (see, e.g., [12]).

For a non empty set X, a collection H of subsets of X is called an m-family ([9]) on X if $\cap H \neq \emptyset$. Let ψ be a generalized neighbourhood system on X, then an m-family H on X converges to $x \in X$ if H is finer than $\psi(x)$ and (X, ψ) is called a generalized neighbourhood space ([9]) (briefly gn-space). The interior and closure operators in gn-space (X, ψ) are defined as below:

$I^*_\psi(A) = \{x \in A : A \in \psi(x)\}$;
$cl^*_\psi(A) = \{x \in X : X - A \notin \psi(x)\}$.

A mapping $I : \exp X \to \exp X$ is called a generalized interior operator ([9]) if $I(A) \subset A$, for all $A \subset X$ and $A \subset B$ implies $I(A) \subset I(B)$, for all $A, B \subset X$. A generalized interior operator I on X is called strong ([9]) if, in addition, $I(I(A)) = I(A)$, for all $A \subset X$. The collection of all strong generalized interior operators on X is denoted by $SGI(X)$.

Let (X, μ) and (Y, ν) be two generalized topological spaces. Then, a map $f : (X, \mu) \to (Y, \nu)$ is called (μ, ν)-continuous if $f^{-1}(U) \in \mu$ for $U \in \nu$. Also if ψ and ψ' be two generalized neighbourhood systems on the spaces X and Y respectively. Then, a map $f : (X, \psi) \to (Y, \psi')$ is called (ψ, ψ')-continuous if for $x \in X$ and $U \in \psi'(f(x))$, there is $V \in \psi(x)$ such that $f(V) \subset U$.

We also use the terminology of categories and functors also. For the standard notions we refer to [1] and [8]. In particular we refer to the following definitions of functor, natural transformation and adjunction between functors:

A functor $F : \mathbf{C} \to \mathbf{D}$ between categories $\mathbf{C}$ and $\mathbf{D}$ is a rule that maps objects and arrows of $\mathbf{C}$ to objects and arrows of $\mathbf{D}$ such that if $g : A \to B$ in $\mathbf{C}$ then $F(g) : F(A) \to F(B)$ in $\mathbf{D}$ and whenever $g \circ h$ is defined in $\mathbf{C}$, $F(g \circ h) = F(g) \circ F(h)$ in $\mathbf{D}$. Further F must preserve identities, that is, if $1_A : A \to A$ is identity of A, then $F(1_A) = 1_{FA}$.

For categories $\mathbf{C}$, $\mathbf{D}$ and functors $F, G : \mathbf{C} \to \mathbf{D}$, a natural transformation $\theta : F \to G$ is a family of arrows in $\mathbf{D}$, $\{\theta_X : FX \to GX\}_{X \in \mathbf{C}}$ such that, for any $f : X \to Y$ in $\mathbf{C}$, $\theta_Y \circ F(f) = G(f) \circ \theta_X$, that is the following diagram is commutative:

$$\begin{array}{ccc} F(X) & \xrightarrow{\theta_X} & G(X) \\ \Big\downarrow{\scriptstyle F(f)} & & \Big\downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow{\theta_Y} & G(Y) \end{array}$$

Diagram 2

Let $\mathbf{C}$ and $\mathbf{D}$ be categories. An adjunction from $\mathbf{D}$ to $\mathbf{C}$ is a triple $\langle F, G, \varphi \rangle : \mathbf{D} \to \mathbf{C}$, where F and G functors

$$\mathbf{D} \underset{\text{G}}{\overset{\text{F}}{\rightleftarrows}} \mathbf{C}$$

while φ is a function which assigns to each pair of objects $d \in \mathbf{D}$ and $c \in \mathbf{C}$, a bijection of sets $\varphi = \varphi_{d,c} : \mathbf{C}(Fd, c) \cong \mathbf{D}(d, Gc)$ which is natural in d and c, where $\mathbf{C}(Fd, c)$ is the bifunctor

$$\mathbf{D}^{op} \times \mathbf{C} \xrightarrow{F^{op} \times 1_{\mathbf{C}}} \mathbf{C}^{op} \times \mathbf{C} \xrightarrow{hom} SETS$$

which sends $\langle d, c\rangle$ to hom set $\mathbf{C}(Fd, c)$ and $\mathbf{D}(d, Gc)$ is a similar bifunctor $\mathbf{D}^{op} \times \mathbf{C} \to SETS$. Therefore, naturality of φ means that for all $k : c \to c'$ and $h : d' \to d$, both the following diagrams are commutative:

$$\begin{array}{ccc} \mathbf{C}(Fd, c) & \xrightarrow{\varphi_{d,c}} & \mathbf{D}(d, Gc) \\ \downarrow{\scriptstyle \mathbf{C}(Fd,k)} & & \downarrow{\scriptstyle \mathbf{D}(d,Gk)} \\ \mathbf{C}(Fd, c') & \xrightarrow{\varphi_{d,c'}} & \mathbf{D}(d, Gc') \end{array} \qquad \begin{array}{ccc} \mathbf{C}(Fd, c) & \xrightarrow{\varphi_{d,c}} & \mathbf{D}(d, Gc) \\ \downarrow{\scriptstyle \mathbf{C}(Fh,c)} & & \downarrow{\scriptstyle \mathbf{D}(h,Gc)} \\ \mathbf{C}(Fd', c) & \xrightarrow{\varphi_{d',c}} & \mathbf{D}(d', Gc) \end{array}$$

Diagram 3

Denote $SETS$ the category in which the objects are sets and arrows are functions. GT is a functor [11] on $SETS$ which assigns to each set X, $GT(X)$, the collection of all generalized topologies on X and to each function $h : X \to Y$, $GT(h) : GT(X) \to GT(Y)$ defined by $GT(h)(\mu) = \{h(A) : A \in \mu\}$. SGI is a functor [11] on $SETS$ which assigns to each set X, $SGI(X)$, the collection of all strong generalized interior operators on X and to each function $h : X \to Y$, $SGI(h) : SGI(X) \to SGI(Y)$ defined by $SGI(h)(I) = h \circ I \circ h^{-1}$, where $h : \exp(X) \to \exp(Y)$ and $h^{-1} : \exp(Y) \to \exp(X)$ are image and inverse image functions respectively. Γ is a functor [11] on $SETS$ which assigns to each set X, $\Gamma(X)$, the collection of all monotonic operators on $\exp X$ and to each function $h : X \to Y$, $\Gamma(h) : \Gamma(X) \to \Gamma(Y)$ defined by $\Gamma(h)(I) = h \circ I \circ h^{-1}$, where $h : \exp(X) \to \exp(Y)$ and $h^{-1} : \exp(Y) \to \exp(X)$ are image and inverse image functions respectively. $1_{\mathbf{C}}$ is identity functor [1] on the category $\mathbf{C}$. exp is the functor on the category $SETS$, which assigns to each set X, $\exp(X)$, the power set of X and to each function $h : X \to Y$, inverse image mapping $h^{-1} : \exp(Y) \to \exp(X)$, defined by $h^{-1}(B) = \{x \in X : h(x) \in B\}$. For objects X, Y in the category $\mathbf{C}$, $Hom_{\mathbf{C}}(X, Y) = Hom(X, Y)$ is the set of all arrows from X to Y. If $h : X \to X'$ is an arrow in $\mathbf{C}$, then $Hom_{\mathbf{C}}(h, Y) = Hom(h, Y) : Hom(X', Y) \to Hom(X, Y)$ is a contravariant functor on category $\mathbf{C}$ defined by $(Hom(h, Y))(g) = g \circ h$. Similarly for the arrow $k : Y \to Y'$ in $\mathbf{C}$, $Hom_{\mathbf{C}}(X, k) = Hom(X, k) : Hom(X, Y) \to Hom(X, Y')$ is a covariant functor on category $\mathbf{C}$ defined by $(Hom(X, k))(g) = k \circ g$.

For a given generalized topology μ on X, ψ_μ and I_μ are generalized neighbourhood system and generalized interior operator on X generated by μ, given by $\psi_\mu(x) = \{A : x \in M \subset A \text{ for some } M \in \mu\}$ and $I_\mu(A) = \cup\{M : M \in \mu, M \subset A\}$ (see, e.g., [4]). For a given generalized neighbourhood system ψ on X, μ_ψ and I_ψ are generalized topology and generalized interior operator on X generated by ψ, given by $\mu_\psi = \{M \subset X : x \in M$ implies the existence of a set $V \in \psi(x)$ satisfying $V \subset M\}$ and $I_\psi(A) = \{x \in X :$ there exists $V \in \psi(x)$ such that $V \subset A\}$

(see, e.g., [6]). For a given generalized interior operator I on X, μ_I and ψ_I are generalized topology and generalized neighbourhood system on X generated by I, given by $\mu_I = \{A \subset X : A = I(A)\}$ and $\psi_I(x) = \{A \subset X : x \in I(A)\}$ (see, e.g., [12]).

2. Strongly generalized closure operator

Now we define strong generalized closure operator and study its various properties.

Definition 2.1. *Let a function $C : \exp X \to \exp X$ satisfying the axioms:*

$(C1)$ $A \subset C(A)$, *for all* $A \in \exp X$;

$(C2)$ $A \subset B \Rightarrow C(A) \subset C(B)$, *for all* $A, B \in \exp X$;

$(C3)$ $C(C(A)) = C(A)$, *for all* $A \in \exp X$;

Then, the function C is called:

(a) *a strong generalized closure operator (shortly sgco) if $(C1)$, $(C2)$, $(C3)$ hold;*

(b) *a generalized closure operator (shortly gco) if $(C1)$, $(C2)$ hold;*

We say (X, C) an sgc (gc)-space when C is sgco (gco).

Lemma 2.1. *Let ψ be a generalized neighbourhood system on X and let $C : \exp X \to \exp X$ be defined as $C(A) = c_\psi(A)$ for each $A \in \exp X$. Then, C is a gco.*

Proof. Proof is obvious by definition of c_ψ. □

The following example shows that the operator c_ψ corresponding to any generalized neighbourhood system ψ need not be sgco.

Example 2.1. Let $X = R$ and $\psi(x) = \{(x-1, x+1)\}$ for each $x \in R$. Let $A = (2, 3)$, then $c_\psi(A) = (1, 4)$ and $c_\psi c_\psi(A) = (0, 5)$. Therefore, $c_\psi c_\psi(A) \neq c_\psi(A)$.

Lemma 2.2. *Let (X, μ) be a generalized topological space and $C : \exp X \to \exp X$ be defined as $C(A) = c_\mu(A)$ for each $A \in \exp X$. Then, C is a sgco.*

Proof. Proof is trivial by definition of c_μ. □

Let X be any set and $\lambda \subset \exp X$. Define $c_\lambda(A) = \cap\{F : F^c \in \lambda, A \subset F\}$.

Lemma 2.3. *$c_\lambda : \exp X \to \exp X$ is an sgco.*

Proof. The proof is immediate by definition of c_λ and hence is omitted. □

Lemma 2.4. *If $C : \exp X \to \exp X$ is an sgco, then $C = c_\lambda$ for some $\lambda \subset \exp X$; λ can be chosen in a manner that it is a generalized topology on X.*

Proof. Let $\lambda = \{L \subset X : L^c = C(L^c)\}$. Let $F^c \in \lambda$, $A \subset F$, then $F = CF \supset CA$ implies $c_\lambda(A) = \cap\{F : F^c \in \lambda, A \subset F\} \supset CA$.

Conversely, since $CCA = CA$, $(CA)^c \in \lambda$ and $A \subset CA$, therefore $c_\lambda(A) \subset CA$.

Also from the definition of λ, it is immediate that λ is a generalized topology on X. □

Lemma 2.5. *If (X, λ) is a generalized topological space and $C : \exp X \to \exp X$ is sgco, then $C = c_\lambda$ implies $\lambda = \{L \subset X : L^c = C(L^c)\}$.*

Proof. Let $L \in \lambda$, since λ is a generalized topology, L^c is λ-closed in X, $c_\lambda(L^c) = L^c \subset C(L^c) = c_\lambda(L^c)$, therefore $L^c = C(L^c)$.

Conversely, $L^c = C(L^c)$ implies $L^c = c_\lambda(L^c)$. Therefore, L^c is λ-closed in X, that is, $L \in \lambda$. □

We denote the collection of all sgco's on X by $SGC(X)$.

Theorem 2.1. *If X is a non empty set. Then, there exists a bijection between $SGC(X)$ and $GT(X)$.*

Proof. Define $\varphi : GT(X) \to SGC(X)$ by $\varphi(\mu) = c_\mu$. Then, φ is well defined. Let $\mu_1, \mu_2 \in GT(X)$ such that $\mu_1 \neq \mu_2$; then there exists $U \in \mu_1$, say $U \notin \mu_2$ and $c_{\mu_1}(U^c) = U^c \neq c_{\mu_2}(U^c)$. Therefore, $c_{\mu_1} \neq c_{\mu_2}$ so that φ is injective. By Lemmas 2.4 and 2.5, φ is surjective. □

Definition 2.2. *Let (X, C) and (Y, C') be two sgc-spaces. Then, $f : (X, C) \to (Y, C')$ is said to be sgc-continuous if $C(f^{-1}B) \subset f^{-1}(C'B)$ for each $B \subset Y$.*

Theorem 2.2. *Let (X, μ) and (Y, ν) be generalized topological spaces. Then, $f : (X, \mu) \to (Y, \nu)$ is (μ, ν)-continuous if and only if $f : (X, c_\mu) \to (Y, c_\nu)$ is sgc-continuous.*

Proof. Let $f : (X, \mu) \to (Y, \nu)$ is (μ, ν)-continuous. Let $B \subset Y$. Since $c_\nu(B)$ is ν-closed in Y, $f^{-1}(c_\nu(B))$ is μ-closed in X, $c_\mu(f^{-1}(c_\nu(B))) \subset f^{-1}(c_\nu(B))$, therefore $c_\mu(f^{-1}(B)) \subset f^{-1}(c_\nu(B))$. Therefore, f is sgc-continuous.

Conversely, if f is sgc-continuous and $U \in \nu$, then U^c is ν-closed in Y, $c_\nu(U^c) = U^c$, therefore $c_\mu(f^{-1}(U^c)) \subset f^{-1}(U^c)$, and so $f^{-1}(U^c)$ is μ-closed in X, that is, $f^{-1}(U) \in \mu$. Therefore, f is (μ, ν)-continuous. □

Theorem 2.3. *Let X be a non empty set and $C : \exp X \to \exp X$ be a gco. Then, $\mu_C = \{A \subset X : C(A^c) = A^c\}$ is a generalized topology on X.*

Proof. Since $CX = X$, $\emptyset \in \mu_C$. Let $U_i \in \mu_C$ for $i \in \Lambda$, where Λ is indexing set. Then, $C((U_i)^c) = (U_i)^c$ for $i \in \Lambda$. $C((\cup U_i)^c) = C(\cap((U_i)^c)) \subset C((U_i)^c) = (U_i)^c$ for $i \in \Lambda$. Therefore, $C((\cup U_i)^c) \subset \cap((U_i)^c) = (\cup U_i)^c$. $\cup U_i \in \mu_C$. Therefore, μ_C is a generalized topology on X. □

Theorem 2.4. *Let X be a non empty set and $C : \exp X \to \exp X$ be a gco. For each $x \in X$, let $\psi_C(x) = \{A \subset X : C(A^c) = A^c, x \in A\}$. Then, $\psi = \{\psi_C(x) : x \in X\}$ is a generalized neighbourhood system on X.*

Proof. Proof is immediate by definition of generalized neighbourhood system and by Theorem 2.3. □

Definition 2.3. *Let (X, C) and (Y, C') be gc-spaces. Then, $f : (X, C) \to (Y, C')$ is said to be gc-continuous if $C(f^{-1}(B)) \subset f^{-1}(C'(B))$, for all $B \subset Y$.*

Theorem 2.5. *Let ψ, ψ' be generalized neighbourhood systems on X, Y respectively. Then, $f : (X, \psi) \to (Y, \psi')$ is (ψ, ψ')-continuous if and only if $f : (X, c_\psi) \to (Y, c_{\psi'})$ is gc-continuous.*

Proof. Let $f : (X, \psi) \to (Y, \psi')$ be (ψ, ψ')-continuous and $B \subset Y$. Let $x \in c_\psi(f^{-1}(B))$ and $x \notin f^{-1}(c_{\psi'}(B))$, $f(x) \notin c_{\psi'}(B)$, there exists $V' \in \psi'(f(x))$ such that $V' \cap B = \emptyset$. Since f is (ψ, ψ')-continuous, there exists $U \in \psi(x)$ such that $f(U) \subset V'$. Since $x \in c_\psi(f^{-1}(B))$, $U \cap f^{-1}(B) \neq \emptyset$, $f(U) \cap B \neq \emptyset$, $V' \cap B \neq \emptyset$. This is a contradiction. Therefore, $x \in f^{-1}(c_{\psi'}(B))$. $c_\psi(f^{-1}(B)) \subset f^{-1}(c_{\psi'}(B))$. $f : (X, c_\psi) \to (Y, c_{\psi'})$ is gc-continuous.

Conversely, let $f : (X, c_\psi) \to (Y, c_{\psi'})$ be gc-continuous, that is, $c_\psi(f^{-1}(B)) \subset f^{-1}(c_{\psi'}(B))$ for each subset B of Y. Let $V \in \psi'(f(x))$ for $x \in X$. Suppose $f(U)$ is not contained in V, for all $U \in \psi(x)$. Then, there exists a point y in $f(U)$ which is not in V, therefore U intersects with $f^{-1}(y)$ for every $U \in \psi(x)$, therefore $x \in c_\psi(f^{-1}(y))$. Since f is gc-continuous, therefore $x \in f^{-1}(c_{\psi'}(y))$, and so $f(x) \in c_{\psi'}(y)$. Therefore, each $V' \in \psi'(f(x))$, contains y, in particular, $y \in V$. This is a contradiction. Therefore, $f(U)$ is contained in V, for some $U \in \psi(x)$. So $f : (X, \psi) \to (Y, \psi')$ is (ψ, ψ')-continuous. □

3. The functor SGC and natural transformation between the functors G and SGC

Since any function $h^{-1} : Y \to X$ is well behaved under arbitrary unions, therefore generalized topologies on Y, should be mapped to generalized topologies on X by the inverse image map $h^{-1} : \exp Y \to \exp X$. In fact we have the following theorem.

Theorem 3.1. *The generalized topologies extend to a contravariant functor on SETS.*

Proof. We construct the functor $G : SETS \to SETS$ by defining $G(X)$ as the collection of all the generalized topologies on X. Further for any $h : X \to Y$, since $h^{-1}(\cup V_\alpha) = \cup h^{-1}(V_\alpha)$, for all $V_\alpha \subset Y$, therefore for a generalized topology ν on Y, $\{h^{-1}(V) : V \in \nu\}$ is a generalized topology on X. Therefore, we define $G(h)(\nu) = \{h^{-1}(V) : V \in \nu\}$. Then, $G : SETS \to SETS$ maps X to $G(X)$ and h to $G(h)$. Further if $h_1 : X \to Y$ and $h_2 : Y \to Z$, then $G(h_1) : G(Y) \to G(X)$

and $G(h_2) : G(Z) \to G(Y)$, $(G(h_1)oG(h_2))(\mu) = G(h_1)(\{h_2^{-1}(V) : V \in \mu\}) = \{h_1^{-1}(h_2^{-1}(V)) : V \in \mu\} = \{(h_2oh_1)^{-1}(V) : V \in \mu\} = (G(h_2oh_1))(\mu)$. Therefore, $G(h_1)oG(h_2) = G(h_2oh_1)$. Finally if $1_X : X \to X$ is identity map then $G(1_X) : G(X) \to G(X)$ is identity map. Therefore, G is a functor. □

Now we establish strong generalized closure operator as a functor in the following theorem.

Theorem 3.2. *The strong generalized closure operators extend to a contravariant functor SGC on SETS.*

Proof. We construct the functor $SGC : SETS \to SETS$ by defining $SGC(X)$ as the collection of all the strong generalized closure operators on X. If $h : X \to Y$ is any function, we define $SGC(h) : SGC(Y) \to SGC(X)$ by $SGC(h)(C) = h^{-1} \circ C \circ h$, where $h : \exp X \to \exp Y$ and $h^{-1} : \exp Y \to \exp X$ are image and inverse image map respectively. C satisfies (C1),(C2) and (C3). For any subset A of X, $(h^{-1} \circ C \circ h)(A) = h^{-1}(C(h(A))) \supset h^{-1}(h(A)) \supset A$, $SGC(h)(C)$ satisfies (C1). If $A_1 \subset A_2$, then $(h^{-1} \circ C \circ h)(A_1) = h^{-1}(C(h(A_1))) \subset h^{-1}(C(h(A_2))) = (h^{-1} \circ C \circ h)(A_2)$, $SGC(h)(C)$ satisfies (C2). $(h^{-1} \circ C \circ h)((h^{-1} \circ C \circ h)(A)) = (h^{-1} \circ C \circ h \circ h^{-1} \circ C \circ h)(A) \subset (h^{-1} \circ C \circ C \circ h)(A) = (h^{-1} \circ C \circ h)(A)$, also $(h^{-1} \circ C \circ h)(A) \supset A \Rightarrow (h^{-1} \circ C \circ h)(h^{-1} \circ C \circ h)(A) \supset (h^{-1} \circ C \circ h)(A)$, so $(h^{-1} \circ C \circ h)(h^{-1} \circ C \circ h)(A) = (h^{-1} \circ C \circ h)(A)$, $SGC(h)(C)$ satisfies (C3). Therefore, $SGC(h)(C) \in SGC(X)$ and so $SGC(h)$ is well defined. If $h_1 : X \to Y$ and $h_2 : Y \to Z$ are any functions then $SGC(h_1) : SGC(Y) \to SGC(X)$ and $SGC(h_2) : SGC(Z) \to SGC(Y)$, $(SGC(h_1) \circ SGC(h_2))(C) = (SGC(h_1))(h_2^{-1} \circ C \circ h_2) = h_1^{-1} \circ h_2^{-1} \circ C \circ h_2 \circ h_1 = (h_2 \circ h_1)^{-1} \circ C \circ (h_2 \circ h_1) = (SGC(h_2 \circ h_1))(C)$. Therefore, $SGC(h_1) \circ SGC(h_2) = SGC(h_2 \circ h_1)$. Also if $1_X : X \to X$ is identity map then $SGC(1_X) : SGC(X) \to SGC(X)$ is identity map. Hence SGC is a functor. □

We have proved a bijection between the collection of generalized topologies on a set X and the collection of all strong generalized closure operators in the Theorem 2.1. In the next theorem, we prove that the bijection is actually a natural isomorphism between the functor G defined above in Theorem 3.1 and the functor SGC defined in Theorem 3.2.

Theorem 3.3. *The functors G and SGC are naturally isomorphic.*

Proof. To prove the existence of a natural transformation $\eta : G \to SGC$, we need to find a collection $\{\eta_X : X \text{ a set}\}$ such that whenever $h : X \to Y$ is any function, the following diagram is commutative.

$$\begin{array}{ccc} G(Y) & \xrightarrow{\eta_Y} & SGC(Y) \\ \Big\downarrow{\scriptstyle G(h)} & & \Big\downarrow{\scriptstyle SGC(h)} \\ G(X) & \xrightarrow{\eta_X} & SGC(X) \end{array}$$

Diagram 4

Define $\eta_X : G(X) \to SGC(X)$ by $\eta_X(\mu) = c_\mu$. Since c_μ is sgco, therefore η_X is well-defined. To check the commutativity of Diagram 4, as for naturality of $\eta = \{\eta_X\}$, if $h : X \to Y$ is any function, then $(\eta_X \circ G(h))(\nu) = \eta_X(\{h^{-1}(V) : V \in \nu\}) = \eta_X(\mu)$, where $\mu = \{h^{-1}(V) : V \in \nu\}$, $\eta_X(\mu) : \exp X \to \exp X$ given by $\eta_X(\mu)(A) = \cap\{F : F^c \in \mu, A \subset F\}$, also $(SGC(h) \circ \eta_Y)(\nu) = SGC(h)(\eta_Y(\nu)) = h^{-1} \circ (\eta_Y(\nu)) \circ h$, where $h^{-1} \circ (\eta_Y(\nu)) \circ h : \exp X \to \exp X$ given by $(h^{-1} \circ (\eta_X(\nu)) \circ h)(A) = h^{-1}(\eta_X(\nu)(h(A)) = h^{-1}(\cap\{E : E^c \in \nu, h(A) \subset E\}) = \cap\{h^{-1}(E) : E^c \in \nu, h(A) \subset E\} = \cap\{h^{-1}(E) : h^{-1}(E^c) \in \mu, h(A) \subset E\} = \cap\{F : F^c \in \mu, A \subset F\}$, therefore $(\eta_X \circ G(h))(\nu) = (SGC(h) \circ \eta_Y)(\nu)$, therefore $(\eta_X \circ G(h)) = (SGC(h) \circ \eta_Y)$. Hence the Diagram 4 is commutative and therefore $\eta = \{\eta_X\}$ is natural transformation. Since G and SGC are functors and η_X is an isomorphism (Theorem 2.1). Therefore, $\eta = \{\eta_X\}$ is a natural isomorphism. □

4. Naturality between the functors GT, CGNS and SGI

Let $\mathcal{C}$ be a category in which the objects are the sets and the morphisms are bijective functions between two sets.

Theorem 4.1. *The complete generalized neighbourhood systems extend to a covariant functor $CGNS$ on $\mathcal{C}$.*

Proof. We construct the functor $CGNS : \mathcal{C} \to \mathcal{C}$ by defining $CGNS(X)$ as the collection of all the complete generalized neighbourhood systems on X. Further for any $h : X \to Y$, we define $CGNS(h) : CGNS(X) \to CGNS(Y)$ by $CGNS(h)(\psi) = \psi'$, where $\psi'(y) = \{h(A) : A \in \psi(x)\}$, for $y = h(x)$, since h is bijective function. Then, $CGNS(h)$ is well defined, since for a complete generalized neighbourhood system ψ on X, ψ' is complete generalized neighbourhood system on Y. Further if $h_1 : X \to Y$ and $h_2 : Y \to Z$, then $CGNS(h_1) : CGNS(X) \to CGNS(Y)$ and $CGNS(h_2) : CGNS(Y) \to CGNS(Z)$, $(CGNS(h_2) o CGNS(h_1))(\psi) = (CGNS(h_2))(\psi') = \psi''$, where $\psi'(y) = \{h_1(A) : A \in \psi(x)\}$, for $y = h_1(x)$ and $\psi''(z) = \{h_2(B) : B \in \psi'(y)\}$, for $z = h_2(y)$. If $z = h_2(y)$ and $y = h_1(x)$, then $\psi''(z) = \{h_2(B) : B \in \psi'(y)\} = \{h_2(h_1(A)) : A \in \psi(x)\} = \{(h_2 o h_1)(A) : A \in \psi(x)\}$. Also $h_2 o h_1 : X \to Z$, then $CGNS(h_2 o h_1) : CGNS(X) \to CGNS(Z)$. Let $(CGNS(h_2 o h_1))(\psi) = \psi_1$, where $(\psi_1)(z) = \{(h_2 o h_1)(A) : A \in \psi(x)\}$, for $z = (h_2 o h_1)(x)$. Therefore, $CGNS(h_2) o CGNS(h_1) = CGNS(h_2 o h_1)$. Finally if $1_X : X \to X$ is identity map of X, then $CGNS(1_X) : CGNS(X) \to CGNS(X)$ is identity map. Therefore, $CGNS$ is a functor. □

We have proved [11] that generalized topologies on a set X extend to a covariant functor GT on the category SETS. If we restrict the category SETS to the category $\mathcal{C}$, then GT is a functor on the category $\mathcal{C}$ also. Now we establish that there exists a natural transformation between the functors GT and $CGNS$ in the following theorem.

Theorem 4.2. *There exists a natural transformation between the functors GT and CGNS on the category $\mathcal{C}$.*

Proof. To prove the existence of a natural transformation $\eta : GT \to CGNS$, we need to find a collection $\{\eta_X : X \text{ a set}\}$ such that whenever $h : X \to Y$ is any bijective function, the following diagram is commutative.

$$\begin{array}{ccc} GT(X) & \xrightarrow{\eta_X} & CGNS(X) \\ \Big\downarrow{\scriptstyle GT(h)} & & \Big\downarrow{\scriptstyle CGNS(h)} \\ GT(Y) & \xrightarrow{\eta_Y} & CGNS(Y) \end{array}$$

Diagram 5

Define $\eta_X : GT(X) \to CGNS(X)$ by $\eta_X(\mu) = \psi_\mu$. Since ψ_μ is complete generalized neighbourhood system, therefore η_X is well-defined. To check the commutativity of the diagram, as for naturality of $\eta = \{\eta_X\}$. If $h : X \to Y$ is any surjective function, then $(\eta_Y \circ GT(h))(\mu) = \eta_Y((GT(h))(\mu)) = \eta_Y(\{h(V) : V \in \mu\}) = \eta_Y(\mu') = \psi_{\mu'}$, where $\mu' = \{h(V) : V \in \mu\}$ and $(\psi_{\mu'})(y) = \{B \subset Y : y \in N \subset B \text{ for some } N \in \mu'\}$. Also $(CGNS(h) \circ \eta_X)(\mu) = CGNS(h)(\eta_X(\mu)) = (CGNS(h))(\psi_\mu) = (\psi_\mu)'$, where $(\psi_\mu)'(y) = \{h(A) : A \in \psi_\mu(x)\}$ for $y = h(x)$. Therefore, it remains to prove that $\psi_{\mu'}(y) = (\psi_\mu)'(y)$. Let $h(A) \in (\psi_\mu)'(y)$, then $A \in \psi_\mu(x)$, where $x \in M \subset A$ for some $M \in \mu$, then $y = h(x) \in h(M) \subset h(A)$ and $h(M) \in \mu'$, therefore $h(A) \in \psi_{\mu'}(y)$. Conversely, if $B \in \psi_{\mu'}(y)$, then $y \in N \subset B$ for some $N \in \mu'$, therefore $N = h(M)$ for some $M \in \mu$, so $y \in h(M) \subset B \Rightarrow x \in M \subset h^{-1}(B)$, since h is injective function. Therefore, $h^{-1}(B) \in \psi_\mu(x) \Rightarrow h(h^{-1}(B)) \in (\psi_\mu)'(y) \Rightarrow B \in (\psi_\mu)'(y)$. Therefore, $(\eta_Y \circ GT(h))(\mu) = (CGNS(h) \circ \eta_X)(\mu)$, and so $\eta_Y \circ GT(h) = CGNS(h) \circ \eta_X$. Hence the diagram is commutative and therefore $\eta = \{\eta_X\}$ is natural transformation. □

Remark 4.1. It has been proved that for each generalized topology μ on X, $\mu = \mu_{\psi_\mu}$ [4] and for a complete generalized neighbourhood system ψ on X, $\psi = \psi_{\mu_\psi}$ [12]. Therefore, if we define $\eta'_X : CGNS(X) \to GT(X)$ by $\eta'_X(\psi) = \mu_\psi$, then $\eta = \{\eta'_X\}$ is a natural transformation such that $\eta'_X \circ \eta_X = 1_{GT(X)}$ and $\eta_X \circ \eta'_X = 1_{CGNS(X)}$, that is, $\eta = \{\eta_X\}$ is natural isomorphism from the functor GT to the functor $CGNS$ and η' is its inverse. In this case, the following diagram is commutative:

$$\begin{array}{ccc} GT(X) & \underset{\eta'_X}{\overset{\eta_X}{\rightleftharpoons}} & CGNS(X) \\ \Big\downarrow{\scriptstyle GT(h)} & & \Big\downarrow{\scriptstyle CGNS(h)} \\ GT(Y) & \underset{\eta'_Y}{\overset{\eta_Y}{\rightleftharpoons}} & CGNS(Y) \end{array}$$

Diagram 6

Remark 4.2. If we define $\psi_X(x) = \dot{x} = \{A \subset X : x \in A\}$, then ψ_X is a complete generalized neighbourhood system on X, that is, $\psi_X \in CGNS(X)$. This operator ψ_X gives a natural transformation between the identity functor and the functor $\exp(\exp X)$ as shown in the following theorem.

Theorem 4.3. *$\psi = \{\psi_X\}$ is a natural transformation from the identity functor 1_{SETS} to the functor* $\exp(\exp)$ *on the category $SETS$.*

Proof. We prove for any function $h : X \to Y$, the following diagram is commutative.

$$\begin{array}{ccc} X & \xrightarrow{\psi_X} & \exp(\exp(X)) \\ \Big\downarrow{\scriptstyle h} & & \Big\downarrow{\scriptstyle \omega(h)} \\ Y & \xrightarrow{\psi_Y} & \exp(\exp(Y)) \end{array}$$

Diagram 7

where $\omega(h) = (h^{-1})^{-1} : \exp(\exp(X)) \to \exp(\exp(Y))$ defined by $\omega(h)(\{A_\alpha\}_\alpha) = \{B \subset Y : h^{-1}(B) \in \{A_\alpha\}_\alpha\}$. Now $(\omega(h) \circ \psi_X)(x) = (\omega(h))(\psi_X(x)) = (\omega(h))(\dot{x}) = \{B \subset Y : h^{-1}(B) \in \dot{x}\} = \{B \subset Y : x \in h^{-1}(B)\} = \{B \subset Y : h(x) \in B\}$. Also $(\psi_Y \circ h)(x) = \psi_Y(h(x)) = \{B \subset Y : h(x) \in B\}$. Therefore, $\omega(h) \circ \psi_X = \psi_Y \circ h$. Hence $\psi = \{\psi_X\}$ is a natural transformation. □

Theorem 4.4. *The neighbourhood system ψ_X, defined as above, satisfies the following properties:*

(i) $I_{\psi^*_X}(h^{-1}(B)) \subset h^{-1}(I_{\psi^*_Y}(B))$ *for* $B \subset Y$.

(ii) $h^{-1}(cl_{\psi^*_Y}(B)) \subset cl_{\psi^*_X}(h^{-1}(B))$ *for* $B \subset Y$.

In addition, if $h : X \to Y$ is bijective function, then the following properties hold:

(iii) $h(I_{\psi^*_X}(A)) \subset I_{\psi^*_Y}(h(A))$ *for* $A \subset X$.

(iv) $cl_{\psi^*_Y}(h(A)) \subset h(cl_{\psi^*_X}(A))$ *for* $A \subset X$.

Proof. (i) Let $x \in I_{\psi^*_X}(h^{-1}(B))$, then $h^{-1}(B) \in \psi_X(x) \Rightarrow B \in \omega(h)(\psi_X(x)) \Rightarrow B \in (\omega(h) \circ \psi_X)(x) = (\psi_Y \circ h)(x) \Rightarrow B \in \psi_Y(h(x)) \Rightarrow h(x) \in I_{\psi^*_Y}(B) \Rightarrow x \in h^{-1}(I_{\psi^*_Y}(B))$. Therefore, $I_{\psi^*_X}(h^{-1}(B)) \subset h^{-1}(I_{\psi^*_Y}(B))$.

(ii) $h^{-1}(cl_{\psi^*_Y}(B)) = h^{-1}(Y - I_{\psi^*_Y}(Y - B)) = h^{-1}(Y) - h^{-1}(I_{\psi^*_Y}(Y - B)) \subset X - I_{\psi^*_X}(h^{-1}(Y - B)) = X - I_{\psi^*_X}(h^{-1}(Y) - h^{-1}(B)) = X - I_{\psi^*_X}(X - h^{-1}(B)) = cl_{\psi^*_X}(h^{-1}(B))$.

(iii) Let $y \in h(I_{\psi^*_X}(A))$, then $y = h(x)$ for some $x \in I_{\psi^*_X}(A)$, that is $A \in \psi_X(x)$. To prove $y \in I_{\psi^*_Y}(h(A))$, that is $h(A) \in \psi_Y(y) = \psi_Y(h(x)) = (\psi_Y \circ h)(x) = (\omega(h) \circ \psi_X)(x)$, that is $h^{-1}(h(A)) \in \psi_X(x)$. Now since h is bijective, $h^{-1}(h(A)) = A \in \psi_X(x)$. Therefore, $y \in I_{\psi^*_Y}(h(A))$. Therefore, $h(I_{\psi^*_X}(A)) \subset I_{\psi^*_Y}(h(A))$.

(iv) Let $y \in cl_{\psi^*_Y}(h(A))$. Then, $Y - h(A) \notin \psi_Y(y) \Rightarrow h^{-1}(Y - h(A)) \notin \psi_X(x)$, where $x \in X$ such that $h(x) = y$. since h is bijective, therefore $h^{-1}(Y - h(A)) = X - A \notin \psi_X(x) \Rightarrow x \in cl_{\psi^*_X}(A) \Rightarrow y \in h(cl_{\psi^*_X}(A))$. Hence $cl_{\psi^*_Y}(h(A)) \subset h(cl_{\psi^*_X}(A))$. □

Remark 4.3. For each complete generalized neighbourhood system ψ on X, I_ψ is a strong generalized interior operator [12]. Also SGI is a functor on the category $SETS$ [11]. Therefore, SGI is a functor on the category $\mathcal{C}$. Now we will prove that there exists a natural transformation from the functor $CGNS$ to the functor SGI on the category $\mathcal{C}$:

Theorem 4.5. *There is a natural transformation from the functor $CGNS$ to the functor SGI on the category $\mathcal{C}$.*

Proof. To prove the existence of a natural transformation $\vartheta : CGNS \to SGI$, we need to find a collection $\{\vartheta_X : X \text{ a set}\}$ such that whenever $h : X \to Y$ is any surjective function, the following diagram is commutative.

$$\begin{array}{ccc} CGNS(X) & \xrightarrow{\vartheta_X} & SGI(X) \\ \Big\downarrow{\scriptstyle CGNS(h)} & & \Big\downarrow{\scriptstyle SGI(h)} \\ CGNS(Y) & \xrightarrow{\vartheta_Y} & SGI(Y) \end{array}$$

Diagram 8

Define $\vartheta_X : CGNS(X) \to SGI(X)$ by $\vartheta_X(\psi) = I_\psi$. Since for a complete generalized neighbourhood system ψ, I_ψ is strong generalized interior operator, therefore ϑ_X is well-defined. To check the commutativity of the diagram, as for naturality of $\vartheta = \{\vartheta_X\}$. If $h : X \to Y$ is any surjective function, then $(\vartheta_Y \circ CGNS(h))(\psi) = \vartheta_Y((CGNS(h))(\psi)) = \vartheta_Y(\psi') = I_{\psi'}$, where $\psi'(y) = \{h(A) : A \in \psi(x)\}$, for $y = h(x)$. Also $(SGI(h) \circ \vartheta_X)(\psi) = (SGI(h))(\vartheta_X(\psi)) = (SGI(h))(I_\psi) = h \circ I_\psi \circ h^{-1}$, where $h : \exp(X) \to \exp(Y)$ and $h^{-1} : \exp(Y) \to$

$\exp(X)$ are image and inverse image mappings respectively. To prove $I_{\psi'} = h \circ I_\psi \circ h^{-1}$. Now $I_{\psi'} : \exp(Y) \to \exp(Y)$ and $h \circ I_\psi \circ h^{-1} : \exp(Y) \to \exp(Y)$ are given by $I_{\psi'}(B) = \{y \in Y :$ there exists $W \in \psi'(y)$ such that $W \subset B\}$ and $(h \circ I_\psi \circ h^{-1})(B) = \{h(x) :$ there exists $V \in \psi(x)$ such that $V \subset h^{-1}(B)\}$. Therefore, $I_{\psi'}(B) = (h \circ I_\psi \circ h^{-1})(B)$. Therefore, $\vartheta_Y \circ CGNS(h) = SGI(h) \circ \vartheta_X$ which proves the diagram is commutative and therefore $\vartheta = \{\vartheta_X\}$ is natural transformation. □

Remark 4.4. R. Shen [12] proved that for a generalized interior operator I on a set X, $I = I_{\psi_I}$. C. Cao et. al. [2] proved that for an ascending generalized neighbourhood system ψ on a set X, $\psi = \psi_{I_\psi}$. Therefore, if we define $\vartheta'_X : SGI(X) \to CGNS(X)$ by $\vartheta'_X(I) = \psi_I$, then ϑ'_X is a natural transformation such that $\vartheta_X \circ \vartheta'_X = 1_{SGI(X)}$ and $\vartheta'_X \circ \vartheta_X = 1_{CGNS(X)}$, that is, in this case, ϑ_X is natural isomorphism. Here ϑ and ϑ' are inverses of each other.

5. More relations between the functors GT and Γ

Remark 5.1. It has been proved [11] that Γ is a covariant functor on the category $SETS$ and $\phi = \{\phi_X\}$ is a natural transformation from the functor GT to the functor Γ, where $\phi_X : GT(X) \to \Gamma(X)$ is defined by $(\phi_X)(\mu) = \gamma_\mu$, $\gamma_\mu : \exp(X) \to \exp(X)$ is monotonic operator defined by $(\gamma_\mu)(A) = \cup\{G : G \in \mu, G \subset A\}$. In the following theorem we prove that left inverse for ϕ_X exists:

Theorem 5.1. *If $\theta_X : \Gamma(X) \to GT(X)$ is defined by $\theta_X(\gamma) = \mu_\gamma$, where $\mu_\gamma = \{A \subset X : A \subset \gamma(A)\}$. Then, $\theta : \Gamma \to GT$ is a natural transformation such that $\theta_X \circ \phi_X = 1_{GT(X)}$.*

Proof. To prove that $\theta : \Gamma \to GT$ is a natural transformation, we need to prove that for each set X, the following diagram is commutative:

$$\begin{array}{ccc} \Gamma(X) & \xrightarrow{\theta_X} & GT(X) \\ \downarrow{\scriptstyle \Gamma(h)} & & \downarrow{\scriptstyle GT(h)} \\ \Gamma(Y) & \xrightarrow{\theta_Y} & GT(Y) \end{array}$$

Diagram 9

Now $(\theta_Y \circ \Gamma(h))(\gamma) = \theta_Y((\Gamma(h))(\gamma)) = \theta_Y(h \circ \gamma \circ h^{-1}) = \theta_Y(\gamma') = \mu_{\gamma'} = \{B \subset Y : B \subset \gamma'(B)\}$, where $\gamma' = h \circ \gamma \circ h^{-1}$ and $(GT(h) \circ \theta_X)(\gamma) = (GT(h))(\theta_X(\gamma)) = (GT(h))(\mu_\gamma) = \{h(A) : A \in \mu_\gamma, A \subset X\}$. To prove $\{B \subset Y : B \subset \gamma'(B)\} = \{h(A) : A \in \mu_\gamma, A \subset X\}$, that is, $\{B \subset Y : B \subset (h \circ \gamma \circ h^{-1})(B)\} = \{h(A) : A \in \mu_\gamma, A \subset X\}$, which is immediate. Therefore, the diagram is commutative and so θ is a natural transformation.

Further $(\theta_X \circ \phi_X)(\mu) = \theta_X(\phi_X(\mu)) = \mu_{\phi_X(\mu)} = \{A \subset X : A \subset (\phi_X(\mu))(A)\} = \{A \subset X : A \subset (\gamma_\mu)(A)\} = \{A \subset X : A \subset \cup\{G : G \in \mu, G \subset A\}\} = \{A \subset X : A = \cup\{G : G \in \mu, G \subset A\}\} = \mu$. Therefore, $(\theta_X \circ \phi_X) = 1_{GT(X)}$. □

In the next two theorems, we establish the natural transformations between the composition functors $GT \circ \Gamma$ and $\Gamma \circ GT$ and identity functor on some suitably chosen categories.

Theorem 5.2. *There exists a natural transformation between the composition functor $GT \circ \Gamma$ and the identity functor 1_{SETS} on the category $SETS$.*

Proof. Define $\xi_X : X \to (GT \circ \Gamma)(X)$ by $\xi_X(x) = \{\emptyset, \{\gamma_x\}\}$, where $\gamma_x : \exp(X) \to \exp(X)$ defined by $\gamma_x(A) = \{x\}$, for all $A \subset X$. To prove $\xi = \{\xi_X\}$ is a natural transformation, we prove that the following diagram is commutative:

$$\begin{array}{ccc} X & \xrightarrow{\xi_X} & (GT \circ \Gamma)(X) \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle (GT \circ \Gamma)(h)} \\ Y & \xrightarrow{\xi_Y} & (GT \circ \Gamma)(Y) \end{array}$$

Diagram 10

$(\xi_Y \circ h)(x) = \xi_Y(h(x)) = \{\emptyset, \{\gamma'_{h(x)}\}\}$, where $\gamma'_{h(x)} : \exp(Y) \to \exp(Y)$ defined by $\gamma'_{h(x)}(B) = \{h(x)\}$, for all $B \subset Y$. Also $(((GT \circ \Gamma)(h)) \circ \xi_X)(x) = ((GT \circ \Gamma)(h))(\xi_X(x)) = ((GT \circ \Gamma)(h))(\{\emptyset, \{\gamma_x\}\}) = (GT(\Gamma(h)))(\{\emptyset, \{\gamma_x\}\}) = \{\Gamma(h)(\emptyset), \Gamma(h)(\{\gamma_x\})\} = \{\emptyset, \{h \circ \gamma_x \circ h^{-1}\}\}$. To prove $\gamma'_{h(x)} = h \circ \gamma_x \circ h^{-1}$. Now for $B \subset Y$, $(h \circ \gamma_x \circ h^{-1})(B) = h(\gamma_x(h^{-1}(B))) = h(\{x\}) = \{h(x)\} = (\gamma'_{h(x)})(B)$. Therefore, $\gamma'_{h(x)} = h \circ \gamma_x \circ h^{-1}$ and so $\xi_Y \circ h = ((GT \circ \Gamma)(h)) \circ \xi_X$, which proves that there is a natural transformation from the identity functor 1_{SETS} to the composition functor $GT \circ \Gamma$ on the category $SETS$. □

Let $\mathbf{C}$ be the category with sets as its objects and injective functions as its arrows. Then, restricting the category $SETS$ to the category $\mathbf{C}$, we can easily prove that the functors Γ and GT are covariant functors on $\mathbf{C}$ such that Γ is left inverse of GT as proved in the following theorem:

Theorem 5.3. *There exists a natural transformation between the functors $\Gamma \circ GT$ and the identity functor $1_{\mathbf{C}}$ on the category $\mathbf{C}$.*

Proof. Define $\rho_X : X \to (\Gamma \circ GT)(X)$ by $\rho_X(x) : \exp(GT(X)) \to \exp(GT(X))$, $(\rho_X(x))(\{\mu_\alpha\}_\alpha) = \{\mu_\alpha \vee x\}_\alpha$, where $\mu_\alpha \vee x = \{A : A \subset G \cup \{x\}$ for some $G \in \mu_\alpha\}$. Since for $\mu_\alpha \in GT(X)$, $\mu_\alpha \vee x \in GT(X)$, therefore $\rho = \{\rho_X\}$ is well defined. To prove $\rho = \{\rho_X\}$ is a natural transformation, we prove that the following diagram is commutative:

$$\begin{array}{ccc} X & \xrightarrow{\rho_X} & (\Gamma \circ GT)(X) \\ \Big\downarrow h & & \Big\downarrow (\Gamma \circ GT)(h) \\ Y & \xrightarrow{\rho_Y} & (\Gamma \circ GT)(Y) \end{array}$$

Diagram 11

$(\rho_Y \circ h)(x) = \rho_Y(h(x))$ and $(((\Gamma \circ GT)(h)) \circ \rho_X)(x) = ((\Gamma \circ GT)(h))(\rho_X(x)) = (\Gamma(GT(h)))(\rho_X(x)) = (GT(h)) \circ (\rho_X(x)) \circ (GT(h))^{-1}$, where $GT(h) : \exp(GT(X)) \to \exp(GT(Y))$ and $(GT(h))^{-1} : \exp(GT(Y)) \to \exp(GT(X))$ are image and inverse image mappings respectively. To prove $\rho_Y(h(x)) = (GT(h)) \circ (\rho_X(x)) \circ (GT(h))^{-1}$.

Now, for each $\{\mu'_\alpha\}_\alpha \in \exp(GT(Y))$, $(\rho_Y(h(x)))(\{\mu'_\alpha\}_\alpha) = \{\mu'_\alpha \vee h(x)\}_\alpha$ and

$$\begin{aligned} &((GT(h)) \circ (\rho_X(x)) \circ (GT(h))^{-1})(\{\mu'_\alpha\}_\alpha) \\ &= ((GT(h)) \circ (\rho_X(x)))(((GT(h))^{-1})(\{\mu'_\alpha\}_\alpha)) \\ &= ((GT(h)) \circ (\rho_X(x)))(\{\mu_\alpha \in GT(X) : (GT(h))(\mu_\alpha) \in \{\mu'_\alpha\}_\alpha\}) \\ &= (GT(h))(\rho_X(x))(\{\mu_\alpha : (GT(h))(\mu_\alpha) \in \{\mu'_\alpha\}_\alpha\}) \\ &= (GT(h))(\{\mu_\alpha \vee x : (GT(h))(\mu_\alpha) \in \{\mu'_\alpha\}_\alpha\}) \\ &= \{(GT(h))(\mu_\alpha \vee x) : (GT(h))(\mu_\alpha) \in \{\mu'_\alpha\}_\alpha\}). \end{aligned}$$

Since h is injective mapping, $(GT(h))(\mu_\alpha) \in \{\mu'_\alpha\}_\alpha$ implies $(GT(h))(\mu_\alpha) = \mu'_\alpha$, that is $\mu'_\alpha = \{h(A) : A \in \mu_\alpha\}$. Therefore, $\{(GT(h))(\mu_\alpha \vee x) : (GT(h))(\mu_\alpha) \in \{\mu'_\alpha\}_\alpha\} = \{h(A) : A \in (\mu_\alpha \vee x), (GT(h))(\mu_\alpha) = \mu'_\alpha\}_\alpha = \{h(A) : A \subset G \cup \{x\}$ for some $G \in \mu_\alpha, (GT(h))(\mu_\alpha) = \mu'_\alpha\}_\alpha = \{h(A) : A \subset G \cup \{x\}$ for some $G \in \mu_\alpha, h(G) \in \mu'_\alpha\}_\alpha = \{h(A) : h(A) \subset h(G) \cup h(\{x\})$ for some $h(G) \in \mu'_\alpha\}_\alpha = \{\mu'_\alpha \vee \{h(x)\}\}_\alpha = (\rho_Y(h(x)))(\{\mu'_\alpha\}_\alpha)$. Therefore, $\rho_Y(h(x)) = (GT(h)) \circ (\rho_X(x)) \circ (GT(h))^{-1}$ and so $(\rho_Y \circ h)(x) = ((\Gamma \circ GT)(h)) \circ \rho_X)(x)$. Hence the diagram is commutative, which proves that there is a natural transformation from the identity functor $1_{\mathbf{C}}$ to the composition functor $\Gamma \circ GT$ on the category $\mathbf{C}$. □

Theorem 5.4. *There exists a transformation $\phi = \phi_{X,Y} : Hom(\Gamma(X), Y) \to Hom(X, GT(Y))$, which is natural in both the components X and Y.*

Proof. Define $\phi = \phi_{X,Y} : Hom(\Gamma(X), Y) \to Hom(X, GT(Y))$ by $\phi(p) : X \to GT(Y)$, $\phi(p)(x) = \{\emptyset, \{p(\gamma_x)\}\}$, for all $p : \Gamma(X) \to Y$.

ϕ **is natural in** $SETS$ **(Component** X**).** $\phi_{-,Y} : Hom(\Gamma -, Y) \to Hom(-, GT(Y))$. For each $h : X \to X'$, we prove the following diagram is commutative:

$$\begin{array}{ccc} Hom(\Gamma(X),Y) & \xrightarrow{\phi_{X,Y}} & Hom(X,GT(Y)) \\ {\scriptstyle Hom(\Gamma(h),Y)}\uparrow & & \uparrow{\scriptstyle Hom(h,GT(Y))} \\ Hom(\Gamma(X'),Y) & \xrightarrow{\phi_{X',Y}} & Hom(X',GT(Y)) \end{array}$$

Diagram 12

For each $p' : \Gamma(X') \to Y$, $(\phi_{X,Y} \circ Hom(\Gamma(h),Y))(p') = \phi_{X,Y}(p' \circ \Gamma(h))$ and $((Hom(h,GT(Y))) \circ \phi_{X',Y})(p') = (Hom(h,GT(Y)))(\phi_{X',Y}(p')) = (\phi_{X',Y}(p')) \circ h$. We have to prove that $\phi_{X,Y}(p' \circ \Gamma(h)) = (\phi_{X',Y}(p')) \circ h$. Now $\phi_{X,Y}(p' \circ \Gamma(h))$ and $(\phi_{X',Y}(p')) \circ h : X \to GT(Y)$.

Firstly, we prove $\phi_{X,Y}(p) = GT(p) \circ \xi_X$. $(GT(p) \circ \xi_X)(x) = GT(p)(\xi_X(x)) = GT(p)(\{\emptyset, \{\gamma_x\}\}) = \{p(\emptyset), \{p(\gamma_x)\}\} = \{\emptyset, \{p(\gamma_x)\}\} = (\phi_{X,Y}(p))(x)$. Therefore, $\phi_{X,Y}(p) = GT(p) \circ \xi_X$.

Now, $(\phi_{X,Y}(p' \circ \Gamma(h)))(x) = (GT(p' \circ \Gamma(h)) \circ \xi_X)(x) = (GT(p' \circ \Gamma(h)))(\xi_X(x)) = (GT(p' \circ \Gamma(h)))(\{\emptyset, \{\gamma_x\}\}) = \{(p' \circ \Gamma(h))(\emptyset), \{(p' \circ \Gamma(h))(\gamma_x)\}\} = \{\emptyset, \{p'(\gamma_{h(x)})\}\}$, since for all $B \in \exp(X')$, $(\Gamma(h))(\gamma_x)(B) = (h \circ \gamma_x \circ h^{-1})(B) = h(\gamma_x(h^{-1}B)) = h(x)$. Also, $((\phi_{X',Y}(p')) \circ h)(x) = (\phi_{X',Y}(p'))(h(x)) = (GT(p') \circ \xi_{X'})(h(x)) = (GT(p'))(\xi_{X'}(h(x))) = (GT(p'))(\{\emptyset, \{\gamma_{h(x)}\}\}) = \{p'(\emptyset), \{p'(\gamma_{h(x)})\}\} = \{\emptyset, \{p'(\gamma_{h(x)})\}\}$. Therefore, $\phi_{X,Y}(p' \circ \Gamma(h)) = (\phi_{X',Y}(p')) \circ h$, which proves that $\phi_{X,Y}$ is natural in component X.

ϕ is natural in $SETS$ (Component Y). $\phi_{X,-} : Hom(\Gamma(X),-) \to Hom(X,GT-)$. For each $k : Y \to Y'$, we prove the following diagram is commutative:

$$\begin{array}{ccc} Hom(\Gamma(X),Y) & \xrightarrow{\phi_{X,Y}} & Hom(X,GT(Y)) \\ \downarrow{\scriptstyle Hom(\Gamma(X),k)} & & \downarrow{\scriptstyle Hom(X,GT(k))} \\ Hom(\Gamma(X),Y') & \xrightarrow{\phi_{X,Y'}} & Hom(X,GT(Y')) \end{array}$$

Diagram 13

For each $p : \Gamma(X) \to Y$, $(\phi_{X,Y'} \circ Hom(\Gamma(X),k))(p) = \phi_{X,Y'}(k \circ p) \in Hom(X,GT(Y'))$ and $((Hom(X,GT(k))) \circ \phi_{X,Y})(p) = (Hom(X,GT(k)))(\phi_{X,Y}(p)) = (GT(k)) \circ (\phi_{X,Y}(p)) \in Hom(X,GT(Y'))$. We have to prove that $\phi_{X,Y'}(k \circ p) = (GT(k)) \circ (\phi_{X,Y}(p))$.

Now, $(\phi_{X,Y'}(k \circ p))(x) = \{\emptyset, \{(k \circ p)(\gamma_x)\}\}$. Also, $((GT(k)) \circ (\phi_{X,Y}(p)))(x) = (GT(k))(\phi_{X,Y}(p))(x) = (GT(k))(\{\emptyset, \{p(\gamma_x)\}\}) = \{\emptyset, \{(k \circ p)(\gamma_x)\}\}$. Therefore, $\phi_{X,Y'}(k \circ p) = (GT(k)) \circ (\phi_{X,Y}(p))$, which proves that $\phi_{X,Y}$ is natural in component Y. □

Remark 5.2. We have proved in the Theorem 5.4 that $\phi(p) = GT(p) \circ \xi_X$. Also, $\xi_X = \phi(1_{\Gamma(X)})$ can be proved, since $1_{\Gamma(X)} : \Gamma(X) \to \Gamma(X)$, $\phi(1_{\Gamma(X)}) : X \to GT(\Gamma(X))$ given by $(\phi(1_{\Gamma(X)}))(x) = \{\emptyset, \{1_{\Gamma(X)}(\gamma_x)\}\} = \{\emptyset, \{\gamma_x\}\} = \xi_X(x)$. Hence ϕ is an adjunction between the functors Γ and GT, that is, the functor Γ is left adjoint of the functor GT, written as $\Gamma \dashv GT$.

Acknowledgement

I am thankful to Dr. Navpreet Singh Noorie for his kind guidance and help.

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Accepted: March 25, 2020

Soft interior ideals over semigroups

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Abstract. The concept of a soft interior ideal over a semigroup S is presented in this paper. Relations between soft left (right) ideals and soft interior ideals are investigated and illustrative examples are provided. Regular and intra-regular semigroups are characterized in terms of soft interior ideals. It is shown that soft interior ideals over a semi-simple semigroup S is soft ideals. Furthermore, semi-simple semigroups are characterized by means of soft interior ideals. Finally, we introduced a characterization of fuzzy interior ideals of S by certain soft interior ideals over S.

Keywords: regular semigroups, soft semigroups, soft left (right) ideals, soft interior ideals.

1. Introduction

Many scientific areas like engineering, medicine and economics involve uncertain concepts. So, ways of traditional mathematics are unsuitable to resolve difficulties in these branches.

In 1965, Zadeh initiated fuzzy sets as a very popular tool for modelling uncertainties in wide fields [20]. In this direction, Molodtsov introduced soft sets [14]. A soft set could be a collection of approximate attributes of an object with no constraints on them. This makes soft set theory simply applicable in practice. We referee the reader to [6, 13, 14, 19] for potential applications of

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soft sets. A semigroup $(S, \star)$ is an algebraic structure consisting of a nonempty set S with associative binary operation $(\star)$ defined on S. Semigroup theory has found significant applications in many fields such as finite state machines, theory of automata, coding theory and others. Therefore, semigroups and related structures are investigated in fuzzy settings [9, 11, 15] and recently in (fuzzy soft) soft settings [1, 2, 4, 5, 8, 10, 12, 17, 18].

Ali et al. [3, 4] introduced the concept of soft semigroups as a collection of subsemigroups of a semigroup and defined soft (left, right, quasi, bi) ideals of a semigroup S. Based on, the smallest soft right (left) ideals over S generated by a soft set over S were studied by the same authors [16]. In this paper, we define the concept of a soft interior ideal over a semigroup S. Relations between soft left (right) ideals and soft interior ideals are investigated and the illustrative examples are provided. Regular and intra-regular semigroups are characterized in terms of soft interior ideals. It is shown that a semigroup S is semi-simple if and only if it is soft semi-simple. Furthermore, semi-simple semigroups are characterized by means of soft interior ideals. Finally, we introduced a characterization of fuzzy interior ideals of S by certain soft interior ideals over S.

2. Preliminaries

Let S be a semigroup. A subset $\emptyset \neq A \subseteq S$ is called a subsemigroup of S if $A^2 \subseteq A$, a left (right) ideal of S if $SA \subseteq A$ ($AS \subseteq A$) and a two-sided ideal (or simply ideal) of S if it is both a left and a right ideal of S. A subsemigroup $A \subseteq S$ is called an interior ideal of S if $SAS \subseteq A$. We write $AB = \{ab : a \in A, b \in B\}$ for some $A, B \subseteq S$.

Definition 2.1 ([14])**.** Let E be a set of parameters, $P(S)$ the power set of S and $A \subseteq E$. The pair (F, A) is called a soft set over S, where F is a mapping $F : A \longrightarrow P(S)$.

Definition 2.2 ([4])**.** Let (F, A) and (G, B) be soft sets over S, then (G, B) is called a soft subset of (F, A), denoted by $(G, B) \subseteq (F, A)$ If $B \subseteq A$ and $G(b) \subseteq F(b)$, for all $b \in B$.

Definition 2.3 ([4])**.** Let (F, A) and (G, B) be two soft sets over S, then:

1. the extended intersection of (F, A) and (G, B), $(F, A) \cap_\varepsilon (G, B)$, is a soft set (H, C) defined as, $\forall c \in C = A \cup B$,

$$H(c) = \begin{cases} F(c), & \text{if } c \in A - B, \\ G(c), & \text{if } c \in B - A, \\ F(c) \cap G(c), & \text{if } c \in A \cap B. \end{cases}$$

2. the restricted intersection of (F, A) and (G, B), $(F, A) \cap_R (G, B)$, is a soft set (H, C) defined as $H(c) = F(c) \cap G(c)$, for all $c \in C = A \cap B$.

Definition 2.4 ([4]). Let (F, A) and (G, B) be two soft sets over S, then:

1. the extended union of (F, A) and (G, B), $(F, A) \cup_\varepsilon (G, B)$, is a soft set (H, C) defined as, $\forall c \in C = A \cup B$,

$$H(c) = \begin{cases} F(c), & \text{if } c \in A - B, \\ G(c), & \text{if } c \in B - A, \\ F(c) \cup G(c), & \text{if } c \in A \cap B. \end{cases}$$

2. the restricted union of (F, A) and (G, B), denoted by $(F, A) \cup_R (G, B)$, is a soft set (H, C)defined as, $H(c) = F(c) \cup G(c)$, for all $c \in C = A \cap B$.

Definition 2.5 ([4]). If (F, A) and (G, B) are soft sets over S. Then, $(F, A) \wedge (G, B)$ is a soft set $(H, A \times B)$ defined as $H(a, b) = F(a) \cap G(b)$, for all $(a, b) \in A \times B$.

Definition 2.6 ([4]). If (F, A) and (G, B) are soft sets over S. Then, $(F, A) \vee (G, B)$, is a soft set $(K, A \times B)$, where $K(a, b) = F(a) \cup G(b)$, for all $(a, b) \in A \times B$.

Definition 2.7 ([4]). If (F, A) and (G, B) are soft sets over $(S, \star)$. Then, $(F, A) \star (G, B)$ is a soft set $(H, A \times B)$ where, $H(a, b) = F(a) \star G(b)$, for all $(a, b) \in A \times B$.

Definition 2.8 ([4]). If (F, A) and (G, B) are two soft sets over S, then the Cartesian product $(F, A) \times (G, B)$, is a soft set $(H, A \times B)$ over $S \times S$ such that $H(a, b) = F(a) \times G(b)$, $\forall (a, b) \in A \times B$.

3. Basic definitions and results of soft ideals

In this section, we recall the definitions of soft semigroups and soft ideals. Some basic results are provided.

Definition 3.1 ([4]). Let (F, A) and (G, B) be two soft sets over S. The soft product of (F, A) and (G, B), $(F, A) \diamond (G, B)$, is defined as the soft set (H, C) where $H(c) = F(c)G(c)$, for all $c \in C = A \cap B$.

Definition 3.2 ([4]). A soft set (F, A) over S is called a soft semigroup if $(F, A) \diamond (F, A) \subseteq (F, A)$.

We denote by **Ssem(S)** to the set of all soft semigroups over S.

Proposition 3.3 ([4]). *A soft set (F, A) is a soft semigroup over S if and only if $\forall a \in A$, $F(a) \neq \emptyset$ is a subsemigroup of S.*

Definition 3.4 ([4]). A soft set (F, A) over S is called a soft left [right] ideal over S, if $(S, A) \diamond (F, A) \subseteq (F, A)$ $[(F, A) \diamond (S, A) \subseteq (F, A)]$ where (S, A) is a soft set over S defined by $S(a) = S$ $\forall a \in A$. A soft set (F, A) is called a soft ideal if it is both a soft left and a soft right ideal over S.

Proposition 3.5 ([4]). *A soft set (F, A) over S is a soft ideal if and only if $\forall a \in A$, $F(a) \neq \phi$ is an ideal of S.*

Example 3.6. Let $S = \{a, b, c, d\}$ be a semigroup defined by the following table

.	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

Define $F : S \longrightarrow P(S)$ by $F(a) = \{a\}$,$F(b) = \{a, b\}$,$F(c) = \{a, b, c\}$, $F(d) = \{a, b, d\}$. It is clear that F (x) is a subsemigroup of S, for all $x \in S$. Hence (F,S) is a soft semigroup over S. Furthermore (F,S) is a soft ideal over S.

Proposition 3.7 ([4]). If (F, A) and (G, B) are two soft semigroups (ideals) over S. Then, $(F, A) \cap_\varepsilon (G, B)$ is a soft semigroup (ideal) over S.

Proposition 3.8 ([4]). If (F, A) and (G, B) are two soft semigroups (ideals) over S such that $A \cap B = \emptyset$. Then, $(F, A) \cup_\varepsilon (G, B)$ is also a soft semigroup (ideals) over S.

Proposition 3.9 ([4]). Let (F, A) and (G, B) be two soft semigroups (ideals) over S. Then, $(F, A) \wedge (G, B)$ is a soft semigroup (ideal) over S.

Proposition 3.10 ([4]). Let (F, A) and (G, B) be two soft semigroups (ideals) over a commutative semigroup $(S, \star)$. Then, $(F, A) \star (G, B)$ is also a soft semigroup (ideal) over S.

Proposition 3.11 ([4]). Let (F, A) and (G, B) be soft semigroups (ideals) over S and T respectively. Then, $(F, A) \times (G, B)$ is a soft semigroup (ideal) over $S \times T$.

4. Soft interior ideals over S

Here, we define the soft interior ideals over a semigroup S. Relations between soft left (right) ideals and soft interior ideals are investigated. Illustrative examples are given.

Definition 4.1. A soft semigroup (F, A) over S is called a soft interior ideal over S if $(S, A) \diamond (F, A) \diamond (S, A) \subseteq (F, A)$.

The notation **Sint(S)** is stands for the set of all soft interior ideals over S.

Example 4.2. Let $S = \{e, a, b, c\}$ be a semigroup defined as following.

.	e	a	b	c
e	e	e	e	e
a	e	e	e	e
b	e	e	e	a
c	e	e	a	b

Consider the soft set $(F,\{b\})$ where $F:\{b\}\to P(S)$ defined as $F(b)=\{e,a\}$. Clearly, $(F,\{b\})\in$ **Ssem(S)** and $(S,\{b\})\diamond(F,\{b\})\diamond(S,\{b\})=(H,\{b\})$ where $H(b)=S(b)F(b)S(b)=S\{e,a\}S=\{e\}\subseteq F(b)$. Therefore, $(F,\{b\})\in$ **Sint(S)**.

Theorem 4.3. *Let (F,A) be a soft set over S, then $(F,A)\in$ **Sint(S)** if and only if $\forall a\in A, F(a)\neq\emptyset$ is an interior ideal of S.*

Proof. Let (F,A) be an element in **Sint(S)**, then (F,A) is a soft semigroup. That is, $F(a)\neq\emptyset$ is a subsemigroup of S, for any $a\in A$. By definition 4.1, we also have $(H,A)=(S,A)\diamond(F,A)\diamond(S,A)\subseteq(F,A)$. Hence,

$$H(a)=S(a)F(a)S(a)=SF(a)S\subseteq F(a).$$

Thus $SF(a)S\subseteq F(a)$, which shows that $F(a)$ is an interior ideal of S, for all $a\in A$. Conversely, suppose that (F,A) is a soft set over S such that, for all $a\in A$, $F(a)\neq\emptyset$, is an interior ideal of S. Then, $F(a)\neq\emptyset$ is a subsemigroup of S, for all $a\in A$. By Proposition 3.3, $(F,A)\in$ **Ssem(S)**. Furthermore, if $F(a)=\emptyset$, for all $a\in A$, then $SF(a)S=\emptyset\subseteq F(a)$; otherwise by hypothesis $F(a)\neq\emptyset$ is an interior ideal of S and so $SF(a)S\subseteq F(a)$. Hence we conclude that $(S,A)\diamond(F,A)\diamond(S,A)\subseteq(F,A)$. This shows that (F,A) is a soft interior ideal over S. □

Lemma 4.4. *Let (F,A) be a soft ideal over S, then $(F,A)\in$ **Sint(S)**.*

Proof. Let (F,A) be a soft ideal over S, then $F(a)$ is an ideal of S, for all $a\in A$. Following, we get $F(a)F(a)\subseteq SF(a)\subseteq F(a)$, that is, $F(a)$ is a subsemigroup of S. Thus $(F,A)\in$ **Ssem(S)**. Now, we have $(S,A)\diamond(F,A)\diamond(S,A)\subseteq(F,A)$. Indeed, since $(F,A)$is a soft ideal over S, we have $F(a)S\subseteq F(a)$. This implies $SF(a)S\subseteq SF(a)\subseteq F(a)$. Therefore, (F,A) is a soft interior ideal over S. □

Generally, the opposite of the result above is not true.

Example 4.5. Let S $=\{0,a,b,c\}$ be a semigroup defined as

.	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	0	0	a
c	0	0	a	b

Consider the soft set $(F,\{b\})$ where $F:\{b\}\to P(S)$ is defined as $F(b)=\{0,b\}$. Clearly,$(F,\{b\})$ is a soft semigroup. Moreover, $(F,\{b\})$ is a soft interior ideal over S because $S\{0,b\}S=\{0\}$,while $(F,\{b\})$ is not a soft ideal over S because $S\{0,b\}=\{0,a\}\not\subseteq F(b)$.

A semigroup S is called regular if for all $a\in S$ there exists $x\in S$ such that $a=axa$.

Theorem 4.6. *In a regular semigroup S, (F, A) is a soft ideal over S if and only if $(F, A) \in$ **Sint(S)**.*

Proof. The direction ($\Longrightarrow$) comes directly by lemma 4.4. For ($\Longleftarrow$), assume that $(F, A) \in$ **Sint(S)**. Then, $F(a)$ is an interior ideal of S, for all $a \in A$. Let $s \in S$ and $x, y \in F(a)$. Since S is regular, there exist $u, v \in S$ such that $x = xux$ and $y = yvy$. Thus we have $sx = sxux = sx(ux) \in F(a)$ and $ys = yvys = (yv)ys \in F(a)$. Thus $SF(a) \subseteq F(a)$ and $F(a)S \subseteq F(a)$. Hence $F(a)$ is an ideal of S, for all$a \in A$. Therefore, (F, A) is a soft ideal over S. □

Recall from [4] that a semigroup S is called intra-regular if for each $a \in S$, there exist $x, y \in S$ such that $a = xa^2y$.

Theorem 4.7. *Let (F, A) be a soft set over an intra- regular semigroup S. Then, the following statements are equivalent.*

(1) *(F, A) is a soft ideal over S.*

(2) *(F, A) is a soft interior ideal over S.*

Proof. The implication (1) $\Longrightarrow$ (2) is clear. For (2) $\Longrightarrow$ (1), assume that $(F, A) \in$ **Sint(S)**, then $F(a)$ is an interior ideal of S, for all $a \in A$. Let $s \in S$ and $x, y \in F(a)$. Since S is intra-regular, there exist $u, v, w, z \in S$ such that $x = ux^2v$ and $y = wy^2z$. Thus, we have $sx = sux^2v = (su)x(xv) \in F(a)$ and $ys = wy^2zs = (wy)y(zs) \in F(a)$. Thus $SF(a) \subseteq F(a)$ and $F(a)S \subseteq F(a)$. Hence, $F(a)$ is an ideal of S, for all $a \in A$. Therefore, (F, A) is a soft ideal over S. □

Proposition 4.8. *If $(F, A), (G, B) \in$ **Sint(S)**. Then, $(F, A) \cap_\varepsilon (G, B) \in$ **Sint(S)**.*

Proof. By Proposition 3.7, $(H, A \cap B) = (F, A) \cap_\varepsilon (G, B)$ is a soft semigroup over S. Let$x \in H(c) = F(c) \cap G(c)$, for all $c \in A \cap B$, then $SxS \subseteq F(c) \cup G(c)$. Thus $SH(c)S \subseteq H(c)$. Therefore, $(H, A \cap B) \in Sint(S)$. □

Proposition 4.9. *If (F, A) and (G, B) are two soft interior ideals over S such that $A \cap B = \emptyset$. Then, $(F, A) \cup_\varepsilon (G, B) \in$ **Sint(S)**.*

Proof. By Proposition 3.8, the proof is straightforward. □

Proposition 4.10. *Let (F, A) and (G, B) be two soft interior ideals over S. Then, $(F, A) \wedge (G, B) \in$ **Sint(S)**.*

Proof. By Proposition 3.9, the proof is straightforward. □

Proposition 4.11. *Let $(F, A) \in$ **Sint(S)**and $(G, B) \in$ **Sint(T)**. Then, $(F, A) \times (G, B)$ is a soft interior ideal over $S \times T$.*

Proof. By Proposition 3.11, the proof is straightforward. □

Proposition 4.12. *Let $(S, \star)$ be a commutative semigroup, and $(F, A), (G, B) \in$ **Sint(S)**. Then, $(F, A) \star (G, B) \in$ **Sint(S)**.*

Proof. By definition 2.7, $(F, A) \star (G, B) = (H, A \times B)$, such that $H(a, b) = F(a) \star G(b)$. By Theorem 4.3, for all $a \in A$,$b \in B$, $F(a)$ and $G(b)$ are interior ideals of S. Thus $F(a)\star G(b)\star F(a)\star G(b) = F(a)\star G(b)\star G(b)\star F(a) \subseteq F(a)\star G(b)\star F(a) \subseteq F(a)\star F(a)\star G(b) \subseteq F(a)\star G(b)$. Thus $(H, A\times B) \in$ **Ssem(S)**. Also, we have $S \star H(a, b) \star S = S \star F(a) \star G(b) \star S = (S \star F(a) \star S) \star G(b) \subseteq F(a) \star G(b)$. This shows that $F(a) \star G(b)$is an interior ideal of S, for all$(a, b) \in A \times B$ and consequently,$(F, A) \star (G, B) \in$ **Sint(S)**. □

The following example shows that the condition of commutativity of S is necessary.

Example 4.13. The semigroup $S = \{1, a, b, c\}$ defined below is non-commutative,

$\star$	1	a	b	c
1	1	a	b	c
a	a	a	a	a
b	b	b	b	b
c	c	b	a	c

Consider the soft semigroups (F, S) and $(G, \{1, a\})$ over S defined by $F(1) = \{1\}$, $F(a) = \{a\}$; $F(b) = \{b\}$, $F(c) = \{c\}$ and $G(1) = \{1, a\}$, $G(a) = \{1, b\}$. By the definition 2.7, we get $(F, S) \star (G, \{1, a\}) = (H, S \times \{1, a\})$ such that $H(x, y) = F(x) \star G(y)$. Obviously, we have $H(c, 1) = F(c) \star G(1) = \{c, a\}$ which is not a subsemigroup of S.

Theorem 4.14. *Let S be a semigroup with identity and (F, A) be a soft ideal over S. Then, the soft set (I, A) defined by*

$$I(a) = \{x \in S : SxS \subseteq F(a)\},$$

for all $a \in A$, is the largest soft interior ideal over S contained in (F, A).

Proof. Firstly, we show that $(I, A) \in$ **Sint(S)**. To this end, let $x, y \in I(a)$, Then, $S(xy)S = (Sxy)S \subseteq F(a)S \subseteq F(a)$ hence $xy \in I(a)$ that is, $I(a)$ is a subsemigroup of S, for all $a \in A$. Thus (I, A) is a soft semigroup over S. Moreover, we have $SI(a)S \subseteq I(a)$, for all $a \in A$. In fact, for $x \in I(a)$ and $t, r \in S$, $rxt \in SI(a)S$. Then, we have $S(rxt)S \subseteq F(a)S \subseteq F(a)$.

Hence $rxt \in I(a)$, that is $SI(a)S \subseteq I(a)$, for all $a \in A$. Therefore, we get $(I, A)) \in$ **Sint(S)**. Since S has an identity element, then$(I, A) \subseteq (F, A)$. To show that (I, A) is the largest soft interior ideal over S contained in (F, A), assume that $(G, A) \in$ **Sint(S)** such that $(G, A) \subseteq (F, A)$. Then, for any $x \in G(a)$, we get $SxS \subseteq G(a) \subseteq F(a) \Longrightarrow x \in I(a)$, for all$a \in A$. Thus $(G, A) \supseteq (I, A)$. This ends the proof. □

5. Soft semi-simple semigroups

It is known that [15] "a semigroup S is called semi-simple if every ideal of S is idempotent". Clearly, S is a semi-simple semigroup if and only if $x \in (SxS)(SxS)$for every $x \in S$, that is, there exist $a, b, c \in S$ such that $x = axbxc$.

Definition 5.1 ([3])**.** A semigroup S is called a soft semi-simple semigroup if for each soft ideal (F, A) over S, $(F, A) \diamond (F, A) = (F, A)$.

Lemma 5.2 ([3])**.** *A semigroup S is soft semisimple if and only if it is semisimple.*

Theorem 5.3. *Every soft interior ideal over a semi-simple semigroup S is a soft ideal over S.*

Proof. Assume that $(F, A) \in$ **Sint(S)**. Let $t \in S$ and $a \in F(\alpha)$ where $\alpha \in A$. Since S is semisimple, then there exist $x, y, z \in S$ such that $a = xayaz$. Thus we have $ta = t(xayaz) = (tx)a(yaz) \in F(\alpha)$ Hence $F(\alpha)$ is a left ideal of S, for all $\alpha \in A$. Similarly, we can prove that $F(\alpha)$ is a right ideal of S, for all $\alpha \in A$. Thus (F, A) is a soft ideal over S. □

Theorem 5.4. *A semigroup S is semi-simple if and only if $(F, A) \cap_\varepsilon R(G, B) = (F, A) \diamond (G, B)$, for all $(F, A), (G, B) \in$ **Sint(S)**.*

Proof. Assume that S is a semisimple semigroup. Let (F, A) and (G, B) be soft interior ideals over S. Following definition 5.1, (F, A) is a soft ideal over S. Then, $(F, A) \cdot (G, B) = (H, A \cap B)$such that $H(c) = F(c)G(c) \subseteq F(c)S \subseteq F(c)$, for all$c \in A \cap B$. Hence $(F, A) \diamond (G, B) \subseteq (F, A)$. Similarly,$(F, A) \cdot (G, B) \subseteq (G, B)$. Thus $(F, A) \diamond (G, B) \subseteq (F, A) \cap_\varepsilon (G, B)$. To prove the reverse inclusion, we assume that $a \in F(c) \cap G(c)$ for $c \in A \cap B$. Since S is semi-simple, there exist$x, y, z \in S$ such that $a = xayaz$. Because $G(c)$ is an ideal of S, then $xay \in F(c)$ and $az \in G(c)$. Hence$a = xayaz \in F(c)G(c)$. This implies that $(F, A) \cap_\varepsilon (G, B) \subseteq (F, A) \diamond (G, B)$. Therefore $(F, A) \cap_\varepsilon (G, B) = (F, A) \diamond (G, B)$. Conversely, let (R, S) be a soft set over S defined by $R(x) = J[x]$ where $J[x] = x \cup xS \cup Sx \cup SxS$ is the principal ideal generated by x. Thus, by hypothesis,$x \in J[x] \cap J[x] = J[x]J[x]$. As a result of a simple argument given in [15], we conclude that $x \in J[x]J[x] \subseteq (SxS)(SxS)$. Therefore, S is a semisimple semigroup. □

The following corollary is a consequence of the above Theorem.

Corollary 5.5. *A semigroup S is semi-simple iff every soft interior ideal over S is idempotent.*

6. Characterization of soft interior ideals by certain soft ideals

From [15], we recall that a fuzzy set $\mu : S \to [0, 1]$ in a semigroup S is called fuzzy subsemigroup of S if for all $x, y \in S$, we have $\mu(xy) \geq \mu(x) \wedge \mu(y)$. A fuzzy

set μ in a semigroup S is called fuzzy left(right) ideal of S if for all$x, y \in S$, $\mu(xy) \geq \mu(y)(\mu(xy) \geq \mu(x))$. A fuzzy subsemigroup μ of a semigroup S is called fuzzy interior ideal of S if for all $x, a, y \in S$, $\mu(xay) \geq \mu(a)$.

Definition 6.1 ([3])**.** Let μ be a fuzzy subset of a semigroup S, define the soft set (F, A) by the map $F : A \to P(S)$ such that $F(t) = \{x \in S : \mu(x) \geq t\}$, for all $t \in A \subseteq [0, 1]$.

Definition 6.2 ([3])**.** Let μ be a fuzzy subset of a semigroup S, define the soft set $(F_q, [0, 1])$ by the map $F_q : A \to P(S)$ such that $F_q(t) = \{x \in S : \mu(x) > 1 - t\}$, for all $t \in [0, 1]$.

Proposition 6.3 ([3])**.** *The soft set $(F, [0, 1])$ is a soft semigroup (soft ideal) over S if and only if the fuzzy set $\mu$is a fuzzy subsemigroup (fuzzy ideal) of S.*

Proposition 6.4 ([3])**.** *The soft set $(F_q, [0, 1])$ is a soft semigroup (soft ideal) over S if and only if the fuzzy set μ is a fuzzy subsemigroup (fuzzy ideal) of S.*

Theorem 6.5. *The soft set $(F, [0, 1])$ is a soft interior ideal over S if and only if the fuzzy set μ is a fuzzy interior ideal of S.*

Proof. ($\Longleftarrow$) Assume that μ is a fuzzy interior ideal of S. By Proposition 6.3, $(F, [0, 1])$ is a soft semigroup over S. Let $a \in F(t)$ and $x, y \in S$, then $\mu(xay) \geq \mu(a) \geq t$ so $xay \in F(t)$, for all $t \in [0, 1]$. Therefore $(F, [0, 1])$ is a soft interior ideal over S.

($\Longrightarrow$)let (F,[0,1]) be a soft interior ideal over S, then F(t) is an interior ideal of S, for all $t \in [0, 1]$. If there are $x, a, y \in S$ such that $\mu(xay) < \mu(a) = t$, then$a \in F(t)$ and $xay \notin F(t)$. This contradicts that F(t) is an interior ideal of S. Hence $\mu(xay) \geq \mu(a)$ holds for all$x, a, y \in S$. □

Theorem 6.6. *The soft set $(F_q, [0, 1])$ is a soft interior ideal over S if and only if the fuzzy set μ is a fuzzy interior ideal of S.*

Proof. ($\Longrightarrow$) Suppose $(F_q, [0, 1])$ is a soft interior ideal over S, then$F_q(t)$ is an interior ideal of S, for all $t \in [0, 1]$. Consider $\mu(xay) < \mu(a)$, for some $x, a, y \in S$ and choose $t \in [0, 1]$ such that $\mu(xay) + t < 1 \leq \mu(a) + t$. Then, $a \in F_q(t)$ but $xay \notin F(t)$. This contradicts that $F_q(t)$ is an interior ideal of S. Hence $\mu(xay) \geq \mu(a)$ holds for all $x, a, y \in S$.

($\Longleftarrow$) Assume that μ is a fuzzy interior ideal of S. Proposition 6.4 implies that $(F_q, [0, 1])$ is a soft semigroup over S. Let $a \in F_q(t)$ and $x, y \in S$, then $\mu(xay) \geq \mu(a) > 1 - t$. So $xay \in F_q(t)$, for all $t \in [0, 1]$. Therefore $(F_q, [0, 1])$ is a soft interior ideal over S. □

7. Conclusion

In this study, we introduced the notion of soft interior ideals over a semigroup. We used this concept to characterize semisimple semigroups . Relations between soft interior and fuzzy interior ideals are introduced. As a future work, soft prime interior ideals may be investigated.

Acknowledgements

The authors would like to thank the referees for their valuable suggestions to publish this paper.

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Accepted: January 03, 2020

z-clean and strongly z-clean rings

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Abstract. In this article we introduce the concept of z-clean and strongly z-clean rings. The ring R is said to be a z-clean ring if every element of R is sum of a zero divisor of R and an idempotent element of R.

We present the necessary and sufficient condition when the clean rings, and z-clean rings become equivalent. We study various properties of the z-clean and strongly z-clean rings.

Keywords: clean ring, z-clean ring, strongly z-clean ring.

1. Introduction

Throughout this paper, R denotes an associative ring with identity, $U(R)$ the group of units, $Id(R)$ the set of idempotents, $Reg(R)$ the set of regulars, $M_n(R)$ the ring of all $n \times n$ matrices over R and

$$Z(R) = \{0 \neq x \in R | \exists 0 \neq y \in R, xy = 0\}.$$

Nicholson in [8] proved some important properties of clean rings. It is important to know when we can decompose some (or all) elements of a ring as a sum of an idempotent and an element with some special properties. For instance, an element x of a ring R is

- clean if $x = u + e$, where $e \in Id(R)$ and $u \in U(R)$ ([9]):
- $r-$clean if $x = r + e$, where $e \in Id(R)$ and $r \in Reg(R)$ ([2]);

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- nill-clean if $x = n + e$,where $e \in Id(R)$ and $n \in N(R)$ ([8]).

Several authors worked on clean rings and investigated properties of clean rings [1, 4, 5, 8].

Further in 1999, Nicholson [10] called an element of a ring R as strongly clean if it is the sum of a unit and an idempotent that commute with each other that means if $a = e + u$, where $e^2 = e$ and u is a unit of R such that $eu = ue$ and R is strongly clean if each of its element is strongly clean. Again clearly from [10], a strongly clean ring is clean, and the converse holds for an abelian ring (that is, all idempotents in the ring are central). Local rings and strongly $\pi-$regular rings are well-known examples of the strongly clean rings. In 1936, Von Neuman defined regular element and regular ring and some properties of regular rings has been studied in [7].

The clean rings were further extended to $r-$clean rings and the $r-$clean rings were introduced by Ashrafi and Nasibi [2, 3] and they defined that an element x of a ring R is $r-$clean if $x = r + e$, where $r \in Reg(R)$ and $e \in Id(R)$. A ring R is said to be $r-$clean if each of its element is $r-$clean. In [3] they proved that every abelian $r-$clean ring is clean.

Garima Sharma and Amit B. Singh in [13] introduced the concept of new class of rings, strongly $r-$clean rings which is generalization of the class of strongly clean rings and stronger class of $r-$clean rings.

Motivated by all above studies, in this paper, we introduce the concept of new class of rings, namely z-clean and strongly z-clean rings.

We call an element x of a ring R is z-clean if $x = z + e$, where $z \in Z(R)$ and $e \in Id(R)$.

A ring R is z-clean if each of its element is z-clean. If $ze = ez$ then x of a ring R is strongly z-clean and a ring R is strongly z-clean.

We will give examples for z-clean ring and cheek the relationship between clean ring, $r-$clean ring with z-clean ring. Moreover, we give the necessary and sufficient condition when the clean rings, strongly clean rings, z-clean rings and strongly z-clean rings become equivalent.

We show that a directly finite of rings $R_i \neq 0$, $i = 1, 2, \cdots, n$ is z-clean if and only if every R_i is z-clean.

Finally we provide some properties of z-clean rings and prove that if R is an z-clean ring then the matrix ring $M_n(R)$ is z-clean for any $n \geq 1$.

2. $z-$clean and strongly z-clean rings

In this section first we define $z-$clean elements, strongly z-clean elements, $z-$clean rings and strongly z-clean rings and we investigate the properties of the $z-$clean ring.

Definition 2.1. *An element x of a ring R is $z-$ clean if $x = z + e$, where $z \in Z(R)$ and $e \in Id(R)$, a ring R is $z-$clean if each of its element is $z-$clean.*

If $ze = ez$ then x of a ring R is strongly z-clean and a ring R is strongly z-clean. From the above definition it is clear that every strongly z-clean ring is z-clean.

Now, in the following example we show that the converse need not be true.

Example 2.1. $R = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a, b \in \mathbb{Z}_2 \right\}$ is a z-clean ring but not strongly z-clean ring.

Proof. Let R: $M_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $M_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $M_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $M_4 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Thus the set of idempotents $Id(R) = \{e_1 = M_1, e_2 = M_2, e_3 = M_3\}$. Also all the elements are z-clean. Now, we prowe that R has at least one element $x = z + e$ such that $ze \neq ez$ for $e \in Id(R)$ and $z \in Z(R)$. Take $M_4 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \neq 0$ so representations of M_4 with commutative condition are $M_4 = e_1 + M_2$ but $e_1 M_2 \neq M_2 e_1$. Thus there exists $M_4 \neq 0$ such that M_4 is z-clean but not strongly z-clean. □

Example 2.2. If p is a prime number, then $\mathbb{Z}_p$ is a field thence $\mathbb{Z}_p$ is not a $z-$clean ring. But for $n = 2k$, $k \in \mathbb{N}$, $\mathbb{Z}_n$ is a strongly z-clean.

Proof The set of zero divisors of $R = \mathbb{Z}_{2n} = \{1, 2, 3, \cdots, 2n-1, 2n\}$ is $Z(R) = \{z, (z, 2n) \neq 1\}$ and the set of idempotents is $Id(R) = \{e, e^2 \overset{2n}{\equiv} e\}$.

Cleary, each of the elements R, writes as follows $x = z+e$ such that $z \in Z(R)$, $e \in I(R)$ and $ze = ez$. Thus $\mathbb{Z}_{2n}$ is strongly z-clean.

Definition 2.2. *The ring R is called Abelian if every idempotent is central, that is, $ae = ea$ for any $e \in Id(R)$, $a \in R$.*

Example 2.3. Every Boolean ring is strongly z-clean.

Example 2.4. Let S be a subset of fixed set U,

$$S = \mathcal{A}(U) = \{X : X \subseteq U\}.$$

We define

$$\forall A, B \in S: \quad A + B = (A - B) \cup (B - A), \quad A \cdot B = A \cap B.$$

In this case $(S, +, \cdot)$ is a strongly z-clean ring.

Remark 2.1. Recall that an element $r \in R$ is regular if $r = ryr$ for some $y \in R$. The ring R is regular if each of its element is regular. Clearly regular rings and clean rings are $r-$clean. But the converse is not true ([2]).

Remark 2.2. Let R be a $r-$clean ring which is not a field. Then R is a z-clean ring.

Proof. Let $x \in R$, since R is $r-$clean therefore $x = r+e$ such that $r \in Reg(R)$, $e \in Id(R)$. Since $r \in Reg(R)$, there exists $y \in R$ such that $r = ryr$. Hence $r - ryr = 0$, $r(1 - yr) = 0$. Since R is not filed, therefore $1 - yr \neq 0$ and this implies that $r \in Z(R)$. Also $x = r + e$ where $r \in Z(R)$, $e \in Id(R)$. Hence R is z-clean ring. □

In the following, we provide an example for which the converse of Remark 2.2, is not true.

Example 2.5. Suppose that G is a additive Abelian group and we define the multiplicative over G as follow:

$$\forall a, b \in G : a \cdot b = 0.$$

In this case G is z-clean but is not $r-$clean ring.

Remark 2.3. Nill-clean rings are z-clean rings. while the converse may not hold.

Proof. Let R be a nill-clean ring and $x \in R$. Then $x = a+e$ such that $a \in N(R)$ and $e \in Id(R)$.

Since a is nillpotent, therefore there exists $n \in \mathbb{N}$, such that $a^n = 0$. Also $a \cdot a^{n-1} = 0$ and this implies that $a \in Z(R)$. Hence R is z-clean ring.

For the converse, pleas see Example 2.3. □

Proposition 2.1. *Every Abelian z-clean ring is strongly z-clean.*

Proof. For every $x \in R$, we write $x = z + e$, where $e \in Id(R)$ and $z \in Z(R)$. Since R is an Abelian ring, so $ez = ze$ for every $z \in R$. Therefore R is strongly z-clean. □

Proposition 2.2. *Let R be a ring and $a \in R$ be strongly $z-$clean. Then we have the following:*

(1) *$-a$ is strongly $z-$clean.*

(2) *$1 - a$ is strongly $z-$clean.*

Proof. (1). Suppose that $a = z + e$ where such that $z \in Z(R)$, $e \in Id(R)$ and $ez = ze$. Because $z \in Z(R)$, hence there exist $b \in R$, such that $b \neq 0$, $z \cdot b = 0$ and this implies that $-z \cdot b = 0$. Hence $-a = -z + (-e)$ that $-e \in Id(R)$ and $(-e)(-z) = (-z)(-e)$ hence $-a$ is strongly z-clean.
(2). Note $1-a = 1-(z+e) = -z+(1-e)$. Clearly $-z \in Z(R)$, $(1-e) \in Id(R)$, $-z \cdot (1 - e) = (1 - e) \cdot (-z)$. Therefore $1 - a$ is strongly z-clean. □

Proposition 2.3. *Let R be a Abelian ring. If $a \in R$ is a strongly $z-$clean and $e \in Id(R)$, then*

(1) *ae is strongly $z-$clean.*

(2) *If the idempotents are orthogonal, then $a+e$ is strongly $z-$clean.*

Proof. It is tivial. □

Proposition 2.4. *Let R be a strongly $z-$clean ring and e centeral idempotent. then*

(1) *eRe and $(1-e)R(1-e)$ are strongly $z-$clean.*

(2) *$\forall I \trianglelefteq R$, $\frac{R}{I}$ is strongly $z-$clean.*

Proof. (1) Let $a' \in eRe$. Then there exits $a \in R$ such that $a' = eae$. Since R is strongly z-clean ring, therefore $a = z + e'$ where $z \in Z(R)$, $e' \in Id(R)$ and $ze' = e'z$. It follows that $a' = e(z+e')e = eze + ee'e = eze = ee'$. Now we need to show that eze is zero divisor, ee' is an idempotent and they commute. For this consider

$$(ee')^2 = (ee')(ee') = e(e'e)e' = e(ee')e' = (ee)(e'e') = e^2e'^2 = ee',$$

since R is $z-$clean, then

$$\exists a \in R, \quad a^{'} = eae = e(z+e) = eze + e.$$

Therefore, ee' is an idempotent. Now consider $z \in Z(R)$, therefore there exit $b \in R$ such that $zb = 0$. $(eze)b = ezeb = ezbe = 0$, hence $eze \in Z(eRe)$. Now we show the commutativity of R,

$$(eze)(ee') = ezeee' = ezee' = eze'e = ee'ze = (ee')(ze) = (ee')(zee) = (ee')(eze).$$

Therefore, it is a strongly z-clean ring. (2). Let $\bar{a} = a + I \in \frac{R}{I}$. Since R is strongly z-clean $a = z + e$ and $ze = ez$, where $z \in Z(R)$, $e \in Id(R)$.

Hence, $\bar{a} = z + (e + I)$ that $(e+I) \in Id(\frac{R}{I})$. Consider

$$(z+I)(e+I) = ze + I = ez + I = (e+I)(z+I)$$

since R is strongly z-clean. This implies that $\frac{R}{I}$ is strongly z-clean. □

Corollary 2.1. *Any homomorphic image of a strongly z-clean ring is strongly z-clean.*

Remark 2.4. In general, the converse of Proposition 2.4, may not be hold. For example, $\frac{\mathbb{Z}}{n\mathbb{Z}} \simeq Z_n$, if $n = 2k$ then z_n is strongly z-clean, but $\mathbb{Z}$ is not z-clean ring.

Proposition 2.5. *A direct product $R = \prod R_i$ of rings R_i is strongly $z-$clean if and only if the same is true for each R_i.*

Proof. Suppose that each R_i is strongly $z-$clean ring. For any $a=(a_i)\in R$, we write $a_i=z_i+e_i$ such that $z_i\in Z(R_i)$ and $e_i\in Id(R_i)$ and $z_ie_i=e_iz_i$. Then $a=z+e$, where $e=(e_i)$ is idempotent in $\prod R_i$ and $z=(z_i)\in Z(R)$ with $a=(a_i)=(e_i)+(z_i)\in\prod R_i$, $(e_i)(z_i)=(z_i)(e_i)$. Thus R is strongly z-clean. Conversely let $R=\prod_{i\in I}R_i$ be strongly z-clean. This implies R_i is strongly z-clean from Corollary 2.1, since R_i is a homomorphic image of R. □

Proposition 2.6. *If R is a strongly $z-$clean ring, and $e\in Id(R)$, then eR and $(1-e)R$ are strongly $z-$clean rings.*

Proof. Since

$$\forall e\in Id(R),\quad R=eR\oplus(1-e)R.$$

Therefore it followes by proposition 2.4. □

Proposition 2.7. *If R is a strongly $z-$clean ring, and $R^{'}$ is the diagonal matrixces ring in $M_n(R)$, then $R^{'}$ is a strongly $z-$clean ring.*

Proof. Suppose that $A=\begin{bmatrix} a_{11} & 0 & \dots & 0\\ 0 & a_{22} & \dots & 0\\ \vdots & \vdots & \dots & \vdots\\ 0 & 0 & \dots & a_{nn}\end{bmatrix}\in R^{'}$. Then since R is a strongly $z-$clean, hence we have $\forall 1\le i\le n:\ a_{ii}=z_{ii}+e_{ii}$ such that $z_{ii}\in Z(R), e_{ii}\in Id(R)$ and $z_{ii}e_{ii}=e_{ii}z_{ii}$.

$$A=\begin{bmatrix} a_{11} & 0 & \dots & 0\\ 0 & a_{22} & \dots & 0\\ \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \dots & a_{nn}\end{bmatrix}=\begin{bmatrix} z_{11} & 0 & \dots & 0\\ 0 & z_{22} & \dots & 0\\ \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \dots & z_{nn}\end{bmatrix}+\begin{bmatrix} e_{11} & 0 & \dots & 0\\ 0 & e_{22} & \dots & 0\\ \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \dots & e_{nn}\end{bmatrix}.$$

Since $z_{ii}\in Z(R)$, there exits $b_{ii}\neq 0\in R$, such that $z_{ii}b_{ii}=0$, and we have

$$\begin{bmatrix} z_{11} & 0 & \dots & 0\\ 0 & z_{22} & \dots & 0\\ \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \dots & z_{nn}\end{bmatrix}\begin{bmatrix} b_{11} & 0 & \dots & 0\\ 0 & b_{22} & \dots & 0\\ \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \dots & b_{nn}\end{bmatrix}=\begin{bmatrix} 0 & 0 & \dots & 0\\ 0 & 0 & \dots & 0\\ \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \dots & 0\end{bmatrix}.$$

Therefore,

$$\begin{bmatrix} z_{11} & 0 & \dots & 0\\ 0 & z_{22} & \dots & 0\\ \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \dots & z_{nn}\end{bmatrix}\in Z(R^{'}),\quad \begin{bmatrix} e_{11} & 0 & \dots & 0\\ 0 & e_{22} & \dots & 0\\ \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \dots & e_{nn}\end{bmatrix}\in Id(R^{'})$$

and

$$\begin{bmatrix} z_{11} & 0 & \dots & 0 \\ 0 & z_{22} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & z_{nn} \end{bmatrix} \begin{bmatrix} e_{11} & 0 & \dots & 0 \\ 0 & e_{22} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & e_{nn} \end{bmatrix}$$
$$= \begin{bmatrix} e_{11} & 0 & \dots & 0 \\ 0 & e_{22} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & e_{nn} \end{bmatrix} \begin{bmatrix} z_{11} & 0 & \dots & 0 \\ 0 & z_{22} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & z_{nn} \end{bmatrix}.$$

Hence $R^{'}$ is a strongly $z-$clean ring. □

Proposition 2.8. *If R is a strongly $z-$clean ring and $R^{'}$ the matrix ring which has a row or a column with elements of R, then $R^{'}$ is a strongly $z-$clean ring.*

Proof. It is clearly. □

Proposition 2.9. *If R is a commutative ring, then $R[x]$ is not z-clean.*

Proof. See [[2], Theorem 12]. □

Proposition 2.10. *Let R be ring. Then the ring $R[[x]]$ is strongly z-clean if and only if so is R.*

Proof. If $R[[x]]$ is strongly z-clean, then by Remark 2.1, R is strongly z-clean.

Conversely, suppose that R is strongly z-clean. Then by proposition 2.5, $R[[x]]$ is strongly z-clean. □

Proposition 2.11. *Suppose that R is Abelian ring which is not field. If R is a clean ring then $R[[x]]$ is strongly z-clean ring.*

Proof. By [[13], Theorem 3.7], if R is a clean ring then $R[[x]]$ is strongly $r-$clean ring. By Remark 2.2, $R[[x]]$ is strongly z-clean ring. □

Proposition 2.12. *Let a be a strongly clean element of R and $a - a^2 \in Z(R)$. Then a is strongly $z-$clean.*

Proof. Let $a = e + u$ be a strongly cleandecomposition in R and $a - a^2$ be a $z-$clean. Then $a^2 = e + 2eu + u^2$ and so $a - a^2 = (1 - 2e - u)u$.
$a - a^2 \in Z(R)$, then

$$\exists 0 \neq b \in R : (a - a^2)b = 0, \qquad (1 - 2e - u)u \cdot b = 0.$$

It follows that

$$(1 - 2e - u) \in Z(R) \qquad and \qquad (-1 + 2e + u) \in Z(R).$$

so

$$a = (1 - e) + (-1 + 2e + u)$$

is a strongly $z-$clean decomposition in R. □

Corollary 2.2. *If $u \in R$ be unit and $1 - u \in Z(R)$, then u is strongly $z-$clean.*

For $a \in R$, the commutant if a in R is denoted by $c(a)$, i.e.,

$$c(a) = \{x \in R : ax = xa\}.$$

Proposition 2.13. *Let $a \in R$ and a be a strongly $z-$clean element of R with a strongly $z-$clean decomposition $a = e + b$. then $c(a) \subseteq c(e)$.*

Proof. since

$$\begin{aligned} a &= \begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} eb & 0 \\ 0 & (1-e)b \end{bmatrix} \\ &= \begin{bmatrix} e & ex(1-e) \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} eb & -ex(1-e) \\ 0 & (1-e)b \end{bmatrix} \\ &= \begin{bmatrix} e & 0 \\ (1-e)xe & 0 \end{bmatrix} + \begin{bmatrix} eb & 0 \\ -(1-e)xe & (1-e)b \end{bmatrix} \end{aligned}$$

are all strongly $z-$clean decomposition of a in R, it follows that $ex(1 - e) = (1 - e)xe = 0$. so $x \in c(e)$. □

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Accepted: January 01, 2020

Some types of multiplication N-group in near rings

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Abstract. In this paper, the authors have defined multiplication and cyclic N-groups in near rings. In addition, their relationships have been found. Also, the symbol weak multiplication N-groups and F^*- weak multiplication N-groups in near rings have been introduced. Finally, it has been shown through examples and demonstration of some theorems that these definitions are not equivalent.

Keywords: cyclic N-group, multiplication N-group, weak multiplication N-group, F^* weak multiplication N-group.

1. Introduction and preliminaries

The concept of near rings was expressed by Dicson in 1905 [4], that followed by other mathematicians, create the concepts and ruler space on the rings in near rings with conditions. Near rings have a great deal of use in dynamic systems and automation systems [5].

A right(left) near ring is a non - empty set N together with two binary operations "+" and "." such that

(a) $(N, +)$ is a group (not necessarily abelian),

(b) $(N, \cdot)$ is a semigroup,

(c) $(n_1 + n_2)n_3 = n_1n_3 + n_2n_3$ $(n_1(n_2 + n_3) = n_1n_2 + n_1n_3)$, $\forall n_1,n_2, n_3 \in N$.

A near ring N is called abelian (commutative) if $n_1 + n_2 = n_2 + n_1$ $(n_1.n_2 = n_2.n_1)$, $n_1, n_2 \in N$.

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In this paper, the word near ring N, means a commutative and zero symmetric near ring with 1, and M is an unitary N-group. It should be noted that all near rings are right near rings. All the following definitions and theorems have been extracted from [5].

Let $(M,+)$ be a group with zero 0, N be a near ring and the map $\mu : N \times M \longrightarrow M$ such that $\mu\ (n,m) = nm$, $\forall\ n \in N$, $\forall\ m \in M$. Then M is called an N-group if M satisfies the following conditions:

(i) $(n+n')m = nm + n'm$,

(ii) $(nn')m = n(n'm)$, $\forall\ N$, $n' \in N$, $\forall\ m \in M$.

It is written ${}_NM$ for the N-group above.

A subgroup Δ of a near ring N with, $\Delta\Delta \subseteq \Delta$ is called a subnear ring of N (notation: $\Delta \leq N$).

A subgroup Δ of an N-group M with, $N\Delta \subseteq \Delta$ is said to be an N-subgroup of M(notation: $\Delta \leq_N M$).

A normal subgroup I of $(N$,+) is called an ideal of N ($I \trianglelefteq N$) if

(i) $IN \subseteq I$,

(ii) $n(n'+i) - nn' \in I$, $\forall\ N$, $n' \in N$, $\forall\ i \in I$.

The normal subgroup R of $(N,\ +)$ with (i) is called the right ideal of N ($R \trianglelefteq_r N$), while normal subgroup L of $(N,\ +)$ with (ii) is said to be the left ideal($L \trianglelefteq_l N$).

A normal subgroup Δ of N-group M is called the ideal of M ($\Delta \trianglelefteq_N M$) if $n(m+\delta) - nm \in \Delta$, $\forall\ n \in N$, $\forall\ m \in M$, $\forall\ \delta \in \Delta$.

An N-homomorphism is a mapping h of N-group M to N-group M' such that:

(i) $h(m_1 + m_2) = h(m_1) + h(m_2)$,

(ii) $h(nm_1) = nh(m_1)$, $\forall\ m_1,\ m_2 \in M$, $\forall\ n \in N$.

If $I \trianglelefteq_N M$, then the canonical map $\pi : M \to \frac{M}{I}$ is an N-epimorphism. As a result, $\frac{M}{I}$ is an N-homomorphic image of M.

A proper N-subgroup Δ of an N-group M is called a prime if $nx \in \Delta$, for $n \in N$ and $x \in M$, then $x \in \Delta$ or $nM \subseteq \Delta$.

Let $\Delta_1, \Delta_2 \leq_N M$. Then, we have

$$Spec(M) = \{\Delta \mid \Delta \text{ is prime } N - \text{subgroup of } M\}$$

$$(\Delta_1 : \Delta_2) = \{n \in N \mid n\Delta_2 \subseteq \Delta_1\}.$$

Let P be an ideal of N. Then P is called a prime ideal of N if for ideals I and J, $IJ \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$.

A near ring N is called a local near ring if $L = \{x \in N \mid x$ has no left inverse$\} \leq_N N$.

If Δ is an N-subgroup of M, such that Δ is an ideal of M, then $(\Delta : M) \trianglelefteq N$.

A near ring N is a local near ring, if and only if N has a unique maximal N-subgroup.

Every proper N-subgroup of M is contained in some maximal N-subgroup of M.

Let M and M' be N-groups. A proper N-subgroup Δ of $M \times M'$ is called a prime if $n(m, m') = (nm, nm') \in \Delta$, then $(m, m') \in \Delta$ or $n \in (\Delta : M \times M')$, $\forall\ n \in N$, $\forall\ (m, m') \in M \times M'$.

Let M and M' be N-groups and I be a ideal of N, we define

$$\begin{aligned} I(M \times M') &= \{\sum_{i=1}^{t} a_i(m_i, m_i') \mid a_i \in I, m_i \in M, m_i' \in M'\} \\ &= \{\sum_{i=1}^{t} (a_i m_i, a_i m_i') \mid a_i \in I, m_i \in M, m_i' \in M'\}. \end{aligned}$$

2. Multiplication N-subgroups in near rings

In this section, the notion of multiplication and cyclic N-groups will be introduced. Our purpose is to find the relation between them and prove some theorems.

Definition 2.1. *Let M be an N-group. Then M is called a multiplication N-group. If for each N-subgroup Δ of M, there is an ideal I of N, such that $\Delta = IM$.*

Example 2.1. Consider the near ring $(N, +, \cdot)$, where $(N, +)$ is the *kleins'* four group $\{0, a, b, c\}$ and "$\cdot$ " is defined as the following:

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	b	b	b
c	0	c	c	c

Then N is a multiplication N-group.

Lemma 2.1. *Let M be an N-group and $x \in M$. Then $K = \{nx \mid n \in N\}$ is an N-subgroup of* M*, called the N-subgroup generated by $\{x\}$ and denoted by $\langle x \rangle$.*

Proof. The proof is clear. □

Definition 2.2. *An N-group M is called the cyclic, if an element as x exists, such that $M = \langle x \rangle$.*

Theorem 2.1. Let M be a cyclic N-group, such that every N-subgroup of M be an ideal. Then M is a multiplication N-group.

Proof. Let M be a cyclic N-group. Then $x_0 \in M$ exists, such that $M = \langle x_0 \rangle$. Suppose that Δ is an N-subgroup of M, by assuming $(\Delta : M)$ is an ideal of N. We have $(\Delta : M)M \subseteq \Delta$. Consider $a \in \Delta$. Since $\Delta \subseteq M = Nx_0$, $n \in N$ exists, $a = nx_0$. We have $nM = nNx_0 = N(nx_0) = Na$, $nM \subseteq N\ \Delta \subseteq \Delta$, so $nM \subseteq \Delta$ or $n \in (\Delta : M)$.

Consequently, $a \in (\Delta : M)M$ and $\Delta \subseteq (\Delta : M)M$. Therefore, M is a multiplication N-group. □

The N-group N of Example 2.1 is a multiplication N-group while N is not cyclic. So, the converse of Theorem 2.1 may not be true.

Lemma 2.2. Let M be a multiplication N-group and $I \leq_N N$. If for every $a \in I$, $1 - a$ has a left inverse and $M = IM$, then $M = \{0\}$.

Proof. Suppose that $x \in M$. We have $\langle x \rangle \leq_N M$. By Definition 2.1, there is an ideal E of N, such that

$$\begin{aligned} \langle x \rangle &= EM = E(IM) = (EI)M \\ &= (IE)M \quad (M \text{ is an } N-\text{group}) \\ &= INx (I \leq_N N, \text{ so } I = NI) \\ &= Ix. \end{aligned}$$

Since $\langle x \rangle = Ix$, so $\exists\ a \in I$ such that $x = ax$. Therefore, $(1 - a)x = 0$ by assuming we get that $x = 0$. □

Lemma 2.3. Let N be a local near ring and $x \in L = \{x \in N \mid x$ has no left inverse$\}$. Then $1 - x$ has a left inverse.

Proof. Suppose that $1 - x$ has no left inverse. By definition of local near ring, we have $1 - x \in L$. As a consequence, $1 \in L$ which is a contradiction. Therefore, $1 - x$ has a left inverse. □

Theorem 2.2. If N is a local near ring, then every multiplication N-group is cyclic.

Proof. Let M be a non-zero multiplication N-group and L be a unique maximal N-subgroup of M. By Lemma 2.2, we have $M \neq LM$, so there is $x \in M - LM$. Hence $\langle x \rangle \leq_N M$ and so $\exists\ I \trianglelefteq N$, $\langle x \rangle = Nx = IM$. Since $I \trianglelefteq N$, we have $I \leq_N N$. If $N \neq I$, therefore $I \subseteq L$ and $\langle x \rangle = IM \subseteq LM$ which is a contradiction. As a result, $I = N$ and $\langle x \rangle = IM = NM = M$. Therefore, M is a cyclic N-group. □

3. F^* weak multiplication N-groups in near rings

In this section, the notion of a weak multiplication N-group, an F^* weak multiplication N-group have been defined, and the relation between of them is found. By bringing some examples and demonstrating some theorems, it has been shown that these definitions are not equivalent.

Definition 3.1. *An N-group M is called a weak multiplication, if $Spec(M)=\emptyset$ or for every prime N-subgroup Δ of M, there is an ideal I of N such that, $\Delta = IM$.*

Example 3.1. Consider the near ring $(N, +, \cdot)$ where $(N, +)$ is the *kleins'* four group $\{0, a, b, c\}$ and "$\cdot$" defined as the following:

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	0	0	a
b	0	a	b	b
c	0	a	b	c

We see that N is a weak multiplication N-group.

Proposition 3.1. Every multiplication N-group is a weak multiplication N-group.

Proof. The proof is clear. □

The Example 2.1 shows that the converse of the above proposition may not be true, because $\{0, b\}$ is an N-subgroup of N, but there is no ideal I of N such that $\{0, b\}=IN$.

Definition 3.2. *An N-group M is called an F^* weak multiplication if it satisfies the following conditions:*

(*i*) *M is a weak multiplication;*

(*ii*) *PM is an ideal of M and $(PM : M) = P$, $\forall\ P \in Spec(N)$.*

By some calculations, it can be seen that Example 2.1 is an F^* weak multiplication N-group.

It is clear that every F^* weak multiplication N-group is a weak multiplication N-group.

Proposition 3.2. Let $\Phi: {}_NM \longrightarrow {}_NM'$ be an N-epimorphism.Then Φ induces a 1-1 correspondence between the prime N-subgroup of M containing $\ker(\Phi)$ and the prime N-subgroup of M'.

Proof. We define $T := \{\ \Delta \mid \Delta$ is a prime N-subgroup of M containing $\ker(\Phi)\}$, $T' := \{\ \Delta' \mid \Delta'$ is a prime N-subgroup of $M'\ \}$ and the map $h: T \longrightarrow T'$ such that $h(\Delta) = \Phi(\Delta)$, $\forall\ \Delta \in T$.

We first prove that h is well-defined.

Suppose that $\Delta \in T$, since $\Delta \neq M$, so there is $m \in M$ such that $m \in M - \Delta$. If $\Phi(\Delta) = M'$, then

$$\Phi(\Delta) = \Phi(M) \Longrightarrow$$
$$\Phi(\Delta - M) = \{0\} \Longrightarrow$$
$$\Delta - M \subseteq Ker(\Phi) \subseteq \Delta \Longrightarrow$$

We have $0 \in \Delta$ and $0 - m \in \Delta$, so $m \in \Delta$ which is a contradiction. Thus $\Phi(\Delta) \neq M'$. We get that $\Phi(\Delta)$ is a proper N-subgroup of M'.

Now, we show that $\Phi(\Delta)$ is a prime N-subgroup of M', i.e., if $nx' \in \Phi(\Delta)$, then $x' \in \Phi(\Delta)$ or $nM' \subseteq \Phi(\Delta)$, $\forall\ n \in N$, $\forall\ x' \in M'$.

Since $x' \in M'$, there is $x \in M$ such that $\Phi(x) = x'$. If $x' \in \Phi(\Delta)$, then $x \in \Delta$. We have $\Delta \in T$. By definition of prime N-subgroup, we get that $nM \subseteq \Delta$ and $\Phi(nM) = n\Phi(M) = nM' \subseteq \Phi(\Delta)$. Therefore, $\Phi(\Delta)$ is a prime N-subgroup of M'.

It is easy to see that if $\Delta_1 = \Delta_2$. As a result, $h(\Delta_1) = h(\Delta_2)$, $\forall\ \Delta_1, \Delta_2 \in T$.

Now, we will prove that h is injective. Consider $\delta_1 \in \Delta_1$, $\Phi(\delta_1) \in \Phi(\Delta_1) = h(\Delta_1) = h(\Delta_2) = \Phi(\Delta_2)$, and so $\exists\ \delta_2 \in \Delta_2$ such that $\Phi(\delta_1) = \Phi(\delta_2)$. Then $\delta_1 - \delta_2 \in \Delta_2$. Consequently, there is $\delta_0 \in \Delta_2$ such that $\delta_1 - \delta_2 = \delta_0$, $\delta_1 = \delta_0 + \delta_2$. Thus $\Delta_1 \subseteq \Delta_2$. Similarly, $\Delta_2 \subseteq \Delta_1$.

We will prove that h is surjective or if $\Delta' \in T'$, then $\Phi(\Phi^{-1}(\Delta')) = \Delta'$. It is enough to prove $\Phi^{-1}(\Delta') \in T$.

Suppose that $nx \in \Phi^{-1}(\Delta')$ for $n \in N$ and $x \in M$. Since $nx \in \Phi^{-1}(\Delta')$, $\Phi(nx) = n\Phi(x) \in \Phi(\Phi^{-1}(\Delta')) = \Delta'$.

If $x \notin \Phi^{-1}(\Delta')$, then $\Phi(x) \notin \Delta'$. $\Phi(nM) = n\Phi(M) = nM' \subseteq \Delta'$, so $\Phi^{-1}(\Phi(nM)) \subseteq \Phi^{-1}(\Delta')$. We know $nM \subseteq \Phi^{-1}(\Phi(nM)) \subseteq \Phi^{-1}(\Delta')$. Consider $x \in Ker(\Phi)$, i.e., $x \in \Phi^{-1}(\{0\})$. We have $\{0\} \subseteq \Delta'$. Thus $x \in \Phi^{-1}(\{0\}) \subseteq \Phi^{-1}(\Delta')$. We get that $Ker(\Phi) \subseteq \Phi^{-1}(\Delta')$.

By the above results, $\Phi^{-1}(\Delta') \in T$, and so h is surjective. □

Theorem 3.1. Every N-homomorphic image of an F^* weak multiplication N-group is an F^* weak multiplication N-group.

Proof. Let $h : {}_NM \longrightarrow {}_NM'$ be an N-homomorphism and M be an F^* weak multiplication N-group. We will prove that $h(M)$ is an F^* weak multiplication N-group.

Firstly, we show that $h(M)$ is a weak multiplication N-group.

Suppose that $Spec(h(M)) \neq \emptyset$, and Δ' is a prime N-subgroup of $h(M)$. By Proposition 3.2, a prime N-subgroup Δ of M containing $Ker(h)$ exists such that $h(\Delta) = \Delta'$. Since M is a weak multiplication N-group, there is an ideal I of N, such that $\Delta = IM$, and $\Delta' = h(\Delta) = h(IM) = Ih(M)$. Therefore, $h(M)$ is a weak multiplication N-group.

We take $P \in Spec(N)$. Since M is an F^* weak multiplication N-group, $PM \trianglelefteq_N M$ and $h(PM) = Ph(M) \trianglelefteq_N h(M)$.

Our next claim is that $(Ph(M) : h(M)) = P$.

Assume that $x \in P = (PM : M)$, so $xM \subseteq PM$ and $h(xM) = xh(M) \subseteq h(PM) = Ph(M)$. Then $x \in (Ph(M) : h(M))$ and so $P \subseteq (Ph(M) : h(M))$. Now suppose that $y \in (Ph(M) : h(M))$, we have $yh(M) = h(yM) \subseteq Ph(M)$. As a result, $h(ym) \in h(PM)$, $\forall\ m \in M$. Therefore, $\sum_{i=1}^{n} p_i m_i \in PM$, for

$p_i \in P$, $m_i \in M$,

$$h(ym) = h(\sum_{i=1}^{n} p_i m_i) \Longrightarrow$$
$$h(ym - \sum_{i=1}^{n} p_i m_i) = 0 \Longrightarrow$$
$$ym - \sum_{i=1}^{n} p_i m_i \in Ker(h),$$
$$(Ker(h) \subseteq PM) \Longrightarrow$$
$$ym \in PM \text{ and } y \in (PM : M) = P.$$

Therefore, $(Ph(M) : h(M)) \subseteq P$.

By the above results, we get that $P = (Ph(M) : h(M))$. □

Proposition 3.3. Let M be an F^* weak multiplication N-group and K be an ideal of M. Then $\frac{M}{K}$ is an F^* weak multiplication N-group.

Theorem 3.2. Let $\{M_i\}$ be a family of N-groups, and $\bigoplus_{i=1}^{n} M_i$ be an F^* weak multiplication N-group. Then M_i is an F^* weak multiplication N-group for each $i \in \{1, 2, ..., n\}$.

Proof. We know that for each $i \in \{1, 2, ..., n\}$, the map $\pi_i : \bigoplus_{i=1}^{n} M_i \longrightarrow M_i$ is an N-homomorphism. Since $\bigoplus_{i=1}^{n} M_i$ is an F^* weak multiplication N-group, by Theorem 3, we get that M_i is an F^* weak multiplication N-group. □

If we consider ${}_N M =_N N$, in Example 2.1, $\{0, a\}$,$\{0\}$ are the prime N-subgroups of M. By some calculations, we see that $\{0\} \times \{0, a\} \cong \{0, a\}$.

Consequently, $\{0\} \times \{0, a\}$ is a prime N-subgroup of $M \times M$, but there is not an ideal of N as I, such that $\{0\} \times \{0, a\} = I(M \times M)$. Thus $M \times M$ is not a weak multiplication N-group, and so $M \times M$ is not an F^* weak multiplication N-group. We get that the converse of Theorem 3.2 may not be true.

Theorem 3.3. Let N be a multiplication N-group. Then N is a F^* weak multiplication N-group.

Proof. It is enough to prove that for each $P \in Spec(N)$, $PN \trianglelefteq_N N$ and $(PN : P) = P$.

Since $P \triangleleft N$, we have $PN \subseteq P$ and $P \subseteq PN(N \text{ has } 1)$. Then $P = PN$ and $PN \trianglelefteq_N N$. Suppose that

$$x \in (PN : N) \Longrightarrow$$
$$xN \subseteq PN \Longrightarrow$$
$$\{x\}N \subseteq PN \subseteq P \Longrightarrow$$
$$\{x\} \subseteq P \text{ or } N \subseteq P.$$

If $N \subseteq P$, it is a contradiction. Then $x \in P$ and $(PN : N) \subseteq P$.
It is clear that $P \subseteq (PN : N)$. □

By Example 3.1 we can see that the converse of the above theorem is not true.

Conclusion

In this paper, we presented the notions of multiplication N-group, weak multiplication N-group, $F*$ weak multiplication N-group and Cyclic N-group and some of their properties proved. In addition, we obtained the relationships between of them.

The following questions can be raised:

What is the relationship between Cyclic N-group and F* weak multiplication N-group?

What properties does weak multiplication N-group M, where M is Cyclic N-group?

These questions could be topics for further research.

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Accepted: January 07, 2020

On almost weak flatness of acts

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Abstract. Laan in (Commun. Algebra 29(2), 829-850 (2001)) introduced Condition (E'), a generalization of Condition (E). Golchin and Mohammadzadeh in (Semigroup Forum, 86(2), 413-430 (2012)) introduced Condition (P'), a generalization of Condition (P). Sedaghatjoo in (Semigroup Forum, 87, 653-662 (2013)) introduced a generalization of weak flatness property, called *almost weak flatness*. In this paper, we give a characterization of monoids by this property of their right acts. We also will give a characterization of almost weakly flat coherent monoids in general and monoids coming from some special classes.

Keywords: Almost Weakly Flat, Condition (W'), Condition (R_{AWF}).

1. Introduction

In studying weak pullback flatness of acts over monoids, Laan [17] introduced Condition (E'), a generalization of Condition (E). He showed that a right S-act A_S is weakly pullback flat if and only if it satisfies Conditions (P) and (E'), and so weak pullback flatness is a generalization of strong flatness. Subsequently Golchin and Mohammadzadeh [4] gave a characterization of monoids by Condition (E') of their right acts. They also in [3] introduced a generalization of Condition (P) called Condition (P'), and gave a characterization of monoids by this property of their right acts. A new flatness property of acts over monoids

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which is a generalization of weak flatness, and is called *almost weak flatness* was introduced in [11]. In section 2, we give some basic results and in section 3 of this paper, we give a characterization of monoids when almost weak flatness of right acts implies other properties and vise versa. In Section 4, we give a characterization of almost weakly flat coherent monoids in general and monoids coming from some special classes.

Throughout this paper S will denote a monoid. We refer the reader to [10] for basic definitions and terminology relating to semigroups and acts over monoids and to [17] for definitions and results on flatness which are used here.

A monoid S is called *right (left) reversible* if for every $s, t \in S$, there exist $u, v \in S$ such that $us = vt$ ($su = tv$). A monoid S is called *left (right) collapsible* if for every $s, t \in S$, there exists $u \in S$ such that $us = ut$ ($su = tu$). A right ideal K_S of a monoid S is called *left stabilizing* (or K_S satisfies *Condition* (LU)), if for every $k \in K_S$, there exists $l \in K_S$ such that $lk = k$.

A non-empty set A is called a *right S-act*, usually denoted A_S, if S acts on A unitarily from the right, that is, there exists a mapping $A \times S \to A$, $(a, s) \mapsto as$, satisfying the conditions $(as)\, t = a(st)$ and $a1 = a$, for all $a \in A$ and $s, t \in S$. A right S-act A_S satisfies *Condition* (E) if for any $a \in A_S$ and $s, t \in S$, $as = at$ implies that there exist $a' \in A_S$ and $u \in S$ such that $a = a'u$ and $us = ut$. A right S-act A_S satisfies *Condition* (E') if for any $a \in A_S$ and $s, t, z \in S$, $as = at$ and $sz = tz$ imply that there exist $a' \in A_S, u \in S$ such that $a = a'u$ and $us = ut$. A right S-act A_S satisfies *Condition* (P) if for any $a, a' \in A_S$, $s, t \in S$, $as = a't$ implies that there exist $a'' \in A_S$, $u, v \in S$ such that $a = a''u$, $a' = a''v$ and $us = vt$. A right S-act A_S satisfies *Condition* (P') if for any $a, a' \in A_S$ and $s, t, z \in S$, $as = a't$ and $sz = tz$ imply that there exist $a'' \in A_S$ and $u, v \in S$ such that $a = a''u$, $a' = a''v$ and $us = vt$. A right S-act A_S satisfies *Condition* (PWP) if for any $a, a' \in A_S$, $s \in S$, $as = a's$ implies that there exist $a'' \in A_S$ and $u, v \in S$ such that $a = a''u$, $a' = a''v$ and $us = vs$. A right S-act A_S satisfies *Condition* (EP) if for any $a \in A_S$ and $s, t \in S$, $as = at$ implies that there exist $a' \in A_S$, $u, v \in S$ such that $a = a'u = a'v$ and $us = vt$. A right S-act A_S satisfies *Condition* $(E'P)$ if for any $a \in A_S$ and $s, t, z \in S$, $as = at$ and $sz = tz$ imply that there exist $a' \in A_S$ and $u, v \in S$ such that $a = a'u = a'v$ and $us = vt$.

We use the following abbreviations,

Weak flatness = WF.

Almost weak flatness = AWF.

Principal weak flatness = PWF.

Weak pullback flatness = WPF.

Weak kernel flatness = WKF.

Principal weak kernel flatness = PWKF.

Translation kernel flatness = TKF.

2. Basic results

In this section we give some basic results. Recall from [10] that a right S-act A_S is called *weakly flat* if the functor $A_S \otimes_S -$ preserves all inclusions of left ideals into S. This is equivalent to say that, $as = a't$, for $a, a' \in A_S$ and $s, t \in S$ implies $a \otimes s = a' \otimes t$ in $A_S \otimes_S (Ss \cup St)$. Also, a right S-act A_S is weakly flat if and only if it is principally weakly flat and satisfies Condition:

(W) If $as = a't$, for $a, a' \in A_S$ and $s, t \in S$, then there exist $a'' \in A_S$ and $u \in Ss \cap St$ such that $as = a't = a''u$.

We recall from [11] the following definition:

Definition 2.1. *A right S-act A_S is called almost weakly flat if it is principally weakly flat and satisfies Condition:*

(W') If $as = a't$ and $Ss \cap St \neq \emptyset$, for $a, a' \in A_S$ and $s, t \in S$, then there exist $a'' \in A_S$ and $u \in Ss \cap St$ such that $as = a't = a''u$.

Clearly, for any right S-act we have the following implications:

$$\mathrm{WF} \Longrightarrow \mathrm{AWF} \Longrightarrow \mathrm{PWF}, \quad (W) \Longrightarrow (W').$$

Example 2.1 ([11])**.** Let X be an infinite set and $S = \mathcal{T}(X)$ its full transformation monoid. S has left zeros and consequently is right reversible. Since S is a regular and right reversible monoid, all right S-acts are principally weakly flat and almost weakly flat ones are weakly flat. In [10], 4.7.9 it is shown that S is not weakly right absolutely flat and hence there exists a principally weakly flat act which is not almost weakly flat. On the other hand for any monoid S, the one element act Θ_S is almost weakly flat and it is well known that Θ_S is (weakly) flat if and only if S is right reversible. Thus, if S is not right reversible, Θ_S is almost weakly flat, but it is not weakly flat.

From the definition of AWF the following lemma is deduced.

Lemma 2.1. *Let S be a monoid. Then*

1. *S_S is AWF.*
2. *The one-element act Θ_S is AWF.*
3. *Let $A_S = \coprod_{i \in I} A_i$, where A_i , $i \in I$ are right S-acts. Then A_S is AWF if and only if A_i is AWF for every $i \in I$.*
4. *Let the family $\{A_i \mid i \in I\}$ of subacts of an act A_S be a chain. Then $\bigcup_{i \in I} A_i$ is AWF if and only if A_i is AWF for every $i \in I$.*

5. *for any family* $\{A_i \mid i \in I\}$ *of right* S*-acts, if* $\prod_{i\in I} A_i$ *is AWF then* A_i *is AWF for every* $i \in I$.

Lemma 2.2 ([10])**.** *Let* A_S *be a right* S*-act and* ${}_SM$ *be a left* S*-act. Then* $a\otimes m = a'\otimes m'$ *for* $a, a' \in A_S$ *and* $m, m' \in {}_SM$*, if and only if there exist* $s_1, \ldots, s_k, t_1, \ldots, t_k \in S, n_1, \ldots, n_k \in SM$*,* $b_1, \ldots, b_{k-1} \in A_S$ *such that*

$$\begin{aligned} & & s_1n_1 &= m \\ as_1 &= b_1t_1 & s_2n_2 &= t_1n_1 \\ b_1s_2 &= b_2t_2 & s_3n_3 &= t_2n_2 \\ &\vdots & &\vdots \\ b_{k-1}s_k &= a't_k & m' &= t_kn_k. \end{aligned}$$

The following lemma is a consequence of lemma 2.2.

Lemma 2.3 ([10])**.** *Let* S *be a monoid. Then the following statements are equivalent:*

1. *The right* S*-act* A_S *is PWF.*
2. *If* $as = a's$*, for* $a, a' \in A_S$ *and* $s \in S$*, then* $a\otimes s = a'\otimes s$ *in the tensor product* $A_S\otimes_S Ss$.
3. *If* $as = a's$*, for* $a, a' \in A_S$ *and* $s \in S$*, then there exist* $s_1, \ldots, s_k, t_1, \ldots, t_k \in S, b_1, \ldots, b_{k-1} \in A_S$ *such that*

$$\begin{aligned} & & s_1s &= s \\ as_1 &= b_1t_1 & s_2s &= t_1s \\ b_1s_2 &= b_2t_2 & s_3s &= t_2s \\ &\vdots & &\vdots \\ b_{k-1}s_k &= a't_k & s &= t_ks. \end{aligned}$$

In the above S-tossing, k is called the *length* of the S-tossing and the minimum

length of the existing S-tossing is denoted by $d_s(a, a')$.

Theorem 2.1. *A right* S*-act* A_S *is AWF if and only if* $as = a't$ *and* $Ss\cap St \neq \emptyset$*, for* $a, a' \in A_S$*,* $s, t \in S$ *imply* $a\otimes s = a'\otimes t$ *in the tensor product* $A_S\otimes_S (Ss\cup St)$.

Proof. *Necessity.* Suppose A_S is AWF, and let $as = a't$ and $Ss \cap St \neq \emptyset$ for $a, a' \in A_S$ and $s, t \in S$. Since A_S satisfies Condition (W') there exist $a'' \in A_S$ and $u \in Ss\cap St$ such that $as = a't = a''u$. Let $u = s's = t't$, for some $s', t' \in S$. Since A_S is PWF, by Lemma 2.3 we get from $as = a''s's$ and $a't = a''t't$ that $a\otimes s = a''s'\otimes s$ in $A_S\otimes_S Ss$, $a'\otimes t = a''t'\otimes t$ in $A_S\otimes_S St$. Hence we have:

$$a\otimes s = a''s'\otimes s = a''\otimes s's = a''\otimes t't = a''t'\otimes t = a'\otimes t$$

in tensor product $A_S \otimes_S (Ss \cup St)$.
Sufficiency. If $as = a's$ for $a, a' \in A_S$ and $s \in S$ then $a \otimes s = a' \otimes s$ in $A_S \otimes_S Ss$ by assumption. Consequently A_S is PWF by Lemma 2.3. Now let $as = a't$ and $Ss \cap St \neq \emptyset$ for $a, a' \in A_S$ and $s, t \in S$. Then $a \otimes s = a' \otimes t$ in the tensor product $A_S \otimes_S (Ss \cup St)$ by assumption. This means that we have an S-tossing

$$\begin{array}{cc} & s_1u_1 = s \\ as_1 = b_1t_1 & s_2u_2 = t_1u_1 \\ b_1s_2 = b_2t_2 & s_3u_3 = t_2u_2 \\ \vdots & \vdots \\ b_{j-2}s_{j-1} = b_{j-1}t_{j-1} & s_ju_j = t_{j-1}u_{j-1} \\ b_{j-1}s_j = b_jt_j & s_{j+1}u_{j+1} = t_ju_j \\ \vdots & \vdots \\ b_{k-1}s_k = a't_k & t = t_ku_k. \end{array}$$

where $s_1, \ldots, s_k, t_1, \ldots, t_k \in S$, $u_1, \ldots, u_k \in Ss \cup St$, $b_1, \ldots, b_{k-1} \in A_S$, by Lemma 2.2. If $s \in St$, take $u = s$ and $a'' = a$, then Condition (W') is satisfied. Otherwise let j be the first index such that $u = s_ju_j \in St$. Since $u_{j-1} \in Ss$ then $u = s_ju_j = t_{j-1}u_{j-1} \in Ss \cap St$ and

$$as = as_1u_1 = b_1t_1u_1 = b_1s_2u_2 = \cdots = b_{j-1}s_ju_j = b_{j-1}u.$$

Take $a'' = b_{j-1}$. Hence Condition (W') is satisfied. □

Proposition 2.1. *Any retract of an AWF right S-act is AWF.*

Proof. Suppose A_S is AWF and let $f : A_S \longrightarrow B_S$ be a retraction. Hence there exists an S-homomorphism $g : B_S \longrightarrow A_S$ such that $fg = id_{B_S}$. Assume $bs = b's$ for $b, b' \in B_S$, $s \in S$. So $g(b)s = g(b')s$. Since A_S is PWF, by Lemma 2.3, there exist $s_1, \ldots, s_k, t_1, \ldots, t_k \in S, a_1, \ldots, a_{k-1} \in A_S$ such that

$$\begin{array}{cc} & s_1s = s \\ g(b)s_1 = a_1t_1 & s_2s = t_1s \\ a_1s_2 = a_2t_2 & s_3s = t_2s \\ \vdots & \vdots \\ a_{k-1}s_k = g(b')t_k & s = t_ks. \end{array}$$

Then,

$$\begin{array}{cc} & s_1s = s \\ f(g(b))s_1 = f(a_1)t_1 & s_2s = t_1s \\ f(a_1)s_2 = f(a_2)t_2 & s_3s = t_2s \\ \vdots & \vdots \\ f(a_{k-1})s_k = f(g(b'))t_k & s = t_ks. \end{array}$$

and so

$$\begin{array}{cc} & s_1 s = s \\ bs_1 = f(a_1)t_1 & s_2 s = t_1 s \\ f(a_1)s_2 = f(a_2)t_2 & s_3 s = t_2 s \\ \vdots & \vdots \\ f(a_{k-1})s_k = b't_k & s = t_k s. \end{array}$$

By Lemma 2.3, B_S is PWF. Now let $bs = b't,\ Ss \cap St \neq \emptyset$ for $b, b' \in B_S,\ s, t \in S$. Hence $g(b)s = g(b')t$. Since A_S satisfies Condition (W') there exist $a'' \in A_S,\ u \in Ss \cap St$ such that $g(b)s = g(b')t = a''u$ and so $bs = b't = f(a'')u$. Consequently, B_S satisfies Condition (W'). □

Theorem 2.2. *Let ρ be a right congruence on S. Then the right S-act S/ρ is AWF if and only if $(xs)\,\rho\,(yt),\ Ss \cap St \neq \emptyset,\ x, y, s, t \in S$, imply the existence of elements $u, v \in S$ such that $us = vt,\ x\,(\rho \vee ker\rho_s)\,u$ and $y\,(\rho \vee ker\rho_t)\,v$.*

Proof. *Necessity.* Let S/ρ be AWF and suppose $(xs)\,\rho\,(yt),\ Ss \cap St \neq \emptyset$, for $x, y, s, t \in S$. Then $[x]_\rho s = [y]_\rho t$. By Condition (W') there exist elements $s', t', w \in S$ such that $s's = t't$ and $[x]_\rho s = [y]_\rho t = [w]_\rho s's = [w]_\rho t't$. Since S/ρ is PWF, we get from $(xs)\,\rho(ws's)$ and $(yt)\,\rho(wt't)$ that $x\,(\rho \vee ker\rho_s)\,ws'$ and $y\,(\rho \vee ker\rho_t)\,wt'$ by [[10], 3.10.7]. Take $u = ws',\ v = wt'$. Then $us = ws's = wt't = vt,\ x\,(\rho \vee ker\rho_s)\,u$ and $y\,(\rho \vee ker\rho_t)\,v$.
Sufficiency. Let $[x]_\rho s = [y]_\rho t,\ Ss \cap St \neq \emptyset$ for $x, y, s, t \in S$. Then $(xs)\,\rho\,(yt)$ and so by assumption, there exist elements $u, v \in S$ such that $us = vt,\ x\,(\rho \vee ker\rho_s)\,u$ and $y\,(\rho \vee ker\rho_t)\,v$. Using lemma [[10], 3.10.6], we get from $x\,(\rho \vee ker\rho_s)\,u$ that $[x]_\rho \otimes s = [u]_\rho \otimes s$ in $(S/\rho)_S \otimes_S Ss$. Analogously we get from $y\,(\rho \vee ker\rho_t)\,v$ that $[y]_\rho \otimes t = [v]_\rho \otimes t$ in $(S/\rho)_S \otimes_S St$. Now in $(S/\rho)_S \otimes_S (Ss \cup St)$ we have

$$(1) \qquad [x]_\rho \otimes s = [u]_\rho \otimes s = [1]_\rho \otimes us = [1]_\rho \otimes vt = [v]_\rho \otimes t = [y]_\rho \otimes t.$$

Hence S/ρ is AWF by Theorem 2.1. □

Corollary 2.1. *For any monoid S and $z \in S$, the following statements are equivalent:*

1. *The principal right ideal zS is WF.*
2. *The principal right ideal zS is AWF.*
3. *For every $x, y, s, t \in S,\ zxs = zyt$ implies the existence of elements $u, v \in S$ such that $us = vt,\ x\,(ker\lambda_z \vee ker\rho_s)\,u$ and $y\,(ker\lambda_z \vee ker\rho_t)\,v$.*
4. *For every $s, t \in S,\ zs = zt$ implies the existence of elements $u, v \in S$ such that $us = vt,\ 1\,(ker\lambda_z \vee ker\rho_s)\,u$ and $1\,(ker\lambda_z \vee ker\rho_t)\,v$.*

Proof. Implication (1) $\Longrightarrow$ (2) is obvious.
(2) $\Longrightarrow$ (3). Since $zS \cong S/ker\lambda_z$, apply Theorem 2.2, for $\rho = ker\lambda_z\ , z \in S$.
(3) $\Longrightarrow$ (4). Put in (3), $x = y = 1$.
(4) $\Longrightarrow$ (1). It follows from [[10], 3.11.8]. □

Theorem 2.3. *Let S be a monoid, $w, t \in S, wt \neq t$ and set $\rho = \rho(wt, t)$. Then the following statements are equivalent:*

1. *S/ρ is a flat right S-act.*
2. *S/ρ is a WF right S-act.*
3. *S/ρ is a AWF right S-act.*
4. *S/ρ is a PWF right S-act.*
5. *The element t is regular in S.*

Proof. Implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) are obvious.
(4) $\Longrightarrow$ (5). It follows from [[10], 3.10.9].
(5) $\Longrightarrow$ (1). It follows from [[10], 3.12.15]. □

We recall from [10] that an element $t \in S$ is called *w-regular* for $w \in S$, if $wt \neq t$ and if for every right cancellable element $c \in S$ and every $u \in S$, $uc \in tS$ implies that $u\rho(wt, t)wu$. From Theorem 2.3 and [[10], 3.8.9, 3.12.19] we have two following propositions.

Proposition 2.2. *Let S be a monoid. Then all torsion free monocyclic right S-acts of the form $S/\rho\,(wt, t)$, $w, t \in S$, $wt \neq t$ are AWF if and only if every w-regular element of S is regular for every $w \in S$.*

Proposition 2.3. *Let S be a monoid, K_S be a proper right ideal of S. Then the following statements are equivalent:*

1. *$S \coprod^{K_S} S$ is a flat right S-act.*
2. *$S \coprod^{K_S} S$ is a WF right S-act.*
3. *$S \coprod^{K_S} S$ is a AWF right S-act.*
4. *$S \coprod^{K_S} S$ is a PWF right S-act.*
5. *K_S is left stabilizing.*

A monoid S is called *left PP* if every principal left ideal of S (as a left S-act) is projective, it is equivalent to say that for every $s \in S$ there exists an idempotent $e \in S$ such that $es = s$ and for all $u, v \in S$, $us = vs$ implies $ue = ve$. A monoid S is called *left PSF* if every principal left ideal of S (as a left S-act) is strongly flat, it is equivalent to say that $us = vs$, $u, v, s \in S$, implies that there exists $r \in S$ such that $rs = s$ and $ur = vr$. It is obvious that every left PP monoid is left PSF.

Theorem 2.4. *Let S be a left PSF monoid. Then the following statements are equivalent:*

1. *A right S-act A_S is AWF.*
2. *For every $a, a' \in A_S$, and $s, t \in S$, $as = a't$, $Ss \cap St \neq \emptyset$ imply that there exist $a'' \in A_S$, $u, v, r, r' \in S$, such that $ar = a''ur$, $a'r' = a''vr'$, $rs = s$, $r't = t$ and $us = vt$.*
3. *For every $a, a' \in A_S$, and $s, t \in S$, $as = a't$, $Ss \cap St \neq \emptyset$ imply that there exist $a'' \in A_S$, $u, v, r, r' \in S$, such that $ar = a''u$, $a'r' = a''v$, $rs = s$, $r't = t$ and $us = vt$.*

Proof. (1) $\Longrightarrow$ (2). Suppose that A_S is AWF and let $as = a't$, $Ss \cap St \neq \emptyset$ for $a, a' \in A_S$ and $s, t \in S$. Since A_S satisfies Condition (W'), there exist $a'' \in A_S$, $u, v \in S$ such that $as = a''us$, $a't = a''vt$ and $us = vt$. Since also A_S is PWF, we have $a \otimes s = a''u \otimes s$ in $A_S \otimes_S Ss$, $a' \otimes t = a''v \otimes t$ in $A_S \otimes_S St$ by Lemma 2.3. Thus there exist $s_1, \ldots, s_k, t_1, \ldots, t_k \in S$, $b_1, \ldots, b_{k-1} \in A_S$ such that

$$\begin{array}{cc} & s_1 s = s \\ as_1 = b_1 t_1 & s_2 s = t_1 s \\ b_1 s_2 = b_2 t_2 & s_3 s = t_2 s \\ \vdots & \vdots \\ b_{k-1} s_k = a''ut_k & s = t_k s. \end{array}$$

Since S is left PSF there exists $r_1 \in S$ such that $r_1 s = s$, and $s_1 r_1 = r_1$. Then $s_2 r_1 s = t_1 r_1 s$ implies that there exists $r_2 \in S$ such that $r_2 s = s$, $s_2 r_1 r_2 = t_1 r_1 r_2$. Continuing this manner, we get $r_3, \ldots, r_{k+1} \in S$ such that $r_i s = s$, $1 \leq i \leq k+1$, $s_i r_1 \ldots r_i = t_{i-1} r_1 \ldots r_i$, $2 \leq i \leq k$ and $r_1 \ldots r_k r_{k+1} = t_k r_1 \ldots r_k r_{k+1}$. If $r = r_1 \ldots r_k r_{k+1}$, then $rs = s$ and

$$a''ur = a''ut_k r = b_{k-1} s_k r = b_{k-1} t_{k-1} r = \cdots = b_1 t_1 r = as_1 r = ar.$$

A similar argument shows that there exists $r' \in S$ such that $r't = t$ and $a''vr' = a'r'$.

(2) $\Longrightarrow$ (3). In assumption put $u^* = ur$ and $v^* = vr'$. Hence $u^*, v^* \in S, ar = a''u^*$, $a'r' = a''v^*$, $rs = s$, $r't = t$ and $u^*s = urs = us = vt = vr't = v^*t$.

(3) $\Longrightarrow$ (1). Suppose that $as = a't$, $Ss \cap St \neq \emptyset$ for $a, a' \in A_S$, $s, t \in S$. By assumption there exist $a'' \in A_S$, $u, v, r, r' \in S$, such that $ar = a''u$, $a'r' = a''v$, $rs = s$, $r't = t$, $us = vt$. Then, we have $a \otimes s = a \otimes rs = ar \otimes s = a''u \otimes s = a'' \otimes us = a'' \otimes vt = a''v \otimes t = a'r' \otimes t = a' \otimes r't = a' \otimes t$ in tensor product $A_S \otimes_S (Ss \cup St)$, and so A_S is AWF, by Theorem 2.1, as required. □

Theorem 2.5. *Let S be a left PP monoid. Then the following statements are equivalent:*

1. *A right S-act A_S is AWF.*
2. *For every $a, a' \in A_S$, and $s, t \in S$, $as = a't$, $Ss \cap St \neq \emptyset$ imply that there exist $a'' \in A_S$, $u, v \in S$, $e, f \in E(S)$ such that $ae = a''ue$, $a'f = a''vf$, $es = s$, $ft = t$, $us = vt$.*

3. *For every* $a, a' \in A_S$, *and* $s, t \in S$, $as = a't$, $Ss \cap St \neq \emptyset$ *imply that there exist* $a'' \in A_S$, $u, v \in S$, $e, f \in E(S)$ *such that* $ae = a''u$, $a'f = a''v$, $es = s$, $ft = t$, $us = vt$.

Proof. (1) $\Longrightarrow$ (2). Suppose that A_S is AWF and let $as = a't$, $Ss \cap St \neq \emptyset$ for $a, a' \in A_S$ and $s, t \in S$. Since every left PP monoid is left PSF, by Theorem 2.4, there exist $a'' \in A_S$, $u, v, r, r' \in S$, such that $ar = a''ur$, $a'r' = a''vr'$, $rs = s$, $r't = t$, $us = vt$. Since A_S is PWF, there exist $e, f \in E(S)$ such that $er = r$, $ae = a''ue$ and $fr' = r'$, $a'f = a''vf$, by [[10], 3.10.16] and so $es = ers = rs = s$, similarly $ft = t$.

(2) $\Longrightarrow$ (3). In assumption put $u^* = ue$ and $v^* = vf$. Hence $u^*, v^* \in S, ae = a''u^*$, $a'f = a''v^*$, $es = s$, $ft = t$ and $u^*s = ues = us = vt = vft = v^*t$.
(3) $\Longrightarrow$ (1). It follows from Theorem 2.4. □

Theorem 2.6. *Let* S *be a regular monoid. Then a right* S*-act* A_S *is AWF if and only if for every* $a \in A_S$ *and* $s, t \in S$, *if* $as = at$, $Ss \cap St \neq \emptyset$ *then there exists* $u \in Ss \cap St$ *such that* $as = at = au$.

Proof. Suppose that A_S is AWF and let $as = at$, $Ss \cap St \neq \emptyset$ for $a \in A_S$ and $s, t \in S$. Then by Condition (W') there exist $a'' \in A_S$ and $w \in Ss \cap St$ such that $as = at = a''w$. Since S is regular there exists $w' \in S$ such that $ww'w = w$. Then $u = sw'w \in Ss \cap St$ and $au = asw'w = a''ww'w = a''w = as$, as required.

Conversely, suppose that A_S is a right S-act and $as = a't$, $Ss \cap St \neq \emptyset$ for $a, a' \in A_S$ and $s, t \in S$. Let $t' \in S$ be such that $tt't = t$, $t'tt' = t'$. Then $as = a'tt't = ast't$. Since $Ss \cap St \neq \emptyset$, hence $Ss \cap Sst't \neq \emptyset$. By assumption there exists $w \in Ss \cap Sst't \subseteq Ss \cap St$ such that $as = ast't = a't = aw$, Thus A_S satisfies Condition (W'). Since S is regular, A_S is principally weakly flat and so A_S is AWF, as required. □

3. Characterization of monoids by almost weak flatness

In this section, we give a characterization of monoids when almost weak flatness property of their right acts imply other properties and vise versa. First see the following definition:

Definition 3.1. *A monoid* S *satisfies Condition* (R_{AWF}), *if for all* $x, y, s, t \in S$, $Ss \cap St \neq \emptyset$ *implies the existence of* $w \in Ss \cap St$ *such that* $w\rho(xs, yt)\, xs$, *where* $\rho(xs, yt)$ *is the monocyclic right congruence generated by* xs *and* yt. *Similarly it satisfies Condition* (L_{AWF}), *if for all* $x, y, s, t \in S$, $sS \cap tS \neq \emptyset$ *implies the existence of* $w \in sS \cap tS$ *such that* $w\lambda(xs, yt)\, xs$, *where* $\lambda(xs, yt)$ *is the monocyclic left congruence generated by* xs *and* yt.

Theorem 3.1. *Let* S *be a monoid. Then the following statements are equivalent:*

1. *All right* S*-acts are AWF.*

2. *All finitely generated right S-acts are AWF.*

3. *All cyclic right S-acts are AWF.*

4. *All monocyclic right S-acts are AWF.*

5. *S is regular and satisfies Condition (R_{AWF}).*

6. *S is regular and satisfies Condition*

$$(\forall s,t\in S)\,(Ss\cap St\neq\emptyset \implies (\exists w\in Ss\cap St;\quad 1\,(ker\lambda_s\vee ker\lambda_t)\,w\ ))\,.$$

Proof. Implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) are obvious.

(4) $\Longrightarrow$ (5). Since AWF implies PWF, regularity of S follows from [[10], 4.6.6]. Let $x,y,s,t\in S$, $Ss\cap St\neq\emptyset$ and $\rho=\rho\,(xs,yt)$. Then $(xs)\rho(yt)$ and Since S/ρ is AWF by assumption, Theorem 2.2 implies the existence of elements $u,v\in S$ such that $us=vt$, $x\,(\rho\vee ker\rho_s)\,u$ and $y\,(\rho\vee ker\rho_t)\,v$. Then $x\,(\rho\vee ker\rho_s)\,u$ implies that there exist elements $k_1,\ldots,k_n,l_1,\ldots,l_n\in S$ such that

$$xs=k_1s,\quad k_1\rho l_1,\quad l_1s=k_2s,\quad k_2\rho l_2,\ \ldots\ ,\quad l_ns=us.$$

Hence

$$xs=k_1s\ \rho\ l_1s=k_2s\ \rho\ l_2s\ \ \ldots\ \ l_ns=us,$$

and so $xs\rho w$ where $w=us=vt\in Ss\cap St$.

(5) $\Longrightarrow$ (3). Let ρ be a right congruence on S. We show that S/ρ is AWF. Let $(xs)\,\rho\,(yt)$ and $Ss\cap St\neq\emptyset$, for $x,y,s,t\in S$. Set $\tau=\rho\,(xs,yt)$. By assumption there exist $u,v\in S$ such that $w=us=vt$ and $w\tau(xs)$. Since S is regular, S/τ is PWF by [[10], 4.6.6]. Hence $(xs)\tau(us)$ implies $x\,(\tau\vee ker\rho_s)\,u$ by [[10], 3.10.7]. Analogously from $(yt)\tau(vt)$ we get $y\,(\tau\vee ker\rho_t)\,v$. Since $\tau\leq\rho$ one has $x\,(\rho\vee ker\rho_s)\,u$, $y\,(\rho\vee ker\rho_t)\,v$. Hence S/ρ is AWF by Theorem 2.2.

(3) $\Longrightarrow$ (1). Let A_S be right S-act, $as=a't$ and $Ss\cap St\neq\emptyset$ for $a,a'\in A_S$ and $s,t\in S$. By assumption, all cyclic right S-acts are AWF, hence all cyclic right S-acts are PWF and so S is regular by [[10], 4.6.6]. Let $s',t'\in S$ be such that $ss's=s$ and $tt't=t$. Then $b(s's)=b(t't)$ where $b=as=a't$. Since bS is AWF by assumption and $Ss\cap St\neq\emptyset$ implies $Ss's\cap St't\neq\emptyset$, we have $b\otimes s's=b\otimes t't$ in $bS\otimes_S(Ss's\cup St't)$. Thus $as\otimes s's=a't\otimes t't$ in $A_S\otimes_S(Ss's\cup St't)$ and so we have

$$a\otimes s=a\otimes ss's=as\otimes s's=a't\otimes t't=a'\otimes tt't=a'\otimes t$$

in $A_S\otimes_S(Ss\cup St)$. Hence A_S is AWF by Theorem 2.1.

(6) $\Longleftrightarrow$ (1). It follows from [[11], 3.4, 3.8]. □

We give a characterization of monoid when this property of their acts implies some others.

Since the one-element act $\Theta_S\cong S/S_S$ satisfyies Condition (W'), three following results are obvious.

Theorem 3.2. *Let S be a monoid. Then the following statements are equivalent:*

1. *All AWF right S-acts are (strongly) faithful.*
2. *All AWF finitely generated right S-acts are (strongly) faithful.*
3. *All AWF cyclic right S-acts are (strongly) faithful.*
4. *All AWF right Rees factor acts of S are (strongly) faithful.*
5. $S = \{1\}$.

Theorem 3.3. *Let S be a monoid. Then the following statements are equivalent:*

1. *All right S-acts satisfying Condition (W') are (strongly) faithful.*
2. *All finitely generated right S-acts satisfying Condition (W') are (strongly) faithful.*
3. *All cyclic right S-acts satisfying Condition (W') are (strongly) faithful.*
4. *All right Rees factor acts satisfying Condition (W') are (strongly) faithful.*
5. $S = \{1\}$.

Let $\mathbf{C}$ be a category. An object $G \in \mathbf{C}$ is called a *generator* in $\mathbf{C}$ if the functor $Mor_{\mathbf{C}}(G, -)$ is faithful, i.e, for any $X, Y \in \mathbf{C}$ and any $f, g \in Mor_{\mathbf{C}}(X, Y)$ with $f \neq g$ there exists $\alpha \in Mor_{\mathbf{C}}(G, X)$ such that $f\alpha \neq g\alpha$. We recall from [[10], 2.3.16] that G_S is a generator if and only if there exists an epimorphism $\pi : G_S \longrightarrow S_S$. Hence S_S is a generator in **Act**-S.

Theorem 3.4. *Let S be a monoid. Then the following statements are equivalent:*

1. *All AWF right S-acts are generator.*
2. *All AWF finitely generated right S-acts are generator.*
3. *All AWF cyclic right S-acts are generator.*
4. *All AWF right Rees factor S-acts are generator.*
5. $S = \{1\}$.

We recall from [10] that a right S-acts A_S is *divisible* if $Ac = A$ for any left cancellable element $c \in S$. By [[10], 3.2.2] and Since S_S is AWF the following result is deduced.

Theorem 3.5. *Let S be a monoid. Then the following statements are equivalent:*

1. *All right S-acts are divisible.*
2. *All AWF right S-acts are divisible.*
3. *All AWF finitely generated right S-acts are divisible.*
4. *All AWF cyclic right S-acts are divisible.*
5. *All AWF monocyclic right S-acts are divisible*
6. *S_S is divisible.*
7. *All left cancellable elements of S are left invertible.*

Theorem 3.6. *Let S be a monoid. Then the following statements are equivalent:*

1. *All right S-acts are regular.*
2. *All AWF right S-acts are regular.*
3. *All AWF finitely generated right S-acts are regular.*
4. *All AWF cyclic right S-acts are regular.*
5. *$S = \{1\}$ or $S = \{0, 1\}$.*

Proof. Implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) are obvious.

(4) $\Longrightarrow$ (5). Since weak flatness implies almost weak flatness, every element of $S\backslash\{1\}$ is right zero, by [[5], 1.14]. Also, by [[5], 1.4], every AWF cyclic right S-act satisfies Condition (E). So, the one-element act Θ_S satisfies Condition (E). Thus S is left collapsible, by [[10], 3.14.3]. Consequently, either $S = \{1\}$ or $S = \{0, 1\}$, by [[5], 1.9].

(5) $\Longrightarrow$ (1). By [[10], 4.14.4], it is obvious. □

Since Condition (P) and flatness implies AWF and by [[10], 4.12.8, 4.10.5], three following results are immediate.

Theorem 3.7. *Let S be a monoid. Then the following statements are equivalent:*

1. *All AWF right S-acts are free.*
2. *All AWF finitely generated right S-acts are free.*
3. *All AWF cyclic right S-acts are free.*

4. *All AWF monocyclic right S-acts are free.*
5. *All AWF right Rees factor acts of S are free.*
6. $S = \{1\}$.

Theorem 3.8. *Let S be a monoid. Then the following statements are equivalent:*

1. *All AWF right S-acts are projective generator.*
2. *All AWF finitely generated right S-acts are projective generator.*
3. *All AWF cyclic right S-acts are projective generator.*
4. *All AWF monocyclic right S-acts are projective generator.*
5. *All AWF right Rees factor acts of S are projective generator.*
6. $S = \{1\}$.

Theorem 3.9. *Let S be a monoid. Then the following statements are equivalent:*

1. *All AWF right S-acts are projective.*
2. *All AWF right S-acts are strongly flat.*
3. $S = \{1\}$.

Theorem 3.10. *Let S be a monoid. Then the following statements are equivalent:*

1. *All AWF cyclic right S-acts are projective.*
2. *All AWF cyclic right S-acts are strongly flat.*
3. *S is left collapsible and right nil.*

Proof. Implication (1) $\Longrightarrow$ (2) is obvious.

(2) $\Longrightarrow$ (3). Since the one-element act Θ_S is AWF, it is strongly flat. Since for cyclic acts strong flatness coincides with Condition (E), hence Θ_S satisfies Condition (E), and so S is left collapsible, by [[10], 3.14.3].

Since flatness implies AWF, hence all flat cyclic right S-acts are strongly flat, by assumption. So, all flat monocyclic right S-acts are strongly flat. Thus S is right nil, by [[10], 4.10.4].

(3) $\Longrightarrow$ (1). Since S is left collapsible, it is right reversible and so, weak flatness coincides with almost weak flatness. But S is right nil, and so all WF cyclic right S-acts are projective, by [[10], 4.11.10]. □

Theorem 3.11. *Let S be a monoid. Then the following statements are equivalent:*

1. *All right S-acts satisfying Condition (W') satisfy Condition (W).*
2. *All AWF right S-acts satisfy Condition (W).*
3. *All AWF right S-acts are WF.*
4. *All finitely generated right S-acts satisfying Condition (W') satisfy Condition (W).*
5. *All finitely generated AWF right S-acts satisfy Condition (W).*
6. *All finitely generated AWF right S-acts are WF.*
7. *All cyclic right S-acts satisfying Condition (W') satisfy Condition (W).*
8. *All cyclic AWF right S-acts satisfy Condition (W).*
9. *All cyclic AWF right S-acts are WF.*
10. *All right Rees factor S-acts satisfying Condition (W') satisfy Condition (W)*
11. *All AWF right Rees factor S-acts satisfy Condition (W).*
12. *All AWF right Rees factor S-acts are WF.*
13. *S is right reversible.*

From Theorem 3.11 and [[10], 4.8.1] we have the following result.

Corollary 3.1. *Let S be a left PP right reversible monoid. If S satisfies Condition*

$$(L'): \ \forall e, f \in E(S), \ \exists z \in eS \cap fS; \ z\lambda(e,f)\, e,$$

where $\lambda(e,f)$ is the monocyclic congruence on ${}_SS$ generated by e and f, then every AWF right S-act is flat.

Definition 3.2. *A monoid* S *is said to be right (left) almost weakly absolutely flat if all right (left) S-acts are AWF, and it is said to be almost weakly absolutely flat if it is both right and left almost weakly absolutely flat.*

Since an almost weakly absolutely flat monoid is both right and left almost weakly absolutely flat the following theorem follows from Theorem 3.1 and its left-right dual.

Theorem 3.12. *For any monoid S the following statements are equivalent:*

1. *S is almost weakly absolutely flat.*

2. *S is regular and satisfies Conditions (R_{AWF}) and (L_{AWF}).*

From Theorem 3.12 and [[10], 4.8.12] we have the following important result.

Theorem 3.13. *For any right reversible monoid S the following statements are equivalent:*

1. *S is absolutely flat*
2. *S is weakly absolutely flat*
3. *S is almost weakly absolutely flat.*
4. *S is regular and satisfies Conditions (R_{AWF}) and (L_{AWF})*
5. *S is regular and satisfies Conditions (R) and (L).*

4. AWF Coherent monoids

In this section we introduce AWF coherent monoids and will give a characterization of AWF coherent monoids in general and monoids coming from special classes.

Theorem 4.1. *Let S be a monoid and $\{A_i\}_{i\in I}$ a family of right S-acts. Then $\prod_{i\in I} A_i$ is AWF if and only if it is PWF and for any $s,t \in S$, if $A_is \cap A_it \neq \emptyset$, and $Ss \cap St \neq \emptyset$, for every $i \in I$, then for every $(w_i)_I \in \prod_{i\in I}(A_is \cap A_it)$, there exist $(a''_i)_I \in \prod_{i\in I} A_i$ and $u \in Ss \cap St$ such that $(w_i)_I = (a''_i)_I u$.*

Proof. *Necessity.* Suppose $\prod_{i\in I} A_i$ is AWF, and let $A_is \cap A_it \neq \emptyset$, $Ss \cap St \neq \emptyset$ and $(w_i)_I \in \prod_{i\in I}(A_is \cap A_it)$, for $i \in I$, $s,t \in S$. Then there exist $a_i, a'_i \in A_i$ such that $w_i = a_is = a'_it$, for every $i \in I$, and so $(a_i)_Is = (a'_i)_It$. Since $\prod_{i\in I} A_i$ is AWF it satisfies Condition (W'), and so there exist $(a''_i)_I \in \prod_{i\in I} A_i$ and $u \in Ss \cap St$ such that $(a_i)_Is = (a'_i)_It = (a''_i)_Iu$. Thus $(w_i)_I = (a''_i)_Iu$.
Sufficiency. Let $(a_i)_Is = (a'_i)_It$, $Ss \cap St \neq \emptyset$, for $(a_i)_I, (a'_i)_I \in \prod_{i\in I} A_i$, and $s,t \in S$. Then $(w_i)_I = (a_i)_Is = (a'_i)_It \in \prod_{i\in I}(A_is \cap A_it)$, and so there exist $(a''_i)_I \in \prod_{i\in I} A_i$ and $u \in Ss \cap St$ such that $(w_i)_I = (a''_i)_Iu$. Thus $(a_i)_Is = (a'_i)_It = (a''_i)_Iu$, and so $\prod_{i\in I} A_i$ satisfies Condition (W'), and hence $\prod_{i\in I} A_i$ is AWF. □

The right S-act $(S\times S)_S$ equipped with the right S-action $(s,t)\,u = (su,tu)$ for $s,t,u \in S$ is called the *diagonal act* of S and is denoted by $D(S)$. Also, if S is a monoid and I a non-empty set, then the set of all maps from I to S, equipped with the right S-action $(\alpha s)\,x = (\alpha\,(x))\,s$, for the mapping $\alpha : I \to S$, $s \in S$ and $x \in I$, is a right S-act, and denoted by $(S^I)_S$. We recall that a left S-act ${}_SB$ is called *locally cyclic*, if every finite subset (or equivalently, every 2-element) of ${}_SB$ is contained in a cyclic subact of ${}_SB$. A locally cyclic left ideal of S is called *locally principal*. As corollaries of Theorem 4.1, [[9], 2.19] and [[9], 2.20] we have:

Corollary 4.1. *The following statements are equivalent:*

1. $(S^I)_S$ *is WF, for any non-empty set* I.
2. $(S^I)_S$ *is AWF, for any non-empty set* I.
3. $(S^I)_S$ *is PWF, for any non-empty set* I, *and for any* $s,t \in S$, $Ss \cap St$ *is either empty or a principal left ideal.*

Corollary 4.2. *The following statement are equivalent:*

1. $D(S)$ *is WF.*
2. $D(S)$ *is AWF.*
3. $D(S)$ *is PWF, and for any* $s,t \in S$, $Ss \cap St$ *is either empty or else a locally principal ideal.*

Definition 4.1. *A monoid* S *is called AWF left (right) coherent if all direct products of non-empty families of AWF right (left)* S*-acts are AWF, and it is called AWF coherent if it is both left and right coherent.*

Definition 4.2. *A monoid* S *is called right finite AWF if for every* $s \in S$, *there exists* $n_s \in \mathbb{N}$ *such that for every AWF right* S*-act* A_S, $d_s(a,a') \leq n_s$ *where* $as = a's$, $a, a' \in A_S$.

Theorem 4.2. *A monoid* S *is AWF left coherent if and only if:*

1. $(S^I)_S$ *is AWF for any non-empty set* I, *and*
2. S *is right finite AWF.*

Proof. Suppose that S is an AWF left coherent monoid. It is obvious that (1) holds. If S is not right finite AWF, then there exists $s \in S$ such that for any $n \in \mathbb{N}$ there exists an AWF right S-act A_n, such that $a_n s = a'_n s$ for some $a_n, a'_n \in A_n$ and $d_s(a_n, a'_n) > n_s$. By assumption $\prod_{n=1}^{\infty} A_n$ is AWF. Thus $(a_n)_{\mathbb{N}} s = (a'_n)_{\mathbb{N}} s$, implies $(a_n)_{\mathbb{N}} \otimes s = (a'_n)_{\mathbb{N}} \otimes s$ in $(\prod_{n=1}^{\infty} A_n)_S \otimes_S Ss$. Suppose that the length of the corresponding tossing is k. Then for each $n \in \mathbb{N}$, there exists an S-tossing of length k connecting (a_n, s) to (a'_n, s) in $A_n \times Ss$. But this contradicts $d_s(a_{k+1}, a'_{k+1}) \geq k+1$.

Conversely, suppose that $\{A_i \mid i \in I\}$ is a family of AWF right S-acts and let $A_S = \prod_{i \in I} A_i$. At first we show that A_S is PWF. Let $(a_i)_I s = (a'_i)_I s$, then $a_i s = a'_i s$ for every $i \in I$. Since S is right finite AWF, there exists $n \in \mathbb{N}$ such that (a_i, s) and (a'_i, s) are connected by an S-tossing of length n in $A_i \times Ss$, for every $i \in I$. Thus for each $i \in I$, there exists an S-tossing of the form

$$\begin{array}{rl} & s_{i_1} s = s \\ a_i s_{i_1} = b_{i_1} t_{i_1} & s_{i_2} s = t_{i_1} s \\ b_{i_1} s_{i_2} = b_{i_2} t_{i_2} & s_{i_3} s = t_{i_2} s \\ \vdots & \vdots \\ b_{i_{n-1}} s_{i_n} = a'_i t_{i_n} & s = t_{i_n} s. \end{array}$$

where $s_{i_1}, \ldots, s_{i_n}, t_{i_1}, \ldots, t_{i_n} \in S$ and $b_{i_1}, \ldots, b_{i_{n-1}} \in A_i$. Let $t_{i_0} = s_{i_{n+1}} = 1$, $b_{i_0} = a_i$ and $b_{i_n} = a'_i$ for $i \in I$. Then $s_{i_j} s = t_{i_{j-1}} s$ for $i \in I$ and $1 \leq j \leq n+1$, and so $\left(b_{i_{j-1}} t_{i_{j-1}}\right)_I \otimes s = \left(b_{i_{j-1}} s_{i_j}\right)_I \otimes s$ in $A_S \otimes_S Ss$. Thus, $(a_i)_I \otimes s = (b_{i_0} t_{i_0})_I \otimes s = (b_{i_0} s_{i_1})_I \otimes s = (b_{i_1} t_{i_1})_I \otimes s = \cdots = (b_{i_n} s_{i_{n+1}})_I \otimes s = (a'_i)_I \otimes s$. in $A_S \otimes_S Ss$.

Now, we have to show that A_S satisfies Condition (W'). Let $(a_i)_I s = (a'_i)_I t$, $Ss \cap St \neq \emptyset$, for $(a_i)_I, (a'_i)_I \in A_S$, and $s, t \in S$. Then for each $i \in I$, there exist $a''_i \in A_i$, $w_i \in Ss \cap St$, such that $a_i s = a'_i t = a''_i w_i$. By Corollary 4.1 there exists $u \in S$ such that $Ss \cap St = Su$. So for each $i \in I$, there exists $v_i \in S$ such that $w_i = v_i u$, and so $(a_i)_I s = (a'_i)_I t = (a''_i v_i)_I u$, where $u \in Ss \cap St = Su$. □

By Theorem 2.4 every left PSF monoid is right finite AWF with $n_s = 2$ for each $s \in S$. So in this case we have the following corollaries as results of Theorem 4.2.

Corollary 4.3. *Let S be a left PSF monoid. Then S is AWF left coherent if and only if $(S^I)_S$ is AWF for any nonempty set I.*

Corollary 4.4. *Let S be a left PP monoid. Then S is AWF left coherent if and only if $(S^I)_S$ is AWF for any nonempty set I.*

Definition 4.3 ([14])**.** *Let S be a monoid. A finitely generated left S-act ${}_SB$ is called finitely definable (FD) if $S^I \otimes B \to B^I$, $(s_i) \otimes b \mapsto (s_i b)$ is injective for every non-empty set I.*

Lemma 4.1. *If $(S^I)_S$ is AWF for any non-empty set I, then for every $x \in S$, Sx is FD.*

Lemma 4.2 ([14])**.** *Let S be a commutative monoid. If $x \in S$ and Sx is FD then the principal left ideal Sx is projective.*

A monoid S is called *weakly left coherent* if direct products of non-empty families of weakly flat right S-acts are weakly flat. In [BULCOH] it is proved that a commutative monoid S is (weakly) coherent if and only if the act S^I is weakly flat for each nonempty set I. Now for AWF left coherent monoids see the following.

Theorem 4.3. *Let S be a commutative monoid. Then S is AWF left coherent if and only if $(S^I)_S$ is AWF for any non-empty set I.*

Proof. *Necessity.* It is clear from Theorem 4.2.
Sufficiency. Let $\left(S^I\right)_S$ be AWF for any non-empty set I, and let $x \in S$. By Lemma 4.1, Sx is FD and so by Lemma 4 it is projective. That is, all principal left ideals of S is projective and so S is a left PP monoid. Now the result follows from Corollary 4.4. □

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Accepted: February 2, 2020

Econometric analysis of Jordanian phosphate industry

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Abstract. The aim of this study is to assess the competitiveness of the Jordanian extractive phosphate industry using the econometrics analysis, i.e., a common integration approach and error correction model, as well as to appraise some of the competitiveness indicators of the Jordan Phosphate Mines Company (JPMC) for the period (1990-2018), on the basis of Porter's index of comparative advantage, technical efficiency, as well as allocative efficiency. The results pointed out that the Jordanian extractive phosphate industry has a comparative advantage, moderate technical efficiency and decreasing returns to scale. However, the results of the econometrics analysis showed that there is no significant long-term statistical relationship between the company's performance and the variables affecting it. Moreover, the results revealed that the profitability of the company fluctuates during the studied period. On that account, the study presented a set of recommendations, including raising the technical efficiency and efficiency of scale through the optimal use of resources, opining new export markets, using modern technologies, in addition to expanding exploration and extraction areas in order to boost productivity, profitability and competitiveness.

Keywords: competitiveness, comparative advantage, technical efficiency, allocative efficiency, co-integration, vector error correction model.

1. Introduction

Competitiveness has been regarded as a principal tool for economic development economy in globalization, liberalization of markets and global co-integration. Hence, a higher degree of competitiveness improves country's per capita GDP, standard of living and welfare economy due to the optimal utilization of resources, high degrees of quality and superiority over other similar countries and global market penetration.

The importance of the competitiveness of companies in general and Jordanian extractive phosphate industry, in particular, is that it leads to the achieve-

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ment of many advantages that reflect on the well-being of the people. This is because that competitiveness leads to overstepping the domestic market by expanding exports, supplying the state with foreign currencies, creating new job opportunities and localizing high-tech technologies by attracting foreign investment, which improves efficiency and productivity levels.

Phosphate mining in Jordan is one of the most important components of the mining sector, which contributed 0.08 of GDP in 2017. Phosphate was discovered in Jordan in 1908 and invested for the first time in Russeifa in 1935. Fourteen years later, the Jordan Phosphate Mines Company (JPMC) in order to exploit phosphate ores in the fertilizer industry. The company has the fifth largest phosphate reserves in the world by about 3.7 billion tons. It is the sixth largest producer of phosphate in the world with a production capacity of more than seven million tons per year.

Phosphate is used in many industries such as detergent industry and the chemical fertilizer industry.It is used in agriculture to feed the plant. It is one of the inputs of the manufacturing industry like extraction of uranium.In the nineties of the last century, the Indian-Jordanian Chemical Company was established to manufacture high-quality phosphoric acid with a production capacity of 224,000 tons per year, and the Japanese-Jordanian Fertilizer Company to produce compound fertilizers with a production capacity of 300,000 tons per year. During the period (2007-2010), the company was invested in several allied companies such as PetroJordan, the Jordanian Indian Company, Jordan Industrial Ports Company and others.

The importance of studying the competitiveness of phosphate metal, as it is linked to global food security, and the increasing global demand for the production of agricultural crops, for which phosphate fertilizers and other fertilizers are important elements. The demand for phosphate is increasing because of the limited global stock of this precious metal and the continuous increase in the world population.

1.1 Problem statement

The dependence of the phosphate sales on export calls for policies to improve international competitiveness. The export of phosphate and the related fertilizers is closely linked to the global market conditions of supply, demand and exchange market of the global fertilizer market, in addition to droughts in agricultural seasons in many regions of the world,competition from countries with cheap energy sources, as well as competition between the prices of compound fertilizersand natural gas-based nitrogen fertilizers.

Consequently, fluctuations in the sale prices of raw phosphate products and compound fertilizers exported to the world market are reflected in the performance and profitability of JPMC, hence, its efficiency and competitiveness. Accordingly, the problem of the study signifies the extent of the ability of JPMC to face global competition by improving its efficiency, maximizing its profits,

raising its competitiveness and absorbing external shocks. Basically, the study seeks to answer the following questions:

1. What is the comparative advantage of JPMC in exporting crude phosphate?

2. Is there a technical competence enjoyed by JPMC?

3. Does the company benefit from economies of scale to reduce costs and raise efficiency?

4. Is there a long-term relationship between the company's performance and profitability with its production and costs?

1.2 Study objectives

The study aims to shed light on:

1. Sources of competitiveness and the challenges facing the Jordanian phosphate industry.

2. The comparative advantage that JPMC possesses according to the Port Competitiveness Index.

3. The competitiveness of JPMC using the efficiency indicators such as technical efficiency and scale efficiency according to the method of production function of cobb- Douglas and translog function

4. Using the co-integration approach and error correction model in exploring the long and short term causal relationship between the JPMC's performance, production and costs in order to measure the allocative efficiency.

1.3 Study hypotheses

Three key hypotheses were postulated in the current study.

1. The company has a comparative advantage in the extraction and production of crude phosphate qualifiesit for exporting phosphate to the world market.

2. The company has a technical efficiency in the extraction and production of crude phosphate, as well as economies of scale increase its competitiveness in global markets.

3. The performance of the company is affected by the size of its production and the value of its costs through a short and long-term causal relationship.

1.4 Data sources

The study was based on annual time series data for the indicators of net profit, production volume, total cost, number of employees and capital size during the period studied (1990-2018). Data were collected from the published annual reports of JPMC and unpublished data issued by the company. Secondary data sources were books, researches and scientific papers published in refereed journals.

1.5 Data processing methods

The descriptive-analytical method was used to apply some indicators such as Porter's comparative advantage index and quantitative analytical analysis to estimate the production function on crude phosphate by Douglas function to determine the type of returns to scale in the Jordanian phosphate industry.Using the Douglas production function for the half-normal distribution and the translog function for truncated-normal distribution to measure the technical efficiency achieved in the Jordanian phosphate industry using frontier 4.0 Software.

The econometric analysis was also used to measure the allacative efficiency in the Jordanian phosphate industry through a co-integration approach, in order to avoid the false results that result from using ordinary least squares method for unstable time series. This methodology was applied by E–views using the following steps:

• First, the stability of time series test. The test was used to ensure long-term stability of time series data for the variables of the study through the null hypothesis test (H0), which assumes unit root presence in the time series versus the alternative hypothesis (H1) which indicates that the time series is stationary using Augmented Dickey–Fuller(ADF). The null hypothesis is rejected if the absolute statistical ADF values are greater than the critical(tabular) value, and if the probability value is less than 0.05. The ADF test is performed in three cases: no constant and no trend, constant and no trend, constant and trend. This is at a level less than 0.05 for the original chains (level). If the chains are not stable at that level, the first differences are taken until they stabilize and integrate from the same degree.

• Second: Determination of the appropriate number of lag. The appropriate lag interval for time series is determined for its direct impact on the results of subsequent tests. Vector autoregressive (VAR) model is applied to select the appropriate length of time slack based on the results of five tests: LR-test, (AIC), (SC), (HQ), and (FPE).

• Third: The Johansen Test for Co-integration (1988). In order to test the existence of a long-term equilibrium relationship between the variables of the study, and to determine the number of vectors and equilibrium relationships in the long term, Johansen test of co-integration will be used, which aims to test the extent of harmony between the variables under study. Johansen proposed two tests. The first one is the impact test, to test the null hypothesis (H0),

which states that the number of co-integrated equations is less than or equal to the rank of the matrix (r = 0) versus the alternative hypothesis that there are several alternatives for co-integration. The second test is the Max Eigen value test, which tests a specific alternative. It tests the null hypothesis (H0), which states that the number of the co-integrated equations equals the rank of the matrix (r = 0) versus the alternative hypothesis that there is a number (H1) of co-integrated trends. Commonly, the null hypothesis is accepted if the impact and the Max Eigen values are less than the critical value at a level of less than 0.05.

• Fourth: Error Correction Models (ECM). In case of co-integration, there is a long-term relationship between the variables. This suggests that there is a causal relationship in at least one direction, so ECM is used to determine the short and long-term causality of the variables under study shown by the results of the co-integration test, and to estimate the speed of reaching long-term equilibrium.

• Fifth: Granger Causality Test. If one of the variables causes the other, the relationship is one-way. In the case of each of the variables causes the other, the relationship is reciprocal. There may be no causal relationship between variables, and the causality test is based on the F-test.

Theoretical framework and Literature Review. Since the 1970s, the concept of competitiveness has been linked to international trade and evolved to relate to the technological policy of countries since the 1990s. In the present century, the concept of global competitiveness has been linked to the state's superiority over its peers in economic activity and raising the standard of living of its citizens. The British definition of the company's competitiveness refers to the ability to produce the right goods and services of good quality, at the right price and at the right time, and that means meeting the needs of consumers more efficiently than other enterprises (Oughton, 1997). Measuring the competitiveness of the enterprise economy is linked to a set of indicators including profitability, cost of manufacture, total factor productivity and market share.

This paper discusses the probability of achieving competitiveness by JPMC through a number of indicators including:

1. Revealed Comparative Advantage: This measure of competitiveness is based on the share of economic activity in the international market (Porter, 1990). The indicator can be computed based on the following equation: (State exports of economic activity / total exports of the State) ÷ (World exports of economic activity / total world exports). When the value of the index is greater than the unit, the state has an apparent comparative advantage for the product (Ghazali, 2003).

2. Allocative Efficiency: Pearce (1992) defined allocative efficiency as "the optimal mix of output resulting from the best efficient mix of inputs." It was measured in this study by estimating net profit as a dependent

variable, via analyzing the time series of the phosphate mines company between (1990-2018), stability test, co-integration and error correction model in the regression model, where the determinants of the company's profitability represented by production and total cost.

3. Technical Efficiency: It represents the ratio of actual output to the maximum potential output. A company achieves its technical efficiency if the output is increased using a specific set of inputs. The technical competence of the JPMC was measured by the Maximum Likelihood Method, estimation of the cobb- Douglas production function (a) assuming the half-normal distribution and estimation of the translog production function (b) assuming truncated-normal, respectively.

   (a) $LnYi = A + B1LnLi + B2LnKi + (ui - vi) \ldots$ (Douglas production function);

   (B) $LnYi = A + B1LnLi + B2LnKi + B3(LnLi) + B4(LnKi) + B5[(LnLi) * LnKi)] + (ui - vi)$.

4. Scale Efficiency: When the company selects a level of production appropriate to the cost conditions, where the average costs fall, production volume increases and the company achieves economies of scale.

1.6 Competitiveness aspects of JPMC

Four key aspects of JPMC Competitiveness were elucidated:

1. The company owns Al-Shidia mines with a production capacity of five million tons annually, Al-Hasa mine with a production capacity of half a million tons annually, and Wad-Alabyad mine with a production capacity of 1.5 million annually.

2. The company has a number of allied companies such as Indian Fertilizer Company, Jordan Industrial Ports Company, Manajemfor Mining Development, Al-Abyad Company for Fertilizers and Jordan Chemicals and others.

3. Jordanian exports of crude phosphate in 2017 constituted 375.7 million dollars, i.e., 0.122 of the global exports of crude phosphate, and ranked second after Morocco, where the value of world exports of phosphates equal 3076.7 million.

4. The company was able to reduce production costs per ton of crude phosphate by about 10 dollars in 2018, equivalent to 0.20 of production costs for 2017.

1.7 Challenges facing the competitiveness of Jordanian crude phosphate:

1. The fluctuation in the selling prices of raw phosphate products and fertilizers exported to the global market, which reflects on the performance and profitability of the company.

2. The Low production capacity of the company due to the depreciation achieved in mining equipment, crushing equipment, in addition to the periodic maintenance required for these devices.

3. The high extraction costs associated with the high costs of clearance back to the layer where the concentration of phosphate ores.

4. High costs of global shipping, and their effect on the marketing of phosphate products.

5. Little spending on promotional activities to market the company's products in international markets, especially in the world markets that have not been penetrated.

6. The global competition that the company's exports face, as the company lost some of its export markets for crude phosphate in some years, which increases the challenges facing the company.

7. The high volume of unjustified expenses and the absence of a rational policy to control expenditures, in order to increase the company's profits and competitiveness.

2. Literature Review

Al-Dabbagh's (1992) study aimed at comparing the productivity of the Jordanian phosphate mining company and the Jordanian cement factories, and found that both the average labor and capital productivity in the phosphate company are greater than in the cement company, and that the overall productivity of the phosphate company was characterized by fluctuation due to increased production costs. This result is consistent with the findings of this study that the company's performance fluctuates during the study period. It also found a positive relationship between profitability and a number of independent variables of total productivity and capital productivity. Studying Jordan's phosphate exports, Azhar (2000) discussed the contribution of the phosphate industry to Jordanian exports, where it maintained its market share in the global market. The study reviewed the places of abundance of raw phosphate in Jordan, such as Alwadi Alabiadh, Rusaifeh, Ras Al-Naqab, Al-Hasal and Al-Qatraneh. Furthermore, the phases of the development of phosphate production have been discussed since the company's establishment.

Li (2008) evaluated the competitiveness of phosphate fertilizer producers and compound fertilizer producers in China, and provided a way to explore the quantitative assessment of the competitiveness of Chinese fertilizer enterprises. The overall economic strength of fertilizer and phosphate producers in China was analyzed by the establishment of a targeted economic valuation system according to the distinctive feature of phosphate and compound fertilizer producers. Alrawashdeh (2008) examined the competitive behavior of the State Mining Corporation (the case of JPMC) and showed that the growth of state-owned companies was reflected in the expansion of JPMC to reduce the current account deficit. One of the determinants of the expansion of exports of JPMC is its role in paying off the external debt obligations of the Jordanian economy.

AL-Sukkar (2010) examined the competitiveness analysis of the JPMC (Case Study of the Least Developed Countries) in order to identify the factors affecting the competitiveness of the Jordanian phosphate industry in the global market, and compare the Jordanian phosphate company globally. Alrawashdeh and Sukkar (2010) investigatedfactors affecting the competitiveness of JPMC and concluded that although some markets were lost in Eastern and Western European countries, new markets had been gained in South Asia. The study showed the company outperformed some of its competitors against lower performance compared to other competitors. Furthermore, Alrawashdeh and Thyabat (2012) measured the links of the mining industry in the Jordanian economy and examined the challenges facing the mining industry in Jordan. It concluded that mining had strong forward and backward linkages to the entire economy, and that there was a significant increase in the fertilizer income multiplier from 2.8 to 6.67. Using Porter analysis,Alrawashdeh (2013)explored the competitiveness of JPMC viaa questionnaire and found that JPMC has a comparative advantage in the bargaining power of suppliers, and does not have a comparative advantage in negotiating ability among buyers.Alrawashdeh and Maxwell (2013) used the MMSD method to identify the variables that most affect the economy and showed that the Jordanian economy was not affected by the Dutch disease.

On the other hand, Al-Thyabat (2014) dealt with the determinants of demand for Jordanian phosphate through time series analysis for the period (1980-2010). Jordan's demand for crude phosphate exports was estimated using time series analysis and a co-integration approach to ensure a long-term equilibrium relationship. The results confirmed the existence of such a relationship between the variables of the study, and recommended the need to raise the competitiveness of the JPMC. In order to identify opportunities and threats facing the mining sector in Jordan, Altarawneh (2016) showed that the importance of crude phosphate as one of the most important elements of the sector's competitiveness, and provided recommendations regarding expanding investment in the mining industries.

Irshad and Xin (2017) examined the determinants of export competitiveness and an applied testof the comparative advantage index in the external sector of Pakistan. It concluded that there was a concentration of exports of Pak-

istan in specific markets and products, and recommended the need to diversify exports.The comparative advantage index of the metal commodity in Pakistan recorded 1.22 and 1.58 during the years 2012 and 2014, respectively.

Using logistic and Gompertz model,Al-Titi et al (2019) found that the peak production of the phosphate in Jordan will be in 2044-2048, where it is expected that the production volume will reach (15.2) million tons in 2048 according to Gompertz model.

On the basis of the above-mentioned literature, this study is characterized one of micro-economic studies that analyzethe institution's economy, while highlighting the economic impact at the macro level through the analysis of competitiveness indicators. Moreover, the study uses various methods of quantitative econometrics analysis to measure competitiveness in extractive industries in the Jordanian economy, according to the best of the researchers' knowledge.

3. Results

1. Using the Porter Relative Advantage Index: The index was calculated according to the available global data for 2017 and 2018.If the value of this index exceeds one, the country has a comparative advantage in the studied commodities.If it were less than 1, it lacks comparative advantage. According to the formula of Porter's comparative advantage index, which indicates : (State exports of economic activity / total exports of the State) ÷ (World exports of economic activity / total world exports). The value of this index for Jordanian crude phosphate was 350, 136.9 for the years 2017 and 2018, respectively. This indicator shows that Jordan enjoys a high comparative advantage in the production and export of crude phosphate.This result is consistent with the position of the Jordanian Phosphate Mines Company as the sixth largest producer and exporter of phosphate in the world with a production capacity of more than 7 million tons annually.

2. Technical efficiency measurement: The technical efficiency of JPMC was measured using Stochastic Frontier (SF) model and Cobb-Douglas production function with the assumption of half-normal and the translog function assumed by truncated–normal and using Frontier 4.0 software. The results revealed that the technical efficiency using half-normal for Cobb-Douglas production function was 0.75 and the technical efficiency using truncated-normal for translog output function was 0.71.This result indicates that phosphate producers can increase their production by 0.029 without increasing the economic resources used in the production process.Stochastic Frontier method of measuring technical competence has been used by many researchers including (e.g., Battese and Coelli, 1995). This result is consistent with the findings of Ajibefun (2007) when mea-

suring technical competence in metal industries in a number of regions of Nigeria where it reached 0.82.

3. Measuring the efficiency of Scale : The efficiency of Scale in the Jordanian phosphate industry was measured according to the Cobb-Douglas production function. It was found that:

   - There is no effect of labor and capital elements on production. This is shown by the estimated parameters of labor and capital elasticity of production, as it came very low as shown in the estimated equation.
   - The results showed that production depends on exports, as most of the crude phosphate production is exported globally.
   - The overall explanatory power of the model was appropriate by testing the determinant coefficient and the overall significance of the model was accepted according to F-test.
   - The statistical significance according to the T test was acceptable with the exception of the labor parameter, and the results revealed that there was no problem of Auto correlation.
   - The results showed that there are decreasing returns to scale in the phosphate industry, where the value of production elasticity is less than 1, where the value of returns to scale is about 0.31, it should be noted that the most of the returns come from exports, because the work component was restructured in the company, so that the number of workers decreased significantly since 1990, it was decreased from 5135 workers to 2570 in 2018.
   - The company can benefit from economies of scale if it makes good use of the available elements of production, and expanded in the extraction and processing of phosphate in order to raise the efficiency of scale and thus improve competitiveness.

     $LogQ = LogA + B1LogL + B2LogK + B3Logx + (ui - vi).$

     $LogQ = 6.38 + 0.0001LogL + 0.0007LogK + 0.30Logx + e.$

     t 9.23 1.66 2.15 3.82

     $R^2 = 0.52$, D.W = 2.06, F = 6.27, P-Value = 0.004

4. Measuring the allocative efficiency: By detecting a long-term relationship between net profit and its determinants, allocative efficiency will be measured based on a co-integration approach and error correction, assuming thatnet profit as a dependent variable is influenced by the volume of crude phosphate production and total costs.In addition to assuming that the net profit as a dependent variable is affected by the value of crude phosphate exports and total costs. Because of the linear interdependence between production volume and the value of exports, it was estimated separately.

- Test the stability of the time series of the study variables: net profit, production volume, export value, total costs. It is clear from the results shown in Table 1 that the values of MacKinnon (1996) calculated for the ADF test for all study variables that there is a fixed limit and a and a general trend less than the critical value at a level less than 0.01, and thus the null hypothesis was accepted, which indicates that the time series suffers from the problem of a unit root, as it is unstable at the level.The series must be converted to the first differences to make sure that the series are stable and static at the first differences where all ADF test values are greater than their critical value, and that all probability values are less than 0.01.Therefore, we reject the null hypothesis and accept the alternative hypothesis that there is no unit root in the series.

- The results of the optimal lag length test: The results of the unrestricted directional gradient vector test, based on the most commonly used Akaike information criterion (AIC) that the length of the appropriate lag time are two periods between the net profit and its determinants.

- Johnson co-integration test:The test is used to discover long-term relationships. The results of the Johansen test for the co-integration of the variables of the study represented by the first equation: net profit, as a dependent variable, affected by the volume of production and total costs.The existence of a single vector for the integration of these variables at the level of 0.05, in contrast, the test results according to the second equation: net profit is a dependent variable and is affected by the value of exports and total costs indicated that there are two vectors for the co-integration between those variables at the level of 0.05. Therefore, the null hypothesis that there is no co-integration at 0.01 level will be rejected. This indicates a long-term relationship in the model, where the statistical values of both the trace test and the maximum-Eigen test were greater than the critical value in both equations, andthe probability is less than 0.05. Based on these results, the short- and long-term relationship will be estimated using the error correction vector (VECM).

Table 1: Results Augmented Dickey -Fuller Statistical test (ADF)

Variables	% level		First differences (1%)		Result
	Intercept	Intercept and Trend	Intercept	Intercept and Trend	
Profit	-2.99	-3.02	-6.02	-5.91	I(1)
	(0.04)	(0.14)	(0.000)	(0.0003)	stationary
Quantity	-1.62	3.73	-5.3	-5.31	I(1)
	(0.45)	(0.04)	(0.0002)	(0.0011)	stationary
Cost	-0.29	-2.5	-7.54	-7.46	I(1)
	(0.91)	(0.33)	(0.000)	(0.000)	stationary
Export	-1.68	-2.63	-6.43	-6.28	I (1)
	(0.42)	(0.27)	(0.0000)	(0.0001)	stationary
critical value	-3.68	-4.32	-3.71	-4.36	

Source: authors' own elaboration and the numbers in parentheses represent the probability.

Table 2: Results of co-integration between net profit, production volume and total costs of JPMC

Hypothesis	Eigen	Trace test			Max Eigen		
		p-value	C.V (5%)	t-stat	p-value	C.V (5%)	t-stat
Non*	0.573	0.02	29.8	33.40	0.03	21.13	22.14
At most 1	0.347	0.19	15.5	11.26	0.15	14.26	11.08
At most 2	0.006	0.67	3.84	0.18	0.67	3.84	0.18

Source: authors' own elaboration. * Null hypothesis is rejected at 5%.

Table 3: Results of the test of the co-integration between the net profit, the value of exports and the total costs of JPMC

Hypothesis	Eigen	Trace test			Max Eigen		
		p-value	C.V (5%)	t-stat	p-value	C.V (5%)	t-stat
Non*	0.64	0.0001	29.8	48.76	0.006	21.13	26.97
At most 1*	0.567	0.005	15.5	21.79	0.003	14.26	21.79
At most 2	0.002	0.989	3.84	0.0002	0.99	3.84	0.0002

Source: authors' own elaboration. Two vectors for co-integration at 5%.

- Results Error Correction Vector Model estimation:In order to determine the direction of the causal relationship in the short and long term, and to estimate the speed of reaching the long-term equilibrium, the error correction model is tested. The appendix (3) shows the results of estimating the error correction vector according to the first equation, where the net profit is affected by the volume of production and costs. The coefficient of error correction was (0.145). Contrary to the expected, the value was positive in addition to the absence of statistical significance, which indicates the absence of a long-term equilibrium relationship from production quantities and costs to profits. The appendix (4) also shows the results of estimating the error correction vector according to the second equation, where the net profit is affected by the value of exports and costs. The error correction coefficient was (-0.325), which means that 32.5% of the imbalance is treated in the subsequent period after any shocks in the independent variables and affects the dependent variable. So, the company's net profit variable needs 32.5% each year to return to equilibrium. It is interesting to note that the correction coefficient lacks statistical significance at a level less than 5%, indicating that the long-term equilibrium relationship that moves from the value of exports and costs to net profit is not significant, which means a weak long-term relationship between net profit and its determinants. We can attribute the presence of the positive signal in the first equation, and the lack of the second equation of statistical significance to several reasons, the most important of which are the large fluctuations in the company's profits over (1990-

2018), the company's reliance on export contracts, the loss of the company for important export markets, in addition to severity competition between global prices of compound fertilizers that based on phosphate products and nitrogen fertilizers which based on natural gas and oil. It should be noted that the JPMC has recorded losses in various years during the studied period, including 1993, 1999, 2016, and 2017. In contrast, during 2008, the company recorded profits of 238.6 million JDs. The company has many projects and future plans to increase production, maximize profits and improve competitiveness.

The direction of the causal relationship of the variables in the short term:

Based on the first equation where the net profit is affected by the volume of production and costs, and based on the Granger causality test (F test) and Chi Square-basedWald's test, the short-term relationship between the variables was as follows:

- There is a short-term relationship between profits and the volume of production in one trend. The increase in profits stimulates the company to produce and extract more phosphate in a number of undiscovered areas. In the contrast, the increase in the volume of production does not cause profits because the profits depend on demand, especially export sales and global demand.

- There is no short-term causal relationship between costs and profits in both trends according to the F-test. This result contradicts the economic theory, but there is a statistically significant relationship between both profits and production in influencing the costs according to Wald's test. Increased profits motivate the company to purchase high-tech exploration and extraction machinery and equipment, thereby reducing costs.

- The existence of a short-term causal relationship in one trend between the volume of production and the costs, since the lower costs, causes an increase in the volume of production, and according to F-test, the high volume of the production does not cause a decrease in costs. This result is consistent with our results on the declining returns to scale in the phosphate industry. Based on the second equation, where net profit is affected by the value of exports and costs, and based on the Granger causality test, the results indicated that there is no short-term or long-term relationship between net profit and its determinants.

Consequently, analyzing the time series according to the co-integration approach in order to measure the allocative efficiency, it was clear that there is no long-term causal relationship between the net profit and its determinants of the JPMC, which means that the company has low allocative efficiency as a measure of competitiveness

4. Conclusion

JPMC has many forms of competitive strength, represented in the production capacity of mines, allied and affiliated companies and the volume of its exports to international markets. However, the company suffers from some challenges such as fluctuating prices of selling raw fertilizer and phosphate products, high extraction costs associated with high landfill costs, high international shipping costs as well as international competition in the company's exports. The Porter Index showed in the revealed comparative advantage that the Phosphate Mines Company has a comparative advantage in the production of raw phosphate, exceeding one, as the index recorded 137 in 2018. The results of the competitiveness measurement showed that the crude extractive phosphate industry has a moderate technical efficiency of 73%, and presence decreasing returns to scale in the phosphate industry. The results of the measurement of allocative efficiency using the co-integration approach and the error correction model indicated that there is no long-term equilibrium relationship moving from production quantities and costs to profits. The results showed that there is no statistically significant long-term relationship between the net profit and its determinants, due to various reasons, including fluctuations in the company's profits, global competition, and the company's dependence on export contracts, which contributes to the weakness of the allocative efficiency enjoyed by the company as a measure of competitiveness. The causality test showed that there is a short-term relationship between production volume and costs in one trend, as well as between profits and production volume.

In the light of the findings of this study, it is recommended to face the challenges facing the company and focus on increasing production capacity and the use of modern machinery and equipment in extraction, mining, and crushing. As well as spending on promotional activities to market the products of Jordan Phosphate Mines Company in international markets, in addition to participating in international exhibitions.

The study recommends the necessity of rationalizing the current and capital expenditures of the company and controlling its disbursements, raising the technical and scale efficiency of the company through optimizing the available production elements and improving the company's performance by opening new markets to the company's exports, in addition to expanding phosphate mining and exploration to cover new target areas in order to increase productivity and maximize profitability.

Finally, the study recommends strengthening the company's hedge fund with financial reserves to meet global developments, uncertainty conditions and unexpected losses.

Appendix 1. *Granger causality test results between net profit, total costs and production volume of Jordan Phosphate Mines Company.*

Null hypothesis	F	Prop.	Results
Production volume doesn?t cause profits	0.147	0.86	Supported
Profits do not cause production	5.93	0.0087	Rejected
Total costs do not cause profits	0.8	0.83	Supported
Profits do not cause total costs	1.1	0.35	Supported
Costs do not cause production volume	8.07	0.002	Rejected
Production volume does not cause costs	0.93	0.4	Supported

Appendix 2. *Short-term causality according to Wald test*

Dependent	Decision	Null- Hypothesis	Chi-Square	Probability	Excluded
D(profit)	Accept	C(4)=c(5)=0	2.53	0.28	D(Quantity)
	Accept	C(6)=c(7)=0	3.03	0.21	D(COST)
D(quantity)	Accept	C(10)=c(11)=0	1.22	0.54	D(Profit)
	Accept	C(14)=c(15)=0	0.67	71	D(COST)
D(cost)	Reject	C(18)=c(19)=0	8.27	0.01	D(Profit)
	Reject	C(20)=c(21)=0	8.14	0.01	D(Quantity)

Source: authors' own elaboration based on e-views results.

Appendix 3. *Co-integration Equation between Profit and quantity, cost.*

Co-integrating Eq:	CointEq1		
PROFIT(-1)	1		
QUANTITY(-1)	0.080283		
	-0.02497		
	[3.21504]		
COST(-1)	-0.658383		
	-0.16097		
	[-4.09021]		
C	-280.9747		
Error Correction:	D(PROFIT)	D(COST)	D(QUANTITY)
CointEq1	0.145038	0.549194	-4.171196
	-0.21616	-0.24169	-2.51237
	[0.67097]	[2.27228]	[-1.66026]
D(PROFIT(-1))	-0.05354	-0.821832	-2.484471
	-0.38751	-0.43328	-4.50388
	[-0.13816]	[-1.89678]	[-0.55163]
D(PROFIT(-2))	-0.595467	-1.245379	3.420935
	-0.41519	-0.46422	-4.82555
	[-1.43421]	[-2.68272]	[0.70892]

Continue table in Appendix 3

D(QUANTITY(-1))	-0.009686	-0.032609	0.256072
	-0.01925	-0.02152	-0.22371
	[-0.50323]	[-1.51520]	[1.14466]
D(QUANTITY(-2))	-0.024071	-0.041989	-0.371883
	-0.01649	-0.01844	-0.19167
	[-1.45960]	[-2.27717]	[-1.94021]
D(COST(-1))	-0.4619	-0.008659	-2.573768
	-0.27179	-0.30389	-3.15892
	[-1.69946]	[-0.02849]	[-0.81476]
D(COST(-2))	-0.017998	0.902218	-1.046856
	-0.29341	-0.32806	-3.41017
	[-0.06134]	[2.75015]	[-0.30698]
C	9.245993	0.286902	165.8472
	-14.5141	-16.2283	-168.692
	[0.63703]	[0.01768]	[0.98314]
Error Correction:	D(PROFIT)	D(COST)	D(QUANTITY)
R-squared	0.475073	0.531263	0.529365
Adj. R-squared	0.270935	0.348976	0.34634
Sum sq. resids	58180.29	72734.7	7859270
S.E. equation	56.85278	63.56742	660.7769
F-statistic	2.327216	2.914438	2.892311
Log likelihood	-137.1641	-140.0666	-200.9408
Akaike AIC	11.16647	11.38974	16.07237
Schwarz SC	11.55357	11.77685	16.45948
Mean dependent	1.207692	16.65769	105.9231
S.D. dependent	66.58382	78.78367	817.295
Determinant resid covariance (dof adj.)	1.25E+12		
Determinant resid covariance	4.14E+11		
Log likelihood	-458.417		
Akaike information criterion	37.33977		
Schwarz criterion	38.64625		

Appendix 4. *Co-integration Equation between Profit and Export, cost.*

Co-integrating Eq :	CointEq1		
PROFIT(-1)	1		
Export (-1)	-1.562126		
	(0.18101)		
	[-8.62998]		
COST(-1)	0.745785		
	(0.08741)		
	8.53224		

Continue table in Appendix 4

C	7.213026		
Error Correction:	D(PROFIT)	D(COST)	D(export)
CointEq1	0.439396	-0.759038	-0.325961
	-0.53294	-0.56414	-0.43986
	[0.82447]	[-1.34548]	[-0.74105]
D(PROFIT(-1))	-0.553529	0.432789	0.396314
	(0.71290)	(0.75464)	(0.58840)
	[-0.77645]	[0.57351]	[0.67355]
D(PROFIT(-2))	-0.016331	0.759763	0.356945
	(0.79733)	(0.84400)	(0.65808)
	[-0.02048]	[0.90019]	[0.54241]
D(COST) (-1))	-0.216702	0.084871	-0.316696
	(0.36833)	(0.38989)	(0.30400)
	[-0.58834]	[0.21768]	[-1.04177]
D(COST) (-2))	0.024851	0.410273	-0.150246
	(0.29916)	(0.31667)	(0.24691)
	[0.08307]	[1.29557]	[-0.60850]
D(EXPORT) (-1))	0.304948	-0.938009	-0.478195
	(0.71855)	(0.76061)	(0.59306)
	[0.42440]	[-1.23323]	[-0.80632]
D(Export) (-2))	-0.53254	-1.228862	-0.763207
	(0.66807)	(0.70718)	(0.55139)
	[-0.79714]	[-1.73770]	[-1.38415]
C	7.870440	28.22875	20.71450
	(18.8756)	(19.9806)	(15.5790)
	[0.41696]	[1.41281]	[1.32964]
Error Correction:	D(PROFIT)	D(COST)	D(EXPORT)
R-squared	0.368598	0.368658	0.462642
Adj. R-squared	0.123053	0.123135	0.253669
Sum sq. resids	87430.26	97966.48	59558.17
S.E. equation	69.69388	73.77386	57.52206
F-statistic	1.501144	1.501525	2.213886
Log likelihood	-142.4589	-143.9381	-137.4684
Akaike AIC	11.57376	11.68755	11.18987
Schwarz SC	11.96087	12.07465	11.57698
Mean dependent	4.326923	16.65769	1.207692
S.D. dependent	74.42316	78.78367	66.58382
Determinant resid covariance (dof adj.)	4.28E+09		
Determinant resid covariance	1.42E+09		
Log likelihood	-384.6532		
Akaike information criterion	31.66563		
Schwarz criterion	32.97211		

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Accepted: May 05, 2020

A novel artificial variable-free simplex algorithm for solving neutrosophic linear programming problems

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Abstract. In this article, we propose a novel fuzzy artificial variable-free version of simplex algorithm (FAVFSA) for solving fuzzy linear programming problems. Also, we develop this algorithm as a simplified neutrosophic artificial variable-free version of simplex algorithm (NAVFSA) for solving neutrosophic linear programming problems. A computational comparison between the proposed algorithm and the traditional two-phase algorithm is introduced. The computational results show that the size and the running time of the new algorithm is less than the size and the running time of the traditional two-phase algorithm. We also use numerical examples to compare between the fuzzy approach and the neutrosophic approach. The results show that the neutrosophic approach is more accurate than the fuzzy approach.

Keywords: fuzzy number, neutrosophic number, artificial variable-free-version of simplex method, two-phase simplex method.

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1. Introduction

One of the serious issues with linear programming is that modeling needs accurate and well defined data. In fact, this is a task that is almost impossible to accomplish in many cases, due to risk or uncertainty in some data. The classic LP problem can be defined using vague numbers. Fuzzy linear programming explains the mathematical model in more realistic way. Fuzzy set does not represent vague and imprecise information efficiently, because it considers only the truthiness function. The concept of intuitionistic fuzzy set is introduced to handle vague and imprecise information, by considering both the truth and falsity function, also intuitionistic fuzzy set does not simulate human decision-making process. Because the proper decision is fundamentally a problem of arranging and explicate facts the concept of neutrosophic set theory was presented by Smarandache to handle vague, imprecise and inconsistent information. Neutrosophic Set (NS) [20] is a generalization of the fuzzy set [29] and intuitionistic fuzzy set [3]. It can deal with uncertain, indeterminate, and incongruous information where the indeterminacy is quantified explicitly and truth membership, indeterminacy membership and falsity membership are completely independent. The simplex method is the common tool for solving linear programs. It is an iterative method that was developed by Dantzig [10, 14, 15]. It is known that the application of the simplex algorithm requires at least one basic feasible solution. The common techniques that are used for determining an initial feasible basis are the Two-Phase and Big-M methods. In 1997, Arsham [4] presented the simplex method without using artificial variables. In the first step, the basic feasible variable set (BVS) is determined to be the empty set. Then, the non-basic variable is chosen to be the basic variable one by one until the BVS is full. After the problem has complete BVS, the simplex method is performed. However, this method has a mistake as shown by Enge and Huhn [21] in 1998. In 2000, Pan [23] proposed the simplex method by avoiding artificial variables. The algorithm starts when the initial basis gives primal and dual infeasible solutions by adjusting negative reduced costs to a single positive value. Then, the dual solution is feasible and the dual simplex method is performed. After the optimal solution is found in this step, the original reduced costs are restored and the simplex method is performed.

Later, in 2006, Corley et.al [13] constructed the relaxed problem for improving the simplex method without using artificial variables. The cosine criterion is used for choosing the suitable constraints to construct the relaxed problem. Then, it is solved by the simplex method. After the optimal solution of the relaxed problem is found, the relaxed constraints will be restored into the current tableau, and the dual simplex method will be performed until the optimal solution is found. However, this algorithm can solve only the problem which has all positive coefficients. Boonperm and Sinapiromsaran, 2014 [12] proposed the non-acute constraint relaxation technique that improves the simplex method without using artificial variables; it can reduce the start-up time to solve the

initial relaxation problem. The algorithm starts by relaxing the non-acute constraints which can guarantee that the relaxed problem is always feasible. So, the relaxed problem can be solved without using artificial variables. After the optimal solution of the relaxed problem is found, the relaxed constraints are restored, and the dual simplex method is used to solve it. Since the relaxed problem can reduce variables or constraints, the computation can be reduced.

For more details about the linear programming, the reader can refer to [5, 8, 9, 15, 24, 25, 26]. On the other hand, for more details about the fuzzy linear programming, the reader is referred to [6, 7]. Finally, for more details about the neutrosophic linear programming, the reader may refer to [2, 16, 19].

In this paper, instead of developing a new method for solving general linear programming problems, we are going to present a novel version of simplex algorithm (fuzzy and neutrosophic artificial variable free simplex algorithm) which has several advantages, for instance, it could start with any feasible or infeasible basis of linear programming problem. It also deals with the artificial variables in an implicit manner therefore, as long as a variable is infeasible its corresponding slack variable would be invisible but implicitly after leaving the basis it would be replaced by the corresponding invisible slack variable.

Also, we compare between the fuzzy approach and the neutrosophic approach by using the same rank function to make the comparison fair.

The remaining parts of this work are organized as follows: In sec. 2, the fundamental concepts of fuzzy and neutrosophic sets have been presented, and a new technique which converts the fuzzy representation to the neutrosophic representation has been proposed. Akanksha Singh et al.'s modifications and a new neutrosophic artificial variable-free simplex algorithm (NAVFSA) are proposed in Section 3. In Section 4, a numerical examples that show the importance of the proposed modification for primal neutrosophic simplex method and they show the superiority of the proposed algorithm (NAVFSA) on the primal neutrosophic simplex method has been shown. Finally, we introduce conclusions and the future work in Section 5.

2. Preliminaries

In this section, three subsections have been introduced. First one is a representation of the fuzzy numbers. Second, the representation of the neutrosophic numbers is shown. Finally, we show how to convert the fuzzy numbers to neutrosophic numbers.

2.1 Fuzzy representation

We review the fundamental notions of fuzzy set theory, initiated by Bellman and Zadeh [11].

Definition 2.1.1. *A convex fuzzy set $\tilde{A}$ on $\mathbb{R}$ is a fuzzy number if the following conditions hold:*

1. Its membership function is piecewise continuous.

2. There exist three intervals $[a, b]$, $[b, c]$, $[c, d]$ such that $\mu_{\tilde{a}}$ is increasing on $[a, b]$, equal to 1 on $[b, c]$, decreasing on $[c, d]$ and equal to 0 elsewhere.

Definition 2.1.2. *Let $\tilde{a} = (a^L, a^U, \alpha, \beta)$ denote the trapezoidal fuzzy number, where $(a^L - \alpha, a^U + \beta)$ is the support of a and $[a^L, a^U]$ its core.*

Remark 1. We denote the set of all trapezoidal fuzzy numbers by $F(\mathbb{R})$ as shown as in figure 1.

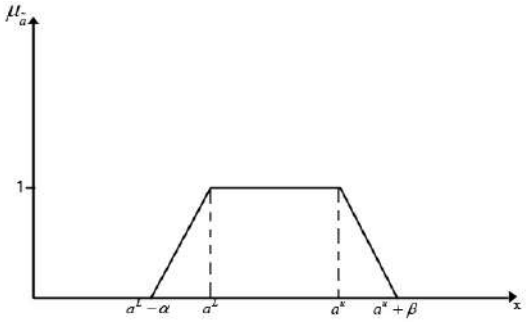


Figure 1: Truth membership function of trapezoidal fuzzy numbers $\tilde{a}$

Definition 2.1.3. *Let $\tilde{a} = (a^L, a^U, \alpha, \beta)$ and $\tilde{b} = (b^L, b^U, \gamma, \theta)$ be two trapezoidal fuzzy numbers, the arithmetic operation on the trapezoidal fuzzy number are defined as:*

$$\begin{aligned} x\tilde{a} &= (xa^L, xa^U, x\alpha, x\beta); x > 0, x \in \mathbb{R}, \\ x\tilde{a} &= (xa^U, xa^L, -x\beta, -x\alpha); x < 0, x \in \mathbb{R}, \\ \tilde{a} + \tilde{b} &= (a^L, a^U, \alpha, \beta) + (b^L, b^U, \gamma, \theta) \\ &= [a^L + b^L, a^U + b^U, \alpha + \gamma, \beta + \theta]. \end{aligned}$$

2.2 Neutrosophic representation

In this subsection, some basic definitions in the neutrosophic set theory are introduced:

Definition 2.2.1 ([1])**.** *the trapezoidal neutrosophic number $\tilde{A}$ is a neutrosophic set in $\mathbb{R}$ with the following truth (T), indeterminacy (I) and falsity (F) membership functions as shown in figure 2:*

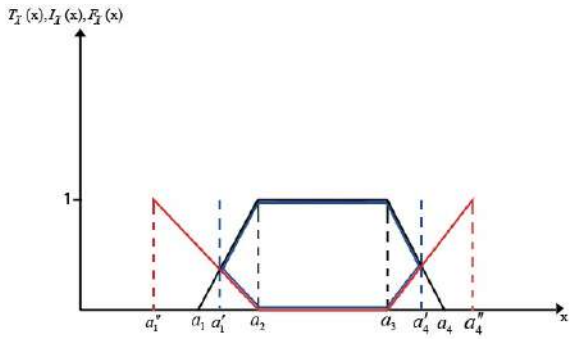


Figure 2: Truth, indeterminacy and falsity membership functions of trapezoidal neutrosophic numbers $\tilde{A}$

$$T_{\tilde{A}}(x) = \begin{cases} \dfrac{\alpha_{\tilde{A}}(x-a_1)}{a_2-a_1}, & a_1 \le x \le a_2 \\ \alpha_{\tilde{A}}, & a_2 \le x \le a_3 \\ \alpha_{\tilde{A}}(\dfrac{x-a_3}{a_4-a_3}), & a_3 \le x \le a_4 \\ 0, & \text{otherwise} \end{cases}$$

$$I_{\tilde{A}}(x) = \begin{cases} \dfrac{(a_2-x+\theta_{\tilde{A}}(x-a_1'))}{a_2-a_1'}, & a_1' \le x \le a_2 \\ \theta_{\tilde{A}}, & a_2 \le x \le a_3 \\ \dfrac{(x-a_3+\theta_{\tilde{A}}(a_4'-x))}{a_4'-a_3}, & a_3 \le x \le a_4' \\ 1, & \text{otherwise} \end{cases}$$

$$F_{\tilde{A}}(x) = \begin{cases} \dfrac{(a_2-x+\beta_{\tilde{A}}(x-a_1''))}{a_2-a_1''}, & a_1'' \le x \le a_2 \\ \beta_{\tilde{A}}, & a_2 \le x \le a_3 \\ \dfrac{(x-a_3+\beta_{\tilde{A}}(a_4''-x))}{a_4''-a_3}, & a_3 \le x \le a_4'' \\ 1, & \text{otherwise} \end{cases}$$

where $\alpha_{\tilde{A}}$, $\theta_{\tilde{A}}$ and $\beta_{\tilde{A}}$ represent the maximum degree of truthiness, minimum degree of indeterminacy and minimum degree of falsity, respectively, $\alpha_{\tilde{A}}$, $\theta_{\tilde{A}}$ and $\beta_{\tilde{A}} \in [0, 1]$.

Definition 2.2.2 ([1]). *The mathematical operations on two trapezoidal neutrosophic numbers.* $\tilde{A} =< a_1, a_2, a_3, a_4; \alpha_{\tilde{A}}, \theta_{\tilde{A}}, \beta_{\tilde{A}} >$ *and* $\tilde{B} =< b_1, b_2, b_3, b_4; \alpha_{\tilde{B}}, \theta_{\tilde{B}}, \beta_{\tilde{B}} >$ *are as follows:*

$$\tilde{A} + \tilde{B} =< (a_1+b_1, a_2+b_2, a_3+b_3, a_4+b_4); \alpha_{\tilde{A}} \wedge \alpha_{\tilde{A}}, \theta_{\tilde{A}} \vee \theta_{\tilde{A}}, \beta_{\tilde{A}} \vee \beta_{\tilde{A}} >,$$
$$\tilde{A} - \tilde{B} =< (a_1-b_4, a_2-b_3, a_3-b_2, a_4-b_1); \alpha_{\tilde{A}} \wedge \alpha_{\tilde{A}}, \theta_{\tilde{A}} \vee \theta_{\tilde{A}}, \beta_{\tilde{A}} \vee \beta_{\tilde{A}} >,$$

$$\tilde{A}^{-1} =< (\frac{1}{a_4}, \frac{1}{a_3}, \frac{1}{a_2}, \frac{1}{a_1}); \alpha_{\tilde{A}}, \theta_{\tilde{A}}, \beta_{\tilde{A}} > where\ \tilde{A} \neq 0),$$

$$\lambda\tilde{A} = \begin{cases} < \lambda a_1, \lambda a_2, \lambda a_3, \lambda a_4; \alpha_{\tilde{A}}, \theta_{\tilde{A}}, \beta_{\tilde{A}} >, \lambda > 0 \\ < \lambda a_4, \lambda a_3, \lambda a_2, \lambda a_1; \alpha_{\tilde{A}}, \theta_{\tilde{A}}, \beta_{\tilde{A}} >, \lambda < 0 \end{cases}$$

$$\tilde{A}\tilde{B} \begin{cases} < (a_1b_1, a_2b_2, a_3b_3, a_4b_4), \alpha_{\tilde{A}} \wedge \alpha_{\tilde{A}}, \theta_{\tilde{A}} \vee \theta_{\tilde{A}}, \beta_{\tilde{A}} \vee \beta_{\tilde{A}} > \\ if\,(a_4 > 0, b_4 > 0) \\ < (a_1b_4, a_2b_3, a_3b_2, a_4b_1), \alpha_{\tilde{A}} \wedge \alpha_{\tilde{A}}, \theta_{\tilde{A}} \vee \theta_{\tilde{A}}, \beta_{\tilde{A}} \vee \beta_{\tilde{A}} > \\ if\,(a_4 < 0, b_4 > 0) \\ < (a_4b_4, a_3b_3, a_2b_2, a_1b_1), \alpha_{\tilde{A}} \wedge \alpha_{\tilde{A}}, \theta_{\tilde{A}} \vee \theta_{\tilde{A}}, \beta_{\tilde{A}} \vee \beta_{\tilde{A}} > \\ if\,(a_4 < 0, b_4 < 0) \end{cases}$$

$$\frac{\tilde{A}}{\tilde{B}} \begin{cases} < (\frac{a_1}{b_4}, \frac{a_2}{b_3}, \frac{a_3}{b_2}, \frac{a_4}{b_1}), \alpha_{\tilde{A}} \wedge \alpha_{\tilde{A}}, \theta_{\tilde{A}} \vee \theta_{\tilde{A}}, \beta_A \vee \beta_{\tilde{A}} > \\ if\,(a_4 > 0, b_4 > 0) \\ < (\frac{a_4}{b_4}, \frac{a_3}{b_3}, \frac{a_2}{b_2}, \frac{a_1}{b_1}), \alpha_{\tilde{A}} \wedge \alpha_{\tilde{A}}, \theta_{\tilde{A}} \vee \theta_{\tilde{A}}, \beta_{\tilde{A}} \vee \beta_{\tilde{A}} > \\ if\,(a_4 < 0, b_4 > 0) \\ < (\frac{a_4}{b_1}, \frac{a_3}{b_2}, \frac{a_2}{b_3}, \frac{a_1}{b_4}), \alpha_{\tilde{A}} \wedge \alpha_{\tilde{A}}, \theta_{\tilde{A}} \vee \theta_{\tilde{A}}, \beta_{\tilde{A}} \vee \beta_{\tilde{A}} > \\ if\,(a_4 < 0, b_4 < 0) \end{cases}$$

2.3 Transfer from fuzzy representation to neutrosophic representation

The goal of this section is to explain how to convert fuzzy numbers representation into neutrosophic numbers representation. This transformation is used for simplicity and to make the comparison fair. There are many types of techniques to transfer from fuzzy to neutrosophic representation such as, rank functions and $\propto$-cut technique.

Definition 2.3.1. *Ranking function is a viable approach for ordering fuzzy numbers and neutrosophic numbers. It is known that there are many rank functions for ordering the fuzzy numbers and neutrosophic numbers.*

In this subsection, we explain how to apply technique to convert from fuzzy number to neutrosophic number:

From Figure 1 and Figure 2 we can illustrate the following relations between the two representations

$$a_1 = a_2 - \alpha, a_2 = a^L, a_3 = a^U \text{ and } a_4 = a_3 + \beta. \tag{1}$$

Assuming that the rank function is used for ordering the fuzzy numbers as follows:

$$R(\tilde{a}) = \frac{a^L + a^U}{2} + \frac{\beta - \alpha}{4}. \tag{2}$$

From (1) we can express the used rank function for ordering the neutrosophic numbers as follows:

$$R(\tilde{a}) = \frac{1}{4}\sum_{i=1}^{4} \tilde{a}_i + (T_{\tilde{a}} - I_{\tilde{a}} - F_{\tilde{a}}). \tag{3}$$

From (1), we can convert fuzzy numbers representation into neutrosophic numbers representation. On the other hand, from (2) and (3), we can use the same function for both fuzzy numbers and neutrosophic numbers to obtain a fair comparison between them.

Assuming that $T_{\tilde{A}} = 1, I_{\tilde{A}} = 0, F_{\tilde{A}} = 0$, then the TrNN $\tilde{a} =< a_1, a_2, a_3, a_4; T_{\tilde{A}}, I_{\tilde{A}}, F_{\tilde{A}} >$ will be transformed into a trapezoidal neutrosophic number $\tilde{a} = < a_1, a_2, a_3, a_4; 1, 0, 0 >$ and hence, in this case,

1. The expression

$$R(\tilde{a}) = \frac{1}{4}\sum_{j=1}^{4} a_i + (T_{\tilde{a}} - I_{\tilde{a}} - F_{\tilde{a}})$$

is equivalent to the expression $R(\tilde{a}) = \frac{1}{4}\sum_{j=1}^{4} a_i + 1$.

Furthermore, it is well known that if $a_1 = a_2 = a_3 = a_4$ then the trapezoidal neutrosophic number $\tilde{A} =< a_1, a_2, a_3, a_4; 1, 0, 0 >$ will be transformed into a real number $A = (a, a, a, a; 1, 0, 0)$ and hence, in this case. Furthermore, it is well known that if $a_1 = a_2 = a_3 = a_4$ then the trapezoidal neutrosophic number $\tilde{A} =< a_1, a_2, a_3, a_4; 1, 0, 0 >$ will be transformed into a real number $A = (a, a, a, a; 1, 0, 0)$ and hence, in this case.

2. The expression

$$R(\tilde{a}) = \frac{1}{4}\sum_{j=1}^{4} a_i + (T_{\tilde{a}} - I_{\tilde{a}} - F_{\tilde{a}})$$

is equivalent to the expression $R(A) = a + 1 \neq a$.

The following table represents the fuzzy rank function, the corresponding neutrosophic rank function and the corresponding real rank function.

Table 1: fuzzy rank function and this corresponding neutrosophic rank function

No	Fuzzy Rank Function	Corresponding Neutrosophic Rank Function	Corresponding Real Rank Function
1	$R(\tilde{a}) = (\frac{a^l + a^u}{2} + \frac{\beta - \alpha}{4})$	$R(\tilde{a}) = \frac{1}{4}\sum_{j=1}^{4} a_i + (T_{\tilde{a}} - I_{\tilde{a}} - F_{\tilde{a}})$	$R(a) = a + 1$

3. Algorithms

In this section; we first present Akanksha Singh *et al.*'s modifications [28] and the proposed modification about the mathematical incorrect assumptions, considered by Abdel-Basset *et al.* [1] in their proposed method to convert from

neutrosophic numbers into real numbers. Second, we propose a new fuzzy artificial variable- free simplex algorithm. Finally, we develop this algorithm in order to solve linear programming with neutrosophic numbers (neutrosophic artificial variable-free simplex algorithm).

3.1 Akanksha Singh *et al.*'s modifications [28]

The following table presents Akanksha Singh et al.'s modifications to convert from neutrosophic number to deterministic number.

Table 2: Akanksha Singh et al.'s modifications

no	NLPP-(Type)	NLPP- (Form)	Exact Crisp LPP
1	The coefficients of the objective function are represented by trapezoidal neutrosophic numbers	$\max \setminus \min[\sum_{j=1}^{n} = \tilde{c}_j x_j]$ $s.t.$ $\sum_{j=1}^{n} a_{ij}x_j \leq, \geq, = b_j,$ $i = 1, 2, \ldots\ldots, m;$ $x_j \geq 0, j = 1, 2, \ldots\ldots, n$	$\max \setminus \min[\sum_{j=1}^{n} R(\tilde{c}_j x_j) - \sum_{j=1}^{n} T_{\tilde{c}_j} x_j$ $+ \sum_{j=1}^{n} I_{\tilde{c}_j} x_j + \sum_{j=1}^{n} F_{\tilde{c}_j} x_j + \min_{1\leq j\leq n}\{T_{\tilde{c}_j} x_j\} -$ $\max_{1\leq j\leq n}\{I_{c_j} x_j\} - \max_{1\leq j\leq n}\{F_{\tilde{c}_j} x_j\}]$ $s.t.$ $\sum_{j=1}^{n} a_{ij}x_j \leq, \geq, = b_j, i = 1, 2, \ldots\ldots, m;$ $x_j \geq 0, j = 1, 2, \ldots\ldots, n.$
2	The coefficients of constraints variables and right hand side are represented by trapezoidal neutrosophic numbers	$\max \setminus \min[\sum_{j=1}^{n} = \tilde{c}_j x_j]$ $s.t.$ $\sum_{j=1}^{n} \tilde{a}_{ij}x_j \leq, \geq, = \tilde{b}_j,$ $i = 1, 2, \ldots\ldots, m;$ $x_j \geq 0, j = 1, 2, \ldots\ldots, n$	$\max \setminus \min \sum_{j=1}^{n} c_j x_j$ $s.t.[\sum_{j=1}^{n} R(\tilde{a}_{ij}x_j) - \sum_{j=1}^{n} T_{\tilde{a}_{ij}} x_j + \sum_{j=1}^{n} I_{\tilde{a}_{ij}} x_j$ $+ \sum_{j=1}^{n} F_{\tilde{a}_{ij}} x_j + \min_{1\leq j\leq n}\{T_{\tilde{a}_{ij}} x_j\} - \max_{1\leq j\leq n}$ $\{I_{\tilde{a}_{ij}} x_j\} - \max_{1\leq j\leq n}\{F_{\tilde{a}_{ij}} x_j\}] \leq, \geq, = R(\tilde{b}_i)$ $x_j \geq 0, j = 1, 2, \ldots\ldots, n.$
3	All parameters are represented by trapezoidal neutrosophic numbers, except variables are exemplified only by real values	$\max \setminus \min[\sum_{j=1}^{n} = \tilde{c}_j x_j]$ $s.t.$ $\sum_{j=1}^{n} \tilde{a}_{ij}x_j \leq, \geq, = \tilde{b}_j,$ $i = 1, 2, \ldots\ldots, m;$ $x_j \geq 0, j = 1, 2, \ldots\ldots, n$	$\max \setminus \min[\sum_{j=1}^{n} R(\tilde{c}_j x_j) - \sum_{j=1}^{n} T_{\tilde{c}_j} x_j +$ $\sum_{j=1}^{n} I_{\tilde{c}_j} x_j + \sum_{j=1}^{n} F_{\tilde{c}_j} x_j + \min_{1\leq j\leq n}\{T_{\tilde{c}_j} x_j\} -$ $\max_{1\leq j\leq n}\{I_{\tilde{c}_j} x_j\} - \max_{1\leq j\leq n}\{F_{\tilde{c}_j} x_j\}]$ $s.t.$ $(\sum_{j=1}^{n} R(\tilde{a})_{ij} x_j) \leq, \geq, = (R\tilde{b}_j),\ i = 1, 2, \ldots\ldots, m;$ $x_j \geq 0, j = 1, 2, \ldots\ldots, n.$
4	The coefficients of objective function and constraints variables are represented by real numbers and right hand side are represented by trapezoidal neutrosophic numbers	$\max \setminus \min[\sum_{j=1}^{n} = c_j x_j]$ $s.t.$ $\sum_{j=1}^{n} a_{ij}x_j \leq, \geq, = \tilde{b}_j,$ $i = 1, 2, \ldots\ldots, m;$ $x_j \geq 0, j = 1, 2, \ldots\ldots, n$	$\max \setminus \min \sum_{j=1}^{n} c_j x_j$ $s.t.$ $R[\sum_{j=1}^{n} (a_{ij}x_j) \leq, \geq, = R(\tilde{b}_j),$ $i = 1, 2, \ldots\ldots, m;$ $x_j \geq 0, j = 1, 2, \ldots\ldots, n.$

Remark 2.

- If $R(a) = a+1$ and the coefficients of the objective function & constraints variables are real, then the fuzzy linear programming problem is equivalent to the neutrosophic linear programming problem.

- $\sim$: *represents the presence* of neutrosophic numbers within the matrices or vectors.
- NLPP: neutrosophic linear programming problem.

3.2 A novel neutrosophic artificial variable-free simplex algorithm

Nayatullah et al [22] proposed a simple direct artificial variable free simplex algorithm (AVFSM) for solving the linear programming problems with real numbers. In this subsection, we propose a new algorithm which solves linear programming with neutrosophic numbers (Neutrosophic artificial variable-Free Simplex Algorithm). The proposed algorithm overcame traditional neutrosophic simplex algorithms.

Algorithm 1 Neutrosophic Artificial variable -Free Simplex Algorithm (NAVFSA)

Step 0: (Initialization)

Step 5: Converting fuzzy numbers into neutrosophic numbers (see section 2.3.1)
Apply Akanksha Singh et al.'s modifications according to (see section 3.1)

Step 1: Let $\tilde{L}$ be a maximal subset of $\tilde{B}$ such that $\tilde{L} = \{i : d_{i0} < 0, i \in B\}$. **if** $\tilde{L} = \varphi$ **then** $D(\tilde{B})$ is primal feasible. Exit.

Step 2: Denote the basic variables x_L by x_L^- and compute infeasibility objective vector $W(\tilde{B}) \in R^{\tilde{N}}$ such that $w_j = \sum_{i \in L} d_{ij}$, $j \in \tilde{N}$
Place w to the top of the reduced short table $D(\tilde{B})$.

Step 3: Let $\tilde{K} \subseteq \tilde{N}$ such that $\tilde{K} = j : w_j < 0, j \in \tilde{N}$. **if** $\tilde{K} = \varphi$ **then** $D(\tilde{B})$ is primal inconsistent. Exit.

Step 4: (Choice of entering variable)
Choose $k \in \tilde{K}$ such that $w_k \leq w_h$, $\forall h \in \tilde{K}$

Step 5: (Choice of leaving variable)

Step 5: Choose $r_1 \in \tilde{L}$ and $r_1 \in \tilde{B} \backslash \tilde{L}$ such that:

$$r_1 = \arg\min\{\{\frac{d_{i0}}{d_{ik}} \mid (d_{i0} \leq 0, d_{ik} < 0)\}, i \in \tilde{L}\}$$

$r_2 = \arg\max\{\{\frac{d_{i0}}{d_{ik}} \mid (d_{i0} \geq 0, d_{ik} > 0)\}, i \in \tilde{B} \backslash \tilde{L}\} Setr := \arg\min\{\frac{d_{r_1 0}}{d_{r_1 k}}, \frac{d_{r_2 0}}{d_{r_2 k}}\}$

Step 6: Make a pivot on (r, k) ($\Rightarrow$ Set $\tilde{B} := (\tilde{B} \cup \tilde{k}) \backslash r$, $\tilde{N} := (\tilde{N} \cup r) \backslash k$ and update $D(\tilde{B})$ along with the supplement $w(\tilde{B})$).

Step 7: If $r \in \tilde{L}$, set $\tilde{L} := \tilde{L} \backslash r$ and $w_r := w_r + 1$, substitute notation of $-x_r^-$ by x_r.

Step 8: Pivot-operation

Step 8: For $r \in B$, $k \in \tilde{N}$ and (r, k) being the position of the pivot element d_{rk} $(\neq 0)$ of D, then one can obtain an updated equivalent short table $D(\tilde{B}')$ with a new basis $\tilde{B}' := (\tilde{B} \cup k) \backslash r$ and the new non-basis $\tilde{N}' := (\tilde{N} \cup r) \backslash k$ by performing the following operations on $D(\tilde{B})$

$$d'_{rk} := \frac{1}{d_{rk}}$$

$$d'_{rj} := \frac{d_{rj}}{d_{rk}}, j \in \tilde{N} \backslash k$$

$$d'_{ik} := -\frac{d_{ik}}{d_{rk}}, i \in \tilde{N} \backslash r$$

$$d'_{ij} := d_{ij} - d_{rj} \times \frac{d_{ik}}{d_{rk}}, i \in \tilde{B} \backslash r, j \in \tilde{N} \backslash k$$

The above replacement is known as a pivot operation on (r, k).

Step 9: if $\tilde{L} = \varphi$ **then** $D(\tilde{B})$ is primal feasible. Exit.

Step 10: If phase 1 ends with an objective value equal to zero, it implies that all artificial variables have attained a value zero (means all infeasibilities have been removed) and our current basis is feasible to the original problem, then we return to the original objective and proceed with simplex phase 2. Otherwise, we conclude that the problem has no solution.

Remark 3. The proposed algorithm can be applied on the fuzzy artificial variable free simplex algorithm.

4. Numerical examples

In this section, we solve some well-known fuzzy and neutrosophic linear programming.

Problems presented in [27] with the traditional simplex method and the proposed method.

4.1 Example 1: case study

A farmer who raises chickens would like to determine the amounts of the available ingredients that would meet certain nutritional requirements. The available ingredients and the nutrients in the ingredients are summarized below (Table 3).

The average cost of each unit of ingredients corn, lime and alfalfa are close to 6, 5 and 3 dollars, respectively. The minimum daily requirement of nutrients protein, carbohydrates and vitamins are approximately 8, 5 and 4 units,

Table 3: The data of Example

Nutrient	Ingredient		
	Corn	Lime	Alfalfa
Protein	3	4	2
Carbohydrates	4	2	1
Vitamins	2	1	3

respectively. The aim is to find the optimal mix. This problem is evidently an uncertain optimization problem due to variations in costs and in minimum daily requirements. So the amount of each unit of ingredients will be uncertain. Hence, the problem is described as an ambiguous linear programming problem. Symmetrically fuzzy numbers are used in the form of Trapezoidal for each uncertain value. The minimum daily requirement of nutrients protein, carbohydrates and vitamins which are approximately 8, 5 and 4 units, is modeled as (6, 10, 3, 3), (4, 6, 2, 2) and (2, 6, 1, 1) respectively. In a similar way, the other parameters also modeled as symmetric trapezoidal fuzzy numbers taking into account the nature of the problem and the other requirements. So the problem is formulated as follows:

$$
\begin{aligned}
(\mathbf{P_1}) \quad \text{Min } \tilde{z} \quad & = (4,8,3,3)x_1 + (4,6,2,2)x_2 + (2,4,1,1)x_3 \\
s.t. \quad & \\
& 3x_1 + 4x_2 + 2x_3 \geq (6,10,3,3) \\
& 4x_1 + 2x_2 + x_3 \geq (4,6,2,2) \\
& 2x_1 + x_2 + 3x_3 \geq (2,6,1,1) \\
& x_1, x_2, x_3 \geq 0.
\end{aligned}
$$

We will solve this problem by using fuzzy and neutrosophic artificial variable-free simplex method uses the same rank function and compare between the results.

1. **Solving the case study (P_1) by using Fuzzy artificial variable-Free Simplex Method**

Putting the last formula into the standard form, we have:

$$
\begin{aligned}
\text{Min } \tilde{z} \quad = \quad & (4,8,3,3)x_1 + (4,6,2,2)x_2 + (2,4,1,1)x_3 \\
s.t. \quad & \\
- \quad & 3x_1 - 4x_2 - 2x_3 + x_4 = (-10,6,3,3) \\
- \quad & 4x_1 - 2x_2 - x_3 + x_5 \leq (-6,-4,2,2) \\
- \quad & 2x_1 - x_2 - 3x_3 + x_6 \leq (-6,-2,1,1) \\
& x_1, x_2, x_3, x_4, x_5, x_6 \geq 0.
\end{aligned}
$$

By adding non-negative slack variables x_4, x_5, x_6, we can construct the associated reduced short table along with row vector w (sum of rows of infeasible

basic variables) of the above problem as

Initial table:

$$\begin{array}{c} \\ w \\ x_4 \\ x_5 \\ x_6 \end{array} \begin{array}{c} \begin{array}{cccc} b & x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{cccc} (-22,-12,6,6) & -9 & -7 & -6 \\ (-10,-6,3,3) & -3 & -4 & -2 \\ (-6,-4,2,2) & -4 & -2 & -1 \\ (-6,-2,1,1) & -2 & -1 & -3 \end{array}\right] \end{array}$$

Clearly, current basic solution is infeasible.
Here $L = 4, 5, 6$, replace $-x_L^- \to x_L$

$$\begin{array}{c} \\ w \\ -x_4- \\ -x_5- \\ -x_6- \end{array} \begin{array}{c} \begin{array}{cccc} b & x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{cccc} (-22,-12,6,6) & -9 & -7 & -6 \\ (-10,-6,3,3) & -3 & -4 & -2 \\ (-6,-4,2,2) & -4* & -2 & -1 \\ (-6,-2,1,1) & -2 & -1 & -3. \end{array}\right] \end{array}$$

Here, $B = \{4, 5, 6\}$ and $N = \{1, 2, 3\}$, according to most negative coefficient rule $k = 1$, so entering basic variable is x_1 and according to new minimum ratio test $r = 5$, the leaving basic variable is '$-x_5^-$'. Perform the pivot operation on (5, 1).
Replace $-x_5^- \to x_5, L = 4, 5, 6 \backslash 5 = 4, 6, w_5 := w_5 + 1$.

$$\begin{array}{c} \\ w \\ x_4- \\ x_1 \\ x_6- \end{array} \begin{array}{c} \begin{array}{ccc} b & x_5 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc} (-13,3/2,21/2,21/2) & -5/4 & -5/2 & -15/4 \\ (-7,-3/2,9/2,9/2) & -3/4 & -5/2 & -5/4 \\ (1,6/4,1/2,1/2) & -1/4 & 1/2 & 1/4 \\ (-4,1,2,2) & -1/2 & 0 & -5/2 \end{array}\right] \end{array}$$

It can be seen that now a single infeasibility is removed.
Iteration 2. Here, $k = 3$ and $r = 6$ perform pivot operation on (6, 3).
Since $r \in L$, replace $-x_6^- \to x_6; L = 4, 6 \backslash 6 = 4, w_6 := w_6 + 1$.

$$\begin{array}{c} \\ w \\ -x_4^- \\ x_1 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} b & x_5 & x_2 & x_6 \end{array} \\ \left[\begin{array}{ccc} (\frac{-29}{2}, \frac{15}{2}, \frac{27}{2}, \frac{27}{2}) & \frac{-1}{2} & \frac{-5}{2} & \frac{-1}{2} \\ (\frac{-15}{2}, \frac{1}{2}, \frac{11}{2}, \frac{11}{2}) & \frac{-1}{2} & \frac{-5}{2} & \frac{-1}{2} \\ (\frac{3}{5}, \frac{8}{5}, \frac{7}{10}, \frac{7}{10}) & \frac{-3}{10} & \frac{1}{2} & \frac{1}{10} \\ (\frac{-2}{5}, \frac{8}{5}, \frac{4}{10}, \frac{4}{10}) & \frac{1}{5} & 0 & \frac{-2}{5}. \end{array}\right] \end{array}$$

Iteration 3. Here, $k = 2$ and $r = 4$ perform pivot operation on (4, 2).
Since $r \in L$, replace $-x_4^- \to x_4$; $L = \{4\} \backslash \{4\} = \{\}$, $w_4 := w_4 + 1$.

$$\begin{array}{c} \\ w \\ x_2 \\ x_1 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} b & x_5 & x_4 & x_6 \end{array} \\ \left[\begin{array}{ccc} (-15,15,19,19) & 0 & 0 & 0 \\ (\frac{-1}{5}, 3, \frac{11}{5}, \frac{11}{5}) & \frac{1}{5} & \frac{-2}{5} & \frac{1}{5} \\ (\frac{-9}{10}, \frac{17}{10}, \frac{9}{5}, \frac{9}{5}) & \frac{-2}{5} & \frac{1}{5} & 0 \\ (\frac{-2}{5}, \frac{8}{5}, \frac{4}{5}, \frac{4}{5}) & \frac{1}{5} & 0 & \frac{-2}{5}. \end{array}\right] \end{array}$$

Now, all the infeasibilities have been removed. Primal feasibility is achieved; the feasible solution is (x_1, x_2, x_3), $x_1 = (\frac{-9}{10}, \frac{17}{10}, \frac{9}{5}, \frac{9}{5})$, $x_2 = (\frac{-1}{5}, 3, \frac{11}{5}, \frac{11}{5})$ and $x_3 = (\frac{-2}{5}, \frac{8}{5}, \frac{4}{5}, \frac{4}{5})$.

Therefore, to obtain the optimal solution we solve **(SM) phase 2**.

Iteration 4.

$$\begin{array}{c} \\ z \\ x_2 \\ x_1 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cccc} b & & x_5 & x_4 \quad x_6 \end{array} \\ \left[\begin{array}{cccc} \frac{1}{5} & (\frac{-38}{5}, 30, \frac{121}{5}, \frac{121}{5}) & \frac{4}{5} & \frac{4}{5} \\ (\frac{-1}{5}, 3, \frac{11}{5}, \frac{11}{5}) & \frac{1}{5} & \frac{-2}{5} & \frac{1}{5} \\ (\frac{-9}{10}, \frac{17}{10}, \frac{9}{5}, \frac{9}{5}) & \frac{-2}{5} & \frac{1}{5} & 0 \\ (\frac{-2}{5}, \frac{8}{5}, \frac{4}{5}, \frac{4}{5}) & \frac{1}{5} & 0 & \frac{-2}{5}. \end{array}\right] \end{array}$$

The optimal solution is (x_1, x_2, x_3), $x_1 = 0$, $x_2 = \frac{1}{5}$ and $x_3 = \frac{-2}{5}$

$$\text{Min}\,\tilde{z} \;=\; (4,8,3,3)x_1 + (4,6,2,2)x_2 + (2,4,1,1)x_3 \;=\; 11.2.$$

We will solve the same example by using neutrosophic artificial variable-free simplex Method uses the same rank function and we will compare between them.

4.1.1 Solving the same example (case study) by using neutrosophic artificial variable-free simplex method

First: We will convert the fuzzy numbers into neutrosophic numbers. After that, we apply the following rank function:

$$R(\check{a}) = \frac{1}{4}\sum_{i=1}^{4} \tilde{a}_i + (T_a - I_{\tilde{a}} - F_{\tilde{a}})$$

$$R(a) = a + 1$$

$$\begin{array}{rcl} \text{Min}\,\tilde{z} & = & R[(1,4,8,11)]x_1 + R[(2,4,6,8)]x_2 + R[(1,2,4,5)]x_3 \\ & s.t. & \\ & - & R[3x_1 + 4x_2 + 2x_3] \geq R[(3,6,10,13)] \\ & - & R[4x_1 + 2x_2 + 1x_3] \geq R[(2,4,6,8)] \\ & - & R[2x_1 + 1x_2 + 3x_3] \geq R[(1,2,6,7)] \\ & & x_1, x_2, x_3 \geq 0. \end{array}$$

$$\begin{array}{rcl} \text{Min}\,\tilde{z} & = & 7x_1 + 6x_2 + 4x_3 - 2 \\ & s.t. & \\ & & 3x_1 + 4x_2 + 2x_3 \geq 8 \\ & & 4x_1 + 2x_2 + x_3 \geq 5 \\ & & 2x_1 + x_2 + 3x_3 \geq 4 \\ & & x_1, x_2, x_3 \geq 0. \end{array}$$

Putting the last formula into the standard form, we have:

$$\begin{aligned} \text{Min}\,\tilde{z} \quad &= \quad 7x_1 + 6x_2 + 4x_3 - 2 \\ s.t. \quad & \\ & -3x_1 - 4x_2 - 2x_3 + x_4 = -8 \\ & -4x_1 - 2x_2 - x_3 + x_5 = -5 \\ & -2x_1 - x_2 - 3x_3 + x_6 = -4 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0. \end{aligned}$$

Then, the first tableau is as follows:

$$\begin{array}{c} \\ w \\ x_4 \\ x_5 \\ x_6 \end{array} \begin{array}{c} \begin{array}{cccc} b & x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{cccc} -17 & -9 & -7 & -6 \\ -8 & -3 & -4 & -2 \\ -5 & -4 & -2 & -1 \\ -4 & -2 & -1 & -3 \end{array}\right] \end{array}$$

Clearly, current basic solution is infeasible.
Here L = {4, 5, 6}, replace $-x_L^- \to x_L$

$$\begin{array}{c} \\ w \\ -x_4^- \\ -x_5^- \\ -x_6^- \end{array} \begin{array}{c} \begin{array}{cccc} b & x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{cccc} -17 & -9 & -7 & -6 \\ -8 & -3 & -4 & -2 \\ -5 & -4 & -2 & -1 \\ -4 & -2 & -1 & -3 \end{array}\right] \end{array}$$

Here, $B = \{4, 5, 6\}$ and $N = \{1, 2, 3\}$, according to most negative coefficient rule $k = 1$, so entering basic variable is x_1 and according to new minimum ratio test r = 5, the leaving basic variable is '$-x_5^-$'. Perform the pivot operation on (5, 1). Replace $-x_5^- \to x_5$, $L = \{4, 5, 6\}\backslash\{5\} = \{4, 6\}$, $w_5 := w_5 + 1$.

$$\begin{array}{c} \\ w \\ -x_4^- \\ x_1 \\ -x_6^- \end{array} \begin{array}{c} \begin{array}{cccc} b & x_5 & x_2 & x_3 \end{array} \\ \left[\begin{array}{cccc} \frac{-23}{4} & \frac{-5}{4} & \frac{-5}{2} & \frac{-15}{4} \\ \frac{-17}{4} & \frac{-3}{4} & \frac{-5}{2} & \frac{-5}{4} \\ \frac{5}{4} & \frac{-1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{-3}{2} & \frac{-1}{2} & 0 & \frac{-5}{2} \end{array}\right] \end{array}$$

It can be seen that now a single infeasibility is removed.
Iteration 2. Here, $k = 3$ and $r = 6$ perform pivot operation on (6, 3). Since $r \in L$, replace $-x_6^- x_6$; $L = \{4, 6\}\backslash\{6\} = 4$, $w_6 := w_6 + 1$.

$$\begin{array}{c} \\ w \\ -x_4^- \\ x_1 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cccc} b & x_5 & x_2 & x_6 \end{array} \\ \left[\begin{array}{cccc} \frac{-7}{2} & \frac{-1}{2} & \frac{-5}{2} & \frac{-1}{2} \\ \frac{-7}{2} & \frac{-1}{2} & \frac{-5}{2} & \frac{-1}{2} \\ \frac{11}{10} & \frac{-3}{10} & \frac{1}{2} & \frac{1}{10} \\ \frac{3}{5} & \frac{1}{5} & 0 & \frac{-2}{5} \end{array}\right] \end{array}$$

Iteration 3. Here, $k = 2$ and $r = 4$ perform pivot operation on (4, 2). Since $r \in L$, replace $-x_4^- \to x_4$; $L = \{4\}\backslash\{4\} = \{\}$, $w_4 := w_4 + 1$.

$$\begin{array}{c} \\ w \\ x_2 \\ x_1 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cccc} b & x_5 & x_4 & x_6 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{7}{5} & \frac{1}{5} & \frac{-2}{5} & \frac{1}{5} \\ \frac{2}{5} & \frac{-2}{5} & \frac{1}{5} & 0 \\ 0 & \frac{1}{5} & 0 & \frac{-2}{5} \end{bmatrix} \end{array}$$

Now, all the infeasibilities have been removed. Primal feasibility is achieved; the feasible solution is $(x_1, x_2, x_3) = (\frac{2}{5}, \frac{7}{5}, 0)$

Therefore, to obtain the optimal solution we solve **(SM) phase 2**.

Iteration 4.

$$\begin{array}{c} \\ Z \\ x_2 \\ x_1 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cccc} b & x_5 & x_4 & x_6 \end{array} \\ \begin{bmatrix} \frac{1}{5} & \frac{56}{5} & \frac{4}{5} & \frac{4}{5} \\ \frac{1}{5} & \frac{7}{5} & \frac{1}{5} & \frac{-2}{5} \\ 0 & \frac{2}{5} & \frac{-2}{5} & \frac{1}{5} \\ \frac{-2}{5} & 0 & \frac{1}{5} & 0 \end{bmatrix} \end{array}$$

The optimal solution is (x_1, x_2, x_3), $x_1 = 0$, $x_2 = \frac{1}{5}$ and $x_3 = -2/5$, $\text{Min}\, z = 7x_1 + 6x_2 + 4x_3 - 2 = \frac{56}{5} - 2 = 9.2$.

By the same way as in example1 (P_1), we can solve both example 2 (P_2) and example 3 (P_3) in the same way by using the proposed algorithm (fuzzy and neutrosophic artificial variable-free simplex algorithm).

Table 4: A comparison between two-phase algorithm, Fuzzy and Neutrosophic AVFSA

Problem	Two-Phase Simplex Algorithm				Fuzzy Art-Free Simplex Algorithm				Neutrosophic Art-Free Simplex Algorithm			
	Z	x_1	x_2	x_3	Z	x_1	x_2	x_3	Z	x_1	x_2	x_3
P_1	11.2	$\frac{2}{5}$	$\frac{7}{5}$	0	11.2	$\frac{2}{5}$	$\frac{7}{5}$	0	9.2	$\frac{2}{5}$	$\frac{7}{5}$	0
P_2	5.6	$\frac{11}{5}$	$\frac{2}{5}$	0	5.6	$\frac{11}{5}$	$\frac{2}{5}$	0	2.46	$\frac{69}{5}0$	$\frac{2}{25}$	0
P_3	19.07	$\frac{16}{7}$	$\frac{15}{28}$	-	19.07	$\frac{16}{7}$	$\frac{15}{28}$	-	19.07	$\frac{16}{7}$	$\frac{15}{28}$	-

In table 4, a good comparisons have been made between two-phase simplex algorithm, fuzzy and neutrosophic artificial variable-free simplex algorithm; we noticed that the neutrosophic approach is more accurate than the fuzzy approach as shown as in problem 1 and problem 2. However, there is one case the fuzzy approach is equivalent to the neutrosophic approach according to (remark 3) as shown as in problem 3.

Conclusion

In this work, a new algorithm (Artificial Variable-Free Simplex Algorithm) has been proposed, which solves linear programming problems with fuzzy and neu-

trosophic numbers. We evaluated the proposed algorithm by using a comparison against the traditional simplex method. We noticed that the artificial variables in the proposed algorithm are virtually present, but their presence is not revealed to the user in the form of extra columns in the simplex table. It also follows the same pivoting sequence as of simplex phase 1 without showing any explicit description of artificial variables or artificial constraints. Therefore, the proposed algorithm reduces the size of the problem and reduces the execution time to solve the problem, then CPU time for the proposed algorithm is also faster than the two-phase simplex method. We also compared between the fuzzy approach and the neutrosophic approach using numerical examples. Finally, the numerical examples show that the neutrosophic approach is more accurate than the fuzzy approach and the proposed method overcomes the traditional simplex method for both the fuzzy and neutrosophic approach. In future work, we can improve this work by proposing a new algorithm (Dual Artificial Variable Free Simplex Algorithm) which solves linear programming problems with fuzzy and neutrosophic numbers.

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Accepted: May 15, 2020

On a class of semi-hypergeneralized quasi-Einstein manifolds

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Abstract. The object of the present paper is to introduce semi-hypergeneralized quasi-Einstein manifolds and the manifolds with semi- hypergeneralized quasi-constant curvature. Ricci recurrent semi-hypergeneralized quasi-Einstein manifolds have also been studied.

Keywords: generalized quasi-Einstein manifold, semi-hypergeneralized quasi-Einstein manifold, semi-hypergeneralized quasi-constant curvature, Ricci recurrent.

1. Introduction

A Riemannian or semi-Riemannian manifold (M^n, g), $n = \dim M \geq 2$, is said to be an Einstein manifold if the condition

$$S = \frac{r}{n}$$

holds on M, where S and r denote the Ricci tensor and the scalar curvature of (M^n, g) respectively. According to A. L. Besse [4], the above condition is called Einstein metric condition. Einstein manifolds play an important role in Riemannian geometry as well as in general theory of relativity. Einstein manifolds have been studied by several authors. For more details see [4]. Tamassay and Binh [25] studied symmetry of Einstein manifolds. Einstein manifolds form a natural subclass of various classes of Riemannian or semi-Riemannian manifolds by a curvature condition imposed on their Ricci tensor [4]. For instance, every Einstein manifold belongs to the class of Riemannian manifolds (M^n, g) realizing the following relation:

$$S(X, Y) = ag(X, Y) + bA(X)A(Y), \tag{1}$$

where a, b are real numbers and A is a non-zero $1-$form defined by $g(X, U) = A(X)$, for all vector fields X and the unit vector field U.

A non-flat Riemannian manifold (M^n, g) $(n > 2)$ is defined to be quasi-Einstein manifold if its Ricci tensor S of type $(0, 2)$ is not identically zero and for any vector fields X, Y it satisfies the equation (1) (see [14]). In the year 2000,

Chaki and Maity [5] defined and studied quasi-Einstein manifolds by taking a, b as scalars. Such a quasi-Einstein manifold of dimension n is denoted by $(QE)_n$. In the paper [15] Ghosh, De and Binh studied certain curvature restrictions on quasi-Einstein manifolds. Quasi-Einstein manifold arose during the study of exact solution of Einstein's field equations as well as the study of quasi umbilical hypersurfaces of Euclidean space.

According to U. C. De and G. C. Ghosh [9] a non-flat Riemannian manifold (M^n, g) $(n > 2)$ is said to be generalized quasi-Einstein manifold if its Ricci tensor S of type $(0, 2)$ is not identically zero and for any vector fields X, Y it satisfies

$$S(X, Y) = ag(X, Y) + bA(X)A(Y) + cB(X)B(Y),$$

where a, b, c are scalars and A, B are non-zero $1-$forms defined respectively by $g(X, U) = A(X)$ and $g(X, V) = B(X)$. The unit vector fields U and V corresponding to the $1-$forms A and B are orthogonal, i.e., $g(U, V) = 0$. U, V are known as the generators of the manifold. Such an $n-$dimensional manifold is denoted by $G(QE)_n$. Quasi-Einstein manifolds have been generalized by several authors in several ways. For instance we mention [6], [15], [17], [19], [20] [22]. Some generalizations of quasi Einstein manifolds appear in the study of hypersurfaces of some manifolds [24].

The notion of hypergeneralized quasi-Einstein manifold was introduced by A. A. Shaikh, C. Ozgur and A Patra [23]. An n-dimensional Riemannian manifold (M^n, g), $(n > 2)$ is called a hypergeneralized quasi-Einstein manifold if its Ricci tensor of type (0, 2) is non zero and satisfies the following condition

$$\begin{aligned} S(X, Y) = {} & ag(X, Y) + bA(X)A(Y) + c[A(X)B(Y) + A(Y)B(X)] \\ & + d[A(X)D(Y) + A(Y)D(X)], \end{aligned}$$

for all $X, Y \in T(M)$, where a, b, c and d are real valued, non-zero scalars functions on (M^n, g).

Later hypergeneralized quasi-Einstein manifolds has been studied by D. Debnath [12].

In the present paper, we like to introduce the notion of semi-hypergeneralized quasi-Einstein manifold by ensuring its existence and physical relevance of study. The present paper is organized as follows:

Section 2 contains some preliminaries. In Section 3, we establish the existence of semi-hypergeneralized quasi-Einstein manifolds. Section 4 is devoted to study the nature of a manifold with semi-hypergeneralized quasi-constant curvature and to prove that such a manifold is semi-hypergeneralized quasi-Einstein manifold. Finally, in Section 5 a necessary and sufficient condition for a semi-hypergeneralized quasi-Einstein manifold to be Ricci recurrent is obtained.

2. Preliminaries

Let M be a submanifold immersed in a $(2n+1)-$dimensional Riemannian manifold $\tilde{M}$, we denote by the same symbol g the induced metric on M. Let TM be

the set of all vector fields tangent to M and $T^{\perp}M$ is the set of all vector fields normal to M. Then Gauss and Weingarten formula are given by [8]

$$\tilde{\nabla}_X Y = \nabla_X Y + B(X,Y), \tag{2}$$

$$\tilde{\nabla}_X N = -A_N X + \nabla_X{}^{\perp} N, \tag{3}$$

for any $X, Y \in TM$ and $N \in T^{\perp}M$, where $\nabla^{\perp}$ is the connection in $T^{\perp}M$. The second fundamental form B and A_N are related by

$$g(A_N X, Y) = g(B(X,Y), N). \tag{4}$$

For a unit normal vector field N of the submanifold we get

$$B(X,Y) = H(X,Y)N, \tag{5}$$

where $H(X,Y)$ is the second fundamental tensor and $B(X,Y)$ is the second fundamental form of the submanifold. The Gauss equation is given by

$$\begin{aligned} \tilde{R}(X,Y,Z,W) &= R(X,Y,Z,W) - g(B(X,W),B(Y,Z)) \\ &\quad + g(B(Y,W),B(X,Z)), \end{aligned} \tag{6}$$

where R is the Riemannian curvature tensor of the submanifold and $\tilde{R}$ is the Riemannian curvature tensor of the manifold.

Definition 2.1. A hypersurface M^{2n} immersed isometrically in a generalized Sasakian-space-form $\tilde{M}^{2n+1}(c)$ is said to be $2-$quasi-umbilical [16] if the second fundamental tensor $H(X,Y)$ satisfies the the equation

$$H(X,Y) = \alpha g(X,Y) + \beta\omega(X)\omega(Y) + \gamma T(X)T(Y), \tag{7}$$

where α, β, γ are scalars and ω, T are $1-$forms defined by $\omega(X) = g(X,U)$ and $T(X) = g(X,\rho)$, and U, ρ are two unit vectors such that $g(U,\rho) = 0$.

If $\gamma = 0$, the hypersurface is called quasi-umbilical and for $\beta = \gamma = 0$, it is known as umbilical.

Recently, P. Alegre, D. Blair and A. Carriazo [1] introduced and studied generalized Sasakian-space-forms. These space-forms are defined as follows:

Given an almost contact metric manifold $\tilde{M}(\phi, \xi, \eta, g)$, we say that $\tilde{M}$ is generalized Sasakian-space-form if there exist three functions f_1, f_2, f_3 on $\tilde{M}$ such that the curvature tensor $\tilde{R}$ of $\tilde{M}$ is given by

$$\begin{aligned} \tilde{R}(X,Y)Z &= f_1\{g(Y,Z)X - g(X,Z)Y\} \\ &+ f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ &+ f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &+ g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}, \end{aligned} \tag{8}$$

for any vector fields X, Y, Z on $\tilde{M}$. In such a case we denote the manifold as $\tilde{M}(f_1, f_2, f_3)$. These kind of manifolds appear as a generalization of the well known Sasakian-space-forms, which can be obtained as a particular case of generalized Sasakian-space-forms by taking $f_1 = \frac{c+3}{4}$, $f_2 = f_3 = \frac{c-1}{4}$. But, it is to be noted that generalized Sasakian-space-forms are not merely generalization of Sasakian-space-forms. It also contains a large class of almost contact manifolds. For example, it is known that [2] any three-dimensional trans Sasakian manifold of type (α, β) with α, β depending on ξ is a generalized Sasakian-space-form. However, we can find generalized Sasakian-space-forms with non-constant functions and arbitrary dimensions. In [1], the authors cited several examples of generalized Sasakian-space-forms in terms of warped product spaces. Again generalized Sasakian-space-forms have been studied in the papers [10], [11], [21].

3. Existence of semi-hypergeneralized quasi-Einstein manifolds

Definition 3.1. A non-flat Riemannian manifold of dimension m $(m > 2)$ will be called a semi-hypergeneralized quasi-Einstein manifold if its Ricci tensor S of type $(0, 2)$ is non-zero and satisfies the condition

$$S(X, Y) = ag(X, Y) + bA(X)A(Y) + cB(X)B(Y) + dT(X)T(Y), \tag{9}$$

where, a, b, c, d are certain non-zero scalars and A, B and T are three non-zero $1-$forms defined by $A(X) = g(U, X)$ and $B(X) = g(X, V)$, and $T(X) = g(X, \rho)$. U, V, ρ are three mutually orthogonal unit vectors. A semi-hypergeneralized quasi-Einstein manifold of dimension m will be denoted by $H(GQE)_m$.

If $d = 0$, then the manifold becomes generalized quasi-Einstein manifold. If $c = d = 0$, the manifold is qusi-Einstein. The manifold is Einstein when $b = c = d = 0$.

Let us consider a $(2n+1)-$dimensional generalized Sasakian-space-form $\tilde{M}$. Now, from (8) we get

$$\begin{aligned} \tilde{R}(X, Y, Z, W) &= f_1\{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\} \\ &+ f_2\{g(X, \phi Z)g(\phi Y, W) - g(Y, \phi Z)g(\phi X, W) \\ &+ 2g(X, \phi Y)g(\phi Z, W)\} \\ &+ f_3\{\eta(X)\eta(Z)g(Y, W) - \eta(Y)\eta(Z)g(X, W) \\ &+ g(X, Z)\eta(Y)\eta(W) - g(Y, Z)\eta(X)\eta(W)\}. \end{aligned} \tag{10}$$

By (5) and (6) we get

$$\tilde{R}(X, Y, Z, W) = R(X, Y, Z, W) - H(Y, Z)H(X, W) + H(X, Z)H(Y, W). \tag{11}$$

Let us consider a $2-$quasi-umbilical hypersurface of $\tilde{M}$. In view of (7) the above equation yields

$$\tilde{R}(X, Y, Z, W) = R(X, Y, Z, W)$$

$$
\begin{aligned}
&- (\alpha g(Y,Z) + \beta\omega(Y)\omega(Z) + \gamma T(Y)T(Z))(\alpha g(X,W) \\
&+ \beta\omega(X)\omega(W) + \gamma T(X)T(W)) \\
&+ (\alpha g(X,Z) + \beta\omega(X)\omega(Z) + \gamma T(X)T(Z))(\alpha g(Y,W) \\
&+ \beta\omega(Y)\omega(W) + \gamma T(Y)T(W)).
\end{aligned} \tag{12}
$$

By virtue of (10) and (12), it follows that

$$
\begin{aligned}
&f_1\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\} \\
&+ f_2\{g(X,\phi Z)g(\phi Y,W) - g(Y,\phi Z)g(\phi X,W) + 2g(X,\phi Y)g(\phi Z,W)\} \\
&+ f_3\{\eta(X)\eta(Z)g(Y,W) - \eta(Y)\eta(Z)g(X,W) \\
&+ g(X,Z)\eta(Y)\eta(W) - g(Y,Z)\eta(X)\eta(W)\} \\
&= R(X,Y,Z,W) \\
&- (\alpha g(Y,Z) + \beta\omega(Y)\omega(Z) + \gamma T(Y)T(Z))(\alpha g(X,W) \\
&+ \beta\omega(X)\omega(W) + \gamma T(X)T(W)) \\
&+ (\alpha g(X,Z) + \beta\omega(X)\omega(Z) + \gamma T(X)T(Z))(\alpha g(Y,W) \\
&+ \beta\omega(Y)\omega(W) + \gamma T(Y)T(W)).
\end{aligned} \tag{13}
$$

In the above equation contracting Y and Z, we obtain

$$
\begin{aligned}
&(2n-1)f_1 g(X,W) + 3f_2 g(\phi X,\phi W) \\
&+ f_3\{(2-2n)\eta(X)\eta(W) - g(X,W)\} \\
&= S(X,W) - (2n\alpha + \beta + \gamma)(\alpha g(X,W) + \beta\omega(X)\omega(W) + \gamma T(X)T(W)) \\
&+ (\alpha^2 g(X,W) + \alpha\beta\omega(X)\omega(W) + \alpha\gamma T(X)T(W) + \alpha\beta\omega(X)\omega(W) \\
&+ \beta^2\omega(X)\omega(W) + \alpha\gamma T(X)T(W) + \gamma^2 T(X)T(W)).
\end{aligned} \tag{14}
$$

After simplification the above equation yields

$$
\begin{aligned}
S(X,W) &= \{(2n-1)f_1 + 3f_2 - f_3 + \alpha(2n\alpha + \beta + \gamma) - \alpha^2\}g(X,W) \\
&+ \{(2-2n)f_3 - 3f_2\}\eta(X)\eta(W) \\
&+ \{\gamma(2n\alpha + \beta + \gamma) - 2\alpha\gamma - \gamma^2\}T(X)T(W) \\
&+ \{\beta(2n\alpha + \beta + \gamma) - 2\alpha\beta - \beta^2\}\omega(X)\omega(W).
\end{aligned} \tag{15}
$$

Therefore we can state the following:

Theorem 3.1. A 2−quasi-umbilical hypersurface of a generalized Sasakian-space-form is a semi-hypergeneralized quasi-Einstein manifold.

If $\beta = 0$, the hypersurface becomes quasi-umbilical. Hence we obtain the following:

Corollary 3.1. A quasi-umbilical hypersurface of a generalized Sasakian-space-form is a generalized quasi-Einstein manifold.

If $\beta = \gamma = 0$, the hypersurface is known as umbilical. Hence we can state the following:

Corollary 3.2. An umbilical hypersurface of a generalized Sasakian-space-form is an Einstein manifold.

Example 3.1: physical example of semi-hypergeneralized quasi-Einstein manifolds. In [1], it is shown that the warped product space $\tilde{M} = \mathbb{R} \times_f \mathbb{C}^m$ is a generalized Sasakian-space-form with

$$f_1 = -\frac{(f')^2}{f^2}, \quad f_2 = 0, \quad f_3 = -\frac{(f')^2}{f^2} + \frac{f''}{f},$$

where $f = f(t), t \in \mathbb{R}$ and f' denotes derivative of f with respect to t. It is known that[3] the space near a massive star or black whole is warped product space. Hence, a $2-$quasi umbilical hypersurface of the manifold under consideration in this example gives a physical example of semi-hypergeneralized quasi-Einstein manifold by Theorem 3.1.

Example 3.2: example of non-trivial Riemannian manifold which is semi-hypergeneralized quasi-Einstein. Let us consider the $4-$dimensional real number space with the Riemennian metric g_{ij} given by

$$g_{ij}dx^i dx^j = (dx^1)^2 + (x^1)^2(dx^2)^2 + (x^1 \sin x^2)^2(dx^3)^2 + (dx^4)^2,$$

$i, j = 1, 2, 3, 4$, $x^1 \neq 0$ and $0 < x^2 < \frac{\pi}{2}$. It can be easily calculated that the non-vanishing components of the Christoffel symbols are

$$\Gamma^1_{22} = -x^1, \Gamma^1_{33} = -x^1(\sin x^2)^2, \Gamma^2_{12} = \frac{1}{x^1} = \Gamma^3_{13}, \Gamma^2_{23} = \cot x^2, \Gamma^2_{33} = -\sin x^2 \cos x^2.$$

Using the above results we find the non-vanishing components of the curvature tensor and Ricci tensor as follows:

$$R_{2332} = -(x^1 \sin x^2)^2, S_{22} = -1, S_{33} = -(\sin x^2)^2.$$

Let

$$a = -\frac{1}{(x^1)^2}, \ b = \frac{1}{(x^1)^2}, \ c = \frac{1}{2(x^1)^2}, \ d = \frac{1}{2}$$

and

$$\begin{aligned} A_i &= 1, \text{for } i = 1 \\ &= 0, \text{otherwise}, \end{aligned}$$

$$\begin{aligned} B_i &= 1, \text{for } i = 4 \\ &= 0, \text{otherwise}. \end{aligned}$$

$$\begin{aligned} T_i &= \tfrac{1}{x}, \text{for } i = 4 \\ &= 0, \text{otherwise}. \end{aligned}$$

From the above results we can verify the following:

$$\begin{aligned}
S_{11} &= 0 &&= ag_{11} + bA_1A_1 + cB_1B_1 + dT_1T_1,\\
S_{22} &= -1 &&= ag_{22} + bA_2A_2 + cB_2B_2 + dT_2T_2,\\
S_{33} &= -(\sin x^2)^2 &&= ag_{33} + bA_3A_3 + cB_3B_3 + dT_3T_3,\\
S_{44} &= 0 &&= ag_{44} + bA_4A_4 + cB_4B_4 + dT_4T_4,\\
S_{12} &= 0 &&= ag_{12} + bA_1A_2 + cB_1B_2 + dT_1T_2,\\
S_{13} &= 0 &&= ag_{13} + bA_1A_3 + cB_1B_3 + dT_1T_3,\\
S_{14} &= 0 &&= ag_{14} + bA_1A_4 + cB_1B_4 + dT_1T_4,\\
S_{23} &= 0 &&= ag_{23} + bA_2A_3 + cB_2B_3 + dT_2T_3,\\
S_{24} &= 0 &&= ag_{24} + bA_2A_4 + cB_2B_4 + dT_2T_4,\\
S_{34} &= 0 &&= ag_{34} + bA_3A_4 + cB_3B_4 + dT_3T_4.
\end{aligned}$$

From the above expressions it is clear that the manifold under consideration is a semi-hypergeneralized quasi-Einstein manifold.

4. Manifold of semi-hypergeneralized quasi-constant curvature

Definition 4.1. A Riemannian manifold is called a manifold of quasi-constant curvature if it is conformally flat and if the curvature tensor $\tilde{R}$ of type (0,4) satisfies the condition

$$\begin{aligned}
\tilde{R}(X,Y,Z,W) &= \alpha[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]\\
&+ \beta[g(X,W)T(Y)T(Z) - g(X,Z)T(Y)T(W)\\
&+ g(Y,Z)T(X)T(W) - g(Y,W)T(X)T(Z)], \qquad (16)
\end{aligned}$$

where $\tilde{R}(X,Y,Z,W) = g(R(X,Y)Z,W)$, R is the curvature tensor of type (1,3), α, β are scalar functions and ρ is a unit vector field defined by $g(X,\rho) = T(X)$. The notion of quasi-constant curvature for Riemannian manifolds were given by Chen and Yano [7]. Again, in the paper [9], the authors have given the notion of manifolds of generalized quasi-constant curvature. In this section we like to give the idea of another generalization, viz., the manifolds of semi-hypergeneralized quasi constant curvature. We also show that such manifolds are semi-hypergeneralized quasi-Einstein manifolds.

Definition 4.2. A Riemannian manifold will be called a manifold of semi-hypergeneralized quasi-constant curvature if it is conformally flat and if the curvature tensor $\tilde{R}$ of type (0,4) satisfies the condition

$$\begin{aligned}
\tilde{R}(X,Y,Z,W) &= \alpha[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]\\
&+ \beta[g(Y,Z)A(X)A(W) - g(X,Z)A(Y)A(W)\\
&+ g(X,W)A(Y)A(Z) - g(Y,W)A(X)A(Z)]\\
&+ \gamma[g(Y,Z)B(X)B(W) - g(X,Z)B(Y)B(W)\\
&+ g(X,W)B(Y)B(Z) - g(Y,W)B(X)B(Z)]\\
&+ \delta[g(Y,Z)T(X)T(W) - g(X,Z)T(Y)T(W)\\
&+ g(X,W)T(Y)T(Z) - g(Y,W)T(X)T(Z)], \qquad (17)
\end{aligned}$$

where $\tilde{R}(X,Y,Z,W) = g(R(X,Y)Z,W)$, R is the curvature tensor of type (1,3), $\alpha, \beta, \gamma, \delta$ are scalar functions and the $1-$forms have the same meanings as in Definition 3.1.

Let us consider a semi-hypergeneralized quasi-Einstein manifold. Then by definition the manifold is conformally flat. Therefore, we have

$$\begin{aligned} \tilde{R}(X,Y,Z,W) &= \frac{1}{n-2}[S(Y,Z)g(X,W) - S(X,Z)g(Y,W) \\ &+ S(X,W)g(Y,Z) - S(Y,W)g(X,Z)] \\ &+ \frac{r}{(n-1)(n-2)}[g(X,Z)g(Y,W) - g(Y,Z)g(X,W)]. \end{aligned} \tag{18}$$

Since the manifold is semi-hypergeneralized quasi-Einstein, by Definition 3.1 and equation (18) it follows that

$$\begin{aligned} \tilde{R}(X,Y,Z,W) &= (\frac{3a}{n-2} - \frac{r}{(n-1)(n-2)})[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ &+ \frac{b}{n-2}[g(Y,Z)A(X)A(W) - g(X,Z)A(Y)A(W) \\ &+ g(X,W)A(Y)A(Z) - g(Y,W)A(X)A(Z)] \\ &+ \frac{c}{n-2}[g(Y,Z)B(X)B(W) - g(X,Z)B(Y)B(W) \\ &+ g(X,W)B(Y)B(Z) - g(Y,W)B(X)B(Z)] \\ &+ \frac{d}{n-2}[g(Y,Z)T(X)T(W) - g(X,Z)T(Y)T(W) \\ &+ g(X,W)T(Y)T(Z) - g(Y,W)T(X)T(Z)]. \end{aligned} \tag{19}$$

From above we get the following:

Theorem 4.1. A semi-hypergeneralized quasi-Einstein manifold is a manifold of semi-hypergeneralized quasi-constant curvature.

Conversely, consider a manifold of semi-hypergeneralized quasi-constant curvature. Then in (17), contracting Y and Z we get

$$\begin{aligned} S(X,W) &= ((n-1)\alpha + \beta + \gamma + \delta)g(X,W) + \beta(n-2)A(X)A(W) \\ &+ \gamma(n-2)B(X)B(Z) + \delta(n-2)T(X)T(Z). \end{aligned} \tag{20}$$

In view of (20), we can easily state the following:

Theorem 4.2. A manifold of semi-hypergeneralized quasi-constant curvature is a semi-hypergeneralized quasi-Einstein manifold.

5. Ricci-recurrent $H(GQE)_n$

Definition 5.1. A Riemannian manifold is called Ricci recurrent if its Ricci tensor satisfies

$$(\nabla_Z S)(X,Y) = \lambda(Z)S(X,Y),$$

where λ is a non-zero 1−form, for any vector fields X, Y, Z.

Let U, V, ρ are parallel vector fields. Then $\nabla_X U = 0$ for all X, which implies that $R(X,Y)U = 0$. Consequently, $S(X,U)=0$. Again, from (9) $S(X,U) = (a+b)A(X)$. Hence, $a+b=0$, because $A(X) \neq 0$ for all X other than V. Similarly, $a+c=0$ and $a+d=0$. In this way, it follows that $b=c=d=-a$. Hence, (9) takes the form

$$S(X,Y) = a[g(X,Y) - A(X)A(Y) - B(X)B(Y) - T(X)T(Y)]. \tag{21}$$

From the above equation we obtain by covariant differentiation

$$\begin{aligned}(\nabla_Z S)(X,Y) &= da(Z)[g(X,Y) - A(X)A(Y) - B(X)B(Y) - T(X)T(Y)] \\ &- a[(\nabla_Z A)(X)A(Y) + A(X)(\nabla_Z A)(Y) \\ &+ (\nabla_Z B)(X)B(Y) + B(X)(\nabla_Z B)(Y)].\end{aligned} \tag{22}$$

Since U, V, ρ are parallel vector fields, $(\nabla_Z A)(X) = 0$, $(\nabla_Z B)(X) = 0$ and $(\nabla_Z T)(X) = 0$. Therefore, the above equation reduces to

$$(\nabla_Z S)(X,Y) = \frac{da(Z)}{a} S(X,Y). \tag{23}$$

Thus, the manifold is Ricci recurrent.

Conversely, let a semi-hypergeneralized quasi-Einstein manifold is Ricci recurrent. Then by Definition 5.1 and equation (9), it follows that

$$\begin{aligned}&da(W)g(X,Y)+db(W)A(X)A(Y)+dc(W)B(X)B(Y)+dd(W)T(X)T(Y) \\ &+ b((\nabla_W A)(X)A(Y) + A(X)(\nabla_W A)Y) \\ &+ c((\nabla_W B)(X)B(Y) + B(X)(\nabla_W B)(Y)) \\ &+ d((\nabla_W T)(X)T(Y) + T(X)(\nabla_W T)(Y)) \\ &= \lambda(Z)(ag(X,Y) + bA(X)A(Y) + cB(X)B(Y) + dT(X)T(Y)).\end{aligned} \tag{24}$$

In the above equation putting $X = U$ and $Y = V$, where U, V are described in the Definition 3.1, we get

$$(\nabla_W A)(V) = 0.$$

By the definition of the 1−form A the above equation yields $g(\nabla_W V, U) = 0$. Hence, either $\nabla_W V = 0$, or, $\nabla_W V$ is orthogonal to U. Since W is arbitrary, $\nabla_W V$ is not orthogonal to U, in general. Therefore, $\nabla_W V = 0$. In the similar way it can be proved that $\nabla_W U = 0$ and $\nabla_W \rho = 0$. Thus, we have the following:

Theorem 5.1. The generators of a semi-hypergeneralized quasi-Einstein manifold is parallel if and only if the manifold is Ricci recurrent.

Acknowledgement

The author is thankful to the referee for his valuable suggestions towards the improvement of the paper.

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Accepted: May 14, 2020

Molecular structure investigation of anticancer drug by topological indices

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Abstract. The various medical observations represent that the properties of the drugs have an adjacent relation to its molecular structures. Drug properties will be determined by analysing and investigating the corresponding molecular structures of the drug. In chemical graph theory, topological index is one of the significant tools which is applied to understand the properties of a chemical compound. Topological indices defined on these chemical molecular structures can help researchers to better understand the physical features, chemical re-activities and biological activities. This work is to demonstrate various topological indices on anticancer drug, Hyaluronic acid (HA) used in the manufacturing of anticancer drugs.

Keywords: atom bond connectivity (ABC) indices, symmetric division index (SDD), F−indices, multiplicative F−indices, F−polynomial indices, anticancer drug.

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1. Introduction

In today's scenario, one of the main ailments that is alarming and fatal is cancer. As improved the pharmaceutical technology improved in several countries, new tremendous anticancer drugs are synthesized in the laboratory. They are tested and released in the market for public. Earlier many examinations were needed to analyse the new drugs to determine the biological characteristic, percentage of toxicity and side-effects on the human anatomy. Consequently, there was more work-load in the laboratories which resulted in various problems in the countries having least financial and worst laboratory situations.

The graph theory offers experts with the powerful tool, i.e. topological indices. In the early chemical experiments, experts compared a vast quantity of experimented results with the compound structure. It was concluded that the atom structure ordering of a compound had inherent relationship with the properties it exhibited. Based on these facts, using the interpretation of the topological indices on the compounds and the respective characteristics, the compounds were used in the drug design in medicine and pharmaceutical science fields. In the computation model, atoms are denoted as vertices and also link between each atom is represented as an edge between two vertices. A graph derived from a chemical compound is called a molecular graph.

This is an attempt to the application of graph theory in anticancer drugs. In this work, the drugs are taken and using the degree based calculations, few topological indices are computed.

Here, anticancer drug considered is Hyaluronic acid(HA), in which paclitaxel conjugate molecular structure is studied. The compound which is available naturally is HA. HA is also called as hyaluronan. It is very unique form in the plasma membrane instead of the Golgi Apparatus. HA contributes most importantly to cell multiplication and relocation. It helps in the development of fewer malignant tumours [8]. HA is a robust anticancer drug because of its distinct, biodegradable, bio-compatible, non-toxic, hydro-philic and non-immutable characters. Also, HA present in organs are experimented using expression on many of the tumour cells. Because of the fast improving competition, the main goal is to manufacture drugs containing HA for cancer treatments [15]. HA is an efficient drug bearer. The constructive drugs which are mostly used in various categories of cancers like ovarian, breast, lung and bladder is Paclitaxel (PTX). PTX has its own disadvantages, they are less soluble and has side effects. Ringsdorf introduced the concept of developing polymeric conjugate, outlined to provide small hydrophobic drug molecules to their sites of action. The important benefits of Hyaluronic acid-Paclitaxel (HA-PTX) conjugate are in the water solubility and recovery of anti tumour efficiently.

The topological index is described as a non-empirical numerical measure, which represent molecular structures and its branching pattern. And also at this end, topological indices will map each molecule structure to a real number and used as a descriptor of a molecule under testing. Various important indices are

applicable in chemical-engineering (e.g., QSPR/QSAR study) for establishing relationship between the molecular structures and the physical and chemical characters (see, [1, 5, 15, 18, 21]).

Usually, the chemical compound is modelled as a graph by considering the atoms as the vertex set and the links connecting them as the edge set (see, [16, 19, 20, 22]). In a similar fashion, these anticancer drugs under this study are considered as chemical compounds and the said topological indices (TI's) are determined.

Applications of graph theory are QSAR, QSPR and QSTR where chemists or pharmacists are welcome to use this data for further research study. There are several degree based indices introduced to test the properties of compounds and drugs, which are used in the manufacture of anticancer drugs.

All the graphs used in this paper are simple graphs, i.e. cycle free and not directed ([5]).

Let $G = (V(G), E(G))$ be a molecular graph, where $V(G)$ and $E(G)$ are vertices and edges which corresponds to atoms set and chemical bonds set.

Definition 1.1. For a graph G, the first $F-index$ and second $F-$index [4, 12], are defined respectively as

$$F_1(G) = \sum_{e=uv\in E(G)} \big[(d_G(u))^2 + (d_G(v))^2\big],$$
$$F_2(G) = \sum_{e=uv\in E(G)} \big[(d_G(u))^2 \times (d_G(v))^2\big].$$

Definition 1.2. The multiplicative first $F-$index was introduced by Bhanumathi [2] and Ghobadi [6], is defined as

$$F_1II(G) = \prod_{e=uv\in E(G)} \big[(d_G(u))^2 + (d_G(v))^2\big].$$

Definition 1.3. For a graph G, the second multiplicative F-index [12] is defined as

$$F_2II(G) = \prod_{e=uv\in E(G)} \big[(d_G(u))^2 \times (d_G(v))^2\big].$$

Definition 1.4. The multiplicative first and second hyper F-index [11, 12] for a graph G are

$$HF_1II(G) = \prod_{e=uv\in E(G)} \big[(d_G(u))^2 + (d_G(v))^2\big]^2,$$
$$HF_2II(G) = \prod_{e=uv\in E(G)} \big[(d_G(u))^2 \times (d_G(v))^2\big]^2.$$

Definition 1.5. For a graph G, both multiplicative sum connectivity and multiplicative product connectivity F-indices [12] are

$$SFII(G) = \prod_{e=uv\in E(G)} \frac{1}{\sqrt{(d_G(u))^2 + (d_G(v))^2}},$$
$$PFII(G) = \prod_{e=uv\in E(G)} \frac{1}{\sqrt{(d_G(u))^2 \times (d_G(v))^2}}.$$

Definition 1.6. For a graph G, general multiplicative first and second F-indices are stated [12, 13] as

$$F_1^k II(G) = \prod_{e=uv\in E(G)} \left[(d_G(u))^2 + (d_G(v))^2\right]^k,$$
$$F_2^k II(G) = \prod_{e=uv\in E(G)} \left[(d_G(u))^2 \times (d_G(v))^2\right]^k.$$

Definition 1.7. The multiplicative atom bond connectivity F-index [12, 14] for a graph G is

$$ABCFII(G) = \prod_{e=uv\in E(G)} \sqrt{\frac{(d_G(u))^2 + (d_G(v))^2 - 2}{(d_G(u))^2 \times (d_G(v))^2}}.$$

Definition 1.8. For a graph G, multiplicative geometric arithmetic F-index [12] can be defined as

$$GAFII(G) = \prod_{e=uv\in E(G)} \frac{2\sqrt{(d_G(u))^2 \times (d_G(v))^2}}{(d_G(u))^2 + (d_G(v))^2}.$$

Definition 1.9. Ghobadi et al., defined the first F- polynomial [6] of a graph as

$$F_1(G,x) = \sum_{e=uv\in E(G)} x^{(d_G(u))^2+(d_G(v))^2}.$$

Definition 1.10. The second F-polynomial, the first and second hyper F-polynomial [6, 13] of a graph are stated as

$$F_2(G,x) = \sum_{e=uv\in E(G)} x^{(d_G(u))^2\times(d_G(v))^2},$$
$$HF_1(G,x) = \sum_{e=uv\in E(G)} x^{\left[(d_G(u))^2+(d_G(v))^2\right]^2},$$
$$HF_2(G,x) = \sum_{e=uv\in E(G)} x^{\left[(d_G(u))^2\times(d_G(v))^2\right]^2}.$$

Definition 1.11. Consider a molecular graph $G = (V, E)$, $d_G(u)$ is the vertex degree of u and $d_G(v)$ is the vertex degree of v then ABC index [7] can be stated as

$$ABC(G) = \sum_{e=uv \in E(G)} \left[\sqrt{\frac{(d_G(u) + d_G(v) - 2)}{(d_G(u)) \times (d_G(v))}} \right].$$

Definition 1.12. In chemical graph theory field, there are some new degree based graph types, which plays an important role. These topological indices are required for finding total surface area and heat-formation of various chemical compounds. These graphs types are as follow symmetric division index [9],

$$SSD(G) = \sum_{e=uv \in E(G)} \left[\frac{P}{Q} + \frac{Q}{P} \right],$$

where $P = \min [d_G(u), d_G(v)]$ and $Q = \max [d_G(u), d_G(v)]$.

In section 2, main results obtained are highlighted with detail proofs and the calculations of topological indices on the family of molecular graphs.

2. Results and discussions

In this segment, our goal is to represent few topological indices of HA-paclitaxel conjugates. The important method used here separates links which results the molecular structure graph in to few parts. In the below discussions, consider G_n as the molecular graph of HA-PTX conjugates with n units of linear iteration. In Figure 2 and 3 denotes their equivalent molecular graphs G_1 and G_3 with $n = 1$ and $n = 3$.

By considering the graph representation and computation, the vertex set and the edge set are respectively $|V(G_n)|$=87n and $|E(G_n)|$=96n.

If δ and Δ are the lowest and highest degrees of the graph G_n, respectively describes:

(i) $E_i = \{e = uv \in E(G_n)/d_G(u) + d_G(v) = i\}$, for every i with $2\delta \leq i \leq 2\Delta$;

(ii) $E_j^* = \{e = uv \in E(G_n)/d_G(u) \times d_G(v) = j\}$, for every j with $\delta^2 \leq j \leq \Delta^2$.

In specific, the sets E_i and E_j^* might be non-empty. Now with Δ= 4 and $\delta \geq 1$, we notice that the G_n sides can be degraded with the following subsets according to E_i and E_j^* definitions:

(i) $E_3 = E_2^* : d_G(u) = 1$ and $d_G(v) = 2$;

(ii) $E_{4,1} = E_3^* : d_G(u) = 1$ and $d_G(v) = 3$;

(iii) $E_{5,1} = E_{4,1}^* : d_G(u) = 1$ and $d_G(v) = 4$;

(iv) $E_{4,2} = E_{4,2}^* : d_G(u) = 2$ and $d_G(v) = 2$;

(v) $E_{5,2} = E_6^* : d_G(u) = 2$ and $d_G(v) = 3$;

(vi) $E_{6,1} = E_8^* : d_G(u) = 2$ and $d_G(v) = 4$;

(vii) $E_{6,2} = E_9^* : d_G(u) = d_G(v) = 3$;

(viii) $E_7 = E_{12}^* : d_G(u) = 3$ and $d_G(v) = 4$;

(ix) $E_8 = E_{16}^* : d_G(u) = d_G(v) = 4$.

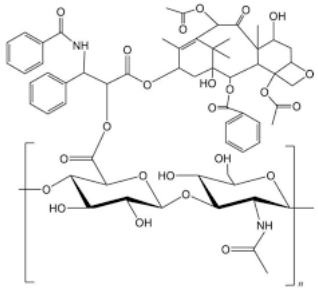


Figure 1: The HA-PTX conjugate molecular structure.

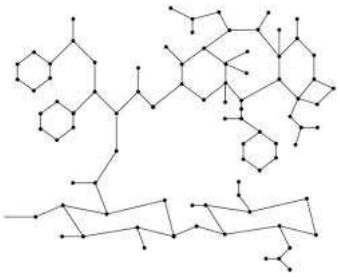


Figure 2: The corresponding molecular graph G_1: n = 1.

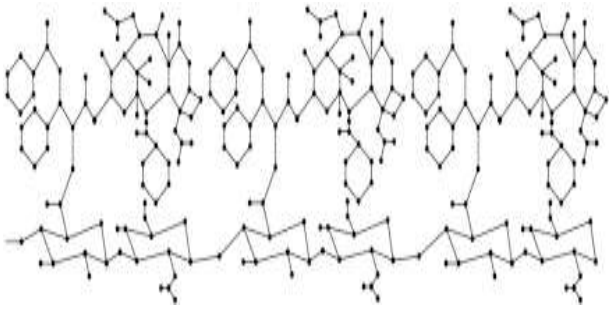


Figure 3: The corresponding molecular graph G_3: n = 3.

Theorem 2.1. *The general multiplicative first F-index of HA-PTX conjugates with n units linear iteration, denoted by molecular graph G_n, is given by*

$$F_1^K II(G_n) = 5^{k(35n)} \times 2^{k(90n+2)} \times 9^{k(19n-1)} \times 13^{k(33n-1)} \times 17^{k(4n)}.$$

Proof. Using the graph representation studies and observations, it is easy to define $|E_3| = |E_2^*| = 2n$, $|E_{4,1} \cap E_3^*| = 16n$, $|E_{5,1} \cap E_{4,1}^*| = 4n$, $|E_{4,2}| = |E_{4,2}^*| = 13n + 1$, $|E_{5,2}| = |E_6^*| = 33n - 1$, $|E_{6,1} \cap E_8^*| = 3n$, $|E_{6,2} \cap E_9^*| = 19n - 1$, $|E_7 \cap E_{12}^*| = 7n$ and $|E_8| = |E_{16}^*| = 2n$. For a graph G_n, the general multiplicative first F- index is

$$\begin{aligned}
F_1^k II(G_n) &= \prod_{e=uv \in E(G)} \left[(d_G(u))^2 + (d_G(v))^2\right]^k \\
&= \prod_{uv \in E_3} \left[(1)^2 + (2)^2\right]^k \times \prod_{uv \in E_3^*} \left[(1)^2 + (3)^2\right]^k \times \prod_{uv \in E_{4,1}^*} \left[(1)^2 + (4)^2\right]^k \\
&\quad \times \prod_{uv \in E_{4,2}} \left[(2)^2 + (2)^2\right]^k \times \prod_{uv \in E_6^*} \left[(2)^2 + (3)^2\right]^k \times \prod_{uv \in E_8^*} \left[(2)^2 + (4)^2\right]^k \\
&\quad \times \prod_{uv \in E_9^*} \left[(3)^2 + (3)^2\right]^k \times \prod_{uv \in E_{12}^*} \left[(3)^2 + (4)^2\right]^k \times \prod_{uv \in E_{16}^*} \left[(4)^2 + (4)^2\right]^k \\
&= 5^{k(35n)} \times 2^{k(90n+2)} \times 9^{k(19n-1)} \times 13^{k(33n-1)} \times 17^{k(4n)}.
\end{aligned}$$

□

We get the following results by using Theorem 2.1.

Corollary 2.1. *The multiplicative first F- index of a graph G_n is*

$$F_1 II(G_n) = 5^{(35n)} \times 2^{(90n+2)} \times 9^{(19n-1)} \times 13^{(33n-1)} \times 17^{(4n)}.$$

Proof. Put $k = 1$ in Theorem 2.1, we obtain the desired result. □

Corollary 2.2. *The multiplicative first hyper F- index of a graph G_n is*

$$HF_1 II(G_n) = 5^{(70n)} \times 2^{(180n+4)} \times 9^{(38n-2)} \times 13^{(66n-2)} \times 17^{(8n)}.$$

Proof. Put $k = 2$ in Theorem 2.1, we obtain the desired result. □

Corollary 2.3. *The multiplicative sum connectivity F- index of a graph G_n is*

$$SFII(G_n) = \left(\frac{1}{\sqrt{5}}\right)^{35n} \times \left(\frac{1}{\sqrt{2}}\right)^{90n+2} \times \left(\frac{1}{\sqrt{9}}\right)^{19n-1} \times \left(\frac{1}{\sqrt{13}}\right)^{33n-1} \times \left(\frac{1}{\sqrt{17}}\right)^{4n}.$$

Proof. Put $k = \frac{-1}{2}$ in Theorem 2.1, we get the desired result. □

Theorem 2.2. *The general multiplicative second F- index of a graph G_n is*

$$F_2^K II(G_n) = 4^{k(100n-1)} \times 9^{k(78n-3)}$$

Proof.

$$\begin{aligned}
F_2^k II(G_n) &= \prod_{e=uv\in E(G)} \left[(d_G(u))^2 \times (d_G(v))^2\right]^k \\
&= \prod_{uv\in E_3} \left[(1)^2(2)^2\right]^k \times \prod_{uv\in E_3^*} \left[(1)^2(3)^2\right]^k \times \prod_{uv\in 4,1^*} \left[(1)^2(4)^2\right]^k \\
&\quad \times \prod_{uv\in E_{4,2}} \left[(2)^2(2)^2\right]^k \times \prod_{uv\in E_6^*} \left[(2)^2(3)^2\right]^k \times \prod_{uv\in E_8^*} \left[(2)^2(4)^2\right]^k \\
&\quad \times \prod_{uv\in E_9^*} \left[(3)^2(3)^2\right]^k \times \prod_{uv\in E_{12}^*} \left[(3)^2(4)^2\right]^k \times \prod_{uv\in E_{16}^*} \left[(4)^2(4)^2\right]^k \\
&= 4^{k(100n-1)} \times 9^{k(78n-3)}.
\end{aligned}$$

□

The following results are obtained by using Theorem 2.2.

Corollary 2.4. *The multiplicative second F- index of a graph G_n is*

$$F_2 II(G_n) = 4^{(100n-1)} \times 9^{(78n-3)}.$$

Proof. Put $k = 1$ in Theorem 2.2, we get the required result. □

Corollary 2.5. *The multiplicative second hyper F- index of a graph G_n is*

$$HF_2 II(G_n) = 4^{(200n-2)} \times 9^{(156n-6)}.$$

Proof. Put $k = 2$ in Theorem 2.2, we get the required result. □

Corollary 2.6. *The multiplicative product connectivity F- index of a graph G_n is*

$$PFII(G_n) = \left(\frac{1}{\sqrt{4}}\right)^{100n-1} \times \left(\frac{1}{\sqrt{9}}\right)^{78n-3}.$$

Proof. Put $k = \frac{-1}{2}$in Theorem 2.2, we get the required result. □

Theorem 2.3. *The multiplicative atom bond connectivity F- index of a graph G_n is*

$$\begin{aligned}
ABCFII(G_n) &= \left(\sqrt{\frac{3}{4}}\right)^{2n} \times \left(\sqrt{\frac{8}{9}}\right)^{16n} \times \left(\sqrt{\frac{15}{16}}\right)^{4n} \\
&\quad \times \left(\sqrt{\frac{6}{16}}\right)^{13n+1} \times \left(\sqrt{\frac{11}{36}}\right)^{33n-1} \\
&\quad \times \left(\sqrt{\frac{18}{64}}\right)^{3n} \times \left(\sqrt{\frac{16}{81}}\right)^{19n-1} \times \left(\sqrt{\frac{23}{144}}\right)^{7n} \times \left(\sqrt{\frac{30}{256}}\right)^{2n}.
\end{aligned}$$

Proof.

$$ABCFII(G_n) = \prod_{e=uv\in E(G)} \left[\sqrt{\frac{(d_G(u))^2+(d_G(v))^2-2}{(d_G(u))^2\times(d_G(v))^2}}\right]$$
$$= \left(\sqrt{\frac{(1)^2+(2)^2-2}{(1)^2(2)^2}}\right)^{2n} \times \left(\sqrt{\frac{(1)^2+(3)^2-2}{(1)^2(3)^2}}\right)^{16n} \times \left(\sqrt{\frac{(1)^2+(4)^2-2}{(1)^2(4)^2}}\right)^{4n}$$
$$\times\left(\sqrt{\frac{(2)^2+(2)^2-2}{(2)^2(2)^2}}\right)^{13n+1} \times\left(\sqrt{\frac{(2)^2+(3)^2-2}{(2)^2(3)^2}}\right)^{33n-1} \times\left(\sqrt{\frac{(2)^2+(4)^2-2}{(2)^2(4)^2}}\right)^{3n}$$
$$\times \left(\sqrt{\frac{(3)^2+(3)^2-2}{(3)^2(3)^2}}\right)^{19n-1} \times \left(\sqrt{\frac{(3)^2+(4)^2-2}{(3)^2(4)^2}}\right)^{7n} \times \left(\sqrt{\frac{(4)^2+(4)^2-2}{(4)^2(4)^2}}\right)^{2n}$$
$$=\left(\sqrt{\frac{3}{4}}\right)^{2n} \times\left(\sqrt{\frac{8}{9}}\right)^{16n} \times\left(\sqrt{\frac{15}{16}}\right)^{4n} \times\left(\sqrt{\frac{6}{16}}\right)^{13n+1} \times\left(\sqrt{\frac{11}{36}}\right)^{33n-1} \times\left(\sqrt{\frac{18}{64}}\right)^{3n}$$
$$\times \left(\sqrt{\frac{16}{81}}\right)^{19n-1} \times \left(\sqrt{\frac{23}{144}}\right)^{7n} \times \left(\sqrt{\frac{30}{256}}\right)^{2n}.$$

□

Theorem 2.4. *The multiplicative geometric-arithmetic F- index of a graph G_n is*

$$\left(\frac{24}{25}\right)^{2n} \times \left(\frac{12}{13}\right)^{16n} \times \left(\frac{4}{5}\right)^{4n} \times \left(\frac{3}{5}\right)^{33n-1} \times \left(\frac{8}{17}\right)^{3n}.$$

Proof.

$$GAFII(G_n) = \prod_{e=uv\in E(G)} \frac{2\sqrt{(d_G(u))^2\times(d_G(v))^2}}{(d_G(u))^2+(d_G(v))^2}$$
$$= \left(\frac{2\sqrt{(1)^2(2)^2}}{(1)^2+(2)^2}\right)^{2n} \times \left(\frac{2\sqrt{(1)^2(3)^2}}{(1)^2+(3)^2}\right)^{16n} \times \left(\frac{2\sqrt{(1)^2(4)^2}}{(1)^2+(4)^2}\right)^{4n}$$
$$\times \left(\frac{2\sqrt{(2)^2(2)^2}}{(2)^2+(2)^2}\right)^{13n+1} \times \left(\frac{2\sqrt{(2)^2(3)^2}}{(2)^2+(3)^2}\right)^{33n-1} \times \left(\frac{2\sqrt{(2)^2(4)^2}}{(2)^2+(4)^2}\right)^{3n}$$
$$\times \left(\frac{2\sqrt{(3)^2(3)^2}}{(3)^2+(3)^2}\right)^{19n-1} \times \left(\frac{2\sqrt{(3)^2(4)^2}}{(3)^2+(4)^2}\right)^{7n} \times \left(\frac{2\sqrt{(4)^2(4)^2}}{(4)^2+(4)^2}\right)^{2n}$$
$$= \left(\frac{24}{25}\right)^{2n} \times \left(\frac{12}{13}\right)^{16n} \times \left(\frac{4}{5}\right)^{4n} \times \left(\frac{3}{5}\right)^{33n-1} \times \left(\frac{8}{17}\right)^{3n}.$$

□

Theorem 2.5. *For the graph G_n, the first F- polynomial is*

$$(2n)x^5 + (16n)x^{10} + (4n)x^{17} + (13n+1)x^8 + (33n-1)x^{13} + (3n)x^{20} + (19n-1)x^{18} + (7n)x^{25} + (2n)x^{32}.$$

Proof.

$$\begin{aligned}
F_1(G_n, x) &= \sum_{e=uv\in E(G)} x^{(d_G(u))^2+(d_G(v))^2} \\
&= (2n)\left[x^{(1+4)}\right] + (16n)\left[x^{(1+9)}\right] + (4n)\left[x^{(1+16)}\right] + (13n+1)\left[x^{(4+4)}\right] \\
&+ (33n-1)\left[x^{(4+9)}\right] + (3n)\left[x^{(4+16)}\right] + (19n-1)\left[x^{(9+9)}\right] + (7n)\left[x^{(9+16)}\right] \\
&+ (2n)\left[x^{(16+16)}\right] \\
&= (2n)x^5 + (16n)x^{10} + (4n)x^{17} + (13n+1)x^8 + (33n-1)x^{13} + (3n)x^{20} \\
&+ (19n-1)x^{18} + (7n)x^{25} + (2n)x^{32}.
\end{aligned}$$

□

Theorem 2.6. *For the graph G_n, the second F- polynomial is*

$$(2n)x^4 + (16n)x^9 + (17n+1)x^{16} + (33n-1)x^{36} + (3n)x^{64} + (19n-1)x^{81} + (7n)x^{144} + (2n)x^{256}.$$

Proof.

$$\begin{aligned}
F_2(G_n, x) &= \sum_{e=uv\in E(G)} x^{(d_G(u))^2\times(d_G(v))^2} \\
&= (2n)\left[x^{(1)(4)}\right] + (16n)\left[x^{(1)(9)}\right] + (4n)\left[x^{(1)(16)}\right] + (13n+1)\left[x^{(4)(4)}\right] \\
&+ (33n-1)\left[x^{(4)(9)}\right] + (3n)\left[x^{(4)(16)}\right] + (19n-1)\left[x^{(9)(9)}\right] + (7n)\left[x^{(9)(16)}\right] \\
&+ (2n)\left[x^{(16)(16)}\right] \\
&= (2n)x^4 + (16n)x^9 + (17n+1)x^{16} + (33n-1)x^{36} + (3n)x^{64} \\
&+ (19n-1)x^{81} + (7n)x^{144} + (2n)x^{256}.
\end{aligned}$$

□

Theorem 2.7. *For the graph G_n, the first hyper F- polynomial is*

$$(2n)x^{10} + (16n)x^{20} + 4n(x^{34}) + (13n+1)x^{16} + (33n-1)x^{26} + (3n)x^{40} + (19n-1)x^{36} + (7n)x^{50} + (2n)x^{64}.$$

Proof.

$$HF_1(G_n, x) = \sum_{e=uv\in E(G)} x^{\left[(d_G(u))^2+(d_G(v))^2\right]^2}$$
$$= (2n)\left[x^{(1+4)^2}\right] + (16n)\left[x^{(1+9)^2}\right] + (4n)\left[x^{(1+16)^2}\right] + (13n+1)\left[x^{(4+4)^2}\right]$$
$$+ (33n-1)\left[x^{(4+9)^2}\right] + (3n)\left[x^{(4+16)^2}\right] + (19n-1)\left[x^{(9+9)^2}\right] + (7n)\left[x^{(9+16)^2}\right]$$
$$+ (2n)\left[x^{(16+16)^2}\right]$$
$$= (2n)x^{10} + (16n)x^{20} + 4n(x^{34}) + (13n+1)x^{16} + (33n-1)x^{26} + (3n)x^{40}$$
$$+ (19n-1)x^{36} + (7n)x^{50} + (2n)x^{64}.$$

□

Theorem 2.8. *For the graph, G_n the second hyper F- polynomial is*

$$(2n)x^8 + (16n)x^{18} + (17n+1)x^{32} + (33n-1)x^{72} + (3n)x^{128} + (19n-1)x^{162}$$
$$+ (7n)x^{288} + (2n)x^{512}.$$

Proof.

$$HF_2(G_n, x) = \sum_{e=uv\in E(G)} x^{\left[(d_G(u))^2\times(d_G(v))^2\right]^2}$$
$$= (2n)\left[x^{(1)(4)}\right]^2 + (16n)\left[x^{(1)(9)}\right]^2 + (4n)\left[x^{(1)(16)}\right]^2 + (13n+1)\left[x^{(4)(4)}\right]^2$$
$$+ (33n-1)\left[x^{(4)(9)}\right]^2 + (3n)\left[x^{(4)(16)}\right]^2 + (19n-1)\left[x^{(9)(9)}\right]^2 + (7n)\left[x^{(9)(16)}\right]^2$$
$$+ (2n)\left[x^{(16)(16)}\right]^2$$
$$= (2n)x^8 + (16n)x^{18} + (17n+1)x^{32} + (33n-1)x^{72} + (3n)x^{128} + (19n-1)x^{162}$$
$$+ (7n)x^{288} + (2n)x^{512}.$$

□

Theorem 2.9. *For a graph G_n, The ABC index will be*

$$ABC(G_n) = 38n - \frac{18}{27}.$$

Proof.

$$ABC(G_n) = \sum_{e=uv\in E(G)} \left[\sqrt{\frac{(d_G(u)+d_G(v)-2)}{(d_G(u))(d_G(v))}}\right]$$
$$= \sum_{uv\in E_3} \left[\sqrt{\frac{(1+2-2)}{(1)(2)}}\right] + \sum_{uv\in E_3^*} \left[\sqrt{\frac{(1+3-2)}{(1)(3)}}\right] + \sum_{uv\in E_{4,1}^*} \left[\sqrt{\frac{(1+4-2)}{(1)(4)}}\right]$$

$$+\sum_{uv\in E_{4,2}}\left[\sqrt{\frac{(2+2-2)}{(2)(2)}}\right]+\sum_{uv\in E_6^*}\left[\sqrt{\frac{(2+3-2)}{(2)(3)}}\right]+\sum_{uv\in E_8^*}\left[\sqrt{\frac{(2+4-2)}{(2)(4)}}\right]$$
$$+\sum_{uv\in E_9^*}\left[\sqrt{\frac{(3+3-2)}{(3)(3)}}\right]+\sum_{uv\in E_{12}^*}\left[\sqrt{\frac{(3+4-2)}{(3)(4)}}\right]+\sum_{uv\in E_{16}^*}\left[\sqrt{\frac{(4+4-2)}{(4)(4)}}\right]$$
$$=51n\left(\sqrt{\frac{1}{2}}\right)+16n\left(\sqrt{\frac{2}{3}}\right)+4n\left(\sqrt{\frac{3}{4}}\right)+(19n-1)\left(\sqrt{\frac{4}{9}}\right)+(7n)\left(\sqrt{\frac{5}{12}}\right)$$
$$+(2n)\left(\sqrt{\frac{6}{16}}\right)$$
$$ABC(G_n)=38n-\frac{18}{27}.$$

□

Theorem 2.10. *The symmetric division index for a graph G_n is $SDD(G_n)=\frac{2369n}{10}-\frac{21}{10}$.*

Proof.

$$SSD(G_n)=\sum_{e=uv\in E(G)}\left[\frac{P(d_G(u),d_G(v))}{Q(d_G(u),d_G(v))}+\frac{Q(d_G(u),d_G(v))}{P(d_G(u),d_G(v))}\right]$$
$$=\sum_{uv\in E_3}\left[\frac{P(1,2)}{Q(1,2)}+\frac{Q(1,2)}{P(1,2)}\right]+\sum_{uv\in E_3^*}\left[\frac{P(1,3)}{Q(1,3)}+\frac{Q(1,3)}{P(1,3)}\right]$$
$$+\sum_{uv\in E_{4,1}^*}\left[\frac{P(1,4)}{Q(1,4)}+\frac{Q(1,4)}{P(1,4)}\right]+\sum_{uv\in E_{4,2}}\left[\frac{P(2,2)}{Q(2,2)}+\frac{Q(2,2)}{P(2,2)}\right]$$
$$+\sum_{uv\in E_6^*}\left[\frac{P(2,3)}{Q(2,3)}+\frac{Q(2,3)}{P(2,3)}\right]+\sum_{uv\in E_8^*}\left[\frac{P(2,4)}{Q(2,4)}+\frac{Q(2,4)}{P(2,4)}\right]$$
$$+\sum_{uv\in E_9^*}\left[\frac{P(3,3)}{Q(3,3)}+\frac{Q(3,3)}{P(3,3)}\right]+\sum_{uv\in E_{12}*}\left[\frac{P(3,4)}{Q(3,4)}+\frac{Q(3,4)}{P(3,4)}\right]$$
$$+\sum_{uv\in E_{16}*}\left[\frac{P(4,4)}{Q(4,4)}+\frac{Q(4,4)}{P(4,4)}\right]$$
$$=2n\left(\frac{5}{2}\right)+16n\left(\frac{10}{3}\right)+4n\left(\frac{17}{4}\right)+(13n+1)(2)$$
$$+(33n-1)\left(\frac{13}{6}\right)+3n\left(\frac{20}{8}\right)$$
$$+(19n-1)(2)+7n\left(\frac{25}{12}\right)+2n(2)=\frac{2369n}{10}-\frac{21}{10}.$$

□

3. Conclusion

The main goal of this work is to demonstrate various topological indices on HA-PTX, an anticancer drug using molecular structure analysis and computations. Various indices are represented as expressions to link the chemical structure of drugs with the physical characteristics. The results obtained here helps the experts to decide the characteristics of the drugs used in the treatment of threatening ailment cancer. Experimental set up is not required as the topological indices help to determine physico-chemical properties of anticancer drugs. Topological indices have many advantages over conducting the experiments.

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Accepted: February 25, 2020

Fuzzy stability of sextic functional equation in normed spaces (direct method)

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Abstract. Instituted stability result with regard to the sextic functional equation

$$f(nx+y)+f(nx-y)+f(x+ny)+f(x-ny)=(n^4+n^2)[f(x+y)+f(x-y)] \\ +2(n^6-n^4-n^2+1)[f(x)+f(y)]$$

in Fuzzy normed space.
Keywords: generalized stability, sextic functional equation (S.Fun.E), fuzzy normed spaces (F.N. spaces), fuzzy Banach spaces (F.B. spaces).

1. Introduction and preliminaries

The theory of Fuzzy space has much advance . We used the concept of F.N space mentioned in the references [1, 5] to test a fuzzy version of generalized stability of the functional equation (this equation is from the reference [3, 4])

$$\begin{aligned} &f(nx+y)+f(nx-y)+f(x+ny)+f(x-ny) \\ &=(n^4+n^2)[f(x+y)+f(x-y)] \\ &+2(n^6-n^4-n^2+1)[f(x)+f(y)] \end{aligned} \tag{1}$$

in the F.N vector space setting.

Definition 1.1 ([2]). *Let be a real linear space. A function $N : X \times \mathbb{R} \longrightarrow [0,1]$ (the so -called fuzzy subset) is said to be a fuzzy norm on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$;*

$(N1)$ $N(x,c) = 0$, *for* $c \leq 0$;

$(N2)$ $x = 0$ *if and only if* $N(x,t) = 1$, *for all* $c > 0$;

$(N3)$ $N(cx,t) = N(x, \frac{t}{|c|})$, *if* $c \neq 0$;

$(N4)$ $N(x+y, s+t) \geq \min\{(x,s),(y,t)\}$;

$(N5)$ $N(x,.)$ *is a non-decreasing function on* $\mathbb{R}$ *and* $\lim_{t\to\infty} N(x,t) = 1$;

$(N6)$ *for* $x = 0$, $N(x, .)$ *is (upper semi) continuous on* $\mathbb{R}$.

The pair $N(X, N)$ *is called a fuzzy normed linear space. One may regard* $N(x, t)$ *as the truth value of the statement the norm of* x *is less than or equal to the real numbert.*

Example 1.1 ([2])**.** Let $(X, \|.\|)$ be a normed linear space. Then

$$N(x,t) := \begin{cases} \dfrac{t}{t+\|.\|}, & \text{if } t > 0, \\ 0, & \text{if } t \leq 0, \end{cases}$$

is a Fuzzy norm on X.

Definition 1.2 ([5])**.** *Let* (X, N) *be a fuzzy normed vector space. A sequence* x_n *in* X *is said to be convergent or converge if there exists an* x *in* X *such that* $\lim_{n\to\infty} N(x_n - x, t) = 1$, *for all* $t > 0$. *In this case* x *is called the limit of the sequence* $\{x_n\}$ *and we denote it by* $N - \lim_{n\to\infty} x_n = x$.

Definition 1.3 ([1])**.** *Let* (X, N) *be a fuzzy normed vector space. A sequence* x_n *in* X *is called Cauchy if for each* $\varepsilon > 0$ *and each* $t > 0$ *there exists an* $n_0 \in \mathbb{N}$ *such that for all* $n \geq n_0$ *and all* $p > 0$, *we have* $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is well-known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed vector space is called a fuzzy Banach space.

2. Fuzzy stability

Theorem 2.1. *Let* X *be linear space and let* $(Z, \aleph')$ *be F.N space, let* $\psi : X \times X \to Z$ *a function such that for some* $0 < x < n^6$

$$\aleph'(\psi(nx, 0), t) \geq \aleph'(\alpha\psi(x, 0), t) \tag{2}$$

and

$$\lim_{\imath\to\infty} \aleph'(\psi(n^\imath x, n^\imath y), n^{6\imath} t) = 1, \tag{3}$$

$\forall x, y \in X$ *and* $t \geq 0$. *Let* $(Y, \aleph)$ *be a fuzzy Banach space and* $f : X \longrightarrow Y$ *be a* ψ*-approximately sextic mapping in the sense that*

$$\aleph(D_s f(x, y), t) \geq \aleph'(\psi(x, 0), t) \tag{4}$$

$\forall x, y \in X$, $t > 0$ *where* $D_s f(x, y) = f(nx + y) + f(nx - y) + f(x + ny) + f(x - ny) - (n^4 + n^2)[f(x + y) + f(x - y)] - 2(n^6 - n^4 - n^2 + 1)[f(x) + f(y)]$. *Then exists sextic mapping* $A : X \longrightarrow Y$ *s.t*

$$\aleph(A(x) - f(x), t) \geq \aleph'(\psi(x, 0), 2t(n^6 - \alpha)) \tag{5}$$

$\forall x \in X$, $t > 0$.

Proof. Impose $y = 0$ in (4)

$$\aleph(2f(nx) - 2n^6 f(x), t) \geq \aleph'(\psi(x, 0), t), \tag{6}$$

$\forall x \in X$, $t > 0$. Replacement x by $n^k x$ in (6) and employ (2) we obtain

$$\aleph(2f(n^{k+1}x)-2n^6 f(n^k x), t)\geq\aleph'(\psi(n^k x, 0), t)\geq\aleph'(\alpha^k\psi(x, 0), t)=\aleph'(\psi(x, 0), \frac{t}{\alpha^k}),$$

$k \geq 0$. Replacement t by $\alpha^k t$ we see that

$$\aleph\left(\frac{f(n^{k+1}x)}{n^{6(k+1)}} - \frac{f(n^k x)}{n^{6k}}, \frac{\alpha^k t}{2n^{6(k+1)}}\right) \geq \aleph'(\psi(x, 0), t), \tag{7}$$

$\forall x \in X$, $t > 0$, $k \geq 0$. From

$$\frac{f(n^j x)}{n^{6j}} - f(x) = \sum_{\imath=0}^{j=1}\left(\frac{f(n^{\imath+1}x)}{n^{6(\imath+1)}} - \frac{f(n^\imath x)}{n^{6\imath}}\right)$$

and (7) perform

$$\begin{aligned}
&\aleph\left(\frac{f(n^j x)}{n^{6j}} - f(x), \sum_{\imath=0}^{j-1}\frac{\alpha^\imath t}{2n^{6(\imath+1)}}\right)\\
&\geq \min\bigcup_{\imath=0}^{j-1}\left(\aleph\left(\frac{f(n^{\imath+1}x)}{n^{6(\imath+1)}} - \frac{f(n^\imath x)}{n^{6\imath}}, \frac{\alpha^\imath t}{2n^{6(\imath+1)}}\right)\right) \qquad (8)\\
&\geq \aleph'(\alpha\psi(x, 0), t),
\end{aligned}$$

$\forall x \in X$, $t > 0$, $j > 0$. By replacement x by $n^m x$ in (8) we observe that

$$\begin{aligned}
&\aleph\left(\frac{f(n^{j+m}x)}{n^{6j+6m}} - \frac{f(n^m x)}{n^{6m}}, \sum_{\imath=0}^{j-1}\frac{\alpha^\imath t}{2n^{6(\imath+6m)}}\right)\\
&\geq \aleph'(\psi(n^m x, 0), t) \geq \aleph'(\alpha^m\psi(x, 0), t) = \aleph'\left(\psi(x, 0), \frac{t}{\alpha^m}\right) \qquad (9)
\end{aligned}$$

whence

$$\aleph\left(\frac{f(n^{j+m}x)}{n^{6j+6m}} - \frac{f(n^m x)}{n^{6m}}, \sum_{\imath=m}^{j-1+m}\frac{\alpha^\imath t}{2n^{6(\imath+1)}}\right) \geq \aleph'(\psi(x, 0), t),$$

$\forall x \in X$, $t > 0$, $j \geq 0$,$m > 0$. Hence

$$\aleph\left(\frac{f(n^{j+m}x)}{n^{6j+6m}} - \frac{f(n^m x)}{n^{6m}}, t\right) \geq \aleph'\left(\psi(x, 0), \frac{t}{\sum_{\imath=m}^{j-1+m}\frac{\alpha^\imath t}{2n^{6(\imath+1)}}}\right), \tag{10}$$

$\forall x \in X$, $t > 0$, $j \geq 0$,$m \geq 0$. Since $0 < \alpha < n^6$ and $\sum_{\imath=0}^{\infty}(\frac{\alpha}{n^6})^\imath < \infty$.

The Cauchy concept for convergence and (N5) show that $\{\frac{f(n^m x)}{n^{6m}}\}$ is a Cauchy sequence in $(Y, \aleph)$. Since $(Y, \aleph)$ is a F.B. space, this sequence converges to some point $A(x) \in Y$.Set$x \in X$ and impose $m = 0$ in (10) to gain

$$\aleph(\frac{f(n^j x)}{n^{6j}} - f(x), t) \geq \aleph' \left(\psi(x, 0), \frac{t}{\sum_{\imath=0}^{j-1} \frac{\alpha^\imath t}{2n^{6(\imath+1)}}} \right),$$

$t > 0$, $j > 0$. From which we obtain

$$\begin{aligned} \aleph(A(x) - f(x), t) &\geq \min\{\aleph \left(C(x) - \frac{f(n^j x)}{n^{6j}}, \frac{t}{2} \right), \aleph \left(\frac{f(n^j x)}{n^{6j}} - f(x), \frac{t}{2} \right)\} \\ &\geq \aleph' \left(\psi(x, 0), \frac{t}{\sum_{\imath=0}^{j-1} \frac{\alpha^\imath t}{2n^{6(\imath+1)}}} \right), \end{aligned}$$

for $\imath$ large enough. Taking the limit an $j \longrightarrow \infty$ and using (N6) we gain

$$\aleph(A(x) - f(x), t) \geq \aleph'(\psi(x, 0), 2t(n^6 - \alpha))$$

replacement x, y by $n^\imath x, n^\imath y$ respectively in (4) to gain

$$\aleph \left(\frac{D_s f(n^\imath x, n^\imath y)}{n^{6\imath}}, \frac{t}{n^{6\imath}} \right) \geq \aleph'(\psi(n^{6\imath} x, n^{6\imath} y), t).$$

Then

$$\aleph \left(\frac{D_s f(n^\imath x, n^\imath y)}{n^{6\imath}}, t \right) \geq \aleph'(\psi(n^\imath x, n^\imath y), n^{6\imath} t),$$

$\forall x \in X$, $\forall t > 0$, since $\lim_{\imath \to \infty} \aleph'(\psi(n^\imath x, n^\imath y), n^{6\imath} t) = 1$ we infer that A complete (5). To state the uniqueness of the S. Fun C, presume exists a S. Fun $D : X \longrightarrow Y$ which satisfies (5) Set $x \in X$. Clearly $A(nx) = n^6 A(x)$ and $D(nx) = n^6 D(x)$ then $A(n^\imath x = n^{6\imath} A(x)$ and $D(n^\imath x = n^{6\imath} D(x)$ $\forall \imath \in \mathbb{N}$. Accompany from (5) that

$$\begin{aligned} \aleph(A(x) - D(x), t) &= \aleph \left(\frac{A(n^\imath x)}{n^{6\imath}} - \frac{D(n^\imath x)}{n^{6\imath}}, t \right) \\ &\geq \min\{\aleph \left(\frac{A(n^\imath x)}{n^{6\imath}} - \frac{f(n^\imath x)}{n^{6\imath}}, \frac{t}{2} \right), \aleph(\frac{A(n^\imath x)}{n^{6\imath}} - \frac{f(n^\imath x)}{n^{6\imath}}, \frac{t}{2}) \\ &\geq \aleph'(\psi(n^\imath x, 0), n^{6\imath} t(n^6 - \alpha)) \geq \aleph' \left(\psi(x, 0), \frac{n^{6\imath} t(n^6 - \alpha)}{\alpha^\imath} \right) \end{aligned}$$

since $\lim_{\imath \to \infty} \frac{n^{6\imath} t(n^6 - \alpha)}{\alpha^\imath} = \infty$, and $0 < \alpha < n^6$ we gain

$$\lim_{\imath \to \infty} \aleph' \left(\psi(x, 0), \frac{n^{6\imath} t(n^6 - \alpha)}{\alpha^\imath} \right) = 1,$$

therefore, $\aleph(A(x) - D(x), t) = 1 \forall t > 0$ whence $A(x) = D(x)$. □

Example 2.1. Let X be a Banach algebra space and Z be a normed space. Denote by $\aleph$ and $\aleph'$ the Fuzzy norms obtained as Example (1.2) on X and Z , respectively . $\psi : X \times X \longrightarrow Z$ realized by $\psi(x, y) = 8n^6(\| x \| + \| y \|)$.

Define $f : X \longrightarrow X$ by $f(x) = x^6 + \| x \| x_0$ for some unit vector $x_0 \in X$. Then

$$\begin{aligned} N(D_s f(x, y), t) &= \frac{t}{t + \| N(D_s f(x, y) \|)} \\ &\geqslant \frac{t}{t + 2(n^6 + n + 2)(\| x \| + \| x \|)} \\ &\geqslant \frac{t}{t + 8n^6(\| x \| + \| x \|)} = \aleph'(\psi(x, y), t) \end{aligned}$$

and

$$\aleph'(\psi(nx, 0), t) = \frac{t}{t + 8n^7 \| x \|} \leqslant \frac{t}{8n^7 \| x \|} = \aleph'(\propto \psi(x, 0), t),$$

$$\lim_{\imath \longrightarrow \infty} \frac{n^{6\imath} t}{n^{6\imath} t + 8n^{6\imath}(\| x \| + \| y \|)} = 1.$$

Hence, conditions of theorem (2) for $\alpha = n$ are complete. Therefore, exists sextic mapping $A : X \longrightarrow X$ s.t

$$\aleph(A(x) - f(x), t) \geqslant \aleph'(\psi(x, 0), 2t(n^6 - n)).$$

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Accepted: April 30, 2020

Analysis for fractional integro-differential equation with time delay

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Abstract. The main purpose of this paper is to solve some linear and nonlinear Caputo fractional Volterra-Fredholm integro-differential equations with delay. In addition, we investigate the convergence analysis for these equations. A semi-analytical method is described and illustrated with examples.
Keywords: convergence, stability, fractional integro-differential equation.

1. Introduction

Fractional calculus (shortly FC) plays a major role in the modeling of mathematical engineering and physical systems. FC is one of the most important branches of mathematical analysis. The basic advantage of this calculus is in a generalization of the concepts of integer-order differentiation and integration [7]. This calculus used to illuminate physical, bioengineering, aerodynamics, biological control, electro-dynamics and electrochemistry numerous models [5, 12, 14, 15].

A delay fractional integro-differential equation is an integro-differential equation in which the fractional derivative of the unknown function at a certain time [1]. In recent years, the fractional differential equations with delays are developed in [6, 16, 18, 19]. If the behavior of systems includes information on the former state, then these systems are called time delay systems. There are various

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fractional differential operators, as Riemann-Liouville, Caputo and Grunwald-Letnikov. Among these operators, we interest to use Caputo fractional operator where, it allows usual initial conditions when operating on integral transform for instance Laplace transform. In particular, Volterra integro- differential equations without delay have been studied [4]. This study contributes to getting a better progressing and closing the above-mentioned powerful information.

There are much more difficult to study the stability analysis for systems. But indeed, the stability is verified for Caputo fractional differential equations and Volterra integro-differential equations in [8, 9, 10, 16, 20, 22].

Decomposition method for solving non-linear integro-differential equations has been presented by Al Khaled [2]. The modified Laplace Adomain decomposition method (MLADM) for solving Volterra integral equation has been discussed by Wazwaz [21]. Then MLADM applied to solve the integro-differential equation [11].

In this paper, we discuss the analysis of fractional constant time delay integro-differential equations with the Caputo fractional derivative. This article focuses on the convergence and stability analysis for the fractional integro-differential equations with constant time delay. Also, we derive the existence and uniqueness of the solution. Finally, we develop and apply MLADM to fractional integro-differential equations with constant time delay.

The outline of this work as follows. Section 1 presents a brief introduction to fractional calculus, recalls the basic notations that help us to get the desired results. Section 2 offers the suggested equation and then investigates the existed unique solution of the proposed problem in this paper. Section 3 states the stability of fractional integro-differential equations with the Caputo derivative. Section 4 proves the convergence of the proposed equation and gives experimental examples.

1.1 Notations and preliminaries

Here, we state some definitions, theorems and preliminary demanded facts that apply to this paper.

Definition 1.1. For $\alpha \in (0,1)$ and a function $u : [0,T] \to \mathbb{R}$ Caputo fractional derivative

$$D^{\alpha}u(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{u^{(1)}(s)ds}{(t-s)^{\alpha}},$$

where $\Gamma(\alpha)$ denotes the gamma function.

Some important mathematical properties of Caputo fractional derivative:

1. The fractional product rule [17]

$$D^{\alpha}[u(t)v(t)] = \sum_{i=1}^{\infty} \binom{\alpha}{i} D^{\alpha-i}u(t)D^{i}v(t), \tag{1.1}$$

where

$$\begin{pmatrix} \alpha \\ i \end{pmatrix} = \frac{\Gamma(\alpha+1)}{\Gamma(i+1)\Gamma(\alpha+1-i)}.$$

2. Linearity of Caputo fractional operater,

$$D^{\alpha}(au(t) \pm bv(t)) = aD^{\alpha}u(t) \pm bD^{\alpha}v(t).$$

3. Caputo fractional operater for composite function,

$$D^{\alpha}u(v(t)) = D^{\alpha}v(t)u^{(1)}v(t). \tag{1.2}$$

4. Caputo fractional differential operator for Riemann fractional integral operator of the same order α is also defined as

$$D^{\alpha}I^{\alpha}u(t) = u(t). \tag{1.3}$$

5. Riemann fractional integral operator of Caputo fractional differential operator for the same order α is also defined as

$$I^{\alpha}D^{\alpha}u(t) = u(t) - \sum_{i=0}^{k-1} u^{(i)}(0)\frac{t^{(i)}}{\Gamma(i+1)},\ k-1 \leq \alpha \leq k \in \mathbb{N}. \tag{1.4}$$

Definition 1.2. The α-order Riemann-Liouville fractional integral operator of u is defined for all $a \leq t \leq b$ by

$$I^{\alpha}u(t) := \begin{cases} u(t), & \alpha = 0, \\ \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t \frac{u(\tau)d\tau}{(t-\tau)^{\alpha-1}}, & \alpha > 0. \end{cases} \tag{1.5}$$

The Riemann fractional integral operator for a negative power α converts into Riemann fractional differential operator, i.e. $\alpha < 0$, $I^{-\alpha} = D^{\alpha}$.

Definition 1.3. $C^1[0,1]$ denotes the space of continuously differentiable functions on [0,1] with the norm

$$\|u(t)\| = \max_{0\leq t\leq 1} |u(t)|. \tag{1.6}$$

Definition 1.4. The functions $G, H \in (C^1[0,1], \mathbb{R})$ satisfy the Lipschitz conditions if there exist real constants ξ, $\mu > 0$, such that

$$|G(u(t)) - G(v(t))| \leq \xi|u-v|,\ |H(u(t)) - H(v(t))| \leq \mu|u-v|, \tag{1.7}$$

where u, $v \in C^1[0,1]$, $0 \leq t \leq 1$.

Definition 1.5. Laplace transform for Caputo fractional derivative and fractional Riemann integral are expressed respectively as follows

$$L(D^{\alpha}u(t)) = s^{\alpha}U(s) - \sum_{j=0}^{n-1} s^{\alpha-j-1}u^{j}(0),\ \alpha \in (n-1,n) \tag{1.8}$$

$$L(I^{\alpha}u(t)) = s^{-\alpha}U(s),\ U(s) = L(u(t)). \tag{1.9}$$

Also, the linearity of Laplace operator

$$L(au(t) + bv(t)) = aL(u(t)) + bL(v(t)). \tag{1.10}$$

Definition 1.6. $L^{\infty}[0,1]$ is the space of all essentially bounded functions with norm

$$\|u\| = \sup_{0\le t\le 1} |u(t)|. \tag{1.11}$$

Theorem 1.1. *Assume that $u, v \in S$, if $(S, \|.\|)$ is a Banach space and $\Phi : S \to S$ satisfying*

$$\|\Phi u - \Phi v\| \le L\|u - \Phi v\| + \chi\|u - v\|, \tag{1.12}$$

where $L \ge 0$, $0 \le \chi < 1$. Suppose that Φ has a fixed point. Then, Φ is a picard $\Phi-$ stable.

2. Statement of the problem

Throughout this paper, we will consider Caputo fractional integro-differential equation of the form:

$$D^{\alpha}u(t) = f\bigg(t, u(t-\tau), \int_0^t G(x,u(x))dx, \int_0^1 H(x,u(x))\ dx\bigg).$$

With another more detailed form:

$$\begin{aligned} D^{\alpha}u(t) = {} & u(t-\tau) + \lambda_1 \int_0^t K(t,x)G(u(x))\ dx + f(t) \\ & + \lambda_2 \int_0^1 M(t,x)H(u(x))\ dx,\ 0 < \alpha \le 1, \end{aligned} \tag{2.1}$$

subject to the initial condition:

$$u(0) = u_0, \tag{2.2}$$

where $D^{\alpha}u$ denotes the Caputo fractional derivative of the unknown function, $u(t) \in S = C([0,1], \mathbb{R}^+)$, τ is the time delay; $\tau > 0$, $K, M : [0,1] \times \mathbb{R} \to \mathbb{R}$ are the kernels of the equation Eq.(2.1), $f(t)$ is an analytic function, $G, H : \mathbb{R} \to \mathbb{R}$ and λ_1, λ_2 are arbitrary constants.

Lemma 2.1. *If the function $f(t)$ is a continuous function, then a nonlinear fractional integro-differential equation Eq.(2.1) is equivalent to the integral equation,*

$$\begin{aligned} u(t) = \; & u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}[u(s-\tau) + f(s)]ds \\ & + \frac{\lambda_1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(\int_0^s K(t,w)G(u(w))dw \right) ds \\ & + \frac{\lambda_2}{\Gamma(\alpha)} \int_0^1 (t-s)^{\alpha-1} \left(\int_0^1 M(t,w)H(u(w))dw \right) ds, \quad t \in [0,1]. \end{aligned} \tag{2.3}$$

Proof. Suppose that $u(t)$ is a solution of Eq.(2.1), let

$$\begin{aligned} w(t) = \; & u(t-\tau) + \lambda_1 \int_0^t K(t,x)G(u(x)) \; dx + f(t) \\ & + \lambda_2 \int_0^1 M(t,x)H(u(x)) \; dx, \end{aligned}$$

then, $w(t) = D^\alpha u(t)$. Since $D^\alpha u(t) = D^\alpha(u(t) - u_0)$, where D^α is a Caputo fractional derivative of order α,

$$w(t) = D^\alpha(u(t) - u_0) = I^{1-\alpha} \frac{d}{dt}(u(t) - u_0).$$

This gives

$$D^{1-\alpha} w(t) = \frac{d}{dt}(u(t) - u_0),$$

upon applying the integral operator I on both sides of the previous relation

$$ID^{1-\alpha} w(t) = u(t) - u_0 + k,$$

where k is a constant of integration. Since $w(t)$ and $u(t) - u_0$ are both continuous functions, for $t = 0$ we obtain $k = 0$. This gives

$$I^\alpha w(t) = u(t) - u_0.$$

By virtue of the definition of $w(t)$, we obtain Eq.(2.3). Conversely, suppose that $u(t)$ is the solution of the Eq.(2.3). Then, it can be written as

$$u(t) = u_0 + I^\alpha w(t),$$

where

$$w(t) \;\; = \;\; u(t-\tau) + \lambda_1 \textstyle\int_0^t K(t,x)G(u(x)) \; dx + f(t) + \lambda_2 \int_0^1 M(t,x)H(u(x)) \; dx,$$

since $w(t)$ is continuous and u_0 is continuous, operating the Caputo fractional differential operator D^α on both sides of Eq.(2.3) then, we get

$$D^\alpha u(t) = D^\alpha u_0 + D^\alpha I^\alpha w(t) = w(t).$$

This gives

$$D^{\alpha}u(t) = u(t-\tau) + \lambda_1 \int_0^t K(t,x)G(u(x))\,dx + f(t) + \lambda_2 \int_0^1 M(t,x)H(u(x))\,dx,$$

From Eq.(2.3), we have $u(0) = u_0$. This completes the proof. □

2.1 Main results

In this section, we will prove the existence and uniqueness of the solution to Eq.(2.1).

Theorem 2.1. *Assume that G, H are nonlinear Lipschitz continuous functions as in* (1.7) *and $f(t)$ is a continuous function, then the non-linear integro-differential equation* (2.1) *has at least a unique solution $u \in S$.*

Proof. First, we examine the continuity and bounded of Φ. Define the operator $\Phi : S \to S$ as follows

$$\begin{aligned}\Phi u(t) &= u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big[u(s-\tau) + f(s)\Big] ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big[\int_0^s K(s,x)F(u(x))dx \\ &+ \int_0^1 M(s,x)G(u(x))dx\Big] ds.\end{aligned}$$

The answer divided into two steps.
Step 1. In fact, for $u \in S$,

$$\begin{aligned}|\Phi u(t)| &\leq |u_0| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big[|u(s-\tau)| + |f(s)|\Big] ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big[\int_0^s |K(s,x)||F(u(x))|dx \\ &+ \int_0^1 |M(s,x)||G(u(x))|dx\Big] ds.\end{aligned}$$

From definition (1.6) and continuity of u

$$|\Phi u(t)| \leq (\|u\|_\infty + \|f\|_\infty)(\frac{t^\alpha}{\Gamma(\alpha+1)}) + \frac{kt^\alpha}{\Gamma(\alpha+1)} \leq \frac{\sigma t^\alpha}{\Gamma(\alpha+1)}.$$

Consequently,

$$|\Phi u(t)| \leq C,$$

where k, σ and C are constants. This mean that Φ is well-defined.
Step 2. Thus,

$$\begin{aligned}|\Phi u(t)-u_0| &\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big[|u(s-\tau)|+|f(s)|\Big]ds\\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big[\int_0^s |K(s,x)||F(u(x))|dx\\ &+ \int_0^1 |M(s,x)||G(u(x))|dx\Big]ds.\end{aligned}$$

From the Lipschitz conditions of the nonlinear functions F, G

$$|\Phi u(t)-u_0| \leq \frac{\omega}{\Gamma(\alpha)}\int_0^t s^{\alpha-1}ds \leq \frac{\omega}{\Gamma(\alpha+1)}t^{\alpha}.$$

Hence, for u bounded, continuous, which leads to Φu is also bounded and continuous.
Note that. Since u is continuous, so Φu is also continuous. Then, Φ is a well defined operator.

Finally, from (2.3), for t_1, $t_2 \in [0,1]$ with $t_2 > t_1$, we gain

$$\begin{aligned}|\Phi u(t_2)-\Phi u(t_1)| &\leq \frac{1}{\Gamma(\alpha)}\Big(|\int_0^{t_2}(t_2-s)^{\alpha-1}\Big[u(s-\tau))+f(s)\Big]ds\\ &- \int_0^{t_1}(t_1-s)^{\alpha-1}\Big[u(s-\tau))+f(s)\Big]ds|\Big)\\ &+ \frac{1}{\Gamma(\alpha)}\Big(|\int_0^{t_2}(t_2-s)^{\alpha-1}\Big[\int_0^s K(s,x)F(u(x))dx\Big]ds\\ &- \int_0^{t_1}(t_1-s)^{\alpha-1}\Big[\int_0^s K(s,x)F(u(x))dx\Big]ds|\Big)\\ &+ \frac{1}{\Gamma(\alpha)}\Big(|\int_0^{t_2}(t_2-s)^{\alpha-1}\Big[\int_0^1 M(s,x)G(u(x))dx\Big]ds\\ &- \int_0^{t_1}(t_1-s)^{\alpha-1}\Big[\int_0^1 M(s,x)G(u(x))dx\Big]ds|\Big).\end{aligned}$$

By some simple calculations,

$$\begin{aligned}|\Phi u(t_2)-\Phi u(t_1)| &\leq \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\Big[|u(s-\tau))|+|f(s)|\Big]ds\\ &+ \frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}\Big[|u(s-\tau))|+|f(s)|\Big]ds\\ &- \frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_2-s)^{\alpha-1}\Big[|u(s-\tau))|+|f(s)|\Big]ds\\ &+ \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\Big[\int_0^s |K(s,x)||F(u(x))|dx\Big]ds\end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \left[\int_0^1 |M(s,x)||G(u(x))|dx \right] ds \\
&+ \quad \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} \left[\int_0^1 |M(s,x)||G(u(x))|dx \right] ds \\
&+ \quad \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} \left[\int_0^s |K(s,x)||F(u(x))|dx \right] ds \\
&- \quad \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_2 - s)^{\alpha-1} \left[\int_0^1 |M(s,x)||G(u(x))|dx \right] ds \\
&- \quad \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_2 - s)^{\alpha-1} \left[\int_0^s |K(s,x)||F(u(x))|dx \right] ds \\
&\to \quad 0,
\end{aligned}
$$

as $t_2 \to t_1$, that is, u is the solution of (2.1). □

On the other hand, we prove the uniqueness of the existed solution Eq. (2.1). Our result is based on the Banach contraction theorem [21].

For $u, v \in S$, we have

$$
\begin{aligned}
|\Phi u(t) - \Phi v(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big[|u(s-\tau) - v(s-\tau)| \Big] ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big[\int_0^s |K(s,x)||F(u(x)) - F(v(x))|dx \\
&+ \int_0^1 |M(s,x)||G(u(x)) - G(v(x))|dx \Big] ds \\
&\leq \frac{\Upsilon}{\Gamma(\alpha)} \|u - v\| \int_0^t s^{\alpha-1} ds \\
&\leq \frac{\Upsilon}{\Gamma(\alpha+1)} \|u - v\| t^{\alpha}.
\end{aligned}
$$

Thus for $\frac{\Upsilon}{\Gamma(\alpha+1)} t^{\alpha} < 1$, we deduce that

$$\|\Phi u(t) - \Phi v(t)\| < \|u(t) - v(t)\|.$$

By the contraction mapping principle, we conclude that Φ has a unique fixed point in S. Consequently, the integro-differential equation (2.1) has a unique solution $u(t)$. In [13], give a definition to check the asymptote stable system.

3. Stability of Caputo fractional integro-differential equations

Definition 3.1. The autonomous system (2.1) with $u(0) = u_0$, is said to be asymptotically stable if and only if

$$\lim_{t \to \infty} \|u(t)\| = 0. \tag{3.1}$$

Theorem 1.1 permits to check stability. First, we investigate the stability as follows:

For $m, n \in \mathbb{N}$

$$\begin{aligned}\|\Phi u_m(t) - \Phi u_n(t)\| = \; & \|\lambda_1 \int_0^t K(t,x)G(u_m(x))\; dx + \lambda_2 \int_0^t M(t,x)H(u_m(x))\; dx \\ & - \lambda_1 \int_0^t K(t,x)G(u_n(x))\; dx - \lambda_2 \int_0^t M(t,x)H(u_n(x))\; dx \\ & + u_m(t-\tau) - u_n(t-\tau)\| \\ & \leq |\lambda_1| \left[\int_0^t \int_0^t K^2(t,s)dsdt\right]^{\frac{1}{2}} \|u_m(t) - u_n(t)\| \\ & + |\lambda_2| \left[\int_0^t \int_0^1 L^2(t,s)dsdt\right]^{\frac{1}{2}} \|u_m(t) - u_n(t)\| \\ & + \|u_m(t-\tau) - u_n(t-\tau)\|,\end{aligned}$$

where

$$\|u_m(t-\tau) - u_n(t-\tau)\| \;\; = \;\; c\|u_m(t) - u_n(t)\|,$$

this leads to

$$\begin{aligned}\|\Phi u_m(t) - \Phi u_n(t)\| \;\; \leq \;\; & \left[|\lambda_1| \left[\int_0^t \int_0^t K^2(t,s)dsdt\right]^{\frac{1}{2}} + c \right. \\ + \;\; & \left. |\lambda_2| \left[\int_0^1 \int_0^1 M^2(t,s)dsdt\right]^{\frac{1}{2}}\right] \|u_m(t) - u_n(t)\|.\end{aligned}$$

Therefore, if $|\lambda_1| < \frac{1}{\zeta_1}$, $|\lambda_2| < \frac{1}{\zeta_2}$, where

$$\zeta_1 = \left[\int_0^t \int_0^t K^2(t,s)dsdt\right]^{\frac{1}{2}}, \; \zeta_2 = \left[\int_0^1 \int_0^1 M^2(t,s)dsdt\right]^{\frac{1}{2}},$$

then, the nonlinear mapping Φ has a fixed point. In particular, for $L = 0$ and

$$\chi = |\lambda_1| \left[\int_0^t \int_0^t K^2(t,s)dsdt\right]^{\frac{1}{2}} + |\lambda_2| \left[\int_0^1 \int_0^1 M^2(t,s)dsdt\right]^{\frac{1}{2}} + c.$$

Theorem 1.1 hold for the nonlinear mapping Φ and hence, it is $\Phi-$ stable. Similarly as in the previous steps, it is easy to indicate that the system is asymptote stable by (3.1).

4. Numerical method for Eq. (2.1)

In this section, we discuss the numerical method with examples.

4.1 Modified Laplace decomposition method [3]

Operating with the Laplace operator L to both sides of Eq. (2.1), yields

$$L[D^{\alpha}u(t)] = L\Bigg[u(t-\tau) + \lambda_1 \int_0^t K(t,x)G(u(x))\,dx + f(t) + \lambda_2 \int_0^1 M(t,x)H(u(x))\,dx\Bigg]. \tag{4.1}$$

By virtue of the linearity property of Laplace transform, we have

$$L[D^{\alpha}u(t)] = L[u(t-\tau)] + \lambda_1 L\Bigg[\int_0^t K(t,x)G(u(x))\,dx\Bigg] + L[f(t)] + \lambda_2 L\Bigg[\int_0^1 M(t,x)H(u(x))\,dx\Bigg], \tag{4.2}$$

hence, from the differentiation rule and formulas of Laplace transform

$$s^{\alpha}U(s) - c = L[u(t-\tau)] + \lambda_1 L\Bigg[\int_0^t K(t,x)G(u(x))\,dx\Bigg] + L[f(t)] + \lambda_2 L\Bigg[\int_0^1 M(t,x)H(u(x))\,dx\Bigg], \tag{4.3}$$

where $c = \sum_{k=1}^{n-1} s^{\alpha-k-1}u^{(k)}(0)$,

on simplifying the previous equation, we note that

$$U(s) = \frac{L[f(t)]+c}{s^{\alpha}} + \frac{\lambda_1}{s^{\alpha}} L\Bigg[\int_0^t K(t,x)G(u(x))\,dx\Bigg] + \frac{\lambda_2}{s^{\alpha}} L\Bigg[\int_0^1 M(t,x)H(u(x))\,dx\Bigg] + \frac{L[u(t-\tau)]}{s^{\alpha}}, \tag{4.4}$$

moreover, the solution $u(t)$ can be rewritten by the infinite series

$$u(t) = \sum_{n=1}^{\infty} u_n. \tag{4.5}$$

Also, the nonlinear functions are decomposed as

$$G = \sum_{n=1}^{\infty} B_n,\ H = \sum_{n=1}^{\infty} D_n, \tag{4.6}$$

where B_n, D_n are the Adomain polynomials of u_0, u_1, ..., u_n. For the nonlinear functions, the Adomain polynomials are given by

$$B_n = \frac{1}{\Gamma(n+1)} \frac{d^n}{d\ell^n}[G(\sum_{i=1}^{\infty} \ell^i u_i)]. \tag{4.7}$$

$$D_n = \frac{1}{\Gamma(n+1)} \frac{d^n}{d\ell^n}[H(\sum_{i=1}^{\infty} \ell^i u_i)], \ n = 0, 1, 2, ... \tag{4.8}$$

The new recursive scheme becomes

$$\begin{aligned} B_0 &= G(u_0), \\ B_1 &= u_1 G'(u_0), \\ B_1 &= u_2 G'(u_0) + \frac{1}{2} u_1^2 G''(u_0), \\ B_2 &= u_3 G'(u_0) + u_1 u_2 G''(u_0) + \frac{1}{3} u_1^3 G'''(u_0), \end{aligned}$$

and so on

$$\begin{aligned} D_0 &= H(u_0), \\ D_1 &= u_1 H'(u_0), \\ D_1 &= u_2 H'(u_0) + \frac{1}{2} u_1^2 H''(u_0), \\ D_2 &= u_3 H'(u_0) + u_1 u_2 H''(u_0) + \frac{1}{3} u_1^3 H'''(u_0), \end{aligned}$$

substituting (4.6), (4.7) into (4.5), we will obtain

$$\begin{aligned} L(\sum_{n=1}^{\infty} u_n) = \ & \frac{L[f(t)] + c}{s^\alpha} + \frac{\lambda_1}{s^\alpha} L\left[\int_0^t K(t,x) \sum_{n=1}^{\infty} B_n \ dx\right] \\ & + \frac{\lambda_2}{s^\alpha} L\left[\int_0^1 M(t,x) \sum_{n=1}^{\infty} D_n \ dx\right] + \frac{L[\sum_{n=1}^{\infty} u_{n-1}(t-\tau)]}{s^\alpha}. \end{aligned} \tag{4.9}$$

Comparing two both sides of (4.9). Then, we have the following iterative algorithm:

$$\begin{aligned} L(u_0) &= \frac{L[f(t)] + c}{s^\alpha}, \ u_0 = T(t), \\ L(u_1) &= \frac{\lambda_1}{s^\alpha} L[\int_0^t K(t,x) B_0 \ dx] + \frac{\lambda_2}{s^\alpha} L[\int_0^1 M(t,x) D_0 \ dx] + \frac{L[u_0(t-\tau)]}{s^\alpha}, \end{aligned}$$

therefore, the general term is given by

$$L(u_n) = \frac{\lambda_1}{s^\alpha} L[\int_0^t K(t,x) B_{n-1} \ dx] + \frac{\lambda_2}{s^\alpha} L[\int_0^1 M(t,x) D_{n-1} \ dx] + \frac{L[u_{n-1}(t-\tau)]}{s^\alpha}.$$

Applying the inverse Laplace transform, the modified Laplace decomposition method suggests that the function $T(t)$ is decomposed into two parts $T(t) = T_1(t) + T_2(t)$, $u_0(t) = T_1(t)$, $n \geq 1$,

$$\begin{aligned} u_1(t) = \ & T_2(t) + L^{-1}[\frac{\lambda_1}{s^\alpha} L(\int_0^t K(t,x) B_0 \ dx)] + L^{-1}[\frac{\lambda_2}{s^\alpha} L(\int_0^1 M(t,x) D_0 \ dx)] \\ & + L^{-1}[\frac{L(u_1(t-\tau))}{s^\alpha}]. \end{aligned}$$

In general,

$$u_n(t) = L^{-1}[\frac{\lambda_1}{s^\alpha} L(\int_0^t K(t,x) B_{n-1}\ dx)] + L^{-1}[\frac{\lambda_2}{s^\alpha} L(\int_0^1 M(t,x) D_{n-1}\ dx)] + L^{-1}[\frac{L[u_{n-1}(t-\tau)]}{s^\alpha}].$$

4.2 Convergence analysis for MLADM

Due to [11], we suppose the following two hypotheses for convergence of MLADM:

i. $(T(u(t)) - T(v(t)), u(t) - v(t)) \geq \beta \|u(t) - v(t)\|^2$; $\beta > 0$, $\forall u,\ v \in C[0,1]$,

ii. for $\varphi > 0$, $u,\ v \in C[0,1]$, there exists a constant $C(\varphi) > 0$ such that $\|u\| \leq \varphi$, $\|v\| \leq \varphi$ we get:

$$(T(u) - T(v), w) \leq C(\varphi) \|u - v\| \|w\| \tag{4.10}$$

for every $w \in C[0,1]$.

Theorem 4.1. *The MLADM applied to the fractional Volterra-Fredholm integro-differential equation without initial condition converges to a particular solution.*

Proof. In the first hand, for the fractional differential operator $T(u)$, $\exists k > 0$, $\forall u, v \in C[0,1]$, we have

$$\begin{aligned} T(u) - T(v) = \ & u(t-\tau) - v(t-\tau) + \lambda_1 \int_0^t K(t,x)[G(u(x)) - G(v(x))]\ dx \\ & + \lambda_2 \int_0^1 M(t,x)[H(u(x)) - H(v(x))]\ dx. \end{aligned}$$

Then, we get

$$\begin{aligned} (T(u) - T(v), u - v) = \ & \Big(u(t-\tau) - v(t-\tau) \\ & + \lambda_1 \int_0^t K(t,x)[G(u(x)) - G(v(x))]\ dx \\ & + \lambda_2 \int_0^1 M(t,x)[H(u(x)) - H(v(x))]\ dx, u - v \Big). \end{aligned}$$

Hence, by using the Schwartz inequality and Eq.1.7 we obtain

$$\begin{aligned} & \Big(u(t-\tau) - v(t-\tau) + \lambda_1 \int_0^t K(t,x)[G(u(x)) - G(v(x))]\ dx \\ & \qquad + \lambda_2 \int_0^1 M(t,x)[H(u(x)) - H(v(x))]\ dx, u - v \Big) \\ & \qquad \leq \Big(\wp \|u - v\| + |\lambda_1| \sup_{0 \leq t \leq 1} K(t,x) \xi \|u - v\| \Big) \Big(\|u - v\| \Big) \\ & \qquad + |\lambda_2| \sup_{0 \leq t \leq 1} M(t,x) \mu \|u - v\| \Big(\|u - v\| \Big). \end{aligned} \tag{4.11}$$

It follows that,

$$(T(u) - T(v), u - v) \geq k\|u - v\|^2.$$

On the other hand,

$$\begin{aligned}(T(u) - T(v), w) = \Big(& u(t-\tau) - v(t-\tau) \\ & + \lambda_1 \int_0^t K(t,x)[G(u(x)) - G(v(x))]\, dx \\ & + \lambda_2 \int_0^1 M(t,x)[H(u(x)) - H(v(x))]\, dx, w\Big).\end{aligned}$$

This implies that

$$(T(u) - T(v), w) \leq C(\varphi)\|u - v\|\|w\|.$$

This completes the proof. □

Now, we will study the convergence theorem of solutions based on the MLADM.

Theorem 4.2. *If the series solution is convergent, then it converges to the exact solution of the fractional Volterra-Fredholm integro-differential equation with constant time delay.*

Proof. For $n \geq m$, assume that S_n, S_m are arbitrary partial sum. We want to prove that S_n is a cauchy sequence, i.e, S_n becomes a convergent series:

$$\|S_n - S_m\|_{L^\infty[0,1]} = \max_{t\in[0,1]} |S_n - S_m| = \max_{t\in[0,1]} |\sum_{k=m+1}^{n} u_k(t)|,$$

where $S_j = \sum_{k=0}^{j} u_k(t)$, $j = 1, 2$.

$$\begin{aligned}\|S_n - S_m\|_{L^\infty[0,1]} = \max_{t\in[0,1]} | & \sum_{k=m+1}^{j} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big[u_k(s-\tau)\Big] ds \\ & + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big[\int_0^s K(s,x) B_k dx + \int_0^1 M(s,x) D_k dx\Big] ds|.\end{aligned}$$

From (4.7) and (4.8), we have

$$\begin{aligned}\sum_{k=m}^{j-1} B_k &= G(S_{n-1}) - G(S_{m-1}), \\ \sum_{k=m}^{j-1} D_k &= H(S_{n-1}) - H(S_{m-1}), \\ \sum_{k=m}^{j-1} u_k &= u(S_{n-1}) - u(S_{m-1}).\end{aligned}$$

Then,

$$\|S_n - S_m\|_{L^\infty[0,1]} = \max_{t\in[0,1]} |\sum_{k=m+1}^{j} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\Big[u_k(s-\tau)\Big]ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\Big[\int_0^s K(s,x)(G(S_{n-1}) - G(S_{m-1}))dx\Big]ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\Big[\int_0^1 M(s,x)(H(S_{n-1}) - H(S_{m-1}))dx\Big]ds|.$$

Since

$$\begin{aligned}\|S_n - S_m\|_{L^\infty[0,1]} &\le \|S_{m+1} - S_m\|_{L^\infty[0,1]}\| + \|\|S_{m+2} - S_{m+1}\|_{L^\infty[0,1]} + ... \\ &\le [\gamma^m + \gamma^{m+1} + ... + \gamma^{n-1}]\|S_1 - S_0\|_{L^\infty[0,1]} \\ &\le \gamma^m[1 + \gamma + \gamma^2 + ... + \gamma^{n-m-1}]\|S_1 - S_0\|_{L^\infty[0,1]} \\ &\le \gamma^m(\frac{1-\gamma^{n-m}}{1-\gamma})\|u_1\|_{L^\infty[0,1]}.\end{aligned}$$

Since $0 < \gamma < 1$, $(1 - \gamma^{n-m}) < 1$, but $|u_1(t)| < \infty$, so $\|S_n - S_m\| \to 0$, as m, $n \to \infty$. This implies that S_n is a cauchy sequence in $C[0,1]$, i.e., the series is also convergent. □

Now, we give examples to illustrate the obtained results.

4.3 Test examples

Consider the following fractional integro-differential equations with time delay.

Example 1.

$$D^\alpha u(t) = u(t-\tau) + \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} + 3\int_0^1 u^2(t)dt - 4\int_0^t \frac{u^3(s)}{t^3}ds, \tag{4.12}$$

subject to the initial conditions $u(0) = 0$, with the exact solution $u(t) = t$ at $\tau = 1$ and $\alpha = 0.75$.

Solution. First, we apply Laplace transform on both sides of (4.12)

$$\begin{aligned}L[D^\alpha u(t)] = L[u(t-\tau)] + L[\frac{t^{0.25}}{\Gamma(1.25)}] + 3L[\int_0^1 u^2(t)dt] \\ - 4L[\int_0^t \frac{u^3(s)}{t^3}ds],\end{aligned} \tag{4.13}$$

using the properties of Laplace transform,

$$\begin{aligned}s^\alpha L[u(t)] - c = L[u(t-\tau)] + \frac{1}{\Gamma(1.25)}L[t^{0.25}] + 3L[\int_0^1 u^2(t)dt] \\ - 4L[\int_0^t \frac{u^3(s)}{t^3}ds].\end{aligned} \tag{4.14}$$

Thus, (4.14) is equivalent to

$$L[u(t)] = \frac{1}{s^{0.75}}L[u(t-\tau)] + \frac{1}{\Gamma(1.25)s^{0.75}}\frac{\Gamma(1.25)}{s^{1.25}} + \frac{3}{s^{0.75}}L[\int_0^1 u^2(t)dt]$$
$$- \frac{4}{s^{0.75}}L[\int_0^t \frac{u^3(s)}{t^3}ds], \tag{4.15}$$

by means of the Laplace decomposition method

$$L[\sum_{n=0}^{\infty} u_n(t)] = \frac{e^{-s}}{s^{0.75}}L[\sum_{n=0}^{\infty} u_n(t)] + \frac{\Gamma(2)}{s^2} + \frac{3}{s^{0.75}}L[\int_0^1 \sum_{n=0}^{\infty} A_n(t)dt]$$
$$- \frac{4}{s^{0.75}}L[\int_0^t \frac{1}{t^3}(\sum_{n=0}^{\infty} B_n(s))ds]. \tag{4.16}$$

This leads to $L(u_0) = \frac{\Gamma(2)}{s^2}$

$$\begin{aligned} L[u_1(t)] &= \frac{e^{-s}}{s^{0.75}}L[t] + \frac{3}{s^{0.75}}L[\int_0^1 t^2dt] - \frac{4}{s^{0.75}}L[\int_0^t \frac{1}{t^3}s^3ds] \\ &= \frac{e^{-s}}{s^{0.75}}L[t] + \frac{1}{s^{0.75}}L[1] - \frac{1}{s^{0.75}}L[t] \\ &= \frac{1}{s^{0.75}}L[t-1] + \frac{1}{s^{0.75}}L(1-t) \\ &= 0 \end{aligned} \tag{4.17}$$

$$L[u_2(t)] = \frac{e^{-s}}{s^{0.75}}L[u_1(t)] + \frac{3}{s^{0.75}}L[\int_0^1 A_1(t)dt] - \frac{4}{s^{0.75}}L[\int_0^t \frac{1}{t^3}(B_1(s))ds]$$
$$\cdots$$
$$\cdots$$
$$\cdots$$

Taking the inverse Laplace transform in the preceding iterative algorithm,

$$\begin{aligned} u_0 &= L^{-1}[\frac{\Gamma(2)}{s^2}] = t, \\ u_1(t) &= L^{-1}[\frac{e^{-s}}{s^{0.75}}L[u_0(t)]] + L^{-1}[\frac{3}{s^{0.75}}L[\int_0^1 A_0(t)dt]] \\ &\quad - L^{-1}[\frac{4}{s^{0.75}}L[\int_0^t (B_0(s))ds]] = 0, \\ &\vdots \end{aligned}$$

Therefore, the solution $u(t) = \sum_{n=0}^{\infty} u_n = t$.

Remark 4.3 For arbitrary constants a, b, c, $d \in \mathbb{R}$, if the non-linear functions $F(u(t))$ and $G(u(t))$ are replaced by $au(t)+b$, $cu(t)+d$ respectively, then Eq. (2.1) becomes a linear Caputo fractional Volterra-Fredholm integro-differential equation.

Example 2.

$$D^{\alpha}u(t) = u(t-\tau) + \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} + 2\int_0^1 u(t)dt - 2\int_0^t \frac{u(s)-1}{t}ds \tag{4.18}$$

with the initial condition $u(0)=0$, with the exact solution $u(t)=t$ at $\tau=3$ and $\alpha=0.5$.

Solution. Utilizing transform of Laplace to (4.18)

$$\begin{aligned} L[D^{\alpha}u(t)] &= L[u(t-\tau)] + L[\frac{t^{0.5}}{\Gamma(1.5)}] + 2L[\int_0^1 u(t)dt] \\ &\quad - 2L[\int_0^t \frac{u(s)-1}{t}ds], \end{aligned} \tag{4.19}$$

hence, by (1.8), (1.10) and the initial conditions

$$\begin{aligned} s^{0.5}L[u(t)] &= e^{-3s}L[u(t)] + \frac{1}{\Gamma(1.5)}L[t^{0.5}] + 2L[\int_0^1 u(t)dt] \\ &\quad - 2L[\int_0^t \frac{u(s)-1}{t}ds]. \end{aligned} \tag{4.20}$$

In order to simplify the above equation, we have

$$\begin{aligned} L[u(t)] &= \frac{e^{-3s}}{s^{0.5}}L[u(t)] + \frac{1!}{s^2} + \frac{2}{s^{0.5}}L[\int_0^1 u(t)dt] \\ &\quad - \frac{2}{s^{0.5}}L[\int_0^t \frac{u(s)-1}{t}ds]. \end{aligned} \tag{4.21}$$

According to MADM

$$\begin{aligned} L[\sum_{i=0}^{\infty} u_i(t)] &= \frac{e^{-3s}}{s^{0.5}}L[\sum_{n=0}^{\infty} u_i(t)] + \frac{1!}{s^2} + \frac{2}{s^{0.5}}L[\int_0^1 \sum_{i=0}^{\infty} u(t)dt] \\ &\quad - \frac{2}{s^{0.5}}L[\int_0^t (\sum_{i=0}^{\infty} \frac{u(s)-1}{t}ds]. \end{aligned} \tag{4.22}$$

That is, $L(u_0) = \frac{1!}{s^2} \Rightarrow u_0 = t$. Consequently,

$$\begin{aligned} u_1(t) &= L^{-1}\left[\frac{e^{-3s}}{s^{0.5}}L[t] + \frac{1}{s^{1.5}} + \frac{2}{s^{1.5}} + \frac{1!}{s^{1.5}}\right] \\ &= L^{-1}\left[\frac{1}{s^{0.5}}L[t-3] + \frac{1}{s^{0.5}}L(3-t)\right] \\ &= 0 \end{aligned} \tag{4.23}$$

Similarly, for $i > 1$, $u_i = 0$. So, the approximate solution is given by

$$u(t) = \sum_{i=0}^{\infty} u_i = t,$$

which is the exact solution.

5. Conclusion

It is obvious that this work represents essential development in modeling the linear and non-linear fractional integro-differential equations. It provides us with an accurate analysis of the solution to the mentioned problem. Throughout it, we can get different results easily. Such previous results will arise, if any terms of Eq. (2.1) are vanished as in [11, 23, 24] and etc.

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Accepted: March 17, 2020

Conformal η-Ricci solitons in Lorentzian para-Sasakian manifold admitting semi-symmetric metric connection

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Abstract. The object of the present research is to study conformal η-Ricci solitons on a semi-symmetric metric connection in a LP-Sasakian manifold of some curvature tensor like conharmonic curvature, W_2 curvature and projective curvature.

Keywords: conformal η-Ricci solitons, LP-Sasakian manifold, semi-symmetry, curvature tensors.

1. Introduction

A Ricci soliton is a natural generalization of an Einstein metric introduced by Hamilton [5] in 1982. A pseudo-Riemannian manifold (M, g) is called a Ricci soliton if it admits a smooth vector field V on (M, g) such that

$$\frac{1}{2}(L_V g(X_1, Y_1)) + S(X_1, Y_1) + \lambda g(X_1, Y_1) = 0, \tag{1}$$

where L_V denotes the lie derivative in the direction of V, S is the Ricci tensor field, λ is a constant and X_1, Y_1 are arbitrary vector fields on M.

The study of η-Ricci soliton was initiated by Cho and Kimura [3]. For a given 1-form η a η-Ricci soliton is a data (g, V, λ, μ) on pseudo-Riemannian manifold

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(M, g) satisfying

$$\frac{1}{2}(L_V g(X_1, Y_1)) + S(X_1, Y_1) + \lambda g(X_1, Y_1) + \mu\eta \otimes \eta = 0, \tag{2}$$

where L_V denotes the lie derivative in the direction of V, S is the Ricci tensor field, λ is a constant and X_1, Y_1 are arbitrary vector fields on M.

In [2] the authors studied the η-Ricci solitons on Hopf hypersurfaces in complex space forms. Later in the year 2015, Basu and Bhattacharya [1] introduced the notion of conformal Ricci soliton as a soliton satisfying

$$L_V g(X_1, Y_1) + 2S(X_1, Y_1) + \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g(X_1, Y_1) = 0. \tag{3}$$

In [4], the authors studied the properties of conformal Ricci soliton in Lorentzian α-Sasakian manifolds. From (2) and (3), the conformal η-Ricci soliton given by

$$L_V g(X_1, Y_1) + 2S(X_1, Y_1) + \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g(X_1, Y_1) + 2\mu\eta \otimes \eta = 0. \tag{4}$$

Particularly if $\mu = 0$ the the data (g, V, λ) is a conformal Ricci soliton. Clearly, the conformal η-Ricci soliton is a generalization of conformal Ricci soliton.

Somashekhara et al. [6], studied some results on invariant sub-manifolds of LP-Sasakian manifolds endowed with semi-symmetric metric conection. Also, they have obtained a condition for totally geodesic by using certain geometrical conditions. In [7], we studied the C-Bochner curvature tensor under D-homothetic deformation in LP-Sasakian manifolds.

In this paper, we study the conformal η-Ricci solitons in LP-Sasakian manifold with semi-symmetric metric connection.

2. Preliminaries

A smooth manifold (M^n, g) is called Lorentzian para-Sasakian manifold, if it admits $(1, 1)$ tensor field ϕ, contravariant vector field ξ, 1-form η and Lorentzian g such that

$$\eta(\xi) = -1, \tag{5}$$

$$\phi^2(X_1) = X_1 + \eta(X_1)\xi, \tag{6}$$

$$g(\phi X_1, \phi Y_1) = g(X_1, Y_1) + \eta(X_1)\eta(Y_1), \tag{7}$$

$$g(X_1, \phi Y_1) = g(\phi X_1, Y_1), \tag{8}$$

$$g(X_1, \xi) = \eta(X_1), \tag{9}$$

$$\nabla_{X_1}\xi = \phi X_1, \tag{10}$$

$$(\nabla_{X_1}\phi)(Y_1) = g(X_1, Y_1)\xi + \eta(Y_1)X_1 + 2\eta(X_1)\eta(Y_1)\xi, \tag{11}$$

where ∇ denotes the covariant differentiation with respect to the Lorentzian metric g.

In an LP-Sasakian manifold the following relations holds:

$$\phi\xi = 0,\ \eta(\phi X_1) = 0,\ \text{Rank}(\phi) = n-1. \tag{12}$$

If we put

$$\Phi(X_1, Y_1) = g(X_1, \phi Y_1), \tag{13}$$

for any vector fields X_1 and Y_1, then the tensor field $\Phi(X_1, Y_1)$ is a symmetric $(0,2)$ tensor field. Also since the 1-form η is closed in a LP-Sasakian manifold, then we have:

$$(\nabla_{X_1}\eta)(Y_1) = \Phi(X_1, Y_1), \Phi(X_1, \xi) = 0, \tag{14}$$

$\forall\ X_1, Y_1\ \in \text{TM}$.

In a LP-Sasakian manifold, the following relations holds:

$$g(R(X_1, Y_1)Z_1, \xi) = \eta(R(X_1, Y_1)Z_1) = g(Y_1, Z_1)\eta(X_1) - g(X_1, Z_1)\eta(Y_1), \tag{15}$$

$$R(\xi, X_1)Y_1 = g(X_1, Y_1)Z_1 - \eta(Y_1)X_1, \tag{16}$$

$$R(X_1, Y_1)\xi = \eta(Y_1)X_1 - \eta(X_1)Y_1, \tag{17}$$

$$R(\xi, X_1)\xi = X_1 + \eta(X_1)\xi, \tag{18}$$

$$S(X_1, \xi) = (n-1)\eta(X_1), \tag{19}$$

$$S(\phi X_1, \phi Y_1) = S(X_1, Y_1) + (n-1)\eta(X_1)\eta(Y_1). \tag{20}$$

for any vector fields X_1, Y, Z_1 where R and S are the Riemannian curvature and Ricci tensors of M, respectively.

A semi-symmetric metric connection $\tilde{\nabla}$ in a LP-Sasakian manifold can be defined by

$$\tilde{\nabla_{X_1}} Y_1 = \nabla_{X_1} Y_1 + \eta(Y_1)X_1 - g(X_1, Y_1)\xi, \tag{21}$$

where ∇ is the Levi-Civita connection on M.

Let M^n, g be a LP-Sasakian manifold. The curvature tensor $\tilde{R}$ of M with respect t0 the semi-symmetric metric connection, the following relations holds:

$$\begin{aligned}\tilde{R}(X_1, Y_1)Z_1 = R(X_1, Y_1)Z_1 &- \left[g(\phi Y_1, Z_1) - \eta(Y_1)\eta(Z_1) - \frac{1}{2}g(Y_1, Z_1)\right] \\ &X_1 + \left[g(\phi X_1, Z_1) - \eta(X_1)\eta(Z_1) - \frac{1}{2}g(X_1, Z_1)\right] Y_1 \\ &- g(Y_1, Z_1)\left[\phi X_1 - \eta(X_1)\xi - \frac{1}{2}X_1\right] \\ &+ g(X_1, Z_1)\left[\phi Y_1 - \eta, (Y_1)\xi - \frac{1}{2}Y_1\right],\end{aligned} \tag{22}$$

$$\tilde{S}(X_1,Y_1)=S(X_1,Y_1)-(n-2)\left[g(\phi X_1,Y_1)-\eta(X_1)\eta(Y_1)-\frac{1}{2}g(X_1,Y_1)\right] - \text{trace}(\alpha)g(X_1,Y_1), \tag{23}$$

$$\tilde{r} = r - 2(n-1)\text{trace}(\alpha), \tag{24}$$

$$\tilde{S}(\phi X_1,\phi Y_1) = S(X_1,Y_1) + \left[\frac{1}{2}\text{trace}(\alpha)\right]\eta(X_1)\eta(Y_1) - (n-2) \left[g(\phi X_1,Y_1) - \eta(X_1)\eta(Y_1) - \frac{1}{2}g(X_1,Y_1)\right] - \text{trace}(\alpha)g(X_1,Y_1). \tag{25}$$

We can use the notion of conformal η-Ricci soliton on LP-Sasakian manifold and we get the following:

$$L_\xi g(X_1,Y_1) + 2S(X_1,Y_1) + \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g(X_1,Y_1) + 2\mu\eta\otimes\eta = 0, \tag{26}$$

using (10) in (26), we get

$$S(X_1,Y_1) = \left(\frac{p}{2} + \frac{1}{n} - \lambda\right) g(X_1,Y_1) - \mu\eta(X_1)\eta(Y_1) - g(\phi X_1,Y_1), \tag{27}$$

$$QX_1 = \left(\frac{p}{2} + \frac{1}{n} - \lambda\right) X_1 - \mu\eta(X_1)\xi - \phi X_1, \tag{28}$$

$$S(X_1,\xi) = \left(\frac{p}{2} + \frac{1}{n} - \lambda + \mu\right)\eta(X_1), \tag{29}$$

$$Q\xi = \left(\frac{p}{2} + \frac{1}{n} - \lambda + \mu\right)\xi. \tag{30}$$

Also the idea of conformal η-Ricci soliton on LP-Sasakian manifold with semi-symmetric metric connection, we have:

$$\tilde{S}(X_1,Y_1) = A_1 g(X_1,Y_1) + A_2\eta(X_1)\eta(Y_1) - (n-1)g(\phi X_1,Y_1), \tag{31}$$

$$\tilde{Q}X_1 = A_1 X_1 + A_2\eta(X_1)\xi - (n-1)\phi X_1, \tag{32}$$

$$\tilde{S}(X_1,\xi) = (A_1 - A_2)\eta(X_1), \tag{33}$$

$$\tilde{Q}\xi = (A_1 - A_2)\xi, \tag{34}$$

where

$$A_1 = \left(\frac{p}{2} + \frac{1}{n} - \lambda + \frac{n-2}{2} - \text{trace}(\alpha)\right), A_2 = (n-2-\mu).$$

3. Conformal η-Ricci soliton in LP-Sasakian manifolds satisfying $\tilde{H}(\xi, Y_1).\tilde{S} = 0$ with respect to semi-symmetric metric connection

Let M be an n-dimensional conformal η-Ricci in LP-Sasakian manifold admitting semi-symmetric metric connection satisfying $\tilde{H}(\xi, Y_1).\tilde{S} = 0$, where $\tilde{H}$ and $\tilde{S}$ are conharmonic curvature tensor and Ricci tensor respectively with semi-symmetric metric connection. The conharmonic curvature tensor $\tilde{H}$ is defined by

$$\tilde{H}(X_1, Y_1)Z_1 = \tilde{R}(X_1, Y_1)Z_1 - \frac{1}{n-2}\left(\tilde{S}(Y_1, Z_1)X_1 - \tilde{S}(X_1, Z_1)Y_1 + g(Y_1, Z_1)\tilde{Q}X_1 - g(X_1, Z_1)\tilde{Q}Y_1\right), \tag{35}$$

we assume that the condition $\tilde{H}(\xi, Y_1).\tilde{S} = 0$. It implies that

$$\tilde{S}(\tilde{H}(\xi, X_1)Y_1, Z_1) + \tilde{S}(Y_1, \tilde{H}(\xi, X_1)Z_1) = 0, \tag{36}$$

substituting $X_1 = \xi$ in (35), we get

$$\tilde{H}(\xi, Y_1)Z_1 = \tilde{R}(\xi, Y_1)Z_1 - \frac{1}{n-2}\left(\tilde{S}(Y_1, Z_1)\xi - \tilde{S}(\xi, Z_1)Y_1 + g(Y_1, Z_1)\tilde{Q}\xi - g(\xi, Z_1)\tilde{Q}Y_1\right), \tag{37}$$

using equation (33) and (37), equation (36) reduces to

$$\begin{aligned}
&(A_1 - A_2)g(X_1, Y_1)\eta(Z_1) - \eta(Y_1)\tilde{S}(X_1, Z_1) \\
&- (A_1 - A_2)g(\phi X_1, Y_1)\eta(Z_1) + \eta(Y_1) \\
&\tilde{S}(\phi X_1, Z_1) - \frac{1}{n-2}(A_1 - A_2)\eta(Z_1)\tilde{S}(X_1, Y_1) \\
&+ \frac{1}{n-2}(A_1 - A_2)\eta(Y_1)\tilde{S}(X_1, Z_1) - \\
&\frac{1}{n-2}(A_1 - A_2)^2\eta(Z_1)g(X_1, Y_1) + \frac{1}{n-2}A_1\eta(Y_1)\tilde{S}(X_1, Z_1) \\
&+ \frac{1}{n-2}A_2(A_1 - A_2) \\
&\eta(X_1)\eta(Y_1)\eta(Z_1) - \frac{n-1}{n-2}\tilde{S}(\phi X_1, Z_1)\eta(Y_1) \\
&+ (A_1 - A_2)g(X_1, Z_1)\eta(Y_1) - \eta(Z_1)\tilde{S}(X_1, Y_1) \\
&- (A_1 - A_2)g(\phi X_1, Z_1)\eta(Y_1) + \eta(Z_1)\tilde{S}(\phi X_1, Y_1) \\
&- \frac{1}{n-2}(A_1 - A_2)\eta(Y_1)\tilde{S}(X_1, Z_1) + \\
&\frac{1}{n-2}(A_1 - A_2)\eta(Z_1)\tilde{S}(X_1, Y_1) - \frac{1}{n-2}(A_1 - A_2)^2\eta(Y_1)g(X_1, Z_1)
\end{aligned} \tag{38}$$

$$+\frac{1}{n-2}A_1\eta(Z_1)$$
$$\tilde{S}(X_1,Y_1)+\frac{1}{n-2}A_2(A_1-A_2)\eta(X_1)\eta(Y_1)\eta(Z_1)$$
$$-\frac{n-1}{n-2}\tilde{S}(\phi X_1,Y_1)\eta(Z_1)=0.$$

Substituting $Z_1=\xi$ and using (5) and (33), we get

$$A_3g(X_1,Y_1)+A_4g(\phi X_1,Y_1)+A_5\eta(X_1)\eta(Y_1)+A_6\tilde{S}(X_1,Y_1)$$
$$+A_7\tilde{S}(\phi X_1,Y_1)=0, \tag{39}$$

where

$$A_3=\left[\frac{(A_1-A_2)^2}{n-2}-(A_1-A_2)\right],$$
$$A_4=(A_1-A_2),$$
$$A_5=\left[\frac{A_1(A_1-A_2)}{n-2}-\frac{(A_1-A_2)^2}{n-2}\right],$$
$$A_6=\left[\frac{A_1}{n-2}+1\right],$$
$$A_7=\left[\frac{1}{n-2}\right].$$

In view of (31), we get

$$A_8g(X_1,Y_1)+A_9\eta(X_1)\eta(Y_1)+A_{10}g(\phi X_1,Y_1)$$
$$+A_6S(X_1,Y_1)+A_7S(\phi X_1,Y_1)=0, \tag{40}$$

where

$$A_8=\left(A_3+\frac{A_6(n-2)}{2}-A_6\text{trace}(\alpha)-A_7(n-2)\right),$$
$$A_9=(A_5+A_6(n-2)-A_7(n-2)),$$
$$A_{10}=\left(A_4-A_6(n-2)+\frac{A_7(n-2)}{2}-A_7\right).$$

Using (27) in (40), we get

$$g(\phi X_1,Y_1)=A_{11}g(X_1,Y_1)+A_{12}\eta(X_1)\eta(Y_1). \tag{41}$$

where

$$A_{11}=-\left[\frac{A_8-A_7+A_6\left(\frac{p}{2}+\frac{1}{n}-\lambda\right)}{A_{10}-A_6+A_6\left(\frac{p}{2}+\frac{1}{n}-\lambda\right)}\right],$$
$$A_{12}=-\left[\frac{A_9-A_7-\mu A_6}{A_{10}-A_6+A_6\left(\frac{p}{2}+\frac{1}{n}-\lambda\right)}\right].$$

Now, again using (27) and (41) in equation (40), we get

$$S(X_1, Y_1) = G_1 g(X_1, Y_1) + G_2 \eta(X_1)\eta(Y_1), \tag{42}$$

where

$$G_1 = -\left[\frac{A_8 - A_7 + A_{11}\left(A_{10} + A_7\left(\frac{p}{2} + \frac{1}{n} - \lambda\right)\right)}{A_6}\right],$$

$$G_2 = -\left[\frac{A_9 - A_7 + A_{12}\left(A_{10} + A_7\left(\frac{p}{2} + \frac{1}{n} - \lambda\right)\right)}{A_6}\right].$$

Hence, we can state the following theorem

Theorem 3.1. *An n-dimensional conformal η-Ricci soliton in LP-Sasakian manifold with semi-symmetric metric connection satisfying $\tilde{H}(\xi, Y_1).\tilde{S} = 0$ is an η-Einstein manifold.*

4. Conformal η-Ricci soliton in LP-Sasakian manifolds satisfying $\tilde{W}_2(\xi, Y_1).\tilde{S} = 0$ with respect to semi-symmetric metric connection

Let M be an n-dimensional conformal η-Ricci soliton in LP-Sasakian manifold admitting semi-symmetric metric connection satisfying $\tilde{W}_2(\xi, Y_1).\tilde{S} = 0$, where $\tilde{W}_2$ and $\tilde{S}$ are W_2 curvature tensor and Ricci tensor respectively with semi-symmetric metric connection. Then the W_2-curvature tensor is defined by

$$\tilde{W}_2(X_1, Y_1)Z_1 = \tilde{R}(X_1, Y_1)Z_1 - \frac{1}{n-1}\left(g(X_1, Z_1)\tilde{Q}Y_1 - g(Y_1, Z_1)\tilde{Q}X_1\right). \tag{43}$$

We assume that the condition $\tilde{W}_2(\xi, Y_1).\tilde{S} = 0$. It implies that

$$\tilde{S}(\tilde{W}_2(\xi, X_1)Y_1, Z_1) + \tilde{S}(Y_1, \tilde{W}_2(\xi, X_1)Z_1) = 0. \tag{44}$$

Substituting $X_1 = \xi$ in (43), we get

$$\tilde{W}_2(\xi, Y_1)Z_1 = \tilde{R}(\xi, Y_1)Z_1 - \frac{1}{n-1}\left(g(\xi, Z_1)\tilde{Q}Y_1 - g(Y_1, Z_1)\tilde{Q}\xi\right). \tag{45}$$

Using (45) in (44), we get

$$\begin{aligned}
&- \eta(Y_1)\tilde{S}(X_1, Z_1) - g(\phi X_1, Y_1)\tilde{S}(\xi, Z_1) \\
&+ \frac{A_1}{n-1}\eta(Y_1)\tilde{S}(X_1, Z_1) + \frac{A_2}{n-1}\eta(X_1)\eta(Y_1) \\
&\tilde{S}(\xi, Z_1) - \frac{(A_1 - A_2)}{n-1}g(X_1, Y_1)\tilde{S}(\xi, Z_1) \\
&- \eta(Z_1)\tilde{S}(X_1, Y_1) - g(\phi X_1, Z_1)\tilde{S}(Y_1, \xi) + \frac{A_1}{n-1} \\
&\eta(Z_1)\tilde{S}(X_1, Y_1) + \frac{A_2}{n-1}\eta(X_1)\eta(Z_1)\tilde{S}(Y_1, \xi) \\
&- \frac{(A_1 - A_2)}{n-1}g(X_1, Z_1)\tilde{S}(\xi, Y_1) = 0.
\end{aligned} \tag{46}$$

Substituting $Z_1 = \xi$ and using (33), we get

$$A_{13}\eta(X_1)\eta(Y_1) + A_{14}g(X_1,Y_1) + A_{15}g(\phi X_1,Y_1) + A_{16}\tilde{S}(X_1,Y_1) = 0, \tag{47}$$

where

$$\begin{aligned}
A_{13} &= \left[(A_2 - A_1) + \frac{A_1(A_1-A_2)}{n-1} - 2\frac{A_2(A_1-A_2)}{n-1} - \frac{(A_1-A_2)^2}{n-1}\right],\\
A_{14} &= \frac{(A_1-A_2)^2}{n-1},\\
A_{15} &= (A_1 - A_2),\\
A_{16} &= \left(1 - \frac{A_1}{n-1}\right).
\end{aligned}$$

Using (31) , we get

$$\begin{aligned}
&(A_{13} - A_{16}(n-2))\,\eta(X_1)\eta(Y_1)\\
&+ g(X_1,Y_1)\left(A_{14} + A_{16}(\frac{n-2}{2} - \text{trace}(\alpha))\right) g(\phi X_1,Y_1)\\
&(A_{15} - A_{16}(n-2)) + A_{16}S(X_1,Y_1) = 0.
\end{aligned} \tag{48}$$

On rearranging and simplification, we have

$$S(X_1,Y_1) = G_3 g(X_1,Y_1) + G_4\eta(X_1)\eta(Y_1), \tag{49}$$

where

$$\begin{aligned}
G_3 &= -\frac{1}{A_{16}}\left[A_{14} - A_{16}\left(\frac{n-2}{2} + \text{trace}(\alpha)\right)\right] + \frac{1}{A_{16}}\left[\frac{A_{15} - A_{16}(n-2)}{A_{15} - A_{16}(n-1)}\right]\\
&\quad \left[A_{14} + A_{16}\left(\frac{n-2}{2} - \text{trace}(\alpha) + \left(\frac{p}{2} + \frac{1}{n} - \lambda\right)\right)\right],\\
G_4 &= -\frac{1}{A_{16}}(A_{13} - (n-2)A_{16})\\
&\quad + \frac{1}{A_{16}}\left[\frac{(A_{15} - A_{16}(n-2))}{A_{15} - A_{16}(n-1)}\right][A_{13} - A_{16}(n-2+\mu)].
\end{aligned}$$

Hence, we can state the following theorem

Theorem 4.1. *An n-dimensional conformal η-Ricci soliton in LP-Sasakian manifold with semi-symmetric metric connection satisfying $\tilde{W}_2(\xi,Y_1).\tilde{S} = 0$ is an η-Einstein manifold.*

5. Conformal η-Ricci soliton in LP-Sasakian manifolds satisfying $\tilde{P}(\xi,Y_1).\tilde{S} = 0$ with respect to semi-symmetric metric connection

Let M be an n-dimensional conformal η-Ricci soliton in LP-Sasakian manifold admitting semi-symmetric metric connection satisfying $\tilde{P}(\xi,Y_1).\tilde{S} = 0$, where $\tilde{P}$

and $\tilde{S}$ are projective curvature tensor and Ricci tensor respectively with semi-symmetric metric connection. Then the Projective curvature tensor is defined by

$$\tilde{P}(X_1,Y_1)Z_1 = \tilde{R}(X_1,Y_1)Z_1 - \frac{1}{n-1}[\tilde{S}(Y_1,Z_1)X_1 - \tilde{S}(X_1,Z_1)Y_1]. \tag{50}$$

We assume that the condition $\tilde{P}(\xi,Y_1).\tilde{S}(Z_1,U) = 0$. It implies that

$$\tilde{S}(\tilde{P}(\xi,Y_1)Z_1,U) + \tilde{S}(Z_1,\tilde{P}(\xi,Y_1)U) = 0. \tag{51}$$

Substituting $X_1 = \xi$ in (50), we get

$$\tilde{P}(\xi,Y_1)Z_1 = \tilde{R}(\xi,Y_1)Z_1 - \frac{1}{n-1}[\tilde{S}(Y_1,Z_1)\xi - \tilde{S}(\xi,Z_1)Y_1]. \tag{52}$$

Using (52) in (51), we get

$$\begin{aligned} &g(Y_1,Z_1)\tilde{S}(\xi,U) - g(\phi Y_1,Z_1)\tilde{S}(\xi,U) \\ &- \eta(Z_1)\tilde{S}(U,Y_1) + \eta(Z_1)\tilde{S}(U,\phi Y_1) - \frac{1}{n-1}\tilde{S}(Y_1,Z_1) \\ &\tilde{S}(\xi,U) + \frac{1}{n-1}\tilde{S}(U,Y_1)\tilde{S}(\xi,Z_1) + g(Y_1,U)\tilde{S}(\xi,Z_1) \\ &- g(\phi Y_1,U)\tilde{S}(\xi,Z_1) - \eta(U)\tilde{S}(Z_1,Y_1) \\ &+ \eta(U)\tilde{S}(Z_1,\phi Y_1) - \frac{1}{n-1}\tilde{S}(Y_1,U)\tilde{S}(\xi,Z_1) + \frac{1}{n-1}\tilde{S}(Z_1,Y_1)\tilde{S}(\xi,U). \end{aligned} \tag{53}$$

Substituting $U = \xi$ and using (33) in (53), we have

$$(A_1 - A_2)g(\phi Y_1,Z_1) - (A_1 - A_2)g(Y_1,Z_1) + \tilde{S}(Y_1,Z_1) - \tilde{S}(\phi Y_1,Z_1) = 0. \tag{54}$$

Using (23) and (27), we get

$$\begin{aligned} &[A_{17}+1]g(Y_1,Z_1) + [A_{19}+1]\eta(Y_1)\eta(Z_1) + \\ &\left[A_{18} - \left(\frac{p}{2} + \frac{1}{n} - \lambda\right)\right] g(\phi Y_1,Z_1) + S(Y_1,Z_1) = 0, \end{aligned} \tag{55}$$

where

$$\begin{aligned} A_{17} &= \left[(A_2 - A_1) + \frac{3}{2}(n-2) - \text{trace}(\alpha)\right], \\ A_{18} &= \left[(A_1 - A_2) - \frac{3}{2}(n-2) + \text{trace}(\alpha)\right], \\ A_{19} &= 2(n-2). \end{aligned}$$

Using (27) in equation (55), we get:

$$g(\phi X_1, Z_1) = -\left(\frac{A_{17}+1+\left(\frac{p}{2}+\frac{1}{n}-\lambda\right)}{A_{18}-1-\left(\frac{p}{2}+\frac{1}{n}-\lambda\right)}\right) g(Y_1, Z_1) - \left(\frac{A_{19}+1-\mu}{A_{18}-1-\left(\frac{p}{2}+\frac{1}{n}-\lambda\right)}\right) \eta(Y_1)\eta(Z_1). \tag{56}$$

In view of (56), equation (55) reduce to

$$S(Y_1, Z_1) = G_5 g(Y_1, Z_1) + G_6 \eta(Y_1)\eta(Z_1), \tag{57}$$

where

$$G_5 = \left(\frac{A_{18}-\left(\frac{p}{2}+\frac{1}{n}-\lambda\right)}{A_{18}-1-\left(\frac{p}{2}+\frac{1}{n}-\lambda\right)}\right)\left(A_{17}+1+\left(\frac{p}{2}+\frac{1}{n}-\lambda\right)\right)-(A_{17}+1),$$

$$G_6 = \left(\frac{A_{18}-\left(\frac{p}{2}+\frac{1}{n}-\lambda\right)}{A_{18}-1-\left(\frac{p}{2}+\frac{1}{n}-\lambda\right)}\right)(A_{19}+1-\mu)-(A_{19}+1).$$

Hence, we can state the following theorem

Theorem 5.1. *An n-dimensional conformal η-Ricci soliton in LP-Sasakian manifold with semi-symmetric metric connection satisfying $\tilde{P}(\xi, Y_1).\tilde{S} = 0$ is an η-Einstein manifold.*

6. Conformal η-Ricci soliton in LP-Sasakian manifolds satisfying $\tilde{R}.\tilde{S} = 0$ with respect to semi-symmetric metric connection

Let M be a n-dimensional conformal η-Ricci soliton in LP-Sasakian manifold admitting semi-symmetric metric connection satisfying $\tilde{R}.\tilde{S} = 0$, where $\tilde{R}$ and $\tilde{S}$ are Riemannian curvature and Ricci tensor respectively with semi-symmetric metric connection. We assume that the condition $\tilde{R}(\xi, Y_1).\tilde{S}(Z_1, U) = 0$. Then

$$\tilde{S}(\tilde{R}(\xi, Y_1)Z_1, U) + \tilde{S}(Z_1, \tilde{R}(\xi, Y_1)U) = 0. \tag{58}$$

Using (22) in equation (58), we get

$$\begin{aligned} &g(Y_1, Z_1)\tilde{S}(U, \xi) - g(\phi Y_1, Z_1)\tilde{S}(U, \xi) - \eta(Z_1)\tilde{S}(U, Y_1) + \eta(Z_1)\tilde{S}(U, \phi Y_1) + \\ &g(Y_1, U)\tilde{S}(Z_1, \xi) - g(\phi Y_1, U)\tilde{S}(Z_1, \xi) - \eta(U)\tilde{S}(Z_1, Y_1) + \eta(U)\tilde{S}(Z_1, \phi Y_1). \end{aligned} \tag{59}$$

Substituting $U = \xi$ and using (33), we get

$$(A_1 - A_2)g(\phi Y_1, Z_1) - (A_1 - A_2)g(Y_1, Z_1) + \tilde{S}(Y_1, Z_1) - \tilde{S}(\phi Y_1, Z_1) = 0. \tag{60}$$

Using (23) in (60), we have

$$A_{20}g(Y_1, Z_1) + A_{21}g(\phi Y_1, Z_1) + A_{22}\eta(Y_1)\eta(Z_1) + S(Y_1, Z_1) - S(\phi Y_1, Z_1) = 0, \tag{61}$$

where

$$A_{20} = \left[(A_2 - A_1) + \frac{3}{2}(n-2) - \text{trace}(\alpha)\right],$$
$$A_{21} = \left[(A_1 - A_2) - \frac{3}{2}(n-2) + \text{trace}(\alpha)\right],$$
$$A_{22} = 2(n-2).$$

Using (27) in equation (61), we get

$$g(\phi Y_1, Z_1) = -g(Y_1, Z_1)\left(\frac{A_{20} + 1 + \left(\frac{p}{2} + \frac{1}{n} - \lambda\right)}{A_{21} - 1 - \left(\frac{p}{2} + \frac{1}{n} - \lambda\right)}\right) - \eta(Y_1)\eta(Z_1)\left(\frac{A_{22} + 1 - \mu}{A_{21} - 1 - \left(\frac{p}{2} + \frac{1}{n} - \lambda\right)}\right). \tag{62}$$

In view of (62) and (27), the equation (61) reduces to

$$S(Y_1, Z_1) = G_7 g(Y_1, Z_1) + G_8 \eta(Y)\eta(Z_1), \tag{63}$$

where

$$G_7 = \left(\frac{A_{21} - \left(\frac{p}{2} + \frac{1}{n} - \lambda\right)}{A_{21} - 1 - \left(\frac{p}{2} + \frac{1}{n} - \lambda\right)}\right)\left(A_{20} + 1 + \left(\frac{p}{2} + \frac{1}{n} - \lambda\right)\right) - (A_{20} + 1),$$
$$G_8 = \left(\frac{A_{21} - \left(\frac{p}{2} + \frac{1}{n} - \lambda\right)}{A_{21} - 1 - \left(\frac{p}{2} + \frac{1}{n} - \lambda\right)}\right)(A_{22} + 1 - \mu) - (A_{22} + 1).$$

Hence, we can state the following theorem

Theorem 6.1. *An n-dimensional conformal η-Ricci soliton in LP-Sasakian manifold with semi-symmetric metric connection satisfying* $\tilde{R}.\tilde{S} = 0$ *is an η-Einstein manifold.*

Acknowledgements

The authors would like to thank the referees for their invaluable comments and suggestions which led to the improvement of the manuscript.

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Accepted: June 28, 2020

The small intersection graph of ideals of a lattice

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Abstract. In this paper, we introduce the small intersection graph of ideals of a lattice $\mathfrak{L}$, denoted by $SG(\mathfrak{L})$, in which the vertex set is being the set of all non-trivial ideals of $\mathfrak{L}$, and two distinct vertices I and J are adjacent if $I \wedge J \ll \mathfrak{L}$. We study basic properties of $SG(\mathfrak{L})$. Also, we investigate the properties of $SG(\mathfrak{L})$ as connectivity, diameter, grith and orthogonal vertices.

Keywords: lattice, small element, small intersection graph.

1. Introduction

The study of algebraic structures, using the properties of graph theory, tends to an exciting research topic in the last decade. There are many papers on assigning a graph to algebraic structures, see for example [3], [4], [5], [9]. The intersection graph of algebraic structures has been investigated by several authors [1], [3], [10], [12]. In 1973, Zelinka [13] introduced the concept of intersection graph of a lattice. In 2014, the intersection graph of ideals of a lattice was considered by Afkhami and Khashyarmanesh [2]. In this paper, we introduce and study the small intersection graph of a lattice, a new kind of intersection graph of lattices.

Now, we recall some definition and notation of graphs and lattices (see [6], [11], [8]). Let G be a graph with vertex-set $V(G)$ and edge-set $E(G)$. The *degree* of a vertex x in the graph G is the number of edges of G incident with x and denoted by $deg(x)$. Also for two distinct vertices x and y in G, the notation $x - y$ means that x and y are adjacent. A *path* from x to y is a series of adjacent vertices $x - x_1 - x_2 - ... - x_n - y$. A graph G is called *connected*, if for any

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vertices x and y of G there is a path between x and y. Otherwise, G is called disconnected. The distance between two vertices x and y, denoted by $d(x, y)$, is the length of shortest path connecting them (if such a path does not exist, then $d(x, y) := \infty$). The *diameter* of a graph G, denoted by $diam(G)$, is equal to $\sup\{d(x, y) : x, y \in V(G)\}$. A graph is *complete* if it is connected with diameter less than or equal to one. We use K_n to denote the *complete* graph with n vertices. The *girth* of a graph G, denoted by $gr(G)$, is the length of a shortest cycle in G, provided G contains a cycle; otherwise, $gr(G) := \infty$. A *clique* of a graph is its maximal complete subgraph and the number of vertices in the largest clique of G, denoted by $\omega(G)$, is called the *clique number* of G. Two vertices x and y of the graph G are *orthogonal*, denote by $x \perp y$, if x and y are adjacent in G and there is no vertex $z \in G$ which is adjacent to both x and y. A graph G is called *complemented* if for each vertex v of G, there is a vertex w of G (called a *complement* of v) such that $v \perp w$.

For r a nonnegative integer, an *r-partite* graph is one whose vertex set can be partitioned into r subsets so that no edge has both ends in any one subset. A *complete r-partite* graph is one in which each vertex is joined to every vertex that is not in the same subset. The *complete bipartite* graph with part sizes m and n is denoted by $K_{m,n}$. We will sometimes call $K_{1,n}$ a *star* graph. Note that a graph whose vertices-set is empty is a *null* graph and a graph whose edge-set is empty is an *empty* graph.

Recall that a *lattice* $\mathfrak{L}$ is a poset $(\mathfrak{L}, \leq)$ in which every pair of elements $a, b \in \mathfrak{L}$ has a *g.l.b.* (called the meet of a and b, and written $a \wedge b$) and a *l.u.b.* (called the join of a and b, and written $a \vee b$). In this paper, we assume that $\mathfrak{L}$ is a a finite lattice. A lattice $\mathfrak{L}$ is called *bounded* if there are elements 0 and 1 such that $0 \wedge a = 0$ and $a \vee 1 = 1$, for all $a \in \mathfrak{L}$. Thus every finite lattice is bounded.

A non-empty subset I of a lattice $\mathfrak{L}$ is called a *semi-ideal* if for $a \in I$ and $b \in L, b \leq a$ implies $b \in I$. A semi-ideal I of a lattice $\mathfrak{L}$ is called an *ideal* of $\mathfrak{L}$ if $a, b \in I$ implies $a \vee b \in I$. An ideal of $\mathfrak{L}$ which consists of all $x \leq a$, for some $a \in \mathfrak{L}$ is called a *principle ideal* generated by a and denoted by $[a]^l$. Clearly in a finite lattice every ideal is principal (see [8]).

An element a of a lattice $\mathfrak{L}$ is said to be *small* (written $a \ll \mathfrak{L}$) if $a \vee b \neq 1$ for every $b \neq 1$. A lattice $\mathfrak{L}$ is called *hollow* if every non-zero element is small in $\mathfrak{L}$. An element $a \in \mathfrak{L}$ is called *compact* if for every subset X of $\mathfrak{L}$ and $a \leq \bigvee X$, there is a finite subset $F \subseteq X$ such that $a \leq \bigvee F$ and $\mathfrak{L}$ is said to be compact if 1 is compact.

Let a and b be two distinct elements of $\mathfrak{L}$. Whenever $a \leq b$ and there is no element l in $\mathfrak{L}$ such that $a \leq l \leq b$, we say b covers a, in notation $a \prec b$. An element $a \in \mathfrak{L}$ is an *atom* of lattice $\mathfrak{L}$ if $0 \prec a$. We denote the set of all atoms in a lattice $\mathfrak{L}$ by $A(\mathfrak{L})$. A lattice $\mathfrak{L}$ is called *semiatomic* if 1 is a join of atoms in $\mathfrak{L}$.

A subset D of $\mathfrak{L}$ is called *upper directed* if each finite subset of D has an upper bound in D. The lattice $\mathfrak{L}$ is called *upper continuous* if $a \wedge (\bigvee D) = \bigvee_{d \in D}(a \wedge d))$

holds for every $a \in \mathfrak{L}$ and upper directed subset $D \subseteq \mathfrak{L}$. A lattice $\mathfrak{L}$ is called *local* if it has a largest element different from 1.

2. Some basic properties of $SG(\mathfrak{L})$

We define the small intersection graph of ideals of a lattice $\mathfrak{L}$, denoted by $SG(\mathfrak{L})$, as an undirected graph with all elements of $\mathfrak{L} \setminus \{0, 1\}$ being the vertices, and two distinct vertices a and b are adjacent if and only if $a \wedge b \ll \mathfrak{L}$. Now, we begin with the following Proposition.

Proposition 2.1. *The small intersection graph $SG(\mathfrak{L})$ is a null graph if and only if 1 is an atom.*

Since all definitions of graph theory are for non-null graph, in this paper all graphs are considered non-null.

Theorem 2.1. *Let a and b be two vertices of $SG(\mathfrak{L})$ such that $a \leq b$, then $deg(b) \leq deg(a)$.*

Proof. Let a and b be two vertices of $SG(\mathfrak{L})$ such that $a \leq b$. Let d be vertex adjacent to b. So $b \wedge d \ll \mathfrak{L}$, which implies $a \wedge d \ll \mathfrak{L}$. Thus d is adjacent to a. Hence $deg(b) \leq deg(a)$. □

Corollary 2.1. *If $SG(\mathfrak{L})$ is a small intersection graph, then $\Delta(SG(\mathfrak{L})) = max\{deg(a) : a \in A(\mathfrak{L})\}$, where $\Delta(SG(\mathfrak{L}))$ is the maximum degree of vertices in $SG(\mathfrak{L})$.*

Proof. Suppose that $b \in \mathfrak{L}$ and $a \in A(\mathfrak{L})$ such that $a \leq b$. Then, by Theorem 2.2, $deg(b) \leq deg(a)$. Thus the result holds. □

It is well-known that, if Q is an ordered set with power set $\mathcal{P}(Q)$ and $\varphi : Q \to \mathcal{P}(Q)$ is given by $\varphi(p) = \{q \in Q : q \leq p\}$, then one can identify Q by the image of φ ordered by inclusion. So every lattice $(\mathfrak{L}, \leq)$ can be embedded to the lattice $(\mathcal{P}(\mathfrak{L}), \subseteq)$. Thus we can consider the graph $SG(\mathfrak{L})$ as a subgraph of $SG(\mathcal{P}(\mathfrak{L}))$. Clearly, in $SG(\mathcal{P}(\mathfrak{L}))$, two distinct elements A and B in $\mathcal{P}(\mathfrak{L})$ are adjacent if and only if $(A \cap B) \cup C \neq \mathfrak{L}$ for every element C in $\mathcal{P}(\mathfrak{L}) \setminus \{\emptyset, \mathfrak{L}\}$.

Proposition 2.2. *Let X be a finite set that contains exactly n elements and S be a subset of X with m elements. Then the neighborhood of S in the small intersection graph of the lattice $(\mathcal{P}(X), \subseteq)$, where $\mathcal{P}(X)$ is the power set of X, is $\mathcal{P}(S^c) \setminus \{\emptyset\}$ and so, the degree of the vertex S in the $SG(\mathcal{P}(X))$ is $2^{n-m} - 1$.*

Proof. For every element A in $\mathcal{P}(X) \setminus \{\emptyset, X\}$, A is adjacent to S if and only if $A \cap S \ll \mathcal{P}(X)$. Clearly, $A \cap S \ll \mathcal{P}(X)$ if and only if $A \cap S = \emptyset$, which is equivalent to $A \subseteq S^c$. Hence the result follows. □

In the next Theorem, we determine the degree of a small element in $SG(\mathfrak{L})$.

Theorem 2.2. *If a is a non-zero small element of $\mathfrak{L}$ and $|SG(\mathfrak{L})| = n$, then $deg(a) = n - 1$, for all $n \in \mathbb{N}$.*

Proof. Suppose that a is a non zero small element of $\mathfrak{L}$. Since for every element $x \neq a$, $a \wedge x \leq a$ and since $a \ll \mathfrak{L}$, $a \wedge x \ll \mathfrak{L}$. Then there is an edge between a and x, for every non-zero element $x \neq a$. Therefore, if $|SG(\mathfrak{L})| = n$, then $deg(a) = n - 1$, for all $n \in \mathbb{N}$. □

The following Proposition is an immediate consequence of Theorem 2.5.

Proposition 2.3. *If $\mathfrak{L}$ is hollow, then $SG(\mathfrak{L})$ is complete graph.*

Corollary 2.2. *Let $\mathfrak{L}$ be a lattice. Then the following hold:*

(i) If $\mathfrak{L}$ is local, then the graph $SG(\mathfrak{L})$ is complete.

(ii) If $\mathfrak{L}$ has at least one non-zero small element, then the induced subgraph on the sets of small elements are clique in $SG(\mathfrak{L})$.

Proof. (i) Since local lattices are hollow, the result follows from Proposition 2.6.

(ii) It follows easily. □

Recall that the one-to-one correspondence $f : \mathfrak{L} \to \acute{\mathfrak{L}}$, for two lattices $\mathfrak{L}$ and $\acute{\mathfrak{L}}$, is a lattice isomorphism if, $f(a \vee b) = f(a) \vee f(b)$ and $f(a \wedge b) = f(a) \wedge f(b)$, for any $a, b \in \mathfrak{L}$.

Proposition 2.4. *Let for two lattices $\mathfrak{L}$ and $\acute{\mathfrak{L}}$, $f : \mathfrak{L} \to \acute{\mathfrak{L}}$ is a lattice isomorphism. Then there exists a graph isomorphism between the small intersection graphs $SG(\mathfrak{L})$ and $SG(\acute{\mathfrak{L}})$.*

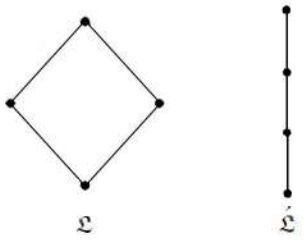
Proof. First Suppose that $\bar{f} : V(SG(\mathfrak{L})) \to V(SG(\acute{\mathfrak{L}}))$ such that for each $x \in V(SG(\mathfrak{L}))$, $\bar{f}(x) = f(x)$. Now, assume that x and y are adjacent in $SG(\mathfrak{L})$, then for each $z \in \mathfrak{L}$, $x \wedge y \ll \mathfrak{L}$ and $(x \wedge y) \vee z \neq 1$. Thus by the lattice isomorphism definition, $(x \wedge y) \vee z \neq 1$ if and only if $f(x \wedge y) \vee f(z) \neq 1$ if and only if $(f(x) \wedge f(y)) \vee f(z) \neq 1$, for every element $z \in \mathfrak{L}$. So $\bar{f}(x)$ is adjacent to $\bar{f}(y)$ in $SG(\acute{\mathfrak{L}})$ and the result holds. □

In the following example shows that, in general, the converse of Proposition 2.8 is not true.

Example 2.1. Assume that $\mathfrak{L}$ and $\acute{\mathfrak{L}}$ are the lattices in Figure 1. Then $\mathfrak{L}$ and $\acute{\mathfrak{L}}$ are not isomorphic, but $SG(\mathfrak{L})$ and $SG(\acute{\mathfrak{L}})$ are isomorphic.

3. Connectivity, diameter, girth and orthogonal vertices of $SG(\mathfrak{L})$

In this section, we study some properties of the graph $SG(\mathfrak{L})$ such as connectivity, diameter, girth and orthogonal vertices. Now, we start with the following Theorem.



Theorem 3.1. *Let $\mathfrak{L}$ be a lattice with at least a non-zero small element. Then $SG(\mathfrak{L})$ is a connected graph with $diam(SG(\mathfrak{L})) \leq 2$.*

Proof. Suppose that a_1 and a_2 are two distinct vertices of the graph $SG(\mathfrak{L})$. Assume that $\mathfrak{L}$ has at least a non-zero small element as b. Now we have the following cases:

Case 1. If $b = a_i$, for $i = 1$ or $i = 2$. Since $a_1 \wedge a_2 \leq a_i$ for $i = 1, 2$, $a_1 \wedge a_2 \ll \mathfrak{L}$. Then there is an edge between a_1 and a_2, thus $d(a_1, a_2) = 1$.

Case 2. If $b \neq a_i$, for $i = 1, 2$. Since $a_i \wedge b \leq b$, $a_i \wedge b \ll \mathfrak{L}$, for $i = 1, 2$, then there exists an a_1, a_2-path to form $a_1 - b - a_2$ and $d(a_1, a_2) = 2$. As the diameter of graph is the maximum distance between any two vertices, consequently $diam(SG(\mathfrak{L})) \leq 2$ and so $SG(\mathfrak{L})$ is a connected graph. □

In the following example, we show that, the above Theorem may not be hold for any lattice without small element.

Example 3.1. Let $X = \{1, 2, 3\}$. Set $\mathfrak{L} := (\mathcal{P}(X), \subseteq)$, where $\mathcal{P}(X)$is the power set of X. Then $\mathfrak{L}$ has no small element and $d(\{1, 3\}, \{2, 3\}) = 3$.

In a lattice $\mathfrak{L}$, $R(\mathfrak{L})$ will indicate the radical of $\mathfrak{L}$, where $R(\mathfrak{L})$ is the intersection of all the maximal elements $\neq 1$ in $\mathfrak{L}$.

Corollary 3.1. *The graph $SG(\mathfrak{L})$ is a connected graph, if one of the following holds:*

(i) The lattice $\mathfrak{L}$ is compact.

(ii) There exists a compact element of $\mathfrak{L}$ contained in $R(\mathfrak{L})$.

Proof. (i) Since $\mathfrak{L}$ is a compact lattice, $R(\mathfrak{L}) \ll \mathfrak{L}$ and results follows from Theorem 3.1.

(ii) If a is a compact element in $\mathfrak{L}$ and $a \leq R(\mathfrak{L})$, then $a \ll \mathfrak{L}$. Hence we can easily check that $SG(\mathfrak{L})$ is connected. □

We say that two vertices x and y of the graph G are orthogonal, written $x \perp y$, if x and y are adjacent in G and there is no vertex $z \in SG(\mathfrak{L})$ which is adjacent to both x and y. In the other word $x \perp y$ if $x \wedge y \ll \mathfrak{L}$ and for every vertex $z \in SG(\mathfrak{L})$, either $x \wedge z$ is not small in $\mathfrak{L}$ or $z \wedge y$ is not small in $\mathfrak{L}$. Also a graph G is called complemented if for each vertex v of G, there is a vertex w of G (called a complement of v) such that $v \perp w$.

Recall that, in a lattice $\mathfrak{L}$, $S(\mathfrak{L})$ denotes the set of all the non-zero small elements of $\mathfrak{L}$.

Proposition 3.1. *Let $\mathfrak{L}$ be a lattice with $|S(\mathfrak{L})| \geq 2$ and $|SG(\mathfrak{L})| \geq 3$, then the following conditions holds:*

(i) Every two small elements of $\mathfrak{L}$ cannot be orthogonal vertices to each other.

(ii) The graph $SG(\mathfrak{L})$ has at least a triangle.

(iii) For $|SG(\mathfrak{L})| = n$ and $|S(\mathfrak{L})| = 2$, if we take m equal to sum of the number of the triangles of this graph, then $(n-2) \leq m \leq \frac{1}{6}n(n-1)(n-2)$.

Proof. (i) Suppose that x and y are two small elements of $\mathfrak{L}$, then $x \wedge y \ll \mathfrak{L}$ and for any vertex $v \in SG(\mathfrak{L})$, since $x \wedge v \leq x$ and $y \wedge v \leq y$, $x \wedge v \ll \mathfrak{L}$ and $y \wedge v \ll \mathfrak{L}$. Thus v is adjacent to both x and y. Therefore, they cannot be orthogonal vertices each other.

(ii) It is clear.

(iii) Let $|SG(\mathfrak{L})| = n$, where $n \geq 3$ and $|S(\mathfrak{L})| = 2$. Then the number of the elements which are not small is equal to $n-2$. If we choose two vertices of the triangles are small elements, then the sum of the triangles is at least equal to $C_2^2 \times C_1^{n-2} = n-2$. Also the sum of the number of all triangles in this graph is at most equal to $C_2^2 \times C_1^{n-2} + C_1^2 \times C_1^{n-2} + C_3^{n-2} = \frac{1}{6}n(n-1)(n-2)$. Therefore, $(n-2) \leq m \leq \frac{1}{6}n(n-1)(n-2)$. □

Theorem 3.2. *Let $\mathfrak{L}$ be a lattice. Then the following statements are equivalent:*

(i) The graph $SG(\mathfrak{L})$ has no triangle.

(ii) Every two adjacent vertices of the graph $SG(\mathfrak{L})$ are orthogonal vertices.

(iii) The lattice $\mathfrak{L}$ has at most a non-zero small element such that the meet of each pair of the non-small element of $\mathfrak{L}$ is not small in $\mathfrak{L}$.

Proof. (i)⇒ (ii) Let a and b be two adjacent vertices of the graph $SG(\mathfrak{L})$ which are not orthogonal vertices. Then by the definition of orthogonality, there exists another vertex c of the graph $SG(\mathfrak{L})$ which is adjacent to both a and b, then there exists a cycle to form $a - b - c$. This means that there is a triangle in the graph $SG(\mathfrak{L})$, a contradiction.

(ii)⇒ (iii) Assume that every two adjacent vertices of the graph $SG(\mathfrak{L})$ are orthogonal vertices. Let there exist at least two non-zero small elements a_1 and a_2 of $\mathfrak{L}$. Since $a_1 \wedge a_2 \ll \mathfrak{L}$, they are adjacent vertices of the graph $SG(\mathfrak{L})$ and also $a_1 \perp a_2$. However, by Proposition 3.4, they cannot be orthogonal to each other, a contradiction.

(iii)⇒ (i) Suppose that $\mathfrak{L}$ has no non-zero small element. Since the meet of each pair of the non-small non-zero element of $\mathfrak{L}$ is not small in $\mathfrak{L}$, $SG(\mathfrak{L})$ has no triangle. Moreover, let z be only non-zero small element of $\mathfrak{L}$. Then for every three arbitrary vertices x_1, x_2 and x_3 of the graph $SG(\mathfrak{L})$, at least two of them are not small in $\mathfrak{L}$. Let $z = x_1$. Since $x_2 \wedge x_3$ is not small in $\mathfrak{L}$, $x_2 - z - x_3$ is a path. Also if $z \neq x_i$ for $i = 1, 2, 3$. Since $x_i \wedge x_j$ is not small in $\mathfrak{L}$, for $i, j = 1, 2, 3$ and $i \neq j$, then x_1, x_2 and x_3 are not adjacent vertices in graph $SG(\mathfrak{L})$. Therefore, there is no triangle in the graph $SG(\mathfrak{L})$. □

For any lattice $\mathfrak{L}$, two elements a and b are comaximal if $a \vee b = 1$.

Proposition 3.2. *Let $\mathfrak{L}$ be a lattice and a, b be two comaximal elements of $\mathfrak{L}$. If $R(\mathfrak{L}) = 0$ and $a \perp b$ in $SG(\mathfrak{L})$, then $1 = a \oplus b$.*

Proof. Since a, b be two comaximal elements of $\mathfrak{L}$ and $a \perp b$ in the graph $SG(\mathfrak{L})$, then $1 = a \vee b$, $a \wedge b \ll \mathfrak{L}$. Hence $a \wedge b \leq R(\mathfrak{L}) = 0$. Consequently, the result follows.. □

We say that a lattice $\mathfrak{L}$ satisfies the covering condition if for all $a, b \in \mathfrak{L}$, $a \wedge b \prec a$ if and only if $b \prec a \vee b$. For the rest of paper, we assume that $\mathfrak{L}$ satisfies the covering condition.

Proposition 3.3. *If $1 = a_1 \vee a_2$, where a_1 and a_2 are two atoms in $\mathfrak{L}$ and it satisfies the covering condition, then $SG(\mathfrak{L})$ is a complete graph.*

Proof. Suppose that $1 = a_1 \vee a_2$ such that a_1 and a_2 are two atoms in $\mathfrak{L}$. Then by covering condition, any non-trivial element of $\mathfrak{L}$ is an atom. Let x and y be two distinct vertices of graph $SG(\mathfrak{L})$. Then they are the non-trivial elements of $\mathfrak{L}$ which are atoms. Moreover, $x \wedge y \leq x, y$ and if $x \wedge y \neq 0$ then $x \wedge y = x = y$, which is a contradiction. Therefore, $x \wedge y = 0 \ll \mathfrak{L}$, thus $SG(\mathfrak{L})$ is a complete graph. □

In the following example, we state that the converse of Proposition 3.7 is not true, in general.

Example 3.2. Assume that $L = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, \{1, 2, 3\}\}$ with two operation $\cup$ and $\cap$. Then, it is easy to see that the graph $SG(\mathfrak{L})$ is complete, but $\{1, 2, 3\}$ can not be written as the disjoint union.

Proposition 3.4. *Let $\mathfrak{L}$ be a lattice with $R(\mathfrak{L}) \neq 0$. Then the following conditions hold:*

(i) If a is a non-zero element of lattice $\mathfrak{L}$ which is a direct summand of 1 with $0 \neq R([a]^l) \ll \mathfrak{L}$, then the graph $SG(\mathfrak{L})$ contains at least one cycle of length 3.

(ii) If x is compact element of $\mathfrak{L}$ contained in $R(\mathfrak{L})$,then $d(x, R(\mathfrak{L})) = 1$ and $d(x, z) = 1$ for every non-zero element z of $\mathfrak{L}$.

Proof. (i) Since a is a direct summand of $\mathfrak{L}$, there exists an element b of $\mathfrak{L}$ such that $a \oplus b = 1$. Then by, $R(\mathfrak{L}) = R([a]^l) \oplus R([b]^l)$. Since $R([a]^l) \leq a$ and $a \wedge R([b]^l) \leq a \wedge b = 0$, by covering condition, $a \wedge R(\mathfrak{L}) = R([a]^l)$. Then $a \wedge R(\mathfrak{L}) \ll \mathfrak{L}$. Also, $R([a]^l) = a \wedge R([a]^l) \ll \mathfrak{L}$ and $R([a]^l) = R([a]^l) \wedge R(\mathfrak{L}) \ll \mathfrak{L}$ and there exist edges $a - R(\mathfrak{L})$, $a - R([a]^l)$ and $R([a]^l) - R(\mathfrak{L})$ in graph $SG(\mathfrak{L})$. Hence $(a, R([a]^l), R(\mathfrak{L}))$ is a cycle of length 3. Consequently, the graph $SG(\mathfrak{L})$ contains at least one cycle of length 3.

(ii) Let x be a compact element of $\mathfrak{L}$, then $x \ll \mathfrak{L}$. Since $x \leq R(\mathfrak{L})$, $x = x \wedge R(\mathfrak{L}) \ll \mathfrak{L}$ and since $x \wedge z \leq x, x \wedge z \ll \mathfrak{L}$ for every non-zero element z of $\mathfrak{L}$. Hence $d(x, R(\mathfrak{L})) = 1$ and $d(x, z) = 1$. □

Theorem 3.3. *If $|S(\mathfrak{L})| \geq 2$, then $SG(\mathfrak{L})$ contains at least a cycle and $gr(SG(\mathfrak{L})) = 3$.*

Proof. Suppose that $|S(\mathfrak{L})| \geq 2$. Then $\mathfrak{L}$ has at least two different small elements, say a_1 and a_2. Since $a_1 \wedge a_2 \leq a_i$ for $i = 1, 2$, $a_1 \wedge a_2 \ll \mathfrak{L}$. Moreover $a_1 \wedge (a_1 \wedge a_2) \ll \mathfrak{L}$ and $a_2 \wedge (a_1 \wedge a_2) \ll \mathfrak{L}$. We consider the following cases:

Case 1. If $a_1 \wedge a_2 \neq 0$, then there exists three edges $a_1 - a_2$, $a_1 - (a_1 \wedge a_2)$ and $a_2 - (a_1 \wedge a_2)$ of graph $SG(\mathfrak{L})$. Hence $a_1 - (a_1 \wedge a_2) - a_2$ is a cycle of length 3. Also $a_1 \vee a_2 \ll \mathfrak{L}$ and since $a_1 \wedge (a_1 \vee a_2) \ll \mathfrak{L}$ and $a_2 \wedge (a_1 \vee a_2) \ll \mathfrak{L}$, $a_1 - (a_1 \vee a_2) - a_2$ is a cycle of length 3. Similarly $a_1 \wedge a_2, a_1, a_1 \vee a_2$ and $a_1 \wedge a_2, a_2, a_1 \vee a_2$ are cycles of length 3 and is a cycle of length 4.

Case 2. If $a_1 \wedge a_2 = 0$, then there exists at least a cycle $a_1 - (a_1 \vee a_2) - a_2$ of length 3 in graph $SG(\mathfrak{L})$. Therefore, the graph $SG(\mathfrak{L})$ contains at least a cycle and since the girth of a graph with cycles is the length of the shortest cycle, $gr(SG(\mathfrak{L})) = 3$. □

Theorem 3.4. *Let $\mathfrak{L}$ be a lattice. Then the following statements hold:*

(i) If $\mathfrak{L}$ is a upper continuous semiatomic lattice which is not simple, then the graph $SG(\mathfrak{L})$ is complemented.

(ii) If $\mathfrak{L}$ is a compact lattice with $R(\mathfrak{L}) \neq 0$, then the graph $SG(\mathfrak{L})$ is not complemented.

Proof. (i) Suppose that $\mathfrak{L}$ is upper continuous semiatomic lattice. Let x be a vertex of the graph $SG(\mathfrak{L})$. Then by [7, Theorem 6.6], every element of $\mathfrak{L}$ is a direct summand of 1. Thus there exists a element y of $\mathfrak{L}$ such that $1 = x \oplus y$. Hence $x \wedge y = 0 \ll \mathfrak{L}$. Since $\mathfrak{L}$ has no non-zero small element, there is no vertex z of $SG(\mathfrak{L})$ such that $z \wedge x \ll \mathfrak{L}$ and $z \wedge y \ll \mathfrak{L}$. Therefore, $SG(\mathfrak{L})$ is complemented.

(ii) Since $\mathfrak{L}$ is a compact lattice, $0 \neq R(L) \ll \mathfrak{L}$. So we have the following cases:

Case 1. If $R(L)$ is an atom, since $R(\mathfrak{L}) = \bigwedge_{i \in I} m_i$ where m_i is the maximal element of $\mathfrak{L}$, for all $i \in I$. We choose $s = \bigwedge_{i \in I - \{1\}} m_i$. Then $m_1 - s - R(\mathfrak{L})$ is a triangle in the graph $SG(\mathfrak{L})$. Then the graph $SG(\mathfrak{L})$ is not complemented

Case 2. If $R(\mathfrak{L})$ is not an atom of $\mathfrak{L}$, then there exists an element x of $\mathfrak{L}$ which $x \leq R(\mathfrak{L})$, then $x \ll \mathfrak{L}$. Thus for each vertex y of $SG(\mathfrak{L})$, $x - y - R(\mathfrak{L})$ is a triangle of this graph. Consequently, the graph $SG(\mathfrak{L})$ is not complemented. □

Proposition 3.5. *Let $1 = a \oplus b$ be a compact lattice such that $R([a]^l) \neq 0$ and $R([b]^l) \neq 0$. Then the following statements hold:*

(i) The vertices a and b are not orthogonal in the graph $SG(\mathfrak{L})$.

(ii) The complete graph K_5 is an induced subgraph of $SG(\mathfrak{L})$.

(iii) The graph $SG(\mathfrak{L})$ is not complemented.

Proof. (i) Suppose that $1 = a \oplus b$. Then $a \wedge b = 0 \ll \mathfrak{L}$ and since $\mathfrak{L}$ is a compact lattice, $R(\mathfrak{L}) \ll \mathfrak{L}$. Hence $a \wedge R(\mathfrak{L}) \ll \mathfrak{L}$ and $b \wedge R(\mathfrak{L}) \ll L$. Thus $SG(\mathfrak{L})$ has a triangle to form $(a, R(\mathfrak{L}), b)$. The result now follows from Theorem 3.5.

(ii) Let $1 = a \oplus b$. Then $R(\mathfrak{L}) = R([a]^l) \oplus R([b]^l)$ and $R([a]^l) \oplus R([b]^l) = 0 \ll \mathfrak{L}$. Also $R([a]^l) \leq a$ and $a \wedge R([b]^l) \leq a \wedge b = 0$, then the covering condition, implies that $a \wedge R(\mathfrak{L}) = R([a]^l)$ and similarly, $b \wedge R(\mathfrak{L}) = R([b]^l)$. Moreover, $a \wedge R([a]^l) = R([a]^l) \ll \mathfrak{L}$, $b \wedge R([b]^l) = R([b]^l) \ll \mathfrak{L}$, $a \wedge R(\mathfrak{L}) = R([a]^l) \ll \mathfrak{L}$, $b \wedge R(\mathfrak{L}) = R([b]^l) \ll \mathfrak{L}$, $R([a]^l) \wedge R(\mathfrak{L}) = R([a]^l) \ll \mathfrak{L}$ and $R([b]^l) \wedge R(\mathfrak{L}) = R([b]^l) \ll \mathfrak{L}$. Hence $a, b, R([a]^l), R([b]^l)$ and $R(\mathfrak{L})$ are adjacent vertices in the graph $SG(\mathfrak{L})$. Therefore the subgraph induced on the vertex set of $\{a, b, R([a]^l), R([b]^l), R(\mathfrak{L})\}$ is the complete graph K_5.

(iii) Let $1 = a \oplus b$. Then, by Theorem 3.11, we are done. □

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Accepted: May 18, 2020

A study on weak pseudoorders in hyper S-posets

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Abstract. In this paper, the concept of weak pseudoorders in a hyper S-poset is introduced, and several related properties are given. Especially, we characterize the structure of factor hyper S-posets by weak pseudoorders. Furthermore, we discuss the relationship between order-congruences and weak pseudoorders on a hyper S-poset, and give some characterizations of order-congruences by the properties of weak pseudoorders. In the sequel, we give some homomorphism theorems of hyper S-posets by weak pseudoorders, which are generalizations of similar results in S-posets and ordered semigroups.

Keywords: ordered semihypergroup, hyper S-poset, order-congruence, weak pseudoorder.

1. Introduction

As we know, an ordered semigroup $(S, \cdot, \leq)$ is a semigroup $(S, \cdot)$ together with an order relation " $\leq$ " which is compatible with the binary operation. Recall that if $(S, \cdot, \leq)$ is an ordered semigroup then a nonempty poset $(A, \leq_A)$ together with a mapping $A \times S \to A, (a, s) \mapsto as$, where as denotes the result of action of s on a, is called a *right S-poset* if (1) $(as)t = a(st)$, (2) $a \leq_A b$ implies $as \leq_A bs$, and (3) $s \leq t$ implies $as \leq_A at$, for all $a, b \in A, s, t \in S$. Left S-posets are defined analogously. S-posets play an important role not only in studying properties of ordered semigroups but also in other mathematical areas, such as graph theory and algebraic automata theory (for example, see [1]). Nowadays many researchers studied different aspects of S-posets, for instance, Bulman-Fleming et al. [2], Laan [11], Liang and Khosravi [12], Liang and Luo [14], Shi [17] and Shi et al. [18], also see [13].

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Algebraic hyperstructures (or hypersystems) represent a natural extension of classical algebraic structures and they were originally proposed in 1934 by a French mathematician Marty [16], at the 8th Congress of Scandinavian Mathematicians. Since then, hyperstructures are widely investigated from the theoretical point of view and for their applications to many branches of pure and applied mathematics, for example, see [4, 5, 7, 22]. One of the main reason which attracts researches towards hyperstructures is its unique property that in hyperstructures composition of two elements is a set, while in classical algebraic structures the composition of two elements is an element. Thus algebraic hyperstructures are natural extension of classical algebraic structures. It is worth pointing out that Heidari and Davvaz [9] applied the theory of hyperstructures to ordered semigroups and introduced the concept of ordered semihypergroups, which is a generalization of the concept of ordered semigroups. The work on ordered semihypergroup theory can be found in [3, 6, 19, 20, 21, 25, 26]. In particular, Tang et al. [20] defined and studied the hyper S-posets over an ordered semihypergroup, and extended some results on S-posets to hyper S-posets. They introduced the concepts of pseudoorders and (strong) order-congruences on a hyper S-poset A, and obtained the relationship between strongly order-congruences and pseudoorders on A. Especially, they proved that there must exist a strong congruence ρ on a hyper S-poset A such that A/ρ is a hyper S-poset (see Theorem 3.7 in [20]). A natural question about hyper S-posets is raised: Is there a congruence ρ on a hyper S-poset A for which A/ρ is a hyper S-poset? As a further study, in the present paper we study the weak pseudoorders in a hyper S-poset in detail and answer the above question.

The rest of this paper is organized as follows. After an introduction, in Section 2 we recall some basic notions and results from the hyperstructure theory which will be used throughout this paper. In Section 3, we introduce the concept of weak pseudoorders on a hyper S-poset A, and illustrate this notion is a generalization of the concept of pseudoorders on A by some examples. Furthermore, the properties of weak pseudoorders on a hyper S-poset are investigated. In particular, we construct a hyper S-poset $A/\underline{\rho}$ by a weak pseudoorder ρ. In Section 4, the relationships between order-congruences and weak pseudoorders on a hyper S-poset is established, and some homomorphism theorems of hyper S-posets by weak pseudoorders are given.

2. Preliminaries and some notations

In this section, we present some definitions and results which will be used throughout this paper.

Let S be a nonempty set. A mapping $\circ : S \times S \to P^\star(S)$, where $P^\star(S)$ denotes the family of all nonempty subsets of S, is called a *hyperoperation* or *hypercomposition* on S. The image of the pair (x, y) is denoted by $x \circ y$. The couple $(S, \circ)$ is called a *hyperstructure*. In the above definition, if $x \in S$ and

A, B are nonempty subsets of S, then $A \circ B$ is defined by

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b.$$

Also $A \circ x$ is used for $A \circ \{x\}$ and $x \circ A$ for $\{x\} \circ A$. A hyperstructure $(S, \circ)$ is a *semihypergroup* [4] if for all $x, y, z \in S$, $(x \circ y) \circ z = x \circ (y \circ z)$, which means that

$$\bigcup_{u \in x \circ y} u \circ z = \bigcup_{v \in y \circ z} x \circ v.$$

Definition 2.1 ([9])**.** An *ordered semihypergroup* (also called *po-semihypergroup* in [9]) $(S, \circ, \leq)$ is a semihypergroup $(S, \circ)$ with an order relation $\leq$ which is compatible with the hyperoperation $\circ$, meaning that for any $x, y, a \in S$, $x \leq y$ implies $a \circ x \leq a \circ y$ and $x \circ a \leq y \circ a$. Here, if $A, B \in P^\star(S)$, then we say that $A \leq B$ if for every $a \in A$ there exists $b \in B$ such that $a \leq b$. In particular, if $A = \{a\}$, then we write $a \leq B$ instead of $\{a\} \leq B$.

Clearly, every ordered semigroup can be regarded as an ordered semihypergroup. Also see [19]. We now recall the notion of hyper S-acts over semihypergroups from [8].

Definition 2.2 ([8])**.** Let $(S, \circ)$ be a semihypergroup and A a nonempty set. If we have a mapping $\mu : A \times S \to P^\star(A) \mid (a, s) \mapsto \mu(a, s) := a * s \in P^\star(A)$, called the *hyper action of* S (or the S*-hyperaction*) on A, such that $a*(s \circ t) = (a*s)*t$, for all $a \in A, s, t \in S$, where

$$(\forall T \subseteq S)\ a * T = \bigcup_{t \in T} a * t;\ (\forall B \subseteq A)\ B * t = \bigcup_{b \in B} b * t,$$

then we call A a *right hyper* S*-act* (also called *right* S*-hypersystem* in [8]).

Left hyper S-acts can be defined analogously, and in this paper we will often use the term hyper S-act to mean right hyper S-act. Furthermore, Tang et al. introduced the concept of hyper S-posets over an ordered semihypergroup.

Definition 2.3 ([20])**.** Let $(S, \circ, \leq)$ be an ordered semihypergroup. A *right hyper* S*-poset* $(A, \leq_A)$, often denoted $A_{\mathcal{H}}$ (or briefly A), is a right hyper S-act A equipped with a partial order $\leq_A$ and, in addition, for all $s, t \in S$ and $a, b \in A$, if $s \leq t$ then $a * s \leq_A a * t$, and if $a \leq_A b$ then $a * s \leq_A b * s$. Here, $a * s$ stands for the result of the hyper action of s on a, and if $A_1, A_2 \in P^\star(A)$, then we say that $A_1 \leq_A A_2$ if for every $a_1 \in A_1$ there exists $a_2 \in A_2$ such that $a_1 \leq_A a_2$.

Analogously, we can define a left hyper S-poset ${}_{\mathcal{H}}A$. Throughout this paper we shall use the term hyper S-poset to mean right hyper S-poset. Clearly, every S-poset over an ordered semigroup can be regarded as a hyper S-poset over an ordered semihypergroup.

Let A be a hyper S-act over a semihypergroup $(S, \circ)$ and ρ a relation on A. If B and C are both nonempty subsets of A, then we set

$$\begin{aligned} B\overrightarrow{\rho}C &\Leftrightarrow (\forall b \in B)(\exists c \in C)\ b\rho c, \\ B\overleftarrow{\rho}C &\Leftrightarrow (\forall c' \in C)(\exists b' \in B)\ b'\rho c', \\ B\overline{\rho}C &\Leftrightarrow B\overrightarrow{\rho}C \text{ and } B\overleftarrow{\rho}C, \\ B\overline{\overline{\rho}}C &\Leftrightarrow (\forall b \in B, \forall c \in C)\ b\rho c. \end{aligned}$$

An equivalence relation ρ on A is called *congruence* [20] if $(\forall x, y \in A)\ x\rho y \Rightarrow x * s\ \overline{\rho}\ y * s$ for all $s \in S$; ρ is called *strong congruence* [20] if

$$(\forall x, y \in A)\ x\rho y \Rightarrow x * s\ \overline{\overline{\rho}}\ y * s \text{ for all } s \in S.$$

We denote by $C(A)$ (resp. $SC(A)$) the set of all congruences (resp. strong congruences) on a hyper S-act A.

Lemma 2.4 ([20]). *Let A be a hyper S-act over a semihypergroup $(S, \circ)$ and ρ an equivalence relation on A. Then*

(1) *If ρ is a congruence, then A/ρ is a hyper S-act with respect to the following hyper S-action: $(a)_\rho \otimes s = \{(x)_\rho \mid x \in a * s\}$, and it is called a factor hyper S-act.*

(2) *If ρ is a strong congruence, then A/ρ is a hyper S-act with respect to the following (hyper) S-action: $(a)_\rho \otimes s = (x)_\rho$ for all $x \in a * s$, and it is called a factor hyper S-act. In particular, if S is a semigroup and the operation on S is defined by $s \circ t := \{st\}$ for all $s, t \in S$, then A/ρ is an S-act.*

Definition 2.5 ([20]). Let $(A, \leq_A)$ be a hyper S-poset over an ordered semihypergroup $(S, \circ, \leq)$. A relation ρ on A is called *pseudoorder* if it satisfies the following conditions:

(1) $\leq_A \subseteq \rho$.

(2) $a\rho b$ and $b\rho c$ imply $a\rho c$, i.e., $\rho \circ \rho \subseteq \rho$.

(3) $a\rho b$ implies $a * s\ \overline{\overline{\rho}}\ b * s$, for all $s \in S$.

Definition 2.6 ([20]). Let $(A, \leq_A)$ and $(B, \leq_B)$ be two hyper S-posets over an ordered semihypergroup $(S, \circ, \leq)$, the "$*$" and "$\diamond$" are hyper S-actions on A and B, respectively, $f : A \to B$ a mapping from A to B. f is called *isotone* if $x \leq_A y$ implies $f(x) \leq_B f(y)$, for all $x, y \in A$. f is called *reverse isotone* if $x, y \in A$, $f(x) \leq_B f(y)$ implies $x \leq_A y$. f is called *homomorphism* if it is isotone and satisfies $f(a) \diamond s = \{f(x) \mid x \in a * s\}$ for all $a \in A, s \in S$. f is called *isomorphism* if it is homomorphism, onto and reverse isotone. The hyper S-posets A and B are called *isomorphic*, in symbol $A \cong B$, if there exists an isomorphism between them.

Remark 2.7. Let $(A, \leq_A)$ and $(B, \leq_B)$ be two hyper S-posets over an ordered semihypergroup $(S, \circ, \leq)$. If f is a homomorphism and reverse isotone mapping from A to B, then $A \cong Im(f)$.

In [20], Tang et al. have illustrated that, for a (strong) congruence ρ on a hyper S-poset A, the factor hyper S-act A/ρ is not necessarily a hyper S-poset, in general. To characterize the structure of factor hyper S-posets, they introduced the concept of (strongly) order-congruences on a hyper S-poset.

Definition 2.8 ([20])**.** Let $(A, \leq_A)$ be a hyper S-poset over an ordered semihypergroup $(S, \circ, \leq)$. A congruence (resp. strong congruence) ρ is called an *order-congruence* (resp. a *strong order-congruence*) if there exists an order relation " $\preceq$ " on A/ρ such that:

(1) $(A/\rho, \preceq)$ is a hyper S-poset, where the S-hyperaction " $\otimes$ " on A/ρ is defined as one in Lemma 2.4.

(2) The canonical epimorphism $\varphi : A \to A/\rho, x \mapsto (x)_\rho$ is isotone, that is, φ is a homomorphism (resp. strong homomorphism) from A onto A/ρ.

The reader is referred to [15, 20, 23] for notation and terminology not defined in this paper.

3. Weak pseudoorders in hyper S-posets

In this section, we shall introduce the concept of weak pseudoorders in a hyper S-poset, and investigate their related properties. In particular, we characterize the structure of factor hyper S-posets by weak pseudoorders.

Let $(A, \leq_A)$ be a hyper S-poset over an ordered semihypergroup $(S, \circ, \leq)$ and ρ a relation on A. If $B, C \in P^\star(A)$, then we set

$$B \ \widetilde{\rho}\ C \ \Leftrightarrow\ (\forall b \in B)(\exists c \in C)\ b\rho c \text{ and } c\rho b \ \&\ (\forall c' \in C)(\exists b' \in B)\ b'\rho c' \text{ and } c'\rho b'.$$

Definition 3.1. Let $(A, \leq_A)$ be a hyper S-poset over an ordered semihypergroup $(S, \circ, \leq)$. A relation ρ on A is called *weak pseudoorder* if it satisfies the following conditions:

(1) $\leq_A \subseteq \rho$.

(2) $a\rho b$ and $b\rho c$ imply $a\rho c$, i.e., $\rho \circ \rho \subseteq \rho$.

(3) $a\rho b$ implies $a * s\ \overrightarrow{\rho}\ b * s$, for all $s \in S$.

(4) $a\rho b$ and $b\rho a$ imply $a * s\ \widetilde{\rho}\ b * s$, for all $s \in S$.

Obviously, the order relation " $\leq_A$ " on a hyper S-poset A is a weak pseudoorder relation on A. Furthermore, one can easily observe that every pseudoorder on a hyper S-poset A is a weak pseudoorder on A. However, the converse is not true, in general, as shown in the following example.

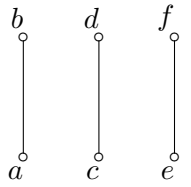
Example 3.2. We consider a set $S := \{a, b, c, d, e, f\}$ with the following hyperoperation " $\circ$ " and the order " $\leq$ ":

$\circ$	a	b	c	d	e	f
a	$\{a\}$	$\{a, b\}$	$\{c\}$	$\{c, d\}$	$\{e\}$	$\{e, f\}$
b	$\{b\}$	$\{b\}$	$\{d\}$	$\{d\}$	$\{f\}$	$\{f\}$
c	$\{c\}$	$\{c, d\}$	$\{c\}$	$\{c, d\}$	$\{c\}$	$\{c, d\}$
d	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$
e	$\{e\}$	$\{e, f\}$	$\{c\}$	$\{c, d\}$	$\{e\}$	$\{e, f\}$
f	$\{f\}$	$\{f\}$	$\{d\}$	$\{d\}$	$\{f\}$	$\{f\}$

$$\leq := \{(a, a), (a, b), (b, b), (c, c), (c, d), (d, d), (e, e), (e, f), (f, f)\}.$$

We give the covering relation "$\prec$" and the figure of S as follows:

$$\prec = \{(a, b), (c, d), (e, f)\}.$$

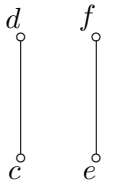


Then $(S, \circ, \leq)$ is an ordered semihypergroup. We now consider the partially ordered set $A = \{c, d, e, f\}$ defined by the order below:

$$\leq_A := \{(c, c), (d, d), (e, e), (f, f), (c, d), (e, f)\}.$$

We give the covering relation "$\prec_A$" and the figure of A

$$\prec_A = \{(c, d), (e, f)\}.$$



Then $(A, \leq_A)$ is a hyper S-poset over S with respect to S-hyperaction on A as above hyperoperation table (see [20]). Let ρ be a relation on S defined as follows:

$$\rho := \{(c, c), (d, d), (e, e), (f, f), (c, d), (c, f), (d, f), (e, f)\}.$$

It is a routine matter to verify that ρ is a weak pseudoorder on A, but it is not a pseudoorder on A. In fact, since $c\rho c$, while $c * b \,\overline{\overline{\rho}}\, c * b$ doesn't hold.

In the following, we shall construct a congruence on a hyper S-poset by a weak pseudoorder, from which we can characterize the structure of factor hyper S-posets.

Theorem 3.3. *Let $(A, \leq_A)$ be a hyper S-poset over an ordered semihypergroup $(S, \circ, \leq)$ and ρ a weak pseudoorder on A. Then, there exists a congruence $\underline{\rho}$ on A such that $A/\underline{\rho}$ is a hyper S-poset over S.*

Proof. We denote by $\underline{\rho}$ the relation on A defined by

$$\underline{\rho} := \{(a, b) \in A \times A \mid a\rho b \text{ and } b\rho a\} \ (= \rho \cap \rho^{-1}).$$

First, we claim that $\underline{\rho}$ is a congruence on A. In fact, for any $a \in A$, clearly, $(a, a) \in \leq_A \subseteq \rho$, so $a\underline{\rho}a$. If $(a, b) \in \underline{\rho}$, then $a\rho b$ and $b\rho a$. Thus $(b, a) \in \underline{\rho}$. Let $(a, b) \in \underline{\rho}$ and $(b, c) \in \underline{\rho}$. Then $a\rho b, b\rho a, b\rho c$ and $c\rho b$. Hence $a\rho c$ and $c\rho a$, which imply that $(a, c) \in \underline{\rho}$. Thus $\underline{\rho}$ is an equivalence relation on A. Now, let $a\underline{\rho}b$ and $s \in S$. Then $a\rho b$ and $b\rho a$. Since ρ is a weak pseudoorder on A, by condition (4) of Definition 3.1, we have $a * s \ \widetilde{\rho} \ b * s$. Thus, for every $x \in a * s$, there exists $y \in b * s$ such that $x\rho y$ and $y\rho x$ which imply that $x\underline{\rho}y$, and for every $y' \in b * s$, there exists $x' \in a * s$ such that $x'\rho y'$ and $y'\rho x'$ which mean that $x'\underline{\rho}y'$. It thus follows that $a * s \ \overrightarrow{\rho} \ b * s$ and $a * s \ \overleftarrow{\rho} \ b * s$. Hence $a * s \ \overline{\rho} \ b * s$. Therefore, $\underline{\rho}$ is indeed a congruence on A. By Lemma 2.4, $A/\underline{\rho}$ is a hyper S-act over S with respect to the following hyper S-action:

$$(\forall (a)_{\underline{\rho}} \in A/\underline{\rho}, \forall s \in S) \ (a)_{\underline{\rho}} \otimes s = \{(x)_{\underline{\rho}} \mid x \in a * s\}.$$

Now, we define a relation $\preceq_\rho$ on $A/\underline{\rho}$ as follows:

$$\preceq_\rho := \{((x)_{\underline{\rho}}, (y)_{\underline{\rho}}) \in A/\underline{\rho} \times A/\underline{\rho} \mid (x, y) \in \rho\}.$$

Then $(A/\underline{\rho}, \preceq_\rho)$ is a poset. Indeed, suppose that $(x)_{\underline{\rho}} \in A/\underline{\rho}$, where $x \in A$. Then $(x, x) \in \leq_A \subseteq \rho$. Hence, $(x)_{\underline{\rho}} \preceq_\rho (x)_{\underline{\rho}}$. Let $(x)_{\underline{\rho}} \preceq_\rho (y)_{\underline{\rho}}$ and $(y)_{\underline{\rho}} \preceq_\rho (x)_{\underline{\rho}}$. Then $x\rho y$ and $y\rho x$. Thus $x\underline{\rho}y$, and we have $(x)_{\underline{\rho}} = (y)_{\underline{\rho}}$. Now, if $(x)_{\underline{\rho}} \preceq_\rho (y)_{\underline{\rho}}$ and $(y)_{\underline{\rho}} \preceq_\rho (z)_{\underline{\rho}}$, then $x\rho y$ and $y\rho z$. Hence, by hypothesis, $x\rho z$, and we conclude that $(x)_{\underline{\rho}} \preceq_\rho (z)_{\underline{\rho}}$.

Furthermore, let $(x)_{\underline{\rho}}, (y)_{\underline{\rho}} \in A/\underline{\rho}$, $(x)_{\underline{\rho}} \preceq_\rho (y)_{\underline{\rho}}$ and $s \in S$. Then $x\rho y$. Since ρ is a weak pseudoorder on A, by condition (3) of Definition 3.1, we have $x * s \ \overrightarrow{\rho} \ y * s$. Thus, for any $a \in x * s$ there exists $b \in y * s$ such that $a\rho b$. This implies that $(a)_{\underline{\rho}} \preceq_\rho (b)_{\underline{\rho}}$. Hence we have

$$(x)_{\underline{\rho}} \otimes s = \{(a)_{\underline{\rho}} \mid a \in x * s\} \preceq_\rho \{(b)_{\underline{\rho}} \mid b \in y * s\} = (y)_{\underline{\rho}} \otimes s.$$

Moreover, let $s, t \in S, s \leq t$ and $(a)_{\underline{\rho}} \in A/\underline{\rho}$. Then, similar to discuss as above, we have $(a)_{\underline{\rho}} \otimes s \preceq_\rho (a)_{\underline{\rho}} \otimes t$. Therefore, $A/\underline{\rho}$ is a hyper S-poset over S. □

Furthermore, we have the following proposition:

Proposition 3.4. *Let $(A, \leq_A)$ be a hyper S-poset over an ordered semihypergroup $(S, \circ, \leq)$ and ρ a weak pseudoorder on A. Then $\underline{\rho}$ is an order-congruence on A, where $\underline{\rho} = \rho \cap \rho^{-1}$.*

Proof. By Theorem 3.3, $(A/\underline{\rho}, \preceq_\rho)$ is a hyper S-poset over S, where the order relation $\preceq_\rho$ is defined as follows:

$$\preceq_\rho := \{((x)_{\underline{\rho}}, (y)_{\underline{\rho}}) \in A/\underline{\rho} \times A/\underline{\rho} \mid (x, y) \in \rho\}.$$

Also, let $x, y \in A$ and $x \leq_A y$. Then, since ρ is a weak pseudoorder on A, $(x, y) \in \leq_A \subseteq \rho$, and thus $((x)_{\underline{\rho}}, (y)_{\underline{\rho}}) \in \preceq_\rho$, i.e., $(x)_{\underline{\rho}} \preceq_\rho (y)_{\underline{\rho}}$. Therefore, $\underline{\rho}$ is an order-congruence on A. □

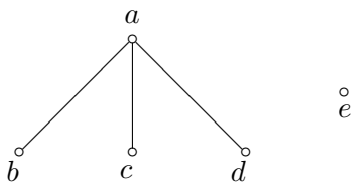
Example 3.5. We consider a set $S := \{a, b, c, d, e\}$ with the following hyperoperation " $\circ$ " and the order " $\leq$ ":

$\circ$	a	b	c	d	e
a	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$
c	$\{a, c\}$	$\{a, c\}$	$\{a, c\}$	$\{a, c\}$	$\{a, c\}$
d	$\{a, d\}$	$\{a, d\}$	$\{a, d\}$	$\{a, d\}$	$\{a, d\}$
e	$\{e\}$	$\{e\}$	$\{e\}$	$\{e\}$	$\{e\}$

$$\leq := \{(a, a), (b, b), (b, a), (c, c), (c, a), (d, a), (d, d), (e, e)\}.$$

The covering relation "$\prec$" and the figure of S are given by:

$$\prec = \{(b, a), (c, a), (d, a)\}.$$

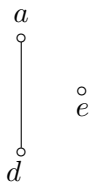


Then $(S, \circ, \leq)$ is an ordered semihypergroup. We now consider the partially ordered set $A = \{a, d, e\}$ defined by the order below:

$$\leq_A := \{(a, a), (d, d), (e, e), (d, a)\}.$$

We give the covering relation "$\prec_A$" and the figure of A.

$$\prec_A = \{(d, a)\}.$$



Then $(A, \leq_A)$ is a hyper S-poset over S with respect to S-hyperaction on A as above hyperoperation table (see [20]).

Let ρ be a weak pseudoorder on A defined as follows:

$$\rho := \{(a,a),(d,d),(e,e),(a,e),(d,a),(d,e),(e,a)\}.$$

Applying Theorem 3.3, we get

$$\underline{\rho} := \{(a,a),(d,d),(e,e),(a,e),(e,a)\}.$$

Then, by Proposition 3.4, $\underline{\rho}$ is an order-congruence on A. Furthermore, $A/\underline{\rho} = \{w_1, w_2\}$, where $w_1 = \{a, e\}, w_2 = \{d\}$. Moreover, $(A/\underline{\rho}, \preceq_\rho)$ is a hyper S-poset over S with the hyper S-action " $\otimes$ " and the order " $\preceq_\rho$ " are given below:

$\otimes$	a	b	c	d	e
w_1	$\{w_1\}$	$\{w_1\}$	$\{w_1\}$	$\{w_1\}$	$\{w_1\}$
w_2	$\{w_1, w_2\}$	$\{w_1, w_2\}$	$\{w_1, w_2\}$	$\{w_1, w_2\}$	$\{w_1, w_2\}$

$$\preceq_\rho := \{(w_1,w_1),(w_2,w_1),(w_2,w_2)\}.$$

Theorem 3.6. *Let $(A, \leq_A)$ be a hyper S-poset over an ordered semihypergroup $(S, \circ, \leq)$, ρ, θ be weak pseudoorders on A such that $\rho \subseteq \theta$. We define a relation θ/ρ on $A/\underline{\rho}$ as follows:*

$$\theta/\rho := \{((x)_{\underline{\rho}}, (y)_{\underline{\rho}}) \in A/\underline{\rho} \times A/\underline{\rho} \mid (x,y) \in \theta\}.$$

Then θ/ρ is a weak pseudoorder on $A/\underline{\rho}$.

Proof. For the simplicity's sake, let

$$\theta' := \{((x)_{\underline{\rho}}, (y)_{\underline{\rho}}) \in A/\underline{\rho} \times A/\underline{\rho} \mid (x,y) \in \theta\}.$$

We claim that θ' is a weak pseudoorder on $A/\underline{\rho}$. To prove our claim, let $((x)_{\underline{\rho}}, (y)_{\underline{\rho}}) \in \preceq_\rho$. Then, by Theorem 3.3, $(x,y) \in \rho \subseteq \theta$, which implies that $((x)_{\underline{\rho}}, (y)_{\underline{\rho}}) \in \theta'$. Thus, $\preceq_\rho \subseteq \theta'$. Now, assume that $((x)_{\underline{\rho}}, (y)_{\underline{\rho}}) \in \theta'$ and $((y)_{\underline{\rho}}, (z)_{\underline{\rho}}) \in \theta'$. Then, $(x,y) \in \theta$ and $(y,z) \in \theta$. It implies that $(x,z) \in \theta$. Hence, $((x)_{\underline{\rho}}, (z)_{\underline{\rho}}) \in \theta'$. Also, let $((x)_{\underline{\rho}}, (y)_{\underline{\rho}}) \in \theta'$ and $s \in S$. Then, $(x,y) \in \theta$ and $s \in S$. Since θ is a weak pseudoorder on A, we have $x * s \overrightarrow{\theta} y * s$. Thus, for every $a \in x * s$, there exists $b \in y * s$ such that $(a,b) \in \theta$. This implies that $((a)_{\underline{\rho}}, (b)_{\underline{\rho}}) \in \theta'$, and thus we have

$$(x)_{\underline{\rho}} \otimes s = \{(a)_{\underline{\rho}} \mid a \in x * s\} \overrightarrow{\theta'} \{(b)_{\underline{\rho}} \mid b \in y * s\} (y)_{\underline{\rho}} \otimes s.$$

Furthermore, let $s \in S$ and $((x)_{\underline{\rho}}, (y)_{\underline{\rho}}) \in A/\underline{\rho}$ be such that $((x)_{\underline{\rho}}, (y)_{\underline{\rho}}) \in \theta'$ and $((y)_{\underline{\rho}}, (x)_{\underline{\rho}}) \in \theta'$. Then $(x,y) \in \theta, (y,x) \in \theta$ and $s \in S$. By hypothesis, we have $x * s \widetilde{\theta} y * s$. Then, similar to the above proof, it can be obtained that $(x)_{\underline{\rho}} \otimes s \widetilde{\theta'} (y)_{\underline{\rho}} \otimes s$. Therefore, $\theta'(= \theta/\rho)$ is indeed a weak pseudoorder on $A/\underline{\rho}$. □

4. Homomorphism theorems of hyper S-posets

In the current section, we establish the relationships between order-congruences and weak pseudoorders on a hyper S-poset, and discuss homomorphism theorems of hyper S-posets by weak pseudoorders.

In order to establish the relationships between order-congruences and weak pseudoorders on a hyper S-poset, the following lemma is essential.

Lemma 4.1. *Let $(A, \leq_A)$ be a hyper S-poset over an ordered semihypergroup $(S, \circ, \leq)$ and σ a relation on A. Then the following statements are equivalent:*

(1) *σ is a weak pseudoorder on A.*

(2) *There exist a hyper S-poset $(B, \leq_B)$ over S and a homomorphism $\varphi : A \to B$ such that*

$$\overrightarrow{ker}\varphi := \{(a, b) \in A \times A \mid \varphi(a) \leq_B \varphi(b)\} = \sigma.$$

Proof. (1) $\Rightarrow$ (2). Let σ be a weak pseudoorder on A. We denote by $\underline{\sigma}$ the congruence on A defined by

$$\underline{\sigma} := \{(a, b) \in A \times A \mid (a, b) \in \sigma, (b, a) \in \sigma\} (= \sigma \cap \sigma^{-1}).$$

Then, by Theorem 3.3, the set $A/\underline{\sigma} := \{(a)_{\underline{\sigma}} \mid a \in A\}$ with the S-hyperaction $(a)_{\underline{\sigma}} \otimes s = \{(x)_{\underline{\sigma}} \mid x \in a * s\}$ for all $a \in A, s \in S$ and the order

$$\preceq_\sigma := \{((x)_{\underline{\sigma}}, (y)_{\underline{\sigma}}) \in A/\underline{\sigma} \times A/\underline{\sigma} \mid (x, y) \in \sigma\}$$

is a hyper S-poset. Let $B = (A/\underline{\sigma}, \preceq_\sigma)$ and φ be the mapping of A onto $A/\underline{\sigma}$ defined by $\varphi : A \to A/\underline{\sigma} \mid a \mapsto (a)_{\underline{\sigma}}$. Then, by Proposition 3.4, φ is a homomorphism from A onto $A/\underline{\sigma}$ and clearly, $\overrightarrow{ker}\varphi = \sigma$.

(2) $\Rightarrow$ (1). Let $(A, \leq_A)$ be a hyper S-poset over an ordered semihypergroup $(S, \circ, \leq)$. If there exist a hyper S-poset $(B, \leq_B)$ over S and a homomorphism $\varphi : A \to B$ such that $\overrightarrow{ker}\varphi = \sigma$, then σ is a weak pseudoorder on A. Indeed, let $(a, b) \in \leq_A$. Then, by hypothesis, $\varphi(a) \leq_B \varphi(b)$. Thus $(a, b) \in \overrightarrow{ker}\varphi = \sigma$, and we have $\leq_A \subseteq \sigma$. Moreover, let $(a, b) \in \sigma$ and $(b, c) \in \sigma$. Then $\varphi(a) \leq_B \varphi(b) \leq_B \varphi(c)$. Hence $\varphi(a) \leq_B \varphi(c)$, i.e., $(a, c) \in \overrightarrow{ker}\varphi = \sigma$. Also, if $(a, b) \in \sigma$, then $\varphi(a) \leq_B \varphi(b)$. Since $(B, \leq_B)$ is a hyper S-poset over S, for any $s \in S$ we have $\varphi(a) \diamond s \leq_B \varphi(b) \diamond s$, where " $\diamond$ " is the S-hyperaction on B. Since φ is a homomorphism from A to B, we have

$$\{\varphi(x) \mid x \in a * s\} = \varphi(a) \diamond s \leq_B \varphi(b) \diamond s = \{\varphi(y) \mid y \in b * s\}.$$

Thus, for every $x \in a*s$ there exists $y \in b*s$ such that $\varphi(x) \leq_B \varphi(y)$, and we have $(x, y) \in \overrightarrow{ker}\varphi = \sigma$, and thus $a * s \ \overrightarrow{\sigma}\ b * s$. Furthermore, let $(a, b) \in \sigma, (b, a) \in \sigma$ and $s \in S$. Then $\varphi(a) \leq_B \varphi(b)$ and $\varphi(b) \leq_B \varphi(a)$. Thus $\varphi(a) = \varphi(b)$, which implies that $\varphi(a) \diamond s = \varphi(b) \diamond s$, i.e., $\{\varphi(x) \mid x \in a * s\} = \{\varphi(y) \mid y \in b * s\}$. Hence, for any $x \in a * s$, there exists $y \in b * s$ such that $\varphi(x) = \varphi(y)$, and

we have $\varphi(x) \leq_B \varphi(y)$ and $\varphi(y) \leq_B \varphi(x)$. It thus follows that $(x, y) \in \sigma$ and $(y, x) \in \sigma$. On the other hand, for any $y' \in b * s$, there exists $x' \in a * s$ such that $\varphi(x') = \varphi(y')$. Hence we have $\varphi(x') \leq_B \varphi(y')$ and $\varphi(y') \leq_B \varphi(x')$, which imply that $(x', y') \in \sigma$ and $(y', x') \in \sigma$. Thus, $a * s \ \widetilde{\sigma} \ b * s$, exactly as required. Therefore, σ is a weak pseudoorder on A. □

In the following, we give a characterization of order-congruences in terms of weak pseudoorders.

Theorem 4.2. *Let $(A, \leq_A)$ be a hyper S-poset over an ordered semihypergroup $(S, \circ, \leq)$ and $\rho \in C(A)$. Then the following statements are equivalent:*

(1) *ρ is an order-congruence on A.*

(2) *There exists a weak pseudoorder σ on A such that $\rho = \sigma \cap \sigma^{-1}$.*

(3) *There exist a hyper S-poset B over S and a homomorphism $\varphi : A \to B$ such that $\rho = ker(\varphi)$, where $ker\varphi = \{(a, b) \in A \times A \mid \varphi(a) = \varphi(b)\}$ is the kernel of φ.*

Proof. (1) ⇒ (2). Let ρ be an order-congruence on A. Then there exist an order relation "$\preceq$" on the factor hyper S-act A/ρ such that $(A/\rho, \preceq)$ is a hyper S-poset over S, and $\varphi : A \to A/\rho$ is a homomorphism. Let $\sigma = \overrightarrow{ker\varphi}$. By Lemma 4.1, σ is a weak pseudoorder on A and it is easy to check that $\rho = \sigma \cap \sigma^{-1}$.

(2) ⇒ (3). For a weak pseudoorder σ on A, by Lemma 4.1, there exist a hyper S-poset B over S and a homomorphism $\varphi : A \to B$ such that $\sigma = \overrightarrow{ker\varphi}$. Then, we have

$$ker\varphi = \overrightarrow{ker\varphi} \cap (\overrightarrow{ker\varphi})^{-1} = \sigma \cap \sigma^{-1} = \rho.$$

(3) ⇒ (1). By hypothesis and Lemma 4.1, $\overrightarrow{ker\varphi}$ is a weak pseudoorder on A. Then, by Theorem 3.3, $\rho = \overrightarrow{ker\varphi} \cap (\overrightarrow{ker\varphi})^{-1}$ is a congruence on A. Thus, by the proof of Lemma 4.1, ρ is an order-congruence on A. □

Remark 4.3. (1) For an order-congruence ρ on A, since the order "$\preceq$" such that $(A/\rho, \preceq)$ is a hyper S-poset is not unique in general, we have the weak pseudoorder σ containing ρ such that $\rho = \sigma \cap \sigma^{-1}$ is not unique.

(2) If σ is a weak pseudoorder on a hyper S-poset A, then $\rho = \sigma \cap \sigma^{-1}$ is the greatest order-congruence on A contained in σ. In fact, if ρ_1 is an order-congruence on A contained in σ, then $\rho_1 = \rho_1 \cap \rho_1^{-1} \subseteq \sigma \cap \sigma^{-1} = \rho$.

Let σ be a weak pseudoorder on a hyper S-poset $(A, \leq_A)$. Then, by Theorem 4.2, $\rho = \sigma \cap \sigma^{-1}$ is an order-congruence on A. We denote by $\rho^\sharp$ the canonical epimorphism from A onto A/ρ, i.e., $\rho^\sharp : A \to A/\rho \mid x \mapsto (x)_\rho$, which is a homomorphism. In the following, we give out a homomorphism theorem of hyper S-posets by weak pseudoorders, which is a generalization of Theorem 12 in [24]. For a similar result about ordered semigroups we refer the readers to Theorem 1 in [10].

Theorem 4.4. *Let $(A, \leq_A)$ and $(B, \leq_B)$ be two hyper S-posets over an ordered semihypergroup $(S, \circ, \leq)$, $\varphi : A \to B$ a homomorphism. Then, if σ is a weak pseudoorder on A such that $\sigma \subseteq \overrightarrow{ker}\varphi$, then there exists the unique homomorphism $f : A/\rho \to B \mid (a)_\rho \mapsto \varphi(a)$ such that the diagram*

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ {\scriptstyle \rho^\sharp}\downarrow & \nearrow_{f} & \\ A/\rho & & \end{array}$$

commutes, where $\rho = \sigma \cap \sigma^{-1}$. Moreover, $Im(\varphi) = Im(f)$. Conversely, if σ is a weak pseudoorder on A for which there exists a homomorphism $f : (A/\rho, \preceq_\sigma) \to (B, \leq_B)$ $(\rho = \sigma \cap \sigma^{-1})$ such that the above diagram commutes, then $\sigma \subseteq \overrightarrow{ker}\varphi$.

Proof. Let σ be a weak pseudoorder on A such that $\sigma \subseteq \overrightarrow{ker}\varphi$, $f : A/\rho \to B \mid (a)_\rho \mapsto \varphi(a)$. Then

(1) f is well defined. Indeed, if $(a)_\rho = (b)_\rho$, then $(a, b) \in \rho \subseteq \sigma$. Since $\sigma \subseteq \overrightarrow{ker}\varphi$, we have $(\varphi(a), \varphi(b)) \in \leq_B$. Furthermore, since $(b, a) \in \sigma \subseteq \overrightarrow{ker}\varphi$, we have $(\varphi(b), \varphi(a)) \in \leq_B$. Therefore, $\varphi(a) = \varphi(b)$.

(2) f is a homomorphism and $\varphi = f \circ \rho^\sharp$. In fact, by Lemma 4.1, there exist an order relation "$\preceq_\sigma$" on the factor hyper S-act A/ρ such that $(A/\rho, \preceq_\sigma)$ is a hyper S-poset and the canonical epimorphism $\rho^\sharp$ is a homomorphism. Moreover, we have

$$\begin{array}{rcl} (a)_\rho \preceq_\sigma (b)_\rho & \Rightarrow & (a, b) \in \sigma \subseteq \overrightarrow{ker}\varphi \\ & \Rightarrow & \varphi(a) \leq_B \varphi(b) \\ & \Rightarrow & f((a)_\rho) \leq_B f((b)_\rho). \end{array}$$

Also, let $(a)_\rho \in A/\rho$ and $s \in S$. Since φ is a homomorphism from A to B, we have

$$f((a)_\rho) \diamond s = \varphi(a) \diamond s = \{\varphi(x) \mid x \in a * s\} = \{f((x)_\rho) \mid (x)_\rho \in (a)_\rho \otimes s\},$$

where "$\diamond$" is the S-hyperaction on B. Furthermore, for any $a \in A$, $(f \circ \rho^\sharp)(a) = f((a)_\rho) = \varphi(a)$, and thus $\varphi = f \circ \rho^\sharp$.

We claim that f is the unique homomorphism from A/ρ to B. To prove our claim, let g is a homomorphism from A/ρ to B such that $\varphi = g \circ \rho^\sharp$. Then, for any $(a)_\rho \in A/\rho$, we have

$$f((a)_\rho) = \varphi(a) = (g \circ \rho^\sharp)(a) = g((a)_\rho).$$

Moreover, $Im(f) = \{f((a)_\rho) \mid a \in A\} = \{\varphi(a) \mid a \in A\} = Im(\varphi)$.

Conversely, let σ be a weak pseudoorder on A, $f : A/\rho \to B$ is a homomorphism and $\varphi = f \circ \rho^\sharp$. Then $\sigma \subseteq \overrightarrow{ker}\varphi$. Indeed, by hypothesis, we have

$$\begin{array}{rcl} (a, b) \in \sigma & \Leftrightarrow & (a)_\rho \preceq_\sigma (b)_\rho \Rightarrow f((a)_\rho) \leq_B f((b)_\rho) \\ & \Rightarrow & (f \circ \rho^\sharp)(a) \leq_B (f \circ \rho^\sharp)(b) \\ & \Rightarrow & \varphi(a) \leq_B \varphi(b) \Rightarrow (a, b) \in \overrightarrow{ker}\varphi, \end{array}$$

where the order $\preceq_\sigma$ on A/ρ is defined as in the proof of Lemma 4.1, that is $\preceq_\sigma := \{((x)_\rho, (y)_\rho) \in A/\rho \times A/\rho \mid (x,y) \in \sigma\}$. □

Corollary 4.5. *Let $(A, \leq_A)$ and $(B, \leq_B)$ be two hyper S-posets over an ordered semihypergroup $(S, \circ, \leq)$ and $\varphi : A \to B$ a homomorphism. Then $A/ker\varphi \cong Im(\varphi)$, where $ker\varphi$ is the kernel of φ.*

Proof. Let $\sigma = \overrightarrow{ker}\varphi$ and $\rho = \overrightarrow{ker}\varphi \cap (\overrightarrow{ker}\varphi)^{-1}$. Then, by Theorems 4.2 and 4.4, ρ is an order-congruence on A and $f : A/\rho \to B \mid (a)_\rho \mapsto \varphi(a)$ is a homomorphism. Moreover, f is inverse isotone. In fact, let $(a)_\rho, (b)_\rho$ be two elements of A/ρ such that $f((a)_\rho) \leq_B f((b)_\rho)$. Then $\varphi(a) \leq_B \varphi(b)$, and we have $(a,b) \in \overrightarrow{ker}\varphi$. Thus, by Lemma 4.1, $((a)_\rho, (b)_\rho) \in \preceq_\sigma$, i.e., $(a)_\rho \preceq_\sigma (b)_\rho$. Clearly, $\rho = ker\varphi$. By Remark 2.7, $A/ker\varphi \cong Im(f)$. Also, by Theorem 4.4, $Im(f) = Im(\varphi)$. Therefore, $A/ker\varphi \cong Im(\varphi)$. □

Remark 4.6. Note that if $(A, \leq_A)$ and $(B, \leq_B)$ are both S-posets, then Corollary 4.5 coincides with Corollary 13 in [24].

Let $(A, \leq_A)$ be a hyper S-poset over an ordered semihypergroup $(S, \circ, \leq)$, ρ and θ weak pseudoorders on A and $\rho \subseteq \theta$. We define a relation on the hyper S-poset $(A/\underline{\rho}, \preceq_\rho)$ denoted by θ/ρ as follows:

$$\theta/\rho := \{((a)_{\underline{\rho}}, (b)_{\underline{\rho}}) \in A/\underline{\rho} \times A/\underline{\rho} \mid (a,b) \in \theta\},$$

where $\preceq_\rho := \{((a)_{\underline{\rho}}, (b)_{\underline{\rho}}) \mid (a,b) \in \rho\}, \underline{\rho} = \rho \cap \rho^{-1}$. By Theorem 3.6, θ/ρ is a weak pseudoorder on $(A/\underline{\rho}, \preceq_\rho)$.

Theorem 4.7. *Let $(A, \leq_A)$ be a hyper S-poset over an ordered semihypergroup $(S, \circ, \leq)$, ρ and θ weak pseudoorders on A and $\rho \subseteq \theta$. Then $(A/\underline{\rho})/\underline{\theta/\rho} \cong A/\underline{\theta}$.*

Proof. Since θ/ρ is a weak pseudoorder on $A/\underline{\rho}$, it can be obtained that the mapping $\varphi : A/\underline{\rho} \to A/\underline{\theta} \mid (a)_{\underline{\rho}} \mapsto (a)_{\underline{\theta}}$ is a homomorphism. In fact:

(1) φ is well-defined. Indeed, let $(a)_{\underline{\rho}} = (b)_{\underline{\rho}}$. Then $(a,b) \in \underline{\rho}$. Thus, by the definition of $\underline{\rho}$, $(a,b) \in \rho \subseteq \theta$ and $(b,a) \in \rho \subseteq \theta$. This implies that $(a,b) \in \underline{\theta}$, and thus $(a)_{\underline{\theta}} = (b)_{\underline{\theta}}$.

(2) φ is a homomorphism. In fact, let $(a)_{\underline{\rho}} \in A/\underline{\rho}$ and $s \in S$. Then, since $\underline{\rho}, \underline{\theta} \in C(A)$, we have

$$(a)_{\underline{\rho}} \otimes_\rho s = \{(x)_{\underline{\rho}} \mid x \in a * s\}, (a)_{\underline{\theta}} \otimes_\theta s = \{(x)_{\underline{\theta}} \mid x \in a * s\},$$

where " $\otimes_\rho$ " and " $\otimes_\theta$ " are the S-hyperaction on $A/\underline{\rho}$ and $A/\underline{\theta}$, respectively. Thus,

$$\varphi((a)_{\underline{\rho}}) \otimes_\theta s = (a)_{\underline{\theta}} \otimes_\theta s = \{(x)_{\underline{\theta}} \mid x \in a * s\} = \{\varphi((x)_{\underline{\rho}}) \mid (x)_{\underline{\rho}} \in (a)_{\underline{\rho}} \otimes_\rho s\}.$$

Also, if $(a)_{\underline{\rho}} \preceq_\rho (b)_{\underline{\rho}}$, then $(a,b) \in \rho \subseteq \theta$. It implies that $(a)_{\underline{\theta}} \preceq_\theta (b)_{\underline{\theta}}$, and thus φ is isotone.

On the other hand, it is easy to see that φ is onto, since

$$Im(\varphi) = \{\varphi((a)_{\underline{\rho}}) \mid a \in A\} = \{(a)_{\underline{\theta}} \mid a \in A\} = A/\underline{\theta}.$$

It thus follows from Corollary 4.5 that $A/\underline{\rho}/Ker\varphi \cong Im(\varphi) = A/\underline{\theta}$.

Furthermore, let $\overrightarrow{ker}\varphi := \{((a)_{\underline{\rho}}, (b)_{\underline{\rho}}) \mid \varphi((a)_{\underline{\rho}}) \preceq_\theta \varphi((b)_{\underline{\rho}})\}$. Then

$$\begin{aligned} ((a)_{\underline{\rho}}, (b)_{\underline{\rho}}) \in \overrightarrow{ker}\varphi &\iff (a)_{\underline{\theta}} \preceq_\theta (b)_{\underline{\theta}} \iff (a,b) \in \theta \\ &\iff ((a)_{\underline{\rho}}, (b)_{\underline{\rho}}) \in \theta/\rho. \end{aligned}$$

Therefore, $Ker\varphi = \overrightarrow{ker}\varphi \cap (\overrightarrow{ker}\varphi)^{-1} = (\theta/\rho) \cap (\theta/\rho)^{-1} = \underline{\theta/\rho}$. We have thus shown that $(A/\underline{\rho})/\underline{\theta/\rho} \cong A/\underline{\theta}$. □

Definition 4.8. Let $(A, \leq_A)$ and $(B, \leq_B)$ be two hyper S-posets over an ordered semihypergroup $(S, \circ, \leq)$, ρ, θ be two weak pseudoorders on A and B, respectively, and the mapping $f : A \to B$ a homomorphism. Then, f is called a (ρ, θ)-*homomorphism* if $(x, y) \in \rho$ implies $(f(x), f(y)) \in \theta$, for all $x, y \in A$.

Lemma 4.9. *Let $(A, \leq_A)$ and $(B, \leq_B)$ be two hyper S-posets over an ordered semihypergroup $(S, \circ, \leq)$, ρ, θ be two weak pseudoorders on A and B, respectively, and the mapping $f : A \to B$ a (ρ, θ)-homomorphism. Then, the mapping $\overline{f} : (A/\underline{\rho}, \preceq_\rho) \to (B/\underline{\theta}, \preceq_\theta)$ defined by*

$$(\forall x \in A)\ \overline{f}((x)_{\underline{\rho}}) := (f(x))_{\underline{\theta}}$$

is a homomorphism of hyper S-posets, where the orders $\preceq_\rho$, $\preceq_\theta$ on $A/\underline{\rho}$ and $B/\underline{\theta}$, respectively, are both defined as in the proof of Lemma 4.1.

Proof. Let $f : A \to B$ be a (ρ, θ)-homomorphism and $\overline{f} : A/\underline{\rho} \to B/\underline{\theta} \mid (x)_{\underline{\rho}} \mapsto (f(x))_{\underline{\theta}}$. Then

(1) $\overline{f}$ is well defined. In fact, let $(x)_{\underline{\rho}}, (y)_{\underline{\rho}} \in A/\underline{\rho}$ be such that $(x)_{\underline{\rho}} = (y)_{\underline{\rho}}$. Then $(x, y) \in \underline{\rho} \subseteq \rho$. Since f is a (ρ, θ)-homomorphism, we have $(f(x), f(y)) \in \theta$. It implies that $((f(x))_{\underline{\theta}}, (f(y))_{\underline{\theta}}) \in \preceq_\theta$. Similarly, since $(y, x) \in \rho$, we have $((f(y))_{\underline{\theta}}, (f(x))_{\underline{\theta}}) \in \preceq_\theta$. Therefore, $((f(x))_{\underline{\theta}} = (f(y))_{\underline{\theta}}$, i.e., $\overline{f}((x)_{\underline{\rho}}) = \overline{f}((y)_{\underline{\rho}})$.

(2) $\overline{f}$ is a homomorphism. Indeed, let $(x)_{\underline{\rho}} \in A/\underline{\rho}$ and $s \in S$. By Theorem 4.2, $\underline{\rho}, \underline{\theta}$ are order-congruences on A and B, respectively. Since f is a homomorphism, by Lemma 2.4 we have

$$\overline{f}((x)_{\underline{\rho}}) \otimes_\theta s = (f(x))_{\underline{\theta}} \otimes_\theta s = \{(f(a))_{\underline{\theta}} \mid a \in x * s\} = \{\overline{f}((a)_{\underline{\rho}}) \mid (a)_{\underline{\rho}} \in (x)_{\underline{\rho}} \otimes_\rho s\}.$$

Also, since f is a (ρ, θ)-homomorphism, we have

$$\begin{aligned} (x)_{\underline{\rho}} \preceq_\rho (y)_{\underline{\rho}} &\Rightarrow (x, y) \in \rho \Rightarrow (f(x), f(y)) \in \theta \\ &\Rightarrow (f(x))_{\underline{\theta}} \preceq_\theta (f(y))_{\underline{\theta}} \Rightarrow \overline{f}((x)_{\underline{\rho}}) \preceq_\theta \overline{f}((y)_{\underline{\rho}}). \end{aligned}$$

Hence $\overline{f}$ is isotone. We have thus shown that $\overline{f}$ is a homomorphism, as required. □

Lemma 4.10. *Let $(A, \leq_A)$ and $(B, \leq_B)$ be two hyper S-posets over an ordered semihypergroup $(S, \circ, \leq)$, ρ, θ be two weak pseudoorders on A and B, respectively, and the mapping $f : A \to B$ a (ρ, θ)-homomorphism. We define a relation on the hyper S-poset $(A/\underline{\rho}, \preceq_\rho)$ denoted by ρ_f as follows:*

$$\rho_f := \{((x)_{\underline{\rho}}, (y)_{\underline{\rho}}) \in A/\underline{\rho} \times A/\underline{\rho} \mid (f(x))_{\underline{\theta}} \preceq_\theta (f(y))_{\underline{\theta}}\}.$$

Then ρ_f is a weak pseudoorder on $A/\underline{\rho}$.

Proof. Assume that $((x)_{\underline{\rho}}, (y)_{\underline{\rho}}) \in \preceq_\rho$. By Lemma 4.9, $\overline{f}$ is a homomorphism. Then $\overline{f}((x)_{\underline{\rho}}) \preceq_\theta \overline{f}((y)_{\underline{\rho}})$, i.e., $(f(x))_{\underline{\theta}} \preceq_\theta (f(y))_{\underline{\theta}}$. It implies that $((x)_{\underline{\rho}}, (y)_{\underline{\rho}}) \in \rho_f$, and thus $\preceq_\rho \subseteq \rho_f$. Now, let $((x)_{\underline{\rho}}, (y)_{\underline{\rho}}) \in \rho_f$ and $((y)_{\underline{\rho}}, (z)_{\underline{\rho}}) \in \rho_f$. Then $(f(x))_{\underline{\theta}} \preceq_\theta (f(y))_{\underline{\theta}}$ and $(f(y))_{\underline{\theta}} \preceq_\theta (f(z))_{\underline{\theta}}$. Thus, by the transitivity of $\preceq_\theta$, $(f(x))_{\underline{\theta}} \preceq_\theta (f(z))_{\underline{\theta}}$. This implies that $((x)_{\underline{\rho}}, (z)_{\underline{\rho}}) \in \rho_f$. Moreover, let $((x)_{\underline{\rho}}, (y)_{\underline{\rho}}) \in \rho_f$ and $s \in S$. Then $(f(x))_{\underline{\theta}} \preceq_\theta (f(y))_{\underline{\theta}}$. Since $(B/\underline{\theta}, \preceq_\theta)$ is a hyper S-poset over S, it can be obtained that $(f(x))_{\underline{\theta}} \otimes_\theta s \preceq_\theta (f(y))_{\underline{\theta}} \otimes_\theta s$, that is, $\overline{f}((x)_{\underline{\rho}}) \otimes_\theta s \preceq_\theta \overline{f}((y)_{\underline{\rho}}) \otimes_\theta s$. Then, since $\overline{f}$ is a homomorphism, we have

$$\{\overline{f}((a)_{\underline{\rho}}) \mid (a)_{\underline{\rho}} \in (x)_{\underline{\rho}} \otimes_\rho s\} \preceq_\theta \{\overline{f}((b)_{\underline{\rho}}) \mid (b)_{\underline{\rho}} \in (y)_{\underline{\rho}} \otimes_\rho s\},$$

which means that $\{(f(a))_{\underline{\theta}} \mid a \in x * s\} \preceq_\theta \{(f(b))_{\underline{\theta}} \mid b \in y * s\}$. Thus, for any $a \in x * s$, there exists $b \in y * s$ such that $(f(x))_{\underline{\theta}} \preceq_\theta (f(y))_{\underline{\theta}}$. Hence $((a)_{\underline{\rho}}, (b)_{\underline{\rho}}) \in \rho_f$. It thus implies that $(x)_{\underline{\rho}} \otimes_\rho s \ \overrightarrow{\rho}_f \ (y)_{\underline{\rho}} \otimes_\rho s$. Furthermore, let $(x)_{\underline{\rho}}, (y)_{\underline{\rho}} \in S/\underline{\rho}$ and $s \in S$ be such that $((x)_{\underline{\rho}}, (y)_{\underline{\rho}}) \in \rho_f$ and $((y)_{\underline{\rho}}, (x)_{\underline{\rho}}) \in \rho_f$. Then, similar to the above proof, it can be verified that $(x)_{\underline{\rho}} \otimes_\rho s \ \widetilde{\rho}_f \ (y)_{\underline{\rho}} \otimes_\rho s$. Therefore, ρ_f is a weak pseudoorder on $A/\underline{\rho}$. □

By Lemmas 4.9 and 4.10, we immediately obtain the following corollary:

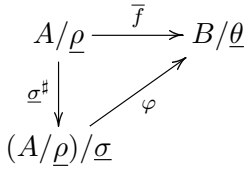
Corollary 4.11. *Let $(A, \leq_A)$, $(B, \leq_B)$ be two hyper S-posets over an ordered semihypergroup $(S, \circ, \leq)$, ρ, θ be two weak pseudoorders on A and B, respectively, and the mapping $f : A \to B$ a (ρ, θ)-homomorphism. Then, the following diagram*

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & B \\ {\scriptstyle \underline{\rho}^\sharp}\downarrow & & \downarrow{\scriptstyle \underline{\theta}^\sharp} \\ A/\underline{\rho} & \xrightarrow[\overline{f}]{} & B/\underline{\theta} \end{array}$$

commutates.

Theorem 4.12. *Let $(A, \leq_A)$, $(B, \leq_B)$ be two hyper S-posets over an ordered semihypergroup $(S, \circ, \leq)$, ρ, θ be two weak pseudoorders on A and B, respectively, and the mapping $f : A \to B$ a (ρ, θ)-homomorphism. If σ is a weak pseudoorder on $A/\underline{\rho}$ such that $\sigma \subseteq \rho_f$, then there exists the unique homomorphism*

$\varphi : (A/\underline{\rho})/\underline{\sigma} \to B/\underline{\theta} \mid ((a)_{\underline{\rho}})_{\underline{\sigma}} \mapsto \underline{f}((a)_{\underline{\rho}})$ *such that the diagram*



commutes.

Conversely, if σ is a weak pseudoorder on $A/\underline{\rho}$ for which there exists a homomorphism $\varphi : (A/\underline{\rho})/\underline{\sigma} \to B/\underline{\theta}$ such that the above diagram commutes, then $\sigma \subseteq \rho_f$.

Proof. The proof is straightforward by Lemmas 4.9, 4.10 and Theorem 4.4, and we omit the details. □

5. Conclusions

In [20], Tang et al. defined and studied the hyper S-posets over an ordered semihypergroup, and introduced the concepts of pseudoorders and (strong) order-congruences on a hyper S-poset A. In particular, they proved that there must exist a strong congruence ρ on a hyper S-poset A such that A/ρ is a hyper S-poset (see Theorem 3.7 in [20]). A natural question about hyper S-posets is raised: Is there a congruence ρ on a hyper S-poset A for which A/ρ is a hyper S-poset? In this paper we studied the weak pseudoorders in a hyper S-poset in detail and answered the above question. To begin with, the concept of weak pseudoorders in a hyper S-poset was introduced, and several related properties was given. Especially, we constructed a hyper S-poset $A/\underline{\rho}$ by a weak pseudoorder ρ.. Furthermore, we provided some homomorphism theorems of hyper S-posets by weak pseudoorders, which are generalizations of similar results in S-posets and ordered semigroups. The foundations that we made through this paper can offer foundation for further study of the hyper S-poset theory.

Acknowledgments

This work is supported by the National Natural Science Foundation of China (Nos. 11801081; 11701504), the Demonstration Project of Grass-roots Teaching and Research Section in Anhui Province (No. 2018jyssf053), Anhui Provincial Excellent Youth Talent Foundation (No. gxyqZD2019043) and the University Natural Science Project of Anhui Province (No. KJ2019A0543).

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Accepted: May 11, 2020

Vague soft separation axioms and vague soft continuous functions in vague soft topological spaces

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Abstract. In this present study, we introduce the notions of vague soft points, vague soft neighborhoods and vague soft functions which pave a way to define vague soft separation axioms in vague soft topological spaces. This research work brings out the following interesting results in vague soft set theory, which are contrary to their respective basic results in set theory and the interrelation between separation axioms in topological spaces.

1. The intersection of a vague soft set and its complement need not be a null vague soft set.
2. The union of a vague soft set and its complement need not be an absolute (universal) vague soft set.
3. Every vague soft normal space need not be a vague soft regular space.
4. Every vague soft regular space need not be a vague soft Hausdorff space.

Counter examples are given to illustrate the above results. Also, we provided an application in the field of medicine using the concept of vague soft functions.

Keywords: vague soft set, vague soft topology, vague soft point, vague soft neighborhood, vague soft $T_i (i = 0, 1, 2, 3, 4)$ spaces, vague soft continuous functions.

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1. Introduction

The most appropriate theory, for dealing with **uncertainties** is the Theory of Fuzzy sets introduced by Zadeh [22]. A fuzzy set F is described by its membership function μ_F. This membership function μ_F associates a real number $\mu_F(x)$ in the interval $[0, 1]$ for every point $x \in X$. The number $\mu_F(x)$ is interpreted for the point as a degree of belonging x to the fuzzy set F. As a generalization of fuzzy sets, Intuitionistic fuzzy (IF) sets [7] and Vague sets [10] are developed. Instead of using point-base membership as in Fuzzy sets, interval-based membership is used in Vague sets which is more expressive in capturing vagueness of data. Vague sets are regarded as a special case of context-dependent fuzzy sets.

A vague set V is defined by a truth-membership function t_V and a false-membership function f_V, where $t_V(x)$ is a lower bound on the grade of membership of x derived from the evidence for x, and $f_V(x)$ is a lower bound on the negation of x derived from the evidence against x. These true membership function value and false membership function value noted as $t_V(x)$ and $f_V(x)$ are associated as a real number in $[0, 1]$ with each point in a basic set X, which satisfies the condition $0 \leq t_V(x) + f_V(x) \leq 1$. The most interesting feature about Vague set V is the approach of bounds for each grade of membership of an object x to a subinterval $[t_V(x), 1 - f_V(x)]$ of [0,1]. The value of the interval characterizes favor, against and abstentions of an object x which belong to the vague set V, whereas in Intuitionistic fuzzy set A, each object has a representation $< t_A(x), f_A(x) >$ which interprets only favor and against of an object x belong to the IF set A. The way of handling the notion of inclusion, union and complement of vague sets in [10] is realistic in nature. When measuring vagueness in practice, it is proved with example that using a Vague set is more nature than using an IF set.

Recently, several authors Bhargavi and Eswarla (2016)[8], Borumand Saeid and Zarandi (2011)[9], Jothi (2017)[13], Mariapresenti and Arockiarani (2016)[14], Mary Margaret and Arockiarani (2016)[15] extended the study of vague sets and they have given many applications in the field of vague set theory. Molodtsov [16] introduced soft set theory as an effective mathematical tool to deal with uncertainties especially in the respective fields like economics, engineering and environment. By combining the notions of vague sets and soft sets, Xu et al. [21] introduced the notion of vague soft sets and illustrated its potential applications. This combination of Vague soft set theory makes descriptions of the object world more realistic, practical and accurate. Vague soft set was applied by Alhazaymeh and Hassan [2, 3, 6] in decision making and further explored interval-valued vague soft set [4], generalized vague soft expert set [5] and Nautrosophic vague soft set [1]. In due course C. Wang and Li [19] introduced the topological structure of vague soft sets to carry further study. Further, C. Wang and Qu [20] examined several properties of vague soft sets and its applications. Adding to it, C. Wang [18] contributed vague soft sets applications in

decision making problems. Recently C. Wang [17] developed the decomposition theorems and representation theorems of vague soft sets.

In this paper, we the authors define new concepts such as vague soft points, vague soft neighborhoods, vague soft T_i $(i = 0, 1, 2, 3, 4)$ spaces in vague soft topological spaces. This research work brings out the characterization of vague soft continuous functions. Finally, we present an application of vague soft functions in the field of medicine to show it's potential use in general situation. It is important to note that most of the results given in this research paper are very basic and introductory to the theory of vague soft topological spaces. This will open new vistas in this area of research.

2. Preliminaries

Definition 2.1 ([16]). *Let X be an initial universe set, $P(X)$ the set of all subsets of X, E a set of parameters, and $A \subseteq E$. A pair (F, A) is called a soft set over X, where F is a mapping given by F: $A \to P(X)$.*

Definition 2.2 ([10]). *A vague set $A = \{(x_i, [t_A(x_i), 1 - f_A(x_i)]) \mid x_i \in X\}$ in the universe $X = \{x_1, x_2, ..., x_n\}$ is characterized by a truth-membership function $t_A : X \to [0, 1]$, and a false-membership function $f_A : X \to [0, 1]$, where $t_A(x_i)$ is a lower bound on the grade of membership of x_i derived from the evidence of x_i, $f_A(x_i)$ is the lower bound on the negation of x_i derived from the evidence against x_i such that $0 \leq t_A(x_i) + f_A(x_i) \leq 1$ for any $x_i \in X$. The grade of membership of x_i in the vague set is bounded to a subinterval $[\, t_A(x_i), 1-f_A(x_i)\,]$ of $[0, 1]$. The vague value $[t_A(x_i), 1 - f_A(x_i)]$ (simply, $V_A(x_i)$) indicates that the exact grade of membership $\mu_A(x_i)$ of x_i may be unknown, but it is bounded by $t_A(x_i) \leq \mu_A(x_i) \leq 1 - f_A(x_i)$, where $0 \leq t_A(x_i) + f_A(x_i) \leq 1$.*

Definition 2.3 ([10]). *A vague set $A = \{(x, [t_A(x), 1 - f_A(x)]) \mid x \in X\}$ with $t_A(x) = 0$, $1 - f_A(x) = 0$ for all $x \in X$ is called the zero vague set.*

Definition 2.4 ([10]). *A vague set $A = \{(x, [t_A(x), 1 - f_A(x)]) \mid x \in X\}$ with $t_A(x) = 1$, $1 - f_A(x) = 1$ for all $x \in X$ is called the unit vague set.*

Remark 2.1. Throughout this paper, we use the notation $\tilde{\emptyset}$ and $\tilde{X}$ for zero vague set and unit vague set respectively.

Definition 2.5 ([21]). *Let X be an initial universe set, $V(X)$ the set of all vague sets on X, E be a set of parameters, and $A \subseteq E$. A pair (F, A) is called a vague soft set over X, where F is a mapping given by $F : A \to V(X)$. The set of vague soft sets on X is denoted by $V\tilde{S}(X, E)$, called vague soft classes.*

Definition 2.6 ([21]). *Let (F, A) and (G, B) be two vague soft sets over X. If $A \subseteq B$ and for all $e \in A$, $F(e)$ is a vague subset of $G(e)$, then (F, A) is called a vague soft subset of (G, B). This relation is denoted by $(F, A) \subseteq (G, B)$.*

Definition 2.7 ([21]). *Two vague soft sets (F, A) and (G, B) over X are said to be vague soft equal if (F, A) is a vague soft subset of (G, B) and (G, B) is a vague soft subset of (F, A). This relation is denoted by $(F, A) = (G, B)$.*

Definition 2.8 ([21]). *The complement of a vague soft set (F, A) is denoted by $(F, A)^c$ and is defined by $(F, A)^c = (F^c, A)$ and is given by $t_{F^c(e)}(x) = f_{F(e)}(x)$, $1 - f_{F^c(e)}(x) = 1 - t_{F(e)}(x)$, for all $e \in A,\ x \in X$.*

Definition 2.9 ([21]). *A vague soft set (F, A) over X is said to be a null vague soft set denoted by $\hat{\emptyset}$, if $\forall\ e \in A$, $t_{F(e)}(x) = 0$, $1 - f_{F(e)}(x) = 0$, $x \in X$.*

Definition 2.10 ([21]). *A vague soft set (F, A) over X is said to be an absolute (Universal) vague soft set denoted by $\hat{X}$, if $\forall\ e \in A$, $t_{F(e)}(x) = 1$, $1 - f_{F(e)}(x) = 1$, $x \in X$.*

Definition 2.11 ([21]). *The union of two vague soft sets (F, A) and (G, B) over a universe X is a vague soft set (H, C), where $C = A \cup B$ and $\forall\ e \in C$,*

$$t_{H(e)}(x) = \begin{cases} t_{F(e)}(x), & e \in A - B,\ x \in X, \\ t_{G(e)}(x), & e \in B - A,\ x \in X, \\ t_{F(e)}(x) \vee t_{G(e)}(x), & e \in A \cap B,\ x \in X, \end{cases}$$

$$1 - f_{H(e)}(x) = \begin{cases} 1 - f_{F(e)}(x), & e \in A - B,\ x \in X, \\ 1 - f_{G(e)}(x), & e \in B - A,\ x \in X, \\ (1 - f_{F(e)}(x) \vee 1 - f_{G(e)}(x)), & e \in A \cap B,\ x \in X. \end{cases}$$

We denote it by $(F, A) \cup (G, B) = (H, C)$.

Definition 2.12 ([21]). *The intersection of two vague soft sets (F, A) and (G, B) over a universe X is a vague soft set (H, C), where $C = A \cup B$ and $\forall\ e \in C$,*

$$t_{H(e)}(x) = \begin{cases} t_{F(e)}(x), & e \in A - B,\ x \in X, \\ t_{G(e)}(x), & e \in B - A,\ x \in X, \\ t_{F(e)}(x) \wedge t_{G(e)}(x), & e \in A \cap B,\ x \in X, \end{cases}$$

$$1 - f_{H(e)}(x) = \begin{cases} 1 - f_{F(e)}(x), & e \in A - B,\ x \in X, \\ 1 - f_{G(e)}(x), & e \in B - A,\ x \in X, \\ (1 - f_{F(e)}(x) \wedge 1 - f_{G(e)}(x)), & e \in A \cap B,\ x \in X. \end{cases}$$

We denote it by $(F, A) \cap (G, B) = (H, C)$.

Definition 2.13 ([19]). *Let X be an initial universe set, E be the nonempty fixed set of parameters and τ be the collection of vague soft sets over X, then τ is said to be a vague soft topology on X if:*

1. *$\hat{\emptyset}_E,\ \hat{X}_E$ belongs to τ.*
2. *the union of any number of vague soft sets in τ belongs to τ.*
3. *the intersection of any two vague soft sets in τ belongs to τ.*

The triplet (X, τ, E) is called a vague soft topological space over X.

Definition 2.14 ([19]). *Let (X, τ, E) be a vague soft topological space over X, then the members of τ are said to be vague soft open set in X. A vague soft set (F, E) over X is said to be a vague soft closed set in X, if its relative complement $(F, E)^c$ belongs to τ.*

Definition 2.15 ([12]). *Let (X, τ, E) be a vague soft topological space and (F, E) be a vague soft set over X. Then vague soft interior of (F, E) is defined by $v\tilde{s}int(F, E) = \cup\{(G, E)/(G, E) \in \tau$ and $(G, E) \subseteq (F, E)\}$.*

Definition 2.16 ([12]). *Let (X, τ, E) be a vague soft topological space and (F, E) be a vague soft set over X. Then vague soft closure of (F, E) is defined by $v\tilde{s}cl(F, E) = \cap\{(H, E)/(H, E) \in \tau^c$ and $(F, E) \subseteq (H, E)\}$.*

3. Vague soft separation axioms

Definition 3.1. *A vague soft set $(F, E) \in V\tilde{S}(X, E)$ is called a **vague soft point**, denoted by e_F if for the element $e \in E$, $F(e) \neq \tilde{\emptyset}$ and $F(e') = \tilde{\emptyset}$ for all $e' \in E \setminus \{e\}$.*

Definition 3.2. *A vague soft point e_F is said to be in a vague soft set (G, E) (or e_F is a vague soft point of (G, E)) if $F(e) \subseteq G(e)$ (i.e, if $e_F \subseteq (G, E)$) and it is denoted by $e_F \in (G, E)$.*

Definition 3.3. *A vague soft point e_F is said to be in an another vague soft point e_G (or e_F is a vague soft point of e_G) if $F(e) \subseteq G(e)$ and it is denoted by $e_F \in e_G$.*

Theorem 3.1. *Let (F, E), (G, E), (H, E), and (S, E) be vague soft sets over X. Then, the following properties hold:*

1. If $(F, E) \cap (G, E) = \hat{\emptyset}_E$ then $(F, E) \subseteq (G, E)^c$.

2. If $(F, E) \subseteq (G, E)$ and $(G, E) \subseteq (H, E)$ then $(F, E) \subseteq (H, E)$.

3. If $(F, E) \subseteq (G, E)$ and $(H, E) \subseteq (S, E)$ then $(F, E) \cap (H, E) \subseteq (G, E) \cap (S, E)$.

4. $(F, E) \subseteq (G, E)$ iff $(G, E)^c \subseteq (F, E)^c$.

Proof. It is obvious from the above Definitions 2.6, 2.8. □

Result 3.1. It is observed that,

1. The union of a vague soft set and its complement need not be the absolute (Universal) vague soft set.

2. The intersection of a vague soft set and its complement need not be the null vague soft set.

The above two results are illustrate in the following Example.

Example 3.2. Let $X=\{x_1, x_2\}$, $E=\{e_1, e_2\}$ and (F, E) be vague soft set defined as follows: $(F,E) = \left\{\left\langle e_1, \frac{[0.2,0.9]}{x_1}, \frac{[0.1,0.5]}{x_2}\right\rangle, \left\langle e_2, \frac{[0.4,0.9]}{x_1}, \frac{[0.3,0.6]}{x_2}\right\rangle\right\}$. Its complement $(F,E)^c$ is given by $(F,E)^c = \left\{\left\langle e_1, \frac{[0.1,0.8]}{x_1}, \frac{[0.5,0.9]}{x_2}\right\rangle, \left\langle e_2, \frac{[0.1,0.6]}{x_1}, \frac{[0.4,0.7]}{x_2}\right\rangle\right\}$. Then $(F,E)\cup(F,E)^c = \left\{\left\langle e_1, \frac{[0.2,0.9]}{x_1}, \frac{[0.5,0.9]}{x_2}\right\rangle, \left\langle e_2, \frac{[0.4,0.9]}{x_1}, \frac{[0.4,0.7]}{x_2}\right\rangle\right\} \neq \hat{X}_E$. and $(F,E) \cap (F,E)^c = \left\{\left\langle e_1, \frac{[0.1,0.8]}{x_1}, \frac{[0.1,0.5]}{x_2}\right\rangle, \left\langle e_2, \frac{[0.1,0.6]}{x_1}, \frac{[0.3,0.6]}{x_2}\right\rangle\right\} \neq \hat{\emptyset}_E$

Proposition 3.1 ([23]). *Let e_F be a soft point and (G, A) be a soft set. If $e_F \in (G, A)$, then $e_F \notin (G, A)^c$.*

Remark 3.3. The following Example 3.4 illustrates that the above Proposition 3.1 fails to satisfy in vague soft set theory.

Example 3.4. Let $X=\{x_1, x_2\}$, $E=\{e_1, e_2\}$, $e_{_1F}=\{< e_1, \frac{[0.2,0.5]}{x_1}, \frac{[0.1,0.3]}{x_2} >\}$ be a vague soft point and (G, E) be vague soft set defined as follows:

$$(G,E) = \left\{\left\langle e_1, \frac{[0.3,0.6]}{x_1}, \frac{[0.4,0.5]}{x_2}\right\rangle, \left\langle e_2, \frac{[0.8,0.9]}{x_1}, \frac{[0.1,0.8]}{x_2}\right\rangle\right\}.$$

And its complement $(G, E)^c$ is given by

$$(G,E)^c = \left\{\left\langle e_1, \frac{[0.4,0.7]}{x_1}, \frac{[0.5,0.6]}{x_2}\right\rangle, \left\langle e_2, \frac{[0.1,0.2]}{x_1}, \frac{[0.2,0.9]}{x_2}\right\rangle\right\}.$$

Then, we note that $e_{_1F} \in (G, E)$ and $e_{_1F} \in (G, E)^c$.

Definition 3.4. *A vague soft point e_F is said to satisfy the condition*

1. *($\mathcal{P}1$), if $t_{F(e)}(x) \leq f_{F(e)}(x)$ for all $x \in X$.*

Proposition 3.2. *If e_F is a vague soft point satisfies the condition ($\mathcal{P}1$), then there exist at-least one vague soft set (G, E) such that $e_F \in (G, E)$ and $e_F \in (G, E)^c$.*

Proof. Since e_F is a vague soft point satisfies the condition ($\mathcal{P}1$), then $t_{F(e)}(x) \leq f_{F(e)}(x)$ for all $x \in X$. Now we construct a vague soft set (G, E) with $G(e)$ as follows: Let us take $t_{G(e)}(x)$ and $f_{G(e)}(x)$ be such that $t_{F(e)}(x) \leq t_{G(e)}(x) \leq f_{F(e)}(x)$ and $t_{F(e)}(x) \leq f_{G(e)}(x) \leq f_{F(e)}(x)$ for all $x \in X$. Since $t_{F(e)}(x) \leq t_{G(e)}(x)$ and $1 - f_{F(e)}(x) \leq 1 - f_{G(e)}(x)$ for all $x \in X$, we have $e_F \in (G, E)$. Also we note that $t_{F(e)}(x) \leq f_{G(e)}(x)$ and $1 - f_{F(e)}(x) \leq 1 - t_{G(e)}(x)$ for all $x \in X$, then $e_F \in (G, E)^c$. □

Corollary 3.1. *If $e_F = (F, E)$ is a vague soft point satisfies the condition ($\mathcal{P}1$), then $e_F \in (F, E)^c$.*

Proof. Since e_F is a vague soft point satisfies the condition ($\mathcal{P}1$), then we have $t_{F(e)}(x) \leq f_{F(e)}(x)$ for all $x \in X$. That is, $1 - f_{F(e)}(x) \leq 1 - t_{F(e)}(x)$ for all $x \in X$. Hence, $e_F \in (F, E)^c$. □

Remark 3.5. In vague soft set theory, let e_F be a vague soft point and (G, E) be a vague soft set. If $e_F \in (G, E)$, then $e_F \cap (G, E)^c = \hat{\emptyset}_E$ is not hold always. It is possible only if $G(e) = \tilde{X}$.

Theorem 3.6. *Every non-null vague soft set can be written as the union of all its vague soft points.*

Proof. Let X be an initial universal set, E be the set of parameters, and (F, E) be a non-null vague soft set. Assume that for each $e \in E$, $\{e_{F_k}\}_{k\in\Lambda}$ is a family of all vague soft points of (F, E). For a fixed $e_i \in E$, $F(e_i) = \bigcup_{k\in\Lambda} F_k(e_i)$ we have $(F, E) = \{(e_i, F(e_i)) : e_i \in E\} = \bigcup_{k\in\Lambda,\ e\in E} e_{F_k}$. □

Theorem 3.7. *For vague soft sets (F, E) and (G, E), $(F, E) \subseteq (G, E)$ if and only if $e_H \in (F, E)$ implies $e_H \in (G, E)$.*

Proof. Let $(F, E) \subseteq (G, E)$. Then, for all e, $F(e) \subseteq G(e)$. If $e_H \in (F, E)$ then $H(e) \subseteq F(e)$. Thus $H(e) \subseteq G(e)$ and so $e_H \in (G, E)$.

Converse is obvious by above Theorem 3.6. □

Definition 3.5. *Let (X, τ, E) be a vague soft topological space. A vague soft set $(G, E) \in V\tilde{S}(X, E)$ is called* ***vague soft neighborhood of the vague soft point*** *$e_F \in (G, E)$ if there exists a vague soft open set (H, E) such that $e_F \in (H, E) \subseteq (G, E)$. The set of all vague soft neighborhood of e_F is called its neighborhood system and is denoted by $\mathcal{N}_\tau(e_F)$.*

Example 3.8. Let $X=\{x_1, x_2, x_3\}$, $E=\{e_1, e_2\}$ and $\tau = \{\hat{\emptyset}_E, (H, E), \hat{X}_E\}$ where $(H, E)=\{< e_1, \frac{[0.1.0.6]}{x_1}, \frac{[0.4,0.4]}{x_2}, \frac{[0,0.5]}{x_3} >, < e_2, \frac{[0.8,0.9]}{x_1}, \frac{[0.2,0.3]}{x_2}, \frac{[0.6,0.7]}{x_3} >\}$ and (X, τ, E) be a vague soft topological space over X. If we assume that the vague soft set $(G, E)=\{< e_1, \frac{[0.3.0.6]}{x_1}, \frac{[0.4,0.6]}{x_2}, \frac{[0.4,0.9]}{x_3} >, < e_2, \frac{[0.8,1]}{x_1}, \frac{[0.4,0.7]}{x_2}, \frac{[0.8,0.9]}{x_3} >\}$ and a vague soft point $e_{_2F}=\{< e_2, \frac{[0,0.1]}{x_1}, \frac{[0.2,0.2]}{x_2}, \frac{[0.5,0.6]}{x_3} >\}$ of (G, E). Then (G, E) is also a vague soft neighborhood of vague soft point $e_{_2F}$, since there exists a vague soft open set (H, E) such that $e_{_2F} \in (H, E) \subseteq (G, E)$.

Definition 3.6. *Let (X, τ, E) be a vague soft topological space. A vague soft set $(F, E) \in V\tilde{S}(X, E)$ is called vague soft neighborhood of a vague soft set (G, E) if there exists a vague soft open set (H, E) such that $(G, E) \subseteq (H, E) \subseteq (F, E)$. The set of all vague soft neighborhood of (G, E) is called its neighborhood system and is denoted by $\mathcal{N}_\tau(G, E)$.*

Theorem 3.9. *Let (X, τ, E) be a vague soft topological space and e_F be a vague soft point. Then $\mathcal{N}_\tau(e_F)$ has the following properties:*

1. *If $(G, E) \in \mathcal{N}_\tau(e_F)$, then $e_F \in (G, E)$.*
2. *If $(G, E) \in \mathcal{N}_\tau(e_F)$ and $(G, E) \subseteq (H, E)$, then $(H, E) \in \mathcal{N}_\tau(e_F)$.*
3. *If $(G, E), (H, E) \in \mathcal{N}_\tau(e_F)$, then $(G, E) \cap (H, E) \in \mathcal{N}_\tau(e_F)$.*
4. *If $(G, E), (H, E) \in \mathcal{N}_\tau(e_F)$, then $(G, E) \cup (H, E) \in \mathcal{N}_\tau(e_F)$.*

Proof. It is obvious from the Definitions 3.5 and 2.13. □

Theorem 3.10. *Let (X, τ, E) be a vague soft topological space. A vague soft set $(F, E) \in V\tilde{S}(X, E)$ is vague soft open if and only if (F, E) is vague soft neighborhood of all its vague soft points.*

Proof. Necessary part is obvious.

Conversely, let (F, E) be a vague soft neighborhood of all its vague soft points in the family $\{e_{F_k}\}_{k\in\Lambda, e\in E}$. Then, for each $e \in E$ and $k \in \Lambda$, there exists a vague soft open set (G_{e_k}, E) such that $e_{F_k} \in (G_{e_k}, E) \subseteq (F, E)$ implying $e_{F_k} \subseteq (G_{e_k}, E) \subseteq (F, E)$. Using Theorem 3.6 we obtain $(F, E) = \bigcup_{k\in\Lambda, e\in E} e_{F_i} \subseteq \bigcup_{k\in\Lambda, e\in E}(G_{e_k}, E) \subseteq (F, E)$.

So, $\bigcup_{k\in\Lambda, e\in E}(G_{e_k}, E) = (F, E)$ is a arbitrary union of vague soft open sets. Hence, (F, E) is vague soft open set. □

Definition 3.7. *Let X be an initial universe set and E be the set of parameters. If $\tau = \{\hat{\emptyset}_E, \hat{X}_E\}$, then τ is called vague soft indiscrete topology on X and (X, τ, E) is **vague soft indiscrete topological space** over X.*

Definition 3.8. *Let X be an initial universe set and E be the set of parameters. If τ be the collection of all vague soft sets which can be defined over X, then τ is called the vague soft discrete topology on X and (X, τ, E) is said to be a **vague soft discrete topological space** over X.*

Definition 3.9. *Two vague soft sets (F, E) and (G, E) in $V\tilde{S}(X, E)$ are said to be vague soft disjoint (written $(F, E) \neq (G, E)$), if $(F, E) \cap (G, E) = \hat{\emptyset}_E$.*

Definition 3.10. *Two vague soft points e_F and e_G are said to be vague soft distinct (written $e_F \neq e_G$), if its corresponding vague soft sets (F, E) and (G, E) are disjoint.*

Remark 3.11. If $e_F \neq e_G$, then neither $F(e) \neq \tilde{X}$ nor $G(e) \neq \tilde{X}$.

Definition 3.11. *A vague soft topological space (X, τ, E) is said to be a **vague soft T_0-space** iff for any two vague soft points $e_M \neq e_N$ there exist vague soft open sets (F, E) and (G, E) such that $e_M \in (F, E)$ and $e_N \notin (F, E)$ or $e_N \in (G, E)$ and $e_M \notin (G, E)$.*

Example 3.12. A vague soft indiscrete topological space over X is not vague soft T_0-space.

Example 3.13. Let X=$\{x_1, x_2\}$, E=$\{e_1, e_2\}$. If we consider the vague soft topologies $\tau_1 = \{\hat{\emptyset}_E, (F_1, E), \hat{X}_E\}$ and $\tau_2 = \{\hat{\emptyset}_E, (F_2, E), \hat{X}_E\}$, where

$$(F_1, E) = \{< e_1, \frac{[1,1]}{x_1}, \frac{[0,0]}{x_2} >, < e_2, \frac{[0,0]}{x_1}, \frac{[1,1]}{x_2} >$$

and

$$(F_2, E) = \{< e_1, \frac{[0.5,0.6]}{x_1}, \frac{[0,0]}{x_2} >, < e_2, \frac{[0,0]}{x_1}, \frac{[1,1]}{x_2} >\}.$$

Then, the triplet (X, τ_1, E) is a vague soft T_0-space. But in τ_2, there is no such a vague soft open set (at-least one vague soft open set containing one vague soft point not containing another vague soft point) for the distinct vague soft points $e_{_1M}$=$\{< e_1, \frac{[0.7,1]}{x_1}, \frac{[0,0]}{x_2} >\}$, $e_{_1N}$=$\{< e_1, \frac{[0,0]}{x_1}, \frac{[0.2,0.4]}{x_2} >\}$. Hence (X, τ_2, E) is not a vague soft T_0-space.

Remark 3.14. If (X, τ, E) is vague soft T_0-space and $\tau \subset \tau'$, then (X, τ', E) is also a vague soft T_0-space.

Example 3.15. Let X=$\{x_1, x_2\}$, E=$\{e_1, e_2\}$ and $\tau = \{\hat{\emptyset}_E, (F_1, E), (F_2, E), \hat{X}_E\}$ where

$$(F_1, E) = \{< e_1, \frac{[1,1]}{x_1}, \frac{[0,0]}{x_2} >, < e_2, \frac{[0,0]}{x_1}, \frac{[1,1]}{x_2} >\},$$
$$(F_2, E) = \{< e_1, \frac{[0.5,0.6]}{x_1}, \frac{[0,0]}{x_2} >, < e_2, \frac{[0,0]}{x_1}, \frac{[1,1]}{x_2} >\}.$$

Since (X, τ_1, E) is a vague soft T_0-space as from the above Example 3.13 and $\tau_1 \subset \tau$, we have (X, τ, E) is also a vague soft T_0-space.

Definition 3.12. *A vague soft topological space (X, τ, E) is said to be a* ***vague soft T_1-space*** *iff for any two vague soft points $e_M \neq e_N$ there exist vague soft open sets (F, E) and (G, E) such that $e_M \in (F, E)$, $e_N \notin (F, E)$ and $e_N \in (G, E)$, $e_M \notin (G, E)$.*

Example 3.16. Let $X = \{x_1, x_2\}, E = \{e_1, e_2\}$ and $\tau = \{\hat{\emptyset}_E, (F_1, E), \ldots, (F_9, E), \hat{X}_E\}$ where

$$(F_1, E) = \{< e_1, \frac{[1,1]}{x_1}, \frac{[0,0]}{x_2} >, < e_2, \frac{[0,0]}{x_1}, \frac{[0.2,0.8]}{x_2} >\},$$
$$(F_2, E) = \{< e_1, \frac{[0,0]}{x_1}, \frac{[1,1]}{x_2} >, < e_2, \frac{[0,0]}{x_1}, \frac{[0,0]}{x_2} >\},$$
$$(F_3, E) = \{< e_1, \frac{[1,1]}{x_1}, \frac{[1,1]}{x_2} >, < e_2, \frac{[0,0]}{x_1}, \frac{[1,1]}{x_2} >\},$$
$$(F_4, E) = \{< e_1, \frac{[1,1]}{x_1}, \frac{[1,1]}{x_2} >, < e_2, \frac{[1,1]}{x_1}, \frac{[0,0]}{x_2} >\},$$
$$(F_5, E) = \{< e_1, \frac{[1,1]}{x_1}, \frac{[0,0]}{x_2} >, < e_2, \frac{[0,0]}{x_1}, \frac{[0.2,0.8]}{x_2} >\},$$
$$(F_6, E) = \{< e_1, \frac{[1,1]}{x_1}, \frac{[0,0]}{x_2} >, < e_2, \frac{[0,0]}{x_1}, \frac{[0,0]}{x_2} >\},$$
$$(F_7, E) = \{< e_1, \frac{[1,1]}{x_1}, \frac{[1,1]}{x_2} >, < e_2, \frac{[0,0]}{x_1}, \frac{[0,0]}{x_2} >\},$$
$$(F_8, E) = \{< e_1, \frac{[1,1]}{x_1}, \frac{[1,1]}{x_2} >, < e_2, \frac{[0,0]}{x_1}, \frac{[0.2,0.8]}{x_2} >\},$$
$$(F_9, E) = \{< e_1, \frac{[1,1]}{x_1}, \frac{[1,1]}{x_2} >, < e_2, \frac{[1,1]}{x_1}, \frac{[0.2,0.8]}{x_2} >\}.$$

Then the vague soft topological space (X, τ, E) is a vague soft T_1-space.

Proposition 3.3. *Every vague soft T_1-space is vague soft T_0-space.*

Proof. If (X, τ, E) is a vague soft T_2-space, then for the distinct vague soft points e_M and e_N, there exist vague soft open sets (F, E) and (G, E) such that $e_M \in (F, E), e_N \notin (F, E)$ and $e_N \in (G, E), e_M \notin (G, E)$. Hence (X, τ, E) is a vague soft T_1-space. □

Remark 3.17. The converse of the above Proposition 3.3 is need not be true as shown in the following example.

Example 3.18. Let us consider the vague soft T_0-space (X, τ, E) as in Example 3.13. Then the vague soft topological space (X, τ, E) is not vague soft T_1-space, since for the pair of distinct vague soft points e_{1M}=$\{< e_1, \frac{[0.4,0.6]}{x_1}, \frac{[0,0]}{x_2} >\}$, e_{1N}=$\{< e_1, \frac{[0,0]}{x_1}, \frac{[0.6,0.7]}{x_2} >\}$ there is no vague soft open set containing e_{1N} and not containing e_{1M}.

Remark 3.19. Based on Proposition 3.2 and Corollary 3.1, the following Theorem 3.20 due to S. Hussain and B. Ahmad need not be true in vague soft topological spaces.

Theorem 3.20 ([11])**.** *A soft topological space (X, τ, E) is a soft T_1-space iff every soft point of X is soft closed.*

Definition 3.13. *Let (X, τ, E) be a vague soft topological space and e_M and e_N be any two vague soft points such that $e_M \neq e_N$. If there exist vague soft open sets (F, E) and (G, E) such that $e_M \in (F, E)$, $e_N \in (G, E)$ and $(F, E) \cap (G, E) = \hat{\emptyset}_E$, then (X, τ, E) is called a* ***vague soft Hausdorff (or vague soft T_2) space.***

Example 3.21. Let X=$\{x_1, x_2\}$, E=$\{e_1, e_2\}$ and $\tau = \{\hat{\emptyset}_E, (F_1, E), \ldots, (F_9, E), \hat{X}_E\}$ where

$$(F_1, E) = \{< e_1, \frac{[1,1]}{x_1}, \frac{[0,0]}{x_2} >, < e_2, \frac{[0,0]}{x_1}, \frac{[1,1]}{x_2} >\},$$
$$(F_2, E) = \{< e_1, \frac{[0,0]}{x_1}, \frac{[1,1]}{x_2} >, < e_2, \frac{[0,0]}{x_1}, \frac{[1,1]}{x_2} >\},$$
$$(F_3, E) = \{< e_1, \frac{[0,0]}{x_1}, \frac{[1,1]}{x_2} >, < e_2, \frac{[1,1]}{x_1}, \frac{[0,0]}{x_2} >\},$$
$$(F_4, E) = \{< e_1, \frac{[0,0]}{x_1}, \frac{[0,0]}{x_2} >, < e_2, \frac{[0,0]}{x_1}, \frac{[1,1]}{x_2} >\},$$
$$(F_5, E) = \{< e_1, \frac{[0,0]}{x_1}, \frac{[1,1]}{x_2} >, < e_2, \frac{[0,0]}{x_1}, \frac{[0,0]}{x_2} >\},$$
$$(F_6, E) = \{< e_1, \frac{[1,1]}{x_1}, \frac{[1,1]}{x_2} >, < e_2, \frac{[0,0]}{x_1}, \frac{[1,1]}{x_2} >\},$$
$$(F_7, E) = \{< e_1, \frac{[1,1]}{x_1}, \frac{[1,1]}{x_2} >, < e_2, \frac{[1,1]}{x_1}, \frac{[0,0]}{x_2} >\},$$

$$(F_8, E) = \{< e_1, \frac{[0,0]}{x_1}, \frac{[1,1]}{x_2} >, < e_2, \frac{[1,1]}{x_1}, \frac{[1,1]}{x_2} >\},$$
$$(F_9, E) = \{< e_1, \frac{[1,1]}{x_1}, \frac{[1,1]}{x_2} >, < e_2, \frac{[0,0]}{x_1}, \frac{[0,0]}{x_2} >\}.$$

Then, the vague soft topological space (X, τ, E) is a vague soft T_2-space.

Example 3.22. A vague soft discrete topological space form a vague soft T_2-space, since every vague soft set is vague soft open.

Proposition 3.4. *Every vague soft T_2-space is vague soft T_1-space.*

Proof. If (X, τ, E) is a vague soft T_2-space, then for the distinct vague soft points e_M and e_N, there exist vague soft open sets (F, E) and (G, E) such that $e_M \in (F, E)$, $e_N \in (G, E)$ and $(F, E) \cap (G, E) = \hat{\emptyset}_E$. Since $(F, E) \cap (G, E) = \hat{\emptyset}_E$, we have $e_M \notin (G, E)$, $e_N \notin (F, E)$. Hence (X, τ, E) is a vague soft T_1-space. □

Remark 3.23. The converse of the above Proposition 3.4 is need not be true as shown in the following example.

Example 3.24. Let us consider the vague soft T_1-space (X, τ, E) as in Example 3.16. Then the vague soft topological space (X, τ, E) is not vague soft T_2-space, since for the pair of vague soft points $e_{2M} = \{< e_2, \frac{[0,0]}{x_1}, \frac{[0.8,0.9]}{x_2} >\}$, $e_{2N} = \{< e_2, \frac{[0.2,0.6]}{x_1}, \frac{[0,0]}{x_2} >\}$ there exist no disjoint vague soft open sets (F, E) and (G, E) containing e_{2M}, e_{2N} respectively.

Remark 3.25. If (X, τ, E) is a vague soft $T_i (i = 1, 2)$-space and $\tau \subset \tau'$, then (X, τ', E) is also a vague soft $T_i (i = 1, 2)$-space.

Definition 3.14. *Let (X, τ, E) be a vague soft topological space, (G, E) be a vague soft closed set in X and $e_K \in V\tilde{S}(X, E)$ such that $e_K \notin (G, E)$. If there exist vague soft open sets (F_1, E) and (F_2, E) such that $e_K \in (F_1, E)$, $(G, E) \subseteq (F_2, E)$ and $(F_1, E) \cap (F_2, E) = \hat{\emptyset}_E$, then (X, τ, E) is called a* ***vague soft regular-space****.*

Definition 3.15. *Let (X, τ, E) be a vague soft topological space. Then (X, τ, E) is said to be a vague soft T_3-space if it is vague soft regular and vague soft T_1-space.*

Remark 3.26. Every vague soft Hausdorff (or vague soft T_2) space need not be vague soft regular-space and vague soft T_3 space.

Example 3.27. Let us consider the vague soft Hausdorff(or vague soft T_2)-space (X, τ, E) as in Example 3.21 and consider a vague soft point $e_K = \{< e_1, \frac{[0,0]}{x_1}, \frac{[0.7,0.8]}{x_2} >\}$ disjoint with vague soft closed set $(F_5, E)^c = \{< e_1, \frac{[1,1]}{x_1}, \frac{[0,0]}{x_2} >, < e_2, \frac{[1,1]}{x_1}, \frac{[1,1]}{x_2} >\}$. Since $\hat{X}_E$ is the only vague soft open set containing $(F_5, E)^c$, which shows that (X, τ, E) is not vague soft regular-space. Hence, (X, τ, E) is not vague soft T_3-space.

Definition 3.16. *Let (X, τ, E) be a vague soft topological space, (F, E) and (G, E) be vague soft closed sets in X such that $(F, E) \cap (G, E) = \hat{\emptyset}_E$. If there exist vague soft open sets (F_1, E) and (F_2, E) such that $(F, E) \subseteq (F_1, E)$, $(G, E) \subseteq (F_2, E)$ and $(F_1, E) \cap (F_2, E) = \hat{\emptyset}_E$, then (X, τ, E) is called a* ***vague soft normal-space****.*

Definition 3.17. *Let (X, τ, E) be a vague soft topological space. Then (X, τ, E) is said to be a vague soft T_4-space if it is vague soft normal and vague soft T_1-space.*

Remark 3.28. Due to the Corollary 3.1, we obtain the following results

1. Every vague soft normal space need not be a vague soft regular space.
2. Every vague soft regular space need not be a vague soft Hausdorff space.

which are illustrated by the following examples.

Example 3.29. Let X=$\{x_1, x_2\}$, E=$\{e_1, e_2\}$ and $\tau = \{\hat{\emptyset}_E, (F_1, E), \ldots, (F_5, E), \hat{X}_E\}$ where

$$(F_1, E) = \{< e_1, \frac{[0.2, 0.8]}{x_1}, \frac{[1, 1]}{x_2} >, < e_2, \frac{[1, 1]}{x_1}, \frac{[0.4, 0.6]}{x_2} >\},$$
$$(F_2, E) = \{< e_1, \frac{[1, 1]}{x_1}, \frac{[0.4, 0.6]}{x_2} >, < e_2, \frac{[0.2, 0.8]}{x_1}, \frac{[1, 1]}{x_2} >\},$$
$$(F_3, E) = \{< e_1, \frac{[0, 0]}{x_1}, \frac{[0.4, 0.6]}{x_2} >, < e_2, \frac{[0.2, 0.8]}{x_1}, \frac{[0, 0]}{x_2} >\},$$
$$(F_4, E) = \{< e_1, \frac{[0.2, 0.8]}{x_1}, \frac{[0, 0]}{x_2} >, < e_2, \frac{[0, 0]}{x_1}, \frac{[0.4, 0.6]}{x_2} >\},$$
$$(F_5, E) = \{< e_1, \frac{[0.2, 0.8]}{x_1}, \frac{[0.4, 0.6]}{x_2} >, < e_2, \frac{[0.2, 0.8]}{x_1}, \frac{[0.4, 0.6]}{x_2} >\}.$$

Here, we note that each vague soft set in τ is both vague soft open as well as vague soft closed. Now for the disjoint vague soft closed sets $(F_3, E) \neq (F_4, E)$, there exist a vague soft open sets (F_3, E) and (F_4, E) such that $(F_3, E) \subseteq (F_3, E)$, $(F_4, E) \subseteq (F_4, E)$ and $(F_3, E) \cap (F_4, E) = \hat{\emptyset}_E$. Therefore the vague soft topological space (X, τ, E) is a vague soft normal-space.

Next we consider a vague soft point e_K=$\{< e_1, \frac{[0.5,0.6]}{x_1}, \frac{[0,0]}{x_2} >\}$ and vague soft closed set (F_3, E) with $e_K \notin (F_3, E)$. Since for this vague soft point e_K and vague soft closed set (F_3, E), there exist no vague soft open sets (G, E) and (H, E) such that $e_K \in (G, E)$, $(F_3, E) \subseteq (H, E)$ and $(G, E) \cap (H, E) = \hat{\emptyset}_E$. Hence (X, τ, E) is not a vague soft regular-space.

Example 3.30. Let X=$\{x_1, x_2\}$, E=$\{e_1, e_2\}$ and $\tau = \{\hat{\emptyset}_E, (F_1, E), (F_2, E), \hat{X}_E\}$ where (F_1, E)=$\{< e_1, \frac{[0,0]}{x_1}, \frac{[1,1]}{x_2} >, < e_2, \frac{[1,1]}{x_1}, \frac{[1,1]}{x_2} >\}$, (F_2, E)=$\{< e_1, \frac{[1,1]}{x_1}, \frac{[0,0]}{x_2} >$, $< e_2, \frac{[0,0]}{x_1}, \frac{[0,0]}{x_2} >\}$ which are both vague soft open and vague soft closed. Then (X, τ, E) is a vague soft regular-space but not a vague soft Hausdorff space.

4. Vague soft continuous functions via vague soft neighborhoods

Definition 4.1. *Let $V\tilde{S}(X,E)$ and $V\tilde{S}(Y,K)$ be two vague soft classes, and let $u : X \to Y$ and $p : E \to K$ be mappings. Then a vague soft function $g_{pu} = (u,p) : V\tilde{S}(X,E) \to V\tilde{S}(Y,K)$ is defined as: for $(F,A) \in V\tilde{S}(X,E)$, the image of (F,A) under g_{pu} denoted by $g_{pu}(F,A) = (u(F),p(A))$, is a vague soft set in $V\tilde{S}(Y,K)$ given by*

$$t_{u(F)(\beta)}(y) = \begin{cases} \sup_{\alpha \in p^{-1}(\beta)\cap A,\ x \in u^{-1}(y)} t_{F(\alpha)}(x), & \text{if } u^{-1}(y) \neq \emptyset, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$1 - f_{u(F)(\beta)}(y) = \begin{cases} \sup_{\alpha \in p^{-1}(\beta)\cap A,\ x \in u^{-1}(y)} 1 - f_{F(\alpha)}(x), & \text{if } u^{-1}(y) \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

for all $\beta \in p(A)$ and $y \in Y$.

Definition 4.2. *Let $V\tilde{S}(X,E)$ and $V\tilde{S}(Y,K)$ be two vague soft classes, and let $g_{pu} = (u,p) : V\tilde{S}(X,E) \to V\tilde{S}(Y,K)$ be a vague soft function and (G,B) be a vague soft set in $V\tilde{S}(Y,K)$. Then the inverse image of (G,B) under g_{pu}, denoted by $g_{pu}^{-1}(G,B) = (u^{-1}(G),\ p^{-1}(B))$ is a vague soft set in $V\tilde{S}(X,E)$ given by $t_{u^{-1}(G)(\alpha)}(x) = t_{G(p(\alpha))}(u(x))$ and $1 - f_{u^{-1}(G)(\alpha)}(x) = 1 - f_{G(p(\alpha))}(u(x))$, for all $\alpha \in p^{-1}(B)$ and $x \in X$.*

Example 4.1. For vague soft classes $V\tilde{S}$(X,E) , $V\tilde{S}(Y,K)$, Let X=$\{x_1,x_2,x_3\}$, E=$\{e_1,e_2,e_3,e_4\}$, Y=$\{y_1,y_2,y_3\}$, K=$\{k_1,k_2,k_3\}$. Define $u : X \to Y$ and $p : E \to K$ as follows:

$u(x_1) = y_1$, $u(x_2) = y_2$, $u(x_3) = y_1$, $p(e_1) = k_1$, $p(e_2) = k_3$, $p(e_3) = k_3$, $p(e_4) = k_1$.

Consider the two vague soft sets (F,A) and (G,K) over X and Y respectively as follows: $(F,A) = \left\{ \left\langle e_1, \frac{[0.3,0.6]}{x_1}, \frac{[0.2,0.5]}{x_2}, \frac{[0.1,0.7]}{x_3} \right\rangle, \left\langle e_2, \frac{[0.4,0.9]}{x_1}, \frac{[0.6,0.8]}{x_2}, \frac{[0.2,0.7]}{x_3} \right\rangle, \left\langle e_3, \frac{[0.8,0.9]}{x_1}, \frac{[0.1,0.8]}{x_2}, \frac{[0.9,1]}{x_3} \right\rangle \right\}$ and $(G,K) = \left\{ \left\langle k_1, \frac{[0.6,0.7]}{y_1}, \frac{[0.9,1]}{y_2}, \frac{[0.5,0.8]}{y_3} \right\rangle, \left\langle k_2, \frac{[0.4,0.8]}{y_1}, \frac{[0.2,0.6]}{y_2}, \frac{[0.7,0.8]}{y_3} \right\rangle, \left\langle k_3, \frac{[0.1,0.8]}{y_1}, \frac{[0.2,0.6]}{y_2}, \frac{[0.2,0.5]}{y_3} \right\rangle \right\}$. Then, the vague soft function $g_{pu} : V\tilde{S}(X,E) \to V\tilde{S}(Y,K)$ is given as: for a vague soft set $(F,E) \in V\tilde{S}(X,E)$, $g_{pu}(F,A) = (u(F),p(A))$ is a vague soft set in $V\tilde{S}(Y,K)$ and is given by $g_{pu}(F,A) = \left\{ \left\langle k_1, \frac{[0.3,0.7]}{y_1}, \frac{[0.2,0.5]}{y_2}, \frac{[0,0]}{y_3} \right\rangle, \left\langle k_3, \frac{[0.9,1]}{y_1}, \frac{[0.6,0.8]}{y_2}, \frac{[0,0]}{y_3} \right\rangle \right\}$.

Next, for the vague soft set (G,K) in $V\tilde{S}(Y,K)$, the vague soft inverse image is given by

$g_{pu}^{-1}(G,K) = (u^{-1}(G),p^{-1}(B))$

$$= \left\{ \begin{array}{l} \left\langle e_1, \frac{[0.6,0.7]}{x_1}, \frac{[0.9,1]}{x_2}, \frac{[0.6,0.7]}{x_3} \right\rangle, \\ \left\langle e_2, \frac{[0.1,0.8]}{x_1}, \frac{[0.2,0.6]}{x_2}, \frac{[0.1,0.8]}{x_3} \right\rangle, \left\langle e_3, \frac{[0.1,0.8]}{x_1}, \frac{[0.2,0.6]}{x_2}, \frac{[0.1,0.8]}{x_3} \right\rangle, \\ \left\langle e_4, \frac{[0.6,0.7]}{x_1}, \frac{[0.9,1]}{x_2}, \frac{[0.6,0.7]}{x_3} \right\rangle \end{array} \right\}.$$

Remark 4.2. The vague soft function g_{pu} is called surjective (resp. injective and constant) if p and u are surjective (resp. injective and constant).

Theorem 4.3. *Let $V\tilde{S}(X,E)$ and $V\tilde{S}(Y,K)$ be two vague soft classes. For the vague soft function $g_{pu}: V\tilde{S}(X,E) \to V\tilde{S}(Y,K)$, the following statements hold:*

1. *$g_{pu}(\hat{\emptyset}_E) \subseteq \hat{\emptyset}_K$ and if p is surjective, the equality holds.*
2. *$g_{pu}(\hat{X}_E) \subset \hat{Y}_K$ and if g_{pu} is surjective the equality holds.*
3. *$(g_{pu}(F,A))^c \subseteq g_{pu}((F,A)^c)$ if g_{pu} is surjective. And if g_{pu} is bijective then its equality holds.*
4. *If $(F_1,A) \subseteq (F_2,A)$, then $g_{pu}(F_1,A) \subseteq g_{pu}(F_2,A)$, $\forall$ $(F_1,A),(F_2,A) \in V\tilde{S}(X,E)$.*

Proof. 1, 2 and 3 are obvious.

4. Since $(F_1,A) \subseteq (F_2,A)$, we have for each $\alpha \in A$ and for all $x \in X$ $t_{F_1(\alpha)}(x) \leq t_{F_2(\alpha)}(x)$, and $1 - f_{F_1(\alpha)}(x) \leq 1 - f_{F_2(\alpha)}(x)$.

Now, for all $\beta \in p(A)$ and $y \in Y$,

$$t_{u(F_1)(\beta)}(y) = \sup_{\alpha \in p^{-1}(\beta)\cap A,\ x \in u^{-1}(y)} t_{F_1(\alpha)}(x) \leq \sup_{\alpha \in p^{-1}(\beta)\cap A,\ x \in u^{-1}(y)} t_{F_2(\alpha)}(x),$$

and

$$\begin{aligned} 1 - f_{u(F_1)(\beta)}(y) &= \sup_{\alpha \in p^{-1}(\beta)\cap A,\ x \in u^{-1}(y)} 1 - f_{F_1(\alpha)}(x) \\ &\leq \sup_{\alpha \in p^{-1}(\beta)\cap A,\ x \in u^{-1}(y)} 1 - f_{F_2(\alpha)}(x) \end{aligned}$$

and hence, $t_{u(F_1)(\beta)}(y) \leq t_{u(F_2)(\beta)}(y)$ and $1 - f_{u(F_1)(\beta)}(y) \leq 1 - f_{u(F_2)(\beta)}(y)$.

This implies that, $g_{pu}(F_1,A) \subseteq g_{pu}(F_2,A)$. □

Theorem 4.4. *Let $V\tilde{S}(X,E)$ and $V\tilde{S}(Y,K)$ be two vague soft classes. For the vague soft function $g_{pu}: V\tilde{S}(X,E) \to V\tilde{S}(Y,K)$, the following statements hold,*

1. *$g_{pu}^{-1}(\hat{\emptyset}_K) = \hat{\emptyset}_E$.*
2. *$g_{pu}^{-1}(\hat{Y}_K) = \hat{X}_E$*
3. *$(g_{pu}^{-1}(G,B))^c = g_{pu}^{-1}((G,B)^c)$*
4. *If $(G_1,B) \subseteq (G_2,B)$, then $g_{pu}^{-1}(G_1,B) \subseteq g_{pu}^{-1}(G_2,B)$, for all $(G_1,B),(G_2,B) \in V\tilde{S}(Y,K)$.*

Proof. 1, 2 and 3 are obvious.

4. Since $(G_1,B) \subseteq (G_2,B)$, we have for each $\beta \in B$ and for all $y \in Y$ $t_{G_1(\beta)}(y) \leq t_{G_2(\beta)}(y)$, and $1 - f_{G_1(\beta)}(y) \leq 1 - f_{G_2(\beta)}(y)$.

Now, for all $\alpha \in p^{-1}(B)$ and $x \in X$, we have $t_{u^{-1}(G_1)(\alpha)}(x) = t_{G_1(p(\alpha))}(u(x)) \leq t_{G_2(p(\alpha))}(u(x)) = t_{u^{-1}(G_2)(\alpha)}(x)$.

Similarly, $1 - f_{u^{-1}(G_1)(\alpha)}(x) = 1 - f_{G_1(p(\alpha))}(u(x)) \leq 1 - f_{G_2(p(\alpha))}(u(x)) = 1 - f_{u^{-1}(G_2)(\alpha)}(x)$. Hence, $g_{pu}^{-1}(G_1, B) \subseteq g_{pu}^{-1}(G_2, B)$. □

Theorem 4.5. *Let (F, A), (G, B) be two vague soft sets in $V\tilde{S}(X, E)$ and $V\tilde{S}(Y, K)$ respectively, and $g_{pu} : V\tilde{S}(X, E) \to V\tilde{S}(Y, K)$ be a vague soft function. Then*

1. *$(F, A) \subseteq g_{pu}^{-1}(g_{pu}(F, A))$ and if g_{pu} is injective, the equality holds.*
2. *$g_{pu}(g_{pu}^{-1}(G, B)) \subseteq (G, B)$ and if g_{pu} is surjective, the equality holds.*

Proof. The proof is obvious from the Definitions 4.1 and 4.2. □

Theorem 4.6. *Let $\{(F_i, E)\}_{i\in I}$ be a family of vague soft sets over X and $g_{pu} : V\tilde{S}(X, E) \to V\tilde{S}(Y, K)$ be a vague soft function. Then 1. $g_{pu}((F_1, E) \cup (F_2, E) = g_{pu}(F_1, E) \cup g_{pu}(F_2, E)$. In general, $g_{pu}(\bigcup_{i\in I}(F_i, E)) = \bigcup_{i\in I} g_{pu}(F_i, E)$.*

2. $g_{pu}((F_1, E) \cap (F_2, E) \subseteq g_{pu}(F_1, E) \cap g_{pu}(F_2, E)$. In general, $g_{pu}(\bigcap_{i\in I}(F_i, E)) \subseteq \bigcap_{i\in I} g_{pu}(F_i, E)$ and if g_{pu} is injective, the equalities holds.

Proof. Obvious. □

Theorem 4.7. *Let $\{(G_j, K)\}_{j\in J}$ be a family of vague soft sets over Y and $g_{pu} : V\tilde{S}(X, E) \to V\tilde{S}(Y, K)$ be a vague soft function. Then:*

1. $g_{pu}^{-1}((G_1, K) \cup (G_2, K)) = g_{pu}^{-1}(G_1, K) \cup g_{pu}^{-1}(G_2, K)$.
In general, $g_{pu}^{-1}(\bigcup_{j\in J}(G_j, K)) = \bigcup_{j\in J} g_{pu}^{-1}(G_j, K)$.
2. $g_{pu}^{-1}((G_1, K) \cap (G_2, K)) = g_{pu}^{-1}(G_1, K) \cap g_{pu}^{-1}(G_2, K)$.
In general, $g_{pu}^{-1}(\bigcap_{j\in J}(G_j, K)) = \bigcap_{j\in J} g_{pu}^{-1}(G_j, K)$.

Proof. 1. Let $(G_1, K) \cup (G_2, K) = (S, K)$. For $\alpha \in p^{-1}(K)$ and $x \in X$, $t_{u^{-1}(S)(\alpha)}(x) = t_{S(p(\alpha))}(u(x))$ and $1 - f_{u^{-1}(G)(\alpha)}(x) = 1 - f_{G(p(\alpha))}(u(x))$.

Since $(S, K) = (G_1, K) \cup (G_2, K)$ we have $t_{S(p(\alpha))}(u(x)) = t_{G_1(p(\alpha))}(u(x)) \vee t_{G_2(p(\alpha))}(u(x)) = t_{u^{-1}(G_1)(\alpha)}(x) \vee t_{u^{-1}(G_2)(\alpha)}(x)$.

Thus, $t_{u^{-1}(S)(\alpha)}(x) = t_{u^{-1}(G_1)(\alpha)}(x) \vee t_{u^{-1}(G_2)(\alpha)}(x)$.

Similarly, $1 - f_{u^{-1}(S)(\alpha)}(x) = 1 - f_{u^{-1}(G_1)(\alpha)}(x) \vee 1 - f_{u^{-1}(G_2)(\alpha)}(x)$ for all $\alpha \in p^{-1}(K)$ and $x \in X$. Hence $g_{pu}^{-1}(S, K) = g_{pu}^{-1}((G_1, K) \cup (G_2, K)) = g_{pu}^{-1}(G_1, K) \cup g_{pu}^{-1}(G_2, K)$.

2. Proof is as similar to the proof of 1. □

Definition 4.3. *Let (X, τ, E) and (Y, σ, K) be two vague soft topological spaces. The vague soft function $g_{pu} : V\tilde{S}(X, E) \to V\tilde{S}(Y, K)$ is called **vague soft continuous at vague soft point** e_F of $V\tilde{S}(X, E)$ if for each $(G, K) \in N_\sigma(g_{pu}(e_F))$, there exist a $(H, E) \in N_\tau(e_F)$ such that $g_{pu}(H, E) \subseteq (G, K)$.*

*If g_{pu} is vague soft continuous at each vague soft point of $V\tilde{S}(X, E)$, then g_{pu} is called **vague soft continuous function**.*

Theorem 4.8. *Let $g_{pu} : (X, \tau, E) \to (Y, \sigma, K)$ be a vague soft function and e_F be a vague soft point. Then g_{pu} is vague soft continuous at e_F if and only if for each $(G, K) \in N_\sigma(g_{pu}(e_F))$, there exist a $(H, E) \in N_\tau(e_F)$ such that $(H, E) \subseteq g_{pu}^{-1}(G, K)$.*

Proof. If g_{pu} is vague soft continuous at e_F, then for each $(G, K) \in N_\sigma(g_{pu}(e_F))$, there exist a $(H, E) \in N_\tau(e_F)$ such that $g_{pu}(H, E) \subseteq (G, K)$. Thus, from the Theorem 4.5 we have $(H, E) \subseteq g_{pu}^{-1}(g_{pu}(H, E)) \subseteq g_{pu}^{-1}(G, K)$.

Conversely, assume that for each $(G, K) \in N_\sigma(g_{pu}(e_F))$, there exist a $(H, E) \in N_\tau(e_F)$ such that $(H, E) \subseteq g_{pu}^{-1}(G, K)$. Now by Theorem 4.5, we get $g_{pu}(H, E) \subseteq g_{pu}(g_{pu}^{-1}(G, K)) \subseteq (G, K)$. Hence from the Definition 4.8, g_{pu} is vague soft continuous at e_F. □

Theorem 4.9. *Let $g_{pu} : (X, \tau, E) \to (Y, \sigma, K)$ be a vague soft function and e_F be a vague soft point. Then g_{pu} is called vague soft continuous at e_F if and only if for each $(G, K) \in N_\sigma(g_{pu}(e_F))$, $g_{pu}^{-1}(G, K) \in \tau$.*

Proof. It is clear from Theorem 4.8. □

Theorem 4.10. *The vague soft function $g_{pu} : (X, \tau, E) \to (Y, \sigma, K)$ is called vague soft continuous if and only if for all $(G, K) \in \sigma$, $g_{pu}^{-1}(G, K) \in \tau$.*

Proof. It is follows from Theorem 4.9. □

Example 4.11. Let $g_{pu} : (X, \tau, E) \to (Y, \sigma, K)$ be a vague soft function.

1. If σ is the vague soft indiscrete topology on Y, then g_{pu} is a vague soft continuous function.

2. If τ is the vague soft discrete topology on X, then g_{pu} is a vague soft continuous function.

Theorem 4.12. *Let $g_{pu} : (X, \tau, E) \to (Y, \sigma, K)$ be a vague soft function. Then the following conditions are equivalent:*

1. *g_{pu} is vague soft continuous function.*

2. *For each vague soft closed set (G, K) over Y, $g_{pu}^{-1}(G, K)$ is a vague soft closed set over X.*

3. *For each vague soft set (F, E) over X, $g_{pu}(v\tilde{s}cl(F, E)) \subseteq v\tilde{s}cl(g_{pu}(F, E))$.*

4. *For each vague soft set (G, K) over Y, $v\tilde{s}cl(g_{pu}^{-1}(G, K)) \subseteq g_{pu}^{-1}(v\tilde{s}cl(G, K))$.*

5. *For each vague soft set (G, K) over Y, $g_{pu}^{-1}(v\tilde{s}int(G, K)) \subseteq v\tilde{s}int(g_{pu}^{-1}(G, K))$.*

Proof. 1 ⇒ 2: Let (G,K) be a vague soft closed set over Y. Then $(G,K)^c \in \sigma$. By 1, we have $g_{pu}^{-1}((G,K)^c) \in \tau$. Now from the Theorem 4.4 (3) we have $g_{pu}^{-1}((G,K)^c) = (g_{pu}^{-1}(G,K))^c \in \tau$. And hence $g_{pu}^{-1}(G,K)$ is a vague soft closed set over X.

2 ⇒ 3: Let (F,E) be a vague soft set over X. Then $(F,E) \subseteq g_{pu}^{-1}(g_{pu}(F,E)) \subseteq g_{pu}^{-1}(v\tilde{s}cl(g_{pu}(F,E)))$.

Since $v\tilde{s}cl(g_{pu}(F,E))$ is vague soft closed set over Y, by 2 $g_{pu}^{-1}(v\tilde{s}cl(g_{pu}(F,E)))$ is vague soft closed set over X. Then, $v\tilde{s}cl(F,E) \subseteq g_{pu}^{-1}(v\tilde{s}cl(g_{pu}(F,E))) \Rightarrow g_{pu}(v\tilde{s}cl(F,E)) \subseteq g_{pu}(g_{pu}^{-1}(v\tilde{s}cl(g_{pu}(F,E)))) \Rightarrow\subseteq v\tilde{s}cl(g_{pu}(F,E))$.

Hence, $g_{pu}(v\tilde{s}cl(F,E)) \subseteq v\tilde{s}cl(g_{pu}(F,E))$.

3 ⇒ 4: Let (G,K) be a vague soft set over Y and let $g_{pu}^{-1}(G,K) = (F,E)$. By 3, we have $g_{pu}(v\tilde{s}cl(F,E)) = g_{pu}(v\tilde{s}cl(g_{pu}^{-1}(G,K))) \subseteq v\tilde{s}cl(g_{pu}(g_{pu}^{-1}(G,K))) \subseteq v\tilde{s}cl((G,K))$ (i.e) $g_{pu}(v\tilde{s}cl(F,E)) \subseteq v\tilde{s}cl(g_{pu}((F,E))) \subseteq v\tilde{s}cl((G,K))$.

$\Rightarrow v\tilde{s}cl(F,E) \subseteq g_{pu}^{-1}(g_{pu}(v\tilde{s}cl(F,E))) \subseteq g_{pu}^{-1}(v\tilde{s}cl((G,K)))$.

Then, $v\tilde{s}cl(g_{pu}^{-1}(G,K)) = v\tilde{s}cl(F,E) \subseteq g_{pu}^{-1}(v\tilde{s}cl((G,K)))$.

Hence, $v\tilde{s}cl(g_{pu}^{-1}(G,K)) \subseteq g_{pu}^{-1}(v\tilde{s}cl(G,K))$.

4 ⇒ 5: Let (G,K) be a vague soft set over Y. Substituting $(G,K)=(G,K)^c$ in 4, $v\tilde{s}cl(g_{pu}^{-1}((G,K)^c)) \subseteq g_{pu}^{-1}(v\tilde{s}cl((G,K)^c))$.

Since $v\tilde{s}int(G,K) = (v\tilde{s}cl((G,K)^c))^c$, we have

$$\begin{aligned} g_{pu}^{-1}(v\tilde{s}int(G,K)) &= g_{pu}^{-1}(v\tilde{s}cl((G,K)^c))^c \\ &= (g_{pu}^{-1}(v\tilde{s}cl((G,K)^c)))^c \qquad \text{by 3 of Theorem 4.4} \\ &\subseteq (v\tilde{s}cl(g_{pu}^{-1}((G,K)^c)))^c \qquad \text{by 4} \\ &= (v\tilde{s}cl((g_{pu}^{-1}(G,K))^c))^c \qquad \text{by 3 of Theorem 4.4} \\ &= v\tilde{s}int(g_{pu}^{-1}(G,K)). \end{aligned}$$

Hence, $g_{pu}^{-1}(v\tilde{s}int(G,K)) \subseteq v\tilde{s}int(g_{pu}^{-1}(G,K))$.

5 ⇒ 1: Let (G,K) be a vague soft open set over Y. Then

$$\begin{aligned} v\tilde{s}int(g_{pu}^{-1}(G,K)) &\subseteq g_{pu}^{-1}(G,K) \\ &= g_{pu}^{-1}(v\tilde{s}int(G,K)) \\ &\subseteq v\tilde{s}int(g_{pu}^{-1}(G,K)) \\ \Rightarrow v\tilde{s}int(g_{pu}^{-1}(G,K)) &= g_{pu}^{-1}(G,K). \end{aligned}$$

That is, $g_{pu}^{-1}(G,K)$ is vague soft open set over X.

Hence, g_{pu} is a vague soft continuous function. □

Theorem 4.13. *Let $g_{pu} : (X,\tau,E) \to (Y,\sigma,K)$ be a bijective vague soft function. Then the following conditions are equivalent:*
1. $g_{pu}^{-1}(v\tilde{s}int(G,K)) \subseteq v\tilde{s}int(g_{pu}^{-1}(G,K))$, for each vague soft set (G,K) over Y.
2. $v\tilde{s}int(g_{pu}(F,E)) \subseteq g_{pu}(v\tilde{s}int(F,E))$, for each vague soft set (F,E) over X.

Proof. Necessary part: Let (F,E) be any vague soft set over X. Then $g_{pu}(F,E)$ is a vague soft set of Y. So by 1, $g_{pu}^{-1}(v\tilde{s}int(g_{pu}(F,E))) \subseteq v\tilde{s}int(g_{pu}^{-1}(g_{pu}(F,E)))$.

Since g_{pu} is one-to-one, $g_{pu}^{-1}(v\tilde{s}int(g_{pu}(F,E))) \subseteq v\tilde{s}int(g_{pu}^{-1}(g_{pu}(F,E))) = v\tilde{s}int(F,E)$. Thus, $g_{pu}(g_{pu}^{-1}(v\tilde{s}int(g_{pu}(F,E)))) \subseteq g_{pu}(v\tilde{s}int(F,E))$. Since g_{pu} is onto, $v\tilde{s}int(g_{pu}(F,E)) = g_{pu}(g_{pu}^{-1}(v\tilde{s}int(g_{pu}(F,E)))) \subseteq g_{pu}(v\tilde{s}int(F,E))$. Hence, $v\tilde{s}int(g_{pu}(F,E)) \subseteq g_{pu}(v\tilde{s}int(F,E))$.

Conversely, let (G,K) be any vague soft set over Y. Then $g_{pu}^{-1}(G,K)$ is a vague soft set of X. Since g_{pu} is onto, $v\tilde{s}int(G,K) = v\tilde{s}int(g_{pu}(g_{pu}^{-1}(G,K))) \subseteq g_{pu}(v\tilde{s}int(g_{pu}^{-1}(G,K)))$.

Since g_{pu} is one-to-one $g_{pu}^{-1}(v\tilde{s}int(G,K)) \subseteq g_{pu}^{-1}(g_{pu}(v\tilde{s}int(g_{pu}^{-1}(G,K)))) = v\tilde{s}int(g_{pu}^{-1}(G,K))$.

Hence, $g_{pu}^{-1}(v\tilde{s}int(G,K)) \subseteq v\tilde{s}int(g_{pu}^{-1}(G,K))$. □

5. Application

In this section, we illustrate an application of vague soft function in the field of medicine. Let us consider a patient with his/her complaints. It can be easily transformed into vague soft sets of possible symptoms and their respective importance. There are six complaints according to the patient. It can be represented by the universal set X={ a (burning in stomach), b (sleeplessness), c (indigestion), d (joint pain), e (depression)}. And it's importance taken to be the set of parameters E={e_1(high importance), e_2(low importance) }.

We know it from the medical experts,

u(burning in stomach) = acidity; u(sleeplessness) = stress;

u(indigestion) = acidity; u(joint pain) = rheumatoid; u(depression) = stress; and

p(high importance) = infrequent high potency;

p(low importance) = frequent low potency.

For the purpose of achieving mathematical manipulation, we denote causes and medical preference for treatment by symbols as follows:

α= acidity; β= stress; γ=rheumatoid;

e_1' = infrequent high potency; e_2' = frequent low potency.

Let Y={α, β, γ} and E'={e_1', e_2'}. Now, there are two vague soft classes (X,E) and (Y,E') denotes the symptoms according to their importance for the patient, the causes with its medical preference for treatments respectively. And we have the mappings $u : X \to Y$ and $p : E \to E'$ as follows,

$u(a) = \alpha$; $u(b) = \beta$; $u(c) = \alpha$; $u(d) = \gamma$; $u(e) = \beta$; and $p(e_1) = e_1'$; $p(e_2) = e_2'$.

Consider Patient's narration into vague soft set (F,E) as follows:

$$(F,E) = \left\{ \begin{array}{l} \left\langle e_1, \frac{[0.6,0.9]}{a}, \frac{[0.5,0.6]}{b}, \frac{[0.8,0.8]}{c}, \frac{[0.3,0.4]}{d}, \frac{[0.2,0.3]}{e} \right\rangle, \\ \left\langle e_2, \frac{[0.3,0.5]}{a}, \frac{[0.4,0.9]}{b}, \frac{[0.4,0.6]}{c}, \frac{[0.1,0.2]}{d}, \frac{[0.7,0.8]}{e} \right\rangle \end{array} \right\}.$$

By calculating the vague soft image of (F,E) under g_{pu} using the Definition 4.1,

we observe

$$g_{pu}(F,E) = \left\{ \left\langle e_1', \frac{[0.8,0.9]}{\alpha}, \frac{[0.5,0.6]}{\beta}, \frac{[0.3,0.4]}{\gamma} \right\rangle, \left\langle e_2', \frac{[0.4,0.6]}{\alpha}, \frac{[0.7,0.9]}{\beta}, \frac{[0.1,0.2]}{\gamma} \right\rangle \right\}.$$

According to the results we have to give medical treatments for the patients having **acidity** and **stress** with their medical preferences infrequent high potency and frequent low potency respectively.

6. Conclusion

In this paper, we have introduced the notions of vague soft points, vague soft neighborhood and vague soft separation axioms. We have investigated the algebraic properties of vague soft sets and obtained an interesting result which is contrary to the fundamental complement laws of set algebra. We have also presented the interconnections between vague soft separation axioms which are also contrary to the properties of the separation axioms in topological spaces. Counter examples are given to illustrate the above results. Also, we provided an application in the field of medicine using the concept of vague soft functions. This research work could be extended by developing computational algorithms to give more applications in the fields of medicine, Engineering and social work, the mysterious results could even be resolved.

Acknowledgements

The authors are thankful to the referees, the editor-in-chief and managing editors for their valuable suggestions and important comments to improve the presentation of the paper.

This study was funded by the National Natural Science Foundation of China (grant numbers 11501444, 11726019).

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Accepted: May 22, 2020

States on hyper equality algebras

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Abstract. In this paper, states on hyper equality algebras are introduced and studied. Two types of states, namely, Bosbach states and Riečan states are considered and some related properties of them are obtained. Also, the existence of two types of states are given by some examples and the relationships between them are discussed. One of important results is that Bosbach states and Riečan states coincide on bounded involutive hyper equality algebras. Moreover, Bosbach states on quotient hyper equality algebras are investigated. By use of θ-invariant Bosbach states, it follows that the induced function $\overline{s}$ by a a Bosbach state s on H is also a Bosbach state on H/θ and the corresponding example is constructed.

Keywords: hyper equality algebra, quotient hyper equality algebra, Bosbach state, Riečan state.

1. Introduction

The hyper structure theory was introduced by Marty [13], at the 8th Congress of Scandinavian Mathematicians. In an algebraic hyper structure, the composition of two elements is not an element but a set. Since then hyper structure theory has been intensively researched such as hyper BCK-algebras [10, 11], hyper MV-algebras [9], hyper BE-algebras [3, 15] and hyper EQ-algebras [1, 4], etc.

Now, hyper structure theory has been applied to many disciplines such as geometry, graphs, automata, cryptography, artificial intelligence, probability

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theory, dismutation reactions and inheritance, etc (see [7, 8]). Recently, Cheng, Xin and Jun [2] have applied the hyper theory to equality algebras in order to introduce hyper equality algebras which are a generalization of equality algebras. They defined some types of hyper filters and hyper deductive systems, moreover discussed the relations between hyper EQ-algebras, hyper BCK-algebras and hyper equality algebras. In addition, they also constructed quotient hyper equality algebras via regular congruence relations.

The notion of states on MV-algebras was introduced by Mundici [14] in 1995 with the intent of capturing the notion of average degree of truth of a proposition in Łukasiewicz logic, and so the states have been used as a semantical interpretation of the probability of fuzzy events a. That is, if s is a state and a is a fuzzy event, then $s(a)$ is presented as averaging of appearing the event a. Hence it is meaningful to extend the notion of states to other algebraic structures. Different approaches to the generalization mainly gave rise to two different notions, namely, Bosbach states and Riečan states. Consequently, Liu studied the existence of Bosbach states and Riečan states on MTL-algebra in [12]. Some examples showed that there exist MTL-algebras having no Bosbach states and Riečan states. Ciungu [6] extended the notions of states to the case of pseudo-BCK algebras, and proved that, in any good pseudo-BCK algebra with pseudo-double negation, Riečan states coincide with Bosbach states. Cheng, Xin and He [5] introduced states on pseudo equality algebras and proved that each Bosbach state is a Riečan state in any good bounded pseudo equality algebra, but the converse may be not true. Recently, Xin and Wang [16] have applied the state theory to hyper structures introducing and studying inf-Bosbach states and hyper measures on hyper BCK-algebras. Furthermore, they considered states on quotient hyper BCK-algebras and got some important results. Then Xin, He and Fu [17] investigated states on hyper MV-algebras and gave some equivalent conditions that a quotient hyper MV-algebra becomes a Boolean algebra. The above are our motivation of studying states on hyper equality algebras.

This paper is organized as follows: in Section 2, we review some basic concepts and results on hyper equality algebras. In Section 3, we introduce Bosbach states and Riečan states on hyper equality algebras and investigate some related properties of them. Also, we give a characterization of Bosbach states and discuss the relationships between Bosbach states and Riečan states. In Section 4, we study states on quotient hyper equality algebras and induce Bosbach states on quotient hyper equality algebras by use of θ-invariant Bosbach states. Moreover we construct a corresponding example.

2. Preliminaries

In [2] hyper equality algebras have been introduced and studied. We recollect some definitions and results from [2] which will be used in the following.

Definition 2.1 ([2]). *A hyper equality algebra $(H;\sim,\wedge,1)$ is a nonempty set H endowed with a binary operation $\wedge$, a binary hyper operation $\sim$ and a top element 1 satisfying the following axioms, for all $x,y,z\in H$:*

$(HE1)$ $(E,\wedge,1)$ *is a meet-semilattice with top element* 1;

$(HE2)$ $x\sim y\ll y\sim x$;

$(HE3)$ $1\in x\sim x$;

$(HE4)$ $x\in 1\sim x$;

$(HE5)$ $x\ll y\ll z$ *implies* $x\sim z\ll y\sim z$ *and* $x\sim z\ll x\sim y$;

$(HE6)$ $x\sim y\ll (x\wedge z)\sim (y\wedge z)$;

$(HE7)$ $x\sim y\ll (x\sim z)\sim (y\sim z)$,

where $x\ll y$ iff $x\wedge y=x$ and $A\ll B$ means that for any $x\in A$, there exists $y\in B$ such that $x\ll y$.

Clearly, an equality algebra can be seen as a hyper equality algebra. In any hyper equality algebra $(H,\sim,\wedge,1)$, define the operation $\to$ by $x\to y:=x\sim (x\wedge y)$ for any $x,y\in H$. Moreover for any nonempty subsets $A,B\subseteq H$, we write $A\wedge B=\{a\wedge b:a\in A,b\in B\}$, $A\circ B=\bigcup_{a\in A,b\in B}a\circ b$, where $\circ\in\{\sim,\to\}$.

Example 2.2 ([2]). Let $H=\{0,a,1\}$ with $0\ll a\ll 1$. Define the operations $\wedge$ and $\sim_1,\sim_2$ on H as follows: $x\wedge y=\min\{x,y\}$ and

$\sim_1$	0	a	1
0	$\{1\}$	$\{0\}$	$\{0\}$
a	$\{0\}$	$\{1\}$	$\{a,1\}$
1	$\{0\}$	$\{a,1\}$	$\{1\}$

$\sim_2$	0	a	1
0	$\{1\}$	$\{0,a\}$	$\{0,a\}$
a	$\{0,a\}$	$\{1\}$	$\{a\}$
1	$\{0,a\}$	$\{0,a\}$	$\{1\}$

Then $(H,\sim_1,\wedge,1)$ and $(H,\sim_2,\wedge,1)$ are hyper equality algebras.

Proposition 2.1 ([2]). *Let $(H,\sim,\wedge,1)$ be a hyper equality algebra. Then for all $x,y,z\in H,A,B,C\subseteq H$, we have:*

$(P1)$ $A\sim B\ll B\sim A$;

$(P2)$ $A\ll B$ *and* $B\ll C$ *imply* $A\ll C$;

$(P3)$ $A\subseteq B$ *implies* $A\ll B$;

$(P4)$ $x\ll y$ *implies* $y\to x=y\sim x$;

$(P5)$ $1\in x\to x$, $1\in x\to 1$, $x\ll x\sim 1$, $x\in 1\to x$;

$(P6)$ $x\sim y\ll x\to y$, $x\sim y\ll y\to x$;

$(P7)$ $x \ll y$ *implies* $1 \in x \to y$;

$(P8)$ $x \ll y \to x$, $A \ll B \to A$;

$(P9)$ $x \ll y$ *implies* $z \to x \ll z \to y$ *and* $y \to z \ll x \to z$;

$(P10)$ $x \to y = x \to x \wedge y$;

$(P11)$ $x \to y \ll (y \to z) \to (x \to z)$.

Definition 2.3 ([2]). *A hyper equality algebra* $(H, \sim, \wedge, 1)$ *is called:*

- *good if* $x = 1 \sim x$ *for any* $x \in H$;
- *bounded if there exists an element* 0 *such that* $0 \ll x$ *for any* $x \in H$.

Proposition 2.2 ([2]). *Let* $(H, \sim, \wedge, 1)$ *be a good hyper equality algebra. Then for any* $x, y \in H$:

(1) $x \ll (x \sim y) \sim y$;

(2) $x \ll (x \to y) \to y$;

(3) $x \ll y \to z$ *iff* $y \ll x \to z$;

(4) $x \to (y \to z) \ll y \to (x \to z)$.

In any bounded hyper equality algebra $(H, \sim, \wedge, 1)$, we denote $x^* := x \sim 0$ for any $x \in H$. Then $x^* = x \to 0$.

Proposition 2.3 ([2]). *Let* $(H, \sim, \wedge, 0, 1)$ *be a bounded hyper equality algebra. Then for all* $x, y, z \in H$:

(1) $0 \in 1^*$ *and* $1 \in 1^{**}$;

(2) $1 \in 0^*$ *and* $0 \in 0^{**}$;

(3) $x \sim y \ll x^* \sim y^*$;

(4) $x \to y \ll y^* \to x^*$;

(5) $x \ll y$ *implies* $y^* \ll x^*$.

Definition 2.4 ([2]). *Let* $(H, \sim, \wedge, 1)$ *be a hyper equality algebra. A subset* D *containing* 1 *of* H *is called a strong hyper deductive system, if* $x \in D$ *and* $x \to y \cap D \neq \emptyset$ *imply* $y \in D$ *for all* $x, y \in H$.

Notice that in any hyper equality algebra, each strong hyper deductive system is an upset.

Lemma 2.1 ([2]). *Let* D *be an upset of a hyper equality algebra* $(H, \sim, \wedge, 1)$. *Then for any nonempty subset* A, B *of* H, $A \cap D \neq \emptyset$ *and* $A \ll B$ *imply* $B \cap D \neq \emptyset$.

Definition 2.5 ([2]). *Let $(H, \sim, \wedge, 1)$ be a hyper equality algebra and θ be an equivalence relation on H.*

(1) *for any $A, B \subseteq H$, $A\overline{\theta}B$ means for any $a \in A$ there exists $b \in B$ such that $a\theta b$ and for any $b \in B$ there exists $a \in A$ such that $a\theta b$;*

(2) *θ is called a congruence relation if for any $x, y, u, v \in H$, $x\theta y$ and $u\theta v$ imply $x \sim u\overline{\theta}y \sim v$ and $x \wedge u\theta y \wedge v$.*

Let θ be a congruence relation on a hyper equality algebra $(H, \sim, \wedge, 1)$. Denote $H/\theta = \{[x]_\theta : x \in H\}$, where $[x]_\theta = \{y \in H : y\theta x\}$ and denote $[A]_\theta = \{[a]_\theta : a \in A\}$. $\overline{\sim}$, $\overline{\Rightarrow}$ and $\overline{\wedge}$ on H/θ are defined by $[x]_\theta\overline{\sim}[y]_\theta = [x \sim y] = \{[a]_\theta : a \in x \sim y\}$, $[x]_\theta\overline{\Rightarrow}[y]_\theta = [x \to y]_\theta = \{[a]_\theta : a \in x \to y\}$ and $[x]_\theta\overline{\wedge}[y]_\theta = [x \wedge y]_\theta$, respectively. $[x]_\theta \ll_\theta [y]_\theta$ iff $[x]_\theta = [x]_\theta\overline{\wedge}[y]_\theta$ iff $x \wedge y\theta x$ for any $[x]_\theta, [y]_\theta \in H/\theta$. For all $A, B \subseteq H/\theta$, $A \ll_\theta B$ means for any $[a]_\theta \in A$, there exists $[b]_\theta \in B$ such that $[a]_\theta \ll_\theta [b]_\theta$. Clearly, $x \ll y$ implies that $[x]_\theta \ll_\theta [y]_\theta$.

Definition 2.6 ([2]). *A congruence relation θ on a hyper equality algebra $(H, \sim, \wedge, 1)$ is called regular if $[x]_\theta \ll_\theta [y]_\theta$ implies $[x]_\theta \ll [y]_\theta$ for all $x, y \in H$.*

Theorem 2.1 ([2]). *Let θ be a regular congruence relation on a hyper equality algebra $(H, \sim, \wedge, 1)$. Then:*

(1) *$[x]_\theta \ll_\theta [y]_\theta$ implies that $[1]_\theta \in [x]_\theta\overline{\Rightarrow}[y]_\theta$ for any $x, y \in H$;*

(2) *$(H/\theta, \overline{\sim}, \overline{\wedge}, [1]_\theta)$ is a hyper equality algebra, which is called as the quotient hyper equality algebra with respect to θ.*

3. States on hyper equality algebras

In this section, we introduce Bosbach states and Riečan states on hyper equality algebras and investigate some related properties of them. Also, we give a characterization of Bosbach states, and discuss the relationships of Bosbach states and Riečan states. By H denote a bounded hyper equality algebra $(H, \sim, \wedge, 0, 1)$, unless otherwise specified.

Definition 3.1. *A function $s : H \to [0, 1]$ is said to be a Bosbach state on H, if it satisfies the following conditions:*

$(BS1)$ *$s(0) = 0$ and $s(1) = 1$;*

$(BS2)$ *$s(x) + s(x \to y) = s(y) + s(y \to x)$ for all $x, y \in H$,*

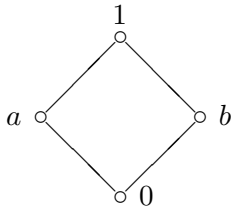
where $s(A) = \sup\{s(a) : a \in A\}$ for all $A \subseteq H$.

Example 3.2. (1) Let $(H, \sim_1, \wedge, 0, 1)$ be a bounded hyper equality algebra in Example 2.2. Define a function $s : H \to [0, 1]$ by $s(0) = 0, s(a) = s(1) = 1$. Then one can calculate that s is a unique Bosbach state on H.

(2) Let $(H; \sim_2, \wedge, 1)$ be a bounded hyper equality algebra in Example 2.2. Define a function $s : H \to [0, 1]$ by $s(0) = 0, s(a) = 0.5, s(1) = 1$. Then one can check that s is a unique Bosbach state on H.

The following example indicates that there exist hyper equality algebras not possessing Bosbach states.

Example 3.3. Let $H = \{0, a, b, 1\}$ in which the Hasse diagram and the operations $\wedge, \sim$ are as follows:



$\wedge$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\sim$	0	a	b	1
0	$\{1\}$	$\{1\}$	$\{b, 1\}$	$\{0, a\}$
a	$\{1\}$	$\{1\}$	$\{a, 1\}$	$\{a\}$
b	$\{b, 1\}$	$\{a, 1\}$	$\{1\}$	$\{b, 1\}$
1	$\{0, a\}$	$\{a\}$	$\{b, 1\}$	$\{1\}$

Then $(H, \sim, \wedge, 0, 1)$ is a bounded hyper equality algebra [2]. Assume that the function $s : H \to [0, 1]$ defined by $s(0) = 0, s(a) = \alpha, s(b) = \beta, s(1) = 1$ is a Bosbach state on H. In (BS2), taking $x = 1, y = b$ and $x = 0, y = b$, we obtain $\beta = 1$ and $\beta = 0$, respectively, which is a contraction. This shows that H admits no Bosbach state.

Proposition 3.1. *Let s be a Bosbach state on H. Then for any $x, y \in H$, the following hold:*

(1) *If $x \ll y$, then $s(x) \leq s(y)$;*

(2) *If $x \ll y$, then $s(y \to x) = 1 + s(x) - s(y) = s(y \sim x) = s(x \sim y)$;*

(3) *$s(x^*) = 1 - s(x)$;*

(4) *$s(A^*) = 1 - s_*(A)$, where $s_*(A) = \inf\{s(a) : a \in A\}$;*

(5) *$s(x \to y) = 1 + s(x \wedge y) - s(x)$;*

(6) *$Ker(s) = \{x \in H : s(x) = 1\}$ is a strong hyper deductive system of H.*

Proof. (1) Let $x \ll y$. Then $1 \in x \to y$ and so $s(x \to y) = s(1) = 1$. Hence by the definition of the Bosbach state, $s(x) + 1 = s(y) + s(y \to x)$. This implies that $s(x) - s(y) = s(y \to x) - 1 \leq 0$ and thus $s(x) \leq s(y)$.

(2) Let $x \ll y$. Then $y \to x = y \sim x$. Using the proof of (1), we have $s(y \sim x) = s(y \to x) = 1 + s(x) - s(y)$.

(3) Since $0 \ll x$, we have $1 \in 0 \to x$ and thus $s(0 \to x) = 1$. Considering $x^* = x \to 0$ and $s(x) + s(x \to 0) = s(0) + s(0 \to x)$, we can get that $s(x^*) = s(0 \to x) - s(x) = s(1) - s(x) = 1 - s(x)$.

(4) Using (3), we get that $s(A^*) = \sup\{s(a^*) : a \in A\} = \sup\{1 - s(a) : a \in A\} = 1 - \inf\{s(a) : a \in A\} = 1 - s_*(A)$.

(5) Since $x \wedge y \ll x$, then from (2) we obtain that $s(x \to y) = s(x \to x \wedge y) = 1 + s(x \wedge y) - s(x)$.

(6) Let $x \to y \cap Ker(s) \neq \emptyset$ and $x \in Ker(s)$ for any $x, y \in H$. Then $s(x) = 1$ and there exists $a \in x \to y$ such that $a \in Ker(s)$. Hence $s(a) = 1$, which implies $s(x \to y) = \sup\{s(t) : t \in x \to y\} = 1$. Since s is a Bosbach state on H, we have $s(y) = s(x) + s(x \to y) - s(y \to x) = 2 - s(y \to x) \geq 1$. Therefore $s(y) = 1$ and thus $y \in Ker(s)$. □

In what follows, we give a characterization of Bosbach states on hyper equality algebras.

Theorem 3.1. *Let $s : H \to [0, 1]$ be a function with $s(0) = 0$. Then the following are equivalent:*

(1) *s is a Bosbach state;*

(2) *If $x \ll y$, then $s(y \sim x) = 1 + s(x) - s(y) = s(y \to x)$;*

(3) *$s(y \to x) = 1 - s(y) + s(x \wedge y)$.*

Proof. (1) ⇒ (2) By Proposition 3.1.

(2) ⇒ (3) Assume (2) holds. Since $x \wedge y \ll y$, then $s(y \to x) = s(y \to x \wedge y) = 1 + s(x \wedge y) - s(y)$.

(3) ⇒ (1) Assume (3) holds. Since $1 \in x \to 1$, then $s(1) = s(x \to 1) = 1 - s(x) + s(x \wedge 1) = 1 - s(x) + s(x) = 1$. Also, $s(x) + s(x \to y) = s(x) + 1 - s(x) + s(x \wedge y) = 1 + s(x \wedge y) = s(y) + 1 - s(y) + s(x \wedge y) = s(y) + s(y \to x)$. Thus s is a Bosbach state on H. □

Next we introduce the notion of Riečan states on hyper equality algebras.

Definition 3.4. *Two elements $x, y \in H$ is said to be orthogonal, if $1 \in x^{**} \to y^*$, we write by $x \perp y$. If $x, y \in H$ is orthogonal, we define a hyper operation $\oplus$ on H by $x \oplus y := y^* \to x^{**}$.*

Proposition 3.2. *In any bounded hyper equality algebra H, the following hold for all $x, y \in H$:*

(1) *$0 \perp x$ and $x^{**} \subseteq 0 \oplus x$;*

(2) *$x \perp 0$ and $x^{**} \subseteq x \oplus 0$.*

Proof. (1) Since $1 \in 0 \to x^* \subseteq 0^{**} \to x^*$, then $0 \perp x$. From Proposition 2.3 $0 \in 0^{**}$, it follows that $x^{**} = x^* \to 0 \subseteq x^* \to 0^{**} = 0 \oplus x$.

(2) Since $1 \in 0^*$ by Proposition 2.3, we have $1 \in x^{**} \to 1 \subseteq x^{**} \to 0^*$. Hence $x \perp 0$. $x^{**} \subseteq x \oplus 0$ can be obtained by $x^{**} \subseteq 1 \to x^{**} \subseteq 0^* \to x^{**}$. □

Definition 3.5. *A Riečan state on a bounded hyper equality algebra H is a function $s : H \to [0,1]$ such that:*

$(RS1)$ $s(1) = 1$;

$(RS2)$ $s(x \oplus y) = s(x) + s(y)$ *whenever* $x \perp y$ *for all* $x, y \in H$, *where* $s(A) = \sup\{s(a) : a \in A\}$ *for any* $A \subseteq H$. *In particular, if* $s(x^*) = 1 - s(x)$ *for all* $x \in H$, *the Riečan state* s *is said to be normal.*

Example 3.6. Let $H = \{0, a, 1\}$. Define the operations $\wedge$ and $\sim$ on H as follows: $x \wedge y = \min\{x, y\}$ and

$\sim$	0	a	1
0	$\{1\}$	$\{0, a\}$	$\{0\}$
a	$\{0, a\}$	$\{1\}$	$\{a\}$
1	$\{0\}$	$\{a\}$	$\{1\}$

Then $(H, \sim, \wedge, 0, 1)$ is a bounded hyper equality algebra [2]. Now we define a function $s : H \to [0,1]$ by $s(0) = 0, s(a) = s(1) = 1$. One can check that s is a Riečan state, but it is not normal since $s(a^*) = s(\{0, a\}) = s(a) = 1 \neq 0 = 1 - s(a)$.

Example 3.7. Let $(H, \sim_1, \wedge, 0, 1)$ be a bounded hyper equality algebra given by Example 2.2, and the function $s : H \to [0,1]$ be defined by $s(0) = 0, s(a) = s(1) = 1$. Then one can calculate that s is a unique normal Riečan state on H.

The following example shows that there exist hyper equality algebras having no (normal) Riečan states.

Example 3.8. Let $(H, \sim, \wedge, 0, 1)$ be a bounded hyper equality algebra given by Example 3.3, and the function $s : H \to [0,1]$ defined by $s(0) = \alpha, s(a) = \beta, s(b) = \gamma, s(1) = 1$ be a Riečan state on H. In (RS2), taking $x = 0, y = a$ and $x = b, y = 0$, we can obtain that $s(a) = s(0 \oplus a) = s(0) + s(a)$ and $1 = s(b \oplus 0) = s(b) + s(0)$, respectively. This implies $\beta = 0$ and $\beta = 1$. On the other hand, taking $x = 1, y = b$ in (RS2), we get that $1 = s(1 \oplus b) = s(1) + s(b)$ and so $\beta = 0$, which is a contraction. Therefore H admits no (normal) Riečan state.

Proposition 3.3. *Let s be a normal Riečan state on H. Then for all $x, y \in H$ and $A \subseteq H$, we have:*

(1) $s(0) = 0$;

(2) $s(x^{**}) \leq s(x)$;

(3) $s(A^*) = 1 - s_*(A)$, *where* $s_*(A) = \inf\{s(a) : a \in A\}$;

(4) $x \ll y$ *implies* $s(x) \leq s(y)$ *and* $s(y^{**} \to x^{**}) = 1 + s(x) - s(y)$.

Proof. (1) Since $0 \in 1^*$, we have $s(\{0\}) \leq s(1^*) = 1 - s(1) = 1 - 1 = 0$.

(2) Since $x^{**} \subseteq x \oplus 0$ from Proposition 3.2, then $s(x^{**}) \leq s(x \oplus 0) = s(x) + s(0) = 0 + s(x) = s(x)$.

(3) Similar to the proof of Proposition 3.1 (4).

(4) Let $x \ll y$ for any $x, y \in H$. Then it follows from Proposition 2.3 that $x^{**} \ll y^{**}$. Hence $1 \in x^{**} \to y^{**}$ and so there is $t \in y^*$ such that $1 \in x^{**} \to t^*$. This implies $x \perp t$ and thus $s(t^* \to x^{**}) = s(x \oplus t) = s(x) + s(t)$. Then we can get that $\sup\{s(t^* \to x^{**}) : t \in y^*\} = s(x) + \sup\{s(t) : t \in y^*\}$. That is $s(y^{**} \to x^{**}) = s(x) + s(y^*) = 1 + s(x) - s(y)$. Furthermore $s(x) - s(y) = s(y^{**} \to x^{**}) - 1 \leq 0$, which implies $s(x) \leq s(y)$. □

Remark 3.9. There exist hyper equality algebras admitting Bosbach states but not possessing Riečan states. Conversely, it is also true that there exist hyper equality algebras admitting Riečan states but not possessing Bosbach states. Indeed, let $(H; \sim_2, \wedge, 1)$ be the bounded hyper equality algebras given by Example 2.2. From (2) of Example 3.2 we know that the function $s_1 : H \to [0, 1]$ defined by $s_1(0) = 0, s_1(a) = 0.5, s_1(1) = 1$ is a unique Bosbach state on H. Also one can calculate that the function $s_2 : H \to [0, 1]$ defined by $s_2(0) = 0, s_2(a) = s_2(1) = 1$ is a unique Riečan state on H.

Definition 3.10. *A hyper equality algebra $(H, \sim, \wedge, 1)$ is said to be involutive if $x^{**} = x$ for all $x \in H$.*

Example 3.11. Let $H = \{0, a, 1\}$ with $0 \ll a \ll 1$. Define the operations $\wedge$ and $\sim$ on H as follows: $x \wedge y = \min\{x, y\}$ and

$\sim$	0	a	1
0	$\{1\}$	$\{0, a\}$	$\{0\}$
a	$\{a\}$	$\{1\}$	$\{a\}$
1	$\{0\}$	$\{a\}$	$\{1\}$

One can easily check that $(H; \sim, \wedge, 1)$ is an involutive hyper equality algebra.

Lemma 3.1. *Let s be a Riečan state (Bosbach state) on H. Then $A \ll B$ implies $s(A) \leq s(B)$ for any $A, B \subseteq H$.*

Proof. Since $A \ll B$, then for any $a \in A$, there exists $b \in B$ such that $a \ll b$. Hence $\sup\{s(a) : a \in A\} \leq \sup\{s(b) : b \in B\}$, namely, $s(A) \leq s(B)$. □

Theorem 3.2. *Let s be a Riečan state on H. Then the following are equivalent:*

(1) *s is a Bosbach state on H;*

(2) *$x \ll y$ implies $s(y \to x) \geq 1 + s(x) - s(y)$ for any $x, y \in H$.*

Proof. (1) $\Rightarrow$ (2) By Theorem 3.1.

(2) $\Rightarrow$ (1) Assume (2) holds and s is a Riečan state on H. Then $s(1) = 1$ and $s(0) = 0$. Now let $x \ll y$ for any $x, y \in H$. Then by Proposition 3.3 we

have $s(y^{**} \to x^{**}) = 1 + s(x) - s(y)$. On the other hand, it follows from $x \ll y$ and Proposition 2.3 that $y \to x \ll x^* \to y^* \ll y^{**} \to x^{**}$. Hence from (P2) $y \to x \ll y^{**} \to x^{**}$. Again using Lemma 3.1, we obtain $s(y \to x) \leq s(y^{**} \to x^{**})$. Therefore $s(y \to x) \leq 1 + s(x) - s(y)$. Considering (2) and Theorem 3.1, we show that s is a Bosbach state on H. □

Theorem 3.3. *Riečan states and Bosbach states coincide on bounded involutive hyper equality algebras.*

Proof. Assume that $(H; \sim, \wedge, 1)$ is a bounded involutive hyper equality algebra. If s is a Riečan state on H and $x \ll y$ for any $x, y \in H$, then $s(y \to x) = s(y^{**} \to x^{**}) = 1 + s(x) - s(y)$ from the proof of Theorem 3.2 and the involutivity of H. It follows from Theorem 3.5 that s is a Bosbach state. Conversely, if s is a Bosbach state on H and $x \perp y$, namely, $1 \in x^{**} \to y^*$ for any $x, y \in H$. Since $s(t) + s(t \to x) = s(x) + s(x \to t)$ for any $t \in y^*$, we get that $sup\{s(t) : t \in y^*\} + sup\{s(t \to x : t \in y^*\} = s(x) + sup\{s(x \to t : t \in y^*\}$. According to the involutivity of H, $s(y^*) + s(y^* \to x^{**}) = s(x) + s(x^{**} \to y^*)$. It follows from $1 \in x^{**} \to y^*$ and $s(y^*) = 1 - s(y)$ that $s(x \oplus y) = s(y^* \to x^{**}) = s(x) + s(y)$. Therefore s is a Riečan state on H. □

Example 3.12. In Example 3.11, define a function $s : H \to [0, 1]$ by $s(0) = 0, s(a) = 0.5, s(1) = 1$. Then one can calculate that s is both a Riečan state and a Bosbach state on H.

4. Bosbach states on quotient hyper equality algebras

In this section, we investigate Bosbach states on quotient hyper equality algebras. By use of θ-invariant Bosbach states, we get the result that the induced function $\overline{s}$ by a a Bosbach state s on H is also a Bosbach state on H/θ and furthermore we construct a corresponding example of the above result.

Definition 4.1. *Let θ be a congruence relation and s be a Bosbach state on H. Then s is called θ-invariant provided that $x\theta y$ implies $s(x) = s(y)$ for all $x, y \in H$.*

Lemma 4.1. *Let θ be a regular congruence relation and s be a θ-invariant Bosbach state on H. Then in the bounded quotient hyper equality algebra $(H/\theta; \overline{\wedge}, \overline{\sim}, [0]_\theta, [1]_\theta)$, the following hold for any $x, y \in H$:*

(1) $[x]_\theta = [y]_\theta$ *implies* $s(x) = s(y)$;

(2) $[x]_\theta \leq_\theta [y]_\theta$ *implies* $s(x \to y) = 1, s(y \to x) = 1$ *and* $s(x) = s(x \wedge y)$.

Proof. (1) It is evident by Definition 4.1.

(2) Let $[x]_\theta \ll_\theta [y]_\theta$ for any $x, y \in H$. Then $[1]_\theta \in [x]_\theta \overline{\Rightarrow} [y]_\theta$ from Proposition 2.1. Hence there exists $a \in x \to y$ such that $a\theta 1$. Since s is a θ-invariant Bosbach state, then $s(a) = s(1) = 1$ and so $s(x \to y) = sup\{s(t) : t \in x \to y\} = 1$. $s(x) = s(x \wedge y)$ follows from $x\theta x \wedge y$. □

Theorem 4.1. *Let θ be a regular congruence relation and s be a θ-invariant Bosbach state on H. Define $\overline{s} : H/\theta \to [0,1]$ by $\overline{s}([x]_\theta) = s(x)$, where $\overline{s}(\{[x]_\theta : x \in A\}) = \sup\{\overline{s}([x]_\theta) : x \in A\}$, for any $x \in H, A \subseteq H$. Then $\overline{s}$ is a Bosbach state on the bounded quotient hyper equality algebra $(H/\theta; \overline{\wedge}, \overline{\sim}, [0]_\theta, [1]_\theta)$.*

Proof. By Lemma 4.1 $\overline{s}$ is well-defined. $\overline{s}([0]_\theta) = s(0) = 0$ and $\overline{s}([1]_\theta) = s(1) = 0$. Since $\overline{s}([x]_\theta \overline{\Rightarrow} [y]_\theta) = \sup\{\overline{s}([t]_\theta) : t \in x \to y\} = \sup\{s(t) : t \in x \to y\} = s(x \to y)$. Hence by (BS2) we have $\overline{s}([x]_\theta) + \overline{s}([x]_\theta \overline{\Rightarrow} [y]_\theta) = \overline{s}([y]_\theta) + \overline{s}([y]_\theta \overline{\Rightarrow} [x]_\theta)$. That is, $\overline{s}$ is a Bosbach state on H/θ. □

In what follows, we construct an example of Theorem 4.1.

Definition 4.2. *A nonempty subset S of H is called $\sim$-reflexive provided that $x \sim y \cap S \neq \emptyset$ implies $x \sim y \subseteq S$.*

Let s be a Bosbach state on H and $Ker(s)$ be $\sim$-reflexive, we define the relation θ_K as follows: $x\theta_K y$ iff $x \sim y \cap Ker(\mu) \neq \emptyset$ for any $x, y \in H$.

Proposition 4.1. *Let s be a Bosbach state on H and $Ker(s)$ be $\sim$-reflexive. Then:*

(1) *θ_K is an equivalence relation on H;*

(2) *θ_K preserves the operation $\wedge$.*

Proof. (1) Let $x, y, z \in H$. Since $1 \in x \sim x$ we have $x \sim x \cap Ker(s) \neq \emptyset$ and thus θ_K is reflexive. The symmetry of θ_K follows from (HE2) and Lemma 2.1. Now we prove the transitivity of θ_K. If $x\theta_K y$ and $y\theta_K z$, then $x \sim y \cap Ker(\mu) \neq \emptyset$ and $y \sim x \cap Ker(s) \neq \emptyset$. From (HE7),(HE2) and (P6) $x \sim y \ll (x \sim z) \sim (y \sim z) \ll (y \sim z) \sim (x \sim z) \ll (y \sim z) \to (x \sim z)$. It follow from (P2) that $x \sim y \ll (y \sim z) \to (x \sim z)$. Using Lemma 2.1 and $x \sim y \cap Ker(\mu) \neq \emptyset$ we have $(y \sim z) \sim (x \sim z) \cap Ker(s) \neq \emptyset$. Hence there exist $a \in x \sim y$ and $b \in y \sim z \subseteq Ker(s)$ such that $a \to b \cap Ker(s) \neq \emptyset$, which implies $b \in Ker(s)$. This shows that $x \sim z \cap Ker(s) \neq \emptyset$, that is, $x\theta_K z$.

(2) Suppose $x\theta_K y$ and $u\theta_K v$ for any $x, y, u, v \in H$. Then $x \sim y \cap Ker(s) \neq \emptyset$ and $u \sim v \cap Ker(s) \neq \emptyset$. By (HE6) we have $x \sim y \ll (x \wedge v) \sim (y \wedge v)$ and $u \sim v \ll (x \wedge u) \sim (x \wedge v)$. Hence $(x \wedge v) \sim (y \wedge v) \cap Ker(s) \neq \emptyset$ and $(x \wedge u) \sim (x \wedge v) \cap Ker(s) \neq \emptyset$. That is, $x \wedge v \theta_K y \wedge v$ and $x \wedge u \theta_K x \wedge v$, which implies $x \wedge u \theta_K y \wedge v$. □

We shall denote the equivalence class of $x \in H$ w.r.t. θ_K by $[x]_K$ and the quotient set associated with θ_K by H/K.

Example 4.3. Assume that $(H, \sim_1, \wedge, 0, 1)$ is a bounded hyper equality algebra given in Example 2.2. One can check that $\theta_K = \{(0,0), (a,a), (1,1), (1,a), (a,1)\}$ is a regular congruence relation and the function s defined by Example 3.2 is a θ_K-invariant Bosbach state on H. The quotient hyper equality algebra $H/K = \{[0]_K, [1]_K\}$, where $[0]_K = \{0\}, [a]_K = [1]_K = \{a, 1\}$ and $\overline{\sim}$ is as follows:

$\overline{\sim_1}$	$[0]_K$	$[1]_K$
$[0]_K$	$\{[1]_K\}$	$\{[0]_K\}$
$[1]_K$	$\{[0]_K\}$	$\{[1]_K\}$

One can easily calculate that $\overline{s} : H/\theta \to [0,1]$ defined by $\overline{s}([0]_K) = s(0) = 0, \overline{s}([1]_K) = s(1) = 1$ is a Bosbach state on H/K.

By Theorem 4.1, we can obtain the following theorem.

Theorem 4.2. *Assume that θ is a regular congruence relation, and s is a θ-invariant Bosbach state on H. Then there is a unique Bosbach state $\overline{s} : H/\theta \to [0,1]$ such that $s = \overline{s} \circ \pi_\theta$, where $\pi_\theta : H \to H/\theta$ defined by $\pi_\theta(x) = [x]_\theta$ is the canonical map.*

Proof. Let $[t]_K \in H/\theta$ for any $t \in H$. Then $\pi_\theta(t) = [t]_\theta$. Setting $s(t) = \overline{s}([t]_\theta)$, where $\overline{s}(\{[t]_\theta : t \in A\}) = \sup\{\overline{s}([t]_\theta) : t \in A\}$ for any $x \in H, A \subseteq H$, the function $\overline{s}$ is a Bosbach state from Theorem 4.1 and further $s(t) = \overline{s}([t]_\theta) = \overline{s}(\pi_\theta(t)) = \overline{s} \circ \pi_\theta(t)$. That is $s = \overline{s} \circ \pi_\theta$. Now we show that $\overline{s} \circ \pi_\theta(t)$ is also a Bosbach state on H. $\overline{s} \circ \pi_\theta(0) = \overline{s}(\pi_\theta(0)) = \overline{s}([0]_\theta) = s(0) = 0$ and similarly $\overline{s} \circ \pi_\theta(1) = s(1) = 1$. $\overline{s} \circ \pi_\theta(x) + \overline{s} \circ \pi_\theta(x \to y) = \overline{s}([x]_\theta) + \overline{s}([x \to y]_\theta) = \overline{s}([x]_\theta) + \overline{s}([x]_\theta \overrightarrow{\to} [y]_\theta) = \overline{s}([y]_\theta) + \overline{s}([y]_\theta \overrightarrow{\to} [x]_\theta) = \overline{s}([y]_\theta) + \overline{s}([y \to x]_\theta) = \overline{s} \circ \pi_\theta(y) + \overline{s} \circ \pi_K(y \to x)$. Finally, we prove that $\overline{s}$ is unique. Suppose there is another Bosbach state ϑ on H/θ such that $s = \vartheta \circ \pi_\theta$. Then for any $[a]_\theta \in H/\theta$, $\vartheta([a]_\theta) = \vartheta(\pi_\theta(a)) = \vartheta \circ \pi_\theta(a) = s(a) = \overline{s}([a]_\theta)$, which implies $\vartheta = \overline{s}$. Combining the above arguments, this proof is finished. □

5. Conclusions

States play an important role in studying fuzzy logics and the related algebraic structures. Recently, states on hyper BCK-algebras and hyper MV-algebras have been studied. In this paper, we introduce states including Bosbach states and Riečan states on hyper equality algebras and obtain some important results. In future work, we shall further study state theory, especially on quotient hyper equality algebras.

Acknowledgments

This research is supported by a grant of National Natural Science Foundation of China (11971384) and PhD Research Start-up Foundation of Xi'an Aeronautical University.

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Accepted: January 07, 2020

Granular reduction based matrix in crisp-fuzzy variable threshold concept lattices

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Abstract. Attribute reduction is a crucial issue in fuzzy formal concept analysis (FFCA for short). Essentially, the volume of the large database can be decreased by the representation of knowledge as far as possible. With respect to reduction, the discernibility matrix and the discernibility function have played a meaningful affect. Nevertheless, the common approach needs to consider the construction of fuzzy concept lattice before searching reducts, and this invariably leads to the large amount of computation during constructing lattice. Considering adequately this problem, granular concepts are taken into account in this paper. First, the intensions of object set, the extensions of attribute set, and object granular matrix based on matrix operations are investigated in detail. Subsequently, the properties of fuzzy sub-contexts and join-irreducible elements derived from a given crisp-fuzzy variable threshold concept lattice are systematically determined, respectively. Furthermore, the notations of granular consistent set, granular reduct, and discernibility attribute matrix, with a given threshold level, are proposed. In the process of reduction, the construction of a variable threshold concept lattice is averted. Hence, the computational complexity is largely reduced, and our discussion is employed by an algorithm of granular reduction on the basis of matrix theory.

Keywords: discernibility attribute matrix, formal fuzzy context, granular reduction, variable threshold.

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1. Introduction

Wille [1, 2] pioneered an order-theoretic mathematical model to show the formal concept analysis (FCA) induced by a context $F = (U, A, I)$. This context F contains an object set U, an attribute set A, and a binary relation $I \subseteq U \times A$ derived from objects and attributes which should be in principle necessary. Focusing on FCA, the essential notion of its is a concept hierarchy denoted as a concept lattice, where a formal concept called a pair (X, B) is composed of extent X (object set) and intent B (attribute set) holden Galois connection. In recent years, it already has a widespread and successful application in the domains of knowledge acquisition, data mining and other orientations [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15].

Nevertheless, in real-word applications, the fuzzy binary relation is more comomn than its crisp counterpart. Therefore, a great many of scholars have applied the theory of classical formal contexts to the formal fuzzy contexts based on fuzzy sets [16]. For instance, R. Bêlohlávek [17, 18, 19] formulated some theories and properties without changing the structure of formal fuzzy contexts from the perspectives of fuzzy logic. After that, Burusco et al. [20, 21] firstly illustrated L-fuzzy concept lattice, and made a comparison of the fuzziness in a given L-fuzzy formal context by utilizing the fuzzy implications of Łukasiewicz, Kleene [22], and Gödel [23]. And concluding that the fuzzy implication of Kleene is fuzzier than the Łukasiewicz and the Gödel applications. Formica [24] proposed an approach to evaluate concept similarity in fuzzy formal contexts. And further, it should be noted that defining the fuzzy operations to obtain fuzzy concepts has been highly successful from different angles. Krajci [25], Yahia and Jaoua [26] all-sidedly investigated an "one-sided fuzzy concept", made identical to a variable threshold concept lattice with a determined threshold $\delta = 1$. Correspondingly, from the granular computing, with a given formal fuzzy context based on the notion of "crisp-fuzzy fuzzy concept", Shao et al. [27] and Shi et al. [28], respectively, demonstrated attribute and object granular reduct approaches in the sense of removing the non-essential attributes and objects while ensuring the lattice structure unchanged. And the proposed method improved greatly the computational efficiency and saved available storage space. Afterwards, Li et al. [30] further studied the judgment theorems of attribute characteristics, i.e. indispensable attributes, relative necessary attributes and dispensable attributes in fuzzy formal contexts for "fuzzy-crisp fuzzy concept". Then the algorithms which are used to compute three types of attributes were proposed. Subsequently, based on formal fuzzy contexts proposed by Shao et al. [27], Mao et al. [31] discussed a new framework to determine all irreducible elements and classify the attribute characteristics applying with the directed graph.

As is well known, Zhang et al. [32] developed three kinds of "variable threshold concept lattices", that is, "crisp-crisp", "crisp-fuzzy", and "fuzzy-crisp" variable threshold concept lattice. If the threshold is $\delta = 1$, the above lat-

tices become "one-sided fuzzy concept lattices". In Zhang et al. [32] notations, the properties are similar to that of a classical concept lattice, and can also be derived from a formal fuzzy context. J Konecny [33] studied a new method to adjust the clarification and reduction approach as an extension of variable threshold concept based on a simple scaling. In addition, Shao et al. [34] generalized the approach of attribute reduction in classical formal contexts proposed by Zhang et al. [35] to variable threshold concept lattices based on the literature [32]. And he proposed knowledge reduction frameworks to construct discernibility matrices which are removed the non-essential attributes and objects without loss of the granular structures. Furthermore, the essential approach of enumerating all the attribute reducts based on discernibility matrix introduced in [35] was applied to namely the three-way concept lattices [37] and the generalized one-sided formal contexts [38].

Compared to "one-sided fuzzy concept" in formal fuzzy contexts, knowledge reduction has always been a meaningful issue. However, little work about computing attribute reduction from the perspectives of granular computing within variable threshold concept lattices parallel to that developed by Shao et al. [27]. In a formal fuzzy context, an object concept is a join of some other object concepts. More importantly, Shao et al. [27] presented a method of granular reduction in the sense of maintaining the extensions of all object concepts. Afterwards, Lin et al. [40] introduced two based-matrix heuristic algorithms of granular reduction by the similar degree about attributes. It should be noticed that constructing the corresponding lattice in variable threshold concept lattices is more complicated. Therefore, the reductions based on the granular computing are more interesting. In this article, based on the matrix theory, a novel framework to calculate reducts is necessarily formulated based on the object granular matrix. Specially, employing the proposed method can be weaken to "one-sided fuzzy concept" when the threshold $\delta = 1$.

The rest of this paper is arranged as follows. Section 2 reviews the notations of variable threshold concept lattices and matrix operations. In the next section, based on the matrix theory, the intensions of object set, extensions of attribute set, and the object granular matrix are developed, respectively. Section 4 and 5 discussed the properties of join-irreducible elements, the discernibility attribute matrix, and the corresponding algorithm for reducing the size of a formal fuzzy context. In the final section, we draw a conclusion with a summary.

2. Preliminaries

In this section, we first briefly review some notions and terminologies of the variable threshold concept lattices and fuzzy logic, and then give some results of matrix operations. Specific information can be found in [32, 34].

2.1 Variable threshold concept lattices and crisp-fuzzy variable threshold concept

Let U be a nonempty and finite set called the universe of discourse and L be a complete lattice. A mapping $\tilde{X} : U \to L$ is described as a L-set of U, and $\tilde{X}(x)$ is denoted as the truth degree of the fact "x belongs to $\tilde{X}$" for all $x \in U$. The set of all L-sets in U is denoted as L^U. For any $\tilde{X}_1$, $\tilde{X}_2 \in L^U$, $\tilde{X}_1 \subseteq \tilde{X}_2$ is given by the expression:

$$\tilde{X}_1 \subseteq \tilde{X}_2 \Leftrightarrow \tilde{X}_1(x) \leq \tilde{X}_2(x), \forall x \in U;$$

and operators $\vee$ and $\wedge$ on L^U are denoted by:

$$(\tilde{X}_1 \vee \tilde{X}_2)(x) = \tilde{X}_1(x) \vee \tilde{X}_2(x);\ (\tilde{X}_1 \wedge \tilde{X}_2)(x) = \tilde{X}_1(x) \wedge \tilde{X}_2(x).$$

Definition 2.1 ([17, 18]). *A structure $\boldsymbol{L} = (L, \wedge, \vee, \otimes, \to, 0, 1)$ is deemed as a complete residuated lattice if*

1. *$(L, \vee, \wedge, 0, 1)$ is a complete lattice with the least element 0 and the greatest element 1;*
2. *$(L, \otimes, 1)$ is a commutative monoid, i.e. $\otimes$ is commutative, associative, and $x \otimes 1 = 1 \otimes x = x$ for any $x \in L$;*
3. *$(\otimes, \to)$ forms a residuated pair in L, i.e. $x \leq y \to c$ if and only if $x \otimes y \leq z$ for all $x, y, z \in L$.*

There are some identities holden in any complete residuated lattice:

$$(a \otimes b) \to c = a \to (b \to c) = b \to (a \to c); \tag{1}$$

$$b \to \wedge_{i \in I} a_i = \wedge_{i \in I}(b \to a_i), \vee_{i \in I} b_i \to a = \wedge_{i \in I}(b_i \to a). \tag{2}$$

According to the above definition, $\vee$ and $\wedge$ are supremum and infimum, $\otimes$ and $\to$ are "fuzzy conjunction" and "fuzzy implication", respectively. A common choice of $\mathbf{L}$ which usually used in fuzzy logic applications is complete residuated lattice with the unit interval $L = [0, 1]$, $\wedge = \min$, $\vee = \max$. And here are three significant pairs of adjoint operators: Łukasiewicz one ($a \otimes b = max(a + b - 1, 0)$, $a \to b = min(1 - a + b, 1)$), Gödel one ($a \otimes b = min(a, b)$, $a \to b = 1$ if $a \leq b$ and $= b$ else), Goguen (product) one ($a \otimes b = a \cdot b$, $a \to b = 1$ if $a \leq b$ *and* $= \frac{b}{a}$ else).

Hereinafter, for convenience of description, let $\otimes$ and $\to$ be the Gödel adjoint operator. A structure $\mathbf{L} = (L, \wedge, \vee, \otimes, \to, 0, 1)$ is a complete residuated lattice, and a formal fuzzy context is a triple $(U, A, \tilde{I})$, where U and A are, respectively, non-empty finite set of objects and attributes, and $\tilde{I} : U \times A \to L$ is a L-fuzzy relation between U and A in [41].

Definition 2.2 ([32]). *Let $(U, A, \tilde{I})$ be a formal fuzzy context with $X \in \mathcal{P}(U)$, $\tilde{B} \in L^A$ and $0 < \delta \leq 1$. Two operators $\triangledown_\delta : \mathcal{P}(U) \to L^A$ and $\triangle_\delta : L^A \to \mathcal{P}(U)$ are denoted as:*

$$X^{\triangledown_\delta}(a) = \delta \to \bigwedge_{x \in X} \tilde{I}(x, a), a \in A; \tag{3}$$

$$\tilde{B}^{\triangle_\delta} = \{x \in U : \bigwedge_{a \in A} (\tilde{B}(a) \to \tilde{I}(x, a)) \geq \delta\}. \tag{4}$$

To simplify notation, if $X = \{x\}$ for any $x \in U$, we will use $x^{\triangledown_\delta}$ instead of $\{x\}^{\triangledown_\delta}$.

By Definition 2.2, its properties are analogous to that of the classical formal context, and it is straightforward to obtain the following properties induced from the formal fuzzy context.

Property 2.1 ([32]). *Let $(U, A, \tilde{I})$ be a formal fuzzy context, $\tilde{X}, \tilde{X}_1, \tilde{X}_2 \in \mathcal{P}(U)$ and $\tilde{B}, \tilde{B}_1, \tilde{B}_2 \in L^A$. Then the following properties hold:*

1. $X_1 \subseteq X_2 \Rightarrow X_2^{\triangledown_\delta} \subseteq X_1^{\triangledown_\delta}$; $\tilde{B}_1 \subseteq \tilde{B}_2 \Rightarrow \tilde{B}_2^{\triangle_\delta} \subseteq \tilde{B}_1^{\triangle_\delta}$.
2. $X \subseteq X^{\triangledown_\delta \triangle_\delta}$; $\tilde{B} \subseteq \tilde{B}^{\triangle_\delta \triangledown_\delta}$.
3. $X^{\triangledown_\delta} = X^{\triangledown_\delta \triangle_\delta \triangledown_\delta}$; $\tilde{B} = \tilde{B}^{\triangle_\delta \triangledown_\delta \triangle_\delta}$.
4. $(X_1 \cup X_2)^{\triangledown_\delta} = X_1^{\triangledown_\delta} \cap X_2^{\triangledown_\delta}$; $(\tilde{B}_1 \cup \tilde{B}_2)^{\triangle_\delta} = \tilde{B}_1^{\triangle_\delta} \cap \tilde{B}_2^{\triangle_\delta}$.

In a formal fuzzy context $(U, A, \tilde{I})$, a pair $(X, \tilde{B}) \in \mathcal{P}(U) \times L^A$ is called a crisp-fuzzy variable threshold concept of $(U, A, \tilde{I})$ such that $X = \tilde{B}^{\triangle_\delta}$ and $\tilde{B} = X^{\triangledown_\delta}$ (see [32]), where X and $\tilde{B}$ are referred to as the extent and intent of $(X, \tilde{B})$, respectively. We denote the set of all crisp-fuzzy variable threshold concepts in $(U, A, \tilde{I})$ as $L_\delta(U, \tilde{A}, \tilde{I})$. From Property 2.1.3, we can see that both $(X^{\triangledown_\delta \triangle_\delta}, X^{\triangledown_\delta})$ and $(\tilde{B}^{\triangle_\delta}, \tilde{B}^{\triangle_\delta \triangledown_\delta})$ are crisp-fuzzy variable threshold concepts. Especially, $(x^{\triangledown_\delta \triangle_\delta}, x^{\triangledown_\delta})$ is interpreted as an object concept for any $x \in U$. For two crisp-fuzzy variable threshold concepts $(X_1, \tilde{B}_1)$ and $(X_2, \tilde{B}_2)$, $(X_1, \tilde{B}_1) \leq (X_2, \tilde{B}_2)$ iff $X_1 \subseteq X_2$ (equivalently $\tilde{B}_2 \subseteq \tilde{B}_1$). Obviously, $L_\delta(U, \tilde{A}, \tilde{I})$ forms a complete lattice, in which the join $\vee$ and the meet $\wedge$ are respectively given by:

$$(X_1, \tilde{B}_1) \vee (X_2, \tilde{B}_2) = ((X_1 \cup X_2)^{\triangledown_\delta \triangle_\delta}, \tilde{B}_1 \cap \tilde{B}_2); \tag{5}$$

$$(X_1, \tilde{B}_1) \wedge (X_2, \tilde{B}_2) = (X_1 \cap X_2, (\tilde{B}_1 \cup \tilde{B}_2)^{\triangle_\delta \triangledown_\delta}). \tag{6}$$

More specifically, a formal fuzzy context can be considered as a data table which is combined of rows and columns corresponding to objects and attributes. In particular, for any $x \in U$ and $a \in A$, a degree $\tilde{I}(x, a) \in L$ is represented as a level to which object x owns attribute a. That is to say, such data table filled with all truth degrees $\tilde{I}(x, a)$ can be seen as a matrix $M_{\tilde{I}} = (c_{ij})_{n \times m}$ called a fuzzy relation matrix, where $|U| = n$ and $|A| = m$. In addition, some matrix operations will be introduced in the following.

Lemma 2.1. *Let $A = (a_{ij})_{n\times m}$, $B = (b_{ij})_{n\times m}$ and $C = (c_{ij})_{m\times p}$ be fuzzy relation matrices, then the following properties hold:*

1. *$A \le B$ iff $a_{ij} \le b_{ij}$, $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, m$;*
2. *$A \cup B = (a_{ij} \vee b_{ij})_{n\times m}$;*
3. *$A \cap B = (a_{ij} \wedge b_{ij})_{n\times m}$;*
4. *$A \cdot C = (e_{ij})_{n\times p}$, where $e_{ij} = \bigvee_{1\le k\le m}(a_{ik} \wedge c_{kj})$;*
5. *$A - B = (a_{ij} \wedge (1 - b_{ij}))_{n\times m}$;*
6. *$\sim A = (1 - a_{ij})_{n\times m}$.*

in which $\vee$ and $\wedge$ are the supremum and the infimum, respectively. Meanwhile, we denote the transposition of a matrix A as A^T.

3. Matrix representation of crisp-fuzzy variable threshold concept lattices

Based on the matrix operations, we will discuss the matrix representation of crisp-fuzzy variable threshold concept lattices in this section. Furthermore, the relationship between derivation operators and the fuzzy sub-contexts is investigated.

Let $U = \{x_1, x_2, \cdots, x_n\}$ and $X \subseteq U$. Denote the characteristic function of X as $\lambda(X)$, where

$$\lambda_X(x_i) \begin{cases} 1, & x_i \in X, \\ 0, & x_i \notin X. \end{cases}$$

Then $\lambda(X) = (\lambda_X(x_1), \lambda_X(x_2), \cdots, \lambda_X(x_n))$.

To facilitate our discussion, let $\otimes$ and $\to$ be the Gödel adjoint operator, i.e. $a \otimes b = min(a, b)$, and $a \to b = 1$ if $a \le b$ and $= b$ else.

Definition 3.1. *Let $A = (a_{ij})_{n\times m}$, $B = (b_{ij})_{m\times p}$ and $C = (c_{ij})_{n\times m}$ be three matrices. Two matrix operators $\odot$ and $\bullet$ are defined as follows:*

1. *$A \odot B = (d_{ij})_{n\times p}$, where $d_{ij} = \bigwedge_{1\le k\le m}(a_{ik} \odot b_{kj})$, and*

$$a_{ik} \odot b_{kj} = \begin{cases} b_{kj}, & a_{ik} > b_{kj}, \\ 1, & a_{ik} \le b_{kj}. \end{cases}$$

2. *$A \bullet C = (e_{ij})_{n\times m}$, where $e_{ij} = 1$ if $a_{ij} \ge c_{ij}$, and $e_{ij} = 0$ else.*

Property 3.1. *Let $A = (a_{ij})_{n\times m}$, $B = (b_{ij})_{n\times m}$, $C = (c_{ij})_{p\times n}$, $D = (d_{ij})_{n\times m}$, $E = (e_{ij})_{m\times q}$, and $F = (f_{ij})_{m\times q}$ are fuzzy matrices. Then the following properties hold:*

1. *If* $A \leq B$, *then* $C \odot A \leq C \odot B$.
2. *If* $A \leq B$, *then* $B \odot E \leq A \odot E$.
3. *If* $A \leq B$,*then* $A \bullet D \leq B \bullet D$.
4. *If* $A \leq B$ *and* $E \leq F$, *then* $C \odot B \odot E \leq C \odot A \odot F$.

Proof. The proof of (1)-(3) is obvious by applying Definition 3.1. Hence, the following item (4) will be illustrated.

(4) Since $A \leq B$, we can have that $C \odot B \odot E \leq C \odot A \odot E$ by item (1) and (2). Then, as $E \leq F$, we get $C \odot A \odot E \leq C \odot A \odot F$. Thus, $C \odot B \odot E \leq C \odot A \odot F$. □

The following theorem shows that the intensions of object set can be represented by matrix operations.

Theorem 3.1. *Let* $K = (U, A, \tilde{I})$ *be a formal fuzzy context and* $0 < \delta \leq 1$. $M_{\tilde{I}}$ *is a fuzzy relation matrix and* $X \in \mathcal{P}(U)$. *Then*

$$X^{\nabla_\delta} = \delta \odot (\sim (\lambda_X \cdot (\sim M_{\tilde{I}}))). \tag{7}$$

Proof. Assume that $\lambda(X) = (\lambda_X(x_1), \lambda_X(x_2), \cdots, \lambda_X(x_n))$. Then $(\alpha_i)_{1\times m} = \sim (\lambda_X \cdot (\sim M_{\tilde{I}}))$, we have that $\alpha_i = 1 - \bigvee_{k=1}^{n} (\lambda_X(x_k) \wedge (1 - c_{ki})) = 1 - \bigvee_{\lambda_X(x_k)=1} (1 - c_{ki}) = 1 - \bigvee_{x\in X} (1 - \tilde{I}(x, a_i)) = \bigwedge_{x\in X} \tilde{I}(x, a_i)$.

Hence, $X^{\nabla_\delta}(a_i) = \delta \to \bigwedge_{x\in X} \tilde{I}(x, a_i)$
$= \delta \odot (\sim (\lambda_X \cdot (\sim M_{\tilde{I}})))$. □

By the following theorem, one can also obtain the extensions of attribute set in a formal fuzzy context. For simplicity, we denote a threshold vector with the size $1 \times n$ as λ_δ. For example, $\delta = 0.6$, then $\lambda_\delta = (0.6, 0.6, \cdots, 0.6)_{1\times n}$.

Theorem 3.2. *Let* $K = (U, A, \tilde{I})$ *be a formal fuzzy context and* $0 < \delta \leq 1$. $M_{\tilde{I}}$ *is a fuzzy relation matrix and* $\tilde{B} \in L^A$. *Then*

$$\lambda(\tilde{B}^{\triangle_\delta}) = (\tilde{B} \odot M_{\tilde{I}}^T) \bullet \lambda_\delta. \tag{8}$$

Proof. It follows from Definition 2.2 that the proof is immediate. □

We denote the following matrix to show the granular matrix of objects.

Definition 3.2. *Let* $K = (U, A, \tilde{I})$ *be a formal fuzzy context and* $0 < \delta \leq 1$. *Each row of* M *corresponds to* $x^{\nabla_\delta \triangle_\delta}$ *of* $x \in U$, *that is to say,* $M(i, :)$ *presents that* $\lambda(x_i^{\nabla_\delta \triangle_\delta})$ *for* $i \leq n$. *Then* $M = (m_{ij})_{n\times n}$ *is called an object granular matrix.*

For convenience of the following description, we write M instead of M_A, and define E_δ and M_δ the diagonal threshold matrix with the size $n \times n$, and the all threshold matrix with the size $n \times n$, respectively. For example, $\delta = 0.6$, $n = 3$, then E_δ and M_δ are obtained:

$$E_\delta = \begin{bmatrix} 0.6 & 0 & 0 \\ 0 & 0.6 & 0 \\ 0 & 0 & 0.6 \end{bmatrix}, M_\delta = \begin{bmatrix} 0.6 & 0.6 & 0.6 \\ 0.6 & 0.6 & 0.6 \\ 0.6 & 0.6 & 0.6 \end{bmatrix}.$$

Theorem 3.3. *Let $K = (U, A, \tilde{I})$ be a formal fuzzy context, $0 < \delta \leq 1$ and $M_{\tilde{I}}$ is a fuzzy relation matrix of K. Then $M = E_\delta \odot M_{\tilde{I}} \odot M_{\tilde{I}}^T \bullet M_\delta$.*

Proof. It follows easily from Eq. (7) and (8). □

To make a clear comprehension on the structure of formal fuzzy contexts. An example in literatures [27, 34] is used to illustrate our discussion.

Example 3.1. Table 1 narrates a formal fuzzy context $K = (U, A, \tilde{I})$ with $U = \{x_1, x_2, x_3, x_4\}$, $A = \{a_1, a_2, a_3, a_4, a_5\}$, and a fuzzy relation is denoted in Table 1. Suppose that $X = \{x_1, x_3, x_4\}$ and $\delta = 0.8$, according to Eq.(7) and (8), we can have:

Table 1: Formal fuzzy context $(U, A, \tilde{I})$

$\tilde{I}$	a_1	a_2	a_3	a_4	a_5
x_1	0.5	0.7	0.7	0.6	0.5
x_2	0.6	0.7	0.9	0.7	1.0
x_3	1.0	0.9	1.0	0.1	0.9
x_4	1.0	0.9	0.9	0.1	0.9

$$X^{\nabla_\delta} = (0.8) \odot (\sim ((1,0,1,1) \cdot \begin{bmatrix} 0.5 & 0.3 & 0.3 & 0.4 & 0.5 \\ 0.4 & 0.3 & 0.1 & 0.3 & 0 \\ 0 & 0.1 & 0 & 0.9 & 0.1 \\ 0 & 0.1 & 0.1 & 0.9 & 0.1 \end{bmatrix}))$$
$$= (0.5, 0.7, 0.7, 0.1, 0.5).$$

$$\lambda(X^{\nabla_\delta \triangle_\delta}) = (0.5, 0.7, 0.7, 0.1, 0.5) \odot \begin{bmatrix} 0.5 & 0.6 & 1.0 & 1.0 \\ 0.7 & 0.7 & 0.9 & 0.9 \\ 0.7 & 0.9 & 1.0 & 0.9 \\ 0.6 & 0.7 & 0.1 & 0.1 \\ 0.5 & 1.0 & 0.9 & 0.9 \end{bmatrix} \bullet (0.8, 0.8, 0.8, 0.8)$$
$$= (1, 1, 1, 1).$$

Then by Theorem 3.3, we can calculate that:

$$M = E_\delta \odot M_{\tilde{I}} \odot M_{\tilde{I}}^T \bullet M_\delta = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

4. Fuzzy sub-contexts and join-irreducible elements

To refer to literatures [32, 34], Zhang et al. and Shao et al. proposed the concept of variable threshold concept lattices and fuzzy sub-contexts based on attribute subsets, respectively. In this section, the relationship between the derivation operators of a formal fuzzy context and that of its fuzzy sub-contexts is researched based on matrix theory.

Definition 4.1 ([34]). *Let $K = (U, A, \tilde{I})$ be a formal fuzzy context, $0 < \delta \leq 1$ and $C \subseteq A$. Here is a formal fuzzy context $K_C = (U, C, \tilde{I_C})$, called a fuzzy sub-context of K, where $\tilde{I}_C = \tilde{I} \cap (U \times C)$. For $X \in \mathcal{P}(U)$ and $\tilde{B} \in L^C$, the operators $\bigtriangledown_\delta^C : \mathcal{P}(U) \to L^C$ and $\bigtriangleup_\delta^C : L^C \to \mathcal{P}(U)$ are given by:*

$$X^{\bigtriangledown_\delta^C}(a) = \delta \to \bigwedge_{x \in X} \tilde{I}(x, a), a \in C; \tag{9}$$

$$\tilde{B}^{\bigtriangleup_\delta^C} = \{x \in U : \bigwedge_{a \in C} (\tilde{B}(a) \to \tilde{I}(x, a)) \geq \delta\}. \tag{10}$$

In Eq.(9) and (10), $X^{\bigtriangledown_\delta^C}$ is the restriction of $X^{\bigtriangledown\delta}$ on the set C, and $\tilde{B}^{\bigtriangleup_\delta^C}$ is the restriction of $\tilde{B}^{\bigtriangleup_\delta}$ on the set C, i.e. $X^{\bigtriangledown_\delta^C}(a) = (X^{\bigtriangledown\delta}|_C)(a)$ and $\tilde{B}^{\bigtriangleup_\delta^C}(x) = (\tilde{B}^{\bigtriangleup_\delta}|_C)(x)$. For brevity, we write $X^{\bigtriangledown\delta}$ the intension of X and $\tilde{B}^{\bigtriangleup_\delta}$ the extension of $\tilde{B}$ instead of $X^{\bigtriangledown_\delta^A}$ and $\tilde{B}^{\bigtriangleup_\delta^A}$, respectively.

Let $K_C = (U, C, \tilde{I}_C)$ be a fuzzy sub-context of the formal fuzzy context K. For $C \subseteq A$, its characteristic function is deemed as $\lambda(C)$. Obviously, $M_{\lambda(C)}$ is called a copy matrix with the size $n \times m$ where each row of $M_{\lambda(C)}$ is characteristic function $\lambda(C)$. According to Example 3.1, suppose $C = \{a_1, a_3, a_5\}$, then we have:

$$M_{\lambda(C)} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

Then by Theorem 3.3, we can also have an object granular matrix $M_C = E_\delta \odot (M_{\tilde{I}} \wedge M_{\lambda(C)}) \odot (M_{\tilde{I}} \wedge M_{\lambda(C)})^T \bullet M_\delta$ in K_C. Based on the above definition, the following property is discussed.

Property 4.1. *Let $K = (U, A, \tilde{I})$ be a formal fuzzy context, $0 < \delta \leq 1$, $C \subseteq A$, and $D = A - C$. For any $X \subseteq U$, then*

1. $X^{\bigtriangledown_\delta^C} = X^{\bigtriangledown\delta} \cap \lambda(C)$;
2. $M \leq M_C$;
3. $X^{\bigtriangledown\delta} = X^{\bigtriangledown_\delta^C} \cup X^{\bigtriangledown_\delta^D}$;

4. $M = M_C \cap M_D$.

Proof. 1. It follows immediately from Eq. (3) and (9).

2. Note that $M_{\tilde{I}} \wedge M_{\lambda(C)} \leq M_{\tilde{I}}$, then by Property 3.1.(1)-(2), we have $E_\delta \odot (M_{\tilde{I}} \wedge M_{\lambda(C)}) \leq E_\delta \odot M_{\tilde{I}}$, and then $E_\delta \odot M_{\tilde{I}} \odot M_{\tilde{I}}^T \leq E_\delta \odot (M_{\tilde{I}} \wedge M_{\lambda(C)}) \odot M_{\tilde{I}}^T$. Since, $E_\delta \odot (M_{\tilde{I}} \wedge M_{\lambda(C)}) \odot (M_{\tilde{I}} \wedge M_{\lambda(C)})^T \leq E_\delta \odot (M_{\tilde{I}} \wedge M_{\lambda(C)}) \odot M_{\tilde{I}}^T$. Assume that $E_\delta \odot (M_{\tilde{I}} \wedge M_{\lambda(C)}) \odot (M_{\tilde{I}} \wedge M_{\lambda(C)})^T = (a_{ij})_{n\times n} < E_\delta \odot (M_{\tilde{I}} \wedge M_{\lambda(C)}) \odot M_{\tilde{I}}^T = (b_{ij})_{n\times n}$. Further, suppose that $E_\delta \odot (M_{\tilde{I}} \wedge M_{\lambda(C)}) = (e_{ij})_{n\times m}$, $(M_{\tilde{I}} \wedge M_{\lambda(C)})^T = (c_{ij})_{m\times n}$, $M_{\tilde{I}}^T = (m_{ij})_{m\times n}$. From Definition 3.1, if $a_{ij} = \wedge_{1\leq k\leq m}(e_{ik} \odot c_{kj}) = 1$, then we can have $b_{ij} = \wedge_{1\leq k\leq m}(e_{ik} \odot m_{kj}) = 1$ which is a contradiction to the assumption and then $E_\delta \odot (M_{\tilde{I}} \wedge M_{\lambda(C)}) \odot (M_{\tilde{I}} \wedge M_{\lambda(C)})^T < E_\delta \odot (M_{\tilde{I}} \wedge M_{\lambda(C)}) \odot M_{\tilde{I}}^T$ does not hold. Therefore, we can obtain that $E_\delta \odot (M_{\tilde{I}} \wedge M_{\lambda(C)}) \odot (M_{\tilde{I}} \wedge M_{\lambda(C)})^T = E_\delta \odot (M_{\tilde{I}} \wedge M_{\lambda(C)}) \odot M_{\tilde{I}}^T$. Thus, an induction complete the proof that $E_\delta \odot M_{\tilde{I}} \odot M_{\tilde{I}}^T \leq E_\delta \odot (M_{\tilde{I}} \wedge M_{\lambda(C)}) \odot (M_{\tilde{I}} \wedge M_{\lambda(C)})^T$, i.e. $M \leq M_C$.

3. From item (1), we obtain $X^{\nabla_\delta^C} \cup X^{\nabla_\delta^D} = (X^{\nabla_\delta} \cap \lambda(C)) \cup (X^{\nabla_\delta} \cap \lambda(D)) = X^{\nabla_\delta} \cap (\lambda(C) \cup \lambda(D)) = X^{\nabla_\delta}$.

4. Follows readily by applying Property 2.1(4) and item (3).

□

On this basis, the following is an example to confirm the fuzzy sub-context and Property 4.1.

Example 4.1. Continued from Example 3.1, let $K_C = (U, C, \tilde{I}_C)$ be a fuzzy sub-context of the formal fuzzy context K, where $C = \{a_1, a_3, a_5\}$, and a fuzzy relation $\tilde{I}_C$ is depicted in Table 2. Then according to Definition 4.1, we can

Table 2: Formal fuzzy context $K = (U, C, \tilde{I}_C)$

$\tilde{I}$	a_1	a_3	a_5
x_1	0.5	0.7	0.5
x_2	0.6	0.9	1.0
x_3	1.0	1.0	0.9
x_4	1.0	0.9	0.9

compute that

$$M_C = E_\delta \odot (M_{\tilde{I}} \wedge M_{\lambda(C)}) \odot (M_{\tilde{I}} \wedge M_{\lambda(C)})^T \bullet M_\delta = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Consequently, we have $M \leq M_C$, which is in accordance with Property 4.1.(2).

Lemma 4.1. *Let $K = (U, A, \tilde{I})$ be a formal fuzzy context, $0 < \delta \leq 1$, $C \subseteq A$, $X \subseteq U$, and $X_i \subseteq U$, $i = 1, 2, \cdots, k$. If $X^{\nabla_\delta} = \bigcap_{i=1}^k X_i^{\nabla_\delta}$, then $X^{\nabla_\delta^C} = \bigcap_{i=1}^k X_i^{\nabla_\delta^C}$.*

Proof.

$$X^{\nabla_\delta^C} = X^{\nabla_\delta} \cap \lambda(C) = (\bigcap_{i=1}^{k} X_i^{\nabla_\delta}) \cap \lambda(C)$$
$$= \bigcap_{i=1}^{k} (X_i^{\nabla_\delta} \cap \lambda(C)) = \bigcap_{i=1}^{k} X_i^{\nabla_\delta^C}.$$

□

Corollary 4.1. *Let $K = (U, A, \tilde{I})$ be a formal fuzzy context, $0 < \delta \leq 1$, $C \subseteq A$, $x \in U$, and $x_i \in U$, $i = 1, 2, \cdots, k$. If $x^{\nabla_\delta} = \bigcap_{i=1}^{k} x_i^{\nabla_\delta}$, then $x^{\nabla_\delta^C} = \bigcap_{i=1}^{k} x_i^{\nabla_\delta^C}$.*

Lemma 4.2. *Let $K = (U, A, \tilde{I})$ be a formal fuzzy context, $0 < \delta \leq 1$, $C \subseteq A$, $X \subseteq U$, and $X_i \subseteq U$, $i = 1, 2, \cdots, k$. If*

$$(X^{\nabla_\delta \triangle_\delta}, X^{\nabla_\delta}) = \bigvee_{i=1}^{k} (X_i^{\nabla_\delta \triangle_\delta}, X_i^{\nabla_\delta}), \tag{11}$$

then

$$(X^{\nabla_\delta^C \triangle_\delta^C}, X^{\nabla_\delta^C}) = \bigvee_{i=1}^{k} (X_i^{\nabla_\delta^C \triangle_\delta^C}, X_i^{\nabla_\delta^C}). \tag{12}$$

Proof. From Eq.(11), we know $X^{\nabla_\delta} = \bigcap_{i=1}^{k} X_i^{\nabla_\delta}$. Then by Lemma 4.1, $X^{\nabla_\delta^C} = \bigcap_{i=1}^{k} X_i^{\nabla_\delta^C}$. Additionally, we have $(\bigcup_{i=1}^{k} X_i^{\nabla_\delta^C \triangle_\delta^C})^{\nabla_\delta^C \triangle_\delta^C} = (\bigcap_{i=1}^{k} X_i^{\nabla_\delta^C \triangle_\delta^C \nabla_\delta^C})^{\triangle_\delta^C}$ $= (\bigcap_{i=1}^{k} X_i^{\nabla_\delta^C})^{\triangle_\delta^C} = X^{\nabla_\delta^C \triangle_\delta^C}$.

□

According to Lemma 4.2, we can know that if a crisp-fuzzy variable threshold concept stemmed from an object set is a join of some crisp-fuzzy variable threshold concepts originated from some finite object sets on U. Therefore, then in any fuzzy sub-context, the corresponding crisp-fuzzy variable threshold concept stemmed from the identical object set can also be expressed as a join of crisp-fuzzy variable threshold concepts originated from these object sets on U, too.

Corollary 4.2. *Let $K = (U, A, \tilde{I})$ be a formal fuzzy context, $0 < \delta \leq 1$, $C \subseteq A$, $x \in U$, and $x_i \in U$, $i = 1, 2, \cdots, k$. If*

$$(x^{\nabla_\delta \triangle_\delta}, x^{\nabla_\delta}) = \bigvee_{i=1}^{k} (x_i^{\nabla_\delta \triangle_\delta}, x_i^{\nabla_\delta}), \tag{13}$$

then

$$(x^{\nabla_\delta^C \triangle_\delta^C}, x^{\nabla_\delta^C}) = \bigvee_{i=1}^{k} (x_i^{\nabla_\delta^C \triangle_\delta^C}, x_i^{\nabla_\delta^C}). \tag{14}$$

Note that any crisp-fuzzy variable threshold concept $(X, \tilde{B})$ can be shown as a join of object concepts of its extension, i.e. $(X, \tilde{B}) = \bigvee_{x \in X}(x^{\nabla_\delta \triangle_\delta}, x^{\nabla_\delta})$.

Shao et al. [27] searched the join-irreducible elements in a formal fuzzy context, i.e. crisp-fuzzy variable threshold concept lattice with threshold $\delta = 1$. In what follows, we will investigate granules and order relations based on objects in crisp-fuzzy variable threshold concept lattices with threshold varied from 0 to 1, and discuss the properties of join-irreducible elements.

Definition 4.2 ([12]). *Let L be a finite lattice and $v \in L$. An element v_* is called a join-irreducible in the case of $v \neq v_*$ if $v_* = \bigvee\{x \in L : x < v\}$.*

Lemma 4.3 ([12]). *Let L be a finite lattice. Then every element in L is composed of a join of some join-irreducible elements.*

Let $K = (U, A, \tilde{I})$ be a formal fuzzy context, $0 < \delta \leq 1$, for any $C \subseteq A$, a binary relation R_C is denoted as follows:

$$R_C = \{(x, y) \in U \times U : x^{\nabla_\delta}(a) \leq y^{\nabla_\delta}(a), \forall a \in C\}. \tag{15}$$

R_C is interpreted as an order relation based on the object set, in which $(x, y) \in R_C$ represents that y is not less than x about attributes C. That is to say, Eq.(15) is evident that $R_C = \bigcap_{a \in C} R_a$. For brevity, if $C = \{a\}$, we will write R_a instead of $R_{\{a\}}$.

For any $x \in U$, we denote the object set whose attribute value includes the intension of object x about all attributes in C as $[x]_{R_C}$: that is, $[x]_{R_C} = \{y \in U : (x, y) \in R_C\}$. Then by Eq.(15), $[x]_{R_C}$ can also be represented as

$$[x]_{R_C} = \{y \in U : x^{\nabla_\delta}(a) \leq y^{\nabla_\delta}(a), \forall a \in C\}. \tag{16}$$

Lemma 4.4. *Let $K = (U, A, \tilde{I})$ be a formal fuzzy context, $0 < \delta \leq 1$, $C \subseteq A$. For any $x \in U$, we can obtain: $[x]^{\nabla_\delta}_{R_C}(a) = x^{\nabla^C_\delta}(a)$.*

Proof. By Eq.(2) and (9), we have

$$[x]^{\nabla_\delta}_{R_C}(a) = \delta \to \bigwedge_{y \in [x]_{R_C}} \tilde{I}(y, a) = \bigwedge_{y \in [x]_{R_C}} (\delta \to \tilde{I}(y, a)) = \bigwedge_{y \in [x]_{R_C}} y^{\nabla_\delta}(a) = x^{\nabla^C_\delta}(a).$$

□

Theorem 4.1. *Let $K_C = (U, C, \tilde{I}_C)$ be a fuzzy sub-context of the formal fuzzy context K, $0 < \delta \leq 1$, and $x \in U$. Afterwards, $([x]_{R_C}, x^{\nabla^C_\delta})$ is a crisp-fuzzy variable threshold concept of K_C and $[x]_{R_C} = x^{\nabla^C_\delta \triangle^C_\delta}$.*

Proof. By Lemma 4.4, we have $[x]^{\nabla_\delta \triangle_\delta}_{R_C} = x^{\nabla^C_\delta \triangle^C_\delta}$. On the one hand, according to Property 2.1(2), we obtain $[x]_{R_C} \subseteq [x]^{\nabla_\delta \triangle_\delta}_{R_C} = x^{\nabla^C_\delta \triangle^C_\delta}$.

On the other hand, for any $y \in x^{\nabla^C_\delta \triangle^C_\delta}$, we can easily obtain $\{y\} \subseteq x^{\nabla^C_\delta \triangle^C_\delta}$. By Property 2.1(1) and (3), we can draw a conclusion that $x^{\nabla^C_\delta} \subseteq y^{\nabla^C_\delta}$. Thus,

$y \in [x]_{R_C}$, and from which $x^{\nabla_\delta^C \triangle_\delta^C} = [x]_{R_C}^{\nabla_\delta \triangle_\delta} \subseteq [x]_{R_C}$. Obviously, $x^{\nabla_\delta^C \triangle_\delta^C} = [x]_{R_C}^{\nabla_\delta \triangle_\delta} = [x]_{R_C}$. Then $([x]_{R_C}, x^{\nabla_\delta^C})$ is a crisp-fuzzy variable threshold concept. □

Notice that $[x]_{R_C} = x^{\nabla_\delta^C \triangle_\delta^C}$ in fuzzy sub-context K_C of K. To avoid confusion, we use m_{ij}^C instead of m_{ij} in the object granular matrix M_C for $x_i, x_j \in U$. By Definition 4.1, we can see that $m_{ij}^C = 1 \Leftrightarrow x_i^{\nabla_\delta} \subseteq x_j^{\nabla_\delta}, \forall a \in C$. That is to say, Eq.(15) is equivalent that

$$R_C = \{(x_i, x_j) \in U \times U : m_{ij}^C = 1\}. \tag{17}$$

Specifically, we write m_{ij} instead of $m_{ij}^A = 1$ in M. Moreover, it is easily obtained that

$$[x_i]_{R_C} = \{x_j \in U : m_{ij}^C = 1\}. \tag{18}$$

Theorem 4.2. *Let $K = (U, A, \tilde{I})$ be a formal fuzzy context, $0 < \delta \leq 1$, $\forall C \subseteq A$, and $x \in U$. If $(x^{\nabla_\delta^C \triangle_\delta^C}, x^{\nabla_\delta^C})$ is a join-irreducible element in $L_\delta(U, \tilde{C}, \tilde{I}_C)$, then $(x^{\nabla_\delta \triangle_\delta}, x^{\nabla_\delta})$ is also a join-irreducible element in $L_\delta(U, \tilde{A}, \tilde{I})$.*

Proof. Suppose that $(x^{\nabla_\delta \triangle_\delta}, x^{\nabla_\delta})$ is not a join-irreducible element in $L_\delta(U, \tilde{A}, \tilde{I})$. By Lemma 4.3, $(x^{\nabla_\delta \triangle_\delta}, x^{\nabla_\delta})$ is a join of some join-irreducible elements in $L_\delta(U, \tilde{A}, \tilde{I})$. That is, there exists $x_i \in U$, $i = 1, 2, \cdots, k$ $(k \geq 2)$ such that $(x^{\nabla_\delta \triangle_\delta}, x^{\nabla_\delta}) = \bigvee_{i=1}^k (x_i^{\nabla_\delta \triangle_\delta}, x_i^{\nabla_\delta})$. Then according to Corollary 4.2, we know that $(x^{\nabla_\delta^C \triangle_\delta^C}, x^{\nabla_\delta^C}) = \bigvee_{i=1}^k (x_i^{\nabla_\delta^C \triangle_\delta^C}, x_i^{\nabla_\delta^C})$. Hence, $(x^{\nabla_\delta^C \triangle_\delta^C}, x^{\nabla_\delta^C})$ is not a join-irreducible element in $L_\delta(U, \tilde{C}, \tilde{I}_C)$, which comes a contradiction. Therefore, we assert that $(x^{\nabla_\delta \triangle_\delta}, x^{\nabla_\delta})$ is a join-irreducible element in $L_\delta(U, \tilde{A}, \tilde{I})$. □

5. Granular reduction based on matrix operations in formal fuzzy contexts

Based on the literature [27], Shao et al. investigated granular reducts based on granular computing in formal fuzzy contexts. In this section, we generalize Shao et al. and Lin et al.'s work [27, 39] to the crisp-fuzzy variable threshold concept lattices and formulate the granular reduction via matrix operations. Thus intuitively speaking, we construct the discernibility matrix and the discernibility function which are employed to calculate attribute reduction.

Definition 5.1. *Let $K = (U, A, \tilde{I})$ be a formal fuzzy context, $0 < \delta \leq 1$, and $\forall C \subseteq A$. The attribute set C is called a granular consistent set of K if $M = M_C$. And further, C is referred to as a granular reduct of K if $M \neq M_B$ for any $B \subset C$.*

Note that M_C is a matrix of extensions of object concepts in the fuzzy sub-context $L_\delta(U, C, \tilde{I}_C)$. According to Definition 5.1, a granular reduct is a minimal attribute subset that preserves the object granules of variable threshold concept lattice unchanged.

Example 5.1. Continued from Example 3.1. Suppose that $C = \{a_1, a_3, a_4\}$, it can easily be concluded that

$$M_C = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

i.e. $M = M_C$, hence, C is a granular consistent set of K.

Property 5.1. *Let $K = (U, A, \tilde{I})$ be a formal fuzzy context, $0 < \delta \leq 1$, and $\forall C \subseteq A$. C is a granular consistent set iff $M_C \leq M$.*

Proof. It follows immediately from Property 4.1(2) and Definition 5.1. □

Property 5.2. *Let $K = (U, A, \tilde{I})$ be a formal fuzzy context, $0 < \delta \leq 1$, $\forall C \subseteq A$, and $D = A - C$. C is a granular consistent set iff $M_C \leq M_D$.*

Proof. It follows immediately from Property 4.1(4) and Definition 5.1. □

All granular reducts of K are denoted as $Red(K)$, in which the attribute set A is classified into three categories based on the significance of the attributes:

- Indispensable (core) attribute set C_K: $C_K = \bigcap Red(K)$;
- Relative necessary attribute set K_K: $K_K = \bigcup Red(K) - \bigcap Red(K)$;
- Dispensable (unnecessary) attribute set I_K: $I_K = A - \bigcup Red(K)$.

In what follows, judgment of indispensable attribute set is discussed based on matrix operations with the purpose of deleting the redundant attributes in a crisp-fuzzy variable threshold concept lattice.

Corollary 5.1. *Let $K = (U, A, \tilde{I})$ be a formal fuzzy context, $0 < \delta \leq 1$. An attribute $a \in A$ is an indispensable attribute if and only if $M \neq M_{A-\{a\}}$.*

Proof. It is easy to prove from Definition 5.1 and the concept of indispensable attribute. □

Example 5.2. Continued from Example 3.1, we can compute that

$$M_{A-\{a_4\}} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Consequently, a_4 is an indispensable attribute because of $M \neq M_{A-\{a_4\}}$.

Definition 5.2. *Let $A = (a_{ij})_{n\times m}$ and $B = (b_{ij})_{n\times m}$ be matrices. The operation is denoted as follows: $A \ominus B = (c_{ij})_{n\times m}$, where*

$$c_{ij} = \begin{cases} 1, & a_{ij} > b_{ij}, \\ 0, & a_{ij} \leq b_{ij}. \end{cases}$$

Hereinafter, for a formal fuzzy context $K = (U, A, \tilde{I})$ with threshold δ, we know that a granular reduct is in fact a minimal granular consistent set of K. In order to build a new framework of attribute reduction, we denote a discernibility attribute matrix similar to [14], which can be given by:

Definition 5.3. *Let $K = (U, A, \tilde{I})$ be a formal fuzzy context, $0 < \delta \leq 1$, and $M = (m_{ij})_{n\times n}$ is an object granular matrix. For any $(x_i, x_j) \in U \times U$, if $m_{ij} = 0$, denote*

$$\lambda(\mathcal{D}(x_i, x_j)) = \lambda(x_i^{\nabla\delta}) \ominus \lambda(x_j^{\nabla\delta}).$$

$\lambda(\mathcal{D}(x_i, x_j))$ is interpreted as the discernibility attribute vector of x_i and x_j with threshold δ in K, and $\boldsymbol{D^*} = (\lambda(\mathcal{D}(x_i, x_j)) : (x_i, x_j) \in U \times U)$ is defined as the discernibility attribute matrix of K with threshold δ.

In the above definition, number 1 of discernibility attribute vector $\lambda(\mathcal{D}(x_i, x_j))$ represents that the ordered pair (x_i, x_j) can be distinguished when $m_{ij} = 0$. More precisely, the more the total numbers of 1 in $\boldsymbol{D^*}$ are, the stronger discernibility ability is.

Theorem 5.1. *Let $K = (U, A, \tilde{I})$ be a formal fuzzy context, $0 < \delta \leq 1$, and $D \subseteq A$. D is a granular consistent set iff $\lambda(D) \cap \boldsymbol{D^*}(k, :) \neq (0, 0, \cdots, 0)_{1\times m}$ for any $\lambda(\mathcal{D}(x_i, x_j)) \neq (0, 0, \cdots, 0)_{1\times m}$, that is, $k \leq q$ (q means the number of rows of $\boldsymbol{D^*}$).*

Proof. ($\Rightarrow$) If D is a granular consistent set, then by Property 5.1, we know that $M_D \leq M$. From Theorem 4.1, we deduce that $[x_i]_{R_D} \subseteq [x_i]_{R_A}, \forall x_i \in U$. For any $\lambda(\mathcal{D}(x_i, x_j)) \in \boldsymbol{D^*}$ with $\lambda(\mathcal{D}(x_i, x_j)) \neq (0, 0, \cdots, 0)_{1\times m}$, then there exists $a \in A$ such that $m_{ij} = 0$. Thus, $x_j \notin [x_i]_{R_A}$, that is to say, $x_j \notin [x_i]_{R_D}$, which represents that $m_{ij}^D = 0$. By Eq.(17), there exists $a \in D$ such that $(x_i, x_j) \notin R_a$, i.e. $x_i^{\nabla\delta}(a) \not\leq x_j^{\nabla\delta}(a)$. Then, that is: $x_i^{\nabla\delta}(a) > x_j^{\nabla\delta}(a)$. From Definition 5.3, we have $\lambda(x_i^{\nabla\delta}(a)) \ominus \lambda(x_j^{\nabla\delta}(a)) = 1$. Therefore, $\lambda(D) \cap \boldsymbol{D^*}(k, :) \neq (0, 0, \cdots, 0)_{1\times m}$.

($\Leftarrow$) Suppose that $\lambda(D) \cap \boldsymbol{D^*}(k, :) \neq (0, 0, \cdots, 0)_{1\times m}$ for any $\lambda(\mathcal{D}(x_i, x_j)) \in \boldsymbol{D^*}$ with $\lambda(\mathcal{D}(x_i, x_j)) \neq (0, 0, \cdots, 0)_{1\times m}$. For any $(x_i, x_j) \in U \times U$, if $x_j \notin [x_i]_{R_A}$, that implies $m_{ij} = 0$. Hence, $\lambda(\mathcal{D}(x_i, x_j)) \neq (0, 0, \cdots, 0)_{1\times m}$. Since $\lambda(D) \cap \boldsymbol{D^*}(k, :) \neq (0, 0, \cdots, 0)_{1\times m}$, there exists $a \in D$ such that $\lambda(x_i^{\nabla\delta}(a)) \ominus \lambda(x_j^{\nabla\delta}(a)) = 1$, i.e. $m_{ij}^D = 0$, and this means that $x_j \notin [x_i]_{R_D}$. Thus, we can obtain that $[x_i]_{R_D} \subseteq [x_i]_{R_A}$. From Property 5.1, it follows that $M_D \leq M$. Consequently, D is a granular consistent set. □

Theorem 5.1 shows a method to judge whether an attribute set is consistent or not.

Theorem 5.2. *Let $K = (U, A, \tilde{I})$ be a formal fuzzy context, $0 < \delta \leq 1$, and an attribute $a_k \in A$ is an indispensable attribute iff there exists $(x, y) \in U \times U$, that is, $i \leq q$ (q means the number of rows of $\boldsymbol{D}^*$) such that $|\boldsymbol{D}^*(i,:)| = 1$ and $\boldsymbol{D}^*(i,k) = 1$.*

Proof. ($\Rightarrow$) Since $a_k \in A$ is an indispensable attribute of K with threshold δ, then $A - \{a_k\}$ is not a granular consistent set. From Theorem 5.1, we have $\lambda(A - \{a_k\}) \cap \boldsymbol{D}^*(i,:) = (0, 0, \cdots, 0)_{1\times m}$. As a consequence, $|\boldsymbol{D}^*(i,:)| = 1$ and $\boldsymbol{D}^*(i,k) = 1$ with 1 representing the position of $a_k \in A$.

($\Leftarrow$) If there exists $i \leq q$ such that $|\boldsymbol{D}^*(i,:)| = 1$ and $\boldsymbol{D}^*(i,k) = 1$ with 1 indicating the position of $a_k \in A$. By Definition 5.2, we can know that $x^{\nabla\delta}(a_k) \ominus y^{\nabla\delta}(a_k) = 1$, and $x^{\nabla\delta}(b) \ominus y^{\nabla\delta}(b) = 0$ for all $b \in A - \{a_k\}$. That is, $(x, y) \notin R_{a_k}$, and $(x, y) \in R_{A-\{a_k\}}$. And from which we have $m_{ij} = 0$ and $m_{ij}^{A-\{a\}} = 1$. We deduce that $m_{ij} \neq m_{ij}^{A-\{a\}}$. That is to say, $M \neq M_{A-\{a_k\}}$, i.e. $a_k \in A$ is an indispensable attribute of K with threshold δ. □

According to Theorem 5.2, by $C_K = \{a_i :| \boldsymbol{D}^*(k,:) |= 1 \wedge \boldsymbol{D}^*(k,i) = 1\}$ we denote the set of indispensable attributes which find the row k-th, column i-th of the discernibility attribute matrix $\boldsymbol{D}^*$. Based on the above discussion, we denote $R(a_i) = \{\boldsymbol{D}^*(j.:) : \boldsymbol{D}^*(:,i) = 1\}$.

Algorithm 1 provides a method to calculate a granular reduct in K. In Step 1, initialization is conducted: *Red* is used to store a reduct, and $\boldsymbol{D}^*$ is stored a discernibility attribute matrix. In Step 2, the diagonal threshold matrix E_δ and the all threshold matrix M_δ are given based on δ. Steps 3-17 are to calculate the discernibility attribute matrix and its time complexity is $O(|U||A| + |U|^2)$. Steps 18-21 are to find all indispensable attributes. And further, if present attribute a is a core attribute, the whole row and column where attribute a is located are deleted, respectively. In Steps 22-25, an attribute with the maximal discernibility ability is added to the above *Red* in each repeat until the repeat fulfills the end condition. Then notice that running time is not than $O(|U|^2|A|)$. Therefore, it is easily seen that the time complexity of Algorithm 1 (ARTM) is $max\{O(|U||A| + |U|^2), O(|A||U|^2)\}$.

In what follows, an example is to confirm our discussion with respect to computing a granular reduct as shown in Algorithm 1.

Example 5.3. Continued from Example 3.1. According to Definition 5.3, we can have the discernibility attribute matrix is

$$\boldsymbol{D}^* = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Algorithm 1: A matrix-based algorithm for finding a granular reduct of K with threshold δ based on matrix theory (ARTM)

Require: A formal fuzzy context $K = (U, A, \tilde{I})$ with threshold δ, where $U = \{x_1, x_2, \cdots, x_n\}$ and $A = \{a_1, \cdots, a_m\}$;
Ensure: A granular reduct Red.
1: Let $M_3 \leftarrow \phi$, $\boldsymbol{D^*} \leftarrow \phi$ and $Red \leftarrow \phi$;
2: Generate the diagonal threshold matrix E_δ and the all threshold matrix M_δ according to δ;
3: **for** $i = 1 : |U|$ **do**
4: Generate copy matrix B inducing from an $|A|$-by-1 tilling of copies of $E_\delta(i, :)$;
5: Let $A = B - M_{\tilde{I}}^T$;
6: **if** $A \leq= 0$ **then**
7: $D = min(C, F) + A$;
8: $E = min(D, [], 2)$ which means that E is used to stored the minimum value of each row in D;
9: $M3 = [M3; E^T]$;
10: **end if**
11: **end for**
12: **for** $i = 1 : |U|$ **do**
13: Generate copy matrix C inducing from an $|U|$-by-1 tilling of copies of $M_3(i, :)$;
14: **if** $H = C - M_3 > 0$ **then**
15: $\boldsymbol{D^*} = [\boldsymbol{D^*}; H]$;
16: **end if**
17: **end for**
18: **if** $\mid \boldsymbol{D^*}(i, :) \mid = 1$ and $\boldsymbol{D^*}(i, j) = 1$ **then**
19: $Red \longleftarrow Red \cup \{a_j\}$ and compute $R(a_j)$. Then all rows of $\boldsymbol{D^*}$ that the sum is 0 and 1 are removed;
20: $\boldsymbol{D^*} = \boldsymbol{D^*} - \cup_{a_j \in Red} R(a_j)$ and updated $\boldsymbol{D^*}$ by deleting columns $\boldsymbol{D^*}(:, j)$, $a_j \in Red$;
21: **end if**
22: **while** $\boldsymbol{D^*} \neq \phi$ **do**
23: $Red = Red \cup \{a_t\}$, where a_i satisfies $|\boldsymbol{D^*}(:, t)| = max\{|\boldsymbol{D^*}(:, j)| : a_j \in A - Red\}$;
24: $\boldsymbol{D^*} = \boldsymbol{D^*} - R(t)$ and then delete corresponding columns in $\boldsymbol{D^*}$;
25: **end while**
26: Output Red.

It should be noted that $Red = \{a_4\}$ in Step 18 according to $\boldsymbol{D^*}(1, 4) = 1$ and $|\boldsymbol{D^*}(1, :)| = 1$ for attribute a_4. and $R(a_4)$ is depicted automatically as shown:

$$R(a_4) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

In Step 20, the discernibility attribute matrix is renewed as follows:

$$\boldsymbol{D}^* = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}. \tag{19}$$

It is easy to see that $\boldsymbol{D}^*(:,1) = \boldsymbol{D}^*(:,2) = 2$ is maximum value in Steps 22-25. A corresponding attribute a_1 is selected automatically. In addition, $\boldsymbol{D}^*$ is updated to empty matrix. Therefore, a granular reduct of K with threshold δ is $Red = \{a_1, a_4\}$.

6. Numerical experiments

In this section, to further explain the effectiveness of the proposed algorithm, we compare it with some existing algorithm with respect to attribute reduction in FFCA. The experiments are performed on a personal computer with Windows 10 and Intel (R) Core (TM) i5-6200U CPU @2.3GHz with 8.0 GB of memory. And these algorithms are performed using Matlab 9.3.

6.1 Real-world datasets

The first set of experiments focuses on the real-word data. All data sets are downloaded from UCI machine learning repository [42] depicted in Table 3, in which $|\cdot|$ means the cardinality of sets.

Table 3: Description of data sets

No.	Data sets	$\lvert U\rvert$	$\lvert A\rvert$
1	Wine	178	14
2	Wpbc	198	34
3	Yeast	1484	8
4	Australian	690	15
5	Crx	690	16
6	Derm	366	35
7	ICU	200	21
8	Sonar	208	61
9	Yale	165	1025
10	Iono	351	35
11	Glass	214	10
12	Wdbc	569	31

6.2 Comparison on finding a granular reduct in crisp-fuzzy variable concept lattices

In the following, we compare Algorithm 1 (called ARTM) with other algorithms from two aspects: the number of reducts and the whole running time. For the first aspect, we compare ARTM with the algorithms COAM [6] and HGRM [39]. For another aspect, we compare ARTM with the algorithm [27] (called TSGR). The results are shown in Table 4 and Table 5. For the ease of discussing, the algorithm ARTM is tested based on the threshold value $\delta = 0.8$.

Table 4: The cardinality of reduct

No.	Data sets	ARTM	COAM	HGRM
1	Wine	13	12	14
2	Wpbc	20	22	21
3	Yeast	7	8	8
4	Australian	9	9	14
5	Crx	6	6	13
6	Derm	33	35	32
7	ICU	5	5	19
8	Glass	7	9	8

Table 4 shows the cardinality of reducts under different algorithms. Intuitively speaking, the cardinality of reducts of ARTM is generally less than COAM and HGRM, respectively.

Table 5: Comparision of the running time

	The time of DAM		The time of reduct	
	ARTM	TSGR	ARTM	TSGR
Glass	0.122	41.871	0.127	41.885
Wine	0.197	27.948	0.201	27.956
Wpbc	0.890	109.484	0.896	109.510
Sonar	1.486	245.081	1.595	245.151
Yale	12.811	1714.114	13.154	1714.588
Iono	3.140	1095.685	3.170	1095.743
Wdbc	11.497	7402.319	11.563	7402.483

Table 5 shows that the running time to compute a discernibility attribute matrix (DAM for short) and find a granular reduct in different algorithms. More precisely, we see that the running time of ARTM is less than that of TSGR. Hence, the proposed algorithm ARTM is more efficient based on Table 4 and 5.

7. Conclusions

As we know, attribute reduction from the viewpoint of granular computing is an important research in FFCA. In this paper, we have introduced systematically a matrix-based approach of granular reduction in crisp-fuzzy variable threshold concept lattices which does not change the extensions of all object concepts. At first, some new matrix representations, such as, intensions of object set, extensions of attribute set, and the object granular matrix have been proposed based on the matrix operations. Subsequently, several notions and properties of fuzzy sub-contexts and join-irreducible elements have been illustrated. Then the attribute reduction method in the sense of constructing the discernibility matrix has been presented by adapting to the extensions of object concepts. With respect to the proposed approach, we have demonstrated how to delete the reductant attributes and search for an optimal attribute subset in a crisp-fuzzy variable threshold concept lattice. In further work, the heuristic approach will be discussed to obtain a granular reduct such that the process and time computation can be speeded up. On the other hand, we will apply this theory to fuzzy-crisp variable threshold concept lattice, three-way concept lattices or generalized one-sided concept lattices. Besides, we should also develop new attribute reduct methods under different requirements in variable threshold concept lattices.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (NO:11871259, 61379021, 11701258), and the Natural Science Foundation of Fujian Province in China (2019J01748, 2017J01507).

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Accepted: May 18, 2020

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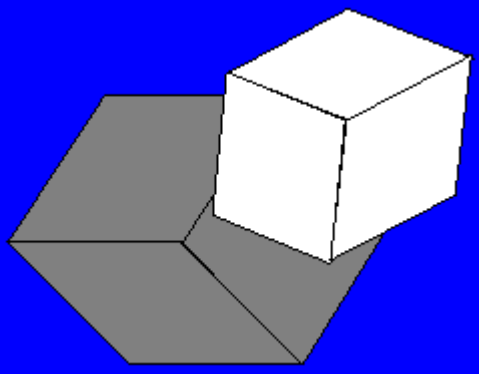
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IJPAM
Italian Journal of Pure and Applied Mathematics
Issue n° 46-2021



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Rivista semestrale: Autorizzazione Tribunale di Udine n. 8/98 del 19.3.98
Direttore responsabile: Piergiulio Corsini
ISSN 2239-0227